INTRODUCTION

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-Francesca Cinelli

—CALCULUS II—

Basic tools 25.09

three-dimensional coordinate system

a point "A" in space can be represented in Cartesian as $A(x,y,z)$ where $\begin{cases} x: \text{absissa of } A \\ y: \text{ordinate of } A \\ z: \text{elevation of } A \end{cases}$

Regions in space

in (x,y) plane, then the equation $x=0$ represents y-axis and $y=0$ represents x-axis.

in (x,y,z) $\begin{cases} x=0 \rightarrow (y,z) \text{ plane} \\ y=0 \rightarrow (x,z) \text{ plane} \\ z=0 \rightarrow (x,y) \text{ plane} \end{cases}$

in (x,y) plane we have 4 quadrants, in (x,y,z) space the coordinate planes $x=0, y=0, z=0$ divide the space into 8 octants. The octant in which the point coordinates are all positive is the 1st octant

Distance and Sphere in Space

the distance between 2 points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is $|AB| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

we use the distance formula in order to find the equation of a sphere: a point $A(x,y,z)$ lies on a sphere of radius r and center $(x_0,y_0,z_0)=C$ if $|CA|=r$

$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} \Rightarrow r^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$

+ unit sphere : $x^2+y^2+z^2=1$

Graphical interpretation of inequalities involving spheres

1. $x^2+y^2+z^2>1$: everything on the exterior of the unit sphere excluding the borders
2. $x^2+y^2+z^2<1$: " " " interior " " " " " "

Vectors in $\mathbb{R}^3$

given a $\vec{v}(v_1,v_2,v_3)$ the magnitude/length/norm $= \sqrt{v_1^2+v_2^2+v_3^2} = \|\vec{v}\|$

a unit vector is a $\vec{v}$ s.t. $\|\vec{v}\|=1$

standard unit vectors are i,j,k $[(1,0,0),(0,1,0),(0,0,1)]$

if $\vec{v}\neq 0$, then $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector

$\rightarrow W = \frac{v}{\|v\|}$ $\|w\|=1$? $\left\|\frac{v}{\|v\|}\right\| = \frac{1}{\|v\|}\|v\| = 1$ ✓

Dot product

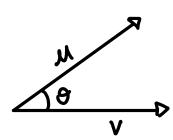
$\vec{u}\cdot\vec{v}$ of $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ is $u_1v_1+u_2v_2+u_3v_3$ and so on for any $\vec{u},\vec{v}\in\mathbb{R}^n$

$\vec{u}\cdot\vec{v} = \|u\|\|v\|\cos\theta \rightarrow \theta = \arccos\left[\frac{\vec{u}\cdot\vec{v}}{\|u\|\|v\|}\right]$

remarks:

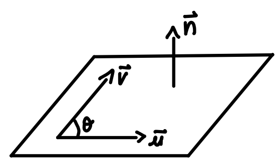
$\rightarrow u\cdot v = v\cdot u$ $\rightarrow u\cdot 0 = 0$ $\rightarrow$ if $u\cdot v=0$ u and v are orthogonal

$\rightarrow u\cdot u = \|u\|^2$ $\rightarrow u\cdot(v+w) = u\cdot v + u\cdot w$



Cross product

$\vec{u}\times\vec{v} = (\|u\|\|v\|\sin\theta)\vec{n}$ where $\vec{n}$ is the normal vector of the (u,v) plane



remarks:

$\rightarrow \vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ $\rightarrow \vec{u}\times\vec{v}=0$ when $\vec{u}\,/\!/\,\vec{v}$ $\rightarrow \vec{u}\times(\vec{v}\times\vec{w}) = (u\cdot w)v-(u\cdot v)w$

$\rightarrow 0\times\vec{u}=0$ $\rightarrow \vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

calculating the cross product as a determinant

assume $\vec{u}(u_1, u_2, u_3)$ and $\vec{v}(v_1, v_2, v_3)$ then $\vec{u} \times \vec{v}$ is

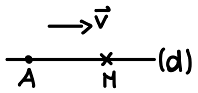
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}(u_2 v_3 - u_3 v_2) + \hat{j}(u_1 v_3 - u_3 v_1) + \hat{k}(u_1 v_2 - u_2 v_1)$$

↳ determinant

Triple scalar product

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$ calculate determinant

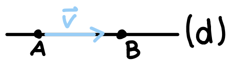
Representing equation of a line



$$\vec{AM} = k\vec{v} \rightarrow \begin{pmatrix} x - x_A \\ y - y_A \\ z - z_A \end{pmatrix} = k \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

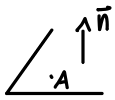
$$\underbrace{\begin{cases} x - x_A = ka \\ y - y_A = kb \\ z - z_A = kc \end{cases}}_{\text{parametric}} \rightarrow \underbrace{\frac{x - x_A}{a} = \frac{y - y_A}{b} = \frac{z - z_A}{c}}_{\text{Cartesian}}$$

$$\vec{v} = \vec{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$



Representing equation of a plane

- normal vector $\vec{n}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \perp$ to (P)
- point $A \in (P)$



let $M \in (P) \rightarrow \vec{AM} \cdot \vec{n} = 0$

$$\begin{pmatrix} x - x_A \\ y - y_A \\ z - z_A \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow (x - x_A)a + (y - y_A)b + (z - z_A)c = 0$$

$$ax + by + cz + \underline{(-ax_A - by_A - cz_A)} = 0$$

$$ax + by + cz + \underline{d} = 0$$

ex.

- find the eqn. of (P) through A and n
- find the eqn. of (P) passing through A, B, C
  - $\rightarrow \vec{n} = \vec{AB} \times \vec{AC}$
  - $\rightarrow \vec{AM} \cdot \vec{n} = 0$

$$\underbrace{\vec{AM} \cdot (\vec{AB} \times \vec{AC})}_{\text{triple scalar product}} = 0 \rightarrow \begin{vmatrix} AM \\ AB \\ AC \end{vmatrix} \overset{\text{determinant}}{=} 0$$

Cylinders and Quadratic surfaces

- circular cylinder $\rightarrow (\text{var}1)^2 + (\text{var}2)^2 = a^2$ (ex. $x^2 + y^2 = 4$ cylinder along z-axis)
- elliptical cylinder $\rightarrow \frac{(\text{var}1)^2}{a^2} + \frac{(\text{var}2)^2}{b^2} = k$ $a \neq b$ (ex. $\frac{x^2}{4} + y^2 = 1$)
- parabolic cylinder $\rightarrow (\text{var}1) = (\text{var}2)^2$ (ex. $y = x^2$)
- ellipsoid $\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ sphere $\rightarrow a = b = c$ (ex. $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} + z^2 = 1$)
- paraboloid $\rightarrow$ var = sum of squares of 2 other vars $\rightarrow \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $\begin{cases} \text{if } a = b \rightarrow \text{circular} \\ \text{if } a \neq b \rightarrow \text{elliptic} \end{cases}$
- cone $\rightarrow$ a cone around z-axis is generated by rotation of an oblique line through fixed angle ϕ with z-axis $\rightarrow (\text{var}1)^2 = a(\text{var}2)^2 + b(\text{var}3)^2$ (ex. $z^2 = x^2 + y^2$)

Functions of several variables - Domain and Range   9.10   exercises

$$f: \begin{array}{l} D\subseteq \mathbb{R}^2 \longrightarrow \mathbb{R} \\ x,y \longrightarrow f(x,y)=z \end{array}$$

Def: suppose D is a set of real numbers $(x_1, x_2, \dots, x_n)$. A real valued function f on D is a rule that assigns a unique real number $w = f(x_1, x_2, \dots, x_n)$ to each element in D. The set D is the domain of the function. The set of values taken by f is the range

Limits and continuity   11.10   exercises

Def: we say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) and write $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $\forall \varepsilon>0\ \exists$ a corresponding $\delta>0$ such that for all (x,y) in the domain of f $|f(x,y)-L|<\varepsilon$ whenever $0<\underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\text{distance}}<\delta$

not $\leq$ because $(x,y)\neq(x_0,y_0)$

ex. show that $\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2} = 0$

$$\sqrt{x^2+y^2}<\delta \to \begin{array}{l} |x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} < \delta \\ |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta \end{array} \qquad \text{also} \quad \frac{x^2}{x^2+y^2} < 1$$

$$|f(x,y)-L|<\varepsilon \to \left|\frac{4x^2y}{x^2+y^2} - 0\right| < \varepsilon \to \left|\frac{4x^2y}{x^2+y^2}\right| < |4y| < 4\delta < \varepsilon \to \delta = \frac{\varepsilon}{4}$$

solution:

let $\varepsilon>0$ and $\delta = \frac{\varepsilon}{4}$ and $\sqrt{x^2+y^2}<\delta$ then $|y|<\sqrt{x^2+y^2}<\delta$ and $|f(x,y)-0| = \left|\frac{4x^2y}{x^2+y^2}\right| <$
$< |4y| < 4\delta = \frac{\cancel{4}\varepsilon}{\cancel{4}} < \varepsilon$

Properties of limits of function of two variables

Assume $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$ then

1. $\lim_{(x,y)\to(x_0,y_0)} (f\pm g) = L\pm M$ (difference and sum rules)

2. $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (k scalar)

3. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = LM$ (product rule)

4. $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ $M\neq 0$ (division rule)

5. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$ (exponent rule)

ex.

$\to \lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{x-y} = \lim \frac{x\cancel{(x-y)}}{\cancel{x-y}}(\sqrt{x}-\sqrt{y}) = 0$

indeterminate form $\left(\frac{0}{0}\right)$

$\to \lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = -3$

determinate form

Limits 2 16.10

Def: A path is any curve passing through the point (x_0, y_0)

Two path test for Non Existence of a Limit

if a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist

ex. show that the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ has no limit as (x,y) approaches $(0,0)$

$\rightarrow$ along x-axis $(y=0)$

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1 = L_1$$

$L_1 \neq L_2 \Rightarrow$ the limit does not exist

$\rightarrow$ along y-axis $(x=0)$

$$\lim_{y\to 0} \frac{-y^2}{y^2} = -1 = L_2$$

$\rightarrow y = mx$

$$\lim \frac{x^2 - m^2x^2}{x^2+m^2x^2} = \lim \frac{1-m^2}{1+m^2}$$

$\hookrightarrow$ depends on m

$m = 0 \rightarrow L_1 = 1$
$m = 1 \rightarrow L_2 = 0$ $\Rightarrow L_1 \neq L_2$

Continuity

Def: A function $f(x,y)$ is continuous at a point (x_0,y_0) if

1) f is defined at (x_0,y_0)
2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

A function is continuous if it is continuous at every point of its domain

Note: polynomials and rational functions of 2 variables are continuous at every point where they are defined

Partial derivatives exercises

$f(x,y) \longrightarrow f_x \; f_y$

Def: Let $z = f(x,y)$

$\rightarrow$ the partial derivative of $f(x,y)$ w.r.t. x at the point (x_0,y_0) is denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \quad f_x(x_0,y_0) \quad \frac{\partial f}{\partial x}(x_0,y_0) \quad \frac{\partial z}{\partial x}(x_0,y_0)$$

is given by $\left.\frac{\partial f}{\partial x}\right|(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$

$\rightarrow$ the partial derivative of $f(x,y)$ w.r.t. y at the point (x_0,y_0) is denoted by

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \quad f_y(x_0,y_0) \quad \frac{\partial z}{\partial y}(x_0,y_0)$$

is given by $\left.\frac{\partial f}{\partial y}\right|(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$

the partial derivative of a function of several variables is its derivative w.r.t. one of those variables while the others hold constant

Implicit differentiation 18.10 exercises

ex. 1

$x^2y + xz + yz^2 = 8$

$z = f(x,y)$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\Rightarrow \frac{\partial}{\partial x}\left(x^2y + xz + yz^2\right) = \frac{\partial(8)}{\partial x}$$

$$2xy + z + x\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(x + 2yz\right) = -\left(2xy + z\right)$$

$$\frac{\partial z}{\partial x} = \frac{-2xy - z}{x + 2yz}$$

$$\boxed{\begin{array}{ll} (x+1)^2 & (x+1) = z \\ 2(x+1)\frac{\partial(x+1)}{\partial x} & \end{array}}$$

Higher order Partial Derivatives 23.10 exercises

Let $z = f(x,y)$, then since both 1st order partial derivatives f_x, f_y are also functions w.r.t. x and y, we could in turn differentiate each w.r.t. x and y. There will be a total of 4 possible second order derivatives

$\rightarrow f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ $\quad\rightarrow f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\rightarrow f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ $\quad\rightarrow f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

mixed partial derivatives

note: the order that we take in performing the derivatives is given by the notation we are using:

- f_{xy}, differentiate from left to right
- $\frac{\partial^2 f}{\partial y \partial x}$, differentiate from right to left

the symmetry of the 2nd derivative is not always true. In general, the symmetry of the 2nd derivatives will always hold at a point if the 2nd partial derivatives are continuous around the point. This is referred to the Schwart's theorem.

Differentials 25.10 exercises

$\rightarrow$ if $y = f(x)$ is a function of one variable, then $dy = f'(x)dx$ is called the differential

$\rightarrow$ given the function $z = f(x,y)$, then the differential dz is given by $dz = f_x\,dx + f_y\,dy$

$\rightarrow$ given the function $w = f(x,y,z)$ then the differential dw is given by $dw = f_x\,dx + f_y\,dy + f_z\,dz$

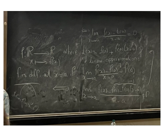
$\rightarrow$ can be extended to more than 3 variables

Partial Derivatives and Continuity exercises

A function $f(x,y)$ can have partial derivatives w.r.t. both x and y at a point without the function being continuous there

Differentiability

- If the partial derivatives f_x and f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R
- If a function $f(x,y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0)

- For 1 variable $\rightarrow$ 

- Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and suppose that the partial derivatives f_x and f_y are defined at the point (a,b). Define $h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
We say that f is differentiable at (a,b) if $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)-h(x,y)}{\|(x,y)-(a,b)\|} = 0$

- If either of the partial derivatives $f_x(a,b)$ or $f_y(a,b)$ doesn't exist or the above limit doesn't exist or not zero, then f is not differentiable at (a,b)

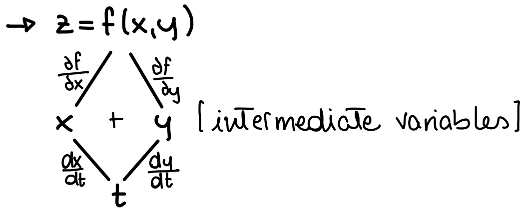
Graphical interpretation of partial derivative 30.10

Let $f: \mathbb{R}^2 \to \mathbb{R}$ fix $y=b$ let $g(x) = f(x,b)$ $\Rightarrow \dfrac{\partial f}{\partial x}(a,b) = \dfrac{\partial g}{\partial x}(a)$ is the slope of the tangent line to the curve that results from the intersection of the plane $y=b$ and the surface at the point (a,b). Fix $x=a$ let $h(y) = f(a,y) \Rightarrow \dfrac{\partial f}{\partial y}(a,b) = \dfrac{\partial h}{\partial y}(b)$, the partial derivative of f w.r.t. y at (a,b) is the slope of the tangent line to the intersection of the graph of f with the plane $x=a$.

Chain rule for multivariable functions

Thm: if $z = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $z = f(x(t), y(t))$ is a differentiable function of t and

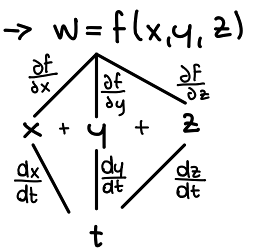
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



$\Rightarrow \dfrac{dz}{dt} = $ above

3 variables

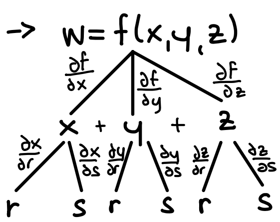
if $w = f(x,y,z)$ is differentiable and x,y,z are differentiable functions of t, then w is differentiable function of t and $\dfrac{dw}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}$



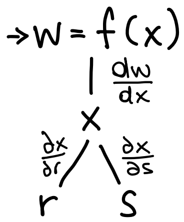
two independent variables and 3 intermediate variables

suppose that $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$ if all are differentiable, then w has partial derivatives w.r.t. r and s

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$



single variable x ($w = f(x)$) and $x = g(r,s)$



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

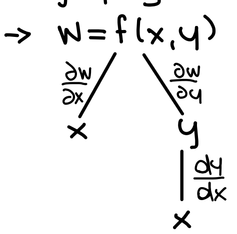
$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

general case

suppose that z is a function of n-variables $x_1, x_2 \dots x_n$ and each one of these variables in turn are functions of m-variables $t_1, t_2 \dots t_m$. then for any variable t_i, $i = 1 \dots m$, we have $\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$

Implicit differentiation with chain rule

Given the function in the form $w = f(x,y) = 0$ where y is implicitly defined function of x, say $y = g(x)$.



$$0 = f_x + f_y \frac{dy}{dx} \rightarrow f_y \frac{dy}{dx} = -f_x \rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$$

This can be extended to functions of three variables

Assume $f(x,y,z) = 0$ and z is implicitly defined as $z = g(x,y)$.
Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$\rightarrow$ Differentiate both sides w.r.t. x:

$$\frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{\to 1} + \frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{\to 0} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$f_x + f_z \frac{\partial z}{\partial x} = 0 \rightarrow \frac{\partial z}{\partial x} = \frac{-f_x}{f_z}$$

and similarly w.r.t. y:

$$\rightarrow \frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$

Directional derivative 6.11

The partial derivative of f w.r.t. x $\left(\frac{\partial f}{\partial x}\right)$ is the slope of the tangent line to the intersection of the graph of f with the plane $y = y_0$ at (x_0, y_0) in the direction of x.
Also $\frac{\partial f}{\partial y}$ gives the slope to the tangent line in y direction.
We can generalize the partial derivatives to calculate the slope in any direction. the result is called directional derivative. The first step in taking a directional derivative is to specify the direction. One way to specify a direction is with a vector $\vec{u}(u_1, u_2)$. For simplicity we will assume that $\vec{u}$ is a unit vector (if not we just normalize it). Sometimes the direction of changing x and y is given as an angle θ. the unit vector that points in this direction is given by $\vec{u}(\cos\theta, \sin\theta)$.

Def the derivative of f in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ called the directional derivative and denoted by $D_{\vec{u}} f(x,y)$ and given by $D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+u_1 h, y+u_2 h) - f(x,y)}{h}$

- The partial derivative $f_x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$ is a directional derivative in the $\vec{i}^{th}$ direction

→ define a function of single variable

$g(z) = f(x_0 + u_1 z, y_0 + u_2 z)$ where x_0, y_0, u_1, u_2 are fixed numbers

$g'(z) = \lim_{h\to 0} \frac{g(z+h) - g(z)}{h}$ $\quad g'(0) = \lim_{h \to 0} \frac{f(x_0 + u_1 h, y_0 + u_2 h) - f(x_0, y_0)}{h}$ $\quad {}^{*}g'(0) = D_{\vec{u}} f(x_0, y_0)$

let's write $g(z) = f(x,y)$ where $\begin{cases} x = x_0 + u_1 z \\ y = y_0 + u_2 z \end{cases}$

$$g'(z) = \frac{dg}{dz} = \frac{\partial f}{\partial x}\underbrace{\frac{dx}{dz}}_{u_1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{dz}}_{u_2}$$

$g'(z) = f_x u_1 + f_y u_2$ $\quad {}^{*}g'(0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2$

→ $D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2$

Graphical interpretation of directional derivative

The equation $z = (x,y)$ represents a surface S. If $z_0 = f(x_0, y_0)$ then the point $(x_0, y_0, z_0) = P$ lies on S. The vertical plane that passes through P and $\vec{u}$ and parallel to (uz) plane intersects S in a curve C. The slope of the tangent to C at P is given by directional derivative.

Gradient vector

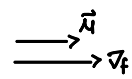
$$D_{\vec{u}} f = \underbrace{(f_x, f_y)}_{\text{gradient vector}} \cdot \underbrace{(u_1, u_2)}_{\vec{u}}$$

Def the g.v. of $f(x,y)$ is $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = (f_x, f_y)$

→ $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

Properties of directional derivative

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$ $\quad$ $\|$unit vector$\| = 1$



→ the maximum value of $D_{\vec{u}} f$ is given by $\|\nabla f\|$ and will occur in the direction of ∇f

→ the lowest value will occur in the direction $-\nabla f$

→ any direction $\vec{u}$ perpendicular to ∇f is a direction of zero change in f because $\theta = \frac{\pi}{2}$ and $\cos\theta = 0$

Algebraic rules for gradient $\quad$ 8.11

1) $\nabla(f \pm g) = \nabla f \pm \nabla g$ $\quad$ (sum-difference rule)
2) $\nabla(kf) = k\nabla f$ $\quad$ (constant multiple rule)
3) $\nabla(fg) = \nabla f\, g + f \nabla g$ $\quad$ (product rule)
4) $\nabla\left(\frac{f}{g}\right) = \frac{\nabla f\, g - f \nabla g}{g^2}$ $\quad$ (quotient rule)

Functions of 3 variables

$f(x,y,z)$ and $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is a unit vector. We have

$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$ and $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = f_x u_1 + f_y u_2 + f_z u_3$

Tangent planes and Normal forms

if $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = 0$ is differentiable

$$\Rightarrow \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dk}{dt}\vec{k}\right)}_{\frac{dr}{dt}} = 0$$

At every point along the curve ∇f is orthogonal to the curve's velocity vector

Def The tangent plane at the point $P_0(x_0, y_0, z_0)$ on $f(x,y,z) = c$ is the plane through P_0 and normal to $\nabla f|_{P_0}$

-> the normal line is the line through P_0 and parallel to $\nabla f|_{P_0}$

Tangent plane is $(x-x_0)f_x(P_0) + (y-y_0)f_y(P_0) + (z-z_0)f_z(P_0) = 0$

Normal line is $\begin{cases} x = x_0 + f_x(P_0)t \\ y = y_0 + f_y(P_0)t \\ z = z_0 + f_z(P_0)t \end{cases}$

Extreme values 13.11

Def: Let $f(x,y)$ be defined on a region R containing the point (a,b), then

1. $f(x,y)$ has a relative minimum at the point (a,b) if $f(a,b) \le f(x,y)$ for all points (x,y) in an open disk centered at (a,b).
2. $f(a,b)$ is a local maximum value if $f(a,b) \ge f(x,y)$ for all points in an open disk centered at (a,b)
3. f has an absolute max at (a,b) if $f(a,b) \ge f(x,y)$ for all (x,y) in R
4. f has absolute minimum at (a,b) if $f(a,b) \le f(x,y)$ for all (x,y) in R

Critical points

A point (a,b) in the domain of a function $f(x,y)$ is called critical point (or stationary point) if

1. $f_x(a,b) = f_y(a,b) = 0$ $(\nabla f(a,b) = 0)$
2. either $f_x(a,b)$ or $f_y(a,b)$ doesn't exist

-> note that both of partial derivatives must be zero at (a,b). If only one of first partial derivatives are zero at the point, then the point will not be critical.

-> the value of the function at a critical point is a critical value.

First derivative test for local extreme values

Thm: if the point (a,b) is a relative extrema of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$ and hence (a,b) is a critical point of $f(x,y)$.

Proof: define $g(x) = f(x,b)$. Assume $f(x,y)$ has a relative extremum at (a,b) then $g(x)$ also has a relative extremum at $x = a$ (of same kind as $f(x,y)$). Then $g'(a) = 0 = f_x(a,b)$. Similarly, you can define $h(y) = f(a,y)$, h will have a relative extremum at b
$\Rightarrow h'(b) = f_y(a,b) = 0$

-> from last lecture we apply the tangent plane formula to $w = f(x,y) - z = 0$ where $x_0 = a$, $y_0 = b$, $z_0 = f(a,b)$, $c = 0$ and $f(x,y,z) = w = F(x,y,z)$. So $F_x = f_x$, $F_y = f_y$ and $F_z = -1$.

-> if we substitute the values $f_x(a,b)=0$ and $f_y(a,b)=0$ into the equation
$f_x(a,b)(x-a)+f_y(a,b)(y-b)-(z-f(a,b))=0$ for the tangent plane to the surface $z=f(x,y)$ at (a,b), then the equation reduces into $z=f(a,b)=c$ => you will have a horizontal tangent plane.
* Not every critical point give rise to a local extremum. (ex. $f(x)=x^3$ $f'(x)=3x^2$, $(0,0)$ is a c.p.)

Saddle point

Def: a function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b), there are domain points (x,y) where $f(x,y)>f(a,b)$ and points (x,y) where $f(x,y)<f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is called a saddle point of the surface.

Second derivative test

Thm: suppose that $f(x,y)$ and its first and second partial derivatives are continuous through a disk centered at (a,b) and that $f_x(a,b)=f_y(a,b)=0$. Define $D=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.
Then

1. If $D>0$ and $f_{xx}(a,b)>0$, f has a local minimum at (a,b)
2. If $D>0$ and $f_{xx}(a,b)<0$, then f has a local maximum at (a,b)
3. If $D<0$, then f has a saddle point
4. If $D=0$, then the test is inconclusive

f_{xx} can also be f_{yy}
↓
notice that if $D>0$, then both f_{xx} and f_{yy} should have same sign

$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ discriminant or Hessian Matrix

Problem solving strategy

1. Determine the critical points (a,b) where $f_x(a,b)=f_y(a,b)=0$
2. Calculate $D=f_{xx}f_{yy}-f_{xy}^2$ at every critical point
3. Apply the four cases

Extreme value Theorem 15.11

Thm: if $f(x,y)$ is continuous function on closed bounded set in $\mathbb{R}^2$, then f has an absolute maximum and an absolute minimum on S.

Closed-Bounded

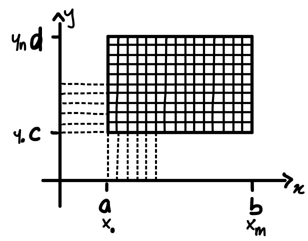
1. A region D in $\mathbb{R}^2$ is called bounded if it lies in a disk of finite radius
2. A region in $\mathbb{R}^2$ is called closed if it includes all its boundary points. A region is called open if it consists entirely of interior points
3. A point (a,b) is an interior point of a region R if it is the center of a disk that lies entirely in D. A point (a,b) is a boundary point if every disk centered at (a,b) contains points that lie outside D as well as points inside D.

Steps exercises

1. List the interior points where f may have local maximum, local minimum, and evaluate f at these points
2. List the boundary points of R where f has local max and min and evaluate f at these points. This usually involves calculus I approach
3. the largest and smallest values found in the two steps are the absolute max and absolute min of the function

Integral calculus 20.11

Double integrals over rectangular region



Let's start with a rectangular region in the (xy) plane $R = \{a \le x \le b,\ c \le y \le d\}$

Divide $a \le x \le b$ into m-subintervals and $c \le y \le d$ into n-subintervals

$\Delta x = \frac{b-a}{m}$; $\Delta y = \frac{d-c}{n}$; $\Delta x \Delta y = \Delta A$

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A \ ; \ V = \lim_{m,n\to\infty} \sum\sum \dots = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

Fubin's Theorem

if $f(x,y)$ is continuous throughout the rectangular region R $a \le x \le b$, $c \le y \le d$

$$\iint_R f(x,y)\,dA = \iint \underline{\ \ dx}dy = \iint \underline{\ \ dy}dx$$

solve this before

these integrals are called <u>iterated integrals</u>

→ notice that the inner differential should match up with the limits on the inner integral and similarly for the outer differentials and outer limit

Double integrals over general region

Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involves variables, not constants

We'll consider two types of Non-rectangular region:

1. $R = \{(x,y) \,/\, a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$
   $$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

note: how to set up integrals → <u>exercises</u>
and switch from dxdy to dydx

2. $R = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$
   $$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

Fubin's theorem for general regions

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

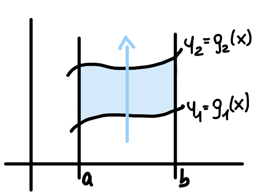
Properties of double integrals

1. $\iint f(x,y) \pm g(x,y)\,dA = \iint f(x,y)\,dA \pm \iint g(x,y)\,dA$
2. $\iint c f(x,y)\,dA = c \iint f(x,y)\,dA$ (where c is a constant)
3. if R can be split into two separate regions R_1 and R_2, then the integral can be written as:
   $$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Geometric interpretation 22.11

A geometric interpretation of a double integral is to find the Area of a given region R:

→ Area of $R = \iint_R dA$



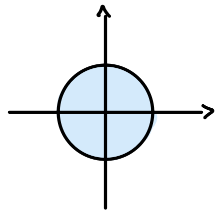
from calculus I: $A_R = \int_a^b (g_2(x) - g_1(x))\,dx$

$$\iint_R dy\,dx = \int_{x=a}^{x=b}\int_{y_1=g_1(x)}^{y_2=g_2(x)} dy\,dx = \int_a^b y\Big|_{y_1}^{y_2} dx = \int_a^b (g_2(x) - g_1(x))\,dx \Rightarrow \text{Proof}$$

Double integrals in polar coordinates 27.11

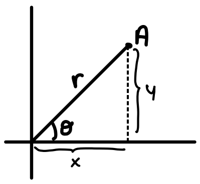
$f(x,y) = e^{-x^2-y^2}$

$\iint_R f(x,y)\, dA$ / where R is the unit circle



$\int_{-1}^{1}\int_{y=-\sqrt{1+x^2}}^{y=\sqrt{1+x^2}} e^{-x^2-y^2}\, dy\, dx$ -> this is a difficult integral

$(x,y) \longrightarrow (r,\theta)$



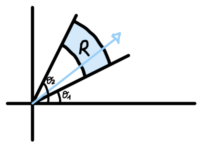
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2=r^2 \end{cases}$$

$$\iint f(x,y)\, \overbrace{dx\,dy}^{dA} \rightarrow \iint_{r_1=g_1(\theta)}^{r_2=g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

note: $dA = r\,dr\,d\theta$

Limits of integration

1. sketch your region
2. draw a line L through the origin and check where it enters R $(g_1(\theta))$ and where it leaves R $(g_2(\theta))$
3.



see exercises

Triple integrals 29.11

Def. the volume of a closed bounded region D in the space is $V = \iiint_D dV$ $\begin{cases} dx\,dy\,dz \\ dx\,dz\,dy \\ dy\,dx\,dz \\ dy\,dz\,dx \\ dz\,dy\,dx \\ dz\,dx\,dy \end{cases}$

How to evaluate triple integral?

→ $f_1(x,y) \le z \le f_2(x,y) \Rightarrow \iiint_{f_1}^{f_2} f(x,y,z)\, dz\, dA$

see exercises

Triple integrals in cylindrical coordinates 4.12

Def: represent a point P in space by triples (r,θ,z) such that
- r and θ are polar coordinates for the vertical projection of P on (xy) plane
- z is the rectangular coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad x^2+y^2=r^2 \quad \tan\theta = \frac{y}{x}$$

$$\iiint f(x,y,z)\, dz\, dy\, dx \Rightarrow \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta$$

see exercises

Triple integrals in spherical coordinates 11.12

A point P in space is determined by (ρ,ϕ,θ):
1. ρ is the distance from P to the origin
2. φ is the angle that $\vec{OP}$ makes with positive z-axis, $0 \le \phi \le \pi$
3. θ is the angle in polar coordinates, $0 \le \theta \le 2\pi$

Transform (x,y,z) into (ρ,ϕ,θ):

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \quad x^2+y^2+z^2=\rho^2 \quad \phi = \cos\left(\frac{z}{\rho}\right)$$

$$\iiint f(x,y,z)\, dV \Rightarrow \iiint f(\rho,\phi,\theta)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Change of variable

Def: let $x = g(u,v)$, $y = h(u,v)$ then the Jacobian is $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$\iint_R f(x,y)\,dx\,dy = \iint f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

let $x = g(u,v)$
$y = h(u,v)$

-> cylindrical coordinates

$$\iint f(x,y)\,dA = \iint f(r,\theta)\,r\,dr\,d\theta \overset{?}{\Rightarrow} \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

-> triple integrals

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases} \quad J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad \iiint_R dV \to \iiint_S |J|\,du\,dv\,dw$$

-> spherical coordinates

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \quad J(\rho,\phi,\theta) = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi\left[\rho^2\cos^2\theta\cos\phi\sin\phi + \rho^2\cos\phi\sin\phi\sin^2\theta\right] + \rho\sin\phi\left[\rho^2\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right] =$$
$$\rho^2\cos^2\phi\sin\phi + \rho^2\sin^2\phi\sin\phi = \rho^2\sin\phi\left[\cos^2\phi + \sin^2\phi\right] = \rho^2\sin\phi$$

Vector valued functions 13.12

-> the coordinates of a point A moving in space are given by $A(x(t),y(t),z(t))$. the position of A is determined by a vector called the position vector $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

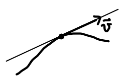
$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad a \le t \le b \quad \Rightarrow \text{parametric equation of the path of A}$$

Def: if $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ defines the position vector of a point M at time t, the velocity of M at t is $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$. the magnitude of $\vec{v}$ is called the speed and given by

$$\text{speed} = \|\vec{v}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

-> the acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$

Note: the velocity $\vec{v}(t)$ represents the direction vector of the tangent line to (C) at any point M at any time t



$\vec{T} = \vec{v}(t)$

Length of a curve

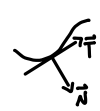
the length of a curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ with $a \le t \le b$ is given by $L = \int_a^b \|\vec{v}\|\,dt$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Unit tangent vector

Since the velocity is the tangent to the curve $\vec{r}(t)$ then the unit tangent vector is $\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$

-> at any point P, the unit normal vector is $\vec{N} = \frac{d\vec{T}}{dt} / \left\|\frac{d\vec{T}}{dt}\right\|$



Vector field

Def: A vector field is a function that assigns to each point (x,y,z) in its domain a vector given by $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

→ the field is continuous / differentiable if M, N, R are continuous / differentiable
→ for a given function $f(x,y,z)$ the gradient vector $\nabla f = (f_x, f_y, f_z)$ is a vector field

Def: A vector field $\vec{F}$ is called a conservative vector field if there exists a function f such that $\vec{F} = \nabla f$
* if $\vec{F}$ is a conservative vector field then the function f is called a potential function of F

Known curves and parametrizations 18.12

- $x^2+y^2=r^2 \Rightarrow \begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \quad 0 \le t \le 2\pi$
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \quad 0 \le t \le 2\pi$
- $x = g(y) \Rightarrow \begin{cases} y = t \\ x = g(t) \end{cases}$
- $(x-a)^2+(y-b)^2=r^2 \Rightarrow \begin{cases} x = a + r\cos t \\ y = b + r\sin t \end{cases}$

Line integral 13.12

→ we perform integration by taking the points (x,y) that lie on a curve (C), this integral is called line or curve integral and is given by $\int_C f(x,y,z)\,ds$, $ds = \|\vec{v}\|\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$
→ assume the curve (C) is given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$:
- the curve is called smooth if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0\ \forall t$

steps

1. parametrize (C), 2. find $\vec{v}$
2. $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,\|\vec{v}\|\,dt$

Line integrals of a vector field 18.12

Given $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ vector field and a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ with $\underline{a \le t \le b}$ then $\int_C \vec{F}\,d\vec{r} = \int_a^b \underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)}_{\text{dot product}}\,dt$ or $\int_C \vec{F} \cdot \vec{T}\,ds$

Thm:
Suppose that C is a smooth curve given by $\vec{r}(t)$ $a \le t \le b$. Assume f is a function whose gradient vector ∇f, is continuous on C, then $\int_C \nabla f\,dr = f(r(b)) - f(r(a))$ where $r(a)$ and $r(b)$ are the endpoints

Proof:
$$\int_C \nabla f\,dr = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt = \int_a^b \frac{d}{dt}(f(\vec{r}(t)))\,dt \underset{\text{fund. thm of integral calculus}}{=} f(r(b)) - f(r(a))$$

Def: Assume $\vec{F}$ is cont. v.f. if $\int_{C_1} \vec{F}\,dr = \int_{C_2} \vec{F}\,dr$ for any 2 paths C_1 and C_2 with the same initial and final points, then $\int_C \vec{F}\,d\vec{r}$ is independent of path

1. $\int \nabla f\,dr$
2. if F is conservative
3. $\int_C F\,dr = 0$ for every closed path C

Test for conservative vector field

Let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ be the v.f. Then F is conservative if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial M}{\partial z}$

finding the potential function:

$\nabla f = F \Rightarrow \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = M\vec{i} + N\vec{j} + P\vec{k}$ → see extra exercises

$\Rightarrow \frac{\partial f}{\partial x} = M \; ; \; \frac{\partial f}{\partial y} = N \; ; \; \frac{\partial f}{\partial z} = P$ (integrate these eqn.)