A Constructive Proof of the Gohberg-Semencul Formula

J. Chun and T. Kailath*
Information Systems Laboratory
Stanford University
Stanford, California 94305

Submitted by Vicente Hernández

ABSTRACT

Gohberg and Semencul gave some elegant formulas for the inverse of a Toeplitz matrix as a difference of products of lower and upper triangular Toeplitz matrices. There are several algebraic and analytic proof of these formulas. Here we give a "constructive" proof for two of the Gohberg-Semencul formulas, under the assumption that the matrices are strongly nonsingular, i.e., all leading minors are nonzero. This assumption is stronger than necessary, but it enables fast $O(n^2)$ constructions for the entries in the Gohberg-Semencul formulas. Our method also gives a new proof of the relation between the reflection coefficients of a Toeplitz matrix and its inverse.

1. INTRODUCTION

If $\mathbf{T}$ is a Toeplitz matrix,

$$\mathbf{T} = (t_{i-j}) \in R^{n \times n},$$

then one can easily show that $\mathbf{T}$ has the following *displacement representa-*

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tion [1, 4–8]:

$$t_0\mathbf{T} = \begin{bmatrix} t_0 & 0 & \cdots & 0 \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{bmatrix} \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ 0 & t_0 & \cdots & t_{2-n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & t_{-1} \\ 0 & 0 & \cdots & t_0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ t_1 & 0 & \cdots & 0 & 0 \\ t_2 & t_1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ t_{n-1} & t_{n-2} & \cdots & t_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ 0 & 0 & t_{-1} & \cdots & t_{2-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & t_{-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (1)$$

It is an interesting fact [4] that $\mathbf{T}^{-1}$ also has a similar displacement representation. To give an explicit formula, we assume that the matrix $\mathbf{T}$ and its $(n-1)\times(n-1)$ leading principal submatrix are nonsingular. Let $\mathbf{z}$ and $\mathbf{v}$ denote the first and last columns of $\mathbf{T}^{-1}$, respectively. Then it turns out that we can write

$$z_1\mathbf{T}^{-1} = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ z_2 & z_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ z_{n-1} & z_{n-2} & \cdots & 0 \\ z_n & z_{n-1} & \cdots & z_1 \end{bmatrix} \begin{bmatrix} v_n & v_{n-1} & \cdots & v_1 \\ 0 & v_n & \cdots & v_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & v_{n-1} \\ 0 & 0 & \cdots & v_n \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ v_1 & 0 & \ddots & 0 & 0 \\ v_2 & v_1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ v_{n-1} & v_{n-2} & \cdots & v_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & z_n & z_{n-1} & \cdots & z_1 \\ 0 & 0 & z_n & \cdots & z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z_n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (2)$$

It also holds that $z_1 = v_n$. This formula was first given by Gohberg and Semencul in 1972 [2, p. 86; 3] as one of a set of slightly different formulas of the type (2) for $\mathbf{T}^{-1}$. We shall describe a variant in Section 2 (see Remark 2 in Section 2). Here it is interesting to note that a simple inspection of (2) yields the following recursive formula for the elements of $\mathbf{T}^{-1}$:

$$z_1[\mathbf{T}^{-1}]_{i+1,j+1} = z_1[\mathbf{T}^{-1}]_{i,j} + z_{i+1}v_{n-j} - v_i z_{n-j+1},$$
$$[\mathbf{A}]_{i,j} \equiv [(i,j) \text{ element of } \mathbf{A}], \tag{3}$$

which was in fact first derived by Trench [14] in 1964. Its relationship to the Gohberg-Semencul formulas and connections with the Christoffel-Darboux formulas for orthogonal polynomials on the unit circle were made in [9].

As one might expect from the above discussion, there can be several ways of establishing Gohberg-Semencul formulas. But all presently known proofs involve a certain amount of algebraic and analytical manipulation. In this short note we shall describe what may be regarded as a "constructive" proof for two Gohberg-Semencul formulas [2, pp. 86, 89] under the rather restrictive condition of *strongly nonsingular* $\mathbf{T}$, i.e., $\mathbf{T}$ with all leading minors nonzero. On the other hand, with this assumption, there are "fast" $O(n^2)$ algorithms for actually computing the Gohberg-Semencul expressions.

Our proof follows the so-called "array method" discussed in [1]. In the array method, the triangular factors of $\mathbf{T}^{-1}$ and $\mathbf{T}$ are *simultaneously* obtained via a sequence of elementary hyperbolic rotations applied to a certain "prearray" of scalars. We show in this note that a slight modification of the prearray leads to Gohberg-Semencul formulas. Our approach can be extended to obtain formulas of the Gohberg-Semencul type for "close-to-Toeplitz" matrices—see [6], of which the present brief note was actually a precursor. This note therefore serves mainly to introduce the basic ideas in a simple context; moreover it contains a simple proof of a result, derived quite differently in [7, 10], relating the reflection coefficients of $\mathbf{T}$ and $\mathbf{T}^{-1}$.

2. A CONSTRUCTIVE PROOF OF TWO GOHBERG-SEMENCUL FORMULAS

Let us consider symmetric positive definite Toeplitz matrix $\mathbf{T} = (t_{i-j}) \in R^{n\times n}$, $t_0 = 1$. Later, we shall indicate simple modifications of the proof for nonsymmetric but strongly nonsingular Toeplitz matrices. Let $\mathbf{L}(\mathbf{x}_1)$ and $\mathbf{L}(\mathbf{x}_2)$ denote lower triangular Toeplitz matrices whose first columns are $\mathbf{x}_1$ and $\mathbf{x}_2$, which are defined as

$$\mathbf{x}_1 \equiv [1, t_1, \ldots, t_{n-1}]^T = \text{the first column of } \mathbf{T},$$
$$\mathbf{x}_2 \equiv [0, t_1, t_2, \ldots, t_{n-1}]^T. \tag{4}$$

Then (1) can be rewritten as

$$\mathbf{T} = \mathbf{L}(\mathbf{x}_1)\mathbf{L}^T(\mathbf{x}_1) - \mathbf{L}(\mathbf{x}_2)\mathbf{L}^T(\mathbf{x}_2). \tag{5}$$

Let us define the *prearray* Δ_0, and a diagonal matrix $\mathbf{J}$,

$$\Delta_0 \equiv \begin{bmatrix} \mathbf{L}(\mathbf{x}_1) & \mathbf{O}_n & \mathbf{L}(\mathbf{x}_2) \\ \mathbf{I}_n & \mathbf{O}_n & \mathbf{I}_n \end{bmatrix} \in R^{2n\times 3n}, \quad \mathbf{J} \equiv \begin{bmatrix} \mathbf{I}_n & & \\ & \mathbf{I}_n & \\ & & -\mathbf{I}_n \end{bmatrix} \in R^{3n\times 3n}, \tag{6}$$

where $\mathbf{O}_n$ and $\mathbf{I}_n$ denote the $n \times n$ null matrix and the $n \times n$ identity matrix, respectively. Suppose that we can find a *J-orthogonal matrix* $\Theta \in R^{3n\times 3n}$, viz., one satisfying $\Theta \mathbf{J} \Theta^T = \mathbf{J}$, such that the *postarray* $\Delta_0\Theta$ has the form

$$\tilde{\Delta} \equiv \Delta_0\Theta = \begin{bmatrix} \mathbf{L} & \mathbf{O}_n & \mathbf{O}_n \\ \mathbf{U} & \mathbf{L}(\mathbf{y}_1) & \mathbf{L}(\mathbf{y}_2) \end{bmatrix}, \tag{7}$$

where $\mathbf{L}$ is a lower triangular matrix, and $\mathbf{L}(\mathbf{y}_1)$, $\mathbf{L}(\mathbf{y}_2)$ are lower triangular Toeplitz matrices whose first columns are some $\mathbf{y}_1$ and $\mathbf{y}_2$, respectively, while $\mathbf{U}$ is not *a priori* constrained in any way. Then, because of the *J*-orthogonality of Θ,

$$\Delta_0 \mathbf{J} \Delta_0^T = \tilde{\Delta}\mathbf{J}\tilde{\Delta}^T,$$

which yields the following indentities:

$$\mathbf{L}(\mathbf{x}_1)\mathbf{L}^T(\mathbf{x}_1) - \mathbf{L}(\mathbf{x}_2)\mathbf{L}^T(\mathbf{x}_2) = \mathbf{T} = \mathbf{L}\mathbf{L}^T, \tag{8a}$$

$$\mathbf{L}(\mathbf{x}_1) - \mathbf{L}(\mathbf{x}_2) = \mathbf{I}_n = \mathbf{L}\mathbf{U}^T, \tag{8b}$$

$$\mathbf{U}\mathbf{U}^T + \mathbf{L}(\mathbf{y}_1)\mathbf{L}^T(\mathbf{y}_1) - \mathbf{L}(\mathbf{y}_2)\mathbf{L}^T(\mathbf{y}_2) = \mathbf{O}_n. \tag{8c}$$

From (8b), we see that $\mathbf{L}^{-1} = \mathbf{U}^{\mathrm{T}}$ (therefore, $\mathbf{U}$ is upper triangular). Now (8a) shows that

$$\mathbf{T}^{-1} = \mathbf{L}^{-T}\mathbf{L}^{-1} = \mathbf{U}\mathbf{U}^T. \tag{9a}$$

Therefore, (8c) implies that $\mathbf{T}^{-1}$ has the following displacement representation:

$$\mathbf{T}^{-1} = \mathbf{L}(\mathbf{y}_2)\mathbf{L}^T(\mathbf{y}_2) - \mathbf{L}(\mathbf{y}_1)\mathbf{L}^T(\mathbf{y}_1). \tag{9b}$$

Next we shall show how to construct Θ so as to obtain a postarray $\tilde{\Delta}$ of the form (7). This can be done in several ways, but perhaps the simplest is to construct Θ as a product of elementary hyperbolic rotations, which are J-orthogonal. (We could also use hyperbolic Householder reflections.) Hyperbolic rotations $\mathbf{H}_{i,j}(\kappa)$ are defined as identity matrices except for the following four entries:

$$[\mathbf{H}_{i,j}(\kappa)]_{i,i} = [\mathbf{H}_{i,j}(\kappa)]_{j,j} = \frac{1}{(1-\kappa^2)^{1/2}},$$

$$[\mathbf{H}_{i,j}(\kappa)]_{i,j} = [\mathbf{H}_{i,j}(\kappa)]_{j,i} = \frac{-\kappa}{(1-\kappa^2)^{1/2}}.$$

We call the pair of indices (i, j) the *plane of rotation*. The matrix $\mathbf{H}_{i,j}(\kappa)$ is real and finite when $|\kappa| < 1$. In signal-processing applications, κ is often called a *reflection coefficient*. Let $\mathbf{w}^T$ be a row vector with $|w_i| > |w_j|$. We can *annihilate* w_j, *pivoting with* w_i, by postmultiplying $\mathbf{w}^T$ with $\mathbf{H}_{i,j}(w_j/w_i)$:

$$\begin{aligned}&[w_1 \quad \cdots \quad w_i \quad \cdots \quad w_j \quad \cdots \quad w_n]\mathbf{H}_{i,j}(w_j/w_i)\\ &\quad = [w_1 \quad \cdots \quad w_i' \quad \cdots w_{j-1} \quad 0 \quad w_{j+1} \quad \cdots \quad w_n].\end{aligned}$$

For a moment, let us assume that the magnitude of *pivoting elements* is always greater than that of *pivoted elements* and therefore that $|\kappa| < 1$. A lemma given in the Appendix shows that this assumption is always valid for a positive definite $\mathbf{T}$.

A Simple Example

Our construction is perhaps best followed with a simple example. Thus, before we describe the general procedure, we shall illustrate the details with a 3×3 symmetric positive definite Toeplitz matrix,

$$\mathbf{T} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1 & \alpha_1 \\ \alpha_2 & \alpha_1 & 1 \end{bmatrix}, \qquad \mathbf{x}_1 = [1, \alpha_1, \alpha_2]^T, \qquad \mathbf{x}_2 = [0, \alpha_1, \alpha_2]^T.$$

We postmultiply the prearray,

$$\Delta_0 = \begin{bmatrix} \mathbf{L}(\mathbf{x}_1) & \mathbf{O}_n & \mathbf{L}(\mathbf{x}_2) \\ \mathbf{I}_n & \mathbf{O}_n & \mathbf{I}_n \end{bmatrix} \in R^{6\times 9},$$

with hyperbolic rotations $\mathbf{H}_{2,7}(\kappa_1)$ and $\mathbf{H}_{3,8}(\kappa_1)$, where $\kappa_1 \equiv \alpha_1$, to annihilate the first subdiagonal of $\mathbf{L}(\mathbf{x}_2)$ pivoting with the diagonal of $\mathbf{L}(\mathbf{x}_1)$ [see (10) below]. To preserve the Toeplitz structure at the (2,3) block, we also apply a "dummy" hyperbolic rotation $\mathbf{H}_{4,9}(\kappa_1)$. This will introduce a nonzero element β_4 at the lower left corner in the (2,2) block. These steps are illustrated below:

$$
\Delta_0 = \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & 1 & & & & & \alpha_1 & & \\
\alpha_2 & \alpha_1 & 1 & & & & \alpha_2 & \alpha_1 & \\
\hline
1 & & & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
 & & 1 & & & & & & 1
\end{array}\right]
$$

$$
\to \underset{\Delta_0 \cdot \mathbf{H}_{2,7}(\kappa_1)}{\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & 1 & & & & \beta_3 & \alpha_1 & \\
\hline
1 & \beta_4 & & & & & \beta_5 & & \\
 & \beta_5 & & & & & \beta_4 & 1 & \\
 & & 1 & & & & & & 1
\end{array}\right]}, \tag{10a}
$$

$$
\underset{\Delta_0 \cdot \mathbf{H}_{2,7}(\kappa_1) \cdot \mathbf{H}_{3,8}(\kappa_1)}{\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & \beta_1 & & & & \beta_3 & & \\
\hline
1 & \beta_4 & & & & & \beta_5 & & \\
 & \beta_5 & \beta_4 & & & & \beta_4 & \beta_5 & \\
 & & \beta_5 & & & & & \beta_4 & 1
\end{array}\right]}
$$

$$
\to \underset{\Delta_0 \cdot \mathbf{H}_{2,7}(\kappa_1) \cdot \mathbf{H}_{3,8}(\kappa_1) \cdot \mathbf{H}_{4,9}(\kappa_1)}{\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & \beta_1 & & & & \beta_3 & & \\
\hline
1 & \beta_4 & & & & & \beta_5 & & \\
 & \beta_5 & \beta_4 & & & & \beta_4 & \beta_5 & \\
 & & \beta_5 & \beta_4 & & & & \beta_4 & \beta_5
\end{array}\right]} \equiv \Delta_1. \tag{10b}
$$

Now we postmultiply Δ_1 with the hyperbolic rotation $\mathbf{H}_{3,7}(\kappa_2)$, where $\kappa_2 \equiv \beta_3/\beta_1$, to annihilate the remaining element β_3 in the (1,3) block of Δ_1. Again to preserve the Toeplitz structure in the (2,3) block, we apply two dummy hyperbolic rotations, $\mathbf{H}_{4,8}(\kappa_2)$ and $\mathbf{H}_{5,9}(\kappa_2)$, to obtain $\tilde{\Delta}$:

$$\underset{\Delta_1 \cdot \mathbf{H}_{3,7}(\kappa_2)}{\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & \gamma_1 & & & & & & \\
\hline
1 & \beta_4 & \gamma_2 & & & & & \gamma_4 & \\
 & \beta_5 & \gamma_3 & & & & \gamma_3 & \beta_5 & \\
 & & \gamma_4 & \beta_4 & & & \gamma_2 & \beta_4 & \beta_5
\end{array}\right]}$$

$$\to \underset{\Delta_1 \cdot \mathbf{H}_{3,7}(\kappa_2) \cdot \mathbf{H}_{4,8}(\kappa_2) \cdot \mathbf{H}_{5,9}(\kappa_2)}{\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & \gamma_1 & & & & & & \\
\hline
1 & \beta_4 & \gamma_2 & & & & \gamma_4 & & \\
 & \beta_5 & \gamma_3 & \gamma_2 & & & \gamma_3 & \gamma_4 & \\
 & & \gamma_4 & \gamma_3 & \gamma_2 & & \gamma_2 & \gamma_3 & \gamma_4
\end{array}\right]} \qquad (10c)$$

$$= \begin{bmatrix} \mathbf{L} & \mathbf{O}_n & \mathbf{O}_n \\ \mathbf{U} & \mathbf{L}(\mathbf{y}_1) & \mathbf{L}(\mathbf{y}_2) \end{bmatrix} = \tilde{\Delta}. \qquad (11)$$

Then as noted before [cf. (9b)], we have

$$\mathbf{T}^{-1} = \mathbf{L}(\mathbf{y}_2)\mathbf{L}^T(\mathbf{y}_2) - \mathbf{L}(\mathbf{y}_1)\mathbf{L}^T(\mathbf{y}_1).$$

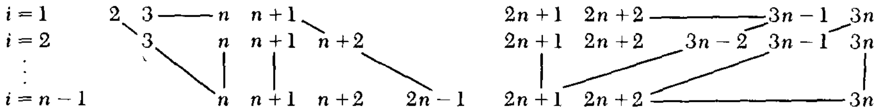
The General Procedure

In general, we successively annihilate the 1st, 2nd, ..., $(n-1)$th subdiagonals of the lower triangular matrix in the (1,3) block of Δ_0, by postmultiplying Δ_0 with a sequence of hyperbolic rotations. The procedure of annihilating the ith subdiagonal will be called the ith *sweep*, and the array obtained after the ith sweep will be denoted Δ_i. In the ith sweep, we apply hyperbolic rotations on the following two sets of planes in Δ_{i-1}:

$$A_i \equiv \{(i+1, 2n+1), (i+2, 2n+2), \ldots, (n, 3n-i)\}, \qquad (12a)$$

$$D_i \equiv \{(n+1, 3n-i+1), (n+2, 3n-i+2), \ldots, (n+i, 3n)\}, \qquad (12b)$$

where A_i and D_i stand for the sets of planes on which "annihilating" hyperbolic rotations and "dummy" hyperbolic rotations are applied, respectively. Thus, if we display (12) for each sweep, we have the pictorial representation



Pivoting columns Pivoted columns

Within a given A_i [or D_i], any *ordering* of rotations can be chosen, because the planes in (12a) [or (12b)] are "disjoint."

Note that Δ_0 has Toeplitz structure in the columns $(1,2,\dots,2n)$ and $(2n+1,2n+2,\dots,3n)$. Suppose that Δ_{i-1} has Toeplitz structure in the columns $(i,i+1,\dots,2n)$ and $(2n+1,2n+2,\dots,3n)$. Then this Toeplitz structure allows us to choose hyperbolic rotations on the set A_i with identical reflection coefficients k_i, to annihilate the ith subdiagonal elements in the (2,3) block of Δ_{i-1}:

$$\Delta'_{i-1} \equiv \Delta_{i-1}\mathbf{H}_{A_i}(\kappa_i),$$

$$\mathbf{H}_{A_i}(\kappa_i) \equiv \mathbf{H}_{i+1,2n+1}(\kappa_i)\mathbf{H}_{i+2,2n+2}(\kappa_i)\cdots\mathbf{H}_{n,3n-i}(\kappa_i). \tag{13a}$$

Furthermore, by applying hyperbolic rotations on the set D_i with the same κ_i, we can keep the Toeplitz structure in the columns of $(i+1,i+2,\dots,2n)$ and $(2n+1,2n+2,\dots,3n)$ in Δ_i:

$$\Delta_i \equiv \Delta'_{i-1}\mathbf{H}_{D_i}(\kappa_i),$$

$$\mathbf{H}_{D_i}(k_i) \equiv \mathbf{H}_{n+1,3n-i+1}(\kappa_i)\mathbf{H}_{n+2,3n-i+2}(\kappa_i)\cdots\mathbf{H}_{n+i,3n}(\kappa_i). \tag{13b}$$

In each sweep, the null (1,2) block is untouched, and the lower triangularity of the (1,1) block is maintained.

Hence, Δ_{n-1} will be the form of $\tilde{\Delta}$ in (7), and will have Toeplitz structure in the columns of $(n,n+1,\dots,2n)$ and $(2n+1,2n+2,\dots,3n)$. Therefore, the diagonal elements of $\mathbf{L}(\mathbf{y}_1)$ will be zero and the first column of $\mathbf{L}(\mathbf{y}_1)$ (i.e., $\mathbf{y}_1$) will be identical to the last column of $\mathbf{U}$ shifted down by one position.

Now we shall show that the displacement representation (9b) obtained by the above construction is the (first) Gohberg-Semencul formula (2). Notice that

$$\mathbf{T}^{-1}\mathbf{e}_1 = [\mathbf{L}(\mathbf{y}_2)\mathbf{L}^T(\mathbf{y}_2) - \mathbf{L}(\mathbf{y}_1)\mathbf{L}^T(\mathbf{y}_1)]\mathbf{e}_1 = \mathbf{L}(\mathbf{y}_2)\mathbf{L}^T(\mathbf{y}_2)\mathbf{e}_1 = [\mathbf{L}(\mathbf{y}_2)]_{1,1}\mathbf{L}(\mathbf{y}_2)\mathbf{e}_1,$$

$$\mathbf{T}^{-1}\mathbf{e}_n = (\mathbf{U}\mathbf{U}^T)\mathbf{e}_n = [\mathbf{U}]_{n,n}\mathbf{U}\mathbf{e}_n,$$

where $\mathbf{e}_i$ denotes the vector with 1 at the ith position and 0 elsewhere. Therefore,

$$[\text{first column of } \mathbf{L}(\mathbf{y}_2)] = \mathbf{L}(\mathbf{y}_2)\mathbf{e}_1 = \frac{\mathbf{T}^{-1}\mathbf{e}_1}{[\mathbf{L}(\mathbf{y}_2)]_{1,1}} \equiv \frac{\mathbf{z}}{z_1^{1/2}},$$

$$[\text{first column of } \mathbf{L}(\mathbf{y}_1)] = \mathbf{L}(\mathbf{y}_1)\mathbf{e}_1 = \mathrm{sd}(\mathbf{U}\mathbf{e}_n) = \frac{\mathrm{sd}(\mathbf{T}^{-1}\mathbf{e}_n)}{[\mathbf{U}]_{n,n}} \equiv \frac{sd(\mathbf{v})}{z_1^{1/2}},$$

where sd(v) denotes the vector v shifted down by one position, and we have used the easily verifiable fact that

$$z_1 = v_n = [\mathbf{L}(\mathbf{y}_2)]_{1,1}^2 = [\mathbf{U}]_{n,n}^2.$$

With these identifications, the displacement representation (9b) is exactly the Gohberg-Semencul formula (2).

Remark 1. The sequence of hyperbolic rotations $\mathbf{H}_{A_i}(\kappa_i)$ [see (13) for the notation] introduces the ith superdiagonal in the (2,1) block, and the ith subdiagonal in the (2,3) block. On the other hand, $\mathbf{H}_{D_i}(\kappa_i)$ introduces the $(n-i)$th subdiagonal in the (2,2) block.

Remark 2. Readers may have noticed that our construction of $\tilde{\Delta}$, and therefore the resulting displacement representation (9b), is not unique. This is because we can use *any* J-orthogonal transformation matrix Θ. In particular, we can apply extra hyperbolic rotations on D_n with *any* $|\kappa| < 1$, i.e., we can replace Δ_{n-1} by

$$\Delta_{n-1}\mathbf{H}_{D_n}(\kappa), \tag{14}$$

and still have the form (7). In fact, the second Gohberg-Semencul formula

[2, p. 89] corresponds to the particular displacement representation (9b) obtained by choosing the reflection coefficient κ in (14) as the last reflection coefficient κ_n of any nonsingular $(n+1)\times(n+1)$ Toeplitz matrix whose $n \times n$ leading principal submatrix is $\mathbf{T}$.

REMARK 3. Gohberg-Semencul formulas for the inverse of strongly nonsingular nonsymmetric Toeplitz matrices can be constructed similarly using *spinors* (see [6]), a generalized form of hyperbolic rotations. A spinor matrix is the identity except for the following four entries:

$$[\mathbf{S}_{i,j}]_{i,i} = [\mathbf{S}_{i,j}]_{j,j} = \frac{1}{(1+\kappa_1\kappa_2)^{1/2}},$$

$$[\mathbf{S}_{i,j}]_{i,j} = \frac{-\kappa_1}{(1+\kappa_1\kappa_2)^{1/2}}, \qquad [\mathbf{S}_{i,j}]_{j,i} = \frac{\kappa_2}{(1+\kappa_1\kappa_2)^{1/2}}.$$

REMARK 4. Note that our construction of the Gohberg-Semencul formula needs $O(n^2)$ operations, because we need to keep track only two columns in Δ_i, and apply only $n-1$ different 2×2 hyperbolic rotations. Our construction here is closely related to the classical Schur algorithm [4, 12, 13] (see Section 3), which has certain advantages for parallel computation [4]. The first and last columns of $\mathbf{T}^{-1}$, which define the matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ in the Gohberg-Semencul formula, can also be obtained with $O(n^2)$ operations by using the Levinson algorithm [4, 11]. In fact, Trench [14] used this algorithm to obtain the "differential form" (3) of the Gohberg-Semencul formula.

3. SCHUR ALGORITHM AND REFLECTION COEFFICIENTS OF $\mathbf{T}^{-1}$

Consider only the upper part of the transformation $\Delta_0 \to \Delta_{n-1}$ in Section 2. Then notice that dummy rotations do not affect the upper part at all, and therefore,

$$[\mathbf{L}(\mathbf{x}_1)\mathbf{O}_n\mathbf{L}(\mathbf{x}_2)]\,\Theta = [\mathbf{L}(\mathbf{x}_1)\mathbf{O}_n\mathbf{L}(\mathbf{x}_2)]\,\Theta_A = [\mathbf{L} \quad \mathbf{O}_n \quad \mathbf{O}], \tag{15}$$

where

$$\Theta = \mathbf{H}_{A_1}(\kappa_1)\mathbf{H}_{D_1}(\kappa_1)\mathbf{H}_{A_2}(\kappa_2)\mathbf{H}_{D_2}(\kappa_2)\cdots\mathbf{H}_{A_{n-1}}(\kappa_{n-1})\mathbf{H}_{D_{n-1}}(\kappa_{n-1}), \tag{16a}$$

$$\Theta_A \equiv \mathbf{H}_{A_1}(\kappa_1)\mathbf{H}_{A_2}(\kappa_2)\cdots\mathbf{H}_{A_{n-1}}(\kappa_{n-1}). \tag{16b}$$

Furthermore Θ_A in (16b) does not touch the middle $\mathbf{O}_n$ in (15), and therefore can be deleted and the indices A_i in (16b) scaled accordingly. Note that this procedure gives the factorization $\mathbf{T} = \mathbf{L}\mathbf{L}^T$.

In general, consider a matrix $\mathbf{A}$ that has the following displacement representation:

$$\mathbf{A} = \mathbf{L}(\mathbf{x}_1)\mathbf{L}^T(\mathbf{x}_1) - \mathbf{L}(\mathbf{x}_2)\mathbf{L}^T(\mathbf{x}_2), \qquad \mathbf{A} \in R^{n\times n}. \tag{17}$$

Matrices that have the representation (17) are called *quasi-Toeplitz*. A quasi-Toeplitz matrix has many displacement representations (see Remark 2, Section 2). However, one can easily check that the displacement representation with zero diagonal in $\mathbf{L}(\mathbf{x}_2)$ is unique. Such a displacement representation is said to be *proper*.

Given a displacement representation (17), the *Schur algorithm* obtains the Cholesky factorization $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ by the procedure

$$[\mathbf{L}(\mathbf{x}_1)\,\mathbf{L}(\mathbf{x}_2)]\Theta = [\mathbf{L}\quad \mathbf{O}],$$

$$\Theta \equiv \mathbf{H}_{A_0}(\kappa_0)\mathbf{H}_{A_1}(\kappa_1)\mathbf{H}_{A_2}(\kappa_2)\cdots\mathbf{H}_{A_{n-1}}(\kappa_{n-1}), \tag{18a}$$

where

$$A_i \equiv \{(i+1,n+1),(i+2,n+2),\ldots,(n,2n-i)\}, \tag{18b}$$

and $\mathbf{H}_{A_i}(\kappa_i)$ is chosen to annihilate the ith subdiagonal of $\mathbf{L}(\mathbf{x}_2)$ (let us call the diagonal the 0th subdiagonal). Note that the Schur algorithm produces a sequence of reflection coefficients

$$\{\kappa_0,\kappa_1,\kappa_2,\ldots,\kappa_{n-1}\}$$

as a by-product. Furthermore, note that the subsequence $\{\kappa_1,\kappa_2,\ldots,\kappa_{n-1}\}$ is unique (although κ_0 is not unique), because of the uniqueness of the proper displacement representation. Such a subsequence will be called a *proper* sequence of reflection coefficients. Sometimes reflection coefficients are of greater interest than the factorization itself (see [5] and references therein).

Now, we shall use the results in Section 2 to give a new constructive proof for the following interesting result [7, 10].

THEOREM. *If the proper sequence of reflection coefficients associated by the above-described Schur algorithm with* $\mathbf{T}$ *is* $\{\kappa_1,\kappa_2,\ldots,\kappa_{n-1}\}$, *then the proper sequence of reflection coefficients associated with* $\mathbf{T}^{-1}$ *is* $\{-\kappa_{n-1}, -\kappa_{n-2}, \ldots, -\kappa_1\}$.

A Simple Example

Again we shall first consider the 3×3 example. We post-multiply $\tilde{\Delta}$ with hyperbolic rotations $\mathbf{H}_{5,9}(\eta_1)$ and $\mathbf{H}_{4,8}(\eta_1)$ to annihilate the first subdiagonal elements (two γ_2's) of $\mathbf{L}(\mathbf{y}_1)$ pivoting with the diagonal element (two γ_4's) of $\mathbf{L}(\mathbf{y}_2)$. Comparing the two arrays in (10c), notice that

$$\mathbf{H}_{5,9}(\eta_1)=\mathbf{H}_{5,9}^{-1}(\kappa_2)=\mathbf{H}_{5,9}(-\kappa_2), \qquad \mathbf{H}_{4,8}(\eta_1)=\mathbf{H}_{4,8}^{-1}(\kappa_2)=\mathbf{H}_{4,8}(-\kappa_2).$$

Now we annihilate the last subdiagonal element, β_4 in the (2,2) block [see (19) below], with hyperbolic rotation $\mathbf{H}_{4,9}(\eta_2)$, pivoting with the last diagonal element, β_4 in (2,3) block. Again comparing the first array in (19) with the second array in (10b), notice that

$$\mathbf{H}_{4,9}(\eta_2)=\mathbf{H}_{4,9}^{-1}(\kappa_1)=\mathbf{H}_{4,9}(-\kappa_1).$$

The above procedure is illustrated below:

$$\tilde{\Delta} \to \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & \gamma_1 & & & & & & \\
\hline
1 & \beta_4 & \gamma_2 & & & & \gamma_4 & & \\
 & \beta_5 & \gamma_3 & & & & \gamma_3 & \beta_5 & \\
 & & \gamma_4 & \beta_4 & & & \gamma_2 & \beta_4 & \beta_5
\end{array}\right]$$
$$\tilde{\Delta}\cdot\mathbf{H}_{5,9}(-\kappa_2)\cdot\mathbf{H}_{4,8}(-\kappa_2)$$

$$\to \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
\alpha_1 & \beta_1 & & & & & & & \\
\alpha_2 & \beta_2 & \gamma_1 & & & & & & \\
\hline
1 & \beta_4 & \gamma_2 & & & & \gamma_4 & & \\
 & \beta_5 & \gamma_3 & & & & \gamma_3 & \beta_5 & \\
 & & \gamma_4 & & & & \gamma_2 & \beta_4 & 1
\end{array}\right] \tag{19}$$
$$\tilde{\Delta}\cdot\mathbf{H}_{5,9}(-\kappa_2)\cdot\mathbf{H}_{4,8}(-\kappa_2)\cdot\mathbf{H}_{4,9}(-\kappa_1)$$

$$=\begin{bmatrix} \mathbf{L} & \mathbf{O} & \underline{\mathbf{O}} \\ \mathbf{U} & \mathbf{O} & \overline{\mathbf{L}} \end{bmatrix} \equiv \overline{\Delta}, \qquad \mathbf{T}^{-1}=\overline{\mathbf{L}}\overline{\mathbf{L}}^T.$$

The General Procedure

In general, to obtain the reflection coefficients of $\mathbf{T}^{-1}$, we annihilate the (2,2) block in $\tilde{\boldsymbol{\Delta}}$ in (7). However, from (12), it is easy to check that we can permute dummy rotations and annihilating rotations as follows:

$$\begin{aligned}&\mathbf{H}_{A_1}(\kappa_1)\mathbf{H}_{D_1}(\kappa_1)\mathbf{H}_{A_2}(\kappa_2)\mathbf{H}_{D_2}(\kappa_2)\cdots\mathbf{H}_{A_{n-1}}(\kappa_{n-1})\mathbf{H}_{D_{n-1}}(\kappa_{n-1})\\&\quad=\mathbf{H}_{A_1}(\kappa_1)\mathbf{H}_{A_2}(\kappa_2)\cdots\mathbf{H}_{A_{n-1}}(\kappa_{n-1})\mathbf{H}_{D_1}(\kappa_1)\mathbf{H}_{D_2}(\kappa_2)\cdots\mathbf{H}_{D_{n-1}}(\kappa_{n-1}).\end{aligned}$$

Therefore, we can annihilate the (2,2) block by postmultiplying $\tilde{\boldsymbol{\Delta}}$ with

$$\begin{aligned}&\left[\mathbf{H}_{D_1}(\kappa_1)\mathbf{H}_{D_2}(\kappa_2)\cdots\mathbf{H}_{D_{n-1}}(\kappa_{n-1})\right]^{-1}\\&\quad=\mathbf{H}_{D_{n-1}}^{-1}(\kappa_{n-1})\mathbf{H}_{D_{n-2}}^{-1}(\kappa_{n-2})\cdots\mathbf{H}_{D_1}^{-1}(\kappa_1),\end{aligned}$$

because, as we noted in Remark 1 in Section 2, the ith subdiagonal was introduced by the set of dummy rotations $\mathbf{H}_{D_{n-i}}(\kappa_{n-i})$. Now, the result follows from the fact $\mathbf{H}^{-1}(\kappa)=\mathbf{H}(-\kappa)$.

4. CONCLUDING REMARKS

We have given constructive proof of Gohberg-Semencul formulas for a positive definite symmetric Toeplitz matrix. This proof clearly shows the connection of the Gohberg-Semencul formula with the Schur algorithm for triangularizing a Toeplitz matrix. The basic ideas here can be extended [6] to more general Toeplitz matrices and to several other matrices with displacement structure.

APPENDIX

LEMMA. *Let* $\mathbf{T}$ *be a symmetric Toeplitz matrix. Then the* $\{\kappa_i\}$ *defined by the construction in Section* 2 *will be less than one in magnitude if and only if* $\mathbf{T}$ *is positive definite.*

Proof. If $|\kappa_i|<1$ for all $1\leqslant i\leqslant n-1$, then we can complete the transformation to get the matrix $\tilde{\boldsymbol{\Delta}}$ in (7), and $\mathbf{T}=\mathbf{L}\mathbf{L}^T$. Hence, $\mathbf{T}$ is positive definite. Now, let us assume that $\mathbf{T}$ is positive definite, and $|\kappa_j|<1$ for

$1 \leqslant j \leqslant i-1$. After the $(i-1)$st sweep, the upper half (Δ^U_{i-1}) of the matrix Δ_{i-1} has the form

$$\Delta^U_{i-1} = \begin{bmatrix} \mathbf{A} & \mathbf{O} & \multirow{2}{*}{$\bigcirc$} & \mathbf{O} & \mathbf{O} \\ \mathbf{B} & \mathbf{C} & & \mathbf{D} & \mathbf{O} \end{bmatrix}, \qquad \Delta^U_{i-1}\mathbf{J}[\Delta^U_{i-1}]^T = \mathbf{T},$$

where $\mathbf{A}$ is a nonsingular lower triangular matrix, and $\mathbf{C}$, $\mathbf{D}$ are lower triangular Toeplitz. Let c and d denote the diagonal elements of $\mathbf{C}$ and $\mathbf{D}$, respectively. Suppose that $|c| \leqslant |d|$, and therefore $|\kappa_i| \geqslant 1$. Then $\mathbf{T}$ cannot be positive definite, because

$$\lambda_{\min}(\mathbf{T}) \leqslant \mathbf{s}^T\Delta^U_{i-1}\mathbf{J}[\Delta^U_{i-1}]^T\mathbf{s} = c^2 - d^2 \leqslant 0,$$

where

$$\mathbf{s}^T = [-\mathbf{b}_1^T\mathbf{A}^{-1}, \mathbf{e}_1^T], \qquad \mathbf{e}_1^T = [1, 0, \dots, 0], \quad \mathbf{b}_1^T \equiv \text{first row of } \mathbf{B},$$

which leads to a contradiction. ■

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