

# A construction of the category of mixed Tate motives using Aomoto polylogarithms and Nori formalism

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# Outline

- ▶ Periods and motives
- ▶ Nori motives
- ▶ Mixed Tate motives
- ▶ Aomoto polylogarithms
- ▶ A construction of mixed Tate motives

# Periods and Motives

## Definition (Kontsevich, Zagier)

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals

$$\int_{\sigma} f(x_1, \dots, x_n) dx_1 \dots dx_n,$$

where  $f$  is a rational function with rational coefficients and  $\sigma \subseteq \mathbb{R}^n$  is given by polynomial inequalities with rational coefficients.

## Examples

$$\sqrt{2} = \int_{2x^2 \leq 1} dx, \quad \pi = \int_{x^2 + y^2 \leq 1} dx dy, \quad \zeta(2) = \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2},$$
$$\log(2) = \int_1^2 \frac{dx}{x}, \quad \zeta(2, 1) = \int_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3}$$

- ▶ Periods form a subring of  $\mathbb{C}$ . We will denote the ring of periods by  $\mathcal{P}^{\text{eff}}$ .
- ▶  $\mathcal{P}^{\text{eff}}$  is countable.
- ▶  $\mathbb{Z} \subset \mathbb{Q} \subset \bar{\mathbb{Q}} \subset \mathcal{P}^{\text{eff}} \subset \mathbb{C}$ .

## Definition

Let  $k$  be a subfield of  $\mathbb{C}$ . A  $k$ -variety is a reduced separated scheme of finite type over  $k$ .

## Definition (Cohomological definition of periods)

Let  $X$  be a smooth  $\mathbb{Q}$ -variety,  $Y \subseteq X$  a normal crossing divisor. The period isomorphism

$$H_{\mathrm{dR}}^i(X, Y) \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow H_{\mathrm{B}}^i(X, Y; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

induces the period pairing

$$H_{\mathrm{dR}}^i(X, Y) \otimes H_i^{\mathrm{B}}(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q}) \rightarrow \mathbb{C}$$
$$\omega \otimes \sigma \mapsto \int_{\sigma} \omega.$$

We call a *period* of  $(X, Y)$  any number in the image of this map.

## Example

- ▶ Let us consider the pair

$$(X, Y) = (\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, \infty\}, \{1, q\}),$$

with  $q \in \mathbb{Q} \setminus \{0, 1\}$ .

- ▶ First singular homology of  $(X(\mathbb{C}), Y(\mathbb{C})) = (\mathbb{C}^*, \{1, q\})$  has a basis  $\{\sigma_1, \sigma_2\}$ , where  $\sigma_1$  is a (counterclockwise) circle around 0 with radius  $r < \min\{1, |q|\}$  and  $\sigma_2$  is the straight line from 1 to  $q$ .
- ▶ First de Rham cohomology of  $(X, Y) = (\text{Spec } \mathbb{Q}[x, x^{-1}], \{1, q\})$  has a basis  $\{\omega_1, \omega_2\}$ , where  $\omega_1 = \frac{dt}{t}$ ,  $\omega_2 = \frac{dt}{q-1}$ .
- ▶ Hence this pair gives the matrix

$$\begin{pmatrix} \int_{\sigma_2} \omega_2 & \int_{\sigma_2} \omega_1 \\ \int_{\sigma_1} \omega_2 & \int_{\sigma_1} \omega_1 \end{pmatrix} = \begin{pmatrix} 1 & \log q \\ 0 & 2\pi i \end{pmatrix}$$

which shows that  $\log$  of rational numbers are periods.

Checking whether two complex numbers are equal or not is not easy. For example

$$\pi\sqrt{163} \text{ and } 3 \cdot \log(640320)$$

both have decimal expansions beginning

$$40.10916999113251\dots$$

but they are not equal.

( $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007\dots$  is known as the Ramanujan constant.)

## Conjecture (Period conjecture)

*If a period has two integral representations, one can pass between them using only the following calculus rules.*

- *Additivity of integral:*

$$\begin{aligned}\int_{\sigma} \omega_1 + \omega_2 &= \int_{\sigma} \omega_1 + \int_{\sigma} \omega_2 \\ \int_{\sigma_1 \cup \sigma_2} \omega &= \int_{\sigma_1} \omega + \int_{\sigma_2} \omega\end{aligned}$$

*where  $\sigma_1 \cap \sigma_2 = \emptyset$ .*

- *Change of variables:*

$$\int_{f(\sigma)} \omega = \int_{\sigma} f^* \omega$$

*where  $f$  is invertible and defined by polynomial equations with rational coefficients.*

- *Stokes' formula:*

$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega.$$

- ▶ The category of *mixed motives*  $\text{MM}(k)$  over a field  $k$  is a conjectural Tannakian category, together with a contravariant functor  $h : \text{Var}_k \rightarrow \text{MM}(k)$  such that any Weil cohomology theory  $H$  factors through  $h$ :

$$\begin{array}{ccc}
 \text{Var}_k & \xrightarrow{H} & k\text{-Vect} \\
 & \searrow h & \nearrow f_H \\
 & \text{MM}(k) & \exists!
 \end{array}$$

- ▶ Let  $\mathcal{P}(M) = \mathbb{Q}(\text{Periods}(M))$  for a motive  $M \in \text{MM}(\mathbb{Q})$ .
- ▶ The following are equivalent.
  - The period conjecture holds.
  - $\text{ev} : \mathcal{P}_{\text{KZ}} \rightarrow \mathbb{C}$  is injective.
  - $\mathcal{P}_{\text{KZ}}$  is an integral domain and for any (Nori) motive  $M$ ,

$$\text{trdeg}[\mathcal{P}(M) : \mathbb{Q}] = \dim G_{\text{mot}}(M),$$

where  $G_{\text{mot}}(M) = \text{Aut}^{\otimes} H_{B|\langle M \rangle}$  is the Galois group of the Tannakian subcategory  $\langle M \rangle$  of  $\text{MM}(\mathbb{Q})$  generated by  $M$ .



# Nori Motives

## Theorem

Let  $D$  be a diagram (quiver, directed graph),  $R$  be a ring and

$$T : D \rightarrow R - \text{Mod}$$

be a (quiver) representation. Then, there is an  $R$ -linear abelian category  $\mathcal{C}(D, T)$  with representation

$$\tilde{T} : D \rightarrow \mathcal{C}(D, T)$$

and a faithful, exact,  $R$ -linear functor

$$f_T : \mathcal{C}(D, T) \rightarrow R - \text{Mod}$$

such that  $T$  factorises as

$$T : D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R - \text{Mod}$$

and  $\mathcal{C}(D, T)$  is universal with this property.

►  $\mathcal{C}(D, T)$  is called the *diagram category*.

► If  $D$  is finite,

$$\mathcal{C}(D, T) = \text{End}(T) - \text{Mod}.$$

► In general,

$$\mathcal{C}(D, T) = 2 - \text{colim}_F \mathcal{C}(F, T|_F),$$

where  $F$  runs through finite full subdiagrams of  $D$ , i.e., the objects of  $\mathcal{C}(D, T)$  are the objects of  $\mathcal{C}(F, T|_F)$  for some  $F$  and the morphisms are

$$\text{Mor}_{\mathcal{C}(D, T)}(X, Y) = \varinjlim_F \text{Mor}_{\mathcal{C}(F, T|_F)}(X_F, Y_F),$$

where  $X_F$  is the image of  $X \in \mathcal{C}(F', T|_{F'})$  in  $\mathcal{C}(F, T|_F)$  for  $F \supseteq F'$ .

► Each object of  $\mathcal{C}(D, T)$  is a subquotient of a finite direct sum of objects from  $\{\tilde{T}_p \mid p \in D\}$ .

- ▶ Let  $X$  be a  $k$ -variety,  $Y \subseteq X$  be a closed subvariety and  $i \in \mathbb{Z}$ . We call  $(X, Y, i)$  an *effective pair*.
- ▶ Let  $\text{Pairs}^{\text{eff}}$  be the diagram whose vertices are effective pairs and edges are the following.
  - For any morphism  $f : X \rightarrow X'$  such that  $f(Y) \subseteq Y'$ , we have an edge  $(X', Y', i) \rightarrow (X, Y, i)$ .
  - For any chain  $X \supseteq Y \supseteq Z$  of closed subvarieties, an edge  $(Y, Z, i) \rightarrow (X, Y, i + 1)$ .
- ▶ The relative singular cohomology

$$H^* : \text{Pairs}^{\text{eff}} \rightarrow \mathbb{Z} - \text{Mod}$$

$$(X, Y, i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$$

is a representation.

- ▶ We define the category of *effective mixed Nori motives* as

$$\mathcal{MM}_{\text{Nori}}^{\text{eff}}(k) := \mathcal{C}(\text{Pairs}^{\text{eff}}, H^*).$$

►  $\mathcal{MM}_{\text{Nori}}^{\text{eff}} := \mathcal{MM}_{\text{Nori}}^{\text{eff}}(k) := \mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$ .

$$\begin{array}{ccc} \text{Pairs}^{\text{eff}} & \xrightarrow{H^*} & \mathbb{Z} - \text{Mod} \\ & \searrow \tilde{T} & \nearrow f_{H^*} \\ & \mathcal{MM}_{\text{Nori}}^{\text{eff}} & \end{array}$$

►  $H_{\text{Nori}}^i(X, Y) := \tilde{T}(X, Y, i) \in \mathcal{MM}_{\text{Nori}}^{\text{eff}}$ .

►  $H_{\text{Nori}}^i(X) := H_{\text{Nori}}^i(X, \emptyset)$ .

## Definition

1. We call an effective pair  $(X, Y, i)$  an *effective good pair* if  $H^j(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) = 0$  for  $j \neq i$  and  $H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$  is free.
2. An effective good pair is called an *effective very good pair* if  $X$  is affine,  $X \setminus Y$  is smooth and either  $\dim(X) = i$ ,  $\dim(Y) = i - 1$  or  $X = Y$  and  $\dim(X) < i$ .
3. We denote the full subdiagram of  $\text{Pairs}^{\text{eff}}$  with effective good pairs by  $\text{Good}^{\text{eff}}$ .
4. We denote the full subdiagram of  $\text{Good}^{\text{eff}}$  with effective very good pairs by  $\text{VGood}^{\text{eff}}$ .

## Lemma (Nori)

Let  $X$  be an affine  $k$ -variety of dimension  $n$  and  $Z \subseteq X$  is a Zariski closed subset with  $\dim(Z) < n$ . Then there is a Zariski closed subset  $Y$  with  $Z \subseteq Y \subseteq X$  and  $\dim(Y) < n$  such that  $(X, Y, n)$  is a good pair.

- By using this lemma iteratively, for any affine variety  $X$  of dimension  $n$ , we can find a filtration

$$\emptyset = F_{-1}X \subset F_0X \subset \dots \subset F_{n-1}X \subset F_nX = X$$

such that each  $(F_jX, F_{j-1}X, j)$  is very good.

- The induced chain complex

$$\dots \rightarrow H^i(F_iX(\mathbb{C}), F_{i-1}X(\mathbb{C}); \mathbb{Z}) \xrightarrow{\delta_i} H^{i+1}(F_{i+1}X(\mathbb{C}), F_iX(\mathbb{C}); \mathbb{Z}) \rightarrow \dots$$

computes the singular cohomology of  $X$ .

## Theorem

$\mathcal{MM}_{\text{Nori}}^{\text{eff}} = \mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$ ,  $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$  and  $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$  are equivalent.

- For good pairs  $(X, Y, i)$  and  $(X', Y', i')$ , let

$$H_{\text{Nori}}^i(X, Y) \otimes H_{\text{Nori}}^{i'}(X', Y') := H_{\text{Nori}}^{i+i'}(X \times X', X \times Y' \cup X' \times Y)$$

in the light of the Künneth formula.

- $\mathcal{MM}_{\text{Nori}}^{\text{eff}} = \mathcal{C}(\text{Good}^{\text{eff}}, H^*)$  is a tensor category.
- $\mathbf{1}(-1) := H_{\text{Nori}}^1(\mathbb{G}_m, \{1\})$ .
- The category  $\mathcal{MM}_{\text{Nori}} := \mathcal{MM}_{\text{Nori}}(k)$  of *mixed Nori motives* is defined as the localization of  $\mathcal{MM}_{\text{Nori}}^{\text{eff}}$  with respect to  $\mathbf{1}(-1)$ .

## Theorem

$\mathcal{MM}_{\text{Nori}}$  is a Tannakian category with the fiber functor  $H^*$ .

- ▶  $\mathbf{1}(-n) := \mathbf{1}(-1)^{\otimes n}$ , for  $n \in \mathbb{Z}$ .
- ▶  $M(-n) := M \otimes \mathbf{1}(-n)$ , for  $n \in \mathbb{Z}$  and  $M \in \mathcal{MM}_{\text{Nori}}$ .
- ▶  $H_{\text{Nori}}^i(\mathbb{P}^N) = \begin{cases} \mathbf{1}(-n), & \text{if } i = 2n \text{ and } N \geq n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$
- ▶ If  $Z$  is a projective variety of dimension  $n$ , then  $H_{\text{Nori}}^{2n}(Z) = \mathbf{1}(-n)$ .

- We work with  $\mathcal{MM}_{\text{Nori}, \mathbb{Q}}$ , the category of mixed Nori motives with rational coefficients.

## Definition

A motive  $M \in \mathcal{MM}_{\text{Nori}, \mathbb{Q}}$  is called *pure of weight*  $n \in \mathbb{Z}$  if it is a subquotient of  $H_{\text{Nori}}^{n+2j}(Y)(j)$  for some  $Y$  smooth and projective and  $j \in \mathbb{Z}$ . A motive is called *pure* if it is a direct sum of pure motives of some weights.

- $\mathbf{1}(-n)$  is pure of weight  $2n$ .

## Theorem

*On every motive  $M \in \mathcal{MM}_{\text{Nori}, \mathbb{Q}}$ , there is a unique bounded increasing filtration  $(W_n M)_{n \in \mathbb{Z}}$  inducing the weight filtration under the Hodge realization. Moreover, every morphism of Nori motives is strictly compatible with this filtration.*

- We call this filtration *weight filtration* and denote

$$\text{gr}_n^W M := W_n M / W_{n-1} M.$$

- $\text{gr}_n^W M$  is pure of weight  $n$ .



# Mixed Tate Motives

- ▶ A Nori motive  $M \in \mathcal{MM}_{\text{Nori}, \mathbb{Q}}$  is called a *mixed Tate Nori motive* if  $\text{gr}_{2n}^W M$  is a direct sum of copies of  $\mathbf{1}(-n)$ .
- ▶ We denote the full subcategory of  $\mathcal{MM}_{\text{Nori}, \mathbb{Q}}$  containing these objects by  $\mathcal{MTM}_{\text{Nori}, \mathbb{Q}}$ .
- ▶  $(\mathcal{MTM}_{\text{Nori}, \mathbb{Q}}, \mathbf{1}(1))$  is a mixed Tate category.

## Example

Let  $B \subseteq \mathbb{G}_m$  be such that  $B = \{x_1, \dots, x_r\}$ . We will find the weight structure of  $H_{\text{Nori}}^1(\mathbb{G}_m, B)$ . We have the following exact sequence

$$0 \rightarrow \underbrace{H_{\text{Nori}}^0(\mathbb{G}_m)}_{\mathbf{1}(0)} \rightarrow \underbrace{H_{\text{Nori}}^0(B)}_{\mathbf{1}(0)^{\oplus r}} \rightarrow H_{\text{Nori}}^1(\mathbb{G}_m, B) \rightarrow \underbrace{H_{\text{Nori}}^1(\mathbb{G}_m)}_{\mathbf{1}(-1)} \rightarrow 0$$

Then,

$$\text{gr}_0^W H_{\text{Nori}}^1(\mathbb{G}_m, B) = \mathbf{1}(0)^{\oplus(r-1)}$$

$$\text{gr}_2^W H_{\text{Nori}}^1(\mathbb{G}_m, B) = \mathbf{1}(-1)$$

Therefore  $H_{\text{Nori}}^1(\mathbb{G}_m, B)$  is mixed Tate.

## Example

Let  $B = M_0 \cup M_1 \cup M_2 \subseteq \mathbb{P}_{\mathbb{C}}^2$ , where  $M_i$  are lines in  $\mathbb{P}_{\mathbb{C}}^2$  in general position such that they are not axis lines  $z_i = 0$ . Then

$$\begin{aligned} \mathrm{gr}_0^W H_{\mathrm{Nori}}^1(B \cap \mathbb{G}_m^2) &= \mathbf{1}(0) \\ \mathrm{gr}_2^W H_{\mathrm{Nori}}^1(B \cap \mathbb{G}_m^2) &= \mathbf{1}(-1)^{\oplus s}, \end{aligned}$$

where  $s$  is the number of intersection of  $B$  with  $\{z_1 = 0\} \cup \{z_2 = 0\}$ .  
Using the exact sequence

$$0 \rightarrow \underbrace{H_{\mathrm{Nori}}^1(\mathbb{G}_m^2)}_{\mathbf{1}(-1)^{\oplus 2}} \rightarrow H_{\mathrm{Nori}}^1(B \cap \mathbb{G}_m^2) \rightarrow H_{\mathrm{Nori}}^2(\mathbb{G}_m^2, B \cap \mathbb{G}_m^2) \rightarrow \underbrace{H_{\mathrm{Nori}}^2(\mathbb{G}_m^2)}_{\mathbf{1}(-2)} \rightarrow 0.$$

we have

$$\begin{aligned} \mathrm{gr}_0^W H_{\mathrm{Nori}}^2(\mathbb{G}_m^2, B \cap \mathbb{G}_m^2) &= \mathbf{1}(0) \\ \mathrm{gr}_2^W H_{\mathrm{Nori}}^2(\mathbb{G}_m^2, B \cap \mathbb{G}_m^2) &= \mathbf{1}(-1)^{\oplus s-2} \\ \mathrm{gr}_4^W H_{\mathrm{Nori}}^2(\mathbb{G}_m^2, B \cap \mathbb{G}_m^2) &= \mathbf{1}(-2) \end{aligned}$$

and  $s - 2 \in \{0, 1, 2, 3, 4\}$ . Here  $\mathbf{1}(-2)$  is coming from the torus  $\mathbb{G}_m^2$  and  $\mathbf{1}(0)$  is coming from the triangle defined by  $B$ .

# Aomoto Polylogarithms

- ▶ There is a generalization of logarithms called *polylogarithms* which is defined inductively by

$$\ell i_1(z) = -\log(1 - z)$$

and

$$d\ell i_n(z) = \ell i_{n-1}(z) \frac{dz}{z},$$

with  $\ell i_n(0) = 0$ .

- ▶ They have the power series expansion

$$\ell i_n(z) = \sum_{1 \leq m} \frac{z^m}{m^n},$$

for  $|z| < 1$ .

- ▶ Fix some  $q \in \mathbb{Q} \setminus \{0, 1\}$ . Let  $z_i$ ,  $i = 0, 1, \dots, n$ , be the homogeneous coordinates on  $\mathbb{P}_{\mathbb{Q}}^n$ .
- ▶ Let  $L_i$  be the hyperplanes defined by  $z_i = 0$  and  $M_i$  be the hyperplanes defined as  $M_0 : z_0 = z_1$ ;  $M_1 : z_0 = z_1 + z_2$ ;  $M_i : z_i = z_{i+1}$  for  $2 \leq i < n$ ; and  $M_n : qz_0 = z_n$ .
- ▶ Let  $M_q = \bigcup M_i$  and  $L = \bigcup L_i$ .
- ▶  $\ell i_n(q)$  is a period of the mixed Tate motive

$$H_{\text{Nori}}^n(\mathbb{P}_{\mathbb{Q}}^n \setminus L, M_q \setminus (L \cap M)).$$

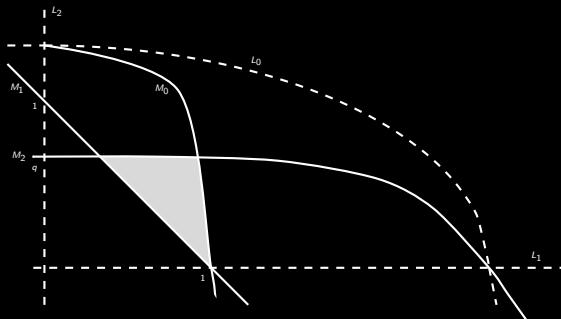
- ▶ We will call the configuration  $(L, M_q)$  as the *polylogarithmic configuration* of  $q$ .

►  $\ell i_2$  is called *dilogarithm*. Its configuration is given by

$$M_0 : z_0 = z_1$$

$$M_1 : z_0 = z_1 + z_2$$

$$M_2 : qz_0 = z_2.$$



Call  $\mathcal{D}(q)$  for the triangle given by  $M_i$ . Then

$$\ell i_2(q) = \int_{\mathcal{D}(q)} \frac{dx}{x} \wedge \frac{dy}{y}.$$

- ▶ We call an  $n$ -simplex a family of  $n + 1$  hyperplanes  $(L_0, \dots, L_n)$  of  $\mathbb{P}_k^n$ .
- ▶ A pair of simplices  $(L, M)$  is said to be *admissible* if they do not have a common face.
- ▶ Let  $(L, M)$  be admissible pair of simplices such that the hyperplanes of  $L$  and  $M$  are in general position. Let

$$\omega_L = d \log(z_1/z_0) \wedge \dots \wedge d \log(z_n/z_0)$$

where  $z_i = 0$  is a homogeneous equation of  $L_i$ . Let  $\Delta_M$  be the simplex whose sides are  $M_i$ . Then

$$a(L, M) = \int_{\Delta_M} \omega_L$$

is a period of

$$H_{\text{Nori}}^n(\mathbb{P}^n \setminus L, M \setminus (L \cap M)).$$

- ▶  $M = H_{\text{Nori}}^n(\mathbb{P}^n \setminus L, M \setminus (L \cap M))$  is a mixed Tate motive with

$$\text{gr}_{2n}^W M = \mathbf{1}(-n),$$

$$\text{gr}_0^W M = \mathbf{1}(0).$$

## Definition

$A_0(k) := \mathbb{Z}$ . For  $n > 0$ , define  $A_n(k)$  as the abelian group generated by  $(L; M)$  where  $(L, M)$  is an admissible pair of simplices in  $\mathbb{P}_k^n$  subject to the following relations:

1. If the hyperplanes of one of  $L$  or  $M$  is not in general position (i.e. degenerate), then  $(L; M) = 0$ .
2. For every  $\sigma \in S_n$ ,

$$(\sigma L; M) = (L; \sigma M) = (-1)^{|\sigma|} (L; M)$$

where  $\sigma L$  and  $\sigma M$ , are defined by the natural action of  $S_n$  on a set indexed by  $1, \dots, n$ .

3. For every family of hyperplanes  $L_0, \dots, L_{n+1}$  and an  $n$ -simplex  $M$ ,

$$\sum (-1)^j (\hat{L}^j; M) = 0,$$

where  $\hat{L}^j = (L_0, \dots, \hat{L}_j, \dots, L_{n+1})$ , and the corresponding relation for the second component.

4. For every  $g \in PGL_{n+1}(k)$ ,

$$(gL; gM) = (L; M).$$



$$A_1(k) \xrightarrow{\sim} k^\times$$

$$(L_0, L_1; M_0, M_1) \mapsto r(L_0, L_1, M_0, M_1)$$

- ▶ The multiplication map  $\mu : A_{n'} \times A_{n''} \rightarrow A_n$ , for  $n' + n'' = n$ , is defined on the generators in the following way. Let  $(L', M')$  and  $(L'', M'')$  be two admissible pairs of non-degenerate simplices from  $\mathbb{P}^{n'}$  and  $\mathbb{P}^{n''}$ , respectively. Also let  $L$  be a non-degenerate simplex from  $\mathbb{P}^n$ . Identify the affine spaces  $\mathbb{P}^n \setminus L_0$  and  $(\mathbb{P}^{n'} \setminus L'_0) \times (\mathbb{P}^{n''} \setminus L''_0)$ . Then  $M' \times M''$  can be seen in  $\mathbb{P}^n \setminus L_0$  and hence in  $\mathbb{P}^n$ . Cutting this product into simplices in  $\mathbb{P}^n$  defines an element in  $A_n$  which is defined as the product of  $(L'; M')$  and  $(L''; M'')$ .
- ▶  $A = \bigoplus A_n$  is a graded Hopf algebra.



- ▶ Let  $G$  be the Galois group of  $\mathcal{MTM}_{\text{Nori}, \mathbb{Q}}$ . Then

$$1 \rightarrow U \rightarrow G \rightarrow \mathbb{G}_m \rightarrow 1$$

is split exact.

- ▶ Here,  $U = \text{Spec} R$ , where  $R = \bigoplus_{d \geq 0} R_d$  is a graded Hopf algebra.
- ▶  $\mathcal{MTM}_{\text{Nori}, \mathbb{Q}}$  is equivalent to the category of graded  $R$ -comodules.

### Conjecture (Beilinson)

*There is a natural isomorphism of graded Hopf algebras*

$$A \otimes \mathbb{Q} \xrightarrow{\sim} R.$$

# A Construction of Mixed Tate Motives

- ▶ We will consider the motives coming from the following configurations.
- ▶ Fix  $n \in \mathbb{N}^{>0}$ . Let

$$B = \bigcup_{1 \leq i \leq m} B_i,$$

where all  $B_i$  are hyperplanes in  $B$  that meet  $x_{i_1} = \dots = x_{i_k} = 0$  properly for all  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ .

- ▶ We call such  $B$  a *nice divisor*.
- ▶ We will be interested in the motives of the form

$$H_{\text{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n).$$

- ▶  $H_{\text{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$  is a mixed Tate motive with

$$\text{gr}_{2n}^W H_{\text{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) = \mathbf{1}(-n).$$

► Let

$$M = \bigoplus_{d \geq 0} M_d$$

where

$$M_d = \mathrm{gr}_{2n-2d}^W \left( \varprojlim_B H_{\mathrm{Nori}}^n (\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \right) \otimes \mathbf{1}(n-d)$$

such that the limit is taken over all nice divisors  $B$  as in the beginning of the section.

► In particular,

$$M_0 = \mathbf{1}(0)$$

and

$$M_n = \mathrm{gr}_0^W \left( \varprojlim_B H_{\mathrm{Nori}}^n (\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \right).$$

- ▶ Viewing  $M$  as a graded  $R$ -comodule, we have a linear map  $\nu : M \rightarrow R \otimes M$ . Let  $\gamma_i : M \rightarrow M_i$  be the restriction map.
- ▶ Since  $M_0 = \mathbf{1}(0)$  is realized as  $\mathbb{Z}$ , there is a natural map  $\ell : M_0 \rightarrow \mathbb{Q}$ .
- ▶ By composing

$$h : M \xrightarrow{\nu} R \otimes M \xrightarrow{\text{id}_R \otimes \gamma_0} R \otimes M_0 \xrightarrow{\text{id}_R \otimes \ell} R \otimes \mathbb{Q} \xrightarrow{\sim} R$$

we have a map  $h : M \rightarrow R$  such that  $h|_{M_0} = \ell$ .

- ▶ This also gives

$$h|_{M_n} : M_n \rightarrow \bigoplus_{i+j=n} R_i \otimes M_j \rightarrow R_n \otimes M_0 \rightarrow R_n \otimes \mathbb{Q} \xrightarrow{\sim} R_n.$$

- ▶ Let  $G_n := S_n \ltimes \mathbb{G}_m^n$ , where  $S_n$  is the symmetric group of order  $n!$ , and the action be given by  $\sigma \cdot (a_1, \dots, a_n) = (\sigma(a_1), \dots, \sigma(a_n))$ .
- ▶ Then  $G_n$  acts on  $\mathbb{G}_m^n$  by

$$(\sigma \cdot a) \cdot x = (-1)^{|\sigma|} \sigma \cdot (ax)$$

for  $\sigma \in S_n$ ,  $a, x \in \mathbb{G}_m^n$ .

- ▶ This action extends on

$$M_n = \mathrm{gr}_0^W \left( \varprojlim_B H_{\mathrm{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \right).$$

- ▶ Let

$$R'_n := H_0(G_n; M_n) = M_n / \langle gx - x \mid g \in G_n, x \in M_n \rangle.$$

## Proposition

$h|_{M_n}$  induces a map  $\varphi_n : R'_n \rightarrow R_n$ .

## Proof.

- ▶  $R_n$  is given by the framed objects and the coaction  $M_n \rightarrow R_n \otimes M_n$  is given by frames

$$\mathbf{1}(0) \rightarrow \mathrm{gr}_0^W H_{\mathrm{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$$

and it corresponds to the periods of  $\mathrm{gr}_0^W H_{\mathrm{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n)$ .

- ▶ WLOG assume  $\mathrm{gr}_0^W H_{\mathrm{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) = \mathbf{1}(0)$ .
- ▶ Its periods are scalar multiples of

$$\rho = \int_B \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}.$$

- ▶  $\rho$  is invariant under the action of both  $S_n$  and  $\mathbb{G}_m^n$ .

□

- ▶ Let  $R'_0 = \mathbb{Z}$  and  $R' = \bigoplus_{n \geq 0} R'_n$ .
- ▶ Tensor product of motives defines a multiplication  $R'_{n'} \otimes R'_{n''} \rightarrow R'_n$ .

### Lemma

Assume  $n' + n'' = n$ . Let  $(L'; B') \in A_{n'}$  and  $(L''; B'') \in A_{n''}$ . Then  $(L'; B') \times (L''; B'') = \sum_i (L; B_i)$ , for some  $(L; B_i) \in A_n$ . Assume that  $L, L', L''$  are given by axis hyperplanes. Then,

$$H_{\text{Nori}}^{n'}(\mathbb{G}_m^{n'}, B' \cap \mathbb{G}_m^{n'}) \otimes H_{\text{Nori}}^{n''}(\mathbb{G}_m^{n''}, B'' \cap \mathbb{G}_m^{n''}) = H_{\text{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n),$$

where  $B$  is the nice divisor given by the union of simplices  $B_i$ .

### Proof.

$$\begin{aligned} & H_{\text{Nori}}^{n'}(\mathbb{G}_m^{n'}, B' \cap \mathbb{G}_m^{n'}) \otimes H_{\text{Nori}}^{n''}(\mathbb{G}_m^{n''}, B'' \cap \mathbb{G}_m^{n''}) \\ &= H_{\text{Nori}}^n(\mathbb{G}_m^n, \mathbb{G}_m^{n'} \times (B'' \cap \mathbb{G}_m^{n''}) \cup (B' \cap \mathbb{G}_m^{n'}) \times \mathbb{G}_m^{n''}) \\ &= H_{\text{Nori}}^n(\mathbb{G}_m^n, (\mathbb{G}_m^{n'} \times B'' \cup B' \times \mathbb{G}_m^{n''}) \cap \mathbb{G}_m^n) \\ &= H_{\text{Nori}}^n(\mathbb{G}_m^n, B \cap \mathbb{G}_m^n). \end{aligned}$$

by the definition of multiplication in  $A$ .



## Theorem

There is an isomorphism of graded algebras  $\phi : R' \rightarrow A$ .

## Idea of proof.

Let  $n > 0$ . Let  $Z = (Z_0, \dots, Z_n)$  be the  $n$ -simplex in  $\mathbb{P}^n$  given by  $Z_i : z_i = 0$ . Define  $A'_n$  as the abelian group generated by  $(B)$  where  $B$  is an  $n$ -simplex in  $\mathbb{P}^n$  such that  $(Z, B)$  is admissible, subject to the following relations:

1. If the hyperplanes of  $B$  are not in general position, then  $(B) = 0$ .
2. For every  $\sigma \in S_n$ ,

$$(\sigma B) = (-1)^{|\sigma|} (B).$$

3. For every family of hyperplanes  $B_0, \dots, B_{n+1}$ ,

$$\sum (-1)^j (\hat{B}^j) = 0.$$

4. For every  $g \in \mathbb{G}_m^n$ ,

$$(gB) = (B),$$

where the action of  $\mathbb{G}_m^n$  is as follows. For  $g = (g_1, \dots, g_n) \in \mathbb{G}_m^n$  and  $p = (z_0 : z_1 : z_2 : \dots : z_n) \in \mathbb{P}^n$ , let  $g \cdot p = (z_0 : g_1 z_1 : g_2 z_2 : \dots : g_n z_n)$ .



## Idea of proof, cont'd.

- ▶ Then,

$$\begin{aligned} A'_n &\rightarrow A_n \\ (B) &\mapsto (Z; B). \end{aligned}$$

is an isomorphism.

- ▶ We will write an isomorphism  $R'_n \rightarrow A'_n$ .
- ▶ We will consider the underlying  $\mathbb{Z}$ -modules of motives.
- ▶ We will work in the homological setting. The category of cohomological motives is isomorphic to the opposite category of homological motives. We denote by  $H_n^{\text{Nori}}(X, Y)$  the corresponding object of  $H_{\text{Nori}}^n(X, Y)$ .

## Idea of proof, cont'd.

- In this case,

$$M_n = \mathrm{gr}_0^W \left( \varinjlim_B H_n^{\mathrm{Nori}} (\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \right),$$

such that the colimit is taken over all nice divisors  $B$ .

- By adding any such  $B$  some hyperplanes, we can divide it into "independent" simplices  $B^i$ .
- So,  $B \subseteq \bigcup B^i$ .
- This gives  $\mathrm{gr}_0^W H_{\mathrm{Nori}}^n (\mathbb{G}_m^n, B \cap \mathbb{G}_m^n) \rightarrow \bigoplus \mathrm{gr}_0^W H_{\mathrm{Nori}}^n (\mathbb{G}_m^n, B^i \cap \mathbb{G}_m^n)$ .
- Define

$$\psi_{B^i} : \mathrm{gr}_0^W H_{\mathrm{Nori}}^n (\mathbb{G}_m^n, B^i \cap \mathbb{G}_m^n) = \mathbf{1}(0) = \mathbb{Z} \rightarrow A'_n$$

as  $\psi_{B^i}(1) = (B^i)$ .

- This extends a map

$$\psi : M_n \rightarrow A'_n.$$

Idea of proof, cont'd.

- ▶  $\psi : M_n \rightarrow A'_n$  is surjective with kernel  $\langle gx - x \mid g \in G_n, x \in M_n \rangle$ .
- ▶ Hence, this gives an isomorphism

$$\phi_n : R'_n = M_n / \langle gx - x \mid g \in G_n, x \in M_n \rangle \xrightarrow{\sim} A'_n \xrightarrow{\sim} A_n.$$

- ▶ By previous lemma,  $\phi = \bigoplus_{n \geq 0} \phi_n$  respects multiplication. Thus  $\phi$  is an isomorphism of graded algebras.



- ▶ The comultiplication on  $A$  can be carried to  $R'$ . This makes  $R'$  a Hopf algebra.
- ▶ Let  $\varphi = \bigoplus \varphi_n : R' \rightarrow R$ .

## Conjecture

$$\varphi \otimes \mathbb{Q} : R' \otimes \mathbb{Q} \rightarrow R$$

*is an isomorphism of graded Hopf algebras.*

Thank you!

