INTERVAL SCHEDULING $O(n \lg n)$

FASTALG(I) :

$O(n \lg n) \longrightarrow$ - SORT THE INTERVALS BY FINISHING TIME

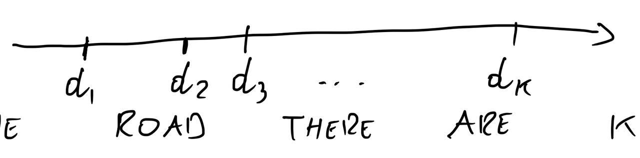
- LET $I = \{I_1, \dots, I_n\}$ S.T. $f(I_1) \le f(I_2) \le \dots \le f(I_n)$
- $T \leftarrow -\infty$ // TIME AT WHICH THE LAST INTERVAL // WE SCHEDULED ENDS
- $S \leftarrow \emptyset$

$O(n) \to$
- FOR $i = 1, 2, \dots, n$
  - IF $s(I_i) \ge T$:
    - $S \leftarrow S \cup \{I_i\}$
    - $T \leftarrow f(I_i)$ } $O(1)$
- RETURN S

$$O(n \lg n) + O(n) \le c\, n \lg n + c' n$$
$$\le (c + c')\, n \lg n = O(n \lg n)$$

$\boxed{\text{INTERVAL PARTITIONING}}$

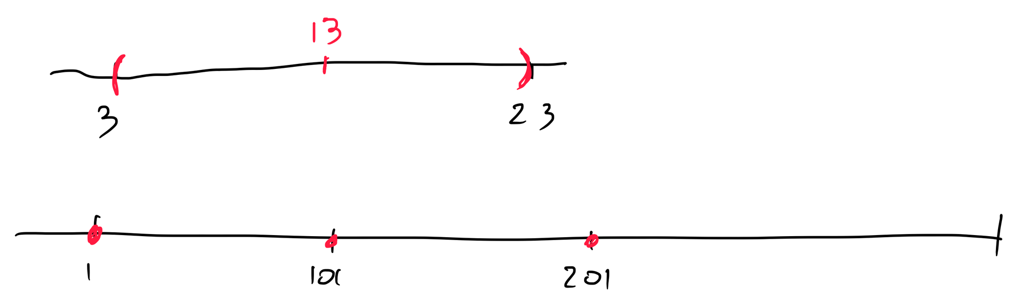
EX.: CONSIDER A RECTILINEAR ROAD OF t KMS.



ALONG THE ROAD THERE ARE K HOUSES, THE iTH OF WHICH LIES AT KM d_i.

A GSM COMPANY WANTS TO INSTALL ANTENNAS SO TO COVER EACH HOUSE WITH ITS GSM NETWORK.

IF AN ANTENNA COVERS A RADIUS 10 KMS, WHAT IS THE SMALLEST NUMBER OF ANTENNAS THAT HAVE TO BE INSTALLED?



INPUT : $d_1, d_2, d_3, \dots, d_n$, R ← RADIUS

- SORT THE d_i'S : $d_1 \le d_2 \le d_3 \le \dots \le d_n$
- ANT = [ ]
- FOR $i = 1, 2, \dots, n$
  - IF LEN(ANT) == 0 OR (ANT[-1] + R < d_i) :
    - ANT.APPEND(d_i + R)
- RETURN LEN(ANT)

LET $O = \{o_1, o_2, \dots, o_m\}$ BE AN OPTIMAL SOLUTION WITH $o_1 \le o_2 \le \dots \le o_m$.

LET $A = \{e_1, e_2, \dots, e_K\}$ BE THE SOLUTION PRODUCED BY OUR GREEDY ALGORITHM

(WE WANT TO MINIMIZE THE NUMBER OF ANTENNAS).

L1: A IS A SOLUTION TO THE PROBLEM.

THUS, $|A| \ge |O|$ (OR $K \ge m$).

L2: (EXCHANGE LEMMA)
SUPPOSE THAT, FOR SOME $j \le m-1$, WE HAVE THAT $e_1 = o_1$, $e_2 = o_2$, ..., $e_j = o_j$.
THEN, $o_{j+1} \le e_{j+1}$, AND $O_{j+1} = \{e_1, e_2, \dots, e_j, e_{j+1}, o_{j+2}, \dots, o_m\}$ IS AN OPTIMAL SOLUTION.

P: LET i BE THE INDEX OF THE FIRST (THE LEFTMOST HOUSE) NOT COVERED BY $e_1, \dots, e_j$.
 $o_1, \dots, o_j$

THEN, $e_{j+1} = d_i + R$. (GREEDY INSTALLS AN ANTENNA IN POSITION $d_i + R$).

RECALL THAT $o_1 \le o_2 \le \dots \le o_m$. THEN, HOUSE i IS COVERED BY SOME $o_\ell \in O$, FOR $\ell \ge j+1$.

BUT, THEN, $o_{j+1} \le o_\ell \le d_i + R = e_{j+1}$. ✓

THE SET O_{j+1} IS JUST SOLUTION O WITH ITS $(j+1)$TH ANTENNA MOVED FROM POSITION o_{j+1} TO e_{j+1}.

BUT THEN, MOVING THE $(j+1)$TH ANTENNA FROM POSITION o_{j+1} TO $e_{j+1} \ge o_{j+1}$,

(I) DOES NOT UNCOVER ANY OF THE FIRST $i-1$ HOUSES;

(II) KEEPS HOUSE i COVERED ($e_{j+1} = d_i + R$, SO $e_{j+1} - d_i \le R$);

(III) DOES NOT UNCOVER ANY OF THE HOUSES $i+1, i+2, \dots, n$.

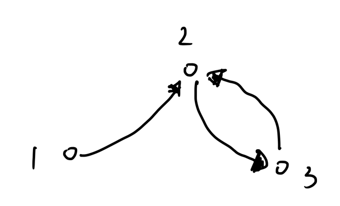
THUS, O_{j+1} IS STILL A VALID SOLUTION (IT COVERS ALL THE HOUSES). MOREOVER $|O_{j+1}| = |O| = m$. ∎

T: A IS AN OPTIMAL SOLUTION (GREEDY ALWAYS RETURNS AN OPTIMAL SOLUTION).

(DIRECTED) GRAPH

A (DI) GRAPH $G(V, E)$ IS COMPOSED OF A SET OF VERTICES V AND OF A SET OF ARCS E, WITH

$$E \subseteq \{(v, w) \mid v \ne w \text{ AND } v, w \in V\}.$$



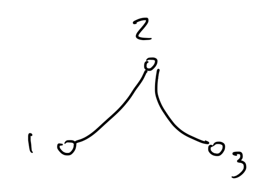
$E = \{(1,2), (2,3), (3,2)\}$

CAN I GO FROM 1 TO 3? YES
1→2→3

CAN I GO FROM 3 TO 1? NO!

AN UNDIRECTED GRAPH $G(V, E)$ IS COMPOSED OF A SET OF NODES V AND OF A SET OF EDGES E, WITH

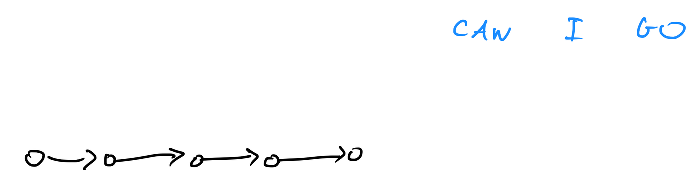
$$E \subseteq \{\{v, w\} \mid v \ne w \text{ AND } v, w \in V\}.$$



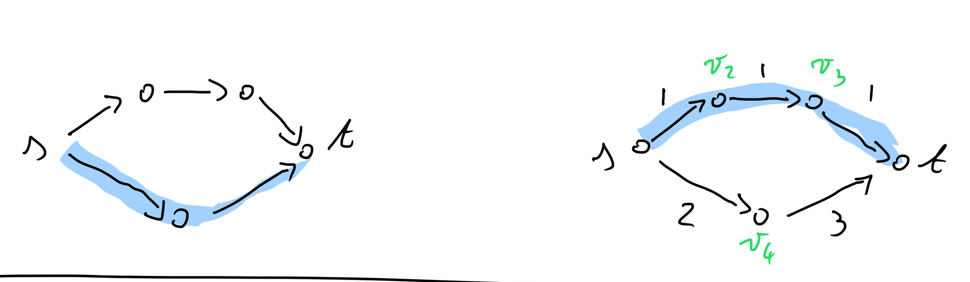
$E = \{\{1,2\}, \{2,3\}\}$

CAN I GO FROM 1 TO 3? YES
1→2→3

CAN I GO FROM 3 TO 1? YES
3→2→1



A WEIGHTED DIGRAPH IS A DIRECTED GRAPH $G(V, E)$ EQUIPPED WITH A FUNCTION $\ell : E \to \mathbb{R}_{\ge 0}$. ($\mathbb{R}_{\ge 0} = \{x \mid x \in \mathbb{R} \text{ AND } x \ge 0\}$)



s, v_2, v_3, t IS A PATH

s, v_4, t IS A PATH

HOW TO FIND THE SHORTEST PATH FROM VERTEX s TO VERTEX t?

DEF.: GIVEN $G(V, E)$, A PATH FROM $s \in V$ TO $t \in V$ IS A SEQUENCE OF NODES $v_1, v_2, \dots, v_K$ (WITH $v_1 = s$ AND $v_K = t$) SUCH THAT $(v_1, v_2) \in E$ AND $(v_2, v_3) \in E$ AND... AND $(v_{K-1}, v_K) \in E$.

DEF.: IF $G(V, E), \ell$ IS A WEIGHTED GRAPH AND IF $\pi = v_1, v_2, \dots, v_K$ IS A PATH IN $G(V, E)$, THE LENGTH OF π IS EQUAL TO $\sum_{i=1}^{K-1} \ell((v_i, v_{i+1}))$.