MATHEMATICAL ANALYSIS I

Summary

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Chapter 1 - Sets of Numbers:

1.6: A subset A of **R** is called **bounded from above** or **upper bounded** if there exists a real number b such that

$$x \leq b, \text{for all } x \in A .$$

Applies for **lower bound** as well.

1.9: Let $A \subset R$ be bounded from above. The **supremum** or **least upper bound** of A is the smallest of all upper bounds of A, denoted by $supA$. The number $s = supA$ is characterised by two conditions:

i) $x \leq s$ for all $x \in A$;
ii) for any real $r < s$, there exists $x \in A$ such that $x > r$.

Applies for **infimum (greatest upper bound)** as well.

Remark: Supremum may exists and not be a maximum, but when a maximum exists, then it is also the supremum of the set.

1.11: $\dfrac{n!}{k!(n-k)!}$ Permutation, $\dbinom{n}{k} = \dfrac{n(n-1)\ldots(n-k+1)}{k!}$ Combination.

Chapter 2 - Functions:

2.1: The absolute value function: $f : R \to R, \quad f(x) = \quad x \quad = \begin{cases} x \ if \ x \geq 0, \\ -x \ if \ x < 0; \end{cases}$

The sign function: $f : R \to Z, \quad f(x) = sign(x) = \begin{cases} +1 \ if \ x > 0, \\ 0 \ if \ x = 0, \\ -1 \ if \ x < 0; \end{cases}$

Floor function (Integer part): $f : R \to Z, \ f(x) = [x] = \text{the greatest integer } \leq x$

The mantissa: $f : R \to R, \ f(x) = M(x) = x - [x]$

2.3: A map with values in Y is called **onto** if $imf = Y$. This means that each $y \in Y$ is the image of one element $x \in X$ at least. The term **surjective** has the same meaning.

A function is called **one-to-one** or **injective** if every $y \in imf$ is the image of a unique element $x \in domf$.

A function is **invertible** if it is **bijective.**

2.13: Even function (with respect to the y axis) if: f (−x) = f (x).
Odd function (with respect to the origin) if: f (−x) = −f (x).

Chapter 3 - Vectors and Complex Numbers:

3.1: Polar Coordinates $rcos\theta,\ y = rsin\theta$

3.2: $r = \sqrt{x^2 + y^2},\ \theta = \begin{cases} arctan\frac{y}{x},\ if.\ x > 0, \\ arctan\frac{y}{x} + \pi,\ ifx < 0,\ y \geq 0, \\ arctan\frac{y}{x} - \pi,\ if\ x < 0,\ y < 0 \\ \frac{\pi}{2},\ if\ x = 0,\ y > 0, \\ \frac{-\pi}{2},\ if\ x = 0,\ y < 0. \end{cases}$

3.3: The sum of vectors: $v + w = (v_1 + w_1, \ldots, v_d + w_d)$.
3.4: The product of vectors: $\lambda v = (\lambda v_1, \ldots, \lambda v_d)$

3.5: The Euclidean norm, or length, of a vector v with end-point P is defined:

$$v \quad = \sqrt{\sum_{i=1}^{d} v_i^2} = \begin{cases} \sqrt{v_1^2 + v_2^2}\ if\ d = 2, \\ \sqrt{v_1^2 + v_2^2 + v_3^2}\ if\ d = 3. \end{cases}$$

3.26: Real and Imaginary part of z:

$$\Re ez = \frac{z + \tilde{z}}{2},\ \Im mz = \frac{z - \tilde{z}}{2i}$$

3.30: Exponential form or **Euler formula**:

$$e^{\theta i} = cos\theta + isin\theta.$$

3.31: Exponential form of z:

$$z = re^{i\theta}.$$

3.38: Additional forms:

$$e^z = e^x e^{iy} = e^x(cosy + isiny).$$

Chapter 4 - Limits and Continuity

4.1: Let be a point $x_0 \in \mathbb{R}$ on the real line and $r > 0$ a real number.

Neighbourhood of x_0 of radius r the open and bounded interval:

$I_r(x_0) = (x_0 - r, x_0 + r) = x \in R : \;\; x - x_0 \;\; < r$

4.5: A sequence $a : n \to a_n$ converges to the limit $\ell \in \mathbb{R}$ (or converges to ℓ or has limit ℓ)

in symbols: $\lim\limits_{n \to \infty} a_n = \ell$

if for any real $\varepsilon > 0$ there exists an integer n_ε such that

$$\forall n \geq n_0, \quad n > n_\varepsilon \Rightarrow \;\; a_n - \ell \;\; < \varepsilon$$

4.7: A sequence $a : n \to a_n$ diverges to $+\infty$ (or tends to $+\infty$ or has limit $+\infty$)

in symbols: $\lim\limits_{n \to \infty} a_n = +\infty$

if for any real $A > 0$ there exists an integer n_A such that

$$\forall n \geq n_0, \quad n > n_A \Rightarrow a_n > A$$

4.16: Let f be defined on a neighbourhood of x_0, excluding the point x_0. If f admits limit $\ell \in \mathbb{R}$ for x tending to x_0 and if

a) f is defined at x_0 but $f(x_0) \neq \ell$,

or

b) f is not defined at x_0

then we say x_0 is a removable discontinuity point for f.

4.25: Discontinuity of the first kind (jump point):

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$$

Chapter 5 - Properties and Computation of Limits:

5.26: Geometric sequences properties:

$$\lim_{x\to\infty} q^n = \begin{cases} 0 \ if \quad q \ < 1 \\ 1 \ if \ q = 1 \\ +\infty \ if \ q > 1, \\ \text{does not exists } if \ q \leq -1. \end{cases}$$

5.27 (Ratio Test): Let a_n be a sequence for which $a_n > 0$ eventually. Suppose the limit

$$\lim_{x\to\infty} \frac{a_{n+1}}{a_n} = q$$

exists, finite or infinite. If $q < 1$ then $\lim_{x\to\infty} a_n = 0$; if $q > 1$ then $\lim_{x\to\infty} a_n = +\infty$.

Chapter 6 - Local comparison of functions:

6.1: If ℓ is finite, we say that f is **controlled by** g **as** x tends to c, and the notation

$f = O(g),\ x \to c,$

This property can be made more precise by distinguishing three cases:

a) If ℓ is finite and non-zero, we say that f has the same order of
magnitude as g (or that it is of the same order of magnitude)
as x tends to c; if so, we write

$f \asymp g,\ x \to c\,.$

As a notable sub-case we have:

b) If, $\ell = 1$, we call f equivalent to g as x tends to c; in this case we
use the notation

$f \sim g,\ x \to c\,.$

c) Finally, if $\ell = 0$, we say that f is negligible with respect to g when
x goes to c; for this situation the symbol

$f = o(g),\ x \to c,$

will be used, read as ‘f is little o of g as x tends to c’

6.2: $f \sim g \quad \longleftrightarrow \quad g = g + o(g)\,.$

6.3: $o(\lambda f) = o(f)$ and $\lambda o(f) = o(f)\,.$

6.5: x^n **as** $x \to 0 : x^n = o(x^m),\ x \to 0, \longleftrightarrow\ n > m$.

6.6: $x \to \pm\infty, \longleftrightarrow , n < m$.

6.7:

a) $o(x^n) \pm o(x^n) = o(x^n)$
b) $o(x^n) \pm o(x^m) = o(x^p)$, with $p = min(n, m)$;
c) $o(\lambda x^n) = o(x^n)$, for each $\lambda \in R \backslash \{0\}$
d) $\varphi(x)o(x^n) = o(x^n)$ if φ is bounded near $x = 0$
e) $x^m o(x^n) = o(x^{m+n})$
f) $o(x^m)o(x^n) = o(x^{m+n})$
g) $[o(x^n)]^k = o(x^{kn})$.

6.8:

$sin x = x + o(x),\ x \to 0;$

$1 - cos x = \frac{1}{2}x^2 + o(x^2), x \to 0,\ or\ cos x = 1 - \frac{1}{2}x^2 + o(x^2), x \to 0;$

$\log(1 + x) = x + o(x),\ x \to 0,\ or\ \log x = x - 1 + o(x - 1),\ x \to 1;$

$e^x = 1 + x + o(x),\ x \to 0;$

$\sqrt{1 + x} = 1 + \frac{1}{2}x + o(x)\ x \to 0.$

6.8.2:

$(1 + x)^\alpha = 1 + \alpha x + o(x),\ x \to 0.$

$x^\alpha = o(e^x),\ x \to +\infty,\ \forall \alpha \in R;$

$e^x = o(\ x\ ^\alpha),\ x \to -\infty, \forall \alpha \in R;$

$\log x = o(x^\alpha),\ x \to +\infty,\ \forall \alpha > 0;$

$\log x = o(\frac{1}{x^\alpha}),\ x \to 0^+,\ \forall \alpha > 0.$

6.9:

Let f , g be two infinitesimals at c.

If $f \asymp g,\ x \to c,$ f and g are said **infinitesimals of the same order**.

If $f = o(g),\ x \to c,$, f is called **infinitesimal of higher order than** g.

If $g = o(f),\ x \to c,$ f is called **infinitesimal of smaller order than** g.

If none of the above are satisfied, f and g are **non-comparable** infinitesimals.

6.10:

Let f and g be two infinite maps at c.
If $f \asymp g,\ x \to c,$ f and g are said to be **infinite of the same order**.

If $f = o(g),\ x \to c,$, f is called **infinite of smaller order than** g.

If $g = o(f),\ x \to c,$ f is called **infinite of higher order than** g.

If none of the above are satisfied, f and g are **non-comparable.**

6.14: Let f be infinitesimal (or infinite) at c. If there exists a real number $\alpha > 0$ such that $f \asymp \varphi^{\alpha},\ x \to c,$ the constant α is called the **order of f at c with respect to the sample infinitesimal** (or **infinite**) φ .

6.15: The function $p(x) = \ell\varphi^{\alpha}(x)$ is called the **principal part of the infinitesimal (infinite) map f at c with respect to the sample infinitesimal (infinite) φ.**

6.4: Asymptotes:

Hole at point $(x_0, f_{semplified}(x_0))$ if plugging the critical point x_0 in the numerator of f gives $\frac{0}{0}$.

Vertical asymptote at a critical point x_0 if:
$\lim_{x \to x_0^-} f(x) = \pm\infty$ (left at $x = x_0$)
$\lim_{x \to x_0^+} f(x) = \pm\infty$ (right at $x = x_0$)

Horizontal asymptote (if domain is unlimited at $\pm\infty$) if:

$\lim_{x \to +\infty} f(x) = k$ (right $y = k$)
$\lim_{x \to -\infty} f(x) = h$ (left $y = h$)

Oblique asymptote (if domain is unlimited at $\pm\infty$) if:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = m \wedge \lim_{x \to +\infty} [f(x) - mx] = q \ \text{(right at } y = mx + q)$$

$$\lim_{x \to -\infty} \frac{f(x)}{x} = m \wedge \lim_{x \to -\infty} [f(x) - mx] = q \ \text{(left at } y = mx + q)$$

Chapter 7 - Global Properties of Continuous Maps:

7.2: (Existence of zeroes) Let f be a continuous map on a closed, bounded interval $[a, b]$. If $f(a)f(b) < 0$, i.e., if the values of f at the interval's end-points have different signs, f admits a zero in the open interval (a, b). If moreover f is strictly monotone on $[a, b]$, the zero is unique.

7.8: (Intermediate value theorem) If a function f is continuous on the closed and bounded interval $[a, b]$, it assumes all values between $f(a)$ and $f(b)$.

7.10: (Weierstrass' Theorem) A continuous map f on a closed and bounded interval $[a, b]$, is bounded on $[a, b]$ and on this interval it admits minimum and maximum

$$m = \min_{x \in [a,b]} f(x) \quad \text{and} \quad M = \max_{x \in [a,b]} f(x)$$

Consequently, $$f([a, b]) = [m, M]$$

7.4: Let f be a continuous map on the interval I and suppose that it admits, as x tends to the end-points of I, non-zero limits (finite or infinite) of different sign. Then f has a zero in I, which is unique if f is strictly monotone on I.

7.14: Let I be a real interval. A map $f : I \to \mathbb{R}$ is called Lipschitz on I if there exists a constant $L \geq 0$ such that $$f(x_1) - f(x_2) \quad \leq L \quad x_1 - x_2 \quad , \quad \forall x_1, x_2 \in I$$
The smallest constant satisfying (7.4) is called Lipschitz constant of f on I.

7.17: A function is called uniformly continuous on I if for any $\varepsilon > 0$, there is a satisfying $$\forall x_1, x_2 \in I, \qquad x_1 - x_2 \quad < \delta \Rightarrow \quad f(x_1) - f(x_2) \quad < \varepsilon$$

7.18: (Heine-Cantor's Theorem) Let f be a continuous map on the closed and bounded interval $I = [a, b]$. Then f is uniformly continuous on I.

Chapter 8 - Differential Calculus:

8.1: A map f defined on a neighbourhood of $x_0 \in \mathbb{R}$ is called differentiable at x_0 if the limit of the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x exists and is finite, as x approaches x_0. The real number

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called (first) derivative of f at x_0.

8.2: Let I be a subset of $dom f$. We say that f is differentiable on I if f is differentiable at each point of I.

8.6: (Algebraic operations)

$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$$

Linearity of the derivative: $(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0)$

Chain rule: $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$

8.12: Derivative of inverse function:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

8.15: If f is even (or odd), f' is odd (resp. even)

Where differentiability fails:

Corner point: $f_+'(x_0) \neq f_-'(x_0)$ e.g. $f(x) = x$

Point with vertical tangent: $f_+'(x_0) = f_-'(x_0) = \pm\infty$ e.g $f(x) = \sqrt[3]{x}$

Cusp point: $f_+'(x_0) = \pm\infty$ and $f_-'(x_0) = \mp\infty$ e.g $f(x) = \sqrt{\ x\ }$

8.23: *Critical point* (or *stationary point*) of f is a point x_0 at which f is differentiable with derivative $f'(x_0) = 0$

8.24: (Fermat's Theorem) Extremum points are critical points.

8.25: (Rolle's Theorem) Let f be continuous on $[a, b]$ and differentiable on (a, b) (at least). If $f(a) = f(b)$, there exists an $x_0 \in (a, b)$ such that $f'(x_0) = 0$

i.e. f admits at least one critical point in (a, b).

8.26: (Mean Value Theorem or **Lagrange's Theorem)**

Let f be continuous on $[a, b]$ and differentiable on (a, b) (at least).

Then, there exists an $x_0 \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(x_0)$

Every such point x_0 is called *Lagrange point for f in (a, b)*

8.28: (Cauchy's Theorem) Let f and g be maps defined on the closed, bounded interval $[a, b]$ and differentiable (at least) on (a, b). Suppose $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$

8.30: Suppose $f : I \to \mathbb{R}$ is differentiable on the interval I with bounded derivative on I, and define $L = \sup_{x \in I} |f'(x)| < +\infty$. Then f is Lipschitz constant L.

8.31: Monotonicity

Monotonically increasing if: $\forall x, y : x \leq y \to f(x) \leq f(y)$

Monotonically decreasing if: $\forall x, y : x \leq y \to f(y) \leq f(x)$

Strictly increasing if: $\forall x, y : x < y \rightarrow f(x) < f(y)$

Strictly decreasing if: $\forall x, y : x > y \rightarrow f(x) > f(y)$

If $f(x) > 0$, then f is strictly increasing.

If $f'(x) < 0$, then f is strictly decreasing.

If $f'(x) = 0$ f is constant.

8.32: Convexity

Convex $\left(\bigcup\right)$ if : $f''(x) > 0$

Concave $\left(\bigcap\right)$ if : $f''(x) < 0$

8.33: Inflection Points

Find the points where $f''(x) = 0$ and check out the signs.

8.36: A map f is said to be **of class** C^k $(k \geq 0)$ on an interval I if f is differentiable k times everywhere on I and its kth derivative $f^{(k)}$ is continuous on I.

Chapter 9 - Taylor Expansions and Applications:

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \frac{f'(a)}{3!}(x-a)^3$$

where n! denotes the factorial of n. In the more compact sigma notation, this can be written as

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where f(n)(a) denotes the nth derivative of f evaluated at the point a. (The derivative of order zero of f is defined to be f itself and $(x-a)o$ and $o!$ are both defined to be 1.)

When $a = o$, the series is also called a Maclaurin series

9.1: (Taylor Formula with Peano's Remainder)

Let n ≥ 0 and f be n times differentiable at x_0. Then the Taylor formula holds

$$f(x) = Tf_{n,x_0}(x) + o(x - x_0)^n, x \to x_0$$

where

$$Tfn, x_0(x) = \sum_{k=0}^{n} \frac{1}{k!} f^k(x_0)(x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{1}{n!} f^n(x_0)(x - x_0)^n.$$

9.2: (Taylor Formula with Lagrange's Remainder)

Let n ≥ 0 and f differentiable n times at x_0, with continuous nth derivative, be given; suppose f is differentiable n + 1 times around x_0, except possibly at x_0. Then the Taylor formula

$f(x) = Tf_{n,x_0}(x) + \dfrac{1}{n+1} f^{(n+1)}\bar{x}, (x - x_0)^{n+1}$ holds, for a suitable x^- between x_0 and x.

9.3: The Maclaurin polynomial of an even (respectively, odd) map involves only even (odd) powers of the independent variable.

9.5: Let $f : (a, b) \to \ \to \mathbb{R}$ be n times differentiable at $x_0 \in (a, b)$. If there exists a polynomial P_n, of degree ≤ n, such that

$$f(x) = P_n + o((x - x_0)^n) \qquad \text{as } x \to x_0$$

then P_n is the Taylor polynomial $T = Tf_{n,x_o}$ of order n for the map f at x_O

Chapter 10 - Integral Calculus:

10.1:

Each function F, differentiable on I, such that
$F'(x) = f(x), \quad \forall x \in I,$
is called a primitive (function) or an antiderivative of f on I

10.3:

If F and G are both primitive maps of f on I, there exist a constant c
$G(x) = F(x) + c, \ \forall x \in I$

10.8: (Linearity of the Integral)

Suppose $f(x), g(x)$ are integrable on I. For any $\alpha, \beta \in \mathbb{R}$ the map $\alpha f(x) + \beta g(x)$ is still integrable on I, and

$$\int (\alpha f(x)dx + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx$$

10.10: (Integration by Parts)

Let $f(x)$,$g(x)$ be differentiable over I. If the map $f'(x)g(x)$ is integrable on I, *then f(x)* $g'(x)$ *,and*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

10.28:

A bounded map f *on* $I=[a,b]$ is said integrable (precisely, Riemann integrable) on I if

$$\underline{\int_I} f = \overline{\int_I} f$$

The common value is called **definite integral** of f on $[a,b]$ and denoted with $\int_I f$ or $\int_a^b f(x)dx$

10.30:

The following functions are integrable on$[a,b]$;
a) Continuous maps on$[a,b]$,
b) Piecewise continuous maps on $[a,b]$,
c) Continuous maps on (a,b), which are bounded on $[a,b]$,
d) Monotone functions on$[a,b]$.

10.33:
Mean Value(Integral) of f on the interval $[a,b]$ the number

$$\text{m}(f;a,b)=\frac{1}{b-a}\int_a^b f(x)dx$$

Jensen Inequality:
$f(tx_1+(1-t)x_2) \leq tf(x_1)+(1-t)f(x_2)$

10.43: (Taylor Formula with Integral Remainder)

Let $n \geq 0$ be an arbitrary integer, f is differentiable $n+1$ times around x_0, with continuous derivative of order n+1. Then

$$f(x) - Tf_{n,x_0}(x) = \frac{1}{n!}\int_{x_0}^{x} f^{n+1}(t)(x-t)^n dt$$

10.45: (Integration by Parts)

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

10.46: (Integration by Substitution)

Take a map $\phi(x)$ defined on $[\alpha, \beta]$ with values in $[a, b]$, this formula that is differentiable with continuous derivative.

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_{\phi(\alpha)}^{\phi(\beta)} f(y)dy$$

If ϕ is a 1-1 correspondence between $[\alpha, \beta]$ and $[a, b]$ this can be written as

$$\int_a^b f(y)dy = \int_{\phi(\alpha)^{-1}}^{\phi(\beta)^{-1}} f(\phi(x))\phi'(x)dx$$

10.48:

If f is an even map

$$\int_{-a}^{f} (x)dx = 2\int_0^f (x)dx$$

if f is odd

$$\int_{-a}^{a} f(x)dx = 0$$

Chapter 11 - Improper Integrals and Numerical Series:

11.1: Let $f \in \mathfrak{R}_{loc}$([a, +∞)),

$$\int_a^{+\infty} f(x)dx = \lim_{c\to+\infty}\int_a^c f(x)dx$$

The symbol on the left is said improper integral of f on [a,+ ∞)

i) If the limit exists and is finite, we say that the map f is integrable over [a, +∞), or equivalently, that its improper integral converges

ii) If the limit exists but is infinite, we say that the improper integral of f diverges.

iii) If the limit does not exist, we say that the improper integral is indeterminate.

11.5: (Comparison Test)

Let $f \in \mathfrak{R}_{loc}$([a, +∞)) be such that $0 \le f(x) \le g(x)$ for all x∈[a, +∞). Then

$$0 \leq \int_{a}^{+\infty} f(x)dx \leq \int_{a}^{+\infty} g(x)dx$$

11.9: (Absolute Converge Test)

Suppose the function $f \in \Re_{loc}([a, +\infty))$ is such that $| f \quad | \in \Re_{loc}([a, +\infty))$. Then $f \in \Re_{loc}([a, +\infty))$ and moreover

$$\int_{a}^{+\infty} (x)dx \;\; \leq \int_{a}^{+\infty} f(x) \;\; dx$$

11.12: (Asymptotic Converge Test)

Suppose the function $f \in \Re_{loc}([a, +\infty))$ is infinitesimal order α, for x $\to +\infty$ with respect to the sample infinitesimal ϕ(x)=$\frac{1}{x}$. Then

i) if $\alpha > 1$, $f \in \Re([0, +\infty))$;

ii) if $\alpha \leq 1$, $\int_{a}^{+\infty} f(x)dx$ de diverges.

11.15: Let $f \in \Re_{loc}([a, ,b))$ and define, formally

$$\int_{a}^{b} f(x)dx = \lim_{c \to b^{-}} \int_{a}^{c} f(x)dx;$$

the left-hand side is called improper integral of f over (a, b).

i) If the limit exists and is finite, one says f is (improperly) integrable on (a, b), or that its improper integral converges.

ii) If the limit exists but is infinite, one says that the improper integral of f diverges.

iii) If the limit does not exist, one says that the improper integral is indeterminate.

11.17: (Comparison Test)

Let $f, g \in \Re_{loc}$ ([a, b)) be such that 0$\leq$ f(x) $\leq$g(x) for any x$\in [a, b)$. Then

$$0 \leq \int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx$$

11.18: (Asymptotic Comparison Test)

If $f \in \Re_{loc}([a, b))$ is infinite of order α as x $\to b^{-}$ with respect to ϕ(x)=$\frac{1}{(b-x)}$, then

i) if $\alpha < 1, f \in \Re([a,b))$;

ii) ii) if $\alpha \geq 1$, $\int_a^b f(x)dx$ diverges.

Chapter 13 - Ordinary Differential Equations:

13.5: An ODE of order n in I is an equation of the form $\mathscr{F}(x, y, y', \ldots, y^{(n)}) = 0$.

A solution to ODE is a function $y \in C^n(I)$ such that

$\mathscr{F}(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$

13.7: Let f be a real valued map defined on a subset of $\mathbb{R}^2$. A solution to the differential equation

$y' = f(x, y)$

Over an interval I of $\mathbb{R}$ is then a differentiable map $y = y(x)$ on I such that $y'(x) = f(x, y(x))$ for any $x \in I$.

13.11.: An **initial-value** problem, also called **Cauchy problem**, for **13.7** on the interval I consists in determining a differentiable function $y = y(x)$ such that

$$\begin{cases} y' = f(x, y) \ in \ I, \\ y(x_0) = y_0 \end{cases}$$

with given points $x_0 \in I, y_0 \in \mathbb{R}$.

Assume that f is continuous, then,

$\exists\ I_\delta(x_0)$ and $\exists!\ y \in C^1(I_\delta(x_0))$ solutions to Cauchy problem in $I_\delta(x_0)$. (Means that the solution exists and it is **unique**.)

13.12: The variables are said “separable” in differential equations of type $y' = g(x)h(y)$, (g, h good enough such as $g, h \in C^1$) which can be integrated $\int \frac{dy}{h(y)} = \int g(x)dx$.

13.16: Homogeneity refers to the form $y' = \varphi(\frac{y}{x})$ (φ good enough such as $\varphi \in C^1$) in which $\varphi = \varphi(z)$ is continuous in the variable z. Make substitution $z' = \frac{y(x)}{x} \leftrightarrow \frac{\varphi(z) - z}{x}$

13.18: A differential equation of type $y' + a(x)y = b(x)$
(a, b good enough such as $a, b \in C$) where a and b are continuous maps on I.

13.20: $A(x)$ denotes a primitive of $a(x)$, i.e.,

$\int a(x)dx = A(x) + C,\ C \in \mathbb{R}$

13.21: We call $B(x)$ a primitive of $e^{A(x)}b(x)$.

13.22: General solution to **13.18** reads

$y(x) = e^{-A(x)}(B(x) + C)$, where A(x) and B(x) are defined by **13.20** and **13.21.**

13.23: The integral is sometimes found in the more telling form

$$y(x) = e^{-\int a(x)dx} b(x)dx \, .$$

13.28: Second order equations: $y'' = f(x, y')$. (f good enough such as $f \in C^1$). Then reduce to the first form by substitution $z = y'$. It then transforms into $z' = f(x, z)$.

13.33: A linear equation of order two with constant coefficients has the form $y'' + ay' + by = g(x)$ (g good enough such as $g \in C^1$). If it is Homogenous ($g = 0$), it is easy to treat.

13.36: We set $\mathscr{L}y = y'' + ay' + by = 0$. Then looking for solutions with the exponential form $y(x) = e^{\lambda x}$ gives $\mathscr{L}(e^{\lambda x} = (\lambda^2 + a\lambda + b) \cdot e^{\lambda x} \longleftrightarrow \lambda^2 + a\lambda + b = 0$.

Second order homogenous differential equations:

<u>First case:</u>

$\lambda_1, \lambda_2 \in \mathbb{R}$, and $\lambda_1 \neq \lambda_2 \rightarrow y_1(x) = e^{\lambda_1 x}$, $y_2(x) = e^{\lambda_2 x}$ (2 independent solutions),
$\longrightarrow S = \{ \ y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} : \ c_1, c_2 \in \mathbb{R} \ \}$ (2 free parameters) is the set of all solutions of (E).

The function $y(x) = y(c_1, c_2; x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ is called **general integral** of the equation.

<u>Second case:</u>

$\lambda_1, \lambda_2 = \lambda \in \mathbb{R} \longrightarrow y_1(x) = e^{\lambda x}, y_2(x) = xe^{\lambda x}$ (2 independent solutions).
$\longrightarrow S = \{ \ y(x) = (c_1 + c_2 x) \cdot e^{\lambda x} : c_1, c_2 \in \mathbb{R}\}$ (2 free parameters) is set of all solutions.

<u>Third case:</u>

$$\lambda_1 = \lambda = -\frac{a}{2} + \frac{\sqrt{\Delta}}{2} i,$$

$$\longrightarrow y_1(x) = e^{\lambda x}, \ y_2(x) = e^{\bar{\lambda} x} \text{ (2 independent solutions)}$$

$$\lambda_2 = \bar{\lambda} = -\frac{a}{2} - \frac{\sqrt{\Delta}}{2} i$$

Set $\sigma = -\dfrac{a}{2}$ and $w = \dfrac{\sqrt{\Delta}}{2}$. Then:

$\longrightarrow S = \{\ y(x) = c_1 e^{\lambda x} + c_2 e^{\lambda x}) : c_1, c_2 \in \mathbb{C}\} = \{y(x) = e^{\sigma x}(c_2 \cos(wx) + c_2 \sin(wx) : c_1, c_2 \in \mathbb{R}$ (2 free parameters) is the set of all solutions of (E).

Second order non-homogenous differential equations:

$(E)\ y'' + ay' + by = g(x)$ (g good enough such as $g \in C^1$). The general integral of (E) is obtained as $y(c_1, c_2; x) = y_0(c_1, c_2; x) + y_p(x)$. The part $y_0(c_1, c_2; x)$ is the general integral of the **homogeneous** equation (i.e. with $g = 0$). $y_p(x)$ part is the particular solution of (E).

Finding particular solutions:

Assume that $g(x) = p_n(x)e^{\mu x}\cos(\theta x)\ or\ g(x) = p_n(x) = e^{\mu x}\sin(\theta x)$ for some $\mu, \theta \in \mathbb{R}, n \in \mathbb{N}$ and p_n, q_n polynomials of degree n.

Then look for solutions in the form:

$y_p(x) = x^m e^{\mu x}(q_{1,n}(x)\cos(\theta x) + q_{2,n}(x)\sin(\theta x))$.

$m = 0$ in most of cases.

When $m \neq 0 \begin{cases} \Delta > 0: \text{ if } \mu = \lambda_1 \text{ and } \theta = 0 \to m = 1 \\ \Delta = 0: \text{ if } \mu = \lambda \text{ and } \theta = 0 \to m = 2 \\ \Delta < 0: \text{ if } \mu = \sigma \text{ and } \theta = w \to m = 1 \end{cases}$

If the second order non-homogenous differential equation is in the exponential form with a degree one polynomial constant, such as $y'' - 2y' + y = e^{3x}$, the particular solution to the problem is $y_p(x) = (\alpha x + \beta)e^{3x}$. Additionally, the homogenous part of the problem is always $y_0(x) = (c_1 + c_2 x)e^x,\ c_1, c_2 \in \mathfrak{R}$. The solution to the second order non-homogenous Cauchy problem varies by coefficients, trigonometry, and polynomials. For instance, $\alpha \sin\theta + \beta\cos\theta$.

However, if we check a different problem $y'' - 2y' + y = -4e^x$, $c_1 = \alpha,\ c_2 = 0$ in the homogenous part of the solution $(c_! + c_2 x)e^x$. Furthermore, it is impossible to choose $y_p(x) = \alpha e^x$ or $y_p(x) = \alpha x e^x$ because they contain the homogenous part of the solution to the equation. So, $y_p(x) = \alpha x^2 e^x$.

Chapter 14: More on Complex Numbers:

Writing in the exponential form $re^{i\theta}$:

$z_1 = 1 - i,\ z_2 = 1 + i\sqrt{3}$. For z_1, $a = 1,\ b = -1$.

$r = \sqrt{a^2 + b^2}$, $\theta = \arctan\frac{b}{a}$. If $a \leq 0 \to \theta = \arctan\frac{b}{a} + \pi$

Euler Form: $r(\cos\theta + i\sin\theta) = z.$

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{\bar{z}}{r^2}$$

<u>Computation of Roots</u>

$$\sqrt[n]{z} = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, k = 0, \ldots, n-1 = \sqrt[n]{r}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}$$

$$e^{\theta i} = \cos\theta + \sin\theta$$

$$\frac{z}{c} = \frac{r_z(\cos\theta + i\sin\theta)}{r_c(\cos\alpha + i\sin\alpha} = \frac{r_z}{r_c}(\cos(\theta-\alpha) + i\sin(\theta-\alpha))$$

$$z\,.\,c = r_z r_c(\cos(\theta+\alpha) + i\sin(\theta+\alpha))$$

Even-Odd Identities

(i) sin(−x) = − sin(x)

(ii) cos(−x) = cos(x)

(iii) tan(−x) = − tan(x)

(iv) cot(−x) = − cot(x)

(v) csc(−x) = − csc(x)

(vi) sec(−x) = sec(x)

Important Limits

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$\lim_{n\to 0}\frac{\sin(n)}{n} = 1$$

$$\lim_{n\to 0}\frac{a^n-1}{n} = \ln(a)$$

$$\lim_{n\to 0}\frac{1-\cos(n)}{n} = 0$$

$$\lim_{n\to\infty}\ln(n) = \infty$$

$$\lim_{n\to 0}\frac{1-\cos(n)}{n^2} = \frac{1}{2}$$

$$\lim_{n\to\infty}\frac{\log_a(1+n)}{n} = \frac{1}{\ln(a)}$$

$$\lim_{n\to 0}\frac{\tan(n)}{n} = 1$$

$$\lim_{n \to 0} \frac{\log_a(1+n)}{n} = \frac{1}{\ln(a)}$$

$$\lim_{n \to 0} \frac{e^n - 1}{n} = 1$$

$$\lim_{n \to \infty} \sqrt[n]{n!} = \infty$$

$$\lim_{n \to \infty} \sqrt[n]{n} = 1$$

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0$$

$$\lim_{x \to \pm\infty} \sinh x = \pm \infty$$

$$\lim_{x \to \pm\infty} \cosh x = + \infty$$

Common Derivatives

$$\frac{d}{dx} x = 1$$

$$\frac{d}{dx}(\ x \ ) = sign(x)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x2}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(ln \ \ x \ ) = \frac{1}{x}, x \neq 0$$

$$\frac{d}{dx}(log_a(x)) = \frac{1}{xln(x)}, \ \ x > 0$$

$$\frac{d}{dx} sin(x) = cos(x)$$

$$\frac{d}{dx} cos(x) = -sin(x)$$

$$\frac{d}{dx} sec(x) = sec(x)tan(x)$$

$$\frac{d}{dx} tan^{-1} = \frac{1}{1+x^2}$$

$$\frac{d}{dx} sinh(x) = cosh(x)$$

$$\frac{d}{dx} tanh(x) = \frac{1}{cosh(x)} = 1 - tanh^2(x)$$

$$\frac{d}{dx} sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx} cosh^{-1} = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx} tanh^{-1} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} cot(x) = -csc^2(x)$$

$$\frac{d}{dx} csc(x) = -csc(x)cot(x)$$

$$\frac{d}{dx} tan(x) = sec^2(x) = tan^2(x) + 1$$

Maclaurin Expansions:

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$sinh = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$cosh = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots + \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \left| \binom{\frac{-1}{2}}{m} \right| \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n} x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

Important Integrals

$$\int k dx = kx$$

$$\int \ln(ax)dx = x\ln(ax) - x$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$$

$$\int x\ln(ax)dx = \frac{x^2}{4}(2\ln(ax) - 1)$$

$$\int \frac{1}{x^n} = \frac{-1}{(n-1)x^{n-1}}$$

$$\int \frac{\ln(ax)}{x}dx = \frac{1}{2}\left(\ln(ax)^2\right)$$

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln\ \ x$$

$$\int a^x dx = \frac{a^x}{\ln(a)}$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax}$$

$$\int e^x dx = e^x$$

$$\int xe^x dx = (x-1)e^x$$

$$\int \log_a(x)dx = x\log_a(x) - x\log_a(e)$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

$$\int \sin(x)dx = -\cos(x)$$

$$\int \frac{1}{\sqrt{x}} = 2\sqrt{x}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$

$$\int \sec(x)dx = \ln\ \ \sec(x) + \tan(x)$$

$$\int \frac{ax+b}{cx+d}dx = \frac{ax}{c} - \frac{ad-bc}{c^2}\ln\ \ cx+d$$

$$\int \csc(x)dx = -\ln\ \ \csc(x) + \cot(x)$$

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$$

$$\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2}$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln\ \ ax+b$$

$$\int \cot^{-1}(x)dx = x\cot^{-1}(x) + \sqrt{1}2\ln\left(1+x^2\right)$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$$

$$\int \frac{1}{(x-a)(x-b)}dx = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right|$$

$$\int \tan^2(x)dx = \tan(x) - x$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln\left|a^2+x^2\right|$$

$$\int \cot^2(x)dx = -\cot(x) - x$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \sec^2(x)dx = \tan(x)$$

$$\int \cot(x)dx = \ln\ \sin(x)$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln\left|a^2+x^2\right|$$

$$\int \frac{1}{1+\cos(x)}dx = \frac{\sin(x)}{1+\cos(x)}$$

$$\int \frac{1}{1-\sin(x)}dx = \frac{\cos(x)}{1-\sin(x)}$$

$$\int \frac{1}{1-\cos(x)}dx = \frac{-\sin(x)}{1-\cos(x)}$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax)$$

$$\int \tan(ax)dx = -\frac{1}{a}\ln(\cos(ax))$$

$$\int x\sin(ax)dx = -\frac{1}{a}x\cos(ax) + \frac{1}{a^2}\sin(ax)$$

$$\int \cosh(x)dx = \sinh(x)$$

$$\int x\cos(ax)dx = \frac{1}{a}x\sin(ax) + \frac{1}{a^2}\cos(ax)$$

$$\int \sinh(x)dx = \cosh(x)$$

$$\int \tanh(x)dx = \ln(\cosh(x))$$

$$\int \coth(x)dx = \ln\ \sinh(x)$$

$$\int \sinh^{-1}(x)dx = x\sinh^{-1}(x) - \sqrt{x^2+1}$$

$$\int \cosh^{-1}(x)dx = x\cosh^{-1}(x) - \sqrt{x^2-1}$$

$$\int \tanh^{-1}(x)dx = x\tanh^{-1}(x) + \frac{1}{2}\ln\left(1-x^2\right)$$

$$\int \coth^{-1}(x)dx = x\coth^{-1}(x) + \frac{1}{2}\ln\left(x^2-1\right)$$

Additional Notes:

$$\frac{1}{\infty} = 0,\ \frac{1}{-\infty} = 0$$

$$\frac{0}{\infty} = 0,\ \frac{0}{-\infty} = 0$$

$$e^{-\infty} = 0$$

$$e^{\infty} = \infty$$

$$\lim_{x\to 0^+} 10^{\frac{1}{x}} = +\infty$$

$$\lim_{x\to 0^-} 10^{\frac{1}{x}} = 0\ (10^{-\infty})$$

Additional Landau Formulas:

$ln(x) = 1 + o(x)$

$\arctan(x) \sim x$ **as** $x \to 0$

$(1 + x)^\alpha = 1 + \alpha x$ **as** $x \to 0$

$\ln(1 + y) \sim y$