Math189R SP19
Homework 5
Monday, Mar 11, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\|\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_i^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d} \lambda_j$ into $\sum_{j=1}^{k} \lambda_j$ and $\sum_{j=k+1}^{d} \lambda_j$.

(a) Consider that $||\vec{v}||^2 = \vec{v}^\top \vec{v}$, we can apply the same premise here;

$$\begin{aligned}
\left\|\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right\|^2 &= \left[\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right]^\top \left[\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right] \\
&= \mathbf{x}_i^\top \mathbf{x}_i - 2\sum_{j=1}^{k} z_{ij}\mathbf{v}_j^\top \mathbf{x}_i + \sum_{j=1}^{k} (z_{ij}\mathbf{v}_j)^\top z_{ij}\mathbf{v}_j \\
&= \mathbf{x}_i^\top \mathbf{x}_i - 2\sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i\mathbf{x}_i^\top \mathbf{v}_j + \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i\mathbf{x}_i^\top \mathbf{v}_j \\
&= \boxed{\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i\mathbf{x}_i^\top \mathbf{v}_j}
\end{aligned}$$

(b) Lets begin a key statement that will prove useful, $\sum \mathbf{x}_i\mathbf{x}_i^T = \mathbf{\Sigma}$, we know that the reconstruction error is;

$$\begin{aligned}
J_k &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i\mathbf{x}_i^\top \mathbf{v}_j\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i - \sum_{i=1}^{n}\sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i\mathbf{x}_i^\top \mathbf{v}_j\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i\right)\sum_{j=1}^{k}\mathbf{v}_j^\top \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^\top\right)\mathbf{v}_j \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i\right) - \sum_{j=1}^{k}\mathbf{v}_j^\top \mathbf{\Sigma}\mathbf{v}_j \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i\right) - \sum_{j=1}^{k}\lambda_j
\end{aligned}$$

(c) We saw in part b that

$$J_d = \sum_{j=1}^{d}\lambda_j = \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i\right) - \sum_{j=1}^{d}\lambda_j$$

We want to find out how much error is introduced for a specific value, J_k. Th expression for this, after partitioning the sum as suggested, in terms of d will be,

$$J_k = \sum_{j=k+1}^{d}\lambda_j + \frac{1}{n}\left(\sum_{i=1}^{n}\mathbf{x}_i^\top \mathbf{x}_i\right) - \sum_{j=1}^{d}\lambda_j$$

This is simply $J_k = \sum_{j=k+1}^{d}\lambda_j$ because $J_d = 0$

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2 (ℓ_1-Regularization) Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

Show that the optimization problem

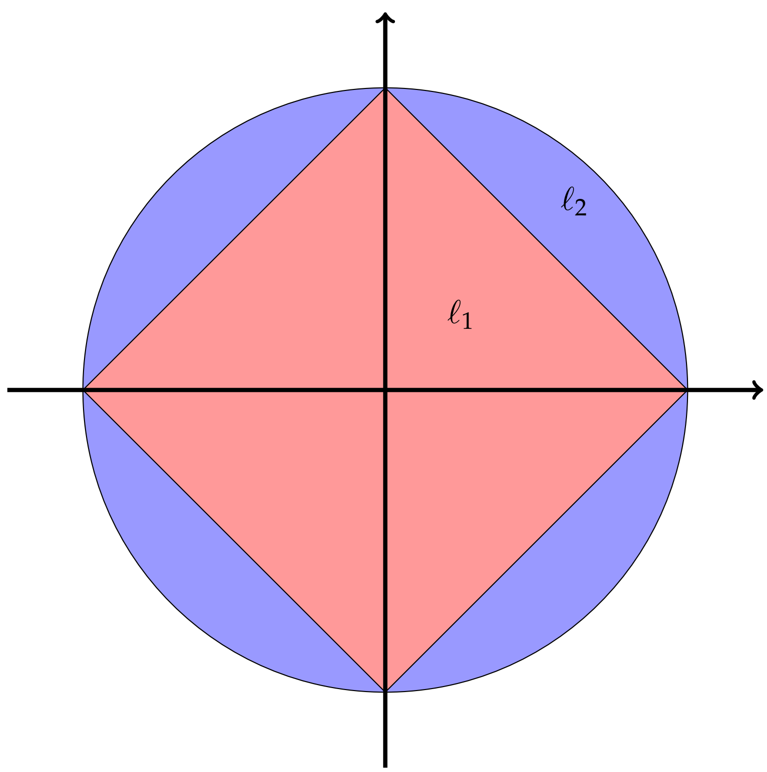
$$\begin{aligned} \text{minimize: } & f(\mathbf{x}) \\ \text{subj. to: } & \|\mathbf{x}\|_p \leq k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.

(a) The desired norm balls



(b) The Lagrange multiplier is defined as $\mathcal{L}(f(x), g(x)) = f(x) - \lambda g(x)$ Where $g(x)$ is our constraint, as such we see that for this problem;

$$\mathcal{L}(f(x), g(x)) = f(x) + \lambda(\|\mathbf{x}\|_p - k)$$

We know that minimizing the Lagrangian is equivalent to minimizing the function $f(x)$ subject to the constraint and that λk is not dependent at all upon x, so we can throw this term away;

$$\text{minimize } \mathcal{L}(f(x), g(x)) = \text{minimize } f(x) + \lambda(\|\mathbf{x}\|_p)$$

Since we have constructed this from a function and constraint given, we can say that the two statements above are equivalent.

(c) We saw in class that an advantage of the ℓ_1 norm is it preference for zeros, and we can think of an optimal solution has residing on a vertex of the norm ball. The ℓ_1 norm will have sparser solutions than the ℓ_2 norm because it has fewer vertexes, and thus fewer optimal solutions.

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Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

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