

RECURSIVE TILINGS AND SPACE-FILLING CURVES WITH LITTLE FRAGMENTATION

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ABSTRACT. This paper defines the Arrwwid number of a recursive tiling (or space-filling curve) as the smallest number a such that any ball Q can be covered by a tiles (or curve fragments) with total volume $O(\text{volume}(Q))$. Recursive tilings and space-filling curves with low Arrwwid numbers can be applied to optimize disk, memory or server access patterns when processing sets of points in \mathbb{R}^d . This paper presents recursive tilings and space-filling curves with optimal Arrwwid numbers. For $d \geq 3$, we see that regular cube tilings and space-filling curves cannot have optimal Arrwwid number, and we see how to construct alternatives with better Arrwwid numbers.

1 Introduction

1.1 The problem

Consider a set of data points in a bounded region U of \mathbb{R}^2 , stored on disk. A standard operation on such point sets is to retrieve all points that lie inside a certain query range, for example a circle or a square. To prevent large delays because of disk head movements while answering such queries, it is desirable that the points are stored on disk in a clustered way [2, 10, 11, 12, 16]. Similar considerations arise when storing spatial data in certain types of distributed networks [21] or when scanning spatial objects to render them as a raster image; in the latter case it is desirable that the pixels that cover any particular object are scanned in a clustered way, so that the object does not have to be brought into cache too often [23]. For ease of explanation, we focus on the application of clustering to storing points on disks.

We could try to achieve a good clustering in the following way. We divide U into *tiles*. The tiles could, for example, form a regular grid of hexagons (see Figure 1). Now we store the points in each tile as a contiguous block on disk. To answer a query, say with a region Q bounded by a circle, we compute which tiles intersect Q . For each intersecting tile, we move the disk read head to the position where the first point in that tile is stored, and then we retrieve all points in the tile by just scanning them sequentially without further delays from disk head movements. Since some of the tiles that intersect Q may lie partially outside Q , some of the points thus retrieved may be *false answers*: they are no answers to our query and need to be filtered out in post-processing.

The approach sketched above may work pretty well, provided the following conditions are met:

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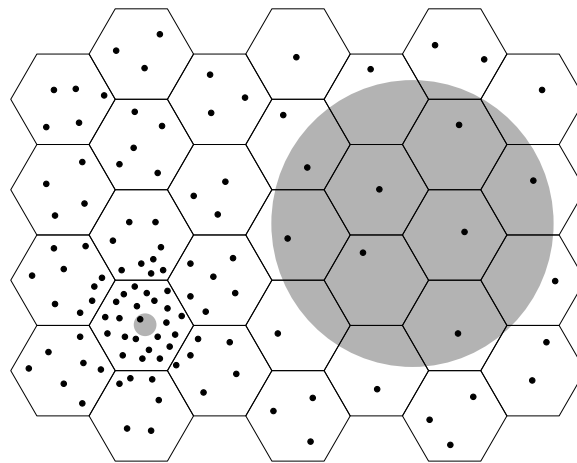


Figure 1: Sorting points into (non-)recursive tiles. Queries with small query ranges (shaded disk on the left) may necessitate scanning the full contents of a crowded tile. Queries with large ranges (shaded disk on the right) may necessitate looking up many tiles.

1. We can compute efficiently which tiles intersect Q .
2. We can figure out efficiently where the contents of these tiles are stored on disk (for example, using a small index to be kept in main memory).
3. A couple of tiles suffice to cover Q (so that we do not have to move the disk read head to the starting point of another tile too often).
4. The tiles that cover Q are not much larger than Q (so that they do not contain too many false answers).

These conditions are subject to a trade-off. On one extreme, we could use only one tile that covers all of U (making the first three conditions trivial), bring the disk head into position only once, and scan all data points—including lots of false answers. On the other extreme, we could make the tiles so small that each of them contains very few points and we get very few false answers, but then we may get large delays because of disk head movements when going from tile to tile; the first and second condition may also be harder to satisfy in this case. The best balance is to be found somewhere between these extremes, but where?

If we choose any fixed tile size, then the third and the fourth condition may be hard to meet if the density of the points stored varies by region or in time (due to updates). In some regions of the data set (or at some times) tiles may then contain many data points. When the query range Q is a small region inside such a crowded tile, we would need to retrieve all points in the tile, including many false answers (see Figure 1). But in other regions (or at other times) there may be many tiles with very few data points each. Queries that cover many such tiles (see Figure 1) may incur an overhead for retrieving many separate tiles that is disproportionally large relative to the number of data points found.

To avoid this problem, we can use *recursive* tilings. A recursive tiling is a subdivision of U into tiles, that are each subdivided into tiles recursively. For example, Figure 2 shows

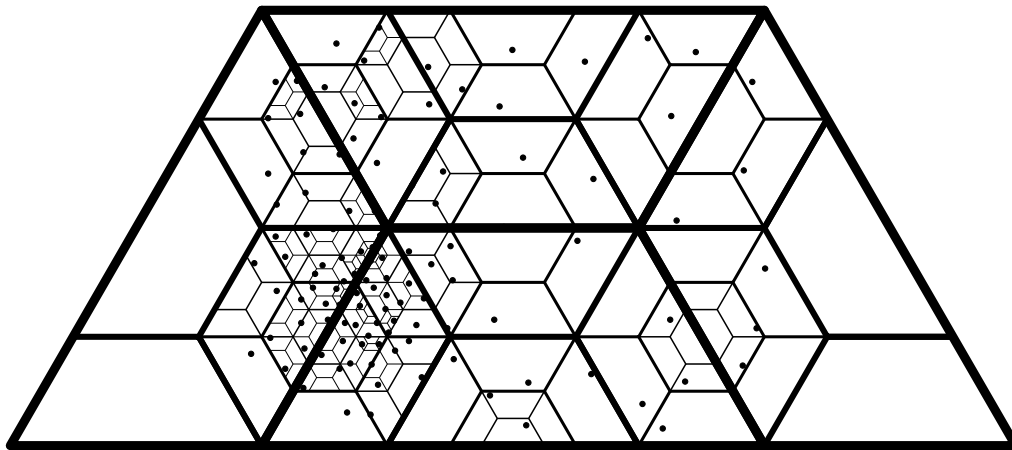


Figure 2: A recursive tiling into trapezoids.

a recursive tiling based on trapezoids, expanded down to the level where each tile contains at most one data point. We store the data points in such a way that for each tile, on any level of recursion, the data points within that tile are stored as a contiguous sequence on the disk. Hopefully, if we get it right, we can now cover the region within any circle Q with a small set of tiles from the level of recursion where the tiles have size proportional to Q . Thus we would satisfy condition three and four at the same time, and because we need to retrieve only few tiles, we could also hope to satisfy condition one and two. Considering that moving the disk head once may easily cost as much time as scanning ten thousands of points from disk, it is really important to keep the number of tiles small. Hence the topic of this study: what recursive tilings make it possible to cover any disk-shaped query region Q with the smallest possible number of tiles, while still making sure that the total size of the tiles is at most some constant times the size of Q ?

In this paper, we study this question for two- and three-dimensional settings, and we give some basic results for higher dimensions. We also study what can be achieved by controlling the order in which tiles are stored. Note that until now, we only required that the data points within each tile are stored contiguously, but we did not assume anything about the order in which the subtiles of any tile are stored with respect to each other. A well-chosen order may have the result that some of the tiles used to cover the query range are stored consecutively on disk, thus eliminating the need to move the disk head when going from one tile to the next.

1.2 The Arrwid number

To be able to make our problem statement more precise, we define the Arrwid number of a recursive tiling of a region U as follows:

The Arrwid number is the smallest number a such that there is a constant c such that any disk Q that lies entirely in U can be covered by a tiles with total area at most $c \cdot \text{area}(Q)$.

Thus the Arrwwid number is the maximum number of tiles needed to cover any disk if we require the tiles to be relatively small. We call the constant c in the definition the *cover ratio*. The definition is essentially from Asano, Ranjan, Roos, Welzl and Widmayer [2], and we name it *Arrwwid number* in their honour. The topic of our study is thus: what recursive tilings have small Arrwwid numbers?

When a recursive tiling is enhanced with a recursive definition of how the subtiles within each tile are ordered relative to each other, the result constitutes the definition of a *recursive scanning order* or a *recursive space-filling curve* (this paper uses these two terms interchangeably; they are explained in more detail in Section 3.1). The Arrwwid number of a space-filling curve that fills a region U is defined as follows:

The Arrwwid number is the smallest number a such that there is a constant c such that any disk Q that lies entirely in U can be covered by a curve fragments (that is, sets of consecutive tiles) with total area at most $c \cdot \text{area}(Q)$.

In the above definitions we could exchange disks for squares: this would only affect the cover ratios c but not the Arrwwid numbers a . To see this, observe that if the Arrwwid number is a with respect to disks, with cover ratio c , then any square Q can be covered by a disk of area $\frac{\pi}{2} \cdot \text{area}(Q)$ which can subsequently be covered by at most a tiles of total area at most $c \cdot \frac{\pi}{2} \cdot \text{area}(Q)$. Therefore the Arrwwid number with respect to squares is at most the Arrwwid number with respect to disks. Furthermore, if the Arrwwid number is a with respect to squares, with cover ratio c , then any disk Q can be covered by a square of area $\frac{4}{\pi} \cdot \text{area}(Q)$, which can subsequently be covered by at most a tiles of total area at most $c \cdot \frac{4}{\pi} \cdot \text{area}(Q)$. Therefore the Arrwwid number with respect to disks is at most the Arrwwid number with respect to squares. It follows that the Arrwwid number is the same regardless of whether disk or square ranges are considered.

The above definitions naturally extend to higher dimensions by replacing disks by balls, area by volume, and squares by (hyper)cubes.

1.3 Related work

Recursive tilings are widely used in quadtrees and its higher-dimensional variants, such as octrees [20]. Typically, tiles are subdivided recursively until each tile only contains at most a constant number of data points. In some variants, the tiles are then stored by sorting them and indexing them along a space-filling curve: this results in a so-called *linear quadtree* [7]. This paper can also be understood as an investigation into whether, for some applications, it could be advantageous to replace the tiling that underlies a quadtree (squares are subdivided into four squares) by another tiling, or to replace the Z-order space-filling curve that usually underlies a linear quadtree by another space-filling curve.

Jagadish, Kumar and Moon et al. studied how well space-filling curves succeed in keeping the number of fragments needed to cover a query range low [10, 11, 12, 16]. The curve quality measures in their work are based on the number of fragments needed to cover a query range, averaged over a selection of query ranges that depends on the underlying tiling. As a result their results can only be used to analyse space-filling curves with the

same underlying tiling; in particular they assume a tiling that subdivides squares into four smaller squares, expanded down to a fixed level of recursion. This class of curves does, in fact, include several well-known curves such as the Hilbert curve [9] and Z-order, also known as Morton order or Lebesgue order [13]. However, it does not include, for example, Peano's curve [18], which is based on subdividing squares into *nine* squares and seems to be the curve of choice in certain applications [3, 23].

In contrast, the Arrwwid number, as defined above, admits a comparison between curves with different underlying tilings. Nevertheless Asano et al., too, only studied curves based on the tiling with four squares per square [2]. The Arrwwid number of such a tiling is four. Asano et al. studied what can be achieved by controlling the order in which the tiles are stored: they presented an ordering scheme, the AR^2W^2 space-filling curve, that guarantees that whenever four tiles are needed, at least two of them are consecutive in the order. Thus these four tiles can be divided into at most three sets such that the tiles within each set are consecutive, and thus the Arrwwid number of the AR^2W^2 space-filling curve is three. Asano et al. also proved that one cannot do better: no ordering scheme of this particular tiling has Arrwwid number less than three.

In practice, the effectiveness of a curve in optimizing disk access time will depend on the cost of “jumping” to another fragment relative to the cost of scanning a false answer. Bugnion et al. [4] studied average disk access times as a function of the cost of jumping relative to scanning. Their work considers scanning orders that follow square grid tilings in such a way that each tile touches the previous tile in the scanning order—all curves mentioned so far, except Z-order, are like that. In such tilings, consecutive tiles may be *horizontally connected* (sharing a vertical edge), *vertically connected* (sharing a horizontal edge) or they may be *diagonally connected* (touching only in a vertex). Bugnion et al. analysed the performance of scanning orders under the assumption that they resemble random walks with a given proportion of horizontal, vertical and diagonal connections (and some further simplifying assumptions). Their results suggest that diagonal connections (as in the AR^2W^2 curve) are harmful for random walks, regardless of the cost of jumping relative to scanning.

1.4 Results

In this paper, we extend the scope of our knowledge on Arrwwid numbers to different tilings, which do not necessarily have four squares per square, and to higher dimensions. The results are the following: in two dimensions, no recursive tiling and no recursive space-filling curve has Arrwwid number better than three if the tiles are simply connected regions in the plane. This paper presents recursive tilings with Arrwwid number three (regardless of the ordering), as well as an alternative square-based space-filling curve with Arrwwid number three but without the diagonal connections that are suspected to harm the performance of the AR^2W^2 curve.

In d dimensions, no recursive tiling has Arrwwid number better than $d+1$. We prove that in three dimensions, putting the tiles in a certain order will not make it possible to get below this bound if the tiles are convex polyhedra. There are recursive tilings with fractal-shaped tiles that have Arrwwid number $d+1$, and tilings with rectangular blocks that have

Arrwid number $2 \cdot (\sqrt{3})^{d-1}$ (if d is odd) or $(\sqrt{3})^d$ (if d is even). In two dimensions, space-filling curves with optimal Arrwid number can be constructed by defining a good order on a regular square-based tiling, but this does not generalize to three (or more) dimensions. Regular (hyper)cube-based tilings have Arrwid number 2^d , and any space-filling curve based on such a tiling has Arrwid number at least $2^d - 1$. For $d \geq 3$ this is not as good as what can be achieved with rectangular blocks.

2 Two-dimensional tilings

2.1 Defining recursive tilings

A recursive tiling of a region U (called the *unit tile*) in the plane is defined by a finite set of recursive tiling rules. Each rule specifies (i) the shape of the region to be tiled; (ii) how this region is tiled with a fixed number of tiles; (iii) which rules should be applied to subdivide each of these tiles recursively. Figure 3 shows a number of examples. Each rule is identified by a letter, and depicted by drawing the shape of its region, the tiles, and within the tiles, the letters of the rules to be applied to them; each letter is rotated and/or mirrored to reflect the transformations that should be applied to the subtiles of the tile. Next to each set of rules we see the tiling that is produced after expanding the recursion down to tiles of a few millimeters.

Simple tilings are recursive tilings that only use one rule (Figure 3(a,b,c)). Using a term coined by Mandelbrot [15], we could also call them *pertilings*. In contrast, *composite* tilings are recursive tilings that use multiple rules (Figure 3(d,e)). We define *uniform* tilings as recursive tilings in which all tiles have the same shape, and each rule subdivides such a shape into an equal number of tiles of equal size (Figure 3(a,b,d)). By *square*, *rectangular*, *triangular* and *fractal* tilings we mean tilings whose tiles are square, rectangular, triangular, or fractal-shaped. *Regular* recursive tilings are recursive tilings that form a fully regular grid or tiling when the recursion is expanded to any fixed depth (Figure 3(a)).

The *size* of a uniform tiling is the number of tiles in each rule. For any given (recursive or non-recursive) tiling, the *degree* of a point p in U is the maximum number of interior-disjoint tiles that meet in p . The *vertex degree* of a (recursive or non-recursive) tiling is the maximum degree of p over all points $p \in U$.

The results in this paper apply to simple as well as composite tilings, and uniform as well as non-uniform tilings. We only require that the number of recursion rules is finite, that each rule that defines the tiling starts with a finite region, and that the transformations that may be applied to the rules preserve similarity.

2.2 The Arrwid number of a tiling: basic bounds

Observation 1. *The uniform square tiling of Figure 3(a) has Arrwid number four.*

Proof. We first prove that the Arrwid number of the uniform square tiling is at most four. Let r be the radius of any disk Q . Since the tiles differ in width by a factor two from one

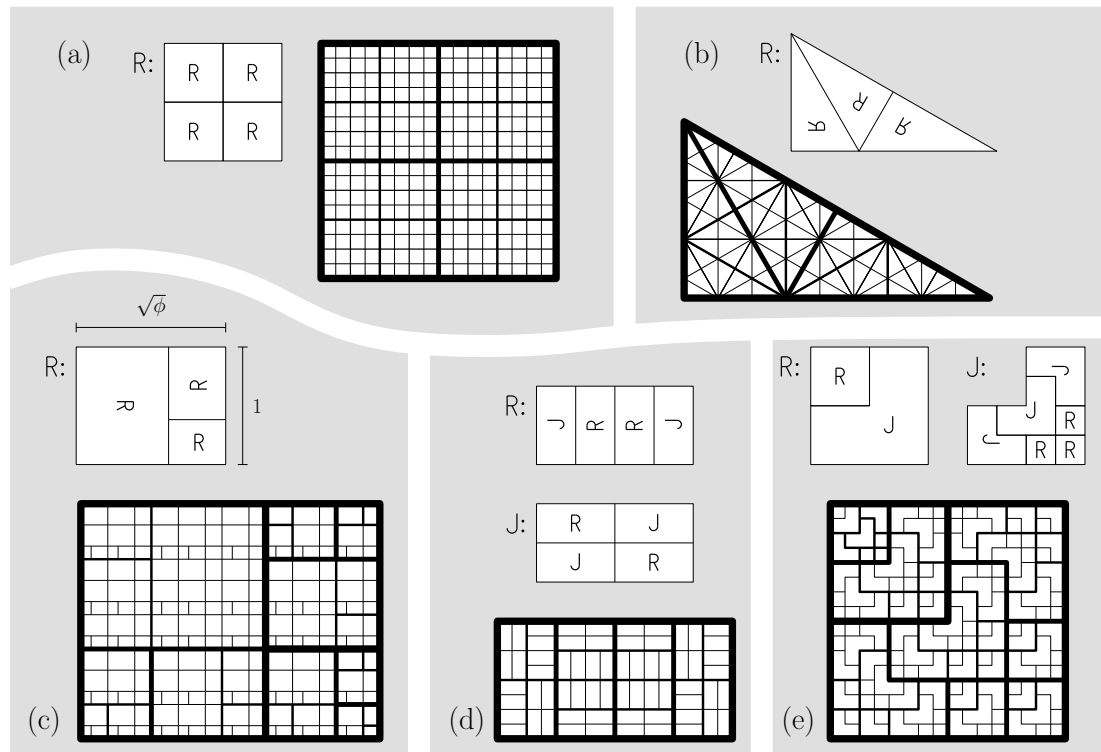


Figure 3: Examples of recursive tilings. (a) A simple, uniform, regular recursive tiling. (b) A simple, uniform recursive tiling. (c) A simple, non-uniform recursive tiling based on the golden ratio $\phi = \frac{1}{2}\sqrt{5} + \frac{1}{2}$. (d) Composite and uniform. (e) Composite, non-uniform.

level of recursion to the next, there is a level of recursion where the tiles have side lengths between $2r$ and $4r$, and hence, area between $4r^2$ and $16r^2$. The distance between any pair of horizontal lines of the grid formed by these tiles is at least $2r$, so Q is crossed by at most one horizontal grid line. By the same argument, Q is crossed by at most one vertical grid line. It follows that at most four tiles of this grid intersect Q , so Q can always be covered by at most four tiles with total area at most $4 \cdot 16r^2 = 64r^2$. Hence the Arrwwid number of the square tiling is at most four (with cover ratio at most $64r^2/\pi r^2 = 64/\pi$).

It remains to show that the Arrwwid number of the uniform square tiling cannot be three or less. Suppose the Arrwwid number would be at most three, so there is a constant c such that any disk of radius r can be covered by at most three tiles with total area at most $c\pi r^2$. Consider a fixed value r , a tile T with area more than $c\pi r^2$, and a disk Q of radius r placed in the middle of this tile T , that is, centered on the spot where the four subtiles of T meet. Thus Q intersects each of the four subtiles of T , and the only way to cover it with less than four tiles is by covering it with T itself—but the area of T is larger than $c\pi r^2$. This contradicts our assumptions and thus proves that the Arrwwid number of the uniform square tiling is at least four. \square

The above arguments illustrate that there is a relation between the vertex degree of a tiling and its Arrwwid number:

Observation 2. *The Arrwwid number of a recursive tiling cannot be lower than its vertex degree.*

This is because any sufficiently small circle centered on a point of maximum degree would yield a contradiction to the assumption that the Arrwwid number is lower.

The subdivision of the unit tile U into its subtiles creates at least one curve \mathcal{C} that forms the boundary between two subtiles and lies in the interior of U . Since tile sizes decrease by at least a constant factor with each level of recursion, further down in the recursion \mathcal{C} must eventually be subdivided by crossings or T-junctions with curves that divide the tiles on each side of \mathcal{C} . Thus we get points on \mathcal{C} of degree at least three. Hence we get the following:

Theorem 1. *Each recursive tiling of a two-dimensional region U has Arrwwid number at least three.*

We now ask: are there actually any recursive tilings with Arrwwid number three? In the next subsections, we answer this question with “yes”.

2.3 Recursifying non-recursive tilings

As observed above, the Arrwwid number of a tiling is at least the vertex degree of the tiling. Therefore, to find a tiling with Arrwwid number three, we need to find a tiling with vertex degree at most three. A regular hexagonal tiling has this property, but that tiling is not recursive: a hexagon cannot be subdivided into similar hexagons. We will now see how a recursive tiling can be obtained by recursively approximating a hexagon by smaller hexagons; this procedure will turn the boundaries of the hexagons into fractals. The technique itself is not new; however, I am currently not aware of any published general description such as the one given below.

The procedure is as follows. We start with a *coarse tiling* which is a regular tiling of large hexagonal tiles. We overlay this with a *fine tiling* of small hexagonal tiles, such that the fine tiling looks the same around each large tile (Figure 4(a)). Now we assign each small tile to a large tile, in such a way that the union of the small tiles assigned to each large tile approximates the shape of the large tile well. To accomplish this we assign each small tile that is completely contained in a single large tile T to T . The remaining small tiles are assigned according to a tie-breaking rule that ensures that for each large tile the union of the small tiles assigned to it has the same size and shape. In Figure 4(b) we do this as follows: when going clockwise around a large tile T , starting at three o'clock, we alternate between giving a small tile to a neighbour of T and assigning a small tile to T . All large tiles of the coarse grid are now replaced by the union of the small tiles assigned to them (Figure 4(c)), thus adding detail to the boundaries of the large tiles. We now replace the small tiles by scaled copies of the large tiles (which are no longer hexagons), and again replace the large tiles by the unions of the small tiles assigned to them. This adds even

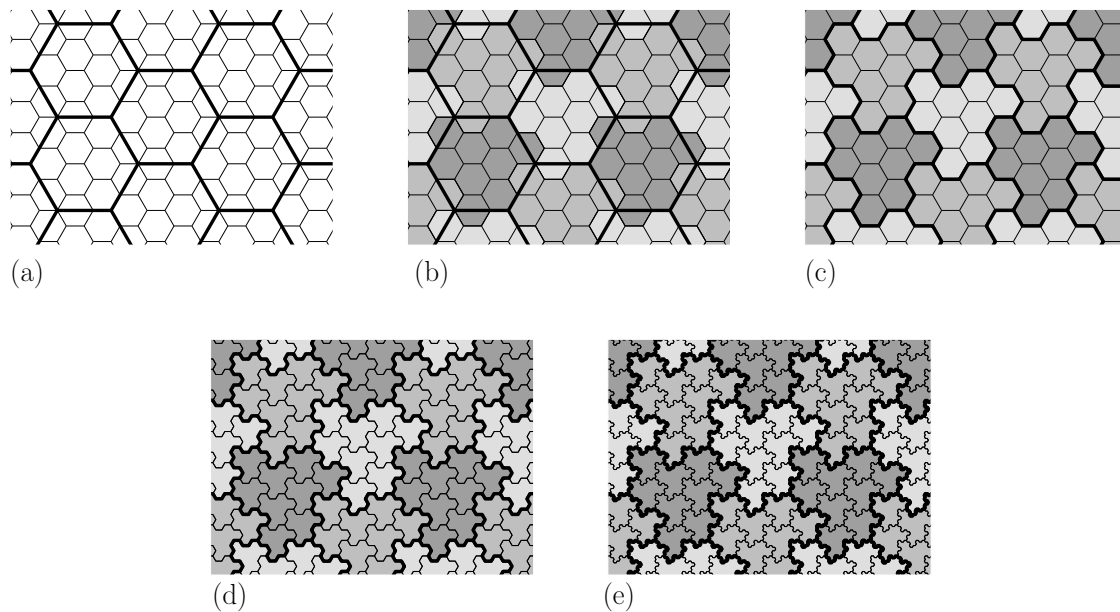


Figure 4: How to turn a hexagonal tiling into a recursive tiling of size nine with Arrwid number three.

more detail to the boundaries of the large tiles (Figure 4(d)). When we repeat this process ad infinitum, the boundaries of the tiles will converge to fractal shapes such that each large tile is tiled by nine scaled-down copies of itself (Figure 4(e)).

The transformation described above changed the shape of the tiles in the coarse tiling, but it did not change its topological structure. The vertex degree of the resulting tiling is still three, but unlike the original hexagonal tiling, our fractal tiling is recursive. In fact, we have produced a fractal tiling with Arrwid number three (we will see how to prove such things below).

The technique described above requires some care: it matters how well the small hexagonal tiles approximate the large hexagonal tiles. Figure 5 shows a recursification with seven small tiles per large tile, resulting in a tiling with shapes called Gosper islands [6]. Figure 6 shows an attempt with four small tiles per large tile. However, it fails to produce a fractal tiling with Arrwid number three: the approximation of large tiles by small tiles is so bad that the hexagonal tiling morphs into a regular grid of rhombuses with vertex degree four. Figure 7 shows another failed attempt: here the result is a tiling whose tiles have disconnected interiors, and it contains vertices where three different tiles meet the three largest connected components of a fourth tile.

Theorem 2. *A recursive tiling with Gosper islands has Arrwid number three.*

Proof. Assume that in the coarse tiling neighbouring vertices are at a distance u from each other. In the first stage of refinement, each segment e of a large tile boundary is replaced by three segments of length $u/\sqrt{7}$, the first and last of which make an angle of α with e

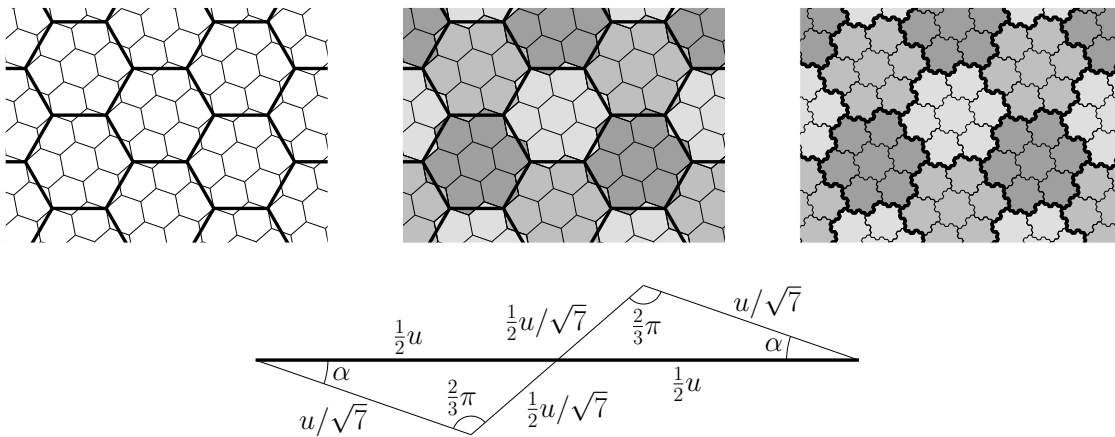


Figure 5: A recursive tiling of Gosper islands.

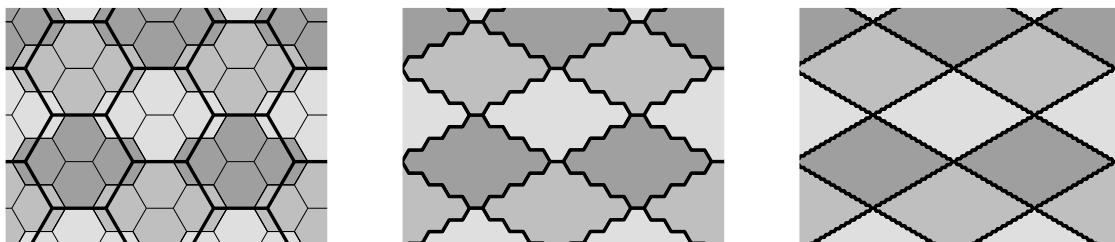


Figure 6: A failed attempt to turn a hexagonal tiling into a recursive tiling of size four with Arrwid number three: the Arrwid number turns out to be four.

where, by the law of sines, $\sin \alpha = \frac{1}{2}\sqrt{3}/\sqrt{7}$ (see Figure 5). The new boundary of the large tile thus stays within a distance $d_1 = \sin \alpha \cdot u/\sqrt{7} = \frac{u}{14}\sqrt{3}$ of the old boundary. Refining the tiles i times recursively thus keeps the boundary within a distance d_i that satisfies $d_i \leq \frac{u}{14}\sqrt{3} + d_{i-1}/\sqrt{7}$; for $i \rightarrow \infty$ this converges to $d_i = \sqrt{7}/(\sqrt{7}-1) \cdot \frac{u}{14}\sqrt{3} < 0.199 \cdot u$. Let s and s' be the radius of the smallest disk that intersects more than three large tiles in the original and in the recursified large tiling, respectively. We have $s = u/2$, and since the boundaries of any tile in the recursified tiling are within a distance of $0.199 \cdot u$ from the boundaries of the corresponding tile in the original tiling, we have $s' > s - 0.199 \cdot u = 0.301 \cdot u$.

Now consider any disk Q of radius r . There is a level of recursion in the recursified tiling where u , the distance between neighbouring vertices in the underlying hexagonal tiling, is between $r/0.301$ and $\sqrt{7} \cdot r/0.301$. Since $r \leq 0.301u < s'$, the disk Q intersects at most three of the tiles on this level of the recursified tiling, and each of these tiles has area $\frac{3}{2}\sqrt{3} \cdot u^2 \leq \frac{3}{2}\sqrt{3} \cdot 7r^2/0.301^2 < 201 \cdot r^2$ (equal to the hexagon whose position they are taking). It follows that Q can always be covered by three tiles of total area at most $603 \cdot r^2 < 192 \cdot \text{area}(Q)$; hence the Arrwid number of the Gosper tiling is three. \square

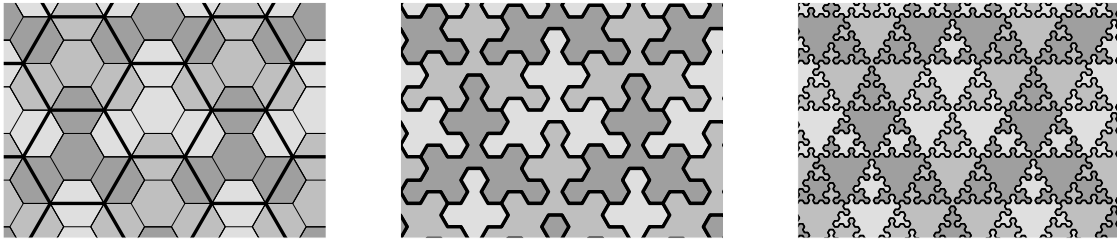


Figure 7: A failed attempt to turn a hexagonal tiling into a recursive tiling of size four with Arrwwid number three: the Arrwwid number turns out to be four.

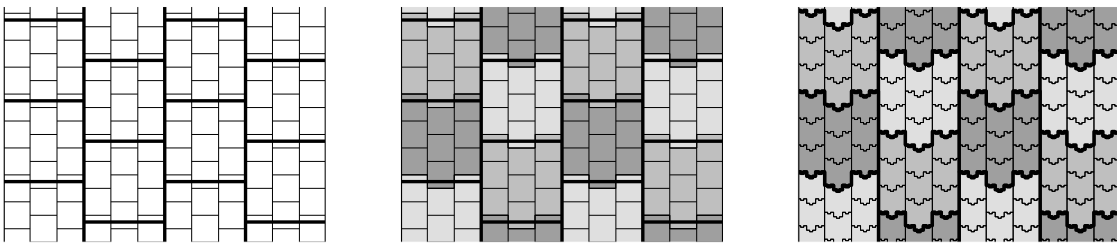


Figure 8: A recursive tiling with Arrwwid number three based on a “shifted” grid of squares.

In a similar way one could prove that the tiling of Figure 4(e) has Arrwwid number three. The recursification technique can also be applied to non-hexagonal tilings. A recursified grid of squares with five small tiles per large tile results in a tiling of cross-like shapes (see Mandelbrot [15], plate 49, or Teachout’s website [22]). Other examples with hexagons can be seen to underly the “generalized Gosper curves” of Akiyama et al. [1]. Figure 8 shows a recursified tiling based on a grid of squares of which each column is shifted by half a square’s height with respect to the next column; it has Arrwwid number three.

2.4 An optimal rectangular tiling

The fractal tilings are beautiful, but have one big practical disadvantage: for any given point it is hard to figure out in which tiles it lies. It would be quite challenging to sort any given set of data points into fractal tiles, so it would be hard to use such tilings to build a data structure. Therefore it may be interesting to look for a tiling with Arrwwid number three that uses simpler tiles, namely rectangles.

In a uniform rectangular tiling, each rule defines a subdivision of a large rectangle into, say, t similar, smaller rectangles which are scaled by a factor $1/\sqrt{t}$ with respect to the large rectangle. Let α be the width/height ratio of the large rectangle; without loss of generality, assume $\alpha \geq 1$.

For any horizontal line ℓ through the large rectangle, the smaller rectangles along ℓ must fill the full width of the large rectangle. This implies that for fixed α and t , the equation

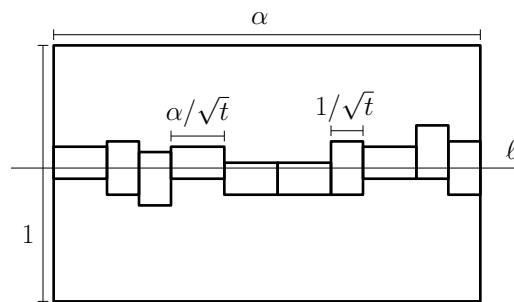


Figure 9: Along any line through the large rectangle, the smaller rectangles fill the full width.

$$\alpha n_{ww} + n_{hw} = \alpha\sqrt{t} \quad (1)$$

has at least one non-negative integer solution for the variables n_{ww} and n_{hw} : here, n_{ww} is the number of small rectangles on ℓ that have the same orientation as the larger rectangle, and n_{hw} is the number of small rectangles on ℓ that are rotated 90 degrees with respect to the orientation of the larger rectangle, see Figure 9. Similarly, considering vertical lines we find that

$$\alpha n_{wh} + n_{hh} = \sqrt{t} \quad (2)$$

must have a non-negative integer solution. In fact, to get an Arrwid number of three, for each of the variables n_{ww} , n_{hw} , n_{wh} and n_{hh} there must be a solution to the above equations where this variable is at least one. Otherwise the only way to subdivide the large rectangle would be to put the tiles in a regular rectangular grid, which would result in an Arrwid number of four.

Lemma 1. *The size t of any uniform rectangular tiling with Arrwid number three is a square number.*

Proof. For the sake of contradiction, assume that t is not a square number. Then any solution to Equation 2 must have $n_{wh} > 0$; this holds in particular for any solution with $n_{hh} > 0$. Fix $n_{wh} > 0$ and $n_{hh} > 0$ such that they satisfy Equation 2. We get:

$$\alpha = \frac{\sqrt{t} - n_{hh}}{n_{wh}}. \quad (3)$$

Substituting α in Equation 1 yields:

$$\frac{n_{ww}}{n_{wh}}\sqrt{t} - \frac{n_{ww}n_{hh}}{n_{wh}} + n_{hw} = \frac{t}{n_{wh}} - \frac{n_{hh}}{n_{wh}}\sqrt{t}. \quad (4)$$

Because \sqrt{t} is irrational, the terms that include \sqrt{t} must be equal, that is, $n_{ww} = -n_{hh}$, but then n_{ww} is negative: a contradiction. Hence t must be a square number. \square

Lemma 2. *The width/height ratio α of the rectangles in a uniform rectangular tiling of size t with Arrwid number three is a rational number with numerator at most \sqrt{t} and denominator less than \sqrt{t} .*

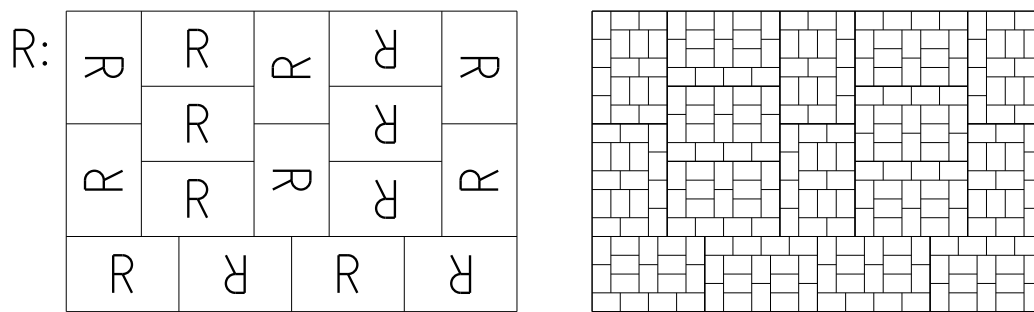


Figure 10: Definition of the Daun tiling, and level-two expansion. Figure 22 at the end of this article shows the level-three expansion.

Proof. We have $\alpha > 1$, otherwise we would get a regular grid of squares with Arrwid number four. Therefore the solution to Equation 2 for which $n_{wh} > 0$, must have $n_{wh} < \sqrt{t}$ and $\alpha = (\sqrt{t} - n_{hh})/n_{wh}$. \square

The above two lemmas brought an exhaustive search for increasing values of t within reach, trying all eligible values of α for each t . This led to the following result:

Theorem 3. *The smallest uniform rectangular tiling (with fewest tiles in the defining rules) with Arrwid number three is the Daun tiling, shown in Figure 10.*

Proof. We need to prove (i) the Daun tiling has Arrwid number three, and (ii) no smaller uniform rectangular tiling has Arrwid number three. The second part is tedious but doable by hand; skeptic readers are welcome try by themselves or to request a photocopy of my notes. We will now prove the first part: the Daun tiling has Arrwid number three.

Consider a disk Q of radius r , and let k_r be the level of recursion on which the tiles have dimensions more than $6r \times 4r$ and at most $24r \times 16r$. We define a level- k vertex, edge or tile as a vertex (a meeting point of three or more tiles), edge or tile that first appears in the tiling on the k -th level of subdivision down from the unit rectangle. So the corners of the unit tile (the level-0 tile) are level-0 vertices, the remaining corners of its sixteen subtiles are level-1 vertices, and so forth. Consider the boundary of any tile to be labelled as follows (see Figure 11(a)): starting from the marked corner (the one that was the lower left corner before rotation) and going in clockwise direction, we divide the boundary of the tile into ten sections of equal length, which we label B, A, B, C, D, E, D, E, F, and A, in that order. We will now prove by induction that there are no points of degree four in the tiling \mathcal{T} that results from expanding the recursion to a depth of k_r levels. Let $H(k)$ be the following claim:

- (i) there is no point of degree four at the starting point or along the interior of an A-section of a level- k tile that coincides with a B-section of another level- k tile;
- (ii) there is no point of degree four at the starting point or along the interior of a D-section of a level- k tile that coincides with an E-section of another level- k tile;

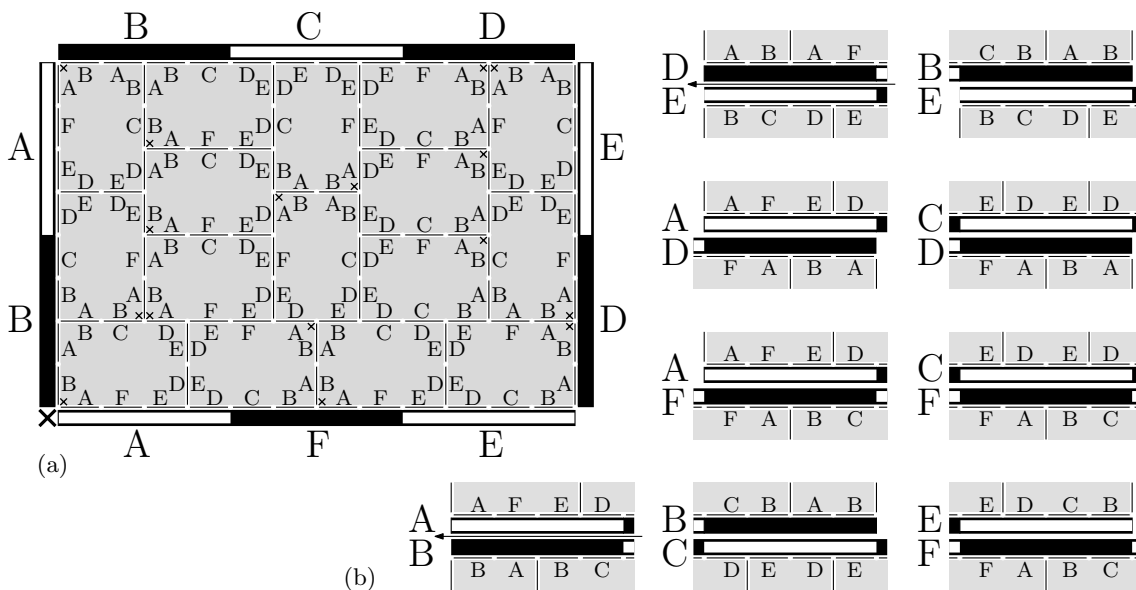


Figure 11: (a) Labelling the boundary of the Daun tiling. (b) Matching sections to each other.

- (iii) for all $(x, y) \in \{(A, D), (A, F), (B, C), (B, E), (C, D), (C, F), (E, F)\}$, there is no point of degree four along or at any endpoint of an x -section of a level- k tile that coincides with a y -section of another level- k tile;
- (iv) subdividing a level- k tile does not create any points of degree four in its interior.

We first prove $H(k_r)$. Part (iv) is trivial because level- k_r tiles are not subdivided further in \mathcal{T} . Part (i) is true because neither the A-section nor the B-section is subdivided further, and at the end of any B-section (which coincides with the starting point of the other tile's A-section) the tile boundary of which it is a part continues in a straight line, so that there cannot be a vertex of degree four there. Part (ii) and (iii) can be verified in a similar way.

We will now prove the following for $k < k_r$: if $H(k+1)$ is true, then $H(k)$ is true. Part (i): As illustrated in Figure 11(a), an A-section s at level k is subdivided into a D-section, an E-section, an F-section and an A-section (going clockwise around the tile) at level $k+1$. A B-section at level k consists of a C-section, a B-section, an A-section, and a B-section (going counterclockwise around the tile). Thus, along an A-section of a level- k tile that coincides with a B-section of another level- k tile, a D-section is matched to a C-section at level $k+1$, E to B, F to A, and A to B (Figure 11(b)). Because $H(k+1)$ is true, this does not create a degree-four vertex at the start or anywhere along the interior of s . Part (ii) and (iii) can be verified in a similar way. For part (iv), it follows from $H(k+1)$, part (iv), that no points of degree four are created in the interior of level- $(k+1)$ tiles, but we have to be careful about points on the boundary of level- $(k+1)$ tiles. Figure 11(a) illustrates that the boundary sections of level- $(k+1)$ tiles inside a level- k tile only coincide in the combinations listed in $H(k+1)$, part (i), (ii) and (iii); therefore the interiors of these level- $(k+1)$ boundary sections do not contain any points of degree four. All end points

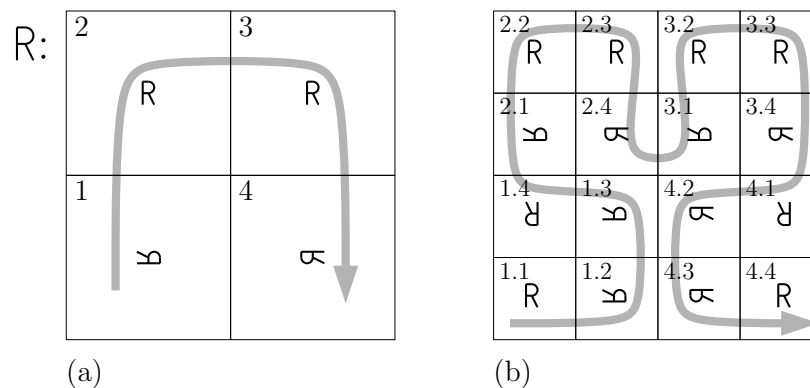


Figure 12: (a) Definition of the Hilbert scanning order. (b) Definition expanded by one level.

of these boundary sections that lie in the interior of a level- k tile are a starting point of a level- $(k+1)$ A-section coinciding with a B-section, a starting point of a level- $(k+1)$ D-section coinciding with an E-section, or any end point of a combination of level- $(k+1)$ sections as listed in part (iii) of $H(k+1)$. Therefore, by $H(k+1)$, these points cannot have degree four either. This proves part (iv) of $H(k)$.

It follows by induction that $H(0)$ holds, that is, \mathcal{T} does not have any degree-four vertices in the interior of the unit rectangle. Obviously there are not any degree-four vertices on its boundary either. Now observe that all tiles of \mathcal{T} consist of six squares of a regular square grid with line spacing more than $2r$. Any disk Q of radius r intersects at most four of these squares. If Q does indeed intersect as many as four squares, then the number of tiles of \mathcal{T} that intersect Q is equal to the degree of the grid point p shared by these four squares. As we have just established, the degree of any point p in \mathcal{T} is at most three. Hence Q can be covered by at most three tiles of \mathcal{T} , which have total area at most $3 \cdot 24r \cdot 16r = 1152r^2 = \frac{1152}{\pi} \text{area}(Q)$. This proves that the Arrwid number of the Daun tiling is three. \square

3 Two-dimensional space-filling curves

3.1 Defining space-filling curves

We define a *recursive scanning order* as a recursive tiling in which the rules specify not only a subdivision into tiles and what rules to apply to the tiles, but also an order of the tiles. One may illustrate this by a curve that visits the tiles recursively in order. Figure 12(a) shows an example: the Hilbert curve [9]. When the tiling is refined by expanding the recursion, so is the curve, see Figure 12(b). When defining a scanning order, the recursion within a tile can be rotated, mirrored and scaled as with recursive tilings. In addition it is possible to apply a recursive rule with reversed order; we indicate a reverse application of a recursive rule by a horizontal stroke above the letter that identifies the rule (see Figure 13 for an example).

One may use a scanning order to define a mapping from the unit interval to the unit tile as follows. For any tile A and any $x \in [0, 1]$, we define the *prefix region* $\text{pre}(A, x)$ as the region within A that has total area $x \cdot \text{area}(A)$ and comes first in the scanning order; intuitively, this region can be constructed by subdividing A recursively to a sufficiently fine level and collecting tiles in scanning order, until tiles with a total area of $x \cdot \text{area}(A)$ have been collected. We can define this more precisely as follows. If $x = 0$, then $\text{pre}(A, x) = \emptyset$. Otherwise let A_1, \dots, A_k be the subtiles of A in order, let a_1, \dots, a_k be their areas relative to A , that is, $a_i = \text{area}(A_i)/\text{area}(A)$, and let c_0, \dots, c_k be their cumulative areas relative to A , that is, $c_i = \sum_{j=1}^i a_j$. Now let i be the largest i such that $c_{i-1} \leq x$. Then $\text{pre}(A, x) = A_1 \cup \dots \cup A_{i-1} \cup \text{pre}(A_i, (x - c_{i-1})/a_i)$. Let the *postfix region* $\text{post}(A, x)$ be defined as $A \setminus \text{pre}(A, x)$, and the *fragment* $A[x, y]$ as $\text{post}(A, x) \cap \text{pre}(A, y)$.

Let U be the unit tile. For fixed y , the fragment $U[x, y]$ shrinks to a point as x approaches y from below; denote this point by $\sigma_{\uparrow}(y)$. Similarly, let $\sigma^{\downarrow}(x)$ be the point to which $U[x, y]$ shrinks when y approaches x from above; see Figure 16 for an example. Together the functions σ_{\uparrow} and σ^{\downarrow} define a “curve” with a discontinuity (a jump) for every x such that $\sigma_{\uparrow}(x) \neq \sigma^{\downarrow}(x)$. These functions also constitute a surjective map from the unit interval to the full set of points within U . Thus they “fill” U : it is a *space-filling curve*.¹

In this paper we identify a scanning order with the space-filling curve defined by it; we use the terms (*scanning*) *order* and (*space-filling*) *curve* interchangeably. The terminology introduced to describe tilings will also apply to curves based on those tilings. Simple curves use one rule of recursion; composite curves use multiple rules. Uniform or square curves are curves based on uniform or square tilings, respectively.

For any fragment $U[x, y]$, we say that $\sigma^{\downarrow}(x)$ is the *entry point* of the fragment, while $\sigma_{\uparrow}(y)$ is its *exit point*. We say that two fragments $U[x, y_1]$ and $U[y_2, z]$ are *consecutive* if $y_1 = y_2$. Two consecutive fragments $U[x, y]$ and $U[y, z]$ *connect* in a point p if $p = \sigma_{\uparrow}(y) = \sigma^{\downarrow}(y)$. Since fragments of space-filling curves fill two-dimensional regions, we measure the size of fragments by area, not by length.

There are other approaches to describing space-filling curves than the one chosen above. For example, Peano, who was the first to invent a space-filling curve, described his curve in an algebraic way [18]. Other authors describe a curve by defining a polygonal approximation of it, with a rule on how to refine each segment of the approximating polyline recursively [6]. Many authors specify the regions filled by fragments of the curve (like we do) together with the location of the entry and exit points of such fragments, but without making reversals explicit (for example Asano et al. [2], Sagan [19] and Wierum [24]). Since we are concerned with the use of space-filling curves as a way to order points in the plane, we choose a method of description that explicitly defines how to order the space inside a

¹Of course real curves do not jump. When $\sigma_{\uparrow}(x) = \sigma^{\downarrow}(x)$ for all x , either function defines a proper space-filling curve. However, when there are jumps, our definitions only provide surjective maps from the unit interval to the points within U , but they are not continuous, so they are not curves, and they do not match the definition of space-filling curves as one would find them in the mathematics literature. Z-order is an example of a scanning order with many jumps. Lebesgue shows how to get a proper (continuous) curve of the Z-order. In effect he defines $\text{pre}(A, x)$ in another way, to include connecting pieces of curve between the tiles [13]. The same technique could be applied to make other curves with jumps continuous. However, for our applications it is more convenient to work with a representation without the connecting pieces.

unit tile [8]. In the following, when definitions are given for curves from other authors, these definitions are the result of analysing the original descriptions to find a definition of a corresponding scanning order in our notation.

3.2 The Arrwid number of a space-filling curve

Consider a space-filling curve that fills a region U . Recall that the Arrwid number of the curve is the smallest number a such that there is a constant c such that any disk Q that lies entirely in U can be covered by a curve fragments with total area at most $c \cdot \text{area}(Q)$. Since every tile in the recursive tiling underlying a space-filling curve is by itself a fragment of the curve, the Arrwid number of a curve is never more than the Arrwid number of the underlying tiling. However, the Arrwid number of a curve may be *less* than the Arrwid number a of the underlying tiling. This can happen if, whenever a tiles are needed to cover a disk, some of these tiles are consecutive along the curve. To illustrate this, a space-filling curve that has a smaller Arrwid number than the underlying tiling is described in the next subsection.

3.3 Square curves with Arrwid number three

Theorem 4. *Dekking's curve (as defined in Figure 13(a)) is a square curve with Arrwid number three.*

Proof. First we establish the entry and exit points of the unit square U . The locations of entry and exit points are determined by the transformations of the recursive rule in the different tiles. The curve drawn in Figure 13(a) does not define the locations of entry and exit points; it is just a sketch for illustration. To prove that the entry and exit points are actually at the locations which the sketch may suggest, we have to study the way in which the tiles are transformed. The first subtile of the unit square to be visited is the one in the lower left corner. Since the recursive tiling and ordering rule **R** is applied to that tile without any rotation, reflection or reversal, the first subtile to be visited within it is again in the lower left corner, and this continues to hold to any depth of recursion. Hence, as y approaches 0 from above, the fragment $U[0, y]$ shrinks to the lower left corner point of U , and this is therefore the entry point of U . The last subtile of U to be visited is the one in the lower right corner. Its exit point is what would be its entry point before the reversal transformation is applied; as a result of the rotation, this point is found in the lower right corner. Therefore the exit point of U is its lower right corner.

Given any disk Q of radius r , consider the square grid \mathcal{T} that is formed by subdividing U recursively until the squares have width at most $10r$ and more than $2r$. Now Q intersects at most one horizontal line and at most one vertical line of \mathcal{T} . If at most one grid line intersects Q , then Q lies in at most two tiles of \mathcal{T} , with total area at most $200r^2 = \frac{200}{\pi} \text{area}(Q)$. Otherwise Q is intersected by a horizontal grid line and a vertical grid line which intersect in a vertex p , and Q intersects the four tiles of \mathcal{T} that share vertex p . Below we prove that two of these tiles are consecutive in the scanning order, so that these four tiles constitute at most three space-filling curve fragments with total area at most

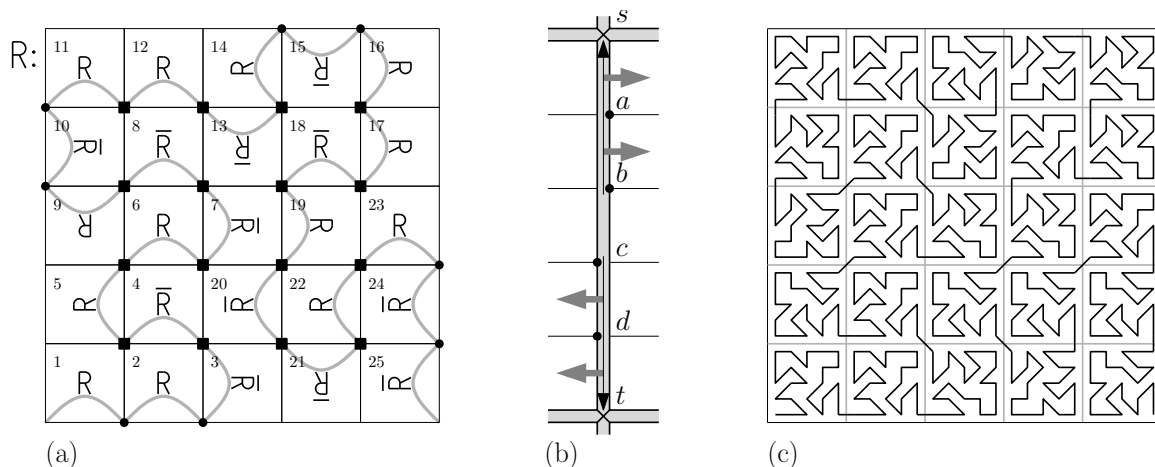


Figure 13: (a) Definition of a scanning order following Dekking's curve [5], with marks added to aid in the proof of Theorem 4. (b) When p lies between two tiles ($p \in \{a, b, c, d\}$), we walk to the closest higher-level vertex $q \in \{s, t\}$ and consider p as one of the vertices of the tile that lies to the right. (c) Sketch of the level-two expansion of the scanning order. This curve looks quite different from Dekking's sketch [5], but the difference is only superficial: our sketch connects the centre points of the tiles, while Dekking's sketch connects the entry and exit points of the tiles.

$400r^2 = \frac{400}{\pi} \text{area}(Q)$. This will prove that the space-filling curve has Arrwid number at most three; together with the lower bound of Theorem 5 in the next subsection this proves that the Arrwid number is in fact exactly three.

For our proof that around any vertex p two of the tiles meeting in p are consecutive in the scanning order, we use the same terminology as in the proof of Theorem 3: a level- k feature (vertex, edge, or tile) is one that first appears when expanding the recursion to a depth of k levels down from the unit tile. Now let k be the level of p , that is, p is a level- k vertex. Now p is of one of two types: it either lies in the interior of a level- $(k-1)$ tile, or it lies on an edge between two level- $(k-1)$ tiles.

If p is of the first type, one may verify in Figure 13(a) (see the vertices marked by squares) that two of the level- k tiles around p , say A and B , connect in p . This implies that if we expand the recursion further until we obtain \mathcal{T} , the tile of $\mathcal{T} \cap A$ that touches p connects to the tile of $\mathcal{T} \cap B$ that touches p .

If p is of the second type, that is, it lies on an edge between two level- $(k-1)$ tiles, then let q be the corner shared by these two tiles which is closest to p . Now consider p as a vertex on the boundary of the level- $(k-1)$ tile that lies to the right when walking from p to q , see Figure 13(b). When Figure 13(a) depicts this tile, then p is one of the vertices marked by black dots (note that here we exploit the fact that the recursive rule for this curve does not use reflections, so that we preserve the left/right orientations for all tiles). Again, one may verify in Figure 13(a) that two level- k tiles connect in p , and hence two tiles of \mathcal{T} connect in p . \square

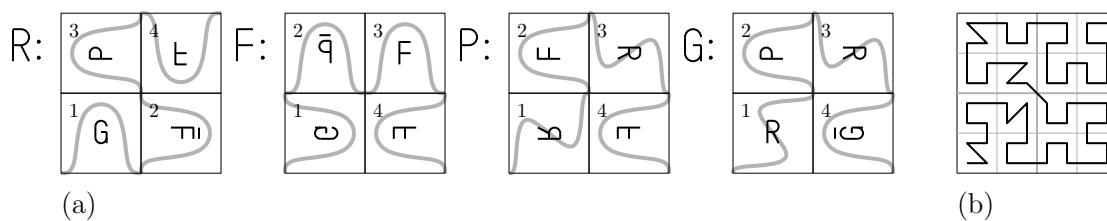


Figure 14: (a) Definition of the AR^2W^2 scanning order. (b) Sketch of the level-three expansion (starting from rule R).

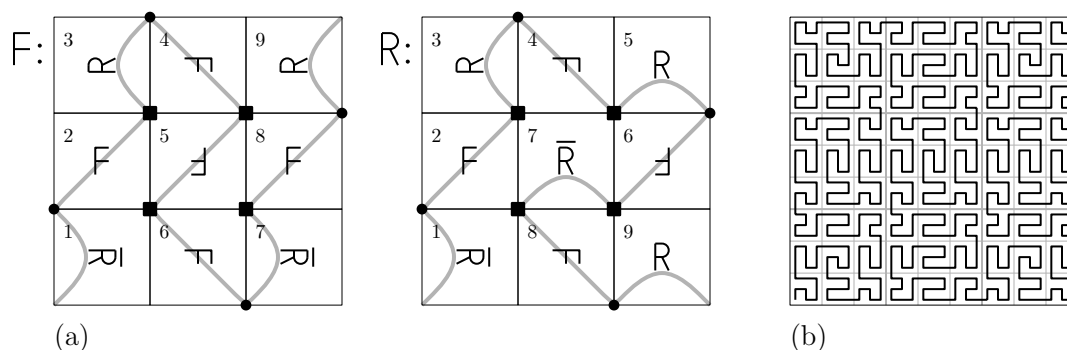


Figure 15: (a) Definition of the Kochel scanning order. (b) Sketch of the level-three expansion (starting from rule F).

The curve analysed above was described by Dekking in the form of a recursive polygonal approximation [5] (he did not give a tiling-based definition or analyse its Arrwid number). There are other uniform square curves with Arrwid number three, with smaller tilings (less squares per square), but these are composite curves, not simple curves like Dekking's. Asano et al. define the AR^2W^2 -curve [2] (see Figure 14) and prove that it has Arrwid number three. The *Kochel curve* (Figure 15) is another curve with Arrwid number three, which can be proven to have Arrwid number three in a way very similar to the proof for Dekking's curve. Unlike the AR^2W^2 -curve and Dekking's curve, the Kochel curve has the property that consecutive tiles in the order always share an edge.

In contrast to the curves mentioned above, the following square curves all have Arrwid number four: Hilbert's curve [9]; Z-order, also known as Lebesgue's curve [13]; the $\beta\Omega$ -curve [24]; Peano's curve [18] and all other simple uniform square curves of size nine, such as Luxburg's variations [14], R-order, and Meurthe order (see Haverkort and Van Walderveen [8] for definitions in our notation).

3.4 A lower bound for two-dimensional space-filling curves

Above we saw that there are uniform square space-filling curves with Arrwid number three, even while the underlying recursive tiling has Arrwid number four. For the specific case of square space-filling curves with four tiles per recursive rule, Asano et al. proved that this is optimal: the Arrwid number cannot be less than three [2]. Below we see a

different proof technique, which we can also generalize to three dimensions later, at least to some extent. With our new proof technique, we see that an Arrwid number of three is in fact optimal for *any* space-filling curve that is based on a recursive tiling whose tiles are topologically equivalent to disks—note that this category includes all tilings we have seen so far, except the one in Figure 7.

Theorem 5. *Any space-filling curve based on a recursive tiling with tiles that are topologically equivalent to disks has Arrwid number at least three.*

Proof. Consider any subdivision of a space-filling curve, filling a unit tile U , into a set F' of k fragments $U[0, x_1], U[x_1, x_2], \dots, U[x_{k-1}, 1]$, such that each fragment is a simply connected region in the plane, that is, topologically equivalent to a disk. Let \bar{U} be the unit tile's complement $\mathbb{R}^2 \setminus U$, and let F be $F' \cup \{\bar{U}\}$.

The boundaries of the fragments in F' form a plane graph \mathcal{G} with face set F . As the edge set E of \mathcal{G} we take the maximal curves that form a boundary between two faces of F . The vertex set V of \mathcal{G} is the set of points where three or more faces of F meet. By $\text{degr}(v)$ we denote the number of edges of E that are incident on the vertex v . By $\text{ends}(v)$ we denote the number of fragments of F' that have v as their entry point plus the number of fragments of F' that have v as their exit point.

Let V' be the vertices of \mathcal{G} that are not on the outer face \bar{U} . I claim that if the space-filling curve has Arrwid number less than three, we must have $\text{degr}(v) - \frac{1}{2}\text{ends}(v) \leq 2$ for every vertex $v \in V'$. To prove this, let's assume, for the sake of contradiction, that there is a vertex $v \in V'$ with $\text{degr}(v) - \frac{1}{2}\text{ends}(v) > 2$. Let F_1, \dots, F_m be the faces of F that meet at v (where $m = \text{degr}(v)$), in the order in which they appear in the scanning order; each face F_i constitutes a fragment $U[f_i, f'_i]$. For $1 \leq i < m$, let B_i (bridge i) be the smallest fragment $U[b_i, b'_i]$ (not necessarily a fragment in F') that starts from v in F_i and ends at v in F_{i+1} , more precisely: B_i is the smallest fragment $U[b_i, b'_i]$ such that $b_i \in [f_i, f'_i]$, $b'_i \in [f_{i+1}, f'_{i+1}]$, and $\sigma^\perp(b_i) = \sigma_\uparrow(b'_i) = v$ (see Figure 16 for an example). Note that B_i degenerates to the point v if and only if F_i and F_{i+1} connect at v , contributing two to the entry and exit points counted in $\text{ends}(v)$. Therefore the number of degenerate bridges at v is at most $\frac{1}{2}\text{ends}(v)$. Since $\frac{1}{2}\text{ends}(v) < \text{degr}(v) - 2$ and there are $\text{degr}(v) - 1$ bridges in total, there are at least two non-degenerate bridges at v . By assumption the space-filling curve has Arrwid number less than three, so there must be a constant c such that any circular query range Q can be covered by at most two fragments (not necessarily from F') with total area at most $c \cdot \text{area}(Q)$. Now consider a circle Q centered at v with area less than $1/c$ times the area of the smallest non-degenerate bridge at v . To cover Q with only two fragments, the fragments must cover Q and all but one of the bridges B_1, \dots, B_{m-1} . Thus at least one non-degenerate bridge is covered, and by the definition of Q , this bridge has area more than $c \cdot \text{area}(Q)$. However, this contradicts the definition of c . Therefore, if the space-filling curve has Arrwid number less than three, we must have $\text{degr}(v) - \frac{1}{2}\text{ends}(v) \leq 2$ for every vertex $v \in V'$.

Summing over all vertices in V' we get:

$$\sum_{v \in V'} \text{degr}(v) \leq 2|V'| + \frac{1}{2} \sum_{v \in V'} \text{ends}(v) < 2|V| + \frac{1}{2} \sum_{v \in V'} \text{ends}(v). \quad (5)$$

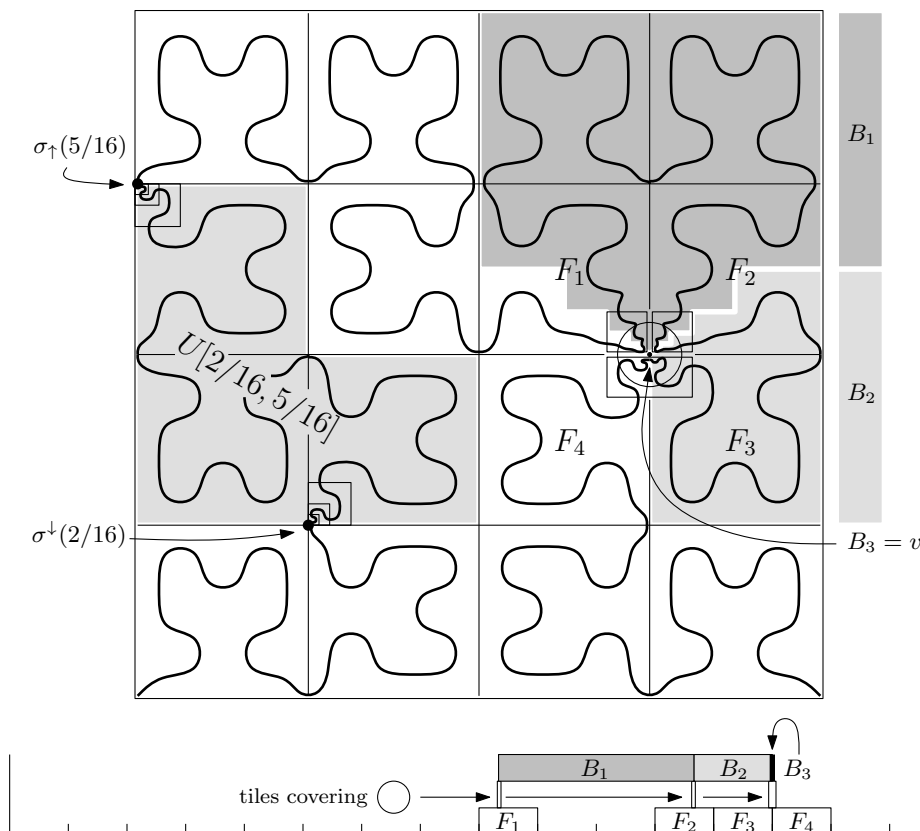


Figure 16: Illustrating the proof of Theorem 5. On top we see a unit square divided into sixteen tiles, and below we see the corresponding unit interval that represents the space-filling curve, divided into sixteen fragments. The four tiles meeting in v are F_1, F_2, F_3 and F_4 , and the bridges around v are B_1 (dark shaded area), B_2 (light shaded area), and B_3 (just the point v). Any disk Q centered on v can be covered with three small fragments (two squares, and one rectangle consisting of two squares). To do with fewer fragments, that is, only two, these fragments need to cover at least one of the bridges B_1 and B_2 .

Let E' be the set of edges with both end points in V' , and let E'' be the remaining edges (those with at least one end point on the outer face). We have $\sum_{v \in V'} \deg(v) > 2|E'| = 2|E| - 2|E''|$ and $\sum_{v \in V'} \text{ends}(v) \leq 2|F| - 2$ (since each face except the outer face has one entry point and one exit point). Thus Equation 5 implies:

$$2|E| - 2|E''| < 2|V| + |F| - 1.$$

Moving $2|E''|$ and F to the other side and subtracting Euler's formula $|E| - |F| = |V| - 2$ twice we get:

$$|F| < 2|E''| + 3. \quad (6)$$

Now take any space-filling curve based on a recursive tiling whose tiles are simply connected regions (topological disks). Consider the subdivision of the unit tile U into tiles

down to a level of recursion on which there is a tile T that does not touch the boundary of U . This level must exist, since the maximum diameter of the tiles decreases geometrically with the level of recursion, so at some point the tiles become so small that the tile that covers the centre point of U does not touch the boundary of U . Now subdivide T into smaller tiles recursively while keeping the tiles outside T as they are. Thus we keep E'' fixed while we continue to increase $|F|$, eventually getting $|F| \geq 2|E''| + 3$. This contradicts Equation 6, so the space-filling curve cannot have an Arrwwid number less than three. \square

Intuitively, the proof can be summarized as follows: if a recursive tiling has vertex degree as low as three, then there will be so many of those vertices that no scanning order can manage to make a connection between two of the three tiles at each vertex. A direct consequence of the above theorem is the following: when a space-filling curve is based on a recursive tiling with simply connected tiles and Arrwwid number three, the Arrwwid number of the space-filling curve is always exactly three as well, regardless of the order in which the curve traverses the tiles; the “curve” does not even need to be continuous.

4 Three-dimensional tilings

The definitions of tilings and Arrwwid numbers generalize naturally to higher dimensions. The Arrwwid number of a three-dimensional recursive tiling of a unit tile U is the smallest number a such that there is a constant c such that any ball Q that lies entirely in U can be covered by a tiles with total volume at most $c \cdot \text{volume}(Q)$. Theorem 1 also generalizes naturally to three dimensions:

Theorem 6. *Each recursive tiling of a three-dimensional region U has Arrwwid number at least four.*

Proof. The proof is analogous to the proof of Theorem 1: because the tile sizes decrease by at least a constant factor with each level of recursion, the subdivision of the unit tile U into its subtiles must eventually create a surface in its interior that forms the boundary between two subtiles S and T ; on the interior of this surface, a curve must eventually appear that forms the boundary between three tiles (one subtile of S and two subtiles of T , or vice versa); on the interior of this curve, a vertex must eventually appear in which four subtiles meet. \square

4.1 Optimal three-dimensional recursified tilings

This subsection describes how to construct a recursified three-dimensional tiling with optimal Arrwwid number, that is, four. The construction is based on a tiling with cubes that are shifted with respect to each other (Figure 17). The coarse tiling consists of horizontal layers that are one cube high; each layer is shifted to the right and to the front with respect to the layer below over $1/3$ of a cube’s width. Each layer consists of columns that are one cube wide; each column is shifted to the back with respect to the column to the left over $1/3$ of a cube’s width. The fine tiling is equal to the coarse tiling scaled by a factor $1/5$ (with

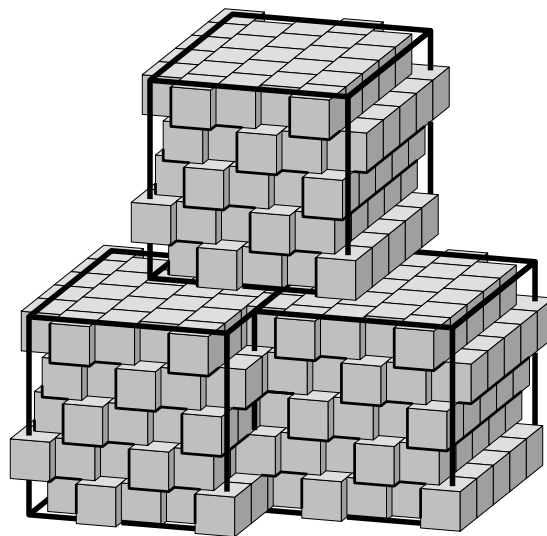


Figure 17: Approximating large cubes in a tiling with vertex degree four by 125 small cubes each, taken from the same tiling scaled by a factor $1/5$.

the centre point of a large cube as its fixed point); each tile of the fine tiling is assigned to the tile of the coarse tiling with which it has the largest overlap.

Theorem 7. *There is a recursified three-dimensional tiling with Arrwid number four.*

Proof. The proof follows the approach of Theorem 2. The recursified tiling is constructed from the coarse and fine tilings of shifted cubes as described above. Let w be the width of a cube in the coarse tiling, and thus, $w' = w/5$ is the width of a cube in the fine tiling. In the coarse tiling, the smallest ball that intersects more than four tiles has radius $w/6$. When we replace a large tile by the union of 125 small tiles, the boundary of the large tile stays within a distance of $\frac{1}{3}\sqrt{2} \cdot w' = \frac{1}{15}\sqrt{2} \cdot w$ from its original location. In recursion with scale factor $1/5$, the movement of the boundary adds up to at most $\frac{5}{4} \cdot \frac{1}{15}\sqrt{2} \cdot w = \frac{1}{12}\sqrt{2} \cdot w$. Thus the smallest ball that intersects more than four large tiles will still have radius at least $(\frac{1}{6} - \frac{1}{12}\sqrt{2})w > \frac{1}{21}w$. The proof can now be completed as in Theorem 2. \square

In fact, there are many ways to make a recursified three-dimensional tiling with Arrwid number four. We can take a periodic tiling with vertex degree four as the coarse tiling, and obtain the fine tiling by scaling it. We only have to choose the scale factor small enough, so that the boundaries of the large tiles are not displaced too much when the large tiles are approximated by a union of small tiles. Thus, we could, for example, obtain a recursified tiling based on turning truncated octahedra into fractals².

²The coarse tiling contains the truncated octahedron whose vertices have coordinates $(0, \pm 5, \pm 10)$, and all permutations of these coordinates. It is the intersection of an axis-parallel cube with diagonal $(-10, -10, -10) - (10, 10, 10)$, and an octahedron with vertices $(\pm 15, 0, 0)$, $(0, \pm 15, 0)$, and $(0, 0, \pm 15)$. Further truncated octahedra are placed at translations $(20k, 20l, 20m)$ and $(20k + 10, 20l + 10, 20m + 10)$, for all $k, l, m \in \mathbb{Z}$. Thus we get a tiling in which all vertices are incident on four tiles. The fine tiling could be made by scaling the coarse tiling by a factor $1/5$, creating 125 small tiles for each large tile.

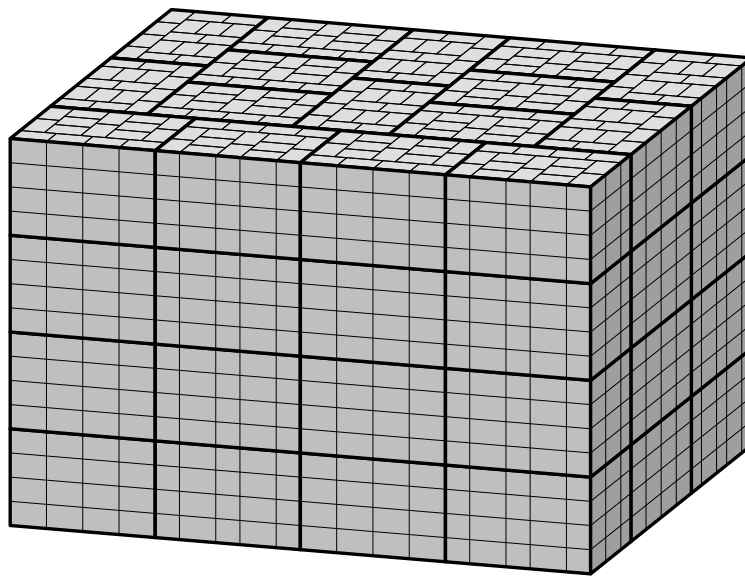


Figure 18: Lifting a Daun tiling into three dimensions.

4.2 Rectangular three-dimensional tilings

The following is a straightforward generalization of Observation 1.

Observation 3. *Any uniform cube tiling has Arrwid number eight.*

With rectangular tiles, that is, with axis-parallel boxes, one can do better. The three-dimensional rectangular tiling with lowest Arrwid number found so far, is the “lifted Daun tiling” shown in Figure 18. It is obtained by adding a third dimension to the two-dimensional Daun tiling from Figure 10. The unit tile of the three-dimensional tiling is a rectangular block with width-to-depth ratio $3/2$ and arbitrary height. It is divided into 64 subtiles, organized into 4 equal layers of 16 tiles each. Each layer, seen from above, shows a Daun tiling. The tiles are rotated around a vertical axis only, in the same way as in the Daun tiling.

Each vertex in the resulting tiling is on the boundary between two layers and is adjacent to three tiles in each layer (because the Daun tiling has vertex degree three), therefore the vertex degree of the three-dimensional tiling is six. From here one can prove (similar to Theorem 3) that the three-dimensional “lifted Daun tiling” has Arrwid number six.

Theorem 8. *There is a three-dimensional rectangular recursive tiling with Arrwid number six.*

We do not know if it is possible to get an Arrwid number of four or five with a rectangular tiling in three dimensions. As in the two-dimensional case, we could derive some properties that the aspect ratios and the numbers of tiles should have in order to allow an Arrwid number of four. Nevertheless the search space is still huge. So far I only managed to search all three-dimensional uniform rectangular tilings with less than 27 tiles; I did not find any recursive tiling with Arrwid number four or five.

5 Three-dimensional space-filling curves

5.1 Rectangular three-dimensional space-filling curves

Theorem 9. *Any space-filling curve based on a uniform cube tiling has Arrwwid number at least seven.*

Proof. Consider the regular cube tiling \mathcal{T} obtained by subdividing a unit cube U recursively into smaller cubes to a certain depth of recursion. Let C be the set of tiles obtained. By $\text{tiles}(v)$ we denote the number of tiles in C that are incident on the vertex v . By $\text{ends}(v)$ we denote the number of tiles in C that have v as their entry point plus the number of tiles in C that have v as their exit point.

Let V' be the set of vertices of \mathcal{T} that do not lie on the boundary of U . If the space-filling curve has Arrwwid number less than seven, we must have $\text{tiles}(v) - \frac{1}{2}\text{ends}(v) \leq 6$ for every vertex $v \in V'$ (the proof of this claim is a straightforward adaptation of the proof of Theorem 5). Note that $\text{tiles}(v) = 8$ for each $v \in V'$, so in fact we must have $\frac{1}{2}\text{ends}(v) \geq 2$ for each $v \in V'$. Since the total number of entry and exit points at vertices is at most $2|C|$, this leads to:

$$|C| \geq \frac{1}{2} \sum_{v \in V'} \text{ends}(v) \geq \sum_{v \in V'} 2 \geq 2|V'|. \quad (7)$$

However, in a uniform cube tiling we have $|V'| = (|C|^{1/3} - 1)^3$, which is more than $\frac{1}{2}|C|$ for a sufficiently deep level of recursion. This contradicts Equation 7, so the space-filling curve cannot have an Arrwwid number less than seven. \square

Theorem 10. *Any space-filling curve based on the lifted Daun tiling of Section 4.2 has Arrwwid number six.*

Proof. The proof is similar to that of Theorem 9. In this case a contradiction is derived from $\text{tiles}(v) - \frac{1}{2}\text{ends}(v) \leq 5$ and $\text{tiles}(v) = 6$ for each $v \in V'$, which gives $\frac{1}{2}\text{ends}(v) \geq 1$ for each $v \in V'$. Summing up over all $v \in V'$ we get $|C| \geq |V'|$. Now consider the tiling obtained by expanding one level of recursion of the lifted Daun tiling. Whenever we subdivide a tile in this tiling, we replace a tile by 64 smaller tiles, and add a number of new vertices: to start with, on each of the three boundaries between the four layers of new tiles, there are 30 new vertices (see Figure 10), at least 24 of which lie in the interior of the unit tile U (since at least one short side and at least one long side of the tile lies in the interior of U). Thus we increase the number of tiles by 63 while increasing the number of vertices in the interior of U by at least 72. Therefore, when we subdivide enough tiles, we get $|V'| > |C|$, which contradicts the above. Hence, the space-filling curve cannot have Arrwwid number less than six. \square

5.2 A lower bound for convex three-dimensional space-filling curves

To prove a lower bound on the Arrwwid number of three-dimensional space-filling curves, we follow the same general approach as in two dimensions. The proof in the two-dimensional

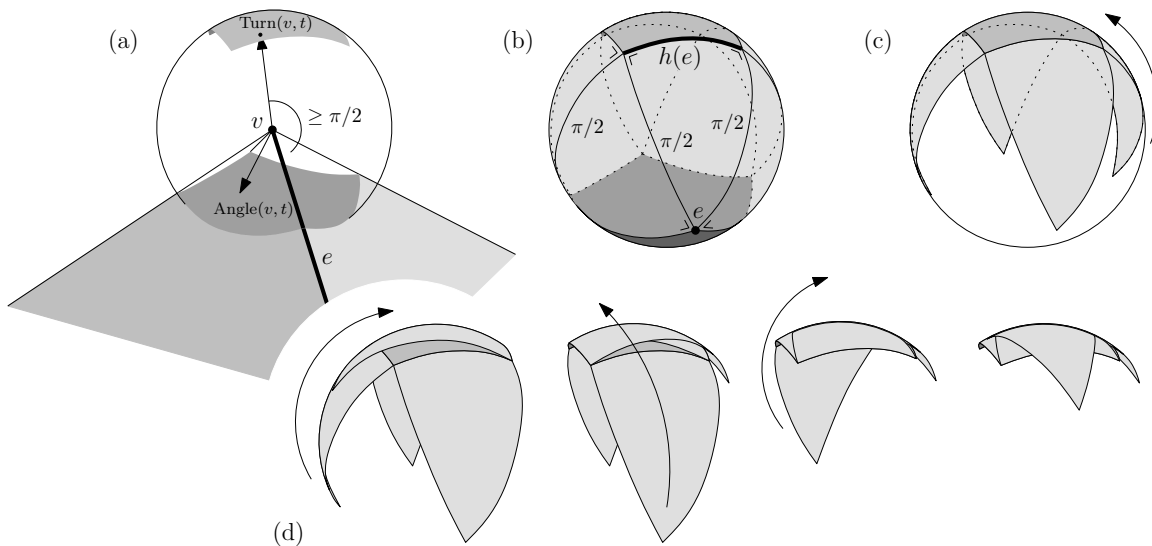


Figure 19: For a vertex v on a polyhedron t , we have that $\text{Angle}(v, t)$ and $\text{Turn}(v, t)$ can be covered by the triangles between them on the unit sphere of directions.

case essentially has two main ingredients: (1) In any subdivision of a two-dimensional unit tile into simply connected tiles, the total number of vertex-tile incidences is roughly twice the number of edges; therefore Euler's formula gives us a relation \mathcal{A} between the number of vertices $|V|$, the number of tiles $|F'|$, and the total number of vertex-tile incidences: the number of incidences is roughly $2|V| + 2|F'|$. (2) For the Arrwid number to be lower than three, there must be another relation \mathcal{B} between the number of vertices, the number of tiles, and the total number of vertex-tile incidences: the number of incidences must be at most roughly $2|V| + |F'|$. For large enough tilings, the word “roughly” in the previous sentences cannot save us, and \mathcal{A} conflicts with \mathcal{B} . Hence an Arrwid number lower than three is impossible.

Unfortunately ingredient (1) does not easily generalize to three dimensions, because besides the number of edges, another unknown enters the equations, namely the number of two-dimensional facets between the tiles. Therefore, to be able to complete our proof, we need another way to establish a relation between the number of vertices, the number of tiles, and the number of vertex-tile incidences in a three-dimensional tiling. We will now see how this can be done when the tiles are convex polyhedra.

Consider a convex polyhedron t , and let v be a point on the boundary of t . In the following, we define the size of a set of vectors S as the area of the projection of the vectors in S on a unit sphere. Let $\text{angle}(v, t)$ be the size of the set $\text{Angle}(v, t)$, which consists of the vectors that point from v into t . Let $\text{turn}(v, t)$ be the size of the set $\text{Turn}(v, t)$, which consists of the vectors that point from v away from t at an angle of at least $\pi/2$ with the boundary of t . Figure 19(a) illustrates the case in which v is a vertex of t . Note that if v is a point in the interior of an edge of t , we have $\text{angle}(v, t) \in (0, 2\pi)$ and $\text{turn}(v, t) = 0$; if v is a point in the interior of a facet of t , we have $\text{angle}(v, t) = 2\pi$ and $\text{turn}(v, t) = 0$. We now have the following:

Lemma 3. $\text{turn}(v, t) + \text{angle}(v, t) \leq 2\pi$. Equality holds if and only if v is a point in the interior of a facet of t , in which case $\text{angle}(v, t) = 2\pi$ and $\text{turn}(v, t) = 0$.

Proof. Let e_1, \dots, e_m be the edges incident to v . For any edge e_i , let $h(e_i)$ be the plane through v that is orthogonal to e_i , and let $H(e_i)$ be the halfspace bounded by $h(e_i)$ that does not contain e_i . Observe that the vectors that point from v away from t at an angle of at least $\pi/2$ are exactly those that point into the intersection of the halfspaces $H(e_1), \dots, H(e_m)$. Thus, on the unit sphere of directions, we find that $\text{Angle}(v, t)$ is represented as a spherical polygon whose vertices correspond to the directions of the edges e_1, \dots, e_m with respect to v , and $\text{Turn}(v, t)$ is represented as a spherical polygon whose edges are segments of $h(e_1), \dots, h(e_m)$, see Figure 19(b). The relation between e_i and $h(e_i)$ as explained above implies that the remaining part of the unit sphere, between $\text{Angle}(v, t)$ and $\text{Turn}(v, t)$, can be triangulated by arcs of length $\pi/2$ that make right angles with the adjacent edges of $\text{Angle}(v, t)$ and $\text{Turn}(v, t)$.

Now consider $\text{Turn}(v, t)$ and the adjacent triangles, see Figure 19(c). Since $\text{Turn}(v, t)$ is smaller than a hemisphere, the triangles would cover $\text{Turn}(v, t)$ completely if folded onto it, see Figure 19(d). Similarly, the triangles adjacent to $\text{Angle}(v, t)$ would cover $\text{Angle}(v, t)$ completely. Hence the total area of the triangles is at least $\text{angle}(v, t) + \text{turn}(v, t)$, so $\text{angle}(v, t) + \text{turn}(v, t)$ is at most 2π , the area of half a unit sphere.

Equality holds only if the triangles adjacent to $\text{Turn}(v, t)$ cover $\text{Turn}(v, t)$ *exactly*, and those adjacent to $\text{Angle}(v, t)$ cover $\text{Angle}(v, t)$ *exactly*. This only happens when both $\text{Angle}(v, t)$ and $\text{Turn}(v, t)$ are either empty, or a full hemisphere. $\text{Angle}(v, t)$ is never empty, but it is possible that $\text{angle}(v, t) = 2\pi$ and $\text{turn}(v, t) = 0$, namely if v is a point in the interior of a facet of t . \square

Consider a subdivision of a bounded convex polyhedral unit tile U into a set C of convex polyhedral tiles, whose shapes come from a fixed set S . Let V be the set of vertices of the subdivision, and let $V' \subset V$ be the vertices that lie in the interior of U . We can now prove:

Lemma 4. $|V'| + (1 + \alpha)|C| < \frac{1}{2} \sum_{v \in V} \text{tiles}(v)$, where $\text{tiles}(v)$ is the number of tiles with v on their boundaries, and $\alpha > 0$ is a constant that depends on S .

Proof. For ease of notation, define $\text{angle}(v, t) = \text{turn}(v, t) = 0$ when v does not lie on the boundary of t . Observe that for every vertex $v \in V'$ the sets $\text{Angle}(v, t)$ of all incident tiles t together cover the full sphere of directions, so we have $\sum_{t \in C} \text{angle}(v, t) = 4\pi$. Also, for every tile $t \in C$ the sets $\text{Turn}(v, t)$ of all vertices v on t together cover the full sphere of directions, so we have $\sum_{v \in V} \text{turn}(v, t) = 4\pi$. With these observations we get:

$$\begin{aligned} |V'| + |C| &= \\ \frac{1}{4\pi} \sum_{v \in V'} \sum_{t \in C} \text{angle}(v, t) + \frac{1}{4\pi} \sum_{t \in C} \sum_{v \in V} \text{turn}(v, t) &< \\ \frac{1}{4\pi} \sum_{v \in V} \sum_{t \in C} (\text{angle}(v, t) + \text{turn}(v, t)). \end{aligned}$$

Since every tile $t \in C$ is a bounded convex polyhedron with a shape from a fixed set S , it must have at least one vertex v with $\text{turn}(v, t) > 0$. Hence, by Lemma 3, for this vertex v

we have $\text{angle}(v, t) + \text{turn}(v, t) \leq 2\pi - \beta$, where $\beta > 0$ is a constant that depends on S . For other vertices v on the boundary of t we have $\text{angle}(v, t) + \text{turn}(v, t) \leq 2\pi$ (by Lemma 3), and for vertices v that are not on the boundary of t we have $\text{angle}(v, t) + \text{turn}(v, t) = 0$ (by definition). Therefore the above equation gives:

$$|V'| + |C| < \left(\frac{1}{4\pi} \sum_{v \in V} \text{tiles}(v) \cdot 2\pi \right) - \frac{\beta}{4\pi} |C| = \left(\frac{1}{2} \sum_{v \in V} \text{tiles}(v) \right) - \frac{\beta}{4\pi} |C|.$$

Setting $\alpha = \beta/4\pi$ proves the lemma. \square

We are now ready to prove the lower bound we are after:

Theorem 11. *Any space-filling curve based on a recursive tiling with convex tiles in three dimensions has Arrwid number at least four.*

Proof. Consider a subdivision \mathcal{T} of the unit tile U into a set C of convex polyhedral tiles, obtained by applying the recursive rules that define the tiling that underlies the space-filling curve. Let V be the set of vertices of \mathcal{T} , and let $V' \subset V$ be the vertices that are not on the boundary of U . Let $\text{tiles}(v)$ be the number of tiles in C that are incident on the vertex v . Let $\text{ends}(v)$ be the number of tiles in C that have v as their entry point plus the number of tiles in C that have v as their exit point.

If the space-filling curve has Arrwid number less than four, we must have $\text{tiles}(v) - \frac{1}{2}\text{ends}(v) \leq 3$ for every vertex $v \in V'$ (again, the proof of this claim is a straightforward adaptation of the proof of Theorem 5). Summing over all vertices in V' gives:

$$\sum_{v \in V'} \text{tiles}(v) \leq 3|V'| + \frac{1}{2} \sum_{v \in V'} \text{ends}(v).$$

Let V'' be the set of vertices of \mathcal{T} on the boundary of U , that is, $V'' = V \setminus V'$. Now Lemma 4 gives $|V'| + (1 + \alpha)|C| - \frac{1}{2} \sum_{v \in V''} \text{tiles}(v) < \frac{1}{2} \sum_{v \in V'} \text{tiles}(v)$ for some fixed constant α . Because all tiles are convex polyhedra, all vertices $v \in V'$ have $\text{tiles}(v) \geq 4$, and therefore $2|V'| \leq \frac{1}{2} \sum_{v \in V'} \text{tiles}(v)$. Note that the total number of entry and exit points at vertices is at most $2|C|$. Therefore the above equation gives us:

$$3|V'| + (1 + \alpha)|C| - \frac{1}{2} \sum_{v \in V''} \text{tiles}(v) < \sum_{v \in V'} \text{tiles}(v) \leq 3|V'| + \frac{1}{2} \sum_{v \in V'} \text{ends}(v) \leq 3|V'| + |C|.$$

Therefore we must have:

$$\alpha|C| < \frac{1}{2} \sum_{v \in V''} \text{tiles}(v). \quad (8)$$

From here we follow the same approach as in the proof of Theorem 5: we construct \mathcal{T} by subdividing the unit tile U into tiles recursively until we get a tile T that does not touch the boundary of U . Then we subdivide T further, keeping the vertices V'' and their incident tiles fixed, while continuing to increase $|C|$, eventually getting $\alpha|C| \geq \frac{1}{2} \sum_{v \in V''} \text{tiles}(v)$. This contradicts Equation 8, so the space-filling curve cannot have Arrwid number less than four. \square

6 More than three dimensions

Many of the results for three dimensions, but not all, generalize to higher dimensions in a straightforward way. These results are given in this section without writing out straightforward generalizations in full detail.

Theorem 12. *Each recursive tiling of a d -dimensional region U has Arrwid number at least $d + 1$.*

Theorem 13. *There is a rectangular recursive tiling in d dimensions with Arrwid number $3^{d/2}$ (if d is even) or $2 \cdot 3^{\lfloor d/2 \rfloor}$ (if d is odd).*

Proof. Given a d_1 -dimensional recursive tiling \mathcal{R}_1 and a d_2 -dimensional recursive tiling \mathcal{R}_2 , we can construct a $(d_1 + d_2)$ -dimensional recursive tiling \mathcal{R} by taking the Cartesian product of \mathcal{R}_1 and \mathcal{R}_2 . More precisely, for any $i \in \{0, 1, 2, \dots\}$, we let the i -th level of recursion of \mathcal{R} be the Cartesian product of the i -th level of recursion of \mathcal{R}_1 and the i -th level of recursion of \mathcal{R}_2 . The reader may now verify that the vertex degree of \mathcal{R} is the product of the vertex degrees of \mathcal{R}_1 and \mathcal{R}_2 , and the Arrwid number of \mathcal{R} is the product of the Arrwid numbers of \mathcal{R}_1 and \mathcal{R}_2 . We already saw one application of this technique in Section 4.2, where, in effect, we took the Cartesian product of a two-dimensional Daun tiling of a horizontal rectangle and a trivial one-dimensional recursive tiling of a vertical line segment.

Thus, if d is even, we can obtain a d -dimensional rectangular recursive tiling with Arrwid number $3^{d/2}$ by taking the Cartesian product of $d/2$ Daun tilings. If d is odd, we take the Cartesian product of $\lfloor d/2 \rfloor$ Daun tilings and lift the result from $d - 1$ into d dimensions by taking the product with a trivial one-dimensional tiling, as we did in Section 4.2; this results in a tiling with Arrwid number $2 \cdot 3^{\lfloor d/2 \rfloor}$. \square

Theorem 14. *There is a recursified d -dimensional tiling with Arrwid number $d + 1$.*

To prove the above theorem, we first define the coarse tiling and the fine tiling; then we prove that the coarse tiling has vertex degree $d + 1$; next we prove that the centre of each small tile lies in the interior of a unique large tile; and then we prove the theorem.

We use a tiling with shifted hypercubes, similar to the one used in Section 4.1. I will describe a way to generalize this construction that is relatively easily seen to result in an Arrwid number of $d + 1$, although it may not give the best cover ratio. Let a k -slab be an axis-parallel rectangular subspace of the d -dimensional space, which is unbounded to both sides in the first k dimensions, and has width 1, occupying the interval $[0, 1]$, in the remaining $d - k$ dimensions. Thus, a 0-slab is just a unit hypercube occupying the space $[0, 1]^d$, and it contains only one tile. A k -slab, with $1 \leq k \leq d$ is tiled by composing tiled slabs S_z for all $z \in \mathbb{Z}$, where each slab S_z is a $(k - 1)$ -slab translated over a vector (t_1, \dots, t_d) with $t_i = z/2^{k-1}$ for $i < k$; $t_k = z$; and $t_i = 0$ for $i > k$. Note that a tiling of a d -slab now constitutes a tiling of the full d -dimensional space.

We define $D = 2^{d-1}$. For the fine tiling, we scale the coarse tiling with scale factor $1/(D + 1)$, taking the centre of an arbitrary tile in the coarse tiling as a fixed point—note that the result is the same, regardless of which tile centre we choose as a fixed point. Then we translate the fine tiling over a distance $\frac{1/2}{D(D+1)}$ in all dimensions.

Lemma 5. *The coarse tiling has vertex degree $d + 1$.*

Proof. Let the coordinates of any point x be given as x_1, \dots, x_d . For any point x in a k -slab S that contains a vertex at point o , let the *snapping set* $Z(x, S)$ be defined as $\{i : 1 \leq i \leq k \text{ and } x_i = o_i \pmod{1/2^{k-1}}\}$. Note that k -slabs are recursively constructed by translating unit hypercubes over vectors whose elements are multiples of $1/2^{k-1}$; thus all vertices in a k -slab have the same coordinates modulo $1/2^{k-1}$ and $Z(x, S)$ does not depend on the choice of o . A direct consequence of this is the following: if x lies in a $(k-1)$ -slab T within a k -slab S , then $Z(x, S) \supseteq Z(x, T)$. We claim that if x is a point in a k -slab S such that m tiles of S meet in x , then $|Z(x, S)| \geq m - 1$. We will now prove this claim by induction on increasing values of k .

The base case $k = 0$ is trivial: since S contains only one tile, m is at most one, and thus we have $|Z(x, S)| \geq 0 \geq m - 1$.

Now assume that the claim holds for $(k-1)$ -slabs, and consider any point x in a k -slab S , with $k > 0$. We consider two cases: (i) x lies in a single $(k-1)$ -slab T within S ; (ii) x lies on the boundary between two $(k-1)$ -slabs T_1 and T_2 within S . In case (i), if m tiles of S meet in x , then all of them must be tiles of T ; thus, by induction, we have $|Z(x, T)| \geq m - 1$ and thus, $|Z(x, S)| \geq |Z(x, T)| \geq m - 1$, QED. In case (ii), for the sake of contradiction, assume there is a coordinate x_i such that $i \in Z(x, T_1) \cap Z(x, T_2)$. Then $1 \leq i \leq k-1$ and x_i equals, modulo $1/2^{k-2}$, the coordinates of the vertices of both T_1 and T_2 . But this is not possible, since T_2 is translated with respect to T_1 over a distance of $1/2^{k-1} = 1/2 \cdot 1/2^{k-2}$ in each of the first $k-1$ dimensions. Hence, $Z(x, T_1) \cap Z(x, T_2) = \emptyset$. Furthermore, since x lies on the boundary between T_1 and T_2 , we have that x_k is equal to the k -th coordinate of all vertices on the boundary between T_1 and T_2 , and thus, $k \in Z(x, S)$. Now, if m tiles of S meet in x , then $m_1 \geq 1$ of these must be tiles of T_1 and $m_2 \geq 1$ of these must be tiles of T_2 , where $m_1 + m_2 = m$. Now we have that $Z(x, S)$ contains the mutually disjoint sets $Z(x, T_1)$, $Z(x, T_2)$, and $\{k\}$, and thus, $|Z(x, S)| \geq |Z(x, T_1)| + |Z(x, T_2)| + 1 \geq (m_1 - 1) + (m_2 - 1) + 1 = m - 1$, QED.

For a point x in a d -slab S (the full d -dimensional space) such that m tiles of S meet in x , we now have $|Z(x, S)| \geq m - 1$. Since by definition, $|Z(x, S)| \leq d$, it follows that $m \leq d + 1$. Therefore, in the coarse tiling, only $d + 1$ tiles can meet in any point. \square

Lemma 6. *The centre of each tile of the fine tiling lies in the interior of a unique tile T in the coarse tiling, at a distance of at least $\frac{1/2}{D(D+1)}$ from the boundary of T .*

Proof. Take the fixed point of the scaling as the origin of our coordinate system. By construction, the centre $x = (x_1, \dots, x_d)$ of each tile in the coarse tiling satisfies $x_i = 0 \pmod{1/D}$ for $1 \leq i \leq d$, and thus, also $x_i = 0 \pmod{1/(D(D+1))}$. The coordinates of the boundary faces of each tile are equal to $1/2$ or $-1/2$, modulo $1/(D(D+1))$, and thus, to 0 , modulo $1/(D(D+1))$.

Each tile in the scaled tiling has its centre at coordinates that are multiples of $1/(D(D+1))$; after translating the tiling, the centre $x = (x_1, \dots, x_d)$ of each tile in the fine tiling must have $x_i = \frac{1}{2}/(D(D+1)) \pmod{1/(D(D+1))}$ for $1 \leq i \leq d$.

It follows that the tile centres in the fine tiling must lie at a distance of at least $\frac{1}{2}/(D(D+1))$ from the boundary of any tile in the coarse tiling. \square

We assign each tile t in the fine tiling to the tile T of the coarse tiling that contains the centre of t . Now, in each dimension, t sticks out of T by half the width of t minus the distance between the centre of t and the boundary of T . Therefore, from the above lemma it follows directly that t sticks out of T by at most $\frac{1}{2}/(D+1) - \frac{1}{2}/(D(D+1)) = \frac{1}{2}(D-1)/(D(D+1))$ in each dimension. Now we make the tiling recursive, following the procedure explained in Section 2.3, and we prove Theorem 14 as follows.

Take a corner of any tile of the coarse tiling as the origin of our coordinate system, and consider the grid of points whose coordinates are multiples of $1/D$. For any such point x , let $C(x)$ be the set of tiles of the coarse tiling that meet in x . Note that all faces of tiles in the coarse tiling are at coordinates that are multiples of $1/D$. Thus, centered on any grid point x there is a relatively open hypercube $R(x)$ of width $2/D$ that does not intersect any tiles of the coarse tiling apart from those in $C(x)$; we denote the width of $R(x)$ by $w = 2/D$. A single recursive refinement step of the tile boundaries displaces the boundaries by at most $\frac{1}{2}(D-1)/(D(D+1))$; over all levels of recursion, the displacement adds up to at most $(D+1)/D \cdot \frac{1}{2}(D-1)/(D(D+1)) = \frac{1}{2}(D-1)/D^2$, which we denote by p . Thus, in the recursified tiling, there is still a relative open hypercube $R'(x)$ of width $w' = w - 2p = 2/D - 2 \cdot \frac{1}{2}(D-1)/D^2 = (D+1)/D^2$ around any grid point x , such that $R'(x)$ does not intersect any tiles apart from those that correspond to $C(x)$. In each dimension, the regions $R'(x)$ and $R'(y)$ of adjacent grid points x and y , at distance $1/D$ from each other, overlap in a relatively open region of width $w' - 1/D = (D+1)/D^2 - 1/D = 1/D^2$. Hence, any ball of diameter less than $1/D^2$ is completely contained in at least one region $R'(x)$, and therefore it intersects only $|C(x)| \leq d+1$ tiles of the recursified tiling, with total volume at most $d+1$. It follows that the Arrwid number of the recursified tiling is at most $d+1$; this concludes the proof of Theorem 14.

Theorem 15. *Any uniform hypercube tiling has Arrwid number 2^d , and any space-filling curve based on it has Arrwid number $2^d - 1$ or 2^d .*

Proof. The proof is analogous to that of Theorem 9: replace 3, 6, 7 and 8 by d , $2^d - 2$, $2^d - 1$, and 2^d , respectively. \square

7 Discussion and conclusions

Arrwid-optimal curves and tilings. This paper shows that in two dimensions the lowest possible Arrwid number is achieved by certain uniform square space-filling curves (such as the AR^2W^2 curve, the Kochel curve, and Dekking's curve) and by a certain rectangular tiling. However, for $d \geq 3$, no uniform cube space-filling curve can have an Arrwid number as low as the best known rectangular space-filling curve, and the best known rectangular space-filling curve does not have an Arrwid number as low as the best known space-filling curve on a fractal tiling. A three-dimensional rectangular recursive tiling with Arrwid number four or five might exist, but so far, it was not found.

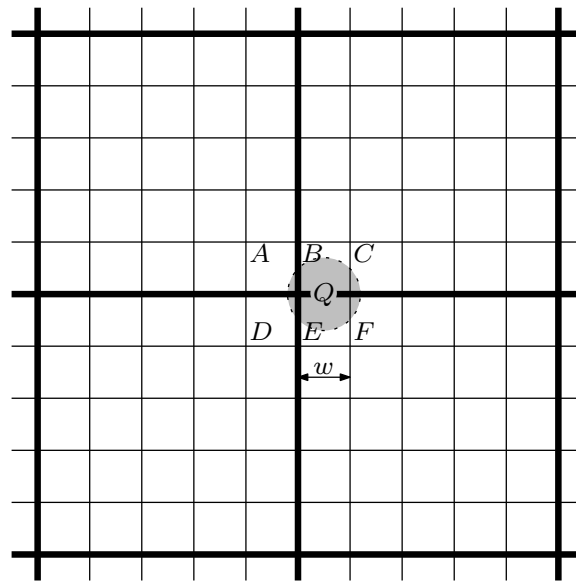


Figure 20: A query range for which straightforward proofs lead to overly pessimistic bounds on the cover ratio of Dekking’s curve.

Our lower bounds for space-filling curves only apply to curves based on tilings with simply connected tiles (in two dimensions) or with convex tiles (in three dimensions). Lifting the restrictions on the shapes of the tiles in these bounds remains a topic for further research. In particular, recursified tilings in three dimensions, such as described in Section 4.1, are not subject to a lower bound for convex tilings. Intuitively, such recursified tilings may nevertheless “inherit” the convex tiling lower bound from the non-recursive tilings from which they are constructed. It may also be interesting to investigate the relation between the Arrwid number of a space-filling curve and its maximum multiplicity (the maximum number of times any point is visited by the curve). Although the Arrwid number and the multiplicity of a curve may differ (Lebesgue’s curve has multiplicity five and Arrwid number four), it is possible that ideas related to the multiplicity of space-filling curves [15] prove useful in deriving bounds on Arrwid numbers, at least for certain classes of curves.

Arrwid numbers versus cover ratios. In principle all proofs of Arrwid numbers come with an upper bound on the cover ratio c . Looking at the proofs in this paper one might get the impression that low Arrwid numbers tend to come with extremely high cover ratios. This may be a shortcoming of the proofs rather than the space-filling curves considered. For example, consider Dekking’s curve, which is based on recursively subdividing squares into 25 squares. Figure 20 shows a disk Q that sticks out just a little bit from the tiles B and E with width w , also intersecting A , C , D and F . The approach from the proof in Section 3.3 would be to cover Q with the parents of these tiles. This results in a cover ratio bound of $100w^2/\frac{\pi}{4}w^2 = 400/\pi$. However, taking the scanning order into account (see Figure 13), we see that Dekking’s curve has the property that if B and C are not adjacent in the order, then E and F must be adjacent, and vice versa. Furthermore, a corner tile and

its neighbour are never far apart in Dekking’s scanning order: there are at most three tiles between them. Note that among A , B , D and E there must also be pair that is consecutive in the scanning order. Therefore Q can be covered with three fragments containing the six tiles A , B , C , D , E , and F , and at most three additional tiles. This results in a cover ratio of at most only $9w^2/\frac{\pi}{4}w^2 = 36/\pi$ (for this particular choice of Q). A more detailed case analysis that takes the scanning order into account may thus give much better bounds on the cover ratios of large tilings than the bounds presented in this paper. With Arrwwid number three, and possibly a cover ratio of at most $36/\pi$, Dekking’s curve might actually have both a better Arrwwid number and a better cover ratio than well-known curves such as Hilbert’s curve [9] or Lebesgue/Z-order [13] (see Observation 1 in Section 2.2).

Worst-case versus average case. Arrwwid numbers only consider the *worst-case* number of tiles or curve fragments that are needed to cover a query range. Therefore, lower Arrwwid numbers do not necessarily give better disk access efficiency on average. It is possible that optimizing the worst case sometimes has an adverse effect on the average-case performance. In the case of square curves of size four there is some intuition to support this concern: getting an Arrwwid number of three requires the use of diagonal connections in this case. For any tile A , let $X(A)$ be the set of query ranges Q such that Q intersects A , but not the next tile of the same size in the scanning order. The total of $|X(A)|$ over all tiles A gives an indication of the probability that after scanning a tile, we must either move the disk head to the next tile that intersects the query range, or we scan a tile with false answers only. Since $X(A)$ is larger when A is connected diagonally to the next tile than when it is connected orthogonally, diagonal connections may lead to decreased performance on average. Note that this does not mean that optimizing the Arrwwid number is bad per se. However, in combination with other constraints—in this case the requirement that each square is subdivided into *four* squares, not nine—rigorously optimizing the (worst-case) Arrwwid number may restrict the set of possible curves so much that it becomes counterproductive. It remains an open question whether it is possible to come up with a precise and sound definition of an *average Arrwwid number*, and whether it is possible to obtain provably correct bounds on that measure for various classes of space-filling curves.

Theory versus practice. In two dimensions, most widely known curves have Arrwwid number four (Peano’s curve [18], Hilbert’s curve [9], Lebesgue/Z-order [13], Sierpiński/Knopp/H-order [17, 19]), while Arrwwid-optimal curves have Arrwwid number three (the AR^2W^2 -curve, the Kochel curve, and Dekking’s curve, all discussed in Section 3.3). One may wonder if there are any practical settings in which it matters much whether three or four fragments are used to cover a query range in the worst case. Average-case performance seems to be more relevant when comparing these curves. I did some rough preliminary experiments for certain ratios between seek time (the cost of “jumping” over a curve fragment outside the query range) and scanning time (the cost of scanning a fragment, relative to its length), but I did not find big differences between the curves. Performance varied by at most 10%, with the simple coil order (Figure 21) giving the best performance, and no apparent advantage for curves with Arrwwid number three rather than four. Proper experiments would be needed to verify these observations and assess their validity.

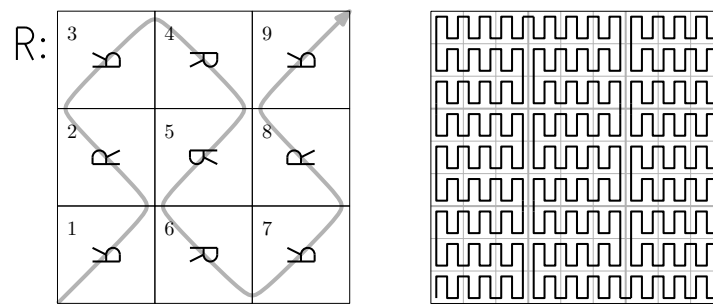


Figure 21: Coil order.

In higher dimensions, the stakes are higher. The gap between the lowest achievable Arrwwid number and the Arrwwid number of regular hypercube tilings increases exponentially with the dimension, and so do the differences in cover ratio that would appear in the analysis of different curves. Thus, in higher dimensions, there is much more room for substantial differences in performance between different curves. Further analysis and possibly experiments would be needed to make proper comparisons.

It is therefore unclear yet whether the research described in this paper may lead to anything that would be useful in practice. This research was an investigation into where optimization of the Arrwwid number leads us. What does it take to optimize the Arrwwid number? How limiting is it to consider only regular square or cube curves? What are tilings and curves with low Arrwwid numbers like? This paper gave some answers to these questions.

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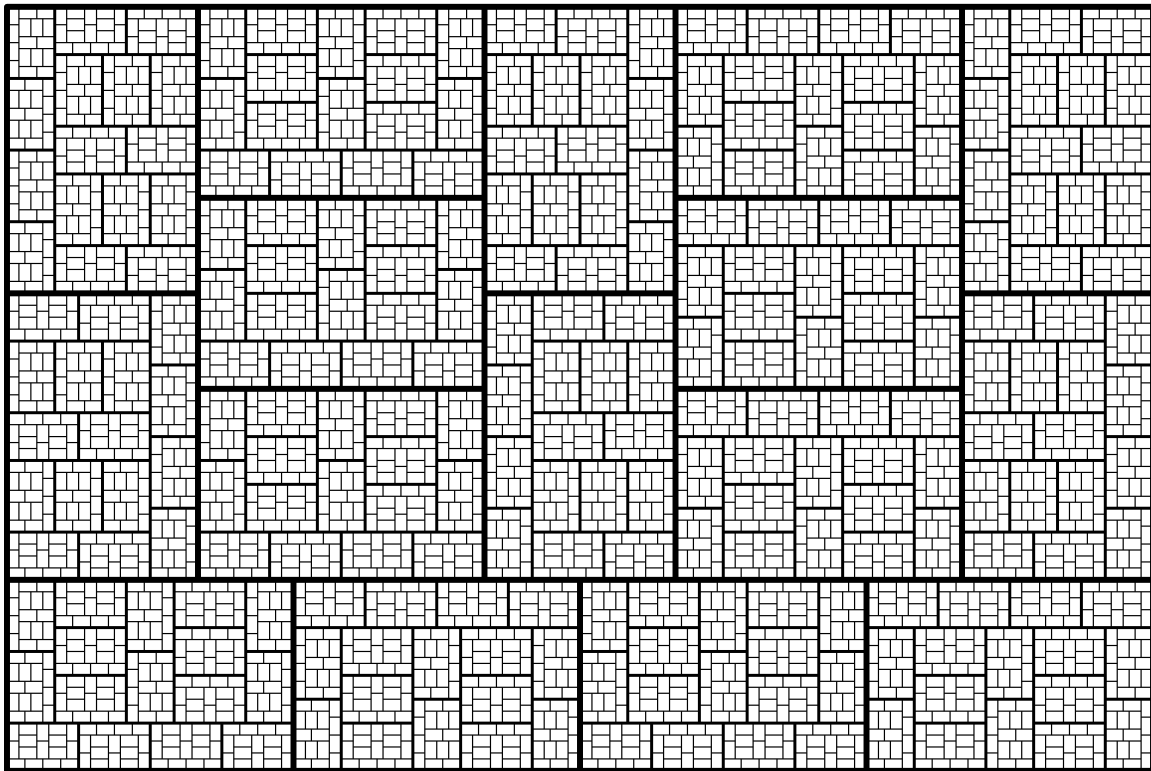


Figure 22: Level-three expansion of the Daun tiling.