INTRODUCTION

A quick disclaimer before reading the notes: they were taken by me during the lectures, they do not replace the professor's work and are not sufficient for passing the exams. Moreover they might contain mistakes, so please double check all that you read. The notes are freely readable and can be shared (always remembering to credit me and to not obscure this page), but **can't** be modified. Thank you and hope these notes are useful!

-Francesca Cinelli

UNIT 2

CALCULUS I

UNIT 2

24 Feb

Taylor Expansion

$$f(x) \rightsquigarrow \sum_{k=0}^{n} a_k (x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n$$

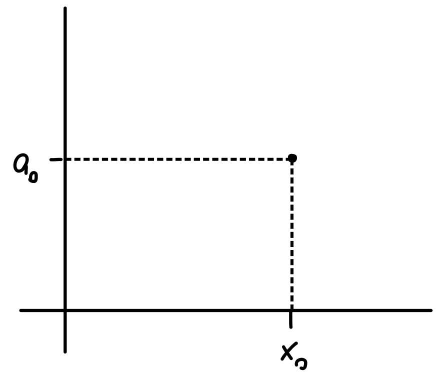
↳ approximation of a function

if $n = 1$

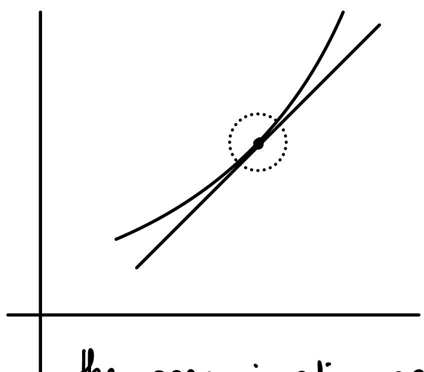
$$f(x) \rightsquigarrow a_0 + a_1(x-x_0)$$

fixed

unknown



First derivative $\rightarrow f'(x) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$



the approximation can only be around a point x_0

$$f(x_0) + a_1(x-x_0) \quad \rightarrow \quad a_0 = f(x_0)$$

↳ all the lines passing through $(f(x_0), x_0)$

but what line is the best for approximation?

↳ tangent line

↳ $a_1 = f'(x_0)$ $\quad \boxed{y = f(x_0) + f'(x_0)(x - x_0)}$

↳ by best we mean a line that in point x_0 has the least ERROR

$\rightarrow \text{Error} = f(x) - [f(x_0) + a_1(x - x_0)]$

function approximated line

$$\lim_{x \to x_0} f(x) - [f(x_0) + a_1(x - x_0)] = 0$$ -> obvious

$$\lim_{x \to x_0} \frac{f(x) - [f(x_0) + a_1(x - x_0)]}{x - x_0} = \left[\frac{0}{0}\right]$$ -> why complicate?

↳ if limit = $\frac{0}{0}$ we need to know who is faster

↳ if limit = 0 -> top is going faster

↳ if limit = 1 -> same speed

↳ we want this to happen

how fast the difference is going to 0

$$\lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\text{incremental ratio}} - \frac{a_1 \cancel{(x - x_0)}}{\cancel{x - x_0}} = f'(x_0) - a_1$$

we want the limit to equal zero:

$f'(x_0) - a_1 = 0$

↳ if $a_1 = f'(x_0)$ then condition is satisfied

Taylor Polynomial

- First degree

  if $\exists f'(x_0)$

$$P_1(x, x_0) = f(x_0) + f'(x_0)(x - x_0)$$

Remainder

$$R_1(x, x_0) = f(x) - P_1(x, x_0)$$ -> "error"

$$\lim_{x \to x_0} \frac{R_1(x, x_0)}{(x - x_0)^1} = 0$$ -> by proof

↳ since top is going to 0 faster than botton

- Second degree

if $\exists$ $f'(x_0), f''(x_0)$

$$P_2(x, x_0) = \underbrace{f(x_0)}_{= a_0} + \underbrace{f'(x_0)}_{= a_1}(x - x_0) + \underbrace{a_2}_{= f''(x_0)\,?}(x - x_0)^2$$

Proof

$$R_2(x, x_0) = f(x) - P_2(x, x_0)$$

$$\lim_{x \to x_0} \frac{R_2(x, x_0)}{(x - x_0)^2} = 0$$ → is this true? does the remainder go faster than $(x - x_0)^2$ to 0?

$$\lim_{x \to x_0} \frac{f(x) - P_2(x, x_0)}{(x - x_0)^2} = \lim_{x \to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x - x_0) + a_2(x - x_0)^2]}{(x - x_0)^2}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} - \frac{a_2 \cancel{(x - x_0)^2}}{\cancel{(x - x_0)^2}} = \left[\frac{0}{0}\right]$$

using DH →

$$= \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{2(x - x_0)} - a_2$$

by definition:

$$f''(x) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

$$= \frac{f''(x_0)}{2} - a_2$$

→ we want this to be $= 0$

$$P_2(x, x_0) = f(x_0) + f'(x_0)(x - x_0) + \underbrace{\frac{f''(x_0)}{2}}_{= a_2}(x - x_0)^2$$

- Third degree

if $\exists\ f'(x_0), f''(x_0), f'''(x_0)$

$$P_3(x, x_0) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + a_3(x-x_0)^3$$

$\hookrightarrow = \frac{f'''(x_0)}{3}$?

$$R_3(x, x_0) = f(x) - P_3(x, x_0)$$

$\lim_{x \to x_0} \frac{R_3(x, x_0)}{(x-x_0)^3} = 0$ -> is this true?

$\hookrightarrow$ use same proof as before

$$\lim_{x \to x_0} \frac{f(x) - P_3(x, x_0)}{(x-x_0)^3} = \lim_{x \to x_0} \frac{f(x) - \left[f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + a_3(x-x_0)^3\right]}{(x-x_0)^3}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{f''(x_0)}{2}(x-x_0)^2}{(x-x_0)^3} - \frac{a_3\cancel{(x-x_0)^3}}{\cancel{(x-x_0)^3}}$$

DH

$$\hookrightarrow = \lim_{x \to x_0} \frac{f'(x) - f'(x_0) - f''(x_0)(x-x_0)}{3(x-x_0)^2} - a_3$$

DH

$$\hookrightarrow = \lim_{x \to x_0} \frac{f''(x) - f''(x_0)}{3 \cdot 2(x-x_0)} - a_3$$

by definition:

$$f'''(x) = \lim_{x \to x_0} \frac{f''(x) - f''(x_0)}{(x-x_0)}$$

- we apply DH until we get incremental ratio or make denominator go away

$$= \frac{f'''(x)}{6} - a_3$$

$\hookrightarrow$ we want this to be 0

$$P_3(x, x_0) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$\hookrightarrow = a_3$

N degree

if $\exists\, f^k(x_0) \quad k = 1 \dots n$

$$\boxed{P_n(x, x_0) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x - x_0)^k} \longrightarrow \text{Taylor polynomial}$$

$$\boxed{R_n(x, x_0) = f(x) - P_n(x, x_0)} \;\text{-->}\; \text{Peano remainder}$$

if

-> then:

$$\lim_{x \to x_0} \frac{R_n(x, x_0)}{(x - x_0)^n} = 0$$

$$\boxed{f(x) = P_n(x, x_0) + R_n(x, x_0)} \;\text{-->}\; \text{Taylor expansion}$$

The very important part is that we are looking in a neighbourhood of x_0 $(x_0 - \delta\,;\, x_0 + \delta)$ so the approximation is there not on all the function.

28 Feb

Mc-laurin = Taylor expansion when $x_0 = 0$

Peano Remainder - little os

$$R_n(x, x_0) = o(x - x_0)^n$$

little o $\neq$ big O, used for finite numbers

└ used to highlight a behaviour, it indicates comparison

└▷ $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$ -> $f(x) = o(g(x))$

$f(x)$ ↓ "faster" $\quad$ $g(x)$ ↓ "slower"

-> if $\lim\limits_{x \to x_0} R_n(x, x_0) = 0$ and

$\lim\limits_{x \to x_0} (x - x_0)^n = 0$ and

$\lim\limits_{x \to x_0} \dfrac{R_n(x, x_0)}{(x - x_0)^n} = 0$ then

$$R_n(x, x_0) = o((x - x_0)^n)$$

$$\lim_{x \to 0} \frac{x^2}{1 - e^x} = \lim_{x \to 0} \frac{2x}{-e^x} = 0$$

↳ this goes to zero faster

Algebra with little o

1. $o(x^n) \pm o(x^n) = o(x^n)$ —> Proof 1

$$\lim_{x\to 0} \frac{o(x^n) \pm o(x^n)}{x^n} = 0 = \lim_{x\to 0} \underbrace{\frac{o(x^n)}{x^n}}_{\downarrow\, 0} \pm \underbrace{\frac{o(x^n)}{x^n}}_{\downarrow\, 0}$$

2. $o(x^n) \pm o(x^m) = o(x^p)$

   $p = \min\{m, n\}$

Proof 2

$n < m$

$$\lim_{x\to 0} \underbrace{\frac{o(x^n)}{x^m}}_{\substack{\downarrow \\ \text{don't know precisely}}} \pm \underbrace{\frac{o(x^m)}{x^m}}_{\downarrow\, 0}$$

can't choose max on bottom but only min so surely will be $= 0$

3. $o(x^n) \cdot o(x^m) = o(x^{n+m})$

Proof 3

$$\lim_{x\to 0} \underbrace{\frac{o(x^n)}{x^n}}_{\downarrow\, 0} \cdot \underbrace{\frac{o(x^m)}{x^m}}_{\downarrow\, 0} = 0$$

4. $\dfrac{o(x^m)}{o(x^n)} = o(x^{m-n})$, $m > n$

   $= o(1)$, $m = n$

5. $c \cdot o(x^n) = o(x^n)$

6. $g(x) \cdot o(x^n) = o(x^n)$,

   g is bounded around $x = 0$

7. $[o(x^n)]^m = o(x^{n \cdot m})$

8. $x^n \cdot o(x^m) = o(x^{n+m})$

Exponential and Taylor

$x_0 = 0$

$f(x) = e^x$ $\quad P_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$ $\quad P_n(x, x_0) = \sum_{k=0}^{n} \frac{x^k}{k!}$

$R_4(x, x_0) = o(x^4)$ $\quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$$

Remark

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \,?$$ does it make any sense?

Exercise

$$f(x) = \begin{cases} e^{\frac{1}{x^2}} & x \neq 0 \\ 0 & \end{cases}$$

Demonstrating special limits with Taylor

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Proof with Taylor

replace with that ↑ up to where you need it

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{x + x^2 + x^3 + o(x^3)}{x} = \lim_{x \to 0} 1 + x + x^2 + \underbrace{\frac{o(x^3)}{x}}_{0} = 1$$

- we can remove the denominators because they are just numbers so multiplied by 0 will = 0

but this is even shorter:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{x + o(x)}{x} = \lim_{x \to 0} 1 + \underbrace{\frac{o(x)}{x}}_{0} = 1$$

- we simplified the 1 with -1

Logarithm and Taylor

$x_0 = 1$

$$\log x = \log 1 + \left(\frac{1}{x}\right)\Big|_{x=1} (x-1) + \left(-\frac{1}{x^2}\right)\Big|_{x=1} \frac{(x-1)^2}{2} + \frac{\cancel{2}}{x^3}\Big|_{x=1} \frac{(x-1)^3}{{}^{3}\cancel{3!}} \left\{ + \left(-\frac{\cancel{2 \cdot 3}}{x^4}\right)\Big|_{x=1} \frac{(x-1)^4}{{}^{4}\cancel{4!}} \right\}$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \left\{ - \frac{(x-1)^4}{4} \right\}$$

$$\log x = \sum_{k=1}^{n} (-1)^{k-1} \frac{(x-1)^k}{k} + o(x-1)^n \implies x_0 = 1$$

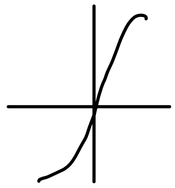
- K starts from 1 because else we would have 0 at denominator

$$\log(x+1) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + o(x^n) \implies x_0 = 0$$

3 March

Sine and Taylor

$x_0 = 0$



odd around 0

$$\sin x = \underset{0}{\sin 0} + \underset{1}{\cos x}\Big|_{x=0} (x) + \underset{0}{(-\sin x)}\Big|_{x=0} \frac{x^2}{2} + \underset{-1}{(-\cos x)}\Big|_{x=0} \frac{x^3}{3!} + \underset{1}{\cos x}\Big|_{x=0} \frac{x^5}{5!} + o(x^6)$$

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$$

Lagrange Remainder - another way to represent remainder

$$f^{n+1}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!}$$

between x and x_0

ex. for sine, if $n=5$

$$-\sin \xi \frac{x^6}{6!}$$

Note:

Lagrange thm

$\forall x, x_0 \ \exists \xi$ between x and x_0, $\frac{f(x)-f(x_0)}{x-x_0} = f(\xi)(x-x_0)$

Cosine and Taylor

$x_0 = 0$

$$\cos x = \underset{1}{\cos 0} - \underset{0}{\sin x}\Big|_{x=0} x - \underset{-1}{\cos x}\Big|_{x=0} \frac{x^2}{2}$$

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$$

Lagrange remainder

$$(-1)^{n+1} \sin \xi \frac{x^{2n+1}}{(2n+1)!}$$

Power function and Taylor

$f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$

$x_0 = 0$

$$(1+x)^\alpha = 1 + \alpha(1+x)^{\alpha-1}\Big|_{x=0} x + \alpha(\alpha-1)(1+x)^{\alpha-2}\Big|_{x=0} \frac{x^2}{2}$$

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} \cdot x^k + o(x^n)$$

Newton binomial:

this simplifies with top at some point

$$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha-k)!} \qquad \binom{\alpha}{0} = 1$$

Lagrange remainder

$$\binom{\alpha}{n+1}(1+\xi)^{\alpha-(n+1)} x^{n+1}$$

Remarks

$$f(x) = (1+x)^\alpha \qquad \frac{f^{k}(0)}{k!} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} = \binom{\alpha}{k}$$

if $\alpha = n \in \mathbb{N}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \text{, remainder} = 0 \qquad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$= \sum_{k=0}^{n-2} \dots + R_{n-2}$$

ex. $\alpha = 3.5$

$$(1+x)^{3.5} + 3.5x + (3.5)(2.5)\frac{x^2}{2} + (3.5)(2.5)(1.5)\frac{x^3}{3!}$$

ex. $\alpha = -1$

$$\frac{1}{1+x} = \binom{-1}{0}x^0 + \binom{-1}{1}x + \binom{-1}{2}x^2 = 1 - x + x^2$$

$$\binom{\alpha}{0} = 1 \qquad \frac{(-1)(-1)}{1!} \qquad \frac{(-1)(-1-1)}{2!}$$

$$\frac{1}{1+x} = \sum_{k=0}^{n} (-1)^k x^k \qquad \binom{-1}{k} = (-1)^k$$

Square root and Taylor

$\alpha = \frac{1}{2}$ $\qquad f(x) = \sqrt{1+x}$

$$\binom{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1} \qquad \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2} = -\frac{1}{8}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}(1+x)^{-\frac{1}{2}}\Big|_{x=0} x + \left(-\frac{1}{4}(1+x)^{-\frac{3}{2}}\right)\Big|_{x=0}\frac{x^2}{2} +$$

$$+ \frac{3}{8}(1+x)^{-\frac{5}{2}}\Big|_{x=0}\frac{x^3}{3\cdot 2} = 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2} + \frac{3}{8}\frac{x^3}{3\cdot 2}$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

Composite functions and Taylor

$n = 2 \quad x_0 = 0$

$$e^x - \sqrt{1+2x} = \underbrace{1 + x + \frac{x^2}{2}}_{e^x = \sum_{k=0}^{2} \frac{x^k}{k!}} + o(x^2) - \underbrace{\left(1 + \frac{2x}{2} - \frac{(2x)^2}{8} + o((2x)^2)\right)}_{\sqrt{1+y} = 1 + \frac{y}{2} - \frac{y^2}{8} + o(y^2)} = x^2 + \underbrace{o(x^2) - o(x^2)}_{o(x^2)}$$

just sum up
the 2 expansions

Product of Taylor

taylor P. of f(x)
" " " g(x)

$$f(x)\cdot g(x) = [P_n(x)+o(x^n)][Q_n(x)+o(x^n)]$$

$$= P_n(x)\cdot Q_n(x) + \underbrace{\underbrace{P_n(x)\cdot o(x^n)}_{o(x^n)} + \underbrace{\underbrace{Q_n(x)\cdot o(x^n)}_{o(x^n)} + \underbrace{o(x^n)\cdot o(x^n)}_{o(x^{2n})}}_{o(x^n)}}_{o(x^n)}$$

ex.

$$e^x\cdot\sqrt{1+2x} = \left(1+x+\frac{x^2}{2}+o(x^2)\right)\cdot\left(1+x-\frac{x^2}{2}+o(x^2)\right)$$

$$= \left(1+x+\frac{x^2}{2}\right)\left(1+x-\frac{x^2}{2}\right)+o(x^2) \quad \rightarrow (a+b)(a-b)=a^2-b^2$$

$$= (1+x)^2-\left(\frac{x^2}{2}\right)^2+o(x^2)$$

$$= 1+2x+x^2-\frac{x^4}{4}+o(x^2)$$

$$= 1+2x+x^2+o(x^2) \quad \checkmark$$

• With short formula you are ignoring stuff that is in the remainder

these are same
1 changes and 1 isn't present

$n=4 \quad x_0=0$

$$e^x\cdot\sqrt{1+2x} = 1+f'(x)\Big|_{x=0}x+f''(x)\Big|_{x=0}\frac{x^2}{2}+f'''(x)\Big|_{x=0}\frac{x^3}{3!}+f^{IV}(x)\Big|_{x=0}\frac{x^4}{4!}$$

$$=1+2x+\cancel{2}\frac{x^2}{\cancel{2}}+\overset{2}{\cancel{4}}\frac{x^3}{3\cdot\cancel{2}}-\cancel{4}\frac{x^4}{\cancel{4}\cdot3\cdot2} = 1+2x+x^2+\frac{2}{3}x^3-\frac{x^4}{6}+o(x^4)$$

• see p.1 of ex. for derivatives

ex.

$$\underset{\substack{\parallel\\ f\circ g(x)}}{e^{\sqrt{1+x}-1}} = 1+\left(\frac{x}{2}-\frac{x^2}{8}+o(x^2)\right)-\underbrace{\frac{\left(\frac{x}{2}-\frac{x^2}{8}+o(x^2)\right)^2}{2}}+\underbrace{o\left(\frac{x}{2}-\frac{x^2}{8}+o(x^2)\right)^2}_{o(x^2)}$$

$$= 1+\frac{x}{2}-\frac{x^2}{8}+o(x^2)+\frac{x^2}{8}+o(x^2)$$

$$= 1+\frac{x}{2}+o(x^2) \quad \checkmark$$

$$f(y)=e^y = 1+y+\frac{y^2}{2}+\frac{y^3}{3!}+o(x^3)$$

$$g(x)=\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+o(x^2)$$

$$\sqrt{1+x}-1 = \frac{x}{2}-\frac{x^2}{8}+o(x^2)$$

7 Mar - Wed 3:30 tutor

Solving limits with Taylor

ex. 1

$$\lim_{x\to 0} \frac{1-e^{3x^2}}{x\sin x} = \frac{0}{0}$$

$$\to \lim_{x\to 0} \frac{-3x^2 - \frac{9x^4}{2} + o(x^4)}{x^2 - \frac{x^4}{3!} + o(x^5)} = \lim_{x\to 0} \frac{-3 - \frac{9}{2}x^2 + \frac{o(x^4)}{x^2}}{1 - \frac{x^2}{3!} + \frac{o(x^5)}{x^2}} = -3$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \dots + o(y^n)$$

$$y = 3x^2 \to 1 + 3x^2 + \frac{(3x^2)^2}{2} + o(x^4)$$

$$\sin x = x - \frac{x^3}{3!} + o(x^4)$$

$$x\sin x = x^2 - \frac{x^4}{3!} + o(x^5)$$

elementary formulas you need to compute the limit

ex. 2

$$\lim_{x\to 0} \frac{\cos 2x - \cos x}{x^2}$$

$$\to \lim_{x\to 0} \frac{-2x^2 + \frac{x^2}{2} + o(x^3)}{x^2} = -\frac{3}{2}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)$$

$$\cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + o(x^5)$$

$$\text{difference} = \cancel{1} - 2x^2 \cancel{-1} + \frac{x^2}{2}$$

ex. 3

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\tan x} = \infty - \infty$$

x^α, $\alpha = -1$ × → function not defined at $x = 0$

$$\lim_{x\to 0} \frac{\tan x - x}{x\tan x} ✓ = \lim_{x\to 0} \frac{o(x^2)}{x^2 + o(x^3)} = \lim_{x\to 0} \frac{\frac{o(x^2)}{x^2}}{1 + \frac{o(x^3)}{x^2}} = 0$$

$$\tan x = 0 + \frac{1}{\cos^2 x}\Big|_{x=0} x + o(x^2) = x + o(x^2)$$

$\frac{1}{\cos^2 x}\Big|_{x=0} = 1$

is odd

$$\tan x - x = o(x^2)$$

$$x\tan x = x^2 + o(x^3)$$

ex. 4

$$\lim_{x\to+\infty} x^3\left(\frac{1}{x}-\sin\left(\frac{1}{x}\right)\right) \overset{y=\frac{1}{x}}{=} \lim_{y\to 0}\frac{1}{y^3}(y-\sin y) = \lim_{y\to 0}\frac{y-\sin y}{y^3} = \lim_{y\to 0}\frac{\frac{y^3}{3!}+o(y^4)}{y^3} = \frac{1}{6}$$

$$\sin y = y - \frac{y^3}{3!} + o(y^4)$$

ex. 5

$$\lim_{x\to 0}\frac{\log(1+x)\arctan x - x\sin x}{\arctan x - 1 - \log(1+x) + \cos x}$$

$$\arctan x = 0 + \left.\frac{1}{1+x^2}\right|_{x=0} x + o(x^2) = x - 2\frac{x^3}{3!} + o(x^4) = x - \frac{x^3}{3} + o(x^4)$$

is odd

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \qquad f'' = \frac{-2x}{(1+x^2)^2} \qquad f''' = \left.\frac{-2(1+x^2)^2 + 2x\cdot 2(1+x^2)2x}{(1+x^2)^4}\right|_{x=0} = -2$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)$$

$$x\sin x = x^2 - \frac{x^4}{3!} + o(x^5)$$

$$\log(1+x)\arctan x = (\cancel{x - \frac{x^2}{2} + o(x^2)})(\cancel{x + o(x^2)}) = \cancel{x^2 + o(x^2)}$$

$$\to x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^4) \qquad \to x - \frac{x^3}{3} + o(x^4) \qquad \to x^2 - \frac{x^3}{2} - \frac{x^3}{2} + o(x^3)$$

we need another derivative

$$\frac{\cancel{x^2} + o(x^2) - \cancel{x^2} \to x^2 - \frac{x^3}{2} - x^2 + o(x^3)}{\cancel{x} + o(x^2) \cdot \cancel{1} - \cancel{x} + \cancel{\frac{x^2}{2}} + o(x^2) + \cancel{1} - \cancel{\frac{x^2}{2}} + o(x^3)} = \frac{o(x^2)}{o(x^2)}$$

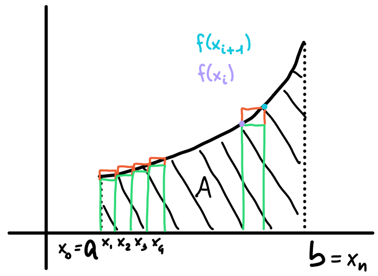
$$\hookrightarrow x - \frac{x^3}{3} + o(x^4) - 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + o(x^3) + 1 - \frac{x^2}{2} + o(x^3)$$

$$= \frac{-\frac{x^3}{2} + o(x^3)}{-\frac{2}{3}x^3 + o(x^3)} = \frac{3}{4}$$

10 March

Integrals

- geometric interpretation
  - way to represent area
- $f[a;b] \to \mathbb{R}$ (bounded function)
- $(a;b)$ (bounded interval) $\to -\infty < a < b < +\infty$



use what we know
- fill my area with rectangles

Reiman integrals
- definite
- indefinite

divide interval into equal parts $\to I_i = [x_i, x_{i+1}]$

- $\frac{b-a}{n} \to x_i = a + \frac{i}{n}(b-a)$
- $x_i - x_{i+1} = a + \frac{1}{n}(b-a) - a - \frac{1}{n}(b-a) = \frac{b-a}{n}$

$$f(x_i) = \min_{x \in [x_i, x_{i+1}]} f(x) \rightsquigarrow \inf_{x \in I_i} f(x)$$

$$f(x_{i+1}) = \max_{x \in [x_i, x_{i+1}]} f(x) \rightsquigarrow \sup_{x \in I_i} f(x)$$

$$\underset{\substack{\downarrow \\ \text{monotone increasing sequence}}}{s_n(f, I)} = \sum_{i=0}^{n-1} \underbrace{f(x_i)(x_{i+1} - x_i)}_{Area(R_i)} \leq Area(A) \leq \sum_{i=0}^{n-1} \underbrace{(x_{i+1} - x_i) f(x_{i+1})}_{Area(R_{i+1})} = \underset{\substack{\downarrow \\ \text{monotone decreasing sequence}}}{S_n(f, I)}$$

↓

if we reduce the base we can get a better approximation

↓

decrease area from above and increase area from below

if we get the same number when passing thru the limit then that is the area, the integral exists and it is this number

$= \sup s_n(f, I_i)$ $\quad$ $= \inf S_n(f, I_i)$

$$s(f, P) = \sum \inf_{x \in (x_i, x_{i+1})} f(x)(x_{i+1} - x_i) \rightsquigarrow \sup_P s(f, P) = s(f) \to \text{lower sum}$$

do not depend on P (partition)

$$S(f, P) = \sum \sup_{x \in (x_i, x_{i+1})} f(x)(x_{i+1} - x_i) \rightsquigarrow \inf_P S(f, P) = S(f) \to \text{upper sum}$$

$P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\} \to$ partition of interval

Def: $f(a,b) \to \mathbb{R}$ bounded (bounded interval) is Reiman integrable if

$$s(f) \underset{\substack{\downarrow \\ \text{identically equal}}}{\equiv} S(f) \underset{\substack{\downarrow \\ \text{definition}}}{:=} \int_a^b f(x)\,dx$$

lower R. integral $s(f) \underset{\substack{\downarrow \\ \text{eventually coincide } (=)}}{\leq} S(f)$ upper R. integral

ex.

$f(x) = x^2$

$a = 0$
$b = 1$
$x_i = \frac{i}{n}$
$i = 0, \dots, n$
$f(x_i) = \left(\frac{i}{n}\right)^2$

$$\sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^2 (x_{i+1} - x_i) \le \text{Area}(A) \le \sum_{i=0}^{n-1} \frac{1}{n}\left(\frac{i+1}{n}\right)^2$$

$\hookrightarrow \frac{i+1}{n} - \frac{i}{n} = \frac{1}{n}$

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{i^2}{n^2} \le \text{Area}(A) \le \frac{1}{n^3} \sum_{i=0}^{n-1} (i+1)^2$$

$$\frac{1}{n^3} \sum_{i=0}^{n-1} i^2$$

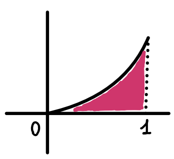
$$\boxed{\sum_{h=1}^{n} h^2 = \frac{n(n+1)(2n+1)}{6}}$$ -> general formula

$$\frac{(n-1)n(2n-1)}{6n^3} \le \text{Area}(A) \le \frac{n(n+1)(2n+1)}{6n^3}$$

$\frac{1}{3} = \sup$ $\qquad$ $\frac{1}{3} = \inf$

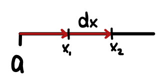
-> the two limits coincide

$$\int_0^1 x^2\, dx = \frac{1}{3} = A$$



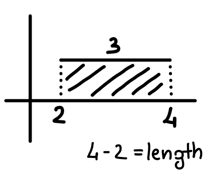
the very small intervals - increments

integral is like a sum of



Integral of a constant function

$$\int_2^4 3\, dx = 6$$



constant

area can be negative

└ we can compute the integral of a negative function

$$s(f, P) = \sum_{i=0}^{n} \overset{3}{\inf} (x_{i+1} - x_i) = 3 \cdot 2 = 6$$

$$S(f, P) = \sum_{i=0}^{n} \overset{3}{\sup} \underbrace{(x_{i+1} - x_i)}_{=2} = 3 \cdot 2 = 6$$

- general formula: $\boxed{\int_a^b c\, dx = c(b-a)}$

Dividing integrals into more

$f[0,2] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in [0,1) \\ 1/2 & x = 1 \\ 0 & x \in (1,2] \end{cases} \qquad \int_0^2 f(x)dx = \int_0^1 1dx = \int_1^2 0 dx = 1$$

$\left\{ \frac{1}{2} \right.$ is a point, it does not have an area $\left. \right\}$

Additivity of the integral - with respect to the interval of integration

- $f[a,b] \to \mathbb{R}$
- Rieman integrable $\Rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $a < c < b$

Additivity of the integral - with respect to the function

- f, g
- Rieman integrable in (a,b) $\Rightarrow \int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Linearity

- $\lambda \in \mathbb{R} \Rightarrow \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$

if the function is negative, $\lambda = -1$ so it will be

$\int_a^b -f(x)dx = -\int_a^b f(x)dx$

Positivity

- $f \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0$

ex. $g-f \geq 0 \quad \int_a^b g-f\, dx \geq 0 \quad \int_a^b g(x)dx + \int_a^b -f(x)dx \quad \int_a^b g(x)dx - \int_a^b f(x)dx \geq$

Monotonicity

- $f \leq g$ on $[a,b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

Absolute value

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Sometimes not there

Note: the extremes are ordered from smallest to biggest

$\int_a^b f(x)dx \quad a < b$

Integral of a point

$$\int_a^a f(x)\,dx = 0$$

Change the extremes

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Null integral

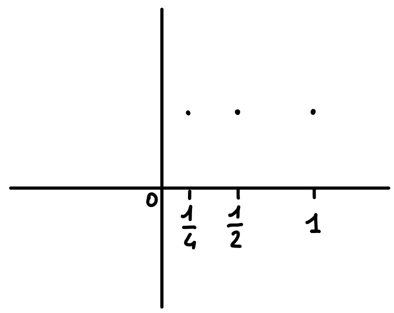
$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = 0$$

---

14 March

ex. Dirichlet function

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}$$



$\inf = 0 = s(f)$
$\neq$
$\sup = 1 = S(f)$

- not R. integrable

Rieman integrable functions:
- continuous functions
- monotone functions (proof)
- continuous except a finite number of points

Proof

$S(f, P_n) - s(f, P_n) =$ $\quad \underbrace{(x_i - x_{i-1})}{} = \frac{b-a}{n}$

not discontinuity points

make this smaller and smaller

$$= \sum [f(x_i) - f(x_{i-1})](x_i - x_{i-1}) = \underbrace{\frac{b-a}{n} \overbrace{(f(b^-) - f(a^+))}^{c}}$$

Def. $f[a;b] \longrightarrow \mathbb{R}$ is R. integrable iff $\forall \varepsilon > 0 \ \exists P_\varepsilon$ s.t. $\underbrace{S(f, P_\varepsilon) - s(f, P_\varepsilon) < \varepsilon}_{\text{means they coincide at some point}}$

$$\varepsilon \rightsquigarrow (x_i - x_{i-1}) < \frac{\varepsilon}{c}$$

Mean value theorem for integrals

$f[a;b] \to \mathbb{R}$ then $\exists\ x_0 \in [a;b]$ s.t. $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \underbrace{f(x_0)}_{c}$ (the function is continuous)

└ Teorema dei valori intermedi

↓

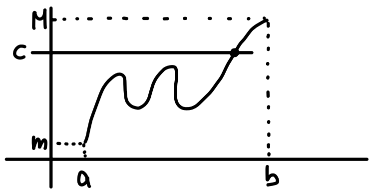
take all values between m and M

Proof

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$$

$\underset{\text{min}}{m} < f(x) < \underset{\text{max}}{M}$

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

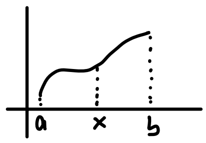


constants

Fundamental theorem of integral calculus

$f[a,b] \to \mathbb{R}$ continuous

$F(x) = \displaystyle\int_a^x f(t)\,dt \qquad a \le x \le b$

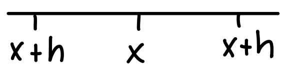


$F(x)$ is differentiable in $(a;b)$ and $F'(x) = f(x)$

$$\begin{tikzcd} f \arrow[r, bend left, "\int"] & F \arrow[l, bend left, "F'"] \end{tikzcd} \quad \{F+c\}$$

Proof

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$



$$= \lim_{h\to 0} \frac{1}{h} \int_a^{x+h} f(t)\,dt - \cancel{\int_a^x f(t)\,dt}$$

↓

$$\cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt}$$

↓ mean value thm

$$= \lim_{h\to 0} f(x_h) \qquad x < x_h < x+h$$

$\downarrow h\to 0$

x

// by continuity

$f(x)$

17 March

Fundamental theorem of integral calculus

G, differentiable

$G'(x) = f(x)$ $\qquad$ $G(x)$ = primitive

$$\int_a^x f(t)\,dt = G(x) - G(a) \rightsquigarrow \int_a^x G'(t)\,dt = G(x) - \underbrace{G(a)}_{\text{constant that depends on extreme}}$$

$$G(x) = F(x) + c,\ c \in \mathbb{R}$$

Proof

$G'(x) = f(x)$

$\underbrace{F'(x) - G'(x)}$ $= f(x) - f(x) = 0$

$\frac{d}{dx}(F(x) - G(x))$ $\forall x$

$\hookrightarrow F(x) - G(x) = c \qquad \forall x \in [a,b]$

$F(x) = G(x) + c$

$\overset{\|}{\int_a^x f(t)\,dt}$ $\qquad F(a) = G(a) + c$

$\overset{\|}{0} \to c = -G(a)$

$\downarrow$ when fix the extremes of integration

indefinite integral

- when we don't fix the extremes, the constant is not a number that can be canceled out, but it is general

ex. $\int 2x = x^2 + c = G(x)$

you get a primitive (function, not a number)

Examples of elementary integrals

1. $\int \sin x\,dx = -\cos x + c$

$\int_0^\pi \sin x\,dx = -\cos\pi - (-\cos 0) = 2$

2. $\int \cos x\,dx = \sin x + c$

3. $\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c,\ \alpha \neq -1$

4. $\int \frac{1}{x}\,dx = \ln|x| + c$

5. $\int e^x\,dx = e^x + c$

6. $\int a^x\,dx = \frac{a^x}{\ln a} + c$

8. $\int \frac{\cos x}{x}\,dx \neq \int \cos x\,dx \cdot \int \frac{1}{x}\,dx$

10. $\frac{1}{2}\int 2x \sin x^2\,dx = -\frac{\cos x^2}{2} + c$

11. $\int g(x)^\alpha \cdot g'(x)\,dx = \frac{g(x)^{\alpha+1}}{\alpha+1}$

12. $\int \frac{(\ln x)^3}{x}\,dx = \frac{(\ln x)^4}{4} + c$

7. $\int \cos x \pm \frac{1}{x}\,dx = \int \cos x\,dx \pm \int \frac{1}{x}\,dx = \sin x \pm \ln|x| + c$

$\int_1^2 \cos x - \frac{1}{x}\,dx = \sin x - \ln|x|\Big|_1^2 = \sin 2 - \ln 2 - (\sin 1 - \ln 1)$

9. $\int 2x(x^2+1)\,dx = \int 2x^3 + 2x\,dx = \frac{x^4}{2} + x^2 + c$

$\int 2x(x^2+1)\,dx = \frac{(x^2+1)^2}{2} + c \qquad = g'(x) \quad = f'(g(x))$

13. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

14. $\int e^{3x} - (x-3)^4\,dx = \frac{e^{3x}}{3} - \frac{(x-3)^5}{5} + c$

15. $\int \sqrt{x+3} - \frac{1}{x}\,dx = \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} - \ln|x| + c = \frac{2}{3}\sqrt{(x+3)^3} - \ln|x| + c$

16. $\int \frac{dx}{x \ln x} = \ln|\ln x|$

Integration by parts

$$\int_a^b f'g \, dx = f(x)\,g(x)\Big|_a^b - \int_a^b f g' \, dx$$

ex.

$$\int x \sin x \, dx = -x\cos x - \int (-\cos x)\, dx = -x\cos x + \sin x + c$$

ex. $\int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x\ln x - x + c$ $\quad \begin{cases} f(x) = \ln x \to f'(x) = \frac{1}{x} \\ g'(x) = 1 \to g(x) = x \end{cases}$

21 March

Methods

- Substitution $\to$ $\int f'(g(x)) \cdot g'(x)\,dx = [f(g(x))]$ $\quad u = g(x)$

$= \int f'(u)\, du = f(u) = f(g(x)) + c$ $\quad du = g'(x)\,dx$

ex. $\int_0^{\frac{3}{4}\pi} \sin^3 x \cos x \, dx = \int u^3 \, du = \frac{u^4}{4} = \frac{(\sin x)^4}{4}\Big|_0^{\frac{3}{4}\pi} = \frac{(\sin(\frac{3}{4}\pi))^4}{4} = \frac{\left(\frac{1}{\sqrt{2}}\right)^4}{4} = \frac{1}{16}$

$u = \sin x$
$du = \cos x \, dx$

- Integration by parts $\to$ $\int f'(x) \cdot g(x)\, dx = \underline{f(x)g(x) - \int f(x) g'(x)\, dx}$

ex. $\int \cos^2 x \, dx = \int \cos x \cdot \cos x \, dx = \cos x \cdot \sin x - \int \sin x(-\sin x)\, dx =$

$g(x) = \cos x \quad g'(x) = -\sin x$
$f(x) = \sin x \quad f'(x) = \cos x$

$\cos x \sin x + \int \sin^2 x \, dx$
$\qquad\qquad \parallel$
$\qquad \int 1 - \cos^2 x \, dx$

$2\int \cos^2 x \, dx = \sin x \cos x + \int 1 \, dx$
$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad x$

$$\int \cos^2 x \, dx = \frac{\sin x \cos x + x}{2} + c$$

$\int f(\cos x, \sin x)\, dx$

$\cos x = \frac{1 - t^2}{1 + t^2}$ $\quad t = \operatorname{tg} \frac{x}{2}$

$x = 2\operatorname{arctg} t$

$\sin x = \frac{2t}{1+t^2}$ $\quad dx = \frac{2}{1+t^2}\, dt$

$$\int \cos^2 x \, dx = \int \frac{(1-t^2)^2}{(1+t^2)^2} \frac{2}{1+t^2}\, dt = \int \frac{2(1-t^2)^2}{(1+t)^3} dt$$

ex.

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

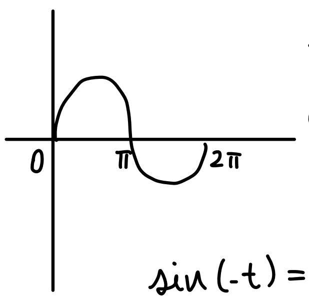
$$\begin{cases} f'(x) = e^x & f(x) = e^x \\ g(x) = \sin x & g'(x) = \cos x \end{cases}$$

$$\begin{cases} f'(x) = e^x & f(x) = e^x \\ g(x) = \cos x & g'(x) = -\sin x \end{cases} \longrightarrow e^x \cos x - \int e^x(-\sin x)\, dx$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + \int e^x \sin x \, dx$$

$$\frac{\cancel{2}}{\cancel{2}} \int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + c$$

ex. $$\int_0^{2\pi} e^{-x} |\sin x| \, dx = \int_0^{\pi} e^{-x} \sin x \, dx - \int_{\pi}^{2\pi} e^{-x} \sin x \, dx =$$



$t = -x$

$dt = -dx$

$$\int e^{-x} \sin x \, dx = \int e^t \sin(-t)(-dt) = \int e^t \sin t \, dt$$

$$= \frac{e^t(\sin t - \cos t)}{2} = \frac{e^{-x}(-\sin x - \cos x)}{2}$$

$\sin(-t) = -\sin t$

$\cos(-t) = \cos(t)$

$$\left.\frac{e^{-x}(-\sin x - \cos x)}{2}\right|_0^{\pi} + \left.\frac{e^{-x}(+\sin x + \cos x)}{2}\right|_{\pi}^{2\pi} = \frac{e^{-\pi}+1}{2} + \left(\frac{e^{-2\pi}}{2} + \frac{e^{-\pi}}{2}\right) = \frac{e^{-\pi} + e^{-2\pi} + 1}{2}$$

$$\left.\frac{e^{-x}(-\sin x - \cos x)}{2}\right|_0^{\pi} - \left.\frac{e^{-x}(-\sin x - \cos x)}{2}\right|_{\pi}^{2\pi}$$

$$= -\left(\frac{e^{-2\pi}(-\sin 2\pi - \cos 2\pi)}{2} - \frac{e^{-\pi}(-\sin \pi - \cos \pi)}{2}\right)$$

$$\frac{e^{-\pi} - 1}{2} - \left(\frac{e^{-2\pi} - \cancel{1}}{2} - \frac{e^{-\pi} + \cancel{1}}{2}\right)$$

$$\frac{e^{-\pi} - 1}{2} - \left(\frac{e^{-2\pi} - e^{-\pi}}{2}\right) = \frac{e^{-\pi} - 1 - e^{-2\pi} + e^{-\pi}}{2} = \frac{2e^{-\pi} - e^{-2\pi} - 1}{2} = e^{-\pi} \; \frac{-e^{-2\pi} - 1}{2}$$

Rational functions and integrals

$$f(x) = \frac{P(x)}{Q(x)}$$

- Degree of $P(x)$ > degree of $Q(x)$

↳ division

ex. $\frac{x^3+x}{x^2+x+1} = (x-1) + \frac{x+1}{x^2+x+1}$

$$\begin{array}{l|l} x^3 + \quad x & x^2+x+1 \\ \hline x^3+x^2+x & x-1 \\ -x^2 & \\ -x^2-x-1 & \\ \hline x+1 & \end{array}$$

General: $\frac{P(x)}{Q(x)} = H(x) + \frac{R(x)}{Q(x)}$

- Degree of $P(x)$ < degree of $Q(x)$

1) $\Delta > 0 \rightarrow Q(x)$ has 2 different real roots

ex. $x^2+x-2=0$

$x = \frac{-1 \pm \sqrt{1+8}}{2}$ ⟨ $x_1 = 1$, $x_2 = -2$

$Q(x) = (x-x_1)(x-x_2)$

$P(x) = x+3$

$\frac{P(x)}{Q(x)} = \frac{A}{(x-x_1)} + \frac{B}{(x-x_2)}$

constants

$$f(x) = \frac{x+3}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)} = \frac{(A+B)x+2A-B}{(x-1)(x+2)}$$

$$\begin{cases} A+B=1 \\ 2A-B=3 \end{cases} \begin{cases} A = 1-B \\ B = 2A-3 \end{cases} \begin{cases} A = 1-2A+3 \\ B = 2A-3 \end{cases} \begin{cases} 3A = 4 \\ B = 2A-3 \end{cases} \begin{cases} A = 4/3 \\ B = \frac{8}{3}-3 \end{cases} \begin{cases} A = 4/3 \\ B = -\frac{1}{3} \end{cases}$$

$$= \frac{4}{3} \cdot \frac{1}{x-1} - \frac{1}{3} \cdot \frac{1}{x+2}$$

$$\int \frac{4}{3(x-1)}dx - \int \frac{1}{3(x+2)}dx = \frac{4}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c = \frac{1}{3}\left(\ln\frac{|x-1|^4}{|x+2|}\right)+c$$

2) $\Delta = 0 \rightarrow Q(x)$ has one real root

$$\frac{P(x)}{Q(x)} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$$

or by <u>substitution</u>

ex. $\int \frac{x+1}{(3x+2)^2}dx$

$$\frac{A}{(3x+2)} + \frac{B}{(3x+2)^2} = \frac{3Ax+2A+B}{(3x+2)^2} \quad \begin{cases} 3A = 1 \\ 2A+B=1 \end{cases} \begin{cases} A = 1/3 \\ B = 1/3 \end{cases}$$

$$\frac{1}{3}\int \frac{1}{(3x+2)}dx + \frac{1}{3}\int \frac{1}{(3x+2)^2}dx = \frac{1}{9}\ln|3x+2| - \frac{1}{9}\frac{1}{(3x+2)} + c$$

3) $\Delta<0 \to Q(x)$ has no real roots

$$\frac{P(x)}{Q(x)}=A\frac{Q'(x)}{Q(x)}+\frac{B}{Q(x)} \longrightarrow \int\frac{P(x)}{Q(x)}dx = A\int\frac{Q'(x)}{Q(x)}dx + B\int\frac{1}{Q(x)}dx$$

$$= A\ln|Q(x)| + B\int\frac{1}{Q(x)}dx + c$$

ex. $f(x)=\dfrac{x}{x^2+2x+4} = A\dfrac{2x+2}{x^2+2x+4}+\dfrac{B}{x^2+2x+4} = \dfrac{2Ax+(2A+B)}{x^2+2x+4}$ $\begin{cases}2A=1\\2A+B=0\end{cases}$ $\begin{cases}A=1/2\\B=-1\end{cases}$

$$\int\frac{x}{x^2+2x+4}dx = \frac{1}{2}\int\frac{2x+2}{x^2+2x+4}dx - \int\frac{1}{x^2+2x+4}dx = \frac{1}{2}\ln|x^2+2x+4| - \underbrace{\int\frac{1}{x^2+2x+4}dx}$$

* $x^2+2x+4=(x+1)^2+3$

$t=\dfrac{(x+1)}{\sqrt{3}}$ $\quad \underbrace{\int\frac{1}{(x+1)^2+3}}dx = \sqrt{3}\int\frac{1}{t^2+1}dt = \sqrt{3}\arctan t = \sqrt{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right)+c$

by substitution

but we can also use the rule:

$$\int\frac{1}{(x+k)^2+m^2} = \frac{1}{m}\arctan\left(\frac{x+k}{m}\right)+c$$

General rule:

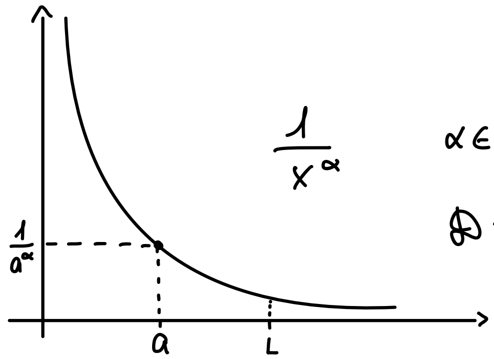
$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

28 March

Improper integral

- $f:(a,b) \to \mathbb{R}$ unbounded
- (a,b)

  ↳ bounded or unbounded



$\frac{1}{x^\alpha}$ $\quad \alpha \in \mathbb{R}^+ \ (\alpha > 0)$

$\mathcal{D} = (0; +\infty)$

$f(x) = \frac{1}{x^\alpha}$ not bounded in $(0; +\infty)$

$f(x) = \frac{1}{x^\alpha}$ $\quad [a; L] \quad a > 0$

$(0; L) \quad \int_a^1 \frac{1}{x^\alpha}\,dx = \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_a^1 = \frac{1}{1-\alpha}(1-a^{1-\alpha})$

$\lim_{a \to 0^+} \frac{1}{1-\alpha}(1-a^{1-\alpha}) \to \begin{cases} \frac{1}{1-\alpha}, & 0 < \alpha < 1 \\ +\infty, & \alpha \geq 1 \end{cases}$

Bounded interval → bounded exponent

Unbounded " → unbounded "

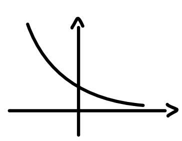
$(L; +\infty) \quad \int_1^L \frac{1}{x^\alpha}\,dx = \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_1^L = \frac{L^{-\alpha+1}}{-\alpha+1} - \frac{1^{-\alpha+1}}{-\alpha+1}$

$\frac{1}{x^\alpha} \begin{cases} (0, L), & 0 < \alpha < 1 \\ (L, +\infty), & \alpha > 1 \end{cases}$

$\lim_{L \to +\infty} \frac{L^{-\alpha+1}}{-\alpha+1} - \frac{1^{-\alpha+1}}{-\alpha+1} \to \begin{cases} +\infty, & 0 < \alpha \leq 1 \\ \frac{1}{\alpha-1}, & \alpha > 1 \end{cases}$

Def:
- $f:(a,b] \to \mathbb{R}$ R. integrable in $[c,b]$ $\forall\, c > a$ there exists the improper integral iff $\exists \lim_{c \to a^+} \int_c^b f(x)\,dx \in \mathbb{R}$
- $f:[a,+\infty) \to \mathbb{R}$ R. integrable in $[a,L]$ $\forall\, L < +\infty$ " " "
  " " " $\exists \lim_{L \to +\infty} \int_a^L f(x)\,dx \in \mathbb{R}$
- $f:[a,b) \to \mathbb{R}$ R. integrable in $[a,c]$ $\forall\, a < c < b$ " " "
  " " " $\exists \lim_{c \to b^-} \int_a^c f(x)\,dx \in \mathbb{R}$
- $f:(-\infty,b] \to \mathbb{R}$ R. integrable in $[L,b]$ $\forall\, -\infty < L < b$ " " "
  " " " $\exists \lim_{L \to -\infty} \int_L^b f(x)\,dx \in \mathbb{R}$

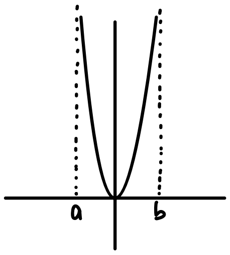
ex. $f(x) = e^{-x} \quad \mathbb{R} \to \mathbb{R}$



$\lim_{L \to +\infty} \int_0^L e^{-x}\,dx = \lim_{L \to +\infty} -e^{-L} + 1 = 1$

$\lim_{L \to -\infty} \int_L^{-3} e^{-x}\,dx = \lim_{L \to -\infty} -e^{3} + e^{-L} = +\infty$

Unbounded interval from left and right



$-\infty \le a < b \le +\infty$

$f(a,b) \to \mathbb{R}$ R. integrable in $[c,d]$ $\quad a < c < x_0 < d < b$; $\quad c, d \in \mathbb{R}$

if $\lim\limits_{c \to a^+} \int_c^{x_0} f(x)\,dx = l_-$, $\lim\limits_{d \to b^-} \int_{x_0}^{d} f(x)\,dx = l_+ \in \mathbb{R}$ then

$$\boxed{\int_a^b f(x)\,dx = l_- + l_+}$$

31 March

Comparison theorem

$f, g\ [a,b) \to \mathbb{R}$ $\forall\ a < c < b \le +\infty$

f, g Reimau Integrable in $[a,c]$.

$0 \le f(x) \le g(x)$ then, if $\int_a^b f(x)\,dx = +\infty$

then $\int_a^b g(x)\,dx = +\infty$. if $\exists \int_a^b g(x)\,dx < +\infty$

then $\int_a^b f(x)\,dx < +\infty$

Proof

$$0 \le \underbrace{\int_a^c f(x)\,dx}_{F(c)} \le \underbrace{\int_a^c g(x)\,dx}_{\overset{\shortparallel}{G(c)}}$$

monotone increasing

$\exists \lim\limits_{c \to b^-} F(c) = +\infty$

$\Downarrow$

$\exists \lim\limits_{c \to b^-} G(c) = +\infty$

ex. $\displaystyle\int_1^{+\infty} \frac{|\cos x|}{x^2}\,dx = \lim_{L \to +\infty} \int_1^L \frac{|\cos x|}{x^2}\,dx$

$$0 \le \int_1^{+\infty} \frac{|\cos x|}{x^2}\,dx \le \int_1^{+\infty} \frac{1}{x^2}\,dx < +\infty$$

Asymptotic comparison test

$f, g\ [a,b) \to \mathbb{R}$

Rieman Integrable $[a,c]$ $\forall\ a < c < b \le +\infty$

$f(x) \ge 0$, $g(x) > 0$

1) $\displaystyle\lim_{x \to b^-} \frac{f(x)}{g(x)} = l > 0 \implies \begin{cases} \int_a^b f(x)\,dx = +\infty = \int_a^b g(x)\,dx \\ \int_a^b f(x)\,dx < +\infty, \ \int_a^b g(x)\,dx < +\infty \end{cases}$

2) $\lim_{x \to b^-} \frac{f(x)}{g(x)} = 0 \Rightarrow \begin{cases} \text{if } \int_a^b f(x)\,dx = +\infty \to \int_a^b g(x)\,dx = +\infty \\ \text{if } \int_a^b g(x)\,dx < +\infty \to \int_a^b f(x)\,dx < +\infty \end{cases}$

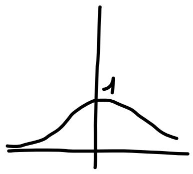
$g(x) > \frac{f(x)}{\varepsilon}$

3) $\lim_{x \to b^-} \frac{f(x)}{g(x)} = +\infty \Rightarrow \begin{cases} \int_a^b g(x)\,dx = +\infty = \int_a^b f(x)\,dx \\ \int_a^b f(x)\,dx < +\infty ,\ \int_a^b g(x)\,dx < +\infty \end{cases}$

$\frac{f(x)}{g(x)} > M \to f(x) > Mg(x)$

ex.

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx$$



$$\lim_{L \to +\infty} \int_0^L e^{-x^2}\,dx < +\infty \,?$$

$$\| $$

$$\lim_{L \to -\infty} \int_L^0 e^{-x^2}\,dx < +\infty \,?$$

even $\to$ just have to verify one

$$\lim_{x \to +\infty} \frac{e^{-x^2}}{e^{-x}} = 0$$

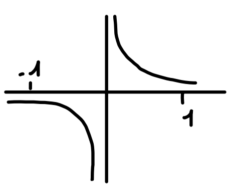
$$\lim_{L \to +\infty} \int_0^L e^{-x}\,dx = -e^{-x}\Big|_0^L = -e^{-L} + 1 \to 1$$

$\hookrightarrow$ converges also for $-\infty$

---

ex.

$$\int_{-1}^{1} \frac{1}{x}\,dx$$



$$= \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0^+} \ln 1 - \ln \varepsilon = +\infty$$

$\hookrightarrow$ diverges $\to$ no improper integrability

ex.

$$f(x)=\frac{1}{x|\ln x|^{\alpha}} \qquad x\in(0;+\infty)\setminus\{1\}, \quad \alpha>0$$

Unbounded at $x = 0, 1$

$$\int_0^{1/2} + \int_{1/2}^{1}$$

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1/2} + \lim_{\varepsilon\to 0^+}\int_{1/2}^{1-\varepsilon}$$

$$\int_1^{+\infty} = \int_1^2 + \int_2^{+\infty}$$

$$\int_0^{+\infty} = \int_0^{1/2} + \int_{1/2}^{1} + \int_1^2 + \int_2^{+\infty}$$

$$\hookrightarrow \int_0^{+\infty} = \underline{\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1/2} + \lim_{\varepsilon\to 0^+}\int_{1/2}^{1-\varepsilon}} + \lim_{c\to 1^+}\int_c^2 + \lim_{L\to+\infty}\int_2^L$$

$$* \int_0^{1/2}\frac{1}{x(-\ln x)^{\alpha}}\,dx + \int_{1/2}^{1}\frac{1}{x(-\ln x)^{\alpha}}\,dx \to -\int_{\varepsilon}^{1/2}\frac{1}{x(\ln x)^{\alpha}}\,dx = -\int_{\varepsilon} t^{-\alpha}\,dt = -\frac{(-\ln x)^{-\alpha+1}}{-\alpha+1}\Bigg|_{\varepsilon}^{1/2} = \frac{(-\ln\frac{1}{2})^{1-\alpha}}{\alpha-1} - \frac{(-\ln\varepsilon)^{1-\alpha}}{\alpha-1}$$

from 0 to 1 ln is negative

$t = \ln x$

$dt = \frac{1}{x}\,dx$

if $1-\alpha<0 \to \alpha>1$

$$\int_0^{1/2}\frac{1}{x(-\ln x)^{\alpha}}\,dx = \frac{(\ln 2)^{1-\alpha}}{\alpha-1}$$

different intervals

↳ so not improper integrable

$$\lim_{\varepsilon\to 0^+}\int_{1/2}^{1-\varepsilon} = \frac{(\ln 2)^{1-\alpha}}{\alpha-1}, \quad 0<\alpha<1$$

Integrale	Primitiva	Integrale	Primitiva
1) $\int k\,dx$ (costante)	$kx + c$	------------------------	------------------------
2) $\int x^n\,dx$ ($n \neq -1$)	$\frac{1}{n+1}\cdot x^{n+1} + c$	$\int [f(x)]^n \cdot f'(x)\,dx$	$\frac{1}{n+1}\cdot f(x)^{n+1} + c$
3) $\int a^x\,dx$	$a^x \cdot \frac{1}{\ln(a)} + c$	$\int a^{f(x)} \cdot f'(x)\,dx$	$a^{f(x)} \cdot \frac{1}{\ln(a)} + c$
4) $\int e^x\,dx$	$e^x + c$	$\int e^{f(x)} \cdot f'(x)\,dx$	$e^{f(x)} + c$
5) $\int \frac{1}{x}\,dx$	$\ln\|x\| + c$	$\int \frac{1}{f(x)} \cdot f'(x)\,dx$	$\ln\|f(x)\| + c$
6) $\int \sin x\,dx$	$-\cos x + c$	$\int \sin f(x) \cdot f'(x)\,dx$	$-\cos f(x) + c$
7) $\int \cos x\,dx$	$\sin x + c$	$\int \cos f(x) \cdot f'(x)\,dx$	$\sin f(x) + c$
8) $\int tg\,x\,dx$	$-\ln\|\cos x\| + c$	$\int tg\,f(x) \cdot f'(x)\,dx$	$-\ln\|\cos f(x)\| + c$
9) $\int cotg\,x\,dx$	$\ln\|\sin x\| + c$	$\int cotg\,f(x) \cdot f'(x)\,dx$	$\ln\|\sin f(x)\| +$

Integrale	Primitiva	Integrale	Primitiva
10) $\int \frac{1}{\cos^2(x)}\,dx$	$tg\,x + c$	$\int \frac{1}{\cos^2 f(x)} \cdot f'(x)\,dx$	$tg\,f(x) + c$
11) $\int \frac{1}{\sin^2(x)}\,dx$	$-cotg\,x + c$	$\int \frac{1}{\sin^2 f(x)} \cdot f'(x)\,dx$	$-cotg\,f(x) + c$
12) $\int \frac{1}{a^2+x^2}\,dx$	$\frac{1}{a}\,arctg\,\frac{x}{a} + c$	$\int \frac{1}{a^2+f^2(x)} f'(x)\,dx$	$\frac{1}{a}\,arctg\,\frac{f(x)}{a} + c$
13) $\int \frac{1}{\sqrt{1-x^2}}\,dx$	$\arcsin x + c$	$\int \frac{1}{\sqrt{1-f^2(x)}} f'(x)\,dx$	$\arcsin f(x) + c$
14) $\int \frac{1}{1+x^2}\,dx$	$arctg\,x + c$	$\int \frac{1}{1+f^2(x)} f'(x)\,dx$	$arctg\,f(x) + c$
15) $\int \sinh x\,dx$	$\cosh x + c$	$\int \sinh f(x) \cdot f'(x)\,dx$	$\cosh f(x) + c$
16) $\int \cosh x\,dx$	$\sinh x + c$	$\int \cosh f(x) \cdot f'(x)\,dx$	$\sinh f(x) + c$
17) $\int tgh\,x\,dx$	$\ln\|\cosh x\| + c$	$\int tgh\,f(x) \cdot f'(x)\,dx$	$\ln\|\cosh f(x)\| + c$
18) $\int cotgh\,x\,dx$	$\ln\|\sinh x\| + c$	$\int cotgh\,f(x) \cdot f'(x)\,dx$	$\ln\|\sinh f(x)\| +$

INTEGRALE	PRIMITIVA
$\int \sin^2 x\,dx$	$\frac{1}{2}(x - \sin x \cos x) + c$
$\int \cos^2 x\,dx$	$\frac{1}{2}(x - \sin x \cos x) + c$
$\int \frac{1}{a^2-x^2}\,dx$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| + c$
$\int \frac{1}{x^2-a^2}\,dx$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + c$
$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx$	$\ln\left\|x+\sqrt{x^2 \pm a^2}\right\| + c$

14 April

Numerical series

sequence:

$$(a_k)_{k\in\mathbb{N}} = \{a_0, a_1, a_2, a_3, \dots, a_k\}$$

$$\sum_{k=0}^{\infty} a_k$$

$$S_n = \underbrace{\sum_{k=0}^{n} a_k}_{\text{partial sum}}$$

sum

$S_0 = a_0$

$S_1 = a_0 + a_1$

$S_2 = a_0 + a_1 + a_2$

$\vdots$

$S_n = a_0 + a_1 + a_2 + \dots + a_n$

$(S_n)_{n\in\mathbb{N}}$

S_n — $\exists \lim s_n = S$ — $\in \mathbb{R}^*$ / $(\pm)\infty$

S_n — $\nexists \lim s_n$ (indeterminated)

- convergent $\to S \in \mathbb{R}$
- divergent $\to S = \pm\infty$

Remainder

$$R_n = \sum_{k=n+1}^{\infty} a_k = S - S_n$$

$$\lim_{n\to\infty} R_n = 0$$

$$\Updownarrow$$

$$\lim_{n\to\infty} S_n = S = \sum_{k=0}^{\infty}$$

Note: if $a_k \geq 0$

S_n is a monotone sequence

so the limit always exists

Geometric series

$x \in \mathbb{R}$, $\displaystyle\sum_{k=0}^{\infty} x^k$

if $x \geq 1 \to = \infty$

if $0 < x < 1$

$x \neq 1, 0$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = S_n$$

$x \neq 1$

$$\sum_{k=1}^{n} x^k = \frac{1-x^{n+1}}{1-x} - 1$$

$$S_n = \frac{1-x^{n+1}}{1-x} \quad \begin{cases} \frac{1}{1-x}, & |x|<1 \\ +\infty, & x>1 \\ \nexists \lim & x<-1 \end{cases}$$

$$\sum_{k=0}^{n} (\pm 1)^k \quad \begin{cases} +\infty, & x = 1 \to S = n+1 \\ \nexists \lim, & x = -1 \end{cases}$$

ex.

$x=2$

$$\sum_{k=0}^{\infty} 2^k$$

$k=0 \quad S_0 = 1 = 2^0$

$k=1 \quad S_1 = 1+2^1$

$k=2 \quad S_2 = 1+2+2^2$

$k=3 \quad S_3 = 1+2+2^2+2^3$

S_n is an increasing monotone sequence

$\Downarrow$

$\exists \lim S_n = \sup_n S_n$

$2^k \geq 1 \quad \forall k$

$R_n = \sum_{k=n+1}^{\infty} 2^k > 1$ for sure

$\downarrow$

limit cannot be equal to 0

ex. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

$S_0 = 1$

$S_1 = 1 + \frac{1}{2}$

$S_2 = 1 + \frac{1}{2} + \frac{1}{4}$

Cauchy sequence

$\lim_{n\to+\infty} a_n = L \in \mathbb{R}$

converging $\iff \forall \varepsilon > 0 \quad \exists \ n_\varepsilon \in \mathbb{N}$

$\forall m, p > n_\varepsilon$

$|a_m - a_p| < \varepsilon$

Proposition

$\sum_{k=1}^{\infty} a_k \in \mathbb{R}$

then $\lim_{k\to+\infty} a_k = 0$

Proof

$S = \lim_{n\to+\infty} s_n \in \mathbb{R}$

S_n Cauchy seq.

$|S_m - S_p| < \varepsilon \quad \forall m, p > n_\varepsilon$

$m = n$
$p = n-1$ $\quad |S_m - S_p| = |a_n| < \varepsilon$

$\forall n > n_\varepsilon$

$\lim_{n\to+\infty} a_n = 0$

$\hookrightarrow$ in this case because $L = 0$

Def of limit:

$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n > n_\varepsilon$

$|a_n - L| < \varepsilon$

Harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k} \qquad a_k = \frac{1}{k}$$

Note: $\frac{1}{n^\alpha}$ $\alpha > 1$ c. $\alpha \leq 1$ d.

$\lim_{k \to +\infty} \frac{1}{k} = 0 \leftarrow$ first do this check

↳ if get 0, go on studying convergence

↳ if not get 0, then have to see if ∞ or $\nexists$ lim

S_n is monotone increasing seq.

$\Downarrow$

$\exists \lim_{n \to +\infty} S_n = S$ < $\mathbb{R}$? / ∞?

if $\sum_{k=0}^{\infty} \frac{1}{k} = s \in \mathbb{R}$

$\Rightarrow S_n$ is a Cauchy sequence -> so lets prove it

$$S_{2k} - S_k = \frac{1}{2k} + \underbrace{\frac{1}{2k-1}}_{> \frac{1}{2k}} + \frac{1}{2k-2} + \dots + \underbrace{\frac{1}{k+1}}_{> \frac{1}{2k}}$$

$m = 2k \quad p = k$

k times

$$\left\{ \begin{aligned} S_k &= \sum_{n=1}^{k} \frac{1}{n} \\ &= 1 + \frac{1}{2} + \dots + \frac{1}{k} \\ S_{2k} &= \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{k}}_{S_k} + \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} \end{aligned} \right\}$$

$$S_{2k} - S_k > \frac{k}{2k} = \frac{1}{2}$$

- so $|S_{2k} - S_k|$ is not $< \varepsilon$, it is bigger than $\frac{1}{2}$ and the series is $+\infty$

↳ $\sum_{k=0}^{\infty} \frac{1}{k} = +\infty$

18 April

First check to do

$\lim_{k \to \infty} a_k = 0$

-> if true < convergence / divergence

-> if false < divergence / indeterminated -> this does not happen if all terms are positive

or $\nexists \lim a_k$

Non-negative series

$\sum a_k \quad (a_k) \geq 0$

$S_n = \sum_{k=1}^{n} a_k$, increasing seq.

$\exists \lim_{n \to +\infty} S_n = \sup_n S_n \begin{cases} \in \mathbb{R} \\ +\infty \end{cases}$

Integral test

$f[1; +\infty) \to \mathbb{R}^+$ decreasing

then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)dx$

have the same behavior

Comparison test

$b_n, a_n \qquad b_n, a_n \geq 0$

$0 \leq a_n \leq b_n$

then if $\sum_{n=1}^{\infty} a_n = +\infty$ then $\sum_{n=1}^{\infty} b_n = +\infty$

if $\sum_{n=1}^{\infty} b_n < +\infty$ then $\sum_{n=1}^{\infty} a_n < +\infty$

Limit comparison test

$\lim_{n \to +\infty} \frac{a_n}{b_n} = L \qquad b_n > 0$

1) $L \in (0; +\infty)$ $\quad \sum a_n, \sum b_n$ have the same behaviour

2) $L = 0$ $\quad \sum b_n < +\infty \Rightarrow \sum a_n < +\infty$ or $\underset{\sum b_n}{\sum a_n} = +\infty \Rightarrow \sum b_n = +\infty$

3) $L = +\infty$ $\quad \sum a_n < +\infty \Rightarrow \sum b_n < +\infty$ or $\sum b_n = +\infty \Rightarrow \sum a_n = +\infty$

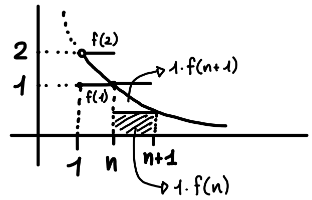
$\downarrow \qquad \downarrow$

$\lim a_n = 0 \qquad \sum b_n < \sum a_n$

Series and integrals

$\sum \frac{1}{n^\alpha} \rightsquigarrow \int_1^{\infty} \frac{1}{x^\alpha} dx$

$\alpha > 0$



for $\alpha \geq 1$ $\quad \sum \frac{1}{n^\alpha} < +\infty$ since

$\int_1^{\infty} \frac{1}{n^\alpha} < +\infty$ for $\alpha \geq 1$

for $0 < \alpha < 1$

$\int_1^{\infty} \frac{1}{n^\alpha} = +\infty \to \sum_{n=1}^{\infty} \frac{1}{n^\alpha} = +\infty$

ex.

$$0 < \frac{1}{n} \leq 1$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

$$0 < \sin x < x \quad , \quad 0 < x < \frac{\pi}{2}$$

necessary condition:

$$\lim_{n \to +\infty} \sin\left(\frac{1}{n}\right) = 0$$

$$\lim_{n \to +\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

the series do the same $\to$ $\sum \frac{1}{n} = +\infty \Rightarrow \sum \sin\left(\frac{1}{n}\right) = +\infty$

$\downarrow$

by limit comparison test

ex.

$$\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^4+4}}$$

$$\lim_{n \to +\infty} \frac{2n+1}{\sqrt{n^4+4}} = \lim_{n \to +\infty} \frac{2+\frac{1}{n}}{\sqrt{\frac{n^4+4}{n^2}}} = 0$$

$$\frac{\frac{2n+1}{\sqrt{n^4+4}}}{\frac{1}{n}} = \frac{2n^2+n}{\sqrt{n^4+4}} \to 2 \in (0; +\infty)$$

$\downarrow$

they do the same

$\hookrightarrow$ since $\sum \frac{1}{n} = +\infty$ then $\sum \frac{2n+1}{\sqrt{n^4+4}} = +\infty$

21 April

Root method

a_n non-negative

if $\exists N \in \mathbb{N}$ s.t. $\sqrt[n]{a_n} \leq L$, $\forall n \geq N$ and $L \in (0,1)$

then series converges.

if $\sqrt[n]{a_n} > L$ $\forall n > N$, $L > 1$ then the series diverges

Proof

$L (0,1)$ $\quad 0 \leq a_n \leq L^n \quad \forall n \in N$

$$s_h - s_{N-1} = \sum_{n=N}^{h} a_n \leq \sum_{n=0}^{\infty} L^n < +\infty$$

$\downarrow$

$h \to +\infty$

$a_n > L^n \quad \forall n > N$

L^n, for $L > 1$ diverges so

a_n surely diverges

Asymptotic/Limit root test

(a_n) non-negative

$\lim_{n\to+\infty} \sqrt[n]{a_n} = L$

- if $L<1 \to$ convergence
- if $L>1 \to$ divergence

Exercise

$$\sum_{n=0}^{+\infty} \frac{3^n+5^n}{2^n+6^n} =$$

$$\sum \frac{3^n}{2^n+6^n} \; ; \; \sum \frac{5^n}{2^n+6^n}$$

splitting and comparison test

1. $\frac{3^n}{2^n+6^n} < \left(\frac{1}{2}\right)^n$ ↳ geometric series

2. $0 < \frac{5^n}{2^n+6^n} < \left(\frac{5}{6}\right)^n$ ↳ geometric series

$\to \frac{1}{2}<1, \frac{5}{6}<1$

$\Downarrow$

convergence

root test

$$\sqrt[n]{\frac{5^n \left(\frac{3}{5}\right)^n + 1}{6^n \left(\frac{2}{6}\right)^n + 1}} = \sqrt[n]{\left(\frac{5}{6}\right)^n \underbrace{\frac{1+\left(\frac{3}{5}\right)^n}{1+\left(\frac{2}{6}\right)^n}}_{1}}$$

$\downarrow$

$\frac{5}{6} < 1$

$\Downarrow$

convergence

Ratio test

$a_n > 0$ if $\exists\, L \in (0,1)$, $N \in \mathbb{N}$ s.t.

$\frac{a_{n+1}}{a_n} \le L \quad \forall n > N$ then $\sum a_n$ converges $\longrightarrow \sum a_n < \sum L^n \left(\frac{a_N}{L^N}\right)$

$\left(\frac{a_N}{L^N}\right)$ ↳ constant

$\sum L^n$ → geometric series

if $\exists\, N$ s.t. $\underline{a_{n+1} \ge a_n}$ $\forall n > N$ then $\sum a_n$ diverges

↳ increasing sequence

Limit ratio test

$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = L$

if $L<1 \to$ convergence

if $L>1 \to$ divergence

Generalized harmonic series

$\alpha > 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

#root test

$$\lim_{n\to+\infty} \sqrt[n]{\frac{1}{n^\alpha}} = \frac{1}{(\sqrt[n]{n})^\alpha} \Rightarrow 1$$

$$\sqrt[n]{n} = n^{\frac{1}{n}} = \infty^0$$
$\downarrow$
1

↳ not included in limit root test

↳ converging $\alpha > 1$

↳ diverging $0 < \alpha \le 1$

#ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^\alpha}}{\frac{1}{n^\alpha}} = \left(\frac{n}{n+1}\right)^\alpha$$

$$\lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^\alpha = 1$$

Exercise

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

#ratio test

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\lim_{n\to+\infty} \frac{1}{n+1} = 0$$

Bertrand series

$$\sum_{n=2}^{\infty} \frac{1}{n^\alpha (\ln n)^\beta} \qquad \alpha, \beta \in \mathbb{R}$$

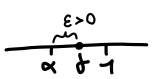
$$* \lim_{n\to+\infty} \frac{n^\gamma}{(\ln n)^\beta} = +\infty \qquad \text{diverging}$$

$\gamma, \beta > 0$

$\alpha < 0$, $\forall \beta \in \mathbb{R} \to$ diverging

$\alpha > 0$

$\alpha < \gamma < 1$



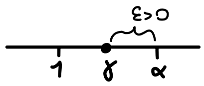
$\alpha = \gamma - \varepsilon$

$$\frac{1}{n^\alpha (\ln n)^\beta} = \frac{n^\varepsilon}{n^\gamma (\ln n)^\beta} > \frac{1}{n^\gamma}$$

$\vee$
$1*$

$\gamma < 1 \to$ diverging

$\alpha > 1$



$\alpha = \gamma + \varepsilon$

$$\frac{1}{n^{\alpha}(\ln n)^{\beta}} = \frac{1}{n^{\gamma}} \cdot \underbrace{\frac{1}{n^{\varepsilon}(\ln n)^{\beta}}}_{\substack{\downarrow n \to +\infty \\ 0}} < \frac{1}{n^{\gamma}}$$

$\gamma > 1$
↳ converging

$\alpha = 1,\ \beta \leq 0$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\beta}} \to \frac{(\ln n)^{-\beta}}{n} > \frac{(\ln 2)^{-\beta}}{n} \quad \forall n \geq 2$$

$$\sum \frac{(\ln n)^{-\beta}}{n} > (\ln n)^{-\beta} \sum \frac{1}{n} = +\infty$$

↳ harmonic series

$\alpha = 1,\ \beta > 0$

$$\sum \frac{1}{n(\ln n)^{-\beta}}$$

$$\begin{cases} 0 < \beta \leq 1 \to \text{divergence} \\ \beta > 1 \to \text{convergence} \end{cases}$$

integral test

$$\int_{2}^{+\infty} \frac{1}{x(\ln x)^{\beta}} = \int_{\ln 2}^{+\infty} \frac{1}{t^{\beta}}\, dt$$

$\downarrow$

$t = \ln x$

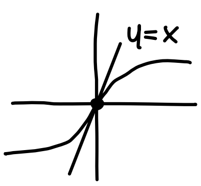
$dt = \frac{1}{x} dx$

Exercises

$$\sum_{k=1}^{\infty} k \arctan \frac{1}{k^2} = \sum_{k=1}^{\infty} \cancel{k} \cdot \frac{1}{k^{\cancel{2}}} = +\infty$$

comparison test

$$\lim_{k \to +\infty} \frac{\arctan \frac{1}{k^2}}{\frac{1}{k^2}} = 1$$



$$\lim_{x \to 0} \frac{\arctan x}{x} \overset{DH}{=} \lim_{x \to 0} \frac{\frac{1}{x^2+1}}{1} = 1$$

Excersises

1) $\sum_{k=1}^{\infty} \log\left(1+\frac{5}{k^2}\right) = \sum_{k=1}^{\infty} \frac{5}{k^2} \rightarrow$ converging

#comparison test

$$\lim_{k\to\infty} \frac{\log\left(1+\frac{5}{k^2}\right)}{\frac{5}{k^2}} = 1$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

2) $\sum_{k=1}^{\infty} \sin\frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$ #comparison test

3) $\sum_{k=1}^{\infty} \frac{\cos^2 k}{k\sqrt{k}} \leq \sum \frac{1}{k^{3/2}} \rightarrow$ converging $(\cos^2 k \leq 1)$

28 Aprile

Series with varying sign

$a_n \in \mathbb{R}$

$\sum_{n=0}^{\infty} a_n$ <
- conditional convergence $\rightarrow \sum a_n$ converges
- absolute convergence
  - $\sum |a_n| < +\infty \rightarrow$ the series converges absolutely
    ≥ 0 ↳ non-negative (apply methods we know)
  - if there is $\rightarrow \sum a_n$ also converges conditionally

Note: $|a+b| \leq |a| + |b|$

$\sum a_n < +\infty$ but $\sum |a_n| \nexists$ could happen

Proof

S_n is the partial sum of $\sum a_n$

$$|S_{n+p} - S_{n-1}| = \left|\sum_{k=n}^{n+p} a_k\right| \leq \sum_{k=n}^{n+p} |a_k| = G_{n+p} - G_{n-1} < \varepsilon \ (G_n \text{ is the p. sum to } \sum |a_k|)$$

converges by assumption

↳ Cauchy sequence

Example

- Alternating series $\sum_{n=0 \text{ or } n=1}^{\infty} (-1)^n a_n$ $a_n \geq 0$

$a_n = \frac{1}{n} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

does not converge absolutely $\rightarrow \sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n} = +\infty$

necessary condition: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ ✓

Leibniz

- $a_n > 0$
- decreasing seq.
- $\lim a_n = 0$

then,

- $\sum (-1)^n a_n$ converges
- $|R_n| = |S - S_n| \leq a_{n+1}$

$$\left\{ R_n = S - S_n \atop \sum_{k=0}^{\infty} a_n \right\}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} < +\infty \quad ; \quad |R_n| \leq \frac{1}{n+1}$$

- $a_n \to 0$
- $a_n = \frac{1}{n} > 0$
- decreasing

converges conditonally

Proof

bigger → decreasing seq.

$$S_{2n} = S_{2n-2} - a_{2n-1} + a_{2n} = S_{2n-2} - \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0} \leq S_{2n-2}$$

↳ decreasing sequence → $S^* \in \mathbb{R}$

$$S_{2n+1} = S_{2n-1} + \underbrace{a_{2n} - a_{2n+1}}_{\geq 0} \geq S_{2n-1}$$

↳ increasing sequence → $\bar{S} \in \mathbb{R}$

$$S^* - \bar{S} = \lim_{n \to +\infty} S_{2n} - S_{2n+1} = \lim_{n\to+\infty} a_{2n+1} = 0$$

$$\lim_{n \to +\infty} S_n = S^* = \bar{S} = S$$

↳ partial sum converges so also the series converges

$$a_0 = S_0 \geq S_{2n} \geq \underset{S_{2n+1} - a_{n+1}}{\overset{\shortparallel}{S_{2n+1}}}$$

$$S_{2n} = S_{2n-1} + a_{2n} > S_{2n-1}$$

$$S_{2n+1} \overset{①}{\leq} S \overset{②}{\leq} S_{2n}$$

$$0 \overset{①}{\leq} S - S_{2n+1} \leq S_{2n+2} - S_{2n+1} = a_{2n+2}$$

$$0 \overset{②}{\leq} S_{2n} - S \leq S_{2n} - S_{2n+1} = a_{2n+1}$$

$$|S - S_k| \leq a_{k+1}$$

k = even $= 2n$ ② → a_{2n+1} ✓

k = odd $= 2n+1$ ① → a_{2n+2} ✓

Ex.

$$\sum_{k=1}^{\infty} \frac{\cos(3k)}{k^3} \qquad a_k = \frac{1}{k^3}$$

$$\left. \begin{array}{ll} k=1 & \cos 3 < 0 \\ k=2 & \cos 6 > 0 \end{array} \right\} \text{not an alternating series}$$

$$\sum \left| \frac{\cos(3k)}{k^3} \right| \leq \sum \frac{1}{k^3} < +\infty$$

↳ comparison test

absolute convergence => conditional convergence

$$\Downarrow$$

$$\sum \frac{\cos(3k)}{k^3} < +\infty$$

2 May

Power series in $\mathbb{R}$

Recall: Taylor Expansion

$$\underset{\substack{\parallel \\ S_n}}{P_n(x,x_0)} = \sum_{k=0}^{n} \underbrace{\frac{f^k(x_0)}{k!}}_{\substack{\shortparallel \\ a_k}} (x-x_0)^k$$

f infinitely many times differentiable

$$a_n = \frac{f^n(x_0)}{n!}$$

$$\lim_{n\to+\infty} S_n$$

- convergence $\Leftrightarrow R_n(x,x_0) \xrightarrow[n\to+\infty]{} 0$

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad x\in\mathbb{R},\ x_0\in\mathbb{R} \text{ fixed}, a_n\in\mathbb{R}$$

$\sum |a_n(x-x_0)^n| < +\infty \rightarrow$ absolute convergence

↳ if we don't we can still have conditional convergence

- similar to geometric series

Theorem

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

1) converges only at $x=x_0$ $\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n = \overset{f(x_0)}{a_0} + a_1(x-x_0)+\dots \right)$

2) converges $\forall x\in\mathbb{R}$

3) $\exists R>0$ s.t. $|x-x_0|<R$ converges absolutely

$|x-x_0|>R$ does not converge

Proof

$x_0=0$

Suppose $x_1\neq 0$ $\quad \sum a_n x_1^n < +\infty$

$$\Downarrow$$

$$\lim_{n\to+\infty} a_n x_1^n = 0 \Rightarrow (a_n x_1^n) \text{ bounded}$$

$$\exists M>0 \quad |a_n x_1^n| < M$$

$$|a_n x^n| = |a_n \tfrac{x^n}{x_1^n} x_1^n| = \underset{\substack{\wedge \\ M \\ \forall n}}{|a_n x_1^n|} \left|\frac{x}{x_1}\right|^n \le M \underset{\substack{\wedge \\ +\infty}}{\sum} \left|\frac{x}{x_1}\right|^n$$

$$\rightarrow \left|\frac{x}{x_1}\right| < 1$$

$$|x|<|x_1|$$

- $R = \sup\{|x|\in\mathbb{R} \text{ s.t. } \sum a_n(x-x_0)^n < +\infty\} \rightarrow 3$ $(0<R<+\infty)$
- if $\nexists x_1\neq x_0 \rightarrow 1$ $(R=0)$
- if $\forall x, \sum a_n(x-x_0)^n < +\infty \rightarrow 2$ $(R=\infty)$

Proposition

$$\sum a_n (x-x_0)^n$$

Assume $\exists \lim_{n\to+\infty} \sqrt[n]{|a_n|} = L \in [0,+\infty]$

R = radius of convergence

then, $R = \begin{cases} \frac{1}{L}, & L\in(0;+\infty) \\ +\infty, & L=0 \\ 0, & L=+\infty \end{cases}$

Proof

$$\sum |a_n(x-x_0)^n| = \sum |a_n||x-x_0|^n$$

- Root test $\Rightarrow \lim \sqrt[n]{|a_n||x-x_0|^n} = |x-x_0| \underbrace{\lim \sqrt[n]{|a_n|}}_{=L}$ < 1 conv. > 1 div.
  - $L\in(0,+\infty)$

- Can also use ratio test

$|x-x_0| L < 1$ convergence

$|x-x_0| < \frac{1}{L} = R$

- $L = 0$ $\quad \lim_{n\to+\infty} \sqrt[n]{|a_n||x-x_0|^n} = 0 \quad \forall x\in\mathbb{R}$ always

$0 < 1$ ↳ convergence

- $L = +\infty$ $\quad \lim_{n\to+\infty} \sqrt[n]{|a_n||x-x_0|} = +\infty \quad \forall x \neq x_0$ except for $x = x_0$

$+\infty > 1$ ↳ divergence

ex.

$$\sum x^n$$

$a_n \equiv 1$

$L = \lim \frac{a_{n+1}}{a_n} = 1$

$R = 1$

$|x| < 1$ convergence

$|x| > 1$ does not converge (diverges or indeterminate)

$|x| = 1 \begin{cases} x = 1 & \text{div.} \\ x = -1 & \text{ind.} \end{cases}$

5 May

Taylor series

f is differentiable ∞-many times

$$\sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k \overset{?}{=} f(x)$$

ex. $f(x) = |x|^3$

$f'(0) = \lim_{x\to 0^+} \frac{|x|^3 - 0}{x} = \frac{x^3}{x} = 0$

$\lim_{x\to 0^-} \frac{|x|^3 - 0}{x} = \frac{-x^3}{x} = 0$

differentiable just 2 times

ex. $f(x) = e^x$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad a_n = \frac{1}{k!}$$

- $L = \lim_{n \to +\infty} \sqrt[n]{\frac{1}{n!}}$
- $L = \lim_{n \to +\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \to +\infty} \frac{1}{n+1} = 0$

$R = \infty$ -> converges absolutely $\forall x \in \mathbb{R}$

$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \xrightarrow[n \to +\infty]{} 0$$

$R_n(x, 0) \xrightarrow[n \to +\infty]{} 0$

$$|R_n(x,0)| = \left| e^{\xi} \frac{x^{n+1}}{(n+1)!} \right| \xrightarrow[n \to +\infty]{} 0 \checkmark \Rightarrow e^x = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$\frac{f^{n+1}(\xi)}{(n+1)!} x^{n+1}$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \qquad (x = 1)$$

9 May

ODE - Ordinary Differential Equation

$$F(x, y(x), y'(x), \dots, y^k(x)) = 0 \quad \forall x \in I \qquad \text{order } k$$

↳ k - derivatives

domain of existence

$y : I \subset \mathbb{R} \ni x \longrightarrow \mathbb{R}$ diff. k times

unknown $= y$

$F(x, y, y') = 0$ - first order o.d.e.
$F(x, y, y', y'') = 0$ - second order o.d.e.
} max of order 2

Equation in normal form

- first o.d.e.

  $y' = f(x, y(x))$

  $ay' + by + c = 0$ [linear case]

- second o.d.e.

  $y'' = f(x, y(x), y'(x))$

  $\rightarrow a_2 y'' + a_1 y' + a_0 y = f(x)$ [linear case]

$y'(x) = a(x) y(x)$ -> linear homogeneous o.d.e.

$f(x, y(x)) = a(x) \cdot y(x)$

$y'(x) = 2y(x)$ $\quad a(x) \equiv 2$ (constant function)

$f(x, y) = 2y$ autonomous equations (does not depend on x)

ex. $y(x) = e^{2x}$ -> $y'(x) = 2y(x) = 2e^{2x}$ (solution is not unique, general solution is called general integral)

- to fix one solution we define an initial data

$$y(x_0) = y_0$$

Cauchy Problem

- initial data

ex. $\begin{cases} y' = 2y \\ y(0) = 1 \end{cases} \to y(x) = e^{2x}$

general integral:

$\begin{cases} y(x) = c e^{2x} \\ y(0) = c = 1 \end{cases}$

By separation of variables

$\frac{y'}{y(x)} = a(x)$ $\qquad y'(x) = g(x) \cdot f(y)$

ex.

$y' = 2x \cdot y(x)$

$y' = x^2 \cdot \ln y$

Solution

$y'(x) = a(x) y(x) \leftarrow y \equiv 0$ is one solution (trivial solution)

to find other cases:

$\frac{y'(x)}{y(x)} = a(x)$

Relation with integrals

$\int_a^b f' = f(b) - f(a)$

$\hookrightarrow$ primitive

$y' = f'(x)$

$\int y' = \int f'(x) dx$

$y(x) = \int f'(x) dx + c$

Fund. thm. integral calculus

$\to \underbrace{\int \frac{y'(x)}{y(x)} dx}_{\ln|y(x)|} = \underbrace{\int a(x) dx}_{A(x) \text{ is primitive of } a} + \underline{c}$

$\underline{y(x)} = \underline{c}\, e^{\underline{A(x)}}$

ex. $y' = 2y$

$\frac{y'}{y} = 2 \to \int \frac{y'}{y} = \int 2 \to \ln|y(x)| = 2x + c$

$e^{\ln|y(x)|} = e^{2x+c}$

$|y(x)| = e^{2x} \cdot e^c$ $\quad c \neq 0$

$y(x) = c e^{2x}$ $\quad c \in \mathbb{R}$ also include

ex.

$$\begin{cases} y'(x) = y^2 \\ y(0) = 1 \end{cases}$$

- $\int \frac{y'}{y^2}\,dx = \int 1\,dx$

$$-\frac{1}{y(x)} = x + c$$

$$y(x) = -\frac{1}{x+c}$$

then fix the constant

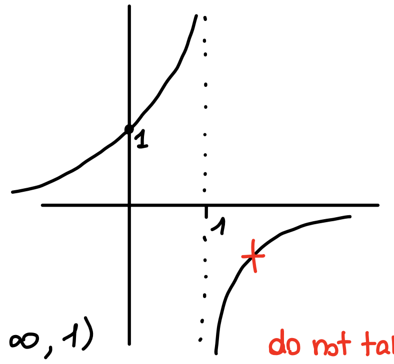
$$\downarrow$$

$$y(0) = -\frac{1}{0+c} = 1$$

$$c = -1$$

this solution is unique $\leftarrow$ $y(x) = \frac{1}{1-x}$

$\hookrightarrow$ not defined in 1



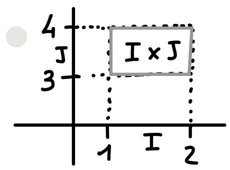
$(-\infty, 1)$

do not take

because initial data was (0,1) so we only take that line

Theorem

$$\begin{cases} y' = g(x) f(y) \\ y(x_0) = y_0 \end{cases} \qquad \begin{matrix} g: I \longrightarrow \mathbb{R} \\ f: J \longrightarrow \mathbb{R} \end{matrix} \quad \to \text{continuous}$$



then, $\forall (x_0, y_0) \in I \times J$ $\exists$ at least a solution in $I_1 \subset I$

12 May

Homogeneous first order ODE

$A(x)$ is the primitive of a

$$A(x) = \int a(x)\,dx$$

$$\to y(x) = c\,e^{A(x)}$$

Inhomogeneous first order ODE

$$y'(x) = a(x) y(x) + b(x)$$

$$\to y(x) = c(x)\,e^{A(x)}$$

$$y'(x) = \underline{c'(x)\,e^{A(x)}} + \underbrace{\underline{c(x)}\,\underline{A'(x)}\,\underline{e^{A(x)}}}_{a(x)\ \ \ \ y} = \underline{a(x) y(x)} + \underline{b(x)}$$

$$c'(x) = b(x)\,e^{-A(x)}$$

$$c(x) = \int b(x)\,e^{-A(x)}\,dx + c$$

$$\to y(x) = e^{A(x)}\left(c + \int b(x)\,e^{-A(x)}\,dx\right)$$

$$= c\,e^{A(x)} + e^{A(x)} \int b(x)\,e^{-A(x)}\,dx$$

ex.

a) $y'(x) = \underbrace{(\tan x)}_{a(x)} y(x) + \underbrace{\sin x}_{b(x)}$

$\left.\begin{array}{l} \tan x \to \mathbb{R} \setminus \{\pm\frac{\pi}{2} + k\pi\} \\ \sin x \to \mathbb{R} \end{array}\right\}$ continuity

depending on initial data we can remove absolute value

$A(x) = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| = -\ln(\cos x) + c$

$\downarrow$ $(-\frac{\pi}{2}; \frac{\pi}{2})$

cancel out: $\frac{e^c}{e^c} = 1$

so can remove

$y(x) = e^{-\ln(\cos x)} e^{c} \left(c + e^{-c} \int b(x) e^{\ln(\cos x)} dx\right)$

$$y(x) = \frac{1}{\cos x}\left(c + \int \sin x \cos x \, dx\right)$$

$$= \frac{1}{\cos x}\left(c + \frac{\sin^2 x}{2}\right)$$

b) $\begin{cases} y'(x) = (\tan x) y(x) + \sin x \\ y(\frac{\pi}{4}) = 2 \end{cases}$

$$y\left(\frac{\pi}{4}\right) = \frac{1}{\cos\frac{\pi}{4}}\left(c + \frac{\sin^2\frac{\pi}{4}}{2}\right) = 2$$

interval of existence: $(-\frac{\pi}{2}; \frac{\pi}{2})$

$$\frac{2}{\sqrt{2}}\left(c + \frac{1}{4}\right) = 2 \longrightarrow c = \sqrt{2} - \frac{1}{4}$$

$$y(x) = \frac{1}{\cos x}\left(\sqrt{2} - \frac{1}{4} + \frac{\sin^2 x}{2}\right)$$ $\to$ solution of Cauchy Problem

Formula to solve with initial data directly

$\hookrightarrow y(x_0) = y_0$

$$y(x) = e^{\int_{x_0}^{x} a(\tau) d\tau}\left(y_0 + \int_{x_0}^{x} e^{-A(\tau)} b(\tau) \, d\tau\right.$$

Separation method

$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$$ <u>not linear</u>

$y \equiv 0$ is a solution

$$\frac{y'}{\sqrt{y}} = 1 \rightarrow \int \frac{y'}{\sqrt{y}} dx = \int dx$$

$$\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = x$$

to guarantee the uniqueness of the solution: boundness in neighbourhood of y_0 of the first derivative

$$2\sqrt{y} = x + c$$

$$y = \left(\frac{x+c}{2}\right)^2$$

$$c = 0 \rightarrow y(x) = \left(\frac{x}{2}\right)^2$$

<u>biforkation</u>

↳ 2 solutions

- $y = 0$
- $y = \left(\frac{x}{2}\right)^2$

16 May

Second order ODE

-> linear homogeneous $a_0, a_1, a_2 \in \mathbb{R}$

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$$y'' = \frac{-a_1}{a_2} y' - \frac{a_0}{a_2} y$$ -> normal form

turn into:

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$ -> characteristic polynomial / equation

3 answers:

1) $\exists \ \lambda_1 \neq \lambda_2 \in \mathbb{R} \quad (\Delta > 0)$ | $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ c_1, c_2 arbitrary constants

2) $\exists \ \lambda_1 \equiv \lambda_2 \in \mathbb{R} \quad (\Delta = 0)$ | $y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$

3) $\nexists \ \mathbb{R}$ solution $(\Delta < 0) \rightarrow$ 3) $\lambda_1 \pm i\lambda_2$ | $y(x) = e^{\lambda_1 x}(c_1 \cos \lambda_2 x + c_2 \sin \lambda_2 x)$

↳ complex numbers

$$\left\{ \begin{array}{l} c_1 e^{(\lambda_1 + i\lambda_2)x} + c_2 e^{(\lambda_1 - i\lambda_2)x} \\ c_1 e^{\lambda_1 x} e^{i\lambda_2 x} + c_2 e^{\lambda_1 x} e^{-i\lambda_2 x} \end{array} \right\}$$

↳ $\cos - \lambda_2 x - i\sin \lambda_2 x$

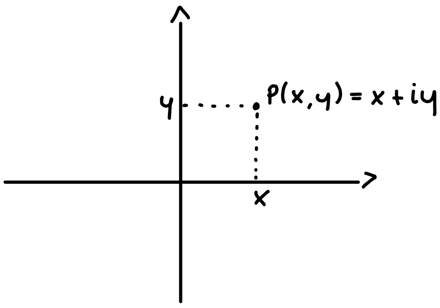
↳ $\cos \lambda_2 x + i\sin \lambda_2 x$

real part

immaginary part

Note on Complex numbers

$\mathbb{R}^2$



$i \to$ immaginary unit

$-i, \frac{1}{2}i, i, 2i, \ldots$

$\rho e^{i\theta} = z = x + iy$

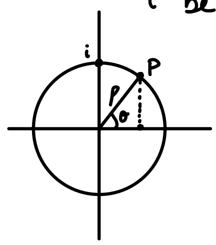
x = Real part of z = $\rho\cos\theta$

y = Imm. part of z = $\rho\sin\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

$i^2 = -1$

i becomes part of a circle



$i \to \rho = 1 \quad \theta = \frac{\pi}{2}$

in solving second order eq. with $\Delta < 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{-(b^2 - 4ac)}}{2a}$$

---

ex.

$$2y'' - 3y' + y = 0$$

$$2\lambda^2 - 3\lambda + 1 = 0$$

$$\frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} \begin{matrix} \nearrow 1 \\ \searrow \frac{1}{2} \end{matrix}$$

$$\lambda_1 = 1 \quad \lambda_2 = \frac{1}{2}$$

$$y(x) = c_1 e^x + c_2 e^{\frac{x}{2}}$$

fix initial data:

$$\begin{cases} a_2 y'' + a_1 y' + a_0 = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

ex.

$$\begin{cases} 2y'' - 3y' + y = 0 \\ y'(0) = 2 \\ y(0) = -2 \end{cases}$$

$y(1) = c_1 + c_2 = -2$

$y'(x) = c_1 e^x + \frac{1}{2} c_2 e^{\frac{x}{2}}$

$y'(0) = c_1 + \frac{c_2}{2} = 2$

$$\Rightarrow \begin{cases} c_1 + c_2 = -2 \\ c_1 + \frac{c_2}{2} = 2 \end{cases} \quad \begin{cases} c_1 = 6 \\ c_2 = -8 \end{cases}$$

ex. $\begin{cases} y'' + 2y' + 3y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$

$\lambda^2 + 2\lambda + 3 = 0$

$\frac{-2 \pm \sqrt{4-12}}{2}$ $\quad \Delta < 0$

$\frac{-2 \pm i\sqrt{8}}{2}$

$-1 \pm i\sqrt{2}$ $\quad \lambda_1 = -1 \quad \lambda_2 = \sqrt{2}$

$y(x) = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$ $\rightarrow$ $y(x) = e^{-x}(\cos\sqrt{2}x + \frac{3\sqrt{2}}{2}\sin\sqrt{2}x)$

$y(0) = c_1 = 1$

$y'(x) = -e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + e^{-x}(-c_1\sqrt{2}\sin\sqrt{2}x + c_2\sqrt{2}\cos\sqrt{2}x)$

$y'(0) = -c_1 + c_2\sqrt{2} = -1 + c_2\sqrt{2} = 2$

$\rightarrow c_2 = \frac{3\sqrt{2}}{2}$

17 May

Inhomogeneous second order ODE

$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad a_2, a_1, a_0 \in \mathbb{R}$

$f(x) = \begin{cases} P_n(x) & \text{polynomial of } n\text{-degree} \\ A\cos ax, \; B\sin bx, \; A\cos ax + B\sin ax & \\ Ce^{Bx} & B \text{ is not root of characteristic polynomial } x \end{cases}$

- Self-similarity

$y_0 \rightarrow$ homogeneous solution from hom. equation

$y = y_0 + y_p \rightarrow$ general integral

$\hookrightarrow$ particular solution

$f(x) = x^2 + x$

$y_p = Ax^2 + Bx + C \quad \rightarrow a_0 \neq 0$ 	have to be of same degree

$y_p = x(Ax^2 + Bx + C) \rightarrow a_0 = 0,\ a_1 \neq 0$

$y_p = x^2(Ax^2 + Bx + C) \rightarrow a_0 = 0,\ a_1 = 0$

ex.

- $y'' - 2y = 1 + x^2$

$\rightarrow y'' - 2y = 0 \quad \lambda^2 - 2 = 0 \quad \lambda = \pm\sqrt{2}$

$y_o(x) = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}$

$\rightarrow y_p = Ax^2 + Bx + C$

$\rightarrow y_p' = 2Ax + B$

$\rightarrow y_p'' = 2A$

$\rightarrow 2A - 2(Ax^2 + Bx + C) = 1 + x^2$

$-2Ax^2 - 2Bx + 2A - 2C = 1 + x^2$

$\begin{cases} -2A = 1 \\ -2B = 0 \\ 2A - 2C = 1 \end{cases} \quad \begin{cases} A = -\frac{1}{2} \\ B = 0 \\ C = -1 \end{cases}$

$\rightarrow y_p = -\frac{x^2}{2} - 1$

ex.

- $y'' + 2y' = x$

$y'' + 2y' = 0 \quad \lambda^2 + 2\lambda = 0 \quad \lambda = -2, 0$

$y_o(x) = c_1 + c_2 e^{-2x}$

$y_p = x(Ax + B) = Ax^2 + Bx$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$2A + 2(2Ax + B) = x \rightarrow 2A + 4Ax + 2B = x$

$\begin{cases} 4A = 1 \\ 2A + 2B = 0 \end{cases} \quad \begin{cases} A = 1/4 \\ B = -1/4 \end{cases}$

$\rightarrow y_p = \frac{x^2}{4} - \frac{x}{4}$

ex.

- $y'' + y' + 2y = 2\cos t$

$y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t - A\sin t + B\cos t + 2(A\cos t + B\sin t) = 2\cos t$

$\cos t(-A + B + 2A) + \sin t(-B - A + 2B) = 2\cos t$

$(A + B)\cos t + (B - A)\sin t = 2\cos t$

$\begin{cases} A + B = 2 \\ B - A = 0 \end{cases} \quad \begin{cases} A = 1 \\ B = 1 \end{cases}$

$\rightarrow y_p = \cos t + \sin t$

$y'' + y' + 2y = 2\cos t$

$\lambda^2 + \lambda + 2 = 0$

$\lambda = \frac{-1 \pm \sqrt{1-8}}{2}$

$y_o(x) = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right)$

$\downarrow$

$y(x) = \quad + \cos x + \sin x$

ex.

$$\begin{cases} y'' + 2y' + 3y = 6\sin 3x \\ y(0) = 1/2 \\ y'(0) = -5/2 - \sqrt{2} \end{cases}$$

homog. solution:

$$\lambda^2 + 2\lambda + 3 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm i\sqrt{8}}{2} = -1 \pm i\sqrt{2} \qquad \lambda_1 = -1 \quad \lambda_2 = \sqrt{2}$$

$$y_0(x) = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$

Particular solution:

$$y_p = A\cos 3x + B\sin 3x$$

$$y'_p = -3A\sin 3x + 3B\cos 3x$$

$$y''_p = -9A\cos 3x - 9B\sin 3x$$

$$-9A\cos 3x - 9B\sin 3x - 6A\sin 3x + 6B\cos 3x + 3A\cos 3x + 3B\sin 3x = 6\sin 3x$$

$$\cos 3x(-9A + 6B + 3A) + \sin 3x(-9B - 6A + 3B) = 6\sin 3x$$

$$\begin{cases} -6A + 6B = 0 \\ -6B - 6A = 6 \end{cases} \begin{cases} B = A \\ A = -1/2 \end{cases} \begin{cases} B = -\frac{1}{2} \end{cases}$$

$$\rightarrow y_p = -\frac{1}{2}\cos 3x - \frac{1}{2}\sin 3x$$

Impose conditions:

$$y(0) = c_1 - \frac{1}{2} = \frac{1}{2} \rightarrow c_1 = 1$$

$$y'(x) = -e^{-x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + e^{-x}(-\sqrt{2}c_1\sin\sqrt{2}x + \sqrt{2}c_2\cos\sqrt{2}x) + \frac{3}{2}\sin 3x - \frac{3}{2}\cos 3x$$

$$y'(0) = -c_1 + \sqrt{2}c_2 - \frac{3}{2} = -1 + \sqrt{2}c_2 - \frac{3}{2} = -\frac{5}{2} - \sqrt{2} = -\frac{5}{2} + \sqrt{2}c_2 = -\frac{5}{2} - \sqrt{2} \rightarrow c_2 = -1$$

$$y(x) = e^{-x}(\cos\sqrt{2}x - \sin\sqrt{2}x) - \frac{1}{2}\cos 3x - \frac{1}{2}\sin 3x$$

ex.

$$\begin{cases} 3y''-2y'-y=e^{2x} \\ y(0)=0 \\ y'(0)=1 \end{cases}$$

Hom. solution:

$3\lambda^2-2\lambda-1=0$

$\lambda = \frac{1\pm\sqrt{1+3}}{3}$ $\begin{cases} 1 \\ -\frac{1}{3} \end{cases}$

Particular solution:

$y_p = Ae^{2x}$

$y'_p = 2Ae^{2x}$

$y''_p = 4Ae^{2x}$

$12Ae^{2x} - 4Ae^{2x} - Ae^{2x} = e^{2x}$

$e^{2x}(12A-4A-A) = e^{2x}$

$(7A)e^{2x} = e^{2x}$

$A = \frac{1}{7}$

$\rightarrow y(x) = c_1e^x + c_2e^{-\frac{1}{3}x} + \frac{e^{2x}}{7}$ $\rightarrow y(0) = c_1+c_2+\frac{1}{7} = 0$

$y'(x) = c_1e^x - \frac{1}{3}c_2e^{-\frac{1}{3}x} + \frac{2}{7}e^{2x}$ $\rightarrow y'(0) = c_1 - \frac{1}{3}c_2 + \frac{2}{7} = 1$

$\rightarrow$ resolve system

$$\begin{cases} c_1+c_2 = -\frac{1}{7} \\ c_1-\frac{1}{3}c_2 = \frac{5}{7} \end{cases} \quad \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -\frac{9}{14} \end{cases}$$

$y(x) = \frac{e^x}{2} - \frac{9}{14}e^{-\frac{1}{3}x} + \frac{e^{2x}}{7}$

19 May

Products of ODE

- $f(x) = P_m(x)e^{ax}$

  $y_p = Q_m e^{ax}$

- $f(x) = P_m(x)\cos\alpha x$

  $P_m(x)\sin\alpha x$

  $y_p = Q_m(x)\cos\alpha x + R_m(x)\sin\alpha x$

- $f(x) = Ae^{ax}\cos\alpha x$

  $y_p = e^{ax}(E\cos\alpha x + F\sin\alpha x)$

- $f(x) = P_m(x)e^{ax}\cos\alpha x$

  $y_p = e^{ax}(Q_m(x)\cos\alpha x + R_m(x)\sin\alpha x)$

Resonance

↳ not included

---

ex.

$y'' + y' - 6y = f(x)$

→ hom. sol.

$\lambda^2 + \lambda - 6 = 0$

$\lambda = \frac{-1 \pm \sqrt{1+24}}{2} \begin{matrix} \nearrow 2 \\ \searrow -3 \end{matrix}$ $\quad \lambda_1 = 2 \quad \lambda_2 = -3$

$y_h(x) = c_1 e^{2x} + c_2 e^{-3x}$

- $f(x) = (3x^2 - x + 2)e^{3x}$ $\quad$ same degree

$y_p = (A_2x^2 + A_1x + A_0)e^{3x}$

$y_p' = 3e^{3x}(A_2x^2 + A_1x + A_0) + e^{3x}(2A_2x + A_1) = e^{3x}(3A_2x^2 + 3A_1x + 3A_0 + 2A_2x + A_1)$

$y_p'' = 3e^{3x}(3A_2x^2 + 3A_1x + 3A_0 + 2A_2x + A_1) + e^{3x}(6A_2x + 3A_1 + 2A_2)$

$= e^{3x}(9A_2x^2 + 9A_1x + 9A_0 + 6A_2x + 3A_1 + 6A_2x + 3A_1 + 2A_2)$

$e^{3x}(9A_2x^2 + x(9A_1 + 12A_2) + (9A_0 + 6A_1 + 2A_2))$

$y_p = e^{3x}(9A_2x^2 + (9A_1 + 12A_2)x + (9A_0 + 6A_1 + 2A_2) + 3A_2x^2 + 3A_1x + 3A_0 + 2A_2x + A_1$

$-6A_2x^2 - 6A_1x - 6A_0) = e^{3x}(6A_2x^2 + 6A_1x + 14A_2x + 6A_0 + 7A_1 + 2A_2$

$$y_p = \cancel{e^{3x}}\left(6A_2x^2 + (6A_1 + 14A_2)x + (6A_0 + 7A_1 + 2A_2)\right) = \cancel{e^{3x}}(3x^2 - x + 2)$$

$$\begin{cases} 6A_2 = 3 \\ 6A_1 + 14A_2 = -1 \\ 6A_0 + 7A_1 + 2A_2 = 2 \end{cases} \quad \begin{cases} A_2 = 1/2 \\ 6A_1 + 14 \cdot \frac{1}{2} = -1 \\ 6A_0 + 7A_1 + 2 \cdot \frac{1}{2} = 2 \end{cases} \quad \begin{cases} A_2 = 1/2 \\ A_1 = -4/3 \\ 6A_0 + 7 \cdot -\frac{4}{3} = 1 \end{cases} \quad \begin{cases} A_2 = 1/2 \\ A_1 = -4/3 \\ A_0 = \frac{31}{18} \end{cases}$$

$$y(x) = \left(\frac{x^2}{2} - \frac{4x}{3} + \frac{31}{18}\right)e^{3x} + c_1e^{2x} + c_2e^{-3x}$$

- $f(x) = e^{-x}\sin 2x$

$$y_p = e^{-x}(A\cos 2x + B\sin 2x)$$

$$y_p' = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x)$$

$$= e^{-x}\left(\cos 2x(-A + 2B) + \sin 2x(-B - 2A)\right)$$

$$y_p'' = -e^{-x}(\cos 2x(2B - A) + \sin 2x(-B - 2A)) + e^{-x}(-2(-A + 2B)\sin 2x + 2(-B - 2A)\cos 2x)$$

$$= e^{-x}((-2B + A)\cos 2x + (B + 2A)\sin 2x + (2A - 4B)\sin 2x + (-2B - 4A)\cos 2x)$$

$$= e^{-x}(\cos 2x(-2B + A - 2B - 4A) + \sin 2x(B + 2A + 2A - 4B))$$

$$e^{-x}(\cos 2x(-4B - 3A - A + 2B - 6A) + \sin 2x(4A - 3B - B - 2A - 6B)) = e^{-x}\sin 2x$$

$$\cancel{e^{-x}}(\cos 2x(-2B - 10A) + \sin 2x(2A - 10B)) = \cancel{e^{-x}}\sin 2x$$

$$\begin{cases} -2B - 10A = 0 \\ 2A - 10B = 1 \end{cases} \quad \begin{cases} A = -\frac{1}{5}B \\ 2 \cdot -\frac{1}{5}B - 10B = 1 \end{cases} \quad \begin{cases} A = -\frac{1}{5}B \\ -\frac{2}{5}B - 10B = 1 \end{cases} \quad \begin{cases} A = 1/52 \\ B = -\frac{5}{52} \end{cases}$$

$$y_p = e^{-x}\left(\frac{\cos 2x}{52} - \frac{5\sin 2x}{52}\right)$$

$$y(x) = e^{-x}\left(\frac{\cos 2x}{52} - \frac{5\sin 2x}{52}\right) + c_1e^{2x} + c_2e^{-3x}$$

- $f(x) = \frac{1}{4}(x-1)e^{3x}\cos x$

$$y_p = e^{3x}[(Ax+B)\cos x + (Ex+F)\sin x]$$

$$y_p' = 3e^{3x}[(Ax+B)\cos x + (Ex+F)\sin x]$$

$$+ e^{3x}[A\cos x - B\sin x - Ax\sin x + E\sin x + Ex\cos x + F\cos x]$$

$$= e^{3x}[\cos x(3Ax+3B+A+F+Ex) + \sin x(3Ex+3F-B+E-Ax)]$$

$$= e^{3x}[\cos x((3A+E)x+3B+A+F) + \sin x((3E-A)x+3F-B+E)]$$

$$y_p'' = 3e^{3x}[\ \overset{\downarrow}{\dots}\ ] + e^{3x}[-\sin x((3A+E)x+3B+A+F) + \cos x(3A+E) + \cos x((3E-A)x+3F-B+E) + \sin x(3E-A)]$$

$$= e^{3x}[\cos x((9A+3E)x+9B+3A+3F) + \sin x((9E-3A)x+9F-3B+3E) + \ ]$$

$$= e^{3x}[\cos x((9A+3E+3E-A)x+9B+3A+3F+3A+E+3F-B+E) + \sin x((9E-3A-3A-E)x + 9F-3B+3E-3B-A-F+3E-A)]$$

$$= e^{3x}[\cos x((8A+6E)x+8B+6A+2E+6F) + \sin x((8E-6A)x-2A-6B+6E-8F)]$$

$$e^{3x}[\cos x((8A+6E+3A+E-6A)x-6B+3B+A+F+8B+6A+2E+6F) + \sin x((8E-6A+3E-A-6E)x-6F+3F-B+E-2A-6B+6E-8F)]$$

$$\cancel{e^{3x}}[\cos x((5A+7E)x+5B+7A+7F+2E) + \sin x((5E-7A)x-2A-7B+7E-11F)]$$

$$= \cancel{e^{3x}}\cos x\left(\frac{1}{4}x - \frac{1}{4}\right)$$

$$\begin{cases} 5A+7E = \frac{1}{4} \\ 7A+5B+2E+7F = -\frac{1}{4} \\ 5E-7A = 0 \\ -2A-7B+7E-11F = 0 \end{cases} \quad \begin{cases} A = \frac{5}{296} \\ B = -\frac{7}{296} \\ E = \frac{7}{296} \\ F = -\frac{11}{259} \end{cases}$$

$$\begin{cases} A = 244 \\ F = -\frac{137616}{109} \\ E = 1708 \\ B = -\frac{122604388}{1635} \end{cases}$$

da rivedere

$F = -\frac{66}{1369}$ $E = \frac{7}{296}$

$B = -\frac{171}{10952}$ $A = \frac{5}{296}$