# ECON622 Notes: Social and Economic Networks

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## 1 Network games

The term "network games" can have many meanings. Generally it refers to "games played on networks", usually a linear-quadratic game where players choose an action $x_i \geq 0$ and have utility

$$u_i(x_1, \ldots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j$$

The function is strictly concave in own effort ($\frac{\partial^2 u_i}{\partial x_i^2} = \sigma_{ii} < 0$ represents costly effort).

For simplicity, assume a symmetry in payoffs so that $\alpha_i = \alpha$ and $\sigma_{ii} = \sigma$. Let $\Sigma = [\sigma_{ij}]$ be the matrix of cross-effects.

Reparametrize the utility function using $\sigma_{\min}$ and $\sigma_{\max}$, $\gamma = -\min\{\sigma_{\min}, 0\} \geq 0$, and $\beta = -\gamma - \sigma$. For simplicity, assume that $\sigma_{ij} = \sigma_{ji} \in \{\sigma_{\min}, \sigma_{\max}\}$.

$$u_i(x_1, \ldots, x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j$$

Which means that $\Sigma = -\beta I - \gamma U + \lambda G$ where $U = [1]$ and $G$ is a symmetric, binary adjacency matrix.

Finally, define

$$M(g, a) = [I - aG]^{-1} = \sum_{t=0}^{\infty} a^k G^k$$

where g is the graph with adjacency $G$, $0 \leq a \leq 1/\lambda_{\max}(G)$. If $M$ exists, then the **bonacich centrality** of the nodes is given by $b(g, a) = [I - aG]^{-1}\mathbf{1}$

Bonacich centrality summarizes the total number of paths in $g$ that start at $i$, where paths of length $k$ are weighted by a factor $a^k$.

Finally, [1] proves that as long as $[\beta I - \lambda G]$ is invertible, (e.g. $\beta > \lambda\lambda_{\max}(G)$), the unique nash equilibrium of the game above is interior, and given by

$$x_i^* = \frac{\alpha}{\beta + \gamma \sum_j b_j(g, \lambda/\beta)} b_i(g, \lambda/\beta)$$

The same holds above if the $\alpha_i$ are not the same, in which case the centrality measure is $b_\alpha(g, \lambda/\beta) = [I - (\lambda/\beta)G]^{-1}\alpha$.

To see why, note that the first order conditions show us that an interior equilibrium exists at the solution to

$$-\Sigma x = [\beta I + \gamma U - \lambda G]x = \alpha = (\sum_i \alpha_i)\mathbf{1}$$

Where the solution follows from the fact that $Ux = (\sum_i x_i)\mathbf{1}$

This is a very powerful result for a number of reasons. Primarily, it gives a centrality measure $b$ that use information about paths in the network to compute actions in equilibrium. Furthermore, this measure is directly related to the spectrum of $G$.

## 1.1 Example: Naive social learning

One of the most commonly used examples of this type of game is one where all players try to reach consensus (see, e.g. [2]). Such a game could be written with utilities

$$u_i(x_i, x_{-i}) = \frac{1}{2}\sum_{j=1}^{n} g_{ij}(x_i - x_j)^2$$

The first-order condition is

$$\frac{\partial u_i}{\partial x_i} = \sum_{j=1}^{n} g_{ij}(x_i - x_j) = dx_i - [Gx]_i = 0$$

which can be stacked into a single vector equation as

$$Dx - Gx = 0$$

where $D$ is a diagonal degree matrix. Then the best responses are simply

$$x^* = D^{-1}Gx$$

The matrix $D^{-1}G$ is called the **random walk matrix** of the graph. It is row-stochastic. If $G$ is symmetric, then it is similar to a symmetric matrix and can be diagonalized as $P\Lambda P^{-1}$, where $P = D^{-1/2}U$ and $U$ are the eigenvectors of $G$. Therefore, the eigenvector centralities will still play an important role.

Note that the equilibrium here is not unique, since any vector proportional to $\mathbf{1}$ is a Nash equilibrium. However, we are especially interested in how agents arrive at a particular equilibrium, which is a process of learning. If players have knowledge of their neighbors states and the structure of the network, then they should use a Bayesian rule to update beliefs. However, this quickly becomes intractable for large networks. A reasonable "naive" rule, studied by [6, 4] would be to update your state in the next time period as a potentially weighted average of your neighbor's states. Conveniently, this "DeGroot" rule can be written as

$$x_{t+1} = D^{-1}Gx_t$$

which is a linear dynamical system in discrete time. [6, 5] studied the properties of this updating process. Noting that $D^{-1}G$ is not generically symmetric even when $G$ is, and the only time when the eigenvectors of $G$ are equal to the singular vectors of $D^{-1}G$ is when $d_i = d_j$ for all agents (or the graph is regular). Therefore, the naive dynamics can be shown to converge to the true population mean only when the graph is undirected, symmetric, and regular. This emits a random walk matrix that is doubly-stochastic and thus orthogonally diagonalizable.

### 1.2 Network formation games

Network formation is a hard problem to model, because the state space of possible networks is $O(2^{n^2})$. This means it is intractable for even small to medium size networks. Econometrics of networks usually uses MCMC methods to fit Gibbs-type distributions (undirected graphical models) to the data. Methods used in this area are a direct application of the methods we covered in the previous sections of the course. See [7] for a direct application of local Gibbs sampling and [3] for an overview of the econometric problem.

## References


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