1 Approximation Algorithm II Homework 1

A Primal-dual algorithm for set cover

Set cover problem: $E = \{e_1, \ldots, e_n\}$ is a set, $S_1, S_2, \ldots, S_m \subset E$, and w_j is a non-negative weight for S_j. The goal is to minimize weighted collection of S_j's containing all E, i.e. find a minimal cover of E.

Consider the following LP:

$$\min \sum_{j=1}^{m} x_j \cdot w_j$$

$$\text{subject to}$$

$$\sum_{j:e_i \in S_j} x_j \geq 1, \qquad i = 1 \ldots n$$

$$x_j \geq 0, \qquad j = 1 \ldots m$$

Problem 1.1 *What is the dual of this LP?*

Solution:

$$\max \sum_{i=1}^{n} y_i$$

$$\text{subject to}$$

$$\sum_{i:e_i \in S_j} y_i \leq w_j, \quad j = 1 \ldots m$$

$$y_i \geq 0, \qquad i = 1 \ldots n$$

□

The primal-dual algorithm

1. $y \leftarrow 0$
2. $I \leftarrow \emptyset$
3. while there is $e_i \notin I$
   - increate y_i until a l hit boundary: $\sum_{j:e_j \in S_l} y_j = w_l$
   - $I \leftarrow I \cup \{S_l\}$
4. return I

Problem 1.2 *In how many iterations of the while loop can a given dual variable be increased?*

Solution: We assume $\cup S_j = E$ or the problem is not solvable. Then any y_i must cause some $\sum_{j:e_j \in S_l} y_j$ increase.

Note if increasing y_i can make $\sum_{j:e_j \in S_l} y_j$ increase, we must have $e_i \in S_l$. Hence in each iteration, at lease one un-covered e_i is added into the sub cover I. So it terminates in at most n times. □

Problem 1.3 *Using Q2, argue the algorithm terminates to a solution.*

Solution: It terminate only when all $e_i \in I$, which is a solution. □

Approximation Ratio Assume each $e_i \in E$ can appear in at most f sets among S_j's.

Problem 1.4 *Recall a tight lower bound between the value of the optimal fractional solution for the dual $val(y^*)$ and the value of the optimal integral solution for the set cover problem OPT.*

Solution: Note the fractional optimal x^* of primal equals to fractional dual

$$val(x^*) = val(y^*)$$

and x^* is the relaxation of the integral problem, so,

$$val(y^*) = val(x^*) \leq OPT$$

. □

Problem 1.5 *Argue that the solution y is feasible for the dual.*

Solution: In the while loop, any increasement will not violate the constraints, so the y_i's are remaining feasible. □

Problem 1.6 *Combine Q4 and Q5, recall a tight lower bound between $val(y)$ and OPT.*

Solution: By definition $val(y) \leq val(y^*)$, hence

$$val(y) \leq val(x^*) \leq OPT$$

. □

In the following, we want to show:

$$\sum_{j:S_j \in I} w_j \leq f \cdot val(y)$$

Problem 1.7 *Consider $S_j \in I$, what is the relationship between w_j and $\sum_{i:e_i \in S_j} y_i$?*

Solution: By the definition of the algorithm, S_j is included only if $\sum_{i:e_i \in S_j} y_i = w_j$ becomes tight. i.e. the equality holds. $\square$

Problem 1.8 *Using Q7, give the relationship between $\sum_{j \in I} w_j$ and y_i.*

Solution:

$$\sum_{j \in I} w_j = \sum_{j \in I} \sum_{i:e_i \in S_j} y_i$$

$\square$

Problem 1.9 *Recall $|\{j : e_i \in S_j\}| \leq f$, using Q8, prove $\sum_{j \in I} w_j \leq f \cdot val(y)$.*

Solution:

$$\begin{aligned}
\sum_{j \in I} w_j &= \sum_{j \in I} \sum_{i:e_i \in S_j} y_i \\
&= \sum_{i} \left(\sum_{j \in I} 1_{e_i \in S_j} \right) y_i \\
&\leq \sum_{i} f \cdot y_i \\
&= f \cdot val(y)
\end{aligned}$$

$\square$

Problem 1.10 *Conclude using Q6 and Q9.*

Solution: From Q9, we have $val(X) = \sum_{j \in I} w_j \leq f \cdot val(y)$, and from Q6 we have $val(y) \leq OPT$, so we have

$$val(X) \leq f \cdot OPT$$

. $\square$