DISCRETE PROBABILITY

$\Omega = \{w_1, w_2, \dots\}$ subset $A \subseteq \Omega$ = event

sample space $\hookrightarrow$ outcomes

$$P(A) = \frac{|A|}{|\Omega|}$$

$\hookrightarrow$ uniform probability!

if $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

else $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \leq P(A) + P(B)$

$A \cup B = \{A \text{ or } B\} = \{\text{either } A \text{ or } B \text{ occurred}\}$ = addition 

$A \cap B = \{A \text{ and } B\} = \{\text{both } A \text{ and } B \text{ occurred}\}$ = multiplication 

$A^c = \Omega \setminus A = \{\text{not } A\} = \{A \text{ has not occurred}\}$ 

$P(A) = 1 - P(\bar{A})$

multiplication rule $|\Omega_1 \times \Omega_2 \times \dots \times \Omega_N| = n_1 \cdot n_2 \cdot \dots \cdot n_n$ → they must not affect each other

Ex. 6 starters, 7 main course, 5 desserts → # possible 3 course meal: $6 \cdot 7 \cdot 5 = 210$

permutations. how many orderings of n elements? $n!$

subsets # ways of choosing k elements from a set of n elements

ORD, REP • ordering with repetition n^k pick the 1st ball, read the number, put it back and repeat k times

ORD, NO REP • ordering without repetitions $\frac{n!}{(n-k)!}$ ordered ways to pick k balls

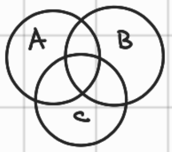
NO ORD, NO REP • without ordering and repetitions $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

if we don't care about the order of the groups we divide by the number of groups with the same cardinality (the factorial, $m!$)

• more generally $\binom{n}{n_1 \dots n_k} = \frac{n!}{n_1! \dots n_k!}$ ex $\binom{n}{m_1\, m_2\, m_3}$ = subdivide the balls in 3 unordered groups of m_1, m_2, m_3 balls each

NO ORD, REP • without ordering, with repetition $\binom{n-1+k}{k} = \frac{(n-1+k)!}{k!\,(n-1)!}$

Inclusion / exclusion



$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

independence

if $P(A\cap B) = P(A)P(B)$ A and B are independent $P(A\cap \bar{B}) = P(A) - P(A\cap B) = P(A) - P(A)P(B)$
$= P(A) - (1-P(B)) = P(A)\cdot P(\bar{B})$

conditional probability

A, B two events

Prob. of A given B $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ if A and B are independent $P(A|B) = P(A)$

law of total probability

$P(A\cap B) = P(A|B)\cdot P(B)$ if they are not independent

$P(A) = P(B)\cdot P(A|B) + P(A|\bar{B})\cdot P(\bar{B})$

Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

PROBABILITY DISTRIBUTION

- **hypergeometric**

$$P\left(\begin{matrix}K \text{ successi in un} \\ \text{campione } m\end{matrix}\right) = \frac{\binom{K}{k}\binom{N-K}{m-k}}{\binom{N}{m}}$$

insuccessi

N = tot popolazione
K = successi nella pop.
m = campione
k = successo desiderato nel campione

- **Bernoulli**

binary experiment repeated n times

$\Omega = \{0, 1\}$ $P(1) = p$ $P(0) = 1-p$

$P(\{w_1, w_2, \dots w_m\}) = p^{\#1}(1-p)^{\#0}$

- **Binomial**

bernoulli distribution where we are also intrested in how many ways we can obtain the success

$p_k = \binom{m}{k} p^k (1-p)^{m-k}$ $0 \le k \le n$ n total number of tosses, k number of 1's (Heads)

- **Geometric**

probability of seeing the 1st success at the k^{th} trial

$P = p\sum_{k=0}^{\infty}(1-p)^k = \dfrac{p}{1-(1-p)} = 1$ $p_k = (1-p)^{k-1}p$

- **Poisson**

describes the # of event in a fixed time slice or space

$P_k = \dfrac{e^{-\lambda}\cdot\lambda^k}{k!}$

RANDOM VARIABLES

quantità numerica il cui valore dipende dall'esito di un fenomeno casuale

$p_x = P(X = x)$

random variable, expected value

ex. fair dice
$\Omega = \{1, \dots 6\}$
$P(X=2) = \frac{1}{6}$

- **expectation** $E(x) = \sum_i x_i P(X = x_i)$ valore medio previsto
- **distribution** descrive come le probabilità sono assegnate ai valori che la variabile può assumere $Im(X)$
- **variance** $V(x) = E(x^2) - E(x)^2$ misura quanto i valori differiscono da quello medio
$V(X) = E[(X - E(X))^2]$

types of random variable

- **CERTAIN** $X(w) = \bar{x} \quad \forall w \in \Omega$ $\quad E(X) = \bar{x}$

  $V(X) = 0$

- **BERNOULLI** $Im(X) = \{0,1\}$ $\quad E(X) = p$

  $X \sim Ber(p)$ $\quad X = \begin{cases} 0 & \text{with prob. } 1-p \\ 1 & \text{" } \quad p \end{cases}$ $\quad V(X) = p(1-p)$

- **BINOMIAL** n tosses of a coin with $p = P(H$ $\quad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

  $X \sim Bin(n,p)$ $\quad \Omega = \{0,1\}^n$ $\quad E(X) = np$

  $V(X) = np(1-p)$

- **GEOMETRIC** machine subject to fault $\quad E(X) = \frac{1}{p}$

  $X \sim Geom(p)$ $\quad t$ at which the 1^{st} fault happens $\quad V(X) = \frac{1-p}{p^2}$

  $P(X=n) = (1-p)^{n-1} p$

  $G(m) = P(X > m) = (1-p)^m$?

- **POISSON** # clients arriving in a queue in a unit of time $\quad E(X) = \lambda$

  $X \sim poisson(\lambda)$ $\quad V(X) = \lambda$

  $P(X=k) = \frac{e^{-\lambda} \lambda^k}{k!}$

  λ = arrival rate

- **HYPERGEOMETRIC** p of obtain a certain number of some kind during a extraction without replacement $\quad E(X) = n \cdot \frac{K}{N}$

  $P(X=k) = \frac{\binom{N}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

  $V(X) = n \cdot \frac{K}{N} \cdot \left(1 - \frac{K}{N}\right) \cdot \frac{N-n}{N-1}$

properties of the expectation

1. if $X > 0$, $E(X) \geq 0$ $\quad E(X) = 0 \Rightarrow P(X=0) = 1$
2. $E(cX) = cE(X)$ if $c \in R$
3. $E(X+Y) = E(X) + E(Y)$

properties of the variance

1. if $E(X) = 0$, then $V(X) = E(X^2)$
2. $V(cX) = c^2 V(X)$
3. $V(X+c) = V(X)$
4. $V(X+Y) = V(X) + V(Y)$

Covariance

$Cov(X,Y) = E(XY) - E(X)E(Y)$

properties

1. $C(X+Z,Y) = Cov(X,Y) + Cov(Z,Y)$
2. $V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$
3. if X,Y are independent $Cov(X,Y) = 0$

Joint and Conditional Distribution

- joint $P(X=x, Y=y) = P(X=x)P(Y=y)$ if independent

- marginal distribution $P(X_i = x) = \sum_{x_1 \dots x_i, \dots x_n} P(X_1 = x_1, \dots X_i = x, \dots X_n = x_n)$ or $P(X=x) = \sum_{y \in S_Y} P(X=x|Y=y)P(Y=y)$

- conditional distribution $P(X=x, Y=y) = \dfrac{P(X=x, Y=y)}{P(Y=y)}$

Continuous Random Variable

$$P([a,b]) = \int_a^b f(x)dx \qquad P([-\infty,+\infty]) = \int_{-\infty}^{+\infty} f(x)dx = 1$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x)dx \qquad V(X) = \int_{-\infty}^{+\infty} [x - E(X)]^2 f(x)dx$$

Distributions

- uniform $f(x) = \dfrac{1}{b-a}$ $a \leq x \leq b$

- gaussian $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\mu \in \mathbb{R}$ mean, $\sigma^2 \in (0,\infty)$ variance

  $E(X) = \mu$ $V(X) = \sigma^2$

  - standardization! $P(a<x<b) = P\left(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$