Algorithms

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CONTINUE of Previous Lecture

PROOF OF THEOREM A

By contradiction, suppose that some "a_i" ends up being matched to a partner other than best(a_i) in M^*

Since the a's propose in decreasing order of preference, there must be a time when some "a" gets rejected by one of its valid matches "b" (rejection can happen right after a proposal or when "b" accepts some other proposal)

Let the pair $\{\bar{a}, \bar{b}\}$ be the first pair (during the execution of G-S algorithm) that is valid, and such that $\bar{b}$ rejects $\bar{a}$.

When this rejection happens, $\bar{b}$ will be paired up with some other $\bar{a}'$ that she likes better than $\bar{a}$

Now, since $\{\bar{a}, \bar{b}\}$ is a valid pair there must exist a stable matching M' such that $\{\bar{a}, \bar{b}\} \in M'$.
Since M' is a stable matching, it must match $\bar{a}'$ to some $\bar{b}'$, $\{\bar{a}', \bar{b}'\} \in M'$

Then observe that $\{\bar{a}', \bar{b}'\}$ is a valid pair.
Given that $\{\bar{a}, \bar{b}\}$ was the first valid pair with a rejection, it must be that $\bar{a}'$ was not rejected by a valid partner before $\bar{a}'$ gets engaged with $\bar{b}$ $\boxed{\text{1st claim}}$

2nd claim

Now, since $\bar{b}'$ is a valid partner of $\bar{a}'$ ($\{\bar{a}', \bar{b}'\} \in M'$), 3rd claim
and since $\bar{a}'$ proposes in decreasing order of preference, it must be that $\bar{a}'$ prefers $\bar{b}$ to $\bar{b}'$

Since $\bar{b}$ prefers $\bar{a}'$ to $\bar{a}$ (she rejected $\bar{a}$ to be with $\bar{a}'$) and since $\{\bar{a}, \bar{b}\}$, $\{\bar{a}', \bar{b}'\} \in M$. It holds that $\{\bar{a}, \bar{b}\}, \{\bar{a}', \bar{b}'\} \in M'$ is an instability of M'. Thus M' is unstable. This is a contradiction.

PROOF OF THEOREM B

Suppose that $\exists \{\bar{a}, \bar{b}\} \in M^*$ s.t. $\bar{a} \neq$ worst $(\bar{b})$.

Then, there exists a stable matching M' s.t. $\{\bar{b}, \bar{a}\} \in M'$ and $\bar{b}'$ likes $\bar{a}'$ less than $\bar{a}$

Suppose that in M', $\bar{a}$ is matched with some $\bar{b}' \neq \bar{b}$ ($\{\bar{a}', \bar{b}'\} \in M'$).

By theorem A, we know that $\bar{b}$ is the best valid partner of $\bar{a}$ thus, $\bar{a}$ likes $\bar{b}$ more than $\bar{b}'$ (or thm A would not hold). that is $\bar{a}: \bar{b} > \bar{b}'$ and $\bar{b}: \bar{a} > \bar{a}'$
but, $\{\bar{a}', \bar{b}\}$, $\{\bar{a}, \bar{b}'\} \in M'$.
Thus, " " form an instability of M', which is then unstable. CONTRADICTION

G-S has given us the opportunity to discuss about a number of questions one encounters when studying algorithm problems.

1. Formulated the problem in a mathematical precise way
2. Ask questions about the mathematical problem
3. Design an efficient algorithm for your problem
4. Prove that your assumption is correct, and bound its runtime

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EFFICIENCY

Def (??) " An algorithm is efficient if, after being implemented, it runs quickly on real input instances. "

(not a formal definition)

$\boxed{\text{" QUICKLY "}}$: "quick" on which hardware? with which programming language? with which implementation? (each of these questions significantly changes the runtime)

We would like our definition to be independent of details

REAL INPUT INSTANCES : what does it mean? How to define them?

To prove something about an algorithm, we need mathematical definitions.

WORST-CASE ANALYSIS

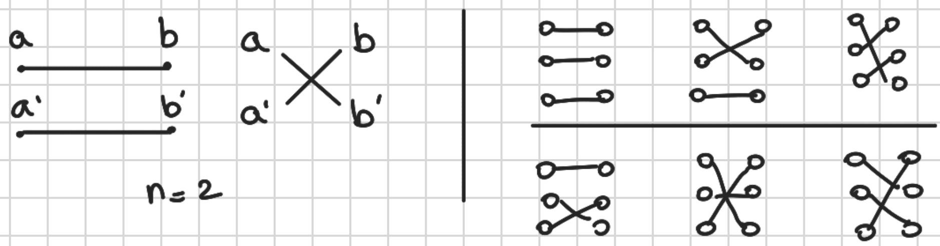
We want the runtime of an algorithm to be bounded in terms of the worst possible input. For instance, in the case of stable matching, we proved that G-S algorithm uses at most n^2 iterations ($n = |A| = |B|$)

Suppose, for instance that we consider inputs in which $|A| = |B| = N$, for $N = 2n$

$$N^2 = (2n)^2 = 4n^2$$

Doubling n changes the runtime by a factor 4

The brute-force algorithm tried each of the $n!$ matches



What would happen to the brute-force algorithm if we doubled the number of people?

$$N! = (2n)! =$$

$$= 2n(2n-1)(2n-2) \dots (n+1) \cdot n(n-1) \cdot 2 \cdot 1$$
$$= 2n(2n-1) \dots (n+1)\, n!$$
$$\geq (n+1)^n \cdot n! \geq 2^n \cdot n!$$

Def: An algorithm having a runtime $\leq c \cdot n^d$, where c and d are constants, on inputs of size n is said to be a polynomial time (POLYTIME) algorithm

If I have a polytime algorithm running in time $c \cdot n^d$ on inputs of size n, and I run it on inputs of size $N = b \cdot n$,

$$c \cdot N^d = c(nb)^d = c \cdot n^d \cdot b^d$$

then the blow-up in the runtime is going to be a constant b^d