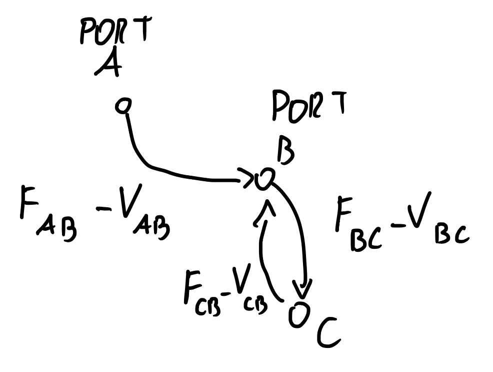
SHORTEST PATHS ON GRAPHS

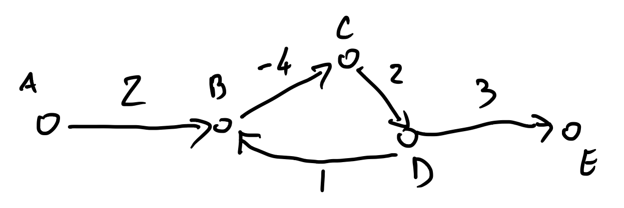
WITH POSITIVE AND NEGATIVE WEIGHTS

CRUISE SHIP / TAXI PROBLEM



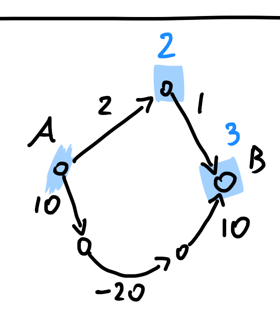
F_{xy} : COST OF THE FUEL NECESSARY TO GO FROM x TO y

V_{xy} = AMOUNT OF MONEY WE GET WHEN GOING FROM x TO y



SHORTEST PATH CAN HAVE A "$-\infty$" LENGTH.

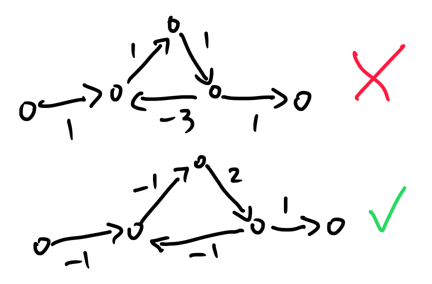
WHAT IS THE SHORTEST PATH FROM A TO E?



DIJKSTRA'S ALGORITHM NEEDS EDGES OF NON-NEGATIVE WEIGHT

(IT FAILS TO FIND A SHORTEST PATH FROM A TO B)

FOR SIMPLICITY LET US ASSUME THAT THERE ARE NO CYCLES HAVING A NEGATIVE TOTAL LENGTH.



L: IF THE GRAPH CONTAINS NO CYCLES OF NEG. TOTAL LENGTH, THEN $\forall s, t$ $\exists$ A SHORTEST PATH π FROM s TO t THAT CONTAINS NO REPEATED NODE, AND THAT IS MADE UP OF AT MOST $n-1$ EDGES.

P: LET π BE AN s-t SHORTEST PATH WITH A MINIMAL NUMBER OF EDGES.

BY CONTRADICTION, SUPPOSE THAT $\exists$ v S.T. $\pi = \pi' v \pi'' v \pi'''$.

SINCE G CONTAINS NO NEGATIVE CYCLE, IT MUST BE THAT $v \pi'' v$ IS A CYCLE HAVING A NON-NEGATIVE TOTAL LENGTH.

LET US NOW CONSIDER THE PATH $\pi' v \pi'''$. THE LATTER PATH HAS FEWER EDGES THAN π.

MOREOVER, THE TOTAL LENGTH OF $\pi' v \pi'''$ CANNOT BE LARGER THAN THE TOTAL LENGTH OF π.

THUS, π CANNOT BE A s-t PATH WITH A MINIMAL NUMBER OF EDGES. CONTRADICTION.

THEN, A SHORTEST PATH WITH A MINIMAL NUMBER OF EDGES CAN CONTAIN NO REPEATED NODE, AND THUS AT MOST $n-1$ EDGES. □

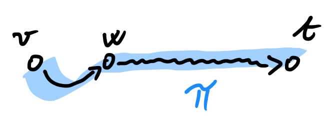
LET $OPT(i, v)$ BE THE LENGTH OF THE SHORTEST PATH (THE PATH OF MINIMUM TOTAL LENGTH) FROM v TO t, AMONG THOSE MADE UP OF AT MOST i EDGES.
($OPT(i,v) = +\infty$ IF $\nexists$ PATH FROM v TO t OF $\leq i$ EDGES).

OUR PROBLEM'S SOLUTION HAS VALUE $OPT(n-1, s)$ (BECAUSE THE LEMMA CLAIMS THAT THERE ALWAYS EXISTS A SHORTEST PATH OF $\leq n-1$ EDGES)

WHAT IS $OPT(0, t)$? $\qquad OPT(0,t) = 0$

WHAT IS $OPT(0, v)$ FOR $v \neq t$? $\qquad OPT(0, v) = +\infty$

NOW, LET π BE A PATH OF i EDGES FROM v TO t WITH THE MINIMUM COST $OPT(i, v)$.



- IF π HAS AT MOST $i-1$ EDGES, THEN
  $OPT(i, v) = OPT(i-1, v)$.
- IF π HAS EXACTLY i EDGES, THEN $\exists w$ SUCH THAT π BEGINS FROM v, MOVES TO w, AND THEN PROGRESSES TO t USING AT MOST $i-1$ EDGES. IN THIS CASE,
  $OPT(i, v) = c_{vw} + OPT(i-1, w)$.

L: $OPT(0, t) = 0$, $OPT(0, v) = +\infty$ $\forall v \neq t$.

IF $i \geq 1$, $OPT(i, v) = \min\left(OPT(i-1, v), \min\limits_{\substack{w \\ (v,w) \in E}} \left(c_{vw} + OPT(i-1, w)\right)\right)$.

THIS DP APPROACH/ALGORITHM IS CALLED THE BELLMAN-FORD ALGORITHM.

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BELLMAN-FORD (G, s, t):
  INITIALIZE  M[0...n-1][v ∈ V(G)]
  M[0][t] = 0
  M[0][v] = +∞   ∀ v ∈ V(G) - {t}
  FOR i = 1, ..., n-1
    FOR v ∈ V(G):
      M[i][v] = M[i-1][v]
      FOR w  S.T.  (v, w) ∈ E(G):
        γ = c_vw + M[i-1][w]
        IF γ < M[i][v]:
          M[i][v] = γ
  RETURN  M[n-1][s]
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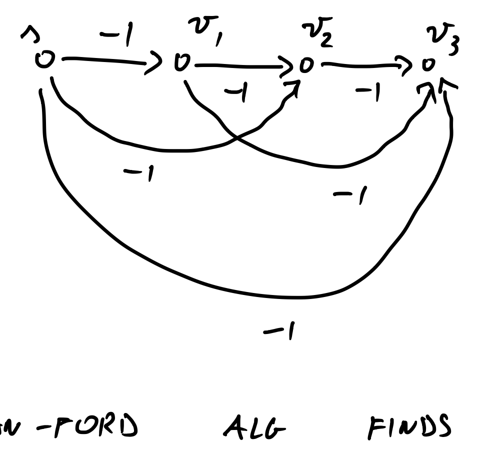
RECURRENCE OF THE LEMMA



THM: THE BELLMAN-FORD ALG FINDS A SHORTEST PATH FROM s TO t, IF THE GRAPH CONTAINS NO NEG. CYCLES.

EX.: FIND A SHORTEST PATH USING THE TABLE M.

EX.: GIVE AN ALGORITHM THAT (i) FINDS A SHORTEST PATH FROM s TO t, OR (ii) CLAIMS THAT NO PATH FROM s TO t EXIST, OR (iii) CLAIMS THAT THERE EXIST PATHS FROM s TO t OF ARBITRARILY SMALL LENGTHS.