Chapter 12. Base Change

This chapter answers the one difficulty arising from backtrack searches : Suppose we have selected an appropriate base, how do we obtain a strong generating set relative to the selected base? We assume we have some base and a strong generating set relative to that base, so we are not in the situation of starting just with the generators of the group. (That situation is discussed in the next chapter.)

Other Bases

Suppose $B = [\beta_1, \beta_2, ..., \beta_k]$ is a base for G, and that S is a strong generating set of G relative to B. What other bases can we easily obtain, and are strong generating sets relative to these other bases also easy to obtain? We present some examples. They are all rather obvious, except the last one.

(1) We can add points to the end of B.

For any point β_{k+1}, not in B, the sequence $\bar{B} = [\beta_1, \beta_2, ..., \beta_k, \beta_{k+1}]$ is a base for G, and S is a strong generating set of G relative to $\bar{B}$.

(2) We can delete redundant base points from the base B.

If $G^{(i)} = G^{(i+1)}$ then β_i is a redundant base point. So $\bar{B} = [\beta_1, \beta_2, ..., \beta_{i-1}, \beta_{i+1}, \beta_{i+2}, ..., \beta_k]$ is a base for G, and S is a strong generating set of G relative to $\bar{B}$.

(2′) We can delete redundant base points from the end of the base B.

If $G^{(k)} = \{id\}$, then $\bar{B} = [\beta_1, \beta_2, ..., \beta_{k-1}]$ is a base for G and S is a strong generating set of G relative to $\bar{B}$.

(3) We can interchange adjacent base points.

The sequence $\bar{B} = [\beta_1, \beta_2, ..., \beta_{j-1}, \beta_{j+1}, \beta_j, \beta_{j+2}, \beta_{j+3}, ..., \beta_k]$ obtained by interchanging the base points β_j and β_{j+1} is a base for G. A strong generating set of G relative to this base is not obtained trivially. A method for constructing a strong generating set is given in a later section.

(4) We can take the image of a base.

If $g \in G$ then the base image $\bar{B} = [\beta_1^g, \beta_2^g, ..., \beta_k^g]$ is a base for G. Furthermore, the conjugate S^g of S is a strong generating set of G relative to the base $\bar{B}$.

To show that the image of the base is a base, and that the conjugate of the strong generating set is the required strong generating set relative to the base image, we show that

$$G_{\beta^g} = g^{-1} \times \left[G_\beta \right] \times g.$$

If h fixes β^g then $g \times h \times g^{-1}$ fixes β, since

$$\beta \xrightarrow{\ g\ } \beta^g \xrightarrow{\ h\ } \beta^g \xrightarrow{\ g^{-1}\ } \beta$$

Hence, $h \in \left[G_\beta \right]^g$. Conversely, if $h \in G_\beta$ then $g^{-1} \times h \times g$ fixes β^g. Hence,

$$G_{\beta^g} = g^{-1} \times \left[G_\beta \right] \times g.$$

Thus, $G_{\beta_1^g, \beta_2^g, \ldots, \beta_k^g}$ is the conjugate of $G_{\beta_1, \beta_2, \ldots, \beta_k}$, which is the identity. Hence, the base image is a base. Furthermore, if the set T generates G_β then the set T^g generates $\left[G_\beta \right]^g$, which is G_{β^g}. Therefore, the conjugate of S contains generating sets for each group in the new stabiliser chain. That is, S^g is a strong generating set relative to $\overline{B}$.

Using (1), (2′), and (3) we can obtain any base, and a strong generating set relative to that base. Using (1), the points of the new base may be appended to the original base. Using (3) repeatedly, these new base points are moved into their correct position, and (2′) deletes the now redundant points of the original base from the end.

We will see later that this approach can be improved through the use of (4).

Interchanging Adjacent Base Points

This section shows how to construct a strong generating set relative to a base obtained by interchanging adjacent base points.

Suppose we are interchanging the base points β_j and β_{j+1}. To make our notation clear, all objects relative to the original base will be as per usual, while those relative to the new base $\overline{B} = [\beta_1, \beta_2, \ldots, \beta_{j-1}, \beta_{j+1}, \beta_j, \beta_{j+2}, \beta_{j+3}, \ldots, \beta_k]$ will have a bar $^-$. Thus,

$$\overline{G}^{(j+1)} = G_{\overline{\beta}_1, \overline{\beta}_2, \ldots, \overline{\beta}_j} = G_{\beta_1, \beta_2, \ldots, \beta_{j-1}, \beta_{j+1}} = G^{(j)}_{\beta_{j+1}}.$$

In fact, $\overline{G}^{(j+1)}$ is the only stabiliser in the new stabiliser chain that is different from its counterpart in the old stabiliser chain. That is, $\overline{G}^{(i)} = G^{(i)}$, for all $i \neq j+1$. The only basic orbits that are different are

$$\overline{\Delta}^{(j)} = \overline{\beta}_j^{\,\overline{G}^{(j)}} = \beta_{j+1}^{\,G^{(j)}}$$

and

$$\overline{\Delta}^{(j+1)} = \overline{\beta}_{j+1}^{\,\overline{G}^{(j+1)}} = \beta_j^{\,\overline{G}^{(j+1)}}.$$

The strong generating set S contains generators for all the stabilisers $\overline{G}^{(i)}$, except perhaps $\overline{G}^{(j+1)}$. Our aim is to construct a set of generators T of $\overline{G}^{(j+1)}$. Then $S \cup T$ is a strong generating set relative to the new base $\overline{B}$.

We know that

$$G^{(j+2)} \leq \overline{G}^{(j+1)} \leq G^{(j)}{}_{\beta_{j+1}} \leq G^{(j)}.$$

An element g of $G^{(j)}$ is in $\overline{G}^{(j+1)}$ if it maps β_j to some point in the basic orbit $\Delta^{(j)}$ and fixes β_{j+1}. We do not care what it does to the remaining base points $\beta_{j+2}, \beta_{j+3}, \ldots, \beta_k$. That is,

$$\begin{array}{ccc} & g & \\ \beta_j & \longrightarrow & \gamma \in \Delta^{(j)} \\ \beta_{j+1} & \longrightarrow & \beta_{j+1} \\ & \cdots & \end{array}$$

One way to think of the problem is as a miniature search : Find elements of G whose base images start with $[\beta_1, \beta_2, ..., \beta_{j-1}, *, \beta_{j+1}]$, where "*" matches any point. A simple algorithm for this based on previous algorithms for searching and for enumerating all the elements of the group is presented as Algorithm 1. The algorithm does not return a set of generators of $\overline{G}^{(j+1)}$, it returns a set of coset representatives of $G^{(j+2)}$ in $\overline{G}^{(j+1)}$. The next algorithm shows it is a simple matter to form a set of generators using the coset representatives.

Algorithm 1 : Miniature Search

Input : a group G;
a base $[\beta_1, \beta_2, ..., \beta_k]$ for G and a strong generating set;
an integer j between 1 and $k-1$;

Output : a set T of coset representatives of $G^{(j+2)}$ in $\overline{G}^{(j+1)}$;

begin

T := empty set;

for each $\gamma \in \Delta^{(j)}$ **do**

find $g_1 \in G^{(j)}$ mapping β_j to γ;

if $\beta_{j+1} \in \left[\Delta^{(j+1)}\right]^{g_1}$ **then**
choose $g_2 \in G^{(j+1)}$ mapping β_{j+1} to $\beta_{j+1}{}^{g_1^{-1}}$;
add $g_2 \times g_1$ to T;
end if;

end for;

end;

Of course, choosing $\gamma = \beta_j$ leads to an element of $G^{(j+2)}$ being added to T, and choosing $\gamma = \beta_{j+1}$ does not lead to a permutation. To obtain a generating set for $\overline{G}^{(j+1)}$ we require generators for $G^{(j+2)}$ as well as the coset representatives of $G^{(j+2)}$ in $\overline{G}^{(j+1)}$. We know that $S^{(j+2)}$ generates $G^{(j+2)}$. Furthermore, as in earlier backtrack searches, we need only consider the points γ that are first in the orbit. In this case, the orbit under $<T>$. Combining these ideas leads to Algorithm 2.

Algorithm 2 : Search for Generators

Input : a group G;
a base $[\beta_1, \beta_2, ..., \beta_k]$ for G and a strong generating set;
an integer j between 1 and $k-1$;

Output : a set T of generators of $\overline{G}^{(j+1)}$;

begin

$T := S^{(j+2)}$; $\Gamma := \Delta^{(j)} - \{\beta_j, \beta_{j+1}\}$;

while $\Gamma \neq$ empty **do**

choose $\gamma \in \Gamma$; find $g_1 \in G^{(j)}$ mapping β_j to γ;
if $\beta_{j+1}^{g_1^{-1}} \in \Delta^{(j+1)}$ **then**
find $g_2 \in G^{(j+1)}$ mapping β_{j+1} to $\beta_{j+1}^{g_1^{-1}}$;
add $g_2 \times g_1$ to T; $\Gamma := \Gamma - \gamma^{<T>}$;
else
$\Gamma := \Gamma - \gamma^{<T>}$;
end if;

end while;

end;

The next improvement comes about because we can determine when the search is finished. We know the order of the subgroup $\overline{G}^{(j+1)}$ that we are searching for because the equations

$$|G| = \prod_{i=1}^{k} |\Delta^{(i)}| = \prod_{i=1}^{k} |\overline{\Delta}^{(i)}|$$

show that

$$|\overline{\Delta}^{(j+1)}| = \frac{|\Delta^{(j)}| \times |\Delta^{(j+1)}|}{|\beta_{j+1}^{G^{(j)}}|}.$$

Hence, the search can be terminated once sufficient generators have been added to T to give an orbit of the correct size. The final algorithm is Algorithm 3. We have presented it as a procedure *interchange* for use by the complete base change algorithm.

Algorithm 3 : Interchange Base Points

Input : a group G;
 a base $[\beta_1, \beta_2, ..., \beta_k]$ for G and a strong generating set;
 an integer j between 1 and $k-1$;

Output : a base $\overline{B} = [\beta_1, \beta_2, ..., \beta_{j-1}, \beta_{j+1}, \beta_j, \beta_{j+2}, \beta_{j+3}, ..., \beta_k]$ for G;
 a strong generating set relative to $\overline{B}$;

procedure *interchange*(**var** B : sequence of points;
 var S : set of elements;
 j : 1..$k-1$);
(* Given a base $B = [\beta_1, \beta_2, ..., \beta_k]$ and a strong generating set S of G,
 return the base obtained by interchanging β_j and β_{j+1},
 and return a strong generating set relative to the new base. *)
begin
 (*find generators T for $\overline{G}^{(j+1)}$ *)
 compute $|\overline{\Delta}^{(j+1)}|$; $T := S^{(j+2)}$; $\Gamma := \Delta^{(j)} - \{\beta_j, \beta_{j+1}\}$;
 $\Delta := \{\beta_j\}$; (* $= \beta_j^{<T>}$ and will grow to be $\overline{\Delta}^{(j+1)}$ *)

 while $|\Delta| \neq |\overline{\Delta}^{(j+1)}|$ **do**

 choose $\gamma \in \Gamma$; find $g_1 \in G^{(j)}$ mapping β_j to γ;
 if $\beta_{j+1}^{g_1^{-1}} \in \Delta^{(j+1)}$ **then**
 find $g_2 \in G^{(j+1)}$ mapping β_{j+1} to $\beta_{j+1}^{g_1^{-1}}$;
 add $g_2 \times g_1$ to T; $\Delta := \beta_j^{<T>}$; $\Gamma := \Gamma - \Delta$;
 else
 $\Gamma := \Gamma - \gamma^{<T>}$;
 end if;

 end while;

 (*return new base and strong generating set*)
 (*only orbits and Schreier vectors at levels j and $j+1$ change*)
 $B := [\beta_1, \beta_2, ..., \beta_{j-1}, \beta_{j+1}, \beta_j, \beta_{j+2}, \beta_{j+3}, ..., \beta_k]$;
 $S := S \cup T$;
end;

Example

Consider the group G of degree 21 and order $3^4 \times 7^3$ which has a base

$$[1, 9, 8, 10, 2, 12]$$

and a strong generating set

s_1=(1,8,9)(2,11,15)(3,10,12)(4,14,19)(5,16,17)(6,21,20)(7,13,18),
s_2=(9,18,20)(12,19,17),
s_3=(10,21,11)(13,16,14),
s_4=(8,13,21)(10,14,16),
s_5=(2,6,3)(4,5,7), and
s_6=(12,20,15)(17,19,18).

The basic indices $|\Delta^{(i)}|$ are

$$27, 7, 7, 3, 3, 3.$$

Suppose we wish to interchange 1 and 9 in the base. Then we require generators for $\overline{G}^{(2)} = G_9$. Since $|9^G| = 21$, we calculate that $|\overline{\Delta}^{(2)}| = 7$. Initially,

$$\Delta = \{1\},\ \Gamma = \{2..8,\ 10..21\}, \text{ and } T = \{s_3, s_4, s_5, s_6\}.$$

Choosing $\gamma = 2$ from Γ gives

$$g_1 = s_1{}^2 \times s_2{}^2 \times s_6 \times s_1 = (1,2,3,4,6,5,7).$$

Then $9^{g_1^{-1}} = 9$, so g_2 is the identity, and we add

$$s_7 = g_1 = (1,2,3,4,6,5,7)$$

to T. Thus,

$$\Delta = \{1..7\},\ \Gamma = \{8,\ 10..21\}, \text{ and } T = \{s_3, s_4, s_5, s_6, s_7\}.$$

Since Δ has the correct size, we are finished. The new base is

[9, 1, 8, 10, 2, 12]

and the new strong generating set is

s_1=(1,8,9)(2,11,15)(3,10,12)(4,14,19)(5,16,17)(6,21,20)(7,13,18),
s_2=(9,18,20)(12,19,17),
s_3=(10,21,11)(13,16,14),
s_4=(8,13,21)(10,14,16),
s_5=(2,6,3)(4,5,7),
s_6=(12,20,15)(17,19,18), and
s_7=(1,2,3,4,6,5,7).

Note that s_2 is redundant in the new strong generating set. It was there to help generate G_1, which is no longer in the stabiliser chain.

Example

Consider now the interchange of 1 and 8, so that the new base will be

$$[9, 8, 1, 10, 2, 12].$$

We require the generators for $\overline{G}^{(3)} = G_{9,8}$. The orbit of 8 under $G^{(2)} = G_9$ is $\{8,10,11,13,14,16,21\}$, and so we calculate that $|\overline{\Delta}^{(3)}| = 7$. Initially,

$$\Delta = \{1\},\ \ \Gamma = \{2..7\}, \text{ and } T = \{s_3, s_5, s_6\}.$$

Choosing $\gamma = 2$ from Γ gives

$$g_1 = s_7 = (1,2,3,4,6,5,7).$$

Then $8^{g_1^{-1}} = 8$, so g_2 is the identity, and we add s_7 to T. Thus,

$$\Delta = \{1..7\},\ \ \Gamma = \text{empty, and } T = \{s_3, s_5, s_6, s_7\}.$$

Since Δ has the correct size, we are finished.

The new strong generating set is the same as the old. By checking not only $S^{(j+2)}$ but also $S^{(j)}$ for elements that fix β_{j+1}, we could have saved ourselves the trouble of duplicating permutations already in S. This simple improvement is used in implementations. We leave it as an exercise for the reader to make the necessary modifications to Algorithm 3.

Analysis of Interchanging Base Points

To analyse Algorithm 3, let

$$\begin{aligned} N_\gamma &= \text{number of choices of } \gamma \in \Gamma \text{ made, and} \\ N_{gen} &= \text{number of generators } g_2 \times g_1 \text{ added to } T. \end{aligned}$$

Then

$$\begin{aligned} N_\gamma &\le |\Delta^{(j)}| - 1, \text{ and} \\ N_{gen} &\le |\overline{\Delta}^{(j+1)}| - 1. \end{aligned}$$

Also note that the size of $\gamma^{<T>}$ in the **else**-clause is bounded by

$$|\Delta^{(j)}| \ - \ |\overline{\Delta}^{(j+1)}|.$$

The cost of initialisation is bounded by

$$\begin{aligned} &|\overline{\Delta}^{(j)}| \times \left[2 \times |S^{(j)}| + 3 \right] + 3 \times |\Omega| && \text{for computing } \beta_{j+1}{}^{G^{(j)}} \\ &+ |\Omega| + 2 && \text{for the assignment } \Gamma := \Delta - \{\beta_j, \beta_{j+1}\} \\ &+ |\Omega| && \text{for the assigment } \Delta := \{\beta_j\}. \end{aligned}$$

The **while**-loop is executed N_γ times. The number of those iterations that execute the **then**-clause is N_{gen}, while $N_\gamma - N_{gen}$ iterations execute the **else**-clause. Hence, the total cost of the **while**-loop is bounded by

N_γ times

$$2 \times |\Delta^{(j)}| \times \left[|\Omega| + 1 \right] \quad \text{to form } g_1$$
$$+ 2 \quad \text{to test the condition } \beta_{j+1}{}^{g_1^{-1}} \in \Delta^{(j+1)}$$

plus N_{gen} times

$$2 \times |\Delta^{(j+1)}| \times \left[\begin{array}{l} |\Omega| + 1 \\ + 2 \times |\Omega| \end{array} \right] \quad \begin{array}{l} \text{to form } g_2 \\ \text{to form the product} \end{array}$$
$$+ |\overline{\Delta}^{(j+1)}| \times \left[\begin{array}{l} 2 \times |T| + 1 \\ + |\Omega| \end{array} \right] + |\Omega| \quad \begin{array}{l} \text{to compute } \Delta \\ \text{to form set difference} \end{array}$$

plus $N_\gamma - N_{gen}$ times

$$\left[|\Delta^{(j)}| - |\overline{\Delta}^{(j+1)}| \right] \times \left[\begin{array}{l} 2 \times |T| + 1 \\ + |\Omega| \end{array} \right] + |\Omega| \quad \begin{array}{l} \text{to form } \gamma^{<T>} \\ \text{to form set difference.} \end{array}$$

The final addition to the cost is to compute the Schreier vector of $\overline{\Delta}^{(j+1)}$. The calculation of

$$\beta_{j+1}{}^{G^{(j)}}$$

already gives a Schreier vector of $\overline{\Delta}^{(j)}$. This adds

$$|\overline{\Delta}^{(j+1)}| \times \left[2 \times |T| + 3 \right] + 3 \times |\Omega|$$

to the total.

The actual total is not enlightening. Bounding the size of the generating sets by $|\Omega|^2$ gives the order of Algorithm 3 as

$$O(|\Omega|^4).$$

In practice, N_{gen} is seldom larger than 1 or 2 and N_γ is generally quite small.

Removing Redundant Strong Generators

We saw in the first example of interchanging base points, that the strong generating set may contain redundancies. In this section we present a quick method of detecting and removing some redundancies. That is, we will look at each strong generator (in some order) and see if it is in the group generated by the previous strong generators. This check is easy if we work up from the bottom of the stabiliser chain. Working from the bottom also guarantees that the result is a strong generating set.

Suppose $T \subseteq S^{(i)}$ and $G^{(i+1)} \leq <T>$. Then we are concerned that T generates $G^{(i)}$. Suppose that $g \in G^{(i)}$. Then $g \in <T>$ if and only if $\beta_i^g \in \beta_i^{<T>}$, because $<T>$ contains $G^{(i+1)}$. This simplifies the redundancy test. The algorithm is Algorithm 4.

Algorithm 4 : Removing Some Redundancies

Input : a base $B = [\beta_1, \beta_2, ..., \beta_k]$;
a strong generating set S of a group G;

Output : a subset T of S that is also a strong generating set of G relative to B;

begin

T := empty;

for $i := k$ **downto** 1 **do**
  for each generator s in $S^{(i)} - S^{(i+1)}$ **do**
    if $\beta_i^s \notin \beta_i^{<T>}$ **then**
      $T := T \cup \{s\}$;
    end if;
  end for;
end for;

end;

The cost of the algorithm is essentially $|T|$ orbit calculations.

Empirical evidence shows that the interchanges of base points introduces many redundancies. The above algorithm applied to the strong generating set produced by "random" base changes (involving several interchanges) often reduces the number of strong generators by 30-60%. (For these figures, the generators of $S^{(i)} - S^{(i+1)}$ were traversed in the reverse order to which they were originally added to the strong generating set.)

Conjugation and the Complete Algorithm

There are two remarks that lead to the work in this section. The first is that we have not used new bases of type (4) - $\bar{B} = [\beta_1^g, \beta_2^g, ..., \beta_k^g]$, where $g \in G$. The second is that the number of interchanges required to position the first point(s) in the new base is(are) generally larger than the number required to position the remaining base points. The first remark can help alleviate the second problem, since the groups we treat are often transitive.

Let B be the old base and $\bar{B}$ be the new base. From chapter 10, we know how to decide whether there exists an element mapping some initial segment of the old base to an initial segment of the new base. We can determine such an element as well. By taking such an element g, and applying a type (4) transformation we have positioned the initial segment of the new base. Interchanges will complete the task.

Algorithm 5 presents the complete base change. Empirical evidence indicates that, on average, the algorithm using conjugation is three times faster than an algorithm using only interchanges.

Algorithm 5 : Complete Base Change

Input : a base $B = [\beta_1, \beta_2, ..., \beta_k]$ and a strong generating set S of a group G;
a sequence $B' = [\beta'_1, \beta'_2, ..., \beta'_{k'}]$ to form an initial segment of the new base;

Output : a base $\overline{B}$ for G that is an extension (or initial segment) of B' and
a strong generating set $\overline{S}$ relative to $\overline{B}$;

begin

(*find a conjugating element g*)
$i := 0$; $g := id$; $more := \beta'_1 \in \Delta^{(1)}$;
while *more* **do**
$i := i + 1$; $g := trace(\beta'^{\,g^{-1}}_i, v^{(i)}) \times g$;
$more := (i+1) \leq \min(k, k')$;
if *more* **then** $more := \beta'^{\,g^{-1}}_{i+1} \in \Delta^{(i+1)}$; **end if**;
end while;

(*conjugate by a nontrivial element*)
if $g \neq id$ **then**
$B := B^g$; $S := S^g$;
translate the basic orbits and Schreier vectors by g;
end if;

(*transpose remaining points into position*)
for $i := i+1$ **to** k' **do**

if $\beta'_i \in B$ **then**
$pos :=$ position of β'_i in B;
else
append β'_i to B; $pos :=$ length of B;
end if;

while $pos \neq i$ **do**
$interchange(B, S, pos-1)$; $pos := pos - 1$;
end while;

end for;

delete redundant base points from the end of B;

(*return results*)
$\overline{B} := B$; $\overline{S} := S$;

end;

Example

Consider the group G of degree 21 and order $3^4 \times 7^3$. Let the original base be

[1, 9, 8, 10, 2, 12].

A base beginning with 9,8 can be obtained by conjugating by $g = s_1{}^2$. The conjugated base is

[9, 8, 1, 3, 15, 10]

and the corresponding strong generating set is

s_1=(1,8,9)(2,11,15)(3,10,12)(4,14,19)(5,16,17)(6,21,20)(7,13,18),
s_2=(8,13,21)(10,14,16),
s_3=(2,3,6)(4,7,5),
s_4=(1,7,6)(4,5,11),
s_5=(12,15,20)(17,18,19), and
s_6=(10,21,11)(13,16,14).

Summary

The base change algorithm is an efficient means of determining a strong generating set relative to a base, provided that some base and strong generating set for the group is known.

Exercises

(1/Easy) Modify Algorithm 3 to initially include in T not only $S^{(j+2)}$ but also those generators of $S^{(j)}$ which fix β_{j+1}.

(2/Easy) Change the base of the group G of degree 21 used in the examples from

[1, 9, 8, 10, 2, 12]

to

[9, 8, 1, 10, 2, 12]

and then to

[1, 2, 8, 9, 10, 12.]

Delete redundant strong generators.

(3/Moderate) For Algorithm 4 develop an algorithm that translates an orbit or Schreier vector by an element g. That is, given an element g of the group, an orbit $\Delta^{(i)}$ or a Schreier vector $v^{(i)}$ of $G^{(i)}$ relative to the generators $S^{(i)}$ compute an orbit $\overline{\Delta}^{(i)}$ or Schreier vector $\overline{v}^{(i)}$ of $\overline{G}^{(i)} = \left[G^{(i)} \right]^g$ relative to the generators $\left[S^{(i)} \right]^g$.

Do this in one scan of the orbit or Schreier vector.

Bibliographical Remarks

The base change algorithm is due to C. C. Sims, "*Determining the conjugacy classes of a permutation group*", **Computers in Algebra and Number Theory** (Proceedings of the Symposium on Applied Mathematics, New York, 1970), G. Birkhoff and M. Hall, Jr (editors), SIAM-AMS Proceedings, volume 4, American Mathematics Society, Providence, Rhode Island, 1971, 191-195. A much fuller description is given in C. C. Sims, "*Computation with permutation groups*", (Proceedings of the Second Symposium on Symbolic and Algebraic Manipulation, Los Angeles, 1971), S. R. Petrick (editor), Association of Computing Machinery, New York, 1971, 23-28. The use of conjugation is investigated in G. Butler, **Computational Approaches to Certain Problems in the Theory of Finite Groups**, Ph. D. Thesis, University of Sydney, 1980, leading to the complete base change algorithm presented in this chapter. This is also the source of our empirical evidence.

The algorithm for removing redundant strong generators was first introduced by Sims in the Los Angeles paper, and later improved by the author in his thesis.

Recently C.A. Brown, L. Finkelstein, and P.W. Purdom, Jr, "*A new base change algorithm for permutation groups*", SIAM Journal of Computing **18**, 5 (1989) 1037-1047, have shown that transposition of base points can be generalized to cyclic right shifts of base points. They present an algorithm and analyse it to be $O(|\Omega|^3)$.