W271-Lab1 Spring 2016

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Question 1

ML = 36
Stat = 28
Awesome = 18
ML ∩ Stat = 22
ML ∩ Awesome = 12
Stat ∩ Awesome = 9
ML ∪ Stat ∪ Awesome = 48

1)

$$ML \cup Stat = ML + Stat - ML \cap Stat = 42$$

$$Awesome.Other = (ML \cup Stat \cup Awesome) - (ML \cup Stat) = 6$$

$$Awesome = Awesome.Other \cup (Stat \cap Awesome) \cup (ML \cap Awesome) - (ML \cap Stat \cap Awesome)$$

$$18 = 6 + 9 + 12 - (ML \cap Stat \cap Awesome)$$

$$ML \cap Stat \cap Awesome = 9$$

$$\mathbf{Pr(ML \cap Stat \cap Awesome) = \frac{ML \cap Stat \cap Awesome}{ML \cup Stat \cup Awesome} = \frac{9}{48}}$$

2)

$$Pr(Awesome|ML) = \frac{Pr(Awesome \cap ML)}{Pr(ML)}$$

$$Pr(Awesome|ML) = \frac{12}{36} = \frac{1}{3}$$

$$\mathbf{Pr(!Awesome|ML) = 1 - Pr(Awesome|ML) = 1 - \frac{12}{36} = 1 - \frac{1}{3} = \frac{2}{3}}$$

3)

$$Pr(ML \cup Stat|Awesome) = \frac{Pr((ML \cup Stat) \cap Awesome)}{Pr(Awesome)}$$

$$Pr(ML \cup Stat|Awesome) = \frac{Pr(ML \cup Awesome) + Pr(Stat \cup Awesome) - Pr(Stat \cap ML \cap Awesome)}{Pr(Awesome)}$$

$$\mathbf{Pr(ML \cup Stat|Awesome) = \frac{12 + 9 - 9}{48} = \frac{12}{48} = \frac{1}{4}}$$

Question 2

Pr(A)=p $\leq \frac{1}{2}$, Pr(B)=q where $\frac{1}{4} < q < \frac{1}{2}$

1)
We know that:

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap \ B)$$

Pr(A ∪ B) is maximized when Pr(A ∩ B) is minimized (A and B are independent and Pr(A ∩ B) = 0). The maximum value is in the range:

$$\mathbf{\frac{1}{4} < Max(Pr(A \cup B)) < 1}$$

From there, we also know that Pr(A ∪ B) is minimized when Pr(A ∩ B) is maximized and A and B are completely overlapping random events and Pr(A ∩ B) = Min (Pr(A), Pr(B)). In these conditions

$$\mathbf{\frac{1}{4} < Min(Pr(A \cup B) < \frac{1}{2}}$$

2)
We know that:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Thus Pr(A|B) is minimized when Pr(A ∩ B) = 0. And

$$\mathbf{Min(Pr(A \cup B)) = 0}$$

Similarly, Pr(A|B) is maximized when the random event B is completely overlapped by the random event A. In such cases

$$\mathbf{Max(Pr(A|B)) = 1}$$

Question 3

1)
Given that the server's lifespan is a randon uniform distribution over the range [0,k], the probability of every additional year of operation is independent of the time elapsed and is equal to

$$\mathbf{Pr(1\ year\ of\ operation) = \frac{1}{k}}$$

2)
We know that:

$$E(g(x)) = \int_{x=0}^{k} g(x) f_x(x) dx$$

Considering g to be our refund function over time t:

$$E(g(t)) = \int_{t=0}^{1} \frac{\theta}{k} dt + \int_{t=1}^{k/2} \frac{A(k-t)^{\frac{1}{2}}}{k} dt + \int_{t=k/2}^{\frac{3k}{4}} \frac{\theta}{10k} dt$$

$$E(g(t)) = \frac{\theta}{k}[t]_0^1 + [\frac{-2A}{3k}(k-t)^{\frac{3}{2}}]_1^{k/2} + [\frac{\theta}{10k}t]_{\frac{k}{2}}^{\frac{3k}{4}}$$

$$\mathbf{E(g(t)) = \frac{\theta}{k} + \frac{\theta}{40} + \frac{2A}{3k}[(k-1)^{\frac{3}{2}} - \frac{k}{2}^{\frac{3}{2}}]}$$

3)
We know that Var(X) = E[X^2] -$E[X]^2$
Thus Var(g(x)) = E[(g(x)^2] - $[E[g(X)]]^2$
We previously computed E(g(x)). We now compute E[g(x)]^2

$$E[(g(x)^2] = \int_0^1 \frac{\theta^2}{k}dt + \int_1^{k/2} \frac{A^2(k-t)}{k}dt + \int_{\frac{k}{2}}^{\frac{3k}{2}} \frac{\theta^2}{100k}dt$$

$$\mathbf{E[(g(x)^2] = \frac{\theta^2}{k} - \frac{\theta^2}{400} - \frac{A^2(3k^2-8k+4)}{8k}}$$

We substract $E(g(x))^2$ as previously computed to obtain the variance, and

$$\mathbf{Var(g(x)) = \frac{\theta^2}{k} - \frac{\theta^2}{400} - \frac{A^2(3k^2-8k+4)}{8k} - [\frac{\theta}{k} + \frac{\theta}{40} + \frac{2A}{3k}[(k-1)^{\frac{3}{2}} - \frac{k}{2}^{\frac{3}{2}}]]^2}$$

Question 4

f(x,y) = 2e^{-x}e^{-2y} for 0 < x < ∞, 0 < y < ∞, 0 otherwise

1)

$$Pr(x > a, y < b) = \int_{y=0}^{b}\int_{x=a}^{\infty} 2e^{-x}e^{-2y}dxdy$$

$$Pr(x > a, y < b) = 2\int_{y=0}^{b} e^{-2y}dy \int_{x=a}^{\infty} e^{-x}dx$$

$$Pr(x > a, y < b) = 2\int_{y=0}^{b} e^{-2y}(1-[-e^{-x}]_0^a)dy$$

$$Pr(x > a, y < b) = 2e^{-a}\int_{y=0}^{b} e^{-2y}dy$$

$$Pr(x > a, y < b) = 2e^{-a}[-\frac{1}{2}e^{-2y}]_0^b$$

$$\mathbf{Pr(x > a, y < b) = e^{-a}(1-e^{-2b})}$$

2)

$$Pr(x < y) = \int_{y=0}^{\infty}\int_{x=0}^{y} 2e^{-x}e^{-2y}dxdy$$

$$Pr(x < y) = 2\int_{y=0}^{\infty} e^{-2y}dy \int_{x=0}^{y} e^{-x}dx$$

$$Pr(x < y) = 2\int_{y=0}^{\infty} e^{-2y}dy[-e^{-x}]_0^y$$

$$Pr(x < y) = 2\int_{y=0}^{\infty} e^{-2y}(1-e^{-y})dy$$

$$Pr(x < y) = 2\int_{y=0}^{\infty} e^{-2y} - e^{-3y}dy$$

$$Pr(x < y) = 2[\frac{1}{6}e^{-3y}(2-3e^{y})]_0^{\infty}$$

$$\mathbf{Pr(x < y) = \frac{1}{3}}$$

3)

$$Pr(X < a) = \int_{x=0}^{a} \int_{y=0}^{\infty} 2e^{-x}e^{-2y}dxdy$$
$$Pr(X < a) = 2\int_{x=0}^{a} e^{-x}dx \int_{y=0}^{\infty} e^{-2y}dy$$
$$Pr(X < a) = 2\int_{x=0}^{a} e^{-x}dx[-\frac{1}{2}e^{-2y}]_{0}^{\infty}$$
$$Pr(X < a) = \int_{x=0}^{a} e^{-x}dx$$
$$\mathbf{Pr(X < a) = 1 - e^{-a}}$$

Question 5

X random variable, x a real number.
Y = a + b (X - x^2)

1)

$$E(Y) = a + bE[(x - x^2)]$$
$$E(Y) = a + bE[X^2 - 2Xx - x^2]$$
$$E(Y) = a + b[E[X^2 - 2xE[X] + x^2]]$$

E(Y) is minimized when $\frac{d}{dx}$E(Y) = 0

$$\frac{d}{dx}E(Y) = -2bE(X) + 2bx$$
$$\frac{d}{dx}E(Y) = 0 \Rightarrow \mathbf{x = E(X)}$$

2)

$$When x = E(X) : E(Y) = a + b[E[X^2] - 2(E[X])^2 + (E[X])^2]$$
$$E(Y) = a + b[E[X^2] - (E[X])^2]$$
$$\mathbf{E(Y) = a + bVar[X]}$$

3)
Y = ax + b(X - x^2)

$$E(Y) = ax + bE[(x - x^2)]$$
$$E(Y) = ax + bE[X^2 - 2Xx - x^2]$$
$$E(Y) = ax + b[E[X^2 - 2xE[X] + x^2]]$$

E(Y) is minimized when $\frac{d}{dx}$E(Y) = 0

$$\frac{d}{dx}E(Y) = a - 2bE(X) + 2bx$$
$$\frac{d}{dx}E(Y) = 0 \Rightarrow \mathbf{x = E(X) - \frac{a}{2b}}$$

Question 6

X, Y independent continuous variables, uniform over [0..1]
Z = X + Y

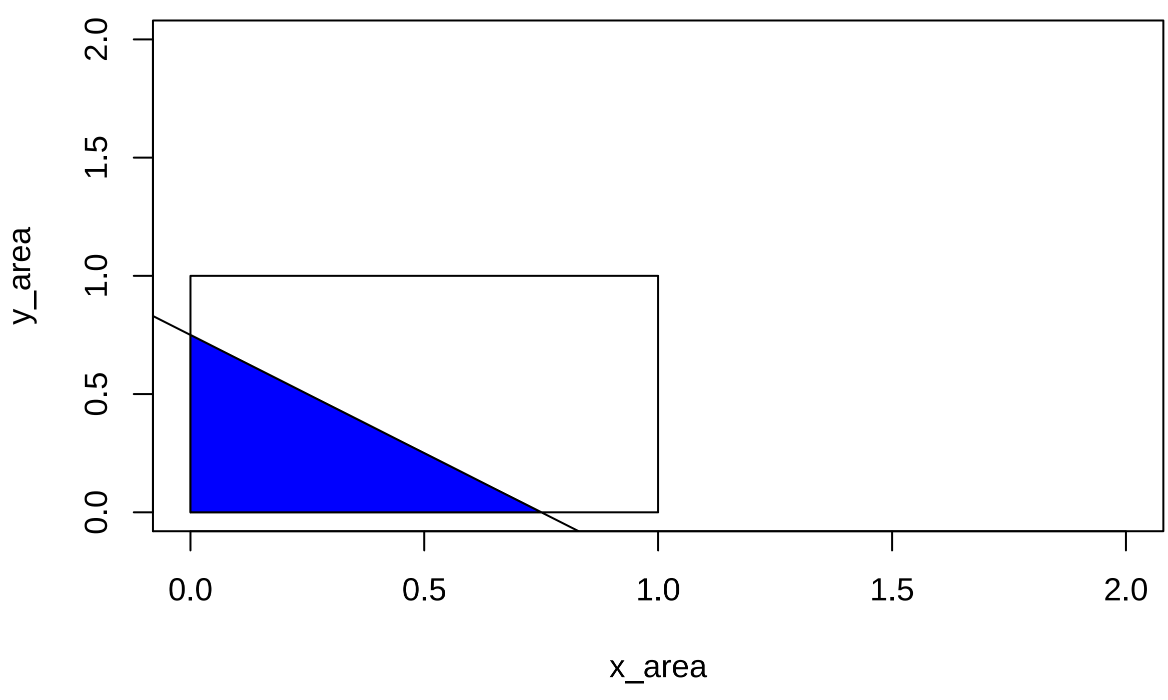
1)

```
x_area = c(0:2)
y_area = c(0:2)

plot(x_area, y_area, type = "n")

xx = c(0, 1, 1, 0)
yy = c(0, 0, 1, 1)
polygon(xx, yy, density = 0, border = "black")

abline(.75, -1)
xz = c (0, .75, 0)
yz = c(0, 0, .75)
polygon(xz, yz, col = "blue", border = "black")
```

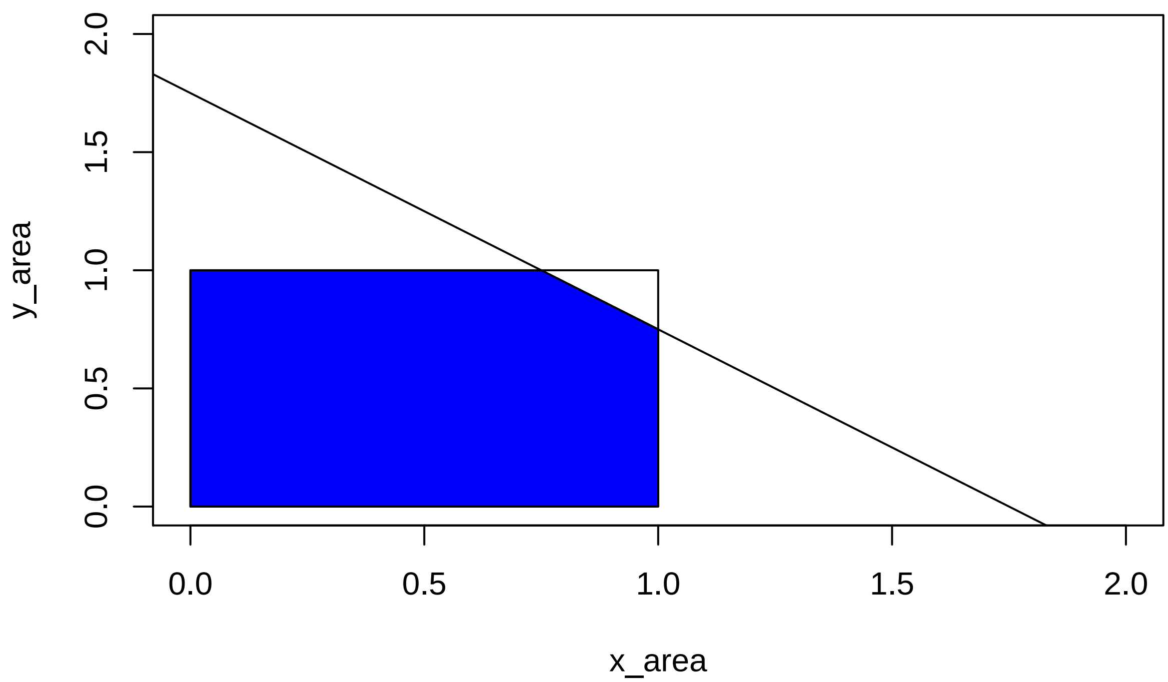



```
plot(x_area, y_area, type = "n")

xx = c(0, 1, 1, 0)
yy = c(0, 0, 1, 1)
```

```r
polygon(xx, yy, density = 0, border = "black")

abline(1.75, -1)
xz = c (0, 1, 1, .75, 0, 0)
yz = c(0, 0, .75, 1, 1, 0)
polygon(xz, yz, col = "blue", border = "black")
```



2)
From the areas above, we derive that:
For 0 <= z <= 1:

$$\mathbf{Pr(Z < z) = \frac{z^2}{2}}$$

For 1 < z <= 2:

$$\mathbf{Pr(Z < z) = 1 - \frac{(2-z)^2}{2}}$$

Hence:
For 0 <= z <= 1:

$$\mathbf{f(z) = \frac{d}{dz}\frac{z^2}{2} = z}$$

For 1 < z <= 2:

$$\mathbf{f(z) = \frac{d}{dz} 1 - \frac{(2-z)^2}{2} = 2 - z}$$

Question 7

1)

Event Class	Sum of Dices	Events in Class	Pr(Sum of Dices)
House wins	2	(1,1)	$\frac{1}{36}$
	3	(1,2)(2,1)	$\frac{1}{18}$
	12	(6,6)	$\frac{1}{36}$
You win	7	(3,4)(4,3)(5,2)(2,5)(1,6)(6,1)	$\frac{1}{6}$
	11	(5,6)(6,5)	$\frac{1}{18}$
X	4	(2,2)(3,1)(1,3)	$\frac{1}{12}$
	5	(2,3)(3,2)(1,4)(4,1)	$\frac{1}{9}$
	6	(3,3)(4,2)(2,4)(5,1)(1,5)	$\frac{5}{36}$
	8	(4,4)(6,2)(2,6)(3,5)(5,3)	$\frac{5}{36}$
	9	(3,6)(6,3)(5,4)(4,5)	$\frac{1}{9}$
	10	(5,5)(6,4)(4,6)	$\frac{1}{12}$

We can now define

$$\begin{gathered}
E(Y_{Playerwins}) = Pr(Playerwinsinone) * 1 + \\
\sum_{n=0}^{\infty}(Pr(4))^2(1-(Pr(4)+Pr(7)))^n(n+2) + \\
\sum_{n=0}^{\infty}(Pr(5))^2(1-(Pr(5)+Pr(7)))^n(n+2) + \\
\sum_{n=0}^{\infty}(Pr(6))^2(1-(Pr(6)+Pr(7)))^n(n+2) + \\
\sum_{n=0}^{\infty}(Pr(8))^2(1-(Pr(8)+Pr(7)))^n(n+2) + \\
\sum_{n=0}^{\infty}(Pr(9))^2(1-(Pr(9)+Pr(7)))^n(n+2) + \\
\sum_{n=0}^{\infty}(Pr(10))^2(1-(Pr(10)+Pr(7)))^n(n+2)
\end{gathered}$$

$$E(Y_{Housewins}) = Pr(Housewinsinone) * 1 + \sum_{n=0}^{\infty} Pr(X)Pr(7)(1-Pr(7))^n(n+2)$$

We know:

$$\begin{gathered}
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \\
\frac{1}{(1-x)^2} - 1 = 2x + 3x^2 + 4x^3 + \cdots \\
\frac{1}{x(1-x)^2} - \frac{1}{x} = 2 + 3x + 4x^2 + 5x^3 \cdots \\
\frac{1-(1-x)^2}{x(1-x)^2} = 2 + 3x + 4x^2 + 5x^3 \cdots
\end{gathered}$$

Thus:

$$E(Y_{PlayerWins}) = Pr(Playerwin)*1 + \Pr(4)^2 \frac{1-(Pr(4)+Pr(7))^2}{(Pr(4)+Pr(7))^2(1-(Pr(4)+Pr(7)))} +$$
$$\Pr(5)^2 \frac{1-(Pr(5)+Pr(7))^2}{(Pr(5)+Pr(7))^2(1-(Pr(5)+Pr(7)))} +$$
$$\Pr(6)^2 \frac{1-(Pr(6)+Pr(7))^2}{(Pr(6)+Pr(7))^2(1-(Pr(6)+Pr(7)))} +$$
$$\Pr(8)^2 \frac{1-(Pr(8)+Pr(7))^2}{(Pr(8)+Pr(7))^2(1-(Pr(8)+Pr(7)))} +$$
$$\Pr(9)^2 \frac{1-(Pr(9)+Pr(7))^2}{(Pr(9)+Pr(7))^2(1-(Pr(9)+Pr(7)))} +$$
$$\Pr(10)^2 \frac{1-(Pr(10)+Pr(7))^2}{(Pr(10)+Pr(7))^2(1-(Pr(10)+Pr(7)))} +$$

$$E(Y_{HouseWins}) = Pr(Housewin)*1 + \Pr(X)Pr(7)\frac{1-(Pr(7)+Pr(X))^2}{(1-(Pr(7)+Pr(X)))(Pr(7)+Pr(X))^2}$$

```r
pr.house.wins.in.one <- 4/36
pr.player.wins.in.one <- 8/36
pr.seven <- 6/36
pr.x.events <- c(3/36, 4/36, 5/36/ 5/36, 4/36, 3/36)
pr.x.plus.seven.events <- pr.x.events + pr.seven
exp.y.player.wins <- pr.player.wins.in.one +
    sum((pr.x.events)^2 * (1 - pr.x.plus.seven.events^2)/((1- pr.x.plus.seven.events)*pr.x.plus.seven.e

exp.y.house.wins <- pr.house.wins.in.one +
    pr.seven*sum(pr.x.events)*(1 - (pr.seven + sum(pr.x.events))^2)/
    ((1 - (pr.seven + sum(pr.x.events)))*(pr.seven + sum(pr.x.events))^2)

print(sprintf("E(Y_Player_Wins) = %f", exp.y.player.wins))
```

```
## [1] "E(Y_Player_Wins) = 0.908914"
```

```r
print(sprintf("E(Y_House_Wins) = %f", exp.y.house.wins))
```

```
## [1] "E(Y_House_Wins) = 0.437681"
```

2)

$$E(Payoff) = 100*Pr(Y=1) + 80*Pr(Y=2) + 60*Pr(Y=3) + 40*Pr(Y=4) + 0*Pr(Y=5)$$
$$Pr(Y=1) = Pr(Playerwinsinone)$$

Question 8

$\mathrm{E}(\mathrm{Y}_1) = \mathrm{E}(\mathrm{Y}_2) = \ldots = \mathrm{E}(\mathrm{Y}_n) = \mu$
$\mathrm{Var}(\mathrm{Y}_1) = \mathrm{Var}(\mathrm{Y}_2) = \ldots = \mathrm{Var}(\mathrm{Y}_n) = \sigma^2$

1)
For W to be un unbiased estimator of μ:

$$E(W) = \mu$$

$$\Rightarrow E(\sum_{i=1}^{n} a_i Y_i) = \mu$$

$$\Rightarrow \sum_{i=1}^{n} a_i E(Y_i) = \mu$$

$$\Rightarrow \sum_{i=1}^{n} a_i \mu = \mu$$

$$\Rightarrow \mu \sum_{i=1}^{n} a_i = \mu$$

$$\Rightarrow \boldsymbol{\sum_{i=1}^{n} a_i = 1}$$

2)

$$Var(W) = Var(\sum_{i=1}^{n} a_i Y_i)$$

$$Var(W) = \sum_{i=1}^{n} a_i^2 Var(Y_i)$$

$$\mathbf{Var(W)} = \sigma^{\mathbf{2}} \boldsymbol{\sum_{i=1}^{n} a_i^2}$$

3)
We know that:

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n} a_i^2$$

Knowing also that:

$$Var(W) = \sigma^2 \sum_{i=1}^{n} a_i^2$$

We conclude:

$$Var(W) \geq \sigma^2 \frac{1}{n}(\sum_{i=1}^{n} a_i)^2$$

When W is unbiased, we know that:

$$\sum_{i=1}^{n} a_i = 1$$

Thus:

$$Var(W) \geq \frac{\sigma^2}{n}$$

and

$$\mathbf{Var(W)} \geq \mathbf{Var(\bar{Y})}$$

Question 9

W$_1$ = $(\frac{n-1}{n})\bar{Y}$
W$_2$ = k$\bar{Y}$

1)

$$bias(W_1) = E((\frac{n-1}{n})\bar{Y})$$

$$bias(W_1) = \frac{n-1}{n}E(\bar{Y}) - \mu$$

$$bias(W_1) = \frac{n-1}{n}\mu - \mu$$

$$\mathbf{bias(W_1)} = \frac{-\mu}{\mathbf{n}}$$

Similarly:

$$bias(W2) = E(k\bar{Y}) - \mu$$

$$bias(W2) = kE(\bar{Y}) - \mu$$

$$bias(W2) = k\mu - \mu$$

$$\mathbf{bias(W2)} = \mu(\mathbf{k-1})$$

W1 is a consistent estimator of μ as bias(W2) tends towards zero when n tends towards infinity.
W2 is not a consistent estimator of μ.

2)

$$Var(W_1) = Var((\frac{n-1}{n})\bar{Y})$$

$$Var(W_1) = \frac{(n-1)^2}{n^2}Var(\bar{Y})$$

$$Var(W_1) = \frac{(n-1)^2}{n^2}\frac{\sigma^2}{n}$$

$$\mathbf{Var(W_1)} = \frac{\mathbf{(n-1)^2}\sigma^{\mathbf{2}}}{\mathbf{n^3}}$$

Similarly:

$$Var(W_2) = Var(k\bar{Y})$$

$$Var(W_2) = k^2Var(\bar{Y})$$

$$Var(W_2) = k^2\frac{\sigma^2}{n}$$

$$\mathbf{Var(W_2)} = \mathbf{k^2}\frac{\sigma^{\mathbf{2}}}{\mathbf{n}}$$

Question 10

$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$

1)

$$E(\bar{Y}) = E[\frac{\sum_{i=1}^{n} Y_i}{n}]$$

We know that E(X) is a linear function, thus:

$$E(\bar{Y}) = \frac{1}{n}\sum_{i=1}^{n} E[Y_i]$$

$$E(\bar{Y}) = \frac{1}{n}(n\mu)$$

$$\Rightarrow \mathbf{E(\bar{Y})} = \mu = \mathbf{E[Y_i], y = 1, \cdots, n}$$

2)

$$Var(\bar{Y}) = Var[\frac{\sum_{i=1}^{n} Y_i}{n}]]$$

We know that $\mathrm{Var}[\sum_{i=1}^{n} \mathrm{a}_i X_i] = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$, thus:

$$Var(\bar{Y}) = \frac{1}{n^2}\sum_{i=1}^{n}(Var[Y_i])$$

$$\Rightarrow \mathbf{Var(\bar{Y})} = \mathbf{\frac{1}{n^2} n}\boldsymbol{\sigma}^{\mathbf{2}} = \mathbf{\frac{1}{n} Var[Y_i], y = 1, \cdots, n}$$

3)

$$E[\widehat{\sigma^2}] = E[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2]$$

$$E[\widehat{\sigma^2}] = \frac{1}{n}\sum_{i=1}^{n} E[(Y_i - \bar{Y})^2]$$

$$E[\widehat{\sigma^2}] = \frac{1}{n}\sum_{i=1}^{n} E(Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2)$$

Using the linearity of the expectation function and the indepedence of identically distributed variables:

$$E[\widehat{\sigma^2}] = \frac{1}{n}\sum_{i=1}^{n} E(Y_i^2) - 2E(Y_i)E(\bar{Y}) + \bar{Y}^2)$$

$$E[\widehat{\sigma^2}] = \frac{1}{n}\sum_{i=1}^{n} E(Y_i^2) - E(Y_i)^2 + E(\bar{Y}^2) - E(\bar{Y})^2$$

$$E[\widehat{\sigma^2}] = \frac{1}{n}\sum_{i=1}^{n} Var(Y_i) + Var(\bar{Y})$$

$$E[\widehat{\sigma^2}] = \frac{1}{n}(n\sigma^2 + n\frac{\sigma^2}{n})$$

$$\mathbf{E}[\widehat{\sigma^2}] = \frac{\mathbf{n+1}}{\mathbf{n}}\sigma^{\mathbf{2}}$$

4)
We have shown that E($\widehat{\sigma^2}$) != σ^2 and we conclude that $\widehat{\sigma^2}$ is a biased estimator for σ^2.

5)
We know that:

$$\widehat{\sigma^2} = \frac{n+1}{n}\sigma^2$$

is a biased estimator. Thus:

$$\frac{n}{n+1}\widehat{\sigma^2}$$

is an unbiased estimator of σ^2

Question 11

X, Y positive random variables. E(Y|X) = θ X

i)
We know that Z = $\frac{Y}{X}$
We first compute E(Z|X)

$$E(Z|X) = E(\frac{Y}{X}|X)$$

Using E(a(X) Y + b(X))|X) = a(X)E(Y|X) + bX, we derive:

$$E(Z|X) = \frac{1}{X}E(Y|X) = \frac{\theta X}{X} = \theta$$

Then, we knwo that E[E(Z|X)] = E(Z). Thus:

$$E(\theta) = E(Z)$$

Θ being a constant:

$$E(\theta) = \theta$$

and

$$\mathbf{E}(\mathbf{Z}) = \theta$$

ii)
$W_1 = n^{-1}\sum_{i=1}^{n}\frac{Y_i}{X_i}(X_i, Y_i) : i = 1, 2, \cdots, n$ is the estimator.
We compute $E(W_1)$:

$$E(W_1) = n^{-1}E[\sum_{i=1}^{n}(\frac{Y_i}{X_i})]$$

$$E(W_1) = n^{-1}\sum_{i=1}^{n}E(Y_i/X_i)$$

$$\mathbf{E}(\mathbf{W_1}) = \mathbf{n^{-1}}[\mathbf{nE}(\mathbf{Z})] = \mathbf{E}(\mathbf{Z}) = \theta$$

We conclude that W_1 is unbiased for θ

iii)
$W_2 = \frac{\bar{Y}}{\bar{X}}$

$$W_2 = \frac{n^{-1}\sum_{i=1}^{n}Y_i}{n^{-1}\sum_{i=1}^{n}X_i}$$

$$\Rightarrow W_2 = \frac{Y_1 + Y_2 + Y_3 + \cdots + Y_n}{X_1 + X_2 + X_3 + \cdots + X_n}$$

whereas:

$$W_1 = \frac{Y_1}{X_1} + \frac{Y_2}{X_2} + \frac{Y_3}{X_3} + \cdots \frac{Y_n}{X_n}$$

$W_2 and W_1$ are thus different estimators

$$E(W_2) = E[\frac{\bar{Y}}{\bar{X}}]$$

We know that

$$E(Y) = E(\bar{Y}) = E[E(Y|X)] = E[\theta X] = \theta E(X)$$

$$\Rightarrow E(W_2) = E[\frac{\theta E(X)}{E(X)}] = E(\theta)$$

$$\Rightarrow \mathbf{E}(\mathbf{W_2}) = \theta$$

Question 12

i)
The null hypothesis is that $\mu = 0$
ii)
The alternative hypothesis hypothesis is that $\mu < 0$
iii)

$$n = 900$$

$$\bar{Y} = -32.8$$

$$s = 466.4$$

$$t = \frac{\bar{Y}}{\frac{s}{\sqrt{n}}} = -2.150538$$

$$p(z \leq t) = 0.0158$$

Thus, we reject the null hypothesis at the 5% significance level as $p(z \leq t) \leq 0.05$ We cannot reject the null hypothesis at the 1% significance level as $p(z \leq t) \geq 0.01$
iv)
The effect size is inferior to 10% of the variance of the State Liquor Consumption variable. That's an indication of a small practical effect. We also compute the correlation coefficient as

$$r = \sqrt{\frac{t^2}{t^2 + DF}} = 0.07$$

The value of R also confirms the small practical effect despite the test being statistically significant because of the high sample size.

Question 13

$Y_i = 1$ Shot made. $Y_i = 0$ Shot missed. $\theta = Pr(Makinga3ptshot)$ Bernouilli distribution. $\bar{Y} = \frac{FGM}{FGA}$ estimator of θ

i)

$$\theta = \frac{188}{429} = .4382284$$

ii)
Because Y has a Bernouilli distribution: $E(Y) = \theta$ Let $Y_i i \in 1, \cdots, n$ be an occurance of a free throw. We know that each Y_i is a Bernouilli variable. We can define:

$$\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

Thus:

$$Var(\bar{Y}) = (\frac{1}{n})^2 Var(\sum_{i=1}^{n} Y_i)$$

$$Var(\bar{Y}) = (\frac{1}{n})^2 \sum_{i=1}^{n} Var(Y_i)$$

Thus:

$$Var(\bar{Y}) = (\frac{1}{n})^2 n\theta(1-\theta)$$

$$\Rightarrow Var(\bar{Y}) = \frac{\theta(1-\theta)}{n}$$

And:

$$sd(Y) = \sqrt{Var(\bar{Y})}$$

$$\Rightarrow sd(Y) = \sqrt{\frac{\theta(1-\theta)}{n}}$$

iii)
We know $se(\bar{\gamma}) = \sqrt{\frac{\bar{\gamma}(1-\bar{\gamma})}{n}}$ And $\frac{\bar{\gamma}-\theta}{se(\bar{Y})} \equiv Normal(0,1)$ We compute:

$$z_Y = \frac{\frac{188}{429} - .5}{se(\bar{Y})} = -2.588303$$

$$\Rightarrow p(z) = 0.0048$$

The p-value is significant at the 1% significance level and we reject the null hypothesis. iv)
Type I error is the probability of a false positive or the probability that the null hypothesis was rejected but it should not have been. It's the probability of saying there is a result when there is not one. v)
The probablity of the type error is the significance level and is 1%. vi)
Type II error is the probability of a false negative, or the probablity of failing to reject the null hypothesis when it should be rejected. vii)
viii)
The power of the test is the probabilty of rejecting the null hypothesis when its is actually false. ix)