1 Approximation Algorithm II Homework 1

2 Approximation Algorithm II Homework 2

A Primal-dual algorithm for Shortest path Problem
Symbols:

- $G = (V, E)$ connected graph
- $w : E \to \mathbb{R}_+$ cost function on edges
- $\mathcal{S} = \{S | S \subset V \text{ and } s \in S \text{ and } t \notin S\}$, which are cuts cross s, t.
- $\delta(S) = \{e \in E | \, |S \cap e| = 1\}$ which are edges crossing S

Question, find minimum cost path from s to t in G.
Form of LP:

$$
\begin{aligned}
&\min && \sum_{e \in E} x_e \cdot w(e) \\
&\text{subject to} && \\
&\forall S \in \mathcal{S} && \sum_{e : e \in \delta(S)} x_e \geq 1 \\
&\forall e \in E && x_e \geq 0
\end{aligned}
$$

Problem 2.1 *What is the dual of this LP?*

Solution:

$$
\begin{aligned}
&\max && \sum_{S \in \mathcal{S}} y_S \\
&\text{subject to} && \\
&\forall e \in E && \sum_{S : e \in \delta(S)} y_S \leq w(e) \\
&\forall S \in \mathcal{S} && y_S \geq 0
\end{aligned}
$$

□

The primal-dual algorithm

1. $F \leftarrow \emptyset$
2. $y \leftarrow 0$
3. while there is not path connecting s, t in F
   - Let C be the connected component of s in $G' = (V, F)$.

- Increase y_C until some constaint become tight: $\exists e' : \sum_{S: e \in \delta(S)} y_S = w(e')$
- $F \leftarrow F \cup \{e'\}$

4. return a path P in F connecting s, t (Prunning, just choose the path, not whole F)

Correctness

Problem 2.2 *In how many iterations of the while loop can a given dual variable be increased?*

Solution: Once. C is chosen only when there are e' crossing its boundary, after that, it's not connected component any more. □

Problem 2.3 *Using Q2, argue the algorithm terminates to a solution.*

Solution: C is strictly increased every iteration. G is connected and finite, hence must stop for $s, t \in C$, hence a solution. □

Approximation Ratio

Lemma 2.1 *At any step, set F is a tree containing s.*

Proof: Induction by $|C|$. When $|C| = 1$, obviouse. Assume when $|C| = i - 1$, F is a tree containing s, then the newly add edge $e' = (u, v)$ must have one vertex in C and the other vertex in $V - C$ because $e' \in \delta(C)$, then next step $F \cup \{(u, v)\}$ is still a tree. □

Problem 2.4 *Recall a tight lower bound between the value of the optimal fractional solution for the dual $val(y^*)$ and the value of the value of shortest path between s, t, denoted by P^*.*

Solution: Note the fractional optimal x^* of primal equals to fractional dual

$$val(x^*) = val(y^*)$$

and x^* is the relaxation of the integral problem, so,

$$val(y^*) = val(x^*) \leq P^*$$

. □

Problem 2.5 *Argue that the solution y is feasible for the dual.*

Solution: In the while loop, any increasement will not violate the constraints, so the y_S's are remaining feasible. $\square$

Problem 2.6 *Combine Q4 and Q5, recall a tight lower bound between $val(y)$ and P^*.*

Solution: By definition $val(y) \leq val(y^*)$, hence

$$val(y) \leq val(x^*) \leq P^*$$

. $\square$

In the following, we want to show:

$$\sum_{e \in P} w(e) \leq val(y)$$

Problem 2.7 *Consider $e \in P$, what is the relationship between $w(e)$ and $\sum_{S: e \in \delta(S)} y_S$?*

Solution: The algorithm ensures the complementary slackness: e is included only if its contraints becomes tight, hence

$$\sum_{S: e \in \delta(S)} y_S = w(e)$$

$\square$

Problem 2.8 *Using Q7, give the relationship between $\sum_{e \in P} w(e)$ and y_S.*

Solution:

$$\sum_{e \in P} w(e) = \sum_{e \in P} \sum_{S: e \in \delta(S)} y_S$$

$\square$

Problem 2.9 *Using Q8, give relation between $\sum_{e \in P} \sum_{S: e \in \delta(S)} y_S$ and y_S and $|P \cap \delta(S)|$.*

Solution:

$$\begin{aligned}
\sum_{e \in P} \sum_{S: e \in \delta(S)} y_S &= \sum_{S \in \mathcal{S}} \sum_{e \in P} 1_{e \in \delta(S)} \cdot y_S \\
&= \sum_{S \in \mathcal{S}} y_S \cdot |P \cap \delta(S)|
\end{aligned}$$

□

We now want to prove the following:

$$\forall S,\, y_S > 0 \Rightarrow |P \cap \delta(S)| = 1$$

Assume not, then exists S, such that $y_S > 0$ but $|P \cap \delta(S)| \geq 2$

Problem 2.10 *Using Lemma2.1 and $y_S > 0$, explain the contradiction*

Solution: $y_S > 0$ then it has to be some C during the iteration. If it exists, then choose maximal such C that $|P \cap \delta(C)| \geq 2$, then exists $e, e' \in P$, cross C, not matter to assume $e = (u, v), e' = (u', v')$ and $u, u' \in C$ and $v, v' \in V - C$, $e \neq e'$.

But P is a path connecting s, t, we now have $s - -uv - -t$ and $s - -u'v' - -t$, hence P must have shape $s - -uv - -u'v' - -t$ and $s - -u'v' - -uv - -t$, i.e. $s - -uv - -u'v' - -uv - -t$, hence P have loop and contradict to $P \subset F$ must be a tree. □

Problem 2.11 *Conclude using Q6, Q9, Q10*

Solution:

$$\begin{aligned}
\sum_{e \in P} w(e) &= \sum_{e \in P} \sum_{S: e \in \delta(S)} y_S \\
&= \sum_{S \in \mathcal{S}} \sum_{e \in P} 1_{e \in \delta(S)} \cdot y_S \\
&= \sum_{S \in \mathcal{S}} y_S \cdot |P \cap \delta(S)| \\
&= \sum_{S \in \mathcal{S}} y_S \\
&\leq val(y)
\end{aligned}$$

□

Problem 2.12 *Explain why the pruning part is important, which question use this?*

Solution: Q10. Or $|P \cap \delta(S)|$ might be great than 1. □

Compare to Dijkstra's algorithm

Let $d(i) = w((s, i)), \forall (s, i) \in E$ and $d(i) = \infty$ for other. Let $D = \{s\}$.

Add $\min d(i)$'s i to D and update all $j \notin D$ connected to i using the following:

$$d(j) = \min(d(i) + w(i,j),\, d(j))$$

Consider the primal-dual, start with $F = \emptyset$,

Increase the dual corresponding to the set of vertices induced by $F \cup \{s\}$ until some constraint becomes tight and adds the corresponding e to F.

Problem 2.13 *Prove (s,i) is the first edge added to F iff i is the second vertex added to D (the first is s).*

Proof: Edge cross the singleton cut $\{s\}$ are just edges of form (s,i). And $y_{\{e\}} \leq w(e)$, the first hitted e must have smallest $w(e) = w(s,i)$, just the second one in Dijkstra algorithm. $\square$

We define a notion of time, initially the time is 0 and after we increased a dual variable by ϵ, the time is ϵ.

We now fix a set C_0 that is a connected component of F containing s at some time in the execution of the primal-dual algorithm.

Denote by $a(e)$ the time at which constaint $\sum_{S:e\in\delta(S)} y_S = w(e)$ would become tight if we never stop to increase y_S (even if some other gets voilated)

Moreover, for all $j \notin C_0$, let $l(j) = \min_{(j,i)\in E} a(e)$.

Problem 2.14 *Using $l(j)$, which vertex of $V - C_0$ will be added to C_0 in the next iteration?*

Solution: Let j be the one getting the minimum of $l(j)$, this will be added to C_0 $\square$

Problem 2.15 *$S' = S \cup \{j\}$, which appear in $\delta(S') - \delta(S)$?*

Solution: Edges of form $(j, V - S')$. $\square$

Problem 2.16 *How to modify $l(k)$ for each $k \notin S'$ after j is added to S.*

Solution:

$$l(k) \leftarrow \min(l(k), l(j) + w(j,k))$$

$\square$

Problem 2.17 *Using Q16 and $d(j)$, conclude about the order in which vertices are added to $G' = (V, F)$ of primal-dual and to D of Dijkstra.*

Solution: $l(i)$ is updated in the same way as $d(i)$, hence the order of adding vertices in primal-dual and Dijkstra are same. $\square$

Complexity:

Problem 2.18 *Based on Q13, what is the best known worst-case complexity for primal-dual algorithm?*

Solution: Same as Dijkstra, hence $O(|E| \cdot |V|)$ or $O(|E| \cdot \log(|V|))$ using a heap. $\square$