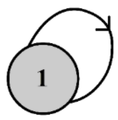
INTRODUCTION TO PHYSIC

Systems

Classical Physic is deterministic, this means that if we know the initial conditions and the equations that describe how the system evolves, we can know the future and vice versa (the system is invertible). Indeed, in classical physics you have a collection of principles and laws that govern all the phenomena without uncertainty. Instead, the model of Quantum Physics is based on uncertainty. In any case from an abstract point of view, physics is concerned with the study of state of systems.

A **dynamical system** is a system that changes with time. We express the state of a system with σ and analyse how it changes over time.

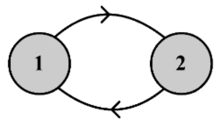
A first example is a system that never changes. It can be expressed with:



$$\sigma(t+1) = \sigma(t)$$

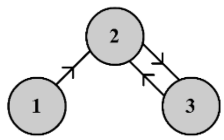
So, the next state is equal to the previous one. It's compatible with the laws of the classical physics because from there we can know the past and the future.

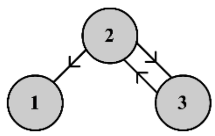
A second example is a system that passes from state 1 to state 2 and vice versa. This one it's compatible with the laws of the classical physics too.



$$\sigma(t+1) = \sigma(t-1)$$

The third case (arrow from state 1 to state 2) is not following the laws of the classical physics, indeed, if we're in the state 2, we can't know if we were coming from the state 1 or 3. The same reasoning can be applied to the fourth case (arrow from state 2 to state 1), because, from the state 2, we can't know if we'll go to the state 1 or 3.





Physics and Measurement

Physics are based on the scientific method. Here the key element is reproducibility (so the results must be consistent over time). We try to express the result with a law, that is a quantitative relation among physical quantities. Physics is based on experimental observations and quantitative measurements. If we are to report the results of a measurement to someone who wishes to reproduce this measurement, a standard must be defined. The variables length, time, and mass are examples of **fundamental quantities**. Most other variables are **derived quantities**. An example of derived quantity is the density defined as its mass per unit volume:

$$\rho \equiv \frac{m}{V}$$

The **dimension** denotes the physical nature of a quantity. Often the brackets [ ] are used to denote the dimensions of a physical quantity. In many situations, you may have to check a specific equation to see if it matches your expectations. A useful procedure for doing that is the **dimensional analysis**.

When certain quantities are measured, the measured values are known only to within the limits of the experimental uncertainty. The **number of significant figures** in a measurement can be used to express something about the uncertainty and is related to the number of numerical digits used to express the measurement.

The number of significant figures with multiplication or division in the final answer is the same as the number of significant figures in the quantity having the smallest number of significant figures.

The number of decimal places in the result when numbers are added or subtracted should equal the smallest number of decimal places of any term in the sum or difference.

Vectors

Many aspects of physics involve a description of a location in space. This description is accomplished with the use of the Cartesian coordinate system, in which perpendicular axes intersect at a point defined as the origin O. Sometimes it is more convenient to represent a point with a polar coordinates system. In a 2-dimensional polar coordinate system the coordinates are r, the distance from the origin, and θ, the angle between a fixed axis and a line drawn from the origin to the point.

We can express Cartesian coordinates in terms of polar coordinates:

$$x = r\cos\theta\,, y = r\sin\theta$$

And polar coordinates in terms of Cartesian coordinates:

$$r = \sqrt{x^2 + y^2}, \theta = \arctan\frac{y}{x}$$

Scalar quantities are those that have only a numerical value and no associated direction.

Vector quantities have both magnitude and direction and obey the laws of vector addition. The magnitude of a vector is always a positive number. Vectors are denoted with $\vec{a}$ and their magnitude with a.

When two or more vectors are added together, they must all have the same units and they all must be the same type of quantity.

We can add two vectors $\vec{a}$ and $\vec{b}$ graphically we the parallelogram rule.

A vector $\vec{a}$ multiplied by a scalar quantity m is a vector that has, the same direction of $\vec{a}$ if m is positive and opposite if m is negative, and magnitude ma.

Vectors can be expressed with rectangular components, that are the projections of its vertex if the vector is pointed at the origin.

Vector quantities often are also expressed in terms of unit vectors. The unit vectors are unit vectors pointing along the positive direction of axes. The $\hat{\imath}$ is the unit vector pointing along the x direction and $\hat{\jmath}$ is the unit vector pointing along the y direction. The hat over the letter tells us the vectors have unit magnitude: $|\hat{\imath}| = |\hat{\jmath}| = 1$.

The components of a vector $\vec{a}$ are equal to the projections of $\vec{a}$ along the x and y axis of a coordinate system so $a_x = a\cos\theta\,, a_y = a\sin\theta$ where a is the magnitude of the vector and θ the angle with the x axis. A vector expressed in unit-vector form as $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$.

Taking advantage of the components expansion we have a second method of adding vectors: the vector resultant of adding two vectors has as components the sum of respective components of the vectors.

Errors

The key element of a measurement is the reliability and we can measure it by evaluating its error. If we know that there is a true value, then the error can be obtained in this way:

$$error = |measure\ value - true\ value|$$

An error can be originated by an instrumental limitation or accidental causes. We can have systematic and statistical errors. The first one can be supposed, while the second ones, no.

An elementary way to evaluate the propagation of error is:

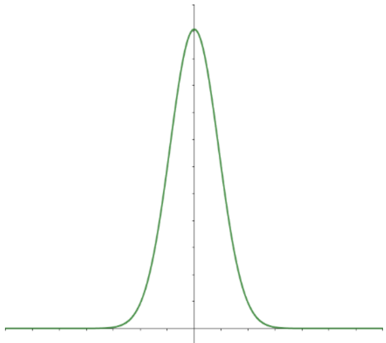
sum/difference: $(l_1 \pm \varepsilon_1) \pm (l_2 \pm \varepsilon_2) = (L \pm \varepsilon_L)$. The error is: $\varepsilon_L = \varepsilon_1 + \varepsilon_2$.

product/division: $(l_1 \pm \varepsilon_1) \cdot (l_2 \pm \varepsilon_2) = (L \pm \varepsilon_L)$. The error is: $\varepsilon_L = \left(\frac{\varepsilon_1}{l_1} + \frac{\varepsilon_2}{l_2}\right) L$

But this method hypothesizes that all values in the range $(l - \varepsilon; l + \varepsilon)$ are equiprobable. However, we can verify that the values at the extremes are less probable than the ones in the middle of the range. More precisely, the distribution of natural phenomena follows the Gaussian distribution.

A statistical measure we use to evaluate the dispersion of that distribution is the **standard deviation**. The lesser we have dispersion, the better it is (the measures are all near to their average and so, we can put more reliability to them). The formula is:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_X)^2}, \qquad where\ \mu_X = \frac{1}{N}\sum_{i=1}^{N} x_i$$

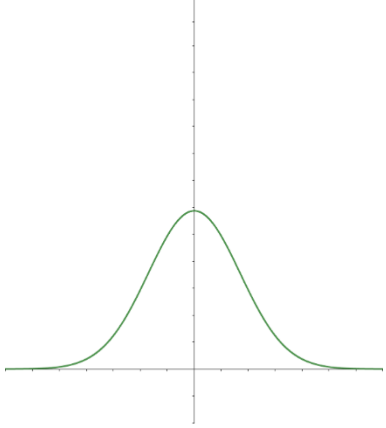


Where N is the total number of the measurements, μ_X is their arithmetic mean.

The analytical form of the **gaussian distribution** is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Where σ is the standard deviation and μ is the average value. In this way we have a graph that has always the same area (this is for standard purposes) and it tells you how precise are the measurements, indeed the higher is the curve, the more precise they're.



Now we can evaluate again the propagation of errors in a more precise way. Given a function of two or more variables ($z = f(x, y, \dots)$), in order to evaluate its standard deviation/error we have to do:

$$\sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 + \cdots$$

So now we can obtain again the formula for the sum/difference and the product/division:

sum/**difference**: $(l_1 \pm \sigma_1) \pm (l_2 \pm \sigma_2) = (L \pm \sigma_L)$. Where: $\sigma_L^2 = \sigma_1^2 + \sigma_2^2$.

product/**division**: $(l_1 \pm \sigma_1) \cdot (l_2 \pm \sigma_2) = (L \pm \sigma_L)$. Where: $\frac{\sigma_L^2}{L^2} = \frac{\sigma_1^2}{l_1^2} + \frac{\sigma_2^2}{l_2^2}$

MOTION

Motion in one dimension

A particle's **position** x is the location of the particle with respect to a chosen reference point that we can consider to be the origin of a coordinate system.

When a particle moves along the x axis from some initial position x_i to some final position x_f, its **displacement** is:

$$\Delta x \equiv x_f - x_i$$

Displacement is an example of a vector quantity.

It is very important to recognize the difference between displacement and distance traveled. **Distance** is the length of a path followed by a particle.

The **average velocity** of a particle during some time interval is the displacement Δx divided by the time interval Δt during which that displacement occurs:

$$v_{x,\text{avg}} \equiv \frac{\Delta x}{\Delta \text{t}}$$

The **average speed** of a particle is equal to the ratio of the total distance it travels to the total time interval during which it travels that distance:

$$v_{\text{avg}} \equiv \frac{d}{\Delta \text{t}}$$

The instantaneous velocity of a particle is defined as the limit of the ratio $\Delta x/\Delta t$ as Δt approaches zero. By definition, this limit equals the derivative of x with respect to t, or the time rate of change of the position:

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta \text{x}}{\Delta \text{t}} = \frac{dx}{dt}$$

Generally, the word velocity designates instantaneous velocity.

The instantaneous speed of a particle is equal to the magnitude of its instantaneous velocity.

When you encounter a new problem, you should identify the fundamental details of the problem and attempt to recognize which model already studied to apply.

The model of a particle under constant velocity is, for example, that of a car moving in a straight line at constant speed. We can obtain the position of the particle as a function of time. Using $v_x = \Delta x/\Delta t$. and $\Delta x = x - x_0$ and setting $t_0 = 0$ we have:

$$x = x_0 + v_x \text{t}$$

In a model of a particle under constant speed through a distance d along a straight line or a curved path in a time interval Δt, the constant speed will be:

$$v = \frac{d}{\Delta \text{t}}$$

The average acceleration of a particle is defined as the ratio of the change in its velocity Δv_x divided by the time interval Δt during which that change occurs:

$$a_{x,\text{avg}} \equiv \frac{\Delta \text{v}_\text{x}}{\Delta \text{t}} = \frac{v_{x,t} - v_{x,0}}{t - t_0}$$

The instantaneous acceleration is equal to the limit of the ratio $\Delta v_x/\Delta t$ as Δt approaches 0:

$$a_x \equiv \lim_{\Delta t \to 0} \frac{\Delta \text{v}_\text{x}}{\Delta \text{t}} = \frac{dv_x}{dt}$$

The model of particle under constant acceleration uses the definition of $a_{x,avg}$. From that and setting $a_{x,avg} = a_x, t_0 = 0$ we have the velocity as a function of time:

$$v_{x,t} = v_{x,0} + a_x t$$

Moreover, under constant acceleration the velocity increases linearly and its average is:

$$v_{x,\text{avg}} = \frac{x - x_0}{t} = \frac{v_{x,0} + v_{x,t}}{2}$$

from this expression we have the position as a function of time, initial and final velocity and initial position:

$$x = x_0 + \frac{1}{2}\left(v_{x,0} + v_{x,t}\right)t$$

Substituting the $v_{x,t}$ we have another expression for the position, now the position is a function of time, initial velocity and initial position:

$$x = x_0 + \frac{1}{2}\Big(v_{x,0} + \big(v_{x,0} + a_x t\big)\Big)t = x_0 + v_{x,0}t + a_x t^2$$

Finally, we can obtain an expression between the velocities without the time variable. Indeed substituting

$$x = x_0 + \frac{1}{2}\left(v_{x,0} + v_{x,t}\right)\left(\frac{v_{x,t} - v_{x,0}}{a_x}\right)$$

and rearranging:

$$v_{x,t}^2 - v_{x,0}^2 = 2a_x(x - x_0)$$

The equation seen so far are known as Kinematic equation, now that Kinematic equations can be derived from calculus. We can bridge this elementary approach to the calculus' one, dividing the interval traveled by a particle in small and small n parts until we have an infinitesimal displacement. In each part, spanning for a Δt_n, we have a different average velocity $v_{xn,\text{avg}}$ and the total displacement travelled from t_0 to t is:

$$\Delta x = \sum_n v_{xn,\text{avg}} \Delta t_n$$

Letting $n \to \infty$ we have:

$$\Delta x = \lim_{n\to\infty} \sum_n v_{xn,\text{avg}} \Delta t_n = \int_{t_0}^{t} v_x(t)dt$$

For example, if we are under the model of constant acceleration $v_x(t) = v_{xi} + a_x t$:

$$\Delta x = \int_{t_0}^{t} (v_{x,0} + a_x t)dt = [v_{x,0}t]_{t_0}^{t} + \left[\frac{1}{2}a_x t^2\right]_{t_0}^{t} = v_{x,0}(t - t_0) + \frac{1}{2}a_x(t - t_0)^2$$

Motion in two dimensions

In one dimension, a single numerical value describes a particle's position, but in two dimensions, we indicate its position by its **position vector** $\vec{r}$ drawn from the origin of some coordinate system to the location of the particle.

The generalization of the displacement of the particle (difference between its final position and its initial position) is the **displacement vector** $\Delta\vec{r}$ that is the difference between its final position vector and its initial position vector:

$$\Delta\vec{r} \equiv \vec{r} - \vec{r}_0$$

Generally, the position vector and its coordinates are function of the parameter time t:

$$\begin{cases} \Delta x = x(t + \Delta t) - x(t) \\ \Delta y = y(t + \Delta t) - y(t) \end{cases}$$

The **average velocity** of a particle during the time interval Δt is defined as the displacement of the particle divided by the time interval:

$$\vec{v}_{\text{avg}} \equiv \frac{\Delta\vec{r}}{\Delta t}$$

The average velocity between points is independent of the path taken. That is because the displacement depends only on the initial and final position vectors and not on the path taken. As with one-dimensional motion, we conclude that if a particle starts its motion at some point and returns to this point via any path, its average velocity is zero for this trip because its displacement is zero.

The **instantaneous velocity** of a particle is defined as the limit of the average velocity as Δt approaches zero:

$$\vec{v} \equiv \lim_{\Delta t\to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

The magnitude of the instantaneous velocity vector $v = |\vec{v}|$ is called the **speed** of the particle, which is a scalar quantity.

The **average acceleration** of a particle is defined as the change in its instantaneous velocity vector divided by the time interval Δt during which that change occurs:

$$\vec{a}_{\text{avg}} \equiv \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

The **instantaneous acceleration** of a particle is defined as the limiting value of the average acceleration as Δt approaches zero:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Motion in two dimensions can be modelled as two independent motions in each of the two perpendicular directions associated with the x and y axes. The position vector for a particle moving in the x,y plane can be written:

$$\vec{r} = x\hat{\imath} + y\vec{\jmath}$$

If the position vector is known, given that the unit vectors remain unchanged, the velocity is given by:

$$\vec{v} = \frac{d\vec{r}}{dt} = v_x\hat{\imath} + v_y\hat{\jmath}$$

For example, if a particle moves with constant acceleration $\vec{a}$ and has velocity $\vec{v}_0$ and position $\vec{r}_0$ at $t = 0$, its velocity and position vectors at time t are:

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

Where each of these vector expressions is equivalent to two component expressions: one for the motion in the x direction and one for the motion in the y direction.

Models of projectile motion

Projectile motion is one important type of two-dimensional motion, exhibited by an object launched into the air near the Earth's surface and experiencing free fall. This common motion can be analyzed by applying the particle under constant velocity model to the motion of the projectile in the x direction and the particle under constant acceleration model ($a_y = -g$) in the y direction, so for the position vector, we apply the expression:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

We can study this vector equation with two independent equations for the horizontal and vertical component of the position vector: $\vec{r} \equiv \big(x(t), y(t)\big)$:

$$\begin{cases} x(t) = x_0 + v_{x0}t \\ y(t) = y_0 + v_{y0}t - \frac{1}{2}gt^2 \end{cases}$$

Where the g has the minus sign because the direction of y axis is opposite of that of gravitational force.

In general, we know at $t = 0$ the magnitude of the velocity v and the angle θ_0 between the vector velocity and the x axis, so we have: $v_{x0} = v_0 \cos\theta_0$, $v_{y0} = v_0 \sin\theta_0$ and the equation become:

$$\begin{cases} x = x_0 + v_0 \cos\theta_0 \, t \\ y = y_0 + v_0 \sin\theta_0 \, t - \frac{1}{2}gt^2 \end{cases}$$

we can set the origin of coordinates at the initial position so $x_0 = 0, y_0 = 0$. Now if we solve t in the first equation and substitute in the second one, we have the y component as function of the x component:

$$y = x\tan\theta_0 - \frac{1}{2}g\frac{x^2}{v_0^2\cos^2\theta_0}$$

that prove the projectile motion is parabolic.

The **maximum height** is determined noticing that at that point $v_y = 0$. At a generic time, t:

$$v_y = v_{y,0} - gt = v_0 \sin\theta - gt$$

and at the peak:

$$v_0 \sin\theta_0 - g t_{\text{peak}} = 0$$

$$t_{\text{peak}} = \frac{v_0 \sin\theta_0}{g}$$

substituting in the component y (setting also $y_0 = 0$):

$$y = v_0 \sin\theta_0 \frac{v_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g}\right)^2 = \frac{v_0^2 \sin^2\theta_0}{g} - \frac{v_0 \sin^2\theta_0}{2g} = \frac{v_0 \sin^2\theta_0}{2g}$$

Now we could also determine the range of the bullet because is the space travelled at $t = 2t_{\text{peak}}$. Alternatively, we solve for $y = 0$ in the:

$$y = x \tan\theta_0 - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\theta_0}$$

and obtain:

$$\frac{1}{2} g \frac{x}{v_0^2 \cos^2\theta_0} - \tan\theta_0 = 0$$

$$x = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} = \frac{v_0^2 \sin(2\theta_0)}{g}$$

Frame of reference

The velocity $\vec{u}_{\text{PA}}$ of a particle measured in a fixed frame of reference $\mathcal{S}_{\text{A}}$ can be related to the velocity $\vec{u}_{\text{PB}}$ of the same particle measured in a moving frame of reference $\mathcal{S}_{\text{B}}$ by

$$\vec{u}_{\text{PA}} = \vec{u}_{\text{PB}} + \vec{v}_{\text{BA}}$$

where $\vec{v}_{\text{BA}}$ is the velocity of $\mathcal{S}_{\text{B}}$ relative to $\mathcal{S}_{\text{A}}$.

Circular motion

An important model of bidimensional motion is the particle in uniform circular motion, that is the motion of a particle with constant speed that follow a circular path.

If R is the radius of the circle, ω the angular velocity (radiant/s) of the particle with:

$$\omega = \frac{2\pi}{T}$$

and if we set the origin at the center of the circle, the position vector is:

$$\vec{r} = r\cos(\omega t)\,\vec{i} + r\sin(\omega t)\,\vec{j}$$

if we derive respect to t, we have the velocity vector

$$\vec{v} = -r\omega\sin(\omega t)\,\vec{i} + r\omega\cos(\omega t)\,\vec{j}$$

and we another derivative we get the acceleration:

$$\vec{a} = -r\omega^2\cos(\omega t)\,\vec{i} - r\omega^2\sin(\omega t)\,\vec{j}$$

comparing $\vec{v}$ with $\vec{r}$ we see that $\vec{r}\cdot\vec{v} = 0$, so they're perpendicular. That's why we define $\vec{v}$ as tangential velocity. Instead, comparing $\vec{a}$ with $\vec{r}$ we can write:

$$\vec{a} = -\omega^2\vec{r}$$

The magnitude of $\vec{r}$ is trivially R. The magnitude of $\vec{v}$ is:

$$v = r\omega$$

and that of acceleration is:

$$a = r\omega^2 = \frac{v^2}{r}$$

A particle in uniform circular motion undergoes a radial acceleration $\vec{a}$ because the direction of $\vec{v}$ changes in time. This acceleration is called centripetal acceleration, and its direction is always toward the center of the circle.

We can calculate the period (the time to complete an oscillation) as function of v:

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

FORCES

Introduction

We define force as that which causes a change in motion of an object. Forces have been experimentally verified to behave as vectors, and so you have to use the rules of vector addition to obtain the net force on an object.

When examined at the atomic level, all the forces are caused by the fundamental field forces: gravitational forces between objects, electromagnetic forces between electric charges, strong forces between subatomic particles, and weak forces that arise in certain radioactive decay processes.

Nevertheless, in developing models for macroscopic phenomena, it is convenient to distinguish contact forces, involved in physical "contact" between two objects and the field forces gravitational and electromagnetic.

The net force (or total force) on an object is the vector sum of all forces acting on the object.

The SI unit of force is the newton (N). A force of 1 N is the force that, when acting on an object of mass 1 kg, produces an acceleration of 1 m/s^2.

Newton's laws

The Newton's first law of motion, sometimes called the law of inertia defines a special set of reference frames called inertial frames. It states that in the absence of an external force, when viewed from an inertial frame, an object at rest remains at rest and an object in uniform motion in a straight line maintains that motion.

Mass is that property of an object that specifies how much resistance an object exhibits to changes in its velocity.

The Newton's second law states that the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.

Newton's third law states that if two objects interact, the force exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force exerted by object 2 on object 1:

$$\vec{F}_{12} = -\vec{F}_{21}$$

The force that object 1 exerts on object 2 is popularly called the action force, and the force of object 2 on object 1 is called the reaction force. An example of a reaction force is the normal force, a force that surfaces produce to compensate an external one.

Examples of forces

Gravitational force: The gravitational force exerted on an object is equal to the product of its mass (a scalar quantity) and the freefall acceleration:

$$\vec{F}_g = m\vec{g}$$

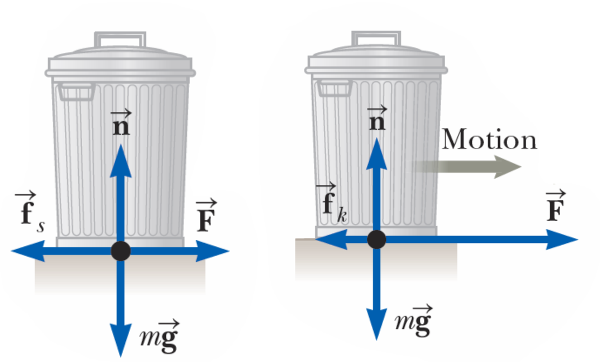
The mass m in these equations determines the strength of the gravitational attraction between the object and the Earth and in this role m is called gravitational mass. The interpretation of mass as resistance to changes in motion in response to an external force is called inertial mass. Even though gravitational mass and inertial mass has different interpretation it is one of the experimental conclusions in Newtonian dynamics that gravitational mass and inertial mass have the same value.

The **weight** of an object is the magnitude of the gravitational force acting on the object:

$$F_g = mg$$

Force of friction: When an object is in motion either on a surface or in a viscous medium such as air or water, there is resistance to the motion because the object interacts with its surroundings. We call such resistance a force of friction.

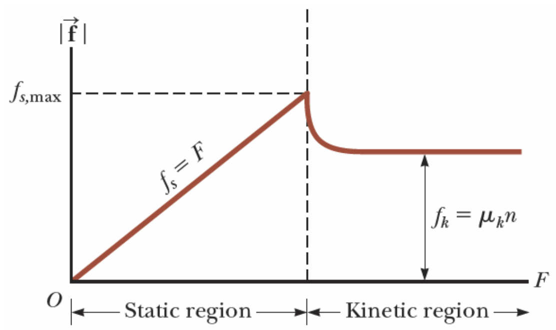
For small applied forces, the magnitude of the force of static friction equals the magnitude of the applied force. When the magnitude of the applied force exceeds the magnitude of the maximum force of static friction, the trash can break free and accelerates to the right.



The force on the object that counteracts $\vec{F}$ and keeps it from moving is called the **force of static friction**.

If we increase the magnitude of force $\vec{F}$ applied to the object, it eventually slips. When it is on the verge of slipping, f_s has its maximum value $f_{s.\max}$. When $\vec{F}$ exceeds $f_{s,\max}$ the object can move and accelerates. We call the friction force for an object in motion the **force of kinetic friction** $\vec{f}_k$ and is given by $f_k = \mu_k n$ where μ_k is the **coefficient of kinetic friction**:

The **maximum force of static friction** $\vec{f}_{s,\max}$ between an object and a surface is proportional to the normal (perpendicular) force acting on the object. In general, $f_s \leq \mu_s n$ where μ_s is the **coefficient of static friction** and n is the magnitude of the normal force.



Examples of coefficients of friction are: rubber on concrete $\mu_s = 1, \mu_k = 0{,}8$, copper on steel $\mu_s = 0.53, \mu_k = 0.36$, ice on ice $\mu_s = 0.1, \mu_k = 0.03$.

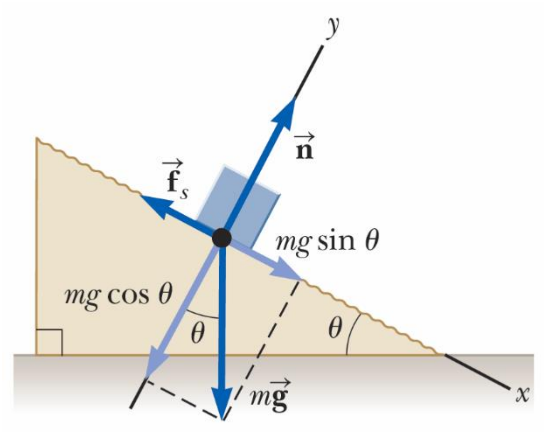
By considering the parallel and perpendicular components of the forces acting on a rough incline we can evaluate the coefficient of static friction (so when the force of static friction is maximum) in this way:

$$\mu_s = \tan\theta$$

Tension: pulling force transmitted axially by the means of a string/rope.

Models

A typical model that involves gravity and friction is that of the block lying on a rough incline:



ENERGY

Work

A **system** is a single particle, a collection of particles, or a region of space, and may vary in size and shape. A system boundary separates the system from the **environment**.

The **work** W done on a system by an agent exerting a constant force $\vec{F}$ on the system is the product of the magnitude Δr of the displacement of the point of application of the force and the component $F \cos \theta$ of the constant force along the direction of the displacement $\Delta\vec{r}$.

$$W \equiv \vec{F} \cdot \Delta\vec{r}$$
$$W \equiv F \Delta r \cos \theta$$

Its unit will be $[\mathrm{N}] \cdot [\mathrm{m}] = [\mathrm{J}]$ (Joule).

If a varying force does work on a particle as the particle moves along the x axis from x_i *to* x_f , the work done by the force on the particle is given by:

$$W = \int_{x_i}^{x_f} F_x dx$$

where F_x is the component of force in the x direction.

A force is **conservative** if the work it does on a moving particle is independent of the path the particle between the starting and ending point. Furthermore, a force is conservative if the work it does on a particle is zero when the particle moves through an arbitrary closed path and returns to its initial position. A force that does not meet these criteria is said to be **non-conservative**.

Kinetic and potential energy

The **kinetic energy** of a particle of mass m moving with a speed v is:

$$K \equiv \frac{1}{2} m v^2$$

$$[Kg] \frac{[m^2]}{[s^2]} = [J]$$

The **potential energy** of an object is the energy held by that object in relation to its position respect to other objects under a field of force influencing them.

A **potential energy function** U can be associated only with a conservative force. If a conservative force $\vec{F}$ acts between members of a system while one member moves along the x axis from x_i to x_f, the change in the potential energy of the system equals the negative of the work done by that force:

$$U_f - U_i = -\int_{x_i}^{x_f} F_x dx$$

$$F_x = -\frac{dU}{dx}$$

If the energy is constant, then $\frac{dE}{dt} = 0$. From here:

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}2mv\frac{dv}{dt} = mva$$

$$\frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt} = -Fv$$

Then:

$$\frac{dE}{dt} = mva - Fv = v(ma - F) = 0$$

$$F = ma$$

So, conservation of energy is a consequence of Newton's second law.

If a particle of mass m is at a distance y above the Earth's surface, the **gravitational potential energy** of the particle–Earth system is:

$$U_g \equiv mgy$$

The **elastic potential energy** stored in a spring of force constant k is:

$$U_s \equiv \frac{1}{2}kx^2$$

The **total mechanical energy of a system** is defined as the sum of the kinetic energy and the potential energy:

$$E_{\text{mech}} \equiv K + U$$

In absence of specifications, we'll set the reference point of the potential energy at an infinite distance from the source of the force. For example, for a gravitational force, since we can't choose as a reference point the center of the mass, because we would have $r = 0$ at the denominator, we select instead $U_i = 0$ for $r = \infty$. Then, U_f will be always < 0 and in this case will be: $\Delta U = U_f = -\frac{GMm}{r}$.

Equilibrium

Systems can be in three types of equilibrium configurations when the net force on a member of the system is zero. Configurations of **stable equilibrium** correspond to those for which $U(x)$ is a local minimum (that means that if I apply a force, the system after a small change, comes back to the initial position).

Configurations of **unstable equilibrium** correspond to those for which $U(x)$ is a maximum.

Neutral equilibrium arises when U is constant as a member of the system moves over some region.

Conservation of Energy

A **non-isolated system** is one for which energy crosses the boundary of the system. An **isolated system** is one for which no energy crosses the boundary of the system.

The **instantaneous power** P is defined as the time rate of energy transfer:

$$\mathcal{P} \equiv \frac{dE}{dt} = \frac{dW}{dt}$$

$$\mathcal{P} = \frac{dW}{dt} = \vec{\mathrm{F}} \cdot \frac{d\vec{\mathrm{r}}}{dt} = \vec{\mathrm{F}} \cdot \vec{\mathrm{v}}$$

For a non-isolated system, we can equate the change in the total energy stored in the system to the sum of all the transfers of energy across the system boundary, which is a statement of conservation of energy. For an isolated system, the total energy is constant.

If a friction force of magnitude f_k acts over a distance d within a system, the change in internal energy of the system is:

$$\Delta E = f_k d$$

The most general statement describing the behaviour of a non-isolated system is the conservation of energy equation:

$$\Delta E_{\text{system}} = \sum T$$

This equation, expressed in term of the most common form of energy, is:

$$\Delta K + \Delta U + \Delta E_{\text{int}} = W + Q + T_{\text{MW}} + T_{\text{MT}} + T_{\text{ET}} + T_{\text{ER}}$$

The total energy of an isolated system is conserved, so:

$$\Delta E_{\text{system}} = 0$$

which can be written as:

$$\Delta K + \Delta U + \Delta E_{\text{int}} = 0$$

If no conservative forces act within the isolated system so $\Delta E_{\text{mech}} = 0$ the equation reduces at:

$$\Delta K + \Delta U = 0$$

COLLISIONS

Concepts and Principles

The linear momentum $\vec{\mathrm{p}}$ of a particle of mass m moving with a velocity $\vec{\mathrm{v}}$ is:

$$\vec{\mathrm{p}} \equiv m\vec{\mathrm{v}}$$

There exists also a relation between momentum and force:

$$\frac{d\vec{\mathrm{p}}}{dt} = \frac{dm\vec{\mathrm{v}}}{dt} = m\vec{\mathrm{a}} = \vec{\mathrm{F}}$$

Equivalently:

$$\vec{\mathrm{p}}_f - \vec{\mathrm{p}}_i = \int_{t_i}^{t_f} \vec{\mathrm{F}} dt$$

We define the impulse as the increment of the momentum due to an external force:

$$\vec{\mathrm{I}} \equiv \vec{\mathrm{p}}_f - \vec{\mathrm{p}}_i$$

If there is no force ($\vec{\mathrm{F}} = 0$), then:

$$\vec{\mathrm{p}}_f - \vec{\mathrm{p}}_i = 0$$

So, the momentum is preserved.

If we pass into a system of particles, its total momentum will be given by the sum of the single momentums:

$$\vec{\mathrm{p}}_{\text{tot}} \equiv \sum_j \vec{\mathrm{p}}_j = \sum_j m_j \vec{\mathrm{v}}_j$$

analogously, if, on the whole system is not applied an external force ($\vec{\mathrm{F}}_{\text{ext}} = 0$), then the total momentum is preserved:

$$\vec{\mathrm{p}}_{\text{tot}} = 0$$

this is interesting because indirectly it tells us that even if there are elastic collisions between particles, the momentum is preserved.

Collisions

An elastic collision is one in which the kinetic energy of the system is conserved. For two particles we have that:

$$\begin{cases} \frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2, & \text{(conservation of kinetic energy)} \\ m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}, & \text{(conservation of momentum)} \end{cases}$$

An inelastic collision is one for which the total kinetic energy of the system of colliding particles is not conserved. A perfectly inelastic collision is one in which the colliding particles stick together after the collision.

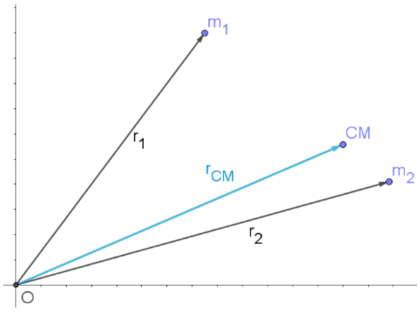
If one or more particles changes direction after the collision, then we're speak of elastic or inelastic scattering.

Center of mass

The position vector of the center of mass of a system of particles is defined as:

$$\vec{\mathrm{r}}_{\mathrm{CM}} \equiv \frac{1}{M}\sum_i m_i \vec{\mathrm{r}}_i$$

where $M = \sum_i m_i$ is the total mass of the system and $\vec{\mathrm{r}}_i$ is the position vector of the ith particle.



If we have an extended object, then the position vector of its center of mass can be obtained from the integral expression:

$$\vec{\mathrm{r}}_{\mathrm{CM}} = \frac{1}{M}\int \vec{\mathrm{r}}dm$$

Where dm is the differential mass element. We call it "element" because it's an infinitesimal quantity with a unit of measure. The definite integral will be extended into the entire length/surface/volume of the object.

For example, if we have a 1-dimensional bar, that extends from the origin along a length L, and with uniform mass distribution, its density will be constant:

$$\lambda = \frac{M}{L} = \text{const}$$

$$M = \lambda L$$

The generic position vector has component x and, since we are in a single axis, the differential element of mass is $dm = \lambda dx$:

$$\vec{\mathrm{r}}_{\mathrm{CM}} = \frac{1}{M}\int \vec{\mathrm{r}}dm = \frac{1}{M}\int_0^L \lambda x dx = \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_0^L = \frac{\lambda}{M}\left(\frac{L^2}{2} - 0\right) = \frac{\lambda}{\lambda L}\frac{L^2}{2} = \frac{L}{2}$$

The velocity of the center of mass for a system of particles is:

$$\vec{\mathrm{v}}_{\mathrm{CM}} \equiv \frac{1}{M}\sum_i m_i \vec{\mathrm{v}}_i = \frac{1}{M}\sum_i \vec{\mathrm{p}}_i$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

Analogously for the acceleration:

$$\vec{\mathrm{a}}_{\mathrm{CM}} = \frac{d\vec{\mathrm{v}}_{\mathrm{CM}}}{dt} = \frac{d\frac{1}{M}\sum_i m_i \vec{\mathrm{v}}_i}{dt} = \frac{1}{M}\sum_i \frac{dm_i \vec{\mathrm{v}}_i}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{\mathrm{v}}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{\mathrm{a}}_i$$

so, the center of mass' acceleration is the weighted average of the particles' acceleration. Moreover, from the Newton's second law we have:

$$\vec{a}_{\mathrm{CM}} = \frac{1}{M}\sum_i m_i\vec{a}_i = \frac{1}{M}\sum_i \vec{F}_i = \frac{1}{M}\vec{F}_{\mathrm{ext}}$$

$$\vec{F}_{\mathrm{ext}} = M\vec{a}_{\mathrm{CM}}$$

So, we have found again the Newton's second law but considering the particles as a single one with mass M and all the forces applied to the particles as a single external force.

Models

If a system interacts with its environment in the sense that there is an external force on the system (non-isolated system), the behavior of the system is described by the **impulse–momentum theorem**:

$$\Delta\vec{p}_{\mathrm{tot}} = \vec{I}$$

In an isolated system (no external forces) the total momentum is conserved:

$$\Delta\vec{p}_{\mathrm{tot}} = 0$$

The system may be isolated in terms of momentum but non-isolated in terms of energy, as in the case of inelastic collisions.

MOMENT

Rotation of a rigid object about a fixed axis

When an extended object rotates about its axis, the motion cannot be analyzed by modeling the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. We can, however, analyze the motion modeling the extended object as a collection of particles, each of which has its own linear velocity and linear acceleration.

Each particle that rotates about an axis can be described with a formula similar to the one of linear motion:

$$\omega_t = \omega_0 + \alpha t$$

$$\theta_t = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Where θ is the angle (like the linear s), ω is the angular velocity (like the linear velocity) and α is the angular acceleration (like the linear acceleration).

In order to pass from the angular quantities to the tangential ones we have to multiply them with the radius:

$$v = r\omega, \qquad a_t = r\alpha$$

In the uniform circular motion $a_t = 0$; in general, we could have an a_t and that implies that the total acceleration is given by the sum of the tangential component and the centripetal component:

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

Rotational kinetic energy

The kinetic energy of a rotating particle is the usual one:

$$K_i = \frac{1}{2}m_i v_i^2$$

where m_i is the mass of the ith particle and v_i is the velocity.

We can explicit the angular velocity, that is the same independently from the distance between the rotational axis and the particle:

$$K_i = \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} (m_i r_i^2) \omega^2$$

We define the **moment of inertia** of a particle as:

$$I = m_i r_i^2$$

And the rotational kinetic energy becomes:

$$K_i = \frac{1}{2} I_i \omega^2$$

so, the I_i takes the place of the m_i for linear motion.

If we have a system of particles:

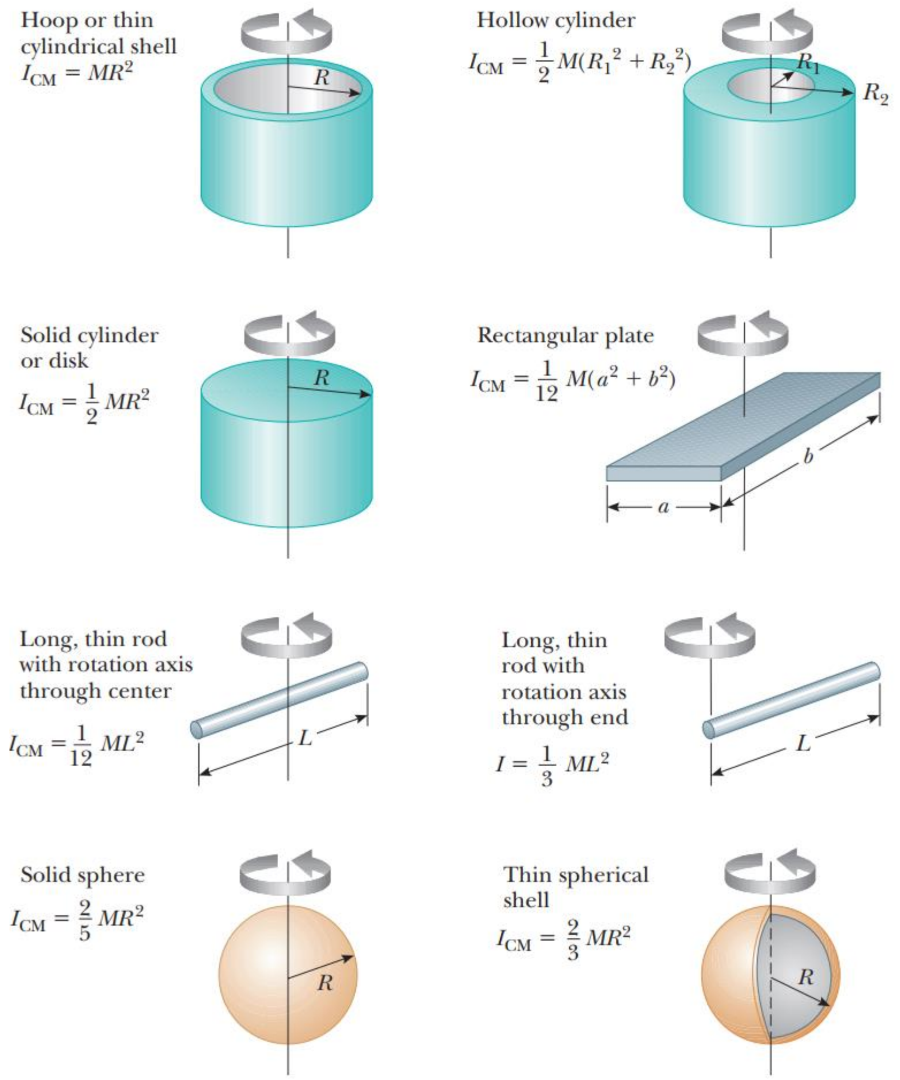
$$I \equiv \sum_i m_i r_i^2$$

$$K = \frac{1}{2} I \omega^2$$

For an extended body instead of the summation we use the integral, so:

$$I = \int r^2 dm$$

Where the integral runs over all the particles' positions. In general, the mass will be a function of a specific position. For example, the moment of inertia of some solids are:



Torque

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a quantity called torque $\vec{\tau}$. We define the torque as:

$$\vec{\tau} = \vec{F} \times \vec{r}$$

And its module as:

$$\tau = rF \sin \varphi = rF_t$$

Where r is the distance between the rotation axis and the force and φ is the fixed angle between the force and the radius. So, the torque is the analogue of the tangential force in a rotational motion. From here we get:

$$\tau = rF_t = r(ma_t) = r(mr\alpha) = mr^2\alpha = I\alpha$$

Since we assume that the mass and the shape of the object don't change, we know that I is constant and consequently the torque is proportional to the angular acceleration. From these equations we can also obtain:

$$W = \int_{\theta_0}^{\theta_f} \tau d\theta \,, \qquad \mathcal{P} = \tau\omega$$

Angular momentum

In this case we want to find the equivalent of the momentum in an angular motion. Since we know that:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

We start with the torque:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \left(\frac{d\vec{p}}{dt}\right)$$

Now we add $\left(\frac{d\vec{r}}{dt}\right) \times \vec{p}$, because we know that is equal to 0 (the derivative is equal to the velocity and we are doing a vectorial product with the velocity, itself and the mass, so $\sin\theta = 0$). Now we recognize that this is the derivative of the product of two functions:

$$\vec{\tau} = \vec{r} \times \left(\frac{d\vec{p}}{dt}\right) + \left(\frac{d\vec{r}}{dt}\right) \times \vec{p} = \frac{d(\vec{r} \times \vec{p})}{dt}$$

Finally, we can define the angular momentum as:

$$\vec{L} = \vec{r} \times \vec{p}$$

Rotating object along a slope

The total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass:

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

WAVES

Introduction

The world is full of waves, the two main types being mechanical waves and electromagnetic waves. In the mechanical ones we need a medium, while the electromagnetic do not require it. All mechanical waves require:

1. some source of disturbance;
2. a medium which contains elements that can be disturbed;
3. some physical mechanism through which elements of the medium can influence each other.

In general, we can divide the waves into longitudinal and transversal waves: in the first ones the particles of the medium are moving parallel to the wave's direction, while in the second ones, the particles are going perpendicular.

A normal wave is just a shift of the previous one, while in the standing wave, you're not moving, but you have fixed nodes and the extremes changing6.

The particle in simple harmonic motion

In this case we are studying a single particle that's moving in a harmonic motion. The mathematical expression that describes this motion is the following:

$$x(t) = A\cos(\omega t + \varphi)$$

Where A is the amplitude, φ is the phase and ω is the angular frequency.

If we derive this equation, then we'll find the velocity and the acceleration:

$$v(t) = -\omega A\sin(\omega t + \varphi)$$

$$a(t) = -\omega^2 A\cos(\omega t + \varphi) = a(t) = -\omega^2 x(t)$$

From these we can deduce:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \varphi)$$

$$F = ma = -\omega^2 mx$$

So, there must be a force that keeps the particle on its trail. Let's assume that this force is the elastic one, so that the particle is connected with a spring; then we would have:

$$F = -\omega^2 mx = -kx$$

where $k = \omega^2 m$. The work/ potential energy will be:

$$\int F(x)dx = \frac{1}{2}kx^2 = \frac{1}{2}A^2\cos^2(\omega t + \varphi)$$

Quench force: We have a weight anchored to a spring that can oscillate inside a tank of water. The spring is pulled down and then released. The forces that act on the weight are:

$$F_g = F_{spring} + F_{quench}$$

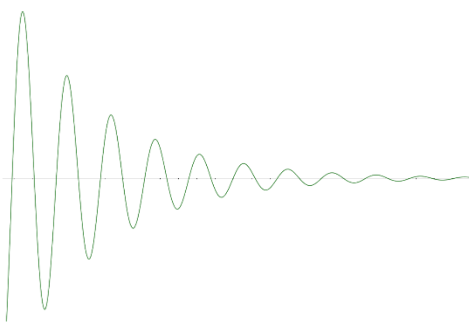
$$ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

This is a second order differential equation, which general solution is:

$$x(t) = Ae^{-\frac{bt}{2m}}\cos\left(\omega_q t + \varphi\right)$$

From here we can see that the amplitude decreases with time and so, the kinetic energy too.



Resonance: In the previous exercise the kinetic energy was decreasing progressively. In order to compensate this loss of energy, we can apply an external force, whose intensity follows the path of a sinusoidal wave, so:

$$F_0 \sin \omega_q t - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

By solving the differential equation again, we observe that this time the exponential factor isn't affecting the wave. This is possible because we have chosen a sine function (that act inversely respect to the cosine of the initial solution) and because we have chosen the same ω.

$$x(t) = A\cos(\omega_q t + \varphi)$$

Where:

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

The travelling wave model

In the previous case we were studying the oscillatory motion of a single particle; here we'll study the motion of an entire wave along a direction. In general, assuming that the shape of the wave doesn't change with time, we have:

$$y(x,t) = f(x - vt)$$

If we set $t = 0$ we get the shape of the wave, indeed: $y(x,0) = f(x)$. If t increases, this shape will be translated towards right with a velocity v.

Wave length: If we have a function like $\sin \lambda x$ we know that its period will be 2π. If we want to find the distance between two points that have the same height h, we'll have that:

$$\sin(\alpha x_1) = \sin(\alpha x_1 + 2\pi)$$

For example, let's take $\alpha x_1 = \frac{\pi}{2}$ $(x_1 = \frac{\pi}{2\alpha})$ and $\alpha x_1 + 2\pi = \alpha x_2 = \frac{5\pi}{2}$ $(x_2 = \frac{5\pi}{2\alpha})$, then:

$$x_2 - x_1 = \frac{5\pi}{2\alpha} - \frac{\pi}{2\alpha} = \frac{2\pi}{\alpha} = 2\pi\lambda$$

$$\sin\left(\frac{2\pi}{\lambda}x_1\right)$$

Where λ is the wave length (the distance between two identical points).

The model. An interesting case is the travel of a sinusoidal wave, described by:

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

So, the shift is described by $x - vt$. The velocity of the wave is $v_{wave} = \frac{\lambda}{T}$. If we solve the product with $\frac{2\pi}{\lambda}$, we get:

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}vt\right]$$

$$y(x,t) = A\sin(kx - \omega t)$$

Where $k = \frac{2\pi}{\lambda}\left[\frac{rad}{m}\right]$ and $\omega = \frac{2\pi}{\lambda}v\left[\frac{rad}{s}\right]$.

Superposition of sinusoidal waves

If we have two waves that are overlapping with a phase φ, we'll get a new wave with a new amplitude:

$$y_1 = A\sin(kx - \omega t), \qquad y_2 = A\sin(kx - \omega t + \varphi)$$

By using the prosthaphaeresis formula:

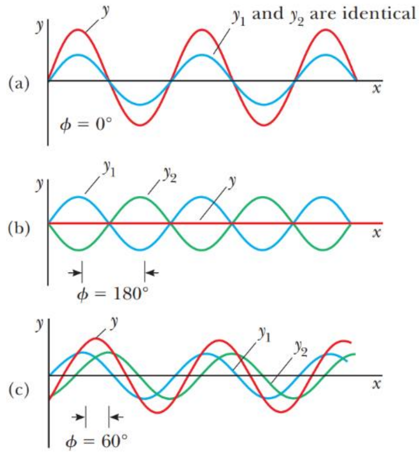
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

We get:

$$2A\sin\left(\frac{kx-\omega t+kx-\omega t+\varphi}{2}\right)\cos\left(\frac{kx-\omega t-kx+\omega t-\varphi}{2}\right)$$

$$2A\sin\left(\frac{2kx-2\omega t+\varphi}{2}\right)\cos\left(-\frac{\varphi}{2}\right)$$

$$2A\sin\left(kx-\omega t+\frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right)$$



Doppler effect

If a point source emits sound waves, they will spread in a spherical way. If the source is moving, then, for a static observer, the frequency will change, because the waves will overlap or will go away from the previous ones emitted. The formula to evaluate this new frequency is:

$$f' = f\frac{v+v_0}{v-v_s}$$

Where v is the speed of the wave/sound, v_0 is the speed of the observer, v_s is the speed of source, f' the frequency perceived and f the natural frequency.

SPECIAL RELATIVITY

Area of application

The theory of special relativity studies the movement of particles through space and time without gravity force and with scales greater than the Compton one, so everything that is greater than its wave length.

Frames of reference

The frame of reference (observational frame of reference) is the point of view of a physical observer that describes the physical observation through the physical unit of measures (defined in its system). If we can observe an isolated particle (so, without external forces) from the frame of reference and this follows a linear path, then the frame is defined inertial frame. Equivalently we define inertial frame the one in which is defined the first Newton's law. When we found the inertial frame, all the others that move with a translational uniform motion respect to that one will be inertial frames, so we can talk about class of congruence of inertial frames.

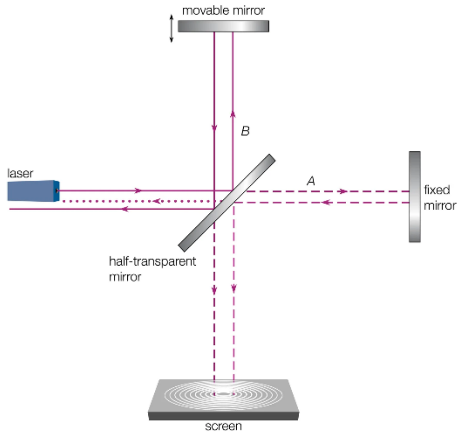
This classical definition of inertial frame, is extended by Einstein that presents it through a principle of simplicity: an inertial frame is the one from which an observer can describe the physical laws in the simplest form.

Special principle of relativity

One of the fundamental assumptions of special relativity is the special principle of relativity. It says that the laws of physical phenomena are observed in the same way in all inertial frames.

Now a problem arises: the observations are made in term of coordinate systems and a problem is how to transform the coordinates of one frame into those of another. This is not a mere mathematical exercise because, accepting the special principle of relativity, the transformations must give unaltered description of universal laws.

Now there is the second fundamental hypothesis of special relativity and regards light. On the basis of numerous experimental observations, it was verified the independence of the speed of light from the speed and direction of the source.



Michelson-Morley experiment: we have a laser that hits the central half-transparent mirror. It divides in two: one part goes up, hits the top mirror and goes back to the screen, the other one, goes straight, hits again the central mirror and goes to the same bottom screen. The two rays merge before hitting the screen. Thanks to this experiment we are demonstrating the isotropy of the propagation of light, because the two rays merge constructively, independently from the direction.

Now we let all the system move along the direction of the laser. By following the Galilean assumptions, we should expect that the vertical ray is not affected at all and the horizontal one goes slower, but, we arrive at the same conclusion.

Therefore, Einstein states the speed of light in a vacuum as an axiom and for the principle of special relativity this constant must be observed in the same way among all inertial frames.

Time and simultaneity of events

The constancy of the speed of light has the consequence that pairs of events detected by the emission of light cannot be evaluated in the same way as simultaneous between inertial frames that are not stationary with each other. That is, the concept of simultaneity is related to a specific inertial frame. And this whenever the two inertial frames have a relative speed even though could have the same origin. In special relativity, the assumption of constancy of the speed of light implies also a relativization of the definition of the unit of measurement of time intervals and length.

Lorentz transformation

Let consider two inertial frames $\mathcal{S}$ and $\mathcal{S}'$with same oriented axes and same origin at $t=0$, but a relative velocity v of $\mathcal{S}'$ respect to $\mathcal{S}$. If at $t=0$ a light ray is emitted from the origins after a period of t has reached a spherical surface of radius ct, so we have:

$$c^2t^2 = x^2 + y^2 + z^2$$

$$c^2t'^2 = x'^2 + y'^2 + z'^2$$

Since the system $\mathcal{S}'$moves with velocity v respect to $\mathcal{S}$ we want to find the relation between t', x', y', z' and t, x, y, z. Out of all the possible transformations, we chose the linear ones and we get to the Lorentz transformations. In the simple case in which the two systems are only moving along the positive x axis, and defining:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

we get to:

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = \mathrm{z}$$

equivalently, if we are in $\mathcal{S}'$, we will look at $\mathcal{S}$, moving with velocity $-v$:

$$t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

The Lorentz transformations, once accepted, will determine a scientific method too. A law will be so if it's invariant respect to the Lorentz transformations.

Time dilation and length contraction

The **proper time** is the one measured on the stationary frame with the clock, while the **improper time** is the one measured on any moving frame. If we are in $\mathcal{S}'$ with $x' = 0$, from the Lorentz transformations we get the relation between proper and improper time:

$$t = \gamma t' = \frac{t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Or with the notation:

$$\Delta t = \gamma \Delta t_0$$

So, the improper time is always greater than the proper. The same relation can reached by the other system $\mathcal{S}'$ when he sees the clock stationary on $\mathcal{S}$ respect to its clock.

Analogously the **proper length** is the one measured on the stationary frame and the **improper length** on the moving one.

To compare the two lengths, we have to take in account that the two extremes of the object are not simultaneous. One extreme of the object is at $O = O' = 0$ on $t = t' = 0$. The other extreme on $\mathcal{S}'$is at $t' = 0$ and x'. But in $\mathcal{S}$ the extreme is at $t = 0$ in x, so

$$x' = \gamma(x - v0)$$

$$x' = \gamma x$$

$$x = \frac{1}{\gamma}x'$$

Or with the notation:

$$\mathrm{L} = \frac{1}{\gamma}L_0$$

Addition of velocities

If a system $\mathcal{S}'$ moves with velocity v respect to $\mathcal{S}$ and an object is moving with velocity u' respect to $\mathcal{S}'$ we have that the velocity of the object respect to $\mathcal{S}$ will be:

$$u = \frac{x}{t} = \frac{u' + v}{\left(1 + \frac{u'v}{c^2}\right)}$$

that is the analogous of the Galilean sum of velocities:

$$u = u' + v$$

Momentum and mass

Like the time and the lengths measurements are relative to the observational frame, also the mass and the momentum depends on the frame:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0$$

$$p = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0 v$$

Where m_0 is the stationary mass.

The equivalence of mass and energy

We can express the relativistic mass with the Taylor expansion:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0 \left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \cdots \right)$$

if we multiply by c^2:

$$mc^2 = m_0 c^2 + m_0 \frac{v^2}{2} + m_0 \frac{3v^4}{8c^2} + \cdots$$

we get the sum between the constant term $m_0 c^2$, that is specific for the particle, and the ordinary kinetic energy. So, we are calling $mc^2$the total energy of a particle e in a stationary frame with the object (so with $v = 0$) we have in any case a rest energy that is:

$$E_0 = m_0 c^2$$

An interpretation of this energy is given by the atomical and subatomical vibrations.

GENERAL RELATIVITY

In the theory of general relativity, we study the motion of particles under the effect of the gravitational field. In this theory we have a model of physics that is drastically different from the gravitation Newtonian one. In the last one, the gravity is a force that acts just like the other forces (electromagnetic, weak and strong); moreover, it should affects only particles with mass. With general relativity, the masses modify the space-temporal field, so a particle moves along a complex and not uniform space. An experimental proof of this vision is that also the photon's trajectory is deflected by a particle with mass.

A key mental experiment is this one: we are in a rocket that moves with an acceleration equivalent to the Earth's one and we're emitting flashes from the top to the bottom of the rocket; each one of those is emitted 1 second apart, but in the bottom of the rocket we'll receive them with a difference of time that's less than 1 second, because the rocket is moving towards the flash. By the principle of equivalence, we can transpose the same effect to a rocket still on the surface of the Earth.

Other experimental tests were the eclipse of Mercury in which the Sun's shape was perceived slightly different in size due to the deflection of light caused by the mass of Mercury. Another one involves the collapse of two massive black holes that causes a deformation of the space-time lattice that propagated to the Earth too.

FLUID MECHANICS

Definition

We define a fluid as a substance that does not have a shape but assumes the container's one, that doesn't have a strong molecular structure and in which the shear stress at rest is not maintained. A fluid can exert a pressure on an object that is put inside it. This pressure will be perpendicular to the object and equal in all its points. The **pressure** P of the fluid on the object is given by:

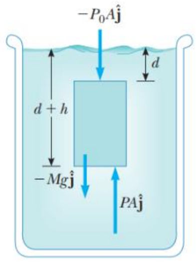
$$P \equiv \frac{F}{A}, \qquad \frac{[N]}{[m^2]} = [Pa]$$

Where F is the force applied on the body (weight of the fluid) and A is the area.

The study of the laws about fluids begins with the model of the ideal fluid. We define ad ideal fluid with these properties:

- **Incompressible**: density remains constant during flow.
- Viscosity is zero (the **viscosity** is the friction between adjacent layers of the fluid).
- **Stationary law**: velocity of each element of the fluid does not vary with time.
- **Irrotational**: Angular momentum is zero at each point.

Pascal law



If we submerge an object in the water, on the y axis we'll have three forces: one is the gravitational force, the other one is the pressure of the water above it and the atmosphere and these two are compensated by the pressure of the water below the object. So, we have that:

$$PA = P_0A + Mg$$

We know that the mass is equal to the Area times the height (so the volume) times its density so we get:

$$P = P_0 + \rho g h$$

So, the pressure does not depend on the shape of the object.

Buoyant Forces and Archimedes' Principle

If we put an object inside a liquid, the liquid will exert on the object an upward force that is the buoyant force:

$$B = PA = (P_{bot} - P_{top})A = \left(\frac{F}{A}\right)A = \left(\frac{M_{fluid}g}{V}h\right)A = \rho_{fluid}gV$$

This force depends only on the fluid and the volume of the object. Clearly, its effect will depend on the object's mass: indeed, if the weight of the object is greater than B, the object will go down, if they're equal, the object is still or else the object will float on the water.

Equation of continuity

If a fluid flows through a pipe with different size sections at the same height, then the amount of liquid that flows inside the two sections is the same:

$$\Delta V_1 \rho = \Delta V_2 \rho$$

$$A_1 \Delta x = A_2 \Delta x$$

$$A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

$$A_1 v_1 = A_2 v_2$$

Or equivalently:

$$\frac{\Delta V_1}{\Delta t} = A_2 v_2$$

So, the larger the area, the slower is the fluid.

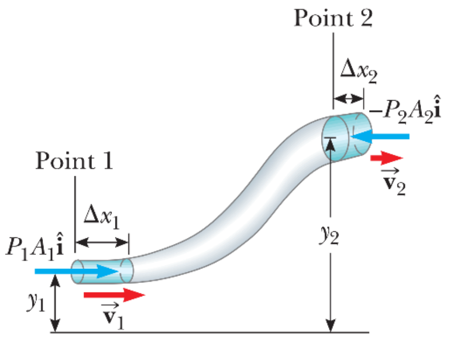
Bernoulli's formula

If we consider now a pipe with different size sections and different heights, we will have from the study of kinetic and potential energy:

$$P + \frac{1}{2}\rho v^2 + \rho g \Delta y = \text{constant}$$

So, if the velocity or the height increases, the pressure has to decrease.

If $v = 0$ we come back to the Pascal's law.



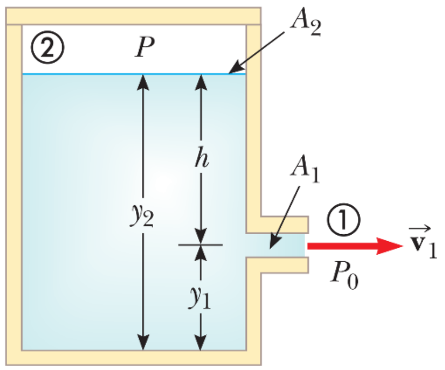
Torricelli's law

If we have a tank of water with a hole in the middle, then the water will flow with a velocity v. We assume that $A_2 \gg A_1$ and that the air pressure P is maintained constant during the flow.

Since $A_2 \gg A_1$, from the equation of continuity $A_1 v_1 = A_2 v_2$ we get that $v_2 = 0$. Now, since the energy is conserved, we have that:

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P + \rho g y_2$$

$$v = \sqrt{\frac{2(P - P_0)}{\rho} + 2gh}$$



THERMODYNAMICS

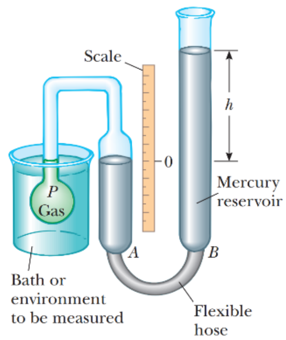
Temperature and its scales

Thermodynamics explains the macroscopic properties of matter starting from the mechanics of the atoms and molecules.

If we have two objects at different temperature, energy between them can be transferred with heat or electromagnetic radiation, until the thermal equilibrium is reached. The principle zero states that if two bodies A, B are separately in thermal equilibrium with body C, then A and B are in thermal equilibrium themselves.

Gas thermometer: in order to realize a gas thermometer, we have to calibrate it in at least two points: usually we use a mixture of ice and water to fix the 0°C and a mixture of steam and water for the 100°C.

On version of the gas thermometer is the fixed-volume one, in which there are two arms one of which contains a gas and is submerged in ice and water, while the other one contains mercury. The level of mercury is adjusted so that it can compensate the pressure of the gas in such a way that the liquid is always on the baseline (0°C). Now we repeat the same measurement but instead of using ice and water we put steam and water so that we can get the pressure of the gas at temperature 0°C and 100°C. By repeating this with other gases we understand that the pressure changes linearly with temperature and also that if we go into negative temperatures, pressure will decrease until, independently from the gas, will be always 0 Pa at -273,15°C. This point is the zero point of another temperature scale, the Kelvin one.



Ideal gas law

Macroscopic definition of perfect gas: it has low density and short-range forces (if two points are very far, then they're not interacting). One mole of any substance is that amount of the substance that contains Avogadro's number ($N_A = 6.022 \cdot 10^{23}$) of constituent particles (atoms or molecules). The number of moles n of a substance is related to its mass m through the expression $n = \frac{m}{M}$, where M is the molar mass of the substance. So, for example two moles of O_2 have $1.2044 \cdot 10^{24}$ molecules and mass of $m = nM = 64g$ ($M = 32\frac{g}{mol}$).

The equation of state for an ideal gas or the **ideal gas law** states that:

$$pV = nRT$$

Where $R = 8{,}314\frac{J}{mol \cdot K}$ and is the **universal gas constant**. Another equivalent expression of the ideal gas law is:

$$pV = \frac{N}{N_A}RT = Nk_BT$$

Where $N = n \cdot N_A$ and $k_B = \frac{R}{N_A} 1{,}38 \cdot 10^{23}\frac{J}{K}$.

The kinetic theory of gases

Relationship between pressure and molecular kinetic energy: In order to study a gas, we can first assume that the gas is composed of a single particle, study it and then extend the properties found to the entire gas.

We start with a particle inside an empty box with side d, whose volume and mass are much greater than the particle ones. We first consider the particle's movement along a single component x. If it hurts a surface, its momentum is inverted. So, its impulse is equal to $I = \Delta p = -2mv_{x_i}$. Now we consider a "complete particle cycle" so from when it hurts the surface of the cube to its return to the same surface when it has travelled the distance $2d$. The time is $\Delta t = \frac{2d}{v_{x_i}}$. The only time in which we have a force is during the impact with the surface. So, the average force is given by:

$$\bar{F}_{x_i} = -\frac{2mv_{x_i}}{\Delta t} = -\frac{mv_{x_i}^2}{d}$$

For the Newton's third law we know that the force exerted by the particle on the wall will be the exact opposite of the one exerted by the wall on the particle:

$$\bar{F}_{w_i} = \frac{mv_{x_i}^2}{d}$$

If we extend this force to all particles we have:

$$\bar{F}_w = \sum_{i=1}^{N}\frac{mv_{x_i}^2}{d} = \frac{m}{d}\sum_{i=1}^{N}v_{x_i}^2$$

We can also express the average velocity as:

$$\overline{v_x^2} = \frac{\sum_{i=1}^{N} v_{x_i}^2}{N}$$

Now, for the Pythagorean theorem we know that: $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$. Also, since the motions of the particles are completely random, we can say that:

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

So:

$$\overline{v^2} = 3\overline{v_x^2}$$

$$\bar{F} = \frac{1}{3}N\frac{m}{d}\overline{v^2}$$

The pressure will be:

$$p = \frac{F}{A} = \frac{F}{d^2} = \frac{1}{3} N \frac{m}{V} \overline{v^2} = \frac{2}{3} \frac{N}{V} \left(\frac{1}{2} m \overline{v^2} \right)$$

This result indicates that the pressure of a gas is proportional to the number of molecules per unit volume and to the average translational kinetic energy of the molecules.

Now we can compare this equation with the ideal gas law:

$$pV = \frac{2}{3} N \left(\frac{1}{2} m \overline{v^2} \right) = N k_B T$$

$$T = \frac{2}{3} \frac{1}{k_B} \left(\frac{1}{2} m \overline{v^2} \right)$$

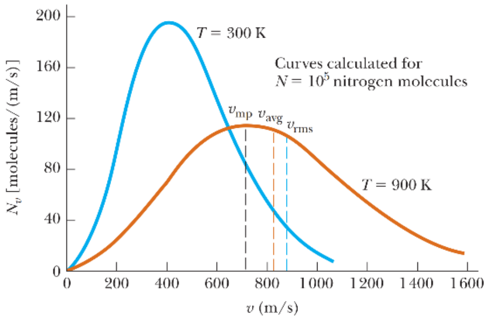
$$E_{tot} = \frac{3}{2} nRT$$

This result tells us that temperature is a direct measure of average molecular kinetic energy. But a gas has also other types of energies, the rotational one and the vibrational (only for non-monoatomic gases).

Distribution of molecular velocities

The fundamental expression that describes the distribution of speeds of N gas molecules is:

$$N_v = 4\pi N \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2k_B T}}$$



Heat

We define with a calory the necessary heat to increase the temperature of 1g water by 1°C:

$$1 \text{ cal} = 4{,}186 J$$

The **heat capacity** C ($\frac{J}{°C}$)of a particular sample is defined as the amount of energy needed to raise the temperature of that sample by 1°C, so:

$$Q = C\Delta T$$

The **specific heat** c ($\frac{J}{°Ckg}$) of a substance is the heat capacity per unit mass. Therefore:

$$Q = mc\Delta T$$

Latent heat: if during a transfer of heat the substance changes phase we notice that the temperature doesn't change linearly but it stops for a moment. This is because of the latent heat, the energy that the mass uses in order to change its phase. Only after all the particles of the body have changed phase, the temperature will rise again. We define latent heat as:

$$L = \frac{Q}{\Delta m}$$

Where Δm is the difference between the starting mass of a certain phase and the ending mass of the same phase.

First principle of thermodynamics

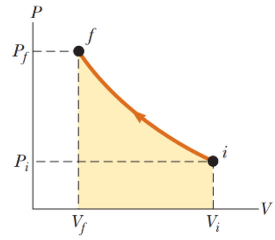
During an internal transformation, there can be work done by or on the system. The difference of this work is defined as:

$$dW = -Fdr = -PAdr$$

But since Adr is the volume, then the work is the integral of the function of pressure relative to volume:

$$\boxed{W = -\int_{V_i}^{V_f} PdV}$$

In the PV diagram this expression corresponds to the negative of the area.



The first law of thermodynamics is a special case of the law of conservation of energy that describes processes in which only the internal energy changes and the only energy transfers are by heat and work:

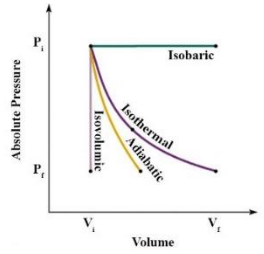
$$\Delta E_{int} = Q + W$$

During a slow compression the system is in equilibrium at every instant t, while in a fast compression is adiabatic, so without exchanging heat.

A special case is an isolated system in which energy transfer by heat takes place and the work done on the system is zero. In this case the internal energy E_{int} remains constant.

Types of transformations

There can be different types of transformations:



Isochoric: $W = 0; \Delta V = 0; \Delta E_{int} = Q$. constant-volume process

Isobaric: $E_{int} = Q + W; \Delta P = 0$. Constant pressure, for example the piston of the container moves freely.

Isothermal: $\Delta E_{int} = 0;\ Q = -W; PV = \text{constant}$. The work is given by:

$$W = -\int_{V_i}^{V_f} PdV = -\int_{V_i}^{V_f} \frac{nRT}{V} dV = -nRT \int_{V_i}^{V_f} \frac{dV}{V} = nRT \ln\left(\frac{V_i}{V_f}\right)$$

Adiabatic: $Q = 0; \Delta E_{int} = W$. No energy enters or leaves the system by heat; such that W is positive the energy and temperature increases. In an adiabatic free expansion also $W = 0$.

Cyclic transformations: $\Delta E_{int} = 0; Q = -W$. In a cy clic process the initial and final state are the same so $\Delta E_{int} = 0$ (because E_{int} is a state variable) but now we must have $Q = -W$.

Molar specific heats

In an order to study that heat exchanged in an arbitrary transformation is useful to consider two special transformation for which the determination of heats is known, the isochoric and the isobaric transformations:

$$Q = nC_V \Delta T \text{ (isochoric = constant volume)}$$

$$Q = nC_P \Delta T \text{ (isobaric)}$$

where C_V is the molar specific heat at constant volume and C_P is the molar specific heat at constant pressure.

For a monoatomic gas, from the kinetic theory of gases, we know that all its internal energy is given by the translational kinetic energy:

$$E_{\text{int}} = \frac{3}{2} nRT$$

If we consider an isochoric transformation $W = 0$ this implies $Q = E_{\text{int}} = \frac{3}{2} nR\Delta T$. So, in this case we can predict the value of the molar specific heat at constant volume for a monoatomic gas:

$$nC_V \Delta T = \frac{3}{2} nR\Delta T$$

$$C_V = \frac{3}{2} R \sim 12{,}5 \frac{J}{mol \cdot K}$$

Now for an isobaric transformation

$$Q = nC_P \Delta T$$

$$\Delta E_{int} = Q + W = nC_P \Delta T - p\Delta V$$

And since p is constant with the $pV = nRT$ becomes:

$$p\Delta V = nR\Delta T$$

so:

$$nC_V\Delta T = nC_P\Delta T - nR\Delta T$$

and we can have the relation between the molar specific heat:

$$C_P - C_V = R$$

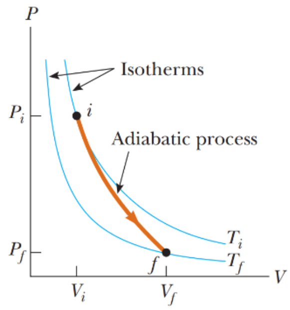
So molar specific heat in isobaric is larger than isochoric.

The ratio of these molar specific heats is a dimensionless quantity denoted by:

$$\gamma = \frac{C_p}{C_V} = \frac{5}{3}$$

Adiabatic transformations

An adiabatic process is one in which no energy is transferred by heat between a system and its surroundings. All three variables in the ideal gas law—P, V, and T— change during an adiabatic process so this process have to be study by infinitesimal variations: dV, dT, etc. The work done on the gas will be $dW = pdV$ hence the first law of thermodynamic takes the form:



$$dE_{\text{int}} = nC_V dT = -pdV$$

that gives us a first relation between dT and dV. Another relation is:

$$pdV + Vdp = -\frac{RpdV}{C_V}$$

substituting $C_P - C_V = R$ and dividing by pV gives:

$$\frac{dV}{V} + \frac{dp}{p} = -\frac{C_p - C_V}{C_V}\frac{dV}{V}$$

$$\frac{dV}{V} + \frac{dp}{p} = (1-\gamma)\frac{dV}{V}$$

this differential equation once solved has the following solution:

$$\ln p + \gamma \ln V = \text{constant}$$

equivalent to:

$$pV^\gamma = \text{consant}$$

From the same relation we get also that:

$$TV^{\gamma-1} = \text{constant}$$

Radiation

Every object radiates electromagnetic waves due to the thermal motion of its molecules. The Stefan's law states that the rate at which an object radiates energy is proportional to the fourth power of its absolute temperature:

$$\mathcal{P} = \sigma AeT^4$$

where $\mathcal{P}$ is the power in watts of electromagnetic waves radiated from the surface of the object, e is the emissivity, T is the surface temperature in kelvins and the constant:

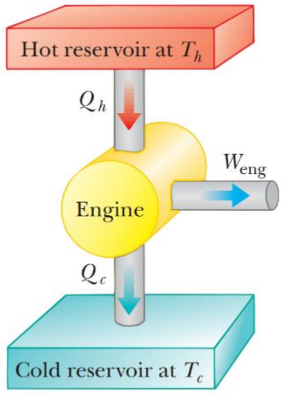
$$[\sigma] = \frac{W}{npK^4} = 5{,}669 \cdot 10^{-8}\frac{W}{m^2K^4}$$

The value of e can vary between zero and unity depending on the properties of the surface of the object. The emissivity is the fraction of the incoming radiation that the surface absorbs. A mirror has very low emissivity and black surface has high emissivity. An ideal absorber referred as a black body has $e = 1$.

Convection

When a gas heats it gets less dense, so, by the Archimedes principle, it goes up and the gas cooler and denser goes down this result in a transfer of energy. For example, if it were not for convection currents, it would be very difficult to boil water.

When the transfer of energy results from differences in density the process is referred to as natural convection. When the heated substance is forced to move by a fan or pump, as in some hot-air and hot water heating systems, the process is called forced convection.



Thermal machines

$$\Delta E_{int} = Q + W$$

A thermal machine is a machine that operates by accumulating energy in form of heat and releasing a fraction of it in the form of work. The source could be a hot reservoir, like a thermostat. The machine works by computing on the gas some kind of cyclic transformations, so that $\Delta E_{int} = 0$. From this we know that:

$$Q = -W = W_{machine}$$

Here we have two types of heat, the one that is given by the hot reservoir (Q_h) and another one that is given to the cold reservoir (Q_c). So:

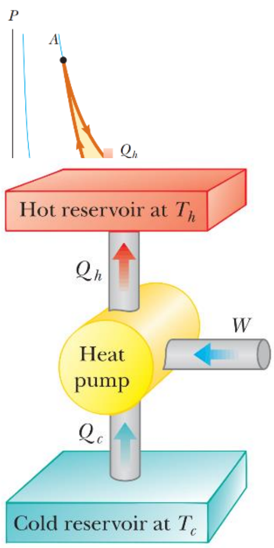
$$W_{machine} = |Q_h| - |Q_c|$$

The more work the machine is capable of doing, the more efficiency we'll have. In particular we define the **efficiency** e as:

$$e = \text{machine efficiency} = \frac{W_{machine}}{|Q_h|} = \frac{|Q_h| - |Q_c|}{|Q_h|} = 1 - \frac{|Q_c|}{|Q_h|}$$

A machine that has $e = 100\%$ is the ideal machine, or the Carnot's one. This can be obtained in three different ways, all of three impossible by the second principle of thermodynamics:

- $Q_c = 0$;
- absorb heat from a cold source spontaneously (impossible);
- $\Delta S > 0$ (variation of entropy).



Refrigeration machine: A heat pump or refrigeration machine, is a machine that works in opposite to the natural transfer of energy, indeed it takes energy from the cold reservoir and, thanks to some work, it gives it to the hot one. The efficiency here is called **performance coefficient** (PC) and it's so defined:

$$PC = \frac{|Q_c|}{W} = \frac{|Q_c|}{|Q_h| - |Q_c|} = \frac{|T_c|}{|T_h| - |T_c|}$$

In this type of machines, the lower is the work that we're using, the highest will be the performance (in the most efficient, impossible case, $W = 0$).

Reversible and irreversible processes

In a **reversible process**, the system undergoing the process can be returned to its initial conditions along the same path on a PV diagram, and every point along this path is an equilibrium state. A process that does not satisfy these requirements is **irreversible** (all the processes in nature are irreversible).

Entropy

Entropy is essentially the measure of disorder of a system: we can give two types of definition of entropy, one microscopic and the other one macroscopic. In the macroscopic definition we have a relation between the change of heat and temperature:

$$dS = \frac{dQ}{T}$$

Where Q is the variation of heat in the reversible transformation.

If we're considering the change of entropy between an initial and a final state, then we'll have:

$$\Delta S = \int_i^f \frac{dQ}{T}$$

For the microscopic scale, by giving at each molecule a part of volume, we could count the number of possible states. So, the number of all possible positions of a molecule is:

$$\frac{V_f}{V_m}$$

That, repeated for all molecules, becomes:

$$\left(\frac{V_f}{V_m}\right)^2$$

Where V_f/V_i are the final and starting volumes, and V_m is the volume occupied by a molecule. From this equation we get that the entropy is:

$$S_f - S_i = nR\ln\left(\frac{V_f}{V_i}\right) = K_B \ln(W)$$

Where W is the number of approximated microstates.

ELECTROMAGNETISM

Law of Coulomb

We can divide the materials in two categories: there're the electrical **conductors**, in which the electrons are freer to move, and the electrical **insulators**, the opposite. By creating a difference of electrons, we can charge a particle. Two particles charged create a force between each other. This force is called the **electric force** and its magnitude is given by the **Coulomb's law**:

$$F_e = k_e \frac{|q_1||q_2|}{r^2}$$

Where q_1 and q_2 are the charges of the particles, r is the distance between them and k_e is the Coulomb constant equal to: $8{,}9876 \cdot 10^9 N \cdot \frac{m^2}{C^2}$

This constant is also written in the form:

$$k_e = \frac{1}{4\pi\epsilon_0}$$

Where ϵ_0 is the permittivity of free space and has the value of:

$$\epsilon_0 = 8{,}8542 \cdot 10^{-12} \frac{C^2}{N \cdot m^2}$$

The charge of an electron is approximately of $-1{,}602 \cdot 10^{-19} C$.

One difference with the gravitational force is that the gravitational one is always attractive, while the electrical one can also be repulsive.

Electric field

Like for the gravitational force, also the electric one produces a field capable of affecting other particles without touching them, and it's so defined:

$$\vec{E} \equiv \frac{\vec{F}_e}{q_0}$$

Here it's important to know that the electric field exists independently of the test charge. So, this charge is put to detect the presence of an electric field. In particular, if the charge is positive, it'll diverge from the source and if it's negative it'll converge. In this way we can also define the direction of $\vec{E}$. A charge is pulled with a force of $\vec{F}_e = q\vec{E}$. From the Coulomb's law we also get:

$$\vec{E} = k_e \sum_i \frac{q_i}{r_i^2} \hat{r}_i$$

Electric flux

$$\phi_E = \oint \vec{E} d\vec{A}$$

Where we have the closed integral (since we are considering a surface) of the electric field evaluated on the perpendicular vector of the area. The result of this operation is a number. The relation between the electrical flux and the charge enclosed by the surface of a sphere is given by the **Gauss theorem**:

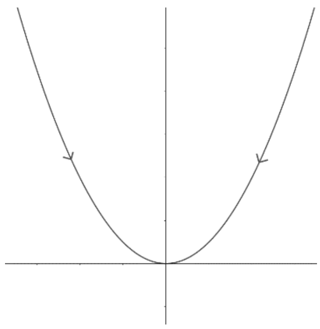
$$\oint \vec{E} d\vec{A} = E \oint 4\pi r^2 = \frac{k_e Q}{r^2} 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

Since we are in a sphere E is constant, so we can take it out from the sphere.

Electric potential

We can recover an analogous concept to the work and potential energy in gravity. The potential energy ΔU is given by a charge q_0 along a path ds, so:

$$\Delta U = -q_0 \int_A^B \vec{E} \cdot d\vec{s}$$



Force and potential energy have opposite direction, so, in the graph, all the arrows point downward because normally they try to minimize the potential energy.

If we divide everything by q_0 we'll get a quantity that only depends on the source and it's the electrical potential:

$$\Delta V \equiv \frac{\Delta U}{q_0} = \int_A^B \vec{E} \cdot d\vec{s}$$

Just like potential energy, also the potential is meaningful when considered as a difference. Its unit is the volt $[V]$. By definition 1 joule of work to move a charge of 1 coulomb from a potential of 1 volt is $1V = 1\frac{Nm}{C}$.

$\vec{E}$ can be interpreted as the rapidity of the change of potential with distance, so:

$$\vec{E} = -\frac{d\Delta V}{ds}$$

Biot-Savart's law

The magnetic field can be generated even with a single moving electric charge or, in the Biot-Savart's law, by a flux of charge in a wire. By following the right hand's rule, we can use the thumb to follow the charge and the other fingers to "draw" the magnetic field. The intensity of this magnetic field will be:

$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

Or:

$$\vec{B} \alpha d\vec{s} \times \vec{r}$$

where r is the distance between the considered point and the wire and I is the current.

Induction, Faraday

We have a coil and a constant magnetic field $\vec{B}$:

$$\phi = \int \vec{B} d\vec{A} = \int B dA \cos\theta = BA\cos\theta$$

$$\varepsilon_{EM} = -\frac{d}{dt}(BA\cos\theta)$$

Lentz's law

The direction of the induced current is such that it tends to compensate the change of the B's flux.

Alternate current

We have a coil on a rotating axis:

$$\phi_B = BA\cos\theta = BA\cos\omega t$$

$$\varepsilon_{EM} = -\frac{d\phi_B N}{dt} = NBA\sin\omega t$$

Where N is the number of elements of the coil.

From this we get that the flux is alternating between a decreasing phase and an increasing one, so also the direction of the current flips.

ε_{EM} will be maximum if $\vec{B}$ is parallel to $\vec{A}$ and minimum if the two are perpendicular (here $\phi_B = 0$).

General formulation of Faraday's law

ε_{EM} here can be considered as tension.

$$\varepsilon_{EM} = -\frac{d\phi_B}{dt} = \oint \vec{E}\, ds$$

$$W_{ext} = \Delta U$$

$$\Delta V = q\varepsilon_{EM}$$

So ΔV is related to $\vec{E}$

Equations of Maxwell

1. $\oint \vec{E}\, d\vec{A} = \frac{q}{\varepsilon_0}$ (Gauss);
2. $\oint \vec{E}\, ds = -\frac{d\phi_B}{dt}$ (Faraday);
3. $\oint \vec{B}\, d\vec{A} = 0$ (Gauss);
4. $\oint \vec{B}\, ds = \mu_0 I + \varepsilon_0\mu_0 \frac{d\phi E}{dt}$ (Modified Ampere), where $\oint \vec{B}\, ds = \mu_0 I$ is normal Ampere and $\varepsilon_0\mu_0 \frac{d\phi E}{dt}$ was added by Maxwell;
5. $F = q(\vec{E} + \vec{v} \times \vec{B})$

1 and 3 are stationary. By 3 we get that there cannot exist a magnetic monopole like for the charges. By 2 and 4 we know that independent fields generate time dependent fields. So E generates B and vice versa, that gives us an electromagnetic wave. In this wave E and B are perpendicular. Its speed is c.

Since the Maxwell equations doesn't cover the case in which $v = 0$, we know that we can't go at the same velocity of light.

This is the integral form of the Maxwell equations. We can also write them in the differential form (we'll consider only one axis instead of the entire 3D space):

$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$ (Faraday);

$\frac{\partial B}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ (Ampere);

In the first one we can write $\frac{\partial B}{\partial t}$ as $\frac{\partial}{\partial t}\frac{\partial B}{\partial x}$ and then substitute the Ampere equation. At the end we'll get:

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

And this is a wave equation. In a wave equation we have that the wave must be of the same shape at each time for all the x, so $f(x) = f(x + v\Delta t)$. The wave equation is indeed:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

So, we have that $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ that experimentally is equal to c. This means that a possible solution to the differential form of the Faraday-Ampere equation is valid for the integral form too. By the consideration on the velocity, we can also say that the electromagnetic wave is made of photons.

One possible solution for this equation is using these two functions:

$$E = E_{max} \cos(kx - \omega t)$$

$$B = B_{max} \cos(\alpha x - \omega t)$$

OPTICS

Energy and pressure

The rate of flow of energy in an electromagnetic wave is represented with the pointing vector_

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}, \qquad \left[\frac{W}{m^2}\right]$$

$$|\vec{S}| = \frac{1}{\mu_0}\vec{E}\vec{B} = \frac{1}{c\mu_0}\vec{E}^2 = \frac{c}{\mu_0}\vec{B}^2$$

Since the light is a wave, we can evaluate it in this way:

$$E = E_{max} \sin(kx - \omega t)$$

The average E will be:

$$\bar{E} = \frac{E_{max}}{\sqrt{2}}$$

The energy density associated to the electric field is $U_E = \frac{1}{2}\varepsilon_0 E^2$ and for the magnetic one: $U_B = \frac{1}{2\mu_0}B^2$.

Since we are in a wave, from the Maxwell's equations we can prove that $U_E = U_B$, so:

$$U = U_E + U_B = \varepsilon_0 E^2 = \frac{B}{\mu_0}$$

From this we get that:

$$E = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} B$$

$$E = cB$$

From the second law of Newton, we know that:

$$F = \frac{\Delta p}{\Delta t}$$

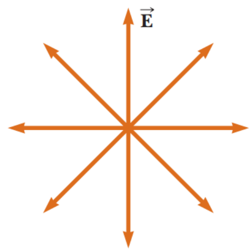
Let's assume that we have the light that incises on the surface and that all the energy is absorbed, so $I \cdot \Delta A \cdot \Delta t$ and $F = \frac{IA}{c}$, we get that:

$$\Delta p = F\Delta t = \frac{IA\Delta t}{c}$$

In order to reflect all the light, we'll need the double of the force. In real life we don't have these two extremes, so:

$$\frac{IA}{c} < F < \frac{2IA}{c}$$

Polarization



We know that a light beam is composed of two perpendicular waves, the electric field and the magnetic field. But since the vector of $\vec{E}$ is fixed only on a plane, there can be multiple light beams on the same plane with different directions. We call this type of beam, an **unpolarized beam**.

There exists a special device, called **polarizer**, that exploits a polarization foil made by long chains of hydrocarbon, that allows to polarize the light. Polarize the light means that we select only the beams whose $\vec{E}$ is fixed within a single plane. If you start from a unpolarized beam and then you polarize it, its intensity will be halved, because you start with all directions and you end only with 2 perpendicular axes.

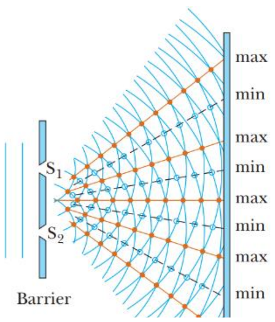
Another important device is the **analyser**, that takes a polarized beam and it changes its direction. In order to do that, the analyser selects only a part of the beam, depending on the angle selected. In particular we have:

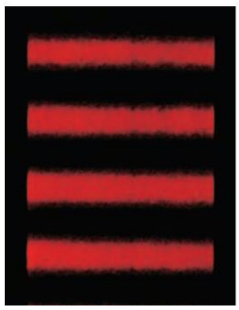
$$I = \frac{1}{2} I_0 \ (polarizer)$$

$$I = I_0 \cos^2 \theta \ \ (analyzer/Malus's\ law)$$

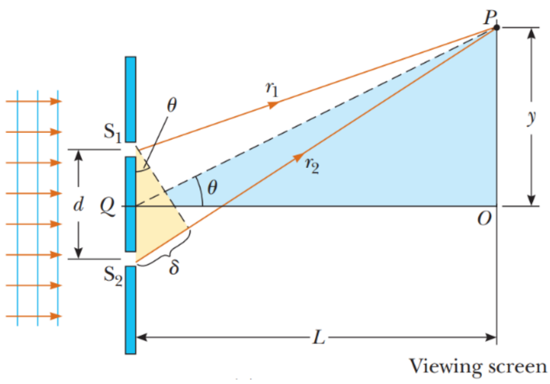
In the first law we are considering the Malus's law with an angle of $\frac{\pi}{4}$, since we're doing an average.

Young's double-slit experiment





In this experiment, there is a panel with two small slits and behind that another panel with neon lights that are turned on when some light hits them. The more light there is, the more intense they'll be. Now some light is let pass through the two slits and in that moment, we get an unexpected behaviour: indeed, the light didn't follow a straight path, but created some parallel bands of interference, called fringes, that alternate between light and dark. This phenomenon is called **diffraction** and it's given to the wave nature of the light.



To prove this, we have to consider a point P and see how the two beams behave. Let's call r_1, r_2 the paths that the two beams travel, d the distance between the two slits and L the distance between the two panels. We can clearly see that the second ray travels δ more distance than the first one. We define $\delta = r_1 - r_2 =$

$d \sin\theta$. Since the two beams are waves, we can have a constructive or a destructive interference in P depending on δ. In particular:

$$\delta = d \sin\theta = m\lambda \; (constructive\ interference)$$

$$\delta = d \sin\theta = \left(m + \frac{1}{2}\right)\lambda \; (destructive\ interference)$$

Where $m \in \mathbb{Z}$ (o $\mathbb{N}$?) and λ is the wave length. From these equations we can get the angular position of the fringes, so θ_{light} and θ_{dark}. Now we want to know the height of each band, so:

$$y_{light} = L \tan\theta_{light}$$

$$y_{dark} = L \tan\theta_{dark}$$

When θ is small, we'll have more clearly defined bands, and also, we can approximate the tangent to the sine, so:

$$y_{light} = \frac{Lm\lambda}{d}$$

Reflection and refraction

When a light changes its medium, then we have two phenomena: reflection and refraction. **Reflection** is when the incident ray doesn't pass through the medium but is indeed reflected with a specular angle θ'. Instead, **refraction** is when the light passes through the medium, but it changes its angle into θ_2. Each medium has its own refraction indexes n, that depends on the velocity v of the light inside that material:

$$n = \frac{c}{v}$$

For reflection we have that $\theta' = \theta$, while for refraction:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

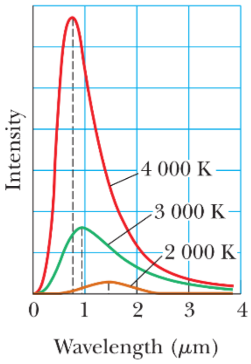
- If $n_1 = n_2$, then $\theta_1 = \theta_2$;
- if $n_1 > n_2$, then $\theta_1 > \theta_2$.

Black body

We define **black body** an idealized object that absorbs all the radiation and emits only thermal radiation λ. The importance of the black body is that, since it only emits thermal radiation, then we can find from this a relation between temperature and electromagnetic waves. Experimentally, Stefan-Boltzmann have determined that the radiation intensity is:

$$P = \sigma A T^4 e$$

Where σ is the Stefan-Boltzmann constant, A is the area, T the temperature and e the emissivity constant that, for black bodies, is equal to 1. From the graph we can see that there is a specific wavelength in which the intensity is maximum and also that the higher the temperature is, the higher intensity we'll have.



According to the classical physics, for small wavelengths, a black body should irradiate infinite I (ultraviolet catastrophe) and this is clearly not possible.

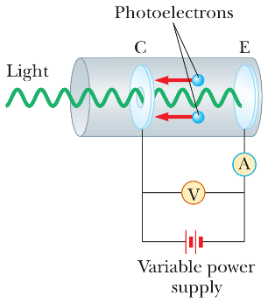
The solution was given by Plank, that stated that the energy carried by the light was discrete, so:

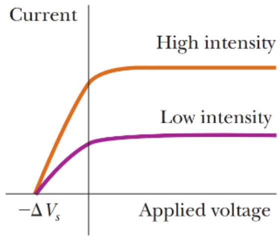
$$E_n = nhf$$

Where $n \in \mathbb{Z}$ and $h = 6{,}62 \cdot 10^{-34} \frac{J}{Hz}$.

Photoelectric experiment

Thanks to this experiment, Einstein gave meaning to the chunks of energy supposed by Plank. In an evacuated tube there are two plates, an emitter of electrons E and a collector of them C, connected to an ammeter, that measures the ΔV. When there's no light, the ammeter doesn't register anything, while if there's some light the current increases. In particular, below some intensity no electrons are emitted, if the intensity is raised then more electrons are emitted with the same velocity, until the plate gets saturated and if we raise the frequency, then the electrons will move with higher velocity. Since the electrons have some kinetic energy, the only way to stop them is to use $\Delta V < 0$, so that the polarity of the two plates is inverted (C becomes negative) and the electrons are repelled.





This explains that light is made of photons or quanta, each one carrying some energy.

By energy conservation, for an electron that is emitted with K_{max}, arriving at C at rest:

$$\Delta K + \Delta U = 0$$

$$-K_{max} - e(-\Delta V_S) = 0$$

$$K_{max} = e\Delta V_S$$

So, we can determine experimentally the K of the electron by measuring the ΔV at which the current dies. Some observations that we can take are that K_{max} is independent on the light intensity, but increases with the frequency, under some threshold, no electrons are emitted and the electrons are emitted almost instantaneously ($\Delta t \sim 10^{-9} s$).

From this we can deduce that the K_{max} depends on the energy of the photon hf and on the extraction energy that links e^- to material:

$$K_{max} = hf - \phi_{max}$$

$$f_{threshold} = \frac{\phi_{max}}{h}$$

if $hf < \phi_{max}$, then no electrons are emitted.

Modern physics

Compton effect

Thanks to this experiment Compton was able to prove that the light is made of quanta: an X ray was on some graphite, emitting, after the elastic scattering an electron and a light ray. According to the classical physics, the light emitted should have had the same λ, but in this case $\lambda' > \lambda$, meaning that the ray has lost some energy. Experimentally he found that:

$$\lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta)$$

De Broglie

All forms of matter have wave-like properties, indeed, if we start from the relation $p = \frac{E}{c}$ and $E = hf$, we'll get that:

$$\lambda = \frac{h}{p}$$

Where $p = mv$ for any particle.

This idea was later confirmed with the Davisson and Germer experiment, that showed a diffraction pattern in electrons too.

Double slit experiment with electrons

We start by doing the experiment with one hole, alternating the two holes (one electron in one hole, the second electron in the second one) and finally one single flux of electrons in both of the holes in the same way.

With one hole, we'll see a distribution of illumination in which in the part right in front of the slit, there will be the most illuminated part. But with two alternating holes we'll see the classical diffraction patterns, typical of the light (they're obtained by the sum of the two patterns of one hole). When we let the flux of electrons pass through the two holes simultaneously, we'll notice again the same pattern. This can be explained if the electron, when it's near the hole, "splits" in two and diffracts with itself.

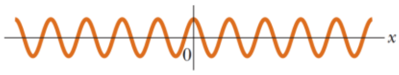
Heisenberg uncertainty principle

According to Heisenberg, we can't know simultaneously the position and the velocity of a particle in the microscopic world:

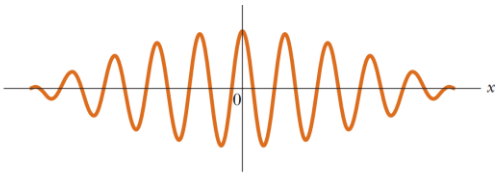
$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Where $\hbar = \frac{h}{2\pi}$.

If we consider a particle in which we know with some precision p, then by De Broglie, we'll have a range of λ and we'll lose all the informations about the position:



On the other side, considering an uncertainity on p, we'll have an image of a packet of a particle, like this:



If we continue adding more precision on x, we'll sum more wavelengths, resulting in a packet of length 0, so we have lost every information about the momentum, but we know exactly where the particle is.

Another form of the uncertainty principle uses the energy E and the time t:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

This principle can also be rewritten in terms of mass:

$$\Delta x \Delta p = \Delta x \Delta v M \geq \hbar$$

$$M = \frac{\hbar}{\Delta x \Delta v}$$

Probabilistic representation of a particle

Since we can't know exactly the position and the velocity of a particle, we have to consider a probabilistic representation. We'll consider the number of photons N over a specific volume V:

$$\frac{Probability}{V} \propto \frac{N}{V}$$

$\frac{N}{V}$ is proportional to the intensity, the intensity is proportional to the electric field squared and the electric field is proportional to the amplitude, so:

$$\frac{Probability}{V} \propto A^2$$

By De Broglie, we know that we can associate a wave also to matter, so in this case we'll have the square of the amplitude of this wave. This amplitude associated with the particle is the probability amplitude, or the wave function:

$$\psi(\vec{r_1}, \vec{r_2}, \dots \vec{r_j}, t) = \psi(\vec{r_j})e^{-i\omega t}$$

So here we are writing the function as a product of a complex function of time and a complex function of space. Since we're dealing with complex number, it's not so obvious that the square of a complex function will give always a real number. We can avoid this problem by doing the product between the conjugates. So, we can't measure ψ, put be can do it to $|\psi|^2$, called the **probability density**. This is the relative probability per unit volume that the particle will be found at any given point in the volume.

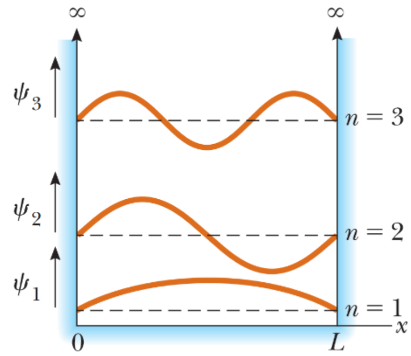
When we are considering a particle in a box, we have to take in account that the particle can be considered as a wave. When it bounces back the wave can overlap, so we have different probabilities inside the box, getting this type of wave function:

$$\psi(x) = A\sin(kx)$$

Where $k = \frac{2\pi}{\lambda}$ and λ is the De Broglie wavelength associated to the particle. When we have a box, we have a fixed x, so L. We know that at the borders of the box, so at 0 and at L, the sine must give 0. This can be possible only when:

$$\frac{2\pi L}{\lambda} = n\pi$$

In the boundary case we get: $y(x) = A\sin\left(\frac{n\pi x}{L}\right)$.



From this we can also get that the momentum and the energy are quantized too:

$$p = \frac{h}{\lambda} = \frac{hn}{2L}$$

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{h^2}{8mL}n^2$$

We can't have $n = 0$, because then we would know both E and p, and, by the uncertainty principle, this is not possible. So, in quantum physics, you can't have still objects.

In this way we discover for example that the electron lays on levels of energy. Every time that an electron jumps from a level to another it releases or absorbs a photon with a different wavelength:

$$\Delta E = hf = \frac{hc}{\lambda}$$

$$\lambda = \frac{hc}{\Delta E}$$

Where ΔE is given by the difference of energy at the two different levels, so by using the corresponding n in this formula: $E = \frac{h^2}{8mL}n^2$.

The Schrödinger Equation

Thanks to Schrödinger, we have the equation to the De Broglie waves:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

This equation applies to a particle of mass m confined to moving along the x axis and interacting with its environment through a potential energy function $U(x)$. This equation is conservative, so, $K + U = E$, with E constant. If we know $U(x)$, we can calculate $\psi(x)$.

By solving the differential equation we know that one possible solution for this equation is $\psi = Ae^{i(kx-\omega t)}$.

Tunneling effect

Let's consider a squared potential energy function divided in three zones: in zone I and III $U = 0$, while in zone II it has some value. Let's assume that we have a moving particle from I to III with some energy $E < U$. According to classical physics, the particle cannot move, but, in quantum physics, because of the uncertainity principle, the particle potentially could be everywhere, just with a different probability. By analyzing the Schrödinger equations at II, we understand that we're violating the energy conservation, because there's a nonzero solution. So we can talk about two different situations: the particle can either be transmitted (T) or reflected (R). In any case, since they're probabilities, we know that $T + R = 1$.

$$T \sim e^{-2CL}$$

$$C = \frac{\sqrt{2m(U-E)}}{\hbar}, \qquad \left[\frac{1}{m}\right]$$

Where L is the length of the barrier

Radioactivity

Description and half-life

After years of experiments, it has been discovered that all the elements, after an amount of time that varies from element to element, emits some type of particles, decaying then in other smaller elements. If we want to represent an element, we can do it by putting on bottom left the number of protons (Z) and on top left the number of protons and neutrons (A). We can study mathematically the radiation; indeed, the rate of decay is:

$$\frac{dN}{dt} = -\lambda N$$

So, the rate of change of nuclei decreases with time. If we integrate, we get:

$$N = N_0 e^{-\lambda t}$$

Where N_0 is the number of undecayed nuclei at $t = 0$.

From these two equations we can get other two important parameters, the activity and the half-life:

The activity refers to the decay rate, so to the number of decays per second:

$$R = \left|\frac{dN}{dt}\right| = \lambda N = \lambda N_0 e^{-\lambda t} = R_0 e^{-\lambda t}$$

The half-life is the time in which we pass from N_0 nuclei to $\frac{N_0}{2}$:

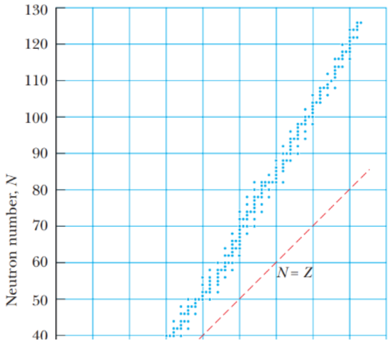
$$\frac{N_0}{2} = N_0 e^{-\lambda T_{1/2}}$$

Where:

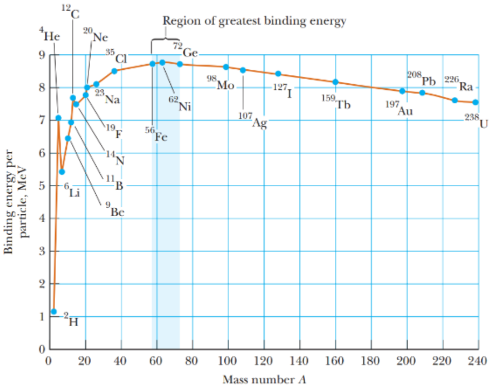
$$T_{1/2} = \frac{\ln 2}{\lambda}$$

An important isotope is the carbon-14, because his half-life is about 5700 years and it's useful to estimate the date of death of an object. Indeed, in an alive creature, the amount of ^{14}C is constant, because it's exchanged with the environment, while, when dead, the ^{14}C is not exchanged anymore and it start to decay.

Nuclear binding energy



In general, two nuclei undergo two opposite forces: the strong force that tends to keep them together, and the electromagnetic force that tends to separate them. The **Coulomb's barrier** is the electric potential energy below which a nucleus will release some energy. If a particle hits the nucleus with an energy that is above this barrier, then the nucleus will absorb it. Remember that, since we're dealing with particles, the exchange of energy occurs in packets, not smoothly.



This reasoning is true also for the protons and neutrons inside a single nucleus. For light atoms, we have that the strong force is enough to compensate the Coulomb's one, so we need the same number of neutrons and protons. For bigger elements (after around 40 protons) we need more neutrons, because the strong force acts on them, but not the Coulomb's one. After $Z = 83$, the strong force starts to not be enough and the elements don't have stable nuclei.

So, inside the nucleus there is a binding energy. This energy is such that the sum of all the energies of the separate components is not equal to the total one. In particular we have that:

$$B\left({}^A_Z X\right) = \left[ZM(H) + Nm_n - M\left({}^A_Z X\right)\right] \times \ 931.494 \text{ MeV} > \text{u}$$

Where $M(H)$ is the mass of the hydrogen, m_n, the mass of the neutron and $M\left({}^A_Z X\right)$ the total mass.

Out of around 3000 elements and isotopes, only 266 of them are stable.

Decays

Alpha decay: In an alpha decay, an element loses an alpha particle, a nucleus of helium with two neutrons:

$${}^A_Z X \to {}^{A-2}_{Z-2} Y + {}^4_2 He$$

This type of decay can be easily blocked by a paper foil.

Beta decay: $\bar{\nu}$ is an anti-neutrino and e^+ is a positron:

$${}^A_Z X \to {}_{Z+1}{}^A Y + e + \bar{\nu}$$

$${}^A_Z X \to {}_{Z-1}{}^A Y + e^+ + \nu$$

In these type of transformations, we have the same mass and electric charge and the leptonic number is maintained the same, thanks to the neutrinos.

In the first case we are gaining a proton and we are losing a neutron, while in the second case it's the opposite. This is because:

$$n \to p + e + \bar{\nu} \to \bar{\beta}$$

$$p \to n + e^+ + \nu \to \beta^+$$

This type of decay can be blocked by a thin metal foil.

Gamma decay: X^* is an excited nuclide and γ is a high-energy photon:

$${}^A_Z X^* \to {}^A_Z Y + \gamma$$

This type of decay is hard to block because we need walls of concrete.

Nuclear reactor

In a nuclear reactor there can be two types of nuclear reactions: fission and fusion. In the fusion the aim is to get energy by the union of two atoms, but still nowadays is not convenient because we would spend more energy to fuse two atoms than the gained one. In the fission it's the opposite, because we want to separate two atoms. For example, in the unstable isotope Uranium-235, we launch a neutron on an atom of this type of uranium, that becomes Uranium-236 that then splits and releases energy K_r, γ rays, neutrinos, nuclear binding energy and another neutron, that will hit another atom of Uranium-235, letting the chain reaction start again.

Mixed – Interesting thoughts

Schwarzschild radius: this radius studies the spacetime singularity and is so defined:

$$r_S = \frac{2G}{c^2}M$$

From the uncertainty principle ($M = \frac{\hbar}{\Delta x \Delta v}$), we get:

$$r_S = \frac{2G}{c^2}\frac{\hbar}{r_S \mathrm{c}}$$

$$r_S = \sqrt{\frac{2G\hbar}{c^3}} \sim 1{,}6 \cdot 10^{-35} m$$

Below this distance, the space-time has a singularity.

Every type of force has a relativistic intensity, a range and a relative particle:

Force	Relativistic int.	Range	Particle
Strong	1	$1 fm$ (the size of a proton)	Gluon
Electromagnetic	10^{-2}	$10^{-3} fm$	Photons
Weak	10^{-4}	$10^{-3} fm$	Z, W bosons
Gravitational	10^{-41}	$10^{-3} fm$	???