Algorithms 2022/2023

Solve the following exercises.

1. Recall that, if $G(V, E)$ is an undirected graph, and $w : E \to \mathbf{R}$ is a function from E to the set of the real numbers $\mathbf{R}$, then $G(V, E, w)$ is an undirected weighted graph. Moreover, a path in $G(V, E, w)$ is a sequence of nodes $v_1, v_2, \ldots, v_t$ such that $\{v_i, v_{i+1}\} \in E$ for each $i = 1, \ldots, t-1$. The length of the path $v_1, v_2, \ldots, v_t$ is equal to $\sum_{i=1}^{t-1} w(\{v_i, v_{i+1}\})$.

   Given $u, v \in V$, a shortest path from u to v in an undirected graph $G(V, E, w)$ is a path of shortest length whose first node is u and whose last node is v.

   Consider the following statement: "For each undirected weighted graph $G(V, E, w)$, and for each $x \in \mathbf{R}$, if $v_1, v_2, \ldots, v_t$ is a shortest path from v_1 to v_t in $G(V, E, w)$, then $v_1, v_2, \ldots, v_t$ is also a shortest path from v_1 to v_t in $G(V, E, w')$, where $w'(e) = w(e) + x$ for each $e \in E$."

   Is this statement true or false? Prove it, or give a counterexample.

   Example: If $V = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, $w(\{1, 2\}) = 3$, $w(\{2, 3\}) = 2$ and $w(\{1, 3\}) = 7$, then the shortest path in $G(V, E, w)$ from 1 to 3 is the path $1, 2, 3$. If $x = 1$, then $w'(\{1, 2\}) = 4, w'(\{2, 3\}) = 3$, $w'(\{1, 3\}) = 8$, and the shortest path from 1 to 3 in $G(V, E, w')$ is still $1, 2, 3$.

2. Let V be an array containing n positive floating point numbers. If i, j are two integers such that $0 \leq i \leq j \leq n-1$, the (i, j)-product $p_{i,j}$ of V equals

   $$p_{i,j} = V[i] \cdot V[i+1] \cdot \cdots \cdot V[j] = \prod_{k=i}^{j} V[k].$$

   Give an algorithm that returns the value of one of the largest $p_{i,j}$ products, for $0 \leq i \leq j \leq n-1$. Prove that your algorithm is correct, and bound its running time. *Larger scores will be awarded to faster solutions.*

   Example 1: if $n = 4$, and $V[0] = 1.5, V[1] = 4.0, V[2] = 2.0, V[3] = 0.1$, then $p_{0,2} = V[0] \cdot V[1] \cdot V[2] = 12.0$ is the unique largest $p_{i,j}$.

   Example 2: if $n = 3$, and $V[0] = 2.0, V[1] = 0.1, V[2] = 2.0$, then there are two largest $p_{i,j}$'s, $p_{0,0} = p_{2,2} = 2.0$.