Linear Algebra

System of linear equation.

- Consistent - a system atleast with one soln.
- Inconsistent - a system without no soln.

Matrix.

if $m = n$ (row, column) → square matrix.

- Equal Matrix → if A and B has same size $(m \times n)$ and if $a_{ij} = b_{ij}$ are equal for every i and j
- Addition - if they have the same size.

$$A + B = [a_{ij} + b_{ij}]$$

- if they don't have the same size A+B is undefined
- Scaler multiplication - if c is scalar

$$cA = [ca_{ij}]$$

- Matrix multiplication - If $A = [a_{ij}]$ with $(m \times n)$ size and $B = [b_{ij}]$ with (n, r) size

eg $C = AB = [c_{ij}]$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj}$

- AB is defined only if column of A = raw of B
- Identity matrix - square matrix that has 1's in the main diagonal and 0's else where

eg $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[1]$

- If any matrix (A) is multiplied with Identity matrix it gives it self (A)

$$AI_m = A, \quad I_m A = A$$

- Transpose of matrix (A^T) - exchanging column and row

$A = \begin{bmatrix} 3 & 1 & 1 \\ 4 & 2 & 5 \end{bmatrix} \to A^T = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$ if $A = (m, n)$, $A^T = (n \times m)$

- square matrix is symetric if $A^T = A$
- Properties. $(A^T)^T = A$, $(AB)^T = B^T A^T$, $(A+B)^T = A^T + B^T$, $(cA)^T = cA^T$
- a matrix A is called skew-matrix or anti-symetric if $~~(A^T)~~$ $A^T = -A$
- The diagonal of skew matrix is 0.
- Trace of Square matrix - the sum of the element on the main diagonal

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

properties.

$$tr(A+B) = tr(A) + tr(B)$$
$$tr(cA) = c\,tr(A)$$
$$tr(A^T) = tr(A)$$
$$tr(AB) = tr(BA)$$

- ✓ zero matrix - Matrix whose all entries are zero.

  $O_{ij} \rightarrow$ zero matrix with $(i \times j)$ size.

  - $A_{mn} + O_{mn} = A_{mn}$
  - $A_{mn} + (-A_{mn}) = O_{mn}$

- ✓ Properties of multiplication.
  - $ABC = A(BC) = (AB)C$
  - $(A+B)C = AC + BC$
  - $A(B+C) = AB + AC$
  - $c(AB) = (cA)B = A(cB)$

- ✓ if $AC = BC \rightarrow A \neq B$

- ✓ Diagonal Matrix - if all entries below and above the ~~entri~~ diagonal are zero.

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} a_1^k & 0 & 0 \\ 0 & a_2^k & 0 \\ 0 & 0 & a_3^k \end{bmatrix}$$

- ✓ Lower triangular matrix - in a square triangle if all entries above the main diagonal are zero
- ✓ Upper triangle - in a square triangle if all entries below the main diagonal are zero.

Row Echelon form (r.e.f)

a matrix in which

1) all rows consisting entirely all zero's at the bottom
2) for row that doesn't contain entirely of zeros ~~entri~~ the 1st non zero entry is 1 (leading 1)
3) For successive non zero rows the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

eq. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

reduced row Echelon form (r.r.e.f)

- Echelon form where every where above and below leading 1 has entries of zero

eq $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- Augmented matrix - matrix derived from coefficients and constants of linear system
- Coefficient Matrix - The matrix consisting only the coefficients of a system

eq $\begin{cases} x - 4y - 3z = 5 \\ -3x + 3y - z = -3 \\ x + z = 4 \end{cases}$ $\begin{bmatrix} 1 & -4 & -3 & 5 \\ -3 & 3 & -1 & -3 \\ 1 & 0 & 1 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -4 & -3 \\ -3 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

system — Augmented — coefficient.

* Elementary raw operations.
  - interchanging raw
  - multiply a row by non zero constant
  - Add a multiple of a row to an other raw.

* Gaussian Elimination with back substitution.
  1. write the augmented matrix
  2. Using elementary operations convert it to r.e.f
  3. ~~write the new simplified system then use back substitution~~ to solve for all variables.

* Gauss-Jordan Elimination.
  - same with the above but continue reduction until r.r.e.f

→ During reduction process, if you get a raw with all zeros except the last one, the system has no soln

→ After the elimination if we have more unknown variables than the no of equations. it means the system has ∞ soln

* Homogenous system of Linear Equations.
  - a system of l. equations in wlc each of the constant terms is zero

eg $\begin{cases} 3x + y - z = 0 \\ 3y + z = 0 \end{cases}$

→ has atleast 1 soln (trivial soln)

→ trivial soln - ~~all en~~ a soln in wlc all variables are 0.
  - all homogeneous system has trivial soln

* Inverse of matrix
  - for square matrix A if there is a matrix B in wlc $AB = BA = I_n$ the A is invertable.

$A^{-1} = B$

  - If a matrix doesn't have inverse it is called singular or non envertible.

→ Using Gauss-Jordan elimination

$[A \vdots I_n] \xrightarrow[\text{of elimination}]{\text{use elementary method}} [I_n \vdots A^{-1}]$

If this doesn't exist it's singular.

→ for 2x2 matrix - $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

A is invertible if $\underbrace{ad - bc}_{\text{determinant}} \neq 0$

$\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

→ properties of inverse

- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = \frac{1}{c}(A)^{-1}, \ c \neq 0$
- $(A^k)^{-1} = (A^{-1})^k$
- $(A^T)^{-1} = (A^{-1})^T$

→ Using the inverse to solve a system

if $Ax = b \rightarrow x = A^{-1}b$

eg: $\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases} \Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

A = coefficient matrix, x, b

using Gauss-Jordan elimination $A^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$

$x = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ $x = 2$ $y = -1$ $z = -2$

⑦ Determinant (det(A) or |A|)

✓ A number associated to every square matrix that can be used for multiple purposes.
eg - finding inverse of a fcn.

- For a (2×2) matrix → $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  $\det(A) = ad - bc$

- For any triangular matrix / Diagonal matrix $A_{(n \times n)}$

  $\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$ { multiple of the diagonal entries (triangular)

- For a matrix with size greater than (2×2) and not diagonal

→ Minors (M_{ij}) - determinant of a matrix obtained by removing i^{th} row and j^{th} column of A.

eg. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

→ Cofactor (C_{ij}) = $(-1)^{i+j} M_{ij} \Rightarrow \begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$ + → if i+j is even, - → else.

- Determinant using minors and cofactors.
  - Determinant = the sum of all entries multiplied by their cofactor in i^{th} row / j^{th} column - you can choose any row or column

if we use 1st row $|A| = a_{11}C_{11} + a_{12}C_{12} \cdots a_{1n}C_{1n}$.

→ choose a row (column) with most 0's to make it easier.

→ properties of determinants → for $A_{(n \times n)}$, c - scaler

- $\det(AB) = \det(A).\det(B)$
- $\det(cA) = c^n \det(A)$
- If $\det(A) \neq 0$ it is invertible. else singular
- $\det(A) = \det(A^T)$
- If $\det(A) \neq 0$ ⇒ $Ax = 0$ has only trivial soln
  $Ax = b$ has unique soln in w/c $x = A^{-1}b$

→ Using Elementary operation to find det

- $\det(A) = 0$ if
  - an entire row/column is 0
  - one row or column is a multiple of another
  - two rows/columns are equal

  one of those is true

- so using elementary operations we can try to make the above cases or make it triangular matrix then multiply the diagonal entries based on...
  - → If B is obtained by exchanging two rows of A
    $\det(B) = -\det(A)$
  - → If B is obtained by adding a multiple of row to another of A
    $\det(B) = \det(A)$
  - → If B is obtained by multiplying row of A by c
    $\det(B) = c\det(A)$

- Adjoint - is the transpose of a matrix of cofactors of A

$$\rightarrow \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \qquad Adj(A) = C^T$$

→ we uses it to find inverse of a matrix

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Cramer's rule - a method to solve linear equations using determinants
  - by expressing each variable as a ratio of two determinants involving matrices derived from the coefficient matrix and constant term.

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A}, \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\text{Constants}}$$

$$\Rightarrow x_i = \frac{|A_i|}{|A|}$$

A_i → matrix in w/c the i^{th} column is the constant terms.

eg. $A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

Vector space

→ V is called a vector space if u and v are its elements then vector addition and scaler multiplication is defined.

- $u + v \in V$ → closure
- $cu \in V$ → if c is a scaler → closure under scaler multiplication

(some vector spaces)

→ $\mathbb{R}^n$ - n is num of spaces → form → $v = (v_1, v_2 \cdots v_n)$
- $\mathbb{R}^1$ - set of real num.

eg. • $(2,3) \in \mathbb{R}^2$
- $(x, x) \in \mathbb{R}^2$
- $(x, x, z) \in \mathbb{R}^3$

Represented as $(n \times 1)$ or $(1 \times n)$ matrix

eg. $\mathbb{R}^4 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, $[v_1, v_2, v_3, v_4]$

- M_{mn} - set of all $m \times n$ matrices
- P_n - set of all polynomials with degree $\le n$
- P - set of all polynomials.
- $C[a,b]$ - set of continuous funs defined on a closed interval $[a,b]$
- $c(-\infty, \infty)$ - " " " " " the real number

→ Let v be any elemet of vector space V, c - scaler.

- $0v = 0$
- $c0 = 0$
- if $cv = 0$ then $c = 0$ or $v = 0$
- $-1(v) = -v$

Subspace of vector spaces.

- Non empty sub set W of a vector V is subspace of V if W is also a vectorspace by itself. ~~under operations of + and scaler * difined in V~~

eg. • W - set of all (2×2) symetric matrics

⇒ W is subspace of $M_{2,2}$

- $W = \{(x_1, x_1 + x_3, x_3) ; x_1, x_3 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}$
- $W \{a_0 + a_1 x + a_2 x^2 ; a_1 > 0\}$ is not a subspace of P_2

→ If V and W are subspaces of U

⇒ $(V \cap W)$ is also subset of U

Spanning Sets and Linear Independence

- A vector v in a vector space V is called Linear combination (LC) of vectors $v_1, v_2, v_3 \cdots v_k$ in V if V can be expressed in the form.

$$V = c_1 v_1 + c_2 v_2 + c_3 v_3 \cdots c_k v_k \quad \} c_1 \cdots c_k \text{ - are scalers.}$$

eg - a set S in $\mathbb{R}^3$ $\quad S = \{\underset{v_1}{(1,3,1)}, \underset{v_2}{(0,1,2)}, \underset{v_3}{(1,0,-5)}\}$

- v_1 is LC of v_2 and v_3 because $v_1 = 3v_2 + v_3$

- for set S in $M_{2,2}$ $\quad S = \left\{ \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \right\}$

- $v_1 = v_2 + v_3 - v_4$ so v_1 is LC of v_2, v_3 and v_4

→ Finding LC

eg write $w = (1,1,1)$ as LC from S. $\quad S = \{(1,2,3), (0,1,2), (-1,0,1)\}$

→ $(1,1,1) = c_1(1,2,3),\ c_2(0,1,2),\ c_3(-1,0,1)$

→ $(1,1,1) = (c_1 + (-c_3), 2c_1 + c_2, 3c_1 + 2c_2 + c_3)$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - c_3 \\ 2c_1 + c_2 \\ 3c_1 + 2c_2 + c_3 \end{bmatrix} \Rightarrow \begin{matrix} c_1 - c_3 = 1 \\ 2c_1 + c_2 = 1 \\ 3c_1 + 2c_2 + c_3 = 1 \end{matrix}$$

solve c_1, c_2 and c_3 using Gauss-Jordan elimination
→ if det ≠ 0 then it is a spanning set.

→ $w = 2v_1 - 3v_2 + v_3$

→ sometimes some vectors cannot be written in LC { when elimination if it is no solution the it cannot be expressed in LC

→ Spanning set - if S is a set in w/c every vector in V can be written as a LC of vectors in S.
- S is a subset of V
→ so S spans V

eg. $S = \{(0,0,1), (0,1,0), (1,0,0)\}$ spans $\mathbb{R}^3$

- $S = \{1, x, x^2\}$ spans P_2

→ Span of a set - if $S = \{v_1, v_2, v_3, v_4 \cdots v_k\}$ is a set of vectors in vector space V.
→ span of S is the set of all LC of the vectors in S

$$\text{span}(S) = \{c_1 v_1, c_2 v_2 \cdots c_k v_k\}$$

→ if $\text{span}(S) = V$ $\quad S$ spans V

→ span(S) is a subspace of V

Linear dependence and Linear independence (LD, LI)

- If S is a set of vectors $S = \{v_1, v_2, v_3 \cdots v_k\}$ in a v.s V and the equation $c_1 v_1 + c_2 v_2 + c_3 v_3 \cdots c_n v_n = 0$ has only trivial soln ($c_1 = c_2 = c_3 = \cdots c_n = 0$) then S is LI.

→ but there is also non trivial soln S is LD.

→ Testing for LI or LD
- Write in $c_1v_1 + c_2v_2 + c_3v_3 \dots c_nv_n = 0$ form
- Check if the system has unique soln or not.

- If the coefficient matrix is a square matrix $(n \times n)$ find the determinant
  - if $\det \neq 0$ → LI
  - if $\det = 0$ → LD

- If coefficient matrix is not square use the augmented matrix and use Gaussian elimination method to find the values $c_1, c_2, c_3 \dots$
  - if $c_1 = c_2 = c_3 \dots c_k = 0$ ⇒ S is LI
  - else S is LD.

→ Any set containing a zero vector is L.D

eg - $S = \{\underbrace{(1,2,3)}_{v_1}, \underbrace{(0,1,2)}_{v_2}, \underbrace{(-2,0,1)}_{v_3}\}$ is LI in $\mathbb{R}^3$

$$c_1(1,2,3) + c_2(0,1,2) + c_3(-2,0,1) = (0,0,0)$$

$$((c_1 - 2c_3), (2c_1 + c_2), (3c_1 + 2c_2 + c_3)) = (0,0,0)$$

$$\begin{cases} c_1 - 2c_3 = 0 \\ 2c_1 + c_2 = 0 \\ 3c + 2c_2 + c_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$ det = −1 so S is LI

- $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ in $M_{4,1}$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 \\ c_2 + 3c_3 + c_4 \\ -c_1 + c_3 - c_4 \\ 2c_2 - 2c_3 + 2c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 2 & 2 & -2 & 2 & 0 \end{array}\right]$$

$c_1 = c_2 = c_3 = c_4 = 0$

S is LI.

→ If two vectors v and u $\in S$ in v.s V, one is the scaler multiple of another S is LD

eg. $S = \{(1,2,-1), (2,4,-2)\}$ is LD.

→ If atleast one of the vectors in S in V can be written as LC then S is LD

Basis and dimentions

→ Basis - a set of vectors $S = \{v_1, v_2 \cdots v_n\}$ in V.S V if S spans V and S is LI
- every vector in V canbe written in one and only one way as LC of vectors in S

→ Standard basis.

→ R^n → $\{(1,0,0 \cdots 0), (0,1,0 \cdots 0), (0,0,1 \cdots 0), \cdots (0,0,0 \cdots 1)\}$

→ P_n → $\{1, x, x^2 \cdots, x^n\}$

→ M_{mn} → has a size of $(m \times n)$ each vector having single 1 and other entries equal to 0.

eg. M_{22} → $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

→ Non standard basis
- Other basis other than the standard one.

eg. $S = \{(1,1), (1,-1)\}$ is a basis for $\mathbb{R}^2$

→ If any set contains more vectors than the basis then it is LD eg → basis → 4 vector if it contains 5 vectors it is LD.

→ All basis of V.S V contains the same number of vectors

→ Dimension of a vector space (dim(V))
- is the number of vector contained in basis of V.S V

eg. $\{(1,0), (0,2)\}$ is basis of $\mathbb{R}^2$ then $\dim(\mathbb{R}^2) = 2$

→ $\dim(R^n) = n$

→ $\dim(P_n) = n+1$

→ $\dim(M_{mn}) = mn$

→ let V be a V.S of dim of 'n'
- If $S = \{v_1, \cdots v_n\}$ is LI, then S is basis for V
- If $S = \{v_1, \cdots v_n\}$ spans V, then S is basis for V

Coordinate representation Relative to basis.

- If B is a basis for V.S V, every vector x in V can be expressed in LC of vectors in B. The coefficients in the LC are coordinates of x relative to B.

$B = \{v_1, v_2, v_3 \cdots v_n\}$ → should be orderd.

$x = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$

cooerdinates = $c_1, c_2, c_3 \cdots c_n$.

→ represented in a column matrix → $[x]_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

if matrice A_{nn} is invertible all colums ~~rows~~ are LI

Rank of a matrix

- if $A = \begin{bmatrix} a_{11} & a_{12} & a & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$

⇒ Row vectors of $A = \{(a_{11}, a_{12} \cdots a_{1n}), (a_{21}, a_{22} \cdots a_{2n}), \cdots (a_{m1}, a_{m2} \cdots a_{mn})\}$
$\in R^n$

⇒ Row vector of $A = \left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} \in R^m$

- Row space (R.S) of A is subspace of R^n spanned by the row vectors of A

- column space (C.S) of A is the subspace of R^m spanned by the column vector of A.

→ How to find basis for ~~row~~ R.S and C.S.

C.S of A = R.S of A^T and vice versa

1. write the matrix in r.e.f.
2. for R.S take the non zero rows
   for C.S take colums with leading 1 if that column is LI in the initial matrix (before r.e.f)
3. write set of vectors using each column/row as individual vectors

to find basis of C.S you can also change it to R.S (A^T) then follow the step as R.S.

→ to apply this method to find basis for subspace spanned by the set $S = \{v_1, v_2, v_3 \cdots v_k\}$ in R^n, use the vectors in S as rows of the apply the same technique.

→ An m×n matrix A has the same row space dimension and C.S dim and called Rank of A (rank(A))

$\text{Rank}(A) = \dim(R.S) = \dim(C.S)$

so Rank is the number of LI raws or raws in the ~~matrix~~ matrix

→ for A_{mn} ⇒ $\text{Rank}(A) \leq \text{minimum}(m, n)$

→ If A is n×n invertible matrix, then $\text{Rank}(A) = n$

→ $\text{Rank}(A^T) = \text{Rank}(A)$

application of Rank

Rouche - Capelli thm.

- if system of L.equations with n variables has a soln if rank of the coefficient matix is equal to rank of the augmented matix $Rank(A) = Rank(A:b)$ o.w. doesn't have soln
  - → if $Rank(A) = n$ → unique soln
  - → o.w → infinite soln

Null space

→ If A_{mn} is matrix, the set of all solns of the homogenous sys of linear equns $\boxed{Ax=0}$ is a subspace of R^n called the nullspace of A ($N(A)$)

$$N(A) = \{ x \in \mathbb{R}^n / Ax = 0 \}$$

→ Nullity of A - is the dimension of null space.

$$\text{Nullity of } A = \dim(N(A))$$

(opposite of rank)

Nullity = # free variables = # columns without leading ones

Nullity(A) + Rank(A) = # column = n

Nullity = ~~Rank(A~~ n - Rank(A)

→ Non homogeneous system $Ax=b$ the set of solns is not subspace b/c zero-vector is never a soln

→ Relation b/n solns of $Ax=0$ & $Ax=b$

$$x = x_p + x_n$$

$\left.\begin{matrix} x \text{ - every soln of the system} \\ x_p \text{ - soln of } Ax=b \\ x_n \text{ - soln of } Ax=0 \end{matrix}\right\}$

eg. find set of all sol of
$\left\{\begin{matrix} x_1 \quad - 2x_3 + x_4 = 5 \\ 3x_1 + x_2 - 5x_3 \quad = 8 \\ x_1 + 2x_2 \quad - 5x_4 = -9 \end{matrix}\right.$

$$\text{ref} = \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - 2x_3 + x_4 = 5 \\ x_2 + x_3 - 3x_4 = -7 \end{matrix}$$

let $x_3 = s$, $x_4 = t$

$x_1 = 5 + 2s - t$, $x_2 = \cancel{-x_3+3} -s + 3t - 7$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - t + 5 \\ -s + 3t - 7 \\ s \\ t \end{bmatrix} = \underbrace{s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{x_n} + \underbrace{\begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}}_{x_p}$$

$$x = \underbrace{sU_1 + tU_2}_{\text{arbitrary vector in the soln space of } Ax=0} + \underbrace{x_p}_{\text{particular soln of } Ax=b}$$

Summary - + Ann square matrix the ff are equivalent:

1) A is invertible

2. $\det(A) \neq 0$

3) $Ax = b$ has unique soln $x = A^{-1}b$

4) $Ax = 0$ has only trivial soln

5) Rank (A) = n

6. n row vectors of A ar LI

7. n Colmn " " " " "

→ The system of linear eqn $Ax = b$ is consistent (has atleast 1 soln) iff b is in column space of A (b can b written as LC of the columns)

Eigen value and eigen vectors

- Let A be an $(n \times n)$ matrix. The scaler λ is called an eigen value of A if there is a non-zero vector x such that $Ax = \lambda x$ then x is eigen vector corresponding to

eg. $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ verify $x_1 = (1, 0)$ is an e.vector of A

$$Ax_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underset{\lambda}{2}\underset{x_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \lambda_1 x_1$$

ex. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. $x_1 = (-3, -1, 1)$

$$Ax_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \lambda_1 x_1 \Rightarrow \lambda_1 = 0$$

- how to find E.value & E.vectors.
  - $\det(\lambda I - A) = 0$
  - $(\lambda I - A)x = 0$

  } use those formulas to find E.value & E.vectors.

eg. $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ $\Rightarrow$ find E.value and E.vectors.

$$\Rightarrow |\lambda I - A| = \left|\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\right| = \begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix}$$

$$= (\lambda-2)(\lambda+5) + 12 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

for $\lambda_1 = -1$ } $(\lambda I - A)x = 0 \Rightarrow \left(-1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\right)x = 0$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we will always have a zero row after solving $\lambda I - A$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow[\text{operation}]{\text{e.row}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$= x_1 - 4x_2 = 0$ $\quad x_1 = 4x_2$ } free var so we use t.

$x_2 = t$, $x_1 = 4t$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ t \neq 0$$

E.vector for $\lambda = -1$

$\rightarrow \beta_1 = \{(4, 1)\}$ = basis for eigen space corresponding to $\lambda = -1$

- You do the same for $\lambda = -2$

- for a triangular matrix - the e.values are the entries on it's main diago

eg. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 1 & 0 \end{bmatrix} \rightarrow \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 0.$

* Eigen space - matrices has infinite number of eigen vectors for eac eigen value.
- every non-zero scaler (c) multiple of x is eigen vector

Diagonalization

✓ an $(n \times n)$ matrix A is diagonizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where is a diagonal matix.

eg. $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is diagonalizable for $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$D = P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ← diagonal

→ A and D are similar for this example.

(square matrices)

✓ similar matrices - A and B are called similar if ther exists an invertible matrix P such that $P^{-1}AP = B$.

- they have same eigen value

✓ a $(n \times n)$ matrix is diagonalizable if it has n LI e.vectors

in the above eg. for $\lambda_1 = 4 \dots P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -2 \dots P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

so $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det(P) = -2 \neq 0$ so all colms are LI

Steps to diagonalizing: let A be $(n \times n)$ matrix.

1. Find n LI e.vectors $P_1, P_2 \dots P_n$ for A for every λ value. if n LI vectors don't exist then A is not diagonalizable.
2. let $P = [P_1 | P_2 | \dots | P_n]$
3. Then evaluate $D = P^{-1}AP$ (diagonal). then D will have $\lambda_1, \lambda_2 \dots \lambda_n$ on its main dia-

⊛ If $(n \times n)$ matrix has n distinct e.values then it is diagonalizable.

Length and dot product of R^n

- length/magnitude/norm of a ~~matrix~~ vector $v = (v_1, v_2, v_3 \dots v_n)$ in R^n is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 \cdots v_n^2}$$

✓ If $\|v\| = 1$ then v is unit vector.

✓ $\|cv\| = |c| . \|v\|$

✓ $u = \dfrac{v}{\|v\|}$ → is the unit vector in the direction of v.

✓ distance b/w two vectors. v and u

$$d(u,v) = \|u - v\|$$

* Dot product of u and v $\{u = (u_1, u_2 \cdots u_n)$, $v = (v_1, v_2 \cdots v_n)\}$

$$u.v = u_1v_1 + u_2v_2 \cdots + u_nv_n$$

Properties.

- $u.v = v.u$
- $v.v = \|v\|^2$
- $u.(v+w) = u.v + u.w$
- $v.v \geq 0$, $v.v = 0$ iff $v = 0$
- $c(u,v) = cu.v = u.cv$

eg. $u = (2,-4)$ $v = (5,8)$ D.P $u.v = 10 - 32 = -22$

* Cauchy Shwarz ~~Identity~~ Inequality.

if $u, v \in R^n$ then $|u.v| \leq \|u\| \|v\|$

* Angle b/w two vectors in R^n

$$\cos\theta = \frac{u.v}{\|u\|\|v\|} \qquad 0 < \theta < \pi$$

* orthagonal vectors / perpendicular.

if $v.u = 0$

* The triangle identity . $v, u \in R^n$

$$\|u + v\| \leq \|u\| + \|v\|$$

if u and v are orthagonal

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$ → Pythagorian them

Dot Product as matrix multiplication.

for $u, v \in R^n$ $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}$

$$u.v = u^T v = [u_1, u_2\ u_3 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1v_1, u_2v_2 \cdots u_nv_n]$$

Inner product space

- let u, v, w be vectors in vectorspace V and c be any scaler
  - An inner product on V is a function that associates a real number $\langle u, v\rangle$ with each pair of vectors u, v an satisfies the ~~ff~~ 4 axioms.

1) $\langle u, v\rangle = \langle v, u\rangle$
2) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$
3) $c\langle u, v\rangle = \langle cu, v\rangle$
4) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v = 0$

eg. • let $u, v \in \mathbb{R}^2$

$\therefore \langle u, v\rangle\ u_1v_1 + 2u_2v_2$ is inner product

- $\langle u, v\rangle = u_1v_1 - 2u_2v_2 + u_3v_3$ is not inner product in $\mathbb{R}^3$ { b/c 4th axiom is not satisfied.

- Norm of u vector in inner product space is

$\|u\| = \sqrt{\langle u, u\rangle}$ if $\|u\| = 1$ → unit vector.

- distance b/n u and v

$d(u,v) = \|u - v\| = \sqrt{\langle u-v, u-v\rangle}$

- angle b/n u and v $\in \mathbb{R}^n$

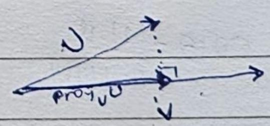
$\cos\theta = \dfrac{\langle u, v\rangle}{\|v\|\|u\|}$

- u and v are orthogonal if $\langle u, v\rangle = 0$.

@ Cauch shwarz inequality, ~~and~~ triangle inequality, pythagorian thm is applicable for u and v.

Orthagonal Projection.

- u, v - non zero vectors. and scaler a.



$\text{proj}_v u$ → Orthagonal projection of u onto v.

$\Rightarrow \text{proj}_v u = av \Rightarrow a = \dfrac{u \cdot v}{v \cdot v}$

as a dot product

if u, v are vectors in inner pdct space V such that $v \neq 0$

$\text{proj}_v u = \dfrac{\langle u, v\rangle}{\langle v, v\rangle} v$

Orthonormal bases: Gram-Shmidt Process

- A set S of vectors in an inner product space V is called orthogonal set if every pair of vector in S is orthogonal (perpendicular). In addition if in the set S each vector is a unit vector then S is orthonormal

$$S = \{v_1, v_2, \ldots, v_n\}$$

1. $\langle v_i, v_j \rangle = 0$ → orthogonal set
2. $\langle v_i, v_j \rangle = 0$ and $\|v_i\| = 1$ → orthonormal

- If S is basis then it's called orthogonal basis or orthonormal basis accordingly

eg standard basis of $\mathbb{R}^n$ is orthonormal

- If V is an inner prdct space of dimension n then any final set of n non-zero vector is basis for V

→ to make it practical

if $\dim(V)$ = # elmts of S then if S is orthogonal (i.p.s) it is a basis for V

※ If $B = \{v_1, v_2 \ldots v_n\}$ is ~~orthogonal~~ orthonormal basis for V then coordinat representation of ~~a~~ vector w relative to B is

$$w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \ldots \langle w, v_n \rangle v_n$$

$$\Rightarrow [w]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Gram-schmidt Orthonormalization

1. begins with a basis of inner product space (doesn't have to be orthogonal)
2. convert the given basis to orthogonal basis.
3. normalize each vector to form orthonormal basis.

Step 1 - let $B = \{v_1, v_2 \ldots v_n\}$ ~~basis for i.p.s~~ V

Step 2 · let $B' = \{w_1, w_2, \ldots w_n\}$

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$w_n = v_n - \frac{\langle v_n w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \cdots - \frac{\langle v_n w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$$

Step 3. let $B'' = \{u_1, u_2, u_3 \ldots u_n\}$ → is orthonormal basis for i.p.s. V

wher $u_i = \dfrac{w_i}{\|w_i\|}$

Introduction to Linear transformation.

* Linear transformation - is a type function - mapping b/n two vector spaces that preserve the properties of vector addition and scaler multiplication.

- domain of T, - codomain of T.

$T: V \to W$ } when $T(U+W) = T(U) + T(W)$, $T(cW) = cT(W)$ } for vectors U and W in V

eq. $T: R^2 \to R^2$ defined by
$T(U_1, U_2) = (U_1 - U_2, U_1 + 2U_2)$

- $T(V) = U$ } is image of V under T; V is pre image of U under T.

bez $T(U+V) = T(U) + T(V)$, $T(cU) = cT(U)$ } - checked by substituting ~~U+V~~ (U_1+V_1, U_2+V_2) in place of U_1, U_2 and same for (cU)

* Properties

① $T(0) = 0$ ③ $T(U - V) = T(U) - T(V)$

② $T(-U) = -T(U)$ ④ $T(c_1 U_1 + c_2 U_2 \dots + c_n U_n) = c_1 T(U_1) + c_2 T(U_2) \dots + c_n T(V_n)$

* Linear Transformation defined by matrix. } $T: R^m \to R^n$

$T(U) = AU$

} A is always $(n \times m)$ matrix and R^m is $(m \times 1)$, R^n is $(n \times 1)$

eq. $T: R^2 \to R^3$

$$T(V) = AV = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

is a Linear transformation.

for $V = (2, -1)$

$$T(V) = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

- Matrices of linear transformation.

- Standard matrix for linear transformation.

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{m1} \end{bmatrix} \quad T(e_2) = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & \dots a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A is called standard matrix.

eq. $T(x_1, x_2, x_3) = (x_1 - 2x_2 + 5x_3, 2x_1 + 3x_3, 4x_1 + x_2 - 2x_3)$

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & -2 \end{bmatrix}$$

Some useful transformations

i → projection on R^3

$T: R^3 \to R^3 \to (x, y, z) \to T(x, y, z) = (x, y, 0)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

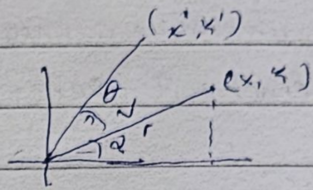
→ Reflection on x axis on R^2

$T: R^2 \to R^2$, $(x, y) \to T(x, y) = (x, -y)$ $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

→ Orthogonal projection on x axis $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (x,0)$ $\quad A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

→ rotation in $\mathbb{R}^2$



let $v = (x,y) \Rightarrow v = (r\cos\alpha, r\sin\alpha)$

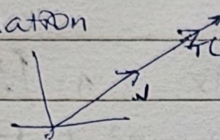
$r = \|\vec{v}\|$ $\quad \theta$ - angle from +ve x axis

$T(v) = A\begin{bmatrix}x\\ y\end{bmatrix} \Rightarrow A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$= Av$

→ contraction/dilation $\quad$ dilation. $\quad A = \begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$

$T(v) = kv$



Composition of Linear transformation.

let $T_1: R^n \to R^m$. $T_2: R^n \to R^p$ Thend

$T = T_2 \circ T_1$ - is composition.

$T(v) = T_2 \circ T_1(v) = T_2(T_1(v))$

~~T(v)~~ then $T: R^n \to R^p$ has a standard matrix A of $A = A_2A_1$

- If $T_2(T_1(v)) = T_1(T_2(v)) = v$ then T_1 is invertible and T_2 is the inverse of T_1

~~means~~ eg. $T(1,4,-5) = (2,3,1)$ and T^{-1} exists then $T^{-1}(2,3,1) = (1,4,-5)$

- If T is invertible then A is also invertible and T is isomorphism {onto and one to one}
- If A is invertible then T is invertible.

eg. let $T: R^3 \to R^3$

$T(x_1,x_2,x_3) = (2x_1+3x_2+x_3,\ 3x_1+3x_2+x_3,\ 2x+4x_2+x_3)$

$A = \begin{vmatrix}2 & 3 & 1\\ 3 & 3 & 1\\ 2 & 4 & 1\end{vmatrix}$ $\quad A$ is invertible $\quad A^{-1} = \begin{vmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 6 & -2 & -3\end{vmatrix}$

∴ T is invertible and it's standard matrix is A^{-1}

Kernel and range of a linear transformation.

Let $T: V \to W$ be a linear transformation.

✓ Kernel of T ($\ker(T)$) - is the set of all vector v in V that satisfy

$$T(v) = 0$$

eg. 1) $T: M_{3,2} \to M_{2,3}$: $T(A) = A^T$

$$\ker(T) = \{A \in M_{3,2} \mid T(A) = 0\}$$
$$= \{A \in M_{3,2} \mid A^T = 0\}$$
$$= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

2) $T: \mathbb{R}^3 \to \mathbb{R}^2$: $T(x) = Ax$ $\quad A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$

$$\ker(T) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$
$$= \left\{ (x, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \left[\begin{array}{ccc:c} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right] \right\}$$
$$= \{ t(1, -1, 1) \mid t \in \mathbb{R} \}$$
$$= \operatorname{span}(1, -1, 2)$$

✓ $\ker(T)$ is a subspace of V

The range of a linear transformation.

let $T: V \to W$ then

$$\text{Range}(T) = \{ w \in W \mid \exists v \in V \text{ such that } T(v) = w \}$$
$$= \{ T(v) \mid v \in V \}$$

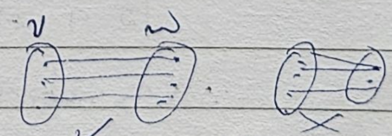
→ $\text{Range}(T)$ is a subspace of W

Nullity of T ($\text{Nullity}(T)$) $= \dim(\ker(T))$ $\Rightarrow \text{Nullity}(T) = \text{Nullity}(A)$

Rank of T ($\text{Rank}(T)$) $= \dim(\text{Range}(T))$ $\Rightarrow$ ~~Nullity~~ $\text{Rank}(T) = \text{Rank}(A)$

let $T_1: \mathbb{R}^m \to \mathbb{R}^n$ ~~$T_2: \mathbb{R}_{mn}$~~

$$\text{Rank}(T_1) + \text{Nullity}(T_1) = m$$



One to One linear transformation.

$T: V \to W$ is 1-1 $\iff$ if $T(v) = T(u) \Rightarrow u = v \;\; \forall\, u, v$ in V

• $T: V \to W$ is 1-1 iff $\ker(T) = \{0\}$

eg. $T: M_{mn} \to M_{nm}$: $T(A) = A^T$ is one to one.

Onto linear transformation.

$T: V \to W$ is onto if $\forall w \in W, \exists v \in V$ such that $T(v) = w$

meaning every elmt in W has a pre image in V

i.e $W = Range(T)$

- T is onto iff $Rank(T) = dim(W)$

If a linear transformation is 1-1 and onto it's called ~~amorphism~~ isomorphism