DEF PRIME(n):
  FOR $i = 2, \dots, \sqrt{n}$:
    IF $n \% i == 0$:
      RETURN FALSE
  RETURN TRUE

$O(1)$ $\quad O(\sqrt{n})$

RUNTIME $\leq O(\sqrt{n})$

$\boxed{\text{RUNTIME} \geq \Omega(1)}$

WORST-CASE RUNNING TIME ON INSTANCES OF SIZE $\leq n$ $\rightarrow R_n = \max_{\substack{\text{INSTANCES } I \\ \text{OF SIZE} \leq n}} \text{RUNTIME}(I)$

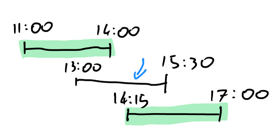
$\Omega\left(\sqrt{\frac{n}{2}}\right) \leq R_n \leq O(\sqrt{n})$

$R_n = \Theta(\sqrt{n})$

GREEDY ALGORITHMS

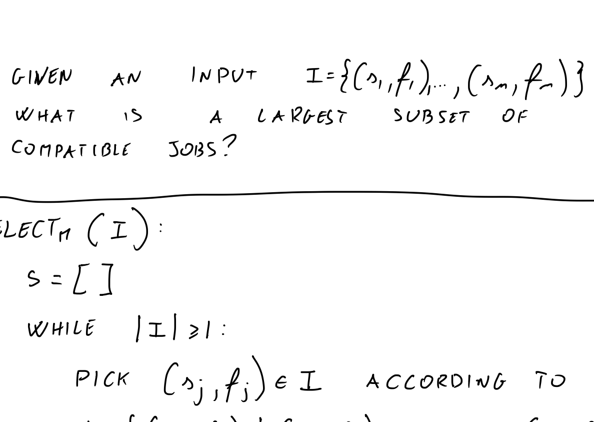
"INTERVAL SCHEDULING"

INPUT: n INTERVALS/JOBS, $I = \{(s_1, f_1), (s_2, f_2), \dots, (s_n, f_n)\}$



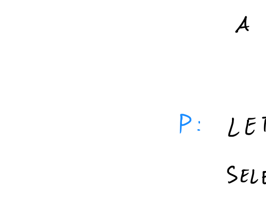
QUESTION: WHAT IS A SET OF "COMPATIBLE" JOBS OF LARGEST CARDINALITY?

DEF.: A SET $S \subseteq I$ OF JOBS IS COMPATIBLE IF $\forall (s_i, f_i), (s_j, f_j) \in S$, $(s_i, f_i) \neq (s_j, f_j)$, THEN $\min(f_i, f_j) \leq \max(s_i, s_j)$.

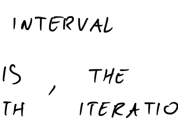


P: GIVEN AN INPUT $I = \{(s_1, f_1), \dots, (s_n, f_n)\}$, WHAT IS A LARGEST SUBSET OF COMPATIBLE JOBS?

```
SELECT_M (I):
    S = [ ]
    WHILE |I| ≥ 1:
        PICK (s_j, f_j) ∈ I ACCORDING TO RULE M
        N = {(s_i, f_i) | (s_i, f_i) ∈ I AND (s_i, f_i) IS
                               INCOMPATIBLE WITH (s_j, f_j)}
        I -= N
        S. APPEND((s_j, f_j))
    RETURN S
```



M: "PICK ONE INTERVAL THAT IS INCOMPATIBLE WITH THE SMALLEST NUMBER OF OTHER INTERVALS"



$\boxed{M^* = \text{"PICK ONE INTERVAL THAT ENDS SOONEST"}}$

"FEASIBILITY" PROOF

L: FOR EACH "RULE" M, SELECT$_M$ RETURNS A SET OF COMPATIBLE INTERVALS.

P: LET (s_j^*, f_j^*) IS THE jTH INTERVAL SELECTED BY SELECT$_M$. (THAT IS, THE INTERVAL SELECTED IN THE jTH ITERATION OF THE LOOP).

SIMILARLY,

- LET $I_0 \triangleq I$ BE THE SET OF INPUT INTERVALS,
- LET $S_0 \triangleq [\,]$ BE THE VALUE OF S BEFORE THE LOOP BEGINS,
- LET I_j, $j \geq 1$, BE THE VALUE OF I RIGHT AFTER THE jTH ITERATION OF THE LOOP,
- LET S_j BE THE VALUE OF S RIGHT AFTER THE jTH ITERATION OF THE LOOP.

THEN $I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_t$, IF THE LOOP RUNS FOR t ITERATIONS. (THE STRICT INCLUSION HOLDS BECAUSE I_j WILL NOT CONTAIN (s_j^*, f_j^*), WHICH WAS IN I_{j-1}).

THEN, $S_{j+1} = S_j \cup \{s_j^*, f_j^*\}$. $\forall j = 0, \dots, t-1$.

$\emptyset = S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_t$.

C: EACH INTERVAL IN I_j IS COMPATIBLE WITH EACH INTERVAL IN S_j, $\forall j \geq 0$.

P: IF $j = 0$, THEN $S_j = S_0 = [\,]$. THUS, "$\forall A \in I_j$ $\forall B \in S_j$: A AND B ARE COMPATIBLE". IS VACUOUSLY TRUE.

WE THEN ASSUME THAT THE CLAIM HOLDS FOR j, AND WE PROVE IT FOR $j+1$.

THAT IS, WE ASSUME THAT $\forall A \in I_j$ $\forall B \in S_j$, A AND B ARE COMPATIBLE. WE WANT TO PROVE THAT $\forall A \in I_{j+1}$ AND $\forall B \in S_{j+1}$, A AND B ARE COMPATIBLE.

RECALL THAT $S_{j+1} = S_j \cup \{(s_{j+1}^*, f_{j+1}^*)\}$. THUS, IF WE PROVE THAT (s_{j+1}^*, f_{j+1}^*) IS COMPATIBLE WITH EACH $A \in I_{j+1}$, WE ARE DONE.

NOW, $I_{j+1} \subseteq I_j$ AND THE GENERIC INTERVAL IN I_j IS ALSO IN I_{j+1} IFF (IF AND ONLY IF) IT IS COMPATIBLE WITH (s_{j+1}^*, f_{j+1}^*). THUS, $\forall A \in I_{j+1}$ IS COMPATIBLE WITH (s_{j+1}^*, f_{j+1}^*). THE INDUCTIVE STEP HAS BEEN PROVED.

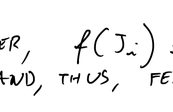
THUS, EACH TIME THE ALGORITHM SELECTS A NEW INTERVAL, THIS INTERVAL WILL BE COMPATIBLE WITH ALL THE INTERVALS PICKED BEFORE. AT ANY TIME, THEN, S WILL BE A SET OF COMPATIBLE INTERVALS

M^* = "PICK AN INTERVAL THAT ENDS SOONEST"

AN "EXCHANGE ARGUMENT" PROOF.

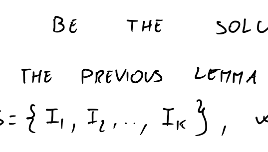
L: THE ALGORITHM GREEDY$_{M^*}$ RETURNS AN OPTIMAL SOLUTION.

P: LET $O \subseteq I$ BE AN OPTIMAL SOLUTION (O IS COMPATIBLE, AND $|O|$ IS AS LARGE AS POSSIBLE).



W.L.O.G, LET $O = \{J_1, J_2, \dots, J_m\}$ (J_i IS AN INTERVAL WITH STARTING TIME $s(J_i)$ AND FINISHING TIME $f(J_i)$ $\forall i$).

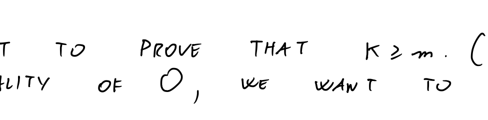
SUPPOSE, FURTHER, THAT THE J_i'S ARE SORTED BY THEIR FINISHING TIME: $f(J_1) \leq f(J_2) \leq \cdots \leq f(J_m)$.



THEN $s(J_i) < f(J_i)$. MOREOVER, $f(J_i) \leq s(J_{i+1})$ $\forall i = 1 \dots m-1$, SINCE O IS AND OPTIMAL (AND, THUS, FEASIBLE) SOLUTION.

THUS,

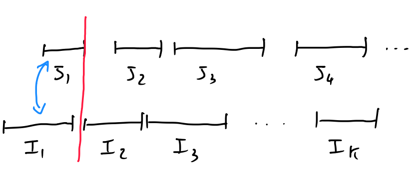
$$s(J_1) < f(J_1) \leq s(J_2) < f(J_2) \leq \cdots \leq s(J_m) < f(J_m).$$



LET S BE THE SOLUTION PRODUCED BY SELECT$_{M^*}$.

THEN, BY THE PREVIOUS LEMMA, S IS COMPATIBLE. SO, LETTING $S = \{I_1, I_2, \dots, I_K\}$, WE HAVE WLOG,

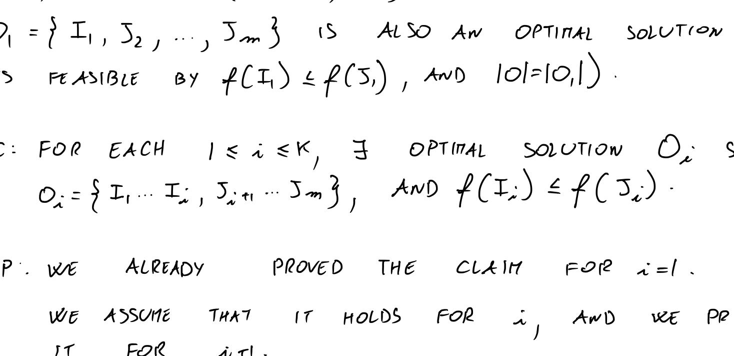
$$s(I_1) < f(I_1) \leq s(I_2) < f(I_2) \leq \cdots \leq s(I_K) < f(I_K).$$



WE WANT TO PROVE THAT $K \geq m$. (IN FACT, BY THE OPTIMALITY OF O, WE WANT TO PROVE THAT $K = m$).

C: $f(I_1) \leq f(J_1)$

P: BY RULE M^*, GREEDY WILL PICK AS ITS FIRST INTERVAL I_1 ONE OF THOSE THAT END SOONEST — THEN, THERE CANNOT BE AN INTERVAL THAT ENDS SOONER THAN I_1. □



THUS, IF $O = \{J_1, J_2, \dots, J_m\}$ IS AN OPTIMAL SOLUTION $O_1 = \{I_1, J_2, \dots, J_m\}$ IS ALSO AN OPTIMAL SOLUTION (IT IS FEASIBLE BY $f(I_1) \leq f(J_1)$, AND $|O| = |O_1|$).

C: FOR EACH $1 \leq i \leq K$, $\exists$ OPTIMAL SOLUTION O_i S.T. $O_i = \{I_1 \dots I_i, J_{i+1} \dots J_m\}$, AND $f(I_i) \leq f(J_i)$.

P: WE ALREADY PROVED THE CLAIM FOR $i = 1$.

WE ASSUME THAT IT HOLDS FOR i, AND WE PROVE IT FOR $i+1$.

THEN, $O_i = \{I_1 \dots I_i, J_{i+1} \dots J_m\}$ WITH $f(I_i) \leq f(J_i)$.

OUR ALG: PICKS AS I_{i+1}, THE ONE INTERVAL, COMPATIBLE WITH $I_1 \dots I_i$, THAT ENDS SOONEST. THUS, $f(I_{i+1}) \leq f(J_{i+1})$.

THUS, $O_{i+1} = \{I_1 \dots I_i\ I_{i+1}\ J_{i+2} \dots J_m\}$ IS AN OPTIMAL SOLUTION, WITH $f(I_{i+1}) \leq f(J_{i+1})$. □

THUS, WE KNOW THAT $O_K = \{I_1 \dots I_K, J_{K+1} \dots J_m\}$. BUT THE GREEDY ALGORITHM STOPS ONLY WHEN THERE ARE NO MORE INTERVALS COMPATIBLE WITH THE ONES CHOSEN SO FAR. GIVEN THAT GREEDY STOPPED AT THE KTH INTERVAL, $J_{K+1} \dots J_m$ CANNOT EXIST. THUS, $K = m$. □

THE QUINTESSENTIAL GREEDY ALGORITHM PROOF HYPOTHESIZES THE EXISTENCE OF SOME OPTIMAL SOLUTION O. AS THE GREEDY ALG. PROGRESSES THE PROOF TRANSFORMS THE CURRENT O INTO OTHER OPTIMAL SOLUTIONS THAT MATCH WHAT GREEDY DID SO FAR.