SAPIENZA university of Rome
Bachelor of science in ACSAI
ALL the LA exam questions + solutions

Elvis Palos

January 19 2024

Exercise 1:

Answer by true or false and justify your answer:

1. Every linear dependent set contains the zero vector. False.
2. The vector $u = (1, 2, 1)$ is a linear combination of $v_1 = (2, 1, 0)$ and $v_2 = (1, -2, 4)$. False.
3. The vector $v = (1, 2, 1)$ is a linear combination of $v_1 = (2, 1, 1)$ and $v_2 = (1, -2, 4)$. True.
4. The set $S = \{(1, -2, 6), (5, -10, 30)\}$ is linearly dependent. True.
5. The vectors $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (2, 0, 2)$ and $\mathbf{v}_3 = (0, 2, 1)$ are linearly independent in R^3. True.
6. The set $S = \{1 + x, -x^2 + 2\}$ is a linearly independent set in $P_2(R)$. True.
7. $J = \{1, 1 + x, 1 + x + x^2\}$ is a linearly independent set in P_2. False.
8. The set $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$ is a linearly independent set in P_2. False.
9. The set $S = \{(-1, 4, 2), (2, 3, 7)\}$ is a basis of R^3. False.
10. The set $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ is a basis of R^3. True.
11. The set $S = \{(-1, 4, 2), (2, 3, 7), (6, 5, 2)\}$ is a basis for R^4. False.
12. $W = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ is a subspace of R^3. False.
13. The set $W = \{(x, y, z) \in R^3$ such that $x = 1$ and $y = z\}$ is a subspace of R^3. False.
14. The set $W = \{A \in M_{2,2} | A^T = A\}$ is a subspace of $M_{2,2}$. True.
15. The set W of 2×2 skew symmetric matrices is a subspace set in P_n. False.
16. The dimension of the subspace $W = \{(x, y, z) | x + y + z = 0\}$ of R^2 is 2. False.

17. The transformation $T : R \to R, T(x) = |x|$ is linear. False.

18. The transformation $T : M_{2,2} \to R$ given by $T(A) = \det(A)$ is linear. False.

19. The map $T : R^2 \to R^3$ defined by $T(x, y) = (x + y, y, x + 1, 3y)$ is a linear transformation. False.

20. The transformation $T : R \to R, T(x) = \sqrt{x}$ is linear. False.

21. If a square matrix A has no zero rows or columns, then it has an inverse. False.

22. If A and B are invertible square matrices, then $(A + B)$ is also invertible. False.

23. There is an invertible matrix A such that $A^2 = 0$. False.

24. If $A^2 = A$ is a non-singular matrix (invertible), then $det(A^2) = 1$. True.

25. Let A be a 4×5 matrix. If nullity $(A^T) = 2$, then $\text{Rank}(A) = 2$. False.

26. If A is a 6×8 matrix such that $\text{Rank}(A^T) = 5$, then $\text{Nullity}(A) = 1$. False.

27. Let A and B be $n \times n$ matrices, then $\text{tr}(AB) = \text{tr}(A) \cdot \text{tr}(B)$. False.

28. Let A and B be any matrices, then $(A - B)(A + B) = A^2 - B^2$. False.

29. If A and B are $n \times n$ skew symmetric matrices, then $A + B$ is skew symmetric. True.

30. if A and B are $(n \times n)$ symmetric matrices, then $A + B$ is also symmetric. True.

31. If $\mathbf{u} = (k, k, 1)$ and $\mathbf{v} = (k, 5, 6)$ are orthogonal, then $k = 1$. False.

32. If u and v are orthogonal vectors such that $||u|| = 6$ and $||v|| = 3$, then $||u + v|| = 9$. False.

33. The function $\langle u, v\rangle = u_1v_1$ defines an inner product on R^2, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$. False.

34. If $\langle x, y\rangle = \langle x, z\rangle$ for vectors x, y, z in an inner product space, then $y - z$ is orthogonal to x. True.

35. The matrix $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$ is orthogonal. False.

36. The coordinates of the vector $v = (4, 2)$ with respect to the basis $u_1 = (3, 2)$ and $u_2 = (2, 3)$ of R^2 are $(2, -1)$. False.

37. The coordinates of the vector $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ with respect to the basis $u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and $u_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ of R^2 are $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. False.

38. If A is a 3×3 matrix such that $|A| = 2$, then $|2A^TA^{-1}| = 2$. False.

39. If P and D are $n \times n$ matrices, then $\det(PDP^{-1}) = \det(D)$. True.

40. The solution of the system $\begin{cases} x - 3y = 2 \\ 5x + y = 1 \end{cases}$ using Cramer's rule is $x = 5,\ y = -9$. False.

Exercise 2:

1. Determine the number of solutions of the following system:

$$\begin{cases} x + 2y - 3z = 4 \\ 4x + y + 2z = 6 \\ x + 2y + (k^2 - 19)z = k \end{cases}$$

depending on the parameter $k \in R$.

2. Determine the number of solutions of the following system:

$$\begin{cases} x + ky + (4k+1)z = 4k + 1 \\ 2x + (1+k)y + (2+7k)z = 7k + 1 \\ 3x + (k+2)y + (3+9k)z = 1 + 9k \end{cases}$$

depending on the parameter $k \in R$.

3. Determine the number of solutions of the following system:

$$\begin{cases} x + ky - kz = 1 \\ -4y + 2z = 1 \\ -x + ky = 1 \end{cases}$$

depending on the parameter $k \in R$ and for the values of k for which the system is compatible, solve it.

4. Determine the number of solutions of the following system:

$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 3 \\ 3x + 2y + kz = 4 \end{cases}$$

depending on the parameter $k \in R$.

5. Determine the number of solutions of the following system:

$$\begin{cases} x + 2y - kz = k \\ -x - y + kz = 0 \\ (2+k)y + (2k+1)z = 0 \end{cases}$$

depending on the parameter $k \in R$ and for the values of k for which the system is compatible, solve it.

Exercise 3:

Considering the following matrices:

1. Find the eigenvalues of A and a basis of each eigenspace of A.
2. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & -1 \\ 0 & 1 & 0 \\ 2 & -4 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Cramer's rule:

Apply Cramer's rule to solve the following system:

$$\begin{cases} 2x + y - z = 3 \\ x + y + z = 1 \\ x + 2y + 3z = 4 \end{cases}$$

Exercises on subspaces:

1. Consider the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -2 \\ 1 & 5 & 6 & 9 & 0 \end{pmatrix}$$

   (a) Find a basis for the row space and column space of A, then deduce the rank and nullity of A.

(b) Find the null space of A and deduce its basis.

2. Let W be the set

$$W = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in M_{2,2} \mid m - 2n = 0 \text{ and } p - 3q = 0 \right\}$$

   (a) Prove that W is a subspace of $M_{2,2}$.

   (b) Find a basis for W and deduce its dimension.

3. Prove or disprove that the subset W of $M_{2,2}$ defined by

$$W = \{A \in M_{2,2} \mid A^T A = I\}$$

   is a subspace of $M_{2,2}$.

Exercises on linear transformations (aka "linear maps"):

1. Let $T : P_2 \to R^2$ be the transformation defined by $T(ax^2 + bx + c) = \begin{pmatrix} a + 3c \\ a - c \end{pmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find the Kernel of T.

2. Let $T : R^2 \to R^2$ be the transformation given by $T(x, y) = (x + 2y, y - 2x)$.

   (a) Show that T is a linear transformation.

   (b) Find the Kernel of T.

Gram-Schimdt orthonormalization process:

1. Apply Gram Schmidt process to transform the basis $B = \{(1, 0, 0), (2, 1, 0), (1, 2, 1)\}$ into an orthonormal Basis of R^3.

Exercises on determinant properties:

1. Let A and B be 2 matrices of size 4×4 such that $|A| = -2$, $|B| = 4$, find $|\frac{1}{2}(A^{-1})^T B^3|$.