1 Introduction to Galois theory assignmens 1

Problem 1.1

Problem 1.2

Problem 1.3 *Which of the following algebras are fields? Products of fields? Describe these fields.*

(a) $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$

(b) $\mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$

(c) $\mathbb{F}_2(\sqrt{T}) \otimes_{\mathbb{F}_2(T)} \mathbb{F}_2(\sqrt{T})$

(d) $\mathbb{F}_4(\sqrt[3]{T}) \otimes_{\mathbb{F}_4(T)} \mathbb{F}_4(\sqrt[3]{T})$

Solution: For (a), it's field, equal to $\mathbb{Q}(\sqrt[3]{2}, \sqrt{2})$.

For (b), it's product of fields, equal to $\mathbb{Q}(\sqrt{2})[X]/(X^2-\sqrt{2})\times\mathbb{Q}(\sqrt{2})[X]/(X^2+\sqrt{2})$. This is from

$$\mathbb{Q}(\sqrt{2})\otimes_{\mathbb{Q}}\mathbb{Q}[X]/(X^4-2) = \mathbb{Q}(\sqrt{2})[X]/(X^4-2) = \mathbb{Q}(\sqrt{2})[X]/(X^2-\sqrt{2})\times\mathbb{Q}(\sqrt{2})[X]/(X^2+\sqrt{2})$$

For (c), it's neither field nor product of field. Actually it's

$$\begin{aligned}
&\mathbb{F}_2(\sqrt{T}) \otimes_{\mathbb{F}_2(T)} \mathbb{F}_2(\sqrt{T}) \\
=&\mathbb{F}_2(\sqrt{T}) \otimes_{\mathbb{F}_2(T)} \mathbb{F}_2(T)[X]/(X^2 - T) \\
=&\mathbb{F}_2(\sqrt{T})[X]/(X^2 - T) \\
=&\mathbb{F}_2(\sqrt{T})[X]/(X - \sqrt{T})^2
\end{aligned}$$

having nilpotents

For (d), it's product of fields, actually

$$\begin{aligned}
&\mathbb{F}_4(\sqrt[3]{T}) \otimes_{\mathbb{F}_4(T)} \mathbb{F}_4(\sqrt[3]{T}) \\
=&\mathbb{F}_4(\sqrt[3]{T}) \otimes_{\mathbb{F}_4(T)} \mathbb{F}_4(T)[X]/(X^3 - T) \\
=&\mathbb{F}_4(\sqrt[3]{T})[X]/(X^3 - T) \\
=&\mathbb{F}_4(\sqrt[3]{T})[X]/((X - \sqrt[3]{T}) \cdot (X^2 + \sqrt[3]{T}X + \sqrt[3]{T^2})) \\
=&\mathbb{F}_4(\sqrt[3]{T})[X]/((X - \sqrt[3]{T}) \times \mathbb{F}_4(\sqrt[3]{T})[X]/(X^2 + \sqrt[3]{T}X + \sqrt[3]{T^2}))
\end{aligned}$$

$\square$