Provably Faster Gradient Descent via Long Steps

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Abstract

This work establishes provably faster convergence rates for gradient descent via a computer-assisted analysis technique. Our theory allows nonconstant stepsize policies with frequent long steps potentially violating descent by analyzing the overall effect of many iterations at once rather than the typical one-iteration inductions used in most first-order method analyses. We show that long steps, which may increase the objective value in the short term, lead to provably faster convergence in the long term. A conjecture towards proving a faster $O(1/T \log T)$ rate for gradient descent is also motivated along with simple numerical validation.

1 Introduction

This work proposes a new analysis technique for gradient descent, establishing provably better convergence rates for smooth, convex optimization than the prior state-of-art textbook proofs. Our theory allows for nonconstant stepsize policies, periodically taking larger steps that may violate the monotone decrease in objective value typically needed by analysis. In fact, contrary to the common intuition, we show periodic long steps, which may increase the objective value in the short term, provably speed up convergence in the long term, with increasingly large gains as longer and longer steps are periodically included. This bears a similarity to accelerated momentum methods, which also depart from ensuring a monotone objective decrease at every iteration.

Establishing this requires new proof machinery capable of analyzing the overall effect of many iterations at once rather than the typical (naive) one-iteration inductions used in most first-order method analyses. Our proofs are based on the Performance Estimation Problem (PEP) ideas of [1,2], which cast computing/bounding the worst-case problem instance of a given algorithm as a Semidefinite Program (SDP). We show that the existence of a feasible solution to a related SDP implies a convergence rate for gradient descent using a corresponding pattern of nonconstant stepsizes. This is in contrast to most existing PEP literature where computer-solves help guide the search for tighter convergence proofs [3–12] and inform the development of new, provably faster algorithms [13–17]. Here the computer output itself constitutes the proof.

We consider gradient descent with a sequence of (normalized) stepsizes $h = (h_0, h_1, h_2, ...)$ applied to minimize a convex function $f: \mathbb{R}^n \to \mathbb{R}$ with L-Lipschitz gradient by iterating

$$x_{k+1} = x_k - \frac{h_k}{L} \nabla f(x_k) \tag{1.1}$$

given an initialization $x_0 \in \mathbb{R}^n$. We assume throughout that a minimizer x_* of f exists and its level sets are bounded $D = \sup\{\|x - x_*\|_2 \mid f(x) \le f(x_0)\}^1$. The classic convergence guarantee [18] for gradient descent is that with constant stepsizes $h = (1, 1, 1, \ldots)$, every T > 0 has

$$f(x_T) - f(x_\star) \le \frac{LD^2}{2(T+1)}$$
 (1.2)

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¹This assumption can likely be relaxed but is used to ease our development herein.

The proof of this result relies on the monotone decrease of the distance $||x_k - x_*||_2$ and the objective $f(x_k)$. Such monotone decreases are guaranteed if and only if $h_k \in (0, 2)$. Taylor et al. [2] conjectured that this rate may be improvable by up to a factor of four by optimally tuning the constant stepsize used between one and two.

This $LD^2/2T$ rate has been the long-standing, best-known convergence guarantee and is used to justify constant stepsize policies. We show that provably faster convergence follows from utilizing nonconstant stepsizes, which periodically take longer steps than one can guarantee descent for. For example, consider gradient descent alternating stepsizes $h = (2.9, 1.5, 2.9, 1.5, \ldots)$. Such a scheme is beyond the reach of traditional descent-based analysis as one may fear the stepsizes of 2.9 can increase the function value individually more than the 1.5 is guaranteed to decrease it. Regardless, this "long step" method is provably faster than any existing constant stepsize guarantee. We show

$$f(x_T) - f(x_*) \le \frac{LD^2}{2.2 \times T} + O(1/T^2)$$

for every even T > 0. See our Theorem 3.2 characterizing many such alternating stepsize methods. The $+O(1/T^2)$ term above is used to suppress two universal constants, namely we show there exist constants \bar{s} and C such that all even $T > 2\bar{s}$ have bound $LD^2/(2.2 \times T - C)$.

Using longer cycles, we derive further performance gains. For example, we show a carefully selected stepsize pattern of length 127 periodically taking stepsizes of 370.0 converges at a rate of $LD^2/(5.8346303 \times T)$. Generally, given a stepsize pattern $h = (h_0, \ldots, h_{t-1}) \in \mathbb{R}^t$, we consider the gradient descent method repeatedly applying the pattern of stepsizes

$$x_{k+1} = x_k - \frac{h_{(k \mod t)}}{L} \nabla f(x_k)$$
 (1.3)

In Theorem 2.1, we give a convergence guarantee for any straightforward stepsize pattern h of

$$f(x_T) - f(x_*) \le \frac{LD^2}{\text{avg}(h)T} + O(1/T^2)$$
 (1.4)

See Section 2 for the formal introduction of this straightforwardness property. Hence the design of provably faster nonconstant gradient descent methods amounts to seeking straightforward stepsize patterns with large average stepsize values. Certifying a given pattern is straightforward can be done via semidefinite programming (see our Theorem 3.1). So the convergence rate analysis of such nonconstant stepsizes is a natural candidate for computer assistance. Table 1 shows straightforward stepsize patterns with increasingly fast convergence guarantees, each proven using a computer-generated, exact-arithmetic semidefinite programming solution certificate.

This work demonstrates that prior convergence theory and its focus on constant stepsizes and monotone objective decreases are insufficient to capture gradient descent's potential. A first provably good path forward is offered.

Future works identifying longer straightforward patterns and other tractable families of nonconstant, periodically long stepsize policies will surely be able to improve on this work's particular rates.

The analysis of such nonconstant, long stepsize gradient descent methods has eluded the literature, with one exception. In 1953, Young [19] showed optimal, accelerated convergence is possible for gradient descent when minimizing a smooth, strongly convex quadratic function by using a careful nonconstant selection of h_i . Namely, Young set $h_0 \dots h_{T-1}$ as one over the roots of the T-degree Chebyshev polynomial². No progress has been made beyond quadratics in accelerating gradient

²A nice summary of this is given by the recent blog post [20].

Pattern Length	"Straightforward" Stepsize Pattern h	Convergence Rate
T dettern Bengun	(longest stepsize marked in bold)	$(+O(1/T^2) \text{ omitted})$
t=2	$(3 - \eta, 1.5) \text{for any } \eta \in (0, 3)$	$\frac{LD^2}{(2.25 - \eta/2) \times T}$
t=3	(1.5, 4.9 , 1.5)	$\begin{bmatrix} \frac{LD^2}{2.63333\times T} \\ LD^2 \end{bmatrix}$
t = 7	(1.5, 2.2, 1.5, 12.0 , 1.5, 2.2, 1.5)	$ \frac{LD^{2}}{3.19999999 \times T} \\ LD^{2} $
t = 15	$ \begin{array}{c} (1.4, 2.0, 1.4, 4.5, 1.4, 2.0, 1.4, 29.7, \\ 1.4, 2.0, 1.4, 4.5, 1.4, 2.0, 1.4) \end{array} $	$3.8599999 \times T$
t = 31	(1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 8.2, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 72.3 ,	$\frac{LD^2}{4.6032258 \times T}$
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 8.2, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4)	LD^2
t = 63	$ \begin{array}{c} (1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, \\ 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 14.2, \\ 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, \end{array} $	$\overline{5.2253968 \times T}$
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 164.0 , 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2,	
	$ \begin{vmatrix} 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 14.2, \\ 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, \\ 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4 \end{vmatrix} $	
t = 127	(1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2,	$\frac{LD^2}{5.8346303 \times T}$
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 12.6, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 23.5,	
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 12.6,	
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 370.0 ,	
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 12.6,	
	$ \begin{vmatrix} 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, \\ 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 23.5, \end{vmatrix} $	
	1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.5, 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 12.6,	
	$ \begin{array}{c} 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4, 7.2, \\ 1.4, 2.0, 1.4, 3.9, 1.4, 2.0, 1.4) \end{array} $	

Table 1: Improved convergence rates for Gradient Descent with stepsizes cycling through a "straightforward" pattern. Each convergence rate is proven by producing a certificate of feasibility for a related SDP, which is sufficient by our Theorems 2.1 and 3.1. Coefficients for $t \geq 7$ rates are slightly smaller than the ideal avg(h) due to rounding to produce an exact arithmetic certificate.

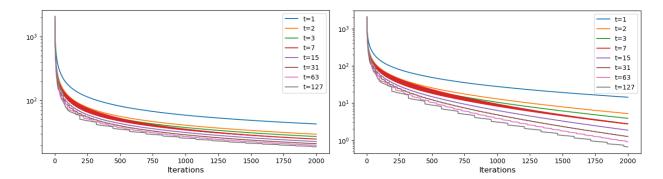


Figure 1: Least squares problems minimizing $||Ax - b||_2^2$ (left) and $||Ax - b||_2^2 + ||x||_2^2$ (right) with i.i.d. normal entries in $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ for n = 4000. Gradient Descent (1.3)'s objective gap is plotted over T = 2000 iterations with h = (1) and with each pattern from Table 1. Note this second objective is substantially more strongly convex, so its faster linear convergence is expected.

descent via long steps. This is partly due to Young's arguments relying fundamentally on framing gradient descent as repeated matrix multiplication, lacking a clear path to generalize. The primary innovation in this work can then be viewed as identifying a tractable analysis technique capable of advancing in this seventy-year-old open direction.

The search for long, straightforward stepsize patterns h is hard; the set of all straightforward patterns is nonconvex, making local searches often unfruitful. Our patterns of length $t = 2^m - 1$ in Table 1 were created by repeating the pattern for $t = 2^{m-1} - 1$ twice with a new long step added in between and (by hand) shrinking the long steps in the length $2^{m-1} - 1$ subpatterns. This recursive pattern has some similarities to the cyclic and fractal Chebyshev patterns for quadratic minimization considered by [21, 22], although we have no provable connection. This doubling procedure consistently increased avg(h) by ≈ 0.6 . We conjecture the following.

Conjecture 1.1. For any t, there exists a straightforward $h \in \mathbb{R}^t$ with $avg(h) = \Omega(\log(t))$.

If true, this would likely yield convergence rates on the order of $O(1/T \log(T))$, strictly improving on the classic O(1/T) guarantee. If such long patterns exist, one natural question is how close to the optimal $O(1/T^2)$ rate attained by momentum methods can be achieved by gradient descent with long steps. The numerics of [12, Figure 2] suggested a $O(1/T^{1.178})$ rate may be possible. The theory of Lee and Wright [23] showed asymptotic o(1/T) convergence for constant stepsize gradient descent, which also motivates the possibility for improved convergence rates like are presented here.

For strongly convex optimization (or, more generally, any problem satisfying a Hölder growth bound), the classic convergence rates for constant stepsize gradient descent are known to improve. We show the same improvements occur for any straightforward stepsize pattern (see Theorem 2.2) with an additional gain of $\operatorname{avg}(h)$. We validate that the convergence speed-ups of Table 1 actually occur on randomly generated least squares problems in Figure 1, seeing gains proportional to $\operatorname{avg}(h)$. These plots also showcase that descent is not ensured within the execution of a straightforward pattern as t=7 rapidly oscillates within each pattern while converging overall.

Outline. In the remainder of this section, we informally sketch how our proof technique proceeds. Then Section 2 formally introduces our notion of straightforward stepsize patterns, showing that any such pattern has a guarantee of the form (1.4). Section 3 shows the existence of a solution to a certain semidefinite program implies straightforwardness. Section 4 concludes by outlining several future directions of interest enabled by and hopefully able to improve on this work.

1.1 Sketch of Proof Technique - Reducing proving eventual descent to an SDP

Our analysis works by guaranteeing a sufficient decrease is achieved after applying the whole pattern $h = (h_0, h_1, \ldots, h_{t-1})$ of t steps (but not necessarily descending at any of the intermediate iterates). Our notion of straightforward stepsize patterns aims to ensure that for all $\delta > 0$ small enough, if $f(x_0) - f(x_*) \leq \delta$, then x_t will always attain a descent of at least

$$f(x_t) - f(x_\star) \le \delta - \frac{\sum_{i=0}^{t-1} h_i}{LD^2} \delta^2$$
 (1.5)

For constant stepsizes equal to one, this amounts to $f(x_0 - \nabla f(x_0)/L) \leq f(x_0) - \delta^2/LD^2$, a classic descent result that holds for all L-smooth convex f. To prove a descent lemma like (1.5), we consider taking a direct combination of several known inequalities. We are given the equalities $x_{k+1} = x_k - (h_k/L)\nabla f(x_k)$ and $\nabla f(x_*) = 0$ and as inequalities an initial distance bound

$$||x_0 - x_\star||_2^2 \le D^2 ,$$

an initial objective gap bound

$$f(x_0) - f(x_\star) \le \delta ,$$

and for any x_i and x_j with $i, j \in \{\star, 0, 1, 2, \dots t\}$, convexity and smoothness imply [24, (2.1.10)]

$$f(x_i) \ge f(x_j) + \nabla f(x_j)^T (x_i - x_j) + \frac{1}{2L} \|\nabla f(x_i) - \nabla f(x_j)\|_2^2$$
.

For any nonnegative multipliers $v(\delta), w(\delta), \lambda_{i,j}(\delta) \geq 0$ (parameterized by the initial objective gap δ), one can combine these inequalities to conclude that on every L-smooth convex function, gradient descent (1.3) with stepsizes h satisfies

$$v(\delta)(\|x_0 - x_\star\|_2^2 - D^2) + w(\delta)(f(x_0) - f(x_\star) - \delta) + \sum_{i,j \in \{\star,0,\dots t\}} \lambda_{i,j}(\delta) \left(f(x_j) + \nabla f(x_j)^T (x_i - x_j) + \frac{1}{2L} \|\nabla f(x_i) - \nabla f(x_j)\|_2^2 - f(x_i) \right) \le 0. \quad (1.6)$$

Our proof then proceeds by showing carefully selected values of $w(\delta), v(\delta), \lambda(\delta)$ reduce this inequality to guaranteeing (1.5). We find to prove our rates in Table 1 it suffices to set $v(\delta) = \frac{\sum_{i=0}^{t-1} h_i}{D^2} \delta^2$, $w(\delta) = 1 - \frac{2\sum_{i=0}^{t-1} h_i}{LD^2} \delta$, and use a linear function $\lambda(\delta) = \lambda + \delta \gamma$. For this choice of $v(\delta)$ and $w(\delta)$, our Theorem 3.1 shows (1.6) implies (1.5) if (λ, γ) is feasible to a certain SDP. Hence proving a t iteration descent lemma of the form (1.5) can be done by semidefinite programming. Our Theorem 2.1 then completes the argument by showing such a periodic descent guarantee implies the rate (1.4).

2 Straightforward Stepsize Patterns

To analyze a given stepsize pattern, we first aim to understand its worst-case problem instance. We do so through the "Performance-Estimation Problem" (PEP) framework of [1]. Given a pattern h and bounds on smoothness L, initial distance to optimal D, and initial objective gap δ , the worst

final objective gap able to be produced by one application of the stepsize pattern is given by

$$p_{L,D}(\delta) := \begin{cases} \max_{x_0, x_{\star}, f} & f(x_t) - f(x_{\star}) \\ \text{s.t.} & f \text{ is convex, } L\text{-smooth} \\ & \|x_0 - x_{\star}\|_2 \le D \\ & f(x_0) - f(x_{\star}) \le \delta \\ & \nabla f(x_{\star}) = 0 \\ & x_{k+1} = x_k - \frac{h_k}{L} \nabla f(x_k) \quad \forall k = 0, \dots, t-1 \ . \end{cases}$$
 (2.1)

The key insight of Drori and Teboulle [1] is that this infinite-dimensional problem can be reformulated as a finite-dimensional semidefinite program. We formalize and utilize this insight in Section 3.

Generally, the worst-case functions f attaining (2.1) can be nontrivial. To find a tractable family of stepsizes for analysis, we focus on ones where this worst-case behavior is no worse than a simple one-dimensional setting: Consider the one-dimensional (nonsmooth) convex function linearly decreasing from x_0 to x_* and constant thereafter. That is, given L, D, δ , consider the problem instance with $f(x) = \max\{\delta x/D, 0\}$, $x_0 = D$, and $x_* = 0$. Provided δ is small enough (i.e., $\delta \leq LD^2/\sum h_i$), the gradient descent iteration $x_{k+1} = x_k - \frac{h_k}{L}f'(x_k)$ has

$$\delta_t = \delta_0 - \frac{\sum_{i=0}^{t-1} h_i}{LD^2} \delta_0^2 \tag{2.2}$$

where $\delta_k = f(x_k) - f(x_*)$ denotes the objective gap. In this example, gradient descent spends all t iterations moving straight forward in a line along the slope. So the descent achieved is just controlled by the objective's slope δ_0/D (squared) and the total length of steps taken $\sum_{i=0}^{t-1} h_i/L$.

We say that a stepsize pattern h is straightforward if its worst-case behavior (over all smooth functions, as defined in (2.1)) is no worse than this one-dimensional piecewise linear setting for δ small enough. Formally, we say that a stepsize pattern h is *straightforward* if for some $\Delta \in (0, 1/2]$,

$$p_{L,D}(\delta) \le \delta - \frac{\sum_{i=0}^{t-1} h_i}{LD^2} \delta^2 \qquad \forall \delta \in [0, LD^2 \Delta]$$

for any L, D > 0. Our analysis of gradient descent with periodic long steps proceeds by:

- (i) Certifying h is straightforward (able to be computer automated) see Theorem 3
- (ii) Solving a resulting recurrence, showing $\delta_T \leq \frac{LD^2}{\operatorname{avg}(h)T} + O(1/T^2)$ see Theorem 2.1. The second step is simpler, so we address it first. Additionally, we show the same factor of rate improvement $\operatorname{avg}(h)$ carries over to more structured domains like strongly convex optimization.

2.1 Convergence Guarantees for Straightforward Stepsize Patterns

Note straightforwardness provides no guarantees on the intermediate objective values of at iterations $k=1,2,\ldots t-1$. We show in our theorem below descent at every tth iteration. To allow for some numerical flexibility, we say a pattern is ϵ -straightforward if for some $\Delta \in (0,1/2]$, all $\delta \in [0,LD^2\Delta]$ have value function bounded by $p_{L,D}(\delta) \leq \delta - \frac{\sum_{i=0}^{t-1}(h_i-\epsilon)}{LD^2}\delta^2$.

Theorem 2.1. Consider any L-smooth, convex f. If $h = (h_0, \ldots, h_{t-1})$ is ϵ -straightforward with parameter $\Delta \in (0, 1/2]$, then gradient descent (1.3) has, for any $s, t \in \mathbb{N}$, after T = st gradient steps

$$f(x_T) - f(x_{\star}) \le \begin{cases} (1 - \sum_{i=0}^{t-1} (h_i - \epsilon) \Delta)^s (f(x_0) - f(x_{\star})) & \text{if } s \le \bar{s} \\ \frac{LD^2}{(\operatorname{avg}(h) - \epsilon)(T - \bar{s}t) + \frac{1}{\Delta}} & \text{if } s > \bar{s} \end{cases}$$
(2.3)

where $\bar{s} = \left[\frac{\log\left(\frac{f(x_0) - f(x_\star)}{LD^2\Delta}\right)}{\log(1 - \sum_{i=0}^{t-1}(h_i - \epsilon)\Delta)}\right]$. In particular, suppressing lower-order terms, this rate is

$$f(x_T) - f(x_*) \le \frac{LD^2}{(\text{avg}(h) - \epsilon)T} + O(1/T^2)$$
.

Proof. We begin by showing $\epsilon \geq 0$ -straightforwardness implies for any $s \in \mathbb{N}$, the recurrence relation

$$\delta_{(s+1)t} \le \begin{cases} \delta_{st} - \frac{\sum_{i=0}^{t-1} (h_i - \epsilon)}{LD^2} \delta_{st}^2 & \text{if } \delta_{st} \le LD^2 \Delta \\ \left(1 - \sum_{i=0}^{t-1} (h_i - \epsilon) \Delta\right) \delta_{st} & \text{if } \delta_{st} > LD^2 \Delta \end{cases}$$
 (2.4)

Consider first s=0. The first case of (2.4) follows from the definition of straightforwardness as $\delta_t \leq p_{L,D}(\delta_0) \leq \delta_0 - \frac{\sum_{i=0}^{t-1} (h_i - \epsilon)}{LD^2} \delta_0^2$. The second case of (2.4) follows from observing that $p_{L,D}(\cdot)$ is concave: Since $p_{L,D}(0) = 0$ and $p_{L,D}(LD^2\Delta) \leq LD^2 \left(\Delta - \sum_{i=0}^{t-1} (h_i - \epsilon)\Delta^2\right)$, every $\delta_0 > LD^2\Delta$ must have

$$\delta_t \le p_{L,D}(\delta_0) \le \frac{\delta_0}{LD^2\Delta} p_{L,D} \left(LD^2\Delta \right) \le \delta_0 \left(1 - \sum_{i=0}^{t-1} (h_i - \epsilon)\Delta \right) .$$

As a result, applying the sequence of steps from the straightforward pattern h from x_0 yields $\delta_t \leq \delta_0$. Hence $||x_t - x_*|| \leq D$ since x_t lies in the initial level set $\{x \mid f(x) \leq f(x_0)\}$. Thus the above reasoning can apply inductively, giving the claimed recurrence.

Next, we show this recurrence implies the claimed convergence rate (2.3). Suppose first, δ_{st} is larger than $LD^2\Delta$. Then applying the pattern h contracts the objective gap³, inductively giving $\delta_{st} \leq (1 - \sum_{i=0}^{t-1} (h_i - \epsilon)\Delta)^s \delta_0$. After at most \bar{s} executions of the stepsize pattern, one must have $\delta_{st} \leq LD^2\Delta$. Afterward, for any $s > \bar{s}$, the objective gap decreases by at least

$$\delta_{(s+1)t} \le \delta_{st} - \frac{\sum_{i=0}^{t-1} (h_i - \epsilon)}{LD^2} \delta_{st}^2.$$

Solving this recurrence with the initial condition $\delta_{\bar{s}t} \leq LD^2\Delta$ gives the claimed sublinear rate

$$\delta_{st} \le \frac{LD^2}{\sum_{i=0}^{t-1} (h_i - \epsilon)(s - \bar{s}) + \frac{1}{\Delta}} . \qquad \Box$$

2.2 Faster Convergence for Straightforward Patterns given Growth Bounds

A function f is μ -strongly convex if $f - \frac{\mu}{2} \| \cdot \|_2^2$ is convex. This condition is well-known to lead to linear convergence for most first-order methods in smooth optimization. More generally, faster convergence occurs whenever f satisfies a Hölder growth or error bound condition. Here we consider settings where all $x \in \mathbb{R}^n$ within the level set $f(x) \leq f(x_0)$ satisfy

$$f(x) - f(x_{\star}) \ge \frac{\mu}{q} \|x - x_{\star}\|_{2}^{q}$$
 (2.5)

Strong convexity implies this condition with q=2 and leads gradient descent with constant $h=(1,1,\ldots)$ to converge at a rate of $O((1-\mu/L)^T)$. When q>2, improved sublinear guarantees of $O((L/\mu^{2/q}T)^{q/(q-2)})$ follow. Below we show that any straightforward stepsize pattern enjoys the same convergence improvements gaining an additional factor of avg(h).

³Interestingly, this contraction factor is independent of L, D. As a result, problem conditioning plays a minimal role in this initial phase of convergence. Instead, only h and the associated straightforwardness parameter Δ matter.

Theorem 2.2. Consider any L-smooth, convex objective f satisfying (2.5). If $h = (h_0, \ldots, h_{t-1})$ is ϵ -straightforward with parameter $\Delta \in (0, 1/2]$, then gradient descent (1.3) has, for any $s, t \in \mathbb{N}$ with $s > \bar{s}$, after T = st gradient steps

$$f(x_T) - f(x_{\star}) \le \begin{cases} q \left(\frac{L}{\mu^{2/q} (q-2) (\operatorname{avg}(h) - \epsilon) (T - \bar{s}t)} \right)^{q/(q-2)} & \text{if } q > 2\\ \left(1 - \frac{\mu(\operatorname{avg}(h) - \epsilon)t}{2L} \right)^{(s-\bar{s})} LD^2 \Delta & \text{if } q = 2 \end{cases}$$

Proof. Let $D_k = \sup\{\|x - x_\star\| \mid f(x) \le f(x_k)\}$ denote the size of each level set visited by gradient descent. The growth bound (2.5) ensures $D_k \le (\frac{q}{\mu}\delta_k)^{1/q}$. Then the recurrence (2.4) implies

$$\delta_{(s+1)t} \le \begin{cases} \delta_{st} - \frac{\mu^{2/q} \sum_{i=0}^{t-1} (h_i - \epsilon)}{Lp^{2/q}} \delta_{st}^{2-2/q} & \text{if } \delta_{st} \le LD_0^2 \Delta \\ \left(1 - \sum_{i=0}^{t-1} (h_i - \epsilon) \Delta\right) \delta_0 & \text{if } \delta_{st} > LD_0^2 \Delta \end{cases}.$$

The analysis of initial linear convergence is unchanged. Once $\delta_0 \leq LD_0^2\Delta$ (after at most \bar{s} applications of the straightforward stepsize pattern), convergence occurs based on the recurrence relation $\delta_{(s+1)t} \leq \delta_{st} - \frac{\mu^{2/q} \sum_{i=0}^{t-1} h_i}{Lq^{2/q}} \delta_{st}^{2-2/q}$. Solving this modified recurrence relation (for example, see [25, Lemma A.1] for this calculation) shows for any $s > \bar{s}$,

$$\delta_{st} \le \begin{cases} q \left(\frac{L}{\mu^{2/q} (q-2) \sum_{i=0}^{t-1} (h_i - \epsilon) (s-\bar{s})} \right)^{q/(q-2)} & \text{if } q > 2\\ \left(1 - \frac{\mu \sum_{i=0}^{t-1} (h_i - \epsilon)}{2L} \right)^{(s-\bar{s})} L D_0^2 \Delta & \text{if } q = 2 \end{cases}.$$

3 Certificates of Straightforwardness

All that remains is to show how one can certify the straightforwardness of a stepsize pattern. We do this in two steps. First, Section 3.1 shows that $p_{L,D}(\delta)$ is upper bounded by an SDP minimization problem using PEP techniques. Hence straightforwardness is implied by showing the SDP corresponding to each $\delta \in [0, LD^2\Delta]$ has a sufficiently good feasible solution. Second, Section 3.2 shows that another semidefinite programming feasibility problem can certify that such an interval of solutions exists.

3.1 The Worst-Case Value Function is Upper Bounded by an SDP

We first reformulate the infinite-dimensional problem (2.1) as a finite-dimensional nonconvex quadratically constrained quadratic problem (see (3.1)), relax that formulation into a SDP (see (3.2)), and then upper bound the SDP by its dual problem (see (3.3)). This process has been detailed by [1,4,12] and is carried out in our setting below (following the notations of [12]).

Step 1: A QCQP reformulation. First, using the interpolation theorem of [2], $p_{L,D}(\delta)$ equals the following finite-dimensional nonconvex problem optimizing over all possible objective values f_k

and gradients g_k at the points x_k with $k \in I_t^* := \{\star, 0, 1, \dots t\}$

$$p_{L,D}(\delta) = \begin{cases} \max_{x_0, f, g} & f_t - f_{\star} \\ \text{s.t.} & f_i \ge f_j + g_j^T (x_i - x_j) + \frac{1}{2L} \|g_i - g_j\|_2^2 & \forall i, j \in I_t^{\star} \\ & \|x_0 - x_{\star}\|_2^2 \le D^2 \\ & f_0 - f_{\star} \le \delta \\ & x_{\star} = 0, f_{\star} = 0, g_{\star} = 0 \\ & x_{i+1} = x_i - \frac{h_i}{L} g_i & \forall i = 0, \dots, t - 1 \end{cases}$$

$$(3.1)$$

where, without loss of generality, we have fixed $x_{\star} = 0, f_{\star} = 0, g_{\star} = 0$.

Step 2: An SDP relaxation. Second, we relax the nonconvex problem (3.1) to the following SDP. Following the notation of [12], define

$$H := [x_0 \mid g_0 \mid g_1 \mid \dots \mid g_t] \in \mathbb{R}^{d \times (t+2)} ,$$

$$G := H^T H \in \mathbb{S}^{t+2}_+ ,$$

$$F := [f_0 \mid f_1 \mid \dots \mid f_t] \in \mathbb{R}^{1 \times (t+1)} ,$$

with the following notation for selecting columns and elements of H and F:

$$\mathbf{g}_{\star} := 0 \in \mathbb{R}^{t+2}, \ \mathbf{g}_{i} := e_{i+2} \in \mathbb{R}^{t+2}, \quad i \in [0:t]$$

$$\mathbf{x}_{0} := e_{1} \in \mathbb{R}^{t+2}, \ \mathbf{x}_{\star} := 0 \in \mathbb{R}^{t+2},$$

$$\mathbf{x}_{i} := \mathbf{x}_{0} - \frac{1}{L} \sum_{j=0}^{i-1} h_{j} \mathbf{g}_{j} \in \mathbb{R}^{t+2}, \quad i \in [1:t]$$

$$\mathbf{f}_{\star} := 0 \in \mathbb{R}^{t+1}, \ \mathbf{f}_{i} := e_{i+1} \in \mathbb{R}^{t+1}, \quad i \in [0:t].$$

This notation ensures $x_i = H\mathbf{x}_i$, $g_i = H\mathbf{g}_i$, and $f_i = F\mathbf{f}_i$. Furthermore, for $i, j \in I_t^*$, define

$$A_{i,j}(h) := \mathbf{g}_j \odot (\mathbf{x}_i - \mathbf{x}_j) \in \mathbb{S}^{t+2} ,$$

$$B_{i,j}(h) := (\mathbf{x}_i - \mathbf{x}_j) \odot (\mathbf{x}_i - \mathbf{x}_j) \in \mathbb{S}_+^{t+2} ,$$

$$C_{i,j} := (\mathbf{g}_i - \mathbf{g}_j) \odot (\mathbf{g}_i - \mathbf{g}_j) \in \mathbb{S}_+^{t+2} ,$$

$$a_{i,j} := \mathbf{f}_j - \mathbf{f}_i \in \mathbb{R}^{t+1}$$

where $x \odot y = \frac{1}{2}(xy^T + yx^T)$ denotes the symmetric outer product. This notation is defined so that $g_j^T(x_i - x_j) = \text{Tr}GA_{i,j}(h)$, $||x_i - x_j||_2^2 = \text{Tr}GB_{i,j}(h)$, and $||g_i - g_j||_2^2 = \text{Tr}GC_{i,j}$ for any $i, j \in I_t^*$. Then the QCQP formulation (3.1) can be relaxed⁴ to

$$p_{L,D}(\delta) \leq \begin{cases} \max_{F,G} & F\mathbf{f}_{t} \\ \text{s.t.} & Fa_{i,j} + \text{Tr}GA_{i,j}(h) + \frac{1}{2L}\text{Tr}GC_{i,j} \leq 0, \quad i, j \in I_{t}^{\star} : i \neq j \\ & -G \leq 0 \\ & \text{Tr}GB_{0,\star} \leq D^{2} \\ & F\mathbf{f}_{0} \leq \delta \end{cases}$$

$$(3.2)$$

with decision variables $F \in \mathbb{R}^{1 \times (t+1)}$ and $G \in \mathbb{R}^{(t+2) \times (t+2)}$.

⁴Under an additional rank condition (that the problem dimension n exceeds t + 2), the QCQP problem (3.1) and SDP (3.2) are actually equivalent. However, this is not needed for our analysis, so we make no such assumption.

Step 3: The upper bounding dual SDP. Third, we note the maximization SDP (3.2) is bounded above by its dual minimization SDP by weak duality (note strong duality is not assumed):

$$p_{L,D}(\delta) \leq \begin{cases} \min_{\lambda,v,w,Z} & D^{2}v + \delta w \\ \text{s.t.} & \sum_{i,j \in I_{t}^{\star}: i \neq j} \lambda_{i,j} a_{i,j} = a_{\star,t} - w a_{\star,0} \\ & v B_{0,\star} + \sum_{i,j \in I_{t}^{\star}: i \neq j} \lambda_{i,j} \left(A_{i,j}(h) + \frac{1}{2L} C_{i,j} \right) = Z \\ & Z \succeq 0 \\ & v, w \geq 0, \ \lambda_{i,j} \geq 0, \quad i, j \in I_{t}^{\star}: i \neq j \ . \end{cases}$$
(3.3)

3.2 An SDP Feasibility Certificate that implies Straightforwardness

The preceding bound (3.3) establishes that $\epsilon \geq 0$ -straightforwardness holds if for some $\Delta \in (0, 1/2]$, every $\delta \in [0, LD^2\Delta]$ has a corresponding dual feasible solution with objective at most $\delta - \frac{\sum_{i=0}^{t-1} (h_i - \epsilon)}{LD^2} \delta^2$. We claim that it suffices to fix L = 1, D = 1 without loss of generality. For any L-smooth f with $||x_0 - x_{\star}|| \leq D$, this follows by instead considering minimizing $\tilde{f}(\tilde{x}) = \frac{1}{LD^2} f(D\tilde{x})$. One can easily verify \tilde{f} is 1-smooth, has $||\tilde{x}_0 - \tilde{x}_{\star}|| \leq 1$ for $\tilde{x}_0 = Dx_0, \tilde{x}_{\star} = Dx_{\star}$, and gradient descent $\tilde{x}_{k+1} = \tilde{x}_k - h_k \nabla \tilde{f}(x_k)$ produces exactly the iterates of $x_{k+1} = x_k - h_k / L \nabla f(x_k)$ rescaled by D. Hence $p_{L,D}(\delta) = LD^2 p_{1,1}(\delta/LD^2)$.

We restrict our search for dual certificates bounding $p_{1,1}(\delta)$ to a special case, which we numerically observed to hold approximately at the minimizers of (3.3): given δ , fix $v = \sum_{i=0}^{t-1} (h_i + \epsilon) \delta^2$ and $w = 1 - 2 \sum_{i=0}^{t-1} h_i \delta$. Noting this fixed variable setting has $v + \delta w = \delta - \sum_{i=0}^{t-1} (h_i - \epsilon) \delta^2$, ϵ -straightforwardness follows if one can show feasible solutions with these fixed values exist.

Given δ and a selection of $\lambda \in \mathbb{R}^{(t+2)\times(t+2)}$ and fixing v as above, we define

$$Z_{h,\epsilon}(\lambda,\delta) := \sum_{i=0}^{t-1} (h_i + \epsilon) \delta^2 B_{0,\star} + \sum_{i,j \in I_t^\star: i \neq j} \lambda_{i,j} \left(A_{i,j}(h) + \frac{1}{2} C_{i,j} \right) . \tag{3.4}$$

Observe that $Z_{h,\epsilon}(\lambda,\delta)$ is nearly linear: the first entry has the only nonlinear behavior, depending quadratically on δ , with the rest depending only linearly on λ . Written in block form, we denote

$$Z_{h,\epsilon}(\lambda,\delta) =: \begin{bmatrix} \sum_{i=0}^{t-1} (h_i + \epsilon)\delta^2 & m_h(\lambda)^T \\ m_h(\lambda) & M_h(\lambda) \end{bmatrix}$$
(3.5)

where $m_h : \mathbb{R}^{(t+2)\times(t+2)} \to \mathbb{R}^{t+1}$ and $M_h : \mathbb{R}^{(t+2)\times(t+2)} \to \mathbb{R}^{(t+1)\times(t+1)}$ are linear functions. Certifying $p(\delta) \leq \delta - \sum_{i=0}^{t-1} (h_i - \epsilon) \delta^2$ for fixed δ then follows by showing the following spectral set is nonempty

$$\mathcal{R}_{h,\epsilon,\delta} = \left\{ \lambda \in \mathbb{R}^{(t+2)\times(t+2)} \mid \begin{array}{l} \sum_{i,j\in I_t^*: i\neq j} \lambda_{i,j} a_{i,j} = a_{\star,t} - \left(1 - 2\sum_{i=0}^{t-1} h_i \delta\right) a_{\star,0} \\ \lambda \geq 0 \\ Z_{h,\epsilon}(\lambda,\delta) \succeq 0 \end{array} \right\} .$$

Lemma 3.1. A stepsize pattern $h \in \mathbb{R}^t$ is $\epsilon \geq 0$ -straightforward if for some $\Delta \in (0, 1/2]$, $\mathcal{R}_{h,\epsilon,\delta}$ is nonempty for all $\delta \in [0, \Delta]$. Straightforwardness of h is implied by each $\mathcal{R}_{h,0,\delta}$ being nonempty.

This lemma alone does not directly enable the computation of a convergence-proof certificate. One would need certificates of feasibility for the infinitely many sets given by each $\delta \in [0, \Delta]$. The following theorem shows that the existence of such solutions can be certified via a single feasible solution to yet another semidefinite program.

Theorem 3.1. A stepsize pattern $h \in \mathbb{R}^t$ is $\epsilon \geq 0$ -straightforward if for some $\Delta \in (0, 1/2]$, $\mathcal{S}_{h,\epsilon,\Delta}$ is nonempty where

$$S_{h,\epsilon,\Delta} = \left\{ (\lambda,\gamma) \in \mathbb{R}^{(t+2)\times(t+2)} \times \mathbb{R}^{(t+2)\times(t+2)} \mid \begin{array}{l} \sum_{i,j\in I_t^*: i\neq j} \lambda_{i,j} a_{i,j} = a_{\star,t} - a_{\star,0} \\ \sum_{i,j\in I_t^*: i\neq j} \gamma_{i,j} a_{i,j} = 2 \sum_{i=0}^{t-1} h_i a_{\star,0} \\ m_h(\lambda) = 0 \\ \lambda \geq 0, \lambda + \Delta \gamma \geq 0 \\ \left[\sum_{i=0}^{t-1} (h_i + \epsilon) & m_h(\gamma)^T \\ m_h(\gamma) & M_h(\lambda) \right] \succeq 0 \\ \left[\sum_{i=0}^{t-1} (h_i + \epsilon) & m_h(\gamma)^T \\ m_h(\gamma) & M_h(\lambda + \Delta \gamma) \right] \succeq 0 \end{array} \right\}.$$

Proof. Let $(\lambda, \gamma) \in \mathcal{S}_{h,\epsilon,\Delta}$. We prove this by showing $\lambda^{(\delta)} := \lambda + \delta \gamma \in \mathcal{R}_{h,\epsilon,\delta}$ for every $\delta \in [0, \Delta]$ by Lemma 3.1. This amounts to verifying the three conditions defining $\mathcal{R}_{h,\epsilon,\delta}$ for each $\lambda^{(\delta)}$.

First, we check $\sum_{i,j\in I_t^*:i\neq j} \lambda_{i,j}^{(\delta)} a_{i,j} = a_{\star,t} - \left(1 - 2\sum_{i=0}^{t-1} h_i \delta\right) a_{\star,0}$. The first equality defining $S_{h,\epsilon,\Delta}$ ensures this for $\lambda^{(0)} = \lambda$. Adding δ times the second equality defining $S_{h,\epsilon,\Delta}$ establishes the equality for every $\lambda^{(\delta)}$ as $\sum_{i,j\in I_t^*:i\neq j} (\lambda_{i,j}^{(\delta)} + \delta \gamma_{i,j}) a_{i,j} = a_{\star,t} - \left(1 - 2\sum_{i=0}^{t-1} h_i \delta\right) a_{\star,0}$. Second, we check nonnegativity $\lambda^{(\delta)} \geq 0$. This follows by noting $\lambda^{(\delta)}$ is a convex combination of λ and $\lambda + \Delta \gamma$, which are nonnegative by construction. Finally, we check the nonlinear (but nearly linear) condition $Z_{h,\epsilon}(\lambda^{(\delta)},\delta) \succeq 0$. We consider the block-form (3.5) of this semidefinite inequality, seeking

$$Z_{h,\epsilon}(\lambda^{(\delta)}, \delta) = \begin{bmatrix} \sum_{i=0}^{t-1} (h_i + \epsilon) \delta^2 & m_h(\lambda^{(\delta)})^T \\ m_h(\lambda^{(\delta)})) & M_h(\lambda^{(\delta)}) \end{bmatrix} \succeq 0.$$

Since $m_h(\lambda) = 0$, using the linearity of m_h and M_h , the above can be expanded to equal

$$\begin{bmatrix} \sum_{i=0}^{t-1} (h_i + \epsilon) \delta^2 & \delta m_h(\gamma)^T \\ \delta m_h(\gamma) & M_h(\lambda) + \delta M(\gamma) \end{bmatrix} \succeq 0.$$

Rescaling the first row and column by $1/\delta$ gives an equivalent condition, which is now linear in δ ,

$$\begin{bmatrix} \sum_{i=0}^{t-1} (h_i + \epsilon) & m_h(\gamma)^T \\ m_h(\gamma) & M(\lambda) + \delta M_h(\gamma) \end{bmatrix} \succeq 0.$$

When $\delta = 0$ or Δ , this condition is explicitly ensured by the definition of $S_{h,\epsilon,\Delta}$. Then the linearity and convexity of this condition imply it holds for all intermediate $\lambda^{(\delta)}$, completing the proof. \square

3.3 Certificates of Straightforwardness Proving Faster Rates in Table 1

To prove a given pattern h converges at rate $LD^2/\text{avg}(h)T$, we only need to show some $\mathcal{S}_{h,0,\Delta}$ is nonempty. The most natural path is to provide an exact member of this set. For all of the stepsizes in Table 1, it was relatively easy to find a feasible solution $\mathcal{S}_{h,0,\Delta}$ in floating point arithmetic via an interior point method. However, exactly identifying a member of $\mathcal{S}_{h,0,\Delta}$ from this can still be hard.

First, we prove the claimed rates for the t=2 and t=3 stepsize patterns of Table 1 by presenting exact members of $S_{h,0,\Delta}$. Then, to handle larger values of t, we present a simple rounding approach able to produce members of $S_{h,\epsilon,\Delta}$, often with ϵ around machine precision $\leq 10^{-9}$. This approach produced rational-valued certificates proving the rest of the claimed convergence rates in Table 1.

The exact rational arithmetic verifying the correctness of all certificates (λ, γ) was done in Mathematica 13.0.1.0. Note that the entries in these certificates for $t \geq 7$ are entirely computer-generated and lack real human insight. As an example for reference, the certificate for t=7 is included in the appendix. Larger certificates are impractical to include here. For example, our t=127 rate is certificate (λ, γ) has 32640 nonzero entries. Certificates for every pattern in Table 1 and exact verifying computations are available at github.com/bgrimmer/LongStepCertificates.

Theorem 3.2. For any $\eta \in (0,3)$, the stepsize pattern $h = (3 - \eta, 1.5)$ is straightforward. Hence gradient descent (1.3) alternating between these two stepsizes has every even T satisfy

$$f(x_T) - f(x_*) \le \frac{LD^2}{(2.25 - \eta/2) \times T} + O(1/T^2)$$
.

Proof. For any $\eta \in (0,3)$, consider the selection of (λ, γ) given by:

$$\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} , \qquad \gamma = \begin{bmatrix} 0 & 3 - \eta & \frac{6 - \eta}{2} & \frac{6 - \eta}{2} \\ 0 & 0 & \frac{-(6 - \eta)}{2} & \frac{-(6 - \eta)}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

It suffices to show for some $\Delta \in (0, 1/2]$, $(\lambda, \gamma) \in \mathcal{S}_{h,0,\Delta}$. One can easily verify the needed equalities and nonnegativities hold for all $0 \le \Delta \le 1/(6-\eta)$. The first positive semidefiniteness condition of $(\lambda, \gamma) \in \mathcal{S}_{h,0,\Delta}$ amounts to checking every $\eta \in (0,3)$ has

$$\begin{bmatrix} \sum_{i=0}^{t-1} (h_i + \epsilon) & m_h(\gamma)^T \\ m_h(\gamma) & M_h(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{9-2\eta}{2} & \frac{-(3-\eta)}{2} & \frac{-(6-\eta)}{4} & \frac{-(6-\eta)}{4} \\ \frac{-(3-\eta)}{2} & \frac{1}{2} & \frac{2-\eta}{4} & \frac{2-\eta}{4} \\ \frac{-(6-\eta)}{4} & \frac{2-\eta}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{-(6-\eta)}{4} & \frac{2-\eta}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \succeq 0.$$

Since this convex condition is linear in η , it suffices to check it at $\eta = 0$ and $\eta = 3$. Moreover, for any $\eta \in (0,3)$, note this matrix has exactly two zero eigenvalues with associated eigenvectors spanning (1/2,1/2,1,0) and (1/2,1/2,0,1). The second positive semidefiniteness condition amounts to checking an update to this matrix of size Δ remains positive semidefinite, namely

$$\begin{bmatrix} \sum_{i=0}^{t-1} (h_i + \epsilon) & m_h(\gamma)^T \\ m_h(\gamma) & M_h(\lambda + \Delta \gamma) \end{bmatrix} = \begin{bmatrix} \frac{9-2\eta}{2} & \frac{-(3-\eta)}{2} & \frac{-(6-\eta)}{4} & \frac{-(6-\eta)}{4} \\ \frac{-(3-\eta)}{2} & \frac{1}{2} & \frac{2-\eta}{4} & \frac{2-\eta}{4} \\ \frac{-(6-\eta)}{4} & \frac{2-\eta}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{-(6-\eta)}{4} & \frac{2-\eta}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-3}{2} & \frac{6-\eta}{4} & \frac{6-\eta}{4} \\ 0 & \frac{6-\eta}{4} & 0 & 0 \\ 0 & \frac{6-\eta}{4} & 0 & 0 \end{bmatrix} \Delta \succeq 0.$$

One can check this added matrix term is positive semidefinite on the subspace spanned by (1/2, 1/2, 1, 0) and (1/2, 1/2, 0, 1) (again by checking when $\eta = 0$ and $\eta = 3$ and then using convexity). As a result, positive semidefiniteness is maintained for Δ small enough. Exact arithmetic verifying all of these claims are given in the associated Mathematica notebook. Hence $(\lambda, \gamma) \in \mathcal{S}_{h,0,\Delta}$, proving the main claim by Theorem 3.1 and the claimed convergence rate by Theorem 2.1.

Theorem 3.3. The stepsize pattern h = (1.5, 4.9, 1.5) is straightforward. Hence gradient descent (1.3) alternating between these three stepsizes has every T = 3s satisfy

$$f(x_T) - f(x_\star) \le \frac{LD^2}{2.63333... \times T} + O(1/T^2)$$
.

Proof. This result is certified with $\Delta = 10^{-4}$ by the following exact values for $(\lambda, \gamma) \in \mathcal{S}_{h,0,\Delta}$ of

$$\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.95 & 0.003 & 0.007 \\ 0 & 0.95 & 0 & 0.5 & 0.5 \\ 0 & 0.006 & 0 & 0 & 0.51 \\ 0 & 0.004 & 0 & 0.013 & 0 \end{bmatrix} , \qquad \gamma = \begin{bmatrix} 0 & 0.005 & 7.825 & 3.9497 & 4.0203 \\ 0 & 0 & -5.24 & -10.555 & 0 \\ 0 & 0 & 0 & 7.9 & -5.315 \\ 0 & 0 & 0 & 0 & 1.2947 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \ \Box$$

Based on numerical exploration, we conjecture that patterns of the form $(3 - \eta, 1.5)$ are the longest straightforward patterns of length two and $(1.5, 5 - \eta, 1.5)$ are the longest of length three.

For larger settings $t \geq 7$, determining an exact member of $S_{h,0,\Delta}$ proved difficult. So we resort to a fully automated construction of a convergence rate certificate by first numerically computing an approximate member of $S_{h,0,\Delta}$ (via an interior point method) and then rounding to a nearby rational-valued exact member of $S_{h,\epsilon,\Delta}$ for some small ϵ . In light of our Theorem 2.1, such rounding only weakens the resulting guarantee's coefficient from avg(h) to $avg(h) - \epsilon$.

Computer Generation of Convergence Rate Proof Certificates. Given a pattern h and Δ . (i) Numerically compute some $(\lambda, \tilde{\gamma})$ approximately in $S_{h,0,\Delta}$,

- (ii) Compute rational $(\hat{\lambda}, \hat{\gamma})$ near $(\tilde{\lambda}, \tilde{\gamma})$ exactly satisfying the three needed equalities,
- (iii) Check in exact arithmetic nonnegativity and positive definiteness of $M_h(\hat{\lambda})$ and $M_h(\hat{\lambda} + \Delta \hat{\gamma})$, (iv) If so, $(\hat{\lambda}, \hat{\gamma}) \in \mathcal{S}_{h,\epsilon,\Delta}$, certifying a $\frac{LD^2}{(\operatorname{avg}(h) \epsilon)T} + O(1/T^2)$ convergence rate, for

$$\epsilon = \frac{\max\{m_h(\hat{\gamma})^T M_h(\hat{\lambda})^{-1} m_h(\hat{\gamma}), \ m_h(\hat{\gamma})^T M_h(\hat{\lambda} + \Delta \hat{\gamma})^{-1} m_h(\hat{\gamma})\}}{t} - \operatorname{avg}(h) \ .$$

The above value of ϵ is the smallest value with $(\hat{\lambda}, \hat{\gamma}) \in \mathcal{S}_{h, \epsilon, \Delta}$, since by considering their Schur complements, the two needed positive semidefinite conditions hold if and only if

$$\sum_{i=0}^{t-1} (h_i + \epsilon) - m_h(\hat{\gamma})^T M_h(\hat{\lambda})^{-1} m_h(\hat{\gamma}) \ge 0 \text{ and } \sum_{i=0}^{t-1} (h_i + \epsilon) - m_h(\hat{\gamma})^T M_h(\hat{\lambda} + \Delta \hat{\gamma})^{-1} m_h(\hat{\gamma}) \ge 0.$$

Theorem 3.4. The patterns of lengths $t \in \{7, 15, 31, 63, 127\}$ in Table 1 are all ϵ -straightforward for $\epsilon \in \{10^{-9}, 10^{-9}, 10^{-11}, 10^{-3}, 10^{-4}\}$ and $\Delta \in \{10^{-5}, 10^{-6}, 10^{-8}, 10^{-7}, 10^{-8}\}$. Hence the convergence rates claimed in Table 1 hold for each corresponding "long step" gradient descent method.

Proof. Certificates $(\hat{\lambda}, \hat{\gamma}) \in \mathcal{S}_{h,\epsilon,\Delta}$ (produced via the above procedure) are available along with exact arithmetic validation at github.com/bgrimmer/LongStepCertificates.

Future Directions 4

We have shown that using nonconstant, long stepsize patterns improves the performance guarantees of gradient descent. This runs contrary to widely held intuitions regarding constant stepsize selections and the importance of monotone objective decreases. Instead, we show that long-run performance improves by periodically taking (very) long steps that may increase the objective value in the short term. We accomplish this by developing a technique based on computer-generatable proof certificates to analyze the collective effect of many varied stepsizes at once. We conclude by discussing a few possible future improvements on and shortcomings of this technique.

Future Improvements in Algorithm Design. The search for long, straightforward stepsize patterns h is hard. The patterns presented in Table 1 resulted from substantial brute force searching. The task of maximizing avg(h) subject to h being straightforward, although nonconvex, may be approachable using branch-and-bound techniques similar to those recently developed by Gupta et al. [12] and applied to a range of PEP parameter optimization problems. Such an approach may yield numerically, globally optimal h for fixed length t. This may also generate insights into the general form of the longest straightforward stepsize patterns for each fixed t.

One practical drawback of the method (1.3) is the requirement that one knows L. Since our analysis in Theorem 2.1 only relies on decreases in objective value after t steps, backtracking linesearch schemes or other adaptive ideas may be applicable. Such an approach could check a sufficient decrease condition every t iterations, halving the estimate of L and repeating those steps if not.

Future Improvements in Analysis Techniques. Almost all iterative optimization methods are analyzed based on deriving a recurrence, ensuring progress at every iteration. Our techniques deviate from this norm, only requiring improvement by a fixed horizon of length t. This deviation is also true of Young's gradient descent method [19] for the strongly convex quadratic setting, which only ensures an optimal rate upon completion of the pattern. Determining which other first-order methods could benefit from collectively analyzing iterations up to a fixed horizon is an interesting open direction. Bregman, inexact, and stochastic variants of gradient descent are natural candidates.

Future works may improve our analysis by considering more sophisticated Lyapunov functions. Our proofs are only concerned with the eventual decrease of the objective gap. The analysis of optimal accelerated and subgradient methods relies additionally on the decreasing distance to a minimizer or a decreasing combination thereof. Identifying more tailored Lyapunovs and stepsize patterns guaranteed to eventually decrease it may lead to stronger guarantees.

Future Improvements in Computational Aspects. We observed that numerically computed primal optimal solutions to (3.2) were rank-one for all considered straightforward patterns. This corresponds to the worst-case objective function being essentially one-dimensional. This property was not used herein but could likely be leveraged to enable customized solvers for evaluating $p_{L,D}(\delta)$ and checking membership of $\mathcal{S}_{h,0,\Delta}$. Such improvements in tractability for SDPs with rank-one solutions have been studied widely [26–32] and may enable the search for longer, provably faster straightforward stepsize patterns than shown here.

As another avenue of improvement, note that any certificates produced by using floating point arithmetic followed by a rounding step (as done here) will likely lose a small ϵ amount in the rate. The use of an algebraic solver, like SPECTRA [33], could enable the automated production of exact certificates of $\epsilon = 0$ —straightforwardness as well as being able to certify when $S_{h,0,\Delta}$ is empty.

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A Computer-Generated Straightforwardness Certificate with $\Delta = 10^{-5}$ and $\epsilon = 10^{-9}$ for h = (1.5, 2.2, 1.5, 12.0, 1.5, 2.2, 1.5)

Below is a certificate $(\hat{\lambda}, \hat{\gamma}) \in \mathcal{S}_{h,\epsilon,\Delta}$, completely computer generated, proving a $LD^2/(3.1999999 \times T)$ rate for the pattern of length t=7 in Table 1. Given the length of these 9×9 matrices, we display their first five and last four columns separately below. Exact calculations verifying the feasibility of these values are given in the associated publicly posted Mathematica notebook.

$$\hat{\lambda}_{6:9} = \begin{bmatrix} 0 & 0 & 0 & 8837407518919583 & 537068840802311 & 960254226721649 \\ 0 & 219152964335457 & 0 & 2305843009213693952 & 144115188075855875 \\ 0 & 27251799813685248 & 4407991053556385 & 0 & 2251799813685248 \\ 0 & 2303843009213693952 & 4407991053556385 & 0 & 2251799813685248 \\ 0 & 13768159860411 & 2695784755734549 & 8068866010524833 & 2251799813685248 \\ 0 & 137203495150241 & 4688926225212825 & 47979775987097 & 118259906842624 \\ 0 & 137203495150241 & 9223372036854775808 & 4611686018427387904 & 495098609957139 & 5204247088958345 & 5379927240703545 & 9223372036854775808 & 4611686018427387904 & 495098609957139 & 5204247088958345 & 5379927240703545 & 9223372036854775808 & 4611686018427387904 & 481991056907609 & 728868819438951 & 12223163611137735 & 6194623724653895 & 1152921504606846976 & 1152921504$$

