

Lecture Notes Calculus Unit 2, ACSAI

Nadia Ansini¹

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¹email: ansini@mat.uniroma1.it

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Chapter 1

Numerical Sequences

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1.1 Sequences of real numbers

A sequence is a function $f : \mathbb{N} \mapsto \mathbb{R}$ that takes values $f(1), f(2), \dots, f(n), \dots$. An intuitive understanding of the definition of a sequence is that a sequence it is an itemized collection of elements, it is a chain of ordered term. Then, the best notation for sequences is rather

$$a_1, a_2, \dots, a_n, \dots$$

where n is the index of the sequence and $(a_n) = \{a_1, a_2, \dots, a_n, \dots\} = f(\mathbb{N})$ is the set of all values assumed by the function f and denoted by $a_n := f(n)$.

Before giving the formal definitions of important notions for sequences, we take a look at the behaviour of the sequence in the example below.

Example 1.1.1. *The sequence of the inverse of the integers:*

$$a_n = \frac{1}{n}, \quad (a_n) = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}.$$

Note that, the sequence a_n is bounded from below and from above; i.e., $0 < a_n \leq 1$. Moreover, $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$; i.e., in this sequence the values are decreasing as n increases and $(1/n)$ seems to “approach” a single point as n increases that is 0.

Indeed, for every arbitrarily fixed $\varepsilon > 0$ we have that

$$0 < \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon},$$

which implies that the sequence $(a_n) \in (0, \varepsilon)$ definitively; i.e., for every $n > 1/\varepsilon$ and for every arbitrarily small $\varepsilon > 0$.

This very simple example suggests the introduction of very important notions already familiar in the context of sets and functions as: boundedness, monotonicity and limits.

First of all we can apply the notion of bounded sets to the sequences; *i.e.*, to the set (a_n) . More precisely, there always exist

$$\sup_n a_n := \sup\{a_n, \forall n \in \mathbb{N}\} \leq +\infty, \quad \inf_n a_n := \inf\{a_n, \forall n \in \mathbb{N}\} \geq -\infty,$$

hence,

- we say that the sequence (a_n) is bounded above if $\sup_n a_n \in \mathbb{R}$ or equivalently, there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for every $n \in \mathbb{N}$;
- we say that the sequence (a_n) is bounded below if $\inf_n a_n \in \mathbb{R}$ or equivalently, there exists $m \in \mathbb{R}$ such that $a_n \geq m$ for every $n \in \mathbb{N}$;
- we say that the sequence (a_n) is bounded if $\inf_n a_n, \sup_n a_n \in \mathbb{R}$ or equivalently, there exists $m, M \in \mathbb{R}$ such that $m \leq a_n \leq M$ for every $n \in \mathbb{N}$.

Definition 1.1.2 (Monotone sequences). *We say that a sequence (a_n) is increasing if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$ (or strictly increasing if $a_n < a_{n+1}$). We say that a sequence (a_n) is decreasing if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$ (or strictly decreasing if $a_n > a_{n+1}$). Sequences which are either increasing or decreasing sequences are called monotone (or strictly monotone).*

Let us now state the formal definition of convergence. We distinguish sequences whose elements approach a single point as n increases from those sequences whose elements do not.

Definition 1.1.3 (Convergence of a sequence). *We say that a sequence (a_n) converges if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a natural number n_ε (depending on $\varepsilon > 0$) such that $a_n \in (L - \varepsilon, L + \varepsilon)$ or, equivalently, $|a_n - L| < \varepsilon$, for all $n \geq n_\varepsilon$.*

It can be easily verified that if such a number L exists then it is unique. In this case, we say that the sequence (a_n) converges to L and we call L the limit of the sequence (a_n) and we denote it

$$L := \lim_{n \rightarrow +\infty} a_n.$$

By Example 1.1.1 we have then the sequence $1/n$ converges to 0 since we may apply the Definition 1.1.3 by taking any natural number $n_\varepsilon > 1/\varepsilon$; hence,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0.$$

Definition 1.1.4 (Divergence of a sequence). *We say that a sequence (a_n) diverges to $+\infty$ if for every $M > 0$, there exists a natural number k such that $a_n > M$ for all $n \geq k$. Similarly, we say that a sequence (a_n) diverges to $-\infty$ if for every $m < 0$, there exists a natural number k such that $a_n < m$ for all $n \geq k$. In these cases, we write*

$$\lim_{n \rightarrow +\infty} a_n = +\infty, \quad \lim_{n \rightarrow +\infty} a_n = -\infty,$$

respectively.

Example 1.1.5. The oscillating sequence $a_n = (-1)^n$ is bounded since $(a_n) = \{1, -1\}$ for every $n \in \mathbb{N}$, it is not monotone and it does not converge since it does not approach a single

point as n increase. Indeed, if there exists a limit $L = \lim_{n \rightarrow +\infty} (-1)^n$ then for a fixed ε we should have by definition

$$L - \varepsilon \leq (-1)^n \leq L + \varepsilon, \quad \forall n \geq n_\varepsilon$$

and

$$a_{n+1} - a_n \leq \left(L + \frac{1}{2}\right) - \left(L - \frac{1}{2}\right) = 1, \quad \forall n \geq n_\varepsilon.$$

This is in general not true when n is odd.

Finally, we note that

$$a_n = \begin{cases} 1 & n = 2k, k \in \mathbb{N} \\ -1 & n = 2k+1, k \in \mathbb{N}. \end{cases}$$

which means that from a_n we can extract two sequences $b_k = a_{2k} = 1$ and $c_k = a_{2k+1} = -1$ and we call b_k and c_k subsequences of a_n . More in general, a *subsequence* of a sequence a_n is a sequence defined as $b_k = a_{n_k}$ where $n_1 < n_2 < \dots < n_k < \dots$ with $k \in \mathbb{N}$.

The two subsequences b_k and c_k trivially converge to 1 and -1 , respectively. Hence, a_n is a bounded sequence that does not converge but we can extract two subsequences converging. This is true not only for this particular example, but in general!

Theorem 1.1.6 (Bolzano-Weierstrass). *Any bounded sequence $(a_n) \subset \mathbb{R}$ has a converging subsequence.*

The following theorem gives a necessary condition for the convergence of a sequence.

Proposition 1.1.7. *Every convergent sequence is a bounded sequence.*

Proof. Let us assume $\lim_{n \rightarrow +\infty} a_n = L$. Then, by Definition 1.1.3 we have that $|a_n| < L + \varepsilon$ for every $n \geq n_\varepsilon$; hence,

$$|a_n| < L + \varepsilon + M, \quad \forall n \in \mathbb{N}$$

with $M = \max\{|a_1|, \dots, |a_{n_\varepsilon-1}|\}$; i.e., (a_n) is bounded. \square

Definition 1.1.8 (Cauchy Sequences). *We say that (a_n) is a Cauchy sequence if for every $\varepsilon > 0$ there exists natural number n_ε such that*

$$|a_m - a_n| \leq \varepsilon, \quad \forall m, n \geq n_\varepsilon.$$

Theorem 1.1.9 (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

We will see soon an important application of the Criterion to the study of converging series.

1.1.1 Limits Theorem

Theorem 1.1.10 (Operations of Limits- I). *Suppose that $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow +\infty$, then*

- $(a_n \pm b_n) \rightarrow a \pm b$,
- $a_n \cdot b_n \rightarrow a \cdot b$,

- if $b_n, b \neq 0$ then $a_n/b_n \rightarrow a/b$.

Remark 1.1.11 (Indeterminate Forms). If the limits are infinite or 0 they may give rise to indeterminate forms as:

$$\infty - \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}, \quad \infty \cdot 0$$

This does not mean that the limits do not necessarily exist but their computation need a further investigation. In case of indeterminate forms of limits of function the De l'Hopital's theorem gives an instrument to compute the limits $0/0$ and ∞/∞ and as a consequence of these two cases also $\infty \cdot 0$ and $\infty - \infty$. Thanks to the next Theorem 1.1.13 we may take advantage of that to study the indeterminate forms for sequences.

Remark 1.1.12 (Operations of Limits- II). Here we collect other cases that can not be included in the previous theorem but for which we still can predict the limit:

- if $a_n \rightarrow +\infty$ and (b_n) is bounded below then $(a_n + b_n) \rightarrow +\infty$,
- if $a_n \rightarrow -\infty$ and (b_n) is bounded above then $(a_n + b_n) \rightarrow -\infty$,
- if $a_n \rightarrow \pm\infty$ and $b_n \rightarrow L \neq 0$ then $(a_n \cdot b_n) \rightarrow \pm\infty$ according to the sign of infinity and of L ,
- if $a_n \rightarrow \pm\infty$ then $(1/a_n) \rightarrow 0$,
- if $a_n \rightarrow \pm\infty$ and $b_n \rightarrow L$ then $(b_n/a_n) \rightarrow 0$,
- if $a_n \rightarrow 0$ then $1/|a_n| \rightarrow +\infty$. In particular, if $a_n > 0$ or < 0 then $1/a_n \rightarrow +\infty$ or $-\infty$.

Theorem 1.1.13 (A bridge between limit of functions and of sequences). Let $L \in \mathbb{R} \cap \{+\infty, -\infty\}$ and $x_0 \in \mathbb{R} \cap \{+\infty, -\infty\}$ then

$$\lim_{x \rightarrow x_0} f(x) = L$$

if and only if for every $x_n \rightarrow x_0$ as $n \rightarrow +\infty$, $x_n \neq x_0$, we have that

$$\lim_{n \rightarrow +\infty} f(x_n) = L.$$

Example 1.1.14. Theorem 1.1.13 is very useful if we have to compute limit of sequences $f(x_n)$ by knowing the limit of the function $f(x)$ as shown the following examples.

- The limit of

$$\lim_{n \rightarrow +\infty} e^{\sqrt{n}} = +\infty,$$

since $\lim_{n \rightarrow +\infty} \sqrt{n} = +\infty$ and $\lim_{x \rightarrow +\infty} e^x = +\infty$.

- The limit of

$$\lim_{n \rightarrow +\infty} n \sin \frac{1}{n} = 1,$$

since, $\lim_{n \rightarrow +\infty} 1/n = 0$ and $\lim_{x \rightarrow 0} \sin x/x = 1$. Note that, if we compute the limit as the product of the limits we would have an indeterminate form $\infty \cdot 0$ or, equivalently, ∞/∞ by writing $n \sin(1/n) = \sin(1/n)/(1/n)$.

Less known application of the theorem is to prove the no existence of a limit of a function. Let us prove, for example, that

$$\nexists \lim_{x \rightarrow +\infty} \sin x.$$

We recall that Theorem 1.1.13 states that the $\lim_{x \rightarrow x_0} f(x) = L$ if and only $\lim_{n \rightarrow +\infty} f(x_n) = L$ for EVERY sequences $x_n \rightarrow x_0$. This implies that if we find two sequences b_n and c_n both converging to x_0 such that $\lim_{n \rightarrow +\infty} f(b_n) \neq \lim_{n \rightarrow +\infty} f(c_n)$ then we can conclude that the limit $\lim_{x \rightarrow x_0} f(x)$ does not exists!

Indeed, if we consider $b_n = 2\pi n$ and $c_n = \pi/2 + 2\pi n$, both they are diverging to $+\infty$ and

$$\lim_{n \rightarrow +\infty} \sin(2\pi n) = 0 \neq \lim_{n \rightarrow +\infty} \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1.$$

Theorem 1.1.15 (Sandwiches Theorem). *Suppose that $a_n \leq b_n \leq c_n$ for every $n \in \mathbb{N}$ (or there exists n_0 such that the inequality is satisfied for every $n \geq n_0$). Then, if there exists $\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} c_n = L \in \mathbb{R}$ then there exists also $\lim_{n \rightarrow +\infty} b_n = L$.*

Theorem 1.1.16 (Comparison Test for sequences). *Suppose that $a_n \leq b_n$ for every $n \in \mathbb{N}$ (or there exists n_0 such that the inequality is satisfied for every $n \geq n_0$). Then,*

- if $\lim_{n \rightarrow +\infty} a_n = +\infty$ then $\lim_{n \rightarrow +\infty} b_n = +\infty$;
- if $\lim_{n \rightarrow +\infty} b_n = -\infty$ then $\lim_{n \rightarrow +\infty} a_n = -\infty$;
- if $|a_n| \leq b_n$ and $\lim_{n \rightarrow +\infty} b_n = 0$ then $\lim_{n \rightarrow +\infty} a_n = 0$.

Theorem 1.1.17 (Ratio test for sequences). *Let (a_n) be a sequence of real numbers such that $a_n > 0$ for all $n \in \mathbb{N}$ and*

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = l.$$

Then

1. if $l < 1$ then $\lim_{n \rightarrow +\infty} a_n = 0$,
2. if $l > 1$ then $\lim_{n \rightarrow +\infty} a_n = +\infty$.

Proof. If $l < 1$ then we can fix ε such that $l + \varepsilon < 1$ and, for every $n \geq n_\varepsilon$, we have that

$$\frac{a_{n+1}}{a_n} \leq (l + \varepsilon) < 1$$

which implies that

$$\begin{aligned} a_{n_\varepsilon+1} &< (l + \varepsilon)a_{n_\varepsilon} \\ a_{n_\varepsilon+2} &< (l + \varepsilon)a_{n_\varepsilon+1} < (l + \varepsilon)^2 a_{n_\varepsilon} \\ &\dots \\ 0 < a_{n_\varepsilon+k} &< (l + \varepsilon)^k a_{n_\varepsilon}. \end{aligned}$$

Since $l + \varepsilon < 1$ then $\lim_{n \rightarrow +\infty} (l + \varepsilon)^k = 0$. By Theorem 1.1.16 we get that $\lim_{n \rightarrow +\infty} a_n = 0$.

If $l > 1$ then we can fix ε such that $l - \varepsilon > 1$ and, for every $n \geq n_\varepsilon$, we have that

$$\frac{a_{n+1}}{a_n} \geq (l - \varepsilon) > 1.$$

Reasoning as above, we have that

$$a_{n_\varepsilon+k} > (l - \varepsilon)^k a_{n_\varepsilon}$$

for every $k \in \mathbb{N}$. By Theorem 1.1.16 we get that $\lim_{n \rightarrow +\infty} a_n = +\infty$. \square

Remark 1.1.18. If $l = 1$ we can not predict the limit of (a_n) . Indeed, if we consider $a_n = n$ then $a_{n+1}/a_n \rightarrow 1$, as $n \rightarrow +\infty$, and $\lim_{n \rightarrow +\infty} a_n = +\infty$, while if we consider $a_n = 3 + 1/n$ then $a_{n+1}/a_n \rightarrow 1$ and $\lim_{n \rightarrow +\infty} a_n = 3$.

In the following proposition we study how fast some sequences approach infinity.

Proposition 1.1.19 (Hierarchy of sequences). *If $\alpha > 0$ and $a > 1$ then we have*

$$\lim_{n \rightarrow +\infty} \frac{\log_a n}{n^\alpha} = 0, \quad \lim_{n \rightarrow +\infty} \frac{n^\alpha}{a^n} = 0.$$

If $a > 0$ then we have

$$\lim_{n \rightarrow +\infty} \frac{a^n}{n!} = 0, \quad \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0. \quad (1.1)$$

Proof. The proof of (1.1) is an application of the ratio test Theorem 1.1.17. \square

Example 1.1.20 (Indeterminate forms: ∞^0 , 0^0 , 1^∞). Another limit of sequence that can give rise to an indeterminate form is

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n}.$$

This limit can be rewritten as $\lim_{n \rightarrow +\infty} \sqrt[n]{n} = \lim_{n \rightarrow +\infty} f(n)^{g(n)}$ where $f(x) = x$ and $g(x) = 1/x$. We recall that if there exist finite $\lim_{x \rightarrow x_0} f(x)$ and $\lim_{x \rightarrow x_0} g(x)$ then $\lim_{x \rightarrow x_0} f(x)^{g(x)} = (\lim_{x \rightarrow x_0} f(x))^{(\lim_{x \rightarrow x_0} g(x))}$. If we apply directly Theorem 1.1.13 we get then an indeterminate form ∞^0 .

Still we can use the same approach by rewriting

$$\sqrt[n]{n} = e^{\ln \sqrt[n]{n}} = e^{\frac{1}{n} \ln n};$$

hence, by Theorem 1.1.13 and Proposition 1.1.19

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n} = e^{\lim_{n \rightarrow +\infty} \frac{1}{n} \ln n} = e^0 = 1. \quad (1.2)$$

As a consequence of the previous result we can prove that

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n} \right)^{\frac{1}{n}} = \lim_{n \rightarrow +\infty} \frac{1}{\sqrt[n]{n}} = 1.$$

Note that in this case, reasoning as above, we could have an indeterminate form by applying directly Theorem 1.1.13 as following $\lim_{n \rightarrow +\infty} f(n)^{g(n)} = 0^0$ with $f(x) = 1/x$ and $g(x) = 1/x$.

Finally, we recall that the Euler number e can be defined as limit of the sequence

$$e := \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n. \quad (1.3)$$

The proof, that such sequence converges, is an interesting application of the convergence of monotone sequence (see Theorem 1.1.21 below) to the sequence (a_{n-1}/a_n) after proving that (a_n) is bounded; i.e., $2 \leq a_n < 4$. Note that, reasoning as above, by a direct application of Theorem 1.1.13 the limit in (1.3) gives rise to the indeterminate form 1^∞ .

More in general, if $\lim_{n \rightarrow +\infty} a_n = \pm\infty$ still we may prove that

$$e = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{a_n}\right)^{a_n}.$$

Theorem 1.1.21 (Limits of monotone sequences). *Suppose (a_n) is a bounded and increasing sequence. Then there exists the limit of (a_n) and*

$$\lim_{n \rightarrow +\infty} a_n = \sup_{n \in \mathbb{N}} a_n \in \mathbb{R}.$$

If (a_n) is not bounded above then $\lim_{n \rightarrow +\infty} a_n = +\infty$.

Suppose (a_n) is a bounded and decreasing sequence. Then there exists the limit of (a_n) and

$$\lim_{n \rightarrow +\infty} a_n = \inf_{n \in \mathbb{N}} a_n \in \mathbb{R}.$$

If (a_n) is not bounded below then $\lim_{n \rightarrow +\infty} a_n = -\infty$.

Proof. Suppose $\sup_{n \in \mathbb{N}} a_n = M$. Then for given $\varepsilon > 0$, there exists n_ε such that $M - \varepsilon \leq a_{n_\varepsilon}$. Since (a_n) is increasing, we have $a_n \geq a_{n_\varepsilon}$ for all $n \geq n_\varepsilon$. This implies that $M - \varepsilon \leq a_n \leq M \leq M + \varepsilon$ for all $n \geq n_\varepsilon$. That is $\lim_{n \rightarrow +\infty} a_n = M$.

For decreasing sequences the proof is similar. □

Example 1.1.22. (1) Let $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$ for $n > 1$. Then use induction to see that $0 \leq a_n \leq 2$ and (a_n) is increasing. Therefore, by previous result (a_n) converges. Indeed, $a_2 = \sqrt{2 + \sqrt{2}}$ and $(a_2)^2 = 2 + \sqrt{2} < 4$; hence $a_2 < 2$. similarly, if $a_{n-1} < 2$ then $(a_n)^2 = 2 + a_{n-1} < 4$ which implies $a_n < 2$. Suppose $\lim_{n \rightarrow +\infty} a_n = L$. Then $\lim_{n \rightarrow +\infty} a_n = L = \sqrt{2 + (\lim_{n \rightarrow +\infty} a_{n-1})} = \sqrt{2 + L}$. This implies that $L = 2$.

(2) Prove that the

$$\lim_{n \rightarrow +\infty} \sqrt{n} - \sqrt{n+1} = 0.$$

The limit can not be computed by applying Theorem 1.1.10, it gives rise to the indeterminate form $\infty - \infty$. But, we can rationalised

$$\begin{aligned} \sqrt{n} - \sqrt{n+1} &= \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n} + \sqrt{n+1}} (\sqrt{n} + \sqrt{n+1}) \\ &= \frac{n - (n+1)}{\sqrt{n} + \sqrt{n+1}} = \frac{1}{\sqrt{n} + \sqrt{n+1}} \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$.

(3) Prove that

$$\lim_{n \rightarrow +\infty} \frac{3^n + n}{n^3 + 1} = +\infty. \quad (1.4)$$

It is convenient to rewrite the sequence as follows

$$\begin{aligned}\frac{3^n + n}{n^3 + 1} &= \frac{3^n}{n^3 + 1} + \frac{n}{n^3 + 1} \\ &= \left(\frac{3^n}{n^3}\right)\left(\frac{n^3}{n^3 + 1}\right) + \frac{1}{\frac{n^3}{n} + \frac{1}{n}}.\end{aligned}$$

By Proposition 1.1.19 we have that $(3^n/n^3) \rightarrow +\infty$, moreover by Remark 1.1.12 we have

$$\frac{n^3}{n^3 + 1} = \frac{1}{1 + \frac{1}{n^3}} \rightarrow 1, \quad \frac{1}{\frac{n^3}{n} + \frac{1}{n}} = \frac{1}{n^2 + \frac{1}{n}} \rightarrow 0,$$

which gives (1.4).

(4) Find the following limit of sequences

$$\lim_{n \rightarrow +\infty} \frac{n+2}{n^2 + 3n}.$$

It is convenient to rewrite the sequence as follows

$$\begin{aligned}0 \leq \frac{n+2}{n^2 + 3n} &= \frac{n}{n^2 + 3n} + \frac{2}{n^2 + 3n} \\ &= \left(\frac{1}{n+3}\right) + \left(\frac{2}{n^2 + 3n}\right);\end{aligned}$$

by Remark 1.1.12 we get that

$$\lim_{n \rightarrow +\infty} \frac{n+2}{n^2 + 3n} = 0.$$

(5) Find the following limit of sequences

$$\lim_{n \rightarrow +\infty} \frac{3n^3 + 2n + 1}{4n^4 + 3n^3 + 2}.$$

We divide the numerator and the denominator by n^3 then

$$\frac{3n^3 + 2n + 1}{4n^4 + 3n^3 + 2} = \frac{3 + \frac{2}{n^2} + \frac{1}{n^3}}{4n + 3 + \frac{2}{n^3}}.$$

By Remark 1.1.12 we get that

$$\lim_{n \rightarrow +\infty} 3 + \frac{2}{n^2} + \frac{1}{n^3} = 3, \quad \lim_{n \rightarrow +\infty} 4n + 3 + \frac{2}{n^3} = +\infty,$$

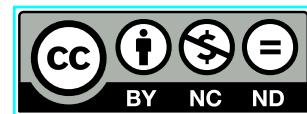
and,

$$\lim_{n \rightarrow +\infty} \frac{3n^3 + 2n + 1}{4n^4 + 3n^3 + 2} = 0.$$

Chapter 2

Riemann Integral

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2.1 Riemann Integral

We assume $f : [a, b] \rightarrow \mathbb{R}$ bounded function defined on a closed bounded interval $[a, b]$; hence, there exists $m, M \in \mathbb{R}$ such that

$$\inf f \leq m \leq f(x) \leq M \leq \sup f \quad \forall x \in [a, b].$$

A partition P of $[a, b]$ is a finite sets of points x_i (not necessarily uniformly spread) for $i = 0, \dots, n$ such that

$$a = x_0 \leq x_1 \leq \dots \leq x_i \leq \dots \leq x_n = b,$$

where we label the endpoints of the subintervals by x_0, x_1, \dots , so that the leftmost point is $a = x_0$ and the rightmost point is $b = x_n$. We denote by $I_i = [x_{i-1}, x_i]$, $i = 1, \dots, n$. There always exist

$$\begin{aligned} m_i &:= \inf\{f(x) : x \in I_i\}, \\ M_i &:= \sup\{f(x) : x \in I_i\}, \end{aligned}$$

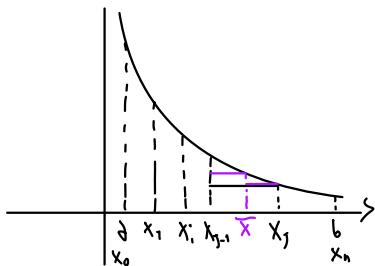
$$\begin{aligned} M &\in \mathcal{E}(x) \in M \\ J_i &= [x_{i-1}, x_i] \quad i=1, \dots, n \\ m_i &= \inf\{f(x) : x \in J_i\} \\ M_i &= \sup\{f(x) : x \in J_i\} \\ m_i, M_i &\in \mathbb{R} \\ s(f, P_n) &= \sum_{i=0}^n m_i (x_i - x_{i-1}) \end{aligned}$$

since f is bounded we have that $m_i, M_i \in \mathbb{R}$ for every $i = 0, \dots, n$. Hence, we define

$$\begin{aligned} s(f, P_n) &:= \sum_{i=1}^n m_i (x_i - x_{i-1}) \\ S(f, P_n) &:= \sum_{i=1}^n M_i (x_i - x_{i-1}), \end{aligned}$$

where $s(f, P_n)$ and $S(f, P_n)$ denote the **lower sum** and the **upper sum**, respectively, of f with respect to the partition P_n . Note that, by definition

$$s(f, P) \leq S(f, P)$$



Indeed, assume for simplicity that we add just one point $\bar{x} \in I_j$; hence, we get a new partition $P_{n+1} = \{a = x_0 \leq x_1 \leq \dots \leq x_i \leq \dots \leq x_{j-1} < \bar{x} < x_j < \dots < x_n = b\}$. We may split the interval $I_j := [x_{j-1}, \bar{x}) \cup [\bar{x}, x_j]$ and we get that

$$m_j \leq m_j^1 := \inf\{f(x) : x \in [x_{j-1}, \bar{x})\}, \quad m_j \leq m_j^2 := \inf\{f(x) : x \in [\bar{x}, x_j]\},$$

moreover,

$$m_j(x_j - x_{j-1}) = m_j(\bar{x} - x_{j-1}) + m_j(x_j - \bar{x}) \leq m_j^1(\bar{x} - x_{j-1}) + m_j^2(x_j - \bar{x}).$$

Therefore,

$$\begin{aligned} s(f, P_n) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{j-1} m_i(x_i - x_{i-1}) + m_j^1(\bar{x} - x_{j-1}) + m_j^2(x_j - \bar{x}) + \sum_{i=j+1}^n m_i(x_i - x_{i-1}) \\ &= s(f, P_{n+1}), \end{aligned}$$

similarly, we get

$$S(f, P_n) \geq S(f, P_{n+1}),$$

monotone incr. and $\lim_{n \rightarrow \infty} s(f, P_n) = \sup s(f, P_n)$ *bounded above* $s(f, P_n) \leq s(f, P_{n+1}) \leq S(f, P_{n+1}) \leq S(f, P_n)$. *monotone decr.* $\lim_{n \rightarrow \infty} S(f, P_n) = \inf S(f, P_n)$ *bounded below*

Hence, it turns out that $s(f, P_n)$ and $S(f, P_n)$ have a monotone behaviour and we may define the **lower** and **upper integral** as

$$s(f) := \sup_n s(f, P_n) \leq S(f) := \inf_n S(f, P_n),$$

respectively.

The following example clarify the role of lower and upper sum and lower and upper integral in the Definition 2.1.5 of the Riemann integral.

Example 2.1.1 (Method of exhaustion to compute the area underneath the graph of a parabola). Let us consider an increasing positive monotone function $f(x) = x^2$ in $[0, 1]$. We investigate the area of the region that is under the graph of $f(x)$ and above the interval $[a, b]$ on the x -axis. The region under the graph of $f(x)$ has such a strange shape, calculating its area is too difficult. But calculating the area of rectangles is simpler! Let's simplify our life by pretending the region is composed of a bunch of rectangles. We divide $[0, 1]$ into n subintervals of length $x_i - x_{i-1} = (1/n)$ and we construct rectangles with base given by $(x_i - x_{i-1})$ and height equal to M_i or m_i . Hence, by adding up the areas of all the rectangles we get

• Intervals are equal, so if I have g intervals, one int. is $\frac{1}{g}$. If I have n int., one int. = $\frac{1}{n}$

$$\begin{aligned} s(f, P_n) &= \sum_{i=1}^n \frac{m_i}{n} = \sum_{i=1}^n m_i \sqrt{\frac{1}{n}(x_i - x_{i-1})} \\ S(f, P_n) &= \sum_{i=1}^n \frac{M_i}{n}, \end{aligned}$$

where $s(f, P_n)$ and $S(f, P_n)$ denote the **lower sum** and the **upper sum**, respectively, of f with respect to the partition P_n . The lower and upper sum $s(f, P_n)$ and $S(f, P_n)$, represent only an approximation to the actual area underneath the graph of f ; i.e.,

$$\sum_{i=1}^n \frac{m_i}{n} \leq \text{Area}(\mathcal{G}_f) \leq \sum_{i=1}^n \frac{M_i}{n}$$

Da FARE!

where $\mathcal{G}_f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \in [a, b], 0 \leq y \leq x^2\}$. Since f is an increasing function we have that $m_i = f(x_{i-1})$ and $M_i = f(x_i)$; hence,

$$s(f, P_n) = \frac{1}{n} \sum_{i=1}^n \left(\frac{i-1}{n} \right)^2 = \left(\frac{1}{n} \right)^3 \sum_{k=0}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6},$$

No

similarly,

$$S(f, P_n) = \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n} \right)^2 = \left(\frac{1}{n} \right)^3 \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Heart

To make the approximation better, we have to increase the number of points x_i which makes increase the number of subintervals; as already observed, we have that $s(f, P_n) \leq s(f, P_{n+1}) \leq S(f, P_{n+1}) \leq S(f, P_n)$. Therefore,

$$s(f, P_n) = \frac{(n-1)n(2n-1)}{6n^3} \leq \text{Area}(\mathcal{G}_f) \leq \frac{n(n+1)(2n+1)}{6n^3} = S(f, P_n),$$

for every $n \in \mathbb{N}$, and

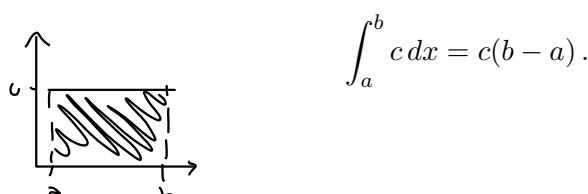
$$\begin{aligned} \sup_n \left(\frac{(n-1)n(2n-1)}{6n^3} \right) &= \lim_{n \rightarrow +\infty} \frac{(n-1)n(2n-1)}{6n^3} \\ &= \frac{1}{3} \\ &= \lim_{n \rightarrow +\infty} \frac{n(n+1)(2n+1)}{6n^3} = \inf_n \left(\frac{n(n+1)(2n+1)}{6n^3} \right). \end{aligned}$$

Hence, by the comparison theorem for sequences, we have that

$$\begin{aligned} \text{Area}(\mathcal{G}_f) &= \sup_n s(f, P_n) \\ &= \inf_n S(f, P_n) \\ &= \frac{1}{3}. \end{aligned}$$

The example shows that when the number of subintervals increase the lower and the upper sum approach the area $\text{Area}(\mathcal{G}_f)$ from above and from below therefore is quite intuitive to reach $\text{Area}(\mathcal{G}_f)$ as the infimum of $S(f, P)$ among all possible partition P as well as the supremum of $s(f, P)$.

Example 2.1.2. Let $f(x) = c$, with $c \in \mathbb{R}$, then f is integrable and the integral is



Indeed,

$$s(f, P) = \sum_{i=1}^n c(x_i - x_{i-1}) = S(f, P) = c(b - a);$$

hence,

$$s(f) = S(f) = c(b - a).$$

If f is bounded but not necessarily continuous in $[a, b]$ we can generalize the definition of $s(f, P)$ and $S(f, P)$ by using inf and sup instead of min and max. More precisely, we denote by

$$m_i := \inf_{x \in [x_{i-1}, x_i]} f(x), \quad M_i := \sup_{x \in [x_{i-1}, x_i]} f(x).$$

We are now ready to define the Riemann integral.

a and b are numbers

Definition 2.1.5. Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function with $-\infty < a < b < +\infty$. We define lower Riemann integral and upper Riemann integral as

$$s(f) := \sup_P s(f, P_n), \quad S(f) := \inf_P S(f, P_n);$$

respectively, where

$$\lim_{n \rightarrow \infty} s(f, P_n) \quad \lim_{n \rightarrow \infty} S(f, P_n)$$

$$s(f, P_n) = \sum_{i=1}^n m_i(x_i - x_{i-1})$$

$$S(f, P_n) = \sum_{i=1}^n M_i(x_i - x_{i-1}),$$

with

$$m_i := \inf_{x \in [x_{i-1}, x_i]} f(x), \quad M_i := \sup_{x \in [x_{i-1}, x_i]} f(x).$$

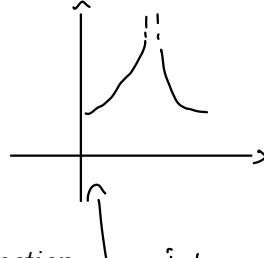
We say that f is **Riemann integrable** if

$$s(f) = S(f).$$

We call the (Riemann) **Integral** of f over $[a, b]$ the real number

$$\int_a^b f(x) dx := s(f) = S(f).$$

Math vocabulary: a and b are also called extremes of integration, f is the integrand, dx states that “it’s summing” all increments of the function f . Since the extremes of integration are fixed we call $\int_a^b f(x) dx$ also definite integral.



Theorem 2.1.7. Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function.

a If f is continuous on $[a, b]$ then f is Riemann integrable.

b If f is monotone on $[a, b]$ then f is Riemann integrable.

c If f has a finitely many discontinuities (i.e. f is continuous on $[a, b]$ except on a finite number of points), then f is Riemann integrable.

To be monotone it can be discontinuous like

2.2 Properties of Riemann Integral

From now on we will refer to Riemann integrable function or Riemann integral as integrable function or integral, respectively.

Let f and g be integrable functions on $[a, b]$ and let $\lambda \in \mathbb{R}$. Then:

- 1) (**Additivity**) the integral of the sum or difference of two functions can be computed by integrated each term separately

$$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx; \quad (2.2)$$

- 2) (**Linearity**) a constant factor $\lambda \in \mathbb{R}$ can be moved outside the integral sign

$$\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx; \quad (2.3)$$

- 3) (**Additivity with respect to the interval of integration**): for every $c \in (a, b)$ we have that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx; \quad (2.4)$$

- 4) (**Positivity**) if $f \geq 0$ (respectively, $f(x) \leq 0$) then

$$\int_a^b f(x) dx \geq 0 \quad (\text{resp. } \leq 0). \quad (2.5)$$

As a consequence of (1)-(2)-(4) we have the following comparison properties between integrals:

5) (**Monotonicity**) If $f \leq g$ on $[a, b]$ then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx ; \quad (2.6)$$

6) (**Modulus**)

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx . \quad (2.7)$$

Finally, note that so far we always assume that $a < b$ for the intervals of integration. In general, we set

- $\int_a^a f(x) dx = 0 ;$
- $\int_b^a f(x) dx = - \int_a^b f(x) dx .$

MEAN VALUE THEOREM

Claim: There exist at least one point z s.t. $f(z) = \frac{\int_a^b f(x) dx}{(b-a)}$ with $z \in [a, b]$

Assume: $f(x)$ is continuous in $[a, b]$

Proof: Since $f(x)$ is continuous in $[a, b]$ then by Weierstrass thm $f(x)$ takes its minimum and maximum value in $[a, b] \Rightarrow m \leq f(x) \leq M$

By integral properties, $\int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx$ interval of a constant $\rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$ divide $\rightarrow m \leq \frac{\int_a^b f(x) dx}{(b-a)} \leq M$

By the intermediate value theorem, the function take at least one time all the values between the minimum and the maximum, so there must exist a point $z \in [a, b]$ s.t. $f(z) = \frac{\int_a^b f(x) dx}{(b-a)}$

Fundamental theorem of integral calculus

Claim: There exist the derivative of the integral function of $f(x)$, that is $F(x) = \int_a^x f(t) dt \quad \forall x \in [a, b]$, and it is $F'(x) = f(x)$

Assume: $f(x)$ is continuous in $[a, b]$

Proof: We increment x with a value $h > 0$ s.t. $a < x+h < b$ and we evaluate $F(x+h) - F(x)$ using integrals

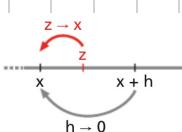
$$F(x+h) - F(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt \rightarrow \int_a^x f(t) dt + \int_x^{x+h} f(t) dt - \int_a^x f(t) dt \rightarrow \int_x^{x+h} f(t) dt$$

By the mean value theorem, $\int_x^{x+h} f(t) dt$ is equal to $h \cdot f(z)$ with $z \in [x; x+h]$ or $[x+h; x]$

$$F(x+h) - F(x) = h \cdot f(z) \rightarrow \frac{F(x+h) + F(x)}{h} = f(z)$$

If $h > 0$ then z is between x and $x+h$, so if $h \rightarrow 0^+$ so $z \rightarrow x^+$, so $\lim_{h \rightarrow 0^+} f(z) = \lim_{z \rightarrow x^+} f(z) = f(x)$

Same reasoning for $h < 0$: $\lim_{h \rightarrow 0^-} f(z) = \lim_{z \rightarrow x^-} f(z) = f(x)$. So $\lim_{h \rightarrow 0} f(z) = \lim_{z \rightarrow x} f(z) = f(x)$

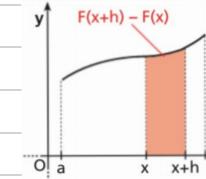
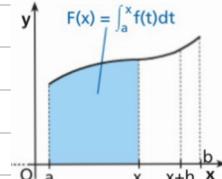
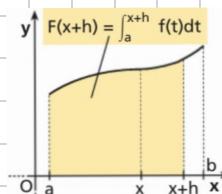


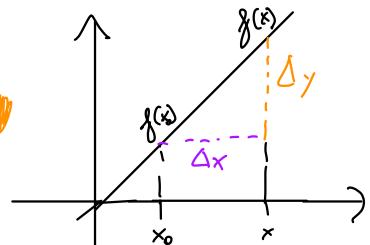
$$\text{We proved that } \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} f(z) = f(x) \rightarrow F'(x) = f(x)$$

Now let $G'(x) = f(x)$, then $F'(x) - G'(x) = 0 \rightarrow F(x) - G(x) = c$. Now fix $x = a$: $F(a) - G(a) = c \rightarrow \int_a^a f(t) dt - G(a) = c$

Now we have that $c = -G(a)$, substitute in the equation: $F(x) = G(x) - G(a) \rightarrow \int_a^x f(t) dt = G(x) - G(a) \quad \forall x \in [a, b]$

In general $\int_a^b f(t) dt = G(b) - G(a) \xrightarrow{\text{derive both}} \int_a^b f'(t) dt = f(b) - f(a)$





$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

$\Delta x \rightarrow 0 \Leftrightarrow x \rightarrow x_0$

CHAPTER 2. RIEMANN INTEGRAL

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

□

Remark 2.2.5. Note that if F and G are both primitive of f then F and G differ by an additive constant; i.e.,

$$G(x) = F(x) + c,$$

$$c \in \mathbb{R}.$$

--

Definition 2.2.6 (Indefinite Integral). *The indefinite integral of f is the set of all possible primitives or antiderivatives of f ; i.e.,*

$$\int f(x) dx = \{F(x) + c, \quad \forall c \in \mathbb{R}\}.$$

Chapter 3

Improper Integral

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The Riemann integral has been defined for functions $f : [a, b] \mapsto \mathbb{R}$ satisfying the following assumptions:

- (a) $[a, b]$ is a bounded interval;
- (b) f is a bounded function.

Now the question is:

What does it happen when we remove one or both assumptions?

From the geometric interpretation of the integral we may immediately observe that we pass from a bounded region (underneath the graph of f) to an unbounded region! Still, is it possible to have (at least in some cases, we don't expect always....) a finite area?

The answer is:

We may still have, in some cases, that the ‘Integral’ is finite!

Let us consider the following function

$$f(x) = \frac{1}{x^\alpha}, \quad x > 0.$$

The function f is not bounded in $(0, 1]$ (or in any interval $(0, a]$) since $\lim_{x \rightarrow 0^+} 1/x^\alpha = +\infty$; hence, we can not perform the Riemann integral on $(0, 1]$ but on $[a, 1]$ for every $a > 0$. More precisely,

$$\int_a^1 \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{1-\alpha}(1-a^{1-\alpha}) & \text{if } \alpha \neq 1 \\ -\ln a & \text{if } \alpha = 1, \end{cases} \quad \begin{aligned} b=0 & \rightarrow \int_b^1 \frac{1}{x^\alpha} dx = \left[\frac{x^{-\alpha}}{-\alpha} \right]_b^1 \\ & \frac{1}{1-\alpha} - \frac{b^{1-\alpha}}{1-\alpha} = \frac{1}{1-\alpha} \left(1 - b^{1-\alpha} \right) \end{aligned}$$

passing to the limit on a we get

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{1-\alpha} & \text{if } \alpha < 1 \\ +\infty & \text{if } \alpha \geq 1. \end{cases} \quad \begin{aligned} \text{if } 1-\alpha > 0 & \rightarrow \lim_{a \rightarrow 0^+} \frac{1}{1-\alpha} (1-a^{1-\alpha}) = \frac{1}{1-\alpha} (1-0^{1-\alpha}) = \frac{1}{1-\alpha} \\ \text{if } 1-\alpha < 0 & \rightarrow \lim_{a \rightarrow 0^+} \frac{1}{1-\alpha} (1-a^{1-\alpha}) = \frac{1}{1-\alpha} (1-\frac{1}{a^{1-\alpha}}) \stackrel{(3.1)}{=} \frac{1}{1-\alpha} (1-\infty) = -\infty \end{aligned}$$

$$37 \quad \alpha = 1 \rightarrow \lim_{a \rightarrow 0^+} -\ln a = +\infty$$

Let us consider the function f on $[1, +\infty)$. Again we can not apply the theory of Riemann integral since $[1, +\infty)$ is an unbounded interval. We may reason as above; hence,

$$\int_1^L \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{1-\alpha}(L^{1-\alpha} - 1) & \text{if } \alpha \neq 1 \\ \ln L & \text{if } \alpha = 1 \end{cases}$$

$$\left| \int_1^L \frac{1}{x^\alpha} dx \right| = \begin{cases} \left| \frac{1}{1-\alpha}(L^{1-\alpha} - 1) \right| & \text{if } \alpha \neq 1 \\ |\ln L| & \text{if } \alpha = 1 \end{cases}$$

passing to the limit on L we get

$$\lim_{L \rightarrow +\infty} \int_1^L \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1. \end{cases} \quad (3.2)$$

This example suggests that we may introduce a notion of integrability for unbounded functions and unbounded intervals such that

$\frac{1}{x^\alpha}$ is “integrable” on $(0, 1]$ for every $\alpha < 1$ and on $[1, +\infty)$ for every $\alpha > 1$.

3.1 Definitions and Examples

Definition 3.1.1 (Improper Integral-def1). *Let $f : (a, b] \mapsto \mathbb{R}$ be a Riemann integrable function in $[c, b]$ for every $c \in (a, b)$. Then the improper integral of f in $(a, b]$ is defined as*

*let y not bounded $\leftarrow \int_a^b f(x) dx := \lim_{c \rightarrow a^+} \int_c^b f(x) dx \in \mathbb{R}$. \nearrow number = improper integral exists
 \searrow $\int_a^b f(x) dx$ doesn't exist*

Similarly, let $f : [a, +\infty) \mapsto \mathbb{R}$ be a Riemann integrable function on $[a, L]$ for every $L > a$. Then the improper integral of f in $[a, +\infty)$ is defined as

*let x not bounded $\int_a^{+\infty} f(x) dx := \lim_{L \rightarrow +\infty} \int_a^L f(x) dx \in \mathbb{R}$. \nearrow number = improper integral exists
 \searrow $\int_a^{+\infty} f(x) dx$ doesn't exist*

Remark 3.1.2. Similarly, if $f : [a, b] \mapsto \mathbb{R}$ is Riemann integrable in $[a, c]$ for every $c \in (a, b)$, then the improper integral of f in $[a, b]$ is given by

$$\int_a^b f(x) dx := \lim_{c \rightarrow b^-} \int_a^c f(x) dx \in \mathbb{R};$$

if $f : (-\infty, b] \mapsto \mathbb{R}$ is Riemann integrable in $[L, b]$ for every $L < b$, then the improper integral of f in $(-\infty, b]$ is given by

$$\int_{-\infty}^b f(x) dx := \lim_{L \rightarrow -\infty} \int_L^b f(x) dx \in \mathbb{R}.$$

Definition 3.1.4 (Improper Integral-def2). *Let $-\infty \leq a \leq b \leq +\infty$ and let $f : (a, b) \mapsto \mathbb{R}$ be a Riemann integrable function in $[c, d]$ for every $[c, d] \subset (a, b)$. Then there exists the improper integral of f in (a, b) if there exist the two following limit*

$$l_- := \lim_{c \rightarrow a^+} \int_c^{x_0} f(x) dx \in \mathbb{R}, \quad l_+ := \lim_{d \rightarrow b^-} \int_{x_0}^d f(x) dx \in \mathbb{R}.$$

and

$$\int_a^b f(x) dx := l_- + l_+,$$

for any arbitrary fixed $x_0 \in \mathbb{R}$. In particular,

$$\int_{\mathbb{R}} f(x) dx = \int_{-\infty}^{+\infty} f(x) dx := \lim_{c \rightarrow -\infty} \int_c^{x_0} f(x) dx + \lim_{d \rightarrow +\infty} \int_{x_0}^d f(x) dx.$$



Theorem 3.2.1 (Comparison Test). *Let $f, g : [a, b] \mapsto \mathbb{R}$ be two Riemann integrable functions on every $[a, c]$ for any $c \in (a, b)$ and $b \leq +\infty$. Let us assume that*

$$0 \leq f(x) \leq g(x), \quad \forall x \in [a, b].$$

Then,

$$0 \leq \int_a^b f(x) dx \leq \int_a^b g(x) dx,$$

and

$$\begin{aligned} \text{if } \int_a^b f(x) dx = +\infty &\implies \int_a^b g(x) dx = +\infty; \\ \text{if } \int_a^b g(x) dx < +\infty &\implies \int_a^b f(x) dx < +\infty. \end{aligned}$$

Example 3.2.2. prove that the following improper integral exists and is finite

$$\int_1^{+\infty} \frac{|\cos x|}{x^2} dx.$$

It is not easy to find the primitive of $|\cos x|/x^2$ but we can establish by comparison if the integral is finite or not. Indeed,

$$0 \leq \frac{|\cos x|}{x^2} \leq \frac{1}{x^2}, \quad \forall x \geq 1,$$

moreover, by (3.2), we have that

$$\int_1^{+\infty} \frac{1}{x^2} dx = 1;$$

hence, by Theorem 3.2.1, we have that

$$\int_1^{+\infty} \frac{|\cos x|}{x^2} dx \leq 1,$$

which implies not only the existence of the improper integral but also an estimate for above.

Exercise 3.2.3. Say if the following improper integral is finite or infinite

$$\int_0^1 \frac{1+e^x}{x} dx.$$

Theorem 3.2.4 (Asymptotic Comparison Test). *Let $f, g : [a, b) \mapsto \mathbb{R}$ be two Riemann integrable function on every $[a, c]$ for any $c \in (a, b)$ and $b \leq +\infty$. Let us assume that $f(x) \geq 0$ and $g(x) > 0$ for every $x \in [a, b)$.*

- Assume that

$$\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = l > 0,$$

then,

$$\begin{aligned} \text{if } \int_a^b f(x) dx = +\infty &\iff \int_a^b g(x) dx = +\infty; \\ \text{if } \int_a^b g(x) dx < +\infty &\iff \int_a^b f(x) dx < +\infty. \end{aligned}$$

- Assume that

$$\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = 0,$$

then,

$$\begin{aligned} \text{if } \int_a^b f(x) dx = +\infty &\implies \int_a^b g(x) dx = +\infty; \\ \text{if } \int_a^b g(x) dx < +\infty &\implies \int_a^b f(x) dx < +\infty. \end{aligned}$$

- Assume that

$$\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = +\infty,$$

then,

$$\begin{aligned} \text{if } \int_a^b g(x) dx = +\infty &\implies \int_a^b f(x) dx = +\infty; \\ \text{if } \int_a^b f(x) dx < +\infty &\implies \int_a^b g(x) dx < +\infty. \end{aligned}$$

Chapter 4

Numerical Series

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Given a sequence of real numbers $(a_n) = \{a_0, a_1, a_2, \dots, a_n, \dots\}$ we associate to (a_n) a recursive sequence (s_n) defined as

$$\begin{aligned} s_0 &= a_0 \\ s_{n+1} &= s_n + a_{n+1}, \quad n \in \mathbb{N}; \end{aligned}$$

that is,

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ s_2 &= a_0 + a_1 + a_2 \\ &\dots \\ s_n &= a_0 + \dots + a_n. \end{aligned}$$

$$(s_n) = \text{Series} = \sum_{h=0}^{\infty} a_h$$

$$S_n = \text{Partial sum} = \sum_{k=0}^n a_k$$

$$(a_n) = \text{Set of sequences} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \right\}$$

$$a_n = \text{Sequence element} = \frac{1}{n}$$

$$\text{or } \forall n \in \mathbb{N} = \text{every single element} = \frac{1}{2} \text{ or } \frac{1}{3} \text{ or } \frac{1}{4} \dots$$

$$\begin{aligned} S_0 &= a_0 & (S) &= \sum_{h=0}^{\infty} a_h \\ S_{n+1} &= S_n + a_{n+1} & \end{aligned}$$

$$\begin{aligned} R_n &= S - S_n = \sum_{h=0}^{\infty} a_h - \sum_{k=0}^n a_k = \sum_{k=n+1}^{\infty} a_k \end{aligned}$$

The sequence (s_n) is called *series* and it is denoted by

$$\sum_{n=0}^{\infty} a_n,$$

the single element of the sequence $s_n = \sum_{k=0}^n a_k$ is called *partial sums*. Then, (s_n) is also called the sequence of partial sums.

Definition 4.0.1. We say that a series is convergent if there exists finite the limit $\lim_{n \rightarrow +\infty} s_n = s \in \mathbb{R}$, it is called the sum of the series and it is denoted by

$$s = \sum_{n=0}^{\infty} a_n. \longrightarrow \lim_{n \rightarrow +\infty} \sum_{k=0}^n a_k = L$$

The series is divergent if $\lim_{n \rightarrow +\infty} s_n = \pm\infty$. The series is undetermined if does not exists $\lim_{n \rightarrow +\infty} s_n$. The remainder associated to the series is defined by

$$R_n := s - s_n = \sum_{k=n+1}^{+\infty} a_k. \quad S = \sum_{k=0}^{\infty} a_k - \sum_{k=0}^n a_k = \sum_{k=n+1}^{\infty} a_k$$

Note that the series converges if and only if $\lim_{n \rightarrow +\infty} R_n = 0$.

Example 4.0.2 (Geometric series). The geometric series is defined as

$$\sum_{k=0}^{\infty} x^k, \quad S_0 = x^0, \quad S_1 = x^0 + x^1, \quad S_2 = x^0 + x^1 + x^2 / S_1 + x^k$$

for every $x \in \mathbb{R}$. Note that $x = x^{k+1}/x^k$; i.e., x represents a common ratio between all terms of the series that is constant. It is easy to prove that the partial sum is given by

- $x = 1$
 $\lim_{n \rightarrow +\infty} n+1 = +\infty$
- $x > 1$
 $\lim_{n \rightarrow +\infty} \frac{1-x^{n+1}}{1-x} = \frac{-\infty}{\text{neg. num.}} = +\infty$
- $x < -1$
 $\lim_{n \rightarrow +\infty} \frac{1-x^{n+1}}{1-x} = \frac{1-(-1)^{n+1}(|x|)^{n+1}}{1-x} = \infty \text{ because of } (-1)^{n+1}$
- $0 \leq x < 1$ (proper fraction)
 $x = \frac{1}{p}$ with $p > 1$
 $\lim_{n \rightarrow +\infty} \frac{1-x^{n+1}}{1-x} = \frac{1-\frac{1}{p^{n+1}}}{1-x} = \frac{1}{1+x}$
- $-1 < x < 0$ → negative proper fraction
 $\lim_{n \rightarrow +\infty} \frac{1-x^{n+1}}{1-x} = \frac{1-(-1)^{n+1}\left(\frac{1}{p}\right)^{n+1}}{1-x} = \frac{1}{1-x}$

REMEMBER: product between a bounded sequence $(-1)^{n+1}$ and a sequence that goes to 0 is 0
 $\Rightarrow (-1)^{n+1} \cdot \left(\frac{1}{p}\right)^{n+1} = 0$

$$s_n = \begin{cases} \frac{1-x^{n+1}}{1-x} & \text{if } x \neq 1 \\ n+1 & \text{if } x = 1. \quad (1)^n \rightarrow S_0=1 \quad S_1=2 \quad S_2=3 \end{cases}$$

$$\lim_{n \rightarrow +\infty} s_n = \begin{cases} \frac{1}{1-x} & -1 < x < 1 \\ +\infty & \text{if } x \geq 1 \\ \# & \text{if } x \leq -1; \end{cases}$$

(4.1)

CONVERGE

DIVERGE



Theorem 4.0.3. CAUCHY CRITERION

A series converges $\iff \forall \varepsilon > 0 \exists n_\varepsilon \in \mathbb{N}$ st.

$$\left| \sum_{k=n_\varepsilon}^{n+p} a_k \right| < \varepsilon \quad \forall n > n_\varepsilon \text{ and } \forall p \in \mathbb{N}$$

PROOF: Assume (s_n) converges

By thm 4.1.9, s_n converges $\iff s_n$ is a Cauchy sequence

$\Rightarrow s_n$ is a Cauchy sequence

$$\downarrow \quad \forall \varepsilon > 0 \exists \gamma \in \mathbb{R} \text{ st. } |s_m - s_n| < \varepsilon \quad \forall m, n > \gamma$$

$$\Rightarrow |s_{m+n} - s_n| < \varepsilon, \Rightarrow \left| \sum_{k=0}^m a_k - \sum_{k=0}^{n-1} a_k \right| < \varepsilon \Rightarrow \left| \sum_{k=n+1}^m a_k \right| < \varepsilon$$

Assume $m > n$, with $m = n+p$

$$\Rightarrow \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \quad \forall n > \gamma \text{ and } \forall p \in \mathbb{N}$$

Corollary: If the series $\sum_{n=0}^{\infty} \alpha_n$ converges then the $\lim_{n \rightarrow \infty} \alpha_n = 0$

Assume: (α_n) converges, so it's a cauchy sequence and $\left| \sum_{k=n+1}^{n+p} \alpha_k \right| < \epsilon \forall n > N$ and $\forall p \in \mathbb{N}$

Proof: $\left| \sum_{k=n+1}^{n+p} \alpha_k \right| < \epsilon \xrightarrow{\text{set } p=1} |\alpha_{n+1}| < \epsilon \xrightarrow{\text{same as}} |\alpha_{n+1} - 0| < \epsilon \xrightarrow{\lim_{n \rightarrow \infty} \alpha_n = 0}$

Definition of converging sequence: $\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |\alpha_n - L| < \epsilon \forall n > N$

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Example 4.0.5. The $\lim_{n \rightarrow +\infty} a_n = 0$ is not sufficient to guarantee that the associated series $\sum_{n=0}^{\infty} a_n$ converges. Indeed, let us consider

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

$$\begin{array}{l} \lim_{n \rightarrow \infty} a_n \neq 0 \\ \text{NECESSARY CONDITION} \end{array} \Rightarrow \sum_{n=0}^{\infty} a_n = \infty$$

$$\lim_{n \rightarrow \infty} a_n = 0 \quad \cancel{\Rightarrow} \quad \sum_{n=0}^{\infty} a_n < \infty$$

it is called harmonic series. Then if we choose $m = 2n$ we have that

So the series converges for $\forall \epsilon > 0$,

but for $\forall \epsilon > 0$.

Then it's not a cauchy sequence \Rightarrow DIVERGES

Cauchy sequence: $\epsilon > |S_m - S_n| \xrightarrow{m=2n} \epsilon > \sum_{k=0}^{2n} \alpha_k - \sum_{k=0}^n \alpha_k \xrightarrow{\epsilon > \sum_{k=n+1}^{2n} \alpha_k} \epsilon > \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq n \frac{1}{2n} = \frac{1}{2}$

4.0.1 Series with non-negative terms

A series $\sum_{n=0}^{\infty} a_n$ with non-negative terms $a_n \geq 0$, $n \in \mathbb{N}$, it is always convergent or divergent since the sequence of partial sums (s_n) is monotone increasing; i.e.,

Never $\cancel{\exists}$ if $s_0 = 3$ and $s_1 = 2$, it means that $s_2 = -1$. In general if $s_{n+1} > s_n$, then $a_n > 0$

$$s_{n+1} = s_n + a_{n+1} \geq s_n, \quad \forall n \in \mathbb{N};$$

hence, as already observed in Theorem 1.1.21, there always exists

$$\lim_{n \rightarrow +\infty} s_n = \sup_{n \in \mathbb{N}} s_n \leq +\infty.$$

if $\sup s_n = +\infty$, the series DIVERGES

if $\sup s_n < +\infty$, $\sup s_n$ is a number, CONVERGES

In particular, a series with non-negative terms converges if and only if the sequence of partial sums (s_n) is bounded. Because monotone and bounded = converges

Here below, we collect some methods that allow to study the divergence or convergence of a series with non negative terms.

Theorem 4.0.6 (Integral test). Let $f : [1, +\infty) \mapsto [0, +\infty)$ be a monotone decreasing function. Then

$$\sum_{n=1}^{\infty} f(n), \quad \text{and} \quad \int_1^{\infty} f(x) dx$$

converge or diverge simultaneously.

Proof. Let us consider a partition of $[1, n]$ in subintervals of width 1; i.e., $x_1 = 1, \dots, x_k = k, x_{k+1} = (k+1), \dots, x_n = n$. Since f is a decreasing function we have that the Riemann Integral on $[1, n]$ can be approximated from below and from above by the following lower and upper sum

$$\sum_{k=1}^{n-1} f(k+1) = \sum_{k=1}^{n-1} m_k \leq \int_1^n f(x) dx = \sum_{k=1}^{n-1} \int_k^{k+1} f(x) dx \leq \sum_{k=1}^{n-1} M_k = \sum_{k=1}^{n-1} f(k); \quad (4.2)$$

respectively. Note that, the sequence of partial sums associated to the series $\sum_{n=1}^{\infty} f(n)$ is given by

$$s_{n-1} = \sum_{k=1}^{n-1} f(k).$$

Similarly,

$$\sum_{k=1}^{n-1} f(k+1) = \sum_{h=2}^n f(h) = s_n - s_1.$$

Therefore,

$$s_n - s_1 \leq \int_1^n f(x) dx \leq s_{n-1},$$

which implies that series and improper integral has the same behaviour; *i.e.*, both converges or diverges. Indeed,

$$\lim_{n \rightarrow +\infty} s_n - s_1 \leq \lim_{n \rightarrow +\infty} \int_1^n f(x) dx \leq \lim_{n \rightarrow +\infty} s_{n-1}.$$

□

Example 4.0.7 (α -series). We consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \quad \alpha > 0.$$

The function $1/x^\alpha$, $\alpha > 0$, satisfies the assumption of Theorem 4.0.6. Moreover, by (3.2), we have that

$$\int_1^{\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1; \end{cases}$$

hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \begin{cases} \text{converges} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1. \end{cases} \quad (4.3)$$

Theorem 4.0.8 (Comparison test). Let (a_n) and (b_n) be nonnegative sequences and assume that there exists $N \in \mathbb{N}$ such that

$$a_n \leq b_n, \quad n \geq N.$$

Then, we have

- if $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges, PROOF
→ $a_n \leq b_n$ → b_n must be $+\infty$
- if $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges. → $a_n \leq L$ → a_n must be a number

Theorem 4.0.9 (Limit Comparison Test). Let (a_n) be a nonnegative sequence, let (b_n) be a positive sequence and assume that

$$\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = L.$$

Then, we have

- if $L \in (0, +\infty)$ then the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge or diverge simultaneously,

- if $L = 0$ then

$$\sum_{n=0}^{\infty} b_n < +\infty \implies \sum_{n=0}^{\infty} a_n < +\infty$$

or

$$\sum_{n=0}^{\infty} a_n = +\infty \implies \sum_{n=0}^{\infty} b_n = +\infty,$$

- if $L = +\infty$ then

$$\sum_{n=0}^{\infty} a_n < +\infty \implies \sum_{n=0}^{\infty} b_n < +\infty$$

or

$$\sum_{n=0}^{\infty} b_n = +\infty \implies \sum_{n=0}^{\infty} a_n = +\infty.$$

Proof. If $L \in (0, +\infty)$, then we may choose $\varepsilon > 0$ such that $L - \varepsilon > 0$; hence,

$$\left| \frac{a_n}{b_n} - L \right| < \varepsilon \implies \underbrace{(L - \varepsilon)b_n < a_n < (L + \varepsilon)b_n}_{\text{COMPARISON}} \quad \forall n \geq n_\varepsilon. \quad \text{comparison on each side}$$

If $L = 0$ then, by definition of limit, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\left| \frac{a_n}{b_n} - 0 \right| < \varepsilon$

$$-\varepsilon < 0 \leq \frac{a_n}{b_n} < \varepsilon \implies \underbrace{a_n < \varepsilon b_n}_{\text{COMPARISON}}, \quad \forall n \geq n_\varepsilon.$$

Finally, if $L = +\infty$ then, for every $M > 0$ there exists $n_\varepsilon \in \mathbb{N}$

$$\frac{a_n}{b_n} > M \implies a_n > Mb_n, \quad \forall n \geq n_\varepsilon.$$

DIVERGES: $\forall M > 0 \exists n_0 \in \mathbb{N}$ s.t. $a_n > M \quad \forall n > n_0$

By Comparison Test 4.0.8, we get the thesis. □

Theorem 4.0.12 (Root test). Let (a_n) be a nonnegative sequence and assume that there exists $L \in [0, 1)$ and $N \in \mathbb{N}$ such that

not $-1 < L < 1$ Cuz $a_n > 0$

$$\sqrt[n]{a_n} \leq L, \quad n \geq N,$$

then, the series converges. If there exists $N \in \mathbb{N}$ such that PROOF: $\sqrt[n]{a_n} \leq L \rightarrow a_n \leq L^n \rightarrow \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} L^n \rightarrow$ GEOMETRICAL series
 $\sqrt[n]{a_n} \geq 1, \quad n \geq N,$ $L > 1 \rightarrow$ DIVERGES

then, the series diverges to $+\infty$.

As an immediate consequence we have that

Corollary 4.0.13 (Limit Root test). Let (a_n) be a nonnegative sequence and assume that

$$\lim_{n \rightarrow +\infty} \sqrt[n]{a_n} = L.$$

Then, we have

- if $L < 1$ then the series converges,

if $L=1$ you don't know
 (s_n) can be $<+\infty$ or $=+\infty$

- if $L > 1$ then the series diverges.

Theorem 4.0.15 (Ratio test). Let (a_n) be a positive sequence and assume that there exists $L \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \leq L, \quad n \geq N,$$

then, the series converges. If

$$a_{n+1} \geq a_n, \quad n \geq N, \Rightarrow \frac{a_{n+1}}{a_n} > L \text{ with } L > 1$$

then, the series diverges to $+\infty$.

PROOF OF LIMIT RATIO TEST

Assume: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1 \iff \forall \epsilon > 0 \exists n_0 \in \mathbb{N} \text{ s.t. } \left| \frac{a_{n+1}}{a_n} - L \right| < \epsilon \forall n > n_0$

Assume $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L \iff \forall \epsilon > 0 \exists n_0 \in \mathbb{N} \text{ s.t. } \left| \frac{a_{n+1}}{a_n} - L \right| < \epsilon \forall n > n_0$

$$L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon$$

$$a_{n+1} < a_n(L + \epsilon)$$

$$a_{n+2} < a_{n+1}(L + \epsilon) < a_n(L + \epsilon)^2$$

⋮

$$a_{n+k} < (L + \epsilon)^k a_n \quad \forall n > n_0 \text{ with } k = n - n_0$$

$$\Rightarrow \sum_{n=n_0}^{\infty} a_n < a_n \sum_{n=n_0}^{\infty} (L + \epsilon)^{n-n_0} \xrightarrow{(L+\epsilon)^n} \frac{(L+\epsilon)^n}{(L+\epsilon)^{n_0}}$$

$$\sum_{n=n_0}^{\infty} a_n < \frac{a_{n_0}}{(L+\epsilon)^{n_0}} \sum_{n=n_0}^{\infty} (L+\epsilon)^n$$

If $L < 1$, we can write $L < 1 - \epsilon$ with a proper ϵ

$$\Rightarrow L + \epsilon < 1$$

$\Rightarrow \sum_{n=n_0}^{\infty} (L + \epsilon)^n$ can be seen as a geometric series +

$$\Rightarrow \sum_{n=n_0}^{\infty} a_n < +\infty \text{ by comparison test}$$

You can do the same considering $L - \epsilon < \frac{a_{n+1}}{a_n}$ with $L > 1$

$$\sqrt[n]{a_n} < L \rightarrow a_n < L^n \rightarrow \sum a_n < \sum L^n$$

As an immediate consequence we have that

Corollary 4.0.16 (Limit Ratio test). *Let (a_n) be a positive sequence and assume that*

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = L.$$

Then, we have

- if $L < 1$ then the series converges,
- if $L > 1$ then the series diverges.

4.0.2 Conditional and absolute convergence

If a series $\sum_{n=0}^{\infty} a_n$ has both positive and negative terms a_n then we can consider a new series by taking the absolute value of each terms; i.e., $\sum_{n=0}^{\infty} |a_n|$ and we may ask if such series converges.

Definition 4.0.19. *If $\sum_{n=0}^{\infty} |a_n|$ converges then we say that $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not, we say that $\sum_{n=0}^{\infty} a_n$ converges conditionally.*

PROOF

Claim: If $\sum a_k$ absolutely converges, then converges

Assume: $\sum |a_k|$ converges

Proof: Rewrite $s_n = |\sum a_k| \geq s_n^+ - s_n^-$ where $s_n^+ = \text{sum of } a_k \text{ terms} > 0$ and $s_n^- = -(\text{sum of } a_k \text{ terms} < 0)$

Note that s_n^+ and s_n^- are monotone increasing

and by triangular inequality $|\sum a_k| \leq \sum |a_k| \Rightarrow s_n^+ \leq \sum |a_k|$ and $s_n^- \leq \sum |a_k|$.

Since by assumption $\sum |a_k|$ converges, $\sum |a_k|$ is bounded and therefore s_n^+ and s_n^- are bounded above. Since s_n^+ and s_n^- are bounded and monotone, then they converge and since $s_n = s_n^+ - s_n^-$ also s_n conv.

Assume $\sum b_k$

bounded

Rewrite $|\sum a_k| \leq s_n^+ + s_n^-$ where $s_n^+ = \text{sum of } a_k > 0 \rightarrow \text{mon. incr.}$ and $s_n^- = -(\text{sum of } a_k < 0 \rightarrow \text{mon. decr.})$

$|\sum a_k| \leq \sum |a_k|$ so s_n^+ and s_n^-

hence, $\sum_{n=0}^{\infty} |a_n|$ converges, by the Cauchy Criterion 4.0.3, for every $\varepsilon > 0$ there exists $n_{\varepsilon} \in \mathbb{N}$ such that for every $n \geq n_{\varepsilon}$ and $p \in \mathbb{N}$ we have

$$|s_{n+p} - s_{n-1}| = \left| \sum_{k=n}^{n+p} a_k \right| \leq \varepsilon,$$

which implies that $\sum_{n=0}^{\infty} a_n$ converges. \square

The Vice versa, it is not true as show the following example.

Example 4.0.21. Let us consider

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}.$$

Such series does not converge absolutely since $|(-1)^n/n| = 1/n$. Nevertheless we can prove that such series converge conditionally thanks to the following theorem.

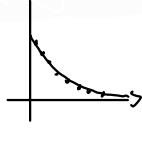
Similarly, if we consider

$$\sum_{n=1}^{\infty} (-1)^n (\sqrt[n]{2} - 1)$$

by Example 4.0.11 we have that the series it is not absolutely convergent. Nevertheless, also in this case we can prove that such series converge conditionally thanks to the following Leibniz Criterion.

ALTERNATING SERIES **Theorem 4.0.22 (Leibniz Criterion).** If you have sequence a_n :

- the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges,
- $|R_n| = |s - s_n| \leq a_{n+1}$, $\forall n \in \mathbb{N}$.

$$\begin{cases} 1) a_n > 0 \quad \forall n \in \mathbb{N} \\ 2) a_n \text{ is monotone decreasing } (a_n > a_{n+1}) \\ 3) \lim_{n \rightarrow \infty} a_n = 0 \end{cases}$$


Remark 4.0.23. It is possible to prove that in particular

$$s_{2n+1} \leq s \leq s_{2n}, \quad \forall n \in \mathbb{N},$$

this implies that we have more information on the approximate the sum of the series s from below with s_{2n+1} and from above with s_{2n} .

By Theorem 4.0.22 we can prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = (-1)^n \cdot \frac{1}{n}$$

\checkmark 1) $\frac{1}{n} > 0 \quad \forall n \in \mathbb{N}$
 \checkmark 2) $\frac{1}{n} > \frac{1}{n+1} \xrightarrow{\text{Isolate } n} n < n+1 = 0 < 1$
 \checkmark 3) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$
CONVERGENT

converges conditionally. The Leibniz criterion is the only method that we are able to provide for series $\sum_{n=0}^{\infty} a_n$ that has both positive and negative terms a_n .

Remark 4.0.24. The geometric series $\sum_{n=0}^{\infty} x^n$ with $x \in \mathbb{R}$ is absolutely convergent for every $|x| < 1$. We can apply either the Ratio or the Root Test to prove it. Indeed,

$$\frac{|x|^{n+1}}{|x|^n} = |x|, \quad \sqrt[n]{|x|^n} = |x|;$$

hence, by Theorems 4.0.15 and 4.0.12 we get that the series is absolutely convergent for every $|x| < 1$ and it is divergent for every $x \geq 1$. Finally, for $x \leq -1$ the series $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n |x|^n$ but the Leibniz Criterion does not apply and we can prove that the series neither converge nor diverge as already observed in (4.1). \checkmark

$$\sum_{n=0}^{\infty} (-1)^n |x|^n$$

\checkmark 1) $|x|^n > 0$
 \times 2) $|x|^n < |x|^m \xrightarrow{n \rightarrow \infty} n > m$
 \times 3) $\lim_{n \rightarrow \infty} |x|^n = \infty$ not 0

Leibniz Criterion Proof

Assume: $a_n > 0$, $a_n > a_{n+1}$, $\lim_{n \rightarrow \infty} a_n = 0$

(Claim: $\sum_{n=1}^{\infty} (-1)^n a_n = S$ (converges))

Proof: Since a_n is decreasing

- $a_{2n} < a_{2n-1} \rightarrow -a_{2n-1} + a_{2n} < 0$ negative
- $a_{2n} > a_{2n+1} \rightarrow a_{2n} - a_{2n+1} > 0$ positive

$S = \sum_{n=0}^{\infty} (-1)^n a_n$

- n is even: we are adding the last element $\rightarrow S_{2n} = S_{2n-2} - a_{2n-1} + a_{2n}$ Then $S_{2n} < S_{2n-2} \rightarrow S_{2n}$ is mon. decreasing
- n is odd: we are subtracting the last element $\rightarrow S_{2n+1} = S_{2n-1} + a_{2n} - a_{2n+1}$ Then $S_{2n+1} > S_{2n-1} \rightarrow S_{2n+1}$ is mon. increasing

Since the series is monotone, we can say that $\lim_{n \rightarrow \infty} S_n$ exists. Now we need to prove that the limit is the same

S_n is mon. decreasing

- $\lim_{n \rightarrow \infty} S_{2n} = \inf S_{2n}$ (either $-\infty$ or L)
- $S_{2n} = S_{2n-1} + a_{2n} > S_{2n-1} \dots \rightarrow S_1 = -a_1 \rightarrow S_{2n}$ is bounded below $\Rightarrow \lim_{n \rightarrow \infty} S_{2n} = S_a \in \mathbb{R}$ (number) = convergent

S_{2n+1} is mon. increasing

- $\lim_{n \rightarrow \infty} S_{2n+1} = \sup S_{2n+1}$ (either $+\infty$ or L)
- $S_{2n+1} = S_{2n} - a_{2n-1} < S_{2n} < \dots S_0 = a_0 \rightarrow S_{2n+1}$ is bounded above $\rightarrow \lim_{n \rightarrow \infty} S_{2n+1} = S^k \in \mathbb{R}$ (number) = convergent

$S_{2n+1} \leq S \leq S_{2n} \rightarrow S^k < \lim_{n \rightarrow \infty} S < S_a \rightarrow \sum_{n=0}^{\infty} (-1)^n a_n = L \in \mathbb{R}$ (converges)

Reminder proof (Claim: $|S - S_n| \leq a_{n+1}$)

Proof: We know that $S_{2n+1} \leq S \leq S_{2n+2}$

- $S \leq S_{2n+2} \rightarrow S - S_{2n+1} \leq S_{2n+2} - S_{2n+1} \rightarrow |S - S_{2n+1}| \leq a_{2n+2}$ include both ways by this property $> |S-3| = |3-S|$
- $-S \leq -S_{2n+1} \rightarrow S_{2n+1} - S \leq S_{2n+1} - S_{2n+2} \rightarrow |S_{2n+1} - S| \leq a_{2n+1}$ difference of 1

Chapter 5

Power series in \mathbb{R}

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Given a sequence of real numbers (a_n) and a fixed $x_0 \in \mathbb{R}$ we define a power series centered at x_0 with coefficients (a_n) as

$$\sum_{n=0}^{+\infty} a_n(x - x_0)^n$$

with $x \in \mathbb{R}$.

Where else do we see such series?

Example 4.0.2: the Geometric series ! Indeed, if we consider $a_n \equiv 1$ and $x_0 = 0$, the power series takes the form

$$\sum_{n=0}^{+\infty} x^n .$$

5.1 Convergence of a power series: the radius of convergence

Theorem 5.1.1. Given $\sum_{n=0}^{+\infty} a_n(x - x_0)^n$. Then, one of the following statements holds true:

- (1) the series converges only at $x = x_0$, $R=0$
- (2) the series converges absolutely for every $x \in \mathbb{R}$, $R=\infty$
- (3) there exists $R > 0$ such that the series converges absolutely for every $|x - x_0| < R$ and it does not converge for every $|x - x_0| > R$. For $|x - x_0| = R$ there is no general statement on the convergence of the series.

CONV: $|x - x_0| < R$
 !CONV: $|x - x_0| > R$
 ? $|x - x_0| = R$

Proof. Without loss of generality we may assume that $x_0 = 0$. This is always possible by a change of variable $y = x - x_0$. Suppose that there exists $x_1 \neq 0$ such that $\sum_{n=0}^{+\infty} a_n x_1^n$ converges. We now prove that the series converges absolutely for every $|x| < |x_1|$. Indeed, since $\sum_{n=0}^{+\infty} a_n x_1^n$ converges we have that $\lim_{n \rightarrow +\infty} a_n x_1^n = 0$ then it is bounded; i.e., there exists $M > 0$ such that

$$|a_n x_1^n| \leq M, \quad \forall n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$

$$|a_n x^n| = |a_n x_1^n| \left| \frac{x}{x_1} \right|^n \leq M \left| \frac{x}{x_1} \right|^n,$$

and, by the comparison test, we have that

$$|x| < |x_1| \left| \frac{x}{x_1} \right| < 1 \implies \sum_{n=0}^{+\infty} \left| \frac{x}{x_1} \right|^n < +\infty, \implies \sum_{n=0}^{+\infty} |a_n x^n| < +\infty. \implies \sum_{n=0}^{\infty} a_n x^n \text{ conv. absolutely}$$

We now set

$$R := \sup \left\{ |x| \in \mathbb{R} : \sum_{n=0}^{+\infty} a_n x^n < +\infty \right\}.$$

Note that, the set is not empty since at least $x = 0$ belongs to it; hence, the definition of R is well posed.

If $R = 0, +\infty$ then (1) and (2) holds true. If $R \in (0, +\infty)$ then the series converges absolutely for every $|x| < R$ and it does not converge not only absolutely but also conditionally for $|x| > R$, otherwise we will have a contraction by the definition of R as supremum. Then also (3) holds true.

In the general case $x_0 \neq 0$, we define

$$R := \sup \left\{ |x - x_0| \in \mathbb{R} : \sum_{n=0}^{+\infty} a_n (x - x_0)^n < +\infty \right\},$$

and we reason as above by making a change of variable $y = x - x_0$. \square

The radius of convergence can be computed by applying the Root and Ratio test for series.

Proposition 5.1.2. Given $\sum_{n=0}^{+\infty} a_n(x - x_0)^n$ assume that there exists

$$\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|} =: L \in [0, +\infty].$$

Then, the radius of convergence of the series is given by

$$R = \begin{cases} \frac{1}{L} & \text{if } L \in (0, +\infty) \\ +\infty & \text{if } L = 0 \\ 0 & \text{if } L = +\infty. \end{cases} \quad (5.1)$$

Proof. For $x \neq x_0$ we have that

$$\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n(x - x_0)^n|} = |x - x_0| \lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = L|x - x_0|.$$

By the Root Test, the series converges absolutely if $L|x - x_0| < 1$; i.e., for every $|x - x_0| < 1/L$ and it does not converge if $|x - x_0| > R = 1/L$. Then $R = 1/L$.

If $L = +\infty$, the series converges only for $x = x_0$ then $R = 0$. Finally, if $L = 0$ then the series converges for every $x \in \mathbb{R}$. \square

Similarly,

Proposition 5.1.3. *Given $\sum_{n=0}^{+\infty} a_n(x - x_0)^n$ assume that there exists*

$$\lim_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| =: L \in [0, +\infty].$$

Then, the radius of convergence of the series is given by

$$R = \begin{cases} \frac{1}{L} & \text{if } L \in (0, +\infty) \\ +\infty & \text{if } L = 0 \\ 0 & \text{if } L = +\infty. \end{cases} \quad (5.2)$$

5.2 Taylor series

The Taylor series of a function f that is infinitely differentiable at a point x_0 is a power series centered at x_0 with coefficients $f^{(n)}(x_0)/n!$; i.e.,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 \\ &\quad + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \dots \end{aligned}$$

where $f^{(n)}(x_0)$ denotes the n -derivative of the function f at x_0 and $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$ ($0! = 1$). Many questions arise naturally!

- (1) What about the convergence of the series?
- (2) The series converges to the function f ?
- (3) What about functions that are not infinitely differentiable but only n -times differentiable?

We will start by answering to *Question 3* !

5.2.1 Taylor polynomials

First of all we prove that if a function is n -differentiable at a point x_0 it can be approximated by (Taylor) polynomials! How? How do we determine the accuracy when we use a (Taylor) polynomial to approximate a function?

Let's start by the case $n = 1$!

Let $f : I \mapsto \mathbb{R}$ be a function differentiable in $x_0 \in I$ then we can state the equation of the tangent line at the point $(x_0, f(x_0))$ as $y = f(x_0) + f'(x_0)(x - x_0)$ where $f'(x_0)$ is the angular coefficient. The tangent line is in particular a polynomial of degree 1 that we can denote by

$$P_1(x; x_0) := f(x_0) + f'(x_0)(x - x_0),$$

which satisfies the following property

$$\lim_{x \rightarrow x_0} \frac{f(x) - P_1(x; x_0)}{(x - x_0)} = 0.$$

Indeed, if we denote by $R_1(x; x_0) := f(x) - P_1(x; x_0)$; i.e., the remainder between the function and the tangent line at $(x_0, f(x_0))$, then by definition of derivative we have that

$$\lim_{x \rightarrow x_0} \frac{R_1(x; x_0)}{(x - x_0)} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{(x - x_0)} - f'(x_0) = 0. \quad (5.3)$$

Formula (5.3) says that *the remainder $R_1(x; x_0)$ goes to zero, as $x \rightarrow x_0$, faster than $(x - x_0)$* which gives us an important information on the capacity to approximate the graph of the function f with the tangent line at the graph around $(x_0, f(x_0))$; i.e., with the polynomial $P_1(x; x_0)$.

We need a language to describe the behaviour of a function that is vanishing as $x \rightarrow x_0$.

Definition 5.2.1. Given two functions f and g defined in an interval I with $x_0 \in I$ such that $f(x_0) = g(x_0) = 0$, we say that f is a “small o ” of g and we write

$$f(x) = o(g(x)) \iff \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0;$$

that is, the function f approaches 0, as $x \rightarrow x_0$, faster than g .

Hence, by Definition 5.2.1 we have that $R_1(x; x_0) = o(x - x_0)$ and

$$\begin{aligned} f(x) &= P_1(x; x_0) + o(x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) + o(x - x_0). \end{aligned}$$

Theorem 5.2.2 (Taylor Formula). If $f : I \rightarrow \mathbb{R}$ has n -derivatives on an open interval I that contains x_0 , we define, for every $x \in I$, the Taylor polynomial $P_n(x; x_0)$ and the remainder $R_n(x; x_0)$ as

$$P_n(x; x_0) := \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, \quad R_n(x; x_0) := f(x) - P_n(x; x_0).$$

Then,

$$\lim_{x \rightarrow x_0} \frac{R_n(x; x_0)}{(x - x_0)^n} = 0,$$

or, equivalently,

$$R_n(x; x_0) = o((x - x_0)^n).$$

As a consequence of the theorem above we have that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^2(x_0)}{2}(x - x_0)^2 + \frac{f^3(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

and the error we incur by approximating f with a polynomial $P_n(x; x_0)$ near x_0 is given by $R_n(x; x_0)$ that goes to zero faster than $(x - x_0)^n$, as $x \rightarrow x_0$.

Corollary 5.2.3 (Lagrange Formula for the remainder). If f has $(n+1)$ -derivatives on an open interval I that contains x_0 then the remainder in Taylor’s theorem can be written as

$$R_n(x; x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

with $\xi \in \mathbb{R}$ between x and x_0 .

Remark 5.2.4 (Maclaurin Formula). If $x_0 = 0$ the Taylor formula is also called Maclaurin Formula.

Example 5.2.5. • There functions that we can not keep taking derivatives as $f(x) = |x|^3$ that is differentiable only two times at $x_0 = 0$:

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{|x|^3}{x} = 0 \\ f^2(0) &= \lim_{x \rightarrow 0} 3|x| = 0 \\ &\not\exists f^3(0). \end{aligned}$$

- The Taylor Polynomial for the exponential function $f(x) = e^x$ with $x_0 = 0$ is given by

$$P_n(x : 0) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \quad (5.4)$$

Since e^x is infinitely differentiable we can use the Lagrange formula for the remainder

$$R_n(x; 0) = e^\xi \frac{x^{n+1}}{(n+1)!} \implies |R_n(x; 0)| = e^\xi \frac{|x|^{n+1}}{(n+1)!} \leq M \frac{|x|^{n+1}}{(n+1)!}$$

where $M = e^x$ for every fixed $x > 0$ (since $0 < \xi < x$) or $M = 1$ for every fixed $x < 0$. It follows then that $\lim_{n \rightarrow +\infty} R_n(x; 0) = 0$ which implies the convergences of the series

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

for every fixed $x \in \mathbb{R}$ as already showed in Examples 5.1.4 (see also Definition 4.0.1). Therefore we can approximate e^x as closely we wish by Taylor polynomials, in particular,

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

5.2.2 From the Taylor polynomials to the Taylor series

In Section 5.2.1 we have computed the Taylor polynomials for a differentiable function and we have used it to approximate that function. We have also observed that if the function is infinitely differentiable we can compute a Taylor series and since it is in particular a power series, we know how to study its convergence (see Section 5.1). By the general theory developed in Section 4 for numerical series we recognise the Taylor polynomials as the partial sums of the Taylor series. This answers to *Question 1*!

Now, we can address also *Question 2*:

When does a function equal its Taylor series?

The function $f(x)$ will equal the Taylor series in x provided the remainder $R_n(x; x_0)$ goes to zero as $n \rightarrow +\infty$; i.e., for every fixed x “near x_0 ” (and of course in the domain of f)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x - x_0)^n \iff \lim_{n \rightarrow +\infty} (f(x) - P_n(x; x_0)) = 0.$$

Example 5.2.6 (VIE: very important example). Let us consider the following function defined on \mathbb{R}

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The function f is infinitely differentiable in $x = 0$. Indeed, by applying L' Hopital rules and the hierarchy between power and exponential, we have that

$$f'(0) = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x} = \lim_{x \rightarrow 0} \frac{2}{x^3} e^{-1/x^2} = 0.$$

If we continue to derive we have that f^n is a sum of terms of the type $e^{-1/x^2}/x^h$ (up to some constants), which implies that passing into the limit as $x \rightarrow 0$ we always get $f^n(0) = 0!$ Therefore, the Taylor series of f at $x_0 = 0$ is identically zero and then, trivially, it converges for every x “near to 0” but the function f equals the Taylor series only at $x = 0$!

We conclude this section by saying:

What can go wrong?

- (1) For some functions the derivatives exist at $x = x_0$ but are too big (they grow too quickly)...Imagine if $f^n(0) = (n!)^2$ then $f^n(0)/n!n!$ and the Taylor series around 0 is

$$\sum_{n=0}^{\infty} n!x^n$$

which has a radius of convergence $R = 0$. There exists such a function but the formula is quite complicated....

- (2) Example 5.2.6 shows that for some functions Taylor series converges but not to the function itself! In this case the polynomials are not the right object to describe around 0 an exponential like e^{-1/x^2} that probably decreases too fast near to zero ...

Example 5.2.7. • Let's study the Taylor series of the sine function $f(x) = \sin x$. We recall that

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, & f^2(x) &= -\sin x, & f^3(x) &= -\cos x, \\ f^4(x) &= \sin x, & \dots & & & & \end{aligned}$$

From the derivative of order 4 the pattern repeats itself then if we consider $x_0 = 0$ we have that the even derivative are always zero at 0 and the odd derivatives are alternatively 1 and -1

$$\begin{aligned} f(0) &= 0, & f^2(0) &= 0, & f^4(0) &= 0, \\ f'(0) &= 1, & f^3(0) &= -1, & f^5(0) &= 1, \dots \end{aligned}$$

Therefore in the Taylor polynomials only odd powers appear and this is not surprising since the sine function is an odd function and it is infinitely differentiable; hence, for every $n \in \mathbb{N}$ we have that

$$\begin{aligned} \sin x &= \sum_{h=1}^n (-1)^{h-1} \frac{x^{2h-1}}{(2h-1)!} + \frac{x^{2n+1}}{(2n+1)!} \left(\frac{d^{2n+1}}{dx} \sin x \right) \Big|_{x=\xi} \\ &= \sum_{h=1}^n (-1)^{h-1} \frac{x^{2h-1}}{(2h-1)!} + (-1)^{(n)} \frac{x^{2n+1}}{(2n+1)!} \cos \xi, \end{aligned}$$

since

$$\left(\frac{d^{2n+1}}{dx} \sin x \right) = (-1)^n \cos x.$$

Therefore,

$$\lim_{n \rightarrow +\infty} |R_{2n-1}(x; 0)| \leq \lim_{n \rightarrow +\infty} \frac{|x|^{2n+1}}{(2n+1)!} = 0,$$

we have that the Taylor series converges to the sine function

$$\sin x = \sum_{h=1}^{+\infty} (-1)^{h-1} \frac{x^{2h-1}}{(2h-1)!}.$$

Similarly, we can prove that

$$\cos x = \sum_{h=0}^n (-1)^h \frac{x^{2h}}{(2h)!} + (-1)^{(n+1)} \frac{x^{2n+2}}{(2n+2)!} \cos \xi;$$

hence,

$$\lim_{n \rightarrow +\infty} |R_{2n}(x; 0)| \leq \lim_{n \rightarrow +\infty} \frac{|x|^{2n+2}}{(2n+2)!} = 0,$$

and

$$\cos x = \sum_{h=0}^{+\infty} (-1)^h \frac{x^{2h}}{(2h)!}.$$

- Let us consider

$$f(x) = \frac{1}{1-x},$$

$$\begin{aligned} f(x) &= \frac{1}{1-x}, \quad f'(x) = \frac{1}{(1-x)^2}, \quad f^2(x) = \frac{2}{(1-x)^3}, \quad f^3(x) = \frac{3!}{(1-x)^4}, \\ &\cdots f^n(x) = \frac{n!}{(1-x)^n}; \end{aligned}$$

hence, there exists ξ between 0 and x such that

$$\frac{1}{1-x} = \sum_{h=0}^n x^h + \frac{x^{n+1}}{(1-\xi)^{n+1}}.$$

Note that, the remainder can be also rewritten in the following form

$$\begin{aligned} R_n(x; 0) &= \frac{1}{1-x} - \sum_{h=0}^n x^h = \frac{1 - \sum_{h=0}^n x^h (1-x)}{1-x} \\ &= \frac{1 - \sum_{h=0}^n (x^h - x^{h+1})}{1-x} \\ &= \frac{1 - (1-x + x - x^2 + x^2 - \cdots + x^n - x^{n+1})}{1-x} \\ &= \frac{x^{n+1}}{1-x}. \end{aligned}$$

Hence, if $|x| < 1$ we have that $\lim_{n \rightarrow +\infty} R_n(x; 0) = 0$ and we have the convergence of the series

$$\frac{1}{1-x} = \sum_{h=0}^{+\infty} x^n, \quad \forall |x| < 1,$$

which agrees with the analysis that we performed for the geometric series in Example 4.0.2.

- We come now to the logarithm function

$$f(x) = \ln(1 + x)$$

which is defined in $(-1, +\infty)$ and on that interval has derivatives of all orders:

$$\begin{aligned} f'(x) &= \frac{1}{1+x}, \quad f^2(x) = -\frac{1}{(1+x)^2}, \quad f^3(x) = \frac{2}{(1+x)^3}, \quad f^4(x) = -\frac{3!}{(1+x)^4}, \\ f^5(x) &= \frac{4!}{(1+x)^5}, \dots \end{aligned}$$

The pattern is clear

$$f^h(x) = (-1)^{h-1} \frac{(h-1)!}{(1+x)^h}, \quad \forall h \geq 1 \implies f^h(0) = (-1)^{h-1} (h-1)!.$$

Therefore,

$$\ln(1+x) = \sum_{h=1}^n \frac{(-1)^{h-1}}{h} x^h + \frac{(-1)^n}{(n+1)} \frac{x^{n+1}}{(1+\xi)^{n+1}}.$$

If $0 < \xi < x$ then $1/(1+\xi) < 1$ and

$$\lim_{n \rightarrow +\infty} |R_n(x; 0)| \leq \lim_{n \rightarrow +\infty} \frac{|x|^{n+1}}{(n+1)} = 0, \quad \forall 0 < x < 1.$$

Actually, we can prove that the remainder by Lagrange formula goes to zero, as $n \rightarrow +\infty$, for every $-1/2 < x < 1$. To recover $(-1, 1)$ we have to apply another formula that allows us to represent the remainder term in integral form. Indeed, by the general theory of power series, we can prove that the radius of convergence of the Taylor series is $R = 1$ and it is possible to prove (by using the integral formula of the remainder) that the sum of series equal the function; *i.e.*,

$$\ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} x^n, \quad \forall |x| < 1.$$

Chapter 6

Ordinary differential equations

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An ordinary differential equation is an equation that relates an unknown function, y , to one or more of its derivatives y', y'', \dots, y^n ; i.e.,

$$F(t, y, y', y'', \dots, y^n) = 0; \quad (6.1)$$

where $F : A \mapsto \mathbb{R}$ is a given function of $(n+2)$ variables $(t, y, z_1, \dots, z_n) \in A \subset \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$, $y : I \mapsto \mathbb{R}$ and $t \in I$. Solving the equation (6.1) means to find an interval $I \subset \mathbb{R}$ and a n -times derivable function $y : I \mapsto \mathbb{R}$ such that $(t, y(t), y'(t), y''(t), \dots, y^n(t)) \in A$ for every $t \in I$ and $F(t, y(t), y'(t), y''(t), \dots, y^n(t)) = 0$. The function y is said a solution to the ordinary differential equation. An ordinary differential equation may have infinitely many solutions, or finitely many solutions or even no solutions at all. We define the general integral of the equation as the following set (which may be infinite, finite, a singleton or empty)

$$\mathcal{I} = \{y : I \mapsto \mathbb{R} \mid I \subset \mathbb{R}, \text{ } y \text{ is a solution to o.d.e.}\}$$

In other words the general integral is the set of all solutions of the equation, everyone defined on its interval of definition. The equation (6.1) is in the so-called non-normal form, that is the highest order derivative, y^n , is not “isolated” with respect to the other derivatives.

In this chapter we will focus on first and second-order partial differential equations ($n = 1, 2$) in normal form.

6.1 First-order ODE

Definition 6.1.1. *A first-order ODE, in normal form, on the unknown y and $f : A \mapsto \mathbb{R}$ given, is*

$$y'(t) = f(t, y). \quad (6.2)$$

Solving the equation (6.2) means to find an interval $I \subset \mathbb{R}$ and a derivable function $y : I \mapsto \mathbb{R}$ such that $(t, y(t), y'(t)) \in A$ and $y'(t) = f(t, y(t))$ for every $t \in I$. The function y is said a solution to the ordinary differential equation.

We define the general integral of the equation as the following set (which may be infinite, finite, a singleton or empty)

$$\mathcal{I} = \{y : I \mapsto \mathbb{R} \mid I \subset \mathbb{R}, \quad y \text{ is a solution to (6.2)}\}.$$

6.1.1 ODE by separation of variables

A very important class of ODE is the one with the f given by

$$f(t, y) = a(t) g(y),$$

where $a : I \mapsto \mathbb{R}$ and $g : J \mapsto \mathbb{R}$ are continuous functions in I and J (intervals of \mathbb{R}), respectively. Given $(t_0, y_0) \in I \times J$ we can also defined the initial value problem, the so-called Cauchy Problem

$$\begin{cases} y'(t) = a(t) g(y), \\ y(t_0) = y_0, \end{cases}$$

where $y(t_0) = y_0$ represents the initial condition.

If $g(y_0) = 0$ then $y(t) \equiv y_0$ is a solution for every $t \in I$ to the Cauchy Problem. If $g(y_0) \neq 0$ then we can solve the differential equation by separation of variables; *i.e.*, we first rewrite the equation as

$$\frac{y'(t)}{g(y(t))} = a(t),$$

then we integrate with respect to the variable t on both sides of the equation and we get an integral equation

$$\int \frac{y'(t)}{g(y(t))} dt = \int a(t) dt. \tag{6.3}$$

In particular,

$$\int \frac{y'(t)}{g(y(t))} dt = \int \frac{1}{g(s)} ds,$$

hence,

$$\int \frac{1}{g(s)} ds = \int a(t) dt.$$

To find the general integral of the equation we have to solve the two indefinite integrals. To find the solution of the Cauchy Problem we can evaluate the general integral in t_0 and fix the constant (of the indefinite integrals) by imposing that $y(t_0) = y_0$. In alternative, we can use directly the initial condition in the integral equation (6.3) by fixing one of the extremes of integration; *i.e.*,

$$\int_{t_0}^t \frac{y'(\tau)}{g(y(\tau))} d\tau = \int_{t_0}^t a(\tau) d\tau \implies \int_{y_0}^{y(t)} \frac{1}{g(s)} ds = \int_{t_0}^t a(\tau) d\tau.$$

If $a(t) \equiv 1$ then we have an autonomous differential equation; *i.e.*,

$$y'(t) = g(y)$$

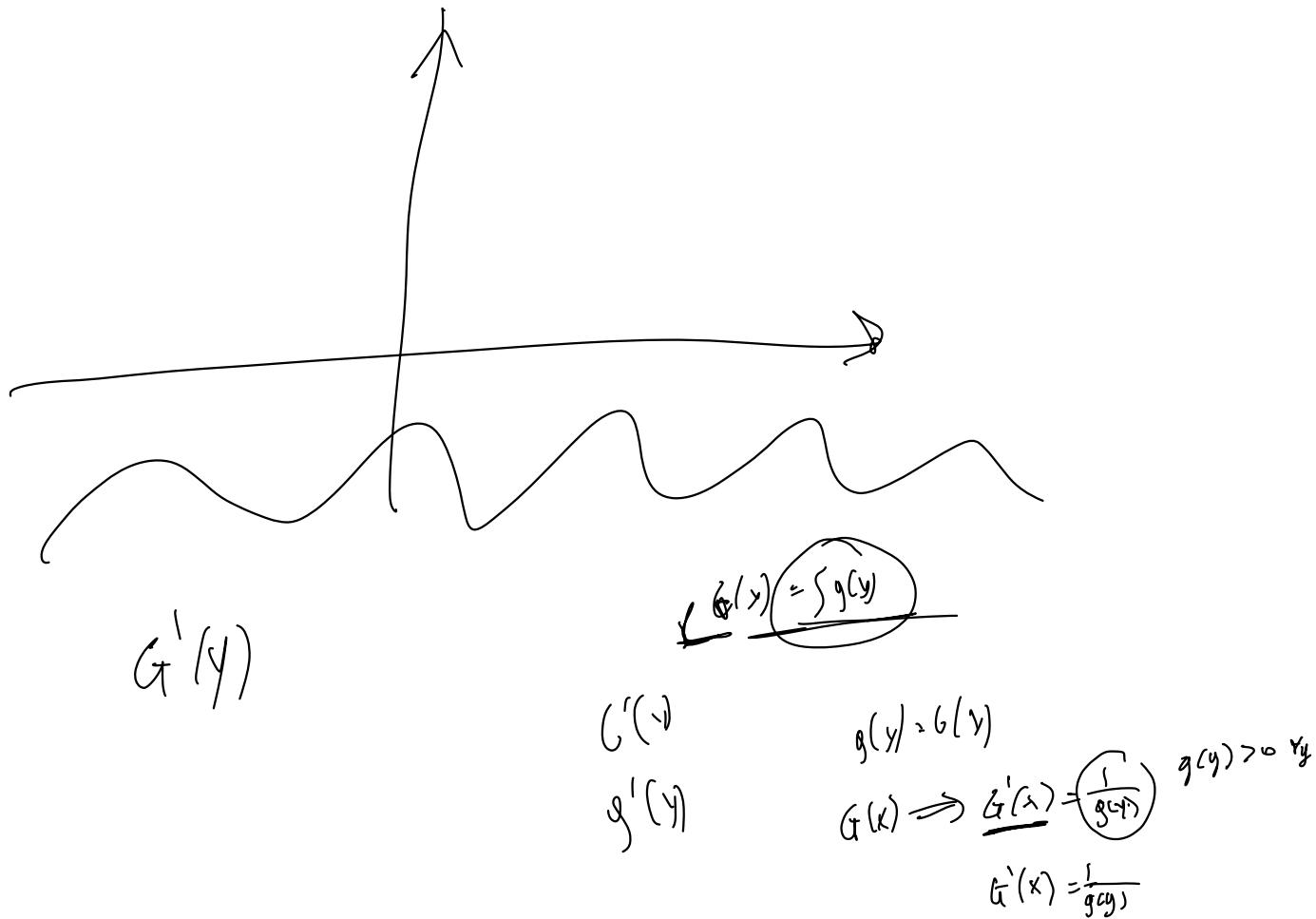
and the rate of change of $y(t)$, the derivative $y'(t)$, does not depends explicitly on t . This implies that given a solution y to the Cauchy Problem

$$\begin{cases} y'(t) = g(y), \\ y(0) = y_0, \end{cases}$$

then, any other solution to

$$\begin{cases} \bar{y}'(t) = g(\bar{y}), \\ \bar{y}(t_0) = y_0, \end{cases}$$

it can be obtained by translating y in the variable t as follows $\bar{y}(t) := y(t - t_0)$. Indeed, $\bar{y}'(t) = y'(t - t_0) = g(y(t - t_0)) = g(\bar{y}(t))$ and satisfies the initial condition $\bar{y}(t_0) = y(0) = y_0$ as show the following example.



Thm Existence and uniqueness of solution
separation of variable

1. Existence

Let $a: I \rightarrow \mathbb{R}$ and $g: J \rightarrow \mathbb{R}$ be two continuous functions

$\Rightarrow \exists$ at least a solution to the Cauchy problem $\begin{cases} y' = a(x) \\ y(x_0) = y_0 \end{cases}$

2. Uniqueness

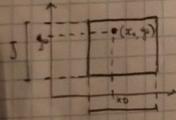
If g is locally Lipschitz $\Rightarrow g \in C^1 \times \begin{cases} g \text{ is continuous} \\ g' \text{ is continuous} \end{cases} \Rightarrow$ locally

\Rightarrow the solution is unique in the maximal interval of existence

PROOF 1

Assume $a: I \rightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$, $g: J \rightarrow \mathbb{R}$, $J \subset \mathbb{R}$

(x_0, y_0) has to be in $I \times J$



Two cases

- $g(y_0) = 0 \Rightarrow$ solution is $y \equiv y_0$
- Example $\begin{cases} y_0 > 0 \\ y(0) = 0 \end{cases} \Rightarrow y = y - 0 \Rightarrow y = 0$ is a sol that satisfied initial data

we found at least a solution
and it's unique because of the initial data

Building of permanence of sign

- $g(y_0) > 0 \Rightarrow \exists J_0 \subset J$ st. $g(y) > 0 \forall y \in J_0$
- $g(y_0) < 0 \Rightarrow \exists J_0 \subset J$ st. $g(y) < 0 \forall y \in J_0$

Since a and g are continuous by assumption $\Rightarrow y'$ is continuous since $y' = a(x)g(y)$
 $\Rightarrow y$ is continuous

Since we have "restricted" J to J_0 , we have to do the same for I

$\exists I_0(x_0) \subseteq I$ st. $y: I_0(x_0) \rightarrow J_0 \Rightarrow y'(y(x)) \neq 0 \forall x \in I_0(x_0)$

Now we separate the variables in the eqn.

$$\frac{y'(x)}{y(x)} = a(x) \quad \forall x \in I_0(x_0) \Rightarrow G(y(x)) := \int \frac{y'(x)}{y(x)} dx = \int a(x) dx := F(x) + C$$

$$\Rightarrow G(y(x)) = F(x) + C$$

We have to prove $\exists G^{-1}$ st.

$$G(y(x)) \cdot G^{-1} = (F(x) + C) \cdot G^{-1}$$

$$\Rightarrow y(x) = G^{-1}(F(x) + C)$$

↳ the solution we are looking for

Note $G'(y) = \frac{1}{g(y)}$ that we said is $\neq 0 \forall y \in J_0$

since $g(y) > 0$ or $< 0 \Rightarrow G(y)$ is strictly monotone (incr or decr.) and continuous

Since monotonicity \Rightarrow invertibility

$\Rightarrow \exists G^{-1}$

$$\Rightarrow \exists y(x) = G^{-1}(F(x) + C)$$

We proved the existence of a solution in both cases

PROOF 2

If $g(y) \in C^1 \Rightarrow \exists$! solution to the Cauchy problem

At $G(y(x)) = F(x) + C$, C can be fixed by imposing initial data
 $\Rightarrow C = G(y_0) - F(x_0)$

Since G' is strictly monotone \Rightarrow invertible \Leftrightarrow injective and continuous

$\Rightarrow G'$ is injective

\Rightarrow one C is fixed

\Rightarrow the solution is unique!

admits at least a solution. If g is locally Lipschitz (for example $g \in C^1$) then the solution is also unique in the maximal interval of existence.

Proof. If $g(y_0) = 0$ then there exists a unique solution $y(x) \equiv y_0$. Hence we focus on the case $g(y_0) \neq 0$. By the Theorem of permanence of sign there exists $J_0 \subseteq J$ such that $g(y) \neq 0$ in particular it maintains the sign of $g(y_0)$ for every $y \in J_0$. Note the solution $y(x)$, we are looking for, is continuous in the interval of existence since $y'(t) = a(t)g(y)$ is continuous by the assumptions on a and g . Therefore, there exists a neighborhood of x_0 , $I_\delta(x_0)$, such that $\overline{I_\delta(x_0)} \subseteq I$ and $g(y(x)) \neq 0$ for every $x \in \overline{I_\delta(x_0)}$. Note that, since y is continuous we have that $y^{-1}(J_0)$ is an open subinterval of I containing x_0 since $y_0 \in J_0$, therefore there exists $I_\delta(x_0) \subseteq y^{-1}(J_0)$ such that the solution $y : I_\delta(x_0) \mapsto J_0$ and $g(y(x)) \neq 0$ for every $x \in \overline{I_\delta(x_0)}$. We now separate the variable in the equation; i.e.,

$$\frac{y'(x)}{g(y)} = a(x), \quad \forall x \in \overline{I_\delta(x_0)}$$

and we integrate with respect to the variable x

$$G(y(x)) := \int \frac{y'(x)}{g(y)} dx = \int a(x) dx := F(x) + c$$

with $G(y)$ and $F(x)$ primitive of $1/g(y)$ in J and $a(x)$ in I , respectively. In order to derive an explicit formula for the solution $y(x)$ we prove that the function $G(y) : J_0 \mapsto \mathbb{R}$ such that $G'(y) = 1/g(y)$ is invertible. Indeed, $G'(y) \neq 0$ for every $y \in J_0$; hence G is continuous and strictly monotone which implies G^{-1} continuous and strictly monotone in $G(J_0)$ as well. Then, there exists

$$y(x) := G^{-1}(F(x) + c), \quad \forall x \in I_\delta(x_0). \quad (6.5)$$

Note that, since $G^{-1} : G(J_0) \mapsto J_0$, there exists a further neighborhood of x_0 , $I_{\delta_1}(x_0) \subset \overline{I_\delta(x_0)} \subset I$, such that $F(x) + c \in J_0$ for every $x \in I_{\delta_1}(x_0)$ with $c = G(y_0) - F(x_0)$; i.e., c fixed by the initial condition $y(x_0) = y_0$. Since, G^{-1} is strictly monotone and therefore injective, it follows that one c is fixed also the solution (6.5) is unique. \square

Let's see some other examples.

Example 6.1.4. • Solve

$$y'(t) = \left(\frac{2}{t}\right)y(t).$$

By separation of variables we have that

$$\int \frac{y'(t)}{y(t)} dt = \int \frac{2}{t} dt, \quad \Rightarrow \ln |y(t)| = \ln t^2 + c_0$$

where the constant $c_0 \in \mathbb{R}$ is due to the computation of two indefinite integrals. Then we get the general integral of the equation

$$y(t) = \pm e^{c_0} t^2 \quad \Rightarrow y(t) = c t^2, \quad c \in \mathbb{R}.$$

If we have also an initial condition then we can fix the constant c by evaluating the general integral at t_0 . For example, if fix the initial condition $y(-1) = 2$ then $c = 2$ and the unique solution to the Cauchy Problem is $y(t) = 2t^2$.

- Solve

$$y'(t) = \sqrt{y(t)}.$$

By separation of variables we have that

$$\int \frac{y'(t)}{\sqrt{y(t)}} dt = \int dt, \quad \Rightarrow \quad 2\sqrt{y(t)} = t + c_0,$$

that is,

$$y(t) = \left(\frac{t + c_0}{2}\right)^2.$$

If we consider the Cauchy problem

$$\begin{cases} y'(t) = \sqrt{y(t)}, \\ y(0) = 0, \end{cases}$$

then $y \equiv 0$ is solution but also $y(t) = (t/2)^2$ with $c_0 = 0$ In this case we DO NOT HAVE UNIQUENESS of the solution! The reason is that the function $g(y) = \sqrt{y}$ is not Lipschitz at $y = 0$; i.e., the derivative g' is not bounded “around” $y = 0$.

In this course we will not further investigate the uniqueness of the solutions. By the way, it is important to be aware that the uniqueness of the solution of a Cauchy problem depends on the regularity of the function $y \mapsto f(t, y)$.

If we change initial condition, for example,

$$\begin{cases} y'(t) = \sqrt{y(t)}, \\ y(0) = 1, \end{cases}$$

then, by evaluating $2\sqrt{y(t)} = t + c_0$ at $t_0 = 0$ we have $2\sqrt{y(0)} = c_0$, which implies $2 = c_0$. Therefore, the solution to the cauchy problem in this case is unique and it is given by

$$y(t) = \left(\frac{t + 2}{2}\right)^2.$$

Finally, note that we can not choose a negative initial condition, $y(0) < 0$, since $g(y) = \sqrt{y(t)}$ is defined only on $J = [0, +\infty)$ and the initial datum $(t_0, y_0) \in I \times J$.

- Solve

$$\begin{cases} y'(t) = 2t\sqrt{1 - y^2} \\ y(0) = 0. \end{cases}$$

The function $g(y) = \sqrt{1 - y^2}$ is defined on $J = [-1, 1]$ and $g(\pm 1) = 0$; hence, $y(t) = \pm 1$ are constant solutions to the equation with initial condition $y_0 = \pm 1$, respectively.

The other solutions can be obtained by separation of variable

$$\int \frac{y'(t)}{\sqrt{1 - y^2}} dt = \int 2t dt = t^2 + c_0.$$

Hence, since $y(0) = 0$ we have that $c_0 = 0$ and

$$\arcsin y(t) = t^2 \in [0, \frac{\pi}{2}) \quad \Rightarrow \quad y(t) = \sin(t^2), \forall t \in (-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}).$$

- Solve

$$\begin{cases} y'(t) = 1 - y^2 \\ y(0) = 0. \end{cases}$$

- Solve

$$y'(t) = 2t y^2.$$

By separation of variables we have from one side that

$$\int \frac{y'(t)}{y^2(t)} dt = \int \frac{1}{s^2} ds = -\frac{1}{y(t)}.$$

On the other side, we have

$$\int 2t dt = t^2 + c_0 \implies -\frac{1}{y(t)} = t^2 + c_0 \implies y(t) = -\frac{1}{t^2 + c_0}.$$

If $c_0 > 0$ then the solution exists for every $t \in \mathbb{R}$; if $c_0 < 0$ then $t^2 + c_0 = 0$ if and only if $t = \pm\sqrt{-c_0}$. It implies that we may have three different type of solutions according to the initial condition $y(t_0) = y_0$; i.e.,

$$\begin{aligned} y(t) &= -\frac{1}{t^2 + c_0}, \quad t \in (-\infty, -\sqrt{-c_0}), \\ y(t) &= -\frac{1}{t^2 + c_0}, \quad t \in (-\sqrt{-c_0}, \sqrt{-c_0}), \\ y(t) &= -\frac{1}{t^2 + c_0}, \quad t \in (\sqrt{-c_0}, +\infty). \end{aligned} \tag{6.6}$$

Finally, if $c_0 = 0$ we have two solutions defined in two different intervals

$$y(t) = -\frac{1}{t^2}, \quad t \in (0, +\infty), \quad y(t) = -\frac{1}{t^2}, \quad t \in (-\infty, 0).$$

Let's make an example by fixing the initial condition $y(0) = 1$. By replacing in

$$y(t) = -\frac{1}{t^2 + c_0} \implies 1 = -\frac{1}{c_0} \implies c_0 = -1;$$

hence, we are in one of the three cases in (6.6). More precisely, since $t_0 = 0$ the interval of existence of the solution should contain 0; i.e.,

$$y(t) = -\frac{1}{t^2}, \quad \forall t \in (-1, 1).$$

- Solve

$$\begin{cases} y'(t) = \frac{t^2 + 1}{2t(t-1)y^2} \\ y(2) = -\frac{\ln 2}{6} \end{cases}$$

By separation of variables we have

$$y^2 y'(t) = \frac{t^2 + 1}{2t(t^2 - 1)} \implies \int y y'(t) dt = \int \frac{t^2 + 1}{2(t^2 - 1)t} dt.$$

Hence,

$$\frac{y^3(t)}{3} = \frac{1}{2} \int \frac{t^2 + 1}{t(t^2 - 1)} dt$$

in particular, we can split in tow parts and solve the first integral by integration by parts the second integral by the method of integration of rational functions (Section 2.4); i.e.,

$$\begin{aligned} \int \frac{t^2 + 1}{2t(t^2 - 1)} dt &= \int \frac{t}{t-1} dt + \int \frac{1}{t(t-1)} dt \\ &= (t-1) \ln(t-1) - 3t + \int \frac{-1}{t} + \frac{1}{t-1} dt \\ &= (t-1) \ln(t-1) - 3t - \ln t + \ln(t-1) + c_0 \\ &= t \ln(t-1) - 3t - \ln t + c_0. \end{aligned}$$

Therefore

$$y^3(t) = \frac{1}{6}(t \ln(t-1) - 3t - \ln t + c_0) \implies y^3(2) = -1 - \frac{\ln 2}{6} + c_0 = -\frac{\ln 2}{6} \implies c_0 = 1,$$

and

$$y(t) = \sqrt[3]{\frac{1}{6}(t \ln(t-1) - 3t - \ln t + 1)}.$$

6.1.2 First-order linear differential equations

A first order linear differential equation can be written in the form

$$y'(t) = a(t)y(t) + b(t) \quad (6.7)$$

with $a, b : I \mapsto \mathbb{R}$ continuous functions.

6.1.3 Homogeneous linear first-order ode

If $b(t) \equiv 0$ then the equation is called *homogeneous*

$$y'(t) = a(t)y(t) \quad (6.8)$$

and we can solved it by separation of variables. In particular,

$$\int \frac{y'(t)}{y(t)} dt = \int a(t) dt \implies \int \frac{1}{s} ds = \int a(t) dt,$$

and, the general integral of the homogeneous equation (6.8) is given by

$$\ln |y(t)| = A(t) + c_0 \implies y_h(t) = c e^{A(t)}, \quad \forall t \in I, c \in \mathbb{R}$$

where

$$A(t) = \int a(t) dt.$$

The solution to the homogeneous ode is called *homogeneous solution*. If we fix the initial condition $y(t_0) = y_0$ then

$$y_h(t) = y_0 e^{A(t)-A(t_0)}, \quad \forall t \in I \quad (6.9)$$

where

$$A(t) - A(t_0) = \int_{t_0}^t a(\tau) d\tau.$$

In particular, if $y_0 = 0$ then $y \equiv 0$.

6.1.4 Inhomogeneous linear first-order ode

The general integral for the equation (6.7) is the sum of the homogeneous solution (the solution to (6.8)) and the so-called *particular solution* that we denote by y_p and that we get by the method of variation of constants. More precisely, we replace the constant c with $C(t)$; i.e.,

$$y_p(t) = C(t) e^{A(t)},$$

then we differentiate

$$y'_p(t) = C'(t) e^{A(t)} + C(t) A'(t) e^{A(t)},$$

and we replace in (6.7); hence,

$$C'(t) e^{A(t)} + C(t) A'(t) e^{A(t)} = a(t) C(t) e^{A(t)} + b(t).$$

since $A'(t) = a(t)$, we have that

$$C'(t) e^{A(t)} = b(t) \implies C'(t) = e^{-A(t)} b(t).$$

Therefore,

$$C(t) = \int e^{-A(\tau)} b(\tau) d\tau \implies y_p(t) = e^{A(t)} \left(\int e^{-A(\tau)} b(\tau) d\tau \right),$$

and the general integral is given by

$$y(t) = y_h + y_p = e^{A(t)} \left(c + \int e^{-A(\tau)} b(\tau) d\tau \right). \quad (6.10)$$

In particular, the solution to the Cauchy Problem

$$\begin{cases} y'(t) = a(t) y(t) + b(t), \\ y(t_0) = y_0, \end{cases}$$

is given, for every $t \in I$, by

$$y(t) = e^{\int_{t_0}^t a(\tau) d\tau} \left(y_0 + \int_{t_0}^t e^{-\int_{t_0}^s a(\tau) d\tau} b(s) ds \right). \quad (6.11)$$

Example 6.1.5. • Find the general integral of

$$y'(t) = (\tan t) y(t) + \sin t.$$

We first note that we have to study the equation in an interval of continuity for $\tan t$ (the function $\sin t$ is continuous in \mathbb{R}). We consider $I = (-\pi/2, \pi/2)$ and we compute the primitive of $a(t) = \tan t$; i.e.,

$$A(t) = \int \tan t dt = -\ln |\cos t| = -\ln(\cos t), \quad \forall t(-\frac{\pi}{2}, \frac{\pi}{2}).$$

Therefore,

$$\begin{aligned} y(t) &= e^{-\ln(\cos t)} \left(c + \int e^{\ln(\cos t)} \sin t dt \right) \\ &= \frac{1}{\cos t} \left(c + \int \cos t \sin t dt \right) \\ &= \frac{1}{\cos t} \left(c - \frac{1}{2} (\cos t)^2 \right). \end{aligned}$$

- Find the solution to the ODE

$$t y'(t) = y(t) + t^2 \cos t.$$

First of all we note that we can always rewrite the equation in normal form

$$y'(t) = \frac{1}{t} y(t) + t \cos t$$

with $t \neq 0$. According to the initial condition $y(t_0) = y_0$, the solution then will exist in an interval contained in $(0, +\infty)$ or $(-\infty, 0)$ if $t_0 \in (0, +\infty)$ or $t_0 \in (-\infty, 0)$, respectively.

General solution

$$y(t) = (c + \sin t)t.$$

- Find the solution of the following ODE:

- $y'(t) = (y(t) - 3) \cos t$
- $y'(t) = e^{t+y}$
- $2t y'(t) + \frac{y^2}{t} = 0$
- $y'(t) + y + e^t = 0$
- $y'(t) + \frac{y}{t} = \sin t$
- $y'(t) + t^2 y = 2t^2$
- $y'(t) = -t y^2$ with $y(0) = 2$
- $y'(t) + t^2 y = 0$ with $y(0) = 3$
- $t y'(t) - y(t) = t^2 \cos t$ with $y(\frac{\pi}{2}) = \pi$.

6.2 Linear Second-order O.D.E. with constant coefficients

6.2.1 The Homogeneous case

The form for the linear second order ode with constant coefficients $a_0, a_1, a_2 \in \mathbb{R}$ is

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0, \quad (6.12)$$

with $a_2 \neq 0$ otherwise the equation would be of first order.

The solution is determined by assuming that there is a solution of the form $e^{\lambda t}$; hence,

$$a_2 \lambda^2 e^{\lambda t} + a_1 \lambda e^{\lambda t} + a_0 e^{\lambda t} = 0 \implies a_2 \lambda^2 + a_1 \lambda + a_0 = 0.$$

The polynomial $a_2 \lambda^2 + a_1 \lambda + a_0$ is called the characteristic polynomial of the ode and we have to solve the characteristic equation

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0. \quad (6.13)$$

Theorem 6.2.1 (Solutions to Homogeneous equation with constant coefficients). *Let $c_1, c_2 \in \mathbb{R}$ be two arbitrary constant and let λ_1, λ_2 be the roots to the characteristic equations (6.13). Then any solutions to (6.12) fulfilled only one of the following cases:*

- if there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$ then

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t};$$

- if there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \equiv \lambda_2$ then

$$y(t) = (c_1 + tc_2) e^{\lambda_1 t};$$

- if there exists $\lambda_1, \lambda_2 \in \mathbb{C}$ then $\lambda_1, \lambda_2 = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$ and

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t).$$

Moreover, for every fixed $t_0, y_0, y_1 \in \mathbb{R}$ there exists a unique solution to the initial value problem

$$\begin{cases} a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0, \\ y(t_0) = y_0, \\ y'(t_0) = y_1. \end{cases} \quad (6.14)$$

Example 6.2.2. • Find the general solution to the following homogeneous equation

$$2y''(t) - 3y'(t) + y(t) = 0.$$

We associate to the ode the characteristic equation

$$2\lambda^2 - 3\lambda + 1 = 0;$$

hence, there exists two real and distinct roots $\lambda_1 = 1, \lambda_2 = 1/2$ and the general integral is

$$y(t) = c_1 e^t + c_2 e^{t/2},$$

for any arbitrary constants c_1, c_2 .

- Find the solution to the initial value problem

$$y''(t) + 2y'(t) + 3y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

We associate to the ode the characteristic equation

$$\lambda^2 + 2\lambda + 3 = 0;$$

hence, there exists $\lambda_1, \lambda_2 = -1 \pm i\sqrt{2}$ and the general integral is

$$y(t) = e^{-t} \left(c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \right).$$

We now find the constants c_1 and c_2 that satisfy the initial conditions; hence, $y(0) = c_1 = 1$ and

$$y'(t) = -e^{-t} \left(c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \right) + e^{-t} \left(-\sqrt{2}c_1 \sin(\sqrt{2}t) + \sqrt{2}c_2 \cos(\sqrt{2}t) \right)$$

and

$$y'(0) = -c_1 + \sqrt{2} c_2 = 2 \implies -1 + \sqrt{2} c_2 = 2 \implies c_2 = \frac{3}{\sqrt{2}}.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = e^{-t} \left(\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}} \sin(\sqrt{2}t) \right).$$

- Find the solution to the initial value problem

$$y''(t) + 4y'(t) + 4y = 0, \quad y(0) = 3, \quad y'(0) = 2.$$

We associate to the ode the characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0 \implies (\lambda + 2)^2 = 0 \implies \lambda_1 \equiv \lambda_2 = -2.$$

Therefore, the general integral is

$$y(t) = (c_1 + tc_2)e^{-2t}.$$

Let us now fix the constants c_1, c_2 . In particular, $y(0) = c_1 = 3$ and $y'(t) = -2e^{-2t}(c_1 + tc_2) + c_2e^{-2t}$, which implies $y'(0) = -6 + c_2 = 2$; i.e., $c_2 = 8$. Therefore, the unique solution to the initial value problem is

$$y(t) = (3 + 8t)e^{-2t}.$$

Exercise 6.2.3. Find the solution to the initial value problem

$$y''(t) + 5y'(t) + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

6.2.2 Inhomogeneous equation: method of undetermined coefficients

We now consider the case of linear second-order o.d.e. with constant coefficients not homogeneous; i.e.,

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t). \quad (6.15)$$

Similarly to the first order inhomogeneous case, the general integral is given by the sum of the solution to the homogeneous equation (that is the one with $f \equiv 0$) with a so-called *particular solution* y_p . More precisely,

$$y(t) = c_1 y_1 + c_2 y_2 + y_p$$

where $c_1, c_2 \in \mathbb{R}$ are two arbitrary constants and y_1, y_2 are as in Theorem 6.2.1.

There are cases where the particular solution y_p belongs to the same class of functions of f . Here below the different cases.

(1) If $f(t) = P_n(t)$ is a polynomial of order n then:

- (a) if $a_0 \neq 0$ then $y_p = Q_n(t)$,
- (b) if $a_0 = 0$ and $a_1 \neq 0$ then $y_p = t Q_n(t)$,
- (c) if $a_0 = 0$ and $a_1 = 0$ then $y_p = t^2 Q_n(t)$.

- (2) If $f(t) = A \cos at$ or $B \sin at$ or $f(t) = A \cos at + B \sin at$ then $y_p(t) = C \cos at + D \sin at$.
- (3) If $f(t) = Ce^{Bt}$ where B is not a root of the characteristic equation then $y_p(t) = Ae^{Bt}$.
Note that the exponential is the same, we may find another constant.

Example 6.2.4 (f is a polynomial). (1) Find the general integral of equation

$$y''(t) - 2y(t) = 1 + t^2.$$

Let's start by the particular solution that will be a polynomial of degree 2 since $f(t) = 1 + t^2$. Note that $a_0 \neq 0$ then $y_p = p_2 t^2 + p_1 t + p_0$. To find the coefficients p_0, p_1, p_2 we have to compute

$$\begin{aligned} y'_p &= 2p_2 t + p_1 \\ &\implies 2p_2 - 2(p_2 t^2 + p_1 t + p_0) = 1 + t^2 \\ y''_p(t) &= 2p_2 \\ &\quad - 2p_2 t^2 - 2p_1 t + 2p_2 - 2p_0 = t^2 + 1. \end{aligned}$$

Therefore,

$$-2p_2 = 1, \quad -2p_1 = 0, \quad 2p_2 - 2p_0 = 1 \implies p_0 = -1, p_1 = 0, p_2 = -1/2;$$

and the particular solution is

$$y_p = -\frac{1}{2}t^2 - 1.$$

The homogeneous solution is obtained by solving $\lambda^2 - 2 = 0$; i.e., $\lambda = \pm\sqrt{2}$; hence,

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} - \frac{t^2}{2} - 1.$$

- (2) Find the general integral of equation

$$y''(t) + 2y'(t) = t.$$

Note that $a_0 = 0$; hence, we expect a particular solution of the type $y_p = t(p_0 + p_1 t)$. More precisely,

$$\begin{aligned} y'_p &= 2p_1 t + p_0 \\ &\implies 2p_1 + 4p_1 t + 2p_0 = t \\ y''_p(t) &= 2p_1 \\ &\quad 4p_1 = 1, 2p_0 + 2p_1 = 0, \implies p_1 = \frac{1}{4}, p_0 = -\frac{1}{4}. \end{aligned}$$

Therefore,

$$y_p = \frac{t}{4}(t - 1).$$

The homogeneous solution is obtained by solving $\lambda^2 + 2\lambda = 0$; i.e., $\lambda = 0, -2$; hence,

$$y(t) = c_1 + c_2 e^{-2t} + \frac{t}{4}(t - 1).$$

It is important to underlying that, at difference of the previous case, now the particular solution is NOT given by a polynomial of the same degree of f .

(3) Find the general integral of equation

$$y''(t) = t.$$

Note that $a_0 = a_1 = 0$ then we expect a particular solution of the type $y_p = t^2(p_0 + p_1 t) = p_0 t^2 + p_1 t^3$. More precisely,

$$\begin{aligned} y'_p &= 3p_1 t^2 + 2p_0 t \\ &\implies 6p_1 t + 2p_0 = t \\ y''_p(t) &= 6p_1 t + 2p_0 \quad 6p_1 = 1, p_0 = 0, \implies p_1 = \frac{1}{6}, p_0 = 0. \end{aligned}$$

Therefore,

$$y_p = t^2 \frac{t}{6} = \frac{t^3}{6}.$$

The homogeneous solution is obtained by solving $\lambda^2 = 0$; i.e., $\lambda = 0$; hence,

$$y(t) = (c_1 + c_2 t) + \frac{t^3}{6}.$$

Example 6.2.5 (f is a combination of cos and sin). (1) Find the general integral of equation

$$y''(t) + y'(t) + 2y = 2 \cos t.$$

Let's start by the particular solution that will be of the form $y_p = A \cos t + B \sin t$. More precisely,

$$\begin{aligned} y'_p &= -A \sin t + B \cos t \\ y''_p(t) &= -A \cos t - B \sin t, \end{aligned}$$

which implies

$$\begin{aligned} &-A \cos t - B \sin t + -A \sin t + B \cos t + 2(A \cos t + B \sin t) \\ &= (A + B) \cos t + (B - A) \sin t \\ &= 2 \cos t. \end{aligned}$$

Therefore,

$$\begin{cases} A + B = 2 \\ B - A = 0, \end{cases} \implies A = B = 1$$

and

$$y_p = \cos t + \sin t.$$

The homogeneous solution is obtained by solving $\lambda^2 + \lambda + 2 = 0$; i.e., $\lambda = (-1 \pm i\sqrt{7})/2$ and

$$y(t) = e^{-t/2} \left(c_1 \cos \left(\frac{\sqrt{7}}{2} t \right) + c_2 \sin \left(\frac{\sqrt{7}}{2} t \right) \right) + \cos t + \sin t.$$

(2) Solve the initial value problem

$$y''(t) + 2y'(t) + 3y = 6 \sin 3t, \quad y(0) = \frac{1}{2}, \quad y'(0) = -\frac{5}{2} - \sqrt{2}.$$

The corresponding homogeneous o.d.e. has a characteristic equation $\lambda^2 + 2\lambda + 3 = 0$ with roots $\lambda = -1 \pm i\sqrt{2}$ so the homogeneous solution is

$$y_h = e^{-t} \left(c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \right).$$

The method of undetermined coefficients says that we should try for a particular solution of the form

$$y_p = C \cos 3t + D \sin 3t.$$

Then,

$$y'(t) = -3C \sin 3t + 3D \cos 3t, \quad y''(t) = -9C \cos 3t - 9D \sin 3t.$$

Substituting into the O.D.E. gives

$$\begin{aligned} & -9C \cos 3t - 9D \sin 3t + 2(-3C \sin 3t + 3D \cos 3t) + 3(C \cos 3t + D \sin 3t) \\ &= (6D - 6C) \cos 3t + (-6D - 6C) \sin 3t \\ &= 6 \sin 3t. \end{aligned}$$

We can now compare the coefficients of $\cos 3t$ and $\sin 3t$ to get

$$(6D - 6C) = 0, \quad (-6D - 6C) = 6 \implies D = C = -\frac{1}{2}.$$

Therefore, the general solution is

$$y(t) = e^{-t} \left(c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \right) - \frac{1}{2} \cos 3t - \frac{1}{2} \sin 3t.$$

We now consider the initial conditions

$$y(0) = c_1 - \frac{1}{2} = \frac{1}{2} \implies c_1 = 1.$$

To consider the initial condition $y'(0)$ we have first to derivate the general integral

$$\begin{aligned} y'(t) &= -e^{-t} \left(c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \right) + e^{-t} \left(-\sqrt{2}c_1 \sin(\sqrt{2}t) + c_2 \sqrt{2} \cos(\sqrt{2}t) \right) \\ &\quad + \frac{3}{2} \sin 3t - \frac{3}{2} \cos 3t \\ y'(0) &= -c_1 + c_2 \sqrt{2} - \frac{3}{2} = -\frac{5}{2} + c_2 \sqrt{2} \\ &= -\frac{5}{2} - \sqrt{2}, \end{aligned}$$

so we get $c_2 = -1$. The unique solution to the initial value problem is then

$$y(t) = e^{-t} \left(\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \right) - \frac{1}{2} \cos 3t - \frac{1}{2} \sin 3t.$$

Example 6.2.6 (f is the exponential function). (1) Solve

$$3y''(t) - 2y'(t) - y = e^{2t},$$

given that

$$y(0) = 0, y'(0) = 1.$$

The corresponding homogeneous o.d.e. has a characteristic equation $3\lambda^2 - 2\lambda - 1 = 0$ with roots $\lambda = 1, -1/3$. Hence the homogeneous solution is $y_h(t) = c_1 e^t + c_2 e^{-t/3}$. We have to look for a particular solution of the form $y_p = C e^{2t}$; hence,

$$y'_p(t) = 2C e^{2t}, \quad y''_p(t) = 4C e^{2t}.$$

Substituting into the O.D.E. gives

$$12C e^{2t} - 4C e^{2t} - C e^{2t} = 7C e^{2t}.$$

We now compare $7C e^{2t}$ with $f(t) = e^{2t}$ and we get $C = 1/7$ and the general integral is given by

$$y(t) = c_1 e^t + c_2 e^{-t/3} + \frac{1}{7} e^{2t}.$$

Now we use the initial conditions to find c_1 and c_2 . Firstly,

$$y(0) = c_1 + c_2 + \frac{1}{7} = 0 \implies c_1 + c_2 = -\frac{1}{7}.$$

Also,

$$y'(t) = c_1 e^t - \frac{1}{3} c_2 e^{-t/3} + \frac{2}{7} e^{2t} \implies y'(0) = c_1 - \frac{1}{3} c_2 + \frac{2}{7} = 1$$

which gives $c_1 - c_2/3 = 5/7$. To find c_1 and c_2 we have to solve the following linear system

$$\begin{cases} c_1 + c_2 = -\frac{1}{7} \\ c_1 - \frac{c_2}{3} = \frac{5}{7}. \end{cases} \implies c_1 = \frac{1}{2}, c_2 = -\frac{9}{14}$$

Therefore, the unique solution to the initial value problem is

$$y(t) = \frac{1}{2} e^t - \frac{9}{14} e^{-t/3} + \frac{1}{7} e^{2t}.$$