Introduction to Linguistic Geometry

Class of Problems

Abstract Board Game (ABG)

is the following eight-tuple

$$< \mathbf{X, P, R_p, \{ON\}, v, S_{\mathit{i}}, S_{\mathit{t}}, TR} >$$

$\mathbf{X = \{x_{\mathit{i}}\}}$ is a finite set of ***points***;

$\mathbf{P} = \mathbf{P_1} \cup \mathbf{P_2}$ is a finite set of ***pieces***, $P_1 \cap P_2 \neq Ø$ (opposing sides);

$\mathbf{R_p(x, y)}$ is a family of binary relations of ***reachability*** in X
($x \in X$, $y \in X$, $p \in P$); y is ***reachable*** from x for p;

ON(p) = x is a partial function of ***placement*** of pieces P into X;

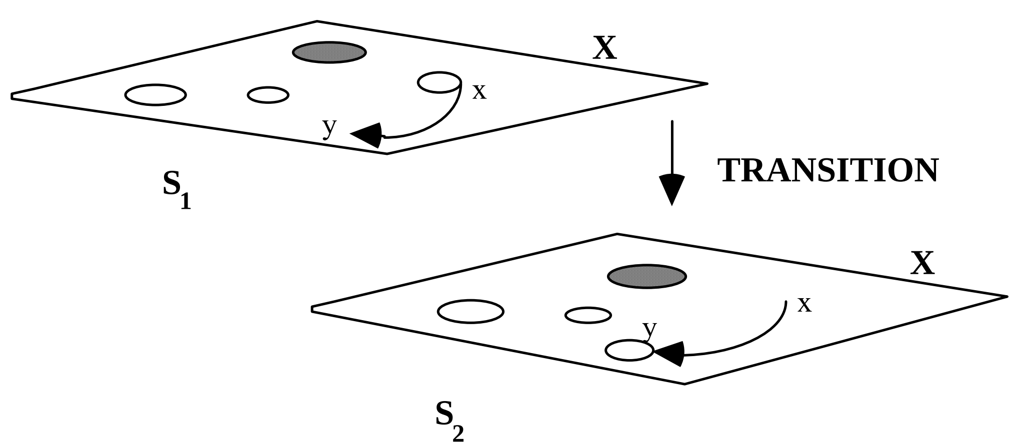
$\mathbf{v} > 0$ is a real function, **v**(p) are the ***values*** of pieces;

$\mathbf{S_{\mathit{i}}}$ is a set of ***initial*** states of the system,
a certain set of formulas $\{ON(p_i) = x_i\}$;

$\mathbf{S_{\mathit{t}}}$ is a set ***target*** states of the system (as S_i);

TR is a set of operators **TRANSITION(p, x, y)** for transition of the system from one state to another described as follows

precondition:	$ON(p) = x \wedge R_p(x, y)$
delete:	$ON(p) = x$, $ON(q) = y$
add:	$ON(p) = y$



Measurement of Distances in the ABG

A **map of the set X**

relative to the point x and element p for the ABG is the mapping:

$$\mathbf{MAP_{x,p}: X \longrightarrow Z_+},$$

(where x is from X, p is from P), which is constructed as follows.

We consider a ***family of reachability areas*** from the point x, i.e., a finite set of the following nonempty subsets of X $\{\mathbf{M^k_{x,p}}\}$**:**

$k = 1$: $\mathbf{M^k_{x,p}}$ is a set of points m ***reachable in one step*** from x: $R_p(x, m)=T$;

$k > 1$: $\mathbf{M^k_{x,p}}$ is a set of points ***reachable in k steps and not reachable in k*****-1** ***steps***, i.e., points m reachable from points of $M^{k\text{-}1}_{x,p}$ and not included in any $M^i_{x,p}$ with numbers i less than k.

Let $\mathbf{MAP_{x,p}(y) = \mathit{k}}$, for y from $\mathbf{M^k_{x,p}}$ (***number of steps from x to y***).

In the remainder points let $\mathbf{MAP_{x,p}(y) = 2n}$, if $y \neq x$**;** $\mathbf{MAP_{x,p}(y) = 0}$, if y=x.

It is easy to verify that **map of the set X** for the element p from P defines an ***asymmetric distance function*** on X:

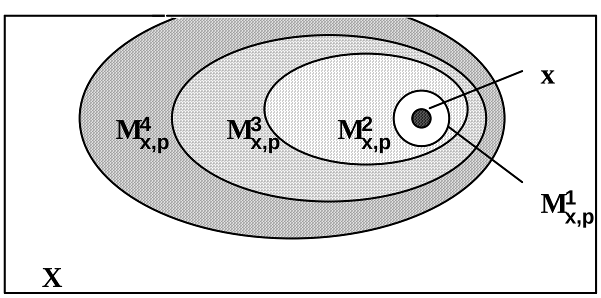
1. $MAP_{x,p}(y) > 0$ for $x \neq y$; $MAP_{x,p}(x) = 0$;
2. $MAP_{x,p}(y) + MAP_{y,p}(z) \geq MAP_{x,p}(z)$.

If R_p is a symmetric relation,

3. $MAP_{x,p}(y) = MAP_{y,p}(x)$,

In this case each of the pieces p from P specifies on X its ***own metric.***

Reachability areas



Values of $\mathbf{MAP_{f6,\ King}}$

5	4	3	2	2	2	2	2
5	4	3	2	1		1	2
5	4	3	2		0	1	2
5	4	3		1	1	1	2
5	4	3	2	2	2		2
5	4	3	3	3	3		3
5	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5

Language of Trajectories. The Shortest Path.

A *trajectory*
for a piece p of P with the beginning at x of X and the end at y of X ($x \neq y$) with a length $\boldsymbol{l}$ is the following string of symbols with parameters, points of X:

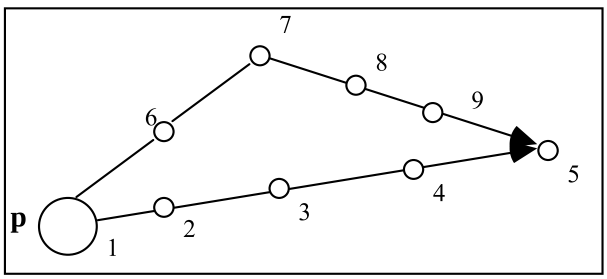
$$t_o = \boldsymbol{a}(x)\boldsymbol{a}(x_1)\ldots\boldsymbol{a}(x_l).$$

Here each successive point x_{i+1} is reachable from the previous point x_i:

$$R_p(x_i, x_{i+1}) \text{ holds for } i = 0, 1, \ldots, \boldsymbol{l}-1;$$

element p stands at the point x: ON(p) = x.
We denote $t_p(x, y, \boldsymbol{l})$ the set of trajectories in which p, x, y, and $\boldsymbol{l}$ are the same. $\boldsymbol{P}(t_o) = \{x, x_1, ..., x_l\}$ is the set of parameter values of the trajectory t.

A ***shortest trajectory*** t
of $t_p(x, y, l)$ is the trajectory of minimum length for the given beginning x, end y and element p.



Interpretation of shortest and admissible trajectories

Reasoning informally, an analogy can be set up: the shortest trajectory is an analogous to a straight line segment connecting two points in a plane. Let us consider an analogy to a k-element segmented line connecting these points.

An ***admissible trajectory of degree k***
is the trajectory which can be divided into k shortest trajectories; more precisely there exists a subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}\}$ of $\boldsymbol{P}(t_o)$, $i_1 < i_2 < \ldots < i_{k-1}$, $k \leq l$, such that corresponding substrings

$$\boldsymbol{a}(x_o)\ldots\boldsymbol{a}(x_{i_1}),\ \boldsymbol{a}(x_{i_1})\ldots\boldsymbol{a}(x_{i_2}),\ \ldots,\ \boldsymbol{a}(x_{i_{k-1}})\ldots\boldsymbol{a}(x_l)$$

are the shortest trajectories.

A ***Language of Trajectories*** $\mathbf{L_t^H(S)}$
for the ABG in a state S is the set of all the trajectories of the length less or equal H.

Controlled grammar of shortest trajectories $G_t^{(1)}$

L	Q	Kernel	F_T	F_F
1	$\boldsymbol{Q_1}$	$\boldsymbol{S}(x, y, l) \to \boldsymbol{A}(x, y, l)$	*two*	ø
2_i	$\boldsymbol{Q_2}$	$\boldsymbol{A}(x, y, l) \to \boldsymbol{a}(x)\boldsymbol{A}(\boldsymbol{next_i}(x, l), y, \boldsymbol{f}(l))$	*two*	3
3	$\boldsymbol{Q_3}$	$\boldsymbol{A}(x, y, l) \to \boldsymbol{a}(\mathbf{y})$	ø	ø

$V_T = \{\boldsymbol{a}\}$

$V_N = \{\boldsymbol{S}, \boldsymbol{A}\}$

V_{PR}

Pred = $\{\boldsymbol{Q_1}, \boldsymbol{Q_2}, \boldsymbol{Q_3}\}$,

$\boldsymbol{Q_1}(x, y, l) = (MAP_{x,p}(y) = l) \quad (0 < l < n)$

$\boldsymbol{Q_2}(l) = (l \geq 1) \quad \boldsymbol{Q_3} = T$

Var = $\{x, y, l\}$

$F = \{\boldsymbol{f}, \boldsymbol{next}_1, \ldots, \boldsymbol{next}_n\}$ $(n = |X|)$,

$\boldsymbol{f}(l) = l - 1$, $D(f) = \mathbf{Z}_+ - \{0\}$, $\boldsymbol{next_i}$ is defined below;

$\boldsymbol{E} = \mathbf{Z}_+ \cup X \cup P$

Parm: $S \to Var$, $A \to Var$, $\boldsymbol{a} \to \{x\}$

$\boldsymbol{L} = \{1, 3\} \cup two$, $two = \{2_1, 2_2, \ldots, 2_n\}$

At the beginning of derivation: $x = x_o$, $y = y_o$, $l = l_o$, $x_o \in X$, $y_o \in X$, $l_o \in \mathbf{Z}_+$, $p \in P$

Function $\boldsymbol{next_i}$ is defined as follows:

$D(\boldsymbol{next_i}) = X \times \mathbf{Z}_+ \times X^2 \times \mathbf{Z}_+ \times P$

$MOVE_l(x)$ is the intersection of the following sets:

$ST_1(x)$, $ST_{l_o - l + 1}(x_o)$ and SUM, where

$SUM = \{v \mid v \text{ from } X, MAP_{x_o,p}(v) + MAP_{y_o,p}(v) = l_o\}$,

$ST_k(x) = \{v \mid v \text{ from } X, MAP_{x,p}(v) = k\}$,

If

$MOVE_l(x) = \{m_1, m_2, \ldots, m_r\} \neq \varnothing$

then

$\boldsymbol{next_i}(x, l) = m_i$ for $i \leq r$;

$\boldsymbol{next_i}(x, l) = m_r$ for $r < i \leq n$,

otherwise

$\boldsymbol{next_i}(x, l) = x$.

Interpretation of the algorithm for *next$_i$* for the grammar $G_t^{(1)}$.

MOVE$_l(x)$ is the intersection of the following sets:

$ST_1(x)$, $ST_{l_o-l+1}(x_o)$, and SUM, where

SUM = $\{v \mid v$ from X, $MAP_{x_o,p}(v) + MAP_{y_o,p}(v) = l_o\}$,

$ST_k(x) = \{v \mid v$ from X, $MAP_{x,p}(v) = k\}$,

If

MOVE$_l(x) = \{m_1, m_2, \dots, m_r\} \neq \emptyset$

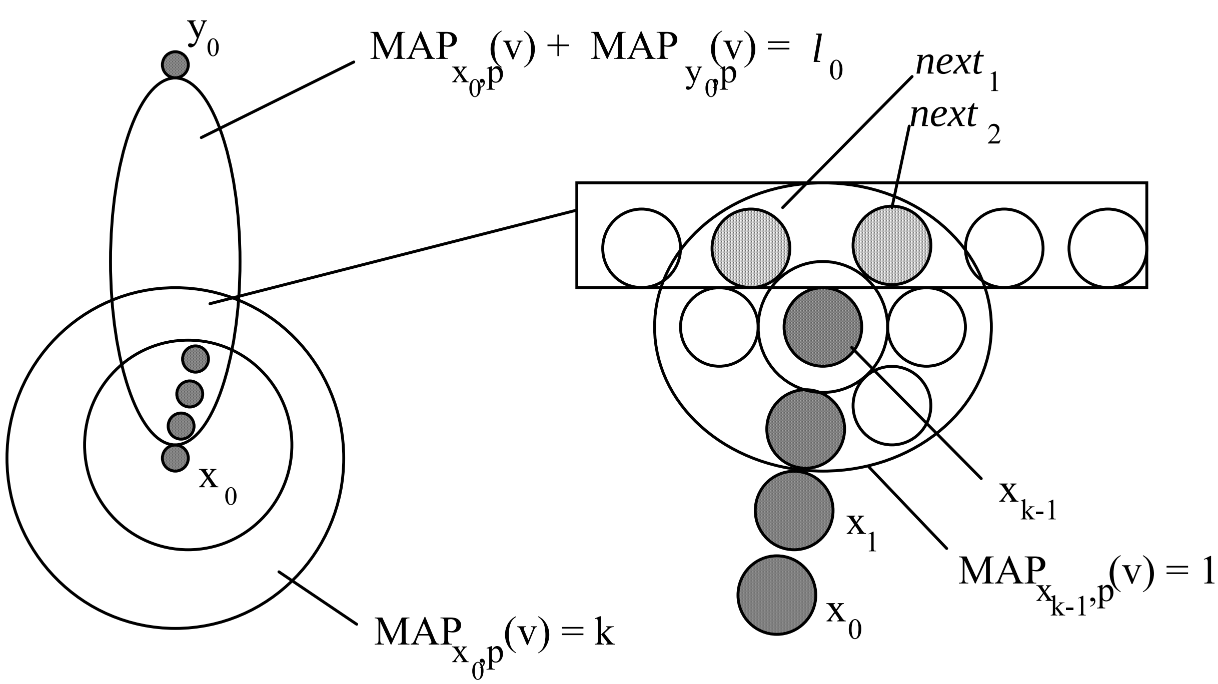
then

$next_i(x, l) = m_i$ for $i \le r$;

$next_i(x, l) = m_r$ for $r < i \le n$,

otherwise

$next_i(x, l) = x$.



Theorem about shortest trajectories

The shortest trajectories from point x_o to point y_o of the length l_o for the element p on x (i.e., ON(p) = x) exist if and only if the distance of these points is equal l_o:

$$MAP_{\mathbf{x_o},p}(y_o) = l_o,$$

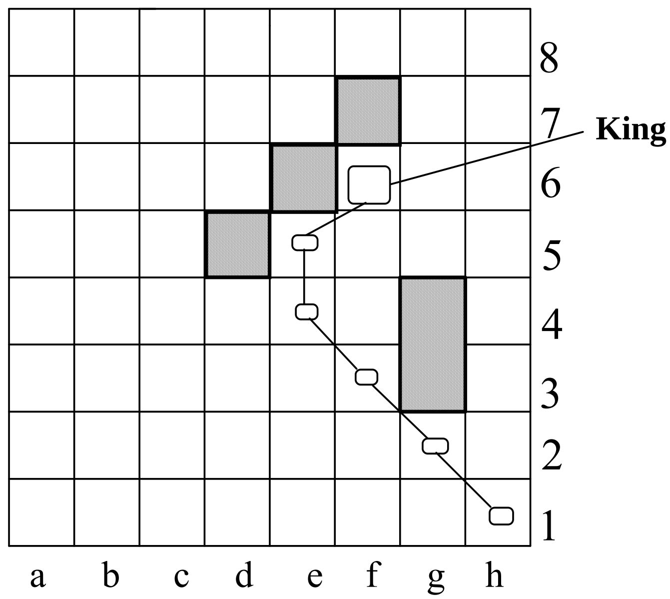
where $l_o < 2n$, n is the number of points in X.

If the relation R_p is symmetric, i.e., for all x from X, y from X and p from P

$$R_p(x, y) = R_p(y, x),$$

then all the shortest trajectories $t_p(x_o, y_o, l_o)$ can be generated by the grammar $\mathbf{G_t^{(1)}}$.

Generation of the shortest trajectory
a(f6)***a***(e5)***a***(e4)***a***(f3)***a***(g2)***a***(h1)
for the robot King



Generation of the shortest trajectory

$$\boldsymbol{S(\mathrm{f6}, \mathrm{h1}, 5) \ {}^{1}\!\!\longrightarrow A(\mathrm{f6}, \mathrm{h1}, 5)^{2} \ {}^{1}\!\!\longrightarrow a(\mathrm{f6})A(next_1(\mathrm{f6}, 5), \mathrm{h1}, 5)}$$

$$\mathbf{SUM} = \{v \mid v \in X,\ \mathrm{MAP}_{\mathrm{f6},\,\mathrm{King}}(v) + \mathrm{MAP}_{\mathrm{h1},\,\mathrm{King}}(v) = 5\}$$

$\mathrm{MAP}_{\mathrm{f6,KING}}$

5	4	3	2	2	2	2	2
5	4	3	2	1		1	2
5	4	3	2		0	1	2
5	4	3		1	1	1	2
5	4	3	2	2	2		2
5	4	3	3	3	3		3
5	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5

$\mathrm{MAP}_{\mathrm{h1,KING}}$

8	7	7	7	6	7	7	7
7	7	6	6	6		6	6
7	6	6	5		5	5	5
7	6	5		4	4	4	4
7	6	5	4	3	3		3
7	6	5	4	3	2		2
7	6	5	4	3	2	1	1
7	6	5	4	3	2	1	0

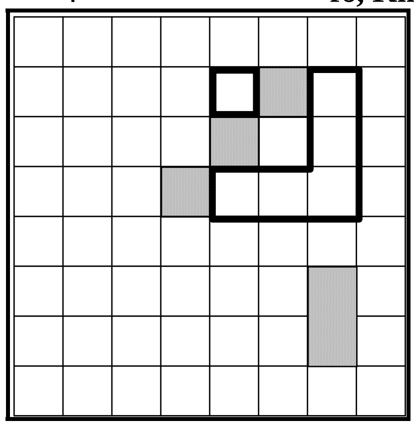
SUM

					5		
				5	5	5	
				5	5		5
					5		5
						5	5
							5

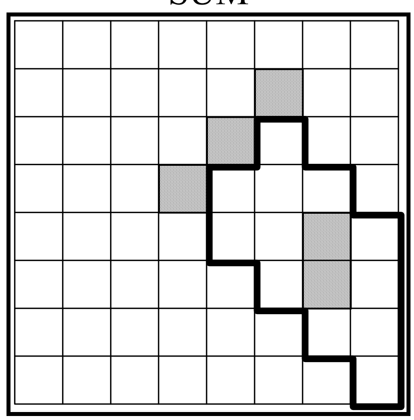
Generation of the shortest trajectory (continued)

$MOVE_5$(f6) is the intersection of
ST_1(f6), ST_{5-5+1}(f6) = ST_1(f6) and SUM

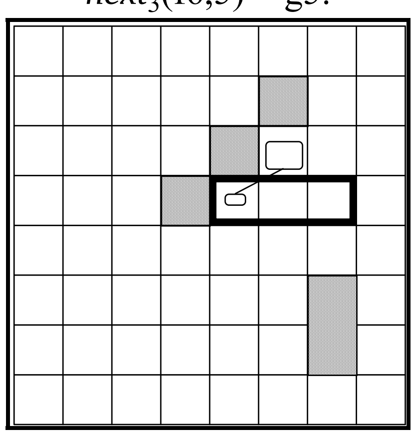
ST_1(f6) ={v | v from X, $MAP_{f6,\ King}(v)$=1}



SUM



$MOVE_5$(f6)={e5, f5, g5}
$next_1$(f6, 5) = e5, $next_2$(f6, 5) = f5,
$next_3$(f6,5) = g5.



Generation of the shortest trajectory (continued)

$a(\mathrm{f6})A(\mathrm{e5},\mathrm{h1},4) \;{}^{2_1}\!\!\longrightarrow a(\mathrm{f6})a(\mathrm{e5})A(next_1(\mathrm{e5},4),\mathrm{h1},3)$

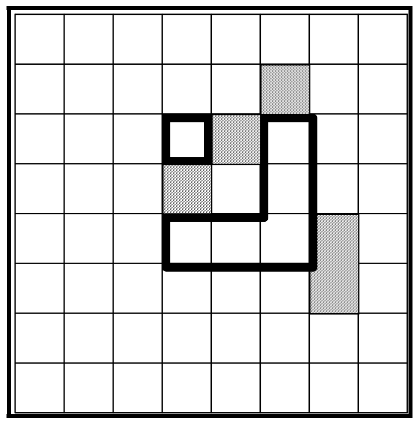
$\mathrm{MOVE}_4(\mathrm{e5})$ is the intersection of

SUM,

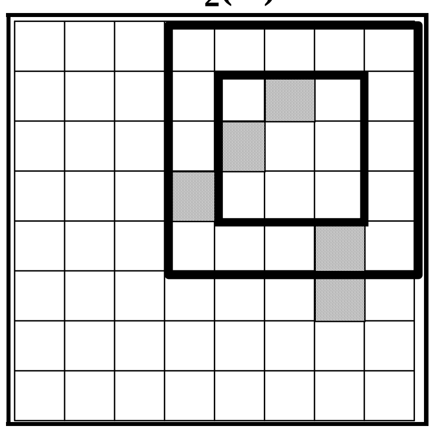
$\mathrm{ST}_1(\mathrm{e5})=\{v \mid v \in X, \mathrm{MAP}_{\mathrm{e5},\,\mathrm{King}}(v)=1\}$,

and $\mathrm{ST}_{5\text{-}4+1}(\mathrm{f6})= \mathrm{ST}_2(\mathrm{f6})=\{v \mid v \in X, \mathrm{MAP}_{\mathrm{f6},\,\mathrm{King}}(v)=2\}$

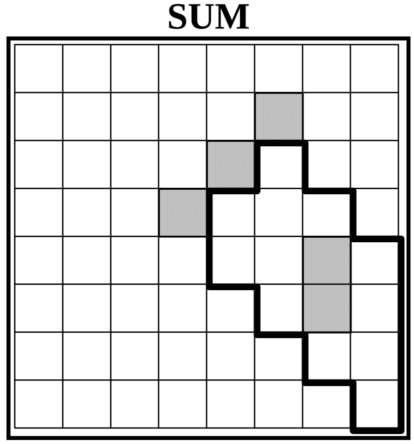
$\mathrm{ST}_1(\mathrm{e5})$



$\mathrm{ST}_2(\mathrm{f6})$



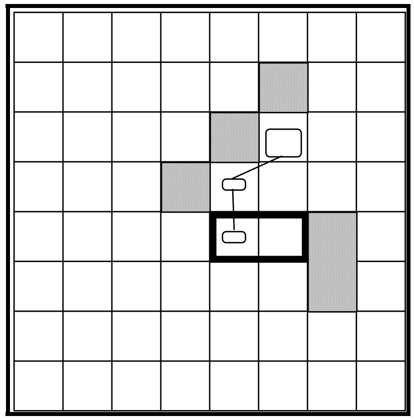
SUM



Generation of the shortest trajectory (continued)

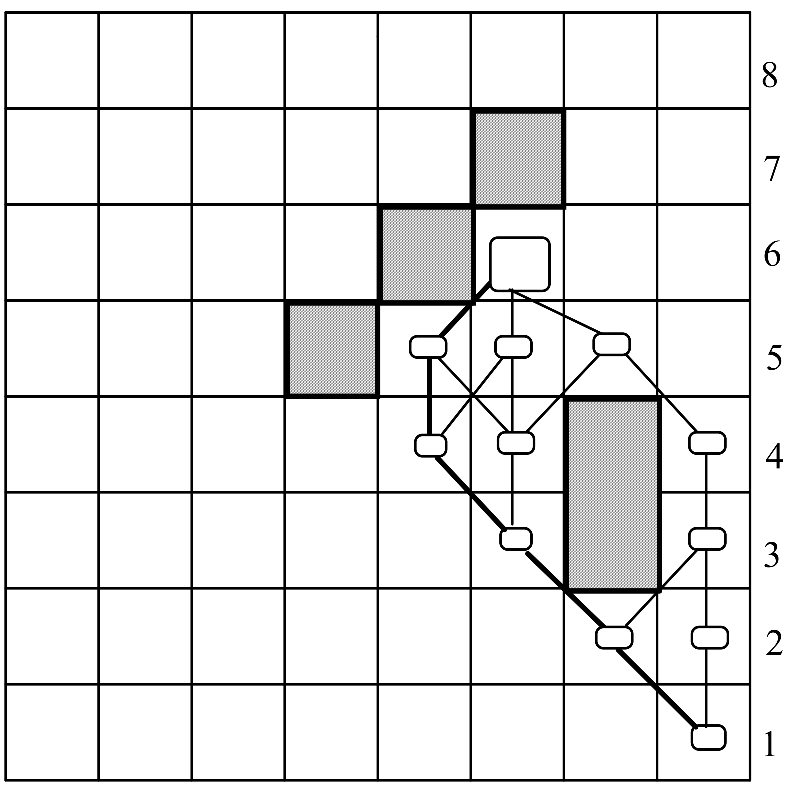
$\text{MOVE}_4(\text{e5}) = \{\text{e4, f4}\}$

$next_1(\text{e5},4) = \text{e4};\ next_2(\text{e5}, 4) = \text{f4}.$



$\boldsymbol{a}(\text{f6})\boldsymbol{a}(\text{e5})\boldsymbol{A}(\text{e4, h1, 3}) \xrightarrow{21} \ldots$

$\boldsymbol{a}(\text{f6})\boldsymbol{a}(\text{e5})\boldsymbol{a}(\text{e4})\boldsymbol{a}(\text{f3})\boldsymbol{a}(\text{g2})\boldsymbol{a}(\text{h1}).$



Tables 15×15

The game of chess can be interpreted as an ABG. To generate trajectories we have to investigate the geometry of the chess system. In chess function $MAP_{x,p}(y)$ yields the number of moves necessary for the piece p from square x to reach square y along the shortest path. Because of the symmetry of the relation R_p in this model, $MAP_{x,p}(y)$ specifies the *metric* on the chess-board different for each kind of piece. Even for a Pawn with more complex symmetry,

$$R_p(x, y) = R_q(y, x),$$

where p and q are the black and white Pawns, a sophisticated symmetry holds:

$$MAP_{x,p}(y) = MAP_{y,q}(x).$$

Hence, MAP as a function can be used as a *ruler* to measure *distances* in this system for arbitrary elements (pieces).

7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
7	6	6	6	6	6	6	6	6	6	6	6	6	6	7
7	6	5	5	5	5	5	5	5	5	5	5	5	6	7
7	6	5	4	4	4	4	4	4	4	4	4	5	6	7
7	6	5	4	3	3	3	3	3	3	3	4	5	6	7
7	6	5	4	3	2	2	2	2	2	3	4	5	6	7
7	6	5	4	3	2	1	1	1	2	3	4	5	6	7
7	6	5	4	3	2	1	0	1	2	3	4	5	6	7
7	6	5	4	3	2	1	1	1	2	3	4	5	6	7
7	6	5	4	3	2	2	2	2	2	3	4	5	6	7
7	6	5	4	3	3	3	3	3	3	3	4	5	6	7
7	6	5	4	4	4	4	4	4	4	4	4	5	6	7
7	6	5	5	5	5	5	5	5	5	5	5	5	6	7
7	6	6	6	6	6	6	6	6	6	6	6	6	6	7
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7

Fig. 1. 15x15 table for a King.

2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	15
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	14
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	13
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	12
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	11
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	10
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	9
1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	8
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	7
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	6
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	5
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	4
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	3
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	1	2	2	2	2	2	2	2	1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	

Fig 2. Superimposition of 8x8 and 15x15 tables for a Rook on c2.

When implementing the Language of Trajectories for the chess problem, it was found necessary to specify the function MAP by a table in order to increase the efficiency of the program PIONEER. The tables were 7 in number, of size 15×15 each. The entries were constructed as follows: 0 is entered into the central square; the remaining squares are filled with numbers equal to the number of moves necessary for the piece to reach the given square from the central square along the shortest path. A number of 15×15 tables are shown in Fig. 1, 2, 3.

These tables may be gathered for uniformity, into a single three-dimensional array T15(v_1, v_2, f) of size 15×15×7. For all x in X, x = (x_1, x_2), x_1 = 1, 2,..., 8, x_2 =1, 2,..., 8, where x_1 and x_2 are the numbers of the files and ranks of the chess-board. Values of v = (v_1, v_2) are the numbers of files and ranks of the respective 15×15 table. Then

$$\mathrm{MAP}_{x,p}(y) = \mathrm{T15}(v_1, v_2, f), \qquad (1)$$

where x = (x_1, x_2), y = (y_1, y_2), $v_1 = 8 - x_1 + y_1$, $v_2 = 8 - x_2 + y_2$, f = f(p) is the type of the piece p (King, Rook, etc.). The seven 15×15 tables specify on X *seven different metrics*.

In order to explain (1) we can imagine the following computation procedure. Assume an 8×8 table superimposed on the 15×15 table in such a way that square x = (x_1, x_2) coincides with the central square of the 15×15 table (Fig. 2). Next, assume that the 8×8 table is transparent, then from the corresponding squares we read off the values of $\mathrm{MAP}_{x,p}$, i.e., the values of the actual distances (in number of moves) of these squares from square x. An example of such a superimposition of tables for x = (3, 2), being c2 and p = Rook is shown in Fig. 2.

1		2		2		2		2		2		2		1
	1		2		2		2		2		2		1	
2		1		2		2		2		2		1		2
	2		1		2		2		2		1		2	
2		2		1		2		2		1		2		2
	2		2		1		2		1		2		2	
2		2		2		1		1		2		2		2
	2		2		2		0		2		2		2	
2		2		2		1		1		2		2		2
	2		2		1		2		1		2		2	
2		2		1		2		2		1		2		2
	2		1		2		2		2		1		2	
2		1		2		2		2		2		1		2
	1		2		2		2		2		2		1	
1		2		2		2		2		2		2		1

Fig. 3. 15x15 table for a Bishop.