Gaussian Integer Continued Fractions

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Contents

Integer continued fractions

Definition

An *integer continued fraction* is an expression of the form

$$[b_1, b_2, b_3, \dots] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots}},$$

where $b_i \in \mathbb{Z}$ for $i = 1, 2, \dots, n$.

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If the sequence of convergents converges to a real number x, we say that the integer continued fraction converges to x, or is an expansion of x.

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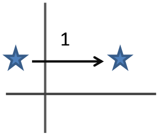

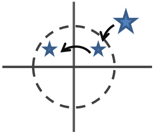

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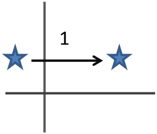

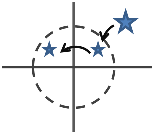

The elements of Γ are the Möbius transformations

$$f(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. They form a discrete group of isometries of the hyperbolic upper half-plane $\mathbb{H}$.

The modular group and continued fractions

Any element of Γ can be written $S^{b_1}TS^{b_2}T\dots S^{b_n}TS^{b_{n+1}}$.

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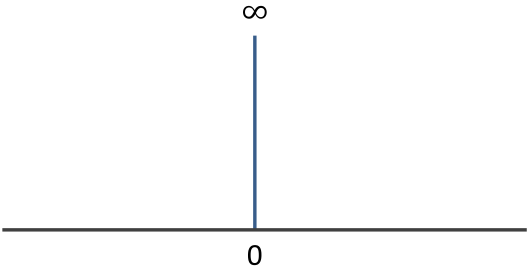
There is a correspondence between elements of Γ and finite integer continued fractions.

The Farey graph

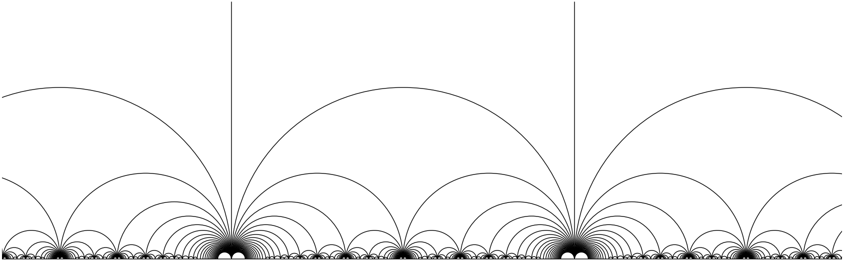
We work in the hyperbolic upper half-plane $\mathbb{H}$. Let L denote the line segment joining 0 to ∞.



The Farey graph, $\mathcal{F}$, is formed as the orbit of L under Γ: Edges are images of L, and vertices are images of ∞, under elements of Γ.

The Farey graph

$\mathcal{F}$ is the 1-skeleton of a tessellation of $\mathbb{H}$ by ideal hyperbolic triangles.



Vertices lie on $\mathbb{R} \cup \{\infty\}$. They are precisely the rational numbers, plus ∞. Each vertex has infinite valency. The vertices neighbouring ∞ are the integers.

Paths in the Farey graph

$$[b_1, b_2, \dots, b_n] = S^{b_1} T S^{b_2} T \dots S^{b_n} T(\infty)$$

so finite integer continued fractions are vertices of $\mathcal{F}$. In particular, each convergent of an infinite integer continued fraction is a vertex of $\mathcal{F}$.

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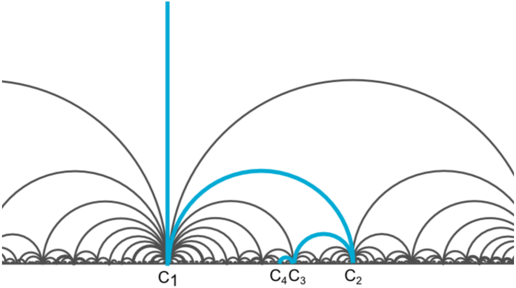
There is a correspondence between integer continued fractions and paths in the Farey graph with initial vertex ∞.

Paths in the Farey graph

Take, for example, $[0, -2, 1, 3, \ldots]$



$$[0] = 0, \quad [0, -2] = \frac{1}{2}, \quad [0, -2, 1] = \frac{1}{3}, \quad [0, -2, 1, 3] = \frac{2}{7}, \ldots$$

The geometry of integer continued fractions

We can reformulate questions about continued fractions into questions about paths. This allows us to

- Interpret the elementary theory of simple continued fractions geometrically.
- Study the convergence of integer continued fractions.
- Investigate the approximation of irrational numbers by rationals.

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Can we develop a similar theory for other classes of continued fractions?

Gaussian integer continued fractions

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A *Gaussian integer continued fraction* is an expression of the form

$$[b_1, b_2, b_3, \dots] = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \dots}},$$

where $b_i \in \mathbb{Z}[i] = \{x + iy \mid x, y \in \mathbb{Z}\}$ for $i = 1, 2, \dots, n$.

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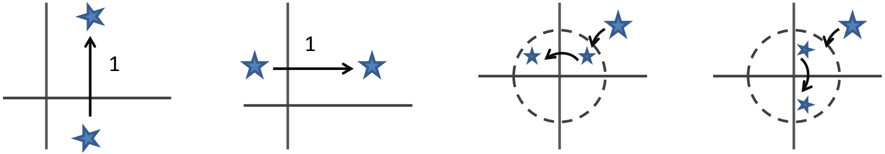

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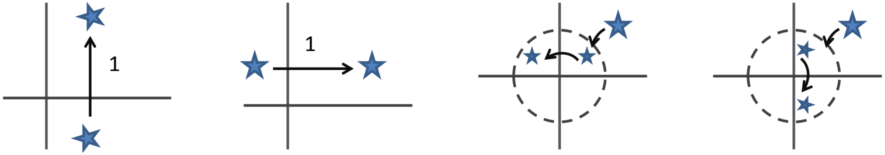

The elements of P are the Möbius transfromations

$$z \mapsto \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{Z}[i]$ and $|ad - bc| = 1$. Their action can be extended via the Poincaré extension to isometries of the hyperbolic upper half-space $\mathbb{H}^3$, and again they form a discrete group.

The Picard group and continued fractions

Given $\alpha \in \mathbb{Z}[i]$ we can define $S_\alpha \in P$ by $S_\alpha(z) = z + \alpha$.

Any element of P can be written $S_{b_1} U S_{b_2} U \dots S_{b_n} U S_{b_{n+1}}$ where each $b_i \in \mathbb{Z}[i]$.

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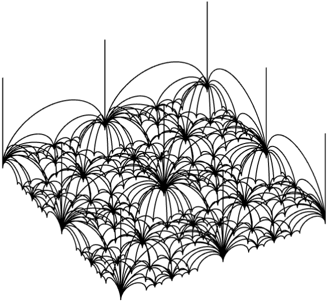
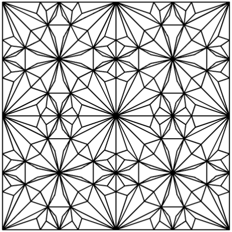
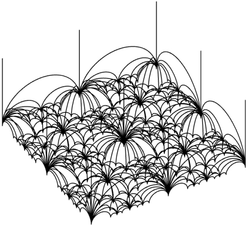
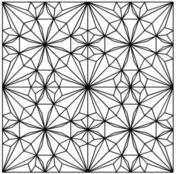
It is a three-dimensional analogue of the Farey graph.





Properties of the Picard graph





- The Picard graph is the 1-skeleton of a tessellation of $\mathbb{H}^3$ by ideal hyperbolic octahedra.
- The vertices $V(\mathcal{G})$ are reduce quotients of Gaussian integers $\frac{a}{c}$: they are precisely those complex numbers with reduced rational real and complex parts, and ∞.
- The edges of $\mathcal{G}$ are hyperbolic geodesics. Two vertices $\frac{a}{c}$ and $\frac{b}{d}$ are joined by an edge in $\mathcal{G}$ if and only if $|ad - bc| = 1$.

Paths in the Picard graph

$$[b_1, b_2, \ldots, b_n] = S_{b_1} U S_{b_2} U \ldots U S_{b_n} T(\infty)$$

so finite Gaussian integer continued fractions are vertices of $\mathcal{G}$. In particular, each convergent C_i of an infinite Gaussian integer continued fraction is a vertex of $\mathcal{G}$.

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There is a correspondence between Gaussian integer continued fractions and paths in the Picard graph with initial vertex ∞.

The geometry of Gaussian integer continued fractions

We can reformulate questions about continued fractions into questions about paths.

- When does a Gaussian integer continued fraction converge?
- Under what conditions is there a unique expansion of every complex number? See, for example, Dani and Nogueira [2].
- Can we investigate the approximation of irrational complex numbers by complex rational numbers?

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When does a Gaussian integer continued fraction converge?

Convergence of Gaussian integer continued fractions

Literature on this topic generally restricts to certain classes of Gaussian integer continued fractions, such as those obtained using algorithms. See, for example, Dani and Nogueira [2].

Can we find a more general condition for convergence that can be applied to all Gaussian integer continued fractions?

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Theorem

An infinite path in $\mathcal{G}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to $z \notin V(\mathcal{G})$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence and has only finitely many accumulation points.

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Corollary

An infinite Gaussian integer continued fraction converges to $z \notin V(\mathcal{G})$ if and only if its sequence of convergents contains no constant subsequence and has only finitely many accumulation points.

Summary

To summarise:

- Gaussian integer continued fractions can be viewed as paths in the Picard graph.
- This technique allows us to find and prove a simple condition for the convergence of Gaussian integer continued fractions.

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- This technique allows us to find and prove a simple condition for the convergence of Gaussian integer continued fractions.

Where next?

- Can we find conditions under which classical theorems of continued fraction theory hold, expanding on the work of Dani and Nogueira in [2]?
- What results can we obtain in the Diophantine approximation of complex numbers? Can we expand on the work of Schmidt [3]?
- Can we use hyperbolic geometry to study other classes of complex continued fractions?

Thank you for your attention

:)

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