Linear algebra theory for exam

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Some things to remember:

- All nilpotent matrices are not invertible (singular matrices). A nilpotent matrix is a matrix that elevated to some exponent gives 0: $A^n = 0$;
- If a matrix a matrix A elevated to some exponent doesn't change, it is called an idempotent matrix: $A^n = A$. Idempotent matrices can have as determinant either 0 or 1, depending on whether they're invertible (not-singular) or not.
- The trace of a matrix, denoted by $tr(A)$ is the **sum** of the elements on the main diagonal.
  - $tr(AB) \neq tr(A) \cdot tr(B)$.
- A matrix A is skew-symmetric (antisymmetric) if $A^T = -A$.

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}$$

For two skew-symmetric matrices A and B, we have:

$$(A+B)^T = A^T + B^T = -A + (-B) = -(A+B).$$

- A matrix A is symmetric if $A^T = A$.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix} = A^T$$

For two symmetric matrices A and B, we have:

$$(A+B)^T = A^T + B^T = A + B.$$

  - The sum of matrices preserves the symmetry or skew-symmetry of the individual matrices.

1 Linear dependence/independence/spanning sets

In linear algebra, a set of vectors is said to be linearly dependent if there exists a non-trivial linear combination of these vectors that equals the zero vector. A non-trivial linear combination means that at least one of the coefficients in the linear combination is not zero. However, this does not imply that the zero vector itself must be a member of the set.

- A set of vector is linearly dependent if: $a_1v_1 + ...a_nv_n = 0$ when at leats one of the coefficients $a_i \neq 0$.
- A set of vectors is linearly independent if: $a_1v_1 + ...a_nv_n = 0$ when $a_1 = ... = a_n = 0$.

1.1 Vectors space

Consider two vectors v_1 and v_2 in a vector space, where $v_2 = 2v_1$. This set of vectors is linearly dependent because $2v_1 - v_2 = 0$, but the set v_1, v_2 does not necessarily contain the zero vector itself.
To check for linear independence, you can either immediately notice just by looking at the vectors, or by solving a system of linear equations (matrix reduction) or by computing the determinant of the nxn corresponding coefficient matrix (if $DET \neq 0$, then the linear system has a unique solution given by $x = A^{-1}b$ and the row and column vectors of the matrix are linearly independent).

The determinant of a matrix can be thought of as a scaling factor for the volume when the matrix is considered as a transformation. If the determinant is non-zero, it means that the matrix, when applied as a transformation, does not collapse the space into a lower dimension, which in turn implies that the rows (or columns) of the matrix span the entire space and none can be written as a linear combination of the others.
In the context of vector spaces, if you have a set of vectors and you arrange these vectors as the rows of a matrix, then:
If the determinant of this matrix is non-zero, the vectors are linearly independent. If the determinant is zero, the vectors are linearly dependent, meaning at least one of the vectors can be expressed as a linear combination of the others.

1. Is the set $S = \{(1, -2, 6), (5, -10, 30)\}$ linearly dependent ? Let's check:
   - $\begin{pmatrix} 1 & -2 & 6 \\ 5 & -10 & 30 \end{pmatrix}$ in REF equals $\begin{pmatrix} 1 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$. Since the last row is made by all 0s, then one of the vectors could be re-written in infinite wayts as a linear combination of the others and the set S is indeed linearly dependent.
2. Is $u = (-1, 7)$ in span $\{(1, 2), (-1, 1)\}$?

   $u = a_1v_1 + a_2v_2 \rightarrow (-1, 7) = a_1(1, 2) + a_2(-1, 1)$

$$\begin{cases} a_1 - a_2 = -1 \\ 2a_1 + a_2 = 7 \end{cases}$$

$$\text{DET}\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix} = 3 \neq 0$$

, hence the linear system has a unique solution. This means that the vectors v_1 and v_2 span indeed the vector u (i.e. v_1 and v_2 are linearly independent).

$$\begin{pmatrix}1 & -1 & -1\\ 2 & 1 & 7\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & 1\\ 0 & 1 & 3\end{pmatrix}$$

$$\begin{cases} a_1 - a_2 = 1 \\ a_2 = 3 \end{cases}$$

$$\begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$$

$$u = 2v_1 + 3v_2$$

1.2 Polynomials space

1. Is the set $S = \{1 + x, -x^2 + 2\}$ a linearly independent set in $P_2(R)$?

   - To quickly determine if this is true, we can look at the degrees of the polynomials. Since one is of degree 1 and the other is of degree 2, and there is no way to write x^2 as a multiple of x or a constant, and similarly, there is no way to write x as a multiple of x^2 or a constant; the set is linearly independent without even needing to solve any equation. This is because their degrees are different, and in a polynomial space, polynomials of different degrees are always linearly independent.
   - Otherwise, one could set a linear system and check.

2. Is the set $J = \{1, 1 + x, 1 + x + x^2\}$ a linearly independent set in P_2 ?

   - We can quickly notice that this set is not linearly independent because the first polynomial is a constant term, which is also present in the other two polynomials. This means that the constant term 1 can be written as a linear combination of the other two polynomials, indicating that there is a dependency among the vectors. Specifically, if we take the polynomial $1+x$ and subtract 1, we get x, which is part of the third polynomial $1+x+x^2$. This immediately shows that the polynomials are not linearly independent, because one of them can be generated by linear combinations of the others.

3. Is the set $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$ a linearly independent set in P_2 ?

- Let's try to verify mathematically this time:
  To determine if this set is linearly independent, we must check if the only solution to the equation $a_1(1-x)+a_2(1-x^2)+a_3(3x^2-2x-1)=0\ \forall x\in R$ is $a_1=a_2=a_3=0$. This requires setting up the equation and solving for a_1,a_2,a_3. If the only solution is the trivial one (where all coefficients are zero), then the set is linearly independent. To check this quickly, let's consider the equation above. We can arrange the terms in the order of decreasing powers of x:

  (a) $x^2(a_3\cdot 3+a_2\cdot -1)+x(a_1\cdot -1+a_3\cdot -2)+(a_1+a_2-a_3)$.

  For this to hold $\forall x$, the coefficients of x^2, x, and the constant term must all be zero.

$$\begin{cases} 3a_3-a_2=0 \quad (1)\\ -2a_3-a_1=0 \quad (2)\\ a_1+a_2-a_3=0 \quad (3)\end{cases}$$

  We have a system of three equations with three unknowns. We can solve this system to find out if the only solution is $a_1=a_2=a_3=0$, which would imply linear independence. Let's solve this system to determine whether the set S is linearly independent.
  The solution to the system of equations is $a_1=-2c$ and $b=3c$. This means that for any non-zero value of C, we can find corresponding values of a and b that satisfy the equation.
  For example, if $c=1$, then $a=-2$ and $b=3$, these values would satisfy the equation. Since there is a combination of a, b and c other than the trivial solution (where all are zero) that makes the linear combination of these polynomials equal to zero, these polynomials are not linearly independent.

2 Subspaces

2.1 Check whether a set of vectors is a subspace

A subset H of a vector space V is called a subspace of V if it satisfies the following three conditions:

1. The zero vector of V is in H: This means that the set H is not empty since it at least contains the zero vector.

2. Closed under vector addition: For every pair of vectors u and v in H, the sum $u+v$ must also be in H. In other words, if you take any two vectors in the subset and add them together, the resultant vector must also be in the subset.

3. Closed under scalar multiplication: For every vector u in H and every scalar c in the field over which the vector space is defined, the product cu must also be in H. That is, multiplying any vector in the subset by a scalar should not produce a vector outside of the subset.

   If a non-empty set of vectors in V satisfies these two closure properties, then it is a subspace of V.

Let's see some examples:

- Is $W = \{(x,y,z) \in R^3; x^2+y^2+z^2 = 1\}$ a subspace of R^3 ? No, because the zero vector $\notin W$.
- Is the set $W = \{(x,y,z) \in R^3; x^2+y^2+z^2 = 1\}$ a subspace of R^3 ?
  1. Looking at the set W defined as $\{(x,y,z) \in R^3 : x^2+y^2+z^2 = 1\}$, we can see that the zero vector does not satisfy the equation $x^2+y^2+z^2 = 1$, because $0^2+0^2+0^2 = 0$, not 1.
  2. The set W describes a unit sphere with radius 1 (centered at the origin), which is not closed under vector addition because if you take two points on the surface of the sphere, their sum will not generally lie on the sphere.
     It is not closed under scalar multiplication because if you take any non-zero scalar and multiply it with a point on the sphere, the result will not lie on the sphere unless the scalar is 1 or -1.
- Is the set $W = \{(x,y,z) \in R^3$ such that $x = 1$ and $y = z\}$ a subspace of R^3 ?
  No, because:
  1. **The zero vector is not in** W: For the zero vector $(0,0,0)$ to be in W, we would need $x = 0$, but the set W is defined such that $x = 1$. Since the zero vector is not included, W cannot be a subspace;
  2. **Not closed under vector addition**: If you take any two distinct vectors from W, say $(1,a,a)$ and $(1,b,b)$, and add them, you get $(2, a+b, a+b)$, which does not satisfy $x = 1$. Hence, W is not closed under addition;
  3. **Not closed under scalar multiplication**: If you multiply any vector in W by any scalar other than 1, the x-component will not be 1 anymore. For example, $2 \cdot (1,a,a) = (2,2a,2a)$, which does not belong to W since x is not 1.
- Is the set $W = \{A \in M_{2,2} | A^T = A\}$ a subspace of $M_{2,2}$?
  1. The zero matrix is in W because it is equal to its transpose.
  2. If A and B are in W, then $A+B$ is also in W because $(A+B)^T = A^T + B^T = A+B$.
  3. If A is in W and c is a scalar, then cA is in W because $(cA)^T = cA^T = cA$.

  Therefore, W is closed under addition and scalar multiplication, and contains the zero vector, which are the necessary and sufficient conditions for a subset to be a subspace of a vector space.

2.2 Find a basis of a subspace

A set of vectors is a basis of a subspace iff:

1. The vectors in the set have the same dimension;
2. The number of vectors in the set is the same as the dimensions of the considered subspace;

3. The vectors in the set are linearly independent (this implies that they span the subspace).

Recall that:

- Basis of row space == basis of subspace;
- Basis of column space == basis of the range.

2.2.1 Vectors subspace

2.2.2 Matrices subspace

1. Let W be the set

$$W = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in M_{2,2} \mid m - 2n = 0 \text{ and } p - 3q = 0 \right\}$$

   (a) Prove that W is a subspace of $M_{2,2}$.

   (b) Find a basis for W and deduce its dimension.

   To prove that W is a subspace of $M_{2,2}$, we must show that it is closed under addition and scalar multiplication, and that it contains the zero vector.

   For closure under addition, take any two matrices $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ and $B = \begin{pmatrix} m' & n' \\ p' & q' \end{pmatrix}$ in W. Then $A + B = \begin{pmatrix} m + m' & n + n' \\ p + p' & q + q' \end{pmatrix}$. We have $m - 2n = 0$ and $p - 3q = 0$ for A, and $m' - 2n' = 0$ and $p' - 3q' = 0$ for B. Adding these equations, we get $(m+m') - 2(n+n') = 0$ and $(p + p') - 3(q + q') = 0$, so $A + B \in W$.

   For closure under scalar multiplication, take any scalar k and matrix $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ in W. Then $kA = \begin{pmatrix} km & kn \\ kp & kq \end{pmatrix}$. Since $m - 2n = 0$ and $p - 3q = 0$, multiplying by k gives $k(m - 2n) = 0$ and $k(p - 3q) = 0$, so $kA \in W$.

   The zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W because $0 - 2 \cdot 0 = 0$ and $0 - 3 \cdot 0 = 0$.

   Thus, W is a subspace of $M_{2,2}$.

   To find a basis for W, we solve the equations $m - 2n = 0$ and $p - 3q = 0$. We can express m as $2n$ and p as $3q$. Hence, every matrix in W can be written as:

$$\begin{pmatrix} 2n & n \\ 3q & q \end{pmatrix} = n \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + q \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}.$$

   Thus, a basis for W is:

$$\left\{ \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \right\}.$$

   The dimension of W is the number of vectors in the basis, which is 2.

2. Prove or disprove that the subset W of $M_{2,2}$ defined by

$$W = \{A \in M_{2,2} \mid A^T A = I\}$$

is a subspace of $M_{2,2}$.

(a) The zero vector (here, the zero matrix) must be in W.

(b) W must be closed under vector addition.

(c) W must be closed under scalar multiplication.

For W, the equation $A^T A = I$ must hold for all $A \in W$. The matrices that satisfy this equation are orthogonal matrices, which means their columns are orthonormal vectors.

Checking the first criterion, the zero matrix does not belong to W because $0^T 0$ does not equal the identity matrix I. Therefore, W is not a subspace of $M_{2,2}$ as it does not contain the zero matrix. There is no need to check the other criteria because W already fails to be a subspace by not satisfying the first condition.

3. Consider the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 1 & 5 & 6 & 9 & 0 \end{pmatrix}$$

(a) Find a basis for the row space and column space of A, then deduce the rank and nullity of A.

(b) Find the null space of A and deduce its basis.

The rank of the matrix is equal to the number of pivot positions (the number of leading 1s in the reduced echelon form), and the nullity of the matrix is the number of free variables, which is the number of columns minus the rank.

For part (b), the null space (or kernel) of A consists of all solutions to the homogeneous system $Ax = 0$. To find a basis for the null space, we solve this system and express the solutions in terms of the free variables.

The row echelon form of the matrix is the following:

$$\begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 1 & 4 & 4 \end{pmatrix}$$

From this, we can deduce the following:

- The basis for the row space of A can be taken from the non-zero rows of the original matrix A, which are the first three rows.

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 4 \end{pmatrix} \right\}$$

- The basis for the column space of A can be taken from the corresponding columns of the original matrix A, which are the first three columns.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

  Note that you can use as basis for the row space either the rows where there are the pivots in the reduced matrix or the corresponding rows in the non-reduced matrix. Same for the column space.

- The rank of A is 3, which is the number of pivot positions.
- Rank(A) + nullity(A) = 5 (n. of columns of A).
  Nullity(A) = 5 - rank(A) = 5 - 3 = 2.
- For part (b), the null space and a basis for it is the following:

$$\begin{cases} x_1 + 3x_2 + x_3 - x_4 = 0, \\ x_2 - 5x_4 - 10x_5 = 0, \\ x_3 + 4x_4 + 4x_5 = 0, \end{cases}$$

  3 equations, 5 unknowns, so: 2 free parameters ($x_4 = s; x_5 = t$).

$$\begin{cases} x_1 + 3x_2 + x_3 - s = 0, \\ x_2 - 5s - 10t = 0, \\ x_3 + 4s + 4t = 0, \end{cases}$$

$$\begin{cases} x_1 = -10x_4 - 26x_5, \\ x_2 = 5x_4 + 10x_5, \\ x_3 = -4x_4 - 4x_5, \\ x_4 = s, \ \text{(free variable)} \\ x_5 = t. \ \text{(free variable)} \end{cases}$$

  A basis for the null space (kernel) is the following:

$$\left\{ s \cdot \begin{pmatrix} -10 \\ 5 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -26 \\ 10 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.2.3 Polynomials subspace

3 Linear transformations (aka linear maps)

Properties of linear maps:

1. $T(x + y) = T(x) + T(y) \rightarrow$ Additivity;
2. $T(cx) = cT(x) \rightarrow$ Homogeneity.

To immediately recognize if a transformation is not linear, you should look for characteristics that violate the two defining properties of a linear transformation: additivity and homogeneity of degree 1. Here are some common indicators that a transformation is not linear:

1. **Constant terms**:If any component of the transformation includes an additive constant (a term that does not depend on the input variables), the transformation is not linear. For example: $T(x) = x + 1$;
2. **Nonlinear operations**:

   Any operation on the variables that is not a first-degree polynomial indicates a non-linear transformation.

   - Powers. $T(x) = x^n$ where $n \neq 1$;
   - Roots (square, cubic ...). $T(x) = \sqrt{x}$;
   - Exponentials and logarithms. $T(x) = e^x$ or $T(x) = log(x)$;
   - Trigonometric functions. $T(x) = sin(x)$ or $T(x) = cos(x)$;
   - Absolute value. $T(x) = |x|$;
   - Products of variables. xy;
   - Determinant. $T(x) = det(x)$. The determinant is associated with square matrices, and it does not preserve the properties of linearity.
3. **Non-proportional scaling**: If different input components are scaled by different amounts, the transformation may not be linear. For example: $T(x, y) = (2x, 3y)$ is linear, but $T(x) = (2x, y^2)$ is not because of the y^2 term;
4. **Division of variables**: If the transformation involves dividing by a variable, it is not linear. For example: $T(x) = \frac{1}{2}$, assuming $x \neq 0$;

5. **Conditional statements**: Any transformation that has different rules for different regions of the input space is not linear. For example:

$$T(x) = \begin{cases} x+1 \ if \ x > 0 \\ x-1 \ if \ x \leq 0 \end{cases}$$

Linear transformations can always be represented by a matrix-vector product, and they transform the zero vector in the domain to the zero vector in the codomain. If a transformation does not have this property, it is also not linear. This means that a transformation is not linear if it moves the origin (zero vector) of the space.

Let's see some examples:

1. Is the map $T : R^2 \to R^3$ defined by $T(x, y) = (x + y, y, x + 1, 3y)$ linear ?

   - **Additivity**: For any vectors $\mathbf{u}, \mathbf{v} \in R^2$, a linear transformation must satisfy $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. However, due to the constant term in the second component of the output $(x + 1)$, this property does not hold;
   - **Homogeneity**: For any scalar c and vector $\mathbf{u} \in R^2$, a linear transformation must also satisfy $T(c\mathbf{u}) = cT(\mathbf{u})$. The presence of the constant term $(x+1)$ again violates this property, as scaling the input by c will not scale the output by c because of the constant addition.

2. Let $T : P_2 \to R^2$ be the transformation defined by $T(ax^2 + bx + c) = \begin{pmatrix} a+3c \\ a-c \end{pmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find the Kernel of T.

   For part (a):

   - Let $f(x) = ax^2 + bx + c$;
   - Let $g(x) = dx^2 + ex + f$.

$$T(f+g) = \begin{pmatrix} (a+d) + 3(c+f) \\ (a+d) - (c+f) \end{pmatrix} = \begin{pmatrix} a+3c \\ a-c \end{pmatrix} = \begin{pmatrix} d+3f \\ d-f \end{pmatrix} = T(f) + T(g).$$

$$T(f+g) = T(kax^2 + kbx + kc) = \begin{pmatrix} ka+3kc \\ ka-kc \end{pmatrix} = k \begin{pmatrix} a+3c \\ a-c \end{pmatrix} = kT(f).$$

   For part (b):

   To find the kernel (or null space) of a transformation T, you need to find all vectors in the domain that map to the zero vector in the codomain.

   Always set and solve an homogeneous linear system.

$$\{f(x) \in P_2 \mid T(f(x)) = 0\}$$

$$\{ax^2 + bx + c \mid \begin{cases} a + 3c = 0 \\ a - c = 0 \end{cases} \}$$

$$\{ax^2 + bx + c \mid \begin{cases} a = -3c \\ c = a \end{cases} \}$$

$$\{ax^2 + bx + c \mid a = c = 0\}$$

The solution to the system is $a = c = 0$. There is no condition on b, which means that b can be any real number.
Therefore, the kernel of T consists of all polynomials of the form bx where b is a real number. In other words, the kernel is the set of all polynomials in P_2 that are linear in x with no constant or quadratic form.

$$Ker(T) = \{bx \mid b \in R\}$$

This is the set of all polynomials of degree one without a constant term, which is a subspace of P_2.

3. Let $T : R^2 \to R^2$ be the transformation given by $T(x, y) = (x + 2y, y - 2x)$.
   (a) Show that T is a linear transformation.
   (b) Find the Kernel of T.

   For part (a):

   - For additivity, consider two vectors $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ in R^2:

   $$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T((x_1 + x_2), (y_1 + y_2)) \\ &= (x_1 + x_2 + 2(y_1 + y_2), (y_1 + y_2) - 2(x_1 + x_2)) \\ &= (x_1 + 2y_1 + x_2 + 2y_2, y_1 - 2x_1 + y_2 - 2x_2) \\ &= (x_1 + 2y_1, y_1 - 2x_1) + (x_2 + 2y_2, y_2 - 2x_2) \\ &= T(x_1, y_1) + T(x_2, y_2). \end{aligned}$$

   - For homogeneity, consider a scalar c and a vector $\mathbf{u} = (x_1, y_1)$ in R^2:

   $$\begin{aligned} T(c\mathbf{u}) &= T(cx_1, cy_1) \\ &= (cx_1 + 2cy_1, cy_1 - 2cx_1) \\ &= c(x_1 + 2y_1, y_1 - 2x_1) \\ &= c \cdot T(x_1, y_1). \end{aligned}$$

   Since T satisfies both additivity and homogeneity, it is a linear transformation.

For part (b):

- To find the kernel of T, we need to solve for all vectors (x, y) such that $T(x, y) = (0, 0)$. This gives us the system of equations:

$$\begin{cases} x + 2y = 0, \\ y - 2x = 0. \end{cases}$$

  Solving this system, we find that $x = 0$ and $y = 0$ are the only solutions. Therefore, the kernel of T is given by:

$$\text{Ker}(T) = \{(0, 0)\}.$$

  This means that the only vector in R^2 that maps to the zero vector under T is the zero vector itself.

4 Cramer's rule

Cramer's rule is a theorem in linear algebra that gives an explicit expression for the solution of a system of linear equations with as many equations as unknowns, provided that the determinant of the coefficient matrix is non-zero (this ensures a unique solution given by: $x = A^{-1}b$), where A is the coefficient matrix.

Let's see 2 examples:

1. Apply Cramer's rule to solve the following system:

$$\begin{cases} 2x + y - z = 3 \\ x + y + z = 1 \\ x + 2y + 3z = 4 \end{cases}$$

If $DET(A) \neq 0$:

$$DET(A) = DET \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = -1$$

$$DET(A_1(x)) = DET \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{pmatrix} = 0$$

$$DET(A_2(y)) = DET \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{pmatrix} = -5$$

$$DET(A_3(z)) = DET\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix} = 0$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{0}{-1} = 0$$

$$y = \frac{DET(A_2)}{DET(A)} = \frac{-5}{-1} = 5$$

$$z = \frac{DET(A_3)}{DET(A)} = \frac{0}{-1} = 0.$$

2. The solution of the system $\begin{cases} x - 3y = 2 \\ 5x + y = 1 \end{cases}$ using Cramer's rule is $x = 5,\ y = -9$.

$$DET(A) = DET\begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix} = 16$$

$$DET(A_1(x)) = DET\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = 5$$

$$DET(A_2(y)) = DET\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix} = -9$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{5}{16} = 0.31$$

$$x = \frac{DET(A_1)}{DET(A)} = \frac{-9}{16} = -0.56.$$

5 Linear systems with parameters

1. Determine the number of solutions of the following system:

$$\begin{cases} x + 2y - kz = k \\ -x - y + kz = 0 \\ (2+k)y + (2k+1)z = 0 \end{cases}$$

depending on the parameter $k \in R$ and for the values of k for which the system is compatible, solve it.

6 Properties of determinants and some transpositions

1. $DET(AB) = DET(A) \cdot DET(B)$;
2. $DET(cA) = c^n \cdot DET(A)$;
3. $DET(A^{-1}) = \frac{1}{DET(A)}$;
4. $DET(A) = DET(A^T)$;

Let's see some examples:

1. Let A and B be two matrices of size 4x4 such that $|A|$ = -2, $|B|$ = 4. Find $\left|\frac{1}{2}(A^{-1})^T B^3\right|$;

$$(\frac{1}{2})^4 \cdot DET(A^{-1}) \cdot DET(B^3)$$
$$\frac{1}{16} \cdot \frac{1}{DET(A)} \cdot DET(B)^3$$
$$\frac{1}{16} \cdot \frac{1}{-2} \cdot 4^3 = -2.$$

2. Let A and B be two matrices of size 3x3 such that $|A|$ = 2, $|B|$ = -3. Find $\left|2(A^2)^T B^{-1}\right|$.

$$2^3 \cdot DET(A^2)^T \cdot DET(B^{-1})$$
$$2^3 \cdot DET(A^2)^T \cdot \frac{1}{DET(B)}$$
$$2^3 \cdot 2^2 \cdot \frac{1}{-3} = -\frac{32}{3}.$$

3. Let A = $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 3 \\ 2 & 2 \end{pmatrix}$.

   - Find X such that $X^T - A = 2B$.

$$X^T = 2B + A$$
$$X = \frac{1}{3}(A^T - B).$$

   - Find X such that $X - B = 3(X + B) = A^T$.

$$X + B = \frac{1}{3}A^T$$
$$X = \frac{1}{3}(A^T - B)$$

4. Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

- Find X such that $(2X + B)^T = A$.

$$2X + B = A^T$$
$$2X = A^T - B$$
$$X = \frac{1}{2}(A^T - B).$$

7 Coordinates of a vector w.r.t a basis

8 Eigendecomposition

9 Inner product spaces

An *inner product space*, in the context of linear algebra, is a vector space equipped with an additional structure known as an *inner product*. This inner product allows for the measurement of angles and lengths within the space. Let's elaborate:

1. **Vector Space:** A collection of objects, called vectors, which can be added together and multiplied by scalars (numbers) in a way that satisfies certain rules (like commutativity, associativity, distributivity, etc.). Examples include R^n (the set of all n-tuples of real numbers) and function spaces.

2. **Inner Product:** A function that takes two vectors from the vector space and returns a scalar. It generalizes the dot product from Euclidean geometry and must satisfy the following properties:

   - *Conjugate Symmetry:* $\langle u, v\rangle = \overline{\langle v, u\rangle}$ (In real vector spaces, this is $\langle u, v\rangle = \langle v, u\rangle$)
   - *Linearity in the First Argument:* $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$ for scalars a, b and vectors u, v, w
   - *Positive-Definiteness:* $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$

3. **Examples of Inner Products:** The dot product in R^n, defined as $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$ for vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, is a familiar example.

4. **Applications and Implications:** An inner product space is fundamental in mathematics and its applications, allowing for the generalization of geometric concepts like orthogonality, angles, and lengths to more abstract settings. In such a space, two vectors are orthogonal if their inner product is zero, and the length (or norm) of a vector v is defined as $\sqrt{\langle v, v\rangle}$.

In summary, an inner product space is a vector space combined with an inner product, extending geometric concepts to abstract mathematical and applied sciences.

9.1 Gram-Schimdt orthonormalization process

A vector space can have many different bases. For example R^3 has the convenient standard basis B $= \{(1,0,0),(0,1,0),(0,0,1)\}$ This set is the standard basis for R^3 because it has special characteristics that are particularly useful. One important characteristic is that the three vectors in the basis are mutually orthogonal (Their dot product equals 0). That is:

$$
\begin{gathered}
(1,0,0) \cdot (0,1,0) = 0 \\
(1,0,0) \cdot (0,0,1) = 0 \\
(0,1,0) \cdot (0,0,1) = 0.
\end{gathered}
$$

A second important characteristic is that each vector in the basis is a unit vector ($\| |v_i| \| = 0$).

The Gram-Schmidt orthonormalization process allows to construct a basis made of mutually orthogonal unit vectors.

- A set S of vectors in an inner product space V is called orthogonal if every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector. then S is called orthonormal.
- If $S = \{v_1, v_2, ..., v_n\}$ is an orthogonal set of non-zero vectors in an inner product space V, then S is linearly independent.

1. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for an inner product space V.
2. Let $B' = \{w_1, w_2, \ldots, w_n\}$, where w_i is given by

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
&\vdots \\
w_n &= v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}.
\end{aligned}
$$

   Then B' is an *orthogonal* basis for V.
3. Let $u_i = \frac{w_i}{\|w_i\|}$. Then the set $B'' = \{u_1, u_2, \ldots, u_n\}$ is an *orthonormal* basis for V. Moreover, $\text{span}\{v_1, v_2, \ldots, v_k\} = \text{span}\{u_1, u_2, \ldots, u_k\}$ for $k = 1, 2, \ldots, n$.

Let's see an example:
Apply the Gram-Schmidt orthonormalization process to the basis for R^3 shown below.

$$B = \{(1,1,0),(1,2,0),(0,1,2)\}$$

$$\begin{aligned}
w_1 &= v_1 = (1,1,0) \\
w_2 &= v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1,2,0) - \frac{3}{2}(1,1,0) = \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\
w_3 &= v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 \\
&= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{1}{2}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = (0,0,2).
\end{aligned}$$

The set $B' = \{w_1, w_2, w_3\}$ is an orthogonal basis for R^3. Normalizing each vector in B' produces

$$\begin{aligned}
u_1 &= \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1,1,0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) \\
u_2 &= \frac{w_2}{\|w_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)
\end{aligned}$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{2}(0,0,2) = (0,0,1).$$

So, $B'' = \{u_1, u_2, u_3\}$ is an orthonormal basis for R^3.

A matrix is said to be orthogonal if the product of the matrix and its transpose results in the identity matrix. Mathematically, a matrix A is orthogonal if $A \cdot A^T = I$, where I is the identity matrix and A^T is the transpose of A.
Consider the following matrix:

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

To check if it's orthogonal, we need to multiply this matrix by its transpose and see if the product is the identity matrix. Let's perform this calculation.
The product of the matrix and its transpose is:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This is not the identity matrix; instead, it is a scalar multiple of the identity matrix. Therefore, the given matrix is not orthogonal.