Math 55

Homework 3
Due Thursday, 9/27/18

1. Prove the identity: For $m \geq 0$ and $n \geq 1$, $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$ by induction on m. (Note: It can also be proved by induction on n, but induction on m is simpler.)

2. Combinatorially prove for $m \geq 0$, $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

3. For $n \geq 1$, conjecture and prove a formula for the n odd-indexed Fibonacci numbers $f_1 + f_3 + f_5 + ... + f_{2n-1}$

a) by induction, and

b) combinatorially (using tilings)

(Note: $f_1 = 1, f_3 = 3, f_5 = 8$, etc.)

4. Use strong induction to prove that every integer n greater than or equal to 2 is the product of prime numbers. (A prime is a number p greater than or equal to 2 whose only positive divisors are p and 1.)

5. A pile with $n \geq 2$ coins is split into two smaller piles, say of size x and y where $x + y = n$. We jot down the numbers x and y. We then continue this process with the pile of x coins and the pile of y coins, always jotting down the sizes of the new piles, until (after $n - 1$ steps) we are left with n piles, each with a single coin. Next we take the $n - 1$ pairs of numbers that we jotted down, and multiply each pair together, then add those products to obtain a single number. Use strong induction to prove that for $n \geq 1$, the number we end up with is $n(n-1)/2$. (For example, if we start with 10 coins, the final number will be 45.)

6. a) How many n-digit positive numbers have their digits in strictly increasing order? b) in Non-decreasing order? (Clarification: an n-digit positive number cannot have a leading digit of zero. In base 4, there are 10 non-decreasing 3-digit numbers: 111, 112, 113, 122, 123, 133, 222, 223, 233, and 333, only one of which (123) is strictly increasing.)

7. a) How many ways are there to pick a collection of 10 balls from a large pile of red balls, blue balls and purple balls, (more than 10 of each color, and balls of the same color are indistinguishable). b) same problem but you must pick at least 5 red balls. c) same, but instead must pick at most 5 red balls.

8. Combinatorially prove that

$$\sum_{k=0}^{m} \left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n+1}{m}\!\!\right).$$