Kripke and Meta-Logical Completeness via Curry-Howard Isomorphism (draft)

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Abstract

This paper provides an alternative proof of Kripke and meta-logical completeness of intuitionistic modal logic. Although the notion of Kripke and meta-logical completeness is located in mathematical logic, this study reduces it to fullness of the standard translation in the theory of λ-calculus via Curry-Howard isomorphism. Our proof saves cost for manufacturing a proof by willingly using some well-known results in both mathematical logic and the theory of λ-calculus. In addition, our proof of meta-logical completeness is purely syntactical. Since meta-logical completeness refers to syntax of intuitionistic modal logic and intuitionistic predicate logic, our proof is considered to be more natural and sophisticated than any other proof touring semantics of them.

Keywords. Modal logic, λ-calculus, Curry-Howard isomorphism, the standard translation, Kripke completeness, meta-logical completeness, fullness.

1 Introduction

In mathematical logic, there exists a trend to obtain deeper understanding of modalities by replacing classical logic as basis by intuitionistic logic, e.g., [14]. On the other hand, in computer science simply typed λ-calculus (considered to be a common core of functional programming languages) has been studied well. Although intuitionistic logic is seemingly-independent from simply typed λ-calculus, it is indeed well-known that these have close connection called *Curry-Howard isomorphism* [9].

In this paper, we contribute to studies on modalities by using accumulated knowledge in the theory of typed λ-calculus. To be concrete, we prove *Kripke and meta-logical completeness* of intuitionistic modal logic by *fullness of the standard translation*. While the notion of Kripke and meta-logical completeness is located in mathematical logic, that of fullness is located in the theory of λ-calculus. That is, this work means that we reduce Kripke and meta-logical completeness in mathematical logic to fullness of the standard translation in the theory of λ-calculus via Curry-Howard isomorphism.

One may think that Kripke completeness can be directly proven by a standard method using *maximal consistent sets* (see Section 2). It is true, but this study provides an alternative solution. We, mathematical logicians, have already known a milestone theorem by Kripke that intuitionistic predicate logic is complete to possible world semantics (see e.g., [13]). In addition, Kripke completeness refers to a relation between intuitionistic modal logic and possible world semantics (formally, see Theorems 2.4 and 2.6). Therefore, it is natural and costless that we try to obtain Kripke completeness by referring to a relation between intuitionistic modal logic and intuitionistic predicate logic. In this paper, we adopt *the standard translation* [6] as such a relation, and indirectly show Kripke completeness by proving *fullness* of the standard translation.

Meta-logical completeness refers to a relation of provability between intuitionistic modal logic and intuitionistic predicate logic (formally, see Theorem 4.3). That is, meta-logical completeness is a purely syntactical statement. However, for example, Simpson proves meta-logical completeness via considering possible world semantics [12]. In contrast, this paper provides a purely syntactical proof method for meta-logical completeness. In this sense, our proof is considered to be more natural and sophisticated.

Our proof also raises an answer about interpretation of the modality of necessity $\Box$ in possible world semantics for intuitionistic modal logic. In possible world semantics for *classical* modal logic, $w \vDash \Box A$ is usually interpreted as $w' \vDash A$ for any $w' \in W$ such that $w\, R\, w'$ where R is a reachability relation for the modality $\Box$. In contrast, we do not have any standard interpretation of the modality of necessity $\Box$ in *intuitionistic* modal logic because intuitionistic modal logic has another reachability relation $\leq$ of implication $\supset$ in possible world semantics. In possible world semantics with mixture of $\leq$ and R, how should the reachability relations $\leq$ and R are affected with each other? This is not obvious at all. Wijesekera gave an interpretation of the modality $\Box$ and showed Kripke completeness of intuitionistic modal predicate logic. Alechina et al. also studied this about Constructive S4, stronger in provability, and gave an answer in [1]. In this paper, we give an interpretation of the modality $\Box$ in intuitionistic modal logic IK, weaker in provability. The interpretation is automatically derived from our proof that willingly reuses completeness of intuitionistic predicate logic, and coincides with Wijesekera and Alechina et al's interpretation. Therefore, our work gives validity to Wijesekera and Alechina et al's interpretation of the modality $\Box$ by *existence of a natural proof* of Kripke completeness using completeness of intuitionistic predicate logic.

Finally, this study contributes to implementation of modal logic in dependent type programming languages (e.g., Agda[1], Coq[2], and Epigram[3]) because the standard translation is regarded as an implementation of intuitionistic modal logic in intuitionistic first-order predicate logic. Fullness of the standard translation can be considered to mean that it provides no junk when modal logic is implemented on such programming languages.

2 Intuitionistic Modal Logic

We define a set of modal formulas in the following grammar:

$$A, B ::= p \mid A \supset A \mid \Box A$$

where p ranges over the set of propositional variables *Var*.

[1] `http://wiki.portal.chalmers.se/agda/`
[2] `http://coq.inria.fr/`
[3] `http://www.e-pig.org/`

A triple $(W, \leq, R)$ is said to be a *frame* if W is a non-empty set, $\leq$ is a partial order on W, and R is a binary relation on W. Elements of W are said to be *worlds*. We write $w \leq w'$ and $w\,R\,w'$ as $(w, w') \in \leq$ and $(w, w') \in R$, respectively. When $\mathfrak{P}(W)$ denotes the set of subsets of M, a function $V\colon \mathit{Var} \to \mathfrak{P}(W)$ is said to be a *valuation* if $w \in V(p)$ and $w \leq w'$ imply $w' \in V(p)$ for any $w, w' \in W$ and $p \in \mathit{Var}$. Such quadruple $(W, \leq, R, V)$ is said to be a *model*. We define that a formula A is *satisfied* at a world $w \in W$ by a model $(W, \leq, R, V)$ (written as $W, \leq, R, V, w \vDash A$) if

$$
\begin{array}{lcl}
W, \leq, R, V, w \vDash p & \iff & w \in V(p) \ , \\
W, \leq, R, V, w \vDash A \supset B & \iff & \text{for any } w' \in W \text{ such that } w \leq w' \\
 & & W, \leq, R, V, w' \vDash A \text{ implies } W, \leq, R, V, w' \vDash B \ , \\
W, \leq, R, V, w \vDash \Box A & \iff & \text{for any } w', w'' \in W \text{ such that } w \leq w' \text{ and } w'\,R\,w'' \\
 & & W, \leq, R, V, w'' \vDash A \ .
\end{array}
$$

Furthermore, we often write $w \vDash A$ when $(W, \leq, R, V)$ is obvious from the context. A is called *true* in a model if A is satisfied at any $w \in W$ in the model.

Proposition 2.1 (Monotonicity)**.** *For any formula A, $w \vDash A$ and $w \leq w'$ imply $w' \vDash A$.*

Proof. By induction on A. □

When a set $\varGamma$ of formulas is given, we define $w \vDash \varGamma$ as $w \vDash A$ for any $A \in \varGamma$. When $n \geq 1$ is given, $\varGamma_1 \mid \cdots \mid \varGamma_n \vdash A$ is said to be a *judgment* (the left side of $\vdash$ called a *context*). We define $w_1 \vDash \varGamma_1 \mid \cdots \mid \varGamma_n \vdash A$ as $w_i \leq w'_i$ $(1 \leq i \leq n)$, $w'_{i-1}\,R\,w_i$ $(1 < i \leq n)$, and $w_i \vDash \varGamma_i$ $(1 \leq i \leq n)$ imply $w_n \vDash A$. We call that $\varGamma_1 \mid \cdots \mid \varGamma_n \vdash A$ is called *true* in a model if $w \vDash \varGamma_1 \mid \cdots \mid \varGamma_n \vdash A$ for any $w \in W$ in the model.

Proposition 2.2. *For any $n \geq 1$, A is true in every model if and only if $\overbrace{\emptyset \mid \cdots \mid \emptyset}^{n} \vdash A$ is true in every model. In particular, A is true in every model if and only if $\emptyset \vdash A$ is true in every model.*

Proof. The only-if part is obvious. We show the if-part. Assume $W, \leq, R, V, w_n \nvDash A$. Then, $W \cup \{\, w_i \mid 1 \leq i \leq n-1 \,\}, \leq, R \cup \{\, (w_i, w_{i+1}) \mid 1 \leq i \leq n-1 \,\}, V, w_n \nvDash A$ holds where $w_1, \ldots, w_{n-1}$ is fresh. It means

$$W \cup \{\, w_i \mid 1 \leq i \leq n-1 \,\}, \leq, R \cup \{\, (w_i, w_{i+1}) \mid 1 \leq i \leq n-1 \,\}, V, w_1 \nvDash \underbrace{\emptyset \mid \cdots \mid \emptyset}_{n} \vdash A \qquad \Box$$

Intuitionistic modal logic can be naïvely conjectured to be sound and complete to the class of frames with reachability relations of intuitionistic implication and modality of necessity. Indeed, Wijesekera defined intuitionistic modal logic like that in sequent calculus style and showed Kripke completeness [14]. In this paper, we adopt the so-called Fitch-style natural deduction (see a detailed comparison to Gentzen-style natural deduction by Bellin et al. [4]), in particular Martini and Masini's natural deduction (based on Prawitz's idea [11] for defining modal logic) since natural deduction is suitable for being translated into a λ-calculus as described in Section 3. Although we change Martini and Masini's notation slightly, there exists no essential difference.

Intuitionistic modal logic IK is as follows,

$$\frac{}{\Gamma_1 \mid \cdots \mid \Gamma_n, A \vdash A}\ (\text{axiom})$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n, A \vdash B}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash A \supset B}\ (\supset \text{I})$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash A \supset B \qquad \Gamma_1 \mid \cdots \mid \Gamma_n \vdash A}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash B}\ (\supset \text{E})$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \mid \emptyset \vdash A}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash \Box A}\ (\Box\, \text{I})$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash \Box A}{\Gamma_1 \mid \cdots \mid \Gamma_n \mid \Gamma_{n+1} \vdash A}\ (\Box\, \text{E}) \quad .$$

Proposition 2.3 (Weakening)**.** *Let $\Gamma_0, \Gamma_i, \Delta_i$ $(1 \le i \le n)$ be sets of formulas such that $\Gamma_1 \mid \cdots \mid \Gamma_i \mid \cdots \mid \Gamma_n \vdash A$ is derivable. Then,*

1. *$\Gamma_1, \Delta_1 \mid \cdots \mid \Gamma_i, \Delta_i \mid \cdots \mid \Gamma_n, \Delta_n \vdash A$ is derivable, and*
2. *$\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_i \mid \cdots \mid \Gamma_n \vdash A$ is derivable.*

Theorem 2.4 (Soundness)**.** *If $\Gamma_1 \mid \cdots \mid \Gamma_n \vdash A$ is derivable, then $\Gamma_1 \mid \cdots \mid \Gamma_n \vdash A$ is true in every model.*

Proof. By induction on derivation. We only show ($\Box$ I) and ($\Box$ E)-cases. Assume $w_i \le w_i'$ $(1 \le i \le n)$ and $w_i' \, R \, w_{i+1}$ $(1 \le i \le n)$ such that $w_i \vDash \Gamma_i$ $(1 \le i \le n)$. Then, $w_{n+1} \vDash A$ holds by induction hypothesis. It means $w_n \vDash \Box A$.

Next, assume $w_i \le w_i'$ $(1 \le i \le n)$ and $w_i' \, R \, w_{i+1}$ $(1 \le i \le n)$ such that $w_i \vDash \Gamma_i$ $(1 \le i \le n+1)$. By induction hypothesis, $w_n \vDash \Box A$ holds. That is, $w_{n+1} \vDash A$ holds. □

Martini and Masini refer to Kripke completeness without any proof for it in [10]. Here, we give a proof for it in a standard manner.

Let Γ_1, Γ_2, and Δ_2 be sets of formulas. A pair (Γ_2, Δ_2) is *Γ_1-consistent* if $\Pi_1 \mid \Pi_2 \vdash A$ is not derivable for any $\Pi_1 \subseteq \Gamma_1$, $\Pi_2 \subseteq \Gamma_2$, and any $A \in \Delta_2$. A pair (Γ_2, Δ_2) is *maximally Γ_1-consistent* if it is Γ_1-consistent and any formula A belongs to Γ_2 or Δ_2.

Lemma 2.5. *If (Γ_2, Δ_2) is Γ_1-consistent, then there exists a maximal Γ_1-consistent $({\Gamma_2}^*, {\Delta_2}^*)$ such that $\Gamma_2 \subseteq {\Gamma_2}^*$ and $\Delta_2 \subseteq {\Delta_2}^*$.*

Proof. Let $B_1, \ldots$ be an enumeration of all formulas. We define a sequence of pairs (Γ_2^m, Δ_2^m) $(m \ge 1)$ as follows,

$$(\Gamma_2^1, \Delta_2^1) = (\Gamma_2, \Delta_2)$$

$$(\Gamma_2^{m+1}, \Delta_2^{m+1}) = \begin{cases} (\Gamma_2^m, \Delta_2^m \cup \{B_m\}) & \text{if } (\Gamma_2^m, \Delta_2^m \cup \{B_m\}) \text{ is } \Gamma_1\text{-consistent} \\ (\Gamma_2^m \cup \{B_m\}, \Delta_2^m) & \text{otherwise} \end{cases}$$

If (Γ_2^m, Δ_2^m) is Γ_1-consistent, so is $(\Gamma_2^{m+1}, \Delta_2^{m+1})$. Otherwise, there exist $\Pi_1 \subseteq \Gamma_1$, $\Pi_2 \subseteq \Gamma_2^m$, and $A \in \Delta_2^m$ such that $\Pi_1 \mid \Pi_2 \vdash B_m$ and $\Pi_1 \mid \Pi_2, B_m \vdash A$ are derivable. Then, $\Pi_1 \mid \Pi_2 \vdash A$ is derivable by ($\supset$ I) and ($\supset$ E)-rules. This contradicts Γ_1-consistency of (Γ_2^m, Δ_2^m). Thus, (Γ_2^m, Δ_2^m) is Γ_1-consistent for any $m \ge 1$, and we obtain a maximal Γ_1-consistent pair $(\bigcup\{\, \Gamma_2^m \mid m \ge 1 \,\}, \bigcup\{\, \Delta_2^m \mid m \ge 1 \,\})$. □

Theorem 2.6 (Completeness)**.** *If A is true in every model, then $\emptyset \vdash A$ is derivable.*

Proof. Assume $\emptyset \vdash A$ is not derivable. By Lemma 2.5, there exists at least a maximal $\emptyset$-consistent (Γ, Δ) such that $A \in \Delta$. We define a model $(W, \leq, R, V)$ where

- W is the set of maximal $\emptyset$-consistent pairs,
- $\leq = \{ ((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)) \in W \times W \mid \Gamma_1 \subseteq \Gamma_2 \}$,
- $R = \{ ((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)) \in W \times W \mid (\Gamma_2, \Delta_2) \text{ is } \Gamma_1\text{-consistent} \}$,
- $V(p) = \{ (\Gamma_1, \Delta_1) \in W \mid p \in \Gamma_1 \}$.

Note that $(W, \leq, R, V)$ is surely a model since $\leq$ is a partial order and V is a valuation.

We show that $B \in \Gamma_1$ if and only if $(\Gamma_1, \Delta_1) \models B$ for any B, Γ_1, and Δ_1. By induction on B.

The case that B is a propositional variable is obvious. Next, let us the case that B is $C \supset D$. Assume $C \supset D \in \Gamma_1$, $(\Gamma_1, \Delta_1) \leq (\Gamma_2, \Delta_2)$ (i.e., $\Gamma_1 \subseteq \Gamma_2$), and $(\Gamma_2, \Delta_2) \models C$. By induction hypothesis, $C \in \Gamma_2$ holds. Then, $D \in \Gamma_2$ holds by $\Gamma_1 \subseteq \Gamma_2$. By induction hypothesis, $(\Gamma_2, \Delta_2) \models D$ holds. This means $(\Gamma_1, \Delta_1) \models C \supset D$. Conversely, suppose $C \supset D \notin \Gamma_1$, i.e., $C \supset D \in \Delta_1$ by the maximality of (Γ_1, Δ_1). Then, $(\{C\}, \{D\})$ is $\emptyset$-consistent. Otherwise, $\emptyset \mid C \vdash D$ is derivable, and so is $\emptyset \mid \emptyset \vdash C \supset D$. This contradicts the $\emptyset$-consistency of (Γ_1, Δ_1). By Lemma 2.5, there exists a maximal $\emptyset$-consistent (Γ_2, Δ_2) such that $\Gamma_1 \subseteq \Gamma_2$, $C \in \Gamma_2$, and $D \in \Delta_2$. By induction hypothesis, $(\Gamma_2, \Delta_2) \models C$ and $(\Gamma_2, \Delta_2) \not\models D$ hold. By $(\Gamma_1, \Delta_1) \leq (\Gamma_2, \Delta_2)$, this means $(\Gamma_1, \Delta_1) \not\models C \supset D$.

Finally, we show the case that B is $\Box C$. Assume $\Box C \in \Gamma_1$, $(\Gamma_1, \Delta_1)\ R\ (\Gamma_2, \Delta_2)$. Then, $C \in \Gamma_2$ holds. Otherwise, $C \in \Delta_2$ holds by the maximality of (Γ_2, Δ_2), and it contradicts the Γ_1-consistency of (Γ_2, Δ_2) since $\Box C \mid \emptyset \vdash C$ is derivable by (axiom) and ($\Box$ E)-rules. By induction hypothesis, $(\Gamma_2, \Delta_2) \models C$ holds. This means $(\Gamma_1, \Delta_1) \models \Box C$. Conversely, suppose $\Box C \notin \Gamma_1$, i.e., $\Box C \in \Delta_1$ by the maximality of (Γ_1, Δ_1). Then, $(\emptyset, \{C\})$ is Γ_1-consistent. Otherwise, $\Pi \mid \emptyset \vdash C$ is derivable for some set $\Pi \subseteq \Gamma_1$, and so is $\emptyset \mid \Pi \vdash \Box C$ by ($\Box$ I)-rule and Proposition 2.3.2. This contradicts the $\emptyset$-consistency of (Γ_1, Δ_1). By Lemma 2.5, there exists a maximal Γ_1-consistent (Γ_2, Δ_2) such that $C \in \Delta_2$. By induction hypothesis, $(\Gamma_2, \Delta_2) \not\models C$ holds. By $(\Gamma_1, \Delta_1)\, R\, (\Gamma_2, \Delta_2)$, this means $(\Gamma_1, \Delta_1) \not\models \Box C$.

Now $A \in \Delta$, that is, $A \notin \Gamma$ holds by the maximality of (Γ, Δ). Therefore, we obtain $(\Gamma, \Delta) \not\models A$. $\Box$

3 The λK and λP-Calculi

We introduce another method for proving Kripke completeness using fullness of translations in the theory of λ-calculus.

First, we translate IK into a λ-calculus in order to deal with IK in the theory of λ-calculus. Via an extended Curry-Howard isomorphism, the following λ-calculus (called λK in [10]) corresponds to intuitionistic modal logic IK.

$$\frac{}{\Gamma_1 \mid \cdots \mid \Gamma_n, x\colon A \vdash x\colon A}$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n, x\colon A \vdash M\colon B}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash \lambda x.M\colon A \supset B}$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash M\colon A \supset B \qquad \Gamma_1 \mid \cdots \mid \Gamma_n \vdash N\colon A}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash MN\colon B}$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \mid \emptyset \vdash M\colon A}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash \mathsf{gen}(M)\colon \Box A}$$

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash M\colon \Box A}{\Gamma_1 \mid \cdots \mid \Gamma_n \mid \Gamma_{n+1} \vdash \mathsf{ungen}(M)\colon A} \quad .$$

The equational relation = of λK is the smallest congruence relation containing

$$\begin{aligned} (\lambda x.M)N &= [N/x]M \\ \mathsf{ungen}(\mathsf{gen}(M)) &= M \quad . \end{aligned}$$

We use various notions of ordinary λ-calculi, e.g., binding, free variable, bound variable, α-conversion, and substitution. The notation is also similar to that in ordinary λ-calculi. In detail, see Barendregt's encyclopedic book [2]. In the following, α-convertible λ-terms are identified syntactically.

Next, we recall Barendregt's λP (equivalent to Harper et al.'s LF [8]) corresponding to an intuitionistic first-order predicate logic (called IP). In this paper, λP's signature is

$$\{\, P\colon W \supset * \mid P \text{ is a unary predicate symbol} \,\} \cup \{R\colon W \supset W \supset *\}$$

where $A \supset B$ is an abbreviation of $\Pi x^A.B$ ($x \notin \mathrm{fv}\, B$). Variables are indexed by integers. We write x_i^A as the i-th variable of the type A. Indices and types often are often omitted when they are obvious from the contexts. We use a, b as worlds and u, v as variables of the type Rab, for readability. The kinds, types, and terms of λP-calculus is as follows,

$$\begin{aligned} K, L &::= * \mid \Pi x^A.K \\ A, B, C &::= x \mid W \mid P \mid R \mid \lambda x^A.A \mid \Pi x^A.A \mid AM \\ M, N &::= x \mid \lambda x^A.M \mid MM \quad . \end{aligned}$$

The judgments of λP-calculus is as follows,

$$\vdash *$$

$$\vdash W \colon *$$

$$\vdash P \colon W \supset *$$

$$\vdash R \colon W \supset W \supset *$$

$$\frac{\Gamma \vdash A \colon * \qquad \Gamma, x \colon A \vdash K}{\Gamma \vdash \Pi x^A.K}$$

$$\frac{\Gamma \vdash A \colon * \qquad \Gamma, x \colon A \vdash B \colon K}{\Gamma \vdash \Pi x^A.B \colon K}$$

$$\frac{\Gamma \vdash K \qquad x \notin \operatorname{dom} \Gamma}{\Gamma, x \colon K \vdash x \colon K}$$

$$\frac{\Gamma \vdash A \colon K \qquad \Gamma \vdash L \qquad x \notin \operatorname{dom} \Gamma}{\Gamma, x \colon L \vdash A \colon K}$$

$$\frac{\Gamma \vdash A \colon K \qquad x \notin \operatorname{dom} \Gamma}{\Gamma, x \colon A \vdash x \colon A}$$

$$\frac{\Gamma \vdash M \colon A \qquad \Gamma \vdash B \colon K \qquad x \notin \operatorname{dom} \Gamma}{\Gamma, x \colon B \vdash M \colon A}$$

$$\frac{\Gamma, x \colon A \vdash B \colon K \qquad \Gamma \vdash \Pi x^A.K}{\Gamma \vdash \lambda x^A.B \colon \Pi x^A.K}$$

$$\frac{\Gamma \vdash A \colon \Pi x^B.K \qquad \Gamma \vdash M \colon B}{\Gamma \vdash AM \colon K}$$

$$\frac{\Gamma, x \colon A \vdash M \colon B \qquad \Gamma \vdash \Pi x^A.B}{\Gamma \vdash \lambda x^A.M \colon \Pi x^A.B}$$

$$\frac{\Gamma \vdash M \colon \Pi x^A.B \qquad \Gamma \vdash N \colon A}{\Gamma \vdash MN \colon [N/x]B}$$

$$\frac{\Gamma \vdash A \colon B \qquad \Gamma \vdash C \colon K \qquad B = C}{\Gamma \vdash A \colon C}$$

where the relation = is the smallest congruence relation containing $(\lambda x.A)M = [M/x]A$ and $(\lambda x.M)N = [N/x]M$. We omit an explanation for notation of λP and its denotation and leave them to e.g., [3, 8] since they are out of the scope of this paper.

The reduction relation $\to$ of λP is defined as the smallest compatible relation containing $(\lambda x.A)M \to [M/x]A$ and $(\lambda x.M)N \to [N/x]M$. Here, we give the following terminologies, for convenience. M is said to *be in normal form* if $M \not\to N$ for any N. M is said to *have a normal form* if $M \to^* N$ and N is in normal form. Of course, $\to^*$ is the reflexive and transitive closure of $\to$. M_0 is *strongly normalizable* if there exists no infinite sequence $M_0, M_1, \ldots, M_n, \ldots$ such that $M_i \to M_{i+1}$ for any $i \in \omega$. A λ-calculus is strongly normalizable if all the typable λ-terms are strongly normalizable. M and M' are called *confluent* if there exists N such that $M \to^* N$ and $M' \to^* N$. A λ-calculus is called confluent if any pair of typable equal λ-terms is confluent.

Under these terminologies the following facts are well-known [3, 8].

Theorem 3.1. *The λP-calculus is strongly normalizable and confluent.*

Corollary 3.2 (Uniqueness)**.** *Any λP-term has a unique normal form.*

4 Completeness

Let us recall *the standard translation* in modal logic (cf. [6]). The translation interprets the modal operator $\Box$ by the universal quantifier $\forall$ (i.e., Π in λP) and the reachability predicate symbol R. The standard translation Φ_{a_n} is formally as follows,

$$\begin{aligned}
\Phi_{a_n}(p) &= Pa_n \\
\Phi_{a_n}(A \supset B) &= \Phi_{a_n}(A) \supset \Phi_{a_n}(B) \\
\Phi_{a_n}(\Box A) &= \Pi b^W.\Pi v^{Ra_nb}.\Phi_b(A)
\end{aligned}$$

where we assume that propositional variables in modal logic one-to-one correspond to predicate symbols in IP. Also, a_n is a variable which denotes a possible world.

We extend the standard translation to a function from not only λK-types but also λK-terms as follows,

$$\begin{aligned}
\Phi_{a_n}(x) &= x \\
\Phi_{a_n}(\lambda x^A.M) &= \lambda x^{\Phi_{a_n}(A)}.\Phi_{a_n}(M) \\
\Phi_{a_n}(MN) &= \Phi_{a_n}(M)\Phi_{a_n}(N) \\
\Phi_{a_n}(\mathsf{gen}(M)) &= \lambda b^W.\lambda v^{Rab}.\Phi_b(M) \\
\Phi_{a_n}(\mathsf{ungen}(M)) &= \Phi_{a_{n-1}}(M)a_n u_1^{Ra_{n-1}a_n} \quad .
\end{aligned}$$

We define interpretations of contexts elementwise. Furthermore, we define those of judgments as follows,

$$\Phi(\Gamma_1 \mid \cdots \mid \Gamma_n \vdash M\colon A) = \begin{array}{ll} a_1\colon W, & \ldots, a_n\colon W, \\ u_1^{Ra_1a_2}\colon Ra_1a_2, & \ldots, u_1^{Ra_{n-1}a_n}\colon Ra_{n-1}a_n, \\ \Phi_{a_1}(\Gamma_1), & \ldots, \Phi_{a_n}(\Gamma_n) \end{array} \vdash \Phi_{a_n}(M)\colon \Phi_{a_n}(A) \quad .$$

Here, under a context of $\Phi(\Gamma_1 \mid \cdots \mid \Gamma_n \vdash M\colon A)$ for some $\Gamma_1, \ldots, \Gamma_n$, M, and A, we can produce a grammar containing the set of λP-terms in normal form of the type A in the image of Φ_{a_n} as follows,

$$\begin{aligned}
I_n &::= x^{\Phi_{a_n}(A)} \mid I_{n-1}a_n^W u_1^{Ra_{n-1}a_n} \mid I_nJ_n \\
J_n &::= I_n \mid \lambda x^{\Phi_{a_n}(A)}.J_n \mid \lambda b^W.\lambda v^{Ra_nb}.J_{n+1} \quad .
\end{aligned}$$

We define a function Ψ_{a_n} on J_n as follows,

$$\begin{aligned}
\Psi_{a_n}(x) &= x \\
\Psi_{a_n}(I_{n-1}a_n^W u_1^{Ra_{n-1}a_n}) &= \mathsf{ungen}(\Psi_{a_{n-1}}(I_{n-1})) \\
\Psi_{a_n}(I_nJ_n) &= \Psi_{a_n}(I_n)\Psi_{a_n}(J_n) \\
\Psi_{a_n}(\lambda x^{\Phi_{a_n}(A)}.J_n) &= \lambda x^A.\Psi_{a_n}(J_n) \\
\Psi_{a_n}(\lambda b^W.\lambda v^{Ra_nb}.J_{n+1}) &= \mathsf{gen}(\Psi_{a_{n+1}}(J_{n+1})) \quad .
\end{aligned}$$

Lemma 4.1. *$\Phi_{a_n} \circ \Psi_{a_n}$ is the identity function.*

Proof. By induction on J_n. □

Theorem 4.2 (Fullness)**.** *If A is typable under $\Phi(\Gamma_1 \mid \cdots \mid \Gamma_n \vdash B)$, then there exists M such that $\Gamma_1 \mid \cdots \mid \Gamma_n \vdash M\colon B$ and $A = \Phi_{a_n}(M)$.*

Proof. Let J_n be the normal form of A. By Lemma 4.1, it is sufficient to take $\Psi_{a_n}(J_n)$ as M. □

Corollary 4.3 (Kripke and Meta-Logical Completeness)**.** *If $\vdash A$ is not derivable in IK, then $\vdash \Phi_{a_n}(A)$ is not derivable in IP, too. Therefore, the contraposition of Theorem 2.6 derives from completeness of IP to the possible world semantics.*

5 Concluding Remark

We showed Kripke and meta-logical completeness for IK using Curry-Howard isomorphism and fullness of the standard translation. Can we apply a similar proof to an extension, e.g., IK with $\Box A \supset A$ (written as IT)? Martini and Masini added the following inference rule

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash M \colon \Box A}{\Gamma_1 \mid \cdots \mid \Gamma_n \vdash \mathsf{T}(M) \colon A} \text{ (T)}$$

for IT [10]. According to the extension, we add a constant $e\colon \Pi a^W.Raa$ to λP and extend the standard translation to one such that

$$\Phi_{a_n}(\mathsf{T}(M)) = \Phi_{a_n}(M)a_n(ea_n) \ .$$

In this case, the set I_n is changed to

$$I_n ::= \cdots \mid I_n a_n^W(ea_n) \ .$$

Then, we can show fullness of the extended standard translation by extending Ψ_{a_n} to

$$\Psi_{a_n}(I_n a_n^W(ea_n)) = \mathsf{T}(\Psi_{a_n}(I_n)) \ .$$

In intuitionistic modal logic, IT with $\Box A \supset \Box\Box A$ (called IS4) is one of the most fascinating logics. Indeed, its Kripke semantics and categorical semantics are exhaustively studied by Bierman and de Paiva [5], and constructive S4 (dealt with in [1] as described in Section 1) is a variant of IS4. Furthermore, Davies and Pfenning clarified that staged computation was realized by the modality of IS4 [7], and opened a new frontier of modalities between mathematical logic and computer science. We can find not only their studies but also other ones about IS4 in some literatures. If our proof method were applied to IS4, we could obtain deeper understandings of modalities in possible world semantics, category theory, and computer science throughout de Paiva et al. and Pfenning et al.'s studies.

However, it is not easy to apply our proof method to IS4. We explain this in the following. Since the class of frame complete to IS4 should be transitive (and reflexive), it is sufficient to add a constant $d\colon \Pi a^W.\Pi b^W.\Pi c^W \Pi u^{Rab}.\Pi v^{Rbc}.Rac$ to λP. Then, one can find more than one proof of Rac depending to a path from the world a to the world c, i.e., a choice of the world b. Here, our proof method seems to require either of the following two approaches:

- to extend the standard translation to the one of the *large* domain (considering difference of paths between worlds), or
- to introduce a relation to equate all the proofs of Rac independent from b.

The former approach is very hard. This makes the former approach to be out of the scope of this paper because this paper aims to provide an easy proof method. The latter approach is also out of the scope of this paper. In fact, the point of our proof method depends on *uniqueness* of proofs (see Corollary 3.2). As we introduce an equational relation, we must give a reduction relation whose closure coincides with the equational relation. What is harder that the reduction relation must be normalizing and confluent. This is far off our policy that reuses the existing results and proves Kripke and meta-logical completeness easily. It is still open whether our proof method is applicable to IS4 and other extensions of intuitionistic modal logic.

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