Distances in random graphs with infinite mean degrees

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Abstract

We study random graphs with an i.i.d. degree sequence of which the tail of the distribution function F is regularly varying with exponent $\tau \in [1, 2]$. In particular, the degrees have infinite mean. Such random graphs can serve as models for complex networks where degree power laws are observed.

The minimal number of edges between two arbitrary nodes, also called the graph distance or the hopcount, is investigated when the size of the graph tends to infinity. The paper is part of a sequel of three papers. The other two papers study the case where $\tau \in (2, 3)$, and $\tau \in (3, \infty)$, respectively.

The main result of this paper is that the graph distance for $\tau \in (1, 2)$ converges in distribution to a random variable with probability mass exclusively on the points 2 and 3. We also consider the case where we condition the degrees to be at most N^α for some $\alpha > 0$, where N is the size of the graph. For fixed $k \in \{0, 1, 2, \dots\}$ and α such that $(\tau + k)^{-1} < \alpha < (\tau + k - 1)^{-1}$, the hopcount converges to $k + 3$ in probability, while for $\alpha > (\tau - 1)^{-1}$, the hopcount converges to the same limit as for the unconditioned degrees. The proofs use extreme value theory.

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1 Introduction

The study of complex networks has attracted considerable attention in the past decade. There are numerous examples of complex networks, such as co-authorship and citation networks of scientists, the World-Wide Web and Internet, metabolic networks, etc. The topological structure of networks affects the performance in those networks. For instance, the topology of social networks affects the spread of information and disease (see e.g., [17, 20]), while the performance of traffic in Internet depends heavily on the topology of the Internet.

Measurements on complex networks have shown that many real networks have similar properties. A first example of such a fundamental network property is the fact that typical distances between nodes are small. This is called the 'small world' phenomenon, see the pioneering work of Watts [21], and the references therein. In Internet, for example, e-mail messages cannot use more than a threshold of physical links, and if the distances in Internet would be large, then e-mail service would simply break down. Thus, the graph of the Internet has evolved in such a way that typical distances are relatively small, even though the Internet is rather large.

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A second, maybe more surprising, property of many networks is that the number of nodes with degree n falls off as an inverse power of n. This is called a 'power law degree sequence'. In Internet, the power law degree sequence was first observed in [8]. The observation that many real networks have the above properties has incited a burst of activity in network modelling. Most of the models use random graphs as a way to model the uncertainty and the lack of regularity in real networks. See [3, 17] and the references therein for an introduction to complex networks and many examples where the above two properties hold.

The current paper presents a rigorous derivation for the random fluctuations of the graph distance between two arbitrary nodes (also called the hopcount) in a graph with i.i.d. degrees having *infinite* mean. The model with i.i.d. degrees is a variation of the *configuration model*, which was originally proposed by Newman, Strogatz and Watts [18], where the degrees originate from a given deterministic sequence. The observed power exponents are in the range from $\tau = 1.5$ to $\tau = 3.2$ (see [3, Table II] or [17, Table II]). In a previous paper [10], the case $\tau > 3$ was investigated, while the case $\tau \in (2, 3)$ was studied in [11]. Here we focus on the case $\tau \in (1, 2)$, and study the typical distances between arbitrary connected nodes. In a forthcoming paper [12], we will survey the results from the different cases for τ, and investigate the connected components of the random graphs.

This section is organized as follows. In Section 1.1, we start by introducing the model, and in Section 1.2, we state our main results. In Section 1.3, we explain heuristically how the results are obtained. Finally, we describe related work in Section 1.4.

1.1 The model

Consider an i.i.d. sequence $D_1, D_2, \ldots, D_N$. Assume that $L_N = \sum_{j=1}^{N} D_j$ is even. When L_N is odd, then we increase D_N by 1, i.e., we replace D_N by $D_N + 1$. This change will make hardly any difference in what follows, and we will ignore it in the sequel.

We will construct a graph in which node j has degree D_j for all $1 \le j \le N$. We will later specify the distribution of D_j. We start with N separate nodes and incident to node j, we have D_j stubs which still need to be connected to build the graph.

The stubs are numbered in an arbitrary order from 1 to L_N. We continue by matching at random the first stub with one of the $L_N - 1$ remaining stubs. Once paired, two stubs form an edge of the graph. Hence, a stub can be seen as the left or the right half of an edge. We continue the procedure of randomly choosing and pairing the next stub and so on, until all stubs are connected.

The probability mass function and the distribution function of the nodal degree are denoted by

$$\mathbb{P}(D_1 = j) = f_j, \quad j = 1, 2, \ldots, \quad \text{and} \quad F(x) = \sum_{j=1}^{\lfloor x \rfloor} f_j, \tag{1.1}$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Our main assumption will be that

$$x^{\tau-1} \left[1 - F(x)\right] \tag{1.2}$$

is slowly varying at infinity for some $\tau \in (1, 2)$. This means that the random variables D_j obey a power law with infinite mean.

1.2 Main results

We define the graph distance H_N between the nodes 1 and 2 as the minimum number of edges that form a path from 1 to 2, where, by convention, this distance equals ∞ if 1 and 2 are not connected. Observe that the graph distance between two randomly chosen nodes is equal in distribution to H_N, because the nodes are exchangeable.

In this paper, we will present two separate theorems for the case $\tau \in (1, 2)$. We also consider the boundary cases $\tau = 1$ (Theorem 1.3) and $\tau = 2$ (Remark 1.4). In Theorem 1.1, we take the sequence $D_1, D_2, \ldots, D_N$ of i.i.d. copies of D with distribution F, satisfying (1.2), with $\tau \in (1,2)$. The result is that the graph distance or hopcount converges in distribution to a limit random variable with mass $p = p_F$, $1-p$, on the values 2, 3, respectively. In the paper the abbreviation **whp**, means that the involved event happens with probability converging to 1, as $N \to \infty$.

Theorem 1.1 *Fix $\tau \in (1,2)$ in (1.2) and let $D_1, D_2, \ldots, D_N$ be a sequence of i.i.d. copies of D. Then,*

$$\lim_{N\to\infty} \mathbb{P}(H_N = 2) = 1 - \lim_{N\to\infty} \mathbb{P}(H_N = 3) = p_F \in (0,1). \tag{1.3}$$

One might argue that including degrees larger than $N-1$ is artificial in a network with N nodes. In fact, in many real networks, the degree is bounded by a physical constant. Therefore, we also consider the case where the degrees are conditioned to be smaller than N^α, where α is an arbitrary positive number. Of course, we cannot condition on the degrees to be at most M, where M is fixed and independent on N, since in this case, the degrees are uniformly bounded, and this case is treated in [10]. Therefore, we consider cases where the degrees are conditioned to be at most a given power of N.

The result with conditioned degrees appears in the Theorem 1.2 . It turns out that for $\alpha > 1/(\tau-1)$, the conditioning has no influence in the sense that the limit random variable is the same as that for the unconditioned case. This is not so strange, since the maximal degree is of order $N^{1/(\tau-1)}$, so that the conditioning does nothing in this case. However, for fixed $k \in \mathbb{N}\cup\{0\}$ and α such that $1/(\tau+k) < \alpha < 1/(\tau+k-1)$, the graph distance converges to a degenerate limit random variable with mass 1 on the value $k+3$. It would be of interest to extend the possible conditioning schemes, but we will not elaborate further on it in this paper.

In the theorem below, we write $D^{(N)}$ for the random variable D conditioned on $D < N^\alpha$. Thus,

$$\mathbb{P}(D^{(N)} = j) = \frac{f_j}{\mathbb{P}(D < N^\alpha)}, \qquad 0 \le j < N^\alpha. \tag{1.4}$$

Theorem 1.2 *Fix $\tau \in (1,2)$ in (1.2) and let $D_1^{(N)}, D_2^{(N)}, \ldots, D_N^{(N)}$ be a sequence of i.i.d. copies of $D^{(N)}$.*

(i) *For $k \in \mathbb{N}\cup\{0\}$ and α such that $1/(\tau+k) < \alpha < 1/(\tau+k-1)$,*

$$\lim_{N\to\infty} \mathbb{P}(H_N = k+3) = 1. \tag{1.5}$$

(ii) *If $\alpha > 1/(\tau-1)$, then*

$$\lim_{N\to\infty} \mathbb{P}(H_N = 2) = 1 - \lim_{N\to\infty} \mathbb{P}(H_N = 3) = p_F, \tag{1.6}$$

where $p_F \in (0,1)$ is defined in Theorem 1.1.

The boundary case $\tau = 1$ and $\tau = 2$ are treated in Theorem 1.3 and Remark 1.4, below. We will prove that for $\tau = 1$, the hopcount converges to the value 2. For $\tau = 2$, we show by presenting two examples, that the limit behavior depends on the behavior of the slowly varying tail $x[1-F(x)]$.

Theorem 1.3 *For $\tau = 1$ in (1.2) and let $D_1, D_2, \ldots, D_N$ be a sequence of i.i.d. copies of D. Then,*

$$\lim_{N\to\infty} \mathbb{P}(H_N = 2) = 1. \tag{1.7}$$

Remark 1.4 *For $\tau = 2$ in (1.2) and with $D_1, D_2, \ldots, D_N$ a sequence of i.i.d. copies of D, the limit behavior of H_N depends on the slowly varying tail $x[1-F(x)]$, $x \to \infty$. In Section 4.2, we present two examples, where we have, depending on the slowly varying function $x[1-F(x)]$, different limit behavior for H_N. We present an example with $\lim_{N\to\infty} \mathbb{P}(H_N \le k) = 0$, for all fixed integers k, as $N \to \infty$, and a second example where $H_N \in \{2,3\}$,* **whp** *, as $N \to \infty$.*

1.3 Heuristics

When $\tau \in (1, 2)$, we consider two different cases. In Theorem 1.1, the degrees are not conditioned, while in Theorem 1.2 we condition on each node having a degree smaller than N^α. We now give a heuristic explanation of our results.

In two previous papers [10, 11], the cases $\tau \in (2, 3)$ and $\tau > 3$ have been treated. In these cases, the probability mass function $\{f_j\}$ introduced in (1.1) has a finite mean, and the the number of nodes on graph distance n from node 1 can be coupled to the number of individuals in the n^{th} generation of a branching process with offspring distribution $\{g_j\}$ given by

$$g_j = \frac{j+1}{\mu} f_j, \tag{1.8}$$

where $\mu = \mathbb{E}[D_1]$. For $\tau \in (1, 2)$, as we are currently investigating, we have $\mu = \infty$, and the branching process used in [10, 11] does not exist.

When we do not condition on D_j being smaller than N^α, then L_N is the i.i.d. sum of N random variables $D_1, D_2, \ldots, D_N$, with infinite mean. It is well known that in this case the bulk of the contribution to $L_N = N^{1/(\tau-1)+o(1)}$ comes from a *finite* number of nodes which have giant degrees (the so-called *giant nodes*). A basic fact in the configuration model is that two sets of stubs of sizes n and m are connected **whp** when nm is at least of order L_N. Since the giant nodes have degree roughly $N^{1/(\tau-1)}$, which is much larger than $\sqrt{L_N}$, they are all attached to each other, thus forming a complete graph of giant nodes. Each stub is **whp** attached to a giant node, and, therefore, the distanc e between any two nodes is, **whp**, at most 3. In fact, this distance equals 2 when the two nodes are connected to the *same* giant node, and is 3 otherwise. In particular, for $\tau = 1$, the quotient $D_{(N)}/L_N$ converges to 1 in probability, and consequently the hopcount converges to 2, in probability.

When we truncate the distribution as in (1.4), with $\alpha > 1/(\tau - 1)$, we hardly change anything since without truncation **whp** all degrees are below N^α. On the other hand, if $\alpha < 1/(\tau - 1)$, then, with truncation, the largest nodes have degree of order N^α, and $L_N \sim N^{1+\alpha(2-\tau)}$. Again, the bulk of the total degree L_N comes from nodes with degree of the order N^α, so that now these are the giant nodes. Hence, for $1/\tau < \alpha < 1/(\tau - 1)$, the largest nodes have degrees much larger than $\sqrt{L_N}$, and thus, **whp,** still constitute a complete graph. The number of giant nodes converges to infinity, as $N \to \infty$. Therefore, the probability that two arbitrary nodes are connected to the *same* giant node converges to 0. Consequently, the hopcount equals 3, **whp**. If $\alpha < 1/\tau$, then the giant nodes no longer constitute a complete graph, so that the hopcount can be greater than 3. For almost every $\alpha < 1/\tau$, the hopcount coverges to a *single* value. The behavior of the hopcount for the cases that $\alpha = 1/(\tau + k)$ for $k \in \mathbb{N} \cup \{0\}$, will be dependent on the slowly varying function in (1.2), as is the case for $\tau = 2$. We do expect that the hopcount converges to at most 2 values in these cases.

The proof in this paper is based on detailed asymptotics of the sum of N i.i.d. random variables with infinite mean, as well as on the scaling of the order statistics of such random variables. The scaling of these order statistics is crucial in the definition of the giant nodes which are described above. The above considerations are the basic idea in the proof of Theorem 1.1. In the proof of Theorem 1.2, we need to investigate what the conditioning does to the scaling of both the total degree L_N, as well as to the largest degrees.

1.4 Related work

The above model is a variation of the configuration model. In the usual configuration model one often starts from a given *deterministic* degree sequence. In our model, the degree sequence is an i.i.d. sequence $D_1, \ldots, D_N$ with distribution equal to a power law. The reason for this choice is that we are interested in models for which all nodes are exchangeable, and this is not the case when the degrees are fixed. The study of this variation of the configuration model was started in [18] for the case $\tau > 3$ and studied by Norros and Reittu [19] in case $\tau \in (2, 3)$.

For a survey of complex networks, power law degree sequences and random graph models for such networks, see [3] and [17]. There, a heuristic is given why the hopcount scales proportionally to $\log N$, which is originally from [18]. The argument uses a variation of the power law degree model, namely, a model where an exponential cut off is present. An example of such a degree distribution is

$$f_j = Cj^{-\tau} e^{-j/\kappa} \tag{1.9}$$

for some $\kappa > 0$. The size of κ indicates up to what degree the power law still holds, and where the exponential cut off starts to set in. The above model is treated in [10] for any $\kappa < \infty$, but, for $\kappa = \infty$, falls within the regimes where $\tau \in (2, 3)$ in [11] and within the regime in this paper for $\tau \in (1, 2)$. In [18], the authors conclude that since the limit as $\kappa \to \infty$ does not seem to converge, the 'average distance is not well defined when $\tau < 3$'. In this paper, as well as in [11], we show that the average distance *is* well defined, but it scales differently from the case where $\tau > 3$.

In [12], we give a survey to the results for the hopcount in the three different re gimes $\tau \in (1, 2)$, $\tau \in (2, 3)$ and $\tau > 3$. There, we also prove results for the connectivity properties of the random graph in these cases. These results assume that the expected degree is larger than 2. This is always the case when $\tau \in (1, 2)$, and stronger results have been shown there. We prove that the largest connected component has **whp** size $N(1 + o(1))$. When $\tau \in (1, \frac{3}{2})$ we even prove that the graph is **whp** connected. When $\tau > \frac{3}{2}$ this is not true, and we investigate the structure of the remaining 'dust' that does not belong to the largest connected component. The analysis makes use of the results obtained in this paper for $\tau \in (1, 2)$. For instance, it will be crucial that the probability that two arbitrary nodes are connected converges to 1.

There is substantial related work on the configuration model for the cases $\tau \in (2, 3)$ and $\tau > 3$. References are included in the paper [11] for the case $\tau \in (2, 3)$, and in [10] for $\tau > 3$. We again refer to the references in [12] and [3, 17] for more details. The graph distance for $\tau \in (1, 2)$, that we study here, has, to our best knowledge, not been studied before. Values of $\tau \in (1, 2)$ have been observed in networks of e-mail messages and networks where the nodes consist of software packages (see [17, Table II]), for which our configuration model with $\tau \in (1, 2)$ can possibly give a good model.

In [1], random graphs are considered with a degree sequence that is *precisely* equal to a power law, meaning that the number of nodes with degree n is precisely proportional to $n^{-\tau}$. Aiello *et al.* [1] show that the largest connected component is of the order of the size of the graph when $\tau < \tau_0 = 3.47875\ldots$, where τ_0 is the solution of $\zeta(\tau - 2) - 2\zeta(\tau - 1) = 0$, and where ζ is the Riemann zeta function. When $\tau > \tau_0$, the largest connected component is of smaller order than the size of the graph and more precise bounds are given for the largest connected component. When $\tau \in (1, 2)$, the graph is **whp** connected. The proofs of these facts use couplings with branching processes and strengthen previous results due to Molloy and Reed [15, 16]. See also [1] for a history of the problem and references predating [15, 16]. See [2] for an introduction to the mathematical results of various models for complex networks (also called massive graphs), as well as a detailed account of the results in [1].

A detailed account for a related model can be found in [5] and [6], where links between nodes i and j are present with probability equal to $w_i w_j / \sum_l w_l$ for some 'expected degree vector' $w = (w_1, \ldots, w_N)$. Chung and Lu [5] show that when w_i is proportional to $i^{-\frac{1}{\tau-1}}$, the average distance between pairs of nodes is proportional $\log N(1+o(1))$ when $\tau > 3$, and equal to $2\frac{\log\log N}{|\log(\tau-2)|}(1+o(1))$ when $\tau \in (2, 3)$. In their model, also $\tau \in (1, 2)$ is possible, and in this case, similarly to $\tau \in (1, \frac{3}{2})$ in our paper, the graph is connected **whp**.

The difference between this model and ours is that the nodes are not exchangeable in [5], but the observed phenomena are similar. This can be understood as follows. Firstly, the actual degree vector in [5] should be close to the expected degree vector. Secondly, for the expected degree vector, we can compute that the number of nodes for which the degree is at least n equals

$$|\{i : w_i \geq n\}| = |\{i : ci^{-\frac{1}{\tau-1}} \geq n\}| \propto n^{-\tau+1},$$

where the proportionality constant depends on N. Thus, one expects that the number of nodes with degree at least n decreases as $n^{-\tau+1}$, similarly as in our model. In [6], Chung and Lu study the sizes of the connected components in the above model. The advantage of working with the 'expected degree model' is that different links are present independently of each other, which makes this model closer to the classical random graph $G(p, N)$.

1.5 Organization of the paper

The main body of the paper consists of the proofs of Theorem 1.1 in Section 2 and the proof of Theorem 1.2 in Section 3. Both proofs contain a technical lemma and in order to make the argument more transparent, we have postponed the proofs of these lemmas to the appendix. Section 4 contains the proof of Theorem 1.3 and two examples for the case $\tau = 2$. Section 5 contains simulation results, conclusions and open problems.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1, which states that the hopcount between two arbitrary nodes has **whp** a non-trivial distribution on 2 and 3. We start with an outline of our proof.

Below, we introduce an event $\mathcal{A}_{\varepsilon,N}$, such that when $\mathcal{A}_{\varepsilon,N}$ occurs, the hopcount between two arbitrary nodes is either 2 or 3. We then prove that $\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) < \varepsilon$, for $N \geq N_\varepsilon$ (see Lemma 2.2 below). For this we need a modification of the extreme value theorem for the k largest degrees, for all $k \in \mathbb{N}$.

We introduce

$$D_{(1)} \leq D_{(2)} \leq \cdots \leq D_{(N)},$$

to be the order statistics of $D_1, \ldots, D_N$, so that $D_{(1)} = \min\{D_1, \ldots, D_N\}$, $D_{(2)}$ is the second smallest degree, etc. Let (u_N) be an increasing sequence such that

$$\lim_{N\to\infty} N\,[1 - F(u_N)] = 1. \tag{2.1}$$

It is well known that the order statistics of the degrees, as well as the total degree, are governed by u_N in the case that $\tau \in (1, 2)$. The following lemma shows this in detail. In the lemma $E_1, E_2, \ldots$ is an i.i.d. sequence of exponential random variables with unit mean and $\Gamma_j = E_1 + E_2 + \ldots + E_j$, hence Γ_j has a gamma distribution with parameters j and 1. Throughout the paper, equality in distribution is denoted by the symbol $\stackrel{d}{=}$, whereas $\stackrel{d}{\to}$ denotes convergence in distribution. .

Lemma 2.1 (Convergence in distribution of order statistics) *For any $k \in \mathbb{N}$,*

$$\left(\frac{L_N}{u_N}, \frac{D_{(N)}}{u_N}, \ldots, \frac{D_{(N-k+1)}}{u_N}\right) \stackrel{d}{\to} (\eta, \xi_1, \ldots, \xi_k)\,, \ \textit{as } N \to \infty, \tag{2.2}$$

where $(\eta, \xi_1, \ldots, \xi_k)$ is a random vector which can be represented by

$$(\eta, \xi_1, \ldots, \xi_k) \stackrel{d}{=} \left(\sum_{j=1}^{\infty} \Gamma_j^{-1/(\tau-1)}, \Gamma_1^{-1/(\tau-1)}, \ldots, \Gamma_k^{-1/(\tau-1)}\right). \tag{2.3}$$

Moreover,

$$\xi_k \to 0 \ \textit{in probability, as } k \to \infty. \tag{2.4}$$

Proof. Because $\tau - 1 \in (0,1)$, the proof is a direct consequence of [14, Theorem 1'], and the continuous mapping theorem [4, Theorem 5.1], which together yield that on $\mathbb{R} \times \mathbb{R}^\infty$, equipped with the product topology, we have

$$(S_N^\#, Z^{(N)}) \xrightarrow{d} (S^\#, Z), \tag{2.5}$$

where $S_N^\# = u_N^{-1} L_N$, $Z^{(N)} = u_N^{-1}(D_{(N)}, \ldots, D_{(1)}, 0, 0, \ldots)$, and $Z_j = \Gamma_j^{-1/(\tau-1)}$, $j \geq 1$.

If we subsequently take the projection from $\mathbb{R} \times \mathbb{R}^\infty \mapsto \mathbb{R}^{k+1}$, defined by

$$\pi(s, z) = (s, z_1, \ldots, z_k), \tag{2.6}$$

i.e., we keep the sum and the k largest order statistics, then we obtain (2.2) and (2.3) from, again, the continuous mapping theorem. Finally, (2.4) follows because the series $\sum_{j=1}^\infty Z_j$ converges almost surely. $\square$

We need some additional notation. In this section, we define the *giant nodes* as the k_ε largest nodes, i.e., those nodes with degrees $D_{(N)}, \ldots, D_{(N-k_\varepsilon+1)}$, where k_ε is some function of ε, to be chosen below. We define

$$\mathcal{A}_{\varepsilon,N} = \mathcal{B}_{\varepsilon,N} \cap \mathcal{C}_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N}, \tag{2.7}$$

where

(i) $\mathcal{B}_{\varepsilon,N}$ is the event that the stubs of node 1 and node 2 are attached exclusively to stubs of giant nodes;

(ii) $\mathcal{C}_{\varepsilon,N}$ is the event that any two giant nodes are attached to each other; and

(iii) $\mathcal{D}_{\varepsilon,N}$ is defined as

$$\mathcal{D}_{\varepsilon,N} = \{D_1 \leq q_\varepsilon, D_2 \leq q_\varepsilon\},$$

where $q_\varepsilon = \min\{n : 1 - F(n) < \varepsilon/8\}$.

The reason for introducing the above events is that on $\mathcal{A}_{\varepsilon,N}$, the hopcount or graph distance is either 2 or 3. Indeed, on $\mathcal{B}_{\varepsilon,N}$, both node 1 and node 2 are attached exclusively to giant nodes. On the event $\mathcal{C}_{\varepsilon,N}$, giant nodes have mutual graph distance 1. Hence, on the intersection $\mathcal{B}_{\varepsilon,N} \cap \mathcal{C}_{\varepsilon,N}$, the hopcount between node 1 and node 2 is at most 3. The event $\mathcal{D}_{\varepsilon,N}$ prevents that the hopcount can be equal to 1, because the probability on the intersection of $\{H_N = 1\}$ with $\mathcal{D}_{\varepsilon,N}$ can be bounded by $q_\varepsilon^2/N \to 0$ (see the first part of the proof of Theorem 1.1 for details). Observe that the expected number of stubs of node 1 is not bounded, since the expectation of a random variable with distribution (1.2) equals $+\infty$. Putting things together we see that if we can show that $\mathcal{A}_{\varepsilon,N}$ happens **whp**, then the hopcount is either 2 or 3. The fact that $\mathcal{A}_{\varepsilon,N}$ happens **whp** is the content of Lemma 2.2, where we show that $\mathbb{P}(\mathcal{A}_{\varepsilon,N}^c) < \varepsilon$, for $N \geq N_\varepsilon$. Finally, we observe that the hopcount between node 1 and 2 is precisely equal to 2, if at least one stub of node 1 and at least one stub of node 2 is attached to the same giant node, and equal to 3 otherwise.

The events $\mathcal{B}_{\varepsilon,N}$ and $\mathcal{C}_{\varepsilon,N}$ depend on the integer k_ε, which we will take to be large for ε small, and will be defined now. The choice of the index k_ε is rather technical, and depends on the distributional limits of Lemma 2.1. Since $L_N/u_N = (D_1 + D_2 + \ldots + D_N)/u_N$ converges in distribution to the random variable η with support $(0, \infty)$, we can find a_ε, such that

$$\mathbb{P}(L_N < a_\varepsilon u_N) < \varepsilon/36, \quad \forall N. \tag{2.8}$$

This follows since convergence in distribution implies tightness of the sequence L_N/u_N ([4, p. 9]), so that we can find a closed subinterval $I \subset (0, \infty)$, with

$$\mathbb{P}(L_N/u_N \in I) > 1 - \varepsilon, \quad \forall N.$$

We next define b_ε, which is rather involved. It depends on ε, the quantile q_ε, the value a_ε defined above and the value of $\tau \in (1,2)$ and reads

$$b_\varepsilon = \left(\frac{\varepsilon^2 a_\varepsilon}{2304 q_\varepsilon}\right)^{\frac{1}{2-\tau}}, \tag{2.9}$$

where the peculiar integer 2304 is the product of 8^2 and 36. Given b_ε, we take k_ε equal to the minimal k such that

$$\mathbb{P}(\xi_k \geq b_\varepsilon/2) \leq \varepsilon/72. \tag{2.10}$$

It follows from (2.4) that such a number k exists. We have now defined the constants that we will use in the proof, and we next claim that the probability of $\mathcal{A}^c_{\varepsilon,N}$ is at most ε:

Lemma 2.2 (The good event has high probability) *For each $\varepsilon > 0$, there exists N_ε, such that*

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) < \varepsilon, \quad N \geq N_\varepsilon. \tag{2.11}$$

The proof of this lemma is rather technical and can be found in appendix A.1. We will now complete the proof of Theorem 1.1 subject to Lemma 2.2.

Proof of Theorem 1.1. As seen in the discussion following the introduction of the event $\mathcal{A}_{\varepsilon,N}$, this event implies the event $\{H_N \leq 3\}$, so that $\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) < \varepsilon$ induces that the event $\{H_N \leq 3\}$ occurs with probability at least $1-\varepsilon$.

The remainder of the proof consist of two parts. In the first part we show that $\mathbb{P}\left(\{H_N = 1\} \cap \mathcal{A}_{\varepsilon,N}\right) < \varepsilon$. In the second part we prove that

$$\lim_{N\to\infty} \mathbb{P}\left(H_N = 2\right) = p_F,$$

for some $0 < p_F < 1$. Since ε is an arbitrary positive number, the above statements yield the content of the theorem.

We turn to the first part. The event $\{H_N = 1\}$ occurs *iff* at least one stub of node 1 connects to a stub of node 2. For $j \leq D_1$, we denote by $\{[1.j] \to [2]\}$ the event that j^{th} stub of node 1 connects to a stub of node 2. Then, with $\mathbb{P}_N$ the conditional probability given the degrees $D_1, D_2, \ldots, D_N$,

$$\mathbb{P}(\{H_N = 1\} \cap \mathcal{A}_{\varepsilon,N}) \leq \mathbb{E}\left[\sum_{j=1}^{D_1} \mathbb{P}_N(\{[1.j] \to [2]\} \cap \mathcal{A}_{\varepsilon,N})\right] \leq \mathbb{E}\left[\sum_{j=1}^{D_1} \frac{D_2}{L_N - 1} \mathbf{1}_{\{\mathcal{A}_{\varepsilon,N}\}}\right] \leq \frac{q_\varepsilon^2}{N-1} < \varepsilon, \tag{2.12}$$

for large enough N, since $L_N \geq N$.

We next prove that $\lim_{N\to\infty} \mathbb{P}\left(H_N = 2\right) = p$, for some $0 < p < 1$. Since by definition for any $\varepsilon > 0$,

$$\max\{\mathbb{P}(\mathcal{B}^c_{\varepsilon,N}), \mathbb{P}(\mathcal{D}^c_{\varepsilon,N})\} \leq \mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) \leq \varepsilon,$$

we have that

$$\begin{aligned}
&|\mathbb{P}(H_N = 2) - \mathbb{P}\left(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}\right)| \\
&\quad \leq \left|\mathbb{P}(H_N = 2)\left(1 - \frac{1}{\mathbb{P}(\mathcal{B}_{\varepsilon,N})}\right)\right| + \left|\frac{\mathbb{P}(H_N = 2) - \mathbb{P}(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \cap \mathcal{B}_{\varepsilon,N})}{\mathbb{P}(\mathcal{B}_{\varepsilon,N})}\right| \\
&\quad \leq \frac{2\mathbb{P}(\mathcal{B}^c_{\varepsilon,N}) + \mathbb{P}(\mathcal{D}^c_{\varepsilon,N})}{\mathbb{P}(\mathcal{B}_{\varepsilon,N})} \leq \frac{3\varepsilon}{1-\varepsilon},
\end{aligned}$$

uniformly in N, for N sufficiently large. If we show that

$$\lim_{N\to\infty} \mathbb{P}\left(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}\right) = r(\varepsilon), \tag{2.13}$$

then there exists a double limit

$$p_F = \lim_{\varepsilon \downarrow 0} \lim_{N\to\infty} \mathbb{P}\left(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}\right) = \lim_{N\to\infty} \mathbb{P}\left(H_N = 2\right).$$

Moreover, if we can bound $r(\varepsilon)$ away 0 and 1, uniformly in ε, for ε small enough, then we also obtain that $0 < p_F < 1$.

In order to prove the existence of the limit in (2.13) we claim that $\mathbb{P}_N(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N})$ can be written as the ratio of two polynomials, where each polynomial only involves components of the vector

$$\left(\frac{D_{(N)}}{u_N}, \dots, \frac{D_{(N-k_\varepsilon+1)}}{u_N}, \frac{1}{u_N}\right). \tag{2.14}$$

Due to (2.2), this vector converges in distribution to $(\xi_1, \dots, \xi_{k_\varepsilon}, 0)$. Hence, by the continuous mapping theorem [4, Theorem 5.1, p. 30], we have the existence of the limit (2.13). We now prove the above claim.

Indeed, the hopcount between nodes 1 and 2 is 2 *iff* both nodes are connected to the same giant node. For any $0 \le i \le D_1$, $0 \le j \le D_2$ and $0 \le k < k_\varepsilon$, let $\mathcal{F}_{i,j,k}$ be the event that both the i^{th} stub of node 1 and the j^{th} stub of node 2 are connected to the node with the $(N-k)^{\text{th}}$ largest degree. Then, conditionally on the degrees $D_1, D_2, \dots, D_N$,

$$\mathbb{P}_N(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}) = \mathbb{P}_N\left(\bigcup_{i=1}^{D_1} \bigcup_{j=1}^{D_2} \bigcup_{k=0}^{k_\varepsilon - 1} \mathcal{F}_{i,j,k} \,\middle|\, \mathcal{B}_{\varepsilon,N}\right),$$

where the right-hand side can be written by the inclusion-exclusion formula, as a linear combination of terms

$$\mathbb{P}_N(\mathcal{F}_{i_1,j_1,k_1} \cap \dots \cap \mathcal{F}_{i_n,j_n,k_n} \,|\, \mathcal{B}_{\varepsilon,N}). \tag{2.15}$$

It is not difficult to see that these probabilities are ratios of polynomials of components of (2.14). For example,

$$\mathbb{P}_N(\mathcal{F}_{i,j,k} \,|\, \mathcal{B}_{\varepsilon,N}) = \frac{D_{(N-k)}(D_{(N-k)} - 1)}{(D_{(N-k_\varepsilon+1)} + \dots + D_{(N)})(D_{(N-k_\varepsilon+1)} + \dots + D_{(N)} - 1)}, \tag{2.16}$$

so that dividing both the numerator and the denominator of (2.16) by u_N^2, we obtain that the right-hand side of (2.16) is indeed a ratio of two polynomials of the vector given in (2.14). Similar arguments hold for general terms of the form in (2.15). Hence, $\mathbb{P}_N(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N})$ itself can be written as a ratio of two polynomials where the polynomial in the denominator is strictly positive. Therefore, the limit in (2.13) exists.

We finally bound $r(\varepsilon)$ from 0 and 1 uniformly in ε, for any $\varepsilon < 1/2$. Since the hopcount between nodes 1 and 2 is 2, given $\mathcal{B}_{\varepsilon,N}$, if they are both connected to the node with largest degree, then

$$\mathbb{P}(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}) \ge \mathbb{E}[\mathbb{P}_N(\mathcal{F}_{1,1,0} \,|\, \mathcal{B}_{\varepsilon,N})],$$

and by (2.16) we have

$$\begin{aligned} r(\varepsilon) &= \lim_{N\to\infty} \mathbb{P}\left(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}\right) \ge \lim_{N\to\infty} \mathbb{E}\left[\tfrac{D_{(N)}(D_{(N)}-1)}{(D_{(N)}+\dots+D_{(N-k_\varepsilon+1)}-1)^2}\right] \\ &= \mathbb{E}\left[\left(\tfrac{\xi_1}{\xi_1+\dots+\xi_{k_\varepsilon}}\right)^2\right] \ge \mathbb{E}\left[\left(\tfrac{\xi_1}{\eta}\right)^2\right]. \end{aligned}$$

On the other hand, conditionally on $\mathcal{B}_{\varepsilon,N}$, the hopcount between nodes 1 and 2 is at least 3, when all stubs of the node 1 are connected to the node with largest degree, and all stubs of the node 2 are connected to the node with the one but largest degree. Hence, for any $\varepsilon < 1/2$ and similarly to (2.16), we have

$$
\begin{aligned}
r(\varepsilon) &= \lim_{N\to\infty} \mathbb{P}\left(\{H_N = 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}\right) \le 1 - \lim_{N\to\infty} \mathbb{P}\left(\{H_N > 2\} \cap \mathcal{D}_{\varepsilon,N} \,|\, \mathcal{B}_{\varepsilon,N}\right) \\
&\le 1 - \lim_{N\to\infty} \mathbb{P}\left(\{H_N > 2\} \cap \mathcal{D}_{\frac{1}{2},N} \,|\, \mathcal{B}_{\varepsilon,N}\right) \\
&\le 1 - \lim_{N\to\infty} \mathbb{E}\left[\left(\prod_{i=0}^{D_1} \frac{D_{(N)}-2i}{D_{(N)}+\cdots+D_{(N-k_\varepsilon+1)}-D_1}\right)\left(\prod_{i=0}^{D_2} \frac{D_{(N-1)}-2i}{D_{(N)}+\cdots+D_{(N-k_\varepsilon+1)}-D_2}\right) \mathbf{1}_{\{\mathcal{D}_{\frac{1}{2},N}\}}\right] \\
&\le 1 - \mathbb{E}\left[\left(\frac{\xi_1\xi_2}{\eta^2}\right)^{q_{\frac{1}{2}}}\right],
\end{aligned}
$$

because: (i) the event $\mathcal{D}_{\frac{1}{2},N}$ implies that both $D_1 \le q_{\frac{1}{2}}$ and $D_2 \le q_{\frac{1}{2}}$, (ii) the event $\mathcal{B}_{\varepsilon,N}$ implies that all stubs of the normal nodes 1 and 2 are connected to stubs of giant nodes, (iii) Lemma 2.1 implies

$$
\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\left(\prod_{i=0}^{q_{\frac{1}{2}}} \frac{D_{(N)}-2i}{D_{(N)}+\cdots+D_{(N-k_\varepsilon+1)}-D_1}\right)\left(\prod_{i=0}^{q_{\frac{1}{2}}} \frac{D_{(N-1)}-2i}{D_{(N)}+\cdots+D_{(N-k_\varepsilon+1)}-D_2}\right)\right] \\
&\quad = \mathbb{E}\left[\left(\frac{\xi_1}{\xi_1+\cdots+\xi_{k_\varepsilon}}\right)^{q_{\frac{1}{2}}}\left(\frac{\xi_2}{\xi_1+\cdots+\xi_{k_\varepsilon}}\right)^{q_{\frac{1}{2}}}\right], \qquad (2.17)
\end{aligned}
$$

and (iv) $\xi_1 + \cdots + \xi_{k_\varepsilon} \le \eta$.

Both expectations

$$
\mathbb{E}\left[\left(\frac{\xi_1}{\eta}\right)^2\right] \qquad \text{and} \qquad \mathbb{E}\left[\left(\frac{\xi_1\xi_2}{\eta^2}\right)^{q_{\frac{1}{2}}}\right], \qquad (2.18)
$$

are strictly positive and independent of ε. Hence, for any $\varepsilon < 1/2$, the quantity $r(\varepsilon)$ is bounded away from 0 and 1, where the bounds are *independent of* ε, and thus $0 < p_F < 1$. This completes the proof of Theorem 1.1 subject to Lemma 2.2. □

3 Proof of Theorem 1.2

In Theorem 1.2, we consider the hopcount in the configuration model with degrees an i.i.d. sequence with a truncated distribution given by (1.4), where D has distribution F satisfying (1.2). We distinguish two cases: (i) $\alpha < 1/(\tau-1)$ and (ii) $\alpha > 1/(\tau-1)$. Since part (ii) is simpler to prove than part (i), we start with part (ii).

Proof of Theorem 1.2(ii). We have to prove that the limit distribution of H_N is a mixed distribution with probability mass p_F on 2 and probability mass $1-p_F$ on 3, where p_F is given by Theorem 1.1.

As before, we denote by $D_1, D_2, \ldots, D_N$ the i.i.d. sequence without conditioning. We bound the probability that for at least one index $i \in \{1, 2, \ldots, N\}$ the degree D_i exceeds N^α, by

$$
\mathbb{P}\left(\bigcup_{i=1}^{N}\{D_i > N^\alpha\}\right) \le \sum_{i=1}^{N} \mathbb{P}(D_i > N^\alpha) = N\mathbb{P}(D > N^\alpha) = N\left[1 - F\left(N^\alpha\right)\right] \le N^{-\varepsilon},
$$

for some positive ε, because $\alpha > 1/(\tau-1)$. We can therefore couple the i.i.d. sequence $\vec{D}^{(N)} = (D_1^{(N)}, D_2^{(N)}, \ldots, D_N^{(N)})$ to the sequence $\vec{D} = (D_1, D_2, \ldots, D_N)$, where the probability of a miscoupling, i.e., a coupling such that $\vec{D}^{(N)} \ne \vec{D}$, is at most $N^{-\varepsilon}$. Therefore, the result of Theorem 1.1 carries over to case (ii) in Theorem 1.2. □

Proof of Theorem 1.2(i). This proof is more involved. We start with an outline of the proof. Fix

$$
\frac{1}{\tau+k} < \alpha < \frac{1}{\tau+k-1}, \qquad (3.1)
$$

with $k \in \mathbb{N} \cup \{0\}$ and define

$$M_N = \sum_{n=1}^{N} D_n^{(N)}. \tag{3.2}$$

From [9, Theorem 1, p. 281], the expected value of M_N is given by

$$\mathbb{E}[M_N] = \frac{N}{F(N^\alpha)} \sum_{i=0}^{N^\alpha - 1} \mathbb{P}(D > i) = N^{1+\alpha(2-\tau)}\ell(N), \tag{3.3}$$

where $N \mapsto \ell(N)$ is slowly varying at infinity. In the sequel, we will use the same $\ell(N)$, for different slowly varying functions, so that the value of $\ell(N)$ may change from line to line.

For the outline, we assume that M_N has roughly the same size as $\mathbb{E}[M_N]$ in (3.3). The proof consists of showing that $\mathbb{P}(H_N \leq k+2) = o(1)$ and $\mathbb{P}(H_N > k+3) = o(1)$. We will sketch the proof of each of these results. To prove that $\mathbb{P}(H_N \leq k+2) = o(1)$, note that **whp** the degrees of nodes 1 and 2 are bounded by q_ε for some large q_ε. Therefore, on this event, the number of nodes that can be reached from node 1 in $l-1$ steps is at most $q_\varepsilon N^{(l-2)\alpha}$, and the number of stubs attached to nodes at distance $l-1$ is at most $q_\varepsilon N^{(l-1)\alpha}$. The probability that one of these stubs is attached to a stub of node 2, making H_N at most l, is of the order $q_\varepsilon^2 N^{(l-1)\alpha}/M_N$. By (3.3) and the assumed concentration of M_N, this is at most $q_\varepsilon^2 \ell(N) N^{(l-3+\tau)\alpha-1} = o(1)$, whenever $\alpha < 1/(l-3+\tau)$. Applying this to $l = k+2$, we see that this probability is $o(1)$ if $\alpha < 1/(k+\tau-1)$.

To prove that $\mathbb{P}(H_N > k+3) = o(1)$, we use the notion of giant nodes in a similar way as in the proof of Theorem 1.1. Due to the conditioning on the degree, Lemma 2.1 no longer holds, so that we need to adapt the definition of a giant node. In this section, a giant node h is a node with degree $D_h^{(N)}$, satisfying that, for an appropriate choice of β,

$$N^\beta < D_h^{(N)} \leq N^\alpha. \tag{3.4}$$

Nodes with degree at most N^β will be called *normal* nodes, and we will denote by K_N the total number of stubs of the normal nodes, i.e.,

$$K_N = \sum_{n=1}^{N} D_n^{(N)} \mathbf{1}_{\{D_n^{(N)} \leq N^\beta\}}. \tag{3.5}$$

Similarly to (3.3), we see that

$$\mathbb{E}[K_N] = N^{1+\beta(2-\tau)}\ell(N). \tag{3.6}$$

To motivate our choice of β, which depends on the value of k, observe that a node with (at least) N^β stubs, which chooses exclusively other *giant* nodes, in $k+1$ steps can reach approximately $N^{(k+1)\beta}$ other nodes. The number of stubs of $N^{(k+1)\beta}$ *giant* nodes is by definition at least $N^{(k+2)\beta}$. Hence, if we take β such that $M_N \sim N^{(k+2)\beta}$, or equivalently, by (3.3), $1+\alpha(2-\tau) \leq (k+2)\beta$, then we basically have all giant nodes on mutual distance at most $k+1$, so that (the non-giant) nodes 1 and 2, given that they both connect to at least one giant node, are on distance at most $k+3$. In the proof, we will see that we can pick any β such that

$$\frac{1+\alpha(2-\tau)}{k+2} < \beta < \alpha,$$

where we use that $\frac{1+\alpha(2-\tau)}{k+2} < \alpha$, precisely when $\alpha > \frac{1}{\tau+k}$. Having this in mind, we choose

$$\beta = \frac{1}{2}\left(\frac{1+\alpha(2-\tau)}{k+2} + \alpha\right). \tag{3.7}$$

Here ends the outline of the proof.

We now turn to the definition of the events involved. This part is similar, but not identical, to the introduction of $\mathcal{A}_{\varepsilon,N}$ in (2.7), because giant nodes no longer are on mutual distance 1. We keep the same notation for the event $\mathcal{B}_{\varepsilon,N}$, the event that the stubs of node 1 and 2 are attached exclusively to stubs of giant nodes, although the definition of a giant node has been changed. We take this slight abuse of notation for granted. The event $\mathcal{D}_{\varepsilon,N} = \{D_1 \leq q_\varepsilon, D_2 \leq q_\varepsilon\}$, where $q_\varepsilon = \min\{k : 1 - F(k) < \varepsilon/8\}$, is identical to the definition in Section 2 (below (2.7)). We define

$$\mathcal{G}_{\varepsilon,N} = \mathcal{B}_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N} \cap \mathcal{H}_{\varepsilon,N}, \tag{3.8}$$

where

$$\mathcal{H}_{\varepsilon,N} = \left\{N^{1+\alpha(2-\tau)}\underline{\ell}(N) \leq M_N \leq N^{1+\alpha(2-\tau)}\bar{\ell}(N)\right\} \cup \{K_N \leq N^{1+\beta(2-\tau)}\ell(N)\}, \tag{3.9}$$

where $\ell(N), \underline{\ell}(N), \bar{\ell}(N)$ are slowly varying at infinity. The event $\mathcal{H}_{\varepsilon,N}$ will enable us to control the distance between any pair of giant nodes, as sketched in the outline.

The following lemma is the counterpart of Lemma 2.2 in Section 2.

Lemma 3.1 (The good event has high probability) *For each $\varepsilon > 0$, there exists N_ε, such that, for all $N \geq N_\varepsilon$,*

$$\mathbb{P}\big(\mathcal{G}^c_{\varepsilon,N}\big) < \varepsilon. \tag{3.10}$$

The proof of Lemma 3.1 is rather technical and can be found in Appendix A.2.

The remainder of the proof of Theorem 1.2 is divided into two parts, namely, the proofs of

$$\mathbb{P}(\{H_N \leq k+2\} \cap \mathcal{G}_{\varepsilon,N}) < \varepsilon/2, \tag{3.11}$$

and

$$\mathbb{P}(\{H_N > k+3\} \cap \mathcal{G}_{\varepsilon,N}) < \varepsilon/2. \tag{3.12}$$

Indeed, if we combine the statements (3.11) and (3.12), then

$$\begin{aligned}\mathbb{P}(H_N = k+3) &= \mathbb{P}(\{H_N = k+3\} \cap \mathcal{G}_{\varepsilon,N}) + \mathbb{P}\big(\{H_N = k+3\} \cap \mathcal{G}^c_{\varepsilon,N}\big) \\ &\geq \mathbb{P}(\{H_N = k+3\} \cap \mathcal{G}_{\varepsilon,N}) - \varepsilon \\ &= 1 - \mathbb{P}(\{H_N > k+3\} \cap \mathcal{G}_{\varepsilon,N}) - \mathbb{P}(\{H_N < k+3\} \cap \mathcal{G}_{\varepsilon,N}) - \varepsilon > 1 - 2\varepsilon,\end{aligned} \tag{3.13}$$

and the conclusion of Theorem 1.2(i) is reached. We will prove (3.11), (3.12) in two lemmas.

Lemma 3.2 (The distance is at least $k+3$ on the good event) *For fixed $k \in \mathbb{N} \cup \{0\}$, and α as in (3.1), for each $\varepsilon > 0$, there exists an integer N_ε, such that*

$$\mathbb{P}(\{H_N \leq k+2\} \cap \mathcal{G}_{\varepsilon,N}) < \varepsilon/2, \qquad N \geq N_\varepsilon.$$

Proof. The inequality of the lemma is proved by a counting argument. We will show that for each $l \in \{1, 2, 3, \ldots, k+2\}$

$$\mathbb{P}(\{H_N = l\} \cap \mathcal{G}_{\varepsilon,N}) < N^{-\delta_l}, \tag{3.14}$$

for some $\delta_l > 0$. Since

$$\mathbb{P}(\{H_N \leq k+2\} \cap \mathcal{G}_{\varepsilon,N}) \leq \sum_{l=1}^{k+2} \mathbb{P}(\{H_N = l\} \cap \mathcal{G}_{\varepsilon,N}) \leq (k+2)N^{-\delta},$$

where $\delta = \min\{\delta_1, \ldots, \delta_{k+2}\} > 0$, the claim of the lemma follows if we choose N_ε, such that $(k+2)N_\varepsilon^{-\delta} \leq \varepsilon/2$.

To prove that $\mathbb{P}(\{H_N = l\} \cap \mathcal{G}_{\varepsilon,N}) < N^{-\delta_l}$ for any $l \leq k+2$, we note that on $\mathcal{G}_{\varepsilon,N}$, the degrees of nodes 1 and 2 are bounded by q_ε. Therefore, on $\mathcal{G}_{\varepsilon,N}$ and using that all degrees are bounded by N^α, the number of nodes that can be reached from node 1 in $l-1$ steps is at most $q_\varepsilon N^{(l-2)\alpha}$, and the number of stubs incident to nodes at distance $l-1$ from node 1 is at most $q_\varepsilon N^{(l-1)\alpha}$. When $H_N = l$, then one of these stubs should be attached to one of the at most q_ε stubs incident to node 2.

Denote by $M_N^{(l)}$ the number of stubs that are not part of an edge incident to a node at distance at most $l-1$ from node 1. Then, conditionally on $M_N^{(l)}$ and the fact that node 2 is at distance at least $l-1$ from node 1, the stubs of node 2 will be connected to one of these $M_N^{(l)}$ stubs uniformly at random. More precisely, conditionally on $M_N^{(l)}$ and the fact that node 2 is at distance at least $l-1$ from node 1, the event $\{H_N = l\}$ occurs precisely when a stub of node 2 is paired with a stub attached to a node at distance $l-1$ from node 1.

We note that, on $\mathcal{G}_{\varepsilon,N}$,

$$M_N^{(l)} \geq M_N - 2q_\varepsilon N^{(l-2)\alpha} = M_N(1+o(1)) \geq \ell(N) N^{1+(2-\tau)\alpha}, \tag{3.15}$$

when $(l-2)\alpha < 1 + (2-\tau)\alpha$, i.e., when $\alpha < 1/(l+\tau-4)$. Since $l \leq k+2$ and $\alpha < 1/(k+\tau-1)$, the latter is always satisfied.

The probability that one of the at most q_ε stubs of node 2 is paired with one of the stubs attached to nodes at distance $l-1$ from node 1 is, on $\mathcal{G}_{\varepsilon,N}$ and conditionally on $M_N^{(l)}$ and the fact that node 2 is at distance at least $l-1$ from node 1, bounded from above by

$$\frac{q_\varepsilon^2 N^{(l-1)\alpha}}{M_N^{(l)}} = \frac{q_\varepsilon^2 N^{(l-1)\alpha}}{M_N}(1+o(1)) \leq \ell(N) N^{(l-3+\tau)\alpha - 1} < N^{-\delta_l}, \tag{3.16}$$

for all $\delta_l < 1 - (l-3+\tau)\alpha$ and N sufficiently large. Here, we use the lower bound on M_N in (3.9). Applying this to $l = k+2$, which gives the worst possible value of δ_l, we see that this probability is bounded from above by $N^{-\delta}$ for any $\delta < 1-(k+\tau-1)\alpha$. Since $\alpha < 1/(k+\tau-1)$, we have that $1-(k+\tau-1)\alpha > 0$, so that we can also choose $\delta > 0$. $\square$

We turn to the proof (3.12), which we also formulate as a lemma:

Lemma 3.3 (The distance is at most $k+3$ on the good event) *Fix $k \in \mathbb{N} \cup \{0\}$, and α as in (3.1). For each $\varepsilon > 0$ there exists an integer N_ε, such that,*

$$\mathbb{P}(\{H_N > k+3\} \cap \mathcal{G}_{\varepsilon,N}) < \varepsilon/2, \qquad N \geq N_\varepsilon.$$

In the proof of Lemma 3.3, we need that the number of giant nodes reachable from an arbitrary giant node h in at most l steps, has a lower bound proportional to $N^{l\beta}$. We denote by $Z_h^{(l)}$ the set of all nodes which are reachable in exactly l steps from a node h:

$$Z_h^{(l)} = \{n = 1, 2, \ldots, N \,:\, d(h,n) = l\} \text{ for } l \in \{0, 1, \ldots\},$$

where $d(h,n)$ denotes the graph-distance between the nodes h and n. The number of giant nodes in $Z_h^{(l)}$ is denoted by $E_h^{(l)}$.

Lemma 3.4 (Growth of the number of giant nodes) *For each $\varepsilon > 0$, $\alpha < 1/(\tau+k-1)$, $l \in \{0, 1, \ldots, k\}$ and β given by (3.7),*

$$\mathbb{P}\left(\bigcap_{h=1}^{N} \left\{(1-\varepsilon)^l N^{l\beta} \leq E_h^{(l)} < N^{l\alpha}\right\} \cap \{h \textit{ is giant}\} \cap \mathcal{G}_{\varepsilon,N}\right) > 1 - N \sum_{s=1}^{k} e^{-3\varepsilon(1-\varepsilon)^s N^{s\beta}/16},$$

for sufficiently large N.

Proof. The upper bound $N^{l\alpha}$ on $E_h^{(l)}$ is trivial, because each node has less than N^α stubs. We will prove by induction with respect to l, that for $l \in \{0, 1, \dots, k\}$, and k fixed,

$$\mathbb{P}\left(\bigcup_{h=1}^{N} \left\{E_h^{(l)} < (1-\varepsilon)^l N^{l\beta}\right\} \cap \{h \,\text{is giant}\} \cap \mathcal{G}_{\varepsilon,N}\right) \leq N \sum_{s=1}^{l} e^{-3\varepsilon(1-\varepsilon)^s N^{s\beta}/16}. \tag{3.17}$$

Denote

$$\mathcal{F}_{\varepsilon,N}^{(l)} = \bigcap_{h=1}^{N} \left\{\{E_h^{(l)} \geq (1-\varepsilon)^l N^{l\beta}\} \cap \{h \,\text{is giant}\}\right\}, \tag{3.18}$$

then it suffices to prove that

$$\mathbb{P}((\mathcal{F}_{\varepsilon,N}^{(l)})^c \cap \mathcal{F}_{\varepsilon,N}^{(l-1)} \cap \mathcal{G}_{\varepsilon,N}) \leq N e^{-3\varepsilon(1-\varepsilon)^l N^{l\beta}/16}. \tag{3.19}$$

Indeed, if (3.19) holds, then (3.17) follows, by the induction hypothesis, as follows:

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{h=1}^{N} \left\{E_h^{(l)} < (1-\varepsilon)^l N^{l\beta}\right\} \cap \{h \,\text{is giant}\} \cap \mathcal{G}_{\varepsilon,N}\right) \qquad (3.20)\\
&\quad = \mathbb{P}(\mathcal{G}_{\varepsilon,N} \cap (\mathcal{F}_{\varepsilon,N}^{(l)})^c) \leq \mathbb{P}(\mathcal{G}_{\varepsilon,N} \cap (\mathcal{F}_{\varepsilon,N}^{(l)})^c \cap \mathcal{F}_{\varepsilon,N}^{(l-1)}) + \mathbb{P}(\mathcal{G}_{\varepsilon,N} \cap (\mathcal{F}_{\varepsilon,N}^{(l-1)})^c)\\
&\quad \leq N e^{-3\varepsilon(1-\varepsilon)^l N^{l\beta}/16} + N \sum_{s=1}^{l-1} e^{-3\varepsilon(1-\varepsilon)^s N^{s\beta}/16}.
\end{aligned}$$

For $l = 0$, (3.19) trivially holds. We therefore assume that (3.19) is valid for $l = m-1$ and we will prove that (3.19) holds for $l = m$.

In this paragraph we will work conditionally given the degrees $D_1, D_2, \dots, D_N$. For h a giant node, we consider only $A_N = E_h^{(m-1)} \lfloor N^\beta \rfloor$ stubs of the nodes in $Z_h^{(m-1)}$. To be more precise: we consider $\lfloor N^\beta \rfloor$ stubs of each of the $E_h^{(m-1)}$ giant nodes in $Z_h^{(m-1)}$. We number these stubs by $i \in \{1, 2, \dots, A_N\}$ and stub i will connect to a stub of a node n_i. Then we denote by $r_{N,i}$, for $i \in \{1, 2, \dots, A_N\}$, the probability that stub i does not connect to a stub of a normal node. We denote by $s_{N,i}$ the probability that stub i does not connect to a stub of a node in $Z_h^{(m-1)}$ (and the total number of stubs of this set is at most $N^{m\alpha}$), and finally, we denote by $t_{N,i,j}$ the probability that stub i does not connect to the giant node h_j previously selected by the stubs $j \in \{1, 2, \dots, i-1\}$ (for each j there are are most $D_{h_j}^{(N)} \leq N^\alpha$ of such stubs). If none of the above attachments happens, then we have a match with a not previously found giant node, and we denote by $q_{N,i}$ the probability of such a match of stub i, i.e.,

$$q_{N,i} = 1 - r_{N,i} - s_{N,i} - \sum_{j=1}^{i-1} t_{N,i,j}.$$

From the number of stubs mentioned between the parenthesis, we can bound this probability from below by

$$q_{N,i} \geq 1 - \frac{K_N}{M_N} - \frac{N^{m\alpha}}{M_N} - \sum_{j=1}^{i-1} \frac{N^\alpha}{M_N}. \tag{3.21}$$

Since, $i - 1 \leq E_h^{(m-1)} \lfloor N^\beta \rfloor \leq N^{\alpha(m-1)} \lfloor N^\beta \rfloor \leq N^{\alpha(m-1)+\beta}$, and $K_N \leq N^{1+\beta(2-\tau)}\ell(N)$, $M_N > \underline{\ell}(N) N^{1+\alpha(2-\tau)}$ on $\mathcal{G}_{\varepsilon,N}$, we can bound $1 - q_{N,i}$ on $\mathcal{G}_{\varepsilon,N}$ from above by

$$1 - q_{N,i} \leq \frac{\ell(N) N^{1+\beta(2-\tau)} + N^{\alpha m} + N^{\alpha(m-1)+\beta+\alpha}}{\underline{\ell}(N) N^{1+\alpha(2-\tau)}}.$$

For sufficiently large N and uniformly in i, we have that $1 - q_{N,i} < \varepsilon/2$, because $\beta < \alpha$, and $m\alpha + \beta \leq k\alpha + \beta < (k+1)\alpha < 1 + \alpha(2-\tau)$.

Introduce the binomially distributed random variable Y_N with parameters B_N and $\varepsilon/2$, where $B_N = \lceil (1-\varepsilon)^m N^{m\beta} \rceil$. On $\mathcal{F}_{\varepsilon,N}^{(m-1)}$, we have that $A_N = E_h^{m-1}\lfloor N^\beta \rfloor \geq B_N$, so that the number of mismatches will be stochastically dominated by Y_N. We need at least $(1-\varepsilon)B_N$ matches, so that

$$\mathbb{P}(\{E_h^{(m)} \geq (1-\varepsilon)^m N^{m\beta}\} \cap \{A_N \geq B_N\} \cap \mathcal{G}_{\varepsilon,N}) \geq \mathbb{P}(Y_N < \varepsilon B_N)\,. \tag{3.22}$$

We will now use the Janson inequality [13], which states that for any $t > 0$,

$$\mathbb{P}(|Y_N - \mathbb{E}[Y_N]\,| \geq t) \leq 2\exp\left(-\frac{t^2}{2(\mathbb{E}[Y_N] + t/3)}\right). \tag{3.23}$$

Since $\mathbb{E}[Y_N] = \varepsilon B_N/2$, we obtain, with $t = \varepsilon B_N/2$,

$$\mathbb{P}(Y_N < \varepsilon B_N) \leq \mathbb{P}(|Y_N - \mathbb{E}[Y_N]\,| > \varepsilon B_N/2) \leq 2\exp\left(-\frac{3\varepsilon B_N}{16}\right).$$

Combining this with (3.22), and since there are at most N giant nodes:

$$\mathbb{P}((\mathcal{F}_{\varepsilon,N}^{(m)})^c \cap \mathcal{F}_{\varepsilon,N}^{(m-1)} \cap \mathcal{G}_{\varepsilon,N}) \leq N\mathbb{P}(Y_N \leq \varepsilon B_N) \leq 2N\exp\left(-\frac{3\varepsilon(1-\varepsilon)^m N^{m\beta}}{16}\right). \tag{3.24}$$

$\square$

Proof of Lemma 3.3. We start with an outline. On the event $\mathcal{G}_{\varepsilon,N}$, each stub of the nodes 1 and 2 is attached to a stub of some giant node. The idea is to show that **whp** the distance between any two giant nodes is at most $k+1$. This implies that the graph distance between nodes 1 and 2, intersected with the event $\mathcal{G}_{\varepsilon,N}$ is **whp** at most $k+3$, and hence Lemma 3.3.

We will extend the event $\mathcal{G}_{\varepsilon,N}$ to include the main event in Lemma 3.4:

$$\mathcal{I}_{\varepsilon,N} = \mathcal{G}_{\varepsilon,N} \cap \mathcal{F}_{\varepsilon,N}^{(k)}, \tag{3.25}$$

where $\mathcal{F}_{\varepsilon,N}^{(k)}$ was defined in (3.18). Then

$$\mathbb{P}(\{H_N > k+3\} \cap \mathcal{G}_{\varepsilon,N}) \leq \mathbb{P}(\{H_N > k+3\} \cap \mathcal{I}_{\varepsilon,N}) + \mathbb{P}\big(\mathcal{G}_{\varepsilon,N} \cap (\mathcal{F}_{\varepsilon,N}^{(k)})^c\big)\,, \tag{3.26}$$

and the second term on the right hand side of (3.26) can be bounded by $\varepsilon/4$ using Lemma 3.4. We use as indicated in the outline of the proof given above, that

$$\begin{aligned}\mathbb{P}(\{H_N > k+3\} \cap \mathcal{I}_{\varepsilon,N}) &\leq \mathbb{P}\left(\bigcup_{h_1,h_2} \{h_1, h_2 \text{ are giant}\} \cap \{d(h_1,h_2) > k+1\} \cap \mathcal{I}_{\varepsilon,N}\right) \\ &\leq \sum_{h_1,h_2} \mathbb{P}(\{h_1, h_2 \text{ are giant}\} \cap \{d(h_1,h_2) > k+1\} \cap \mathcal{I}_{\varepsilon,N})\,, \end{aligned} \tag{3.27}$$

where the sum is taken over all pairs of nodes, and where, as before, $d(h_1, h_2)$ denotes the graph-distance between h_1 and h_2. Indeed, on $\mathcal{I}_{\varepsilon,N}$, the nodes 1 and 2 are connected to giant nodes, so that when $H_N > k+3$, there must be giant nodes h_1, h_2 at mutual distance at least $k+1$.

Clearly for any pair of nodes h_1 and h_2,

$$\{d(h_1,h_2) > k+1\} \subseteq \{d(h_1,h_2) > k\}\,,$$

which implies that for any pair of nodes h_1 and h_2,

$$\begin{aligned}&\mathbb{P}_N(\{d(h_1,h_2) > k+1\} \cap \{h_1, h_2 \text{ are giant}\} \cap \mathcal{I}_{\varepsilon,N}) \\ &\qquad \leq \mathbb{P}_N\big(\{d(h_1,h_2) > k+1\} \cap \{h_1, h_2 \text{ are giant}\} \cap \mathcal{I}_{\varepsilon,N} \,|\, d(h_1,h_2) > k\big).\end{aligned}$$

On the event $\{d(h_1,h_2)>k\}\cap\{h_1,h_2 \text{ are giant}\}$, the giant node h_2 is not attached to one of the nodes at distance k from the node h_1. More precisely, the giant node h_2 is not attached to one of the $\cup_{l=0}^{k-1} Z_{h_1}^{(l)}$ nodes. We have less than $M_N - \sum_{l=0}^{k-1} E_{h_1}^{(l)} N^\beta$ stubs to choose from, and the event $\{d(h_1,h_2)>k+1\}$ conditioned on $\{d(h_1,h_2)>k\}$ implies that no stubs of the giant node h_2 will attach to one of the at least $E_{h_1}^{(k)} N^\beta$ free stubs of $Z_{h_1}^{(k)}$. Therefore, we have, almost surely,

$$\begin{aligned}
&\mathbb{P}_N\big(\{h_1,h_2 \text{ are giant}\}\cap\{d(h_1,h_2)>k+1\}\cap\mathcal{G}_{\varepsilon,N}^{(N)} \mid d(h_1,h_2)>k\big)\\
&\leq \prod_{i=0}^{D_{h_2}^{(N)}-1}\left(1-\frac{E_{h_1}^{(k)}N^\beta}{M_N-\sum_{j=0}^{k-1}E_{h_1}^{(j)}N^\beta-2i+1}\right)\mathbf{1}_{\{\mathcal{I}_{\varepsilon,N}\}}\\
&\leq \left(1-\frac{E_{h_1}^{(k)}N^\beta}{M_N}\right)^{D_{h_2}^{(N)}}\mathbf{1}_{\{\mathcal{I}_{\varepsilon,N}\}} \leq \left(1-\frac{\varepsilon(1-\varepsilon)^k N^{\beta(k+1)}}{N^{1+\alpha(2-\tau)}\bar{\ell}(N)}\right)^{N^\beta}\\
&\leq \exp\left\{-\frac{\varepsilon(1-\varepsilon)^k N^{\beta(k+2)}}{N^{1+\alpha(2-\tau)}\bar{\ell}(N)}\right\} \leq \exp\left\{-\varepsilon(1-\varepsilon)^k N^\delta\right\},
\end{aligned} \tag{3.28}$$

where we used the inequality $1-x\leq e^{-x}$, $x\geq 0$, in the one but last inequality, and where $0<\delta<\beta(k+2)-(1+\alpha(2-\tau))$. If we substitute this upper bound in the right hand side of (3.27), then we end up with

$$\mathbb{P}(\{H_N>k+3\}\cap\mathcal{I}_{\varepsilon,N}) \leq N^2\exp\left(-\varepsilon(1-\varepsilon)^k N^\delta\right) < \varepsilon/2.$$

This completes the proof of Lemma 3.3 and hence of Theorem 1.2. □

4 The cases $\tau=1$ and $\tau=2$

4.1 Proof of Theorem 1.3

It is well known, see e.g. [7, 8.2.4], that when $1-F(x)$ is slowly varying, the quotient of the maximum and the sum of N i.i.d. random variables with distribution F, converges to 1 in probability, i.e.,

$$\frac{D_{(N)}}{L_N}\to 1, \quad \text{in probability.} \tag{4.1}$$

Therefore, we obtain that **whp**, both node 1 and node 2 are connected to the node with maximal degree, which gives the stated result.

□

4.2 Two examples with $\tau=2$

In the following two examples we show that for $\tau=2$, the limit hopcount distribution is sensitive to the slowly varying function.

Example 1. Let, for $x\geq 2$,

$$1-F(x)=\frac{2(\log 2)^2}{(\lfloor x\rfloor)(\log\lfloor x\rfloor)^2}. \tag{4.2}$$

Then we show that for all k fixed,

$$\mathbb{P}\left(H_N>k\right)=1+o(1), \qquad \text{as } N\to\infty. \tag{4.3}$$

We first prove (4.3) for $k=2$. We show this in two steps. In the first step we show that for any $\varepsilon>0$, there exists $v_\varepsilon\in\mathbb{N}$ such that with probability at least $1-\varepsilon$ all nodes at distance at most 1 from nodes 1 and 2 have degrees at most v_ε. In the second step we show that there exists $N_v\in\mathbb{N}$, such that for any $N \geq N_v$, with probability at least $1-\varepsilon$, any two given nodes with degrees at most v_ε, are disconnected. Both steps together clearly imply (4.3).

The second step is similar to (2.12), and is omitted here.

To obtain the first step we consider the event $\mathcal{D}_{\varepsilon,N}$, defined below (2.7). Then, for any $v \in \mathbb{N}$, the probability that within the first q_ε stubs of node 1 or node 2 there is a stub connected to a stub of node with degree at least $v+1$ is at most

$$\mathbb{E}\left[\frac{2q_\varepsilon}{L_N}\sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i>v\}}\right].$$

It remains to show that the above expectation is at most $\varepsilon/2$ for some $v = v_\varepsilon$ large enough. For this, we need that the first moment of the degree distribution for this example is finite. Indeed, from (4.8)

$$\mathbb{E}[D_1] = 1 + \sum_{x=2}^{\infty} \frac{2(\log 2)^2}{x(\log x)^2} \leq 2 + \int_2^\infty \frac{2(\log 2)^2}{u(\log u)^2}\,du = 1 + 2(\log 2)^2 \int_{\log 2}^\infty \frac{dy}{y^2} < \infty. \tag{4.4}$$

Then, from the Law of Large Numbers applied to $L_N = D_1 + \cdots + D_N$, we obtain

$$\mathbb{P}\left(L_N \geq \mu_\varepsilon N\right) \leq \frac{\varepsilon}{12 q_\varepsilon}, \tag{4.5}$$

for $\mu_\varepsilon > \mathbb{E}[D_1]$. Due to (4.4), (4.5) and the Markov inequality

$$\begin{aligned}
&\mathbb{E}\left[\tfrac{2q_\varepsilon}{L_N}\textstyle\sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i>v\}}\right] \leq \tfrac{\varepsilon}{6} + 2q_\varepsilon \mathbb{P}\left(2q_\varepsilon \textstyle\sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i>v\}} \geq \tfrac{\varepsilon L_N}{6}\right)\\
&\leq \tfrac{\varepsilon}{6} + 2q_\varepsilon \mathbb{P}\left(L_N \geq \mu_\varepsilon\right) + 2q_\varepsilon \mathbb{P}\left(2q_\varepsilon \sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i>v\}} \geq \frac{\varepsilon}{6}\mu_\varepsilon N\right)\\
&\leq \tfrac{\varepsilon}{3} + \tfrac{24 q_\varepsilon^2}{\varepsilon \mu_\varepsilon}\mathbb{E}\left(D_i \mathbf{1}_{\{D_i>v\}}\right) \leq \tfrac{\varepsilon}{2},
\end{aligned}$$

for large enough v, and hence we have the second step, since $\mathbb{P}(\mathcal{D}^c_{\varepsilon,N}) \leq 2\mathbb{P}(D_1 > q_\varepsilon) \leq \varepsilon/4$.

In a similar way we can show that, for any $\varepsilon > 0$, there exists $v_\varepsilon \in \mathbb{N}$ such that with probability at least $1-\varepsilon$ all nodes at distance at most 2 from nodes 1 and 2 have degrees at most v_ε. This statement implies that $\mathbb{P}(H_N > 4) \to 1$. Similarly, we obtain that for any $\varepsilon > 0$ there exists $v_\varepsilon \in \mathbb{N}$ such that with probability at least $1-\varepsilon$ all nodes at distance at most k from nodes 1 and 2 have degrees at most v_ε, which implies that for any fixed integer k,

$$\lim_{N\to\infty} \mathbb{P}(H_N > 2k) = 1, \tag{4.6}$$

i.e., the probability mass of H_N drifts away to $+\infty$ as $N \to \infty$. This behavior of H_N for $\tau = 2$, is in agreement with the behavior of H_N for the case $\tau \in (2,3)$, (see [11]), where we show, among other things, tightness of the sequence

$$H_N - \frac{\log\log N}{|\log(\tau-2)|}. \tag{4.7}$$

□

Example 2. Let

$$1 - F(x) = c\frac{(\log x)^{\log\log x - 1}\log\log x}{x}, \qquad x \geq x^*, x \in \mathbb{N}. \tag{4.8}$$

where x^* is chosen such that for $x \geq x^*$, the right side of (4.8) is a non-increasing function, and c is such that $1 - F(x^*) = 1$. We will show that

$$\mathbb{P}\left(H_N \in \{2,3\}\right) = 1 + o(1), \qquad \text{as } N \to \infty. \tag{4.9}$$

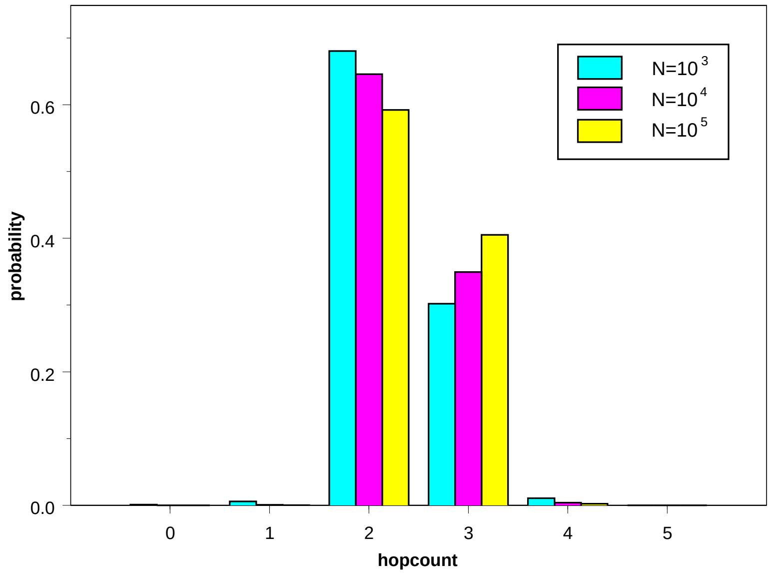

Figure 1: Empirical probability mass function of the hopcount for $\tau = 1.8$ and $N = 10^3, 10^4, 10^5$, for the unconditioned degrees.

Thus, we see entirely different behavior as in the first example.

Define *giant* nodes as nodes with degree at least $N^{\frac{1}{2}+\delta}$, for some $\delta > 0$, to be determined later on. The nodes with degree at most $N^{\frac{1}{2}+\delta} - 1$ we call *normal*. Define the event $\mathcal{A}_{\varepsilon,N}$ as in (2.7), where, in the definition of $\mathcal{B}_{\varepsilon,N}$, we use the above definition of the giant node. In Appendix A.3, we will prove the following lemma, which is similar to Lemma 2.2:

Lemma 4.1 *For each $\varepsilon > 0$, there exists N_ε, such that for all $N \geq N_\varepsilon$,*

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) < \varepsilon. \tag{4.10}$$

We now complete the proof of (4.3) subject to Lemma 4.1, which is straightforward. By (2.12), we obtain that $\mathbb{P}\left(\{H_N = 1\} \cap \mathcal{A}_{\varepsilon,N}\right) = o(1)$. Moreover, when $\mathcal{A}_{\varepsilon,N}$ occurs, all stubs of nodes 1 and 2 are connected to giant nodes due to $\mathcal{B}_{\varepsilon,N}$, and the giant nodes form a complete graph due to $\mathcal{C}_{\varepsilon,N}$, so that $\mathbb{P}\left(\{H_N > 3\} \cap \mathcal{A}_{\varepsilon,N}\right) = 0$. This proves (4.3). $\square$

5 Simulation and conclusions

To illustrate Theorems 1.1 and 1.2, we have simulated our random graph with degree distribution $D = \lceil U^{-\frac{1}{\tau-1}} \rceil$, where U is uniformly distributed over $(0, 1)$. Thus,

$$1 - F(x) = \mathbb{P}(U^{-\frac{1}{\tau-1}} > x) = x^{1-\tau}, \quad x = 1, 2, 3, \ldots$$

In Figure 1, we have simulated the graph distance or hopcount with $\tau = 1.8$ and the values of $N = 10^3, 10^4, 10^5$. The histogram is in accordance with Theorem 1.1: for increasing values of N we see that the probability mass is divided over the values $H_N = 2$ and $H_N = 3$, where the probability $\mathbb{P}(H_N = 2)$ converges.

As an illustration of Theorem 1.2, we again take $\tau = 1.8$, but now condition the degrees to be less than N, so that $\alpha = 1$. Since in this case $(\tau - 1)^{-1} = \frac{5}{4}$, we expect from Theorem 1.2 case (i), that in the limit the hopcount will concentrate on the value $H_N = 3$. This is indeed the case as is shown in Figure 2.

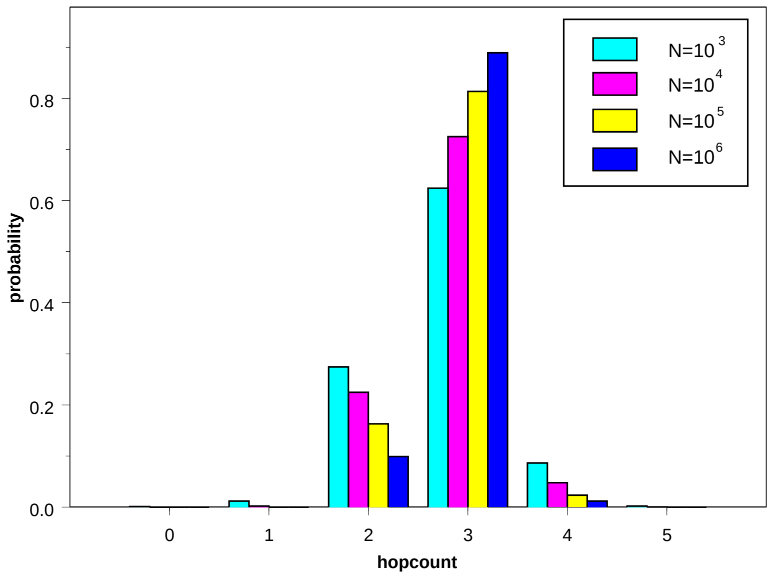

Figure 2: Empirical probability mass function of the hopcount for $\tau = 1.8$ and $N = 10^3, 10^4, 10^5, 10^6$, where the degrees are conditioned to be less than N, $(\frac{1}{\tau} < \alpha = 1 < \frac{1}{\tau-1})$.

Our results give convincing asymptotics for the hopcount when the mean degree is infinite, using extreme value theory. Some details remain open:

(i) It is possible to compute upper and lower bounds on the value p_F, based on Lemma 2.1. We presented two such bounds in (2.18). These bounds can be obtained from simulating the random variables $\Gamma_1, \Gamma_2, \ldots$ in (2.3). It should be possible to obtain much sharper upper and lower bounds, and possibly even numerical values, depending on the specific degree distribution F.

(ii) In the boundary cases $\alpha = 1/(\tau + k)$, $k \in \mathbb{N} \cup \{0\}$, it is natural to conjecture that the specific limit behavior of H_N will depend on the slowly varying function, as is the case for $\tau = 2$ and $\alpha > \frac{1}{\tau-1} = 1$ as described in Section 4.2.

A Appendix.

In the appendix we prove Lemma 2.2, Lemma 3.1 and Lemma 4.1. The proofs of Lemma 3.1 and 4.1 are both adaptations of the proof of Lemma 2.2 in Section A.1 below.

A.1 Proof of Lemma 2.2

In this section we restate Lemma 2.2 and then give a proof.

Lemma A.1.1 *For each $\varepsilon > 0$, there exists N_ε such that*

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) < \varepsilon, \quad N \geq N_\varepsilon. \tag{A.1.1}$$

Proof. We start with an outline of the proof. By (2.7),

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) \leq \mathbb{P}(\mathcal{B}^c_{\varepsilon,N}) + \mathbb{P}(\mathcal{C}^c_{\varepsilon,N}) + \mathbb{P}(\mathcal{D}^c_{\varepsilon,N}), \tag{A.1.2}$$

and an obvious way to prove result (A.1.1) would be to show that each of the three terms on the right-hand side of (A.1.2) is smaller than $\varepsilon/3$. This direct approach is somewhat difficult and instead we introduce an additional event $\mathcal{E}_{\varepsilon,N}$, which controls the total degree L_N in part (c), the degree of the giant nodes in part (b), and the total degree of all normal (non-giant) nodes in part (a):

$$\begin{aligned} \mathcal{E}_{\varepsilon,N} = \quad & \left\{ \sum_{n=1}^{N-k_\varepsilon} D_{(n)} \le \tfrac{\varepsilon}{8q_\varepsilon} L_N \right\} && (a) \\ & \cap \{ D_{(N-k_\varepsilon+1)} \ge c_\varepsilon u_N \} && (b) \\ & \cap \{ L_N \le d_\varepsilon u_N \}, && (c) \end{aligned} \tag{A.1.3}$$

where q_ε is the ε-quantile of F used in the definition of $\mathcal{D}_{\varepsilon,N}$ and where $c_\varepsilon, d_\varepsilon > 0$ are defined by

$$\mathbb{P}\left(\xi_{k_\varepsilon} < c_\varepsilon\right) < \varepsilon/24 \quad \text{and} \quad \mathbb{P}(\eta > d_\varepsilon) < \varepsilon/24,$$

respectively. Observe that c_ε is a lower quantile of ξ_{k_ε}, whereas b_ε defined in (2.9) and (2.10) is an upper quantile of ξ_{k_ε}. Furthermore, d_ε is an upper quantile of η, whereas a_ε defined in (2.8) is a lower quantile of η. Intersection with the additional event $\mathcal{E}_{\varepsilon,N}$, facilitates the bounding of both $\mathcal{B}^c_{\varepsilon,N}$ and $\mathcal{C}^c_{\varepsilon,N}$. Therefore, we write

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) \le \mathbb{P}(\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N} \cap \mathcal{E}_{\varepsilon,N}) + \mathbb{P}(\mathcal{C}^c_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N} \cap \mathcal{E}_{\varepsilon,N}) + \mathbb{P}(\mathcal{D}^c_{\varepsilon,N}) + \mathbb{P}(\mathcal{E}^c_{\varepsilon,N}), \tag{A.1.4}$$

and our strategy to prove the lemma is that we show that each of the four terms on the right-hand side of (A.1.4) is at most $\varepsilon/4$.

Nodes 1 and 2 are connected to giant nodes only. On $\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N}$ at least one of the $2q_\varepsilon$ stubs is attached to a stub of the nodes $D_{(1)}, \ldots, D_{(N-k_\varepsilon)}$. Hence, the first term on the right side of (A.1.4) satisfies

$$\mathbb{P}(\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N} \cap \mathcal{E}_{\varepsilon,N}) \le 2q_\varepsilon \mathbb{E}\left[\frac{1}{L_N} \sum_{n=1}^{N-k_\varepsilon} D_n \mathbf{1}_{\{\mathcal{E}_{\varepsilon,N}\}} \right] \le \varepsilon/4,$$

due to point (a) of $\mathcal{E}_{\varepsilon,N}$.

The giant nodes form a complete graph. We turn to the second term of (A.1.4). Recall that $\mathcal{C}^c_{\varepsilon,N}$ induces that no stubs of at least two giant nodes are attached to one another. Since we have at most N^2 pairs of giant nodes h_1 and h_2, the items (b), (c) of $\mathcal{E}_{\varepsilon,N}$ imply

$$\begin{aligned} \mathbb{P}(\mathcal{C}^c_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N} \cap \mathcal{E}_{\varepsilon,N}) \quad &\le \quad \mathbb{E}\left[N^2 \prod_{i=0}^{\lfloor D_{h_1}/2 \rfloor - 1} \left(1 - \frac{D_{h_2}}{L_N - 2i - 1} \right) \mathbf{1}_{\{h_1, h_2\,\text{giant}\}} \right] \\ &\le \quad N^2 \left(1 - \frac{c_\varepsilon u_N}{d_\varepsilon u_N} \right)^{c_\varepsilon u_N/2} \le N^2 \exp\left(-\frac{c_\varepsilon^2 u_N}{2d_\varepsilon} \right) \le \varepsilon/4, \end{aligned} \tag{A.1.5}$$

for large enough N, because $u_N = N^{1/(\tau-1)+o(1)}$.

Nodes 1 and 2 have small degree. The third term on the right-hand side of (A.1.4) is at most $\varepsilon/4$, because

$$\mathbb{P}(\mathcal{D}^c_{\varepsilon,N}) \le 2\mathbb{P}(D_1 > q_\varepsilon) \le 2\varepsilon/8 = \varepsilon/4. \tag{A.1.6}$$

The order statistics. It remains to estimate the last term on the right side of (A.1.4). Clearly,

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}^c_{\varepsilon,N}\right) \leq\ & \mathbb{P}\left(\sum_{n=1}^{N-k_\varepsilon} D_{(n)} > \tfrac{\varepsilon}{8q_\varepsilon} L_N\right) && (a)\\ &+\mathbb{P}\left(D_{(N-k_\varepsilon+1)} < c_\varepsilon u_N\right) && (b)\\ &+\mathbb{P}\left(L_N > d_\varepsilon u_N\right). && (c)\end{aligned} \tag{A.1.7}$$

We will consequently show that each term in the above expression is at most $\varepsilon/12$. Let a_ε and $b_\varepsilon > 0$ be as in (2.8) and (2.9), then we can decompose the first term on the right-hand side of (A.1.7) as

$$\begin{aligned}\mathbb{P}\left(\sum_{n=1}^{N-k_\varepsilon} D_{(n)} > \frac{\varepsilon}{8q_\varepsilon} L_N\right) &\leq \mathbb{P}\left(L_N < a_\varepsilon u_N\right) + \mathbb{P}\left(\sum_{n=1}^{N-k_\varepsilon} D_{(n)} > \frac{\varepsilon}{8q_\varepsilon} a_\varepsilon u_N\right)\\ &\leq \mathbb{P}\left(L_N < a_\varepsilon u_N\right) + \mathbb{P}\left(D_{(N-k_\varepsilon+1)} > b_\varepsilon u_N\right)\\ &\qquad +\mathbb{P}\left(\sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i < b_\varepsilon u_N\}} > \frac{\varepsilon}{8q_\varepsilon} a_\varepsilon u_N\right).\end{aligned} \tag{A.1.8}$$

From the Markov inequality,

$$\mathbb{P}\left(\sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i < b_\varepsilon u_N\}} > \frac{\varepsilon}{8q_\varepsilon} a_\varepsilon u_N\right) \leq \frac{8q_\varepsilon N \mathbb{E}\left[D\mathbf{1}_{\{D<b_\varepsilon u_N\}}\right]}{\varepsilon a_\varepsilon u_N}. \tag{A.1.9}$$

Since $1 - F(x)$ varies regularly with exponent $\tau - 1$, we have, by [9, Theorem 1(b), p. 281],

$$\mathbb{E}\left[D\mathbf{1}_{\{D<b_\varepsilon u_N\}}\right] = \sum_{k=0}^{\lfloor b_\varepsilon u_N \rfloor} \left[1 - F(k)\right] \leq 2(2-\tau) b_\varepsilon u_N \left[1 - F(b_\varepsilon u_N)\right], \tag{A.1.10}$$

for large enough N. Due to (2.1), for large enough N, we have also

$$N\left[1 - F(u_N)\right] \leq 2. \tag{A.1.11}$$

Substituting (A.1.10) and (A.1.11) in (A.1.9), we obtain

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{N} D_i \mathbf{1}_{\{D_i < b_\varepsilon u_N\}} > \frac{\varepsilon}{8q_\varepsilon} a_\varepsilon u_N\right) &\leq \frac{16 q_\varepsilon N (2-\tau) b_\varepsilon u_N \left[1 - F(b_\varepsilon u_N)\right]}{\varepsilon u_N a_\varepsilon}\\ &\leq \frac{32 q_\varepsilon (2-\tau) b_\varepsilon \left[1 - F(b_\varepsilon u_N)\right]}{\varepsilon a_\varepsilon \left[1 - F(u_N)\right]},\end{aligned} \tag{A.1.12}$$

for large enough N. From the regular variation of $1 - F(x)$,

$$\lim_{N\to\infty} \frac{1 - F(b_\varepsilon u_N)}{1 - F(u_N)} = (b_\varepsilon)^{1-\tau}.$$

Hence the right-hand side of (A.1.12) is at most

$$\frac{64 q_\varepsilon (2-\tau) (b_\varepsilon)^{2-\tau}}{\varepsilon a_\varepsilon} \leq \varepsilon/36,$$

for sufficiently large N, by the definition of b_ε in (2.9). We now show that the second term on the right side of (A.1.8) is at most $\varepsilon/36$. Since $D_{(N-k_\varepsilon+1)}/u_N$ converges in distribution to ξ_{k_ε}, we find from (2.10),

$$\mathbb{P}\left(D_{(N-k_\varepsilon+1)} > b_\varepsilon u_N\right) \leq \mathbb{P}\left(\xi_{k_\varepsilon} > b_\varepsilon/2\right) + \varepsilon/72 \leq \varepsilon/36,$$

for large enough N. Similarly, by the definition of a_ε, in (2.8), we have

$$\mathbb{P}\left(L_N < a_\varepsilon u_N\right) \le \varepsilon/36.$$

Each of the three terms on the right side of (A.1.8) is at most $\varepsilon/36$, so that the term (A.1.7)(a) is at most $\varepsilon/12$.

The upper bound for (A.1.7)(b), i.e., the bound

$$\mathbb{P}\left(D_{(N-k_\varepsilon+1)} < c_\varepsilon u_N\right) < \varepsilon/12,$$

is an easy consequence of the distributional convergence of $D_{(N-k_\varepsilon+1)}/u_N$ to ξ_{k_ε} and the definition of c_ε. Similarly, we obtain the upper bound for the term in (A.1.7)(c), i.e.,

$$\mathbb{P}\left(L_N > d_\varepsilon u_N\right) < \varepsilon/12,$$

from the convergence in distribution of L_N/u_N to η and the definition of d_ε.

Thus we have shown that $\mathbb{P}(\mathcal{E}^c_{\varepsilon,N}) < \varepsilon/4$. This completes the proof of Lemma 2.2. $\square$

A.2 Proof of Lemma 3.1

In this section we restate Lemma 3.1 and give a proof.

Lemma A.2.1 *For each $\varepsilon > 0$, there exists N_ε such that for all $N \ge N_\varepsilon$,*

$$\mathbb{P}(\mathcal{G}^c_{\varepsilon,N}) < \varepsilon. \tag{A.2.1}$$

Proof. From (3.8),

$$\mathbb{P}\big(\mathcal{G}^c_{\varepsilon,N}\big) < \mathbb{P}\big(\mathcal{D}^c_{\varepsilon,N}\big) + \mathbb{P}\big(\mathcal{H}^c_{\varepsilon,N}\big) + \mathbb{P}\big(\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{H}_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N}\big)\,. \tag{A.2.2}$$

We will bound each term on the right hand side of (A.2.2) separately.

From (A.1.6), and because the definition of $\mathcal{D}_{\varepsilon,N}$ is unaltered, the bound $\mathbb{P}\big(\mathcal{D}^c_{\varepsilon,N}\big) < \varepsilon/4$ is immediate.

For $\mathbb{P}\big(\mathcal{H}^c_{\varepsilon,N}\big)$, we will show that the total number of stubs M_N is of the order $\ell(N)N^{1+\alpha(2-\tau)}$ and that the total number K_N of stubs attached to normal nodes is of order $\ell(N)N^{1+\beta(2-\tau)}$. We start with the first statement. Bound

$$M_N = \sum_{i=1}^N D_i^{(N)} \ge \frac{1}{2}N^\alpha \sum_{i=1}^N \mathbf{1}_{\{D_i^{(N)} > \frac{1}{2}N^\alpha\}}.$$

The sum of indicators is distributed as a binomial random variable V_N with parameters N and $N^{\alpha(1-\tau)}\ell(N)$, because

$$\mathbb{P}\left(D^{(N)} > \frac{1}{2}N^\alpha\right) = N^{\alpha(1-\tau)}\ell(N).$$

We use Janson's inequality (compare (3.23)), on the binomial random variable V_N, with expectation $N^{1+\alpha(1-\tau)}\ell(N)$, and with $t = N^{1+\alpha(1-\tau)}\ell(N)/2$, to obtain:

$$\mathbb{P}(|V_N - \mathbb{E}[V_N]| \ge t) \le 2\exp\left(-\frac{t^2}{2\left(\mathbb{E}[V_N] + t/3\right)}\right) = 2\exp\left(-\frac{3}{28}\ell(N)N^{1+\alpha(1-\tau)}\right) < \varepsilon/8,$$

for N sufficiently large. Therefore, with probability at least $1-\varepsilon/8$, we have, for all $\delta > 0$,

$$M_N > \frac{1}{2}N^\alpha V_N \ge N^{1+\alpha(2-\tau)}\underline{\ell}(N), \tag{A.2.3}$$

for N sufficiently large and some slowly varying function $\underline{\ell}(N)$.

The mean degree $\mathbb{E}[M_N]$ is given by (3.3). Thus, by the Markov inequality,

$$\mathbb{P}\left(\sum_{n=1}^{N} D_n^{(N)} > \frac{8}{\varepsilon}\ell(N)N^{1+\alpha(2-\tau)}\right) \leq \frac{\varepsilon}{8},$$

so that with probability at least $1-\varepsilon/8$, we have that

$$M_N \leq N^{1+\alpha(2-\tau)}\bar{\ell}(N), \tag{A.2.4}$$

for some slowly varying function $\bar{\ell}(N)$. Similarly, the mean degree of a normal node is

$$\mathbb{E}\Big[D^{(N)}\mathbf{1}_{\{D\leq N^{\beta}\}}\Big] = \sum_{n=1}^{\lfloor N^{\beta}\rfloor} \mathbb{P}(D \geq n | D < N^{\alpha}) = N^{\beta(2-\tau)}\ell(N),$$

so that in exactly the same way, we find from the Markov inequality, that with probability at least $1-\varepsilon/8$,

$$K_N \leq N^{1+\beta(2-\tau)}\ell(N). \tag{A.2.5}$$

The inequalities (A.2.3),(A.2.4) and (A.2.5) together imply that

$$\mathbb{P}\big(\mathcal{H}^c_{\varepsilon,N}\big) \leq 3\varepsilon/8.$$

We finally turn to $\mathbb{P}\big(\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{H}_{\varepsilon,N}\big)$. From the derivation above, we find that, on $\mathcal{H}_{\varepsilon,N}$, the fraction of the contribution of stubs from normal nodes and giant nodes is at most

$$\frac{\ell(N)N^{1+\beta(2-\tau)}}{\bar{\ell}(N)N^{1+\alpha(2-\tau)}} = \frac{\ell(N)}{\bar{\ell}(N)}N^{(2-\tau)(\beta-\alpha)}.$$

Since $\beta < \alpha$ and $\tau \in (1,2)$ the above ratio tends to 0, as $N \to \infty$. Thus the total number K_N of stubs of the normal nodes is negligible with respect to M_N on the event $\mathcal{H}_{\varepsilon,N}$. This implies that, with probability at least $1-\varepsilon/4$, each stub of nodes 1 and 2 is attached to a stub of a giant node on the event $\mathcal{H}_{\varepsilon,N}$. Therefore, we have showed

$$\mathbb{P}\big(\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{H}_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N}\big) < \varepsilon/4. \tag{A.2.6}$$

Since $2\varepsilon/4 + 3\varepsilon/8 < \varepsilon$, the lemma is proved. □

A.3 Proof of Lemma 4.1

In this section we restate Lemma 4.1 and give a proof.

Lemma A.3.1 *For each $\varepsilon > 0$, there exists N_ε such that for all $N \geq N_\varepsilon$,*

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) < \varepsilon. \tag{A.3.1}$$

Proof. The proof is a slight adaptation of the proof of Lemma 2.2 in Section A.1. We use that

$$\mathbb{P}(\mathcal{A}^c_{\varepsilon,N}) \leq \mathbb{P}(\mathcal{B}^c_{\varepsilon,N} \cap \mathcal{D}_{\varepsilon,N}) + \mathbb{P}(\mathcal{C}^c_{\varepsilon,N}) + \mathbb{P}(\mathcal{D}^c_{\varepsilon,N}), \tag{A.3.2}$$

and bound each of the three terms. The bound on $\mathbb{P}(\mathcal{D}^c_{\varepsilon,N})$ is identical to the one in (A.1.6), and will be omitted here.

We next show that $\mathbb{P}(\mathcal{C}^c_{\varepsilon,N}) \leq \frac{\varepsilon}{3}$. First observe that since $\tau = 2$,

$$\mathbb{P}(L_N \geq N^{1+\delta}) \leq \varepsilon/6.$$

Recall that $\mathcal{C}^c_{\varepsilon,N}$ implies that no stubs of at least two giant nodes are attached to one another. Since there are at most N^2 pairs of giant nodes h_1 and h_2, we can use a similar bound as in (A.1.5), to obtain

$$\begin{aligned}\mathbb{P}(\mathcal{C}^c_{\varepsilon,N}) &\leq \mathbb{E}\left(N^2 \prod_{i=0}^{\lfloor D_{h_1}/2\rfloor -1}\left(1-\tfrac{D_{h_2}}{L_N-2i-1}\right)\right) \leq N^2\left(1-\tfrac{N^{\frac{1}{2}+\delta}}{N^{1+\delta}}\right)^{\frac{1}{2}N^{\frac{1}{2}+\delta}} + \tfrac{\varepsilon}{6}\\ &\leq N^2 e^{-\frac{1}{2}N^{\delta}} + \tfrac{\varepsilon}{6} \leq \tfrac{\varepsilon}{3},\end{aligned}$$

for large enough N.

We finally show that $\mathbb{P}(\mathcal{B}^c_{\varepsilon,N}\cap\mathcal{D}_{\varepsilon,N}) \leq \frac{\varepsilon}{3}$. The event $\mathcal{B}^c_{\varepsilon,N}$ occurs if there exists a stub at node 1 or node 2 which is connected to a stub of a normal node. For $i=1,2$ and $j\leq D_i$, let $\{[i.j]\to[n]\}$ denote the event that the j^{th} stub of the i^{th} node is connected to a stub of a normal node. Let K_N denote the total number of stubs of the normal nodes. Then, clearly,

$$\mathbb{P}(\mathcal{B}^c_{\varepsilon,N}\cap\mathcal{D}_{\varepsilon,N}) \quad \leq 2\mathbb{P}\left(\mathcal{D}_{\varepsilon,N}\cap\bigcup_{j=1}^{D_1}\{[1.j]\to[n]\}\right) \leq 2\mathbb{E}\left[D_1\tfrac{K_N}{L_N}\mathbf{1}_{\mathcal{D}_{\varepsilon,N}}\right] \leq 2q_\varepsilon\mathbb{E}\left[\tfrac{K_N}{L_N}\right].$$

Therefore, it suffices to prove that $\mathbb{E}\left[\frac{K_N}{L_N}\right]\to 0$. This is what we will do in the remainder of this proof. We first bound

$$L_N \geq D_{(N)} \geq \varepsilon_N u_N, \tag{A.3.3}$$

where u_N is such that (2.1) holds, and $\varepsilon_N\downarrow 0$ will be determined later on. To compute u_N, we use (4.8) to obtain

$$N[1-F(u_N)] = N\frac{\ell(u_N)}{u_N} = 1+o(1). \tag{A.3.4}$$

A tedious computation using

$$\ell(u_N) = \frac{(\log u_N)^{\log\log u_N -1}\log\log u_N}{u_N},$$

yields

$$u_N = Ne^{(\log\log N)^2-\log\log N+o(\log\log N)}. \tag{A.3.5}$$

Furthermore, since $K_N\leq L_N$,

$$\mathbb{E}\left[\frac{K_N}{L_N}\right] \leq (\varepsilon_N u_N)^{-1}\mathbb{E}[K_N] + \mathbb{P}(L_N\leq\varepsilon_N u_N). \tag{A.3.6}$$

The second term is $o(1)$ for any $\varepsilon_N\downarrow 0$, and for the first term, we compute

$$\mathbb{E}[K_N] \leq N\sum_{i=1}^{N^{\frac{1}{2}+\delta}}[1-F(i)]. \tag{A.3.7}$$

We now use that for any $y>x^*$,

$$\begin{aligned}\sum_{i=x^*}^{y}[1-F(i)] &= c\sum_{i=x^*}^{y}\frac{(\log i)^{\log\log i-1}\log\log i}{i} \leq c\int_{x^*-1}^{y}\frac{(\log x)^{\log\log x-1}\log\log x}{x}dx\\ &= c\int_{\log(x^*-1)}^{\log y}(\log y)y^{\log y-1}dy \leq c'e^{(\log\log y)^2}+\mathcal{O}(1).\end{aligned} \tag{A.3.8}$$

Applying this to $y=N^{\frac{1}{2}+\delta}$, we obtain

$$\mathbb{E}[K_N] \leq c'Ne^{(\log\log(N^{\frac{1}{2}+\delta}))^2}+\mathcal{O}(N) = Ne^{(\log\log N)^2+2\log(\frac{1}{2}+\delta)\log\log N+\mathcal{O}(1)}, \tag{A.3.9}$$

so that, using (A.3.5),

$$(\varepsilon_N u_N)^{-1}\mathbb{E}[K_N] = \varepsilon_N^{-1} \exp\Big[\big(2\log(\frac{1}{2}+\delta)+1\big)\log\log N + o(\log\log N)\Big] = o(1), \tag{A.3.10}$$

when $\delta < 1$ is so small that $2\log(\frac{1}{2}+\delta)+1<0$ and we take

$$\varepsilon_N = \exp\Big[\frac{1}{2}\big(2\log(\frac{1}{2}+\delta)+1\big)\log\log N\Big] \to 0. \tag{A.3.11}$$

This completes the proof that $\mathbb{P}(\mathcal{B}^c_{\varepsilon,N}\cap\mathcal{D}_{\varepsilon,N}) = o(1)$, and thus the proof of Lemma A.3.1. $\square$

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