This coloring book is both digital and on paper.

The paper copy is where the coloring is done - color through the concepts to explore symmetry and the beauty of math.

The digital copy brings the concepts and illustrations to life in interactive animations.

Digital copy: http://coloring-book.co

The illustrations in this book are drawn by algorithms. The algorithms follow the symmetry rules for the illustrated groups. Many of these algorithms also add components of randomness so that each set of illustrations is unique.

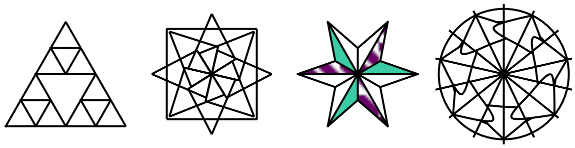
Illustrating Group Theory

A Coloring Book

Math is about more than just numbers. In this “book” the story of math is visual, told in shapes and patterns.

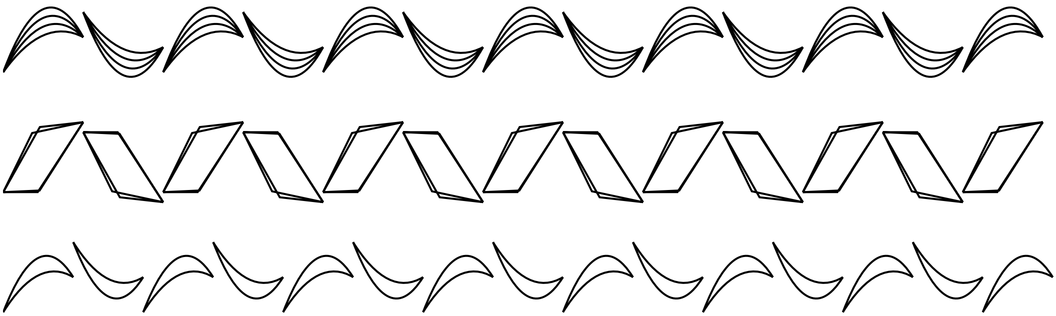
Group theory is a mathematical study with which we can explore symmetry.

We'll start coloring through the basics of **SHAPES & SYMMETRIES**



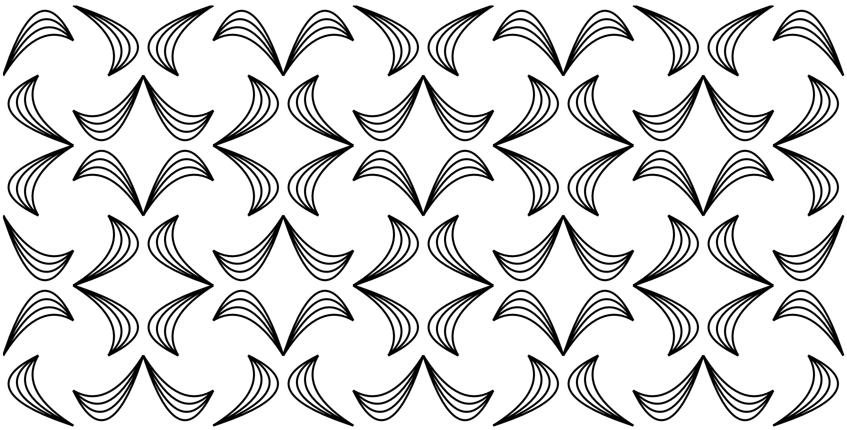
to build an understanding for more patterns and groups,

such as the **FRIEZE PATTERNS**



They start with a single shape that transforms and then repeats forever in opposite directions.

WALLPAPER PATTERNS have infinite repetitions and symmetries in even more directions.



WHO THIS BOOK IS FOR

This book is for children and adults alike. It is for math nerds, experts, and people who avoid the subject. It is for coloring enthusiasts as well as those who would prefer to simply read through or play with patterns. It is for educators and students, parents and children, and casual readers just looking to have a good time.

This book is for you.

WHAT THIS “BOOK” IS AND IS ABOUT

This is a “coloring book about math” that is both digital and on paper.

It is a playful book. The mathematical concepts it presents show themselves in illustrations that are interactive and animated online, and can be colored on paper. Throughout the book there are visual puzzles and coloring challenges to further engage the reader.

The book is about symmetry. Group theory is used as the mathematical foundation to discuss its content and interactive visuals are used to help communicate the concepts.

Group theory and other mathematical studies of symmetry are traditionally covered in college level or higher courses. This is unfortunate because these exciting parts of mathematics can be introduced with language that is visual, and with words that avoid jargon. Such an introduction is the intention of this “book”.

HOW TO USE THIS “BOOK”

This book is both on paper and online.

The two formats complement each other, and can be used together. Their content is the same, but they provide different ways to more deeply engage or play with it.

Color the illustrations on paper. Only on paper can the coloring challenges be fully completed and realized in color.
Play with the illustrations online. They come to life with interactive animations that show the symmetries that generate them.

This book can be used as a playful educational tool to serve as an additional resource in the classroom or home. For educators, the challenges within the pages of the book can be used as “problem sets”.
This book can be used as a relaxing coloring book.
This book can be used to entertain your mathematical intuition or interests.

Contents

SHAPES & SYMMETRIES

INTRODUCTION

Symmetry presents itself in nature.

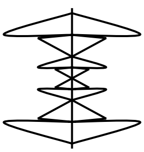
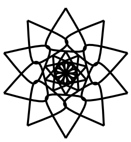
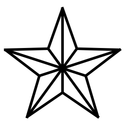
Landscape reflected in water

We can see symmetry in the repetitions, reflections, and turns in life around us, but these symmetries often have imperfections.

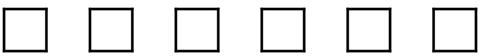


Moth *Sunflower* *Starfish*

Math creates a space where perfect symmetry can be explored.

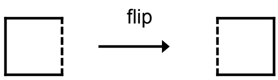
In our real physical world, lines may not be perfectly straight, and squares may not be perfectly square, but mathematics allows us to believe in straight lines and perfect squares.



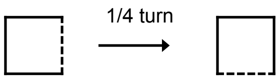
Throughout this book, we will pretend we are in that mathematical space. We will ignore the imperfections in our drawings, and see shapes and patterns as if they are composed of perfect lines and curves. We will play with our shapes and patterns, using color to manipulate their symmetries, and even destroy them at times, all in order to better understand them.

Let's talk about symmetry. See, some shapes have more symmetry than others.

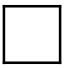
If while you blinked, a square was flipped,



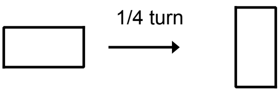
or turned a quarter of the way around,



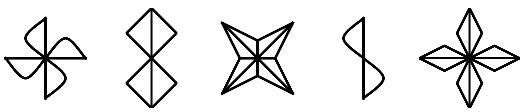
you would then still see the same square and not know.



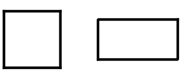
Yet this is not the case for a rectangle...



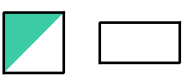
Check in: Which of these shapes can be rotated by a ¼ turn without changing in appearance?



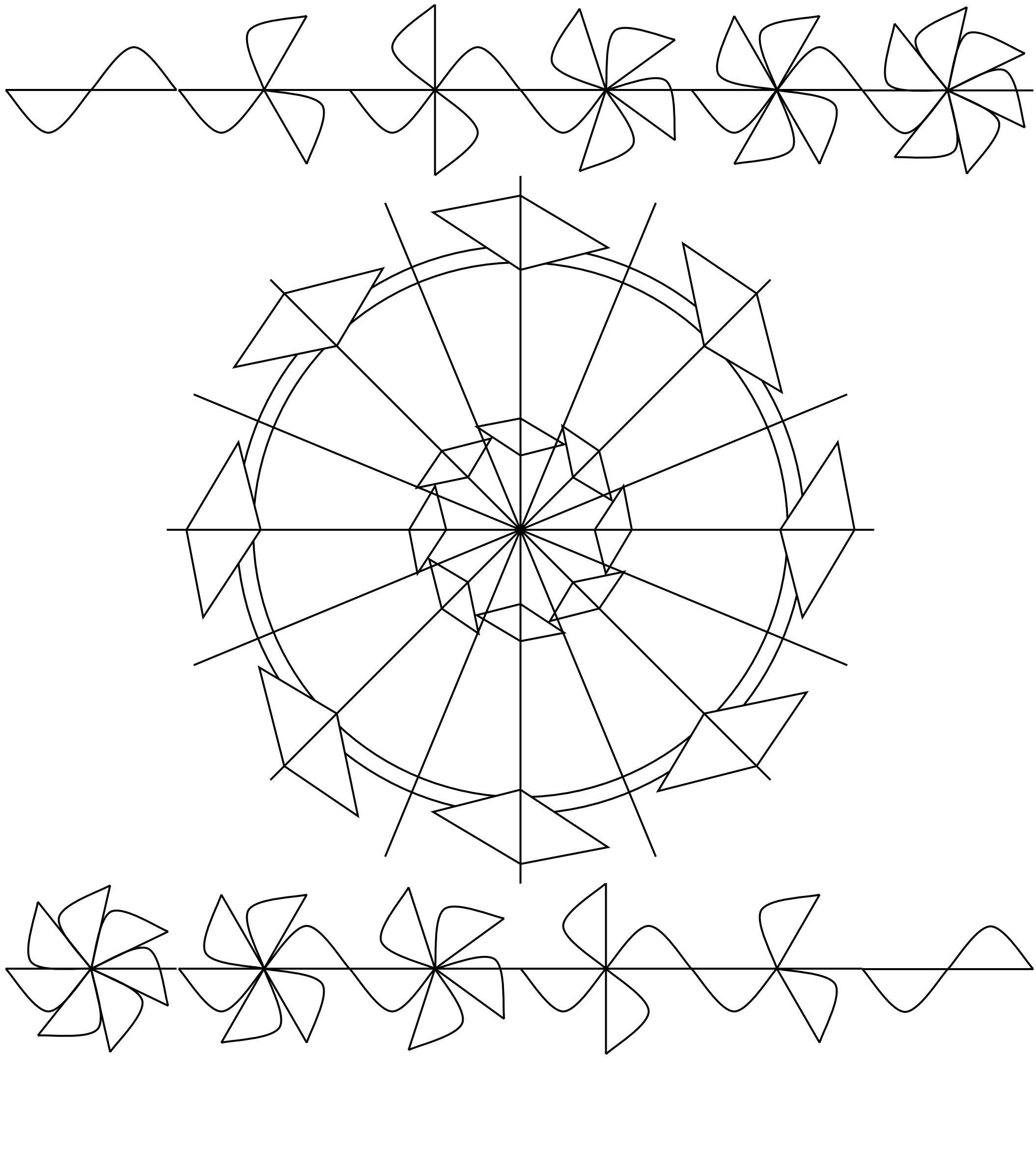
The symmetries of our shapes are the transformations that leave our shapes unchanged. We can see that a ¼ turn is a symmetry of a square but not for a rectangle, and we can intuitively see that a square is "more symmetric" than a rectangle because it can be flipped and turned in more ways.



We will also see how this can change once color is added.

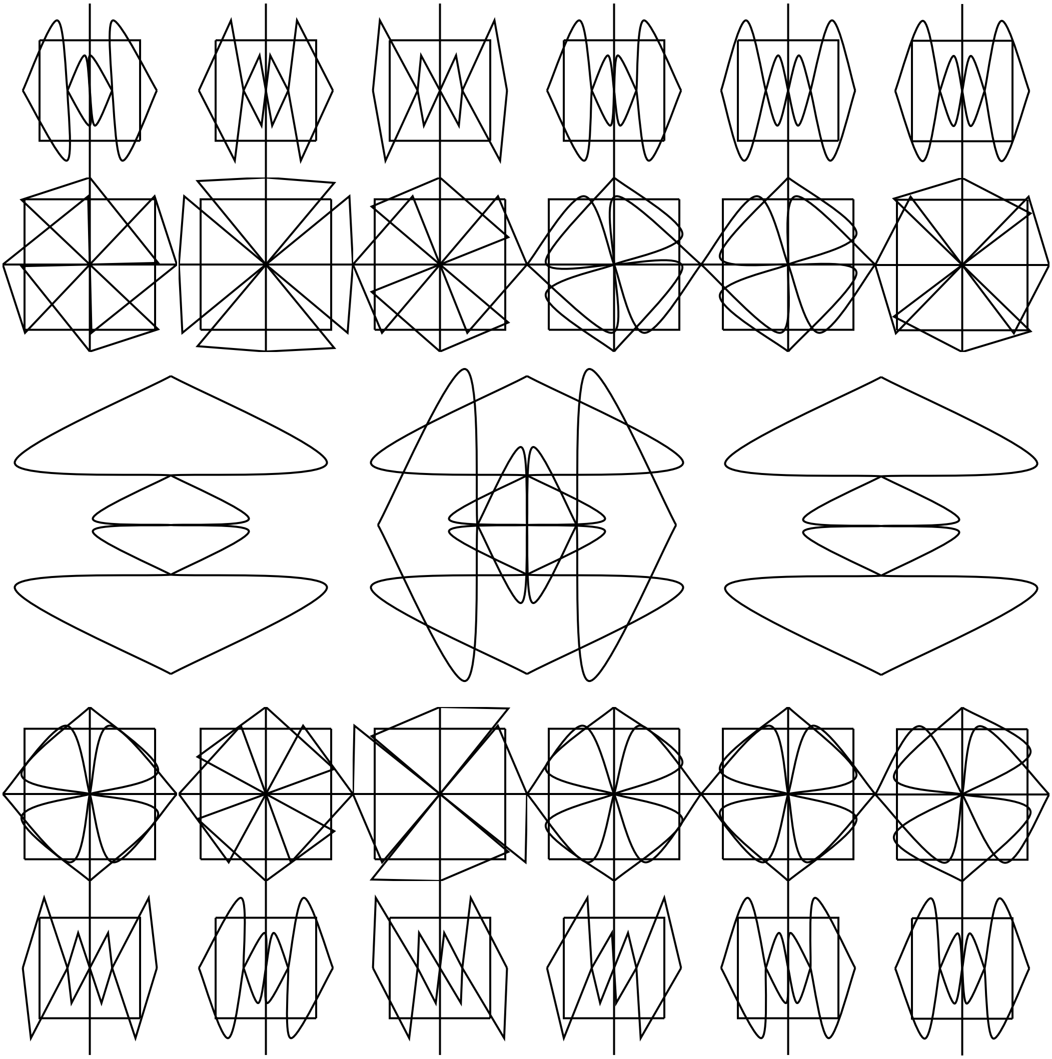


Can you color the shapes to make them "less symmetric"?



Can you see which shapes have ¼ turns and which do not?

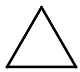
Color the shapes with ¼ turns with a different set of colors than the shapes that do not have ¼ turns.



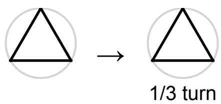
shapes with ½ turns and shapes with ¼ turns

ROTATIONS

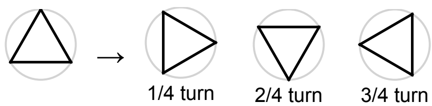
A regular triangle has equal side lengths and equal angles.



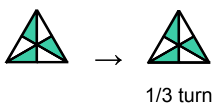
What’s more, it can rotate ⅓ of the way around a circle and appear unchanged. Had our eyes been closed when it rotated, we would not have noticed a difference.



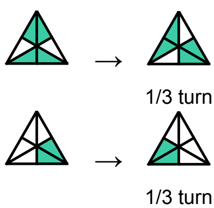
If the triangle instead rotates by an arbitrary amount, like ¼ of the way around a circle, it will then appear changed, since it is oriented differently.



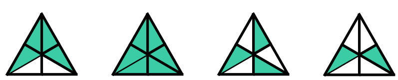
We can even find ways to color the triangle so that a ⅓ turn still does not change it.



While this will not work for other ways.

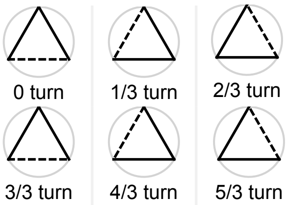


Check in: Which of the following colored triangles can be rotated by a ⅓ turn without changing in appearance?

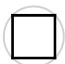


Our triangle can also rotate by more than a ⅓ turn without changing. It can rotate by twice that much - ⅔ of the way around the circle - or by 3 times that much, which is all the way around the circle.

We can keep rotating - by 4 times that much, 5 times that much, 6 times... and keep going. The triangle seems to have an infinite number of rotations! But after 3 they become repetitive.



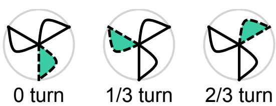
Check in: How many ways can a square rotate without changing before the ways become repetitive?



The triangle has only 3 unique rotations, so we'll talk about rotations that are less than a full turn. When we say our triangle 'has 3 rotations' we mean it can be rotated by these 3 different turns and appear unchanged.

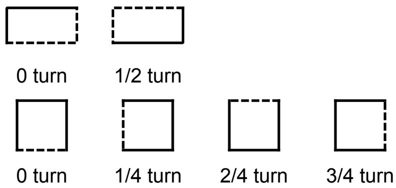
Other shapes have these same 3 rotations. For this reason, we can say they all share the same symmetry group.

However, their rotations can be removed by adding color.

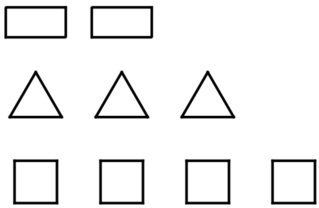


Now when our shape is rotated, its color shows it.

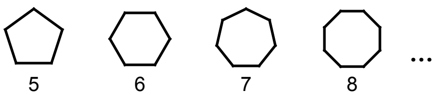
Now that we can count rotations, we can be more precise when we say a square has more symmetry than a rectangle.



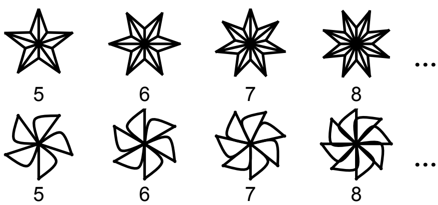
We can also see that a square has more rotational symmetry than a triangle, which in turn has more than a rectangle: A rectangle has only 2 unique rotations, while our triangle has 3, and a square has 4.



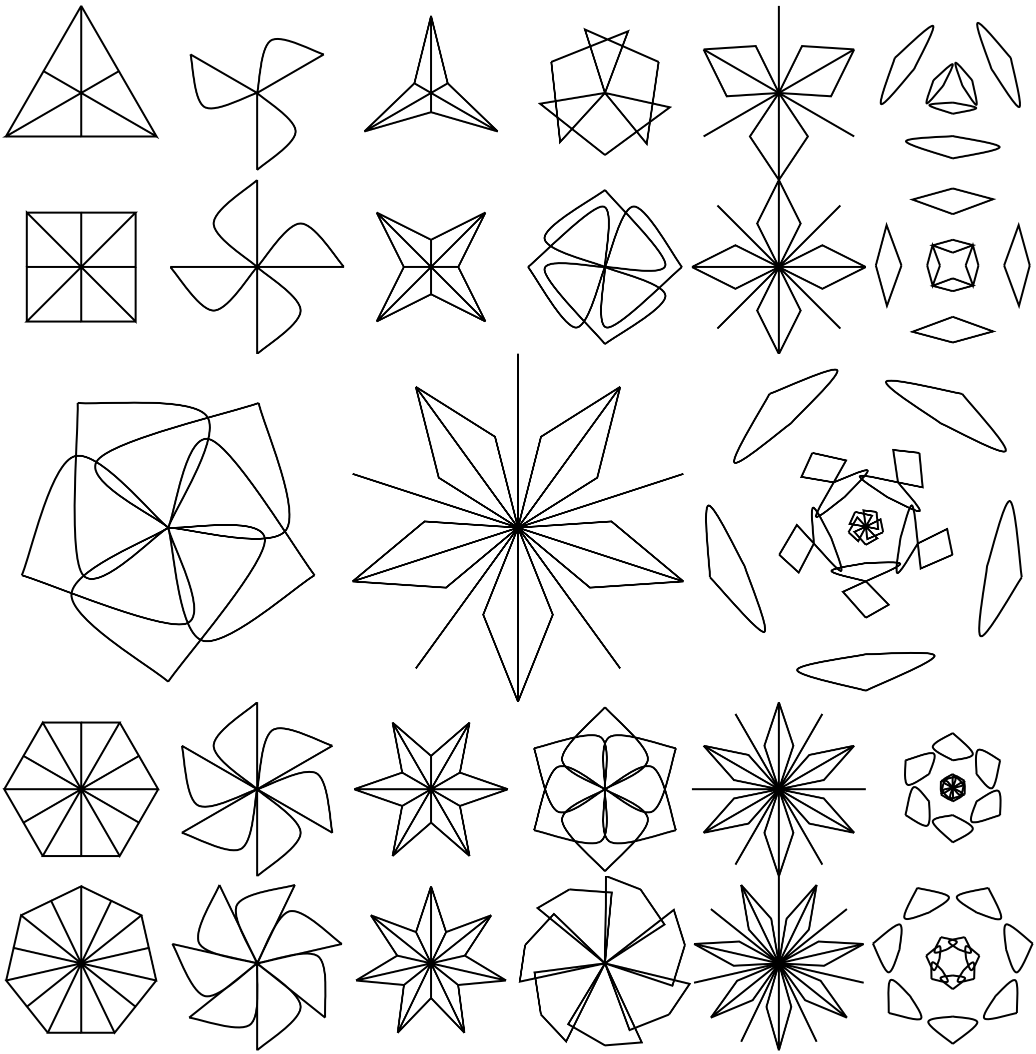
We don't need to stop at 4 rotations. We can find shapes with 5 rotations, 6 rotations, 7, 8, ... and keep going towards infinity.



And these shapes don't even need to be so simple.

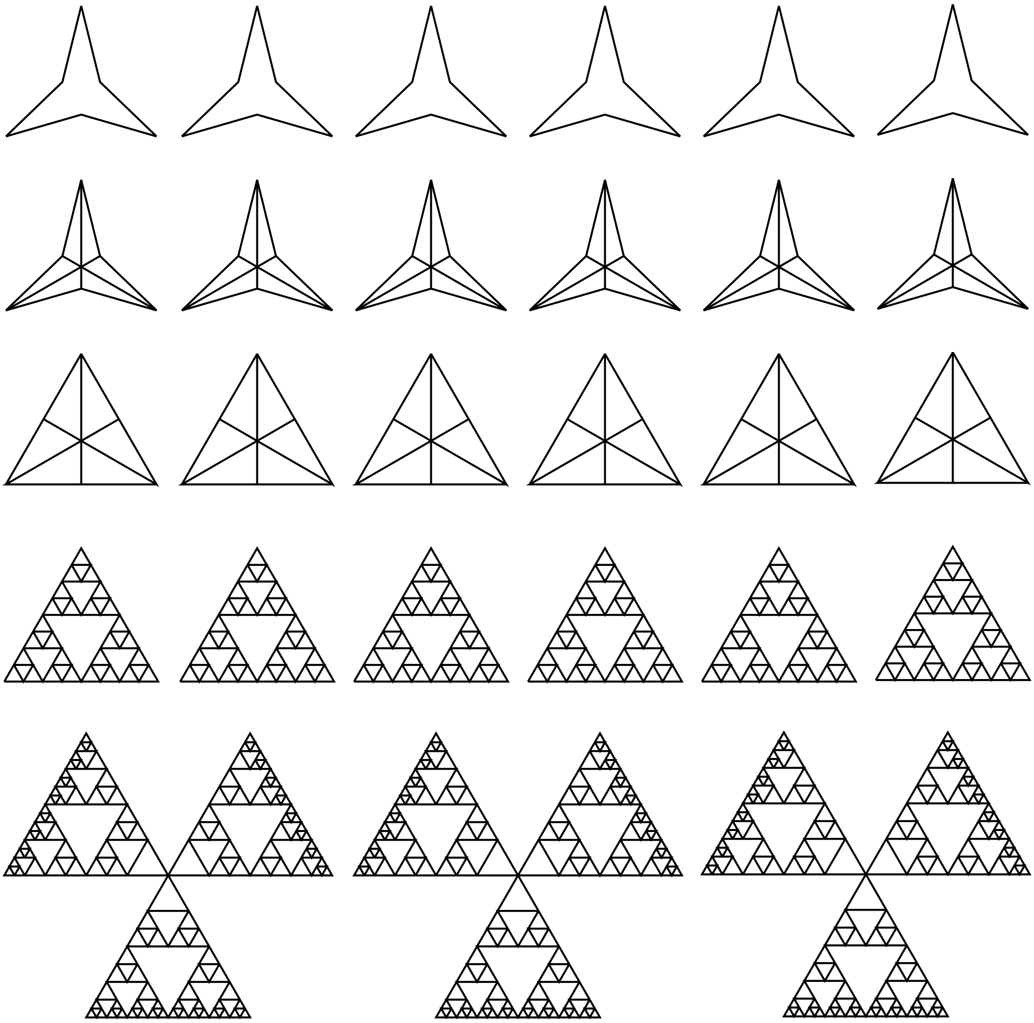


Can you find all of the shapes with 7 rotations?
Color the shapes so that they no longer have any rotations.



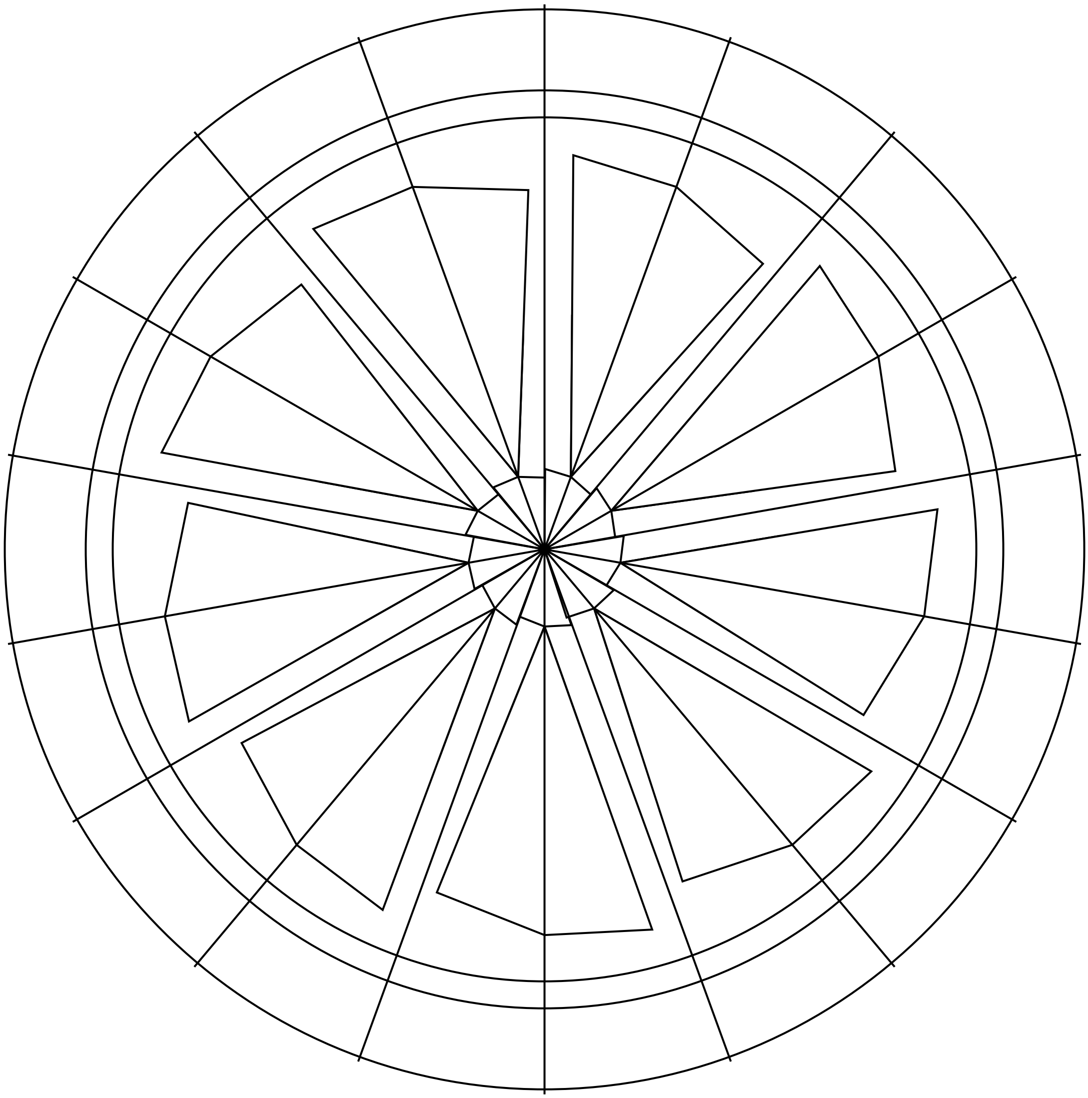
shapes with 3, 4, 5, 6, 7 rotations

Color the shapes so that a ⅓ turn continues to leave their appearance unchanged.



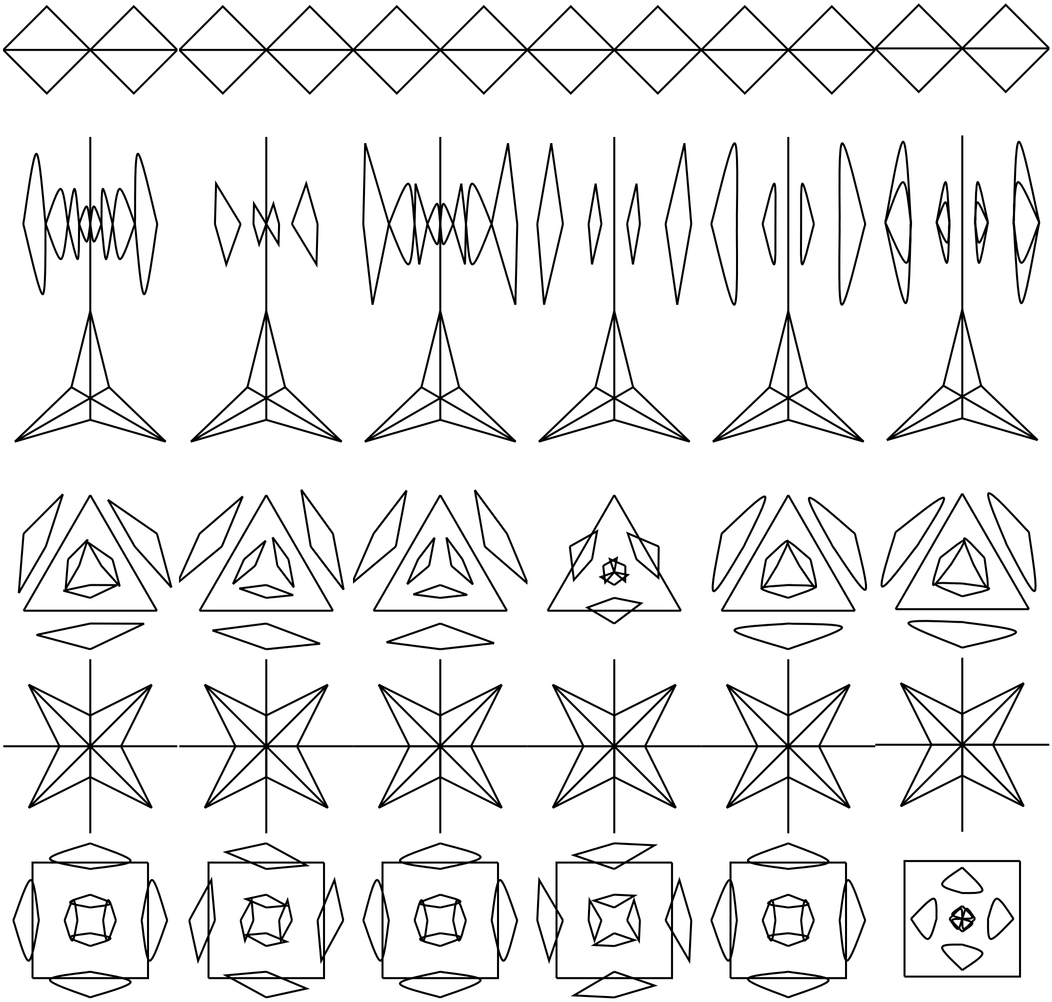
shapes with ⅓ turns and sierpinski triangles

Can you see all of the rotations for this shape?
Color the shape so that it has only 3 unique rotations.



circular pattern with 9 rotations

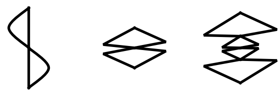
Color the shapes with 4 rotations so that they have only 2 rotations.

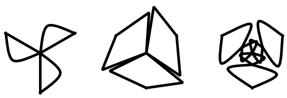


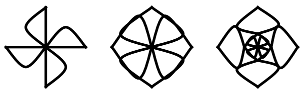
shapes with 2, 3, 4 rotations

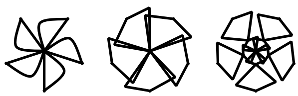
The rotations we have been finding for shapes are symmetries of these shapes - they are transformations that leave the shapes unchanged. When shapes have the same symmetries, they share a symmetry group.

Giving names to the groups that our shapes share will help us talk about and play with them later. We can call the group with 2 rotations C2, and call the group with 3 rotations C3, call the group with 4 rotations C4, and so on...

C2: 

C3: 

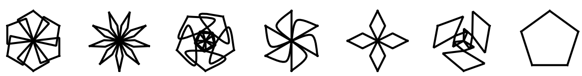
C4: 

C5: 

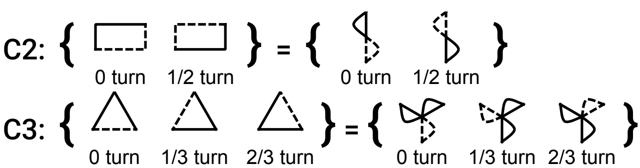
...

These groups are called the cyclic groups.

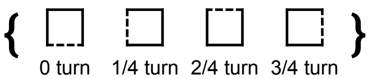
Check in: Which shapes illustrate C6?



Our shapes help us see our groups, but **the members of the groups are the rotations**, not the shapes.

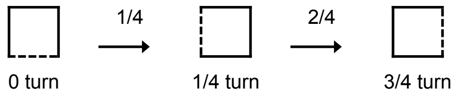


The rotations within each group are related to each other...

C4: 

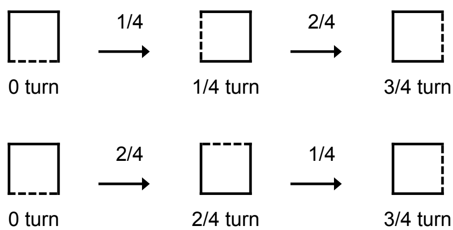
Another way to think about rotating a C4 shape by a $\frac{3}{4}$ turn is to rotate it by a $\frac{1}{4}$ turn and then rotate it again by a $\frac{2}{4}$ turn.

C4: $\frac{1}{4}$ turn $\ast$ $\frac{2}{4}$ turn $\rightarrow$ $\frac{3}{4}$ turn



Notice that the order in which these rotations are combined does not matter. For this reason we say the cyclic groups are <u>commutative</u>.

C4: $\frac{1}{4}$ turn $\ast$ $\frac{2}{4}$ turn = $\frac{2}{4}$ turn $\ast$ $\frac{1}{4}$ turn

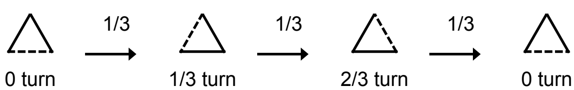


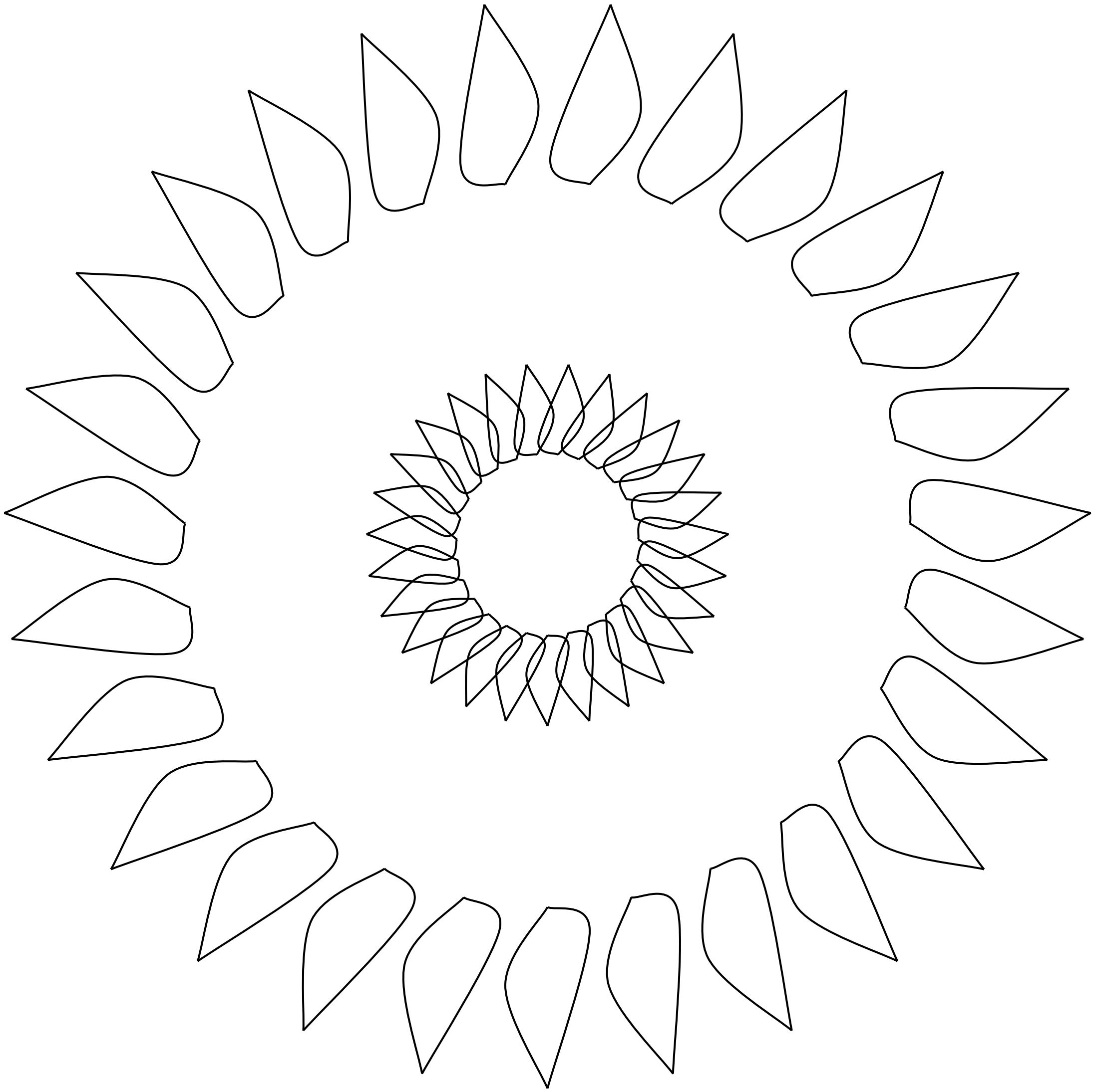
Similarly, for our C3 group, a $\frac{2}{3}$ turn is the same as combining a $\frac{1}{3}$ turn with another $\frac{1}{3}$ turn.

C3: $\frac{1}{3}$ turn $\ast$ $\frac{1}{3}$ turn $\rightarrow$ $\frac{2}{3}$ turn

Adding another $\frac{1}{3}$ turn brings the shape back to its starting position - the 0 turn.

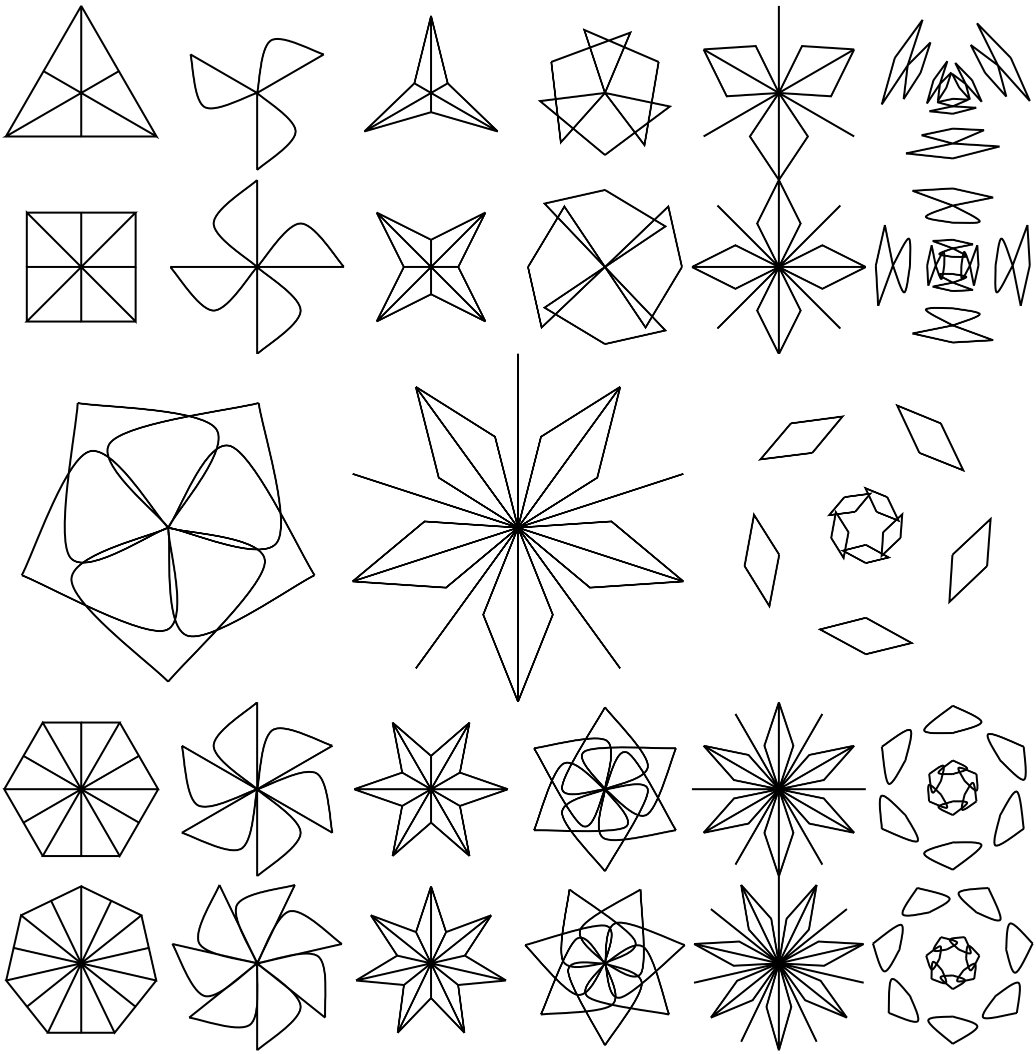
C3: $\frac{1}{3}$ turn $\ast$ $\frac{1}{3}$ turn $\ast$ $\frac{1}{3}$ turn $\rightarrow$ 0 turn





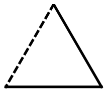
C27 shape (circular pattern)

Can you find all of the C5 and C6 shapes?
Color the C4 shapes with as many colors as possible while keeping them as C4 shapes.

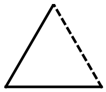


C3, C4, C5, C6, C7 shapes

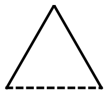
We saw that the ⅓ turn did something special for our C3 group. We were able to combine it with itself again and again in order to generate all of the rotations of C3 - it is a <u>generator</u> for our C3 group.

⅓ turn $\longrightarrow$ 

1/3 turn

⅓ turn $\ast$ **⅓ turn** $\longrightarrow$ 

2/3 turn

⅓ turn $\ast$ **⅓ turn** $\ast$ **⅓ turn** $\longrightarrow$ 

0 turn

In the same way that a ⅓ turn is a generator for our C3 group, we can see that a ¼ turn is a generator for our C4 group.

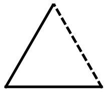
C3: **⅓ turn** $\rightarrow$ {  }

0 turn 1/3 turn 2/3 turn

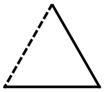
C4: **¼ turn** $\rightarrow$ {  }

0 turn 1/4 turn 2/4 turn 3/4 turn

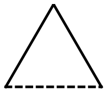
We might even choose different generators to end up with the same result...

⅔ turn ⟶ 

2/3 turn

⅔ turn ✷ ⅔ turn ⟶ 

4/3 = 1/3 turn

⅔ turn ✷ ⅔ turn ✷ ⅔ turn ⟶ 

6/3 = 0 turn

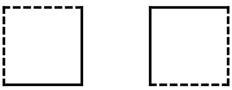
See, we can use a ⅔ turn as a generator and still end up with our C3 group.

C3: ⅔ turn → {  }

0 turn 1/3 turn 2/3 turn

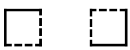
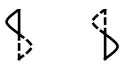
But beware we must be careful: not all members of our groups are generators.

For example, a 2/4 turn does not generate all of the rotations of our C4 group.



0/4 turn 2/4 turn

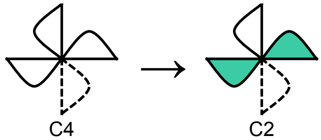
Instead a 2/4 turn generates a smaller group - our C2 group.

2/4 turn → {  } = {  }

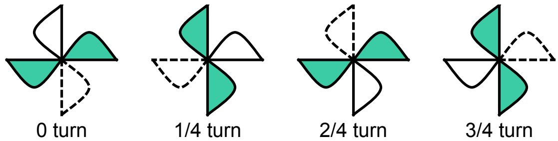
0/4 turn 2/4 turn 0/2 turn 1/2 turn

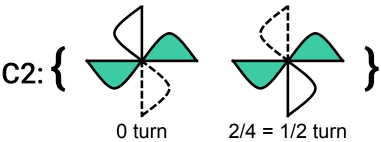
Another way to see this is with color...

We can transform a C4 shape into a C2 shape by coloring it.

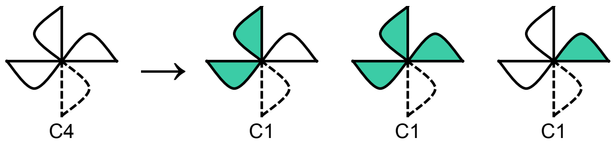


Now the only rotations that leave this colored shape unchanged are those of C2.



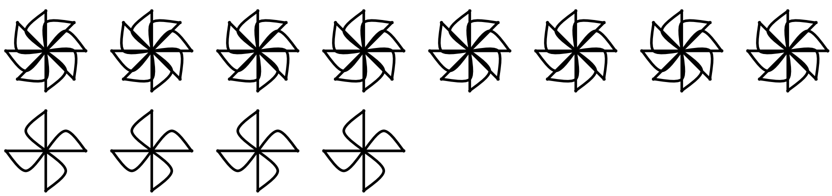


Again we must be careful. Not all colorings of our C4 shapes will transform them into C2 shapes. Some will remove their rotations altogether and leave them with just the 0 turn.

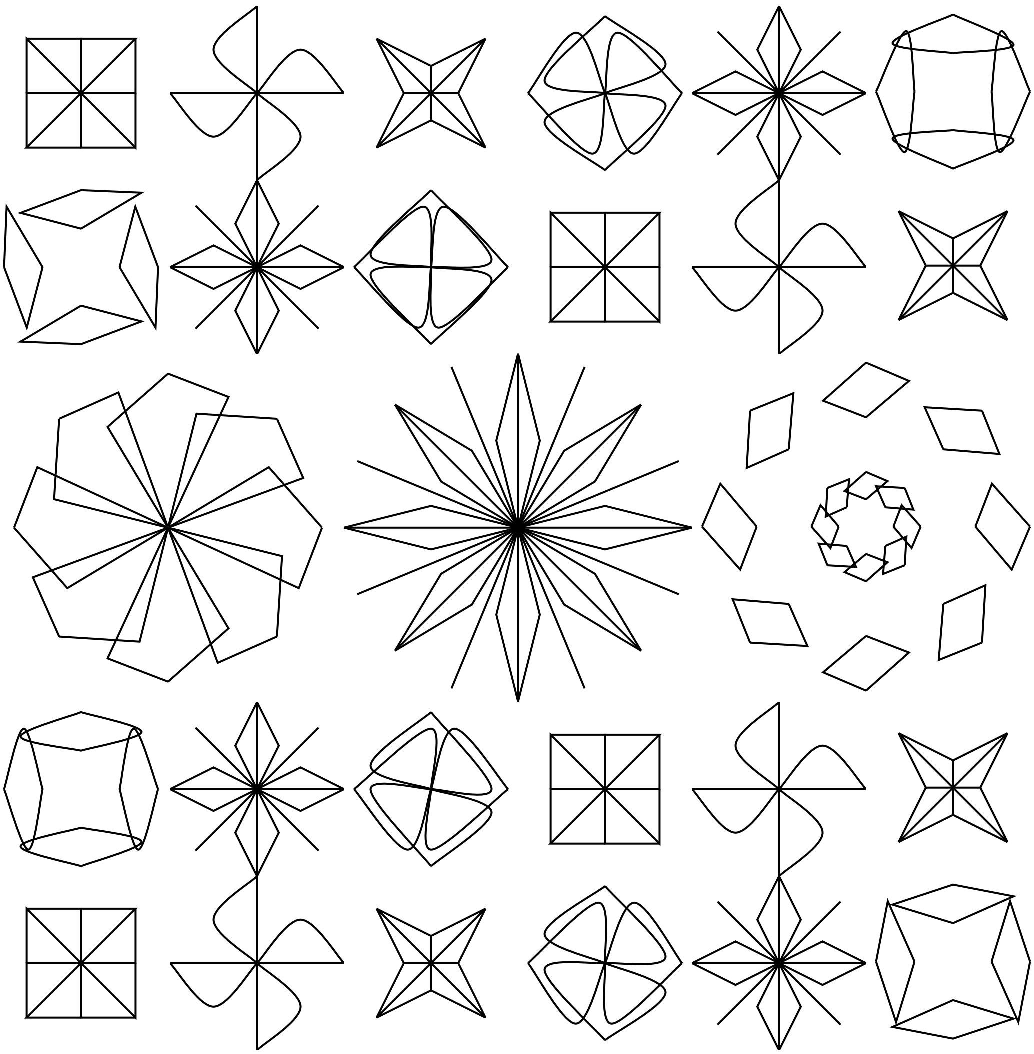


Challenge: Find all the generators for C4 and C8.

Challenge: Which rotations of C8 generate our C4 group but not C8?

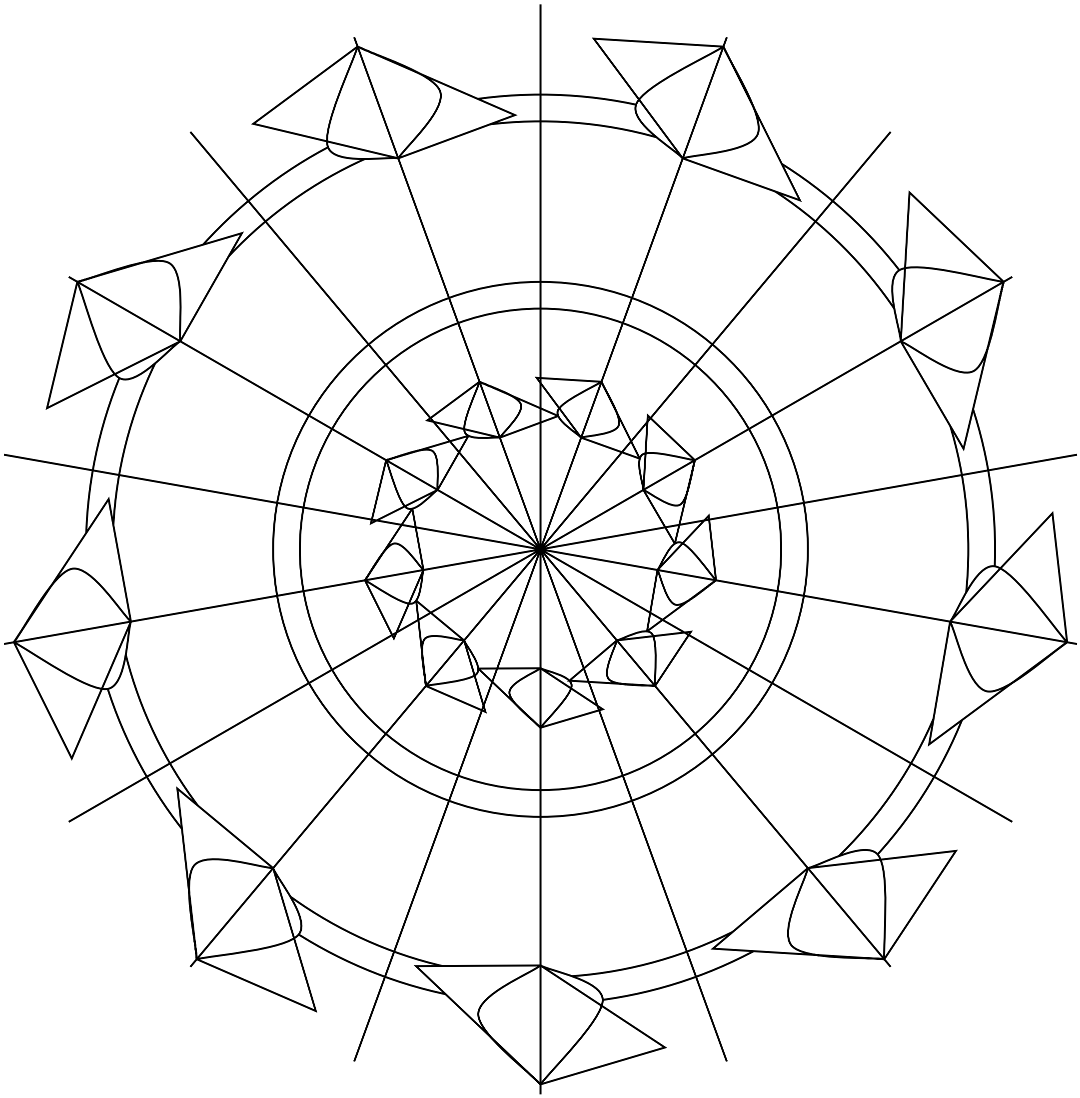


Can you use color to transform the uncolored shapes into C2 shapes?



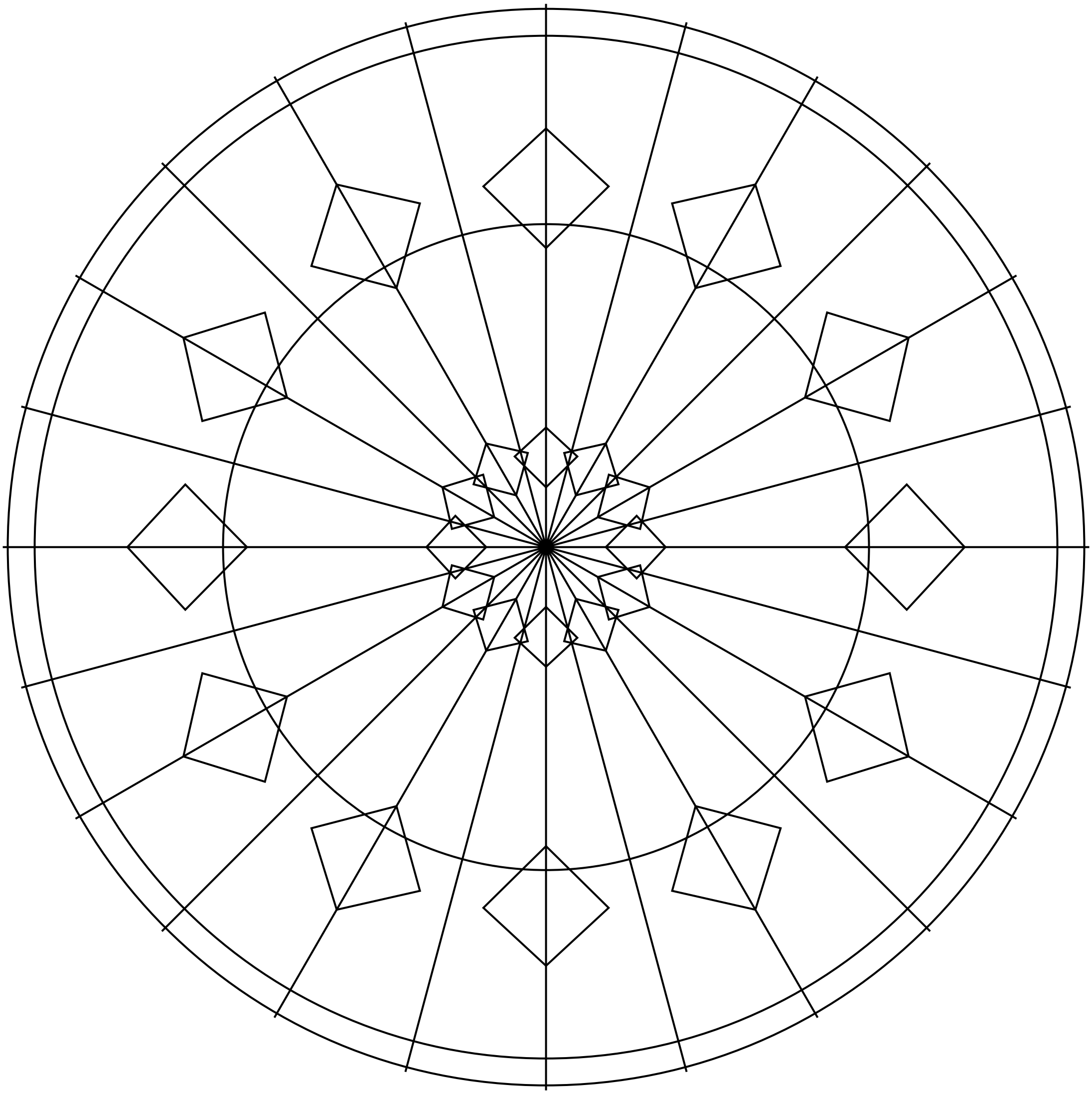
C4 and C8 shapes

The C9 shape below is made up of pieces that repeat around a circle. Go clockwise around the circle, coloring every other repeated piece in the same way. That is, color a piece, skip a piece, color the next piece the same way as the first, and keep going. Do you end up coloring every piece? Can you use this to prove a 2/9 turn is or isn't a generator for C9?



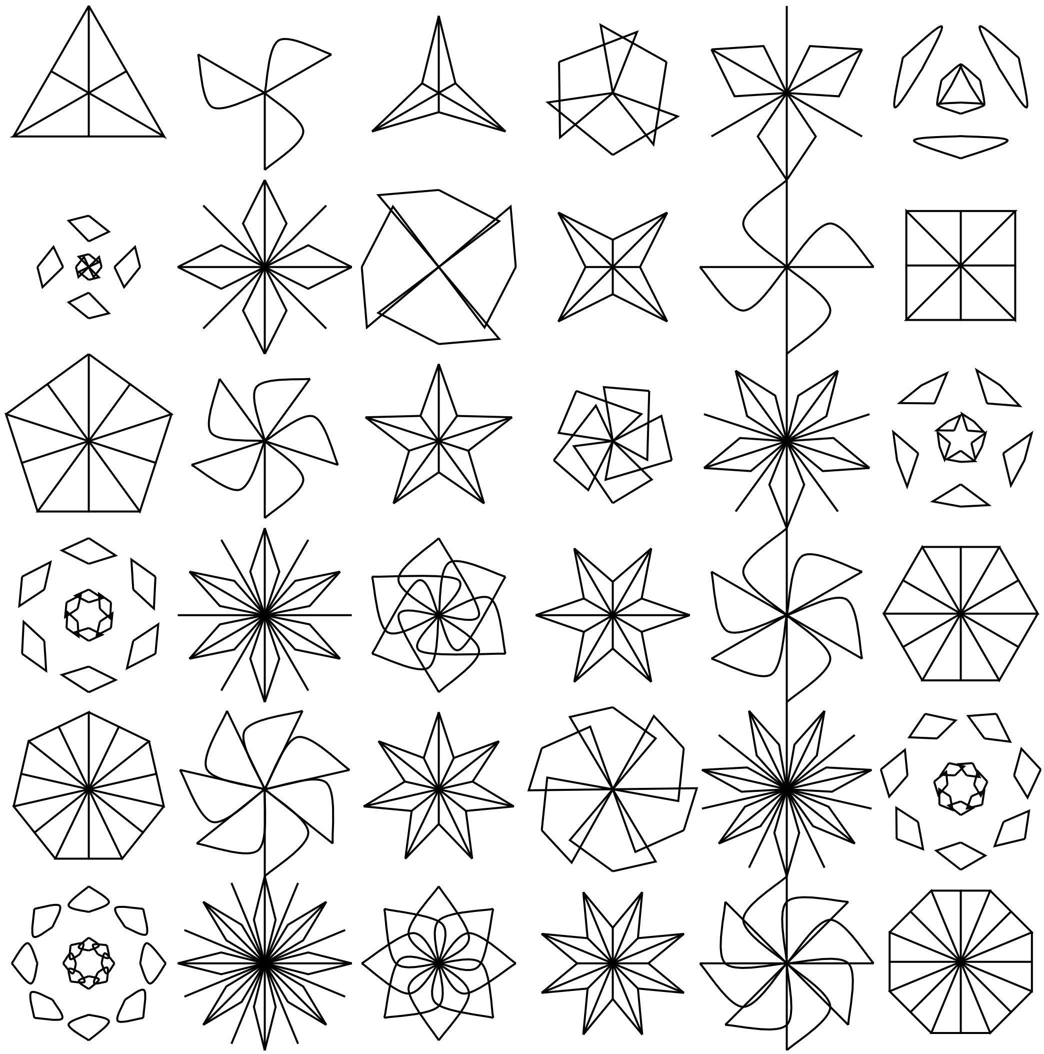
C9 shape (circular tessellation)

Can you show that a 3/12 turn is not a generator for C12 by coloring every other 3 pieces in the same way? What group does the 3/12 turn generate?



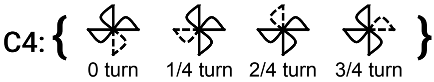
C12 shape (circular tessellation)

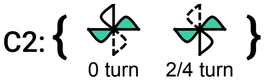
For some cyclic groups, any of their transformations can be used as generators. Which groups are these?

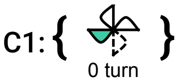


shapes with 3, 4, 5, 6, 7, 8 rotations

Color can reduce C4 shapes to C2 or C1 shapes because C2 and C1 are <u>subgroups</u> of C4. A <u>subgroup</u> is a group contained within a group.

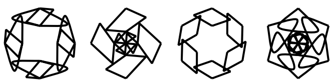




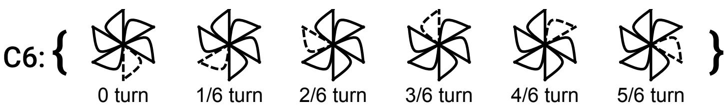


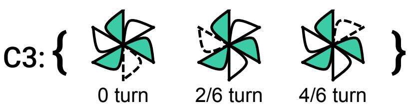
Similarly, C1, C2, and C3 are all <u>subgroups</u> of C6.

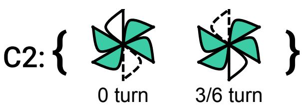
Check in: Can you see how color can reduce the C4 and C6 shapes to C1 or C2 shapes?



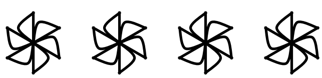
It is easy to see that a group has all of the rotations of its subgroups,





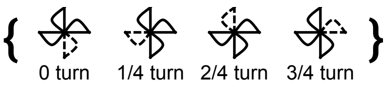


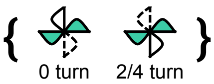
but we cannot simply pick out a few rotations from a group and call them a subgroup. See for yourself: Try to color a C6 shape so that it has only the rotations of C4.



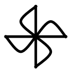
It can't be done - C4 is not a <u>subgroup</u> of C6. There is more to it than that...

When we use color to reduce our shapes to represent smaller groups, we give them a new set of rotations.

C4: 

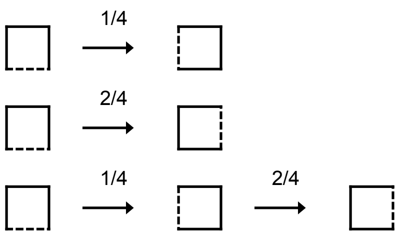
C2: 

Not all sets of rotations are groups, and therefore cannot be <u>subgroups</u>. Try to color a shape in a way so that it has *only* a 0 turn, $\frac{1}{4}$ turn, and a $\frac{2}{4}$ turn.



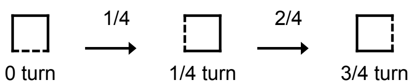
It's impossible without also giving the shape a $\frac{3}{4}$ turn. **{0 turn, $\frac{1}{4}$ turn, $\frac{2}{4}$ turn }** is not a group, but **{0 turn, $\frac{1}{4}$ turn, $\frac{2}{4}$ turn, $\frac{3}{4}$ turn }** is. Why? This brings us back to combining rotations.

In order for a set of rotations to be a group, any combination of rotations in the set must also be in the set. This rule is called <u>group closure</u>, and we can see it by looking at our shapes. If transforming our shape by *either* a $\frac{1}{4}$ turn or a $\frac{2}{4}$ turn leaves our shape unchanged, then transforming our shape by a $\frac{1}{4}$ turn and *then* a $\frac{2}{4}$ turn must also leave our shape unchanged.

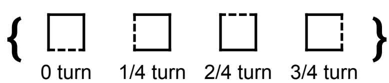


But we already saw that this is the same as just transforming the shape by the combination of these turns! Remember, the elements in our groups are the transformations that leave our shapes unchanged, so this combination must also be in our group.

C4: $\frac{1}{4}$ turn $*$ $\frac{2}{4}$ turn = $\frac{3}{4}$ turn



For this same reason, once we have a generator in our group, we have all the other transformations that it generates.

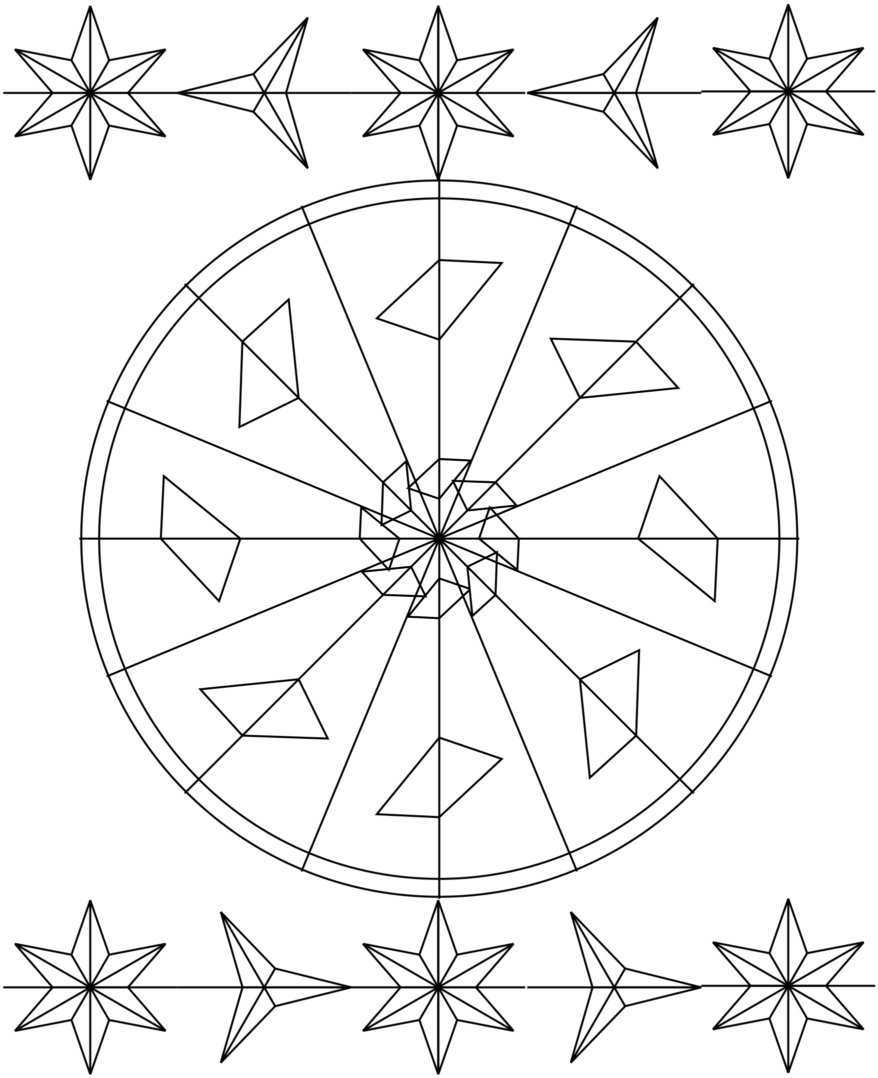
C4: **$\frac{1}{4}$ turn →** 

So far we've only been talking about groups of rotations. These groups are <u>cyclic</u>. They can be created by combining just one rotation - a <u>generator</u> - multiple times with itself.

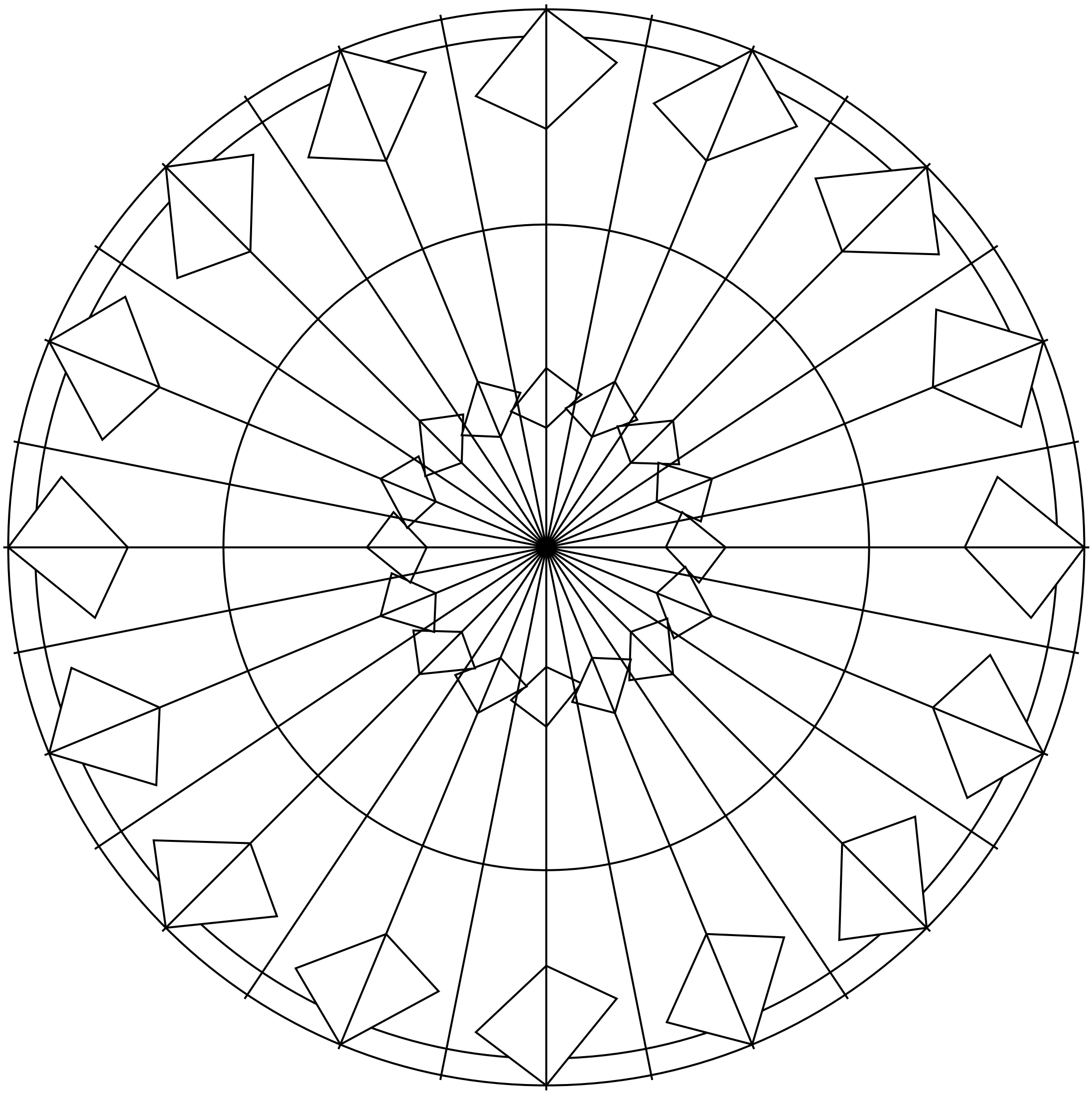
C3: $\frac{1}{3}$ **turn** → {

}

Our next groups have even more <u>generators</u> and symmetries to play with, such as reflections.

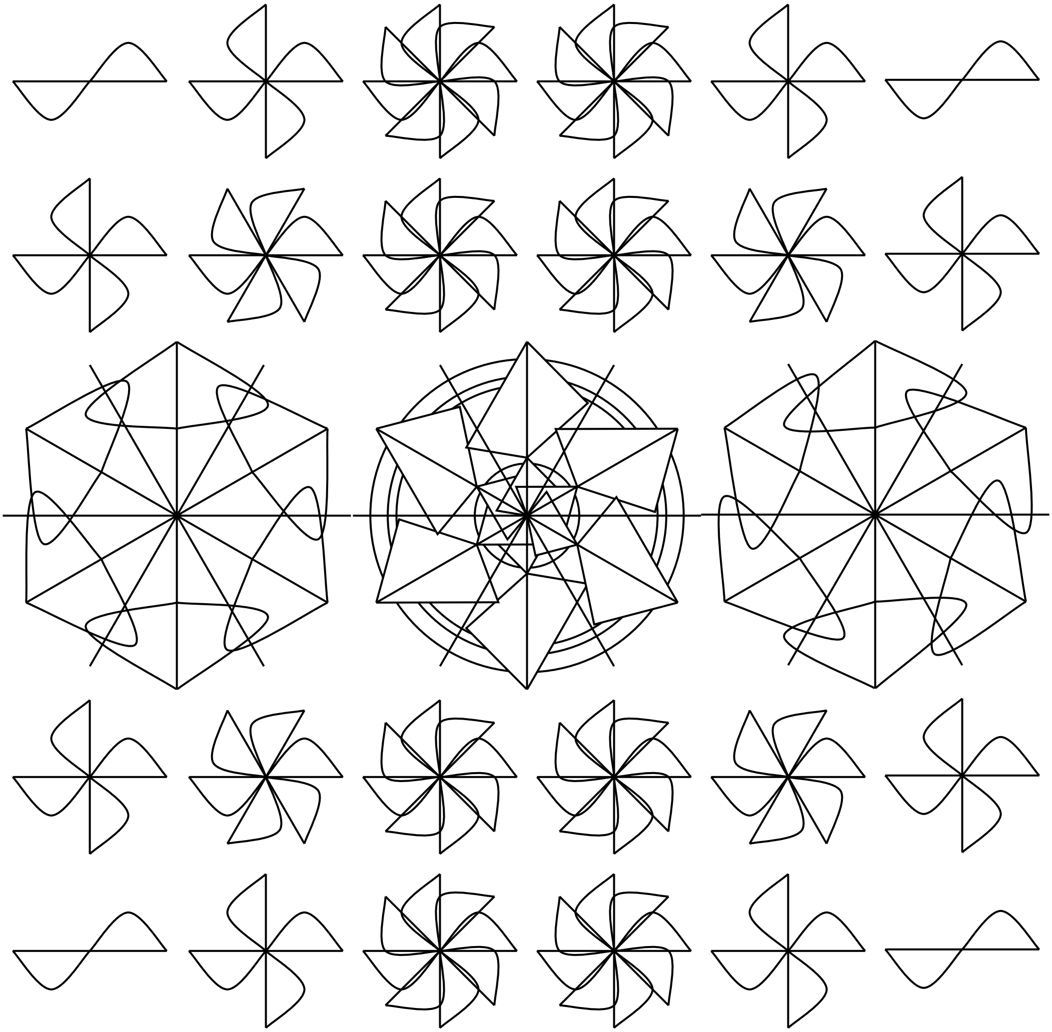


Color the shape to reduce it to a C8 shape. Then add more color to reduce it to a C4 shape. Can you again add more color to reduce it to show an even smaller subgroup? What are the subgroups of C16?



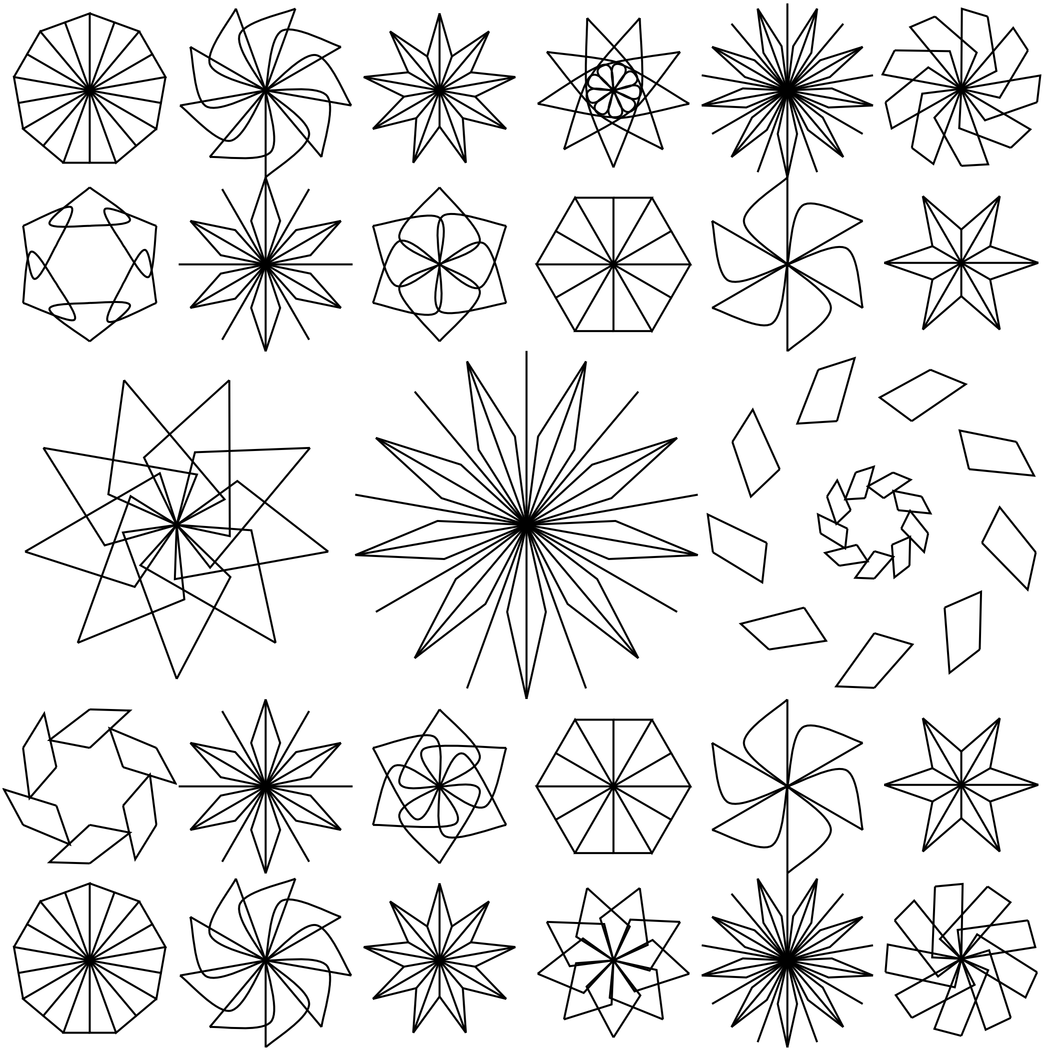
C16 shape (circular tessellation)

Color the shapes to make them all C2 shapes while using as many colors as possible. How many rotations did you remove with color? How many colors were you able to use?



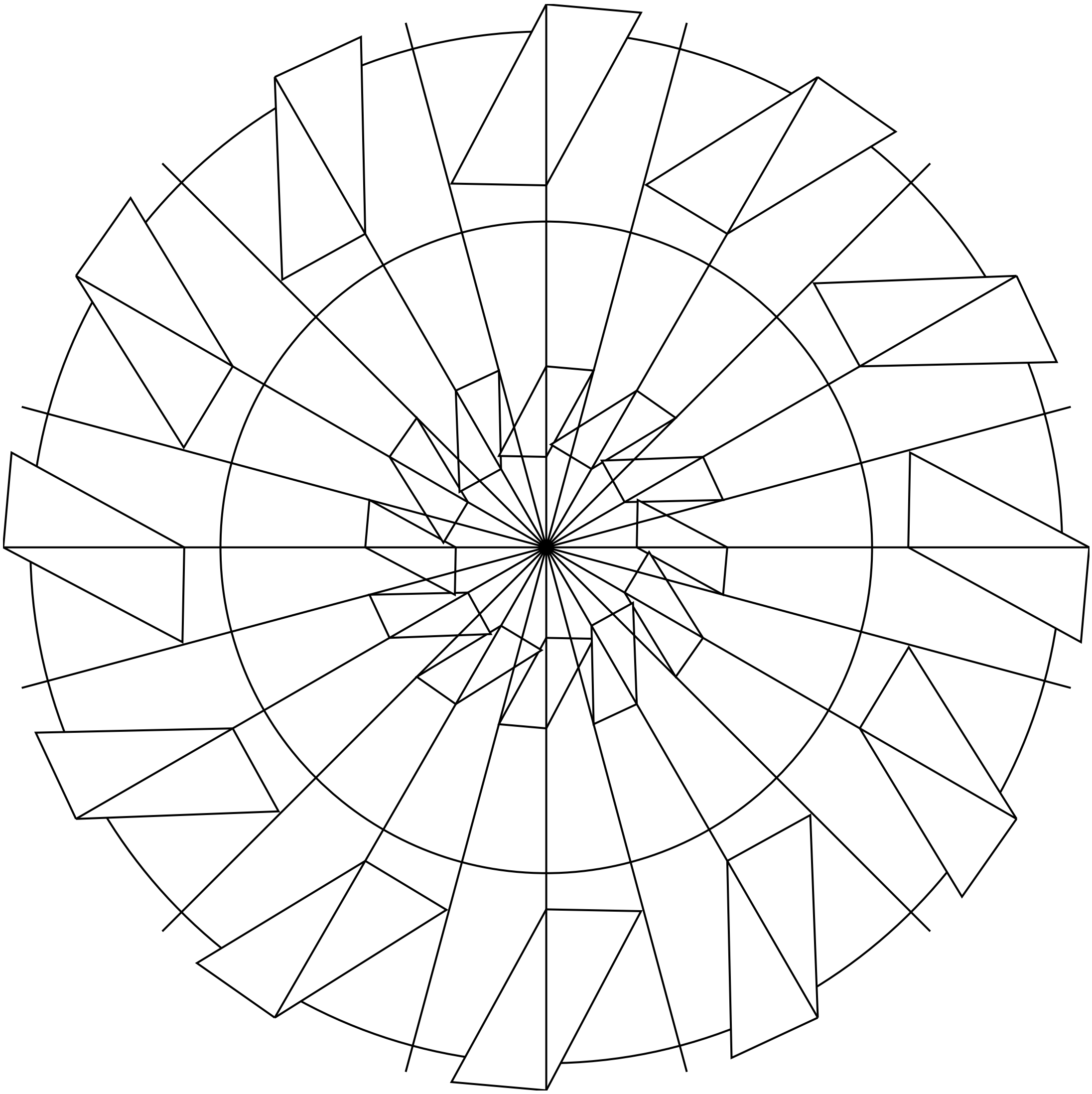
C2, C4, C6, C8 shapes

These shapes illustrate groups that share a common subgroup. Can you color the shapes to remove rotations so that they illustrate their common subgroup?

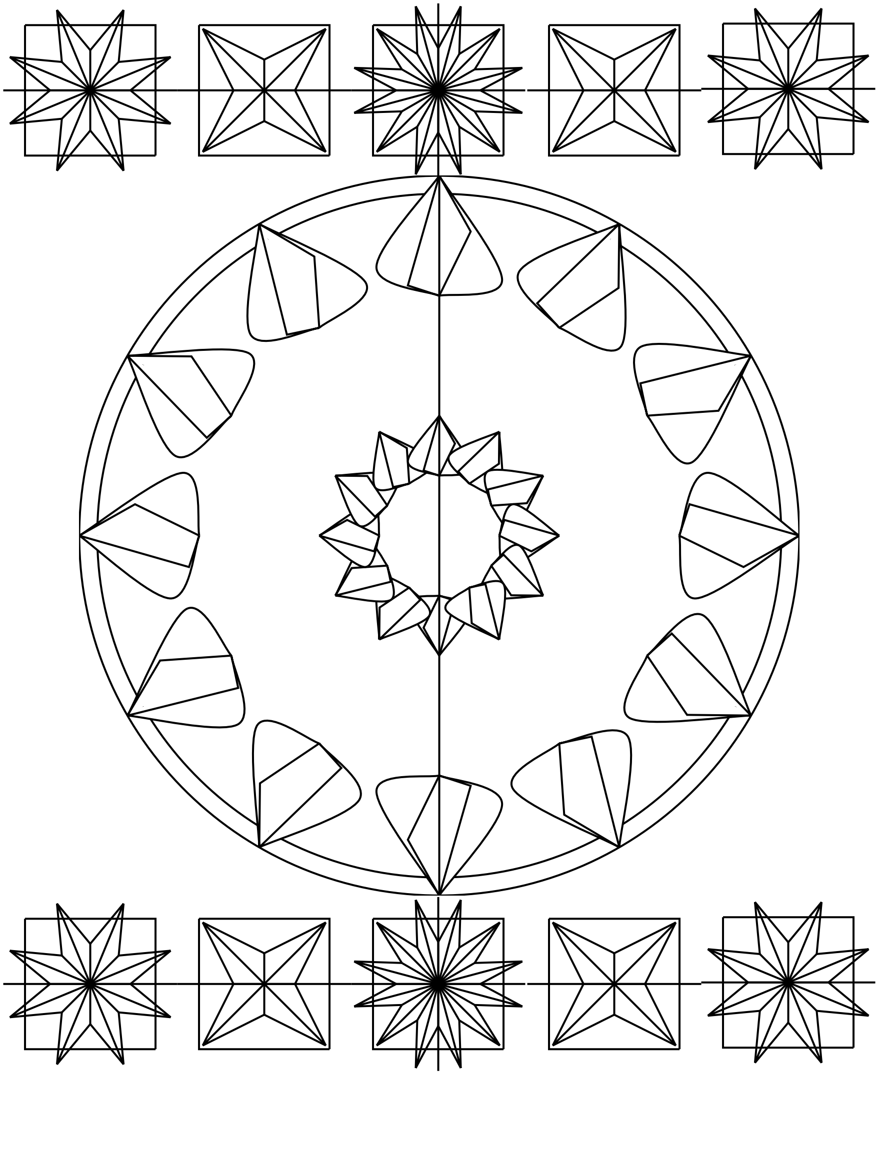


shapes with 6 rotations and shapes with 9 rotations

Use color to reduce the C12 shape to a C6 shape. Is it possible to add more color to reduce it to a C4 shape? What about a C3 shape?

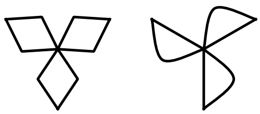


C12 shape (circular tessellation)

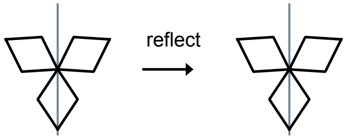


REFLECTIONS

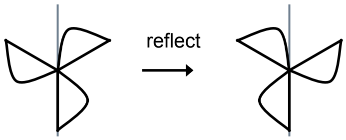
Even when two shapes have the same number of rotations, one can still have more symmetry than the other.



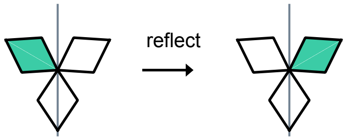
Some shapes have mirrors - they can reflect across internal, invisible lines without changing in appearance.



While others cannot.

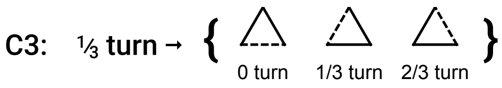


These mirrors are symmetries of our shapes, and we'll see how they can be removed by adding color.

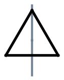


First, let's generate more mirrors.

We saw that a single generator, the ⅓ turn, could generate the entire group of rotations of a regular triangle. This was our C3 group.

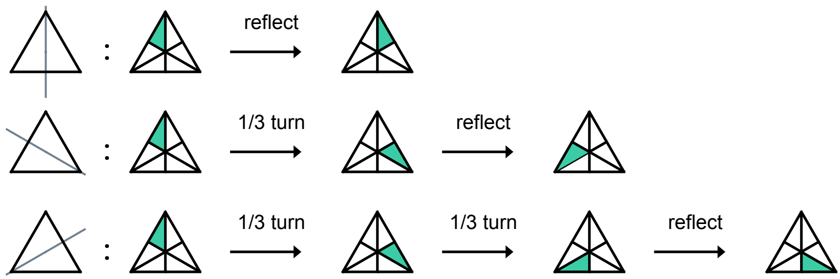


We can also reflect this triangle across a vertical mirror through its center.

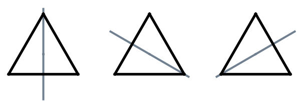


By combining this mirror with a rotation, we can generate even more mirrors, for even larger groups.

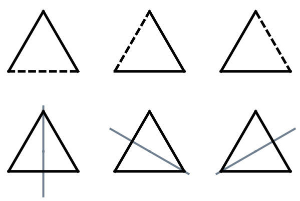
This will be easier to see if we use color.



The triangle has 3 unique mirrors in total.

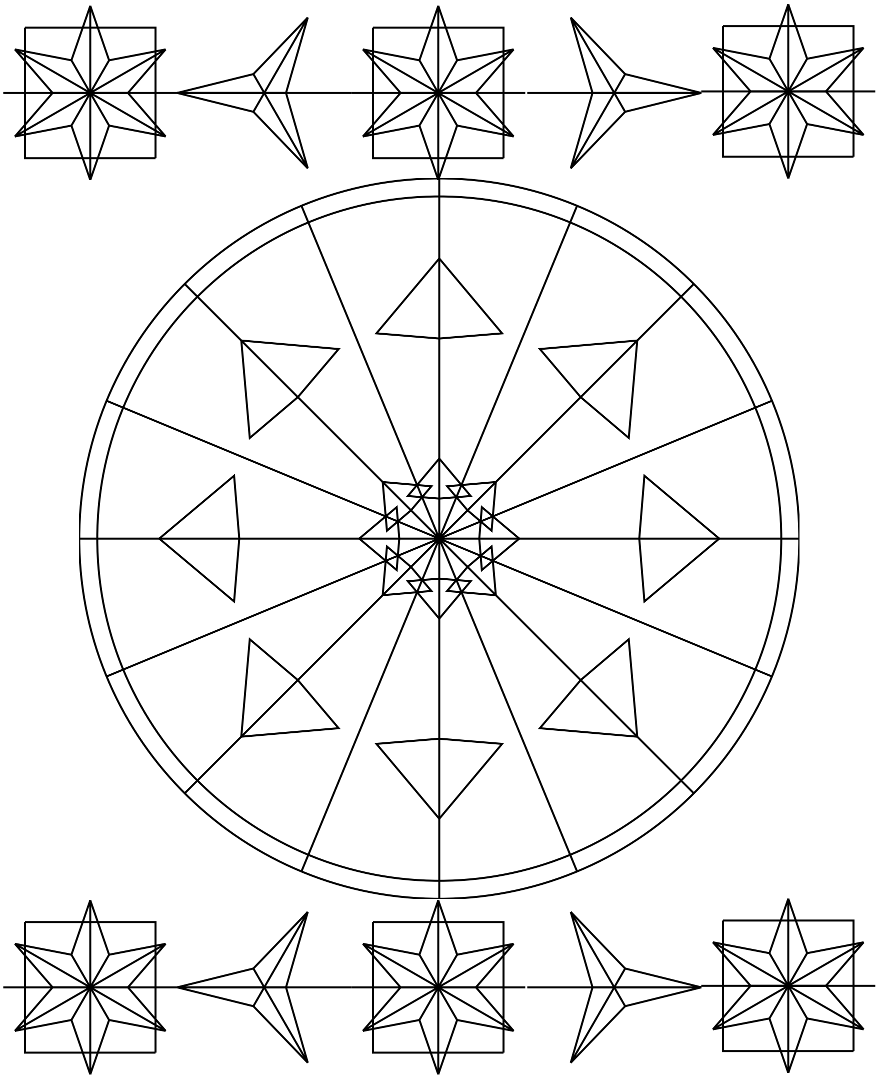


With just a rotation and a mirror as generators, we generated a new, larger group of symmetries for a regular triangle.



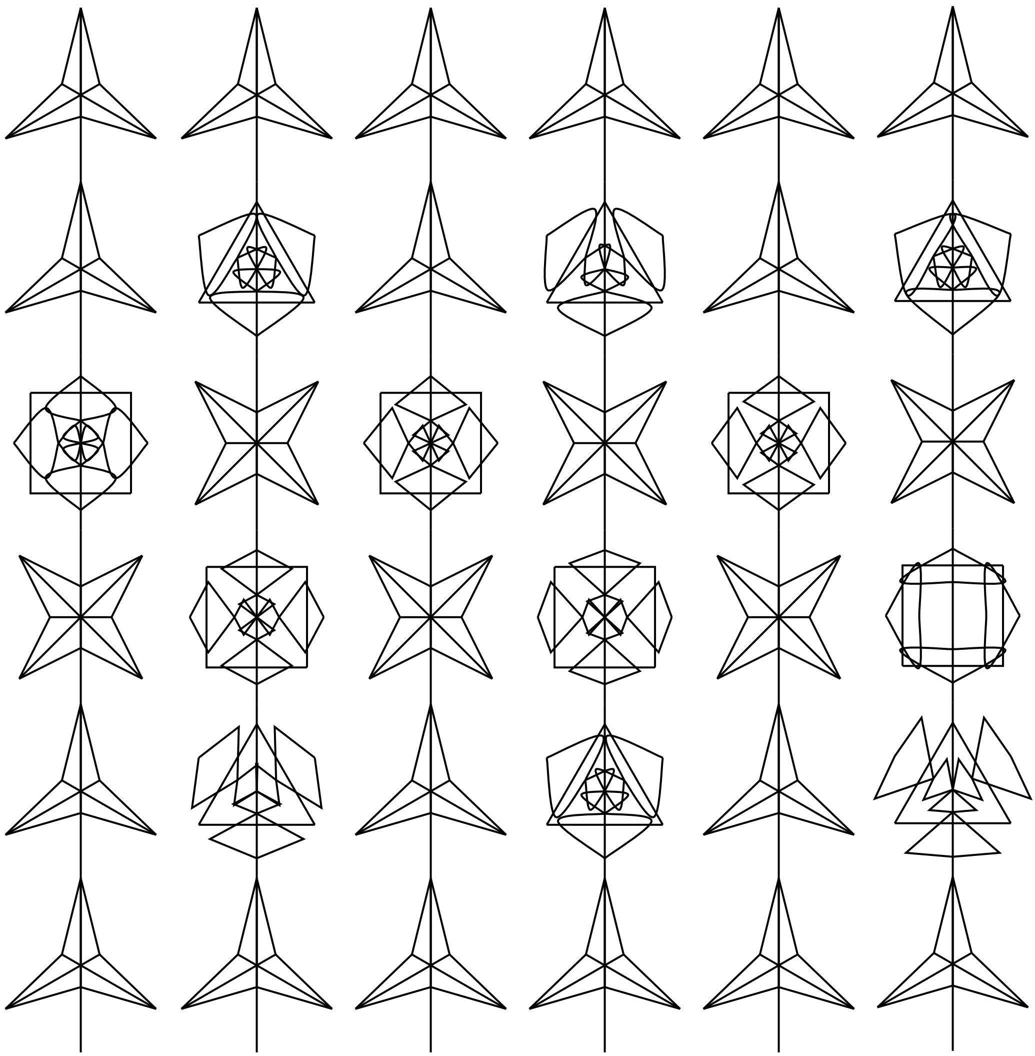
We can do the same with other shapes, to see even bigger groups.

These shapes have mirrors, and so does the illustration as a whole. Can you add color to remove all of the mirrors?



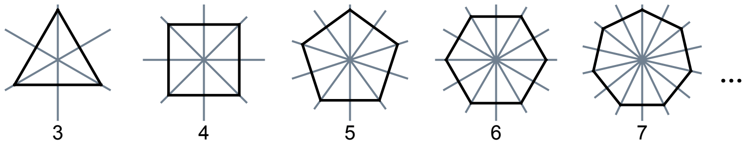
mirrors

These shapes have mirrors. Maintain these mirrors as you color.

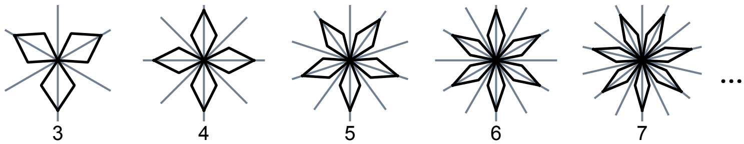


shapes with mirror reflections

Our regular triangle has 3 unique rotations and 3 unique reflections, a square has 4, and we can find shapes with 5, 6, 7, and keep going...

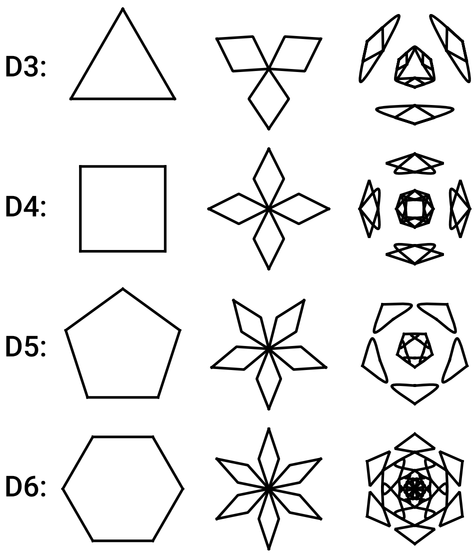


Shapes that are not regular polygons can have these same symmetries.



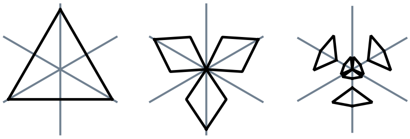
We already saw how shapes that share the same set of symmetries share a symmetry group, but then we only considered rotations. Symmetry groups can have both rotations and reflections.

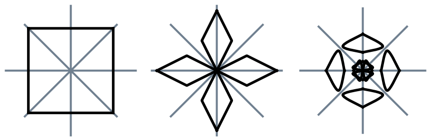
We'll call the symmetry group that contains the 3 rotations and 3 reflections of a regular triangle D3. And we'll call the symmetry group with the 4 rotations and 4 reflections of a square D4, while we call the symmetry group with 5 rotations and 5 reflections D5, and so on...

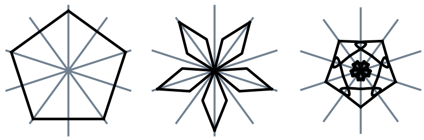


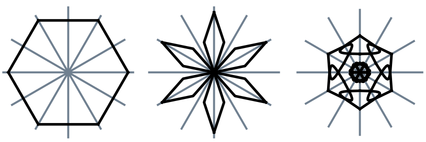
...

This series of groups is called the dihedral groups. Again, these groups contain symmetries, not shapes - the shapes just help us see them.

D3: 

D4: 

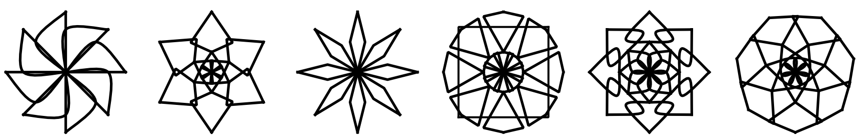
D5: 

D6: 

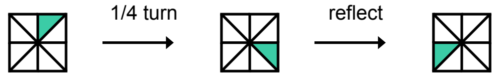
...

These shapes that share a symmetry group may look different, but when viewed through the lens of group theory, they look the same. Only their symmetries - the rotations and reflections that leave them unchanged - are seen.

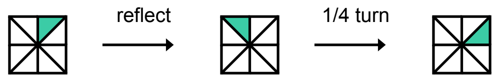
Check in: Which of these shapes have 8 mirrors?



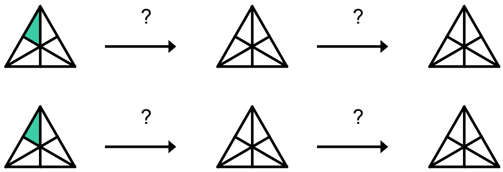
We saw that the cyclic groups are <u>commutative</u>. The order in which we combined rotations did not matter - the result was always the same. The dihedral groups are not commutative. We can see this in our D4 shapes: rotating our D4 shapes by a ¼ turn and then reflecting across a vertical mirror,



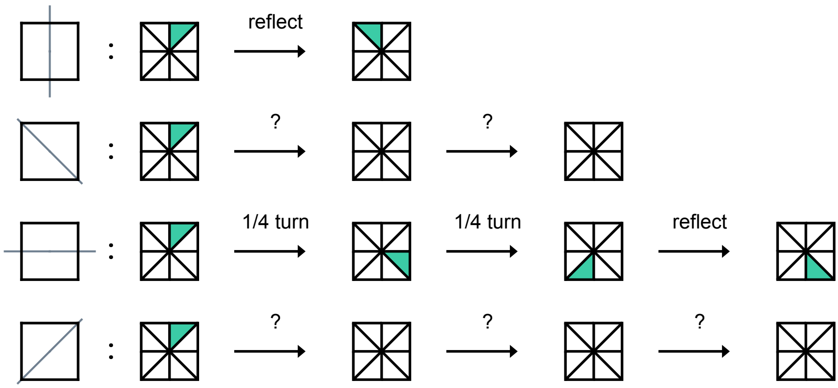
Is not the same as reflecting across a vertical mirror and *then* rotating by a ¼ turn.



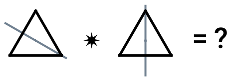
Challenge: Show that D3 is not commutative. Find 2 symmetries of our triangle where transforming the triangle by one symmetry and then the next is not the same as applying the transformations in the reversed order.



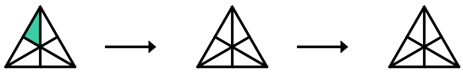
Challenge: We showed how the ⅓ turn and a vertical mirror could be used as generators for D3 and generate all of the other mirrors of a regular triangle. Show how the ¼ turn and a vertical mirror can be used to generate all of the other mirrors of a square.



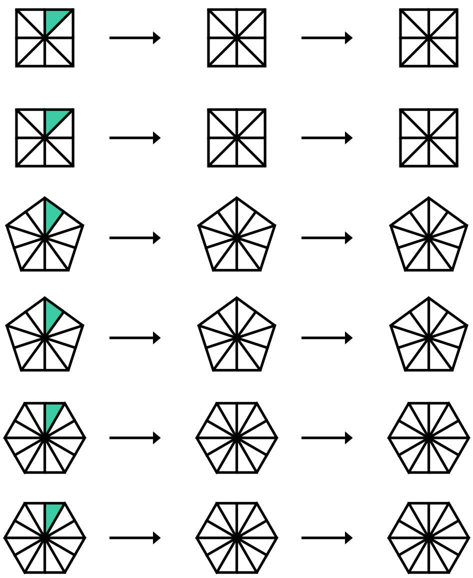
Challenge: What is the result of combining two different mirrors?



Is the result a reflection or a rotation?

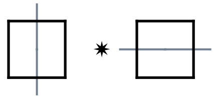


Is this always the case?

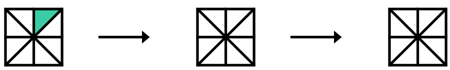


(go ahead and draw more shapes)

For our D4 group, we can see that the result of applying a vertical mirror and then a horizontal mirror is a ½ turn.

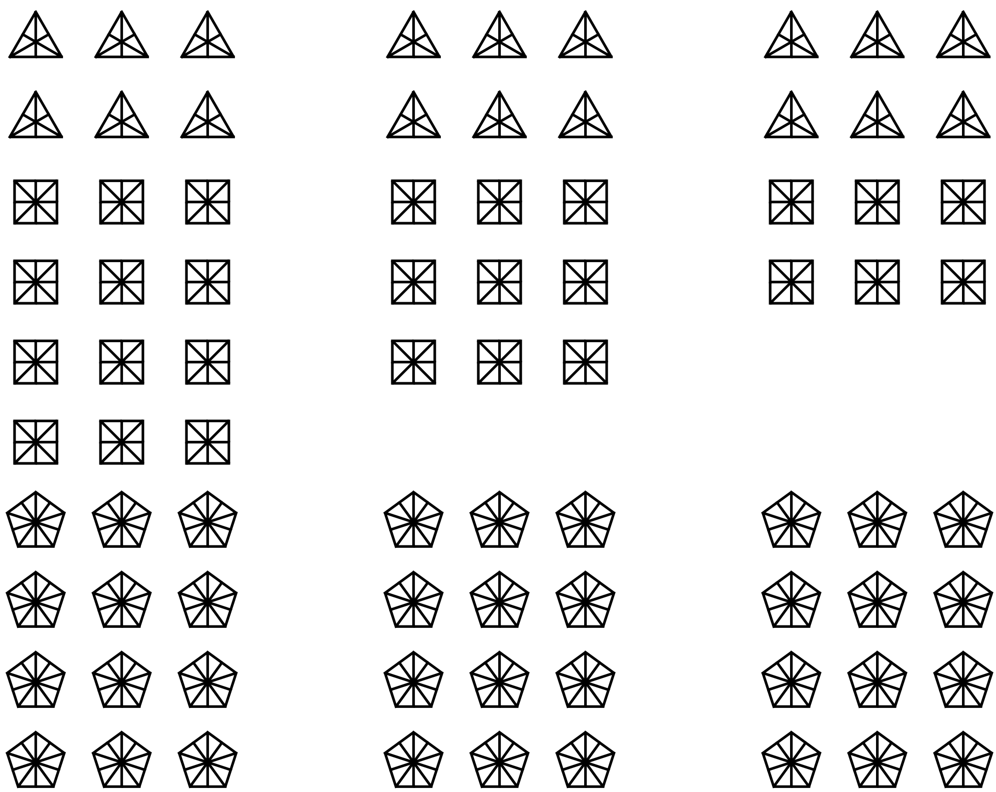
D4:  **= ½ turn**

Challenge: Can you find 2 mirrors where applying one and then the other results in a ¼ turn in our D4 group?

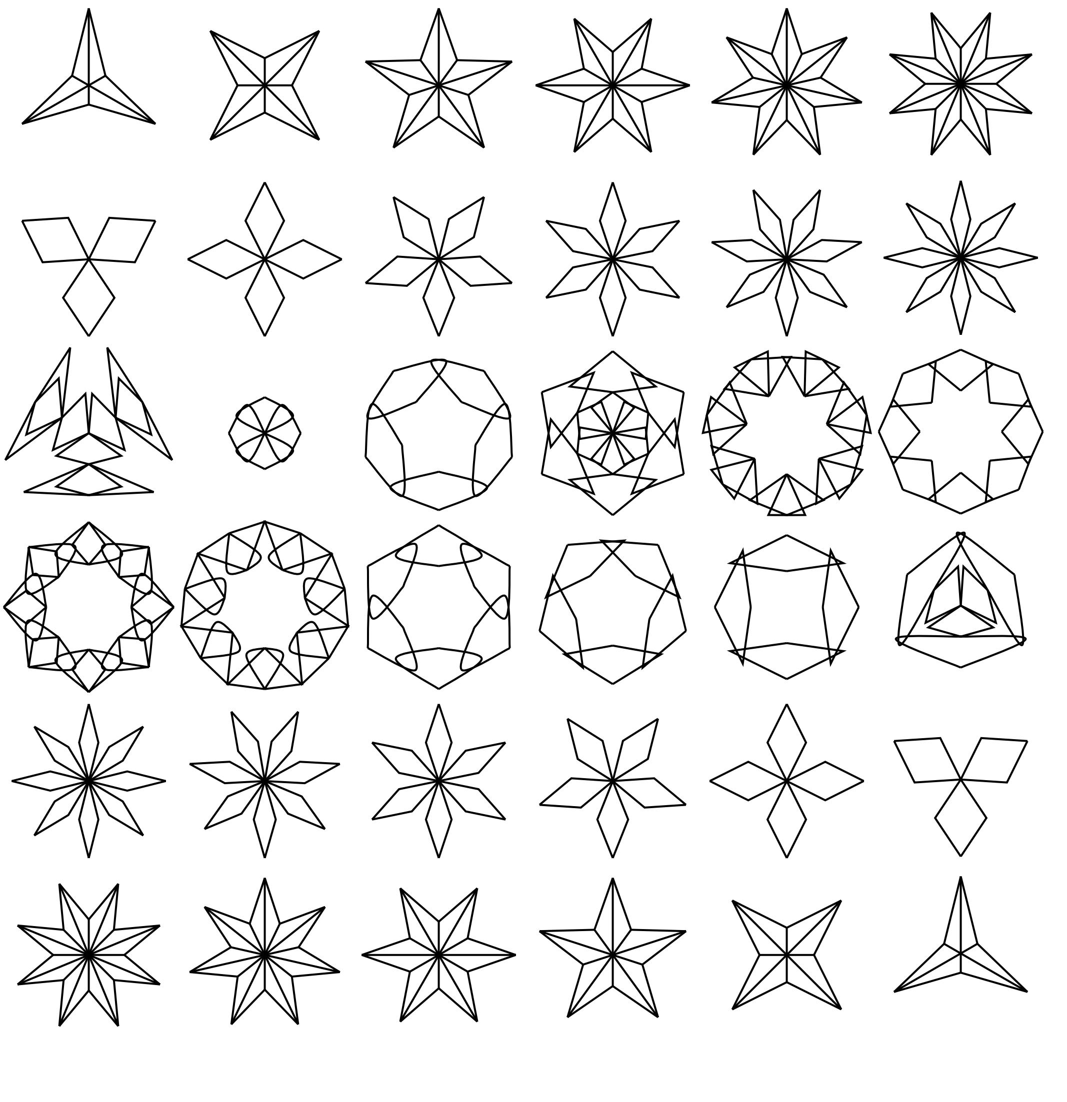


Is it possible to use 2 mirrors to generate all of the symmetries of our D4 group? What about our other dihedral groups?

Here are some shapes for you to puzzle over.

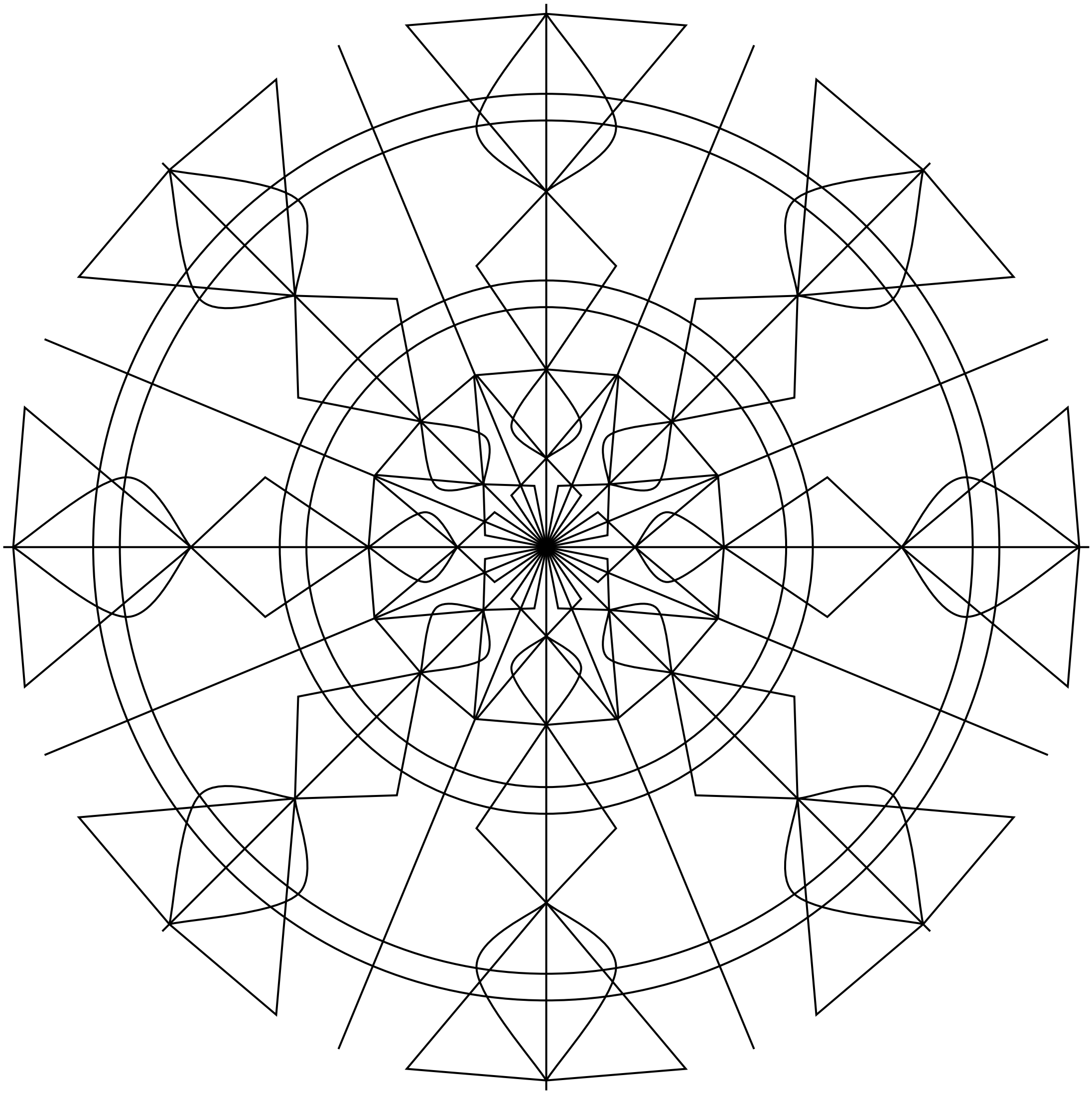


How many unique rotations and reflections does each shape have? Color all of the D8 shapes so that they are no longer D8 shapes but still have at least one mirror reflection.



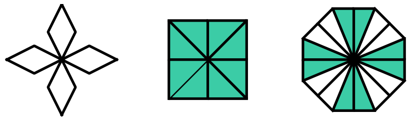
D3, D4, D5, D6, D7, D8 shapes

The D8 shape below is made up of pieces that repeat around a circle. Show that a ¼ turn and a vertical mirror cannot be used as generators for our D8 group by coloring a piece, and then coloring other pieces if and only if they can be reached by a ¼ turn or mirror reflection from an already colored piece. What are the symmetries of the colored shape you end up with?



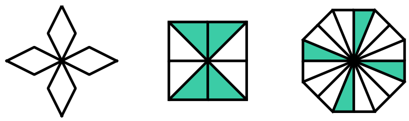
D8 shape (circular tessellation)

By looking for rotations and reflections, we can see when shapes share a symmetry group,



D4 D4 D4

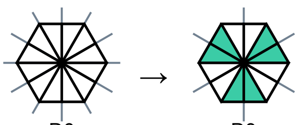
Or when they do not.



D4 D2 C4

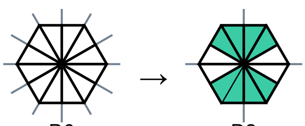
And now that we have groups with more symmetries, there are more interesting subgroups to find.

We can again use color to reduce the amount of symmetry a shape has. For example, a D6 shape has 6 mirrors and 6 rotations, but with color we can remove 3 of these mirrors and 3 of these rotations to reduce it to a D3 shape.



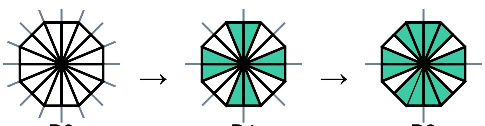
D6 D3

Alternatively, we could have reduced the D6 shape to a D2 shape.



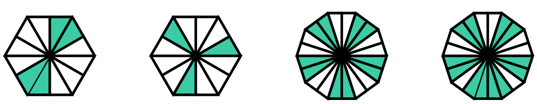
D6 D2

This is possible because D3 and D2 are <u>subgroups</u> of D6. Similarly, D4 is a <u>subgroup</u> of D8, and D2 is a <u>subgroup</u> of both D4 and D8.

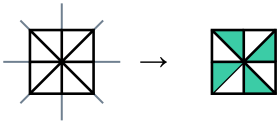


D8 D4 D2

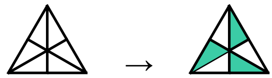
Check in: What are the symmetry groups for these colored shapes?



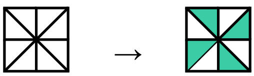
What happens when color is added to remove only mirrors and not rotations?



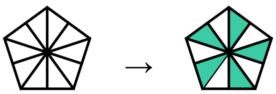
The dihedral groups have mirror reflections, while the cyclic groups do not. When these mirrors are removed, we can see the cyclic groups are subgroups of the dihedral groups.



D3 C3

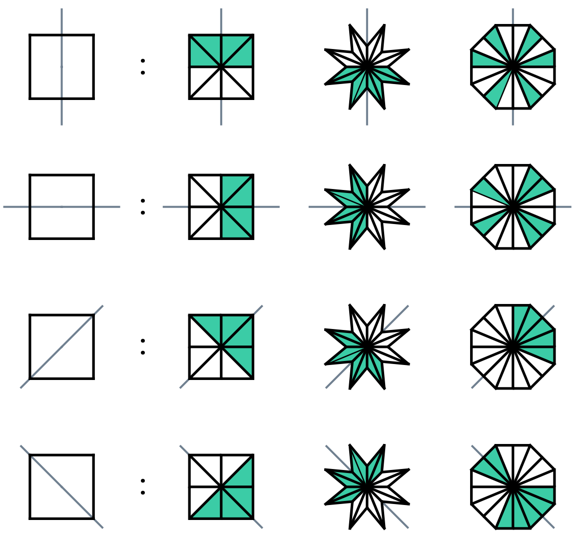


D4 C4



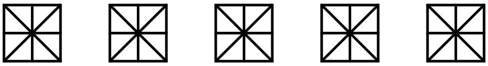
D5 C5

Color can also take away a shape's rotations to show us subgroups with only mirror reflections.



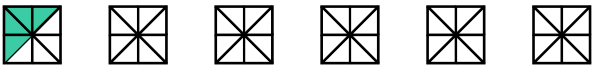
Here is an example where a D4 shape is colored with 2 colors so that it has only 1 mirror and that mirror is horizontal.

Challenge: Find other ways to color the D4 shape with 2 colors so that its only mirror is the horizontal mirror.



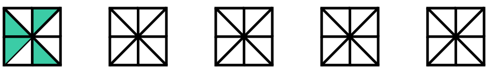
After being colored in this way, the shape no longer has the symmetries of D4, instead it illustrates a subgroup of D4.

Challenge: Can you find different ways to color the D4 shape with 2 colors so that it has only 1 mirror and that mirror is diagonal?



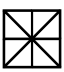
Our D4 group has multiple subgroups that have just 2 symmetries. One of those subgroups is the group of symmetries with just the horizontal mirror and the 0 turn (the 0 turn is also known as the identity).

Challenge: Can you find the other subgroups of our D4 group that have just 2 symmetries?

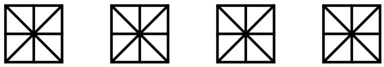


There are multiple ways to color our D4 shape to reduce it to a shape with only a 0 turn, a ½ turn, and 2 mirrors. This is just our D2 group! Here is an example where we keep the 2 diagonal mirrors.

Challenge: Color the D4 shape to remove the diagonal mirrors but keep the horizontal and vertical mirrors.



Challenge: Is it possible to color a D4 shape to remove the horizontal mirror while keeping the vertical mirror and ¼ turn? (Hint: Think about group closure)

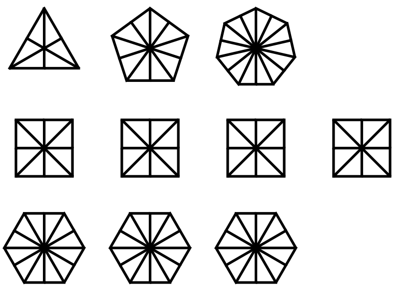


There is a relationship between the number of symmetries in our dihedral groups, the number of symmetries in their subgroups, and the maximum number of colors we can use to reduce our dihedral shapes to show those subgroups.

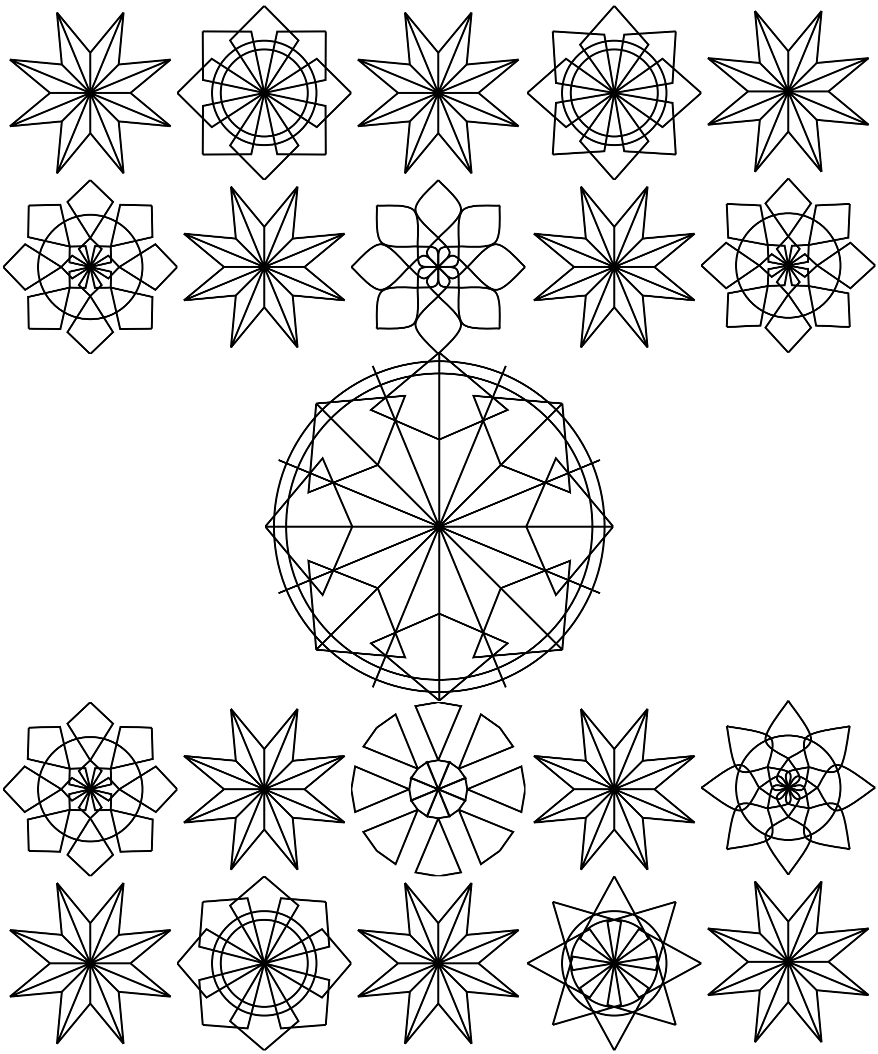
When we reduce our D4 shapes to D2 shapes, we reduce their number of symmetries from 8 (4 mirrors, 4 rotations) to 4 (2 mirrors, 2 rotations). This is also the case when we reduce our D4 shapes to C4 shapes: Shapes go from having 4 mirrors and 4 rotations to having just 4 rotations.

D2 C4

Challenge: In any of the cases where we remove half the symmetries of our dihedral shapes, what is the maximum number of colors we can use?

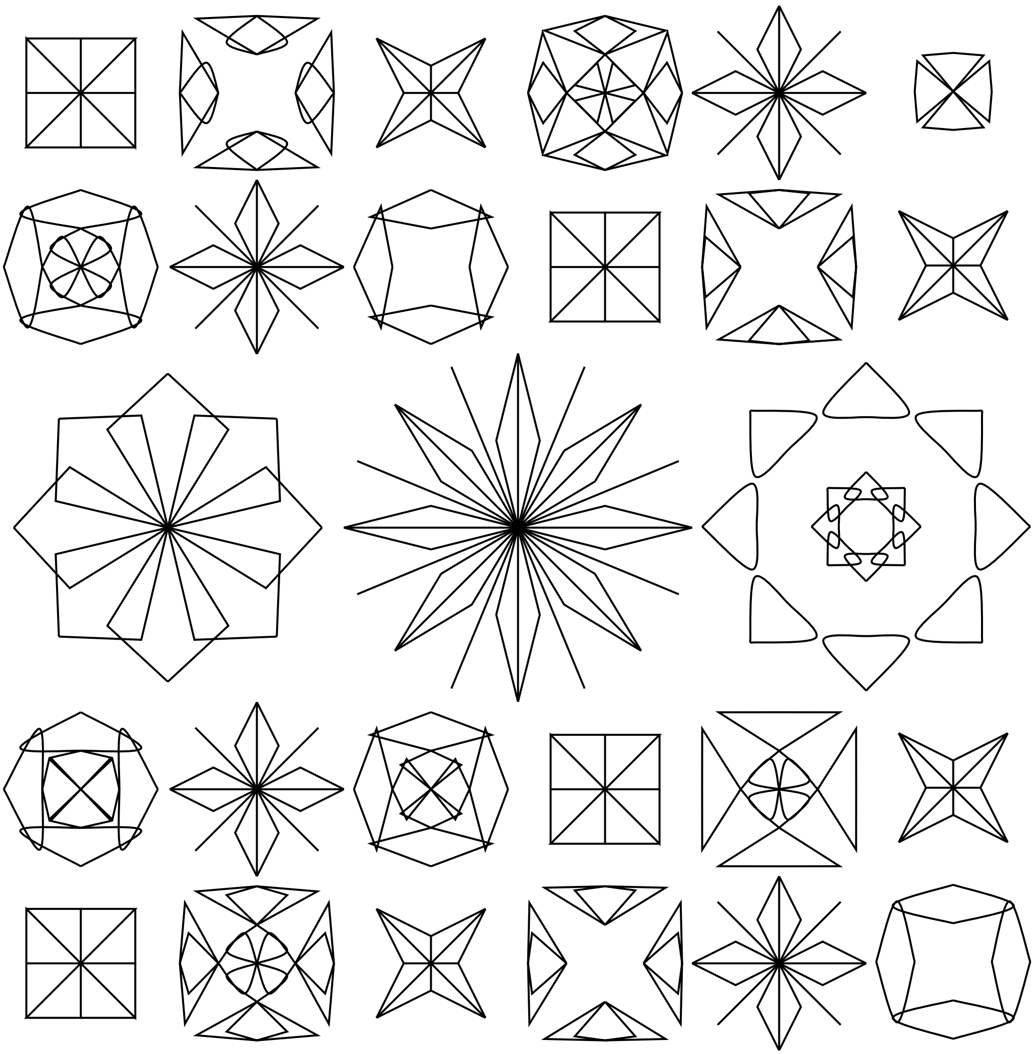


Can you find different ways to remove half of the symmetries of the D8 shapes? Color some of the shapes to remove their mirrors while keeping their rotations. Color others to remove half of their mirrors and half of their rotations.



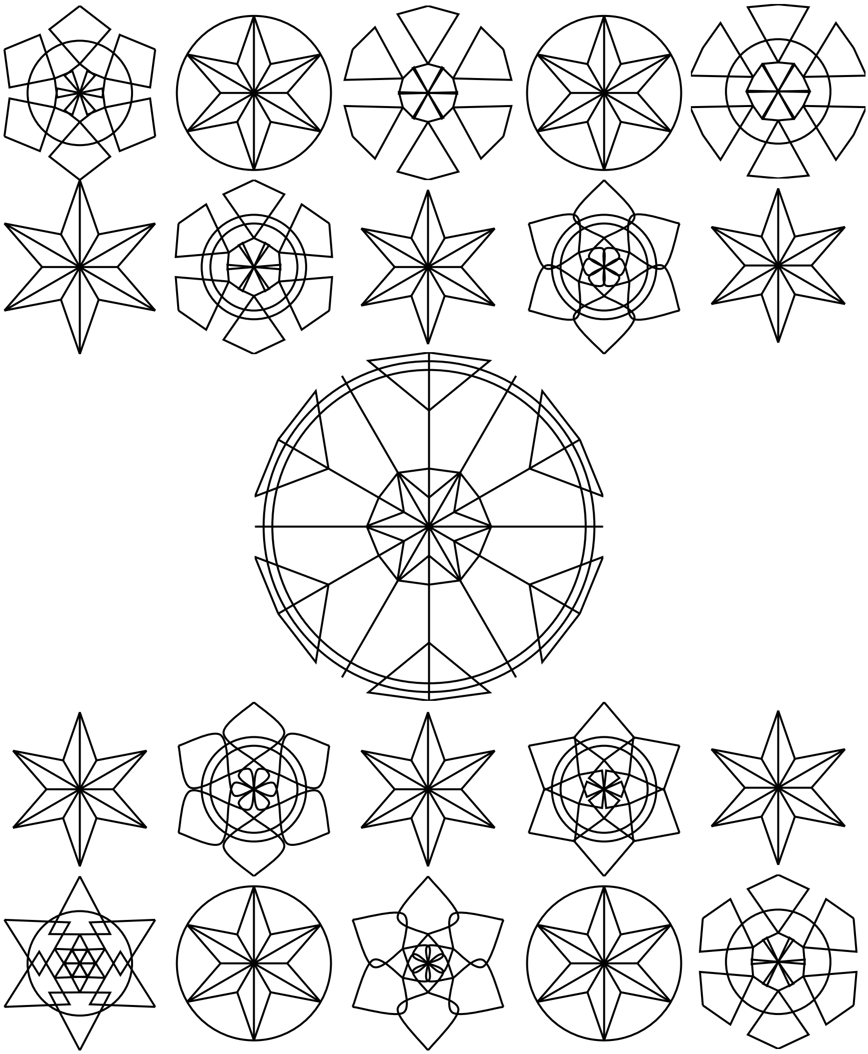
D8 shapes

Use color to reduce the D4 and D8 shapes to D2 shapes.



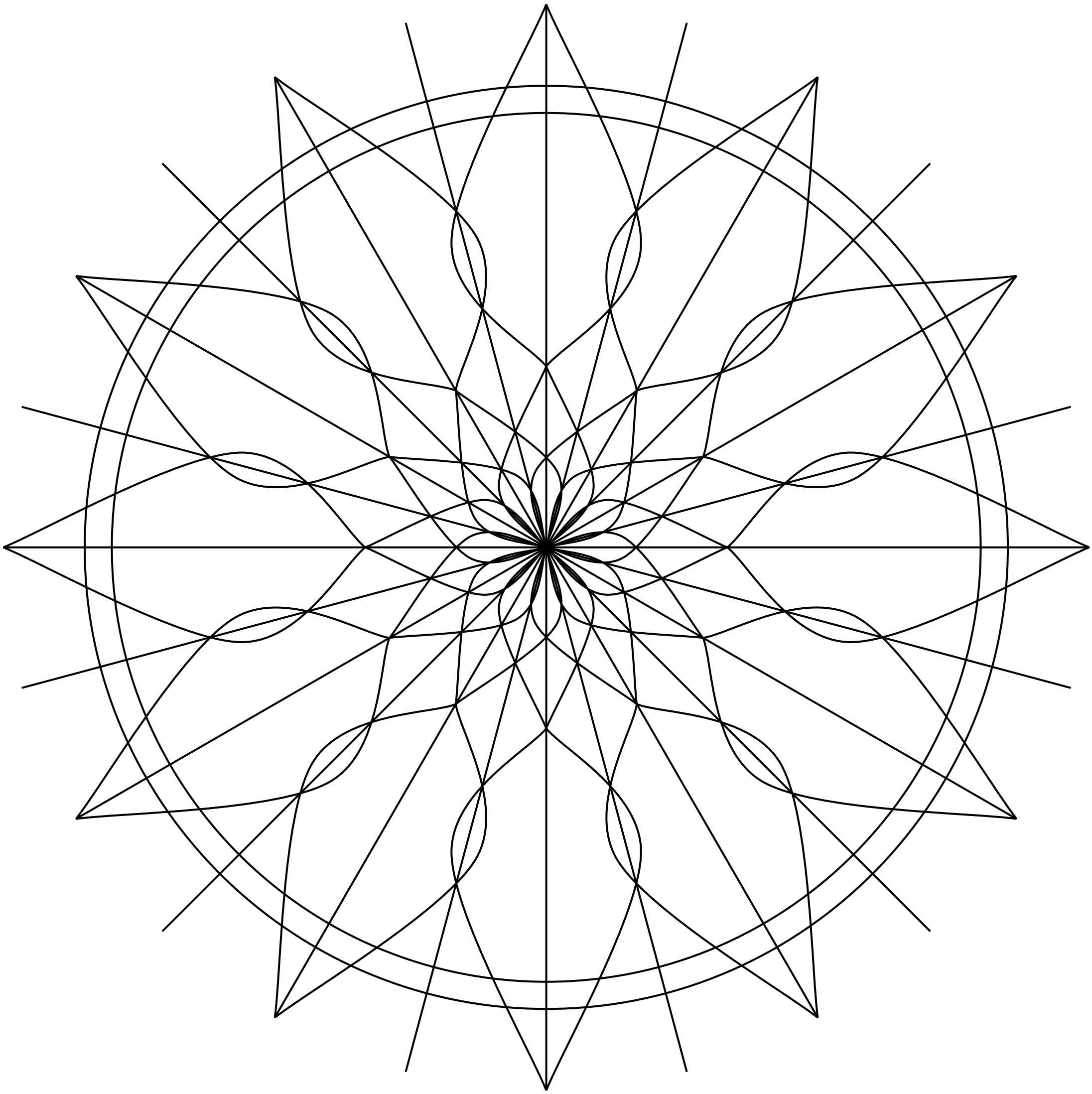
D4 and D8 shapes

Can you add color to reduce the D6 shapes to D3 shapes? Then add more color to reduce them to C3 shapes.



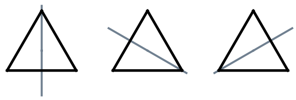
D6 shapes

Can you color the D12 shape to reduce it to a D6 shape? Then add more color to reduce it to a C6 shape. And add more color again to further reduce it to a C3 shape.

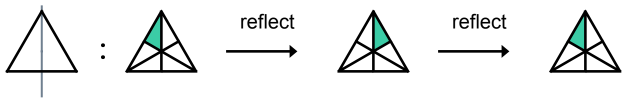


D12 shape (circular tessellation)

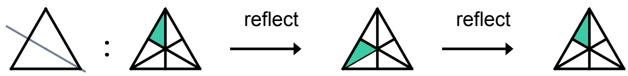
There is something about mirrors that you may have already noticed.

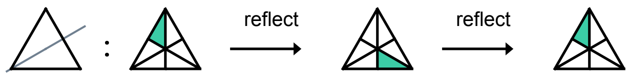


Reflecting a shape across the same mirror twice in a row is the same as not reflecting it at all.



The second reflection reverses the work of the first reflection. The same can be said for all of the mirrors we found.



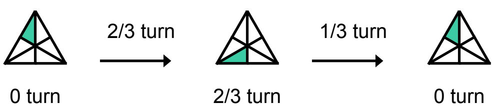


You may have also noticed that our rotations can be reversed as well. When our triangle is rotated by a $\frac{1}{3}$ turn, rotating again by a $\frac{2}{3}$ turn brings it back to the position it started in. The result is the same as a 0 turn.

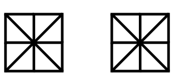
$\frac{1}{3}$ turn ⁎ $\frac{2}{3}$ turn = 0 turn

The same can be said the other way around.

$\frac{2}{3}$ turn ⁎ $\frac{1}{3}$ turn = 0 turn



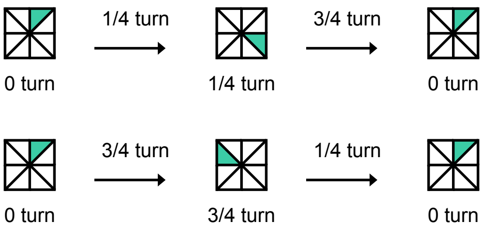
Check in: Which rotation in C4 is the reverse of the $\frac{2}{4}$ turn?



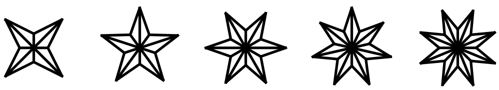
When one transformation, like a ⅓ turn, reverses the work of another transformation, it's called an inverse.

The ⅓ turn is the inverse of the ⅔ turn in C3 and D3. Similarly, the ¼ turn and ¾ turn are inverses in C4 and D4.

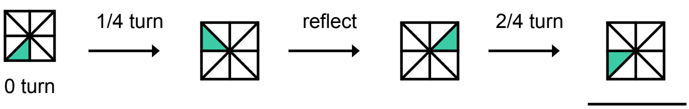
¼ turn ∗ ¾ turn = 0 turn = ¾ turn ∗ ¼ turn



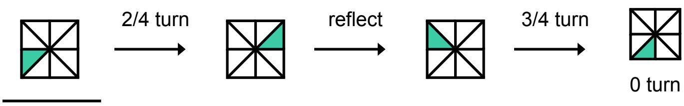
Check in: What is the inverse of a horizontal reflection? What is the inverse of any reflection?



All of the symmetries in our cyclic and dihedral groups have inverses. Even when a shape undergoes a combination of reflections and rotations,



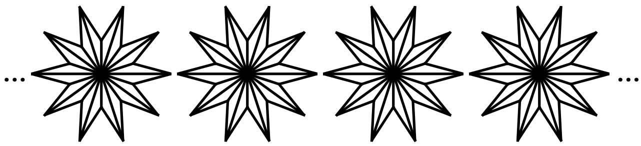
The transformations can be reversed and the shape can end back in the position it started.



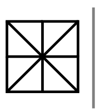
¼ turn ∗ reflect ∗ 2⁄4 turn ∗ 2⁄4 turn ∗ reflect ∗ ¾ turn = 0 turn

This is a rule in group theory: Any member of a group has an inverse that is also in the group. And remember, the members of our groups are the symmetries of our shapes - they are the reflections and rotations that leave our shapes unchanged.

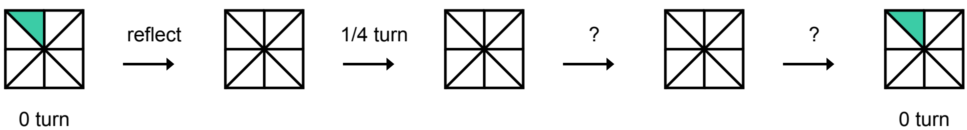
So far we have been focusing on only the symmetries of shapes, but there are even more types of symmetry to see and even bigger groups to talk about - groups of infinite size. Next we'll see transformations that take our illustrations beyond shapes and generate patterns that repeat forever...

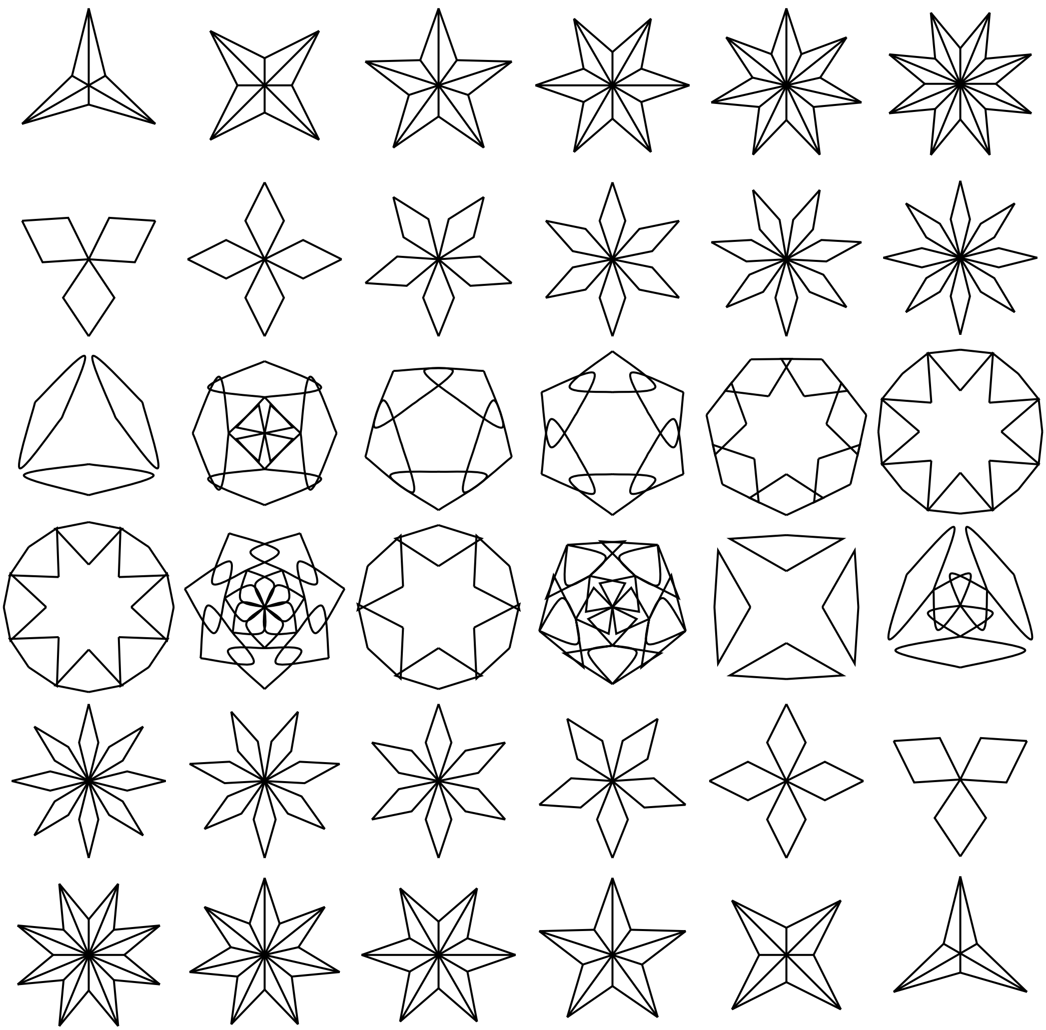


Challenge: What would happen if you reflected a shape across a mirror that sat next to the shape rather than through its center?



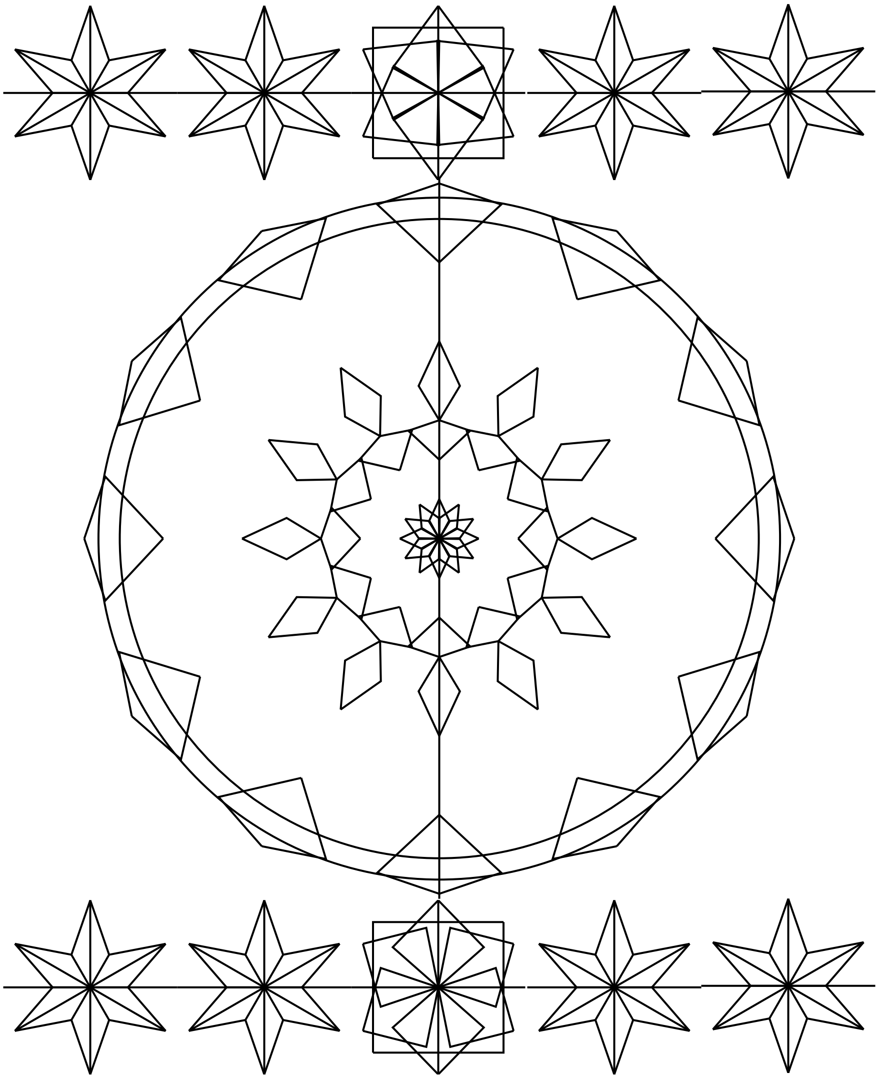
Challenge: Color the squares to show the result of reflecting across a vertical mirror and then rotating by a ¼ turn. Then find the combination of transformations that brings the square back to its starting position.





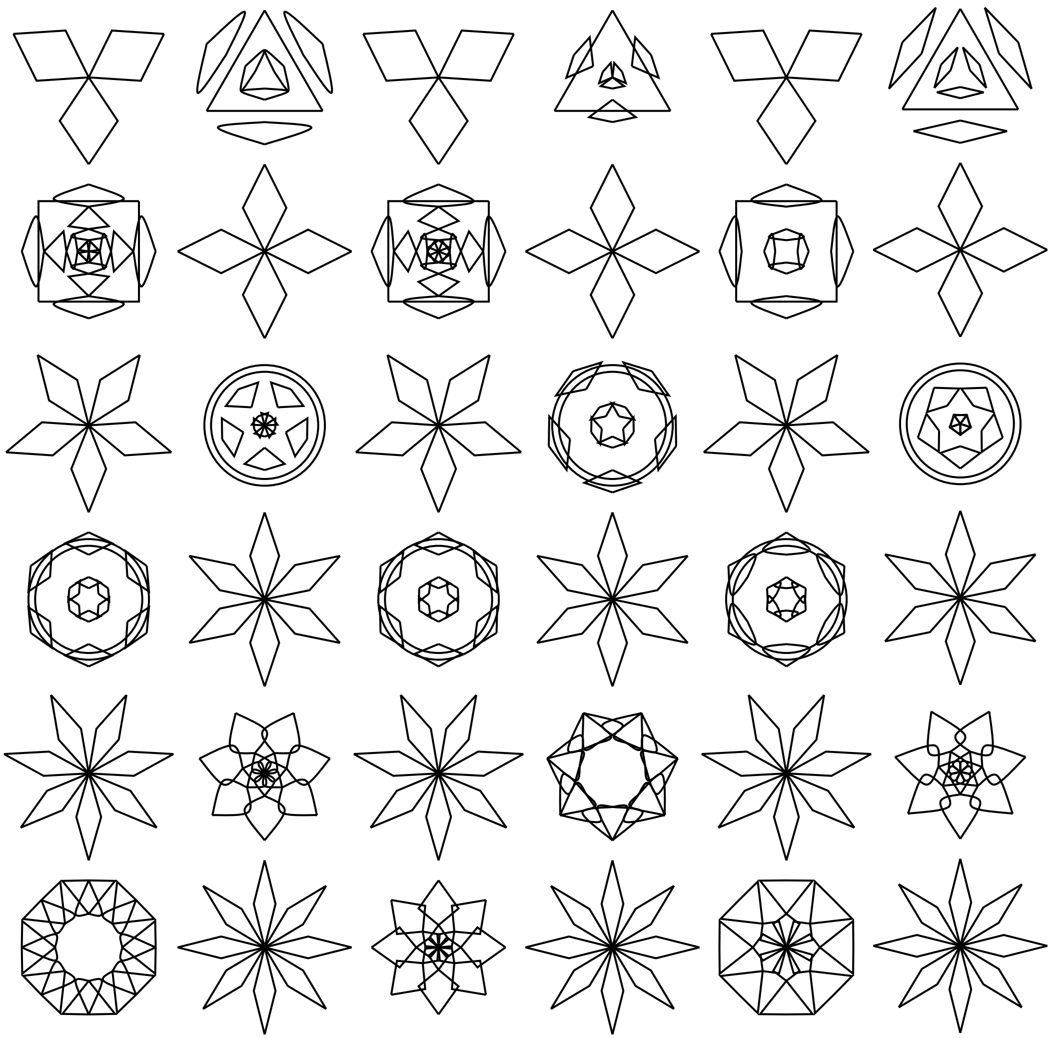
D3, D4, D5, D6, D7, D8 shapes

This entire illustration has a mirror and a ½ turn. Can you use color to remove the mirror while maintaining the ½ turn?



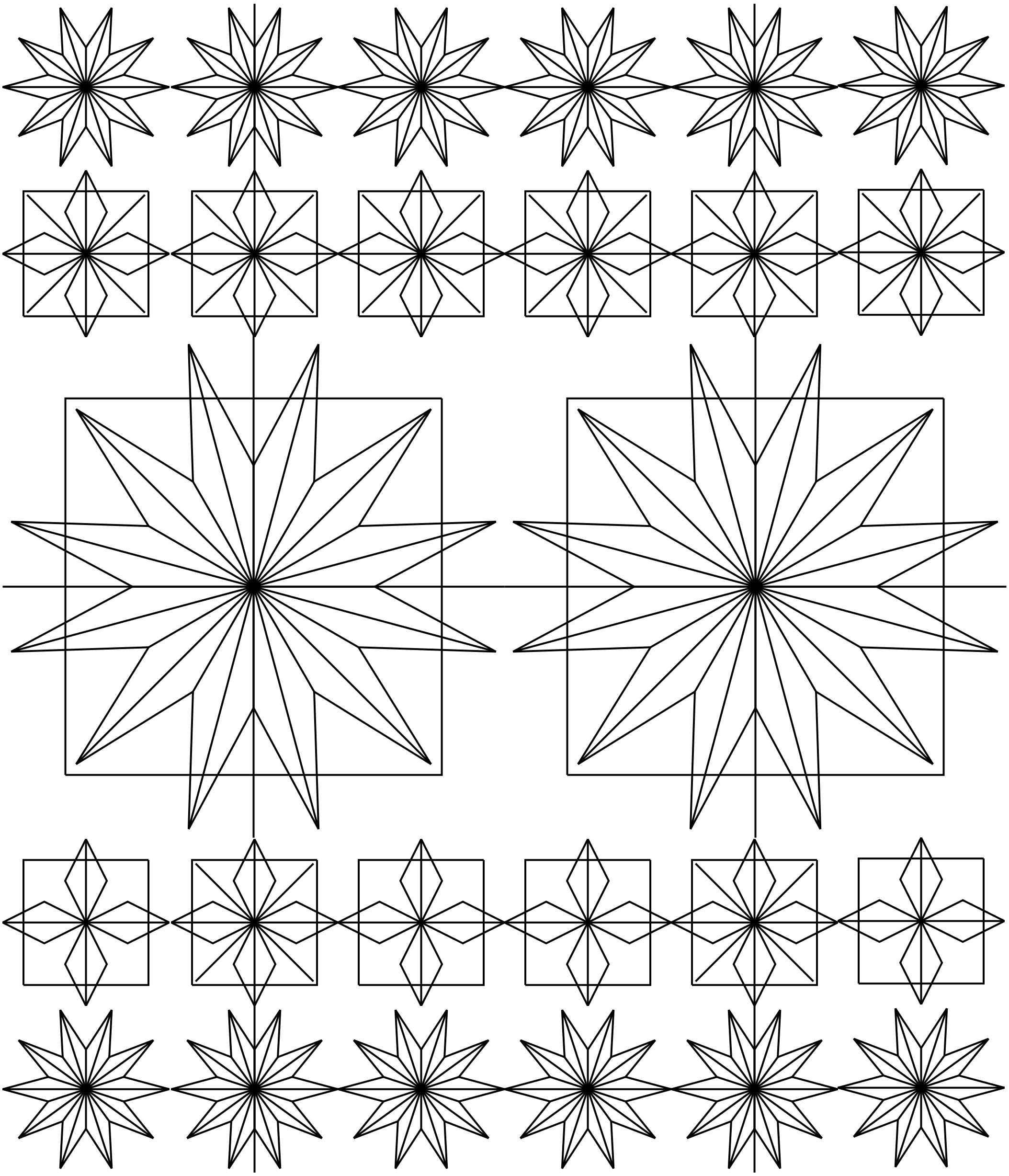
dihedral shapes

Can you use color to remove all of the rotations for these dihedral shapes? Which symmetry groups do they end up representing?



D3, D4, D5, D6, D7, D8 shapes

This entire illustration has 2 mirrors and a ½ turn. Can you use color to remove the mirrors while maintaining the ½ turn? Extra challenge: Use as many colors as possible - how many can you use?

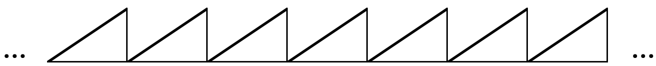


patterns of repeated shapes with mirror reflections and rotations

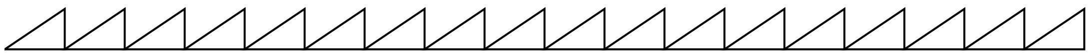
INFINITELY REPEATING PATTERNS

FRIEZE GROUPS

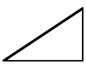
The **Frieze Groups** can be seen in patterns that repeat infinitely in opposite directions.



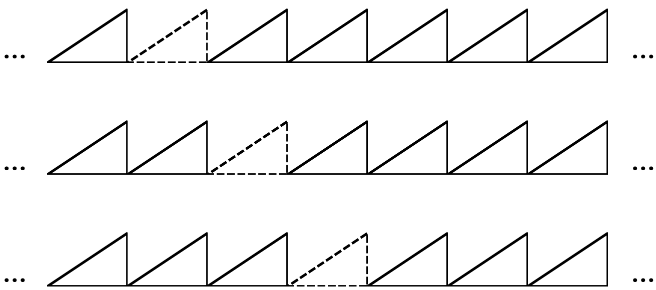
A page cannot do these patterns justice. It cuts them off when really they continue repeating forever...



Consider the smallest repeating piece of this pattern as a unit.



We can see the entire pattern can shift over by this unit. Each piece shifts on to an identical piece and there is always more behind to replace what was shifted,

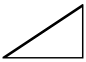


so that the shift leaves the entire pattern unchanged. Such is the nature of infinite repetition...

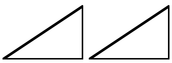
This shift is a symmetry called **translation**.

Translations are the only symmetries in our simplest group of frieze patterns, so this group can be generated by **translations** alone.

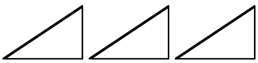
We can see this by starting with a single piece

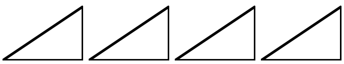


that is copied and then translated

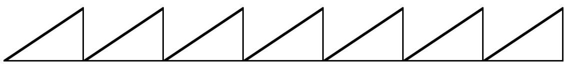


again and again...



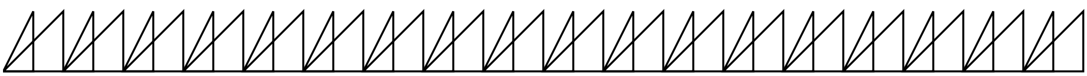


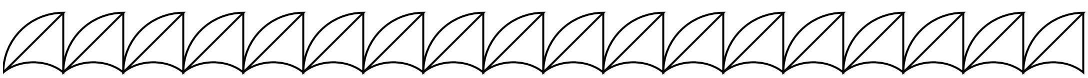
...an infinite number of times...

...  ...

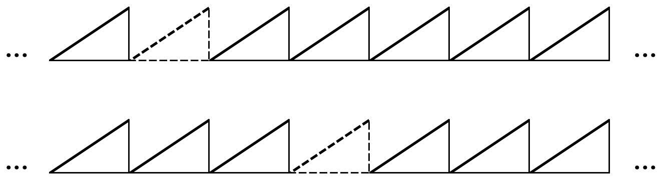
to result in a pattern with **translation** as a symmetry that leaves the entire pattern unchanged.

Check in: Can you see the translations in these patterns? Can you extend your imagination to see these as infinitely repeating patterns that repeat beyond the page borders?

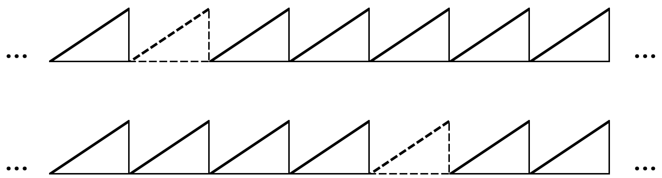




Like any of the symmetries we have seen, **translations** can be combined and the result of the combination will still be a symmetry. We can see this by combining a **translation** with another **translation** so that in the same way a pattern can shift over by 1 unit and remain unchanged, it can also shift over by 2 units and remain unchanged.

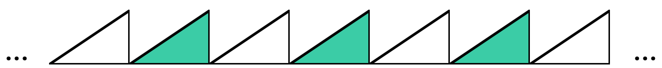


We can keep combining **translations** to see larger and larger shifts...



Or we can use color to take them away.

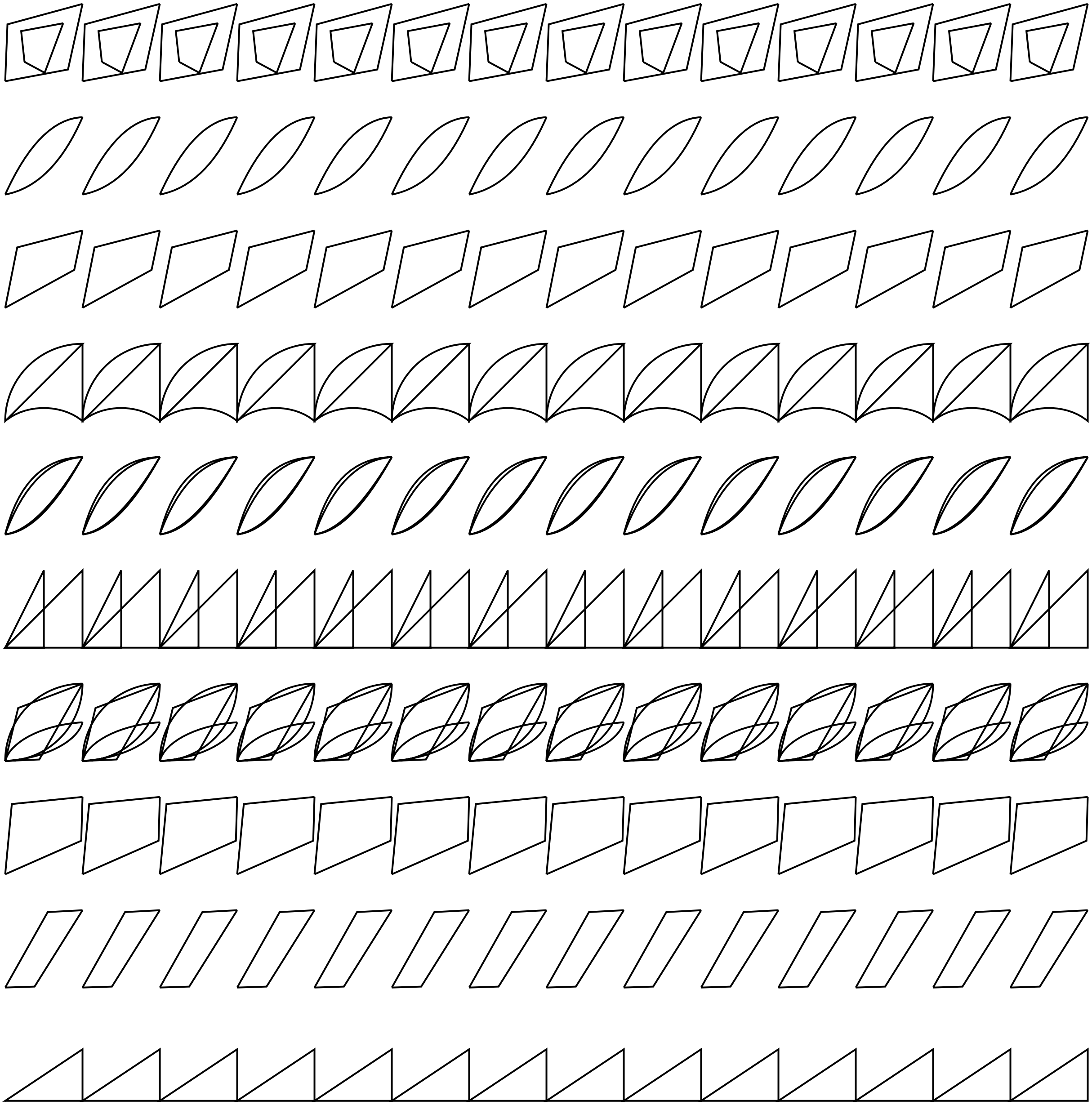
By coloring every other unit in this pattern, we can double the shortest possible distance of **translation** in the pattern from 1 unit to 2.



Now only shifting by an even number of units leaves the pattern unchanged in appearance. The pattern still repeats infinitely, and there are still an infinite number of **translations** that will leave it unchanged. By adding color, we took away ½ of its **translations**, but ½ of infinity is still infinity.

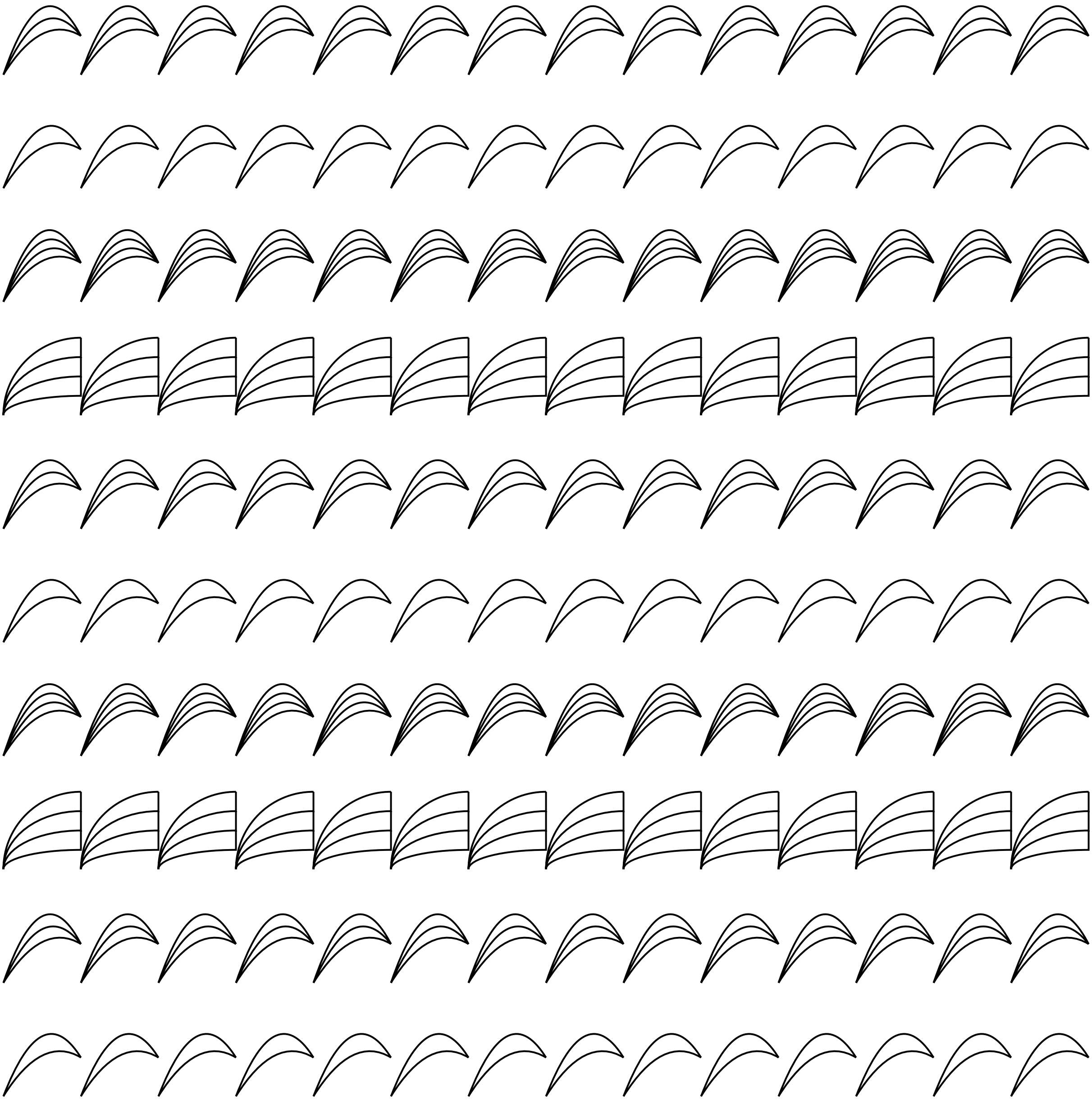
Challenge: What is the <u>*inverse*</u> *of a translation that shifts our pattern a unit to the right?*

Color the patterns in a way that maintains all of their translations.



frieze patterns (∞∞)

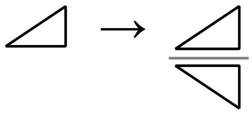
Can you color the patterns so that their shortest possible translation distance triples? Use only 2 colors.



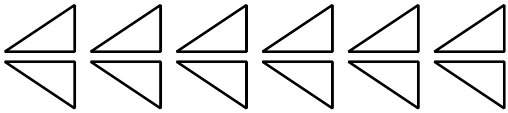
frieze patterns ($\infty\infty$)

Our patterns can have more symmetries than just **translations**.

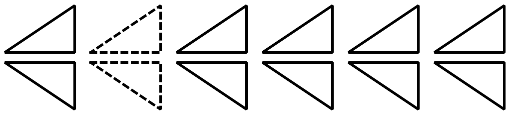
Reflecting a piece across a horizontal mirror before translating it,

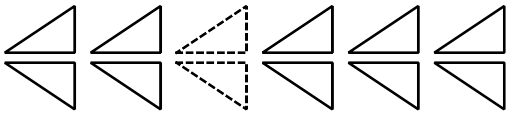


generates a new pattern, with more symmetry than the one before.

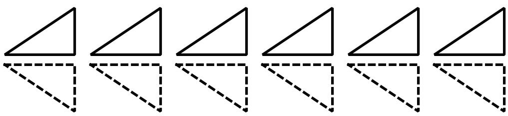


The pattern still has **translations** - it can still shift over without changing.

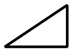


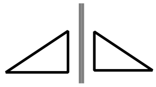


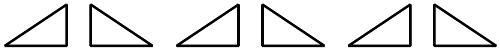
But it also has a **horizontal mirror:** The entire pattern can reflect across the same mirror that transformed our first piece, and appear unchanged.



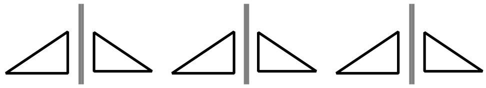
Patterns can have **vertical mirrors** as well.



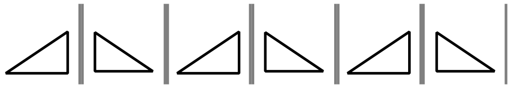




These mirrors shift over with each repeated translation, so once a pattern has one vertical mirror, it has an infinite number of vertical mirrors.

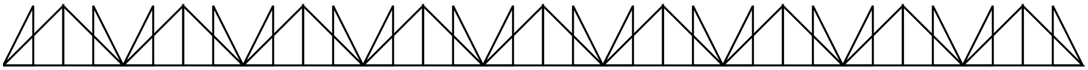


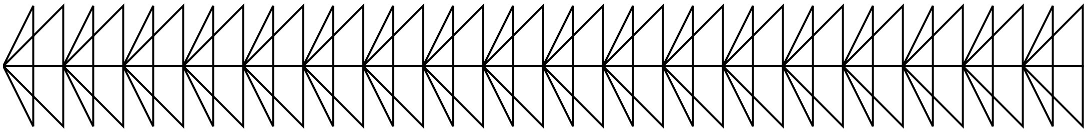
Twice that many, really.



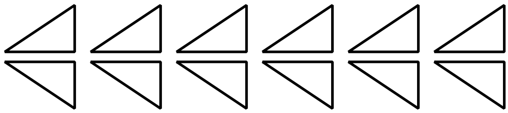
Even though we start with a vertical mirror on one side of each piece, as the pattern repeats, another different vertical mirror shows itself.

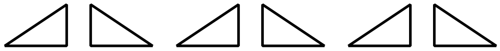
Check in: Can you see the mirrors in the following patterns?



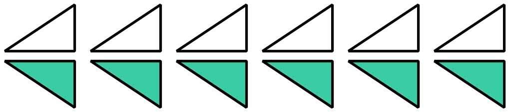


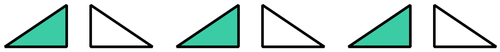
All of the mirrors in our frieze patterns can be removed with color.





With color, we can reduce the patterns so that translations are their only symmetries.

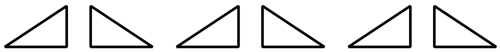




Why can we do this? This brings us back to subgroups.

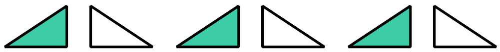
Our patterns with vertical mirror reflections belong to a symmetry group with translations and vertical mirrors.

vertical mirror reflection & translation:

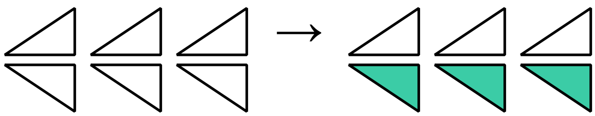


Naturally, the group with only translations is a subgroup.

translation:

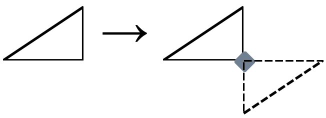


The same goes for our patterns with horizontal mirrors. Color can remove their mirrors as well, and reduce them to patterns with only translations.

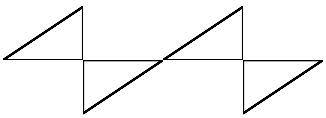


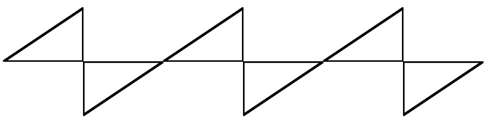
Frieze patterns can also have **½ turns** as symmetries.

We can see how they are generated by looking at a single piece that rotates by a ½ turn around a point,

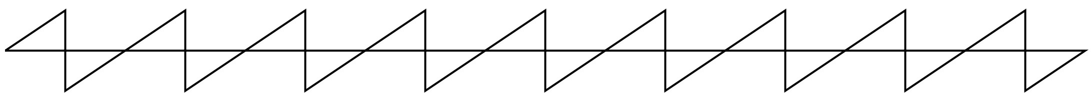


before translating.

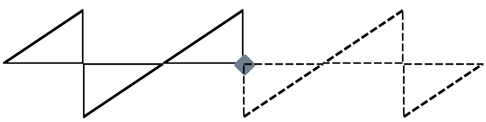




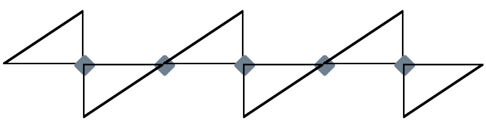
...



The entire pattern can then be rotated by a ½ turn around that rotation point.

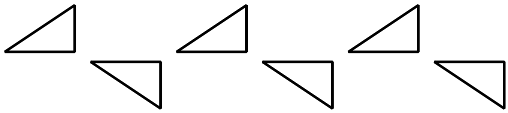


And just as we saw for vertical mirrors, once there is one point of rotation, there are infinitely many more, on either side of each piece,

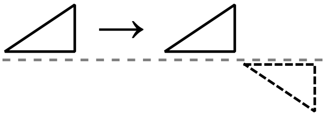


That the entire pattern can rotate around, yet remain unchanged.

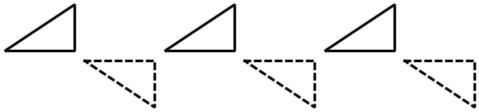
There is another type of symmetry called **glide reflection**.



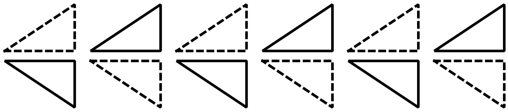
A **glide reflection** is a transformation that reflects across a mirror line at the same time as translating along it.



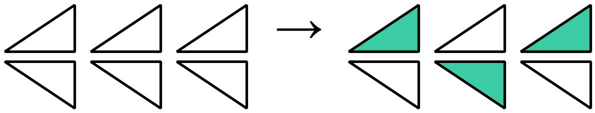
By continuing to translate or glide, a pattern with glide reflection is generated.



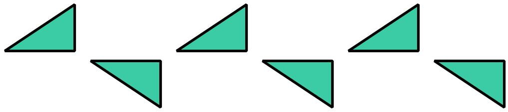
Glide reflections show themselves in other patterns as well. The patterns we generated with horizontal mirrors have glide reflections too,



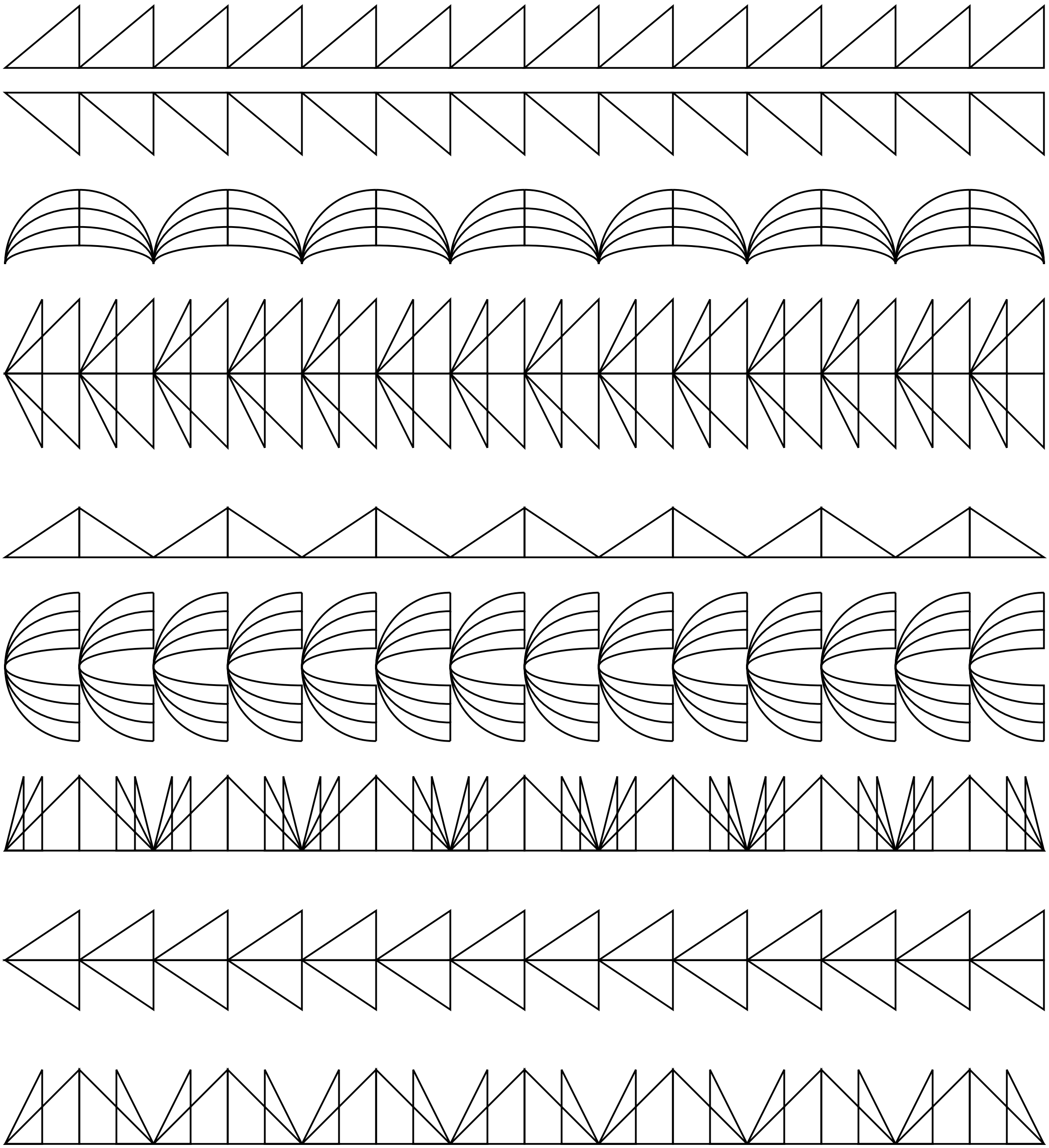
and color can reduce them



to patterns with glide reflections only.

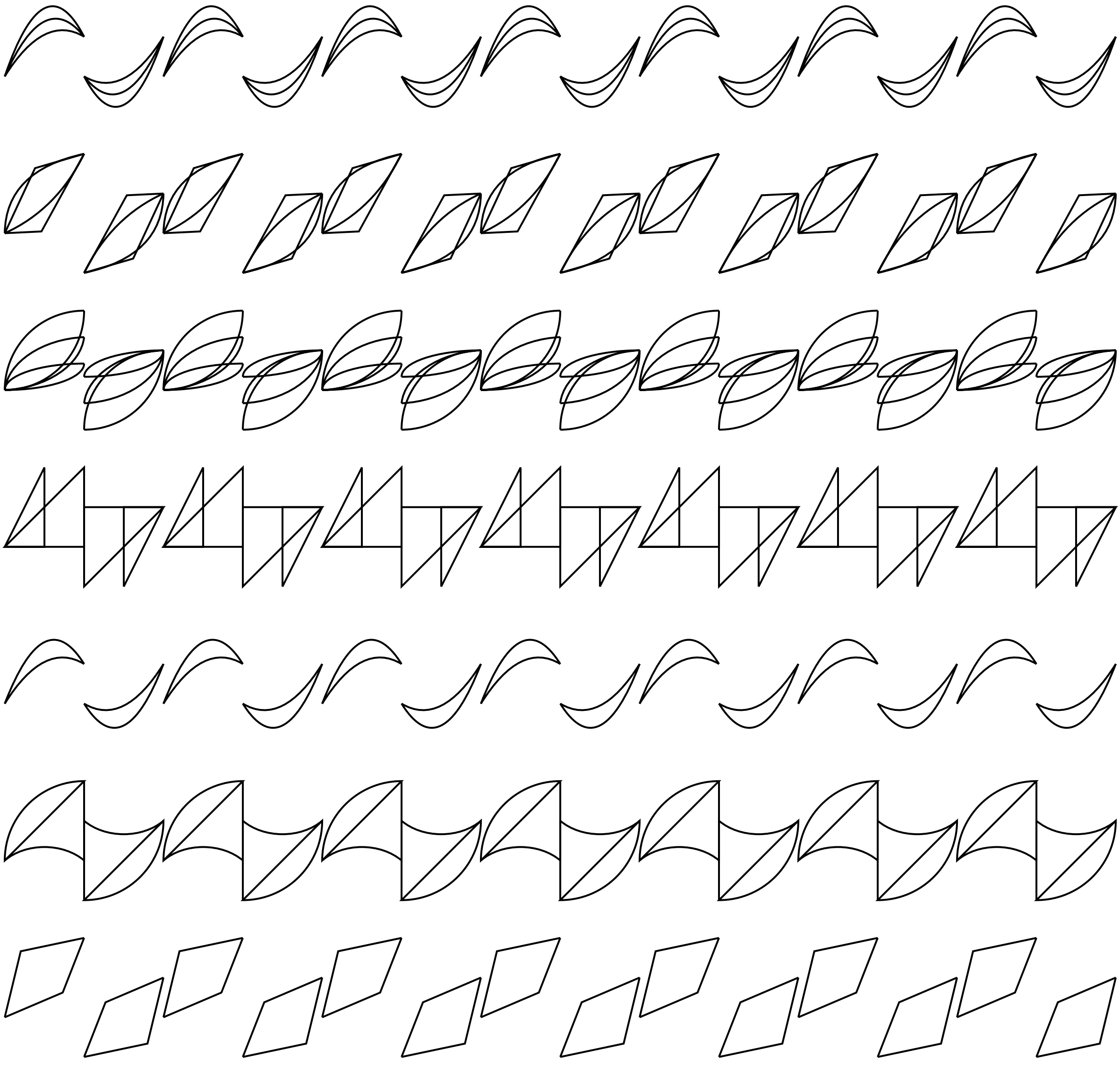


Can you see which patterns have horizontal mirrors and which have vertical mirrors? Use color to remove all of the vertical mirrors.



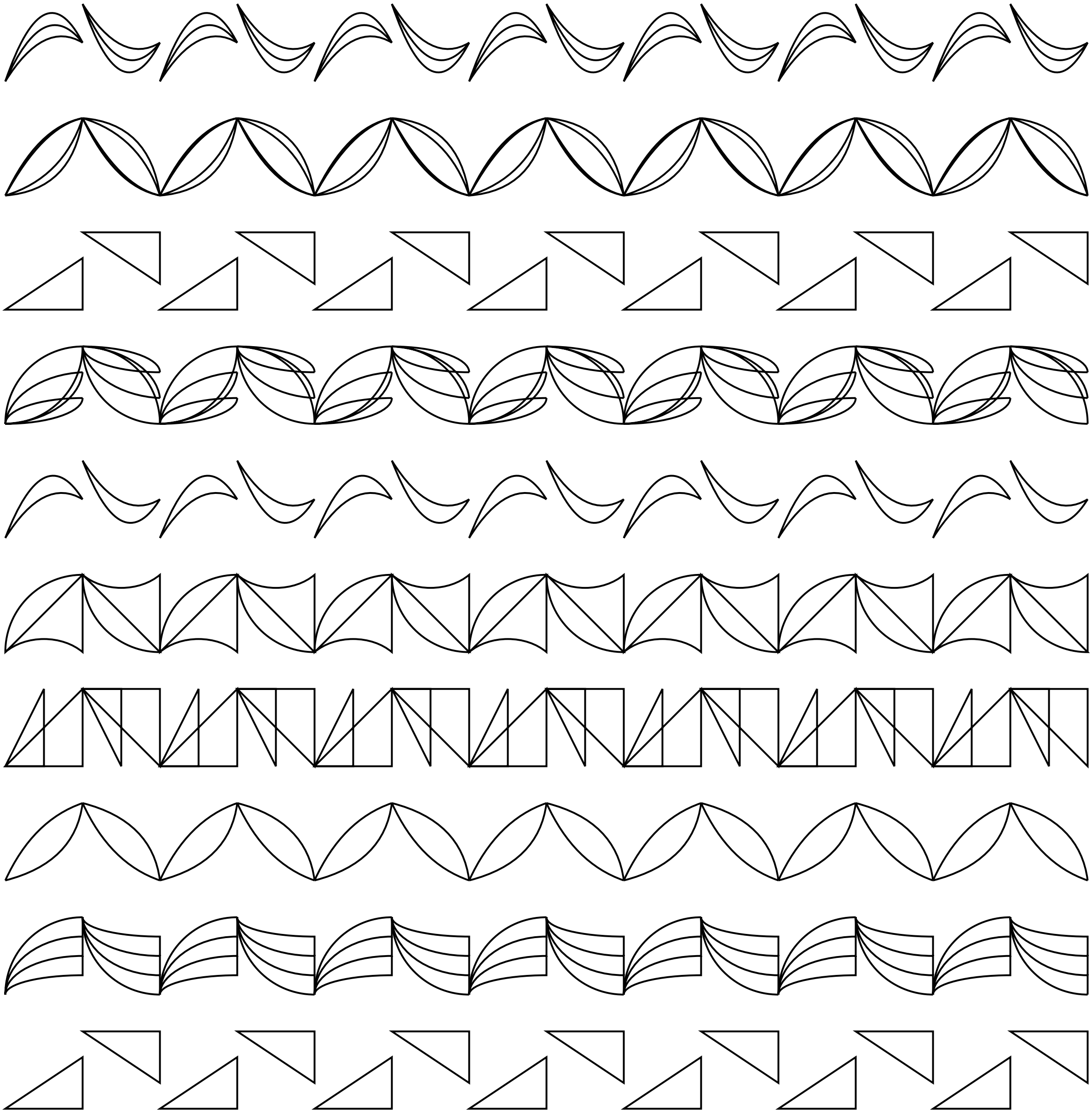
frieze patterns with horizontal mirrors, and frieze patterns with vertical mirrors (∞∗ and ∗∞∞)

Can you see all of the ½ turns in these frieze patterns? Use color to double the shortest possible distance of translation for each pattern while maintaining some of the ½ turns. How does the number of ½ turns change?



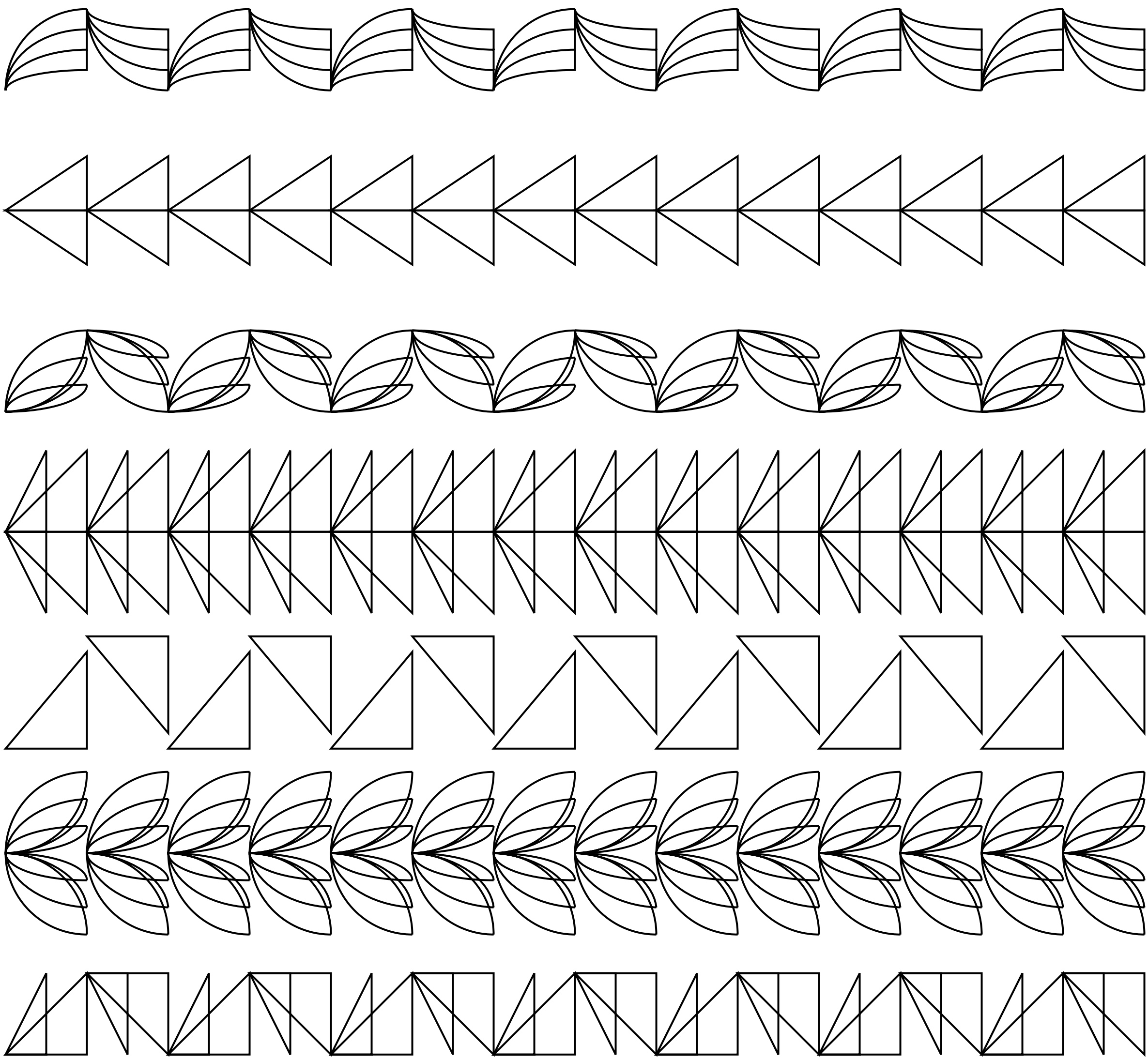
frieze patterns with ½ turns (22∞)

Can you see the glide reflections in these patterns? Use color to double the shortest possible distance of translation in the patterns, while making sure they still have glide reflections.



frieze patterns with glide reflections (∞×)

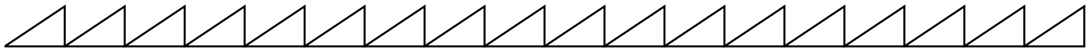
Can you see which patterns have horizontal mirrors, and which patterns have glide reflections? Use color to transform the patterns with horizontal mirrors into patterns with glide reflections only, so that all of the patterns have glide reflections.



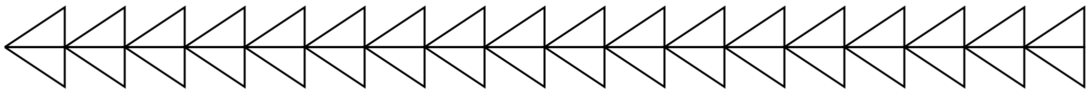
frieze patterns with horizontal mirrors and frieze patterns with glide reflections ($\infty{*}$ and $\infty\times$)

We have now seen patterns with each of the frieze group symmetries.

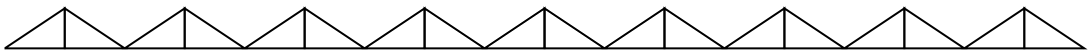
translation:



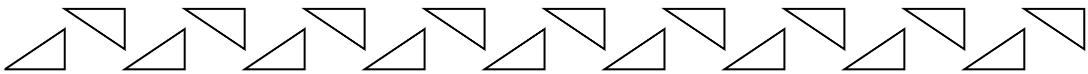
horizontal mirror reflection & translation:



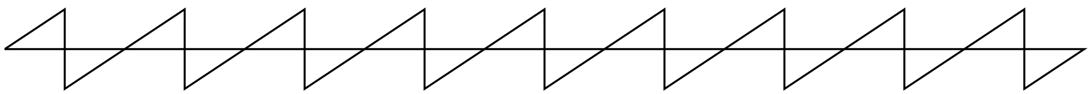
vertical mirror reflection & translation:



glide reflection & translation:



½ turn rotation & translation:

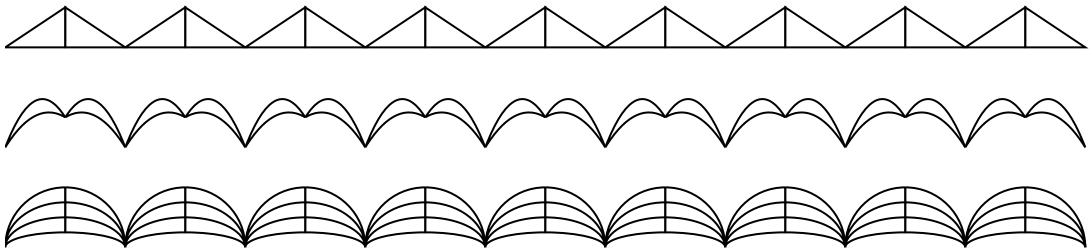


They all have **translations**, and all but the simplest have an additional <u>generator</u> of either a **horizontal mirror**, **vertical mirror**, **glide reflection**, or **½ turn**.

Let's clarify what we've been talking about and coloring...

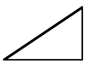
The frieze patterns illustrate the **frieze groups**. These groups contain symmetries, not patterns - the patterns just help us see them.

For example, **vertical mirror reflections** and **translations** are symmetries in a group that can be seen with the patterns:

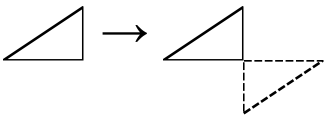


And we can come up with infinitely more pattern designs to illustrate it.

This is the case for all of our pattern groups. As long as a pattern has units

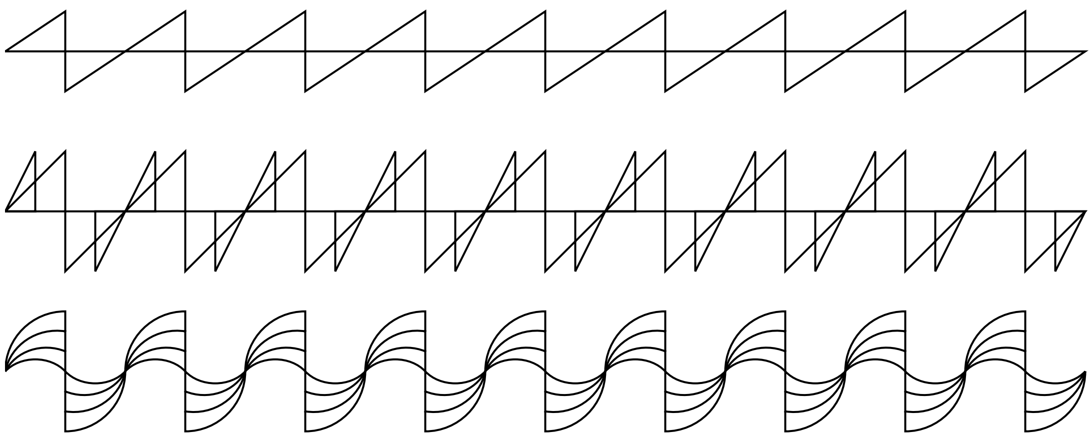


where applying the same symmetries to any unit leaves the entire pattern unchanged,

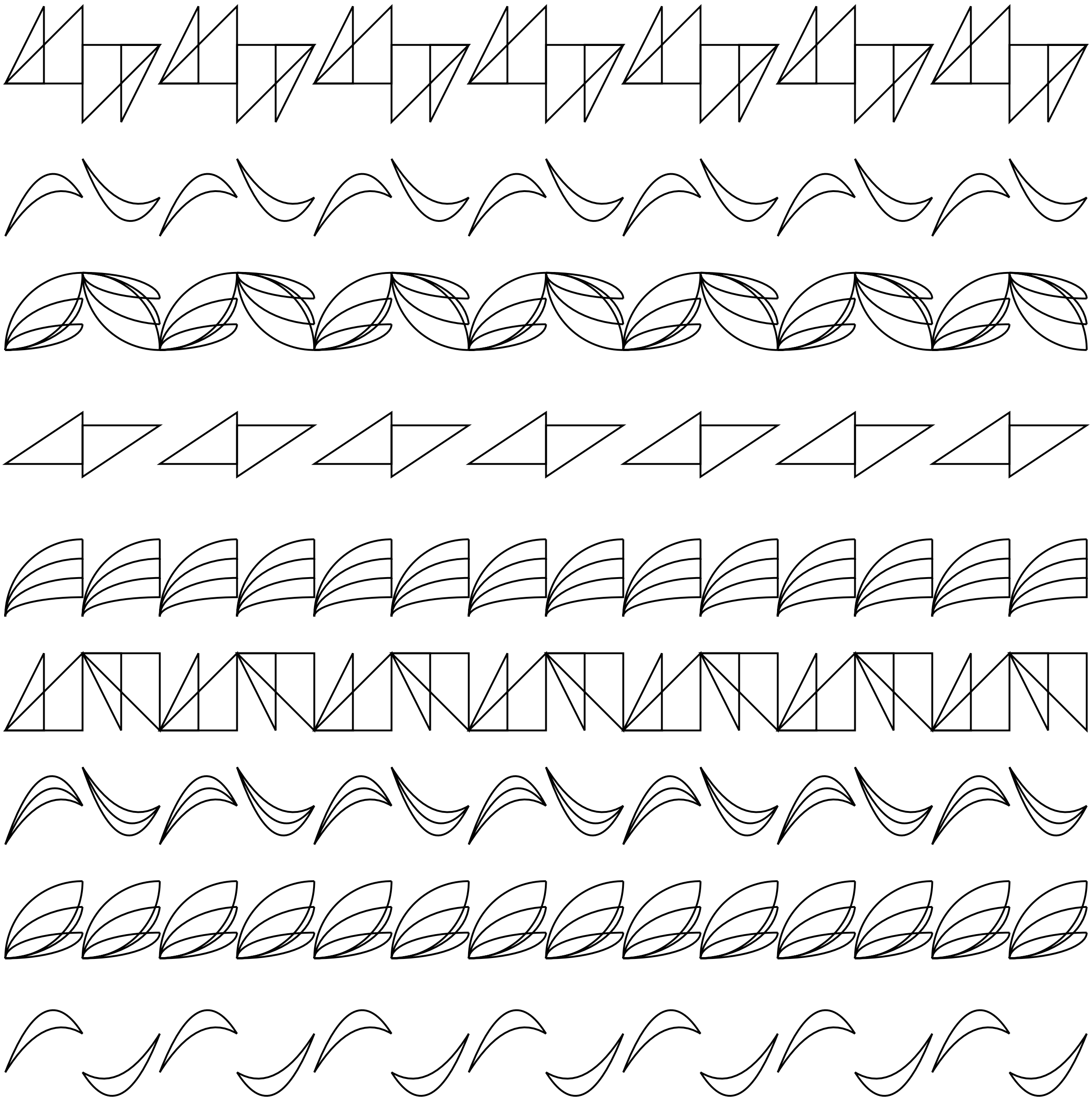


then the pattern illustrates the same group as any other patterns with the same symmetries.

½ turn rotation & translation:

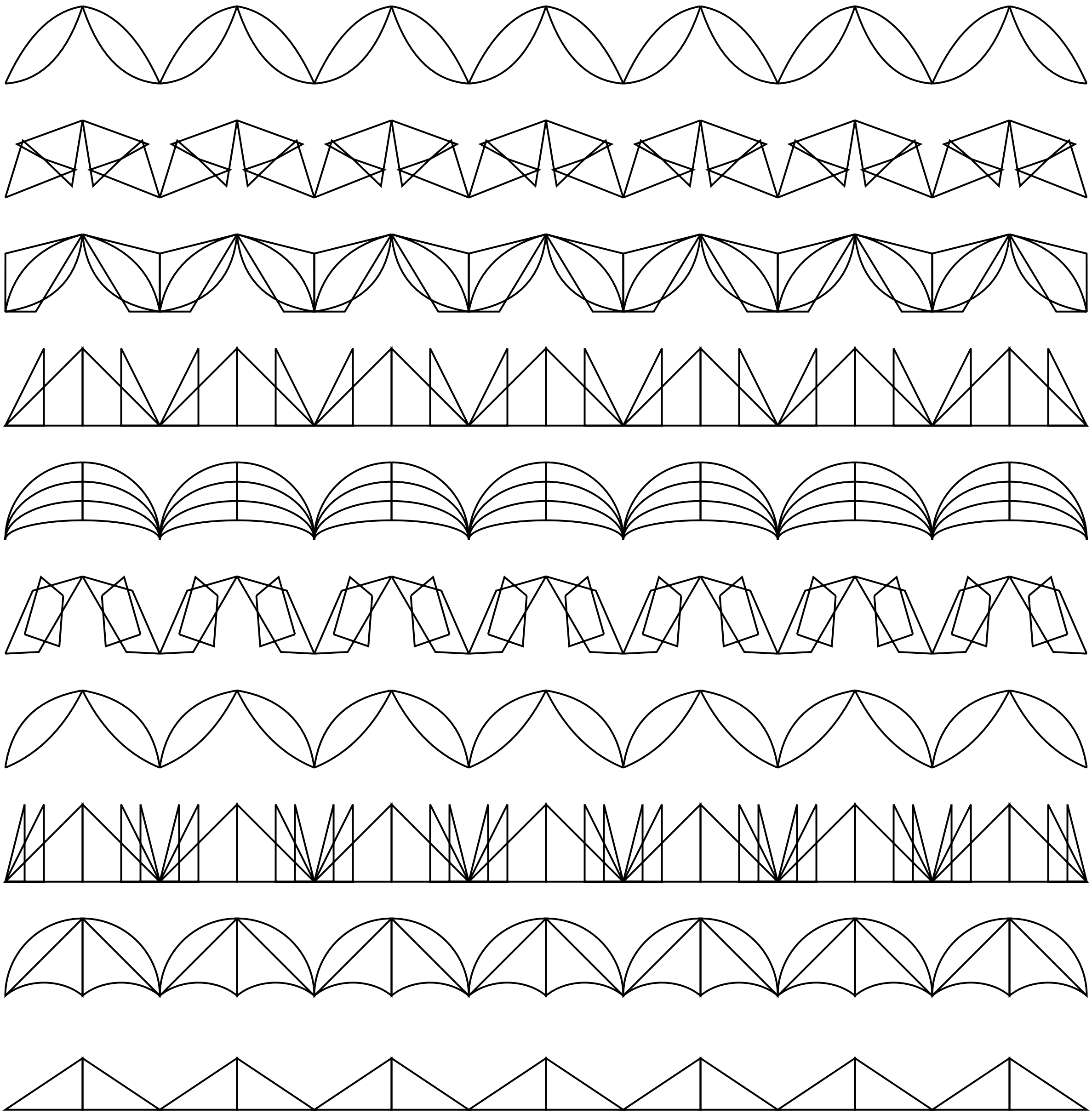


Color the patterns that share the same types of symmetries with the same sets of colors.



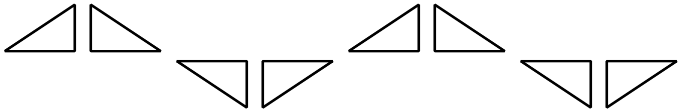
frieze patterns with ½ turns, glide reflections, and translation (22∞, ∞×, ∞∞)

Can you color the patterns to remove half of their mirrors? Use only 2 colors.



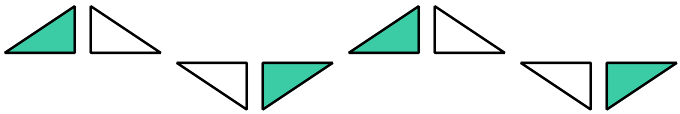
frieze patterns with vertical mirrors ($*\infty\infty$)

Combining the frieze group symmetries yields even more groups of patterns. For example, we can make patterns with **glide reflection, vertical mirror reflection, ½ turn rotation, translation:**

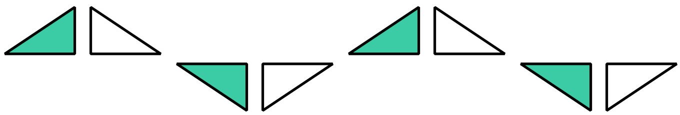


And color can again reduce the symmetry in these patterns so that they share the same symmetry groups as the simpler patterns we already colored.

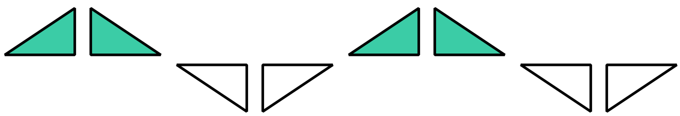
½ turn rotation & translation:



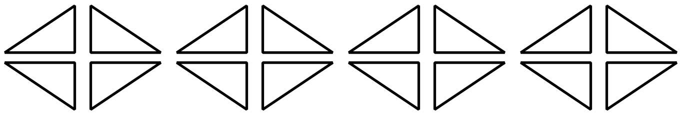
glide reflection & translation:



vertical mirror reflection & translation:

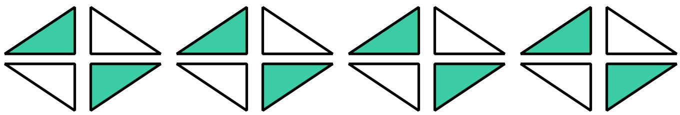


Patterns illustrating the frieze group with all possible symmetries
(glide reflection, horizontal mirror reflection, vertical mirror reflection, ½ turn rotation, translation)

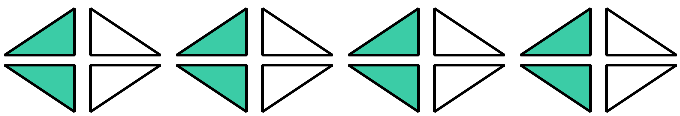


Can be reduced to each of the pattern groups we have already seen.

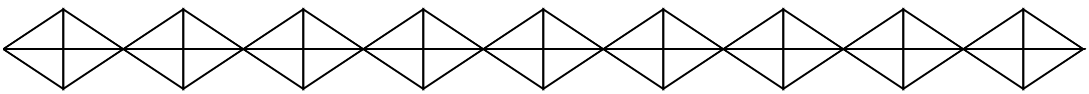
½ turn rotation & translation:



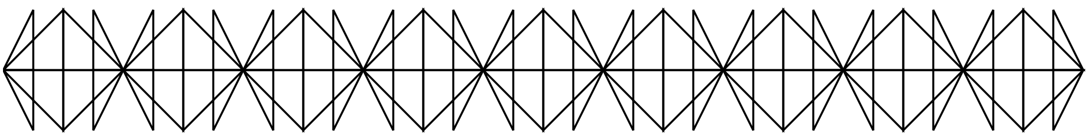
horizontal mirror reflection & translation:

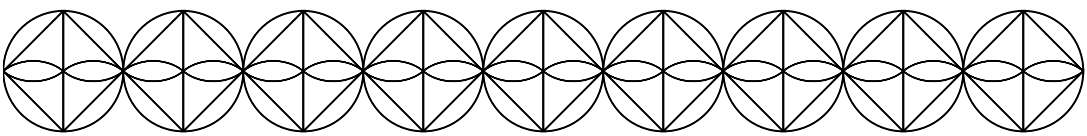


You can find the rest!

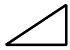


Check in: Can you see ½ turns in these patterns? What about vertical mirrors?

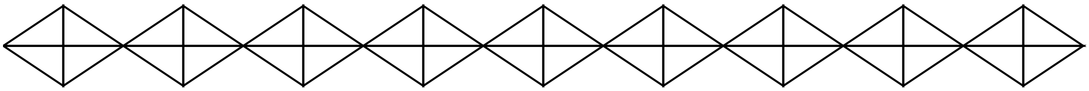




There are 7 Frieze Groups, and we have now colored all of them. There are no other ways to combine our symmetries to generate patterns that repeat forever in one direction. Surprised? Then try to generate more by again starting with a single piece.

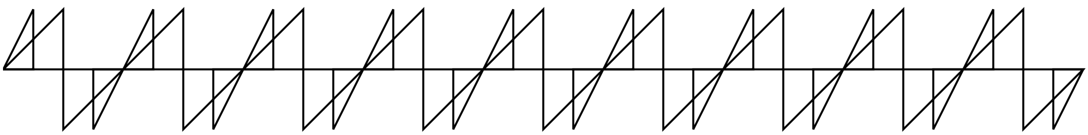


Or use color to reduce a pattern to one with a combination of symmetries that we did not yet see, like a pattern with just **horizontal mirror reflection, vertical mirror reflection, translation**.

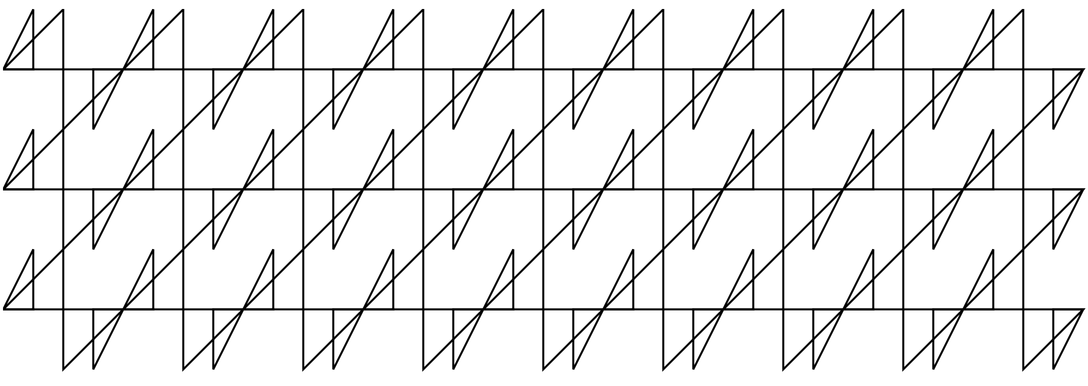


You will have to give up. Combining a **horizontal mirror** with a **vertical mirror** brings about a **½ turn rotation**. This is just one example of how combining symmetries results in other symmetries, and brings us back to pattern groups we already have.

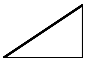
Yet we can still find more repeating patterns. Frieze patterns are limited to repetition along one dimension, but Wallpaper patterns do not have that limit.



When that limit is removed for the Wallpaper patterns, the number of possible patterns and amount of symmetry within them grows beyond what we have colored.

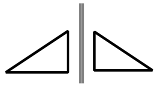


Challenge: What happens when you start with a single piece and then transform it with both rotation and glide reflection? What other symmetries emerge?

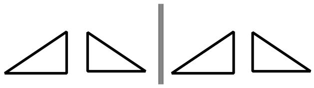


Aside from our simplest frieze pattern group that has just translation, we can see how different types of symmetries can be used to generate the same pattern.

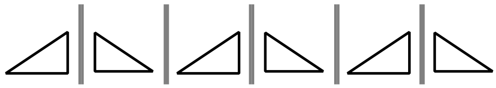
See, we can reflect across one mirror,



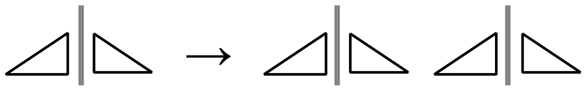
And then across another different mirror,



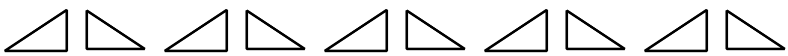
And keep reflecting across these alternating mirrors,



To generate a pattern that can also be generated by just one mirror and a translation.

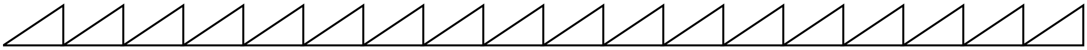


This is an example of how various sets of generators - two different mirrors versus one mirror and a translation - can be used to generate the same pattern group.

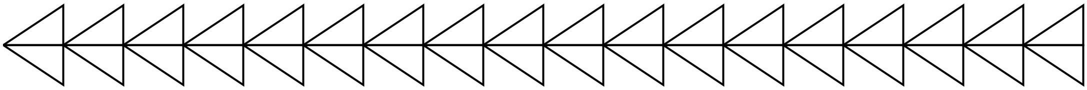


Challenge: For each of the frieze pattern groups, what are the various sets of symmetries that can be used to generate the entire pattern group?

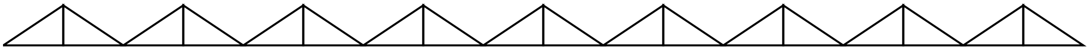
translation:



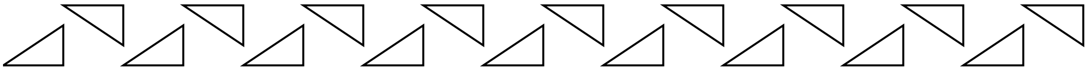
horizontal mirror reflection & translation:



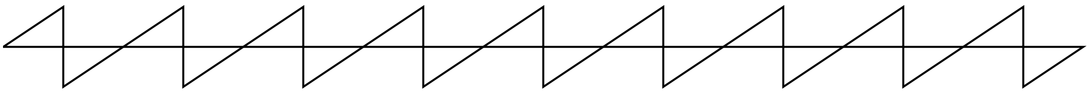
vertical mirror reflection & translation:



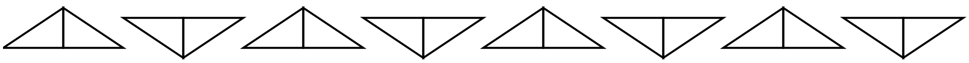
glide reflection & translation:



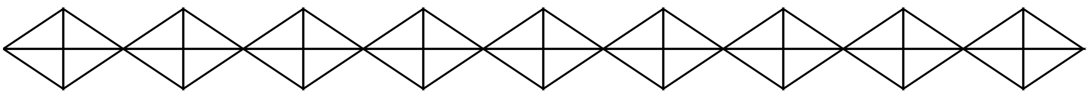
½ turn rotation & translation:



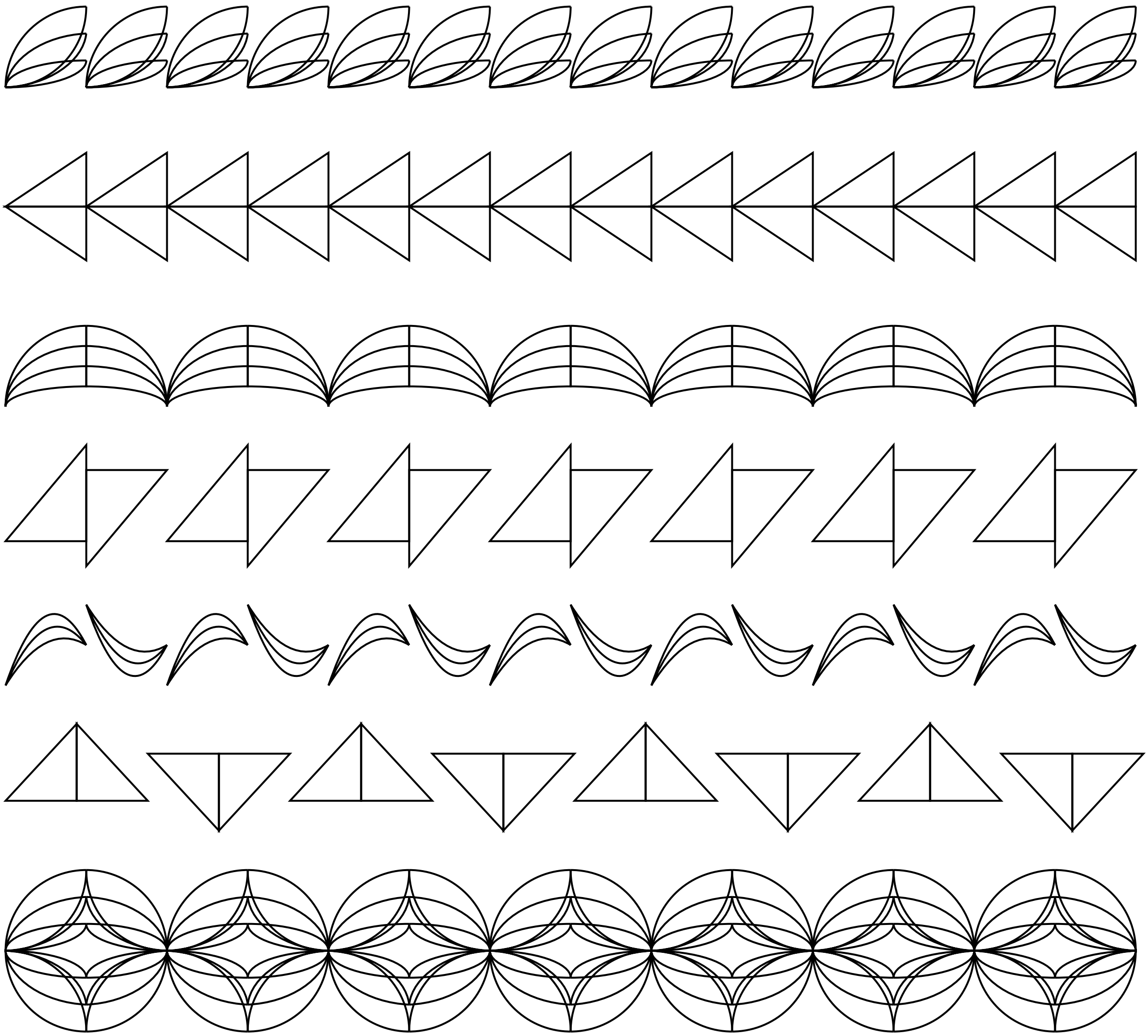
glide reflection, vertical mirror reflection, ½ turn rotation & translation:



horizontal and vertical mirror reflection, ½ turn rotation & translation:



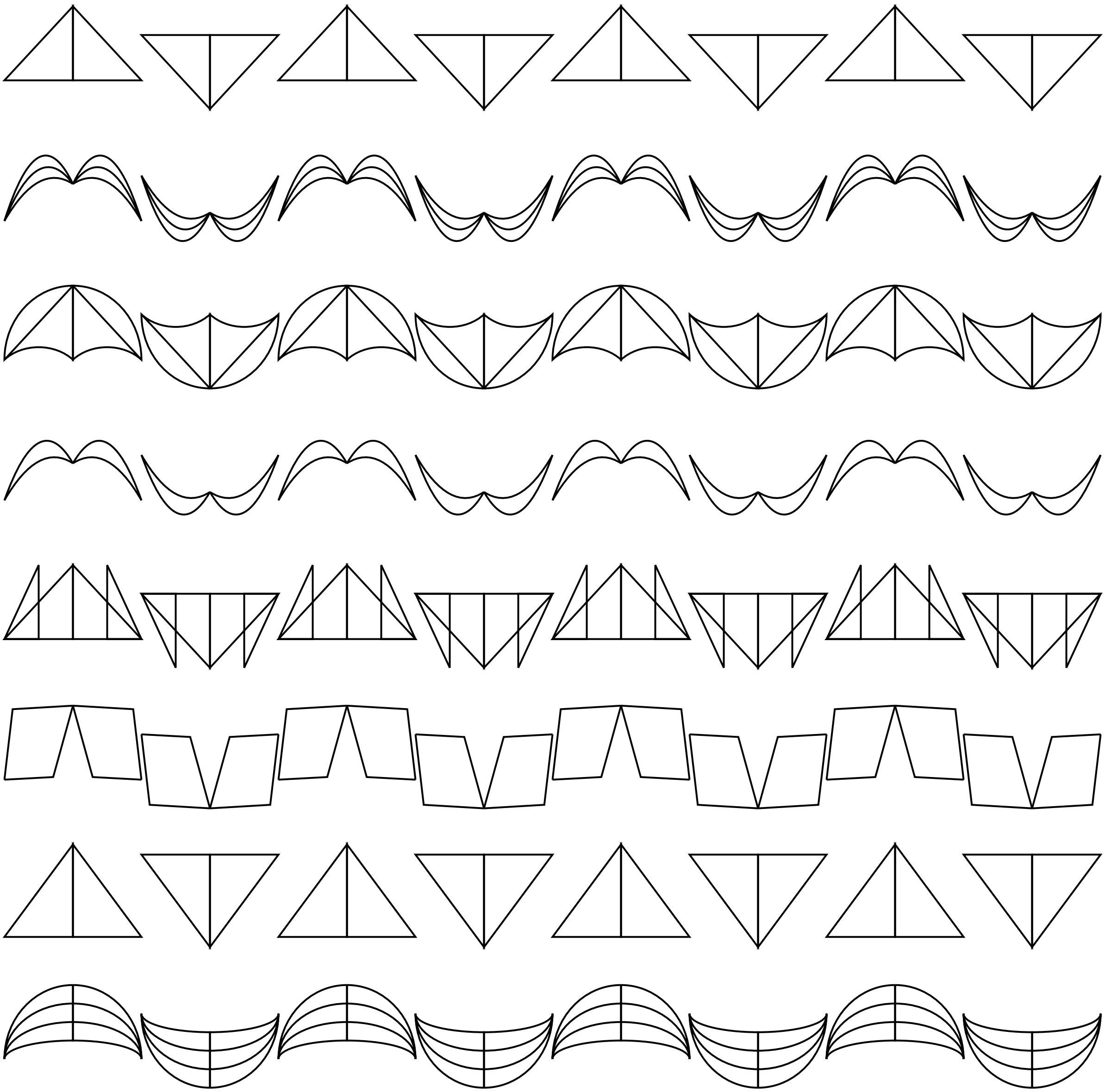
Can you identify the symmetries in each of the patterns?



patterns from each of the 7 frieze groups

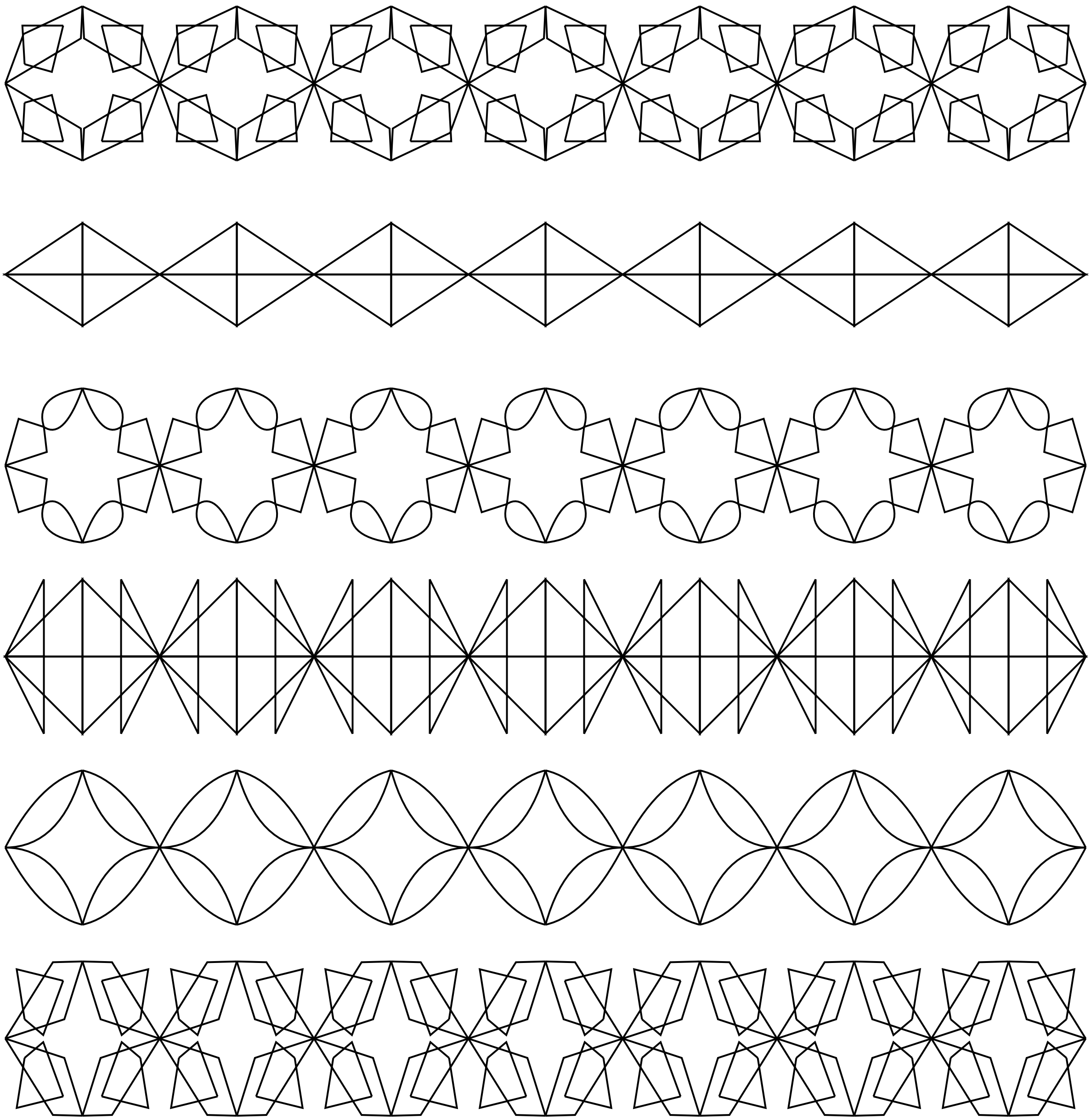
$(\infty\infty,\ \infty{*},\ {*}\infty\infty,\ 22\infty,\ \infty\times,\ 2{*}\infty,\ {*}22\infty)$

Use color to reduce the amount of symmetry in the patterns so that they only have vertical mirrors and translations, and do not have ½ turns.

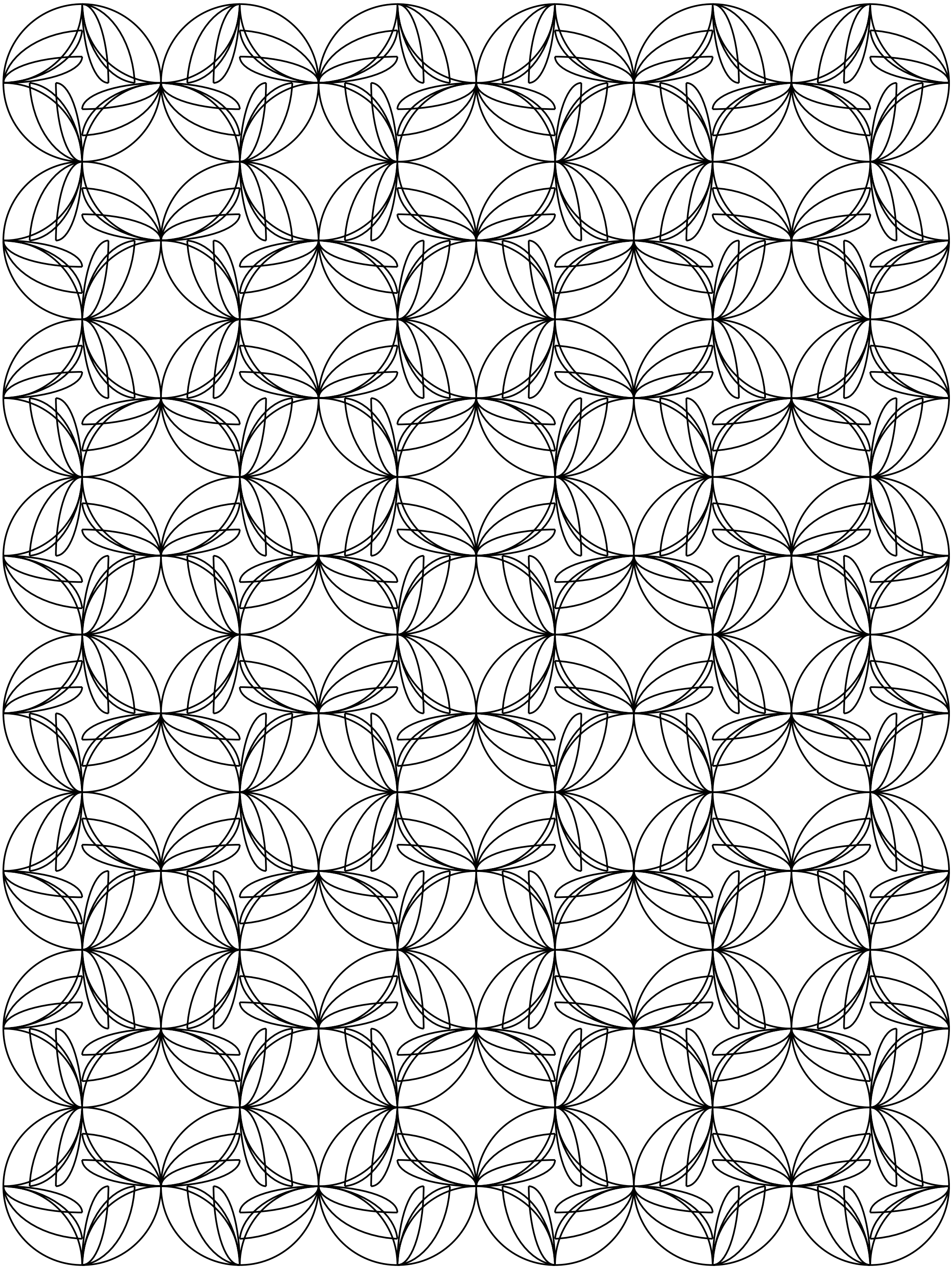


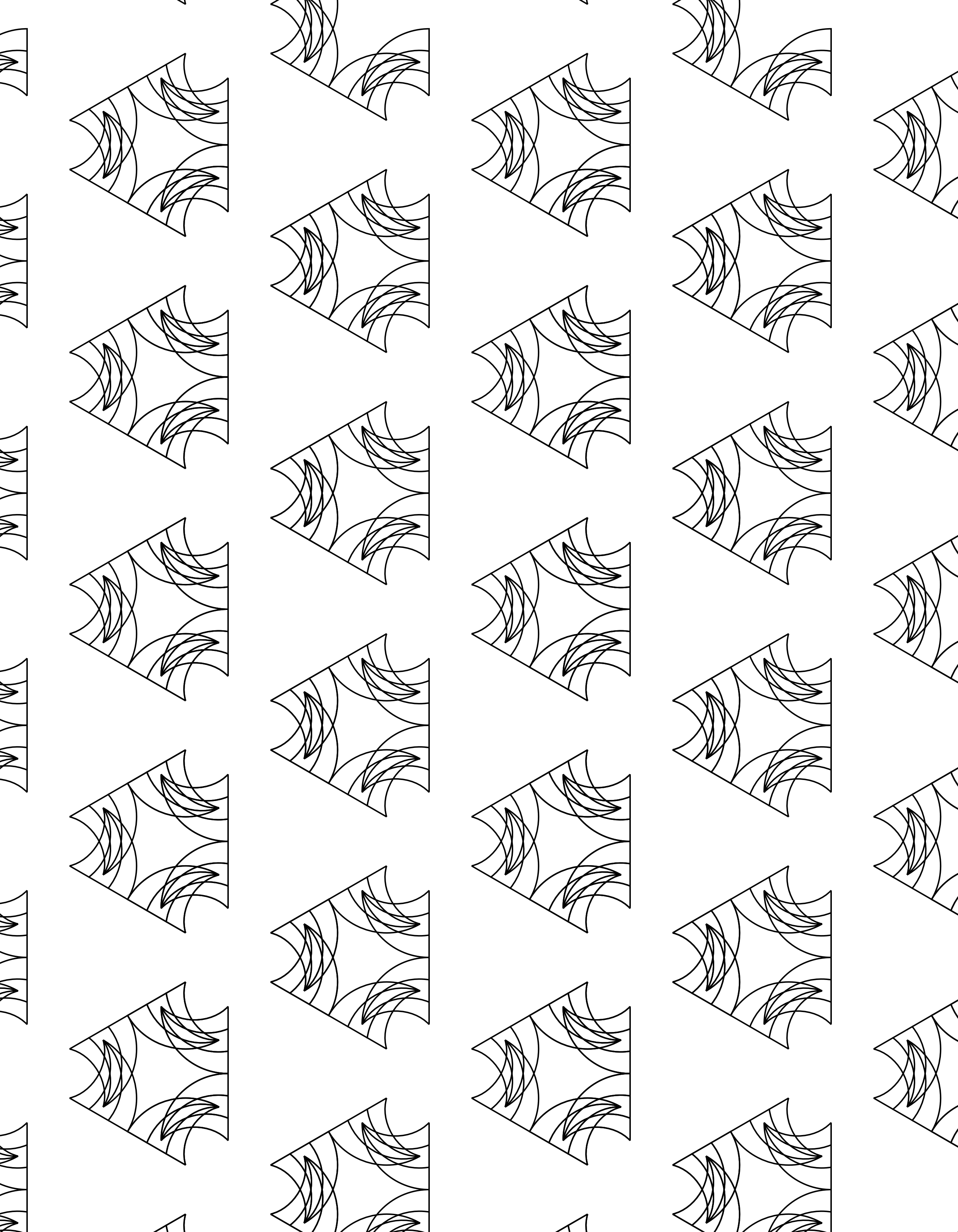
frieze patterns with glide reflections, vertical mirrors, ½ turns, and translations ($2{*}\infty$)

Use color to reduce these patterns to patterns that have vertical mirrors, glide reflections, and ½ turns, but not horizontal mirrors.



frieze patterns with ½ turns, glide reflections, horizontal mirrors, vertical mirrors, and translations (∗22∞)





THEORY REFERENCE

Group theory helps define abstract structures discussed in algebra. The groups in this coloring book are only a window into the groups explored in other realms of mathematics.

There are some rules and definitions that pertain to all groups, not just ours.

Group

A group G is a set coupled with a binary operator ✶ that satisfies 4 requirements:

See the details of each rule for examples.

> Closure: G is closed under ✶; i.e., if a and b are in G, then **a✶b** is in G.
>
> Identity element: There exists an identity element e in G; i.e., for all a in G we have **a ✶ e = e ✶ a = a.**
>
> Inverse element: Every element in G has an inverse in G; i.e., for all a in G, there exists an element -a in G such that **a✶(-a) = (-a)✶a = e.**
>
> Associativity: The operator ✶ acts associatively; i.e., for all a,b,c in G, **a ✶ (b ✶ c) = (a ✶ b) ✶ c.**

Associative Property

When an operator ✶ for a group G is associative, the way elements in G are grouped when the operator is applied does not matter. I.e., for all a,b,c in G, **a ✶ (b ✶ c) = (a ✶ b) ✶ c.**

One example of this is adding numbers: **1 + (2 + 3) = (1 + 2) + 3.**
Notice that subtraction of numbers is not associative: **1 − (2 − 3)** does not equal **(1 − 2) − 3.**

Our groups of rotations have an associative operator: Our operator here is combining rotations.
For C3, **(⅓ turn ✶ ⅓ turn) ✶ ⅔ turn** = **⅓ turn ✶ (⅓ turn ✶ ⅔ turn).** That is, rotating twice by a ⅓ turn and then rotating the result by a ⅔ turn is the same as combining a ⅓ turn with the result of rotating by a ⅓ turn and then by a ⅔ turn.

Binary Operator

A **binary operator** ✶ combines 2 elements, a and b, from a set S to give a third element: **a ✶ b.**

An example is addition over the set of counting numbers: + is a binary operator that combines 2 numbers to create their sum: 1 + 2 = 3.
Our binary operator combines the transformations that act on our symmetry groups. For symmetry elements a and b, **a ✶ b** says "do a, and then do b". For example, if transformation a is "rotate by a ¼ turn" and b is "reflect horizontally", then **a ✶ b** is "rotate by a ¼ turn and then reflect horizontally".

Closure

A set S is **closed** under an operator ∗ if combining any 2 elements in S with ∗ results in an element that is also in S; i.e, for any a and b in S, a∗b is also in S.

For example, the set of all counting numbers 0,1,2,3,... is closed under the addition operator + because adding any two counting numbers results in another counting number.
Coming back to our sets of rotations, the set **{ ¼ turn, $\frac{2}{4}$ turn }** is not closed because combining the ¼ turn with the $\frac{2}{4}$ turn results in the ¾ turn which is not in this set.

Commutative Property

A binary operator ∗ is **commutative** if the order in which it combines elements does not matter.

I.e., for any 2 elements a & b, **a∗b = b∗a.**

For example, addition is commutative because **1 + 2 = 2 + 1,** but subtraction is not commutative because **1 − 2 ≠ 2 − 1.**
A group with a **commutative** binary operator ∗ is called a **commutative group**. This means that the order in which any 2 of the group's elements are combined does not matter.

For example, our groups with only rotations are commutative groups because the order in which any 2 rotations are combined does not matter. e.g. **¼ turn ∗ $\frac{2}{4}$ turn = $\frac{2}{4}$ turn ∗ ¼ turn = ¾ turn**.
However, our groups with both rotations and reflections are not commutative because the order in which a rotation and a reflection are combined *does* matter. e.g. **¼ turn ∗ reflect ≠ reflect ∗ ¼ turn**.

Cyclic Group

A group G is called **cyclic** if it can be generated by a single element.

Our groups of rotations are cyclic groups because they can be generated by their smallest nonzero element. For example, our C2 group was **{0 turn, ½ turn}**, and it was generated by the ½ turn.
There are many other cyclic groups out there. Another C2 group that may look different, is the group **{1, −1}** where the members of the group are the numbers 1 and −1 and the way of combining these members is with multiplication. It can be generated by −1.
The term **cyclic** may be misleading. Our cyclic groups had a finite number of elements, and combining them again and again created cycles. However, there are cyclic groups with infinite elements, such as the integers under addition.

Generator

Generators of a group are a set of elements that when combined with themselves, or each other, can produce all the other elements of the group.

For example, −2 and 2 are **generators** that when combined with addition, generate the entire group of even integers.

Identity Element

An **identity element** is a neutral element - when it's combined with other members in the group, it does not change them.

For our groups of rotations, the identity element is the 0 turn: rotating by the 0 turn is the same as doing nothing at all.
For the group of integers under addition, the identity element is 0: **0 + 2 = 2.**

Inverse Element

An **inverse element** is the reverse of another element.

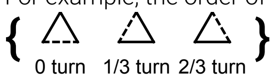
More formally, for a set, S with a binary operator, ∗, and a and b in S: a is the **inverse** of b if
a∗b = b∗a = e, where e is the identity element.

For our groups of rotations, each rotation's inverse element is the rotation that undoes it. For example, the inverse of the ⅓ turn is the ⅔ turn because ⅓ turn ∗ ⅔ turn → full turn. The full turn is the same as the 0 turn which is our identity element.
For addition on the integers, each integer's inverse element is it's negative: −1 is the inverse of 1 because **−1 + 1 = 0.**

Order

The **order** of a group G is the number of elements in G. The **order** of G is sometimes written as **|G|**.

For example, the order of our C3 group of rotations is 3 because C3 has 3 elements:



Set

A **set** is a collection of distinct elements.

For example, the set **{blue, red, blue}** is the same set as the set **{blue, red}**.
For our sets of rotations, the set **{0 turn, ⅓ turn, $\frac{4}{3}$ turn}** is the same as the set **{0 turn, ⅓ turn}** because a ⅓ turn means the same thing as a $\frac{4}{3}$ turn - they are not distinct.

Subgroup

Given a group G, a **subgroup** of G is a group with the same binary operator as G and whose members are all also in G.

For example, the group of even integers under addition **{... −2, 0, 2, 4,...}, +** is a subgroup of the group of all integers under addition **{... −2, −1, 0, 1, 2,...}, +**.
However, the same cannot be said for odd integers. The set of odd integers under addition **{... −3, −1, 1, 3, 5,...}, +** is not closed and therefore cannot be a group: Combining odd integers with addition produces even integers (e.g. 1 + 3 = 4), which are clearly not in the set of odd integers.
Notice that a group and its subgroups always have the same identity element.