SAPIENZA university of Rome
Bachelor of science in ACSAI
ALL the LA exam questions + solutions

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Exercise 1:

Answer by true or false and justify your answer:

1. Every linear dependent set contains the zero vector. False.

   Not necessarily.
   Consider two vectors v_1 and v_2 in a vector space, where $v_2 = 2v_1$. This set of vectors is linearly dependent because $2v_1 - v_2 = 0$, but the set v_1, v_2 does not necessarily contain the zero vector itself.

2. The vector $\mathbf{u} = (1, 2, 1)$ is a linear combination of $\mathbf{v_1} = (2, 1, 0)$ and $\mathbf{v_2} = (1, -2, 4)$. False.

3. The vector $\mathbf{v} = (1, 2, 1)$ is a linear combination of $\mathbf{v_1} = (2, 1, 1)$ and $\mathbf{v_2} = (1, -2, 4)$. False.

   The vector $\mathbf{v}$ could be a linear combination only if there exist two coefficients a and b such that $a \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2 = \mathbf{v}$.

   To determine if this is true or false, we can solve the following system of linear equations:

$$\begin{cases} 2a + b = 1 \\ a - 2b = 2 \\ a + 4b = 1 \end{cases} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{pmatrix} \rightarrow \text{Impossible, no solutions !}$$

4. The set $S = \{(1, -2, 6), (5, -10, 30)\}$ is linearly dependent. True.

   - $\begin{pmatrix} 1 & -2 & 6 \\ 5 & -10 & 30 \end{pmatrix} \xrightarrow{REF} \begin{pmatrix} 1 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$. Since the last row is made by all 0s, then one of the vectors could be re-written in infinite ways as a linear combination of the others and the set S is indeed linearly dependent.

5. The vectors $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (2, 0, 2)$ and $\mathbf{v}_3 = (0, 2, 1)$ are linearly independent in R^3. True.

   In R^3, if each vector in a set has a zero in a different coordinate position where the other vectors have non-zero entries, it implies that they are linearly independent. No combination of the other vectors can recreate the zero in a specific position without making the entire vector zero, which would violate the definition of linear independence.

6. The set $S = \{1 + x, -x^2 + 2\}$ is a linearly independent set in $P_2(R)$. True.

   To quickly determine if this is true, we can look at the degrees of the polynomials. Since one is of degree 1 and the other is of degree 2, and there is no way to write x^2 as a multiple of x or a constant, and similarly, there is no way to write x as a multiple of x^2 or a constant; the set is linearly independent without even needing to solve any equation. This is because their degrees are different, and in a polynomial space, polynomials of different degrees are always linearly independent.

7. $J = \{1, 1 + x, 1 + x + x^2\}$ is a linearly independent set in P_2. True.

   We can re-write the equation as:

$$a_0(1) + a_1(1 + x) + a_2(1 + x + x^2) = 0$$

   For this to be true for all x, each coefficient must be zero:

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_1 + a_2 = 0 \\ a_2 = 0 \end{cases}$$

   It is clear that $a_0 = a_1 = a_2 = 0$.

8. The set $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$ is a linearly independent set in P_2. False.

$$a_1(1 - x) + a_2(1 - x^2) + a_3(3x^2 - 2x - 1) = 0$$

$$\begin{cases} a_1 + a_2 - a_3 = 0 \\ -a_1 - 2a_3 = 0 \\ -a_2 + 3a_3 = 0 \end{cases}$$

   We can solve this system to find out if the only solution is $a_1 = a_2 = a_3 = 0$, which would imply linear independence.

   - The determinant of the coefficients matrix $\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & -1 & 3 \end{pmatrix} = 0$ (NO unique solution, linearly dependent set).
   - The RREF of the coefficients matrix is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$, hence the set is linearly dependent.
   - As previously said, look at the degree of the polynomials.

9. Is the set $S = \{(1+x-2x^2), (2+5x-x^2), (x+x^2)\}$ a linearly independent set in P_2 ? False.

$$a_1(1+x-2x^2) + a_2(2+5x-x^2) + a_3(x+x^2) = 0$$

$$\begin{cases} a_1 + 2a_2 = 0, \\ a_1 + 5a_2 + a_3 = 0, \\ -2a_1 - a_2 + a_3 = 0, \end{cases}$$

DET $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 1 \end{pmatrix} = 0$, hence NO unique solution for the system and l.d. set.

The RREF of the coefficients matrix is: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$, hence the set of vectors is linearly dependent.

10. The set $S = \{(-1,4,2), (2,3,7)\}$ is a basis of R^3. False.

    A set of vectors is a basis of a subspace iff:

    (a) The vectors in the set have the same dimension;
    (b) The number of vectors in the set is the same as the dimensions of the considered subspace;
    (c) The vectors in the set are linearly independent (this implies that they span the subspace).

    In this case, there are only 2 vectors of R^3, hence the set could not be a basis of R^3.

11. The set $S = \{(1,5,3), (0,1,2), (0,0,6)\}$ is a basis of R^3. True.

$$\xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{hence the vectors are l.i.}$$

    Considering that there are three linearly independent vectors of R^3, they form indeed a basis of R^3.

12. The set $S = \{(-1,4,2), (2,3,7), (6,5,2)\}$ is a basis for R^4. False.

13. $W = \{(x,y,z) \in R^3; x^2+y^2+z^2 = 1\}$ is a subspace of R^3. False.

    - Looking at the set W defined as $\{(x,y,z) \in R^3 : x^2+y^2+z^2 = 1\}$, we can see that the zero vector does not satisfy the equation $x^2+y^2+z^2 = 1$, because $0^2+0^2+0^2 = 0$, not 1.

- The set W describes a unit sphere with radius 1 (centered at the origin), which is not closed under vector addition because if you take two points on the surface of the sphere, their sum will not generally lie on the sphere.
  It is not closed under scalar multiplication because if you take any non-zero scalar and multiply it with a point on the sphere, the result will not lie on the sphere unless the scalar is 1 or -1.

14. The set $W = \{(x, y, z) \in R^3$ such that $x = 1$ and $y = z\}$ is a subspace of R^3. False.

    - **The zero vector is not in** W: For the zero vector $(0, 0, 0)$ to be in W, we would need $x = 0$, but the set W is defined such that $x = 1$. Since the zero vector is not included, W cannot be a subspace;
    - **Not closed under vector addition**: If you take any two distinct vectors from W, say $(1, a, a)$ and $(1, b, b)$, and add them, you get $(2, a+b, a+b)$, which does not satisfy $x = 1$. Hence, W is not closed under addition;
    - **Not closed under scalar multiplication**: If you multiply any vector in W by any scalar other than 1, the x-component will not be 1 anymore. For example, $2 \cdot (1, a, a) = (2, 2a, 2a)$, which does not belong to W since x is not 1.

15. The set $W = \{A \in M_{2,2} \mid A^T = A\}$ is a subspace of $M_{2,2}$. True.

    To show that W is a subspace of $M_{2,2}$, we need to verify three properties:

    (a) The zero vector (in this context, the zero matrix) is in W.
    (b) W is closed under vector addition (matrix addition in this context).
    (c) W is closed under scalar multiplication.

    If all three properties are satisfied, W is indeed a subspace of $M_{2,2}$.

    Let's consider these properties:

    (a) The zero matrix is symmetric since its transpose is itself.
    (b) If you take any two symmetric matrices A and B (where $A^T = A$ and $B^T = B$), their sum $A + B$ is also symmetric because $(A + B)^T = A^T + B^T = A + B$.
    (c) If you take a symmetric matrix A and any scalar c, the product cA is also symmetric because $(cA)^T = cA^T = cA$.

    Since all three properties hold, the set W is indeed a subspace of $M_{2,2}$.

16. The set W of 2×2 skew symmetric matrices is a subspace set in P_n. False.

    It's clear why the set W is not a subspace.

17. The dimension of the subspace $W = \{(x, y, z) \mid x + y + z = 0\}$ of R^2 is 2. False.

    (x, y, z) represents vectors of R^3, not of R^2.

18. The transformation $T : R \to R, T(x) = |x|$ is linear. False.

$$DET(A) + DET(B) \neq DET(A + B).$$

19. The transformation $T : M_{2,2} \to R$ given by $T(A) = \det(A)$ is linear. False.

20. The map $T : R^2 \to R^3$ defined by $T(x, y) = (x + y, x + 1, 3y)$ is a linear transformation. False.

    For a map to be a linear transformation, it must satisfy two properties for any vectors u, v in the domain and any scalar c:

    (a) Additivity: $T(u + v) = T(u) + T(v)$.

    (b) Homogeneity: $T(c \cdot u) = c \cdot T(u)$.

    Let's check these properties for the given map T.

    For additivity, we need to check if $T((x_1, y_1) + (x_2, y_2)) = T(x_1, y_1) + T(x_2, y_2)$.

    For homogeneity, we need to check if $T(c \cdot (x, y)) = c \cdot T(x, y)$.

    Given the definition of T, we can verify these properties through algebraic manipulations. However, we can already observe that the second component of $T(x, y)$ includes a "+1" which is a constant term. This term will not behave linearly under scalar multiplication or addition, so we can conclude without further calculation that T is not a linear transformation.

21. The transformation $T : R \to R \mid T(x) = \sqrt{x}$ is linear. False.

    (a) **Additivity Violation:** For the square root function to be additive, the following condition must hold for any x and y:

    $$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

    However, this is not true in general. For example, take $x = 1$ and $y = 4$. According to the additivity property, we would expect that $\sqrt{1+4} = \sqrt{1} + \sqrt{4}$, but this is not the case since $\sqrt{5} \neq 1 + 2$.

    (b) **Scalar Multiplication Violation:** For the square root function to satisfy scalar multiplication, the following condition must hold for any scalar c and vector x:

    $$\sqrt{cx} = c\sqrt{x}$$

    This condition is also not generally true. For instance, let $c = 4$ and $x = 9$. According to the scalar multiplication property, we would expect that $\sqrt{4 \times 9} = 4\sqrt{9}$, but this is false since $\sqrt{36} = 6$ which is not equal to $4 \times 3 = 12$.

    Therefore, the square root function violates both the additivity and scalar multiplication properties of linear transformations, proving that it is not a linear transformation.

22. If a square matrix A has no zero rows or columns, then it has an inverse. False.

    It may be that even if a matrix has no zero rows or columns, its determinant is still zero.

    $$DET \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = 0$$

23. If A and B are invertible square matrices, then $(A + B)$ is also invertible. False.

24. There is an invertible matrix A such that $A^2 = 0$. False.

    No, if a matrix A is invertible, it cannot become the zero matrix when raised to any power. An invertible matrix, by definition, has a non-zero determinant, and an inverse matrix exists such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

    When you raise an invertible matrix to any positive integer power n, you're essentially multiplying the matrix by itself n times. Since the matrix is invertible, none of these multiplications will result in the zero matrix. This is because an invertible matrix corresponds to a transformation that can be "undone" by its inverse, and multiplying by the zero matrix cannot be "undone" as it maps all vectors to the zero vector, losing all original information.

    In other words, if $A^n = 0$ for some positive integer n, then A would be singular (not invertible), as the only matrix that maps to the zero matrix when multiplied by itself is the zero matrix, which is singular.

    Moreover, if A were to become the zero matrix after repeated multiplication (raising to a power), it would imply that A is nilpotent. However, a nilpotent matrix is not invertible, as its determinant is zero. Thus, it's a contradiction to have an invertible matrix that becomes the zero matrix when raised to a power.

25. If $A^2 = A$ is a non-singular matrix (invertible), then $det(A^2) = 1$. True.

    - If A is idempotent and invertible (non-singular), then $\det(A)$ must be 1. This is because for an invertible matrix A, there exists an inverse A^{-1} such that $AA^{-1} = I$. Since $A^2 = A$, multiplying both sides by A^{-1} gives $A = I$, and hence $\det(A) = \det(I) = 1$.
    - If A is idempotent but not invertible (singular), then $\det(A)$ must be 0. This is because a singular matrix does not have an inverse, and the determinant of a singular matrix is always 0.

26. Let A be a 4×5 matrix. If nullity $(A^T) = 2$, then $\text{Rank}(A) = 2$. False.

    By definition: $rank(A) + nullity(A) =$ n. of columns.

    n. of columns $- nullity(A) = 3$ and not 2.

27. If A is a 6×8 matrix such that $\text{Rank}(A^T) = 5$, then $\text{Nullity}(A) = 1$. False.

    Recall that by definition $rank(A) = rank(A^T)$.

    The rank of a matrix is defined as the maximum number of linearly independent column vectors in the matrix, which is also the dimension of the column space (or column rank). For the transpose of the matrix, A^T, the column vectors become the row vectors of A. Since the row space and the column space of a matrix have the same dimension, the rank of A and A^T must be equal. This means that the number of linearly independent rows is equal to the number of linearly independent columns in any matrix.

    n. of columns $- rank(A^T) = 3$ and not 1.

28. Let A and B be $n \times n$ matrices, then $\text{tr}(A \cdot B) = \text{tr}(A) \cdot \text{tr}(B)$. False.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 0$$

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 0$$

$$A \cdot B = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$tr(A \cdot B) = 29$$
$$tr(A) = 5,\ tr(B) = 5$$
$$tr(A) \cdot tr(B) = 25$$
$$29 \neq 25.$$

29. Let A and B be any matrices, then $(A - B)(A + B) = A^2 - B^2$. False.

30. If A and B are $n \times n$ skew symmetric matrices, then $A + B$ is skew symmetric. True.

    The sum of matrices preserves the symmetry or skew-symmetry of the individual matrices.

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

31. if A and B are $(n \times n)$ symmetric matrices, then $A + B$ is also symmetric. True.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

32. If $\mathbf{u} = (k, k, 1)$ and $\mathbf{v} = (k, 5, 6)$ are orthogonal, then $k = 1$. False.

Two vectors are orthogonal if their dot product equals zero.

Let $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, 5, 6)$.

The dot product $\mathbf{a} \cdot \mathbf{b}$ is calculated as follows:

$$\mathbf{a} \cdot \mathbf{b} = (1 \times 1) + (1 \times 5) + (1 \times 6) = 12 \neq 0$$

33. If u and v are orthogonal vectors such that $||u|| = 6$ and $||v|| = 3$, then $||u + v|| = 9$. False.

This statement is incorrect according to the properties of orthogonal vectors. For orthogonal vectors, by the Pythagorean theorem, the magnitude of their sum is given by:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Given $\|\mathbf{u}\| = 6$ and $\|\mathbf{v}\| = 3$, we have:

$$\|\mathbf{u} + \mathbf{v}\|^2 = 6^2 + 3^2 = 36 + 9 = 45$$

Taking the square root to find the magnitude of $\mathbf{u} + \mathbf{v}$:

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{45} \approx 6.71.$$

This result is not 9.

34. The function $\langle u, v\rangle = u_1 v_1$ defines an inner product on R^2, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$. False.

This function does not define a valid inner product on R^2 because it does not satisfy all the properties required for an inner product. The properties that an inner product must satisfy are:

(a) Positivity: $\langle u, u\rangle \geq 0$ for all u, and $\langle u, u\rangle = 0$ if and only if $u = 0$.

(b) Symmetry: $\langle u, v\rangle = \langle v, u\rangle$ for all u, v.

(c) Linearity in the first argument (or the second, depending on the definition):
   - $\langle au, v\rangle = a\langle u, v\rangle$ for any scalar a.
   - $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$ for all u, v, w.

The function $\langle u, v\rangle = u_1 v_1$ only considers the first component of each vector and ignores the second component. This means that a vector with $u_1 = 0$ and $u_2 \neq 0$ would have an inner product of zero with itself, which violates the positivity property since u is not the zero vector. Therefore, the given function cannot be considered a valid inner product on R^2. A proper inner product in R^2 would be the standard dot product $\langle u, v\rangle = u_1 v_1 + u_2 v_2$.

35. If $\langle x, y\rangle = \langle x, z\rangle$ for vectors x, y, z in an inner product space, then $y - z$ is orthogonal to x. True.

The statement asserts that if for vectors x, y, z in an inner product space the inner products $\langle x, y\rangle$ and $\langle x, z\rangle$ are equal, then the vector $y - z$ is orthogonal to x.

To demonstrate why this is true, consider the definition of orthogonality in an inner product space: a vector u is orthogonal to a vector v if their inner product $\langle u, v\rangle$ is zero.

Given $\langle x, y\rangle = \langle x, z\rangle$, we can investigate the inner product of x with $y - z$:

$$\langle x, y - z\rangle = \langle x, y\rangle - \langle x, z\rangle$$

Since $\langle x, y\rangle$ is equal to $\langle x, z\rangle$, their difference is zero:

$$\langle x, y\rangle - \langle x, z\rangle = 0$$

Therefore, $\langle x, y - z\rangle = 0$, which means that $y - z$ is orthogonal to x by the definition of orthogonality in inner product spaces.

36. The matrix $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$ is orthogonal. False.

To verify whether this matrix is orthogonal, we need to check if the product of the matrix and its transpose equals the identity matrix.

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This is not the identity matrix, which would have ones on the diagonal and zeros elsewhere.

An orthogonal matrix should satisfy the equation $A^T A = AA^T = I$.

37. The coordinates of the vector $\mathbf{v} = (4, 2)$ with respect to the basis $\mathbf{u}_1 = (3, 2)$ and $\mathbf{u}_2 = (2, 3)$ of R^2 are $(2, -1)$. False.

Basically you have to check if the vector $\mathbf{v}$ can be written as a linear combination of the basis vectors and some coefficients.

$$a \cdot \mathbf{u}_1 + b \cdot \mathbf{u}_2 = \mathbf{v}$$

$$a \cdot (3, 2) + b \cdot (2, 3) = (4, 2)$$

$$\begin{cases} 3a + 2b = 4 \\ 2a + 3b = 2 \end{cases} \rightarrow \begin{pmatrix} 3 & 2 & | & 4 \\ 2 & 3 & | & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & | & \frac{8}{5} \\ 0 & 1 & | & -\frac{2}{5} \end{pmatrix}$$

The coordinates of the vector $\mathbf{v}$ w.r.t that basis are $(\frac{8}{5}, -\frac{2}{5})$.

38. The coordinates of the vector $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ with respect to the basis $u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and $u_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ of R^2 are $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. False.

39. If A is a 3×3 matrix such that $|A| = 2$, then $|2A^T A^{-1}| = 2$. False.

$$2^3 \cdot |A| \cdot \frac{1}{|A|} = 8 \cdot 2 \cdot \frac{1}{2} = 8.$$

40. If P and D are $n \times n$ matrices, then $\det(PDP^{-1}) = \det(D)$. True.

A property of determinants in linear algebra:

If P and D are $n \times n$ matrices, then $\det(PDP^{-1}) = \det(D)$.

This property is true due to the multiplicative property of determinants, which says that the determinant of a product of matrices equals the product of their determinants. That is:

$$\det(AB) = \det(A)\det(B)$$

for any square matrices A and B of the same size.

Given P is invertible (since P^{-1} exists), we have:

$$\det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1})$$

And since the determinant of an inverse matrix P^{-1} is the reciprocal of the determinant of the matrix P (i.e., $\det(P^{-1}) = \frac{1}{\det(P)}$), it follows that:

$$\det(P)\det(P^{-1}) = \det(P) \cdot \frac{1}{\det(P)} = 1$$

Therefore:

$$\det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1}) = \det(D) \cdot \det(P)\det(P^{-1}) = \det(D) \cdot 1 = \det(D)$$

This shows that the determinant of the original matrix D is unchanged by the conjugation with P. This is a fundamental result in the study of similar matrices and is often used in the field of eigenvalues and eigenvectors.

41. The solution of the system $\begin{cases} x - 3y = 2 \\ 5x + y = 1 \end{cases}$ using Cramer's rule is $x = 5,\ y = -9$. False.

Cramer's rule is a theorem in linear algebra that gives an explicit formula for the solution of a system of linear equations with as many equations as unknowns, provided that the system has a unique solution. It states that the solution can be found by using determinants.

To verify the solution using Cramer's rule, we would calculate the determinant of the coefficient matrix and the determinants of matrices formed by replacing one column of the coefficient matrix by the column vector of the right-hand sides of the equations. Let's calculate the determinants to check if the solution provided is correct.

$$\text{If } DET(A) \neq 0:$$

$$DET(A) = DET\begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix} = 16$$

$$DET(A_1(x)) = DET\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = 5$$

$$DET(A_2(y)) = DET\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix} = -9$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{5}{16} \approx 0.3125$$

$$y = \frac{DET(A_2)}{DET(A)} = \frac{-9}{16} \approx -0.5625.$$

Exercise 2:

1. Determine the number of solutions of the following system:

$$\begin{cases} x + 2y - 3z = 4 \\ 4x + y + 2z = 6 \\ x + 2y + (k^2 - 19)z = k \end{cases}$$

depending on the parameter $k \in R$.

$$\rightarrow \begin{pmatrix} 1 & 2 & -3 & | & 4 \\ 4 & 1 & 2 & | & 6 \\ 1 & 2 & (k^2-19) & | & k \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & | & \frac{8k+25}{7(k+4)} \\ 0 & 1 & 0 & | & \frac{2(5k+27)}{7(k+4)} \\ 0 & 0 & 1 & | & \frac{1}{k+4} \end{pmatrix}$$

$$x = \frac{8k+25}{7k+28}$$

$$y = \frac{10k+54}{7k+28}$$

$$z = \frac{1}{k+4}$$

However, these solutions are valid as long as the denominators $7k + 28$ and $k + 4$ are not equal to zero. If $k = -4$, the denominator for z becomes zero, which means the system does not have a solution for this value of k. Similarly, if $k = -\frac{28}{7}$, then the denominators for x and y become zero, and the system also does not have a solution for this value of k.

So, for all $k \in R$ except $k = -4$ and $k = -\frac{28}{7}$, the system has a unique solution. For $k = -4$ or $k = -\frac{28}{7}$, the system has no solution.

2. Determine the number of solutions of the following system:

$$\begin{cases} x + ky + (4k+1)z = 4k+1 \\ 2x + (1+k)y + (2+7k)z = 7k+1 \\ 3x + (k+2)y + (3+9k)z = 1+9k \end{cases}$$

depending on the parameter $k \in R$.

3. Determine the number of solutions of the following system:

$$\begin{cases} x + ky - kz = 1 \\ -4y + 2z = 1 \\ -x + ky = 1 \end{cases}$$

depending on the parameter $k \in R$ and for the values of k for which the system is compatible, solve it.

4. Determine the number of solutions of the following system:

$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 3 \\ 3x + 2y + kz = 4 \end{cases}$$

depending on the parameter $k \in R$.

5. Determine the number of solutions of the following system:

$$\begin{cases} x + 2y - kz = k \\ -x - y + kz = 0 \\ (2+k)y + (2k+1)z = 0 \end{cases}$$

depending on the parameter $k \in R$ and for the values of k for which the system is compatible, solve it.

Exercise 3:

Considering the following matrices:

1. Find the eigenvalues of A and a basis of each eigenspace of A.

2. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ 2 & -4 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$$

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & -1 & -1 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{pmatrix}$$

$$= (\lambda - 1)\left[(\lambda - 1)(\lambda - 4) - (-2)(1)\right]$$

For $\lambda = 1$:

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_1 = -R_1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{R_2 = R_2 - R_1} \xrightarrow{R_3 = -R_3} \begin{pmatrix} 0 & 4 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 0 & 4 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} y + z = 0 \\ 2z = 0 \\ x = t \end{cases} \implies \begin{cases} y = 0 \\ z = 0 \\ x = t \end{cases} \implies t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 3$:

$$\begin{pmatrix} -2 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1=-R_1} \xrightarrow{R_2=-R_2} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{swap} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \xrightarrow{R_3=R_3-2R_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1=\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{cases} x = 0 \\ y = -t \\ z = t \end{cases} \implies t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = 2$:

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_1=-R_1} \xrightarrow{R_2=-R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_3=R_3-R_2}$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{cases} x - t = 0 \\ y + 2t = 0 \\ z = t \end{cases} \implies \begin{cases} x = t \\ y = -2t \\ z = t \end{cases} \implies t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

To find P^{-1}, put on the right of the P matrix, the identity matrix. Perform Gauss reduction until the identity matrix will be on the left side instead of the right one.

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & -1 & -1 \end{pmatrix}.$$

Cramer's rule:

Apply Cramer's rule to solve the following system:

$$\begin{cases} 2x + y - z = 3 \\ x + y + z = 1 \\ x + 2y + 3z = 4 \end{cases}$$

If $DET(A) \neq 0$:

$$DET(A) = DET \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = -1$$

$$DET(A_1(x)) = DET \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{pmatrix} = 0$$

$$DET(A_2(y)) = DET \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{pmatrix} = -5$$

$$DET(A_3(z)) = DET \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix} = 0$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{0}{-1} = 0$$

$$y = \frac{DET(A_2)}{DET(A)} = \frac{-5}{-1} = 5$$

$$z = \frac{DET(A_3)}{DET(A)} = \frac{0}{-1} = 0.$$

Exercises on subspaces:

1. Consider the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 1 & 5 & 6 & 9 & 0 \end{pmatrix}$$

   (a) Find a basis for the row space and column space of A, then deduce the rank and nullity of A.

   (b) Find the null space of A and deduce its basis.

$$\xrightarrow{REF} \begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 1 & 4 & 4 \end{pmatrix}$$

- The basis for the row space of A can be taken from the non-zero rows of the original matrix A, which are the first three rows.

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 4 \end{pmatrix} \right\}$$

- The basis for the column space of A can be taken from the corresponding columns of the original matrix A, which are the first three columns.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Note that you can use as basis for the row space either the rows where there are the pivots in the reduced matrix or the corresponding rows in the non-reduced matrix. Same for the column space.

- The rank of A is 3, which is the number of pivot positions.
- Rank(A) + nullity(A) = 5 (n. of columns of A).
  Nullity(A) = 5 - rank(A) = 5 - 3 = 2.
- For part (b), the null space and a basis for it is the following:

$$\begin{cases} x_1 + 3x_2 + x_3 - x_4 = 0, \\ x_2 - 5x_4 - 10x_5 = 0, \\ x_3 + 4x_4 + 4x_5 = 0, \end{cases}$$

3 equations, 5 unknowns, so: 2 free parameters ($x_4 = s; x_5 = t$).

$$\begin{cases} x_1 + 3x_2 + x_3 - s = 0, \\ x_2 - 5s - 10t = 0, \\ x_3 + 4s + 4t = 0, \end{cases}$$

$$\begin{cases} x_1 = -10x_4 - 26x_5, \\ x_2 = 5x_4 + 10x_5, \\ x_3 = -4x_4 - 4x_5, \\ x_4 = s, \text{ (free variable)} \\ x_5 = t. \text{ (free variable)} \end{cases}$$

A basis for the null space (kernel) is the following:

$$\left\{ s \cdot \begin{pmatrix} -10 \\ 5 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -26 \\ 10 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. Let W be the set

$$W = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in M_{2,2} \mid m - 2n = 0 \text{ and } p - 3q = 0 \right\}$$

(a) Prove that W is a subspace of $M_{2,2}$.

(b) Find a basis for W and deduce its dimension.

3. Prove or disprove that the subset W of $M_{2,2}$ defined by

$$W = \{A \in M_{2,2} \mid A^T A = I\}$$

is a subspace of $M_{2,2}$.

To prove that W is a subspace of $M_{2,2}$, we must show that it is closed under addition and scalar multiplication, and that it contains the zero vector.

For closure under addition, take any two matrices $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ and $B = \begin{pmatrix} m' & n' \\ p' & q' \end{pmatrix}$ in W. Then $A + B = \begin{pmatrix} m+m' & n+n' \\ p+p' & q+q' \end{pmatrix}$.

We have $m - 2n = 0$ and $p - 3q = 0$ for A, and $m' - 2n' = 0$ and $p' - 3q' = 0$ for B.

Adding these equations, we get $(m + m') - 2(n + n') = 0$ and $(p + p') - 3(q + q') = 0$, so $A + B \in W$.

For closure under scalar multiplication, take any scalar k and matrix $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ in W.

Then $kA = \begin{pmatrix} km & kn \\ kp & kq \end{pmatrix}$.

Since $m-2n = 0$ and $p-3q = 0$, multiplying by k gives $k(m-2n) = 0$ and $k(p-3q) = 0$, so $kA \in W$.

The zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W because $0 - 2 \cdot 0 = 0$ and $0 - 3 \cdot 0 = 0$.

Thus, W is a subspace of $M_{2,2}$.

To find a basis for W, we solve the equations $m - 2n = 0$ and $p - 3q = 0$. We can express m as $2n$ and p as $3q$. Hence, every matrix in W can be written as:

$$\begin{pmatrix} 2n & n \\ 3q & q \end{pmatrix} = n \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + q \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}.$$

Thus, a basis for W is:

$$\left\{ \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \right\}.$$

The dimension of W is the number of vectors in the basis, which is 2.

Exercises on linear transformations (aka "linear maps"):

1. Let $T : P_2 \to R^2$ be the transformation defined by $T(ax^2 + bx + c) = \begin{pmatrix} a + 3c \\ a - c \end{pmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find the Kernel of T.

   For part (a):

- Let $f(x) = ax^2 + bx + c$;
- Let $g(x) = dx^2 + ex + f$.

$$T(f+g) = \begin{pmatrix} (a+d) + 3(c+f) \\ (a+d) - (c+f) \end{pmatrix} = \begin{pmatrix} a+3c \\ a-c \end{pmatrix} = \begin{pmatrix} d+3f \\ d-f \end{pmatrix} = T(f) + T(g).$$

$$T(f+g) = T(kax^2 + kbx + kc) = \begin{pmatrix} ka + 3kc \\ ka - kc \end{pmatrix} = k \begin{pmatrix} a+3c \\ a-c \end{pmatrix} = kT(f).$$

For part (b):

To find the kernel (or null space) of a transformation T, you need to find all vectors in the domain that map to the zero vector in the codomain.

Always set and solve an homogeneous linear system.

$$\{f(x) \in P_2 \mid T(f(x)) = 0\}$$

$$\{ax^2 + bx + c \mid \begin{cases} a + 3c = 0 \\ a - c = 0 \end{cases} \}$$

$$\{ax^2 + bx + c \mid \begin{cases} a = -3c \\ c = a \end{cases} \}$$

$$\{ax^2 + bx + c \mid a = c = 0\}$$

The solution to the system is $a = c = 0$. There is no condition on b, which means that b can be any real number.
Therefore, the kernel of T consists of all polynomials of the form bx where b is a real number. In other words, the kernel is the set of all polynomials in P_2 that are linear in x with no constant or quadratic form.

$$Ker(T) = \{bx \mid b \in R\}$$

This is the set of all polynomials of degree one without a constant term, which is a subspace of P_2.

2. Let $T : R^2 \to R^2$ be the transformation given by $T(x, y) = (x + 2y, y - 2x)$.

   (a) Show that T is a linear transformation.

   (b) Find the Kernel of T.

   For part (a):

- For additivity, consider two vectors $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ in R^2:

$$
\begin{aligned}
T(\mathbf{u} + \mathbf{v}) &= T((x_1 + x_2), (y_1 + y_2)) \\
&= (x_1 + x_2 + 2(y_1 + y_2), (y_1 + y_2) - 2(x_1 + x_2)) \\
&= (x_1 + 2y_1 + x_2 + 2y_2, y_1 - 2x_1 + y_2 - 2x_2) \\
&= (x_1 + 2y_1, y_1 - 2x_1) + (x_2 + 2y_2, y_2 - 2x_2) \\
&= T(x_1, y_1) + T(x_2, y_2).
\end{aligned}
$$

- For homogeneity, consider a scalar c and a vector $\mathbf{u} = (x_1, y_1)$ in R^2:

$$
\begin{aligned}
T(c\mathbf{u}) &= T(cx_1, cy_1) \\
&= (cx_1 + 2cy_1, cy_1 - 2cx_1) \\
&= c(x_1 + 2y_1, y_1 - 2x_1) \\
&= c \cdot T(x_1, y_1).
\end{aligned}
$$

Since T satisfies both additivity and homogeneity, it is a linear transformation.

For part (b):

- To find the kernel of T, we need to solve for all vectors (x, y) such that $T(x, y) = (0, 0)$. This gives us the system of equations:

$$
\begin{cases}
x + 2y = 0, \\
y - 2x = 0.
\end{cases}
$$

Solving this system, we find that $x = 0$ and $y = 0$ are the only solutions. Therefore, the kernel of T is given by:

$$
\mathrm{Ker}(T) = \{(0, 0)\}.
$$

This means that the only vector in R^2 that maps to the zero vector under T is the zero vector itself.

Gram-Schimdt orthonormalization process:

1. Apply Gram Schmidt process to transform the basis $B = \{(1, 0, 0), (2, 1, 0), (1, 2, 1)\}$ into an orthonormal Basis of R^3.

$$
\begin{aligned}
w_1 &= v_1 = (1, 0, 0) \\
w_2 &= v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (2, 1, 0) - 2(1, 0, 0) = (0, 1, 0) \\
w_3 &= v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 \\
&= (0, 2, 1) - 2\,(0, 1, 0) = (0, 0, 1).
\end{aligned}
$$

The set $B' = \{w_1, w_2, w_3\}$ is already an orthonormal basis for R^3, so we don't have to normalize the vectors (make them have length 1).

2. Apply the Gram-Schmidt orthonormalization process to the basis for R^2 shown below.

$$B = \{\mathbf{v}_1 = (1,1), \mathbf{v}_2 = (0,1)\}$$

The Gram-Schmidt orthonormalization process produces

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 = (1,1) \\
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 \\
&= (0,1) - \frac{1}{2}(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right).
\end{aligned}$$

The set $B' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for R^2. By normalizing each vector in B', you obtain

$$\begin{aligned}
\mathbf{u}_1 &= \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \\
\mathbf{u}_2 &= \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).
\end{aligned}$$

Exercises on determinant properties:

$$DET(AB) = DET(A) \cdot DET(B);$$
$$DET(cA) = c^n \cdot DET(A);$$
$$DET(A^{-1}) = \frac{1}{DET(A)};$$
$$DET(A) = DET(A^T);$$
$$DET(A^n) = DET(A)^n.$$

1. Let A and B be 2 matrices of size 4×4 such that $|A| = -2$, $|B| = 4$, find $|\frac{1}{2}(A^{-1})^T B^3|$.
   $|B| = 4$. Find $\left|\frac{1}{2}(A^{-1})^T B^3\right|$;

$$(\frac{1}{2})^4 \cdot |A^{-1}| \cdot |B^3|$$
$$\frac{1}{16} \cdot \frac{1}{|A|} \cdot |B|^3$$
$$\frac{1}{16} \cdot \frac{1}{-2} \cdot 4^3 = -2.$$