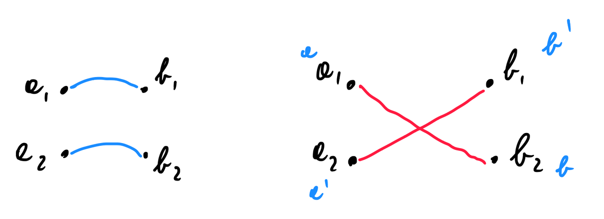
STABLE MATCHING

LET A, B BE TWO DISJOINT SETS, WITH $|A| = |B| = n$.

EACH $e \in A$ HAS A PREFERENCE LIST OVER ALL THE $b \in B$; LIKEWISE, EACH $b \in B$ HAS A PREFERENCE LIST OVER ALL THE $e \in A$.

$e_1 : b_1 > \cdots$
$e_2 : b_2 > \cdots$
$\vdots$
$e_n : b_n > \cdots$

$b_1 : e_1 > \cdots$
$b_2 : e_2 > \cdots$
$\vdots$
$b_n : e_n > \cdots$



$e_1 : b_1 > b_2$ $\quad$ $b_1 : e_1 > e_2$
$e_2 : b_1 > b_2$ $\quad$ $b_2 : e_2 > e_1$

LET M BE A PERFECT MATCHING BETWEEN A AND B.
TWO DISTINCT PAIRS $\{e, b\}, \{e', b'\} \in M$ EXHIBIT AN INSTABILITY IF

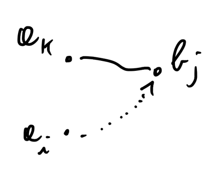
- e PREFERS b' TO b, AND
- b' PREFERS e TO e'.

M IS A STABLE MATCHING IF IT CONTAINS NO INSTABILITIES.

$e_1 : b_1 > \cdots$
$e_2 : b_1 > \cdots$
$\vdots$
$e_n : b_1 > \cdots$

GALE-SHAPLEY ()

- INITIALLY, EACH $e_i \in A$ AND EACH $b_j \in B$ IS FREE
- WHILE THERE EXISTS A FREE $e_i \in A$ THAT HAS NOT YET PROPOSED TO ALL $b_j \in B$:
  - LET e_i BE A FREE ITEM OF A.
  - LET $B' \subseteq B$ BE THE SET OF THE b_j'S THAT e_i HAS NOT YET PROPOSED TO.
  - LET $b_j \in B'$ BE THE ELEMENT OF B' THAT RANKS HIGHEST IN e_i'S PREFER. LIST
  - IF b_j IS FREE:
    - MATCH UP e_i AND b_j (WHO ARE THEN NOT FREE ANYMORE)
  - ELSE:
    - LET e_k BE THE CURRENT PARTNER OF b_j
    - IF b_j PREFERS e_k TO e_i:
      - NOTHING CHANGES (e_i REMAINS FREE, AND $\{b_j, e_k\}$ REMAIN TOGETHER)
    - ELSE:
      - "BREAK UP" $\{b_j, e_k\}$
      - "MATCH" $\{b_j, e_i\}$
      - e_k BECOMES FREE.



L1: EACH $b \in B$ REMAINS MATCHED FROM THE 1ST TIME SHE GETS A PROPOSAL, UNTIL THE END OF THE EXECUTION.
ALSO, THE PARTNERS OF b GET BETTER (FROM b'S PERSPECTIVE) OVER TIME.

L2: FOR EACH $e \in A$, THE SEQUENCE OF PROPOSALS OF e GET WORSE OVER TIME.

THM 1: THE ALGORITHM ENDS AFTER AT MOST n^2 ITERATIONS OF THE WHILE LOOP.

L3: THE ALGORITHM OUTPUTS A MATCHING.

P. IF e MAKES A PROPOSAL, HE IS FREE (AND e CAN ONLY BE MATCHED AFTER A PROPOSAL OF HIS).
MOREOVER, WHEN b ACCEPTS A PROPOSAL, SHE IS EITHER FREE, OR SHE BREAKS UP WITH HER PREVIOUS PARTNERS.

L4: IF $e \in A$ IS FREE AT SOME POINT DURING THE EXECUTION OF THE ALGORITHM, THEN THERE EXISTS A $b \in B$ THAT e HAS NOT PROPOSED TO.

P: BY CONTRADICTION, SUPPOSE THAT — AT SOME POINT DURING THE EXECUTION — $e^* \in A$ IS FREE AND THAT HAS ALREADY PROPOSED TO ALL THE b's IN B.

BY L1, EACH $b \in B$ IS ENGAGED FROM THE TIME OF THE 1ST PROPOSAL SHE GETS UNTIL THE END.

BUT THEN FOR e^* TO BE FREE AFTER $|B| = n$ PROPOSALS, IT MUST BE THAT — AT HIS LAST (n TH) PROPOSAL — ALL THE bs IN B ARE MATCHED.

BY L3, THE ENGAGEMENTS FORM A MATCHING. THUS, FOR ALL THE bs IN B TO BE MATCHED, ALL THE es IN A SHOULD BE MATCHED (INDEED, $|A| = |B| = n$).

ON THE OTHER HAND, WE ASSUMED THAT e^* IS FREE — THUS, AT MOST $n-1$ es CAN BE MATCHED. CONTRADICTION. □

L5: THE ALGORITHM OUTPUTS A PERFECT MATCHING.

P: BY L3, THE ALGORITHM OUTPUTS A MATCHING.
IF THIS MATCHING IS NOT PERFECT, THERE MUST EXIST AT LEAST ONE FREE $e \in A$, AND ONE FREE $b \in B$ (SINCE $|A| = |B| = n$).
BUT THE EXISTENCE OF A FREE $e \in A$ IN THE RETURNED MATCHING CONTRADICTS L4. THUS, THE ALGORITHM OUTPUTS A PERFECT MATCHING. □

T2: THE ALGORITHM OUTPUTS A STABLE MATCHING.

P: BY L5, WE KNOW THAT THE OUTPUT MATCHING M^* IS PERFECT.

BY CONTRADICTION, SUPPOSE THAT $\{e_i, b_j\}, \{e_k, b_\ell\} \in M^*$ WITH THE PAIR $\{e_i, b_j\}, \{e_k, b_\ell\}$ BEING UNSTABLE.
THEN, W.L.O.G.,

1. e_i PREFERS b_ℓ TO b_j, AND
2. b_ℓ PREFERS e_i TO e_k.

BY THE ALGORITHM DEFINITION, e_i'S LAST PROPOSAL WAS TO b_j.
WE CONSIDER NOW TWO CASES:

- e_i PROPOSED TO b_ℓ BEFORE PROPOSING TO b_j. SINCE b_ℓ ENDED UP WITH e_k, L1 ENTAILS THAT b_ℓ PREFERS e_k TO e_i. CONTRADICTION.
- e_i HAS NOT PROPOSED TO b_ℓ BEFORE PROPOSING TO b_j. THEN, e_i HAS NOT PROPOSED TO b_ℓ AT ALL (b_j IS e_i'S LAST PROPOSAL). BUT, THEN, e_i PREFERS b_j TO b_ℓ. CONTRADICTION.

THEN, M^* IS A STABLE MATCHING. □

WE SAY THAT

DEF.: $b_j \in B$ IS A VALID MATCH FOR $e_i \in A$ IF $\exists$ A STABLE MATCHING M SUCH THAT $\{e_i, b_j\} \in M$.

DEF: LET $\text{best}(e_i)$, FOR $e_i \in A$, BE THE VALID MATCH $b_j \in B$ OF e_i, THAT e_i LIKES THE BEST.

THM.: THE GALE-SHAPLEY ALGORITHM RETURNS
$$M^* = \{ \{e_i, \text{best}(e_i)\} \mid \forall e_i \in A \}.$$

DEF.: $e_i \in A$ IS A VALID MATCH FOR $b_j \in B$ IF $\exists$ A STABLE MATCHING M SUCH THAT $\{e_i, b_j\} \in M$.

DEF: LET $\text{worst}(b_j)$, FOR $b_j \in B$, BE THE VALID MATCH $e_i \in A$ OF b_j, THAT b_j LIKES THE WORST.

THM.: THE GALE-SHAPLEY ALGORITHM RETURNS
$$M^* = \{ \{b_j, \text{worst}(b_j)\} \mid \forall b_j \in B \}.$$