ALGORITHMS IN COMBINATORIAL OPTIMIZATION

- INPUT IS GIVEN AT THE OUTSET (THE ALGORITHM KNOWS ALL THE INPUT AT ANY POINT DURING ITS EXECUTION).

ALGORITHMIC TECHNIQUES

- LOCAL SEARCH
- GREEDY
- DIVIDE-ET-IMPERA
- DYNAMIC PROGRAMMING

APPROXIMATION ALGORITHM

IMPOSSIBILITY RESULTS

- NP-COMPLETENESS

ONLINE PROBLEMS

"EXPERT PROBLEM"

PREDICTIONS OVER TIME

- WILL THE STOCKS OF COMPANY X INCREASE IN PRICE TODAY?
- WILL IT RAIN TODAY?

IN MACHINE LEARNING,

- YOU USUALLY HAVE SOME "MODELS" (NEURAL NETWORK OR SOME OTHER MODEL) EACH OF WHICH MAKES A GUESS (WILL IT RAIN? IS THIS PICTURE SHOWING A HUMAN FACE?)

  YOU HAVE TO SELECT THE GUESS OF ONE OF THESE MODELS SO TO MAKE THE USER HAPPY.

- THESE "MODELS" ARE USUALLY CALLED "EXPERTS"

THE "EXPERT" MODEL

FOR $t = 1, 2, \dots, T$

- WE ARE GIVEN A PREDICTION VECTOR $(Y_{1t}, Y_{2t}, \dots, Y_{mt})$, WHERE $Y_{it} \in \{0,1\}$ IS THE PREDICTION OF EXPERT i AT TIME t.
- WE THEN HAVE TO "BET" ON WHETHER NATURE WILL, AT TIME t, DECIDE FOR 0 OR 1. IN OTHER WORDS, WE HAVE TO CHOOSE OUR PREDICTION $z_t \in \{0,1\}$
- NATURE WILL SHOW US THE REAL VALUE FOR t, $x_t \in \{0,1\}$
- WE PAY 1€ IF $x_t \neq z_t$

WHAT SHOULD AN ALGORITHM DO?

	$t=1$	$t=2$	$t=3$	...			
EX_1	1	1	1	1	1	1	1
EX_2	0	0	0	0	0	0	0
EX_3	0	0	0	0	0	0	0
z_t	0	0					
x_t	1	1	1	1	1	1	1

HALVING ALGORITHM

$S \leftarrow [m] = \{1, 2, \dots, m\}$

FOR $t = 1, 2, \dots, T$

  LET z_t BE THE MAJORITY CHOICE IN $\{Y_{it} \mid i \in S\}$ (BREAKING TIES ARBITRARILY)

  WE "SEE" x_t (WE PAY 1€ IF $z_t \neq x_t$)

  $S \leftarrow S - \{i \mid Y_{it} \neq x_t\}$

THM: IF $\exists$ A PERFECT EXPERT (IF $\exists i^* \in [m]$ S.T. $Y_{i^*t} = x_t \ \forall t$), THEN THE HALVING ALGORITHM IS GOING TO MAKE AT MOST $m \leq \lfloor \lg_2 n \rfloor$ MISTAKES

P (SKETCH): IF WE MAKE A MISTAKE AT TIME t, AT LEAST HALF OF THE TRUSTWORTHY EXPERTS MADE A MISTAKE AT TIME t. AT THE OUTSET, THERE ARE n TRUSTWORTHY EXPERTS — IN THE END, THERE WILL BE AT LEAST ONE (i^*). THUS, WE ARE GOING TO MAKE AT MOST $\lg_2 n$ MISTAKES. ∎

WEIGHTED MAJORITY$_\varepsilon$():

ASSIGN A WEIGHT $w_i = 1$ TO EACH EXPERT i

FOR $t = 1, 2, \dots, T$

  LET $A_t \leftarrow \sum_{i: Y_{it}=1} w_i$ AND $B_t \leftarrow \sum_{i: Y_{it}=0} w_i$

  IF $A_t \geq B_t$:
    SET $z_t \leftarrow 1$
  ELSE:
    SET $z_t \leftarrow 0$

  "SEE" x_t (AND PAY 1€ IF $x_t \neq z_t$)

  FOR $i = 1, 2, \dots, n$
    IF $Y_{it} \neq x_t$:
      $w_i \leftarrow \frac{w_i}{2}$ $(w_i \leftarrow (1-\varepsilon)\, w_i)$

THM: SUPPOSE THAT m^* IS THE NUMBER OF MISTAKES MADE BY THE BEST EXPERT (THE EXPERT THAT MAKES THE SMALLEST # OF MISTAKES). THEN,

WM MAKES $m \leq 2.41\,(m^* + \lg_2 n)$ MISTAKES.

$\left(m \leq (2+4\varepsilon)\, m^* + \frac{2}{\varepsilon} \lg_2 n\right)$

P: LET w_i^t BE THE WEIGHT OF EXPERT i AT THE END OF ROUND t. ALSO, $w_i^0 = 1 \quad \forall i \in [n]$.

WE DEFINE A "POTENTIAL" $W^t = \sum_{i=1}^{n} w_i^t$.

(A) $W^0 = \sum_{i=1}^{n} w_i^0 = \sum_{i=1}^{n} 1 = n$.

(B) $W^t \geq W^{t+1}$

$\left(W^t = \sum_{i=1}^{n} w_i^t \geq \sum_{i=1}^{n} w_i^{t+1} = W^{t+1}\right)$

(C) IF $z_t \neq x_t$ (IF WM MADE A MISTAKE AT TIME t), THEN $W^t \leq \frac{3}{4} W^{t-1}$.

IF $z_t \neq x_t$, THEN — AT THE BEGINNING OF ROUND t — THE TOTAL WEIGHT I^{t-1} OF THE EXPERTS THAT ARE GOING TO MAKE A MISTAKE IN ROUND t SATISFIES $I^{t-1} \geq \frac{W^{t-1}}{2}$ (RECALL THAT WM FOLLOWS THE WEIGHTED MAJORITY).

THEN,

$$W^t = \frac{I^{t-1}}{2} + \left(W^{t-1} - I^{t-1}\right) =$$

$$= W^{t-1} - \frac{I^{t-1}}{2} \leq W^{t-1} - \frac{W^{t-1}/2}{2}$$

$$= W^{t-1} - \frac{1}{4} W^{t-1} = \frac{3}{4} W^{t-1}.$$

(D) AFTER ROUND T, IF OUR ALGORITHM HAS MADE m MISTAKES, THEN

$$W^T \leq \left(\frac{3}{4}\right)^m W^0 = \left(\frac{3}{4}\right)^m \cdot n$$

(E) $W^T = \sum_{i=1}^{n} w_i^T \geq w_{i^*}^T$ WHERE i^* IS THE EXPERT THAT HAS MADE THE SMALLEST # OF MISTAKES IN THE FIRST T ROUNDS.

(F) ALSO, $w_{i^*}^T = 2^{-m^*}$.

THEN,

$$2^{-m^*} = w_{i^*}^T \leq W^T \leq \left(\frac{3}{4}\right)^m \cdot n$$

THUS,

$$\left(\frac{3}{4}\right)^m \cdot n \geq 2^{-m^*}$$

$$m \lg_2 \frac{3}{4} + \lg_2 n \geq -m^*$$

$$-m \lg_2 \frac{4}{3} + \lg_2 n \geq -m^*$$

$$m^* + \lg_2 n \geq m \lg_2 \frac{4}{3}$$

FINALLY,

$$m \leq \frac{1}{\lg_2 \frac{4}{3}} \left(m^* + \lg_2 n\right) \leq 2.41 \left(m^* + \lg_2 n\right). \quad ∎$$

RANDOMIZED WEIGHTED MAJORITY$_\varepsilon$():

ASSIGN A WEIGHT $w_i = 1$ TO EACH EXPERT i

FOR $t = 1, 2, \dots, T$:

  WE CHOOSE RANDOMLY AN EXPERT i.
  IN PARTICULAR, WE PICK EXPERT i W.P. $\frac{w_i}{\sum_{j=1} w_j}$.

  SET $z_t \leftarrow Y_{i,t}$

  "SEE" x_t (AND PAY 1€ IF $x_t \neq z_t$)

  FOR $i = 1, 2, \dots, n$
    IF $Y_{it} \neq x_t$:
      $w_i \leftarrow (1-\varepsilon)\, w_i$

THM: RWM$_\varepsilon$ MAKES AT MOST $m \leq (1+\varepsilon)\, m^* + \frac{1}{\varepsilon} \ln n$ MISTAKES.