Lesson 1

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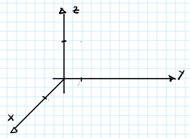
Plot the points $A(2,2,4)$ and $B(1,3,-2)$



Three dimensional coordinate system

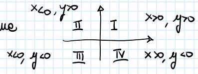
1) A point "A" in space can be represented in Cartesian as $A(x,y,z)$, where $\begin{cases} x: \text{absisse} \\ y: \text{ordinate} \\ z: \text{elevation} \end{cases}$

Region in space

In (x,y) plane, then the equation $x=0$ represents the y axis and the eqn $y=0$ represents the x axis

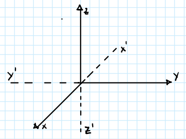
In (x,y,z) $\begin{cases} x=0 \to (z,y) \text{ plane} \\ y=0 \to (x,z) \text{ plane} \\ z=0 \to (x,y) \text{ plane} \end{cases}$

In (x,y) plane



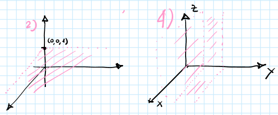
In space, the coordinate planes $x=0$, $y=0$, $z=0$ divides the spaces in 8 cells called octant

The octant in which the point coordinates are all positive is called 1st octant



Ex Specify the region

1) $z \geq 0$ → upper half space consisting of all points on and above the (xy) plane
2) $z=1$ → equation of the plane parallel to the (xy) plane passing through $(0,0,1)$
3) $z=0$, $x \geq 0$, $y \geq 0$ → 1° quadrant of (x,y) plane
4) $2 \leq y \leq 4$ ($y=2$ is eqn of the plane // to (xz) plane at $(0,2,0)$)
5) $x^2+y^2=9$, $yz=0$ → circle centered at $(0,0,0)$ of radius 3
6) $y=2$ and $z=2$ → line made by the intersection of the two planes



Distance and sphere in space

- The distance between 2 points $A(x,y,z)$ and $B(x_2,y_2,z_2)$ is $|AB| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

Ex $A(2,1,5)$, $B(-2,3,0)$ $\quad |AB| = \sqrt{4^2+2^2+5^2} = \sqrt{45}$

- We use the distance formula to write equations for spheres in space

A point A lies on a sphere of radius r and center $C(x_0,y_0,z_0)$, where $|CA|=r$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

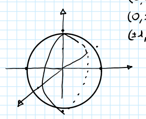
Ex $(x+1)^2+y^2+(z-2)^2=9$
$C=(-1,0,2)$, $r=3$

- $x^2+y^2+z^2=1$ → sphere of radius 1 centered in $(0,0,0)$

for $x=0$ $\quad y^2+z^2=1$ CIRCLE $\begin{cases} y=0 & z=\pm 1 \\ z=0 & y=\pm 1 \end{cases}$

for $y=0$ $\quad x^2+z^2=1$ $\begin{cases} z=0 & x=\pm 1 & (x \text{ intercepts}) \\ x=0 & z=\pm 1 & (z \text{ intercepts}) \end{cases}$

$(0,0,\pm 1)$
$(0,\pm 1,0)$
$(\pm 1,0,0)$



Graphical interpretations of inequalities involving sphere

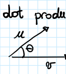
1) $x^2+y^2+z^2>1$ → exterior of a sphere of radius 1 and centered in $(0,0,0)$
2) $x^2+y^2+z^2<1$ → interior of the sphere

Vectors in R^3

- Given $\vec{v}(v_1,v_2,v_3)$ the magnitude (lenght or norm) is given by $\|v\| = \sqrt{v_1^2+v_2^2+v_3^2}$
- A vector is called unit vector if $\|v\|=1$. The standard unit vector in R^3 are $\begin{cases} \vec{i}\ (1,0,0) \\ j\ (0,1,0) \\ k\ (0,0,1) \end{cases}$
- if $v \neq 0$ $\quad w = \frac{v}{\|v\|}$ $\quad w$ is a unit vector

Dot product

- The dot product $\vec{u} \cdot \vec{v}$ of $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ is $u \cdot v = u_1v_1+u_2v_2+u_3v_3$



$\vec{u} \cdot \vec{v} = \|u\| \cdot \|v\| \cdot \cos\theta$ $\quad$ so the angle between two non zero vectors $\vec{u}$ and $\vec{v}$ is given by $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|u\|\|v\|}$

Two vector are said to be orthogonal ($\perp$) if their dot product is zero

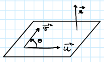
Remarks

1) $u \cdot v = v \cdot u$
2) $u \cdot u = \|u\|^2$
3) $u \cdot 0 = 0$
4) $u \cdot (v+w) = u \cdot v + u \cdot w$

Cross product

$u \times v = (\|u\|\|v\| \cdot \sin\theta)\vec{n}$

where $\vec{n}$ is the normal vector to the plane



1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2) $0 \times u = 0$
3) $\vec{u} \times \vec{v} = 0$ iff u and v are //
4) $u \times (v+w) = u \times v + u \times w$
5) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

Calculating the cross product as determinant

Assuming $\vec{u}(u_1,u_2,u_3)$ and $v(v_1,v_2,v_3)$, then $u \times v$ is given by:

$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}(u_2v_3-u_3v_2) - \vec{j}(u_1v_3-u_3v_1) + \vec{k}(u_1v_2-u_2v_1)$$

Triple scalar product

$\underbrace{(u \times v)}_{\text{vector}} \cdot \underbrace{w}_{\text{vector}} \Rightarrow$ scalar

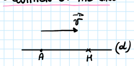
$u = u_1, u_2, u_3$
$v = v_1, v_2, v_3$
$w = w_1, w_2, w_3$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$u = (1,2,-1)$
$v = (-2,0,3)$
$w = (0,7,-4)$

$$(u \times v) \cdot w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23$$

Equation of the line



$H(x,y,z) \in d$

$\vec{v}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad A\begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}$

AH and v are //

$\vec{AH} = k\vec{v}$

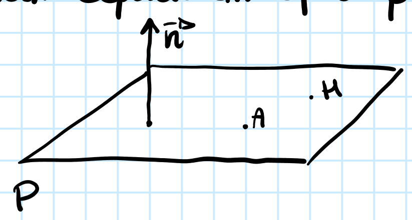
$AH\begin{pmatrix} x-x_A \\ y-y_A \\ z-z_A \end{pmatrix} = k\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{cases} x-x_A = ka \\ y-y_A = kb \\ z-z_A = kc \end{cases} \rightsquigarrow \frac{x-x_A}{a} = \frac{y-y_A}{b} = \frac{z-z_A}{c}$

ex find the equation of the line through $A(-3,2,-3)$, $B(1,-1,4)$

$\vec{v} = \vec{AB} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} = \begin{matrix} a \\ b \\ c \end{matrix}$

PLANES IN SPACE

To find an equation of a plane (P), we need a point $A \in (P)$ and a normal vector $\vec{n}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$



Let $M(x, y, z) \in (P)$

$\vec{AM} \cdot \vec{n} = 0$

where $AM \begin{pmatrix} x - x_a \\ y - y_a \\ z - z_a \end{pmatrix}$ and $\vec{n}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\Rightarrow$ $a(x - x_a) + b(y - y_a) + c(z - z_a) = 0$

$ax + by + cz + (\underbrace{-ax_a - by_a - cz_a}_{d}) = 0$

$ax + by + cz + d = 0$

ex Find the eq. of P through $A(-3, 0, 7)$ and $\vec{n}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

$5x + 2y - z + 22 = 0$ $\quad -(5\cdot -3 + 2\cdot 0 + 7\cdot -1) = 22$

ex Find the eq of P passing through $A(0,0,1)$, $B(2,0,0)$, $C(0,3,0)$

$\vec{n} = \vec{AB} \times \vec{AC}$ $\quad AM \cdot n = 0$ $\quad AM\,(AB \times AC) = 0 \rightarrow$

$AM = \begin{pmatrix} x \\ y \\ z-1 \end{pmatrix}$ $\quad AB = (2, 0, -1)$ $\quad AC = (0, 3, -1)$ $\quad \begin{vmatrix} x & y & z-1 \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 0$

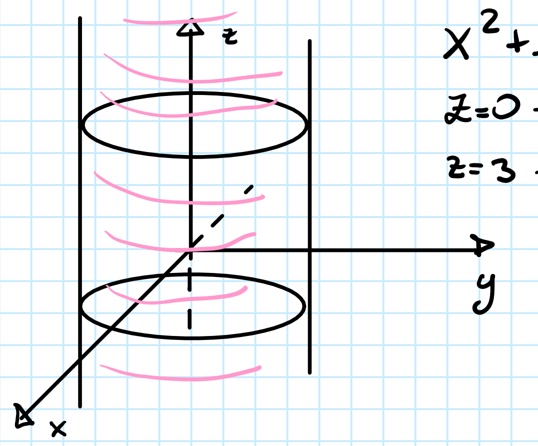
$3x + 2y + 6z - 6 = 0$

SHAPES

I. CYLINDERS
- circular
- eliptical
- parabolic

① CIRCULAR CYLINDER

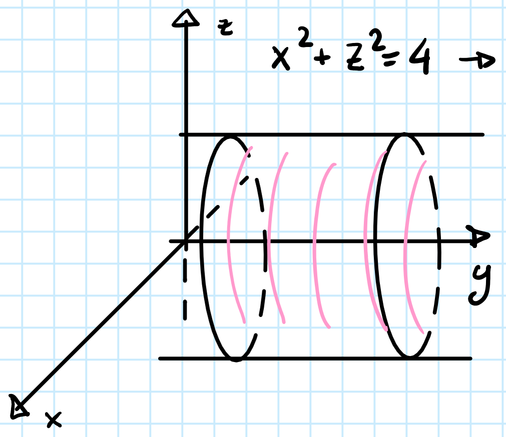
$1\,(\text{VARIABLE } 1)^2 + 1\,(\text{VARIABLE } 2)^2 = a^2$



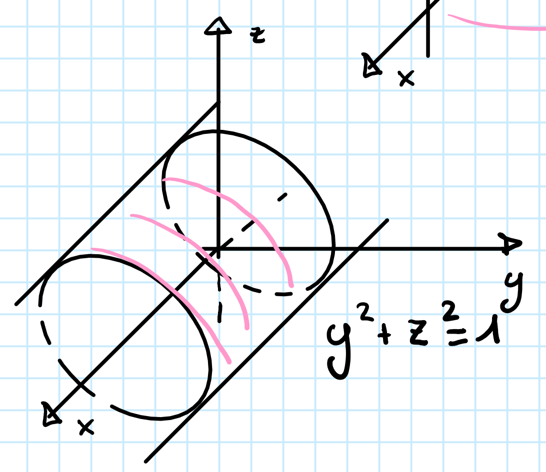
$x^2 + y^2 = 4$

$z = 0 \rightarrow$ circle in $(0,0,0)$

$z = 3 \rightarrow$ circle in $(0,0,3)$



$x^2 + z^2 = 4 \rightarrow$ circular cylinder around y-axis



$y^2 + z^2 = 1$

② ELLIPTICAL CYLINDER

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = c$ $\quad a \neq b$, otherwise it will be a circular cilinder

ex $\frac{x^2}{4} + y^2 = 1$ (elliptical cylinder along z-axis)

$z = 0 \rightarrow$ ELLIPSE OF CENTER $(0, 0, 0)$

$(\pm 2, 0, 0) \rightarrow y = 0 \rightarrow \frac{x^2}{4} = 1 \quad x^2 = \pm 2$

$(0, \pm 1, 0) \rightarrow y^2 = 1 \quad y = \pm 1$

$\frac{(x-2)^2}{4} - y^2 = 1$ $\quad$ ellipse of center $(2, 0, 0)$ along z-axis

③ PARABOLIC CYLINDER

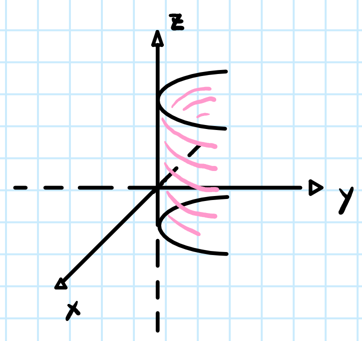
$(\text{VARIABLE}) = (\text{VARIABLE})^2 \xrightarrow{\text{it could be}} \begin{cases} x = y^2 \\ y = x^2 \\ x = (y-1)^2 \\ z = x^2 \end{cases}$

reminder!
$y = a(x-h)^2 + k$
(h, k)

ex
$y = x^2 \rightarrow$ parabolic cylinder around z-axis

$\underline{z = 0}$ $\quad y = x^2 \rightarrow$ parabola in the (x, y) plane whose vertex is $(0,0)$ and opens in the y-axis

$\hookrightarrow$ a plane



Lesson 3

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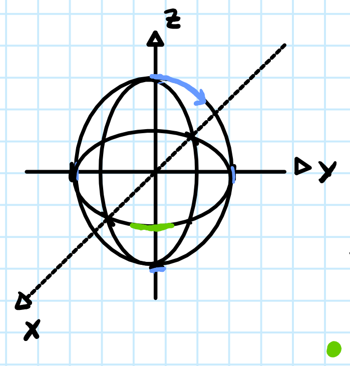
II. Ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \rightarrow$ ellipsoid of centre $(0,0,0)$
$\hookrightarrow$ the intercepts are $(\pm a,0,0)$, $(0,\pm b,0)$, $(0,0,\pm c)$

If $a=b=c \rightarrow$ sphere (special case of ellipsoid)

ex $x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$ $(\pm 1,0,0)$, $(0,\pm 2,0)$, $(0,0\pm 3)$



- for $x=0$ $\frac{y^2}{4}+\frac{z^2}{9}=1$ (ellipse in yz plane)
  - $z=0$ $\frac{y^2}{4}=1$ $y=\pm 2$ (y-intercepts)
  - $y=0$ $z^2=9$ $z=\pm 3$ (z-intercepts)
- for $y=0$ $x^2+\frac{z^2}{9}=1$ (ellipse in xz plane)
- for $z=0$ $x^2+\frac{y^2}{4}=1$ (ellipse in xy plane)

ex $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{4}+z^2=1 \rightarrow$ equation of ellipsoid centered at $(2,-1,0)$

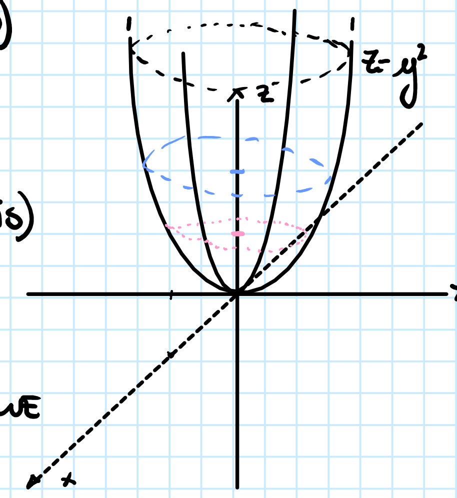
for $x=2$ $\frac{(y+1)^2}{4}+z^2=1$ (ellipse in the plane $x=2$)
$y=-1 \rightarrow z^2=1$ $z=\pm 1$ $(2,-1,\pm 1)$
$z=0$ $(y^2+1)^2=4$ $y=1 \vee y=-3$ $(2,1,0) \vee (2,-3,0)$

III. Paraboloid

(variable = sum of the squares of 2 other variable)

$\frac{z}{c}=\frac{x^2}{a}+\frac{y^2}{b}$ — If $a=b$ circular paraboloid; If $a\neq b$ elliptic paraboloid

ex1 $z=x^2+y^2$ (circular parabola) $z\geq 0\ \forall$
for $x=0$, $z=y^2$ (parabola with $V(0,0,0)$ open in vertical z-axis)
for $y=0$ $z=x^2$ (zx plane)



$z=1$ $x^2+y^2=1$ CIRCLE CENTRED AT $(0,0,1)$

$z=4$ $x^2+y^2=4$

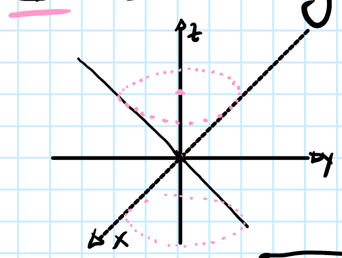
ex2 $x^2+\frac{y^2}{9}=\frac{z}{2}$ $z=2x^2+\frac{2}{9}y^2$ INSTEAD of CIRCLE WE WILL HAVE ELLIPSE!

IV. Cone

A cone around z-axis is generated by rotation of an oblique line through a fixed angle θ with z-axis

$(\text{VARIABLE }1)^2=a(\text{VARIABLE }2)^2+b(\text{VARIABLE }3)^2$

ex $z^2=x^2+y^2$
$z=0 \rightsquigarrow (0,0,0)$
$z=1 \rightsquigarrow x^2+y^2=1$ $(0,0,1)$
for $x=0$ $z^2=y^2 \rightarrow z=\pm y$
for $y=0$ $z=\pm x$



ex2 $z=\sqrt{x^2+y^2}$ $z\geq 0$

Function of several variables

$f: D\subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$
$(x,y) \longrightarrow f(x,y)=z$

Def: Suppose D is a set of real numbers $(x_1,x_2,\dots\dots x_n)$.
A real valued function f on D is a rule that assign a unique real number $w=f(x_1,\dots x_n)$ to each element in D.
The set D is the domain of the function, the set w-values taken by f is the Range.

ex1 $f(x,y)=x^2+y^2$ — DOMAIN: (xy) plane $\mathbb{R}^2$; RANGE: $[0;+\infty)$

$f(x,y)=\frac{x}{y}$ — DOMAIN: $y\neq 0$ (xy) plane exept x-axis; RANGE $(-\infty,+\infty)$

$f(x,y)=\sin(xy)$ — DOMAIN (xy) plane; RANGE $[-1,1]$

Lesson 4

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LIMITS

$f: D \subseteq \mathbb{R} \to \mathbb{R}$
$x \to f(x)$ $\qquad \lim_{x \to a} f(x) = L$

$\forall \varepsilon > 0, \exists \delta > 0 \;/\; |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\mathbb{R}^3$

$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to f(x,y)$ $\qquad \lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$

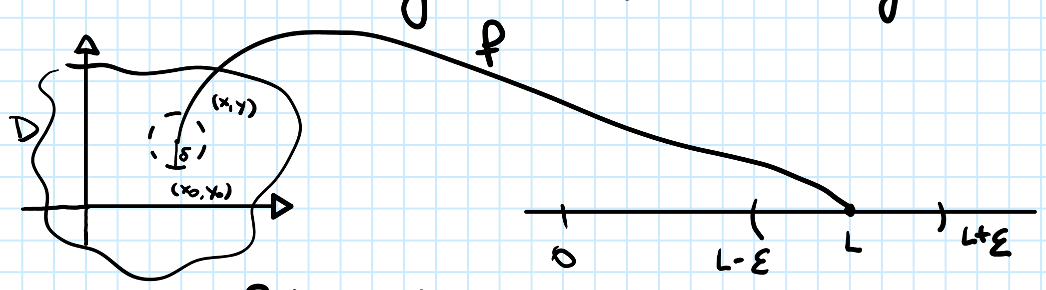
LIMITS AND CONTINUITY

Def. We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) and we write

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

if $\forall \varepsilon > 0$ $\exists$ a corresponding $\delta > 0$ such that for all (x,y) in the domain of function $|f(x,y) - L| < \delta$ whenever $0 < \underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{d((x,y) \to (x_0,y_0))} < \delta$ $\qquad (x,y) \neq (x_0,y_0)$

This definition says that the distance between $f(x,y)$ and L becomes arbitrary small whenever the distance from (x,y) to (x_0,y_0) is made sufficiently small (but not 0)



δ is the radius of the disk centered at (x_0,y_0), for all the points (x,y) within the disk, the function values $f(x,y)$ lies inside the corresponding interval $(L-\varepsilon, L+\varepsilon)$

PROPERTIES

Assume $\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y) \to (x_0,y_0)} f(x,y) = M$, then:

- $\lim_{(x,y) \to (x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M$
- $\lim K f(x,y) = KL$
- $\lim_{(x,y) \to (x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$

$\lim_{(x,y) \to (x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$

$\lim [f(x,y)]^m = L^m$

Ex 1: show that $\lim_{(x,y) \to (0,0)} \dfrac{4x^2y}{x^2+y^2} = 0$

Let $\varepsilon > 0$ $\exists \delta > 0$?

$|f(x) - L| = \left| \dfrac{4x^2y}{x^2+y^2} - 0 \right| < \varepsilon$ $\qquad \sqrt{x^2+y^2} < \delta$

$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} < \delta$

$|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta$

$\left| \dfrac{4x^2y}{x^2+y^2} \right| \leq |4y| < 4\delta < \varepsilon$ $\qquad$ choose $\delta = \dfrac{\varepsilon}{4}$

Let $\varepsilon > 0$ and $\delta = \dfrac{\varepsilon}{4}$ and $\sqrt{x^2+y^2} < \delta$
then $|y| < \sqrt{x^2+y^2} < \delta$ and $|f(x,y) - 0| < \varepsilon$
$|f(x,y) - 0| = \ldots\ldots$

Lesson 5

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DEF: a path is any curve passing through the point (x,y)

- Two paths test for NON EXISTENCE of a limit
- If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then $\lim f(x,y)$ doesn't exist

CONTINUITY

DEF: A function $f(x,y)$ is continuos at point (x_0,y_0) if:

1) f is defined at (x_0,y_0)
2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist
3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

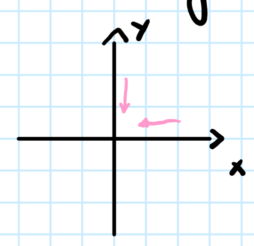
so, a continuos function is continuos if it is at every point of its domain

NOTE: Polynomials and rational functions of 2 variables are continuos at every point where they are defined

EX1 Let $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2-2xy-3y^2} & x \neq -y \\ 5 & \text{if } x=-y \end{cases}$

1. $f(1,-1)=5$
2. $\lim_{(x,y)\to(1,-1)} f(x,y) = \dfrac{(x-y)(x+y)}{(x+y)(x-3y)} = \dfrac{1+1}{1+3} = \dfrac{1}{2}$
3. $f(1,-1)=5 \neq \lim f(x,y) = \frac{1}{2}$ f is not continuos at $(1,-1)$

EX1 Show that the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ has no limit as (x,y) approaches $(0,0)$



- ALONG x-AXIS $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} \underset{y=0}{\Longrightarrow} \lim_{x\to 0} \dfrac{x^2}{x^2} = 1 = L_1$
- ALONG y-AXIS $(x=0)$ $\lim_{y\to 0} \dfrac{-y^2}{y^2} = -1 = L_2$

$L_1 \neq L_2 \to$ the limit does not exist

if $L_1 = L_2$, then you have to check with other methods

EX2: Show that $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$ doesn't exist

- Considering the curve $y=mx^2$, $x\neq 0$
- Along this curve, the function $f(x,y)=\dfrac{2x^2mx^2}{x^4+m^2x^4} = \dfrac{2m}{1+m^2}$

the limit varies with the path of approach

if $m=1$, along the parabola $y=x^2$, we have $\lim = 1 = L_1$

if $m=0$ (x,y) approaches $(0,0)$ along the x-axis $\lim = 0 = L_2$

$L_1 \neq L_2$, so it doesn't exist

PARTIAL DERIVATIVES

$f: R \to R$

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

DEF: Let $z=f(x,y)$

the partial derivative of $f(x,y)$ w.r. to x at the point (x_0,y_0), is denoted by:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \quad \text{or} \quad f_x(x_0,y_0) \quad \text{or} \quad \frac{\partial z}{\partial x}(x_0,y_0)$$

is given by:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

In general, if $f: U \subseteq R^m \to R$ is a function. Then the partial derivative of f at the point $a=(a_1,a_2,\dots,a_m)$ w.r.t. the i^{th} variable x is defined as:

$$\frac{\partial f(a)}{\partial x_i} = \lim_{h\to 0} \frac{f(a_1,a_2,\dots,a_{i-1},a_{i}+h,\dots,a_m)-f(a_1,a_2,\dots,a_m)}{h}$$

Lesson 6

mercoledì 18 ottobre 2023 17:28

PARTIAL DERIVATIVES

→ the partial derivative of a function of several variables is its derivative w.r to one of those variables while the others held constant

IMPLICIT DIFFERENTIATION

ex $x^2y + xz + yz^2 = 8$ where $z = f(x,y)$

· find $\frac{dz}{dx}$, $\frac{dz}{dy}$

z is a function, not a constant

$\frac{d}{dx}(x^2y + xz + yz^2) = 0$ $\quad 2xy + z + x\frac{dz}{dx} + 2yz\frac{dz}{dx} = 0$ $\quad 2xy + z(x + 2yz)\frac{dz}{dy} = 0$

$\frac{dz}{dx} = \frac{-2xy - z}{x + 2yz}$

ex1 $f(x,y) = 3x - x^2y^2 + 2x^3y$

$f_x(x,y) = 3 - 2xy^2 + 6x^2y$

$f_y(x,y) = -2x^2y + 2x^3$ (3x goes away cause it's a constant)

$f(x,y) = y \cdot \sin(xy)$

$f_x(x,y) = y^2 \cdot \cos(xy)$

$f_y(x,y) = \frac{\partial}{\partial y} y \cdot \sin(xy) + y\frac{\partial}{\partial y}\sin(xy)$

$= \sin xy + y x \cos(xy)$

Lesson 7

lunedì 23 ottobre 2023 16:30

HIGHER ORDER PARTIAL DERIVATIVES

$z = f(x,y) < \begin{matrix} f_x \\ f_y \end{matrix}$

Let $z = f(x,y)$, then since both 1st order partial derivatives f_x, f_y are also functions with respect to x and y, we could in turn differentiate each w.r.t. x and y. There will be a total of 4 possible 2nd order derivatives.

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ $\quad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ $\quad (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

MIXED PARTIAL DERIVATIVE (order is important)

NOTE: f_{xy} · diff from left to right. 1st w.r.t x and then y

But $\frac{\partial^2 f}{\partial y \partial x}$ → diff from right to left 1st diff w.r.t x and then y

SCHWARTZ'S THEOREM

The symmetry of the 2nd deriv. is not always true. In general the symmetry will always hold at a point if the 2nd part. der. are continuous around the point.

ex $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ $\quad$ show that is cont. at (0,0)

claim: $\frac{\partial f}{\partial x \partial y}(0,0) \neq \frac{\partial f}{\partial y \partial x}(0,0)$

$f(0,0) = 0$

$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2}$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$ $\quad x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta)$

$f(r,\theta) = \frac{\cancel{r^2}\cos\theta\sin\theta\,(r^2\cos(2\theta))}{\cancel{r^2}} = \frac{r^2}{2}\sin(2\theta)\cos(2\theta) = \frac{r^2}{4}\sin(4\theta)$

$\lim_{(r,\theta)\to(0,0)} \frac{r^2}{4}\sin(4\theta) = 0$ → IT'S CONT.

show that 2nd part. der. are not cont.

$\frac{df}{dx} = \begin{cases} \text{for } (x,y) \neq (0,0) \\ \frac{\partial f}{\partial x} = \frac{[y(x^2-y^2) + xy(2x)][x^2+y^2] - 2x \cdot xy(x^2-y^2)}{(x^2+y^2)^2} = \frac{y(x^2+4x^2y^2-y^4)}{(x^2+y^2)^2} \\ \text{for } (x,y) = (0,0) \\ \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h+0,0) - f(0,0)}{h} = 0 \quad f(h,0) = 0 \end{cases}$

$\frac{df}{dy} = \begin{cases} \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)} \\ 0 \end{cases}$

$\frac{d^2 f(0,0)}{dy\,dx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0} \frac{\frac{df}{dx}(0,h) - \frac{df}{dx}(0,0)}{h} = \lim_{h\to 0} \frac{h(-h^4)}{h^4} \cdot \frac{1}{h} = -1$

$\frac{d^2 f}{\partial x \partial y} = \begin{cases} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

Lesson 8

mercoledì 25 ottobre 2023 17:47

- If $y=f(x)$ is a function of one variable, then $dy = f'(x)\,dx$ is called the differentials
- Given the function $z=f(x,y)$ then the differential dz is given by $dz = f_x\,dx + \underbrace{f_y}_{\text{partial derivative}}\,dy$
- This can be extended to function of 3 or more variables $\to w = g(x,y,z) \Rightarrow dw = f_x dx + f_y dy + f_z dz$

ex $\quad w = \dfrac{x^3y^6}{z^2}$

$$dw = \frac{3x^2y^6}{z^2}dx + \frac{6y^5x^3}{z^2}dy - \frac{2x^3y^6}{z^3}dz$$

PARTIAL DERIVATIVES AND CONTINUITY

→ A function $f(x,y)$ can have partial derivatives w.r.t. both x and y at one point without the function being continuos there

→ If the partial derivatives f_x and f_y of a function $f(x,y)$ are continuos throughout an open region R, then f is differentiable at every point of R.

→ If a function $f(x,y)$ is differentiable at (x_0,y_0), then f is continuos at (x_0,y_0)

ex $\quad f(x,y) = x^2y^3 \qquad \left.\begin{array}{l} f_x = 2xy^3 \\ f_y = 3x^2y^3 \end{array}\right\}$ both are cont., so f is diff.

$f: \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ $\qquad f$ is differentiable at $x=a$ if $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a) \iff \lim_{x\to a}\frac{f(x)-\overbrace{f(a)-f'(a)(x-a)}^{L(x)}}{x-a} = 0$

$$\iff \lim_{x\to a}\frac{f(x)-L(x)}{x-a} = 0$$

→ Let $f: \mathbb{R}^2 \to \mathbb{R}$ and suppose that the partial derivatives f_x and f_y are defined at the point (a,b). Define $h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

→ We say that f is differentiable at (a,b) if $\displaystyle\lim_{(x,y)\to(a,b)}\frac{f(x,y)-h(x,y)}{\|(x,y)-(a,b)\|} = 0$

If either one of the partial derivatives $f_x(a,b)$ or $f_y(a,b)$ doesn't exist or the limit doesn't exist or not 0 then f is not diffe. at (a,b)

ex $\quad$ Let $f(x,y) \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases} \qquad \dfrac{\partial f(0,0)}{\partial x}, \dfrac{\partial f(0,0)}{\partial y}$ EXIST

$f(0,0) = 1$ $\qquad$ but $f(x,y)$ is not cont. at $(0,0)$

along $y=x$ $\quad \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim 0 = 0$

$x=0$ $\quad \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim 1 = 1$

$\}$ f not cont. at $(0,0)$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0 \qquad \frac{\partial f(0,0)}{\partial y} = \lim\frac{f(0,h)-f(0,0)}{h} = 0$$

ex1

study the differentiability of the following function at $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{x\sqrt{x}}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \frac{0-0}{h} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = 0$$

$$h(x,y) = f(0,0) + \underbrace{f_x(0,0)(x-0)}_{0} + \underbrace{f_y(0,0)(y-0)}_{0}$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-h(x,y)}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)}\frac{\frac{x\sqrt{x}}{x^2+y^2} - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}\frac{x\sqrt{y}}{x^2+y^2\sqrt{x^2+y^2}}$$

ALONG $x=0 \to \lim 0 = 0$

ALONG $y=x \to \displaystyle\lim_{x\to 0}\frac{x\sqrt{x}}{2x^2+\sqrt{2x^2}} = \frac{1}{2x+\sqrt{2x}} = \infty$

$\}$ f is not differentiable

ex2 $\quad$ show that the function $f(x,y) = xy+2x+y$ is diff. at $(0,0)$

$f_x = y+2 \qquad f_x(0,0) = 2 \qquad f(0,0) = 0$

$f_y = x+1$ (2x is a const.! so it disappear) $\quad f_y(0,0) = 1$

$$h(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0 + 2x + y \to f(x,y) = 2x+y$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-h(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}\frac{xy+2x+y-2x-y}{\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}} \to \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$

$$\to \lim_{r\to 0}\frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2}} = \lim_{r\to 0}\frac{r}{2}\sin(2\theta) = 0$$

since the limit is 0 the function is diff. at $(0,0)$

Lesson 9

lunedì 30 ottobre 2023 16:24

Graphical interpretation of partial derivatives

→ let $f: \mathbb{R}^3 \to \mathbb{R}$
fix $y = b$ (now we have a function of 1 variable)
let $g(x) = (x, b)$

→ $\frac{\delta f}{\delta x}(a,b) = \frac{\delta g}{\delta x}(a)$ → It's the slope of the tangent line to the curve that results from the intersection of the plane $y=b$ and the surface at pt (a,b)

→ fix $x=a$, let $h(y) = f(a,y)$
$\frac{\delta f}{\delta y}(a,b) = \frac{dh}{dy}(b)$ → the partial derivative of f w.r.t. y at (a,b) is the slope of the tangent line to the intersection of the graph of f with the plane $x=a$

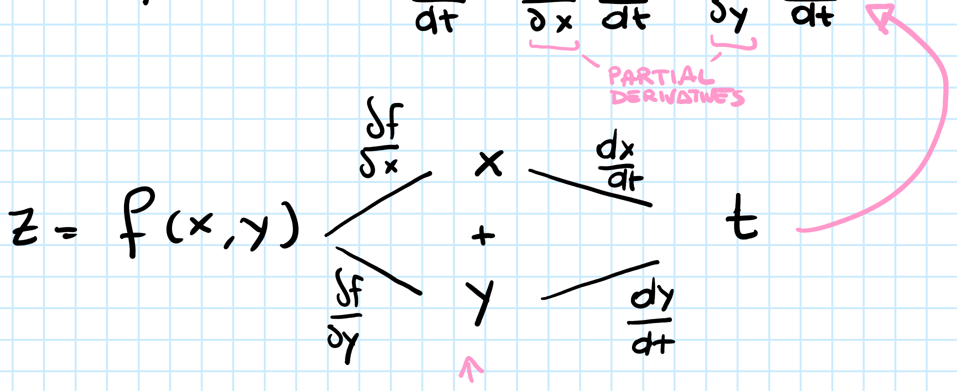
EX the plane $x=1$, intersects the paraboloid $z = x^2y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,2,5)$

the slope is given by: $\left.\frac{\delta z}{\delta y}\right|_{(1,2)}$ $\qquad \frac{\delta z}{\delta y} = 2y\Big|_{(1,2)} = (2)(2) = 4$

$x=1 \to$ parabola $\quad z = 1+y^2$

Chain rule for multivariable functions

thm: If $z = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable function of t, then the composite $z = f(x(t), y(t))$ is differentiable fct. of t and $\frac{dz}{dt} = \underbrace{\frac{\partial f}{\partial x}}_{\text{PARTIAL DERIVATIVES}} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$



intermediate variables

ex Find the derivative of $z = xy$ w.r.t. t along the path $x = \cos t$, $y = \sin t$

$$\frac{dz}{dt} = y(-\sin t) + y\cos t = -\sin^2 t + \cos^2 t = \cos(2t)$$

ex2 $z = x^2y^3 + y\cos x$
$x = \ln(t^2)$
$y = \sin 4t$

$$\frac{dz}{dt} = \frac{\delta z}{\delta x}\frac{dx}{dt} + \frac{\delta z}{\delta y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy^3 - y\sin x)\left(\frac{2t}{t^2}\right) + (3y^2x^2 + \cos x)(4\cos 4t) =$$

$= 2\ln(t^2)\sin^3 4t - \sin 4 + \sin(\ln \ldots\ldots$

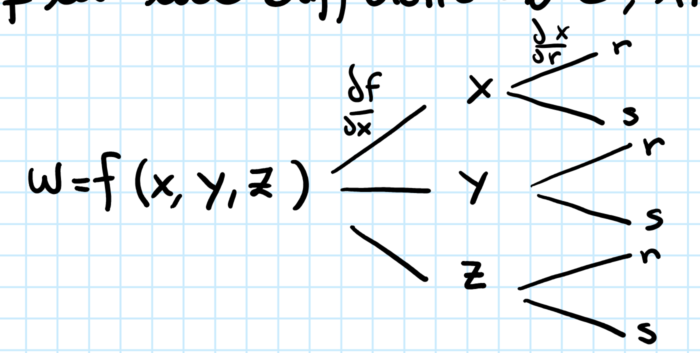
Function of 3 variables

If $w = f(x,y,z)$ is differentiable and x, y, z are diff fct. of t, then w is differentiable fct. of t and $\frac{\partial w}{\partial t} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt} + \frac{\delta f}{\delta z}\frac{dz}{dt}$

Ex: $w = \tan(xz) + \frac{z}{y}$ $\qquad x = t,\ y = t^2,\ z = \sin t$

$$\frac{\partial w}{\partial t} = \left(\frac{z}{1+x^2z^2}\right)(1) + \left(-\frac{z}{y^2}\right)(2t) + \left(\frac{x}{1+x^2z^2} + \frac{1}{y}\right)(\cos t)$$

Thm: Chain Rule for two independent variables and 3 intermediate variables
→ Suppose that $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$.
If all are differentiable, then w has partial derivatives w.r.t r and s



$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

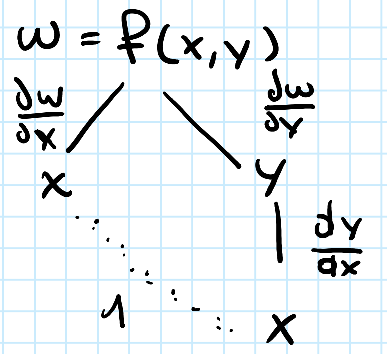
$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Chain rule

Suppose that z is a function of m-variables $x_1, x_2, \ldots, x_m$ and each one of these variables in turn are fct. of m-variables $t_1, t_2, \ldots, t_m$. then, for any variable t_i, $i = 1, 2, \ldots, m$ we have $\frac{dz}{dt_i} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$

Implicit differentation

Given the function in the form $w = f(x,y) = 0$ where y is implicitly defined fct. of x, say $y = g(x)$



$$0 = f_x + f_y\frac{dy}{dx} \to f_y\frac{dy}{dx} = -f_x \to \frac{dy}{dx} = -\frac{f_x}{f_y}$$

ex $\underbrace{2x^2 + 3\sqrt{xy} - 2y - 4 = 0}_{f(x,y)}$ $\qquad y = g(x)$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{4x + \frac{3y}{2\sqrt{xy}}}{\frac{3x}{2\sqrt{xy}} - 2} = -\frac{8x\sqrt{xy} + 3y}{3x - 4\sqrt{xy}}$$

ex2 $x\cos(3y) + x^3y^5 = 3x - e^{xy}$ and $y = g(x)$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\cos(3y) + 3x^2y^5 - 3 + y\,e^{xy}}{-3x\sin(3y) + 5y^4x^3 + x\,e^{xy}}$$

Lesson 10

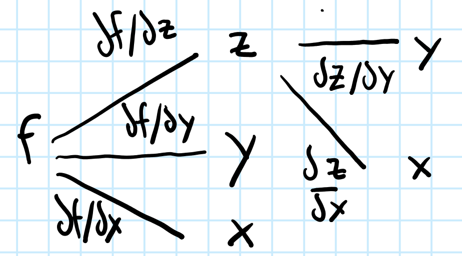
martedì 7 novembre 2023 15:50

- The implicit differentiation for functions of two variables can be easily extended to three variables.

Assume $f(x,y,z)=0$ and z is implicitly defined as $z=g(x,y)$ and we want to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

$f(x,y,z)=0$ Diff. both sides w.r.t. x

$$\frac{\partial f}{\partial x}\cdot\underbrace{\cancel{\frac{\partial x}{\partial x}}}_{1}+\frac{\partial f}{\partial y}\cdot\underbrace{\cancel{\frac{\partial y}{\partial x}}}_{0}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0 \rightarrow f_x+f_z\frac{\partial z}{\partial x}=0 \rightarrow \boxed{\frac{\partial z}{\partial x}=-\frac{f_x}{f_z}} \text{ and similarly } \boxed{\frac{\partial z}{\partial y}=-\frac{f_y}{f_z}}$$



EX $x^2e^{xy}+xy-x^2z+yz^2=0$ and $z=g(x,y)$

$$\frac{\partial z}{\partial x}=\frac{-f_x}{f_z}=\frac{-2xe^{xy}+x^2ye^{xy}+y-2xz}{-x^2+2yz}$$

$$\frac{\partial z}{\partial y}=\frac{-f_y}{f_z}=\frac{x^3e^{xy}+x+z^2}{-x^2+2yz}$$

EX 2 $x^2\sin(2y-5z)-y\cos(6zx)-1=0$

$$\frac{\partial z}{\partial x}=\frac{2x\sin(2y-5z)+6zy\sin(6zx)}{-5x^2\cos(2y-5z)+6yx\sin(6zx)}$$

DIRECTIONAL DERIVATIVE

→ The partial derivative of f w.r.t. x $\left(\frac{\partial f}{\partial x}\right)$ is the slope of the tangent line to the intersection of the graph of f with the plane $y=y_0$ at (x_0,y_0) in the direction of x. Also $\frac{\partial f}{\partial y}$ gives the slope to the tangent line in the y-direction.

→ We can generalize the partial derivatives to calculate the slope in any direction. The result is called directional derivative. It also allow us to find the rate of change of f if we allow both x and y to change simultaneously.

→ The first step in taking a directional derivative is to specify the direction. One way to specify a direction is with a vector $\vec{u}=(u_1,u_2)$ that points in the direction in which we want to compute the slope. For simplicity we will assume that $\vec{u}$ is a unit vector.

→ Sometimes the direction of changing x and y is given as an angle θ. The unit vector that points in this direction is given by $\vec{u}=(\cos\theta,\sin\theta)$

DEF: The derivative of f in the direction of the unit vector $\vec{u}=u_1\vec{i}+u_2\vec{j}$ is called the directional derivative and denoted by $D_{\vec{u}}f(x,y)$ and given by:

$$D_{\vec{u}}f(x,y)=\lim_{h\to0}\frac{f(x+u_1h,\,y+u_2h)-f(x,y)}{h}$$

$$f_x(x_0,y_0)=\lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \rightarrow \text{is a directional derivative in the } \vec{i} \text{ direction.}$$

→ In practice it could be difficult to compute the limit, so we need an easier way to evaluate the directional derivative. Let's define a new function of single variable.

$g(z)=f(x_0+u_1z,\,y_0+u_2z)$ where x_0, y_0, u_1, u_2 are some fixed numbers.

Then $g'(z)=\lim_{h\to0}\frac{g(z+h)-g(z)}{h}$ and $g'(0)=\lim_{h\to0}\frac{g(h)-g(0)}{h}=\lim_{h\to0}\frac{f(x_0+u_1h,\,y_0+u_2h)-f(x_0,y_0)}{h}=D_{\vec{u}}f(x_0,y_0)$

so $g'(0)=D_{\vec{u}}f(x_0,y_0)$

Let's write $g(z)=f(x,y)$ where $\begin{cases} x=x_0+u_1z \\ y=y_0+u_2z\end{cases}$

$$g'(z)=\frac{dg}{dz}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dz}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=f_x(x,y)\,u_1+f_y(x,y)\,u_2$$

$$g'(0)=f_x(x_0,y_0)u_1+f_y(x_0,y_0)u_2 \rightarrow D_{\vec{u}}f(x_0,y_0)=f_x(x_0,y_0)u_1+f_y(x_0,y_0)u_2 \rightarrow D_{\vec{u}}f(x,y)=f_x(x,y)u_1+f_y(x,y)u_2$$

EX 1 Using the definition find the derivative of $f(x,y)=x^2+xy$ at $P=(1,2)$ in the direction of unit vector $\vec{u}=\frac{1}{\sqrt2}\vec{i}+\frac{1}{\sqrt2}\vec{j}$

$$D_{\vec{u}}f=\lim_{h\to0}\frac{f(x+1/\sqrt2\cdot h,\,y-1/\sqrt2\cdot h)-f(x,y)}{h}$$

$$=\lim_{h\to0}\frac{\left(x+\frac{h}{\sqrt2}\right)^2+\left(x+\frac{h}{\sqrt2}\right)\left(y+\frac{h}{\sqrt2}\right)-x^2-xy}{h}$$

$$=\lim_{h\to0} h+\frac{3}{\sqrt2}x+\frac{y}{\sqrt2}=\frac{3x+y}{\sqrt2}$$

$$D_{\vec{u}}(1,2)=\frac{3}{\sqrt2}+\frac{2}{\sqrt2}=\frac{5}{\sqrt2}$$

EX 2 Find $D_{\vec{u}}f(2,0)$ where $f(x,y)=xe^{xy}+y$ where $\vec{u}$ is the unit vector in the direction of $\theta=\frac{2\pi}{3}$

$$\vec{u}=\left(\cos\frac{2\pi}{3},\sin\frac{2\pi}{3}\right)=\left(-\frac12,\frac{\sqrt3}{2}\right)$$

$f_y=e^{xy}+xye^{xy}$ $\quad$ $f_y=x^2e^{xy}+1$

$$D_{\vec{u}}f=(e^{xy}+xye^{xy})\left(-\frac12\right)+\frac{\sqrt3}{2}(x^2e^{xy}+1)$$

$$D_uf(2,0)=(1+0)\left(-\frac12\right)+\frac{\sqrt3}{2}(4+1)=\frac{5\sqrt3-1}{2}$$

Lesson 10 pt2

venerdì 10 novembre 2023 12:04

GRAPHICAL INTERPRETATION OF DIRECTIONAL DERIVATIVE

The equation $z = f(x,y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the pt $(x_0, y_0, z_0) = P$ lies on S. The vertical plane that passes thought P and P_0 and parallel to the (uz) plane intersects S in a curve C.

The slope of the tangent to C at P is given by directional derivative $D_{\vec{u}} f = f_x u_1 + f_y u_2 = \underbrace{(f_x, f_y)}_{\nabla f\text{, GRADIENT VECTOR.}} \cdot \underbrace{(u_1, u_2)}_{\vec{u}}$

GRADIENT VECTOR

the gradient vector of $f(x,y)$ at a pt $(x_0, y_0) = P_0$ is the vector:

$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = (f_x, f_y)$ → obtained by evaluating the partial derivatives of f at P_0.

$\nabla f = (f_x, f_y, f_z)$ $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

PROPERTIES OF THE DIRECTIONAL DERIVATIVE:

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \cdot \underbrace{\|\vec{u}\|}_{1} \cos\theta = \|\nabla f\| \cos\theta$ → where θ is the angle between the vectors $\vec{u}$ and ∇f

1) The maximum value of $D_{\vec{u}} f$ (hence the max. rate of change of the function $f(x,y)$) is given by $\|\nabla f\|$ and will occur in the direction of ∇f. So, at each point in the domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is $D_{\vec{u}} f = |\nabla f|$.

2) Similarly f decreases most rapidly in the direction $-\nabla f$. The derivative in this direction is $D_u f = |\nabla f| \cdot \cos\pi = -|\nabla f|$

3) Any direction orthogonal to a gradient vector $\nabla f \neq 0$ is a direction of zero change in f because $\theta = \frac{\pi}{2}$, then $D_u f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = |\nabla f| \cdot 0 = 0$

ALGEBRAIC RULES FOR GRADIENT

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(Kf) = K\nabla f$
3) $\nabla(fg) = f\nabla g + g\nabla f$
4) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Lesson 11

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FUNCTIONS OF 3 VARIABLES

For a differentiable function $f(x,y,z)$ and a unit vector $u = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ we have

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} \quad \text{and} \quad D_{\vec{u}} f = \nabla f \cdot \vec{u} = f_x u_1 + f_y u_2 + f_z u_3$$

TANGENT PLANES AND NORMAL LINE

If $r(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = 0$ is differentiable

$$\frac{d}{dt}\left(f(g(t), h(t), k(t))\right) = \frac{d(c)}{dt} \quad \text{differentiable both sides w.r.t. } t$$

$$\frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dh}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)}_{\nabla f}\cdot\underbrace{\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dk}{dt}\vec{k}\right)}_{\frac{dr}{dt}} = 0$$

At every point of the curve ∇f is orthogonal to the curve's velocity vector ($\nabla f \cdot r'(t) = 0$)

Def: the tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$

→ the normal line of the surface at P_0 is the line throught P_0 and parallel to $\nabla f|_{P_0}$

TANGENT PLANE is $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$ where $(f_x(P_0), f_y(P_0), f_z(P_0)) = \nabla f|_{P_0}$

↳ normal vector

NORMAL LINE is
$$\begin{cases} x = x_0 + f_x(P_0)t \\ y = y_0 + f_y(P_0)t \\ z = z_0 + f_z(P_0)t \end{cases}$$

Lesson 12

lunedì 13 novembre 2023 16:26

EXTREME VALUES

Def: Let $f(x,y)$ be defined on a region R containg the point (a,b), then:

1) $f(x,y)$ has a relative minimum at the point (a,b), if $f(a,b) \leq f(x,y)$ for all points (x,y) in an open disk centered at (a,b)
2) $f(a,b)$ is a local maximum if $f(a,b) \geq f(x,y)$ for all points in an open disk centered at (a,b).
3) f has an absolute max at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in R.
4) f has an absolute minimum at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) in R.

Def: CRITICAL POINTS

A point (a,b) in the domain of a function $f(x,y)$ is called a CRITICAL POINT (or stationary point) if:

1> $f_x(a,b) = f_y(a,b) = 0$ (i.e. $\nabla f(a,b) = 0$)

2> Either $f_x(a,b)$ or $f_y(a,b)$ doesn't exist

Note that both of the partial derivatives must be zero at (a,b). If only one is zero at the point, then the point will not be critical.

→ The value of the function at a critical point is a critical value.

Ex Find the critical point for $f(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

$\frac{\partial f}{\partial y} = f_x = 2x + 2y + 4 = 0$

$f_y = 2x - 8y - 6 = 0$

$\begin{cases} 2x = -2y - 4 \\ -2y - 4 - 8y - 6 = 0 \end{cases}$ $\quad y = -1$, $x = -1$ $\quad f(-1,-1)$ ↓ CRITICAL POINT

Thm: First derivative test for local extreme values

→ If the point (a,b) is a relative extreme of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$ and hence (a,b) is a critical point

proof:

· Define $g(x) = f(x,y)$ and assume $f(x,y)$ has a relative extremum at (a,b), then $g(x)$ also has a relative extremum at $x = a$ (of same kind as $f(x,y)$). then $g'(a) = 0 \rightarrow f_x(a,b) = 0$

Similarly you can define $h(y) = f(a,y)$ h will have a relative extremum at b

$$h'(b) = f_y(a,b) = 0$$

→ If we sustitute the values $f_x(a,b) = 0$ and $f_y(a,b) = 0$ into the eq. $f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$ for the tangent plane to the surface $z = f(x,y)$ at (a,b), then the eq. reduces into:

$z = f(a,b) = c \rightarrow$ you will have a horizonal tangent plane

Ex find the local extreme values for $f(x,y) = y^2 - x^2$

$f_x = -2x = 0$

$f_y = 2y = 0$ → $(0,0)$ is the critical point

$f(0,0) = 0$

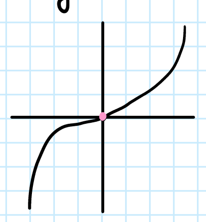
Along the positive x-axis

$f(x,0) = -x^2 < 0 = f(0,0)$

positive y-axis

$f(0,y) = y^2 > 0$

* Not every critical point give rise to a local extremum



$f(x) = x^3$

$f'(x) = 3x^2$

$(0,0)$ is a critical point

Lesson 12 pt.2

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Def: SADDLE POINT

A function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b), there are domain point (x,y) where $f(x,y) > f(a,b)$ and points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a, b, f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface

Ex:

Thm: Second derivative test

Suppose that $f(x,y)$ and its first and second partial derivatives are continuos through a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$.

Define $D = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$, then:

1) If $D>0$ and $f_{xx}(a,b) > 0$, f has a local minimum at (a,b)
2) If $D>0$ and $f_{xx}(a,b) < 0$, f has a local maximum at (a,b)
3) If $D<0$, then f has a saddle point
4) If $D=0$, then the test is inconclusive

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$ DISCRIMINANT HESSIAN

for this course $f_{xy} = f_{yx}$

Notice that if $D>0$ then both f_{xx} and f_{yy} should have the SAME SIGN

PROBLEM SOLVING STRATEGY

1) Determine the critical points (a,b) where $f_x(a,b) = f_y(a,b) = 0$
2) Calculate $D = f_{xx} f_{yy} - f^2_{xy}$ at every critical point
3) Apply the 4 cases

Lesson 13

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Thm: EXTREME VALUE THEOREM.

If $f(x,y)$ is continuos function on closed buonded set in R^2. then f has an absolute maximum and minimum on S.

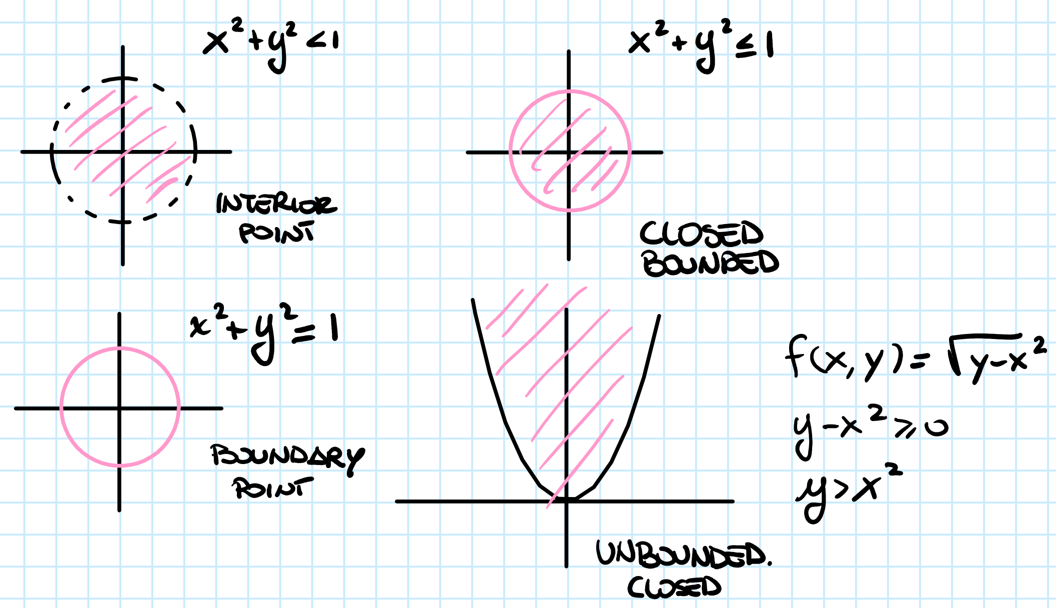
Def. : CLOSED BOUNDED

1) A region D in R^2 is called bounded if it lies in a disk of finite radius

2) A region in R^2 is called closed if it includes all its boundary points. A region is called open if it consists entirely of Interior points.

3) A point (a,b) is an interior point of a region R, if it is the center of a disk that lies entirely in D.

4) A point is a boundary point if every disk centered at (a,b) contains points that lie outside D as well as points inside D.



Steps:

1) List the interior points where f may have local max, local min, and evaluate f at those points

2) List the boundary points of R when f has local max and min and evaluate f at these points. this usually involve Calculus 1 approach.

3) The largest and smallest values found in the two steps are the absolute max and absolute min of the function.

Lesson 14

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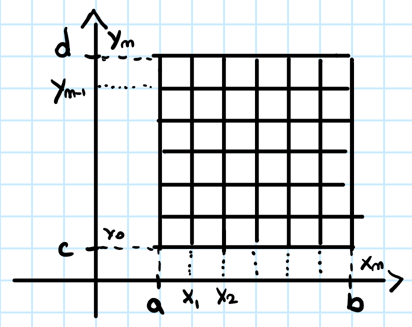
Double integrals

Let's start with a rectangular region in the (xy) plane

$R = \{a \le x \le b \ , \ c \le y \le d\}$



Divide $a \le x \le b$ into n- subintervals and $c \le y \le d$ into m- subintervals

$\Delta x = \frac{b-a}{n} \quad \Delta y = \frac{d-c}{m} \quad \Delta x \Delta y = A$

volume → $V \approx \sum_{i=1}^{m} \sum_{j=1}^{m} f(x_j, y_i)\, \Delta A$

$V = \lim_{m/n \to +\infty} \sum \sum \ldots \quad = \iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy$

thm: Fubin's thm

If $f(x,y)$ is continuos throughout the Rectangular Region R $a \le x \le b$, $c \le y \le d$

$$\iint_R f(x,y)\, dA = \int_{y=c}^{y=d} \left(\int_a^b f(x,y)\, dx \right) dy = \int_{y=a}^{y=b} \left(\int_c^d f(x,y)\, dy \right) dx$$

These integrals are called Iterated integrals.

→ Notice that the inner differential should match up with the limits on the inner integral and similarly for the outer differentials and outer limits

Double integrals over general region

Since the region of integration may have boundaries other than line segments parallel to the coordinates axes, the limits of integration often involves variables not constants.

We will consider two types of non- rectangular regions

CASE 1: $R = \{ (x,y) \ / \ a \le x \le b \ , \ g_1(x) \le y \le g_2(x) \}$

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

CASE 2: $R = \{ (x,y) \ / \ h_1(y) \le x \le h_2(y) \ , \ c \le y \le d \}$

$$\iint_R f\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy$$

ex

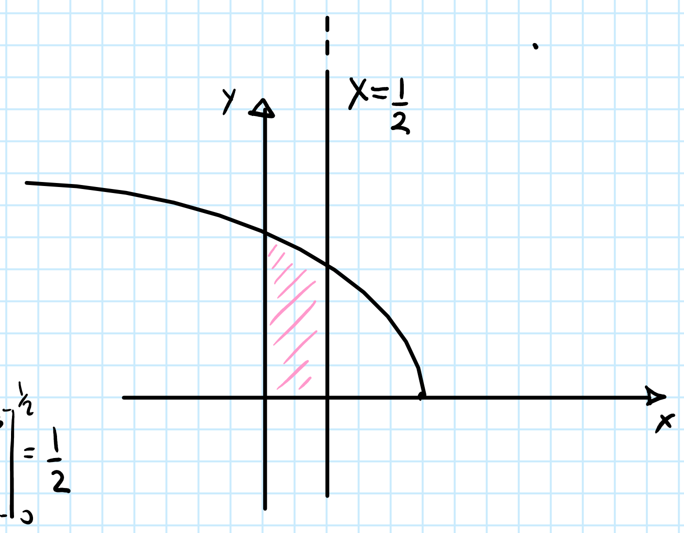
$\iint_R x e^{xy}\, dA$ where $R: [-1,2] \times [0,1]$

- $\int_{-1}^{2} \left(\int_0^1 x e^{xy}\, dx \right) dy = \int_{-1}^{2} \frac{x}{x} \left[e^{xy} \right]_0^1 dx$

$= \int_{-1}^{2} (e^x - 1)\, dx = \left[e^x - x \right]_{-1}^{2} = e^2 - e^{-1} - 3$

- $\int_0^1 \left(\int_{-1}^{2} x e^{xy}\, dx \right) dy = \int_0^1 \left[\frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy} \right]_{-1}^{2} dy$

$\int_0^1 \frac{2}{y} e^{2y} - \frac{1}{y^2} e^{2y} \ldots\ldots$



ex $\iint_R 2xy\, dA$ where $R: \{ (x,y) \ / \ 0 \le x \le \frac{1}{2} \ , \ 0 \le y \le \sqrt{1-x} \}$

$$\int_0^{1/2} \int_0^{\sqrt{1-x}} (2xy)\, dy\, dx = \int_0^{\frac{1}{2}} \frac{2}{2} x \left[y^2 \right]_0^{\sqrt{1-x}} dx = \int_0^{\frac{1}{2}} x \left[1 - x - 0 \right] dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^{\frac{1}{2}} = \frac{1}{2}$$

Fubin's thm for general region

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy$$

Properties of double integral

1) $\iint f(x,y) \pm g(x,y)\, dA = \iint f\, dA \pm \iint g\, dA$

2) $\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA$ where c is a cst.

3) If R can be split into 2 separate regions R_1 and R_2, then the integral can be written as:

$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$ [where R is the union of two non overlapping regions R_1 and R_2]

→ the second Geometric interpretation of a double integral is:

$\text{Area of } R = \iint_R dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_a^b g_2(x) - g_1(x) = \text{Area}$

Lesson 15

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EVALUATE THE FOLLOWING INTEGRALS

1) $\iint_R \frac{x}{1+xy}\,dy\,dx$ where $R: \{(x,y) \,/\, \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}\}$ $\quad \int \frac{1}{1+xy}$ $\quad u = 1+xy$, $du = x\,dy$ $\quad \int \frac{u'}{u} = \ln(u)$

$$\int_0^1 \int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 x\left[\frac{x}{x}\ln(1+xy)\right]_0^1 = \int_0^1 \ln|1+x|\,dx = (1+x)\ln(1+x) - (1+x)\Big|_0^1 = \ln(4) - 1$$

2) $\iint_R \frac{\ln y}{y}\,dA$, where $R = \{(x,y) \,/\, 0 \le x \le \pi$ and $e^{-2x} \le y \le e^{\cos x}\}$

cause x is cst.

$$\int_0^{\pi}\int_{e^{-2x}}^{e^{\cos x}} \frac{\ln y}{y}\,dy\,dx = \int_0^{\pi}\int_{-2x}^{\cos x} u\,du\,dx = \int_0^{\pi} \frac{u^2}{2}\Big|_{-2x}^{\cos x} dx = \frac{1}{2}\int_0^{\pi} \underbrace{\cos^2 x}_{\frac{1+\cos(2x)}{2}} - 4x^2\,dx = \frac{1}{2}\left[x + \frac{1}{4}\sin 2x - \frac{4}{3}x^3\right]_0^{\pi} = \frac{\pi}{2} - \frac{2\pi^3}{3}$$

let $u = \ln y$

$du = \frac{1}{y}\,dy$

$y = e^{-2x}$ $\quad u = -2x$

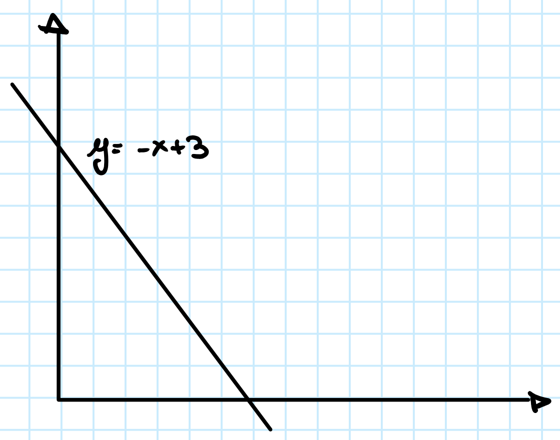
$y = e^{\cos x}$ $\quad u = \cos x$

3) $\iint_R \frac{1}{xy}\,dA$ where $R: \{(x,y) \,/\, 1 \le y \le e$ and $y \le x \le y^2\}$

$$\int_1^e \int_y^{y^2} \frac{1}{xy}\,dx\,dy = \int_1^e \frac{1}{y}\left[\ln x\right]_y^{y^2} dy = \int_1^e \frac{1}{y}\left[\ln y^2 - \ln y\right] dy = \int_1^e \frac{\ln y}{y} = \int_0^1 u\,du = \frac{1}{2}$$

$\ln e = 1$, $\ln 1 = 0$

4) $\iint_R 4x^3\,dy\,dx$ where R is the region bounded by $y = (x-1)^2$ and $y = -x+3$



Ex

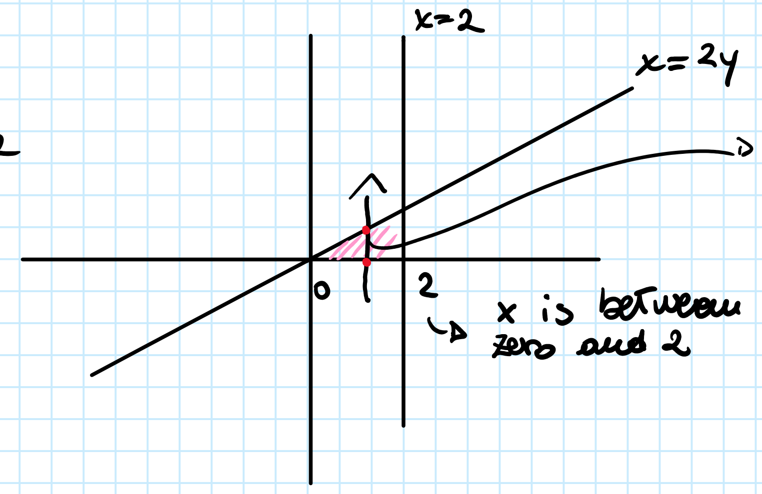
$$I = \int_0^1 \int_{2y}^2 e^{-x^2}\,dx\,dy$$

↓ reverse the order of integration

$$\int_0^2 \int_{y=0}^{y=\frac{1}{2}x} e^{-x^2}\,dy\,dx$$

R: $2y \le x \le 2$
$0 \le y \le 1$

$x = 2y \to y = \frac{1}{2}x$



draw an arrow an check where it enters and where it leaves the region.

↓

here it enters at ZERO and leaves at $y = \frac{1}{2}x$

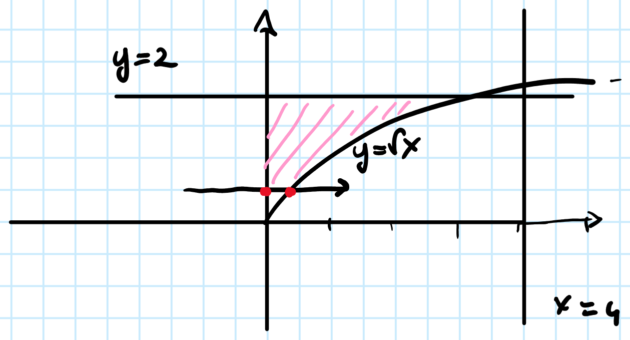
x is between zero and 2

$$= \int_0^2 e^{-x^2}\left[y\right]_0^{\frac{1}{2}x} dx = \int_0^2 \frac{1}{2}x e^{-x^2} dx = \dots\dots = -\frac{1}{4}\left[e^{-4} - 1\right]$$

Ex2

$$I = \int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx$$

→ convinient to reverse so for the 1st integral everything is constant



R: $\sqrt{x} \le y \le 2$
$0 \le x \le 4$

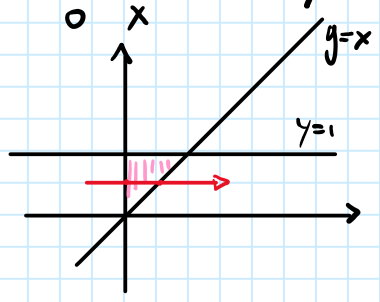
$y = u^3 \quad du = 3u^3$

$x = 0 \to u = 0$
$x = 2 \to u = y$

$$\int_0^2 \int_0^{x=y^2} \sin(y^3)\,dx\,dy = \int_0^2 \sin(y^3)\left[x\right]_0^{y^2} dy = \int_0^2 y^2 \sin(y^3)\,dy = -\frac{1}{3}\cos u \Big|_0^8 = \frac{1-\cos 8}{3}$$

Ex 3

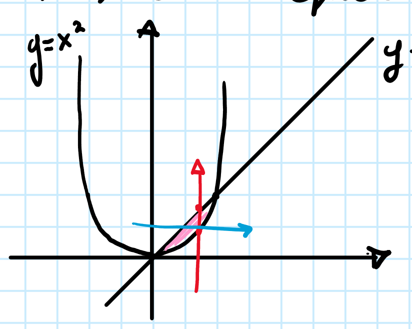
$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^{x=y} e^{y^2}\,dx\,dy = \int_0^1 e^{y^2} \cdot y\,dy = \int_0^1 \frac{1}{2} e^{u}\,du = \frac{1}{2}\left[e - 1\right]$$



R: $x \le y \le 1$
$0 \le x \le 1$

Ex 4

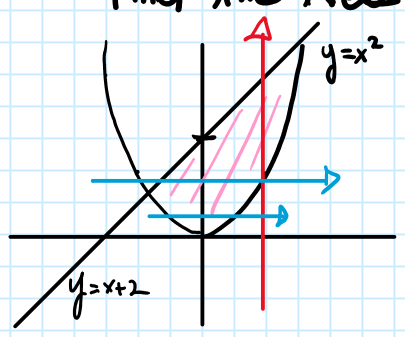
Find the area of the region in the plane bounded by $y = x$ and $y = x^2$ in the 1st quadrant → $A = \iint_R dA$!



$$\int_0^1 \int_{y=x^2}^{y=x} dy\,dx \quad \text{or} \quad \int_0^1 \int_y^{\sqrt{y}} dx\,dy$$

Ex 5

Find the Area of the region enclosed by the parabola $y = x^2$ and $y = x + 2$



→ $x^2 = x + 2$
$x^2 - x + 2 = 0$ ⟨ $x = 2$, $y = 4$; $x = -1$ $y = 1$

cause we have 2 region, we split the integral.

$$\int_{-1}^2 \int_{x^2}^{y=x+2} dy\,dx \quad \text{or} \quad \int_0^2 \int_{-\sqrt{y}}^{+\sqrt{y}} dx\,dy + \int_0^2 \int_{y-2}^{\sqrt{x}} dx\,dy$$

Lesson 16 pt. 2

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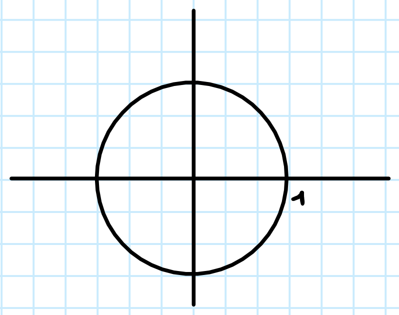
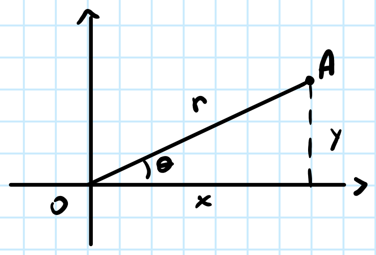
DOUBLE INTEGRALS IN POLAR COORDINATES

$f(x,y) = e^{-x^2-y^2}$

$\iint_R f(x,y)\, dA$ → where R is the unit circle





$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

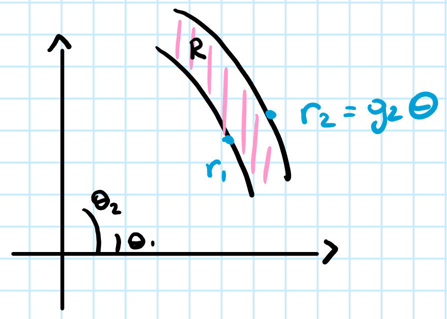
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

"trust me bro"

$$\iint f(x,y) \overbrace{dx\,dy}^{dA} \longrightarrow \int\int_{r=g(\theta)}^{r=g(\theta)} f(r\cos\theta, r\sin\theta) \underbrace{r\,dr\,d\theta}_{dA}$$

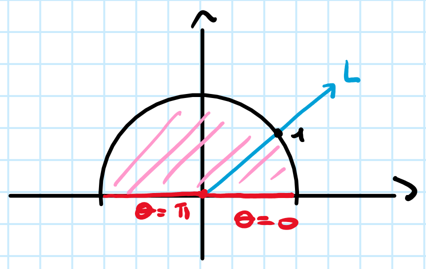
LIMITS OF INTEGRATION

1) Sketch your region
2) Draw a line L through the origin and check where it enters R ($g_1(\theta)$) and where it leaves R ($g_2(\theta)$)



Ex

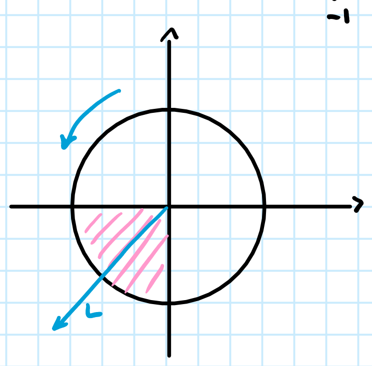
Evaluate $I = \iint_R e^{x^2+y^2} dx\,dy$ where R is the Region bounded by x-axis and the curve $y = \sqrt{1-x^2}$



$$I = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1} r e^{r^2} dr\, d\theta \qquad u = r^2 \quad du = 2r\,dr \longrightarrow \int_0^{\pi} \frac{1}{2}(e-1)\, d\theta = \frac{1}{2}(e-1)\pi$$

Ex2

Evaluate $I = \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$



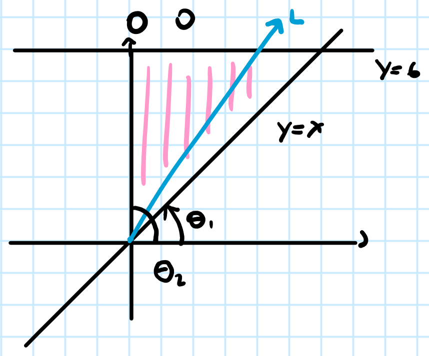
R: $-\sqrt{1-x^2} \le y \le 0$

$-1 \le x \le 0$

$$\int_{\pi}^{\frac{3\pi}{2}}\int_0^1 \frac{1}{\cancel{r}} \cancel{r}\, dr\, d\theta = \dots\dots \frac{\pi}{2}$$

Ex3

$$I = \int_0^6\int_0^y x\, dx\, dy = \int_0^6 \frac{x^2}{2}\Big|_0^y dy = \int_0^6 \frac{y^2}{2} dy = \frac{y^3}{6}\Big|_0^6 = 36$$



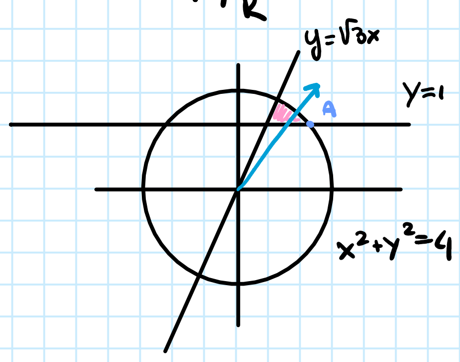
$y = 6 = r\sin\theta \quad r = \frac{6}{\sin\theta} = 6\,\mathrm{cosec}\,\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{r=\frac{6}{\sin\theta}} r\cos\theta\, r\, dr\, d\theta$$

Ex 4

Find the area of the region R in (xy) plane enclosed by the circle $x^2+y^2=4$ above the line $y=1$ and below the line $y = \sqrt{3}x$

$$A = \iint_R dA \curvearrowright \iint_R r\,dr\,d\theta$$



A = ∩ between circle and y=1

$x^2+y^2=4$ and $y=1$

$x^2 = 3 \quad x = \pm\sqrt{3}$

$\tan\theta_1 = \frac{1}{\sqrt{3}}$ → cause the point A is $(\sqrt{3}, 1)$

$\tan\theta_2 = \sqrt{3}$ → B is $(1, \sqrt{3})$

$$\int_{\pi/6}^{\pi/3}\int_{r=\frac{1}{\sin\theta}}^{r=2} r\,dr\,d\theta$$

$y = 1 = r\sin\theta$

Lesson 17

mercoledì 29 novembre 2023 17:25

Triple Integral

def. The volume of a closed bounded region D in the space is

$$V = \iiint_D dV$$ could be $dx\,dy\,dz$, dx, dz, dy, dz, dx, dy ..... and so on

$$D: \begin{array}{l} f_1(x,y) \le z \le f_2(x,y) \\ g_1(x) \le y \le g_2(x) \\ x_1 \le x \le x_2 \end{array} \longrightarrow \int_{x_1}^{x_2}\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx$$

How to evaluate triple integral?

$$f_1(x,y) \le z \le f_2(x,y) \longrightarrow \iiint_{f_1}^{f_2} f(x,y,z)\,dz\,dA$$

ex

$$\int_0^2\int_0^{\frac{\pi}{2}}\int_0^3 x y^2 \cos(z)\,dy\,dz\,dx =$$

$$\int_0^2\int_0^{\frac{\pi}{2}} x\cos(z)\left.\frac{y^3}{3}\right]_0^3 dz\,dx = \int_0^2\int_0^{\frac{\pi}{2}} 9x\cos(z)\,dz\,dx = \int_0^2 9x \sin(z)\Big]_0^{\frac{\pi}{2}} dx = \int_0^2 9x\,dx = \frac{9}{2}x^2\Big|_0^2 = 18$$

Ex 2

Find the volume of the region D enclosed by $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

$$V = \iiint_D dV$$

1° step z - limits $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$ → WHY THIS ORDER?

- take any point in the region and substitute

$z_1 = x^2 + 3y^2$ at $(0,0) = 0$

$z_2 = 8 - x^2 - y^2$ at $(0,0) = 8$

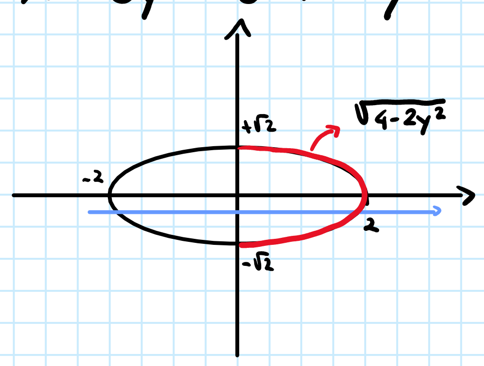
$z_1 < z_2$ (lower, upper)

$$V = \iint_R \int_{z_1}^{z_2} dz\,dA$$

2° step. Find R, the projection of D on (xy) plane

$z = z$

$$x^2 + 3y^2 = 8 - x^2 - y^2 \rightarrow 2x^2 + 4y^2 = 8 \rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$$



$2x^2 + 4y^2 = 8$

$x^2 = 4 - 2y^2$ $\quad x = \pm\sqrt{4-2y^2}$

$$\int_{-2}^{2}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dx\,dy$$

Lesson 18

lunedì 4 dicembre 2023 17:27

Ex Set up the triple integral of $f(x,y,z)$ over T bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{1-x^2-y^2}$

the z-limit

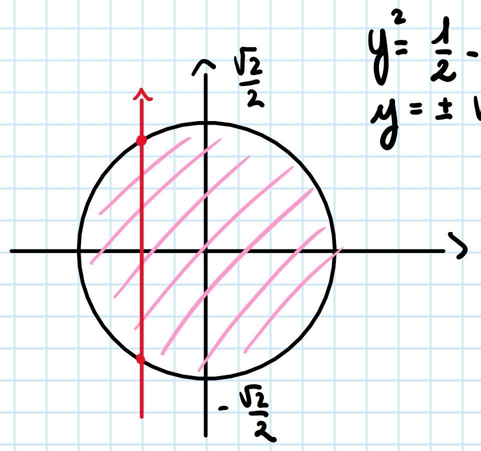
$\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$

R projection on (x,y) plane $z=z$

$\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2}$

$x^2+y^2 = 1-x^2-y^2$

$x^2+y^2=\frac{1}{2}$

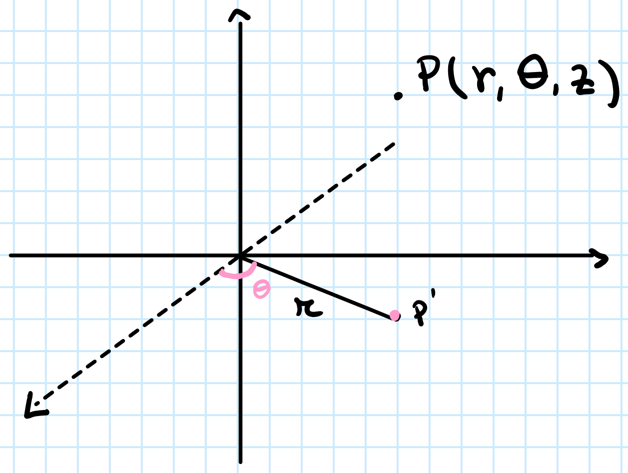


$y^2=\frac{1}{2}-x^2$

$y=\pm\sqrt{\frac{1}{2}-x^2}$

$$\int_{-\frac{\sqrt2}{2}}^{\frac{\sqrt2}{2}}\int_{-\sqrt{\frac12-x^2}}^{\sqrt{\frac12-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

Triple Integral in Cylindrical Coordinates



Def: cylindrical coordinates represents a point P in space by triplets (r,θ,z) such that:

1) r and θ are polar coordinates for the vertical projection of P on the (x,y) plane

2) z is the rectangular coordinate

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$$

$x^2+y^2=r^2$

$\tan\theta=\frac{y}{x}$

$$\iiint f(x,y,z)\,dz\,dy\,dx$$

$$\downarrow$$

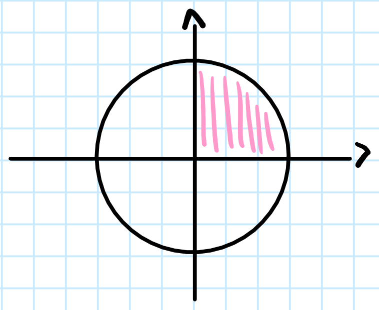
$$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} r f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Ex1 $\iiint_D y\,dV$ where D is bounded by $z=4-x^2-y^2$ in the 1^{st} octant $(x\ge0, y\ge0, z\ge0)$

$0\le z\le 4-x^2-y^2$

R projection on the (x,y) plane

$4-x^2-y^2=0$ $x^2+y^2=4$, $r=2$



$$\iint\int_0^{4-x^2-y^2} y\,dz\,dA$$

$$\int_0^{\frac{\pi}{2}}\int_0^2\int_0^{4-r^2} r^2\sin\theta\,dz\,dr\,d\theta$$

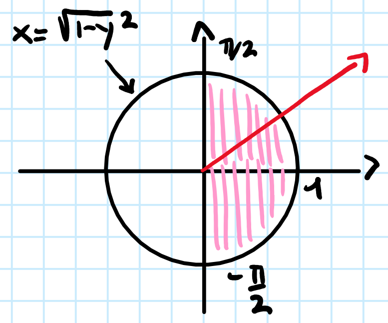
$$\int_0^{\frac{\pi}{2}}\int_0^2 r^2\sin\theta(4-r^2)\,dr\,d\theta = \dots = \frac{64}{15}$$

Ex Convert $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$

$x^2+y^2\le z\le\sqrt{x^2+y^2}$ $r^2\le z\le r$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_{r^2}^{r} r\cos\theta\cdot r\sin\theta\, z\cdot r\cdot dz\,dr\,d\theta$$

$0\le y\le\sqrt{1-y^2}$ $-1\le y\le1$

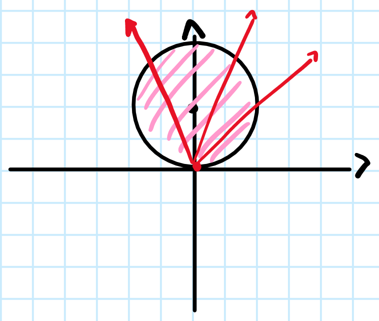


Ex finds the limits of integration over the region D bounded below by $z=0$, $x^2+(y-1)^2=1$ and above by $z=x^2+y^2$

z limits → $0\le z\le x^2+y^2$

$0\le z\le r^2$

R circle centered at $(0,1)$ and $r=1$



$x^2+(y-1)^2=1$

$x^2+y^2-2y=0$ → $r^2-2r\sin\theta=0$

$r(r-2\sin\theta)=0 \begin{cases} r=0 \\ r=2\sin\theta \end{cases}$

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r f(r,\theta,z)\,dz\,dr\,d\theta$$

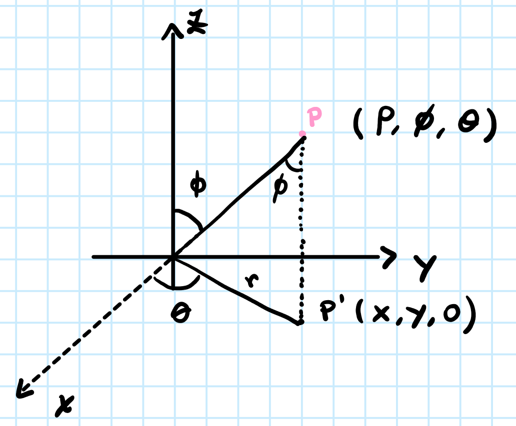
Lesson 19

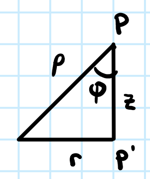
lunedì 11 dicembre 2023 16:28

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

· A point P in space is determined by (ρ, ϕ, θ)

1) ρ is the distance from P to the origin
2) ϕ is the angle that OP makes with positive z-axis $\quad 0 \le \phi \le \pi$
3) θ is the angle in polar coordinates $\quad 0 \le \theta \le 2\pi$





$x = r\cos\theta$
$y = r\sin\theta$
$\sin\theta = \frac{r}{\rho}$

$\cos\theta = \frac{z}{\rho} \rightarrow r = \rho\sin\phi$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \rightarrow \quad x^2+y^2+z^2=\rho^2, \quad \phi = \cos\left(\frac{z}{\rho}\right)$$

$$\iiint f(x,y,z)\,dV \rightarrow \iiint f(\rho,\phi,\theta)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Change of variable

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_c^d f(u)\,du \qquad u = g(x), \quad du = g'(x)\,dx$$

Def. Let $x = g(u,v)$ $\quad y = h(u,v)$, then the jacobian is:

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$\iint_R f(x,y)\,dx\,dy = \iint f(g(u,v), h(u,v))\,\underbrace{|J(u,v)|}_{\text{why we have } r!}\,du\,dv$$

Let $x = g(u,v)$
$y = h(u,v)$

$$\iint f(x,y)\,dA = \iint f(r,\theta)\,r\,dr\,d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad J(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

In a plane:

$$\iiint_R dV \longrightarrow \iiint_S |J|\,du\,dv\,dw$$

$R \longmapsto S$

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases} \qquad J(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad J(\rho,\phi,\theta) = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

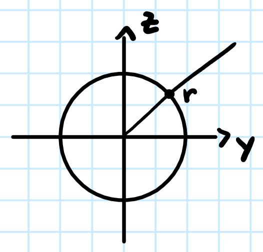
$$= \cos\phi\left[\overbrace{\rho^2\cos^2\theta\cos\phi\sin\phi + \rho^2\cos\phi\sin\phi\sin^2\theta}^{\rho^2\cos\phi\sin\phi}\right] + \rho\sin\phi\left[\underbrace{\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta}_{\rho\sin\phi}\right] = \rho^2\sin\phi$$

Ex. 1 Find the volume of the sphere

$V = \iiint_D \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$D: x^2+y^2+z^2 = r^2$

set $x = 0 \rightarrow y^2+z^2=r^2$

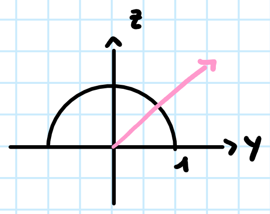


$0 \le \rho \le r$
$0 \le \phi \le \pi$
for $z=0$ $(x^2+y^2=1)$ $0 \le \theta \le 2\pi$

Ex 2

Evaluate $\iiint_D y\,dV$ where D is bounded between $z = \sqrt{1-x^2-y^2}$ and (xy) plane

$$\iiint_D y\,dV = \iiint (\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

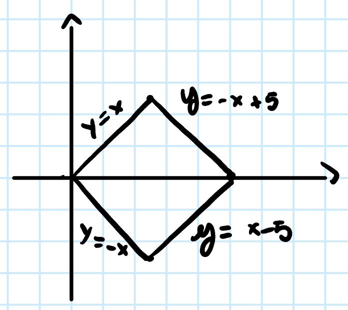


for $x=0 \rightarrow z = \sqrt{1-y^2}$
$0 \le \rho \le 1$
$0 \le \phi \le \frac{\pi}{2}$

for $z=0 \rightarrow x^2+y^2=1 \quad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Ex Evaluate $\iint_R (x+y)\,dA$ where R is the trapezoid with vertices $(0,0)$, $(5,0)$, $\left(\frac{5}{2},\frac{5}{2}\right)$ and $\left(\frac{5}{2},-\frac{5}{2}\right)$ using the transformation $\begin{cases} x = 2u+3v \\ y = 2u-3v \end{cases}$



Along $y=x$
$2u+3v = 2u-3v \rightarrow v=0$

Along $y=-x$
$2u-3v = -2u-3v \rightarrow u=0$

Along $y=-x+5$
$2u-3v = -2u-3v+5 \rightarrow u = \frac{5}{4}$

Along $y=x-5$
$v = \frac{5}{6}$

$$J(u,v) = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = -12$$

$$\int_0^{5/6}\int_0^{5/4} 12\,(2u+3v+2u-3v)\,du\,dv = \frac{125}{4}$$

Lesson 20

mercoledì 13 dicembre 2023 17:27

VECTOR VALUED FUNCTIONS

→ the coordinates of a pt A moving in space are given by $A(x(t), y(t), z(t))$.
The position of A is determined by a vector called the position vector $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad a \le t \le b \quad \to \text{parametric eq. of the path of } A$$

Def:

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ defines the position vector of a point M at time t, the velocity of M at t is:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

the magnitude of v is called speed and it is given by:

$$\text{speed} = \|\vec{v}\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

the acceleration: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$

Note: the velocity $\vec{v}(t)$ represents the direction vector of the tangent line to (c) at any point M at any time t

Ex 1 $r(t) = \cos(t)\vec{i} + \sin(t)\vec{j} \quad 0 \le t \le 2\pi$

$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $x^2 + y^2 = \cos^2 t + \sin^2 t$

C is a circle (0,0), r=1

Ex 2 $r(t) = \cos t\,\vec{i} + \sin t\,\vec{j} \quad 0 \le t \le \pi$

$x^2 + y^2 = 1$ → semicircle due to the values of t!

Ex 3 $r(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} \quad 0 \le t \le 2\pi$

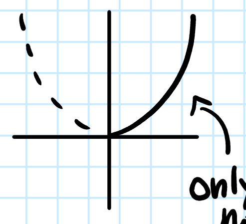
$\begin{cases} x = 2\cos t \to \cos t = \frac{x}{2} \\ y = \sin t \end{cases}$ $\cos^2 t + \sin^2 t = 1$

$\frac{x^2}{4} + y^2 = 1$ ↳ ellipse

Ex 4 $r(t) = t\vec{i} + t^2\vec{j} \quad t \ge 0$

$\begin{cases} x = t \\ y = t^2 \end{cases}$ $y = x^2$

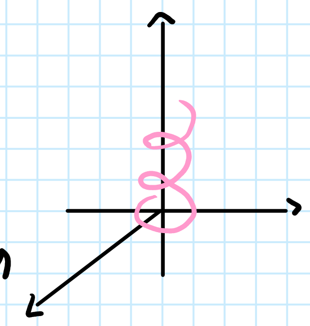
$t \ge 0 \to x \ge 0$



Ex 5 $r(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k} \quad t \ge 0$

$x = \cos t$
$y = \sin t$
$z = t$

$x^2 + y^2 = 1$
$z = t$
↓
circular cylinder



Lenght of a curve

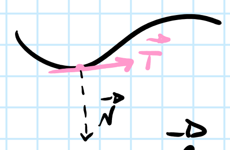
The lenght of a curve $F(t) = x(t)\vec{i} + y(t)\vec{j}$ with $a \le t \le b$ is given by

$$L = \int_a^b \|\vec{v}\|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Unit tangent vector

Since the velocity is the tangent to the curve $\vec{r}(t)$ then the unit tangent vector is

$$T = \frac{\vec{v}}{\|\vec{v}\|}$$



→ at any point P, the unit normal vector is $\vec{N} = \frac{\frac{dT}{dt}}{\left\|\frac{dT}{dt}\right\|}$

VECTOR FIELD

Def: A vector field is a function that assigns to each point (x, y, z) in its domain a vector given by:

$$\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

→ the field is continuous / differentiable if M, N, R are continuous / differentiable

→ for a given function $f(x,y,z)$ the gradient vector $\nabla f = (f_x, f_y, f_z)$ is a vector field

Def: A vector field $\vec{F}$ is called a conservative vector field if there exists a function f such that $\vec{F} = \nabla f$

→ if $\vec{F}$ is a conservative vector field then the function f is called a potential function of F

LINE INTEGRAL

→ we perform integration by taking the points (x,y) that lie on a curve (c), this integral is called line or curve integral and is given by

$$\int_c f(x,y,z)\,ds \quad , \quad ds = \|\vec{v}\|dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

→ assume the curve is given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ → the curve is called smooth if $\vec{r}(t)$ is cont. and $\vec{r}'(t) \ne 0 \ \forall t$

Steps:
1. parametrize (c)
2. $\int_c f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,\|\vec{v}\|\,dt$

Lesson 21

lunedì 18 dicembre 2023 16:27

Curve	Parametrization
$x^2+y^2=r^2$	$x = r\cos t$, $y = r\sin t$, $0 \le t \le 2\pi$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x = a\cos t$, $y = b\cos t$, $0 \le t \le 2\pi$
$x = g(y)$	$y = t$, $x = g(t)$
$(\underbrace{x-a}_{r\cos t})^2+(\underbrace{y-b}_{r\sin t})^2=r^2$	$x = a + r\cos t$, $y = b + r\sin t$

Line integral for a vector field

Given $\vec{F}(x,y,z) = H(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ vector field and smooth curve $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$ $\quad a \le t \le b$

$$\int_C \vec{F}\,d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot r'(t)\,dt$$

↳ dot product

Thm. Suppose that C is a smooth curve given by $\vec{r}(t)$ $a \le t \le b$. Assume f is a function, whose gradient vector ∇f is continuous on C. Then $\int_C \nabla f\,dr = f(r(b)) - f(r(a))$ where $(r(a)) \sim$ initial pt, $(r(b)) \sim$ end pt.

proof: $\int_C \nabla f\,dr = \int_a^b \nabla f(\vec{r}(t)) \cdot r'(t)\,dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt = \int_a^b \frac{d}{dt}(f(r(t)))dt = f(r(b)) - f(r(a))$

$\nabla f \to (f_x, f_y, f_z)$, $\quad r'(t) \to \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$

$f(x(t), y(t), z(t))$

$\frac{\partial f}{\partial x}$: x, y, z

t

Def: Assume $\vec{F}$ is continuous vector field

if $\int_{C_1} \vec{F}\,dr = \int_{C_2} \vec{F}\,d\vec{r}$ for any 2 paths C_1 or C_2 with the same initial and final pts, then $\int_C \vec{F}\,dr$ is indep. of path

remarks

1) $\nabla f\,dr$ is indep. of path
2) If F is conservative, then $\int_C F\,dr$ is independent of path
   ↳ $F = \nabla f$ $\quad \int F\,dr = \int \nabla f\,dr$

Test for conservative vector field

Let $F = H(x,y,z)\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\vec{k}$ be field

Then F is conservative if:

$$\frac{\partial H}{\partial y} = \frac{\partial N}{\partial x} \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \qquad \frac{\partial P}{\partial x} = \frac{\partial H}{\partial z}$$

$F = \nabla f$

$F = H\vec{i} + N\vec{j} + P\vec{k} = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(f_z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}(f_y) = \frac{\partial N}{\partial z}$$

EX $\int_C f\,ds$ / $f(x,y,z) = x - 3y^2 + z$ over the line segment joining the pt $O(0,0,0)$ to $A(1,1,1)$

$\vec{OA} = (1,1,1)$

$\begin{cases} x = t \\ y = t \\ z = t \end{cases}$ $0 \le t \le 1$ $\qquad \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$

$\vec{r}(t) = t\vec{i} + t\vec{j} + t\vec{k}$

$\vec{r}'(t) = \vec{i} + \vec{j} + \vec{k}$

$v = \|r'(t)\| = \sqrt{3}$

$f(\vec{r}(t)) = t - 3t^2 + t = 2t - 3t^2$

$\int_C f\,ds = \int_0^1 (2t - 3t^2)\sqrt{3}\,dt =$

$= \sqrt{3}\,[t^2 - t^3]_0^1 = 0$

EX 2 $\int_C xy^4\,ds$ where C is the right half circle of $x^2+y^2=16$

$\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\vec{r}(t) = 4\cos t\,\vec{i} + 4\sin t\,\vec{j}$

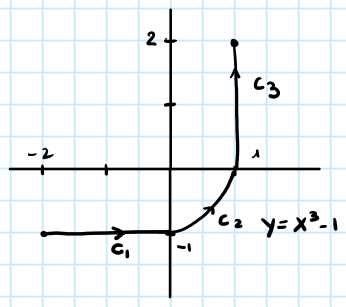
$\vec{r}'(t) = -4\sin t\,\vec{i} + 4\cos t\,\vec{j}$

$\|\vec{r}'(t)\| = 4$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4\,4\,dt$

$u = \sin t$

EX3 $\int_C 4x^3\,ds$ where C is the curve $C = C_1 \cup C_2 \cup C_3$



$C_1: \begin{cases} x = t \\ y = -1 \end{cases}$ $\quad -2 \le t \le 0$

$\vec{r}(t) = t\vec{i} + \vec{j}$

$r'(t) = \vec{i}$

$\|v\| = 1$

$f(\vec{r}(t)) = 4t^3$

$\int_C f\,ds = \int_{-2}^{0} 4t^3\,1\,dt$

(C_2): $y = x^3 - 1$

$\begin{cases} x = t \\ y = t^3 - 1 \end{cases}$ $\quad 0 \le t \le 1$

$\vec{v} = \vec{i} + 3t^2\vec{j}$

$\|v\| = \sqrt{1 + 9t^2}$

$f(r(t)) = 4t^3$

$\int_{C_2} f\,ds = \int_0^1 4t^3\sqrt{1+9t^2}\,dt = \frac{2}{27}(\sqrt{10} - 1)$

C_3: $x = 1$

$\begin{cases} x = 1 \\ y = t \end{cases}$ $\quad 0 \le t \le 2$

$v = \vec{j}$

$\|v\| = 1$

$f(r(t)) = 4$

$\int_{C_3} = \int_0^2 4 \cdot 1\,dt = 8$

ex $\int_C \vec{F}\,dr$ where $F(x,y,z) = 8x^2yz\,\vec{i} + 5z\,\vec{j} - 4xy\,\vec{k}$ and C is given by $\vec{r}(t) = t\,\vec{i} + t^2\vec{j} + t^3\vec{k}$ $\quad 0 \le t \le 1$

$F(t, t^2, t^3) = 8t^7\vec{i} + 5t^3\vec{j} - 4t^3\vec{k}$

$r'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\vec{k}$

$\int_C F\,dr = \int_0^1 F(r(t)) \cdot r'(t)\,dt = \int_0^1 8t^7 + (0t^4 - 12t^5)\,dr$