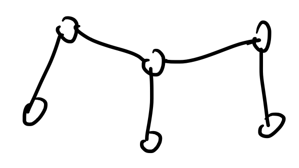
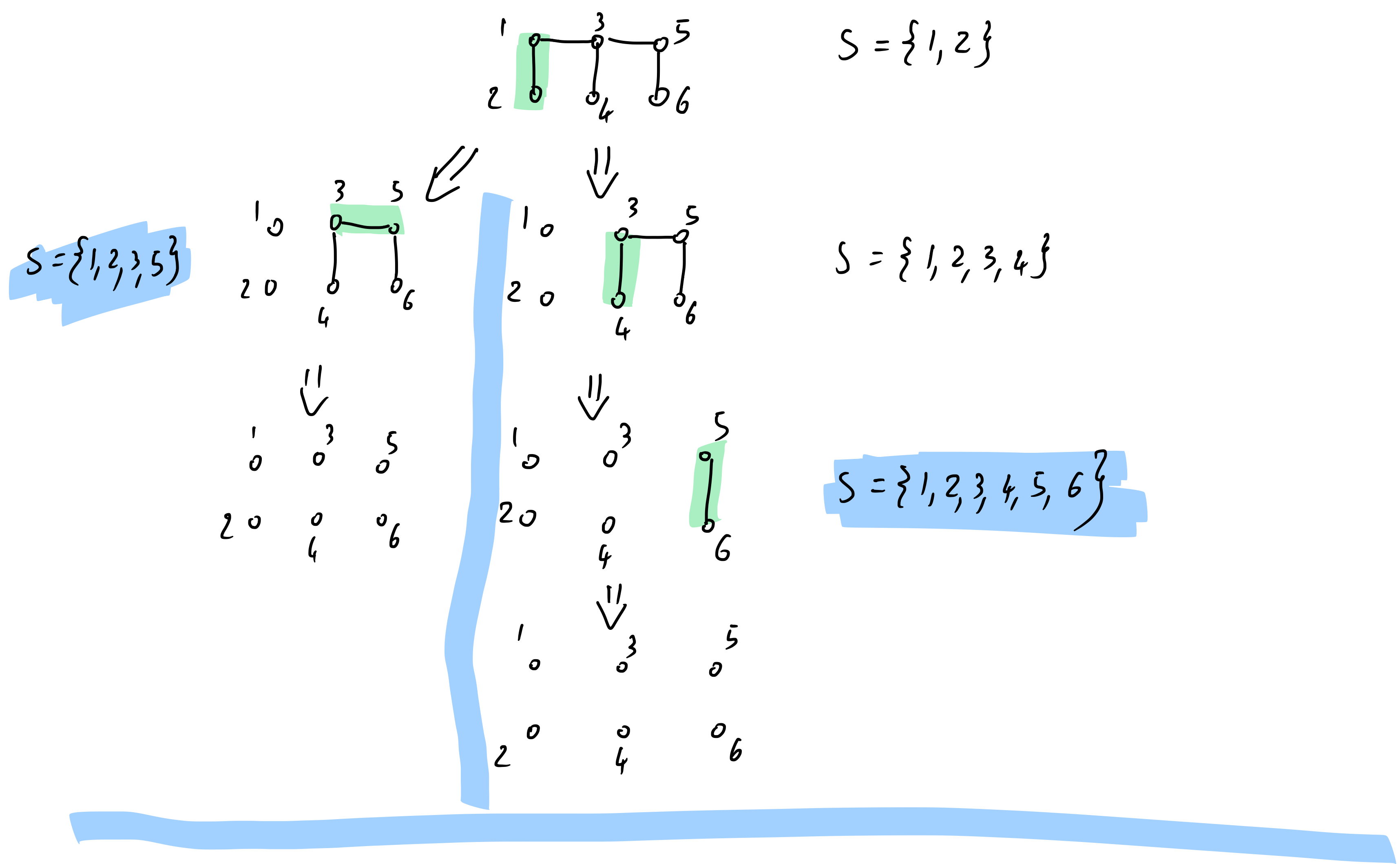
VERTEX-COVER IS NP-COMPLETE

WE CANNOT REALLY EXPECT TO SOLVE IT IN POLYTIME. (IF WE DO, WE'D HAVE A BREAKTHROUGH RESULT).



APPROX-VC $(G(V,E))$

$S \leftarrow \emptyset$
WHILE $E \neq \emptyset$:
PICK AN EDGE $e = \{u, v\} \in E$
$S \leftarrow S \cup \{u, v\}$
REMOVE FROM E EACH EDGE THAT IS INCIDENT ON u OR v
RETURN S



THM: APPROX-VC RETURNS A SET OF NODES THAT
(I) IS A VERTEX COVER, AND WHICH
(II) HAS A CARDINALITY NOT LARGER THAN TWICE THE SMALLEST VERTEX COVER.

MOREOVER, (III) THE ALGORITHM TAKES POLYTIME.

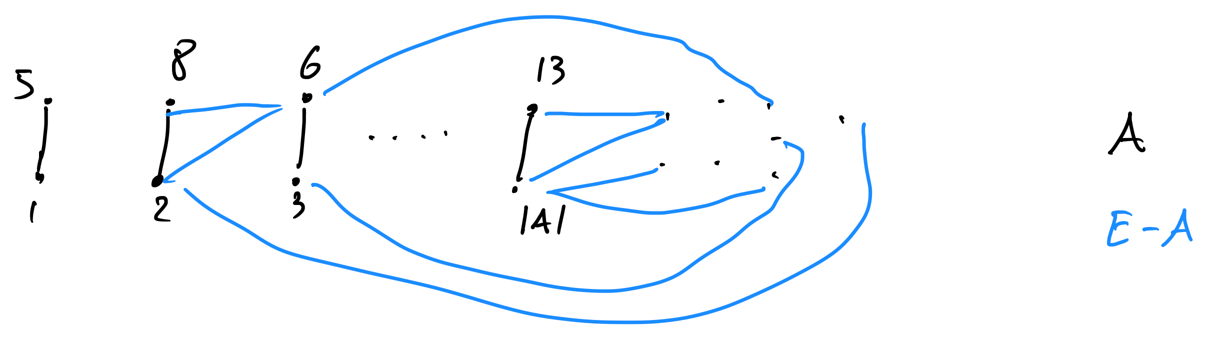
P: THE ALGO ITERATES AS LONG AS THERE ARE EDGES IN THE GRAPH.
IF THE ALG. PICKS EDGE $\{u,v\}$ THEN, BEFORE THE ITERATION ENDS, (I) IT REMOVES FROM THE GRAPH ALL THE EDGES INCIDENT ON u OR v, (II) IT ADDS u AND v TO S.
THUS, AT LEAST $\{u,v\}$ IS REMOVED FROM E (SO THE ALGORITHMS END). MOREOVER THE RETURNED SET S IS A VERTEX COVER (WE REMOVE AN EDGE ONLY IF WE COVER IT). THUS, (I) IS PROVED.

AS FOR (II), LET $A \subseteq E$ BE THE SET OF EDGES PICKED BY THE ALG.

OBSERVE THAT IF $\{u,v\}, \{u',v'\} \in A$, WITH $\{u,v\} \neq \{u',v'\}$, THEN $\{u,v\} \cap \{u',v'\} = \emptyset$. (THE FIRST OF THESE TWO EDGES TO BE PICKED RESULTS IN THE REMOVAL OF EACH OF ITS ADJACENT EDGES — THEREFORE, GIVEN THAT BOTH EDGES WERE PICKED THEY CANNOT SHARE ANY ENDPOINT).

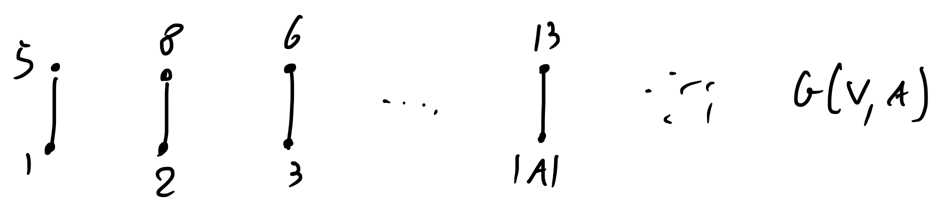
THEN, IF S IS THE RETURNED SOLUTION,
$|S| = 2|A|$ (THE ALGORITHM ADDS TO S TWO NEW NODES FOR EACH EDGE IT PICKS).

RECALL THAT $A \subseteq E$. WE CONSIDER THE GRAPH $G(V,A)$:



SINCE $A \subseteq E$, IF $T \subseteq V$ IS A VERTEX COVER FOR $G(V,E)$, IT MUST HOLD THAT T IS ALSO A VERTEX COVER FOR $G(V,A)$. (IF T COVERS ALL THE EDGES IN E, IT MUST ALSO COVER ALL THE EDGES IN ANY SUBSET OF E).

IN $G(V,A)$ NO TWO EDGES SHARE AN ENDPOINT



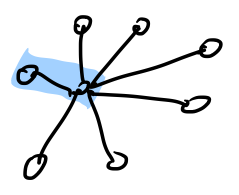
THUS, ONE NEEDS AT LEAST $|A|$ MANY NODES TO COVER ALL THE EDGES OF A.

GIVEN THAT $E \supseteq A$, ONE NEEDS AT LEAST $|A|$ NODES TO COVER ALL THE EDGES OF E.

IN OTHER WORDS THE MINIMUM VERTEX COVER FOR $G(V,E)$ CONTAINS AT LEAST $|A|$ NODES.

GIVEN THAT OUR VERTEX COVER S CONTAINS $|S| = 2|A|$ NODES, OUR SOLUTION IS NO WORSE THAN A 2-APPROXIMATION. ◻

VERTEX COVER IS NP-COMPLETE, BUT IT CAN BE 2-APPROXIMATED IN POLYNOMIAL TIME.



INDEPENDENT SET IS NP-COMPLETE, AND IT IS ALSO NP-COMPLETE TO APPROXIMATE TO ANYTHING BETTER THAN $n^{0.999\ldots}$