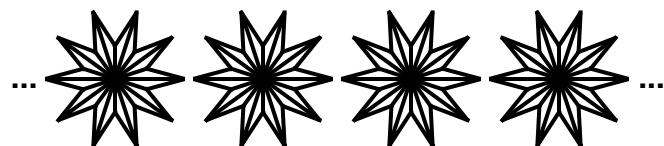
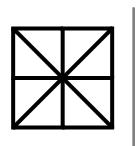
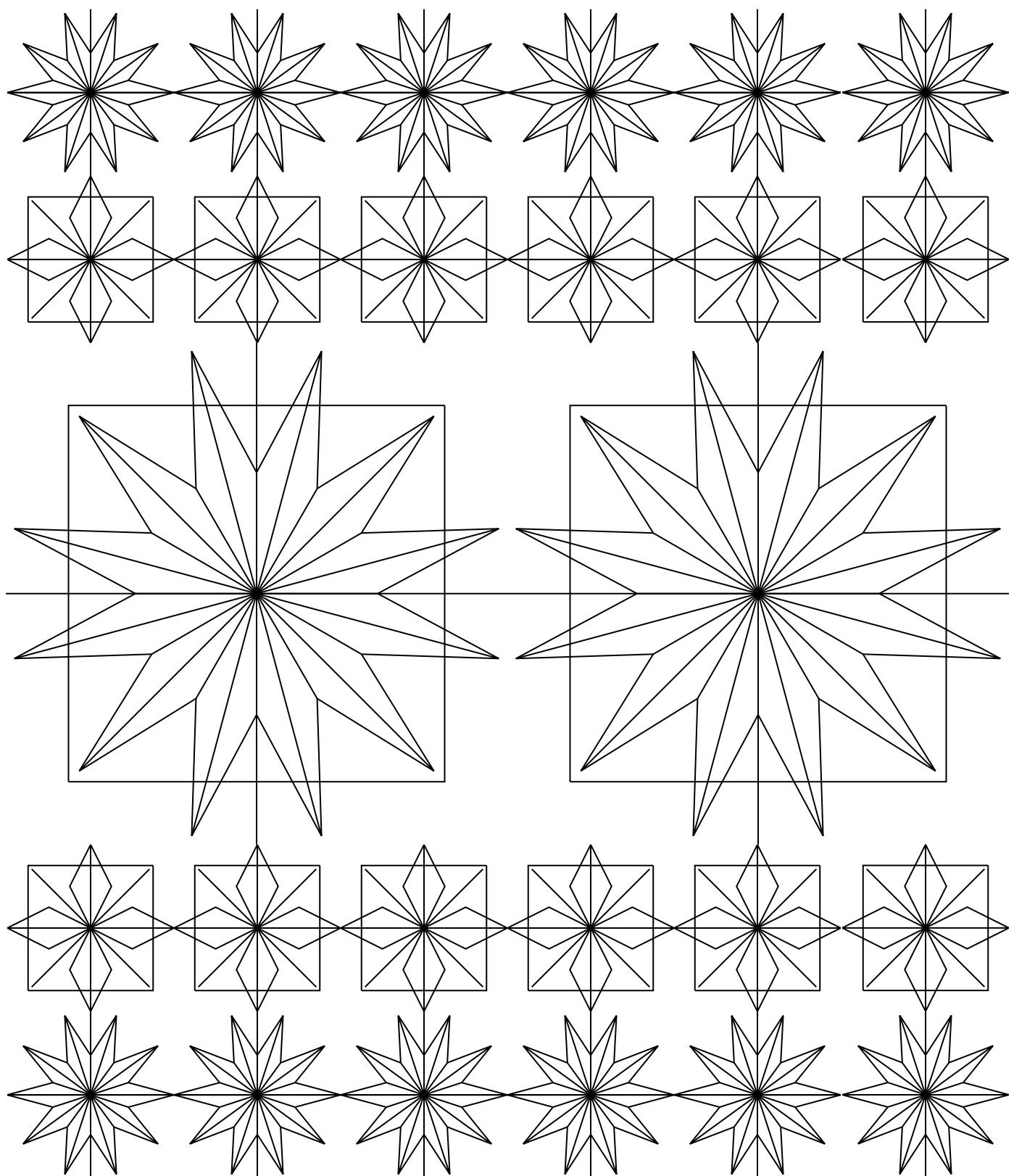


There are bigger groups to see and more types of symmetry to talk about. There are even transformations that take our illustrations beyond shapes and generate patterns that repeat forever.



Challenge: What would happen if you reflected a shape across a mirror that sat next to the shape rather than through its center?

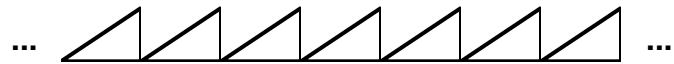




patterns of repeated shapes with mirror reflection

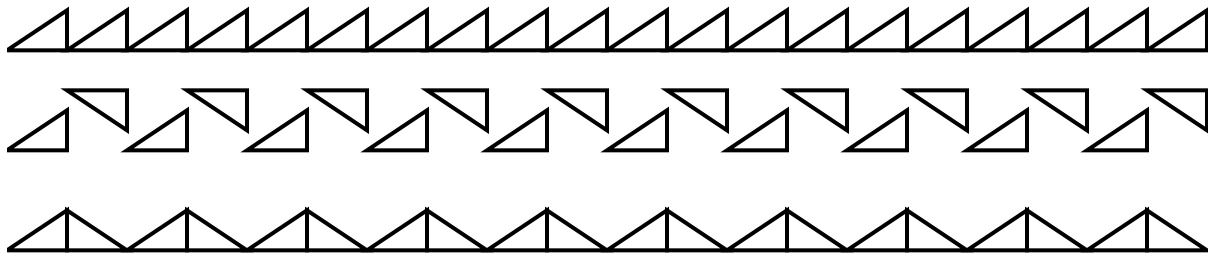
FRIEZE GROUPS

The **Frieze Groups** can be seen in patterns that repeat infinitely in opposite directions.

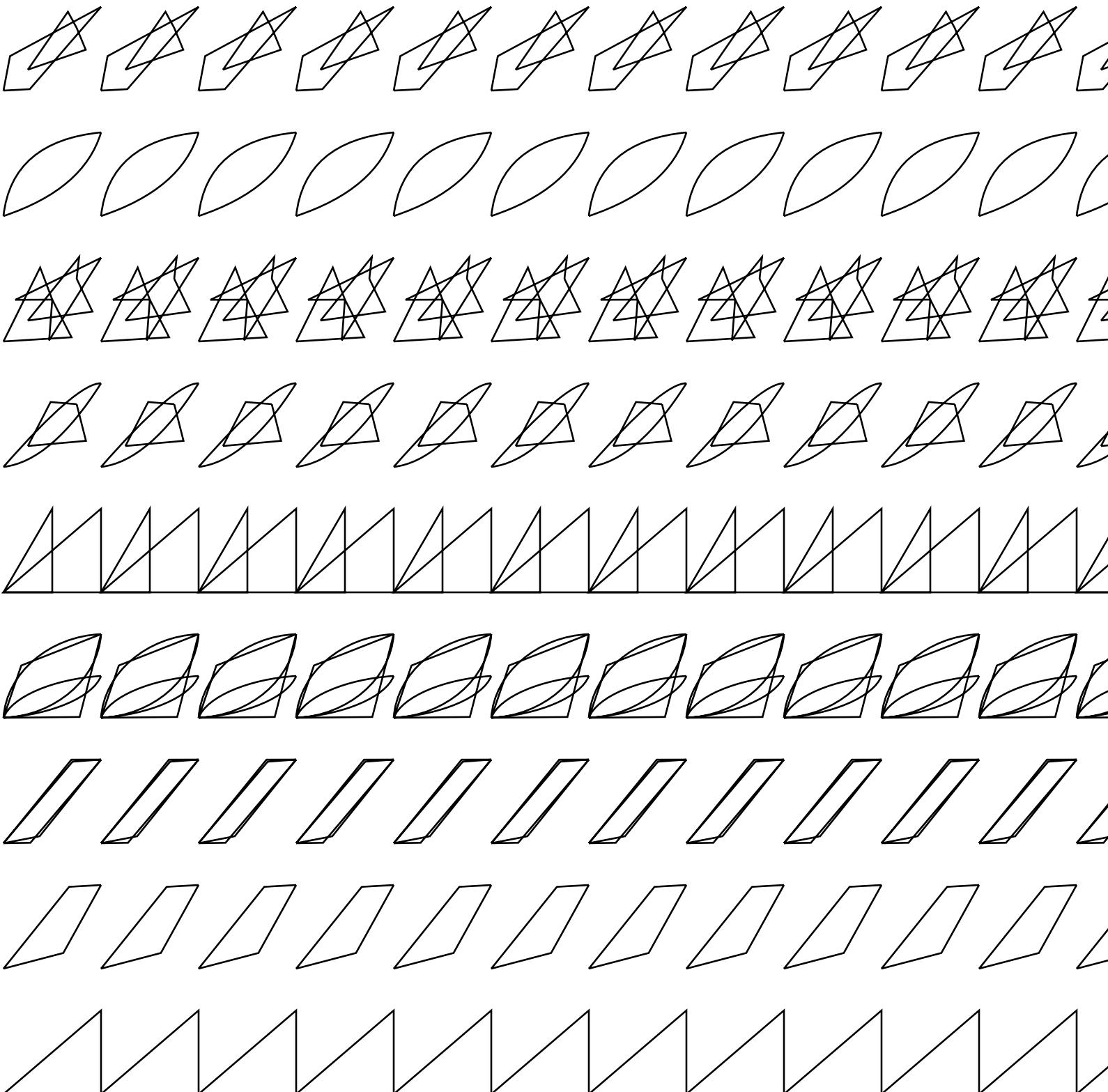


A page cannot do these patterns justice. It cuts them off when really they continue repeating forever...

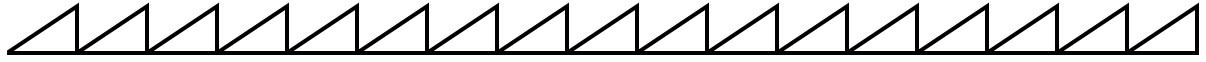
Challenge: Can you see how the patterns repeat across the page? Can you extend your imagination to see these as infinitely repeating patterns?



Coloring Challenge: Color the patterns in a way that maintains their repetition.



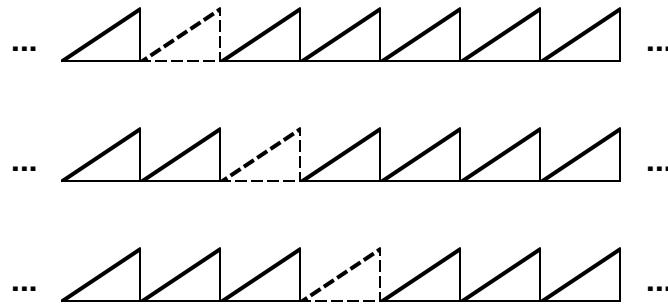
frieze patterns



Consider the smallest repeating piece of this pattern as a unit.



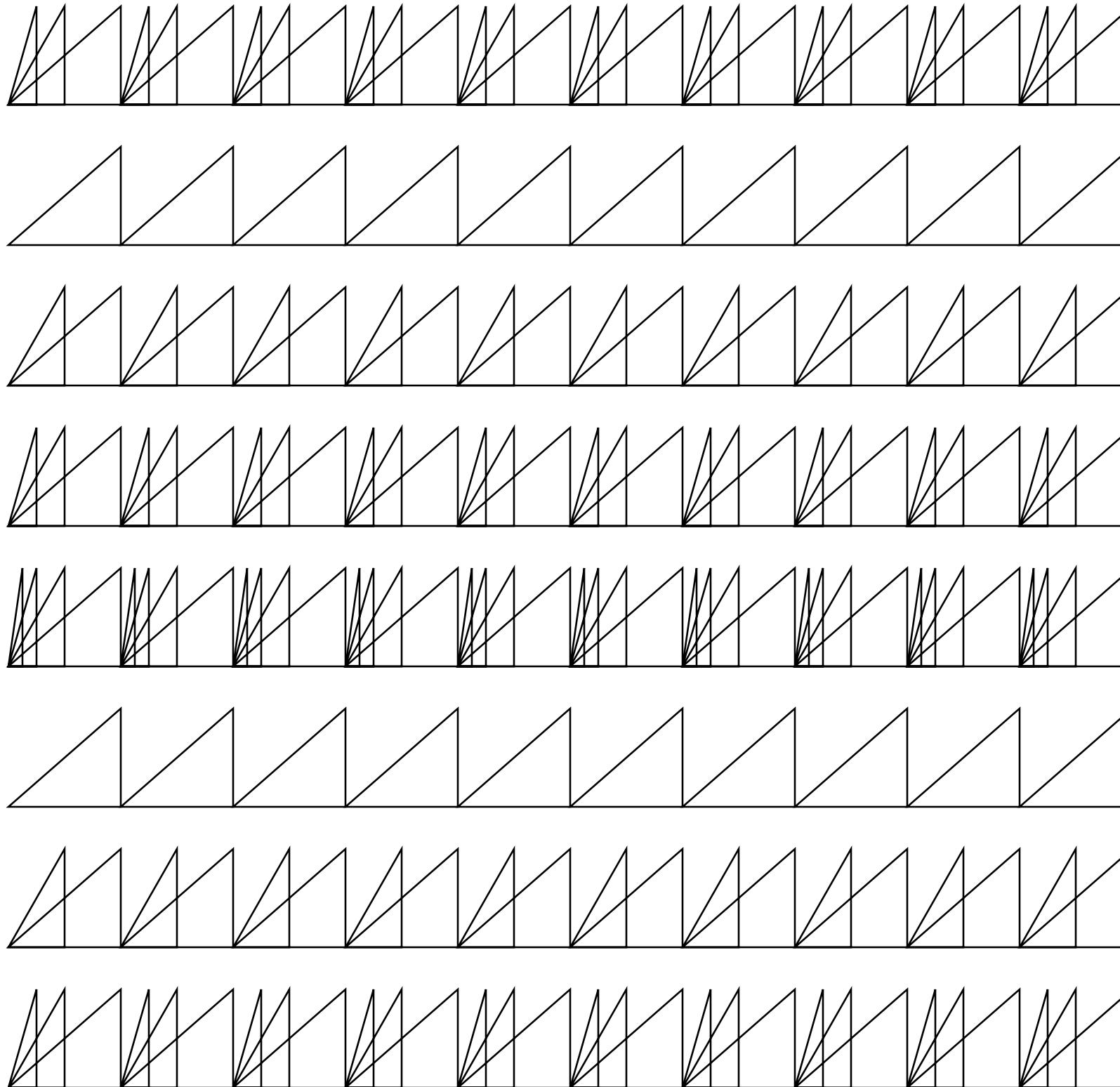
The entire pattern can shift over by this unit and there is always more behind it to replace what was shifted



so that the shift leaves the entire pattern unchanged. Such is the nature of infinite repetition...

This shift is a symmetry called **translation**.

Coloring Challenge: Color the patterns to make them less repetitive.



frieze patterns with triangles in their fundamental domains

P1

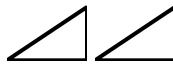
Translation

The simplest frieze group has **translation** as its only symmetry - it is generated by **translation** alone.

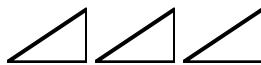
We can see this by starting with a single piece



that is copied and then translated



again and again... ...an infinite number of times...

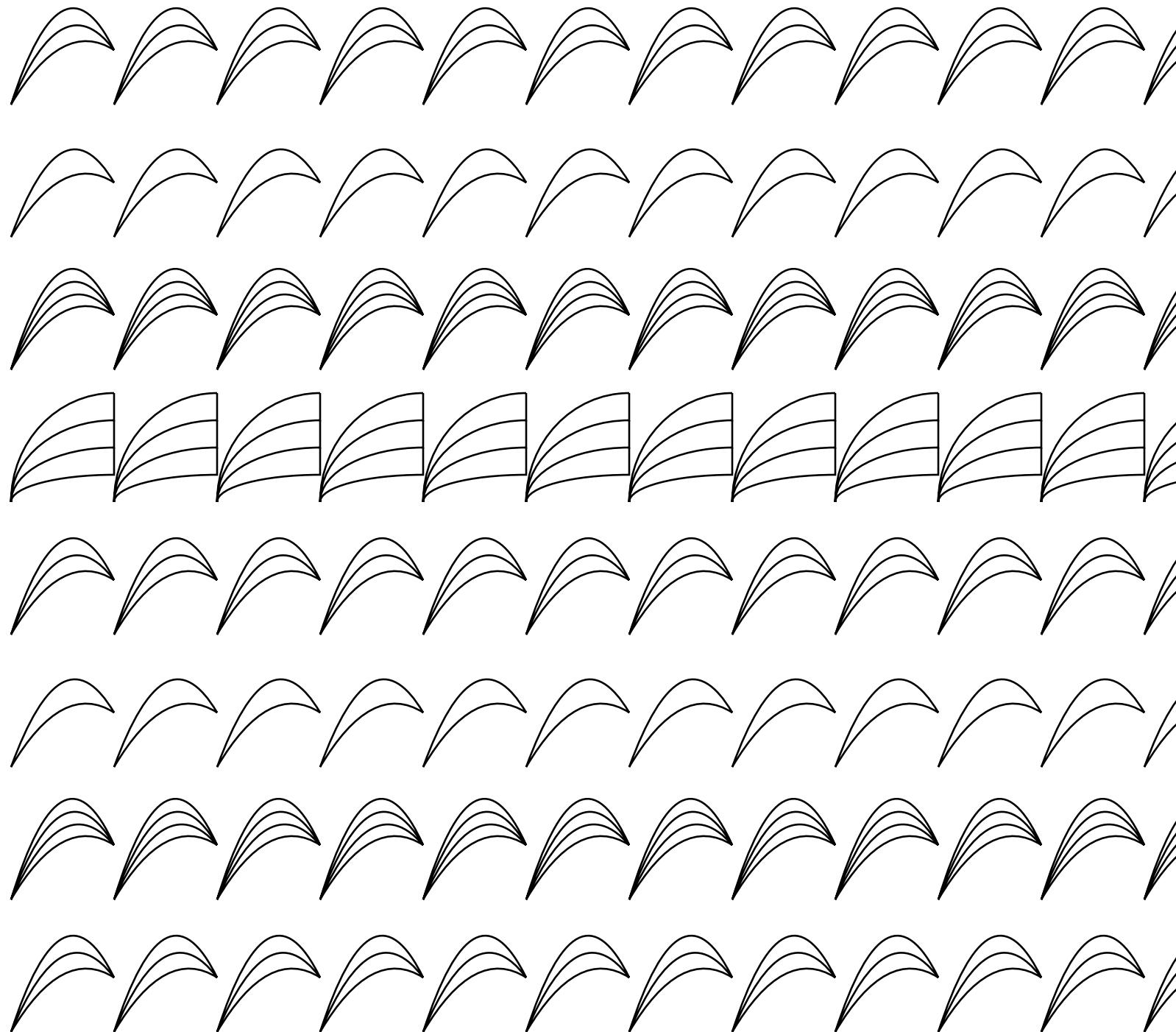


...

to result in a pattern with **translation** as a symmetry that leaves the entire pattern unchanged.

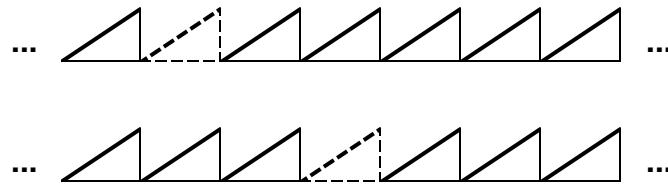


Challenge: Is the frieze group with just translation an example of a cyclic group?

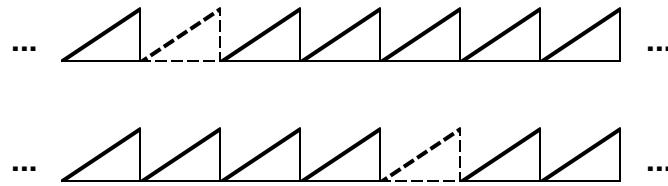


p1 frieze patterns: translation

A **translation** can be composed with another **translation** so that in the same way a pattern can shift over by 1 unit and remain unchanged, it can also shift over by 2 units and remain unchanged.



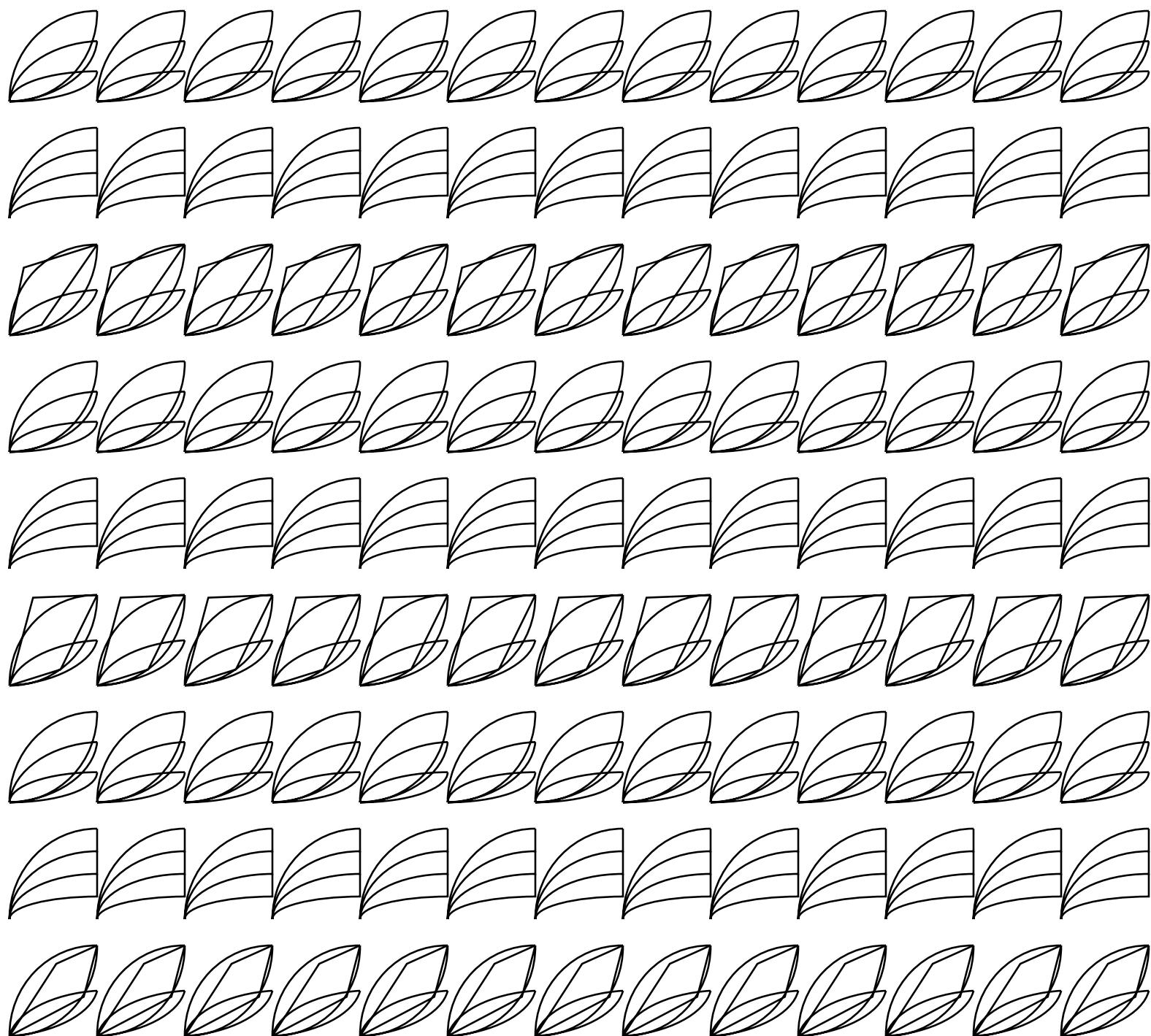
We can keep composing **translations** to see larger and larger shifts...



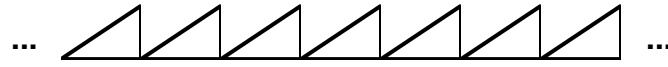
Or we can use color to take them away...

Coloring Challenge: Color the pattern so that it is not possible to translate it by 1 triangle unit without changing its appearance.





p1 frieze patterns: translation



By coloring every other unit in this pattern, we can double the shortest possible distance of **translation** in the pattern from 1 unit to 2.

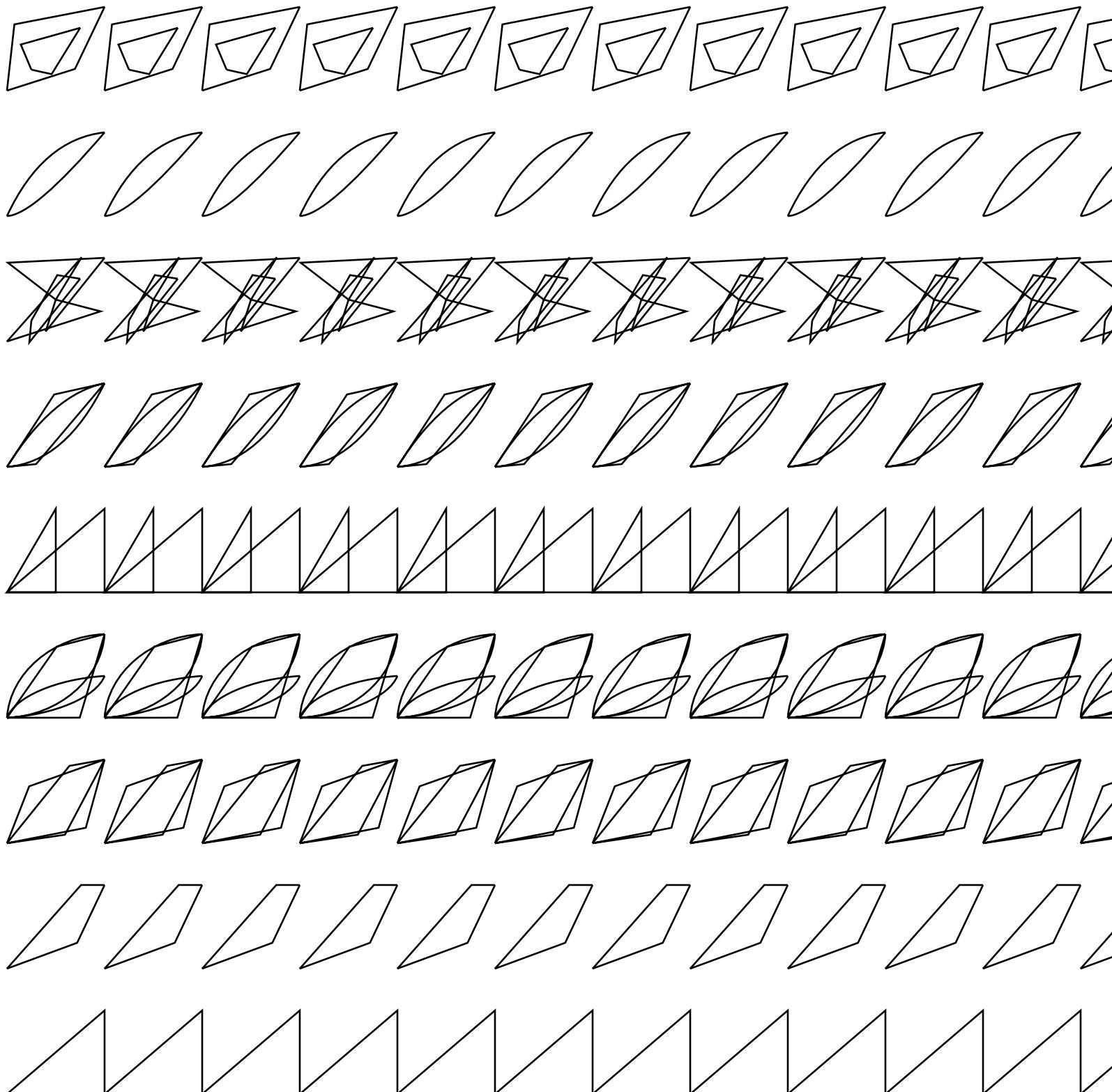


Now only translating by an even number of units leaves the pattern unchanged in appearance.

The pattern still repeats infinitely, and there are still an infinite number of **translations** that will leave it unchanged. By adding color, we took away $\frac{1}{2}$ of its **translations**, but $\frac{1}{2}$ of infinity is still infinity.

Coloring Challenge: Can you color the patterns so that the shortest possible translation distance triples?





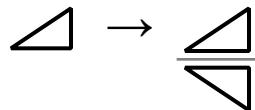
p1 frieze patterns: translation

p11m

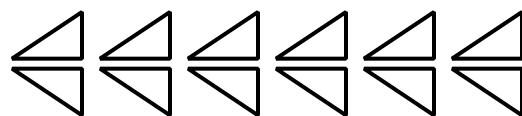
Horizontal Reflection & Translation

Our patterns can have more symmetry than just translation.

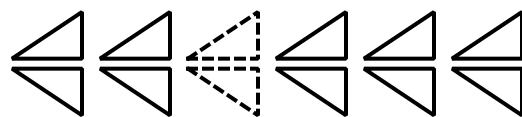
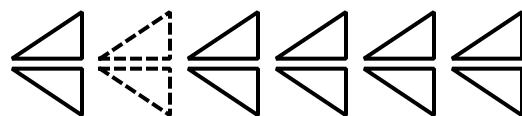
Reflecting a piece across a horizontal mirror before translating it,



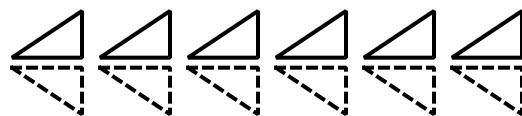
generates a new pattern, with more symmetry than the one before.



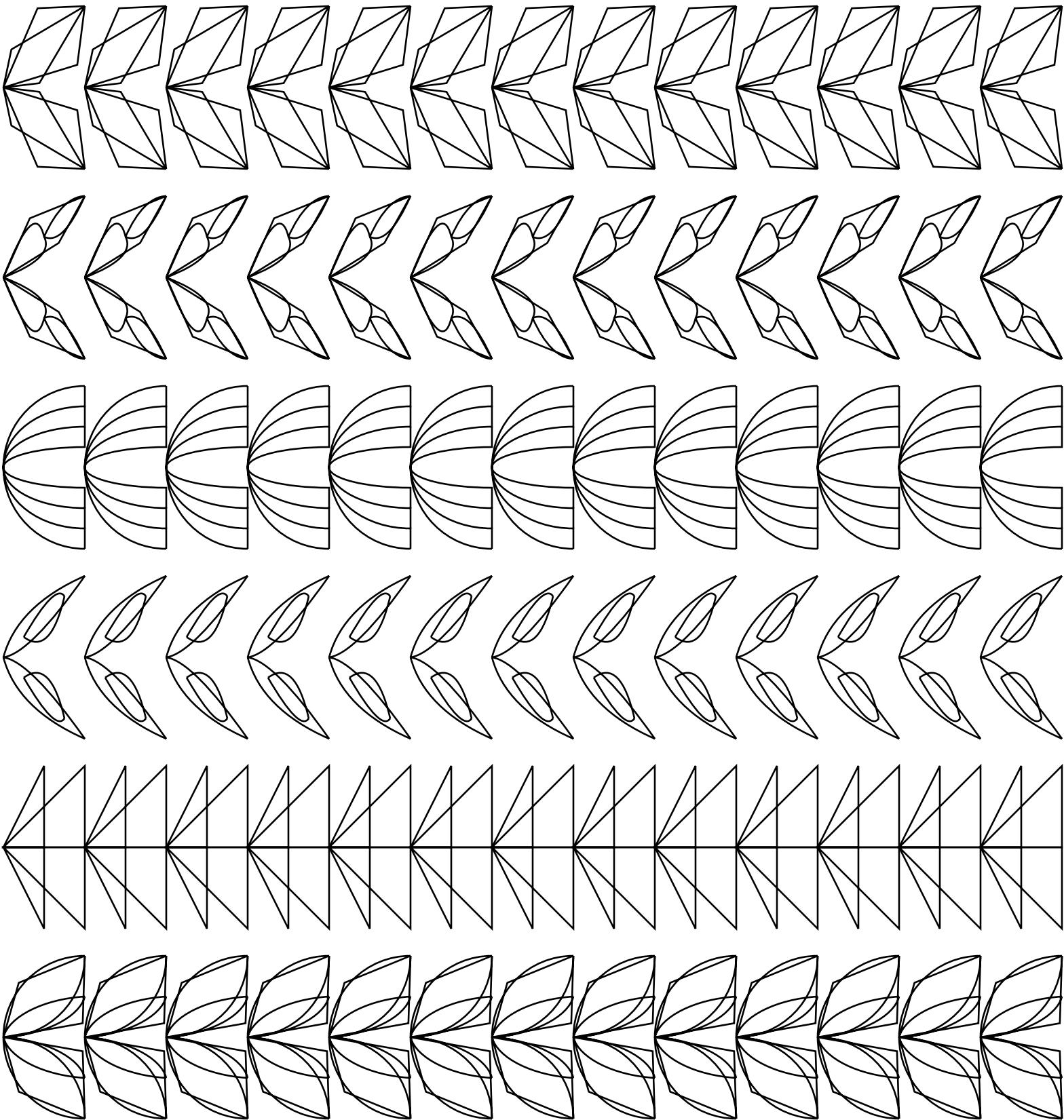
It still has **translation** - it can still shift over without changing.



But it also has a **horizontal reflection**: The whole pattern can reflect across the same mirror that transformed our first piece, and appear unchanged.



Challenge: Can you find the mirrors in the following patterns?

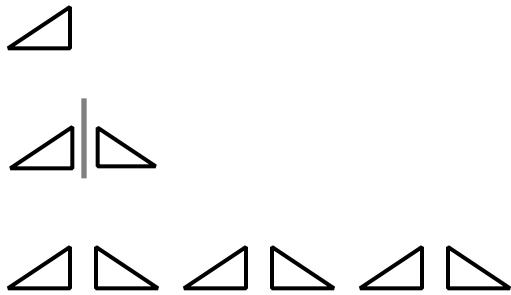


p11m frieze patterns: horizontal reflection, translation

p1m1

Vertical Reflection & Translation

Patterns can have **vertical mirrors** as well.



These mirrors shift over with each repeated translation, so once a pattern has one vertical mirror, it has an infinite number of vertical mirrors.



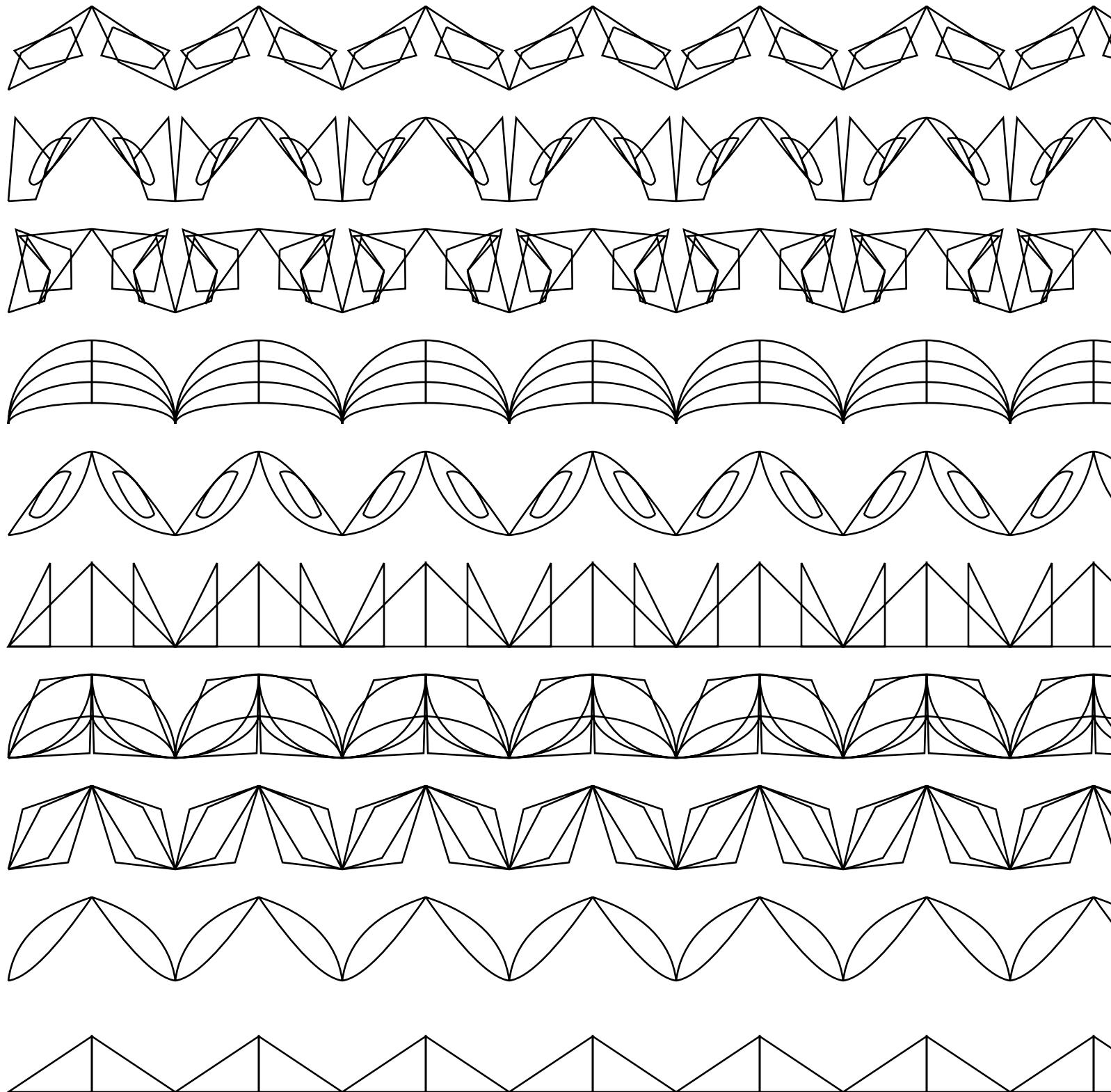
Twice that many, really.



Even though we start with a vertical mirror on one side of each piece, as the pattern repeats, another different vertical mirror shows itself.

Challenge: Can you find the mirrors in the following patterns?

Coloring Challenge: Can you color the patterns to remove the mirrors?



p1m1 frieze patterns: vertical reflection, translation



All of these mirrors can be removed with color.



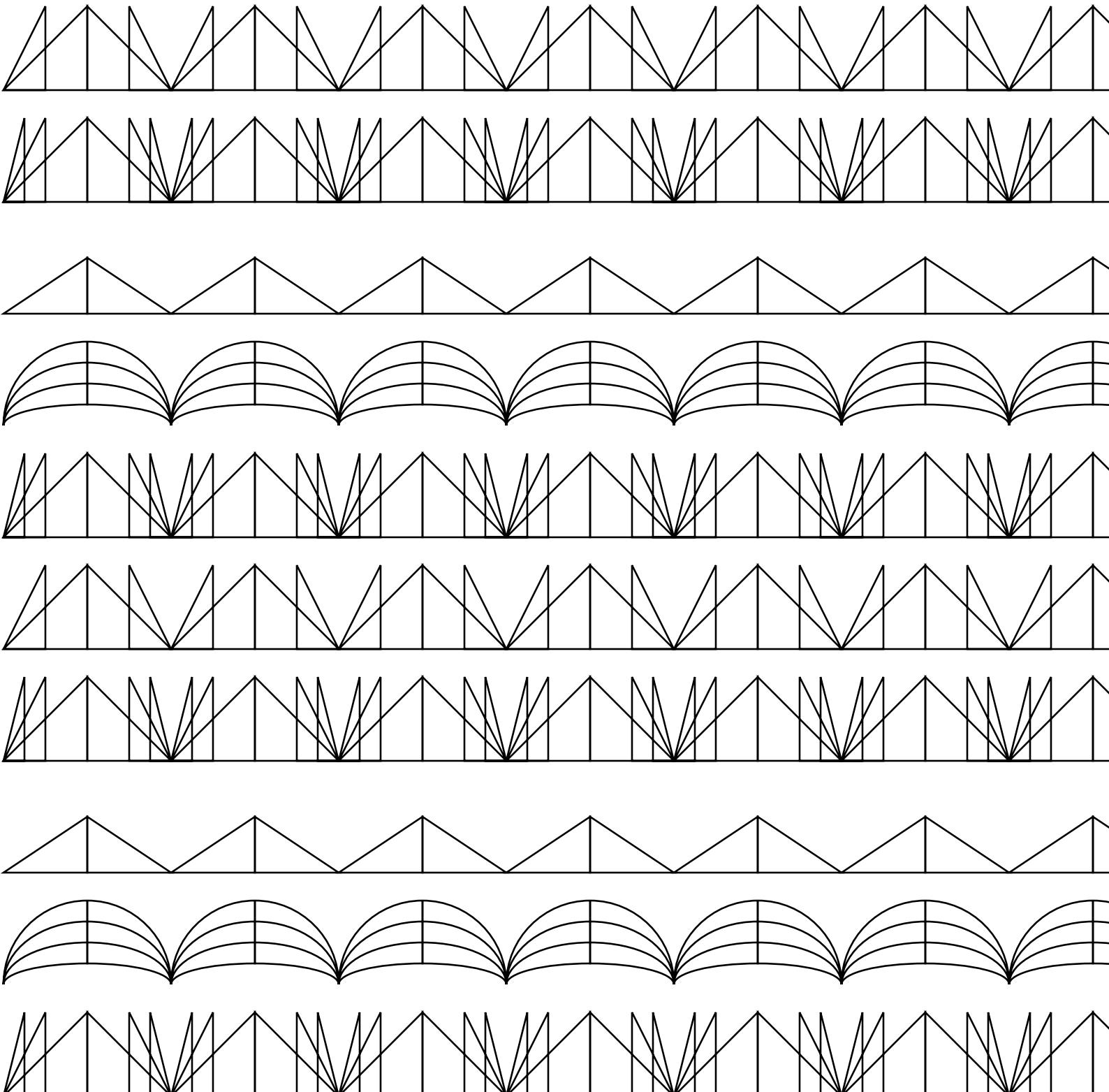
With color, we can reduce the patterns so that translation is their only symmetry.



Why can we do this?

This brings us back to subgroups.

Coloring Challenge: Color the patterns to remove any mirrors so that translation is their only symmetry.



p1m1 frieze patterns: vertical reflection, translation

Our patterns with vertical reflection belong to a symmetry group with translation and vertical reflection.

{vertical reflection, translation}:



Naturally, the group with only translation is a subgroup.

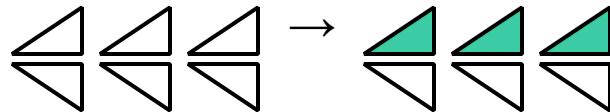
{vertical reflection, translation}:



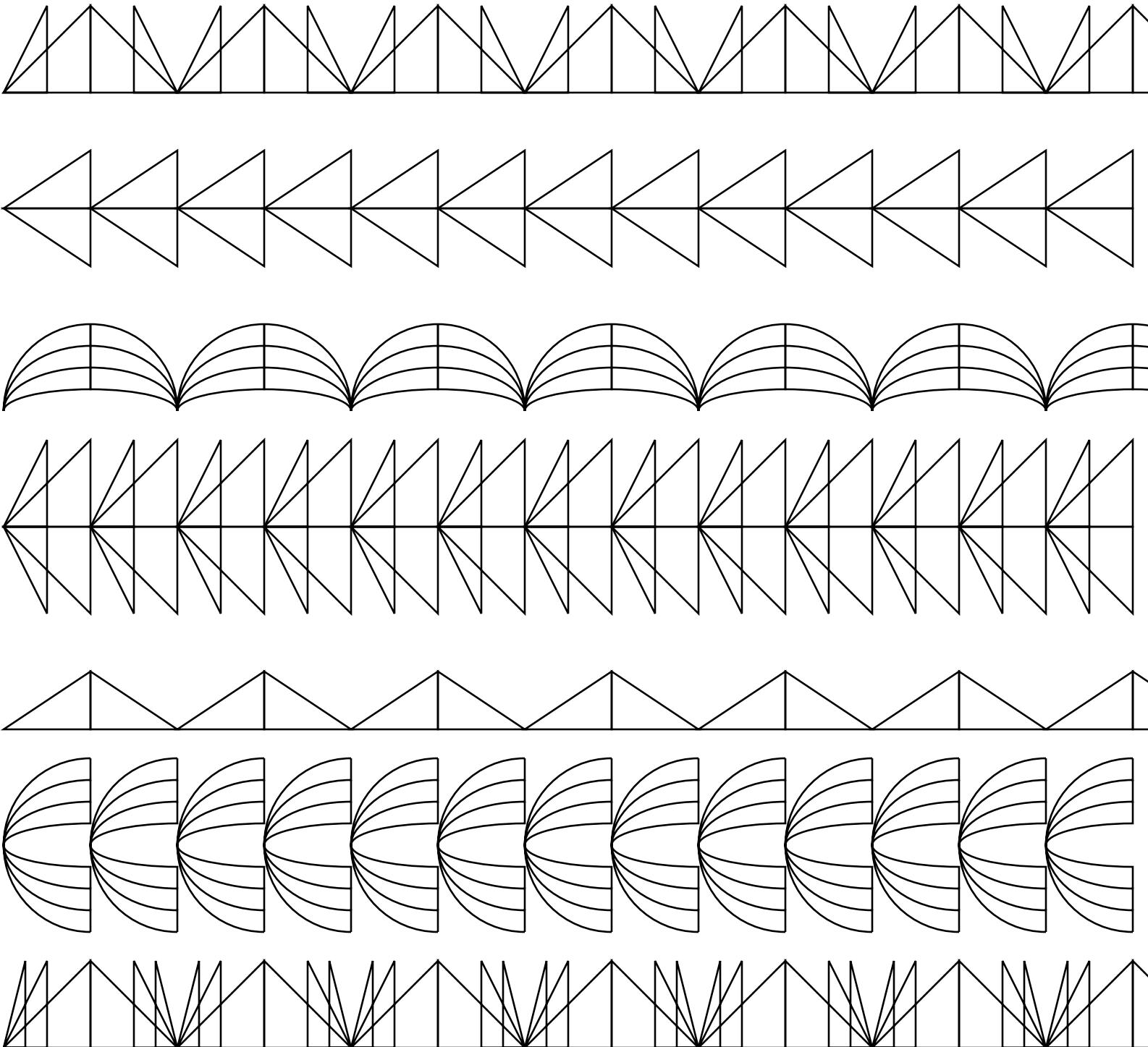
{translation}:



The same goes for our patterns with horizontal reflection. Color can remove their mirrors as well, and reduce them to patterns with only translation.



Coloring Challenge: Can you find all of the mirrors in these patterns and use color to remove them?



frieze patterns with reflections and translation

p2

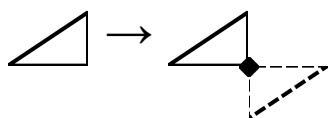
Rotation & Translation

Frieze patterns can also have $\frac{1}{2}$ turns.

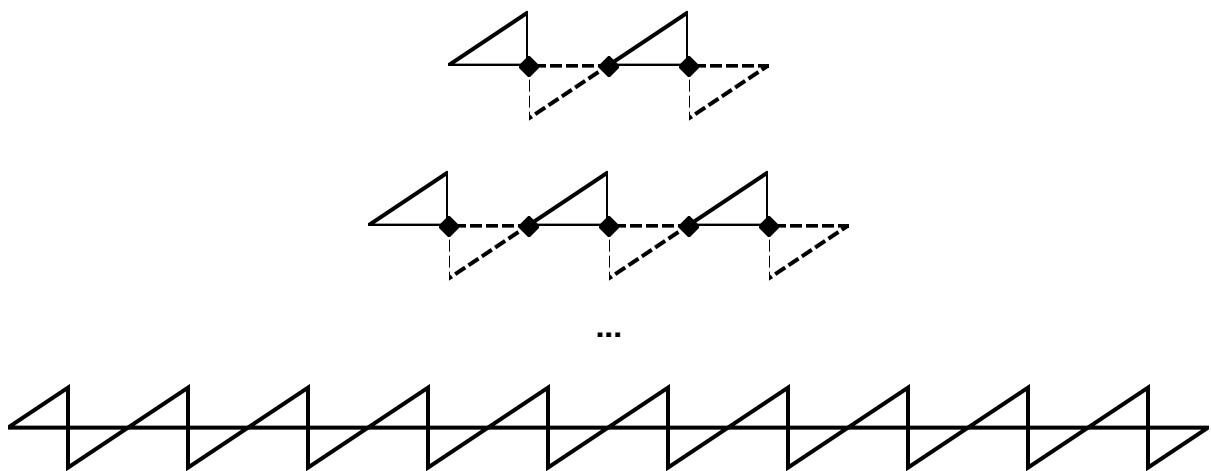
They can be generated by a single piece,



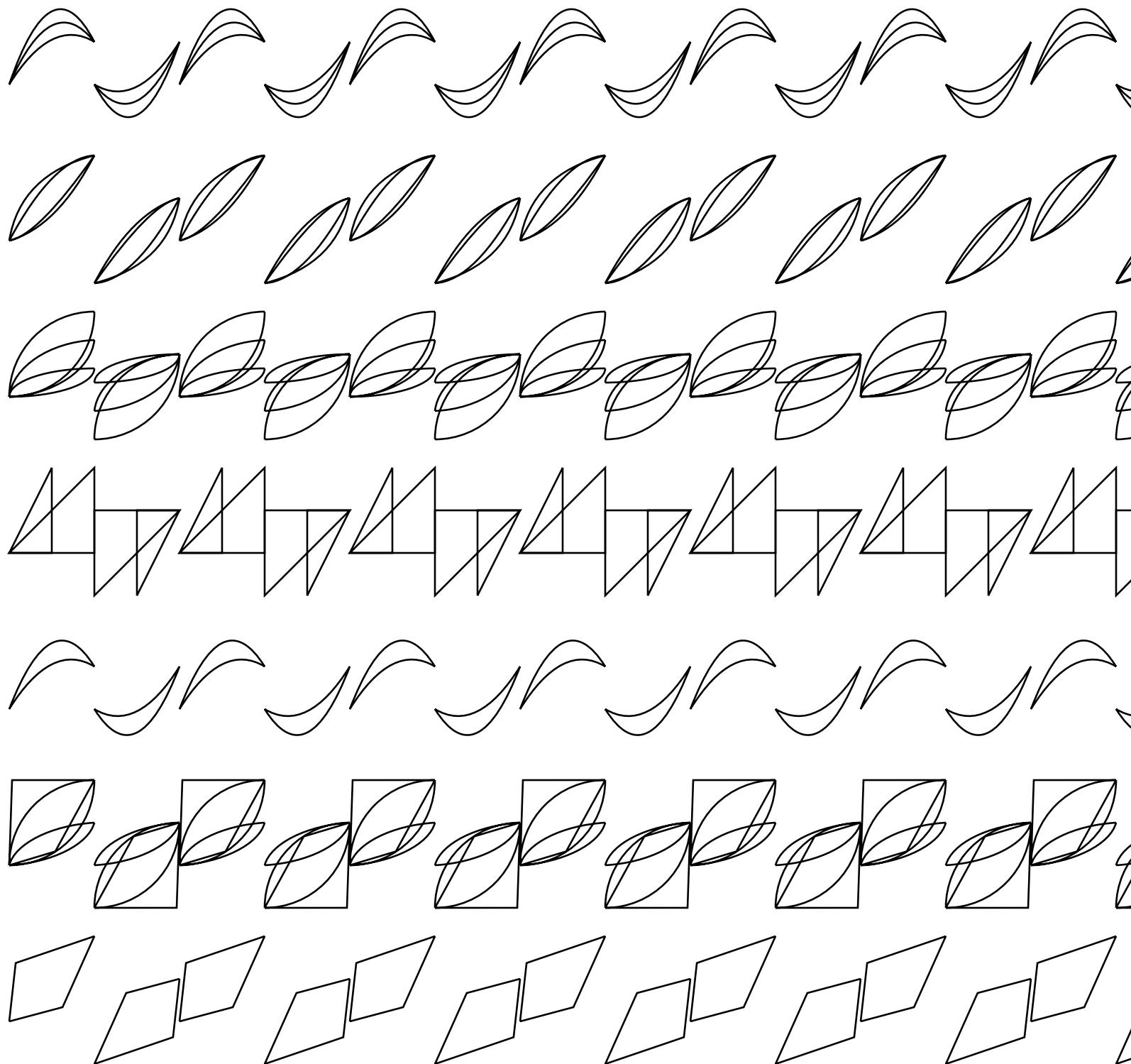
that rotates by a $\frac{1}{2}$ turn around a point,



before translating.



Challenge: Can you find the $\frac{1}{2}$ turns in the following patterns?

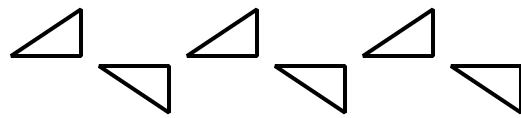


p2 frieze patterns: $\frac{1}{2}$ turn rotation, translation

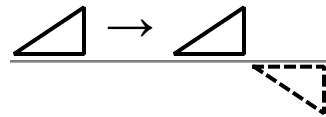
p11g

Glide Reflection & Translation

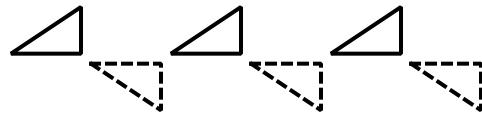
There is another type of symmetry called **glide reflection**.



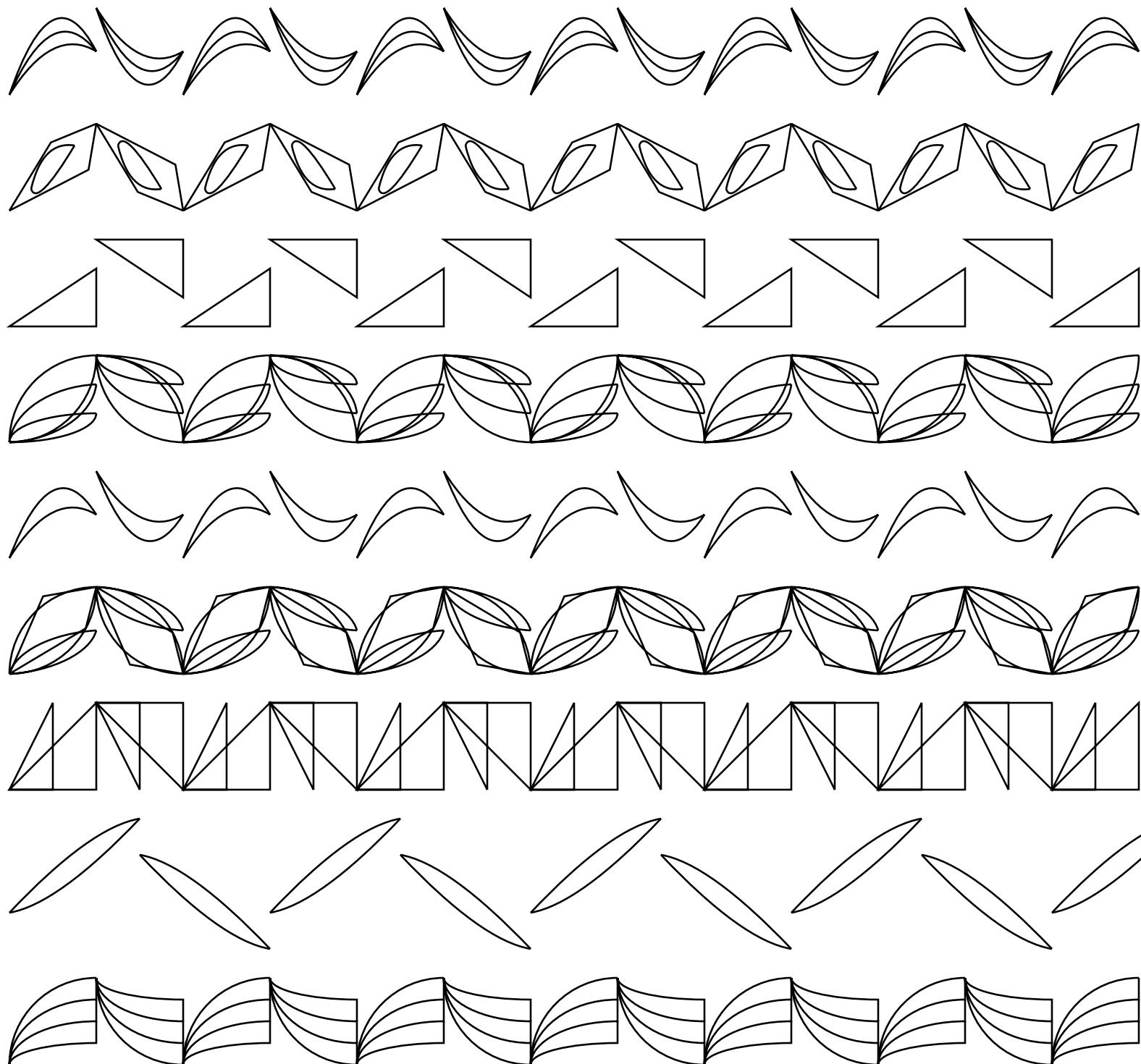
A glide reflection is a transformation that reflects across a mirror line at the same time as translating along it.



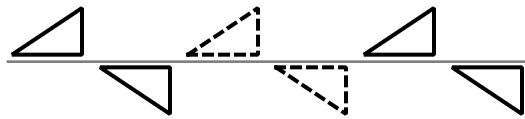
By continuing to translate or glide, a pattern with glide reflection is generated.



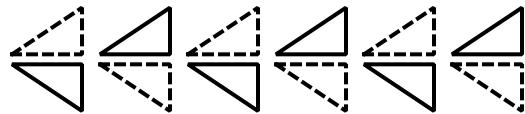
Challenge: Can you see the glide reflections in the following patterns?



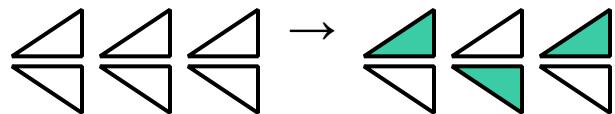
p11g frieze patterns: glide reflection, translation



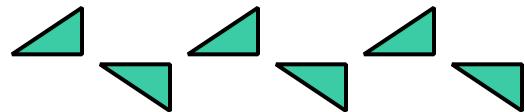
Glide reflections show themselves in other patterns as well. The patterns we generated with horizontal reflection have glide reflection too,



and color can reduce them

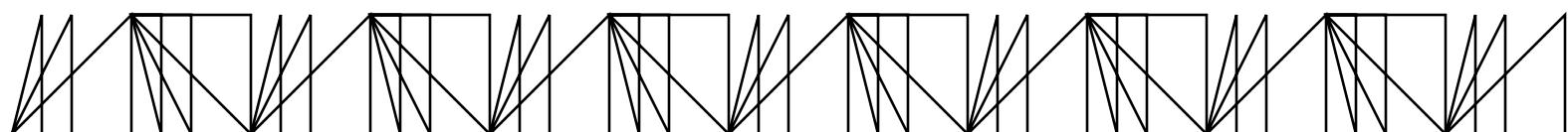
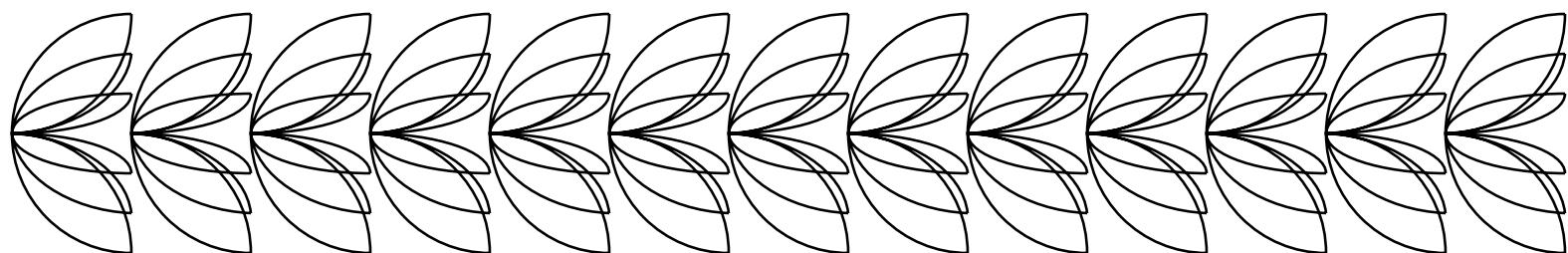
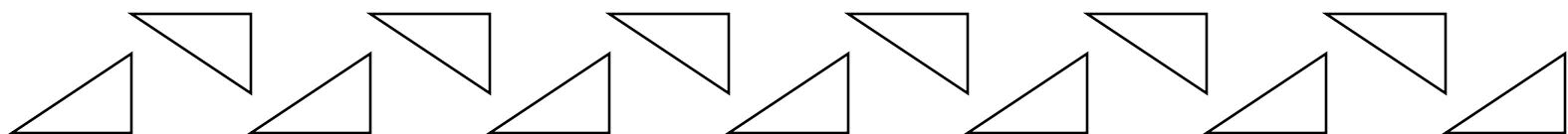
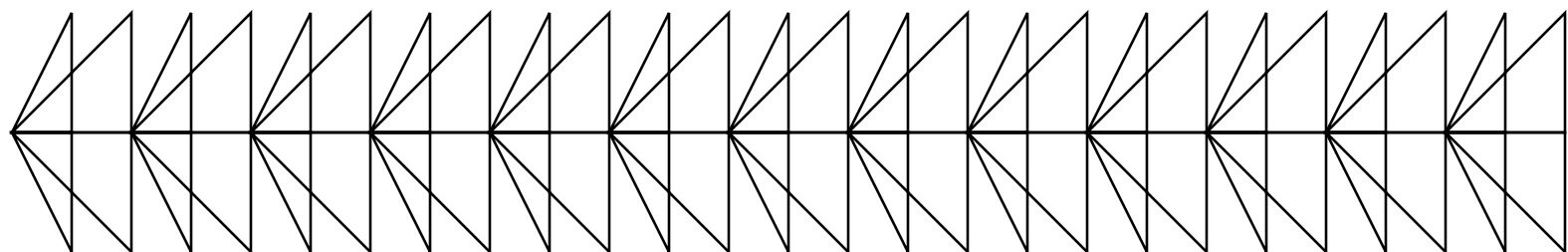
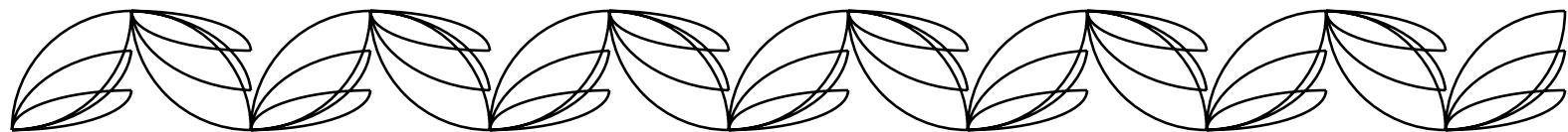
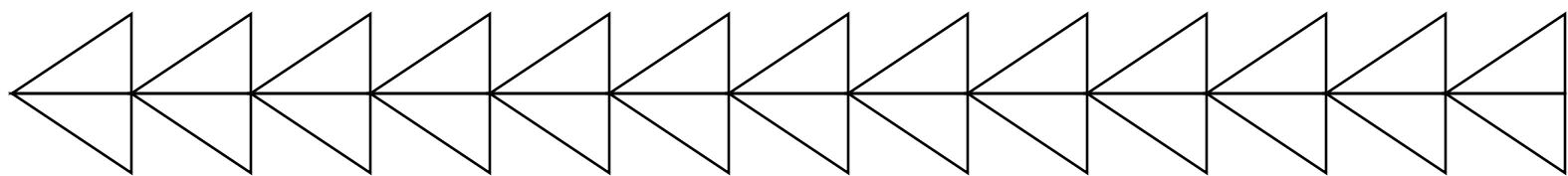
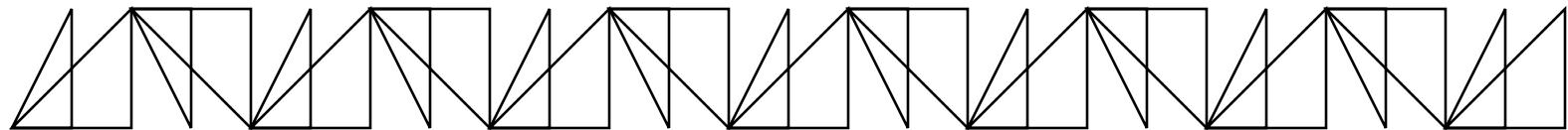
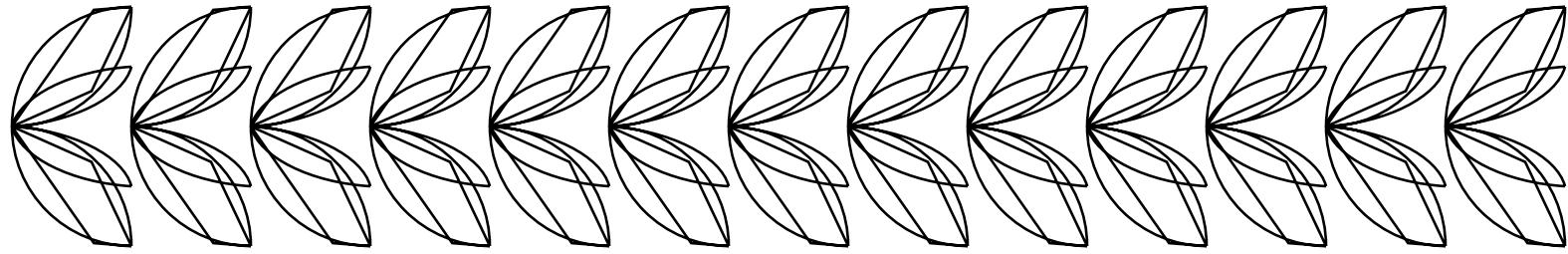


to patterns with glide reflection only.



Coloring Challenge: Use color to reduce the patterns with horizontal reflection so that they only have glide reflection.

Coloring Challenge: Can you add more color to then remove the glide reflections?



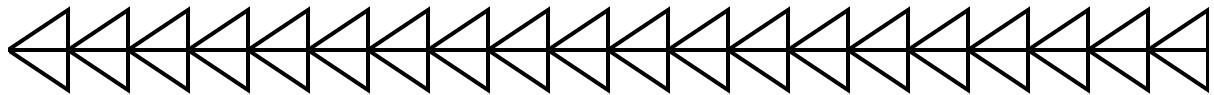
frieze patterns with glide reflections and mirror reflections

We have now seen patterns with each of the frieze group symmetries:

{translation}:



{horizontal reflection, translation}:



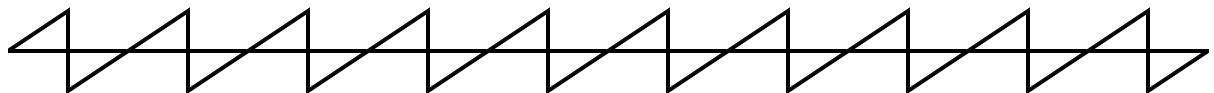
{vertical reflection, translation}:



{glide reflection, translation}:

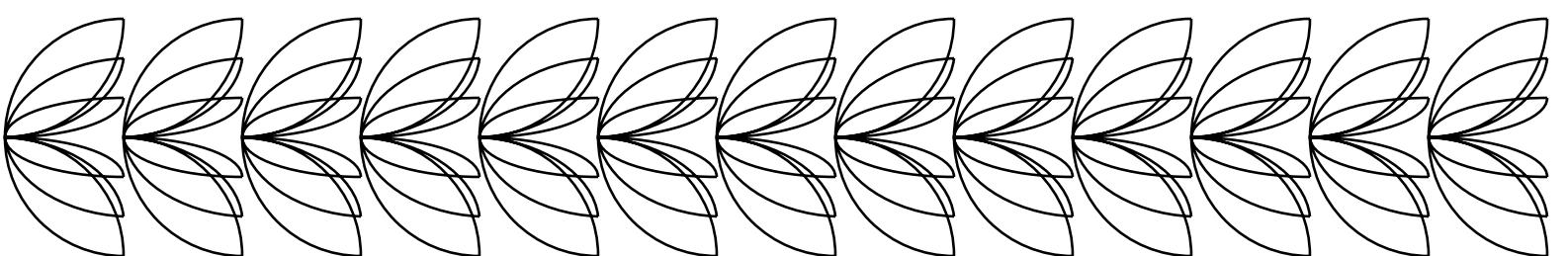
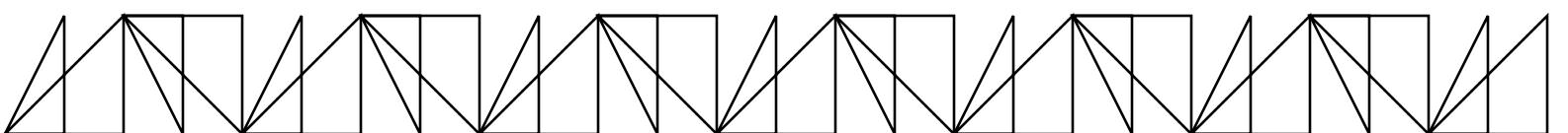
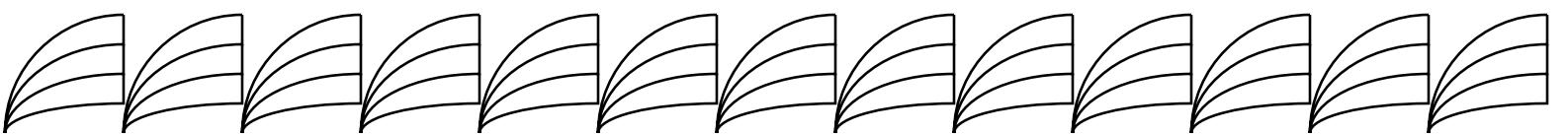
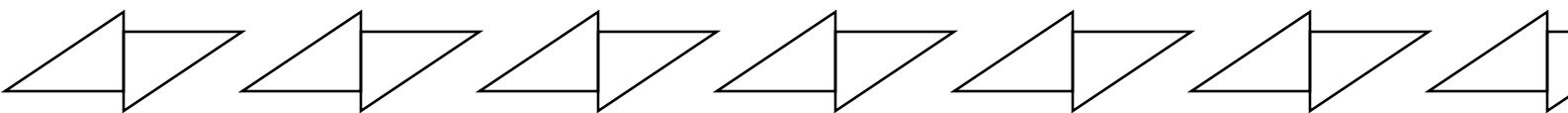
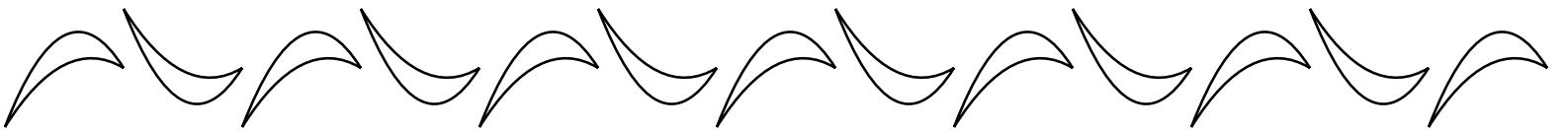
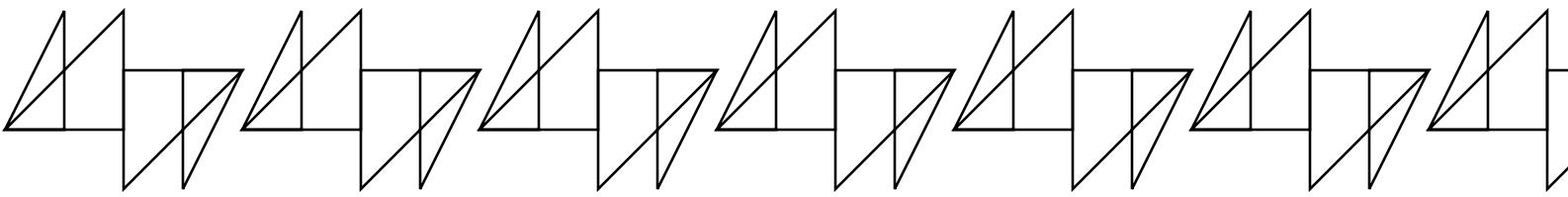


{ $\frac{1}{2}$ turn rotation, translation}:



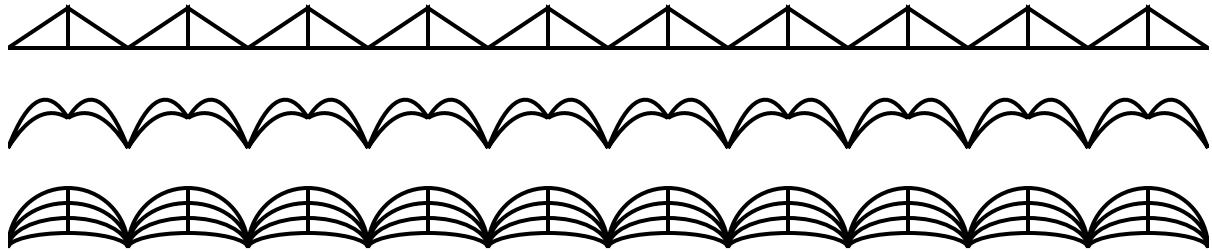
They all have **translation**, and all but the simplest have an additional generator of either **horizontal reflection**, **vertical reflection**, **glide reflection**, or **rotation**.

Let's clarify what we've been talking about and coloring...

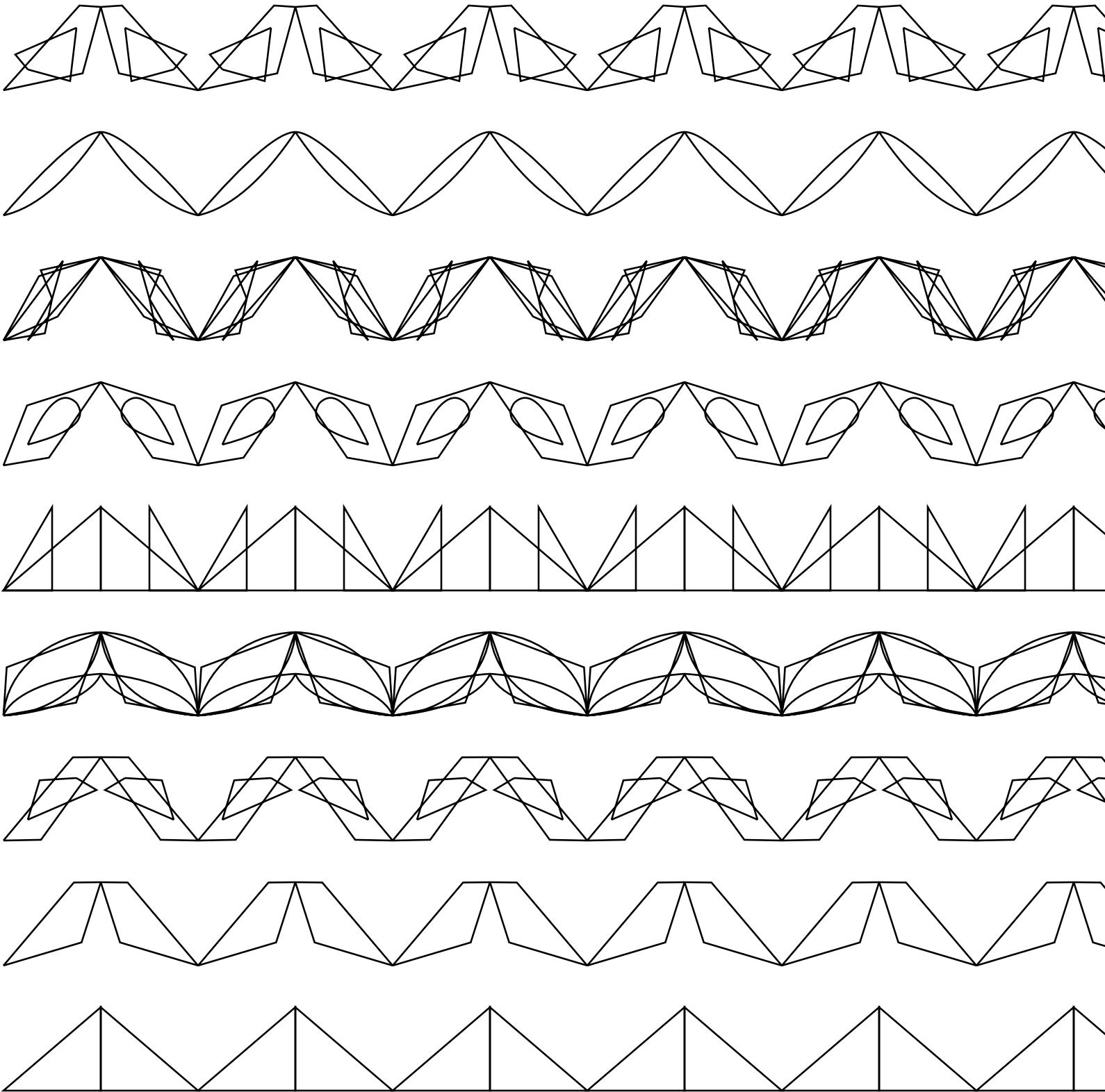


The **frieze patterns** illustrate the **frieze groups**. These groups contain symmetries, not patterns - the patterns just help us see them.

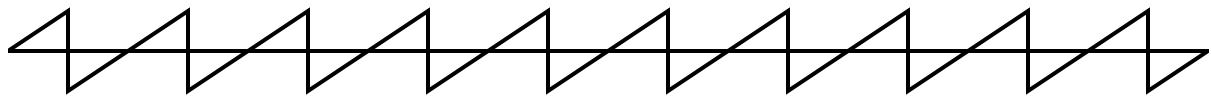
For example, **{vertical reflection, translation}** are symmetries in a group that can be seen with the patterns:



And we can come up with infinitely more pattern designs to illustrate it.



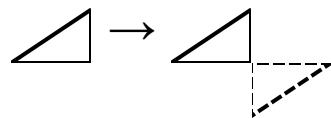
p1m1 frieze patterns: vertical reflection, translation



As long as the pattern has units

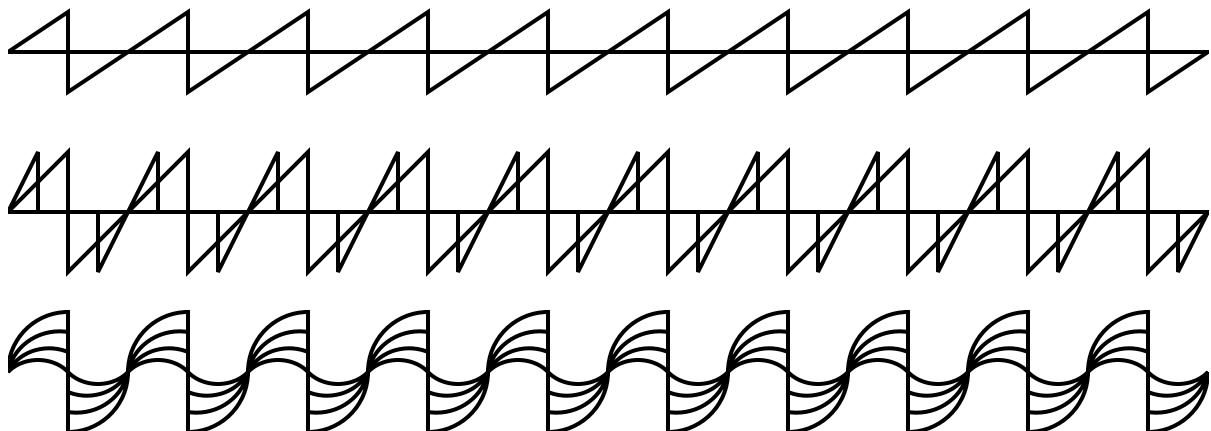


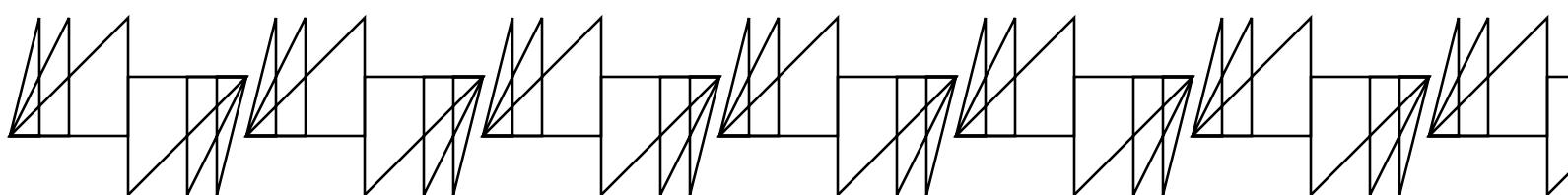
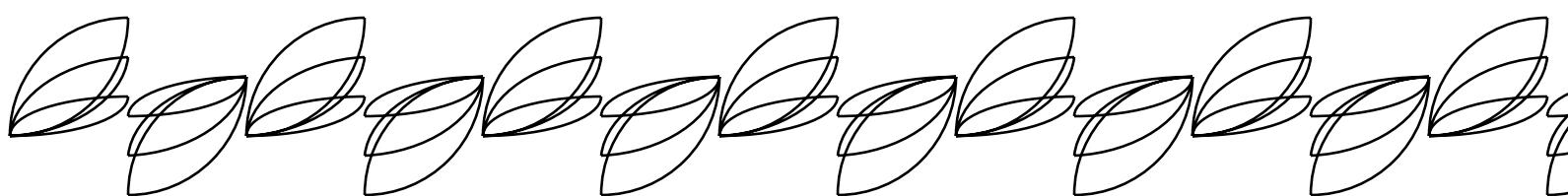
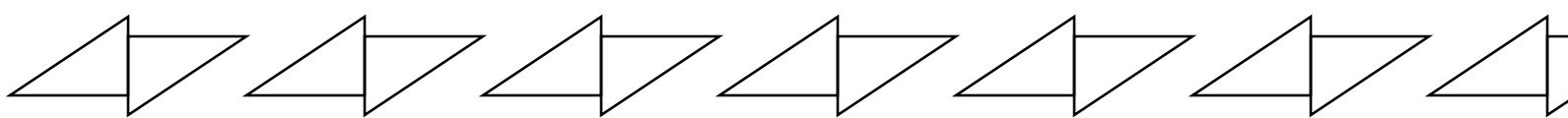
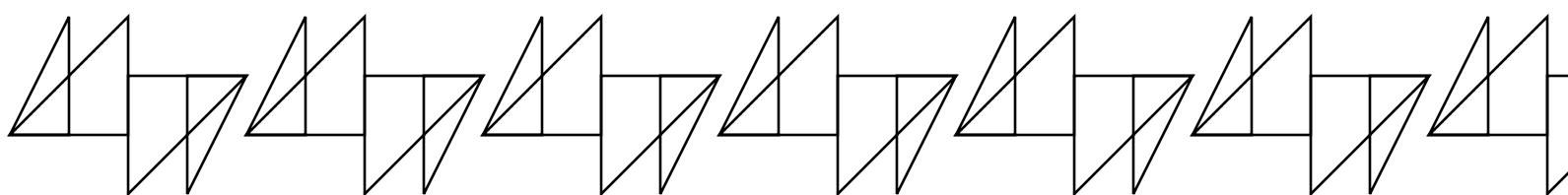
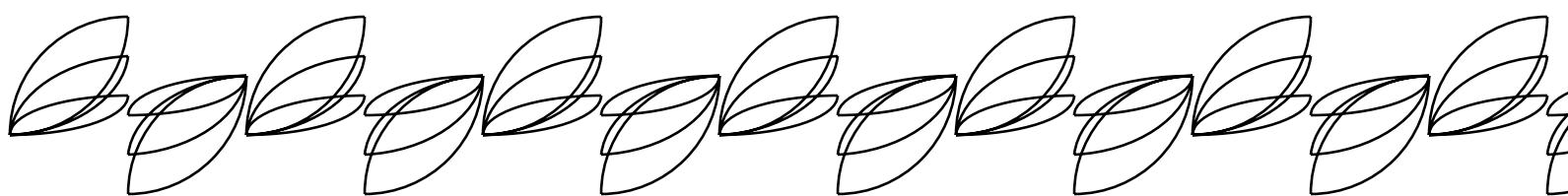
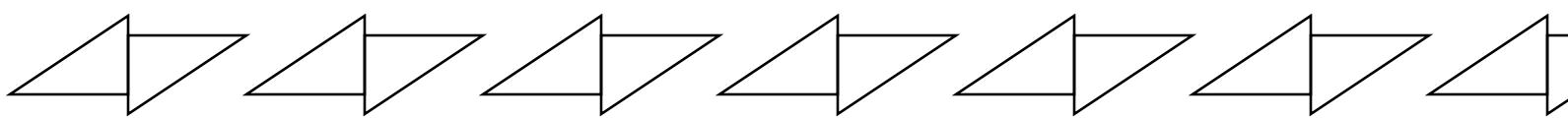
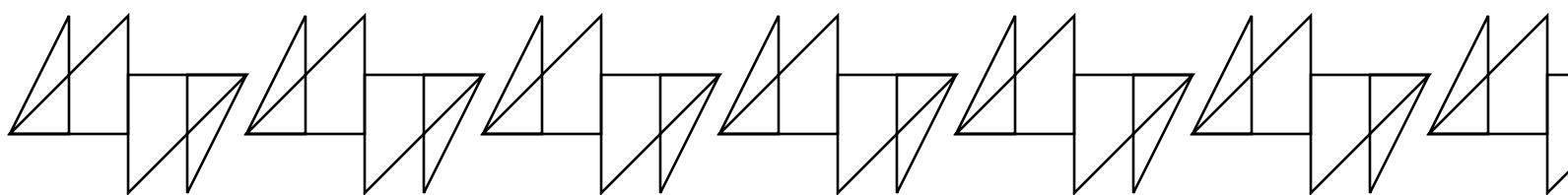
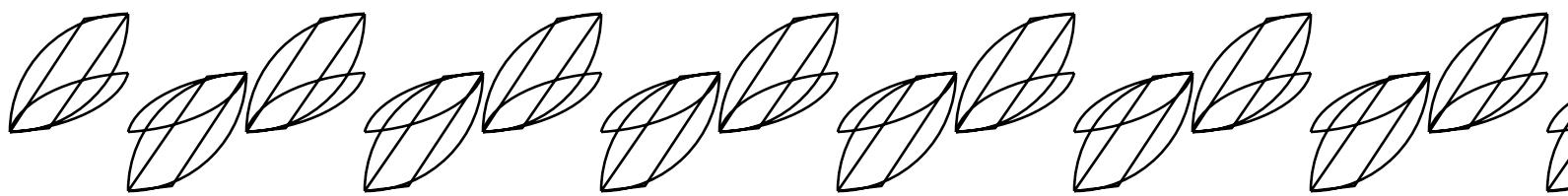
where applying the same symmetries to any unit leaves the entire pattern unchanged



then the pattern illustrates the same group as any others with the same symmetries:

{ $\frac{1}{2}$ turn rotation, translation}





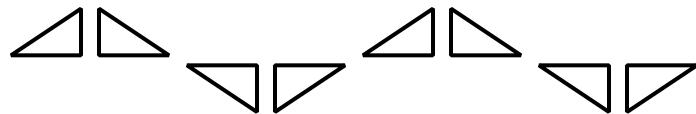
p2 frieze patterns: $\frac{1}{2}$ turn rotation, translation

p2mg

Glide Reflection & Vertical Reflection & Rotation & Translation

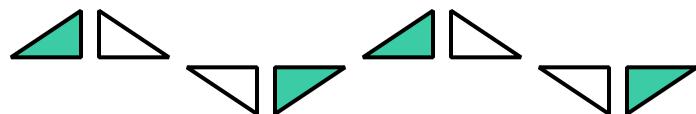
Combining the frieze group symmetries yields even more groups of patterns.
Take the group with

{glide reflection, vertical reflection, rotation, translation}.

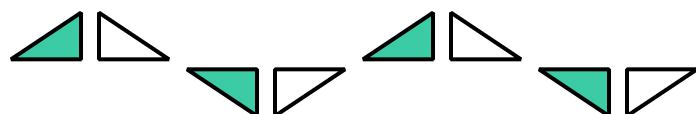


Color can again reduce this pattern to have the same symmetries as the simpler patterns we already colored.

{rotation, translation}:

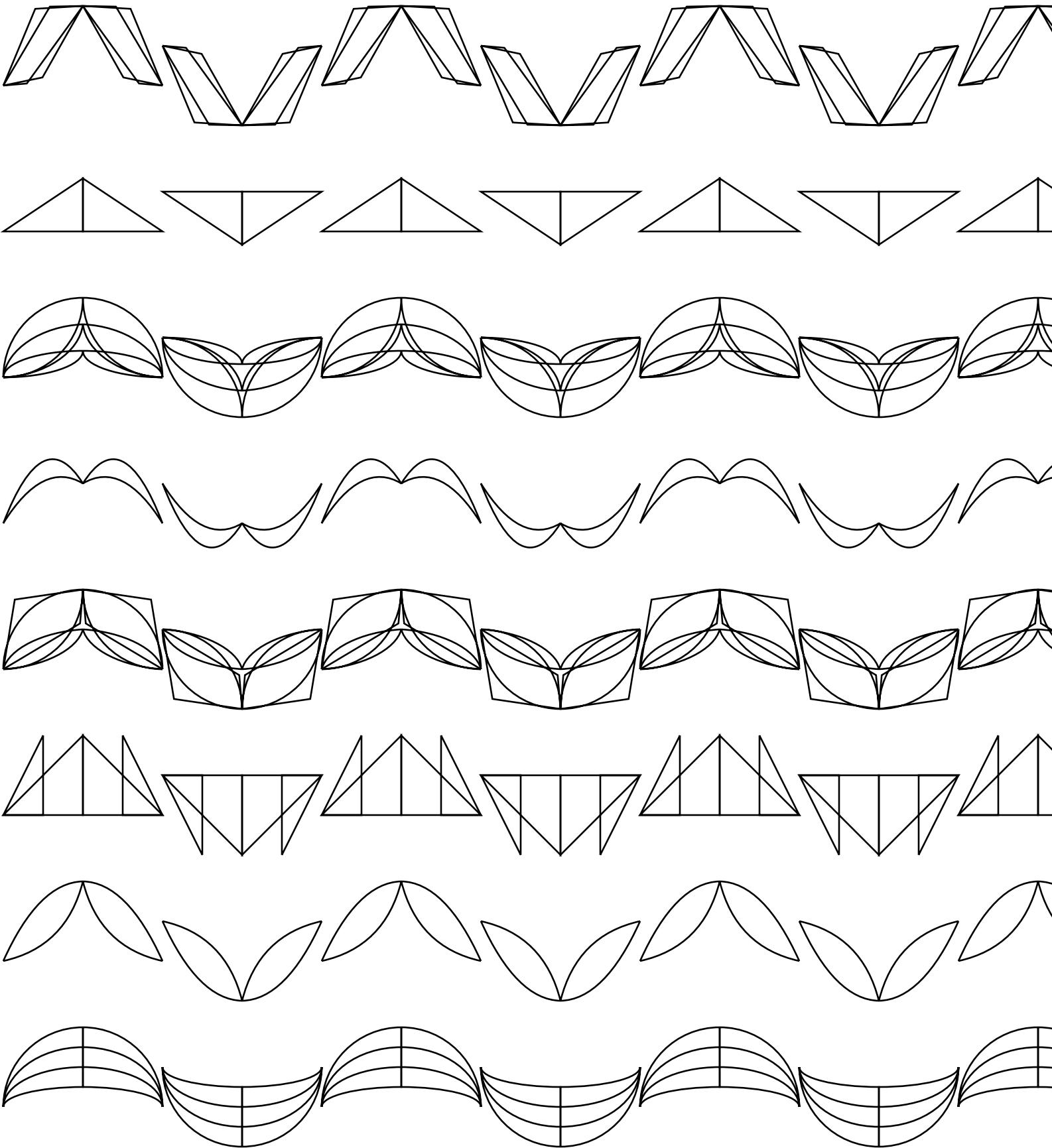


{glide reflection, translation}:



Coloring Challenge: Use color to reduce the amount of symmetry in the patterns so that they only have vertical reflection and translation.





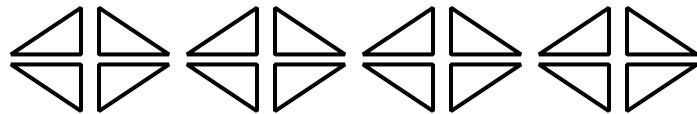
p2mg frieze patterns: glide reflection, horizontal reflection, $\frac{1}{2}$ turn rotation, translation

p2mm

Horizontal Reflection & Vertical Reflection & Rotation & Glide Reflection & Translation

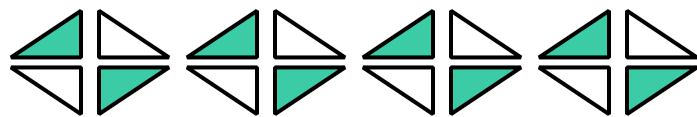
Patterns illustrating the group with all of our symmetries,

{horizontal reflection, vertical reflection, rotation, glide reflection, translation}.

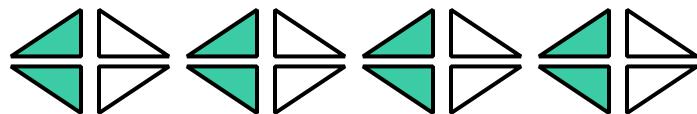


can be reduced to each of the pattern groups we have already seen.

{rotation, translation}:



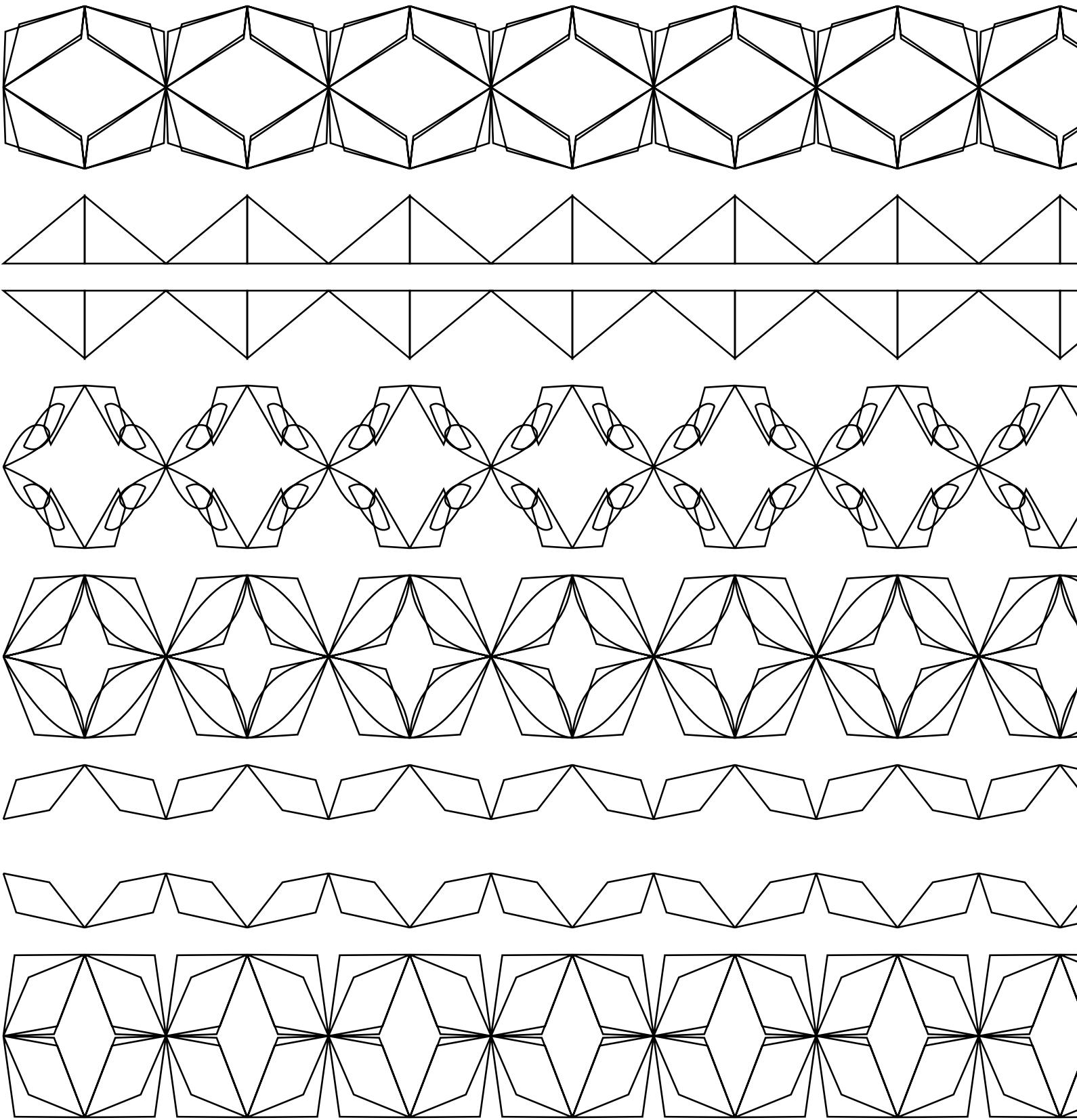
{horizontal reflection, translation}:



You can find the rest.



Coloring Challenge: Use color to transform the patterns of the group {horizontal reflection, vertical reflection, rotation, glide reflection, translation} into each of the simpler pattern groups we have seen.



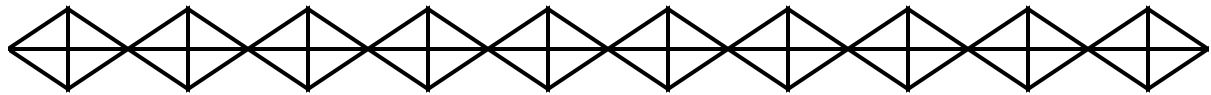
p2mm frieze patterns: horizontal reflection, vertical reflection, glide reflection, $\frac{1}{2}$ turn rotation, translation

We have now colored all 7 Frieze Groups. There are no other groups of patterns that repeat forever in one direction.

Surprised? Then try to generate more by again starting with a single piece.

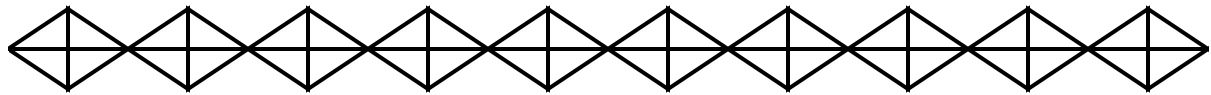


Or use color to reduce a pattern to one with combinations of symmetries that we did not yet see, like **{horizontal reflection, vertical reflection, translation}**.

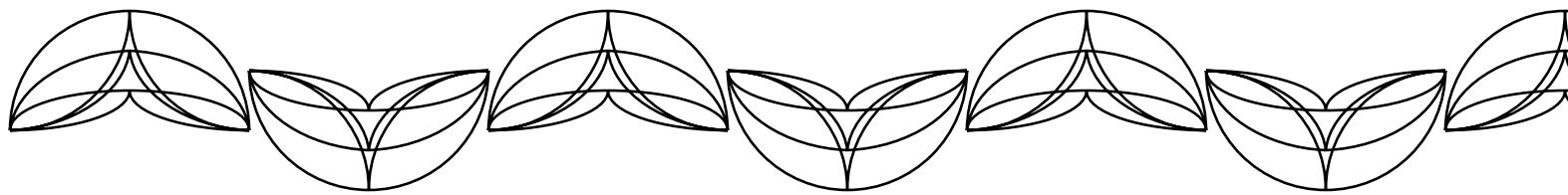
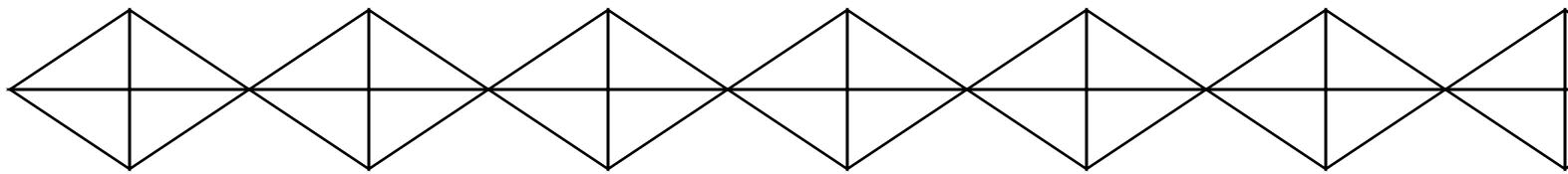
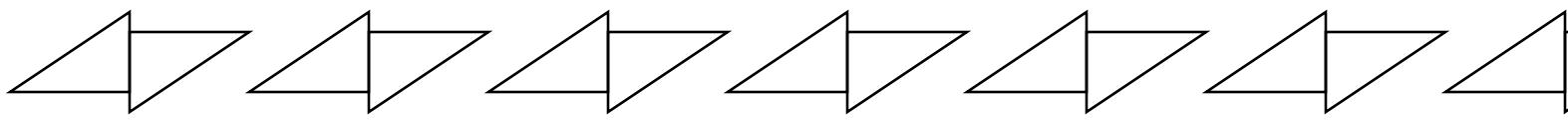
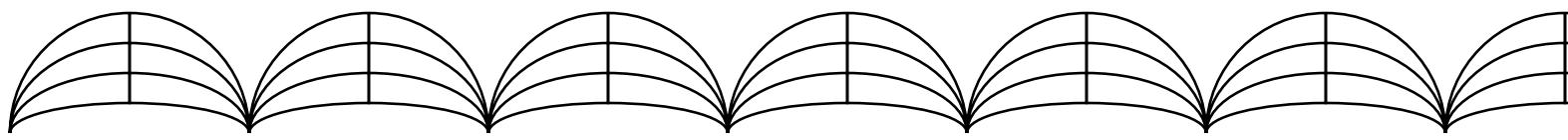
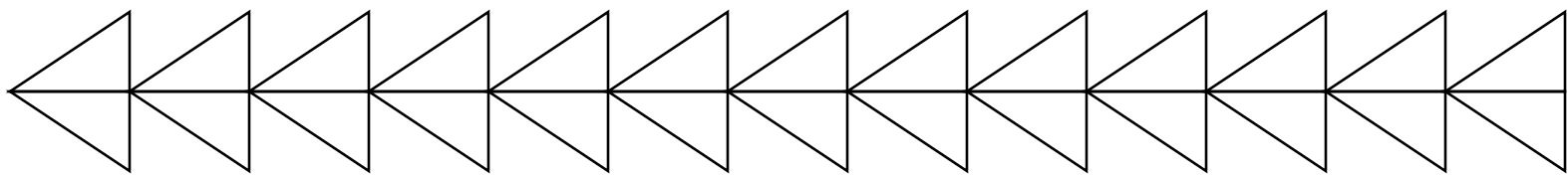
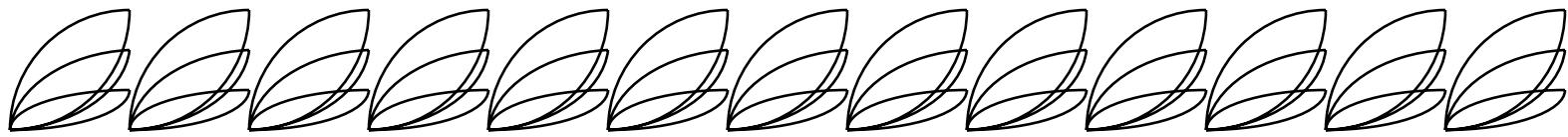


You will have to give up.

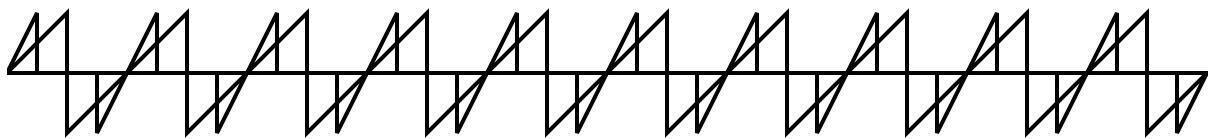
Combining **horizontal reflection** and **vertical reflection** brings about **rotation**. This is just one example of how combining symmetries brings us back to patterns we already have.



Challenge: What happens when you combine glide reflection with rotation?

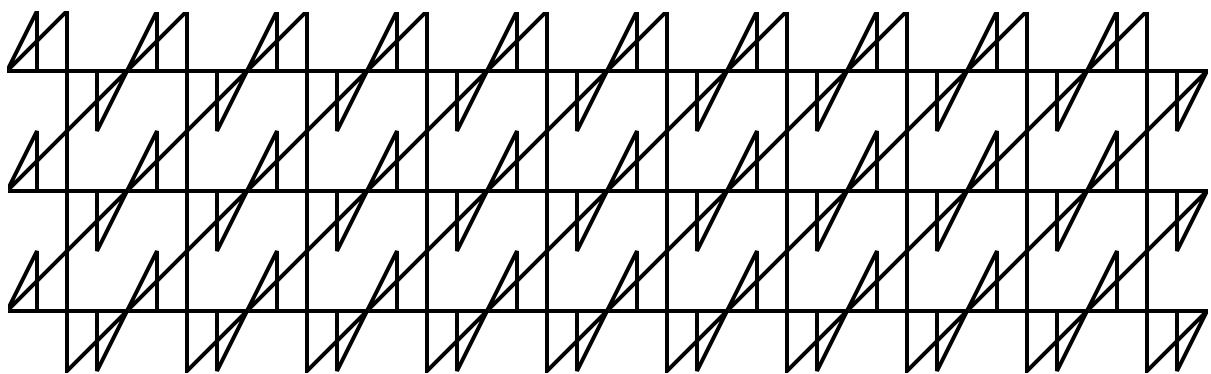


patterns from each of the 7 frieze groups

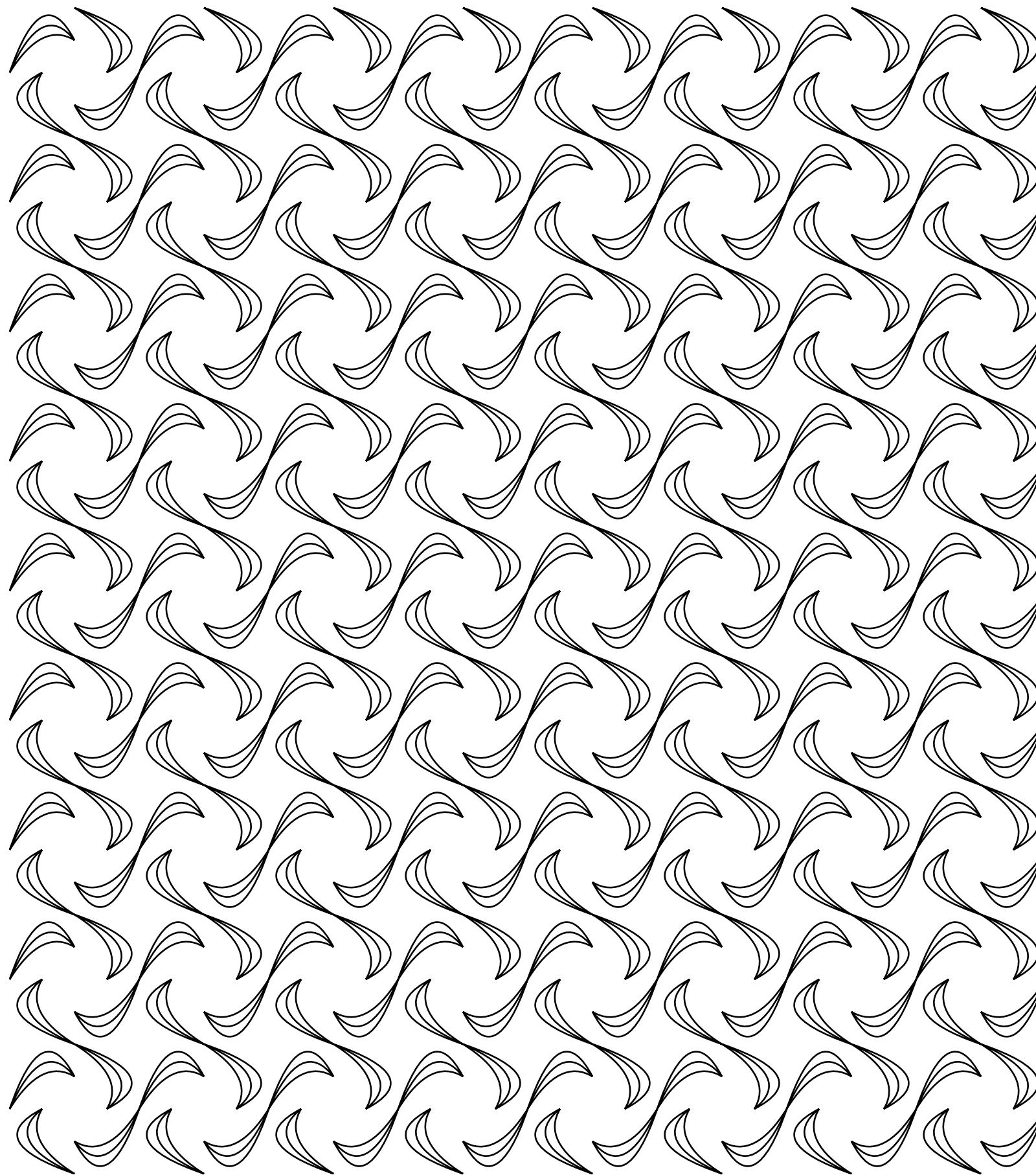


Frieze patterns are limited to repetition along one dimension, but Wallpaper patterns do not have that limit.

When that limit is removed for the Wallpaper Groups, the number of possible patterns and amount of symmetry within them grows beyond what we have colored.



Challenge: Can you find all of the directions in which the pattern repeats?

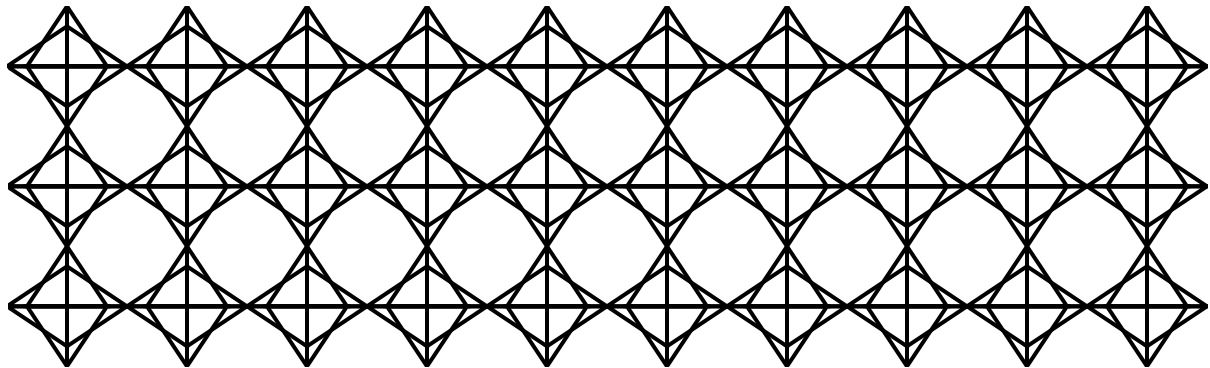


wallpaper pattern with $\frac{1}{4}$ turn rotation, $\frac{1}{2}$ turn rotation, and translation

WALLPAPER GROUPS

Wallpaper patterns repeat along 2 dimensions, and with more dimensions come more symmetries.

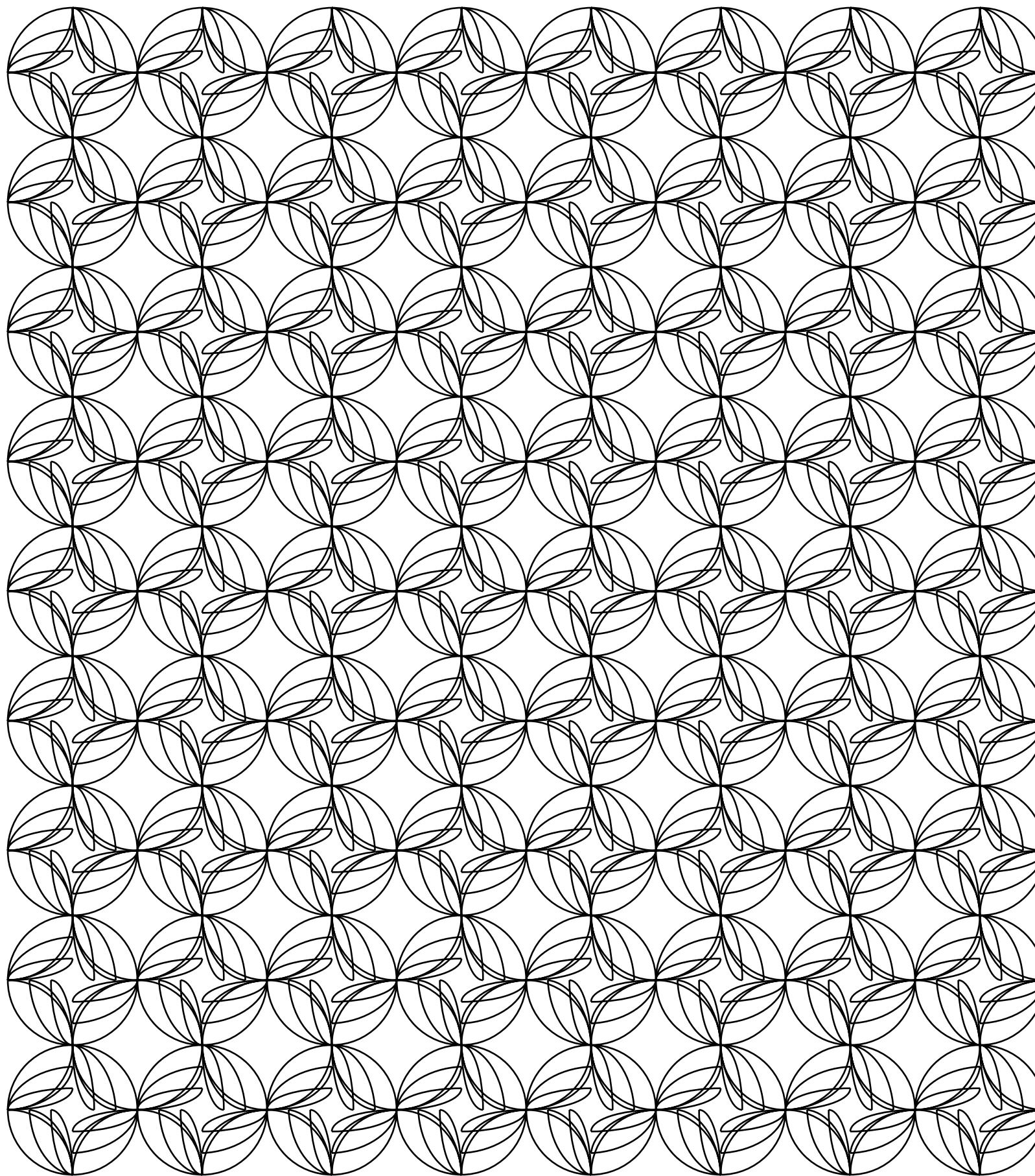
All of these 2-dimensional patterns can be classified by their symmetries into the 17 **wallpaper groups**, and we will color through each of them.



But first, again see how a page cuts off our patterns when really they continue repeating infinitely beyond the page borders...

Challenge: Can you see the repetitions in the pattern?

Challenge: Can you extend your imagination to see the pattern repeat infinitely beyond the page borders?

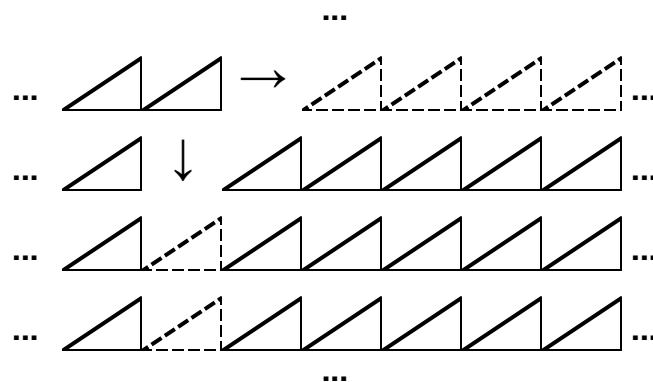


wallpaper pattern

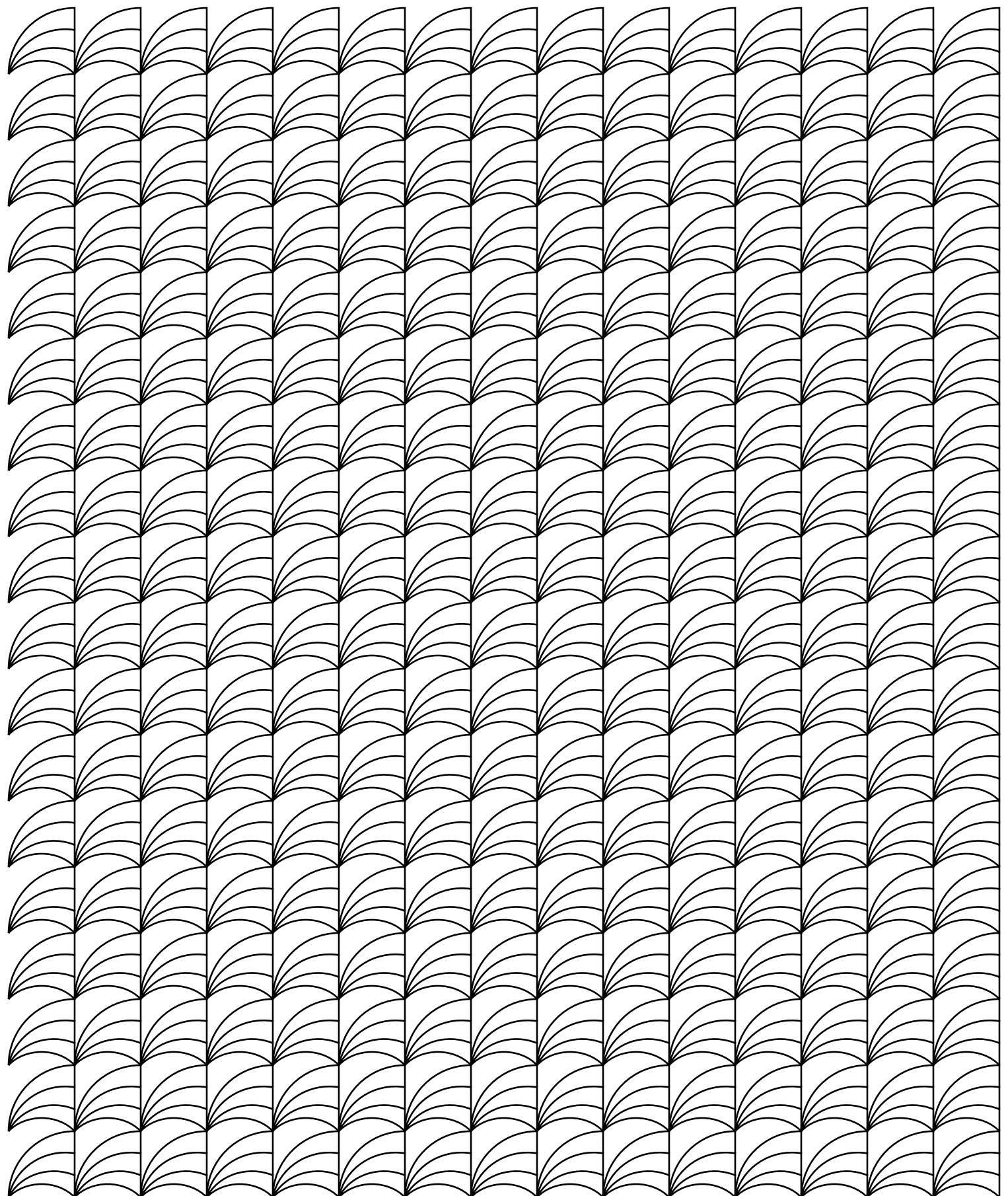
Like the frieze patterns, all wallpaper patterns have an infinite number of translations that can be seen by focusing on a single piece that shifts over.



But this time the translations are not just in one direction.

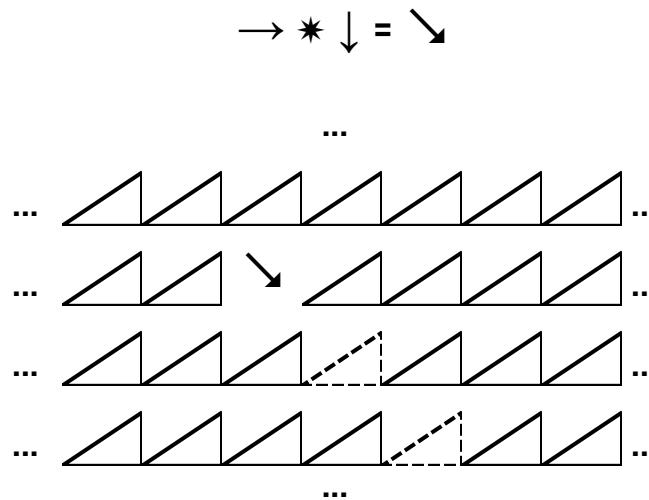


Again we can see how we can shift the entire pattern with these translations, and because each piece is followed by infinitely more pieces, the entire pattern is left unchanged.



wallpaper pattern

These translations in different directions are symmetries of our wallpaper patterns and we can combine them to see such translations in even more directions.

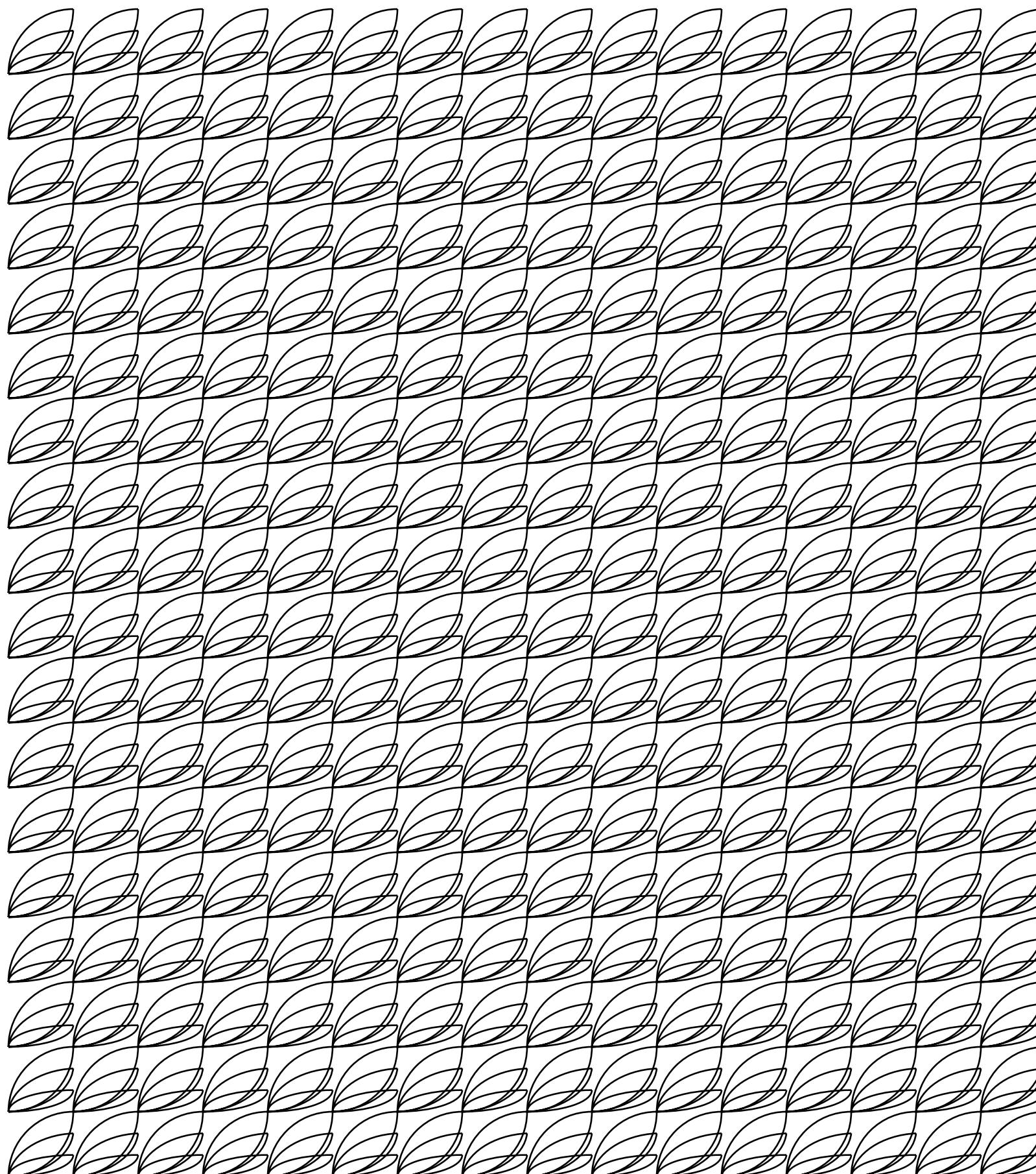


Any 2 different directions of translation can be combined with each other or with themselves, in any number of ways, again and again, to produce more,

$$\rightarrow * \rightarrow * \downarrow * \downarrow * \downarrow \dots$$

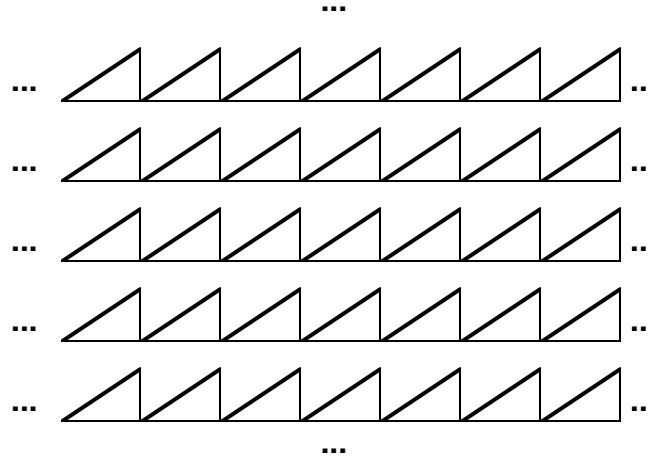
and in this way these 2 translations can be the generators for a group of translations that span across all directions of the wallpaper patterns.

Challenge: Can you see all of the directions of translation in the pattern?

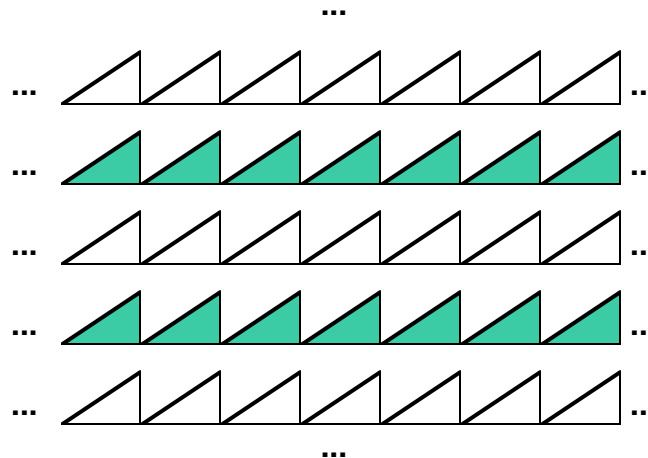


wallpaper pattern

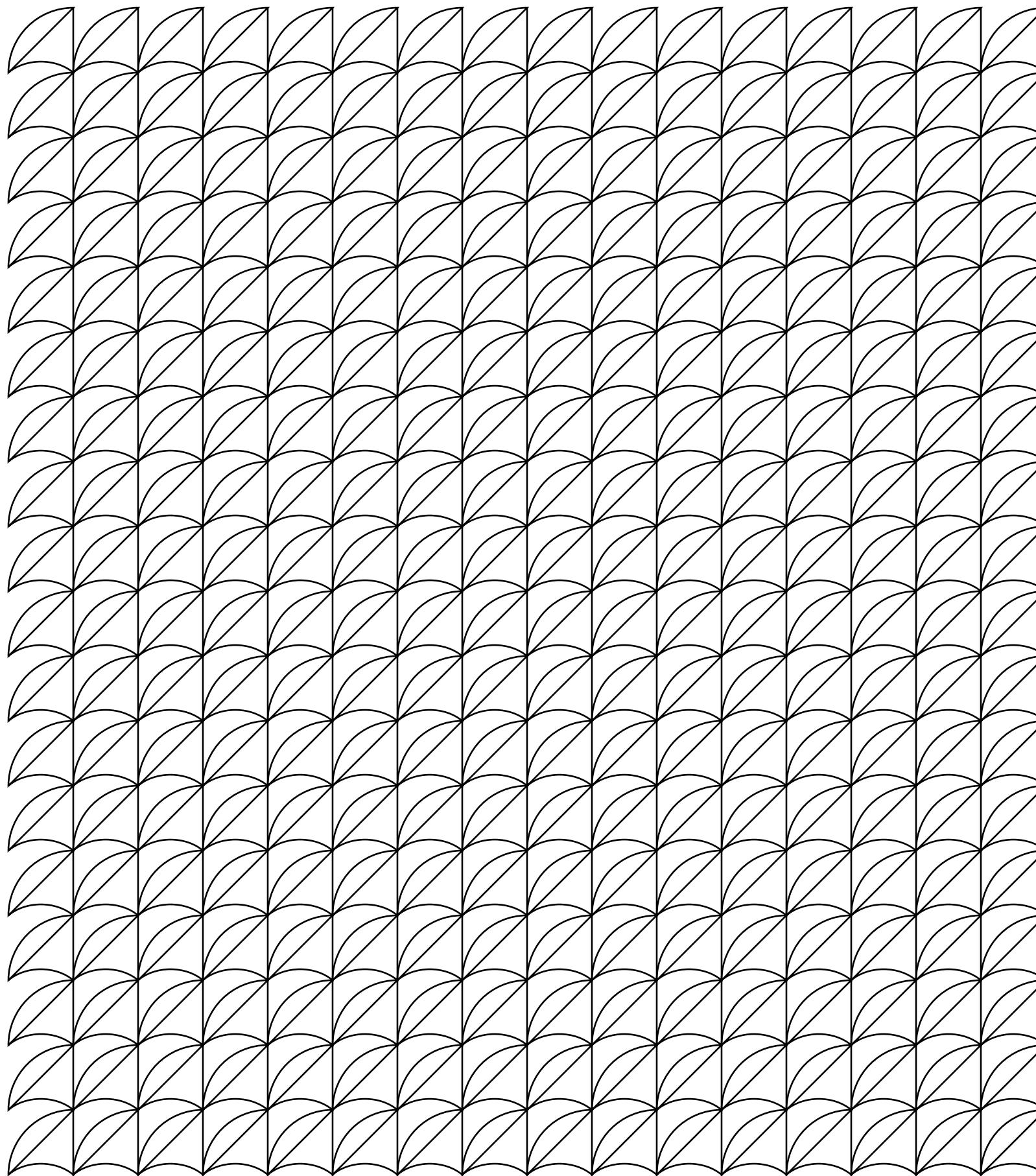
We can again use color to alter our patterns



such as doubling the shortest distance a pattern can translate vertically.



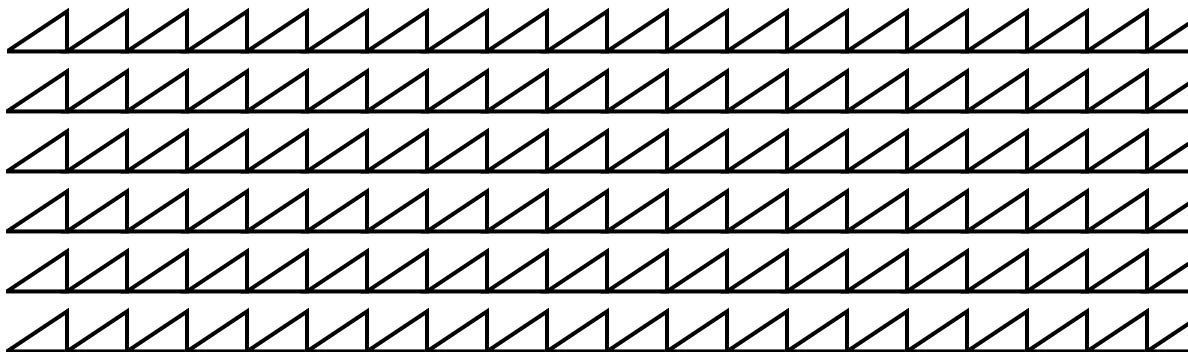
Coloring Challenge: Color the pattern so that the shortest possible distance of horizontal translation is tripled.



wallpaper pattern

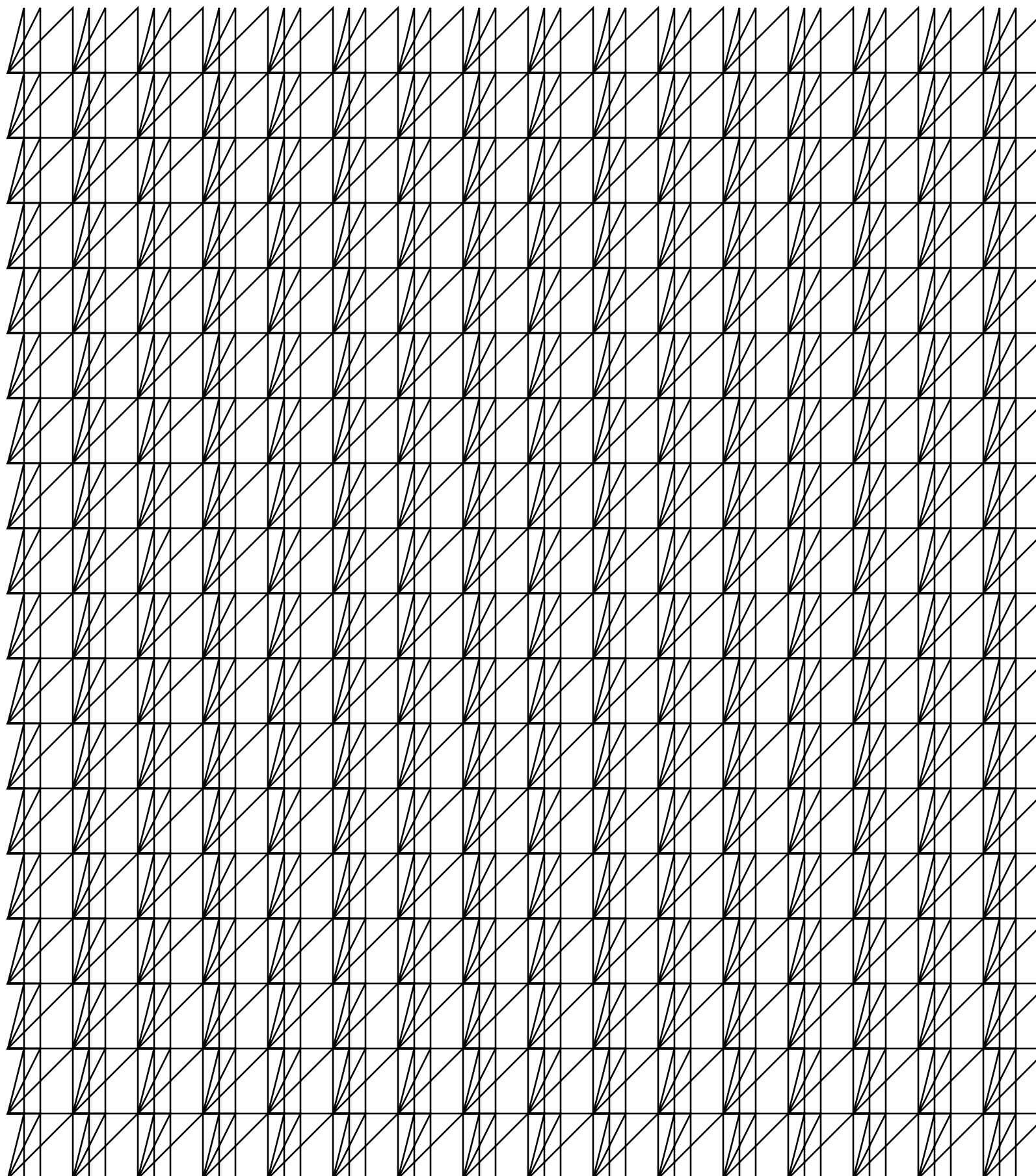
p1

The simplest group of wallpaper patterns has translation as the only symmetry.



All wallpaper patterns have many directions of translation - they get more interesting when we consider groups of more complex symmetries...

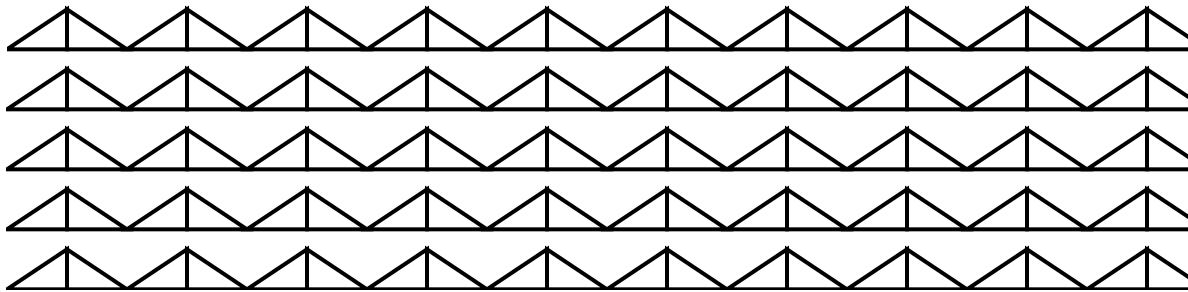
Coloring Challenge: Color the pattern so that the shortest possible translation is in a diagonal direction.



p1 wallpaper pattern: translation

pm

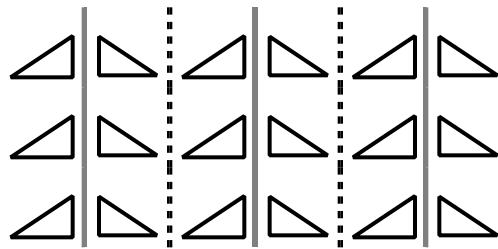
This pattern group has mirror reflections.



We can see a single piece of the pattern reflect across any one of its mirrors

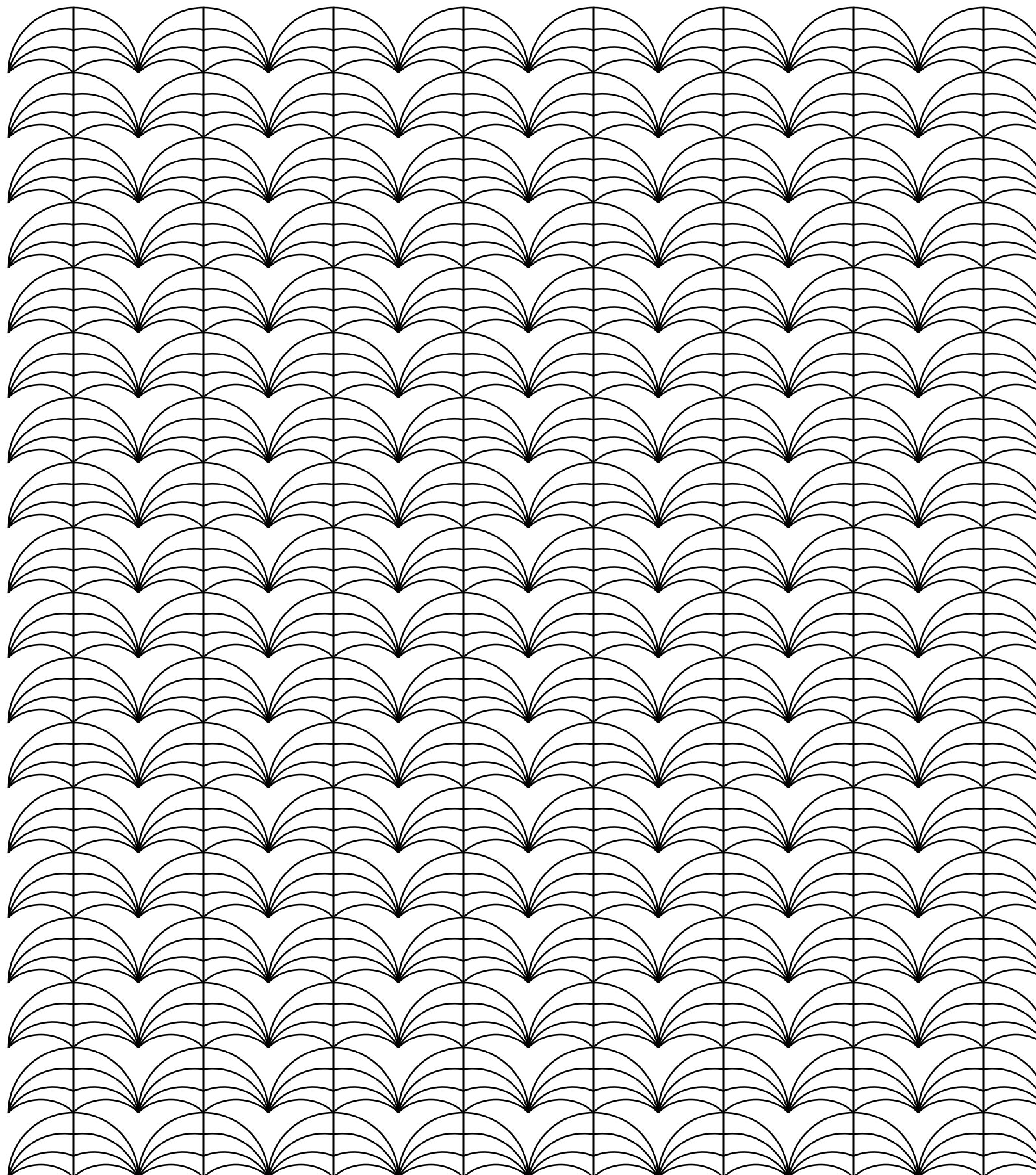


or see the entire pattern reflect across them.



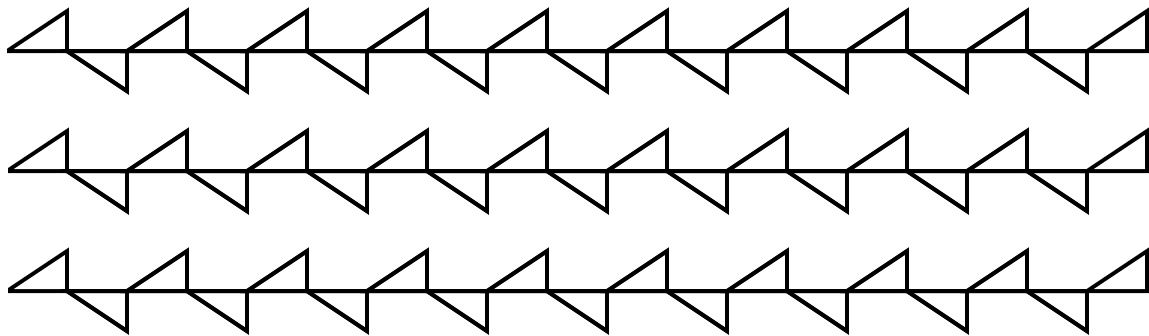
Challenge: Can you see the pattern's different parallel vertical mirrors?

Coloring challenge: Color the pattern to remove its mirrors.

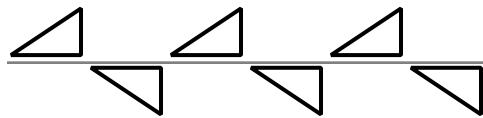


pm wallpaper pattern: mirror reflection, translation

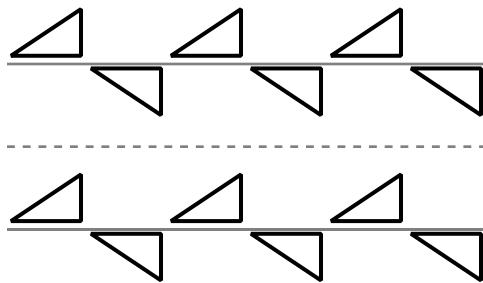
pg



This pattern group has glide reflections



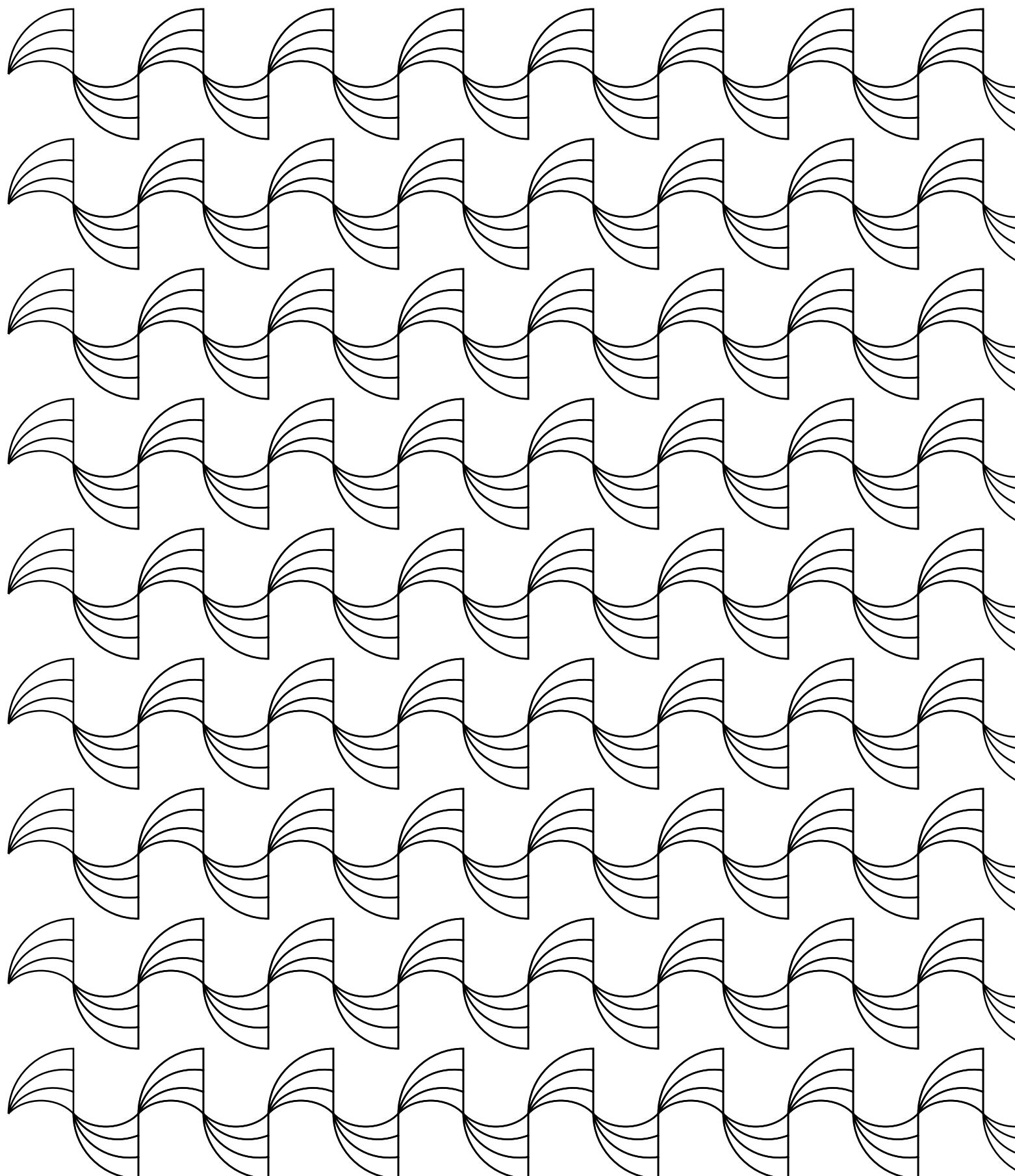
along parallel axes.



These axes shift over with its translations, because the combination of the glide reflection and translation symmetries in this pattern group must also be in the group.

Challenge: Can you see the different parallel axes of the glide reflections?

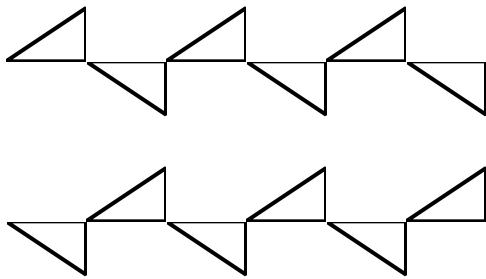
Coloring Challenge: Color the pattern to remove the glide reflection so the only symmetry is translation.



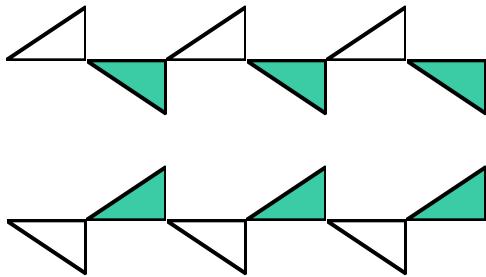
pg wallpaper pattern: glide reflection, translation

cm

This pattern has parallel axes of both glide and mirror reflections,



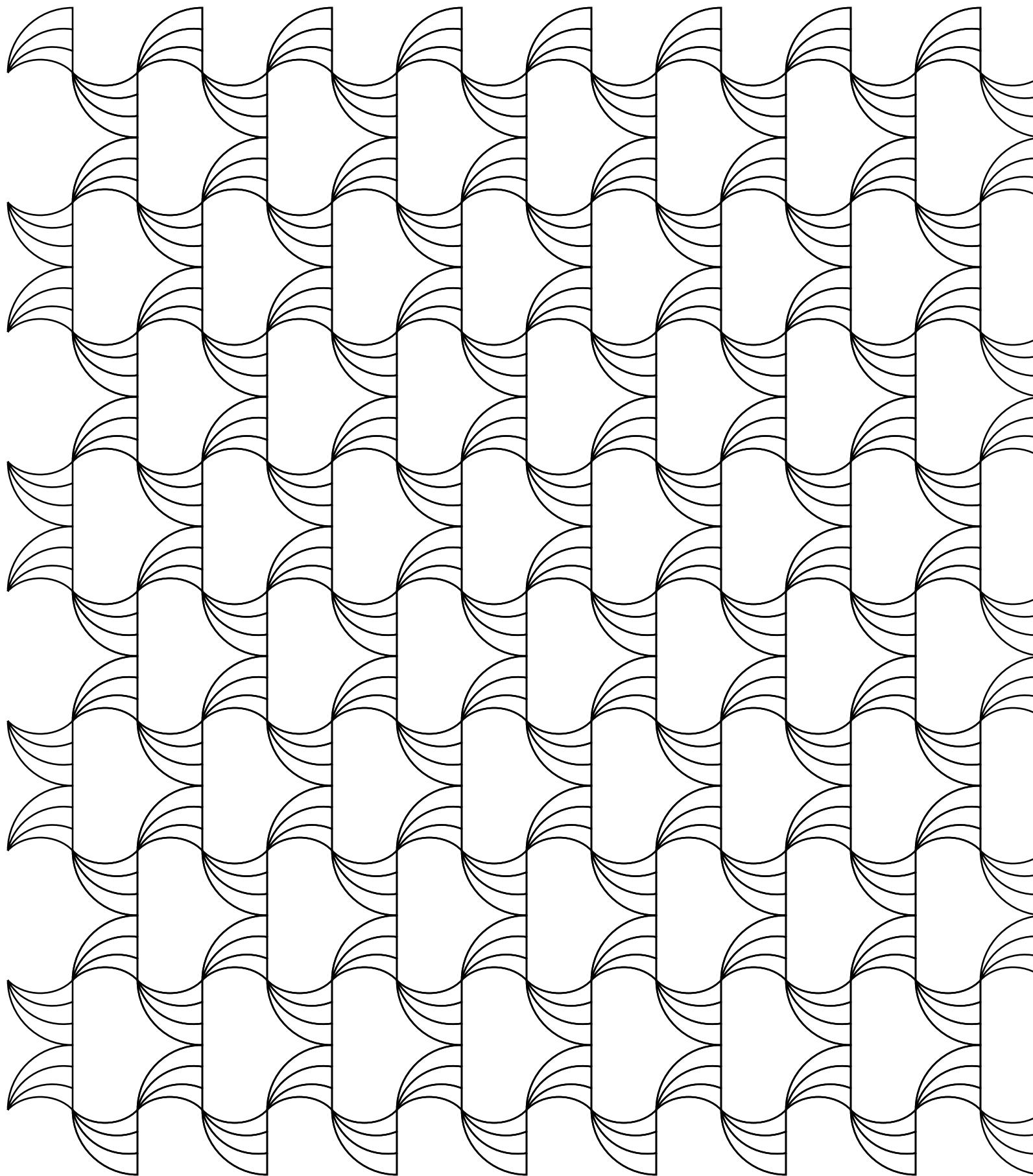
and we can again use color to reduce it to simpler patterns we previously saw



such as by coloring away its glide reflections while keeping its mirror reflections.

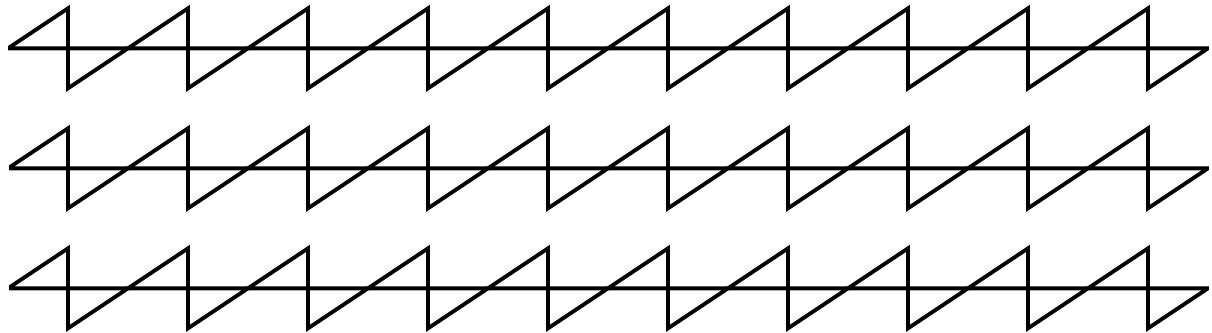
Challenge: Can you see the different parallel axes of glide reflection?

Coloring Challenge: Color the pattern to remove some glide reflections while keeping others.

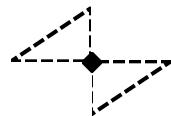


cm wallpaper pattern: mirror reflection, glide reflection, translation

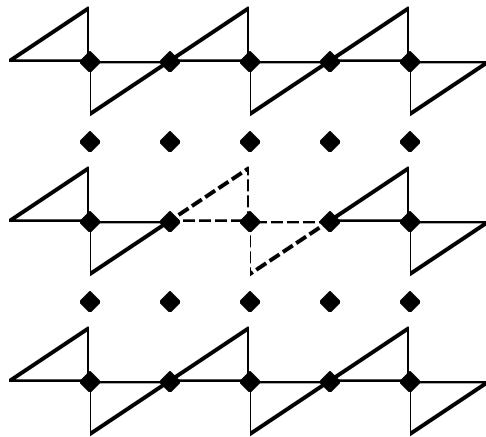
p2



This pattern has $\frac{1}{2}$ turn rotations. There are points that we can see a single piece make a $\frac{1}{2}$ turn around

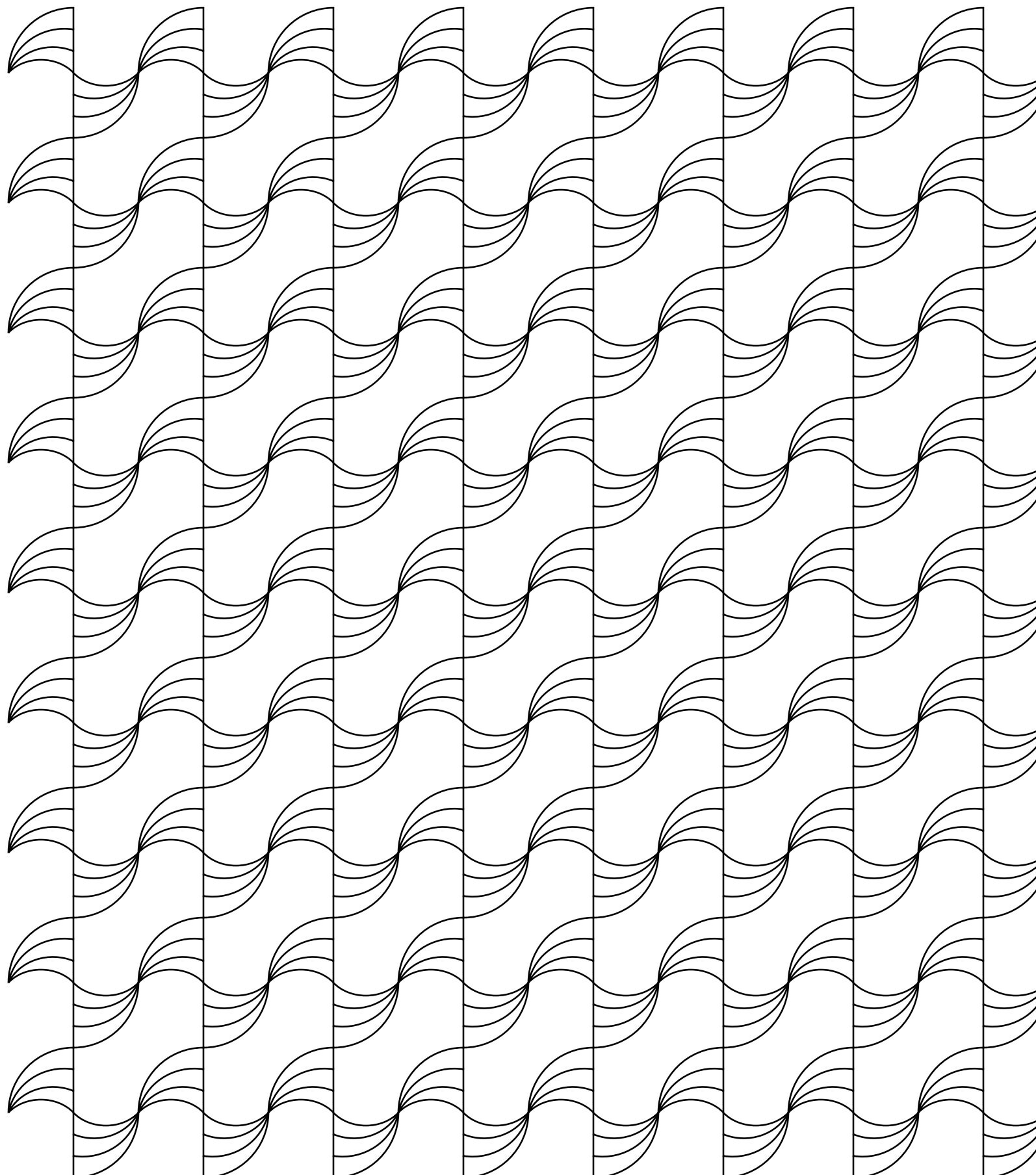


and that the entire pattern can turn around.



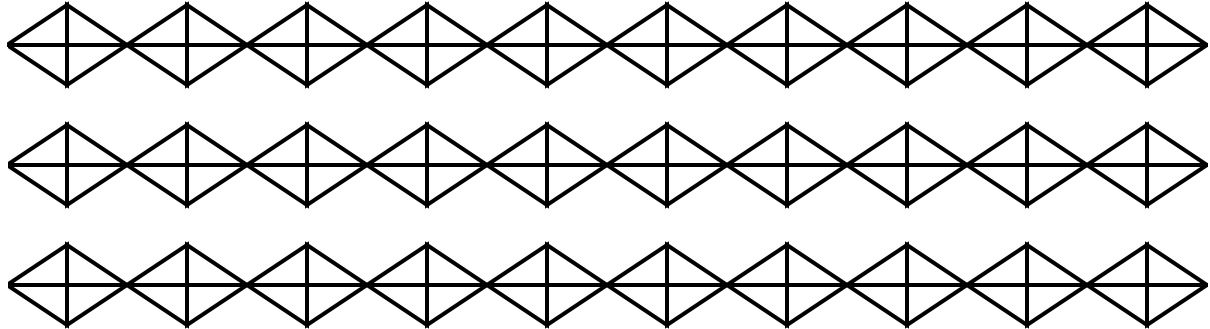
Challenge: Can you see the many points of rotation in the pattern?

Coloring Challenge: Color the pattern to remove some points of rotation while keeping others.

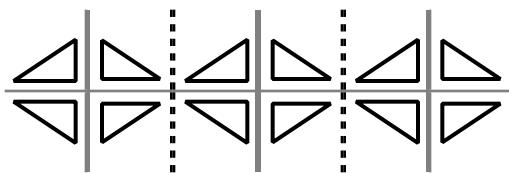


p2 wallpaper pattern: $\frac{1}{2}$ turn rotation, translation

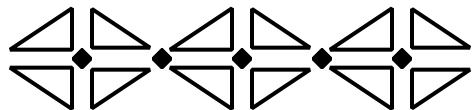
pmm



This pattern has perpendicular axes of mirror reflection



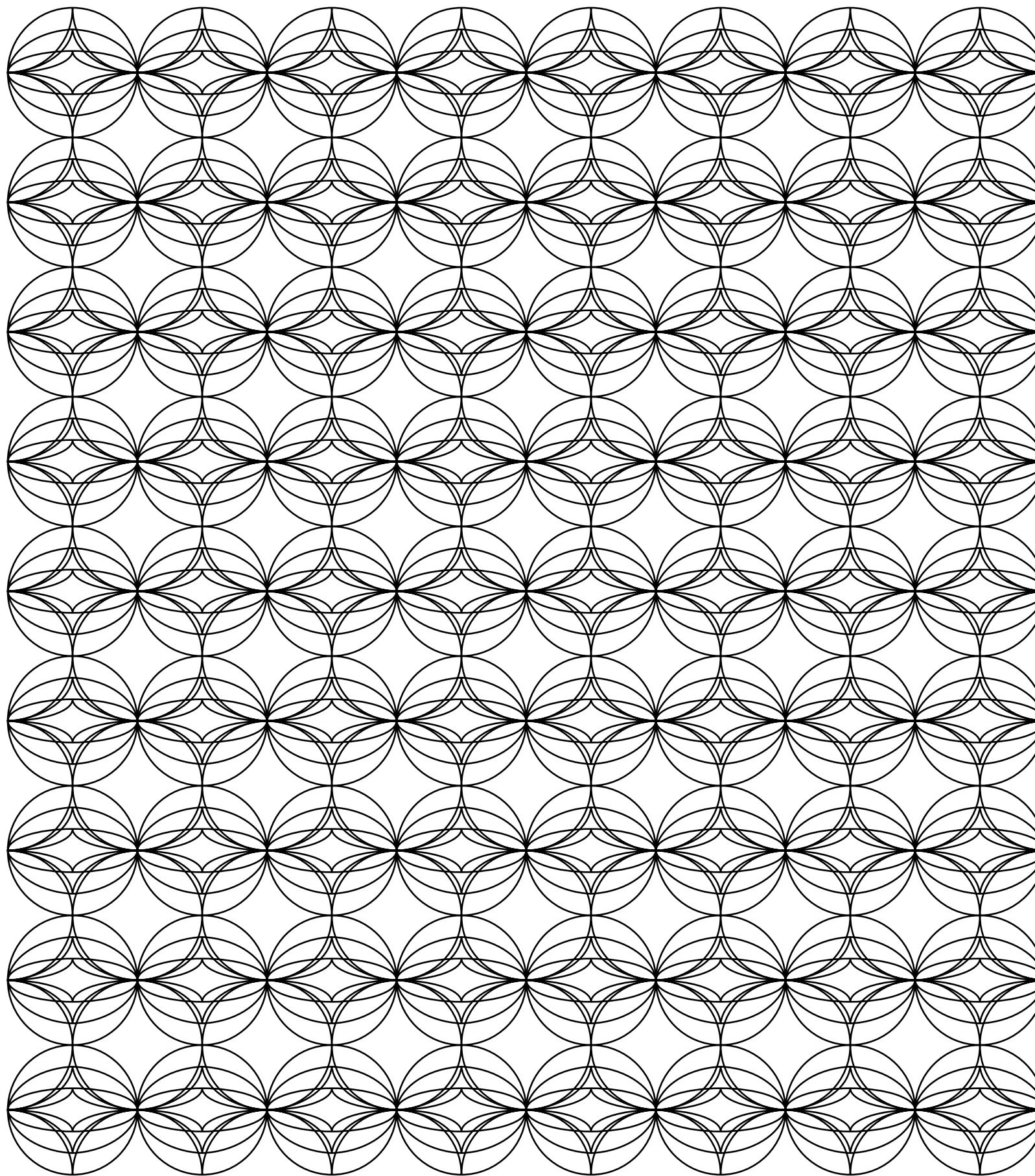
with $\frac{1}{2}$ turn rotations where the axes intersect.



This is about to get more complicated...

Challenge: Is it possible to have a pattern with perpendicular axes of mirror reflection that does not also have $\frac{1}{2}$ turns? Hint: What is the result of combining perpendicular reflections?

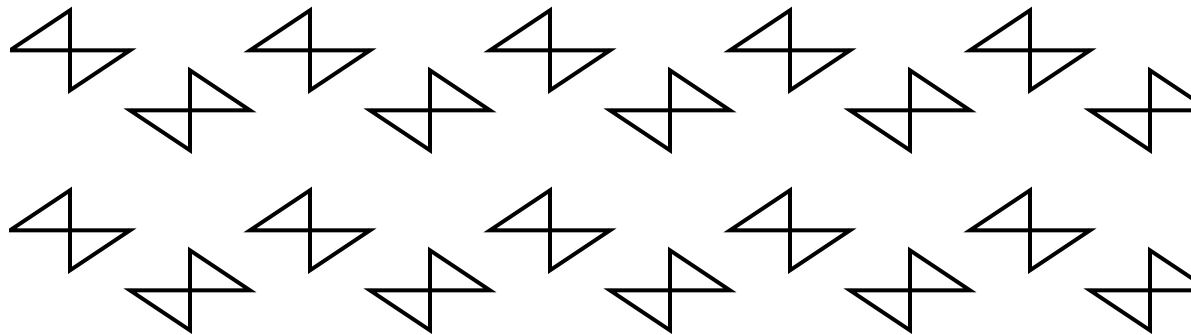
Coloring Challenge: Color the pattern to remove its mirrors while maintaining its $\frac{1}{2}$ turns.



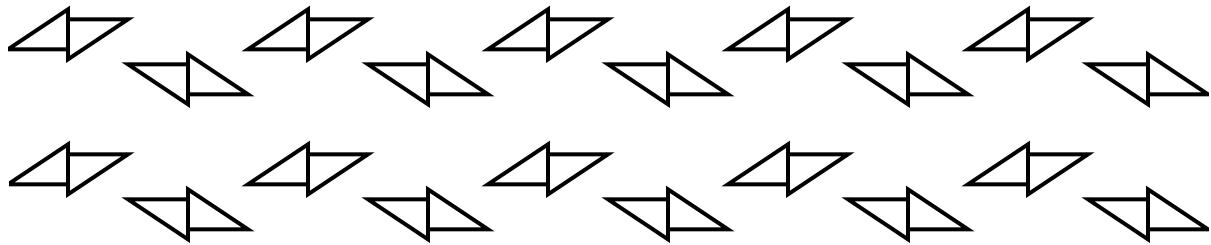
pmm wallpaper pattern: mirror reflection, $\frac{1}{2}$ turn rotation, translation

pgg

This pattern group has glide reflections and $\frac{1}{2}$ turn rotations, but no mirror reflections. The glide reflections have perpendicular axes, and the rotation centers do not lie on their intersection.

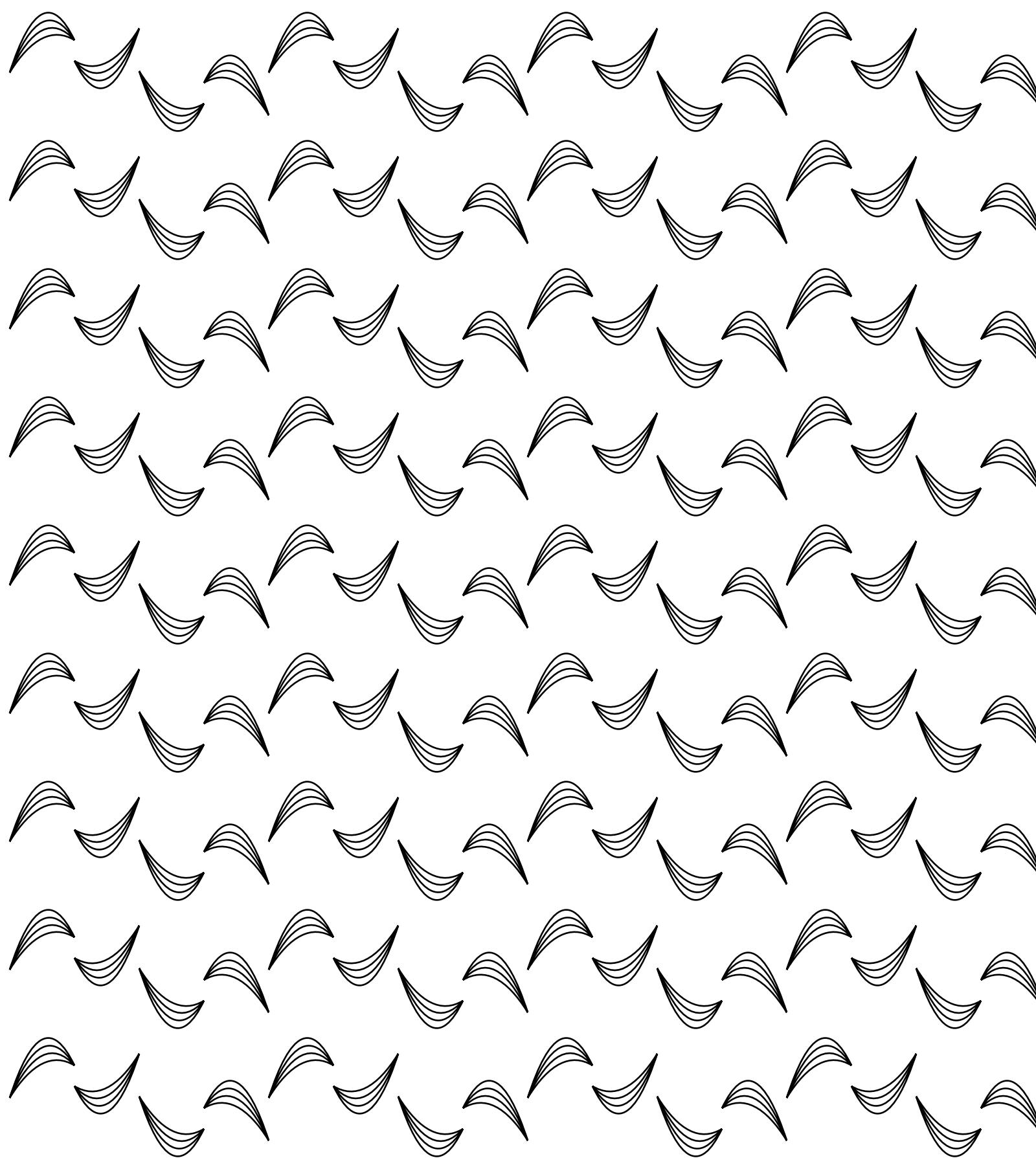


We can shift these axes and yet have a pattern with the same symmetries, and so it's in the same wallpaper group.



Challenge: Can you see the many different axes of glide reflection?

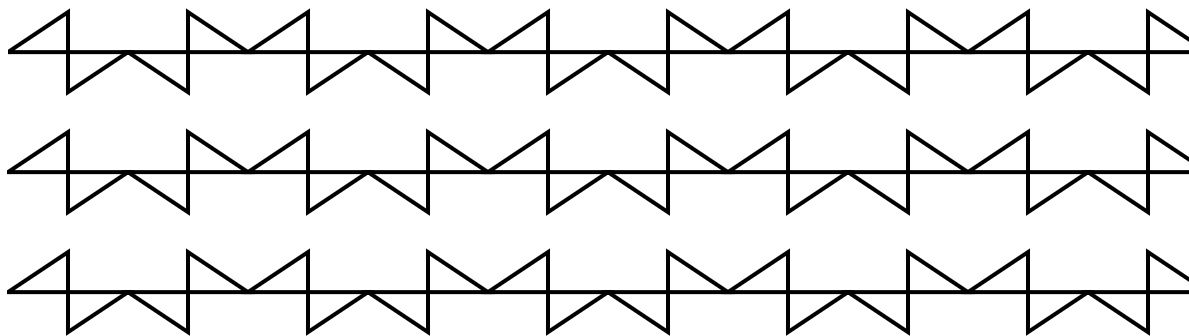
Coloring Challenge: Color the pattern to remove the horizontal axes of glide reflection while maintaining the vertical glide reflections.



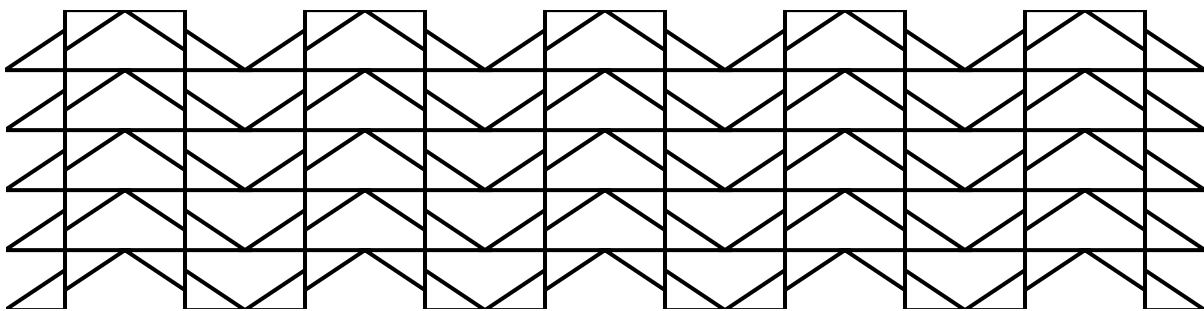
pgg wallpaper pattern: glide reflection, $\frac{1}{2}$ turn rotation, translation

pmg

This pattern group contains both mirror and glide reflections where the axes of the glide reflections are perpendicular to those of the mirror reflections. It also has $\frac{1}{2}$ turn rotations on the glide reflection axes, halfway between the mirror reflections.

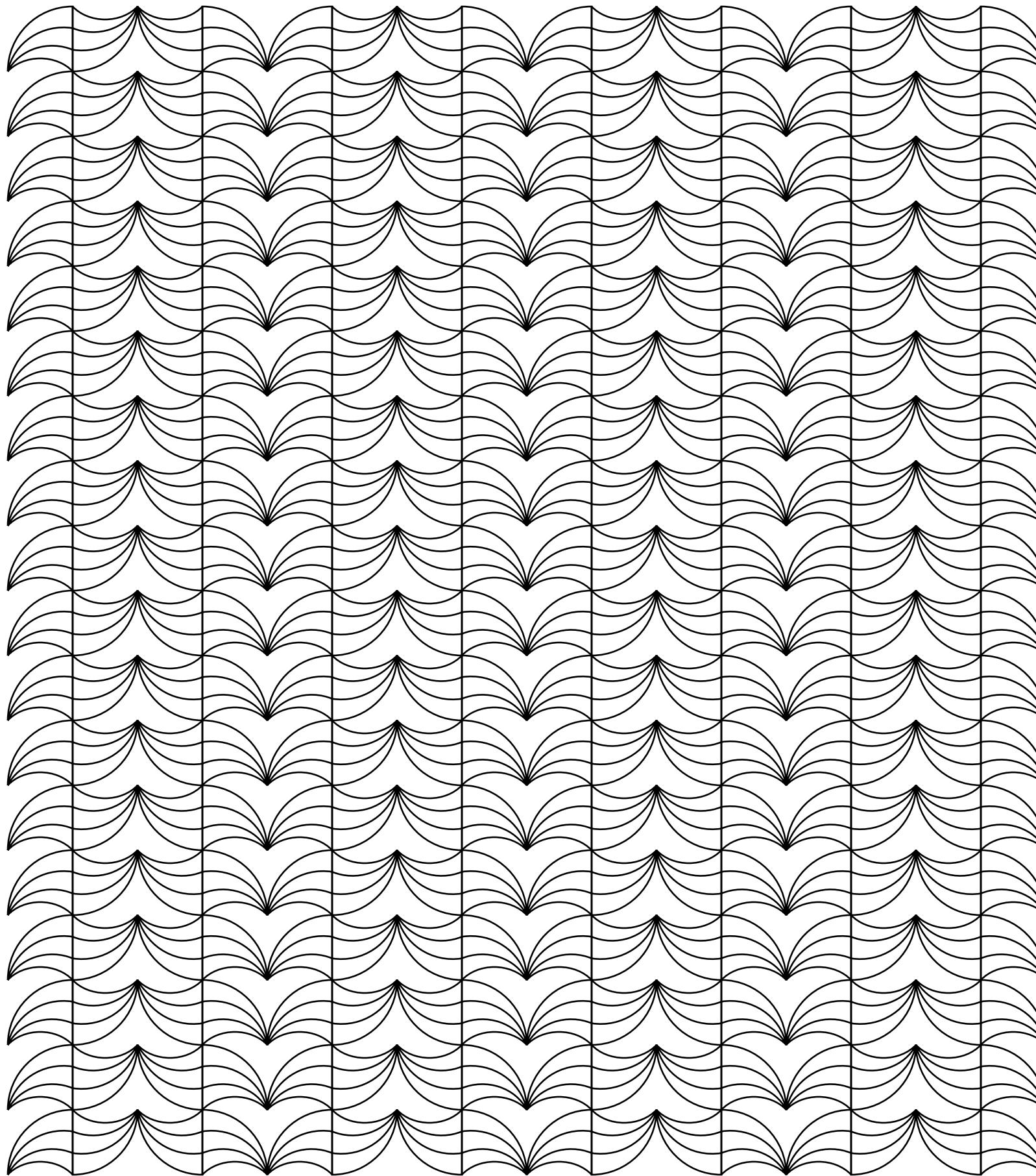


We can again shift the axes to see a pattern with the same symmetries.



Challenge: Can you see glide reflections and the points of rotation?

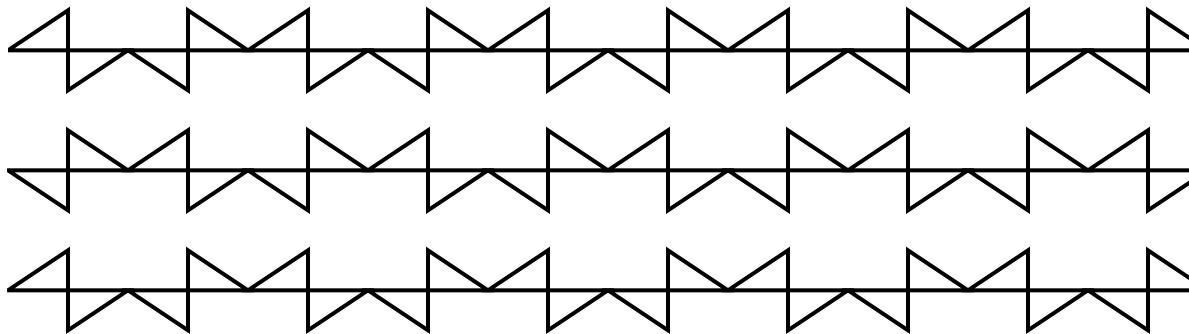
Coloring Challenge: Color the pattern to remove the glide reflections while maintaining the mirror reflections. What happens to the rotations?



pmg wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{2}$ turn rotation, translation

cmm

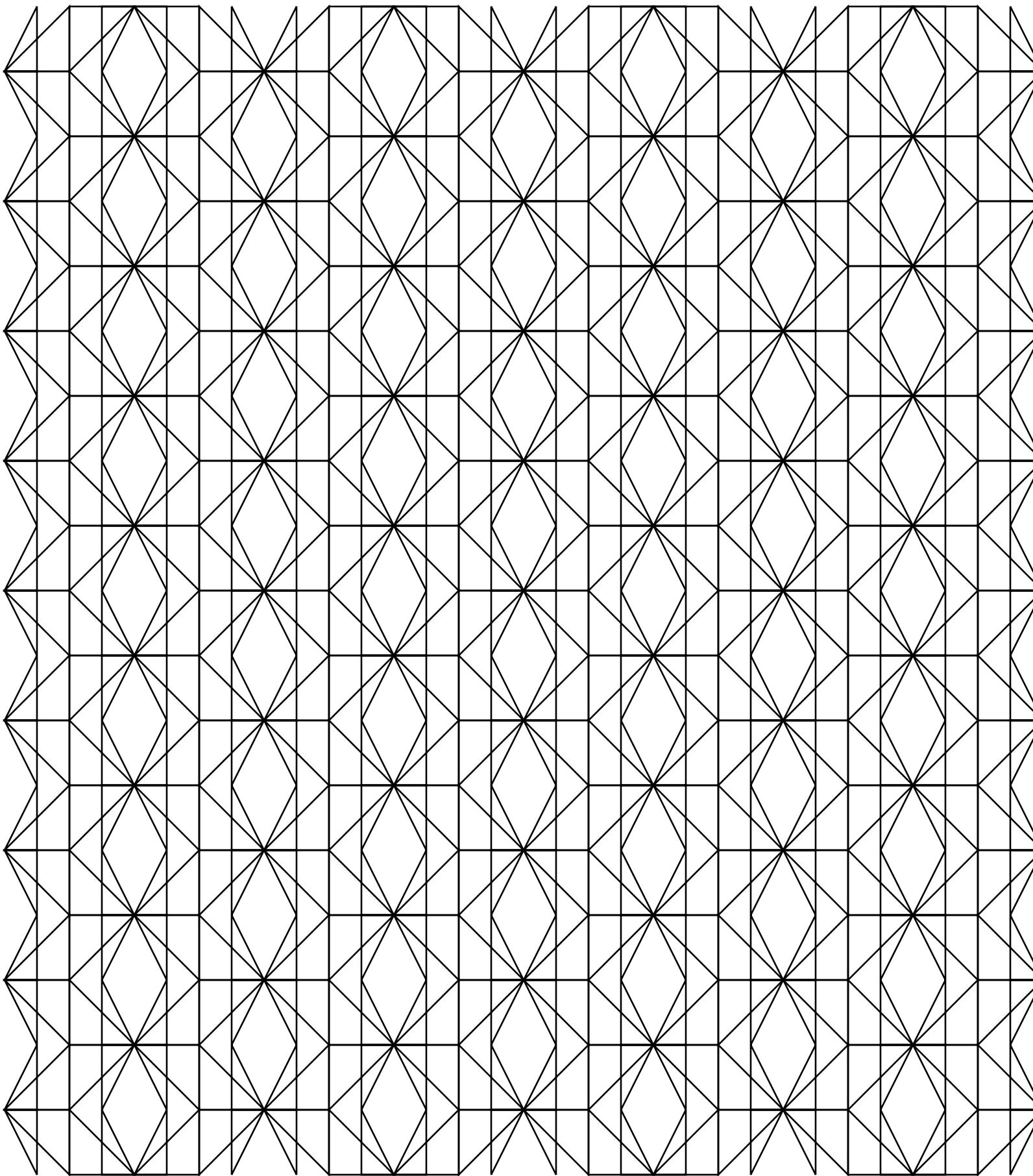
Like another pattern group we already colored, this one has perpendicular reflection axes with $\frac{1}{2}$ turn rotations at their intersections.



However it also has additional rotations that do not lie on the intersection of the reflections.

Challenge: Can you see the points of $\frac{1}{2}$ turn rotations that lie on the mirror reflection axes as well as those that do not?

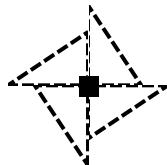
Coloring Challenge: Color the pattern to remove all mirror reflections.



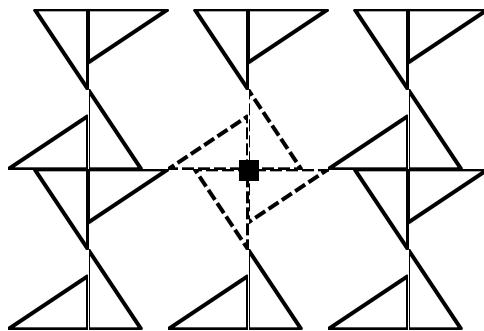
cmm wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{2}$ turn rotation, translation

p4

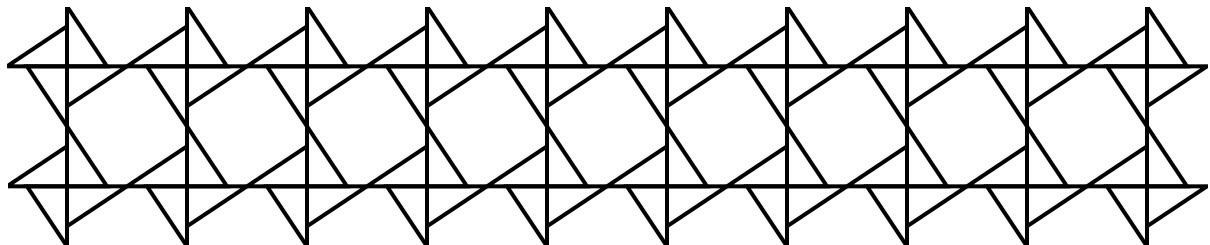
Wallpaper patterns can also have $\frac{1}{4}$ turn rotations. In the same way a single piece can rotate 4 times around a point,



an entire pattern can as well.

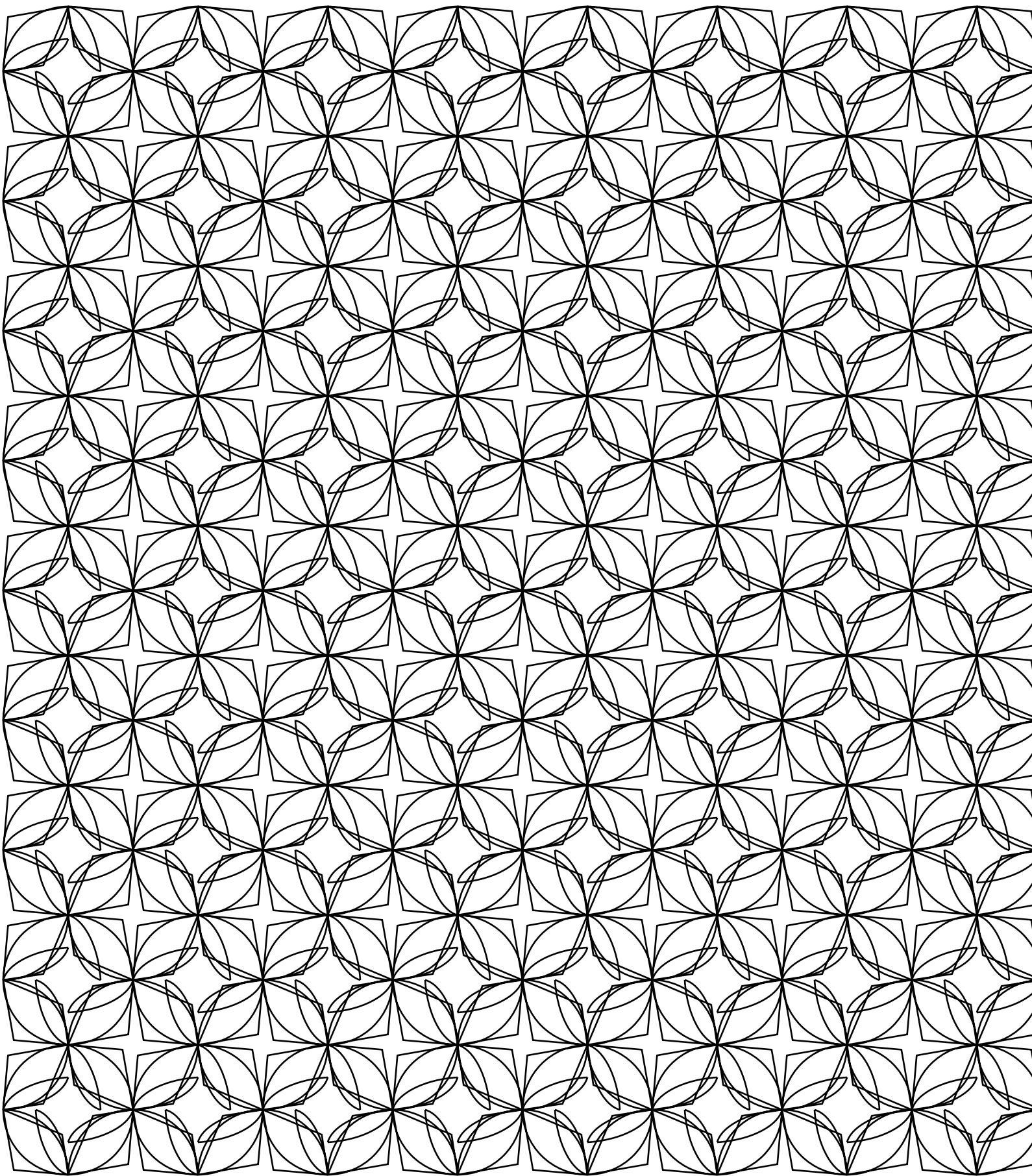


This pattern group has $\frac{1}{4}$ turn and $\frac{1}{2}$ turn rotations.



Challenge: Can you see the many $\frac{1}{4}$ turn rotation points in the pattern?

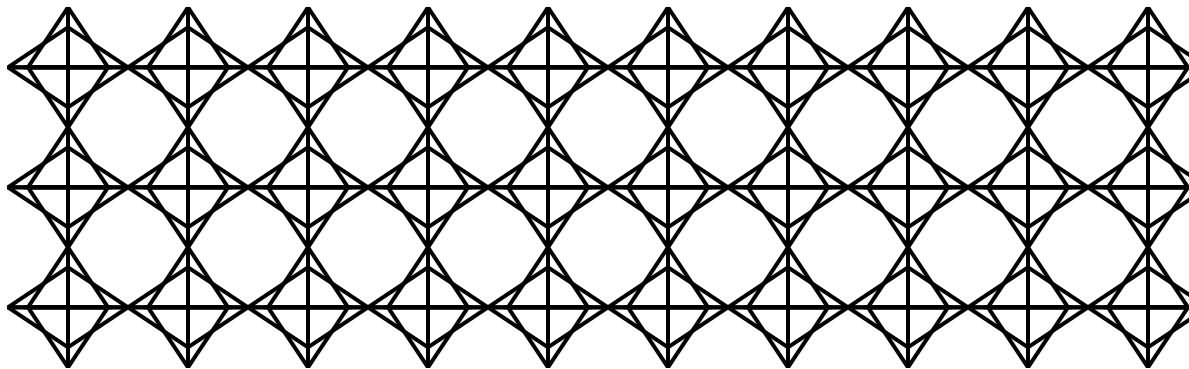
Coloring challenge: Color the pattern to reduce the $\frac{1}{4}$ turns to $\frac{1}{2}$ turns.



p4 wallpaper pattern: $\frac{1}{4}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

p4m

This pattern group has $\frac{1}{2}$ turn and $\frac{1}{4}$ turn rotations, as well as reflections with axes that intersect in ways that are both perpendicular and diagonal.

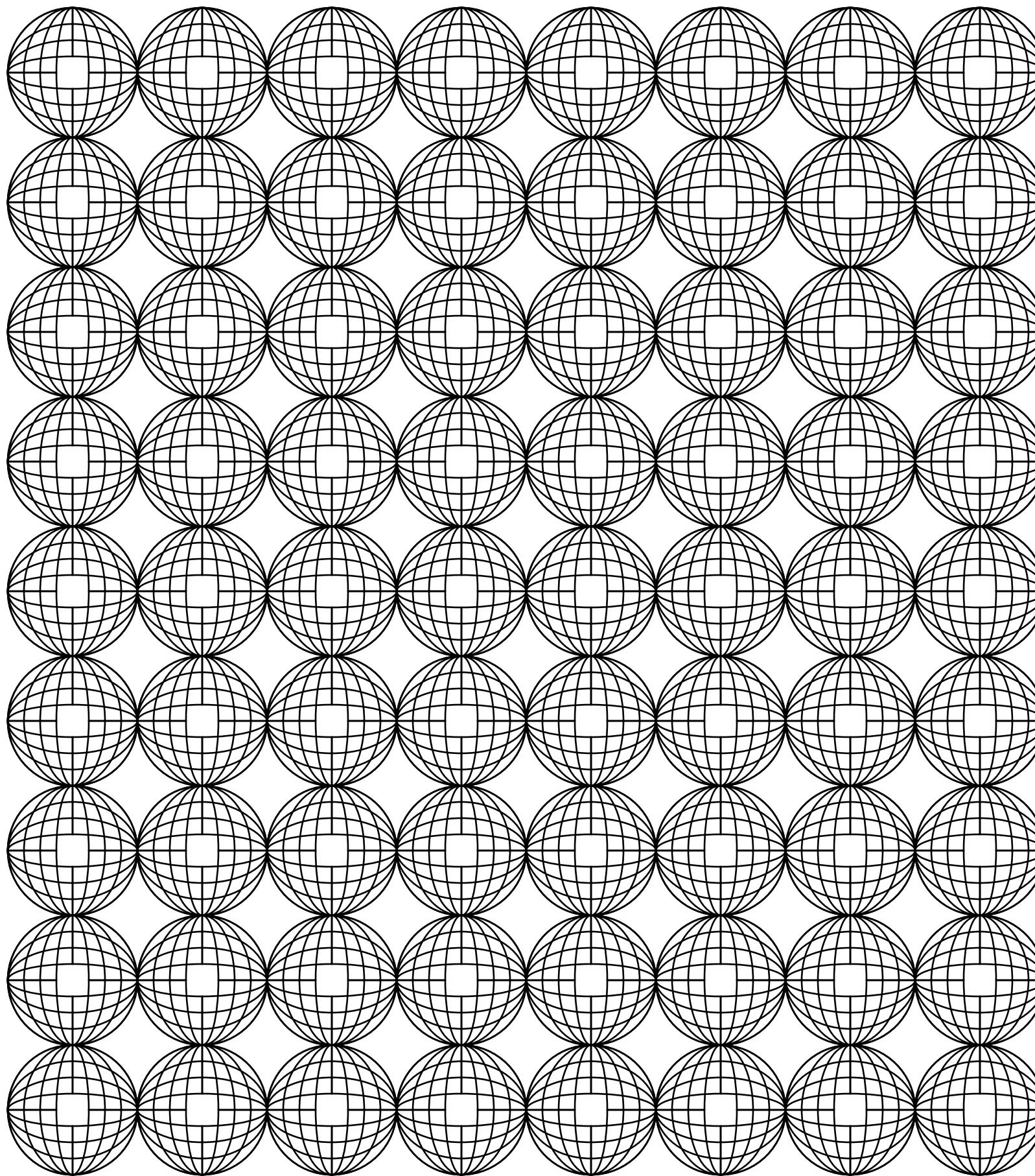


Each of its rotation centers lie on multiple reflection axes:

The centers of the $\frac{1}{4}$ turns are at the intersection of 4 mirror reflection axes. The centers of the $\frac{1}{2}$ turns sit on the intersection of 2 mirror reflection axes and 2 glide reflection axes.

Challenge: Can you see the $\frac{1}{2}$ turns as well as the $\frac{1}{4}$ turns? Can you find the many different axes of reflection?

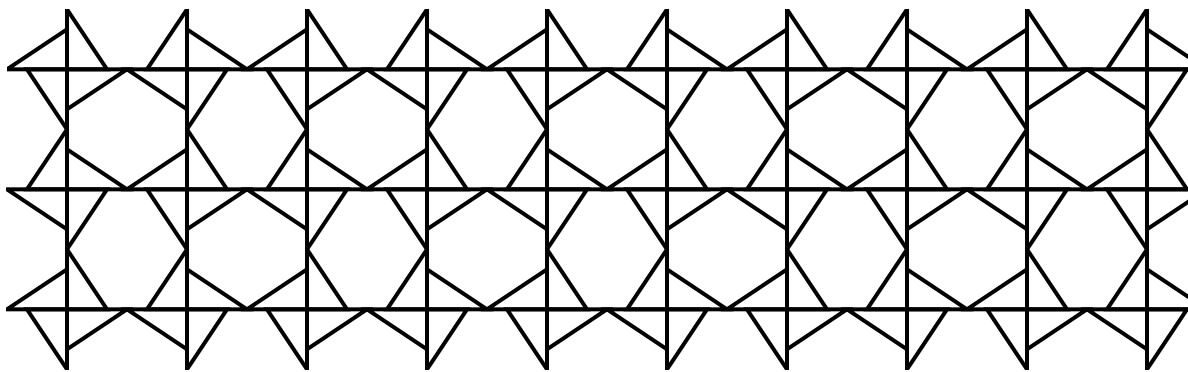
Coloring challenge: Color the pattern to remove its reflections so that rotations are its only symmetries.



p4m wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{4}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

p4g

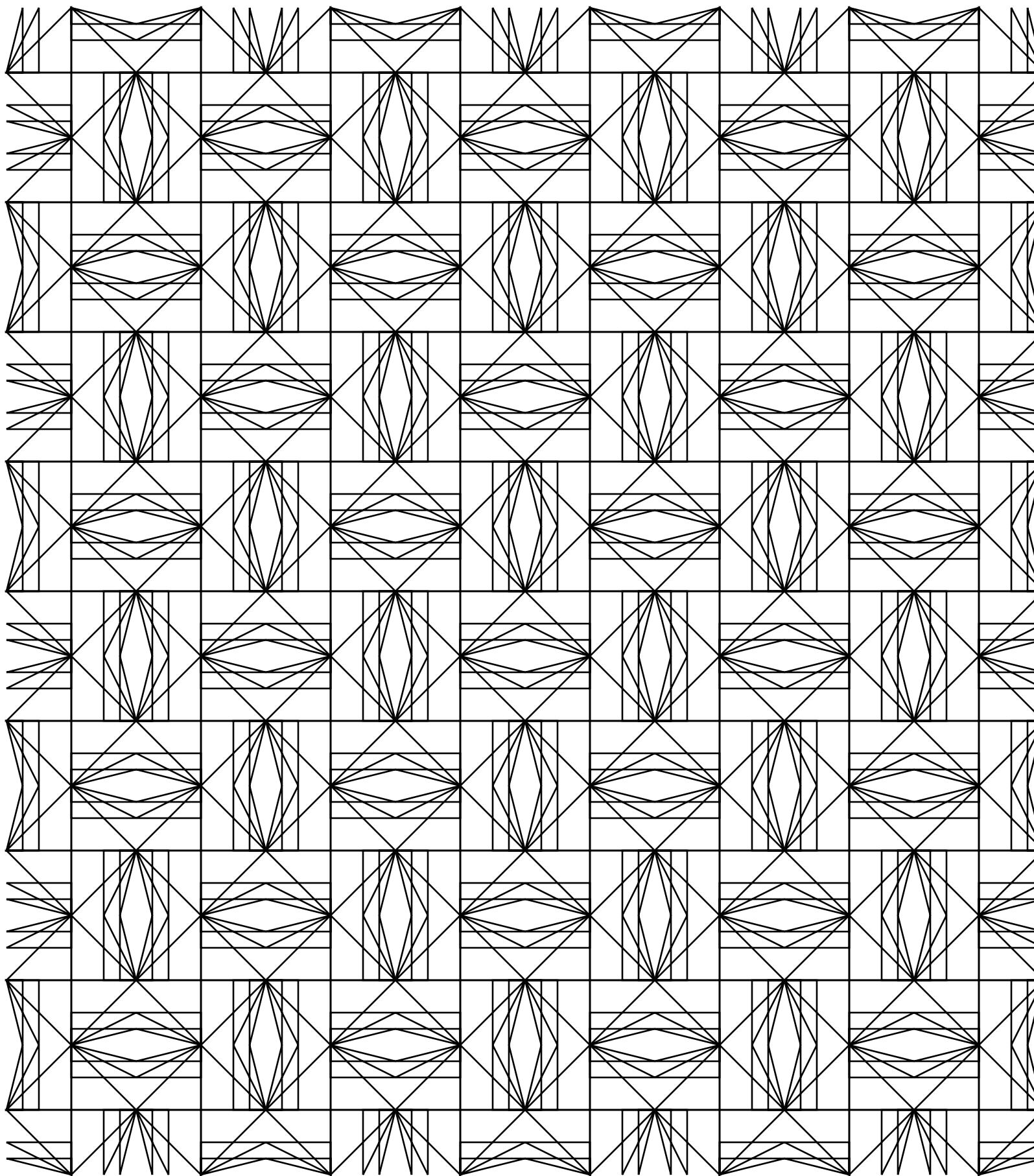
This pattern group again contains $\frac{1}{2}$ turn and $\frac{1}{4}$ turn rotations as well as both mirror and glide reflections, but this time with more glide reflections - there are 4 directions of glide reflection.



Each $\frac{1}{2}$ turn rotation sits on the intersection of 2 perpendicular mirror reflection axes and the $\frac{1}{4}$ turn rotations do not sit on any reflection axes.

Challenge: Can you see the many different axes of glide reflection?

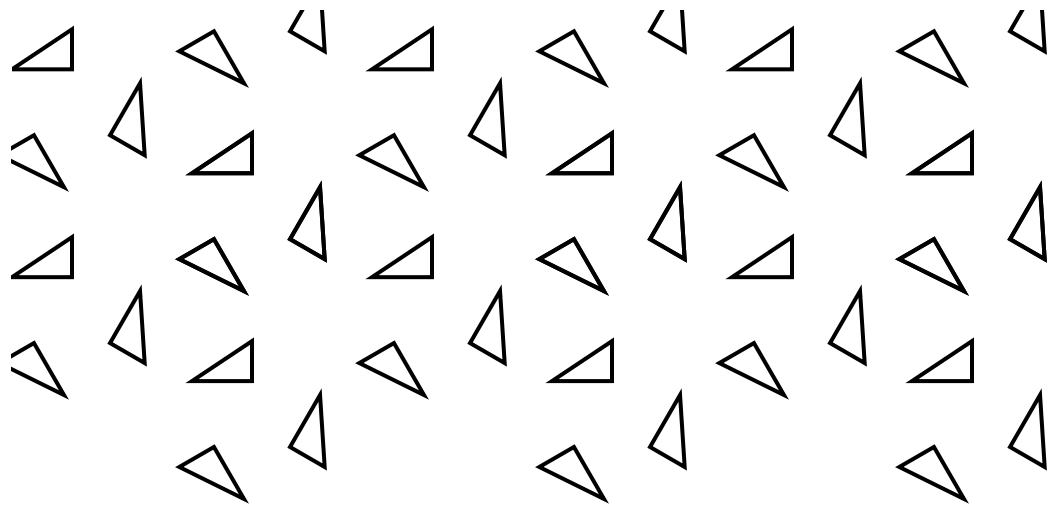
Coloring Challenge: Color the pattern to remove the $\frac{1}{4}$ turns while keeping the $\frac{1}{2}$ turns.



p4g wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{4}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

p3

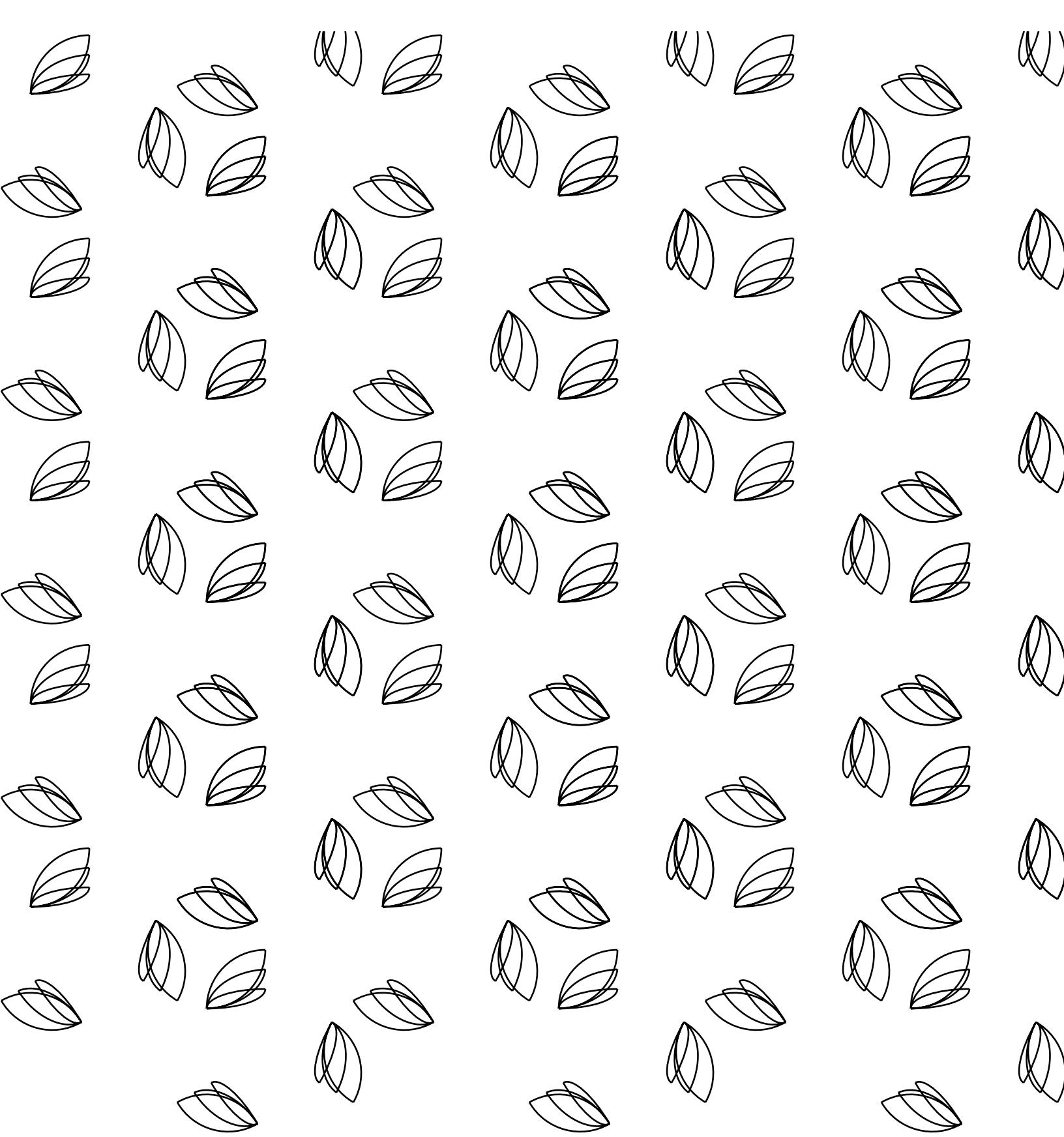
Wallpaper patterns can have $\frac{1}{3}$ turn rotations too.



This is the simplest wallpaper pattern group that contains a $\frac{1}{3}$ turn rotation. It has no reflections, but others can...

Challenge: Can you see the points of $\frac{1}{3}$ turn rotation?

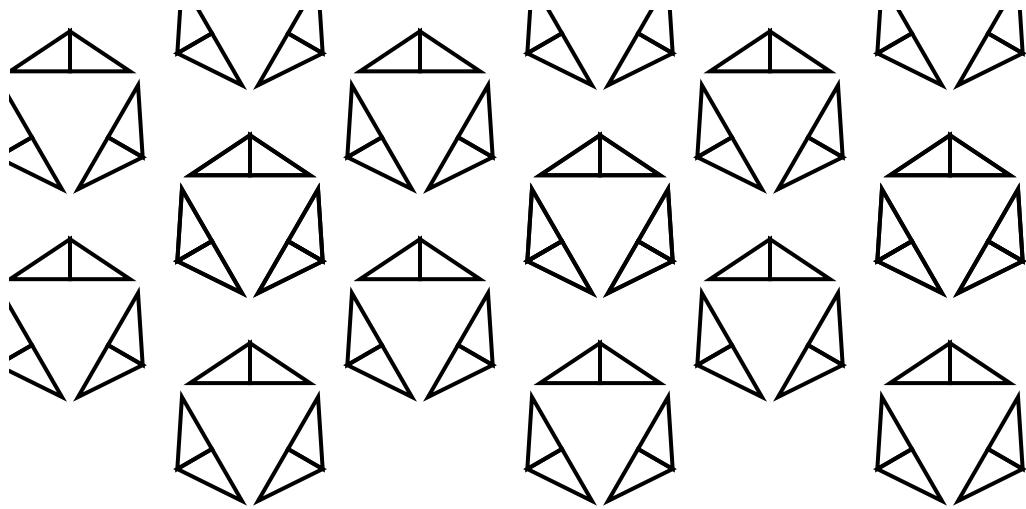
Coloring Challenge: Color the pattern to remove its rotations so that translation is its only symmetry.



p3 wallpaper pattern: $\frac{1}{3}$ turn rotation, translation

p31m

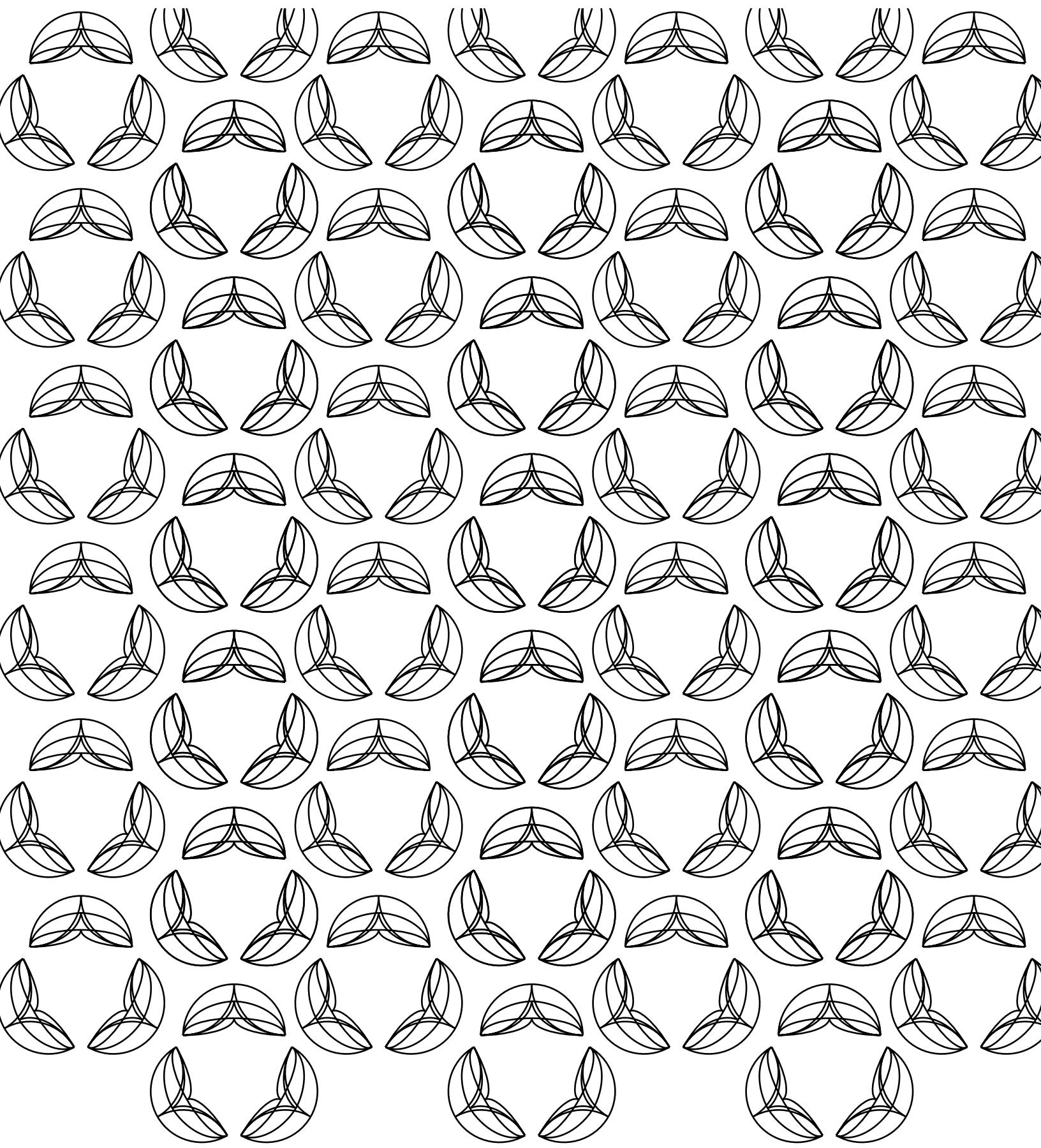
This pattern group contains mirror reflections, glide reflections, and $\frac{1}{3}$ turn rotations.



Some of the centers of rotation lie on the reflection axes, and some do not.

Challenge: Can you see the rotation centers that are both on and off the reflection axes?

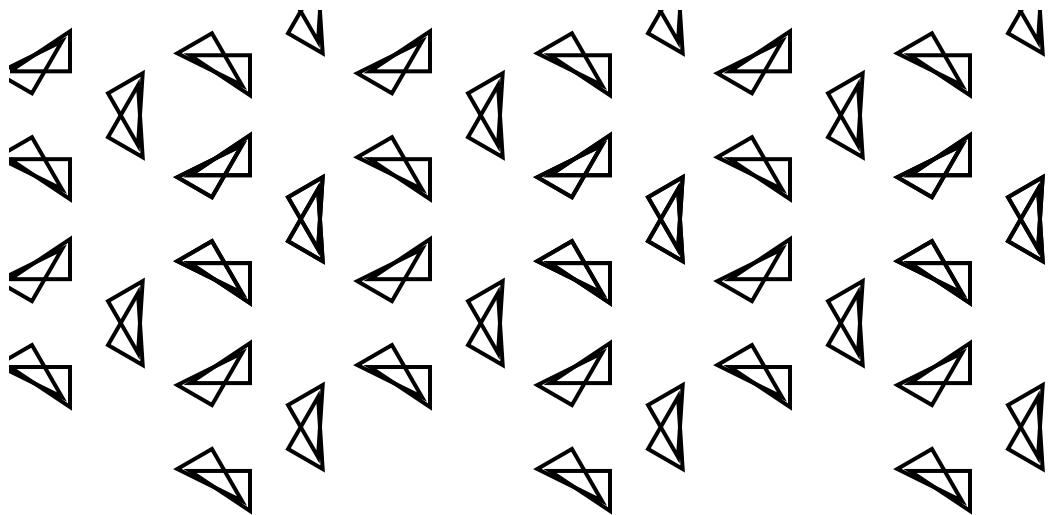
Coloring Challenge: Color the pattern to remove the reflections while keeping the $\frac{1}{3}$ turns.



p31m wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{3}$ turn rotation, translation

p3m1

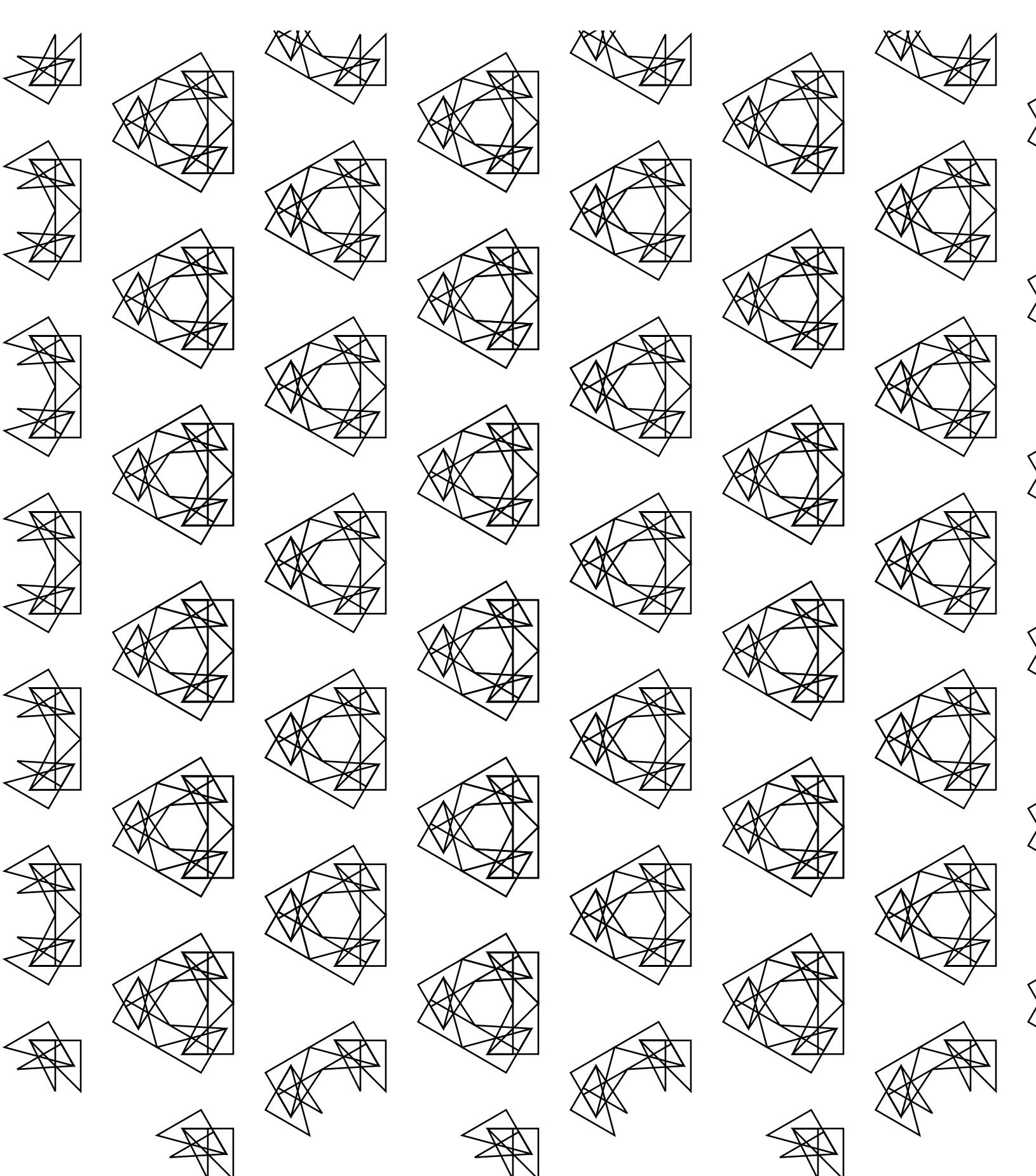
This pattern group also has mirror reflections, glide reflections, and $\frac{1}{3}$ turn rotations,



and this time all of the centers of rotation lie on the reflection axes.

Challenge: Can you see the glide reflections?

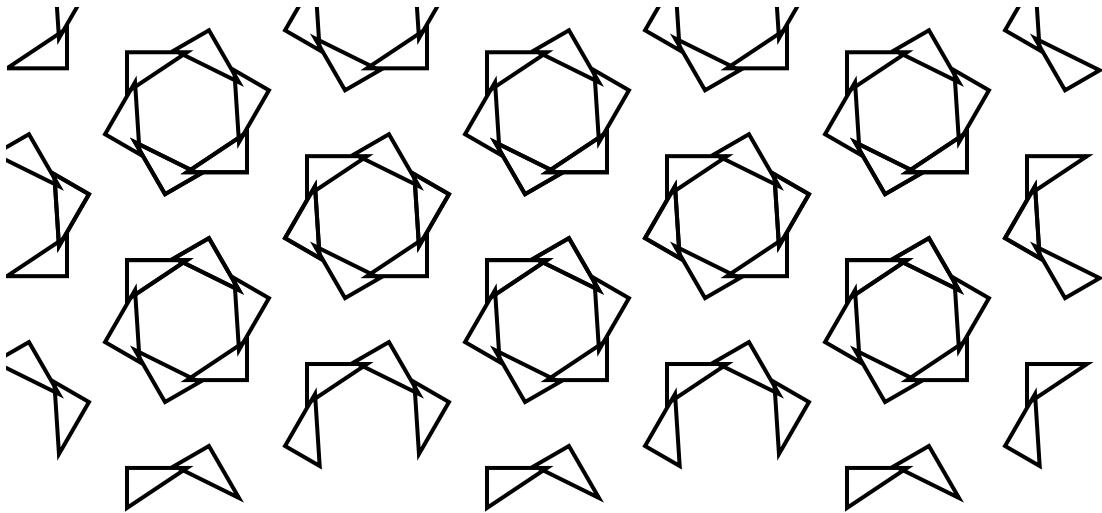
Coloring Challenge: Color the pattern to again remove the reflections while keeping the $\frac{1}{3}$ turns.



p3m1 wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{3}$ turn rotation, translation

p6

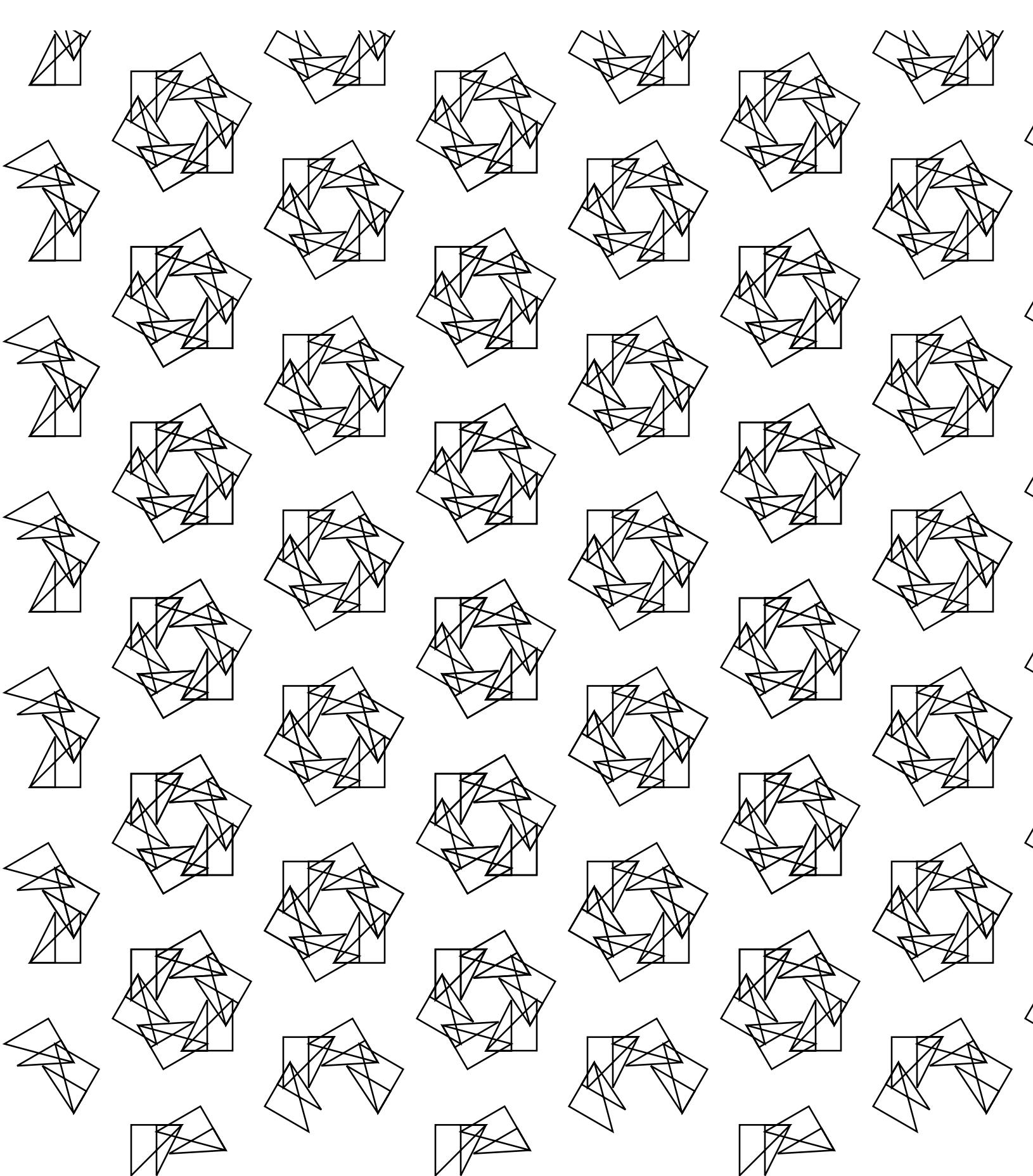
Any group with both $\frac{1}{2}$ turns and $\frac{1}{3}$ turns must have all of their combinations, including $\frac{1}{6}$ turns...



This pattern group has $\frac{1}{2}$ turn, $\frac{1}{3}$ turn, and $\frac{1}{6}$ turn rotations but no reflections.

Challenge: Can you see the $\frac{1}{6}$ turns? Can you see the $\frac{1}{2}$ turns?

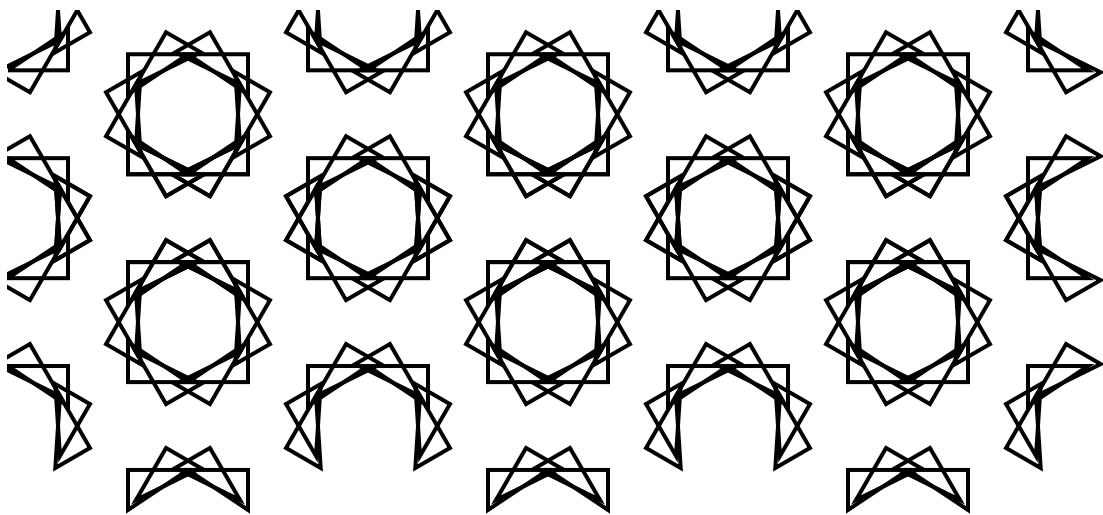
Coloring Challenge: Color the pattern to remove the $\frac{1}{2}$ turn and $\frac{1}{6}$ turn rotations while maintaining the $\frac{1}{3}$ turns.



p6 wallpaper pattern: $\frac{1}{6}$ turn rotation, $\frac{1}{3}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

p6m

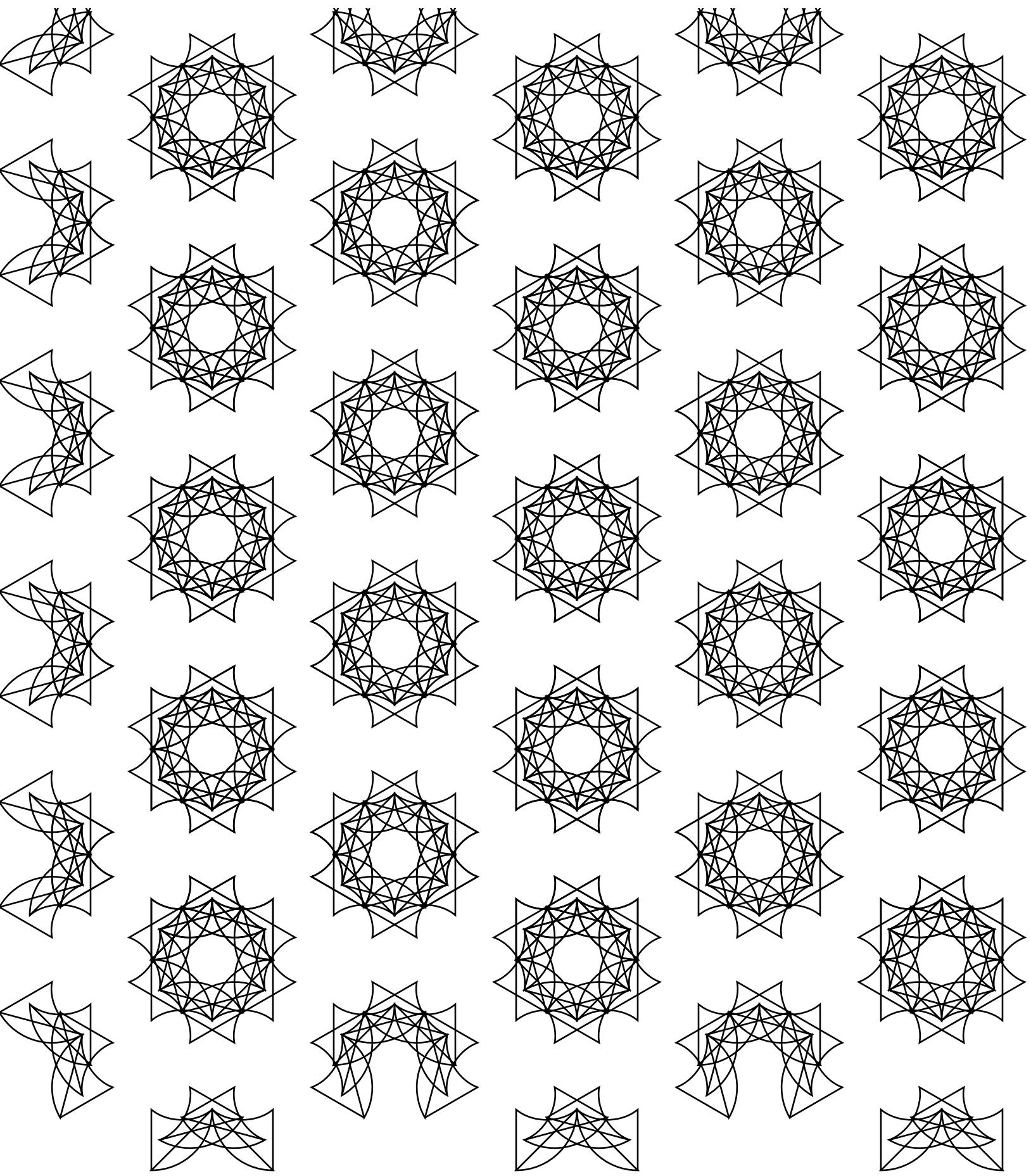
This pattern group has $\frac{1}{6}$ turn, $\frac{1}{3}$ turn, and $\frac{1}{2}$ turn rotations, as well as mirror and glide reflections.



Challenge: How many axes of mirror reflection intersect at the centers of the $\frac{1}{6}$ turn rotations?

Challenge: Can you see the glide reflections?

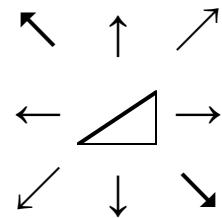
Coloring Challenge: Color the pattern to remove its $\frac{1}{6}$ turns while keeping its reflections and $\frac{1}{3}$ turns.



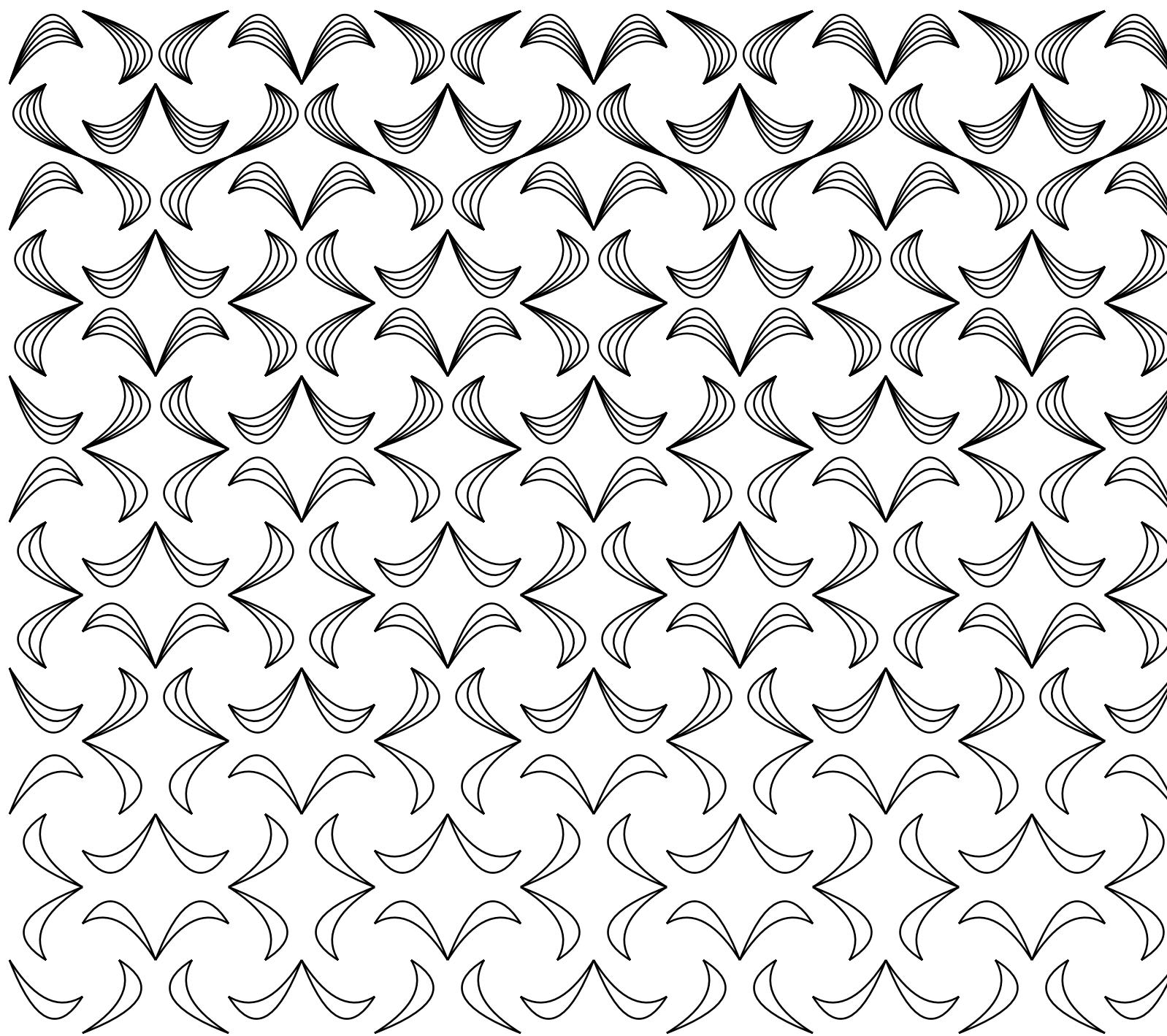
p6m wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{6}$ turn rotation, $\frac{1}{3}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

There are 17 wallpaper pattern groups and we have now colored all of them.

There are no other ways to combine the symmetries in patterns that repeat along 2 dimensions.



Yet each wallpaper group has infinitely many more patterns that could represent it...

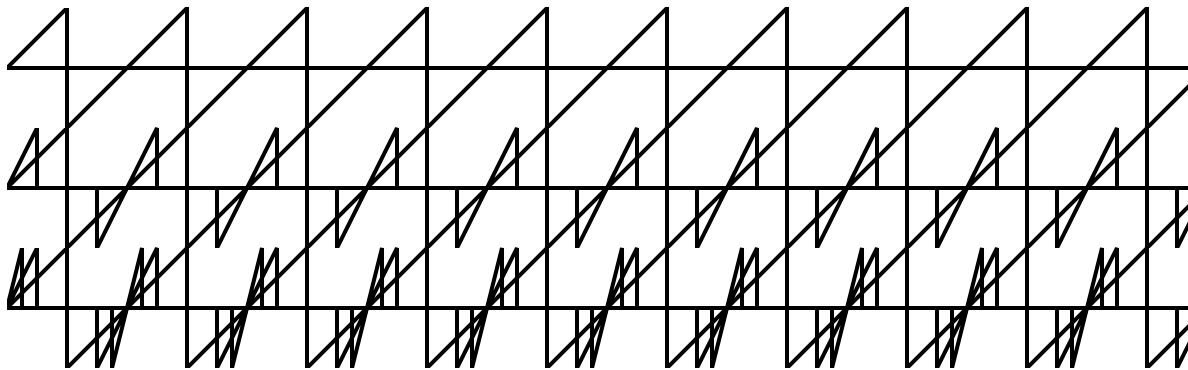


p4g wallpaper pattern: mirror reflection, glide reflection, $\frac{1}{4}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

Like we saw before, it doesn't matter what the starting piece of a pattern is.

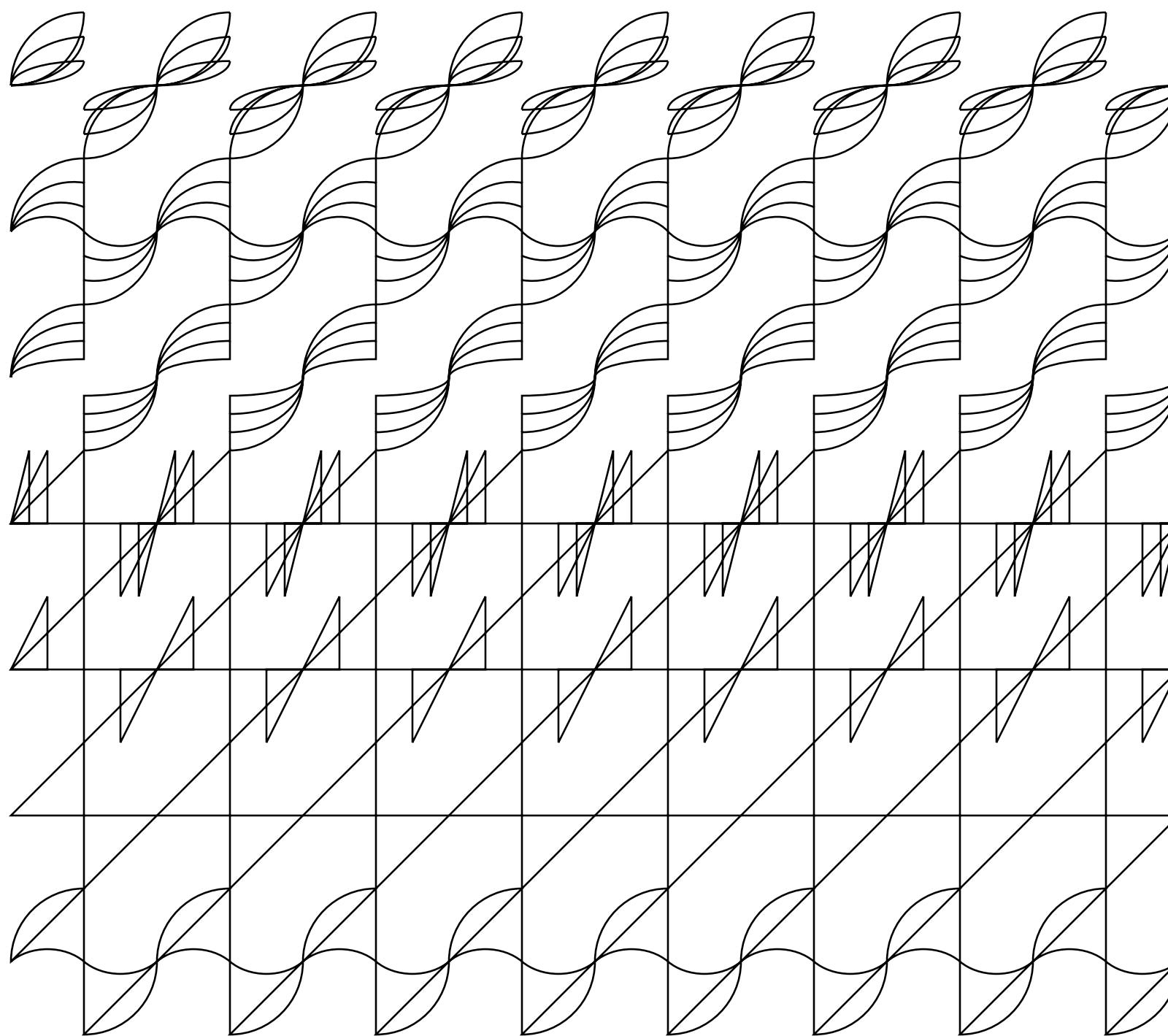


as long as we can see the symmetries in the patterns between those pieces.

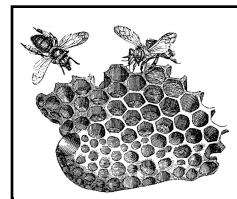


Challenge: Can you see the $\frac{1}{2}$ turns in the pattern?

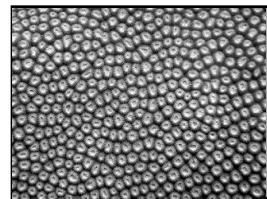
Coloring Challenge: Color the pattern so that translation is its only symmetry.



We can see these symmetries in the patterns of the physical world around us,



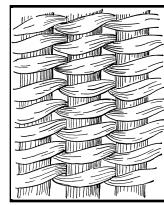
Beehive



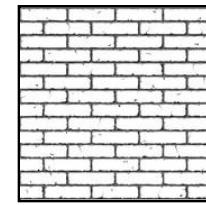
Coral



Office building

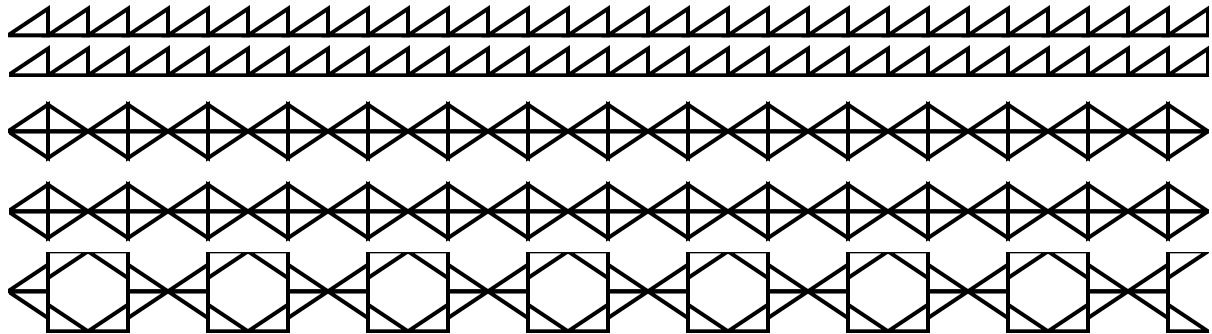


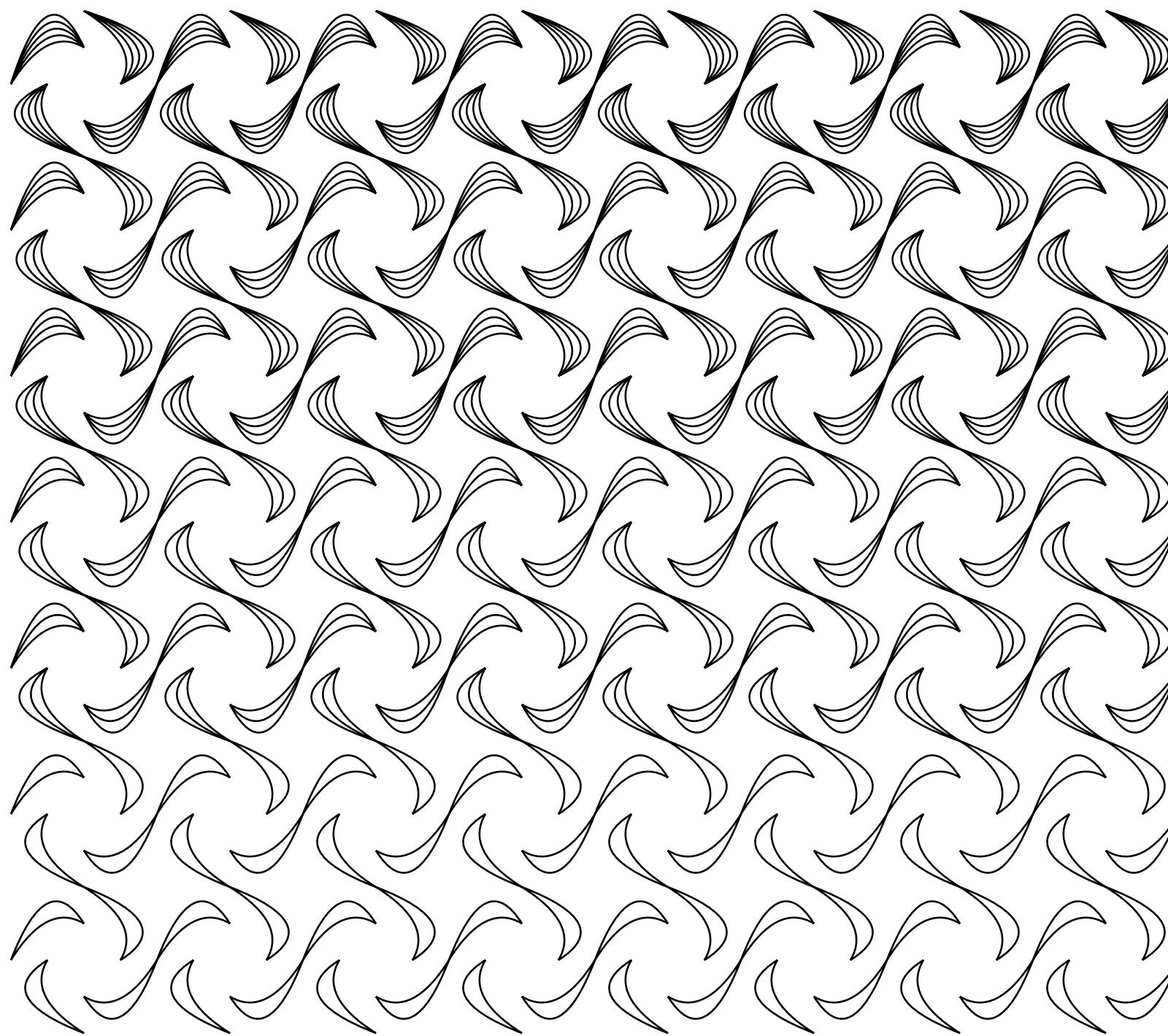
Basket Weave



Bricks

or stay in the world of perfect mathematics.





p4 wallpaper pattern: $\frac{1}{4}$ turn rotation, $\frac{1}{2}$ turn rotation, translation

Here are some wallpaper patterns to play with: <http://coloring-book.co/wallpaper>

THEORY REFERENCE

Group theory helps define abstract structures discussed in algebra. The groups in this coloring book are only a window into the groups explored in other realms of mathematics.

There are some rules and definitions that pertain to all groups, not just ours.

Group

A group G is a set coupled with a binary operator $*$ that satisfies 4 requirements:
See the details of each rule for examples.

Closure: G is closed under $*$; i.e., if a and b are in G, then $a * b$ is in G.

Identity element: There exists an identity element e in G; i.e., for all a in G we have $a * e = e * a = a$.

Inverse element: Every element in G has an inverse in G; i.e., for all a in G, there exists an element $-a$ in G such that $a * (-a) = (-a) * a = e$.

Associativity: The operator $*$ acts associatively; i.e., for all a,b,c in G, $a * (b * c) = (a * b) * c$.

Associative Property

When an operator $*$ for a group G is associative, the way elements in G are grouped when the operator is applied does not matter. I.e., for all a,b,c in G, $a * (b * c) = (a * b) * c$.

One example of this is adding numbers: $1 + (2 + 3) = (1 + 2) + 3$.

Notice that subtraction of numbers is not associative: $1 - (2 - 3)$ does not equal $(1 - 2) - 3$.

Our groups of rotations have an associative operator: Our operator here is combining rotations.

For C₃, $(\frac{1}{3} \text{ turn} * \frac{1}{3} \text{ turn}) * \frac{2}{3} \text{ turn} = \frac{1}{3} \text{ turn} * (\frac{1}{3} \text{ turn} * \frac{2}{3} \text{ turn})$. That is, rotating twice by a $\frac{1}{3}$ turn and then rotating the result by a $\frac{2}{3}$ turn is the same as combining a $\frac{1}{3}$ turn with the result of rotating by a $\frac{1}{3}$ turn and then by a $\frac{2}{3}$ turn.

Binary Operator

A binary operator $*$ combines 2 elements, a and b, from a set S to give a third element: $a * b$.

An example is addition over the set of counting numbers: $+$ is a binary operator that combines 2 numbers to create their sum: $1 + 2 = 3$.

Our binary operator combines the transformations that act on our symmetry groups. For symmetry elements a and b, $a * b$ says "do a, and then do b". For example, if transformation a is "rotate by a $\frac{1}{4}$ turn" and b is "reflect horizontally", then $a * b$ is "rotate by a $\frac{1}{4}$ turn and then reflect horizontally".

Closure

A set S is closed under an operator $*$ if combining any 2 elements in S with $*$ results in an element that is also in S; i.e., for any a and b in S, $a * b$ is also in S.

For example, the set of all counting numbers 0,1,2,3,... is closed under the addition operator + because adding any two counting numbers results in another counting number.

Coming back to our sets of rotations, the set $\{\frac{1}{4} \text{ turn}, \frac{2}{4} \text{ turn}\}$ is not closed because combining the $\frac{1}{4}$ turn with the $\frac{2}{4}$ turn results in the $\frac{3}{4}$ turn which is not in this set.

Commutative Property

A binary operator $*$ is commutative if the order in which it combines elements does not matter. I.e., for any 2 elements a & b, $a * b = b * a$.

For example, addition is commutative because $1 + 2 = 2 + 1$, but subtraction is not commutative because $1 - 2 \neq 2 - 1$.

Cyclic Group

A group G is called cyclic if it can be generated by a single element.

Our groups of rotations are cyclic groups because they can be generated by their smallest nonzero element. For example, our C₂ group was $\{0 \text{ turn}, \frac{1}{2} \text{ turn}\}$, and it was generated by the $\frac{1}{2}$ turn.

There are many other cyclic groups out there. Another C₂ group that may look different, is the group $\{1, -1\}$ where the members of the group are the numbers 1 and -1 and the way of combining these members is with multiplication. It can be generated by -1.

The term cyclic may be misleading. Our cyclic groups had a finite number of elements, and combining them again and again created cycles. However, there are cyclic groups with infinite elements, such as the integers under addition.

Generator

Generators of a group are a set of elements that when combined with themselves, or each other, can produce all the other elements of the group.

For example, -2 and 2 are generators that when combined with addition, generate the entire group of even integers.

Identity Element

An identity element is a neutral element - when it's combined with other members in the group, it does not change them.

For our groups of rotations, the identity element is the 0 turn: rotating by the 0 turn is the same as doing nothing at all.

For the group of integers under addition, the identity element is 0: $0 + 2 = 2$.

For the group of integers under multiplication, the identity element is 1: $1 \times 2 = 2$.

Inverse Element

An inverse element is the reverse of another element.

More formally, for a set, S with a binary operator, $*$, and a and b in S: a is the inverse of b if $a * b = b * a = e$, where e is the identity element.

For our groups of rotations, each rotation's inverse element is the rotation that undoes it. For example, the inverse of the $\frac{1}{3}$ turn is the $\frac{2}{3}$ turn because $\frac{1}{3}$ turn $* \frac{2}{3}$ turn \rightarrow full turn. The full turn is the same as the 0 turn which is our identity element.

For addition on the integers, each integer's inverse element is its negative: -1 is the inverse of 1 because $-1 + 1 = 0$.

Order

The order of a group G is the number of elements in G. The order of G is sometimes written as $|G|$.

For example, the order of our C_3 group of rotations is 3 because C_3 has 3 elements:



Set

A set is a collection of distinct elements.

For example, the set **{blue, red, blue}** is the same set as the set **{blue, red}**.

For our sets of rotations, the set **{0 turn, $\frac{1}{3}$ turn, $\frac{4}{3}$ turn}** is the same as the set **{0 turn, $\frac{1}{3}$ turn}** because a $\frac{1}{3}$ turn means the same thing as a $\frac{4}{3}$ turn - they are not distinct.

Subgroup

Given a group G, a subgroup of G is a group with the same binary operator as G and whose members are all also in G.

For example, the group of even integers under addition **{..., -2, 0, 2, 4, ...}**, **+** is a subgroup of the group of all integers under addition **{..., -2, -1, 0, 1, 2, ...}**, **+**.

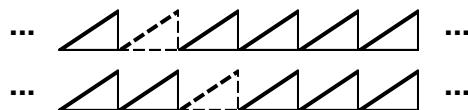
However, the same cannot be said for odd integers. The set of odd integers under addition **{..., -3, -1, 1, 3, 5, ...}**, **+** is not closed and therefore cannot be a group: Combining odd integers with addition produces even integers (e.g. $1 + 3 = 4$), which are clearly not in the set of odd integers.

Notice that a group and its subgroups always have the same identity element.

SYMMETRIES

OF PATTERNS IN THE PLANE

Translation is a shift in a given direction.



An infinitely repeating pattern has translation as a symmetry - the entire pattern can be shifted over without changing in appearance:



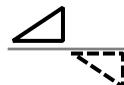
Mirror Reflection is a reflection across an imaginary line.



A pattern has mirror reflection as a symmetry if the entire pattern can reflect across a line yet remain unchanged:



Glide Reflection composes translation and mirror reflection: A **glide reflection** reflects across a mirror line at the same time as translating along it.

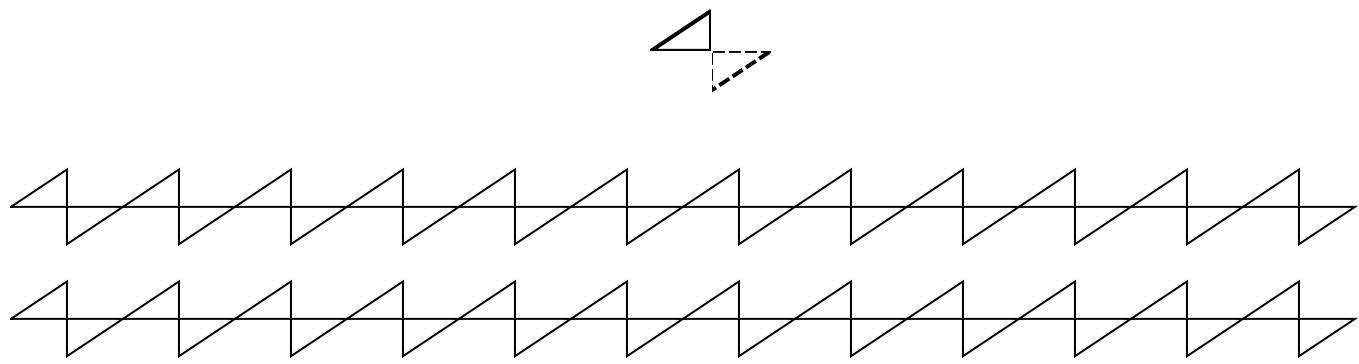


Infinitely repeating patterns can have glide reflections:

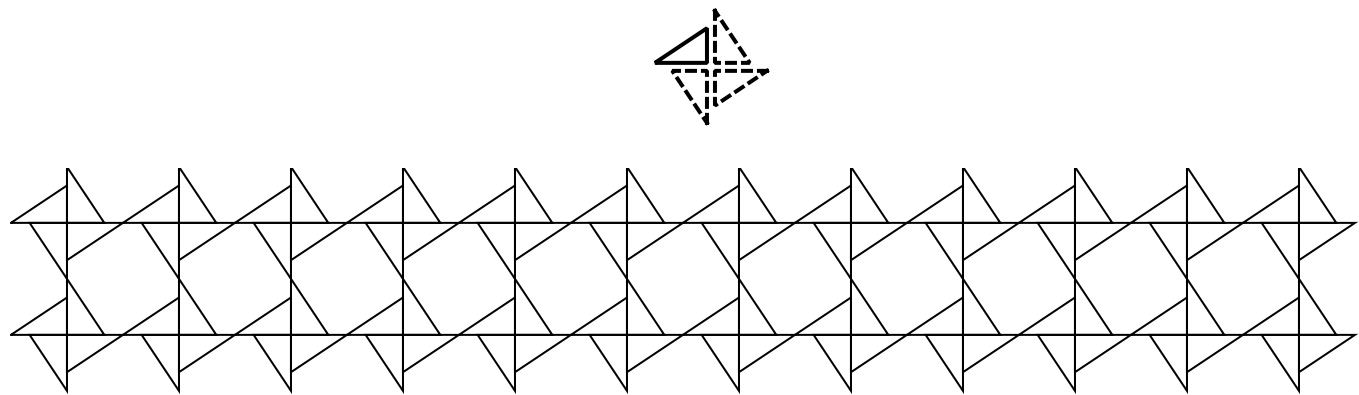


Rotation is a symmetry that turns a pattern around a point.

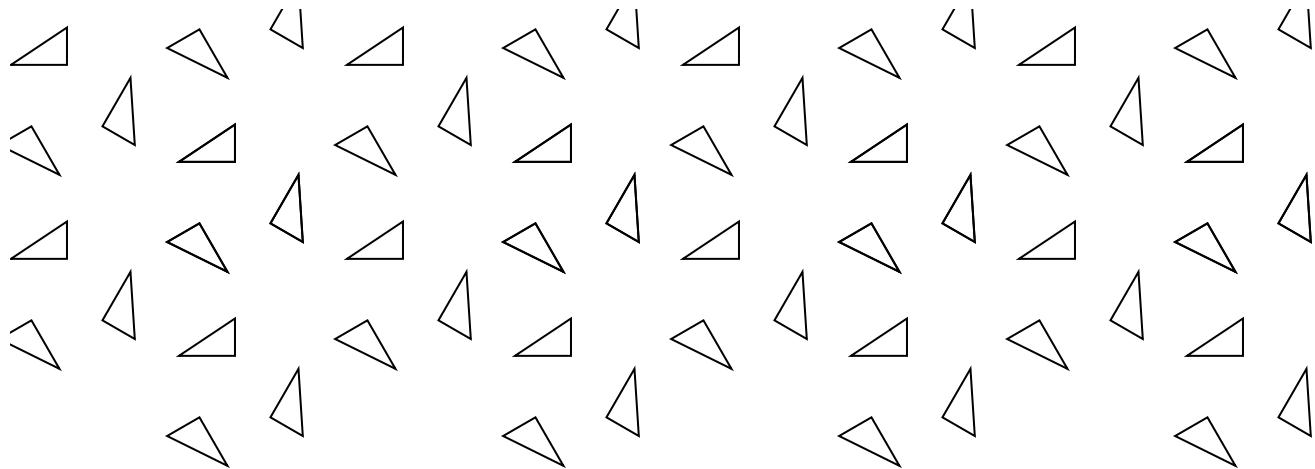
$\frac{1}{2}$ turn rotation:



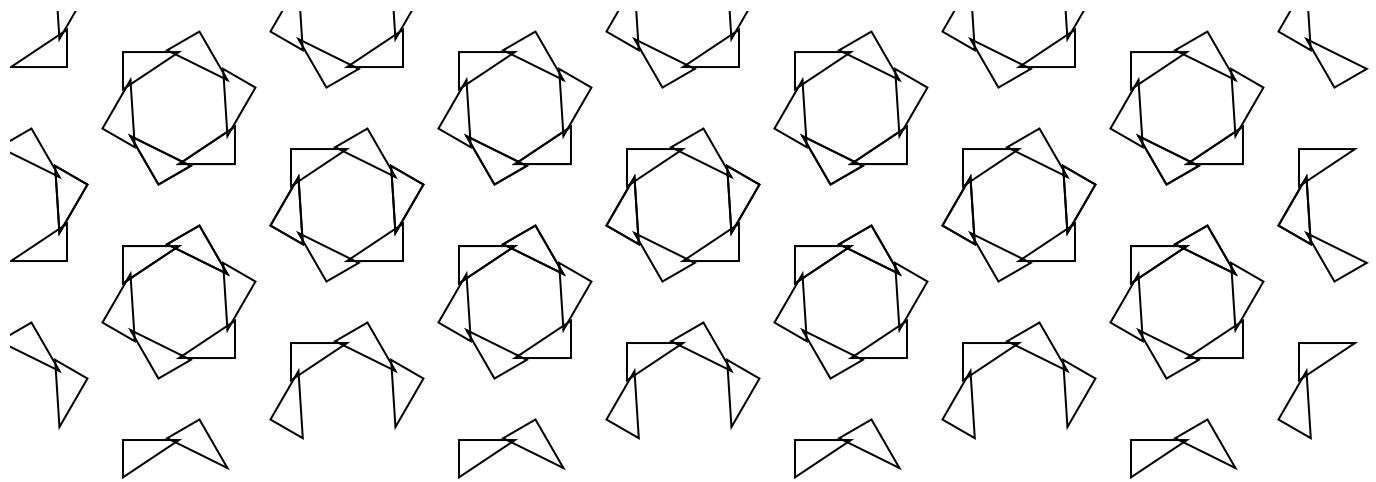
$\frac{1}{4}$ turn rotation:



$\frac{1}{3}$ turn rotation:



$\frac{1}{6}$ turn rotation:



These symmetries can be combined in patterns that repeat infinitely. Play with their combinations: <http://coloring-book.co/wallpaper>.

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CONTACT

If you are an **educator** I want to hear from you.

If you want to collaborate or host an event with this material, please reach out.

If you just read/colored/played with the content in this book, I want to know what you think.

Contact me directly at colorbymath@gmail.com.

Or submit feedback in this form: <http://coloring-book.co/form>

CODE

The code for this project is open source and welcomes contributions:

<https://github.com/aberke/coloring-book>

ABOUT THE AUTHOR & PROJECT

Alex Berke received degrees in mathematics and computer science from Brown University. After graduating college, she lived in New York City working as a technologist, with a focus on social impact. Most recently she worked as a software engineer at Google on the Search team's news credibility effort to fight disinformation, before entering graduate school at MIT.



She has benefited from her education in computer science, but only found computer science in college by pursuing her passion for mathematical thinking. She was only aware of this passion due to all the time she spent as a kid staring at mathematical designs and playing with logic puzzles, and then having a few adults around to suggest that the concepts she loved in these images and games were what math was all about. She feels lucky she had those resources and adults around her.

This project was produced to provide a resource for others to discover the beauty of mathematics. It intends to show that the best parts of math are not about numbers, and make those parts more accessible.