Index

1 Fundamental Theorem of Integral Calculus

2 Generic Construction of a Riemann Integral

3 Definition of Numerical Series

4 Fundamental Theorem of Calculus

5 Taylor Series for sin(x)

5.1 Remainder

6 Proof for Asymptotic Comparison Test

7 Numerical Series

7.1 Cauchy Criterion

7.1.1 Use to prove the necessary condition

7.2 Integral Test and its proof

8 Taylor Expansion

9 Taylor Series

10 Proof of e^x being its Taylor Series

The Taylor series of a function $f(x)$ around $x = a$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For $f(x) = e^x$, the Taylor series expansion around $x = 0$ (Maclaurin series) is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We need to prove that this series converges to e^x for all x.
1. **Taylor Series Expansion**:
The exponential function e^x has derivatives $f^{(n)}(x) = e^x$. At $x = 0$:

$$f^{(n)}(0) = e^0 = 1$$

Thus, the Maclaurin series for e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. **Convergence of the Series**:
To show that the series converges to e^x, we use the ratio test. Consider the n-th term of the series $a_n = \frac{x^n}{n!}$.
The ratio of successive terms is:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right|$$

As n approaches infinity, $\left| \frac{x}{n+1} \right|$ approaches 0 for all x. Since this limit is less than 1, the ratio test confirms that the series converges for all x.
3. **Equality with e^x**:
To prove that the series equals e^x, consider the function:

$$g(x) = e^x - \sum_{n=0}^{N} \frac{x^n}{n!}$$

Taking the limit as N approaches infinity, if the series converges uniformly to e^x, then:

$$\lim_{N \to \infty} g(x) = e^x - \sum_{n=0}^{\infty} \frac{x^n}{n!} = 0$$

Thus, we have shown that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Hence, the proof is complete.