Mixed Tate Motives and Aomoto Polylogarithms

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Outline

- Periods and motives
- Nori motives
- Mixed Tate motives
- Aomoto polylogarithms
- A construction of mixed Tate motives

Periods and Motives

Definition (Kontsevich, Zagier)

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals

$$\int_{\sigma} f(x_1, \ldots, x_n) dx_1 \ldots dx_n,$$

where f is a rational function with rational coefficients and $\sigma \subseteq \mathbb{R}^n$ is given by polynomial inequalities with rational coefficients.

Examples

$\sqrt{2} = \int_{2x^2 \leq 1} dx, \quad \pi = \int_{x^2+y^2 \leq 1} dxdy, \quad \zeta(2) = \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2},$
$\log(2) = \int_1^2 \frac{dx}{x}, \quad \zeta(2,1) = \int_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3}$

- Periods form a subring of $\mathbb{C}$. We will denote the ring of periods by $\mathcal{P}^{\mathrm{eff}}$.
- $\mathcal{P}^{\mathrm{eff}}$ is countable.
- $\mathbb{Z} \subset \mathbb{Q} \subset \bar{\mathbb{Q}} \subset \mathcal{P}^{\mathrm{eff}} \subset \mathbb{C}$.

Definition

Let k be a subfield of $\mathbb{C}$. A *k-variety* is a reduced separated scheme of finite type over k.

Definition (Cohomological definition of periods)

Let X be a smooth $\mathbb{Q}$-variety, $Y \subseteq X$ a normal crossing divisor. The period isomorphism

$$H^i_{\mathrm{dR}}(X, Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^i_{\mathrm{B}}(X, Y; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

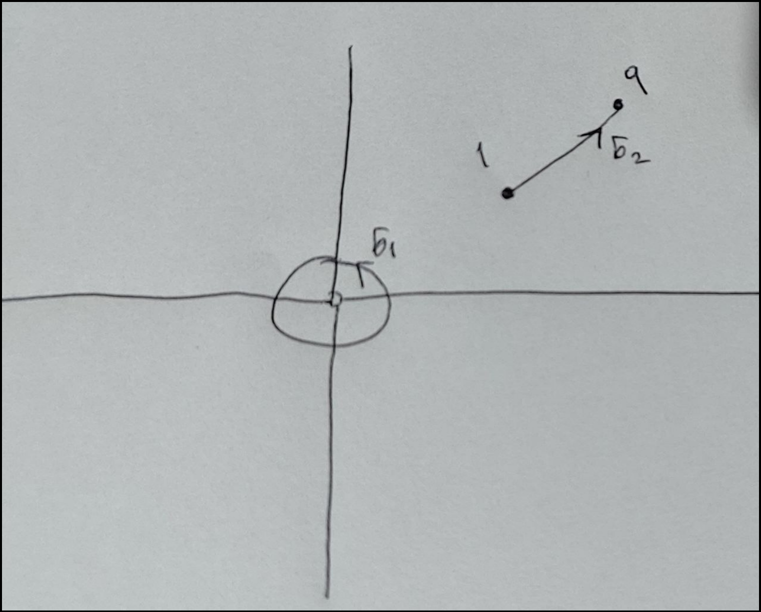
induces the period pairing

$$\begin{aligned} H^i_{\mathrm{dR}}(X, Y) \otimes H_i^{\mathrm{B}}(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q}) &\to \mathbb{C} \\ \omega \otimes \sigma &\mapsto \int_\sigma \omega. \end{aligned}$$

We call a *period* of (X, Y) any number in the image of this map.

Example

- Let us consider the pair
$$(X, Y) = (\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}, \{1, q\}),$$
with $q \in \mathbb{Q} \setminus \{0, 1\}$.
- First singular homology of $(X(\mathbb{C}), Y(\mathbb{C})) = (\mathbb{C}^*, \{1, q\})$ has a basis $\{\sigma_1, \sigma_2\}$, where σ_1 is a (counterclockwise) circle around 0 with radius $r < \min\{1, |q|\}$ and σ_2 is the straight line from 1 to q.
- First de Rham cohomology of $(X, Y) = (\mathrm{Spec}\mathbb{Q}[x, x^{-1}], \{1, q\})$ has a basis $\{\omega_1, \omega_2\}$, where $\omega_1 = \frac{dt}{t}, \omega_2 = \frac{dt}{q-1}$.
- Hence this pair gives the matrix
$$\begin{pmatrix} \int_{\sigma_2} \omega_2 & \int_{\sigma_2} \omega_1 \\ \int_{\sigma_1} \omega_2 & \int_{\sigma_1} \omega_1 \end{pmatrix} = \begin{pmatrix} 1 & \log q \\ 0 & 2\pi \mathrm{i} \end{pmatrix}$$
which shows that log of rational numbers are periods.


a
1
$б_2$
$б_1$

Cheking whether two complex numbers are equal or not is not easy. For example

$$\pi\sqrt{163} \text{ and } 3 \cdot \log(640320)$$

both have decimal expensions beginning

$$40.10916999113251...$$

but they are not equal.
($e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007...$ is known as the Ramanujan constant.)

Conjecture (Period conjecture)

If a period has two integral representations, one can pass between them using only the following calculus rules.

- *Additivity of integral:*

$$\int_\sigma \omega_1 + \omega_2 = \int_\sigma \omega_1 + \int_\sigma \omega_2$$

$$\int_{\sigma_1 \cup \sigma_2} \omega = \int_{\sigma_1} \omega + \int_{\sigma_2} \omega$$

  where $\sigma_1 \cap \sigma_2 = \emptyset$.

- *Change of variables:*

$$\int_{f(\sigma)} \omega = \int_\sigma f^* \omega$$

  where f *is invertible and defined by polynomial equations with rational coefficients.*

- *Stokes' formula:*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

The category of *mixed motives* $\mathrm{MM}(k)$ over a field k is a conjectural Tannakian category, together with a contravariant functor $h : \mathrm{Var}_k \to \mathrm{MM}(k)$ such that any Weil cohomology theory H factors through h:

$$\begin{tikzcd}
\mathrm{Var}_k \arrow[rr, "H"] \arrow[dr, "h"'] & & k - \mathrm{Vect} \\
& \mathrm{MM}(k) \arrow[ur, dashed, "f_H", "\exists!"'] &
\end{tikzcd}$$

- Singular cohomology and de Rham cohomology induce functors
$$f_B, f_{dR} : \mathrm{MM}(\mathbb{Q}) \to \mathbb{Q} - \mathrm{Vect}\,.$$
- Then for any motive $M \in \mathrm{MM}(\mathbb{Q})$, the period pairing yields a pairing
$$f_{dR}(M) \otimes f_B(M)^\vee \to \mathbb{C}.$$
- Let $\mathcal{P}(M)$ be the subfield of $\mathbb{C}$ generated by the image of the pairing.
- The following are equivalent.
  - The period conjecture holds.
  - $\mathrm{ev} : \mathcal{P}_{\mathrm{KZ}} \to \mathbb{C}$ is injective.
  - $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain and for any (Nori) motive M,
$$\mathrm{trdeg}[\mathcal{P}(M) : \mathbb{Q}] = \dim G_{\mathrm{mot}}(M),$$
    where $G_{\mathrm{mot}}(M) = \mathrm{Aut}^{\otimes} H_{B|\langle M\rangle}$ is the Galois group of the Tannakian subcategory $\langle M\rangle$ of $\mathrm{MM}(\mathbb{Q})$ generated by M.

Nori Motives

Theorem

Let D be a diagram (quiver, directed graph), R be a ring and

$$T : D \to R - \mathrm{Mod}$$

be a (quiver) representation. Then, there is an R-linear abelian category $\mathcal{C}(D, T)$ with representation

$$\tilde{T} : D \to \mathcal{C}(D, T)$$

and a faithful, exact, R-linear functor

$$f_T : \mathcal{C}(D, T) \to R - \mathrm{Mod}$$

such that T factorises as

$$T : D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R - \mathrm{Mod}$$

and $\mathcal{C}(D, T)$ is universal with this property.

- $\mathcal{C}(D, T)$ is called the *diagram category*.
- If D is finite,
$$\mathcal{C}(D, T) = \mathrm{End}(T) - \mathrm{Mod}\,.$$
- In general,
$$\mathcal{C}(D, T) = 2 - \mathrm{colim}_F\, \mathcal{C}(F, T\big|_F),$$
where F runs through finite full subdiagrams of D, i.e., the objects of $\mathcal{C}(D, T)$ are the objects of $\mathcal{C}(F, T\big|_F)$ for some F and the morphisms are
$$\mathrm{Mor}_{\mathcal{C}(D,T)}(X, Y) = \varinjlim_F \mathrm{Mor}_{\mathcal{C}(F,T|_F)}(X_F, Y_F),$$
where X_F is the image of $X \in \mathcal{C}(F', T\big|_{F'})$ in $\mathcal{C}(F, T\big|_F)$ for $F \supseteq F'$.
- Each object of $\mathcal{C}(D, T)$ is a subquotient of a finite direct sum of objects from $\{\tilde{T}p \mid p \in D\}$.

- Let X be a k-variety, $Y \subseteq X$ be a closed subvariety and $i \in \mathbb{Z}$. We call (X, Y, i) an *effective pair*.
- Let $\mathrm{Pairs}^{\mathrm{eff}}$ be the diagram whose vertices are effective pairs and edges are the following.
  - For any morphism $f : X \to X'$ such that $f(Y) \subseteq Y'$, we have an edge $(X', Y', i) \to (X, Y, i)$.
  - For any chain $X \supseteq Y \supseteq Z$ of closed subvarieties, an edge $(Y, Z, i) \to (X, Y, i+1)$.
- The relative singular cohomology

$$\begin{aligned} H^* : \mathrm{Pairs}^{\mathrm{eff}} &\to \mathbb{Z} - \mathrm{Mod} \\ (X, Y, i) &\mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) \end{aligned}$$

  is a representation.
- We define the category of *effective mixed Nori motives* as

$$\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k) := \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*).$$

- $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} := \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k) := \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$.

$$\begin{tikzcd}
\mathrm{Pairs}^{\mathrm{eff}} \arrow[rr, "H^*"] \arrow[dr, "\tilde{T}"'] & & \mathbb{Z}-\mathrm{Mod} \\
& \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \arrow[ur, "f_{H^*}"'] &
\end{tikzcd}$$

- $H^i_{\mathrm{Nori}}(X,Y) := \tilde{T}(X,Y,i) \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$.
- $H^i_{\mathrm{Nori}}(X) := H^i_{\mathrm{Nori}}(X,\emptyset)$.

Definition

1. We call an effective pair (X,Y,i) an *effective good pair* if $H^j(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z}) = 0$ for $j \neq i$ and $H^i(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z})$ is free.
2. An effective good pair is called an *effective very good pair* if X is affine, $X \setminus Y$ is smooth and either $\dim(X) = i$, $\dim(Y) = i-1$ or $X = Y$ and $\dim(X) < i$.
3. We denote the full subdiagram of $\mathrm{Pairs}^{\mathrm{eff}}$ with effective good pairs by $\mathrm{Good}^{\mathrm{eff}}$.
4. We denote the full subdiagram of $\mathrm{Good}^{\mathrm{eff}}$ with effective very good pairs by $\mathrm{VGood}^{\mathrm{eff}}$.

Lemma (Nori)

Let X be an affine k-variety of dimension n and $Z \subseteq X$ is a Zariski closed subset with $\dim(Z) < n$. Then there is a Zariski closed subset Y with $Z \subseteq Y \subseteq X$ and $\dim(Y) < n$ such that (X, Y, n) is a good pair.

- By using this lemma iteratively, for any affine variety X of dimension n, we can find a filtration

  $$\emptyset = F_{-1}X \subset F_0X \subset ... \subset F_{n-1}X \subset F_nX = X$$

  such that each $(F_jX, F_{j-1}X, j)$ is very good.
- The induced chain complex

  $$\cdots \to H^i\left(F_iX(\mathbb{C}), F_{i-1}X(\mathbb{C}); \mathbb{Z}\right) \xrightarrow{\delta_i} H^{i+1}\left(F_{i+1}X(\mathbb{C}), F_iX(\mathbb{C}); \mathbb{Z}\right) \to \cdots$$

  computes the singular cohomology of X.

Theorem

$\mathcal{MM}^{eff}_{Nori} = \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$, $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ and $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ are equivalent.

- For good pairs (X, Y, i) and (X', Y', i'), let

$$H^{i}_{\mathrm{Nori}}(X, Y) \otimes H^{i'}_{\mathrm{Nori}}(X', Y') := H^{i+i'}_{\mathrm{Nori}}(X \times X', X \times Y' \cup X' \times Y)$$

  in the light of the Künneth formula.
- $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} = \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ is a tensor category.
- $\mathbf{1}(-1) := H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{1\})$.
- The category $\mathcal{MM}_{\mathrm{Nori}} := \mathcal{MM}_{\mathrm{Nori}}(k)$ of *mixed Nori motives* is defined as the localization of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ with respect to $\mathbf{1}(-1)$.

Theorem

$\mathcal{MM}_{\mathrm{Nori}}$ is a Tannakian category with the fiber functor H^*.

- $\mathbf{1}(-n) := \mathbf{1}(-1)^{\otimes n}$, for $n \in \mathbb{Z}$.
- $M(-n) := M \otimes \mathbf{1}(-n)$, for $n \in \mathbb{Z}$ and $M \in \mathcal{MM}_{\text{Nori}}$.
- $H^i_{\text{Nori}}(\mathbb{P}^N) = \begin{cases} \mathbf{1}(-n), & \text{if } i = 2n \text{ and } N \geq n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$
- If Z is a projective variety of dimension n, then $H^{2n}_{\text{Nori}}(Z) = \mathbf{1}(-n)$.

- We work with $\mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$, the category of mixed Nori motives with rational coefficients (i.e. replace $\mathbb{Z}-\mathrm{Mod}$ by $\mathbb{Q}-\mathrm{Mod}$).

Definition

A motive $M \in \mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$ is called *pure of weight* $n \in \mathbb{Z}$ if it is a subquotient of $H^{n+2j}_{\mathrm{Nori}}(Y)(j)$ for some Y smooth and projective and $j \in \mathbb{Z}$. A motive is called *pure* if it is a direct sum of pure motives of some weights.

- $\mathbf{1}(-n)$ is pure of weight $2n$.

Theorem

On every motive $M \in \mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$, there is a unique bounded increasing filtration $(W_nM)_{n\in\mathbb{Z}}$ inducing the weight filtration under the Hodge realization. Moreover, every morphism of Nori motives is strictly compatible with this filtration.

- We call this filtration *weight filtration* and denote

$$\mathrm{gr}^W_n M := W_nM/W_{n-1}M.$$

- $\mathrm{gr}^W_n M$ is pure of weight n.

Mixed Tate Motives

- A Nori motive $M \in \mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$ is called a *mixed Tate Nori motive* if $\mathrm{gr}^W_{2n} M$ is a direct sum of copies of $\mathbf{1}(-n)$.
- We denote the full subcategory of $\mathcal{MM}_{\mathrm{Nori},\mathbb{Q}}$ containing these objects by $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$.
- $(\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}, \mathbf{1}(1))$ is a mixed Tate category.

Example

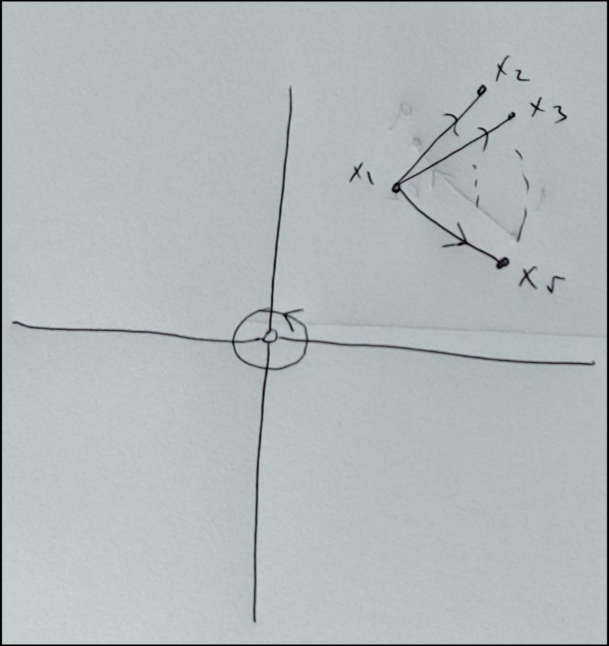
Let $B \subseteq \mathbb{G}_m$ be such that $B = \{x_1, \cdots, x_r\}$. We will find the weight structure of $H^1_{\mathrm{Nori}}(\mathbb{G}_m, B)$. We have the following exact sequence

$$0 \to \underbrace{H^0_{\mathrm{Nori}}(\mathbb{G}_m)}_{\mathbf{1}(0)} \to \underbrace{H^0_{\mathrm{Nori}}(B)}_{\mathbf{1}(0)^{\oplus r}} \to H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) \to \underbrace{H^1_{\mathrm{Nori}}(\mathbb{G}_m)}_{\mathbf{1}(-1)} \to 0$$

Then,

$$\mathrm{gr}^W_0 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) = \mathbf{1}(0)^{\oplus(r-1)}$$
$$\mathrm{gr}^W_2 H^1_{\mathrm{Nori}}(\mathbb{G}_m, B) = \mathbf{1}(-1)$$

Therefore $H^1_{\mathrm{Nori}}(\mathbb{G}_m, B)$ is mixed Tate.


x_2
x_3
x_1
x_5

Example

Let $B = M_0 \cup M_1 \cup M_2 \subseteq \mathbb{P}^2_{\mathbb{C}}$, where M_i are lines in $\mathbb{P}^2_{\mathbb{C}}$ in general position such that they are not axis lines $z_i = 0$. Then

$$\begin{aligned}
\mathrm{gr}^W_0 H^1_{\mathrm{Nori}}\left(B \cap \mathbb{G}^2_m\right) &= \mathbf{1}(0) \\
\mathrm{gr}^W_2 H^1_{\mathrm{Nori}}\left(B \cap \mathbb{G}^2_m\right) &= \mathbf{1}(-1)^{\oplus s}
\end{aligned}$$

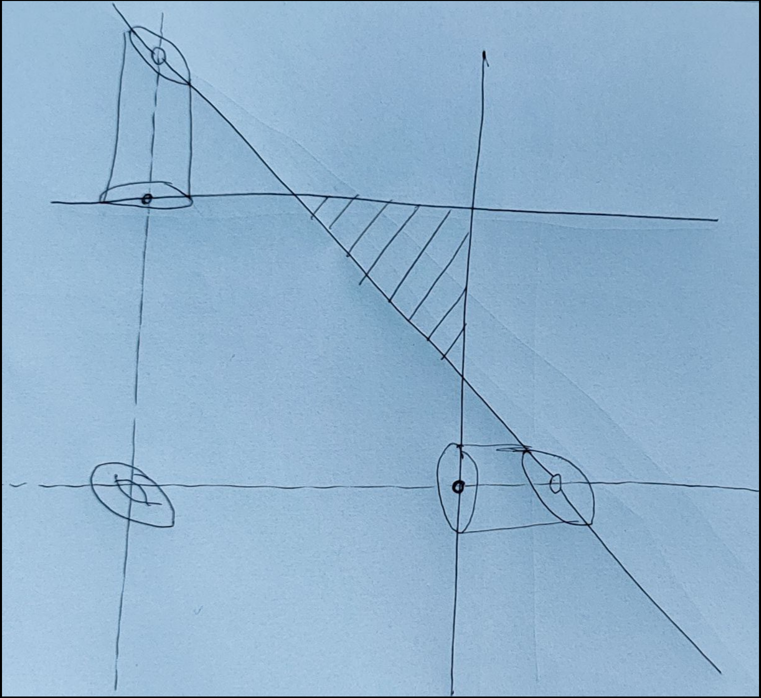
where s is the number of intersection of B with $\{z_1 = 0\} \cup \{z_2 = 0\}$. Using the exact sequence

$$0 \to \underbrace{H^1_{\mathrm{Nori}}\left(\mathbb{G}^2_m\right)}_{\mathbf{1}(-1)^{\oplus 2}} \to H^1_{\mathrm{Nori}}\left(B \cap \mathbb{G}^2_m\right) \to H^2_{\mathrm{Nori}}\left(\mathbb{G}^2_m, B \cap \mathbb{G}^2_m\right) \to \underbrace{H^2_{\mathrm{Nori}}\left(\mathbb{G}^2_m\right)}_{\mathbf{1}(-2)} \to 0$$

we have

$$\begin{aligned}
\mathrm{gr}^W_0 H^2_{\mathrm{Nori}}\left(\mathbb{G}^2_m, B \cap \mathbb{G}^2_m\right) &= \mathbf{1}(0) \\
\mathrm{gr}^W_2 H^2_{\mathrm{Nori}}\left(\mathbb{G}^2_m, B \cap \mathbb{G}^2_m\right) &= \mathbf{1}(-1)^{\oplus s-2} \\
\mathrm{gr}^W_4 H^2_{\mathrm{Nori}}\left(\mathbb{G}^2_m, B \cap \mathbb{G}^2_m\right) &= \mathbf{1}(-2)
\end{aligned}$$

and $s - 2 \in \{0, 1, 2, 3, 4\}$. Here $\mathbf{1}(-2)$ is coming from the torus $\mathbb{G}^2_m$ and $\mathbf{1}(0)$ is coming from the triangle defined by B.



Aomoto Polylogarithms

- There is a generalization of logarithms called *polylogarithms* which is defined inductively by

$$\ell i_1(z) = -\log(1-z)$$

and

$$d\ell i_n(z) = \ell i_{n-1}(z)\frac{dz}{z},$$

with $\ell i_n(0) = 0$.

- They have the power series expansion

$$\ell i_n(z) = \sum_{1\leq m} \frac{z^m}{m^n},$$

for $|z| < 1$.

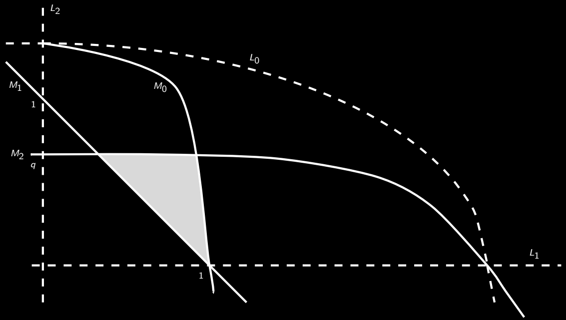
- Fix some $q \in \mathbb{Q} \setminus \{0, 1\}$. Let z_i, $i = 0, 1, ..., n$, be the homogeneous coordinates on $\mathbb{P}^n_{\mathbb{Q}}$.
- Let L_i be the hyperplanes defined by $z_i = 0$ and M_i be the hyperlanes defined as $M_0 : z_0 = z_1$; $M_1 : z_0 = z_1 + z_2$; $M_i : z_i = z_{i+1}$ for $2 \leq i < n$; and $M_n : qz_0 = z_n$.
- Let $M_q = \bigcup M_i$ and $L = \bigcup L_i$.
- $\ell i_n(q)$ is a period of the mixed Tate motive

$$H^n_{\text{Nori}}\left(\mathbb{P}^n_{\mathbb{Q}} \setminus L, M_q \setminus (L \cap M_q)\right).$$

- We will call the configuration (L, M_q) as the *polylogarithmic configuration* of q.

- ℓi_2 is called *dilogarithm*. Its configuration is given by

$$
\begin{aligned}
M_0 &: z_0 = z_1 \\
M_1 &: z_0 = z_1 + z_2 \\
M_2 &: qz_0 = z_2.
\end{aligned}
$$



Call $\mathcal{D}(q)$ for the triangle given by M_i. Then

$$\ell i_2(q) = \int_{\mathcal{D}(q)} \frac{dx}{x} \wedge \frac{dy}{y}.$$

- We call an n-*simplex* a family of $n+1$ hyperplanes $(L_0, \dots, L_n)$ of $\mathbb{P}^n_k$.
- A pair of simplices (L, M) is said to be *admissible* if they do not have a common face.
- Let (L, M) be admissible pair of simplices such that the hyperplanes of L and M are in general position. Let

$$\omega_L = d\log(z_1/z_0) \wedge \dots d\log(z_n/z_0)$$

where $z_i = 0$ is a homogeneous equation of L_i. Let Δ_M be the simplex whose sides are M_i. Then

$$a(L, M) = \int_{\Delta_M} \omega_L$$

is a period of

$$H^n_{\mathrm{Nori}}(\mathbb{P}^n \setminus L, M \setminus (L \cap M)).$$

- $M = H^n_{\mathrm{Nori}}(\mathbb{P}^n \setminus L, M \setminus (L \cap M))$ is a mixed Tate motive with

$$\mathrm{gr}^W_{2n} M = \mathbf{1}(-n),$$
$$\mathrm{gr}^W_0 M = \mathbf{1}(0).$$

Definition

$A_0(k) := \mathbb{Z}$. For $n > 0$, define $A_n(k)$ as the abelian group generated by $(L; M)$ where (L, M) is an admissible pair of simplices in $\mathbb{P}^n_k$ subject to the following relations:

1. If the hyperplanes of one of L or M is not in general position (i.e. degenerate), then $(L; M) = 0$.
2. For every $\sigma \in S_n$,
$$(\sigma L; M) = (L; \sigma M) = (-1)^{|\sigma|}(L; M)$$
where σL and σM, are defined by the natural action of S_n on a set indexed by $1, ..., n$.
3. For every family of hyperplanes $L_0, ..., L_{n+1}$ and an n-simplex M,
$$\sum (-1)^j (\hat{L}^j; M) = 0,$$
where $\hat{L}^j = (L_0, ..., \hat{L}_j, ..., L_{n+1})$, and the corresponding relation for the second component.
4. For every $g \in PGL_{n+1}(k)$,
$$(gL; gM) = (L; M).$$

- $$A_1(k) \xrightarrow{\sim} k^\times$$
$$(L_0, L_1; M_0, M_1) \mapsto r(L_0, L_1, M_0, M_1)$$
- The multiplication map $\mu : A_{n'} \times A_{n''} \to A_n$, for $n' + n'' = n$, is defined on the generators in the following way. Let (L', M') and (L'', M'') be two admissible pairs of non-degenerate simplices from $\mathbb{P}^{n'}$ and $\mathbb{P}^{n''}$, respectively. Also let L be a non-degenerate simplex from $\mathbb{P}^n$. Identify the affine spaces $\mathbb{P}^n \setminus L_0$ and $(\mathbb{P}^{n'} \setminus L'_0) \times (\mathbb{P}^{n''} \setminus L''_0)$. Then $M' \times M''$ can be seen in $\mathbb{P}^n \setminus L_0$ and hence in $\mathbb{P}^n$. Cutting this product into simplices in $\mathbb{P}^n$ defines an element in A_n which is defined as the product of $(L'; M')$ and $(L''; M'')$.
- $A = \oplus A_n$ is a graded Hopf algebra.

- Let G be the Galois group of $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$. Then

$$1 \to U \to G \to \mathbb{G}_m \to 1$$

  is split exact.
- Here, $U = \mathrm{Spec} R$, where $R = \bigoplus_{d \geq 0} R_d$ is a graded Hopf algebra.
- $\mathcal{MTM}_{\mathrm{Nori},\mathbb{Q}}$ is equivalent to the category of graded R-comodules.

Conjecture (Beilinson)

There is a natural isomorphism of graded Hopf algebras

$$A \otimes \mathbb{Q} \xrightarrow{\sim} R.$$

A Construction of Mixed Tate Motives

- We will consider the motives coming from the following configurations.
- Fix $n \in \mathbb{N}^{>0}$. Let
$$B = \bigcup_{1 \leq i \leq m} B_i,$$
where all B_i are hyperplanes in B that meet $x_{i_1} = \ldots = x_{i_k} = 0$ properly for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.
- We call such B a *nice divisor*.
- We will be interested in the motives of the form
$$H^n_{\text{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right).$$
- $H^n_{\text{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right)$ is a mixed Tate motive with
$$\mathrm{gr}^W_{2n} H^n_{\text{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right) = \mathbf{1}(-n).$$

- Let
$$M = \bigoplus_{d \geq 0} M_d$$
where
$$M_d = \mathrm{gr}^W_{2n-2d}(\varprojlim_B H^n_{\mathrm{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right)) \otimes \mathbf{1}(n-d)$$
such that the limit is taken over all nice divisors B as in the beginning of the section.
- In particular,
$$M_0 = \mathbf{1}(0)$$
and
$$M_n = \mathrm{gr}^W_0(\varprojlim_B H^n_{\mathrm{Nori}}\left(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m\right)).$$

- Viewing M as a graded R-comodule, we have a linear map $\nu : M \to R \otimes M$. Let $\gamma_i : M \to M_i$ be the restriction map.
- Since $M_0 = \mathbf{1}(0)$ is realized as $\mathbb{Z}$, there is a natural map $\ell : M_0 \to \mathbb{Q}$.
- By composing

$$h : M \xrightarrow{\nu} R \otimes M \xrightarrow{\mathrm{id}_R \otimes \gamma_0} R \otimes M_0 \xrightarrow{\mathrm{id}_R \otimes \ell} R \otimes \mathbb{Q} \xrightarrow{\sim} R$$

  we have a map $h : M \to R$ such that $h|_{M_0} = \ell$.
- This also gives

$$h|_{M_n} : M_n \to \bigoplus_{i+j=n} R_i \otimes M_j \to R_n \otimes M_0 \to R_n \otimes \mathbb{Q} \xrightarrow{\sim} R_n.$$

- Let $G_n := S_n \ltimes \mathbb{G}_m^n$, where S_n is the symmetric group of order $n!$, and the action be given by $\sigma \cdot (a_1, \dots, a_n) = (\sigma(a_1), \dots, \sigma(a_n))$.
- Then G_n acts on $\mathbb{G}_m^n$ by
$$(\sigma \cdot a) \cdot x = (-1)^{|\sigma|} \sigma \cdot (ax)$$
for $\sigma \in S_n,\ a, x \in \mathbb{G}_m^n$.
- This action extends on
$$M_n = \mathrm{gr}_0^W(\varprojlim_B H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)).$$
- Let
$$R'_n := H_0(G_n; M_n) = M_n / \langle gx - x \mid g \in G_n, x \in M_n \rangle.$$

Proposition

$h|_{M_n}$ induces a map $\varphi_n : R'_n \to R_n$.

Proof.

- R_n is given by the framed objects and the coaction $M_n \to R_n \otimes M_n$ is given by frames

$$\mathbf{1}(0) \to \mathrm{gr}_0^W H^n_{\mathrm{Nori}}\left(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n\right)$$

  and it corresponds to the periods of $\mathrm{gr}_0^W H^n_{\mathrm{Nori}}\left(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n\right)$.
- WLOG assume $\mathrm{gr}_0^W H^n_{\mathrm{Nori}}\left(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n\right) = \mathbf{1}(0)$.
- Its periods are scalar multiples of

$$\rho = \int_B \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}.$$

- ρ is invariant under the action of both S_n and $\mathbb{G}_m^n$.

$\square$

- Let $R'_0 = \mathbb{Z}$ and $R' = \bigoplus_{n\geq 0} R'_n$.
- Tensor product of motives defines a multiplication $R'_{n'} \otimes R'_{n''} \to R'_n$.

Lemma

Assume $n' + n'' = n$. Let $(L'; B') \in A_{n'}$ and $(L''; B'') \in A_{n''}$. Then $(L'; B') \times (L''; B'') = \sum_i (L; B_i)$, for some $(L; B_i) \in A_n$. Assume that L, L', L'' are given by axis hyperplanes. Then,

$$H^{n'}_{Nori}(\mathbb{G}^{n'}_m, B' \cap \mathbb{G}^{n'}_m) \otimes H^{n''}_{Nori}(\mathbb{G}^{n''}_m, B'' \cap \mathbb{G}^{n''}_m) = H^n_{Nori}(\mathbb{G}^n_m, B \cap \mathbb{G}^n_m),$$

where B is the nice divisor given by the union of simplices B_i.

Proof.

$$\begin{aligned} & H^{n'}_{\mathrm{Nori}}(\mathbb{G}^{n'}_m, B' \cap \mathbb{G}^{n'}_m) \otimes H^{n''}_{\mathrm{Nori}}(\mathbb{G}^{n''}_m, B'' \cap \mathbb{G}^{n''}_m) \\ =& H^{n}_{\mathrm{Nori}}(\mathbb{G}^{n}_m, \mathbb{G}^{n'}_m \times (B'' \cap \mathbb{G}^{n''}_m) \cup (B' \cap \mathbb{G}^{n'}_m) \times \mathbb{G}^{n''}_m) \\ =& H^{n}_{\mathrm{Nori}}(\mathbb{G}^{n}_m, (\mathbb{G}^{n'}_m \times B'' \cup B' \times \mathbb{G}^{n''}_m) \cap \mathbb{G}^{n}_m) \\ =& H^{n}_{\mathrm{Nori}}(\mathbb{G}^{n}_m, B \cap \mathbb{G}^{n}_m). \end{aligned}$$

by the definition of multiplication in A. □

Theorem

There is an isomorphism of graded algebras $\phi : R' \to A$.

Idea of proof.

Let $n > 0$. Let $Z = (Z_0, \ldots, Z_n)$ be the n-simplex in $\mathbb{P}^n$ given by $Z_i : z_i = 0$. Define A'_n as the abelian group generated by (B) where B is an n-simplex in $\mathbb{P}^n$ such that (Z, B) is admissible, subject to the following relations:

1. If the hyperplanes of B are not in general position, then $(B) = 0$.
2. For every $\sigma \in S_n$,
$$(\sigma B) = (-1)^{|\sigma|}(B).$$
3. For every family of hyperplanes $B_0, ..., B_{n+1}$,
$$\sum (-1)^j (\hat{B}^j) = 0.$$
4. For every $g \in \mathbb{G}_m^n$,
$$(gB) = (B),$$
where the action of $\mathbb{G}_m^n$ is as follows. For $g = (g_1, \ldots, g_n) \in \mathbb{G}_m^n$ and $p = (z_0 : z_1 : z_2 : \ldots : z_n) \in \mathbb{P}^n$, let $g \cdot p = (z_0 : g_1 z_1 : g_2 z_2 : \ldots : g_n z_n)$.

Idea of proof, cont'd.

- Then,

$$\begin{aligned} A'_n &\to A_n \\ (B) &\mapsto (Z; B). \end{aligned}$$

  is an isomorphism.
- We will write an isomorphism $R'_n \to A'_n$.
- We will consider the underlying $\mathbb{Z}$-modules of motives.
- We will work in the homological setting. The category of cohomological motives is isomorphic to the opposite category of homological motives. We denote by $H_n^{\mathrm{Nori}}(X, Y)$ the corresponding object of $H^n_{\mathrm{Nori}}(X, Y)$.

Idea of proof, cont'd.

- In this case,

$$M_n = \mathrm{gr}_0^W(\varinjlim_B H_n^{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)),$$

  such that the colimit is taken over all nice divisors B.
- By adding any such B some hyperplanes, we can divide it into "independent" simplices B^i.
- So, $B \subseteq \bigcup B^i$.
- This gives $\mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \to \bigoplus \mathrm{gr}_0^W H_n^{\mathrm{Nori}}(\mathbb{G}_m^n, B^i \cap \mathbb{G}_m^n)$.
- Define

$$\psi_{B^i} : \mathrm{gr}_0^W H^n_{\mathrm{Nori}}(\mathbb{G}_m^n, B^i \cap \mathbb{G}_m^n) = \mathbf{1}(0) = \mathbb{Z} \to A'_n$$

  as $\psi_{B^i}(1) = (B^i)$.
- This extends a map

$$\psi : M_n \to A'_n.$$

Idea of proof, cont'd.

- $\psi : M_n \to A'_n$. is surjective with kernel $\langle gx - x \mid g \in G_n, x \in M_n \rangle$.
- Hence, this gives an isomorphism

$$\phi_n : R'_n = M_n / \langle gx - x \mid g \in G_n, x \in M_n \rangle \xrightarrow{\sim} A'_n \xrightarrow{\sim} A_n.$$

- By previous lemma, $\phi = \bigoplus_{n \geq 0} \phi_n$ respects multiplication. Thus ϕ is an isomorphism of graded algebras.

$\square$

- The comultiplication on A can be carried to R'. This makes R' a Hopf algebra.
- Let $\varphi = \bigoplus \varphi_n : R' \to R$.

Conjecture

$$\varphi \otimes \mathbb{Q} : R' \otimes \mathbb{Q} \to R$$

is an isomorphism of graded Hopf algebras.

Thank you!