Lab 2: Q1-Q3 Solutions

Part 1: Broken Rulers

(a) Find the conditional expectation of Y given X, $E(Y|X)$.

Once a value $X = x$ is given, Y is distributed uniformly in $[0, x]$. Given that the conditional distribution of Y is uniform, the conditional expectation will be the average of the maximum and minimum values of Y. In other words:

$$E(Y|X = x) = \frac{\mathbf{x}}{\mathbf{2}}$$

(b) Find the unconditional expectation of Y.

Using the law of iterated expectation, we have,

$$E(Y) = E(E(Y|X)) = E(\frac{X}{2}) = \frac{E(X)}{2}$$

Since X is uniformly distributed on $[0, 1]$, its expectation is at the midpoint of this interval. Therefore,

$$E(Y) = \frac{\frac{1}{2}}{2} = \frac{\mathbf{1}}{\mathbf{4}}$$

(c) Write down the expression for the joint probably density function of X and Y, $f_{X,Y}(x, y)$.

It is clear that the probability density is only positive for $0 \leq y \leq x \leq 1$. In this region, we use the multiplication rule:

$$\begin{aligned} f_{X,Y}(x, y) &= f_X(x) * f_{Y|X}(y|x) \\ &= 1 * \frac{1}{x} = \frac{1}{x} \end{aligned}$$

This gives the final answer,

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{x}, & 0 \leq y \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

(d) Find the conditional probably density function of X given Y, $f_{X|Y}$.

We first need to find the marginal distribution of Y. In the region $0 \leq y \leq 1$, we have:

$$
\begin{aligned}
f_Y(y) &= \int_0^1 f_{X,Y}(x,y)dx \\
&= \int_0^y f_{X,Y}(x,y)dx + \int_y^1 f_{X,Y}(x,y) \\
&= 0 + \int_y^1 \frac{1}{x}dx \\
&= \Big[ln(x)\Big]_y^1 \\
&= -ln(y)
\end{aligned}
$$

Outside of $[0,1]$, the marginal probability density of Y is 0. We now have all the components needed to find the conditional PDF, which is only defined for $y \in [0,1]$:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{\frac{1}{x}}{-ln(y)} \\
&= -\frac{\mathbf{1}}{\mathbf{xln(y)}}
\end{aligned}
$$

(e) Find the expectation of X given that $Y = \frac{1}{2}$, $E(X|Y = \frac{1}{2})$.

We first find the conditional probability distribution of X when $Y = \frac{1}{2}$:

$$
f_{X|Y}(x|y = 1/2) = -\frac{1}{xln(\frac{1}{2})} = \frac{1}{xln(2)}
$$

Now we can find the conditional expectation:

$$
\begin{aligned}
E(X|Y = 1/2) &= \int_{1/2}^1 x \times \frac{1}{xln(2)}dx \\
&= \int_{1/2}^1 \frac{1}{ln(2)}dx \\
&= \frac{1}{ln(2)}\Big[x\Big]_{1/2}^1 \\
&= \frac{1}{2ln(2)} \\
&= \mathbf{0.7213}
\end{aligned}
$$

Part 2: Investing

Let P represent total payoff, $P = aA + bB + cC$ for our investments $a + b + c = 1$. We want to minimize $var(P)$. Before determining the minimum, let's expand $var(P)$ in terms of A, B, and C:

$$
var(P) = var(aA + bB + cC) = a^2var(A) + b^2var(B) + c^2var(C)
$$

We can now represent $var(P)$ in terms of the variance of only one of the returns. Let's represent it as a function of only $var(A)$:

$$var(P) = a^2 var(A) + b^2 var(B) + c^2 var(C) = a^2 var(A) + \frac{1}{2} b^2 var(A) + \frac{1}{3} c^2 var(A) = (a^2 + \frac{b^2}{2} + \frac{c^2}{3}) var(A)$$

We want to minimize this quantity subject to the constraint $a + b + c = 1$. One especially elegant way to do this is with a Lagrange multiplier:

$$F = a^2 + \frac{b^2}{2} + \frac{c^2}{3} + \lambda(a + b + c - 1),$$

where λ is the Lagrange multiplier. To solve for this, we set

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = \frac{\partial F}{\partial c} = \frac{\partial F}{\partial \lambda} = 0$$

and arrive at the following set of equations:

$$\begin{aligned} 2a + \lambda = 0 &\Rightarrow a = -\frac{\lambda}{2} \\ b + \lambda = 0 &\Rightarrow b = -\lambda \\ \frac{2c}{3} + \lambda = 0 &\Rightarrow c = -\frac{3\lambda}{2} \\ a + b + c - 1 = 0 & \end{aligned}$$

Solving these,

$$\begin{aligned} -\frac{\lambda}{2} - \lambda - \frac{3\lambda}{2} - 1 = 0 \\ -\frac{6\lambda}{2} = 1 \Rightarrow \lambda = -\frac{1}{3} \end{aligned}$$

This gives a minimum at $[\frac{1}{6}, \frac{1}{3}, \frac{1}{2}]$. We may also compute the variance for these values:

$$\begin{aligned} var(P) &= ((\frac{1}{6})^2 + \frac{(\frac{1}{3})^2}{2} + \frac{(\frac{1}{2})^2}{3}) var(A) \\ &= (\frac{1}{36} + \frac{1}{18} + \frac{1}{12}) var(A) \\ &= (\frac{1 + 2 + 3}{36}) var(A) \\ &= \frac{var(A)}{6} \end{aligned}$$

Part 3: Turtles

(a) Write down the likelihood function, $l(\theta)$ in terms of $y_1, y_2, ..., y_n$.

Since the lifespan of an individual turtle is normally distributed on $[0, \theta]$, the probability density function for each y_i is given by,

$$f(y_i, \theta) = \begin{cases} \frac{1}{\theta}, & 0 \le y_i \le \theta \\ 0, & otherwise \end{cases}$$

Since we have a random sample, the y_i are independent, and we can write the joint probability density as the product of the marginal probability densities,

$$\begin{aligned} l(\theta) &= \prod_{i=1}^{n} f_Y(y_i|\theta) \\ &= \begin{cases} \frac{1}{\theta^n}, & \theta \geq max(y_i) \\ 0, & otherwise \end{cases} \end{aligned}$$

(b) Based on the previous result, what is $\hat{\theta_{ml}}$, the maximum-likelihood estimator of θ?

Examining the previous result, $l(\theta)$ is positive but decreasing for $\theta \geq max(y_i)$. Therefore,

$$\hat{\theta_{\mathbf{ml}}} = \mathbf{max}(\mathbf{y_1}, \dots, \mathbf{y_n})$$

(c) For the simple case that $n = 1$, what is the expectation of $\hat{\theta_{ml}}$, given θ?

We can determine the expectation based on the result from part (b):

$$\begin{aligned} E(\hat{\theta_{ml}}) &= E(max(y_1)) \\ &= E(y_1) \\ &= \frac{\theta}{\mathbf{2}} \end{aligned}$$

(d) Is the MLE biased?

For the $n = 1$ case, **the MLE is biased** since $E(\hat{\theta_{ml}}) \neq \theta$. The bias in this case is $-\frac{\theta}{2}$.

(e) For the more general case that $n \geq 1$, what is the expectation of $\hat{\theta_{ml}}$?

We now compute the general expectation of $\hat{\theta_{ml}}$. For notational simplicity, let $M = max(y_1, \dots, y_n)$. Let f_M be the probability density function of this variable, and F_M be the cumulative distribution.

Let's first find the cummulative distribution of M. For $0 \leq m \leq \theta$, we have:

$$\begin{aligned} F_M(m) &= P(max(y_1, ..., y_n) < m) \\ &= P(y_1 < z, ..., y_n < m) \\ &= \prod_{i=1}^{n} P(y_i < m) \\ &= \prod_{i=1}^{n} \frac{m}{\theta} \\ &= \frac{m^n}{\theta^n} \end{aligned}$$

Giving the result,

$$F_M(m) = \begin{cases} 0, & m < 0 \\ \dfrac{m^n}{\theta^n}, & 0 \leq m \leq \theta \\ 1, & otherwise \end{cases}$$

Taking the derivative,

$$f_M(m) = \begin{cases} \dfrac{nz^{n-1}}{\theta^n}, & 0 \leq m \leq \theta \\ 0, & otherwise \end{cases}$$

Finally, we can compute the expectation,

$$\begin{aligned} E(Z) &= \int_0^\theta m f_M(m) dm \\ &= \int_0^\theta m. \frac{nm^{n-1}}{\theta^n} dm \\ &= \frac{n}{\theta^n} \int_0^\theta m^n dm \\ &= \frac{n}{\theta^n} \left[\frac{m^{n+1}}{n+1} \right]_0^\theta \\ &= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} \\ &= \frac{n\theta}{n+1} \\ &= \frac{\theta}{\mathbf{1 + \frac{1}{n}}} \end{aligned}$$

We can check that for $n = 1$ this is indeed $\dfrac{\theta}{2}$. We can also check the bias in the general case which is now given by the expression $-\dfrac{\theta}{n+1}$.

(f) Is the MLE consistent?

Given a positive ϵ, we can check that

$$P(|\theta - \hat{\theta_{ml}}| > \epsilon) = P(\hat{\theta_{ml}} < \theta - \epsilon) = F_M(\theta - \epsilon) \leq \frac{(\theta - \epsilon)^n}{\theta^n}$$

This tends to 0 as $n- > \infty$. Thus, **the MLE is consistent**.