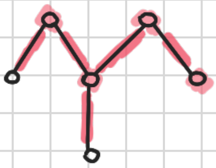
Algorithms

DATE: 19/05/22

APPROXIMATION ALGORITHMS

$O(n W)$ $\quad W = 2^n$



APPROX - VC (G(V,E))

$S \leftarrow \emptyset$

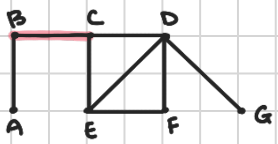
WHILE $E \neq \emptyset$:

PICK ANY EDGE $e = \{u, v\} \in E$

$S \longleftarrow S \cup \{u, v\}$

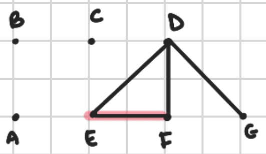
REMOVE EACH EDGE $e' \in E$ S.T. e' is INCIDENT on u OR v

RETURN S



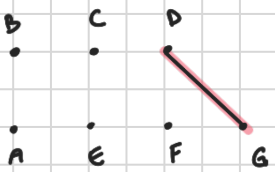
$S = \{B, C\}$

$\Downarrow$

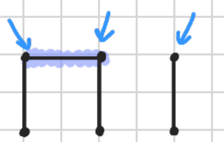


$S = \{B, C, E, F\}$

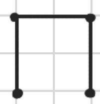
$\Downarrow$



$S = \{B, C, E, F, D, G\}$ $\quad |S| = 6$

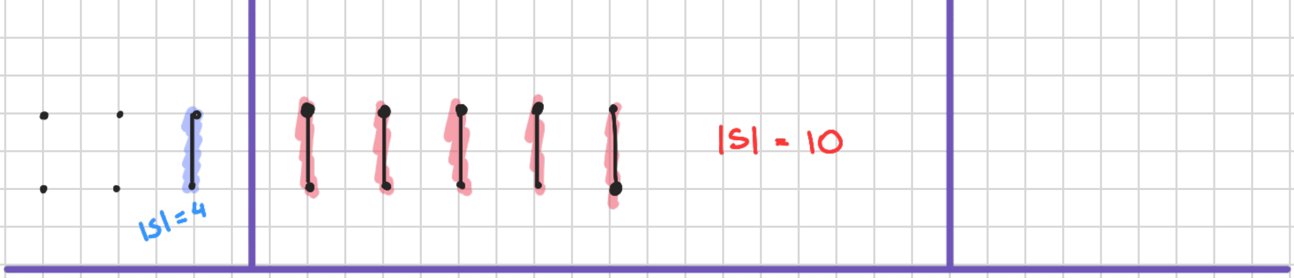


min VC = 5



$|S| = 2$

minVC = 2



THM: APPROX-VC returns a set of nodes that:

① is a vertex cover, and

② that has a cardinality that is not larger than twice the cardinality of a smallest VC.

Moreover, ③ the algorithm runs in polytime.

P: The algorithm iterates for as long as there are edges in the graph.
Whenever the algorithm picks an edge $\{u, v\}$, it adds both "u" and "v" to the partial solution S.
In doing so, it removes from the graph all the edges incident on "u" or "v".

Thus, the final solution is a vertex cover.
(① is proved)

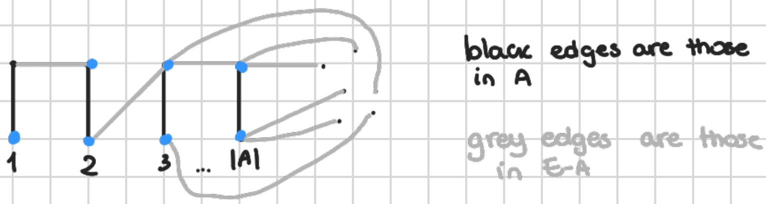
Let $A \subseteq E$ is the set of edges picked by the algorithm.

Observe that if $\{u, v\}, \{x, y\} \in A$, with $\{u, v\} \neq \{x, y\}$, then $\{x, y\} \cap \{u, v\} = \emptyset$.

(Indeed, if, W.L.O.G., $\{u, v\}$ is picked before $\{x, y\}$ then all the edges incident on u, or v, are removed from the graph — Thus, when $\{x, y\}$ is picked, it cannot be that $x = u$, or $x = v$, or $y = u$, or $y = v$.)

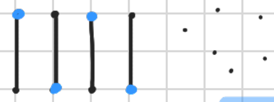
Then, the solution returned by the algorithm has size $2|A|$: The algorithm adds to "S" 2 new vertices for each edge of A.

Recall that $A \subseteq E$. Let us consider the graph $G(V, A)$.



Since $A \subseteq E$, if $T \subseteq V$ is a vertex cover for $G(V, E)$, then T must be a " " " $G(V, A)$ as well.

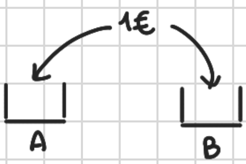
Now, in $G(V,A)$ no two edges share an end point.
Therefore each vertex cover T for $G(V,A)$ must contain at least one node per edge — that is, at least $|A|$ nodes.

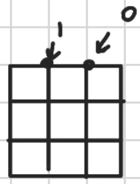


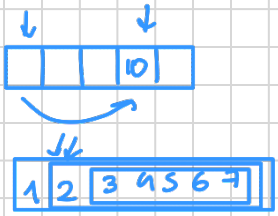
Thus, the minimum VC for $G(V,E)$ must contain at least $|A|$ nodes.
Since our algorithm returns a solution with $2|A|$ nodes, ② is proved. ■

It is NP-HARD to approximate Independent Set to $n^{0.99}$.

RANDOMIZED ALGORITHM







ALG A ( )

OPEN BOX A

0 €

ALG B ( )

OPEN BOX B

ALG' ( )

FLIP A COIN

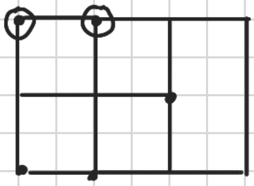
IF THE COIN COMES UP HEADS :

OPEN BOX A

ELSE :

OPEN BOX B

$$E[X] = \frac{1}{2}(0€) + \frac{1}{2}(1€) = 0,5\ €$$



$\Longrightarrow$

$\{B_0, B_1, B_2, B_3\} = \{1, 10, 100, 1000\}$

$= \{1, 1, 1, 1\}$