1 Introduction to Galois theory assignmens 1

Problem 1.1

Problem 1.2 *Set* $\zeta = e^{\frac{2i\pi}{7}}$ *and let* $L = \mathbb{Q}(\zeta)$. *Let* $M = L \cap \mathbb{R}$.

(a) *Let* p *be prime. Prove* $X^{p-1} + X^{p-2} + \cdots + X + 1 = \frac{X^p - 1}{X - 1}$ *is irreducible over* $\mathbb{Q}$ *(hint: Eisenstein)*

(b) *Find the minimal polynomial of* ζ *over* $\mathbb{Q}$ *and the degree of* L *over* $\mathbb{Q}$.

(c) *Find the minimal polynomial of* ζ *over* M *(hint:* $\zeta + \frac{1}{\zeta}$*) and the degree* $[L : M], [M : \mathbb{Q}]$.

(d) *Let* f *be an automorphism of* L *over* $\mathbb{Q}$. *List all possibilities for* $f(\zeta)$ *then for* $f(cos(2\pi/7))$.

Solution: For (a), Let $X = y + 1$, then the polynomial is

$$\frac{(1+y)^p - 1}{1 + y - 1} = \frac{y^p + py^{p-1} + \cdots + py + 1 - 1}{y} = y^{p-1} + py^{p-2} + \cdots + p$$

Using Eisenstein criterion, it's irreducible, hence original polynomial must be irreducible.

For (b), it's a root of $X^7 - 1 = 0$, using (a), we know the minimal polynomial is $X^6 + X^5 + \cdots + 1$. And $[L : \mathbb{Q}] = 6$.

For (c), Because two dimension space need at most two independent vectors to generate. ζ have only degree 2 over $\mathbb{R}$. Actually because $\zeta + \frac{1}{\zeta} = \gamma \in L \cap \mathbb{R}$ we have $\zeta^2 + 1 = \gamma\zeta$ which means it's a minimal polynomial over $M = L \cap \mathbb{R}$. Then $[M : \mathbb{Q}] = 2$ and $[L : M] = 3$.

For (d), $f(\zeta)$ must be a root of $X^6 + X^5 + \cdots + 1$. And using stem field structure, any $\zeta \mapsto \zeta^i, i = 1, \ldots, 6$ exists. So all possible $f(\zeta)$ are $\zeta^i, i = 1, \ldots, 6$ And $f(\cos(2\pi/7)) = f(\frac{\zeta + \zeta^{-1}}{2})$, hence all possibilities are $\cos(2k\pi/7), k = 1, \ldots, 6$.

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