Linear algebra theory for exam

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January 19 2024

Some things to remember:

- All nilpotent matrices are not invertible (singular matrices). A nilpotent matrix is a matrix that elevated to some exponent gives 0: $A^n = 0$;
- If a matrix a matrix A elevated to some exponent doesn't change, it is called an idempotent matrix: $A^n = A$. Idempotent matrices can have as determinant either 0 or 1, depending on whether they're invertible (not-singular) or not.
- The trace of a matrix, denoted by $tr(A)$ is the **sum** of the elements on the main diagonal.
  - $tr(AB) \neq tr(A) \cdot tr(B)$.
- A matrix A is skew-symmetric (antisymmetric) if $A^T = -A$.

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}$$

  For two skew-symmetric matrices A and B, we have:

$$(A + B)^T = A^T + B^T = -A + (-B) = -(A + B).$$

- A matrix A is symmetric if $A^T = A$.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix} = A^T$$

  For two symmetric matrices A and B, we have:

$$(A + B)^T = A^T + B^T = A + B.$$

  - The sum of matrices preserves the symmetry or skew-symmetry of the individual matrices.

1 Linear dependence/independence/spanning sets

In linear algebra, a set of vectors is said to be linearly dependent if there exists a non-trivial linear combination of these vectors that equals the zero vector. A non-trivial linear combination means that at least one of the coefficients in the linear combination is not zero. However, this does not imply that the zero vector itself must be a member of the set.

- A set of vector is linearly dependent if: $a_1v_1 + ...a_nv_n = 0$ when at leats one of the coefficients $a_i \neq 0$.
- A set of vectors is linearly independent if: $a_1v_1 + ...a_nv_n = 0$ when $a_1 = ... = a_n = 0$.

1.1 Vectors space

Consider two vectors v_1 and v_2 in a vector space, where $v_2 = 2v_1$. This set of vectors is linearly dependent because $2v_1 - v_2 = 0$, but the set v_1, v_2 does not necessarily contain the zero vector itself.
To check for linear independence, you can either immediately notice just by looking at the vectors, or by solving a system of linear equations (matrix reduction) or by computing the determinant of the nxn corresponding coefficient matrix (if $DET \neq 0$, then the linear system has a unique solution given by $x = A^{-1}b$ and the row and column vectors of the matrix are linearly independent).

The determinant of a matrix can be thought of as a scaling factor for the volume when the matrix is considered as a transformation. If the determinant is non-zero, it means that the matrix, when applied as a transformation, does not collapse the space into a lower dimension, which in turn implies that the rows (or columns) of the matrix span the entire space and none can be written as a linear combination of the others.
In the context of vector spaces, if you have a set of vectors and you arrange these vectors as the rows of a matrix, then:
If the determinant of this matrix is non-zero, the vectors are linearly independent. If the determinant is zero, the vectors are linearly dependent, meaning at least one of the vectors can be expressed as a linear combination of the others.

1. Is the set $S = \{(1, -2, 6), (5, -10, 30)\}$ linearly dependent ? Let's check:
   - $\begin{pmatrix} 1 & -2 & 6 \\ 5 & -10 & 30 \end{pmatrix}$ in REF equals $\begin{pmatrix} 1 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$. Since the last row is made by all 0s, then one of the vectors could be re-written in infinite wayts as a linear combination of the others and the set S is indeed linearly dependent.
2. Is $u = (-1, 7)$ in span $\{(1, 2), (-1, 1)\}$?

   $u = a_1v_1 + a_2v_2 \rightarrow (-1, 7) = a_1(1, 2) + a_2(-1, 1)$

$$\begin{cases} a_1 - a_2 = -1 \\ 2a_1 + a_2 = 7 \end{cases}$$

$$\text{DET}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = 3 \neq 0$$

, hence the linear system has a unique solution. This means that the vectors v_1 and v_2 span indeed the vector u (i.e. v_1 and v_2 are linearly independent).

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\begin{cases} a_1 - a_2 = 1 \\ a_2 = 3 \end{cases}$$

$$\begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$$

$$u = 2v_1 + 3v_2$$

1.2 Polynomials space

1. Is the set $S = \{(1+x), (-x^2+2)\}$ a linearly independent set in $P_2(R)$?

   - To quickly determine if this is true, we can look at the degrees of the polynomials. Since one is of degree 1 and the other is of degree 2, and there is no way to write x^2 as a multiple of x or a constant, and similarly, there is no way to write x as a multiple of x^2 or a constant; the set is linearly independent without even needing to solve any equation. This is because their degrees are different, and in a polynomial space, polynomials of different degrees are always linearly independent.
   - Otherwise, one could set a linear system and check.

2. Is the set $J = \{(1), (1+x), (1+x+x^2)\}$ a linearly independent set in P_2 ?

   - We can quickly notice that this set is not linearly independent because the first polynomial is a constant term, which is also present in the other two polynomials. This means that the constant term 1 can be written as a linear combination of the other two polynomials, indicating that there is a dependency among the vectors. Specifically, if we take the polynomial $1+x$ and subtract 1, we get x, which is part of the third polynomial $1+x+x^2$. This immediately shows that the polynomials are not linearly independent, because one of them can be generated by linear combinations of the others.

3. Is the set $S = \{(1-x), (1-x^2), (3x^2-2x-1)\}$ a linearly independent set in P_2 ?

- Let's try to verify mathematically this time:
  To determine if this set is linearly independent, we must check if the only solution to the equation $a_1(1-x)+a_2(1-x^2)+a_3(3x^2-2x-1)=0 \; \forall x \in R$ is $a_1=a_2=a_3=0$. This requires setting up the equation and solving for a_1, a_2, a_3. If the only solution is the trivial one (where all coefficients are zero), then the set is linearly independent. To check this quickly, let's consider the equation above.

  (a) $a_1(1-x)+a_2(1-x^2)+a_3(3x^2-2x-1)=0.$

  For this to hold $\forall x$, the coefficients of x^2, x, and the constant term must all be zero.

$$\begin{cases} a_1+a_2-a_3=0 \\ -a_1-2a_3=0 \\ -a_2+3a_3=0 \end{cases}$$

  We have a system of three equations with three unknowns. We can solve this system to find out if the only solution is $a_1=a_2=a_3=0$, which would imply linear independence. Let's solve this system to determine whether the set S is linearly independent.
  The solution to the system of equations is $a_1=-2c$ and $b=3c$. This means that for any non-zero value of C, we can find corresponding values of a and b that satisfy the equation.
  For example, if $c=1$, then $a=-2$ and $b=3$, these values would satisfy the equation. Since there is a combination of a, b and c other than the trivial solution (where all are zero) that makes the linear combination of these polynomials equal to zero, these polynomials are not linearly independent.

  Before solving the linear system of equations, we could compute the determinant of the square coefficient matrix or perform Gauss reduction of the coefficients matrix to check for linear dependence/independence.

  - The determinant of the coefficients matrix $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix} = 9$ (unique solution, linearly independent set).
  - The RREF of the coefficients matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, hence the set is linearly independent.

4. Is the set $S=\{(1+x-2x^2),(2+5x-x^2),(x+x^2)\}$ a linearly independent set in P_2 ?

$$a_1(1+x-2x^2)+a_2(2+5x-x^2)+a_3(x+x^2)=0$$

$$\begin{cases} a_1+2a_2=0, \\ a_1+5a_2+a_3=0, \\ -2a_1-a_2+a_3=0, \end{cases}$$

DET $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 1 \end{pmatrix} = 0$, hence NO unique solution for the system and l.d. set.

The RREF of the coefficients matrix is: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$, hence the set of vectors is linearly independent.

1.3 Matrices space

1. Check if the following set is linearly independent

$$S = \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \right\}.$$

$$a_1 \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2a_1 + 3a_2 + a_3 & a_1 \\ 2a_2 + 2a_3 & a_1 + a_2 \end{pmatrix}$$

$$\begin{cases} 2a_1 + 3a_2 + x_3 - x_4 = 0, \\ x_2 - 5x_4 - 10x_5 = 0, \\ x_3 + 4x_4 + 4x_5 = 0, \end{cases} \quad \rightarrow$$

If the coeff. matrix is square, do DET check or Gauss reduction.

The solution of the system above is clearly $a_1 = a_2 = a_3 = 0$, hence the set is linearly independent.

2 Subspaces

2.1 Check whether a set of vectors is a subspace

A subset H of a vector space V is called a subspace of V if it satisfies the following three conditions:

1. The zero vector of V is in H: This means that the set H is not empty since it at least contains the zero vector.

2. Closed under vector addition: For every pair of vectors u and v in H, the sum $u + v$ must also be in H. In other words, if you take any two vectors in the subset and add them together, the resultant vector must also be in the subset.

3. Closed under scalar multiplication: For every vector u in H and every scalar c in the field over which the vector space is defined, the product cu must also be in H. That is, multiplying any vector in the subset by a scalar should not produce a vector outside of the subset.

   If a non-empty set of vectors in V satisfies these two closure properties, then it is a subspace of V.

Let's see some examples:

- Is $W = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ a subspace of R^3 ? No, because the zero vector $\notin W$.

- Is the set $W = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ a subspace of R^3 ?

  1. Looking at the set W defined as $\{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1\}$, we can see that the zero vector does not satisfy the equation $x^2+y^2+z^2 = 1$, because $0^2+0^2+0^2 = 0$, not 1.
  2. The set W describes a unit sphere with radius 1 (centered at the origin), which is not closed under vector addition because if you take two points on the surface of the sphere, their sum will not generally lie on the sphere.
     It is not closed under scalar multiplication because if you take any non-zero scalar and multiply it with a point on the sphere, the result will not lie on the sphere unless the scalar is 1 or -1.

- Is the set $W = \{(x, y, z) \in R^3$ such that $x = 1$ and $y = z\}$ a subspace of R^3 ?
  No, because:

  1. **The zero vector is not in** W: For the zero vector $(0, 0, 0)$ to be in W, we would need $x = 0$, but the set W is defined such that $x = 1$. Since the zero vector is not included, W cannot be a subspace;
  2. **Not closed under vector addition**: If you take any two distinct vectors from W, say $(1, a, a)$ and $(1, b, b)$, and add them, you get $(2, a+b, a+b)$, which does not satisfy $x = 1$. Hence, W is not closed under addition;
  3. **Not closed under scalar multiplication**: If you multiply any vector in W by any scalar other than 1, the x-component will not be 1 anymore. For example, $2 \cdot (1, a, a) = (2, 2a, 2a)$, which does not belong to W since x is not 1.

- Is the set $W = \{A \in M_{2,2} | A^T = A\}$ a subspace of $M_{2,2}$?

  1. The zero matrix is in W because it is equal to its transpose.
  2. If A and B are in W, then $A+B$ is also in W because $(A+B)^T = A^T+B^T = A+B$.
  3. If A is in W and c is a scalar, then cA is in W because $(cA)^T = cA^T = cA$.

  Therefore, W is closed under addition and scalar multiplication, and contains the zero vector, which are the necessary and sufficient conditions for a subset to be a subspace of a vector space.

2.2 Find a basis of a subspace

A set of vectors is a basis of a subspace iff:

1. The vectors in the set have the same dimension;
2. The number of vectors in the set is the same as the dimensions of the considered subspace;
3. The vectors in the set are linearly independent (this implies that they span the subspace).

Recall that:

- Basis of row space == basis of subspace;
- Basis of column space == basis of the range.

2.2.1 Vectors subspace

It's easy, just do Gauss reduction (RREF) and consider the non-zero rows as basis.

2.2.2 Matrices subspace

1. Let W be the set

$$W = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in M_{2,2} \mid m - 2n = 0 \text{ and } p - 3q = 0 \right\}$$

   (a) Prove that W is a subspace of $M_{2,2}$.

   (b) Find a basis for W and deduce its dimension.

   To prove that W is a subspace of $M_{2,2}$, we must show that it is closed under addition and scalar multiplication, and that it contains the zero vector.

   For closure under addition, take any two matrices $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ and $B = \begin{pmatrix} m' & n' \\ p' & q' \end{pmatrix}$ in W. Then $A + B = \begin{pmatrix} m + m' & n + n' \\ p + p' & q + q' \end{pmatrix}$.

   We have $m - 2n = 0$ and $p - 3q = 0$ for A, and $m' - 2n' = 0$ and $p' - 3q' = 0$ for B.

   Adding these equations, we get $(m + m') - 2(n + n') = 0$ and $(p + p') - 3(q + q') = 0$, so $A + B \in W$.

   For closure under scalar multiplication, take any scalar k and matrix $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ in W.

   Then $kA = \begin{pmatrix} km & kn \\ kp & kq \end{pmatrix}$.

   Since $m - 2n = 0$ and $p - 3q = 0$, multiplying by k gives $k(m - 2n) = 0$ and $k(p - 3q) = 0$, so $kA \in W$.

   The zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W because $0 - 2 \cdot 0 = 0$ and $0 - 3 \cdot 0 = 0$.

   Thus, W is a subspace of $M_{2,2}$.

To find a basis for W, we solve the equations $m - 2n = 0$ and $p - 3q = 0$. We can express m as $2n$ and p as $3q$. Hence, every matrix in W can be written as:

$$\begin{pmatrix} 2n & n \\ 3q & q \end{pmatrix} = n \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + q \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}.$$

Thus, a basis for W is:

$$\left\{ \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \right\}.$$

The dimension of W is the number of vectors in the basis, which is 2.

2. Let $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2} \mid b + 2a = 0 \right\}$

   (a) Verify that W is a subspace

   - Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ where both $A, B \in W$
   - $A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

     2(a+e) + (b+f) = 0
     (b+2a) + (f+2e) = 0
     A+B ∈ W

   - Let $k \in R$, then $kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \quad \Rightarrow \quad 2(ka) + kb = k(2a + b) = 0$ so $kA \in W$

   (b) Find the dimension of W

   $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $b + 2a = 0$

   $b = -2a$

   $$A = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   $$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

3. Prove or disprove that the subset W of $M_{2,2}$ defined by

$$W = \{A \in M_{2,2} \mid A^T A = I\}$$

   is a subspace of $M_{2,2}$.

   (a) The zero vector (here, the zero matrix) must be in W.

   (b) W must be closed under vector addition.

(c) W must be closed under scalar multiplication.

For W, the equation $A^T A = I$ must hold for all $A \in W$. The matrices that satisfy this equation are orthogonal matrices, which means their columns are orthonormal vectors.

Checking the first criterion, the zero matrix does not belong to W because $0^T 0$ does not equal the identity matrix I. Therefore, W is not a subspace of $M_{2,2}$ as it does not contain the zero matrix. There is no need to check the other criteria because W already fails to be a subspace by not satisfying the first condition.

4. Consider the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 1 & 5 & 6 & 9 & 0 \end{pmatrix}$$

(a) Find a basis for the row space and column space of A, then deduce the rank and nullity of A.

(b) Find the null space of A and deduce its basis.

The rank of the matrix is equal to the number of pivot positions (the number of leading 1s in the reduced echelon form), and the nullity of the matrix is the number of free variables, which is the number of columns minus the rank.

For part (b), the null space (or kernel) of A consists of all solutions to the homogeneous system $Ax = 0$. To find a basis for the null space, we solve this system and express the solutions in terms of the free variables.

The row echelon form of the matrix is the following:

$$\begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 1 & 4 & 4 \end{pmatrix}$$

From this, we can deduce the following:

- The basis for the row space of A can be taken from the non-zero rows of the original matrix A, which are the first three rows.

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 4 \end{pmatrix} \right\}$$

- The basis for the column space of A can be taken from the corresponding columns of the original matrix A, which are the first three columns.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

  Note that you can use as basis for the row space either the rows where there are the pivots in the reduced matrix or the corresponding rows in the non-reduced matrix. Same for the column space.

- The rank of A is 3, which is the number of pivot positions.
- Rank(A) + nullity(A) = 5 (n. of columns of A).
  Nullity(A) = 5 - rank(A) = 5 - 3 = 2.
- For part (b), the null space and a basis for it is the following:

$$\begin{cases} x_1 + 3x_2 + x_3 - x_4 = 0, \\ x_2 - 5x_4 - 10x_5 = 0, \\ x_3 + 4x_4 + 4x_5 = 0, \end{cases}$$

  3 equations, 5 unknowns, so: 2 free parameters ($x_4 = s; x_5 = t$).

$$\begin{cases} x_1 + 3x_2 + x_3 - s = 0, \\ x_2 - 5s - 10t = 0, \\ x_3 + 4s + 4t = 0, \end{cases}$$

$$\begin{cases} x_1 = -10x_4 - 26x_5, \\ x_2 = 5x_4 + 10x_5, \\ x_3 = -4x_4 - 4x_5, \\ x_4 = s, \text{ (free variable)} \\ x_5 = t. \text{ (free variable)} \end{cases}$$

  A basis for the null space (kernel) is the following:

$$\left\{ s \cdot \begin{pmatrix} -10 \\ 5 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -26 \\ 10 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.2.3 Polynomials subspace

Let $W = \{a + bx + cx^2 \mid a + b + c = 0\}$

1. Show that W is a subspace of P_2.
   - $0 = 0 + 0x + 0x^2 \in W$
   - Let $p = a + bx + cx^2$ where $a + b - c = 0$
   - Let $q = d + ex + fx^2$ where $d + e - f = 0$
     Then:
     $$p + q = (a + d) + (b + e)x + (c + f)x^2$$
     where:
     $$(a + d) + (b + e) - (c + f) = (a + b - c) + (d + e - f) = 0$$
     hence $p + q \in W$
   - Let $k \in R$, then $kP = ka + kbx + kcx^2$ where:
     $$ka + kb - kc = k(a + b + c) = 0 \Rightarrow kP \in W$$

2. Find a basis for W
   $p = a + bx + cx^2$ where $a + b - c = 0$
   $c = a + b$
   $p = 1a + bx + (a + b)x^2$
   $p = a(1 + x^2) + b(x + x^2) \quad \Rightarrow \quad \dim(W) = 2$

2.3 Image and pre-image

For any vector $\mathbf{v} = (v_1, v_2)$ in R^2, let $T : R^2 \to R^2$ be defined by

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2).$$

(a) Find the image of $\mathbf{v} = (-1, 2)$.
(b) Find the preimage of $\mathbf{w} = (-1, 11)$.

(a) For $\mathbf{v} = (-1, 2)$ you have
$$T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3).$$

(b) If $T(\mathbf{v}) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$, then
$$\begin{cases} v_1 - v_2 = -1 \\ v_1 + 2v_2 = 11. \end{cases}$$

This system of equations has the unique solution $v_1 = 3$ and $v_2 = 4$. So, the preimage of $(-1, 11)$ is the set in R^2 consisting of the single vector $(3, 4)$.

3 Linear transformations (aka linear maps)

Properties of linear maps:

(a) $T(x+y) = T(x) + T(y) \rightarrow$ Additivity;

(b) $T(cx) = cT(x) \rightarrow$ Homogeneity.

To immediately recognize if a transformation is not linear, you should look for characteristics that violate the two defining properties of a linear transformation: additivity and homogeneity of degree 1. Here are some common indicators that a transformation is not linear:

(a) **Constant terms**:If any component of the transformation includes an additive constant (a term that does not depend on the input variables), the transformation is not linear. For example: $T(x) = x + 1$;

(b) **Nonlinear operations**:

Any operation on the variables that is not a first-degree polynomial indicates a nonlinear transformation.

- Powers. $T(x) = x^n$ where $n \neq 1$;
- Roots (square, cubic ...). $T(x) = \sqrt{x}$;
- Exponentials and logarithms. $T(x) = e^x$ or $T(x) = log(x)$;
- Trigonometric functions. $T(x) = sin(x)$ or $T(x) = cos(x)$;
- Absolute value. $T(x) = |x|$;
- Products of variables. xy;
- Determinant. $T(x) = det(x)$. The determinant is associated with square matrices, and it does not preserve the properties of linearity.

(c) **Non-proportional scaling**: If different input components are scaled by different amounts, the transformation may not be linear. For example: $T(x, y) = (2x, 3y)$ is linear, but $T(x) = (2x, y^2)$ is not because of the y^2 term;

(d) **Division of variables**: If the transformation involves dividing by a variable, it is not linear. For example: $T(x) = \frac{1}{2}$, assuming $x \neq 0$;

(e) **Conditional statements**: Any transformation that has different rules for different regions of the input space is not linear. For example:

$$T(x) = \begin{cases} x + 1 \ if \ x > 0 \\ x - 1 \ if \ x \leq 0 \end{cases}$$

Linear transformations can always be represented by a matrix-vector product, and they transform the zero vector in the domain to the zero vector in the codomain. If a transformation does not have this property, it is also not linear. This means that a transformation is not linear if it moves the origin (zero vector) of the space.

Let's see some examples:

(a) Is the map $T : R^2 \to R^3$ defined by $T(x, y) = (x + y, y, x + 1, 3y)$ linear ?

- **Additivity**: For any vectors $\mathbf{u}, \mathbf{v} \in R^2$, a linear transformation must satisfy $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. However, due to the constant term in the second component of the output $(x + 1)$, this property does not hold;
- **Homogeneity**: For any scalar c and vector $\mathbf{u} \in R^2$, a linear transformation must also satisfy $T(c\mathbf{u}) = cT(\mathbf{u})$. The presence of the constant term $(x + 1)$ again violates this property, as scaling the input by c will not scale the output by c because of the constant addition.

(b) Let $T : P_2 \to R^2$ be the transformation defined by $T(ax^2 + bx + c) = \begin{pmatrix} a + 3c \\ a - c \end{pmatrix}$.

i. Show that T is a linear transformation.

ii. Find the Kernel of T.

For part (a):

- Let $f(x) = ax^2 + bx + c$;
- Let $g(x) = dx^2 + ex + f$.

$$T(f + g) = \begin{pmatrix} (a + d) + 3(c + f) \\ (a + d) - (c + f) \end{pmatrix} = \begin{pmatrix} a + 3c \\ a - c \end{pmatrix} = \begin{pmatrix} d + 3f \\ d - f \end{pmatrix} = T(f) + T(g).$$

$$T(f + g) = T(kax^2 + kbx + kc) = \begin{pmatrix} ka + 3kc \\ ka - kc \end{pmatrix} = k \begin{pmatrix} a + 3c \\ a - c \end{pmatrix} = kT(f).$$

For part (b):

To find the kernel (or null space) of a transformation T, you need to find all vectors in the domain that map to the zero vector in the codomain.

Always set and solve an homogeneous linear system.

$$\{f(x) \in P_2 \mid T(f(x)) = 0\}$$

$$\{ax^2 + bx + c \mid \begin{cases} a + 3c = 0 \\ a - c = 0 \end{cases} \}$$

$$\{ax^2 + bx + c \mid \begin{cases} a = -3c \\ c = a \end{cases} \}$$

$$\{ax^2 + bx + c \mid a = c = 0\}$$

The solution to the system is $a = c = 0$. There is no condition on b, which means that b can be any real number.
Therefore, the kernel of T consists of all polynomials of the form bx where b is a real number. In other words, the kernel is the set of all polynomials in P_2 that are linear in x with no constant or quadratic form.

$$Ker(T) = \{bx \mid b \in R\}$$

This is the set of all polynomials of degree one without a constant term, which is a subspace of P_2.

(c) Let $T : R^2 \to R^2$ be the transformation given by $T(x, y) = (x + 2y, y - 2x)$.

  i. Show that T is a linear transformation.

  ii. Find the Kernel of T.

For part (a):

- For additivity, consider two vectors $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ in R^2:

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T((x_1 + x_2), (y_1 + y_2)) \\ &= (x_1 + x_2 + 2(y_1 + y_2), (y_1 + y_2) - 2(x_1 + x_2)) \\ &= (x_1 + 2y_1 + x_2 + 2y_2, y_1 - 2x_1 + y_2 - 2x_2) \\ &= (x_1 + 2y_1, y_1 - 2x_1) + (x_2 + 2y_2, y_2 - 2x_2) \\ &= T(x_1, y_1) + T(x_2, y_2). \end{aligned}$$

- For homogeneity, consider a scalar c and a vector $\mathbf{u} = (x_1, y_1)$ in R^2:

$$\begin{aligned} T(c\mathbf{u}) &= T(cx_1, cy_1) \\ &= (cx_1 + 2cy_1, cy_1 - 2cx_1) \\ &= c(x_1 + 2y_1, y_1 - 2x_1) \\ &= c \cdot T(x_1, y_1). \end{aligned}$$

Since T satisfies both additivity and homogeneity, it is a linear transformation.

For part (b):

- To find the kernel of T, we need to solve for all vectors (x, y) such that $T(x, y) = (0, 0)$. This gives us the system of equations:

$$\begin{cases} x + 2y = 0, \\ y - 2x = 0. \end{cases}$$

Solving this system, we find that $x = 0$ and $y = 0$ are the only solutions. Therefore, the kernel of T is given by:

$$\text{Ker}(T) = \{(0, 0)\}.$$

This means that the only vector in R^2 that maps to the zero vector under T is the zero vector itself.

(d) Let $T : R^3 \to R^3$ be a linear map such that:

$$\begin{aligned} T(1, 0, 0) &= (2, -1, 4) \\ T(0, 1, 0) &= (1, 5, -2) \\ T(0, 0, 1) &= (0, 3, 1) \end{aligned}$$

- Find $T(2,3,-2)$.

$$\begin{aligned}
2T(1,0,0) &= 2(2,-1,4)\ + \\
3T(0,1,0) &= 3(1,5,-2)\ - \\
2T(0,0,1) &= -2(0,3,1) = (7,7,0).
\end{aligned}$$

4 Cramer's rule

Cramer's rule is a theorem in linear algebra that gives an explicit expression for the solution of a system of linear equations with as many equations as unknowns, provided that the determinant of the coefficient matrix is non-zero (this ensures a unique solution given by: $x = A^{-1}b$), where A is the coefficient matrix.

Let's see 2 examples:

(a) Apply Cramer's rule to solve the following system:

$$\begin{cases} 2x + y - z = 3 \\ x + y + z = 1 \\ x + 2y + 3z = 4 \end{cases}$$

If $DET(A) \neq 0$:

$$DET(A) = DET \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = -1$$

$$DET(A_1(x)) = DET \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{pmatrix} = 0$$

$$DET(A_2(y)) = DET \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{pmatrix} = -5$$

$$DET(A_3(z)) = DET \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix} = 0$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{0}{-1} = 0$$

$$y = \frac{DET(A_2)}{DET(A)} = \frac{-5}{-1} = 5$$

$$z = \frac{DET(A_3)}{DET(A)} = \frac{0}{-1} = 0.$$

(b) The solution of the system $\begin{cases} x - 3y = 2 \\ 5x + y = 1 \end{cases}$ using Cramer's rule is $x = 5,\ y = -9.$

$$DET(A) = DET \begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix} = 16$$

$$DET(A_1(x)) = DET \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = 5$$

$$DET(A_2(y)) = DET \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix} = -9$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{5}{16} = 0.31$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{-9}{16} = -0.56.$$

5 Linear systems with parameters

(a) Determine the number of solutions of the following system:

$$\begin{cases} x + 2y - kz = k \\ -x - y + kz = 0 \\ (2+k)y + (2k+1)z = 0 \end{cases}$$

depending on the parameter $k \in R$ and for the values of k for which the system is compatible, solve it.

To determine the number of solutions depending on the parameter k and solve the system for those values of k for which the system is compatible, we can use matrix methods such as Gaussian elimination. We'll form the augmented matrix for this system and then perform row operations to reach row-echelon form. From there, we can determine the conditions on k for which the system has a unique solution, infinitely many solutions, or no solution.

The row-echelon form of the augmented matrix for the given system of equations is:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-k^3-4k^2-k}{2k+1} \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & \frac{-k^2-2k}{2k+1} \end{pmatrix}$$

This form indicates that the system has a unique solution for each k except when the denominator in the solution expressions equals zero. The denominators are $2k+1$, which implies that the system does not have a solution when $k = -\frac{1}{2}$ because it would make the denominator zero and hence the system would be undefined.

For all other values of k, the system has a unique solution given by:

$$x = \frac{-k^3 - 4k^2 - k}{2k+1}, \quad y = k, \quad z = \frac{-k^2 - 2k}{2k+1}$$

These solutions are valid as long as $k \neq -\frac{1}{2}$.

6 Matrices with parameters

(a) For which values of the parameter k is the following matrix not invertible ?

$$A = \begin{pmatrix} k & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & k \end{pmatrix}$$

Hint: a matrix is not invertible (singular) if its determinant is zero. Calculate the determinant of matrix A and find the values of k for which the determinant equals zero.

(b) For which values of the parameter k is the following matrix invertible ?

$$B = \begin{pmatrix} 1 & k & 3 \\ k & 5 & 6 \\ 7 & 8 & k \end{pmatrix}$$

Hint: a matrix is invertible (non-singular) if its determinant is NOT zero. Calculate the determinant of matrix B and find the values of k for which the determinant is not zero.

7 Adjoint method

The adjoint method, or adjugate method, in linear algebra is primarily used for finding the inverse of a square matrix. The concept of an adjoint matrix is closely related to determinants and minors of a matrix.

Here's a brief overview of how it works:

(a) **Minors and Cofactors:** For each element of a square matrix, you compute its minor, which is the determinant of the matrix that remains after removing the row and column of that element. Then, each minor is adjusted with a sign (+ or $-$) according to its position in the matrix, resulting in a cofactor.

(b) **Adjoint or Adjugate Matrix:** The adjoint (or adjugate) of a matrix is the transpose of the cofactor matrix. Each element of the adjoint matrix is the cofactor of the corresponding element in the original matrix.

(c) **Inverse of a Matrix:** The inverse of a matrix A is calculated using the formula $A^{-1} = \frac{1}{\det(A)} \times \mathrm{adj}(A)$, where $\det(A)$ is the determinant of A and $\mathrm{adj}(A)$ is its adjoint. This method is particularly useful when the matrix is not too large, as calculating determinants and cofactors for very large matrices can be computationally intensive.

(d) **Applications:** The adjoint method is useful in solving systems of linear equations, in theoretical work involving matrices, and in various applications across physics and engineering, where the concept of matrix inversion is relevant.

Let's see an example:

If $DET(A) \neq 0 \rightarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a23 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow even \; + \; odd \; - \rightarrow C = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} +|..| & -|..| & +|..| \\ -|..| & +|..| & -|..| \\ +|..| & -|..| & +|..| \end{pmatrix}$.

Remove rows and columns starting from row 1, column 1 and compute the determinants of the minor matrices. Don't forget the signs of the determinants given by the C matrix.

$$ADJ(A) = C^T$$

$$A^{-1} = \frac{1}{DET(A)} \cdot ADJ(A)$$

8 Properties of determinants and some transpositions

(a) $DET(AB) = DET(A) \cdot DET(B)$;

(b) $DET(cA) = c^n \cdot DET(A)$;

(c) $DET(A^{-1}) = \frac{1}{DET(A)}$;

(d) $DET(A) = DET(A^T)$;

(e) $DET(A^n) = DET(A)^n$

Let's see some examples:

(a) Let A and B be two matrices of size 4x4 such that $|A|$ = -2, $|B|$ = 4. Find $\left|\frac{1}{2}(A^{-1})^T B^3\right|$;

$$(\frac{1}{2})^4 \cdot DET(A^{-1}) \cdot DET(B^3)$$
$$\frac{1}{16} \cdot \frac{1}{DET(A)} \cdot DET(B)^3$$
$$\frac{1}{16} \cdot \frac{1}{-2} \cdot 4^3 = -2.$$

(b) Let A and B be two matrices of size 3x3 such that $|A| = 2$, $|B| = $ -3. Find $\left|2(A^2)^T B^{-1}\right|$.

$$2^3 \cdot DET(A^2)^T \cdot DET(B^{-1})$$

$$2^3 \cdot DET(A^2)^T \cdot \frac{1}{DET(B)}$$

$$2^3 \cdot 2^2 \cdot \frac{1}{-3} = -\frac{32}{3}.$$

(c) Let A $= \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ and B $= \begin{pmatrix} -1 & 3 \\ 2 & 2 \end{pmatrix}$.

- Find X such that $X^T - A = 2B$.

$$X^T = 2B + A$$

$$X = \frac{1}{3}(A^T - B).$$

- Find X such that $X - B = 3(X + B) = A^T$.

$$X + B = \frac{1}{3}A^T$$

$$X = \frac{1}{3}(A^T - B)$$

(d) Let A $= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}$ and B $= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

- Find X such that $(2X + B)^T = A$.

$$2X + B = A^T$$

$$2X = A^T - B$$

$$X = \frac{1}{2}(A^T - B).$$

9 Coordinates of a vector w.r.t a basis

(a) Check if the coordinates of the vector $v = (2, 1)$ with respect to the basis $u_1 = (3, 2)$ and $u_2 = (2, 3)$ of R^2 are $(4, -1)$.

Basically you have to check if the vector v can be written as a linear combination of the basis vectors and some coefficients.

$$au_1 + bu_2 = v$$

$$a(3, 2) + b(2, 3) = (2, 1)$$

$$\begin{cases} 3a + 2b = 2 \\ 2a + 3b = 1 \end{cases} \rightarrow RREF \rightarrow \begin{pmatrix} 1 & 0 & \frac{4}{5} \\ 0 & 1 & -\frac{1}{5} \end{pmatrix}$$

The coordinates of the vector v w.r.t that basis are $(\frac{4}{5}, -\frac{1}{5})$.
Recall that: if the coordinates of a vector w.r.t. a non-standard basis $B = \{(1,0),(1,2)\}$ are $(3,2)$, then
the coordinates w.r.t. the standard basis are given by:

$$3(1,0) + 2(1,2) = (5,4).$$

10 Eigendecomposition

Eigendecomposition is a crucial technique in linear algebra, particularly useful in the analysis of linear transformations. The process involves decomposing a matrix into a set of its eigenvectors and eigenvalues. There are several reasons why eigendecomposition is important:

(a) Simplifying Linear Transformations: Eigenvectors and eigenvalues provide a simpler way to understand and visualize linear transformations represented by matrices. An eigenvector of a matrix does not change direction under the associated linear transformation, and the eigenvalue represents the factor by which it is stretched or compressed.

(b) Diagonalization: If a matrix can be eigendecomposed, it can be written as a product of its eigenvectors and eigenvalues. This process, known as diagonalization, transforms the matrix into a diagonal form, making calculations (like raising the matrix to a power) simpler and more efficient.

(c) Solving Differential Equations: In many areas of physics and engineering, differential equations are used to model real-world phenomena. Eigendecomposition is a key tool in solving systems of linear differential equations.

(d) Principal Component Analysis (PCA): In statistics and machine learning, PCA is a method used for dimensionality reduction. It involves transforming data to a new coordinate system where the greatest variance lies on the first axis (the first principal component), the second greatest variance on the second axis, and so on. Eigendecomposition is used to compute these principal components.

(e) Spectral Theorem: The spectral theorem, which applies to normal matrices, states that any normal matrix can be diagonalized using a basis of eigenvectors. This is foundational in quantum mechanics, where operators representing physical observables are often normal matrices.

(f) Stability Analysis: In systems theory and control theory, the stability of a system can often be analyzed by looking at the eigenvalues of its system matrix. If the real parts of all eigenvalues are negative, the system is stable.

(g) Data Compression and Image Processing: Eigendecomposition is used in algorithms for data compression and image processing. For example, the JPEG image compression standard uses a form of eigendecomposition.

Considering the following matrix:

(a) Find the eigenvalues of A and a basis of each eigenspace of A.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

There are several scenarios in which a matrix is guaranteed to be diagonalizable:

(a) **Distinct Eigenvalues:** If a matrix has n distinct eigenvalues (where n is the size of the matrix), it is diagonalizable. Distinct eigenvalues ensure that there are enough linearly independent eigenvectors to form the matrix P.

(b) **Normal Matrices:** In the field of complex numbers, a matrix is diagonalizable if it is normal. A normal matrix is one that commutes with its conjugate transpose, i.e., $A^*A = AA^*$. This includes subclasses such as Hermitian, skew-Hermitian, unitary, and orthogonal matrices.

(c) **Symmetric Matrices (Real-Valued):** In real number space, a symmetric matrix (where $A = A^T$) is always diagonalizable. This is a special case of the normal matrices in real numbers.

(d) **Full Set of Eigenvectors:** If a matrix has a complete set of linearly independent eigenvectors, it is diagonalizable. This is often the case when the algebraic multiplicity (the number of times an eigenvalue appears in the characteristic equation) equals the geometric multiplicity (the dimension of the eigenspace corresponding to the eigenvalue) for each eigenvalue.

(e) **Matrices with Simple Spectrum:** If the characteristic polynomial of the matrix splits and all the eigenvalues are simple (each with algebraic multiplicity one), then the matrix is diagonalizable.

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$$

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & -1 & -1 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{pmatrix}$$

$$= (\lambda - 1)\left[(\lambda - 1)(\lambda - 4) - (-2)(1)\right]$$

For $\lambda = 1$:

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_1=-R_1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \xrightarrow{R_3=-R_3} \begin{pmatrix} 0 & 4 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{R_3=R_3-R_2} \begin{pmatrix} 0 & 4 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} y+z=0 \\ 2z=0 \\ x=t \end{cases} \implies \begin{cases} y=0 \\ z=0 \\ x=t \end{cases} \implies t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 3$:

$$\begin{pmatrix} -2 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1=-R_1} \xrightarrow{R_2=-R_2} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{swap} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \xrightarrow{R_3=R_3-2R_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1=\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{cases} x=0 \\ y=-t \\ z=t \end{cases} \implies t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = 2$:

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_1=-R_1} \xrightarrow{R_2=-R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_3=R_3-R_2}$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{cases} x-t=0 \\ y+2t=0 \\ z=t \end{cases} \implies \begin{cases} x=t \\ y=-2t \\ z=t \end{cases} \implies t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

To find P^{-1}, put on the right of the P matrix, the identity matrix. Perform Gauss reduction until the identity matrix will be on the left side instead of the right one.

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & -1 & -1 \end{pmatrix}.$$

11 Inner product spaces

An *inner product space*, in the context of linear algebra, is a vector space equipped with an additional structure known as an *inner product*. This inner product allows for the measurement of angles and lengths within the space. Let's elaborate:

1. **Vector Space:** A collection of objects, called vectors, which can be added together and multiplied by scalars (numbers) in a way that satisfies certain rules (like commutativity, associativity, distributivity, etc.). Examples include R^n (the set of all n-tuples of real numbers) and function spaces.
2. **Inner Product:** A function that takes two vectors from the vector space and returns a scalar. It generalizes the dot product from Euclidean geometry.

   Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms:

   (a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
   (b) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
   (c) $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$
   (d) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

   Remark: A vector space V with an inner product is called an *inner product space*. Whenever an inner product space is referred to, assume that the set of scalars is the set of real numbers.
3. **Examples of Inner Products:** The dot product (Euclidean inner product) in R^n, defined as $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$ for vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, is a familiar example.

   To distinguish between the standard inner product and other possible inner products, use the following notation:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n\text{)}$$
$$\langle \mathbf{u}, \mathbf{v} \rangle = \text{general inner product for vector space } V$$

4. **Applications and Implications:** An inner product space is fundamental in mathematics and its applications, allowing for the generalization of geometric concepts like orthogonality, angles, and lengths to more abstract settings.

In summary, an inner product space is a vector space combined with an inner product, extending geometric concepts to abstract mathematical and applied sciences.
Let's see some examples:

1. Show that the following function defines an inner product on R^2, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$$

   (a) Because the product of real numbers is commutative,

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 = v_1u_1 + 2v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle.$$

(b) Let $\mathbf{w} = (w_1, w_2)$. Then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2) \\ &= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2 \\ &= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2) \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle.\end{aligned}$$

(c) If c is any scalar, then

$$\begin{aligned}c\langle \mathbf{u}, \mathbf{v} \rangle &= c(u_1v_1 + 2u_2v_2) \\ &= (cu_1)v_1 + 2(cu_2)v_2 \\ &= \langle c\mathbf{u}, \mathbf{v} \rangle.\end{aligned}$$

(d) Because the square of a real number is nonnegative,

$$\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 + 2v_2^2 \geq 0.$$

Moreover, this expression is equal to zero if and only if $\mathbf{v} = \mathbf{0}$ (that is, if and only if $v_1 = v_2 = 0$).

The positive constants c are called weights. If a constant is negative or 0, then a function does not define an inner product.

2. Show that the following function is not an inner product on R^3, where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - 2u_2v_2 + u_3v_3$$

Observe that Axiom 4 is not satisfied. For example, let $\mathbf{v} = (1, 2, 1)$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = (1)(1) - 2(2)(2) + (1)(1) = -6$, which is less than zero.

3. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be matrices in the vector space $M_{2,2}$. The function

$$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$$

is an inner product on $M_{2,2}$. The verification of the four inner product axioms is left to you.

4. Check if the function $\langle u, v \rangle = u_1v_1$ defines an inner product on R^2, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

5. Check if $\langle x, y \rangle = \langle x, z \rangle$ for vectors x, y, z in an inner product space, then $y - z$ is orthogonal to x.

11.1 Properties of inner products

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space V, and let c be any real number.

1. $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

11.2 Formulas to remember

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. The norm (or length) of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{v_1^2 + v_2^2 ....}$
2. The distance between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
3. The angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is given by:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}, \quad 0 \le \theta \le \pi.$$

4. The unit vector in the direction of v is given by: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$.
5. Cauchy-Schwarz Inequality: $|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$.
6. Triangle Inequality: $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.
7. Pythagorean Theorem: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.
8. The orthogonal projection of a vector $\mathbf{u}$ onto $\mathbf{v}$ is given by:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}.$$

Remark: If $\mathbf{v}$ is a unit vector, then $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 1$, and the formula for the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ takes the simpler form

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v}.$$

11.3 Gram-Schimdt orthonormalization process

A vector space can have many different bases. For example R^3 has the convenient standard basis B $= \{(1,0,0), (0,1,0), (0,0,1)\}$ This set is the standard basis for R^3 because it has special characteristics that are particularly useful. One important characteristic is that the three vectors in the basis are mutually orthogonal (Their dot product equals 0). That is:

$$\begin{gathered}(1,0,0) \cdot (0,1,0) = 0 \\ (1,0,0) \cdot (0,0,1) = 0 \\ (0,1,0) \cdot (0,0,1) = 0.\end{gathered}$$

A second important characteristic is that each vector in the basis is a unit vector ($\|\mathbf{v_i}\| = 1$).

The Gram-Schmidt orthonormalization process allows to construct a basis made of mutually orthogonal unit vectors.

- A set S of vectors in an inner product space V is called orthogonal if every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector. then S is called orthonormal.

- If $S = \{v_1, v_2, ..., v_n\}$ is an orthogonal set of non-zero vectors in an inner product space V, then S is linearly independent.

1. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for an inner product space V.
2. Let $B' = \{w_1, w_2, \ldots, w_n\}$, where w_i is given by

$$\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \\
w_3 &= v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 \\
&\vdots \\
w_n &= v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}.
\end{aligned}$$

   Then B' is an *orthogonal* basis for V.

3. Let $u_i = \frac{w_i}{\|w_i\|}$. Then the set $B'' = \{u_1, u_2, \ldots, u_n\}$ is an *orthonormal* basis for V. Moreover, $\text{span}\{v_1, v_2, \ldots, v_k\} = \text{span}\{u_1, u_2, \ldots, u_k\}$ for $k = 1, 2, \ldots, n$.

Let's see an example:
Apply the Gram-Schmidt orthonormalization process to the basis for R^3 shown below.

$$B = \{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}$$

$$\begin{aligned}
w_1 &= v_1 = (1, 1, 0) \\
w_2 &= v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\
w_3 &= v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 \\
&= (0, 1, 2) - \frac{1}{2}(1, 1, 0) - \frac{1}{2}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = (0, 0, 2).
\end{aligned}$$

The set $B' = \{w_1, w_2, w_3\}$ is an orthogonal basis for R^3. Normalizing each vector in B' produces

$$\begin{aligned}
u_1 &= \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) \\
u_2 &= \frac{w_2}{\|w_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) \\
u_3 &= \frac{w_3}{\|w_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1).
\end{aligned}$$

So, $B'' = \{u_1, u_2, u_3\}$ is an orthonormal basis for R^3.

A matrix is said to be orthogonal if the product of the matrix and its transpose results in the identity matrix. Mathematically, a matrix A is orthogonal if $A \cdot A^T = I$, where I is the identity matrix and A^T is the transpose of A.
Consider the following matrix:

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

To check if it's orthogonal, we need to multiply this matrix by its transpose and see if the product is the identity matrix. Let's perform this calculation.
The product of the matrix and its transpose is:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This is not the identity matrix; instead, it is a scalar multiple of the identity matrix. Therefore, the given matrix is not orthogonal.