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The Sherrington–Kirkpatrick model: a challenge for mathematicians

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Summary. The Sherrington–Kirkpatrick (SK) model for spin glasses is deceptively simple to state. Yet its rigorous study represents a considerable challenge. We report here some modest progresses (obtained through elementary methods). Even in the supposedly simple high temperature region, a number of basic questions remain unsolved.

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1. Introduction

In mathematical terms, the SK model is the study of a certain random measure on $\{-1,1\}^N$.

In the framework of statistical mechanics (at the mention of these formidable terms, let us hurry to emphasize that the present paper requires no knowledge whatsoever of any physics, and is self contained) to each sequence $\epsilon \in \Sigma_N = \{-1,1\}^N$ we associate its energy $H(\epsilon) \in \mathbb{R}$. The function H is called the Hamiltonian. An element ϵ of Σ_N is often called a *configuration* ($\epsilon = (\epsilon_i)_{i \leq N}$ describes the configuration of the individual "spins" ϵ_i).

Given a parameter β, we consider the Gibbs measure G on Σ_N given by

$$G(\{\epsilon\}) = \frac{\exp(-\beta H(\epsilon))}{Z} \tag{1.1}$$

where $Z = \sum_\epsilon \exp(-\beta H(\epsilon))$. The parameter β physically represents the inverse of the temperature of the system (so "high temperature" means "small β").

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In the SK model (with external field) we define

$$H(\boldsymbol{\epsilon}) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{ij}\epsilon_i\epsilon_j + \sum_{1 \le i \le N} h\epsilon_i \ . \tag{1.2}$$

The numbers $(g_{ij})_{1 \le i < j \le N}$ are realizations of an independent sequence of standard $N(0,1)$ r.v; they represent the (random) interactions between the spins (and the disorder of the system). The Gibbs measure is then a random measure, and one tries to understand its structure for the generic realization of the g_{ij}'s. The r.v. (g_{ij}) are called the *quenched* r.v; what this means is that their realization (that represents randomness) is fixed at the beginning of the experiment and that their values are not permitted to change any more, while, on the other hand, the system is subject to thermal fluctuations. We do not study these, but only the "expected state" of the system, that is, Gibbs' measure. Expected value and expectation with respect to the quenched variables are denoted respectively by E and P.

In (1.2), h is a positive number. It represents the strength of an external field, that tends to push the spins upwards.

The reason for considering the last term in (1.2) is not the desire for generality, but reflects the fact that this term does induce new effects. It makes sense in physics to consider the more general Hamiltonian

$$H(\boldsymbol{\epsilon}) = \sum_{1 \le i < j \le N} \left(\frac{g_0}{N} + \frac{g_{ij}}{\sqrt{N}}\right)\epsilon_i\epsilon_j + \sum_{1 \le i \le N} h\epsilon_i$$

where $g_0 \ge 0$ is a fixed number. We have not done this, because we do not expect that introducing the parameter g_0 requires new mathematical techniques.

Many authors have considered much more general random variables than Gaussian to represent the disorder. Much of what is said in the present paper could be extended to such cases; the proofs are however often more technical. We have considered only the Gaussian case, because we feel that it is not yet the time to develop purely technical points when there is such a shortage of good ideas.

We will consider many integrals with respect to Gibbs' measure. According to the tradition, these will be denoted by $\langle \ \rangle$. That is, for a function $A\colon \Sigma_N \to \mathbb{R}$, we write

$$\langle A \rangle = \int A(\boldsymbol{\epsilon})\, dG(\boldsymbol{\epsilon}) = Z^{-1} \sum_{\boldsymbol{\epsilon}} A(\boldsymbol{\epsilon}) \exp(-\beta H(\boldsymbol{\epsilon})) \ . \tag{1.3}$$

It turns out to be quite useful to think of $\{-1,1\}^N$ as a subset of $\mathbb{R}^N$. Given a probability measure G on $\{-1,1\}^N$, the two most important characteristics are its barycenter

$$\langle \boldsymbol{\epsilon} \rangle = (\langle \epsilon_i \rangle)_{i \le N} = \int \boldsymbol{\epsilon}\, dG(\boldsymbol{\epsilon})$$

and the quadratic form on $\mathbb{R}^N$ given by

$$Q(\boldsymbol{x},\boldsymbol{y}) = \langle(\boldsymbol{x}\cdot\boldsymbol{\epsilon})(\boldsymbol{y}\cdot\boldsymbol{\epsilon})\rangle \ .$$

There of course $\boldsymbol{x}\cdot\boldsymbol{\epsilon}$ denotes the dot product of the vectors $\boldsymbol{x}$ and $\boldsymbol{\epsilon}$ of $\mathbb{R}^N$.
Thus

$$Q(\boldsymbol{x},\boldsymbol{y}) = \sum_{i,j\le N} x_i y_j \langle\epsilon_i\epsilon_j\rangle \ .$$

One of the most important quantities associated to the matrix $\langle\epsilon_i\epsilon_j\rangle$ is the square of its Hilbert-Schmidt norm, namely $\sum_{i,j\le N}\langle\epsilon_i\epsilon_j\rangle^2$. It is quite reassuring that both this quantity, and the norm of $\langle\boldsymbol{\epsilon}\rangle$ play an essential role in the physicist's study of the model.

Simple observations often are essential, and its seems appropriate to make one such observation with pleasant geometric overtones at this early stage.

Lemma 1.1.

$$\sum_{i,j\le N}\langle\epsilon_i\epsilon_j\rangle^2 = \langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle_2 \ . \tag{1.4}$$

Comment. The right hand side term is

$$\int(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\, dG(\boldsymbol{\epsilon})\, dG(\boldsymbol{\epsilon}') \ .$$

The subscript 2 indicates that the integral is on $\Sigma_N\times\Sigma_N$ rather than on Σ_N.

Proof. Since $\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' = \sum_{i\le N}\epsilon_i\epsilon_i'$, we have

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 = \sum_{i,j\le N}\epsilon_i\epsilon_i'\epsilon_j\epsilon_j' = \sum_{i,j\le N}\epsilon_i\epsilon_j\epsilon_i'\epsilon_j' \ .$$

We now observe (and will use many times) that for any function A on Σ_N,

$$\langle A(\boldsymbol{\epsilon})A(\boldsymbol{\epsilon}')\rangle_2 = \langle A(\boldsymbol{\epsilon})\rangle^2 \ . \tag{1.5}$$

□

Formula (1.4) displays the importance of considering certain functions on Σ_N^2 or even Σ_N^n. The physicists call this "the replica method". They do not hesitate to consider a number $n>0$ of replicas, n integer, $n\to 0$ [M-P-V], a procedure that, to say the least, is difficult to understand mathematically. We will use mostly order two replicas as above, but we have also succeeded in efficiently using higher order replicas in several instances.

Of crucial importance is the (random) function

$$F_N = \log 2^{-N} Z_N \tag{1.6}$$

that is called the *free energy*. (In physics, one would define $F_N = -\beta^{-1}\log Z_N$. The present definition simplifies notation.) Of course, F_N depends on β, h, so we write $F_N(\beta,h)$ when need arises to clear ambiguities. The usefulness of F_N stems in particular from the fact that taking derivatives makes Z appear as a denominator; then we have, as a typical example

$$\frac{\partial F_N}{\partial \beta} = \langle -H(\epsilon) \rangle \ . \tag{1.7}$$

In the first part of the paper, we focus on the simplest case, that is $h = 0, \beta < 1$.

Theorem 1.2. *If $\beta < 1$, there is a constant $K(\beta)$, depending on β only, such that, for each N and each $u > 0$*

$$P\left(F_N(\beta) < \frac{N\beta^2}{4} - u \right) \le K(\beta) \exp\left(-\frac{u^2}{K(\beta)} \right) \tag{1.8}$$

$$P\left(F_N(\beta) > \frac{N\beta^2}{4} + u \right) \le K(\beta) \exp(-2u) \ . \tag{1.9}$$

Before we comment upon this result, it must be said that the understanding of the fluctuations of F_N (and of other quantities) seems to be rather important. It should be quite obvious that $F_N(\beta)$, as a function of the point $(g_{ij})_{i<j}$ of $\mathbb{R}^{N(N-1)/2}$, has a Lipschitz constant at most $\beta\sqrt{(N-1)/2}$, so that by a general principle [I-S-T], for each $t > 0$

$$P(|F_N - EF_N| > t) \le 2 \exp - \frac{t^2}{\beta^2 (N-1)} \ . \tag{1.10}$$

The important (and hardest) part of Theorem 1.2 is (1.8). It provides handy lower bounds for the crucial quantity $Z_N(\beta)$. Inequality (1.8) improves upon a result of [T1], which had a worse bound $K(\beta)\exp(-\frac{u^2}{NK(\beta)})$, following essentially from (1.10).

It is proved in [A-L-R], [C-N] that $F_N(\beta) - \frac{N\beta^2}{4}$ converges in distribution to a (non standard) normal limit; this shows that (1.8) is quite optimal. It is however unclear to me how a result about convergence in distribution is relevant to the supposed physical contents of the model unless a rate of convergence is provided (which is not yet the case). Moreover, such a result does not describe correctly the upper tails of F_N. It is easily seen, as B. Derrida pointed out that, given $\beta > 0$, there exists $k > 0$ such that

$$\lim_{N\to\infty} E\left(\left(2^{-N} Z_N \exp - \frac{N\beta^2}{4} \right)^k \right) \to \infty \ ; \tag{1.11}$$

this shows that in the right-hand side of (1.9), one cannot replace u by a faster growing function.

We will analyze the "moment explosion" described by (1.11) and we will show that this explosion comes from an exceptional set of the quenched variables. In fact, we have the following:

Theorem 1.3. *If $\beta < 1$ there is a constant $K(\beta)$ such that, for each N we have*

$$0 < u \le \sqrt{N} \Rightarrow P\left(F_N(\beta) > \frac{N\beta^2}{4} + u \right) \le K(\beta) \exp\left(-\frac{u^2}{K(\beta)} \right) \ . \tag{1.12}$$

To provide perspective for this result, we should observe that

$$P\left(F_N(\beta) > \frac{N\beta^2}{2}\right) \geq \exp(-NK(\beta)) \ ,$$

an inequality that holds even if one replaces $Z_N(\beta)$ by one of the 2^N terms of which it is the sum. Thus (1.12) cannot hold for $u \gg \sqrt{N}$. This also shows the rather surprising fact that $-\log P(F_N(\beta) > N\beta^2/4 + u)$ stays of the same order as u increases from $\sqrt{N}$ to N.

A rough way to describe the SK model (when $h = 0$) is as follows. When $\beta < 1$, the direction of the vector $\boldsymbol{\epsilon}$ (distributed according to G) is rather random; On the other hand, if $\beta > 1$, $\boldsymbol{\epsilon}$ has a tendency to point towards a specific direction (or its opposite). One says then that polarization occurs. If $\boldsymbol{\epsilon}'$ is an independent copy of $\boldsymbol{\epsilon}$, it also has a tendency to point in the same (or the opposite) direction, and $(\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2$ has a tendency to be large. Thereby, the number $\langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \rangle_2$ is a measure of the polarization of the system. The more polarization, the more order; so $\langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon})^2 \rangle_2$, or better, in the proper normalization, $\tau_N = \langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 / N^2 \rangle_2$, somehow measures the order of the system (and hence is called an order parameter in physics).

Theorem 1.4. *If $\beta < 1$, there exists $K(\beta) < \infty$ such that*

$$E\left\langle \exp \frac{(\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2}{K(\beta)N} \right\rangle_2 \leq K(\beta) \tag{1.12}$$

and hence

$$E \exp \frac{N}{K(\beta)} \tau_N \leq K(\beta) \ . \tag{1.13}$$

This expresses in a strong way that $\tau_N = \langle (\frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'}{N})^2 \rangle_2$ is "macroscopically zero" i.e., $o(1)$.

The following very interesting question remains, concerning the behavior of the quadratic form (1.4).

Problem 1.5. *What is the order of*

$$E \sup_{\|\boldsymbol{x}\|=1} \langle (\boldsymbol{x} \cdot \boldsymbol{\epsilon})^2 \rangle ? \tag{1.14}$$

One can (optimistically) hope for a positive answer to the following.

Problem 1.6. *Is it true that if $\beta < 1$ there is a constant $K(\beta) < \infty$ such that*

$$E \sup_{\|\boldsymbol{x}\| \leq 1} \left\langle \exp \frac{1}{K(\beta)} (\boldsymbol{x} \cdot \boldsymbol{\epsilon})^2 \right\rangle < K(\beta) ?$$

It is simple to see (*since* $\|\boldsymbol{\epsilon}'\| = \sqrt{N}$) *that this would improve upon* (1.12).

We should observe that (proceeding as in Lemma 1.1), if $\|\boldsymbol{x}\| \leq 1$, we have

$$\big\langle (\boldsymbol{x}\cdot\boldsymbol{\epsilon})^2 \big\rangle = \sum_{i,j\leq N} x_i x_j \langle \epsilon_i \epsilon_j \rangle \leq \Big(\sum_{i,j\leq N} \langle \epsilon_i \epsilon_j \rangle_2^2 \Big)^{1/2} = \langle (\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 \rangle_2^{1/2} \tag{1.15}$$

using Cauchy-Schwarz, so that the quantity (1.14) is at most of order $N^{1/2}$ by Theorem 1.4. However, the bound (1.15), that amounts to bound the operator norm of a certain matrix by its Hilbert-Schmidt norm, is very crude. At the expense of considerable work, we will improve upon (1.15) as follows.

Theorem 1.7. *If $\beta < 1$, there is a function $k(N)$ (depending upon β) such that $\lim_{N\to\infty} k(N) = \infty$, and*

$$E\Bigg(\big(\sup_{\|\boldsymbol{x}\|=1} \langle (\boldsymbol{x}\cdot\boldsymbol{\epsilon})^2 \rangle\big)^{k(N)} \Bigg) \leq N^2 \ .$$

This implies in particular that the quantity (1.14) is $o(N^\eta)$ for each $\eta > 0$.

One simple idea (that is only partially correct) is that (when $\beta < 1$) the Gibbs measure G somewhat looks like the uniform measure on $\{-1,1\}^N$. This is not at all true globally, but is quite accurate if one looks at a "small piece" of G. For a subset I of N , we denote by G_I the projection of G on $\{-1,1\}^I$; by μ_I the uniform probability on $\{-1,1\}^I$, and by $|G_I - \mu_I|$ the variation distance.

Theorem 1.8. *If $\beta < 1$, there exists $K(\beta) < \infty$ such that*

$$E|G_I - \mu_I| \leq K(\beta)\frac{\operatorname{card} I}{\sqrt{N}} \ .$$

In particular G_I is close to μ_I (on average) as soon as $\operatorname{card} I \ll N^{1/2}$. We do not know whether this is optimal.

If, rather than trying to take $\operatorname{card} I$ large, one fixes its cardinal, and let $N \to \infty$, a more precise result can be obtained.

Theorem 1.9. *For $k \leq N$, $\beta < 1$ there exists a constant $K = K(k,\beta)$, depending on β, k only such that if $\operatorname{card} I = k$ and $t > 0$,*

$$P(|G_I - \mu_I| \geq t) \leq K \exp\Big(-\frac{t^{4/3}N^{1/3}}{K}\Big) + K\exp\Big(-\frac{N^{1/4}}{K}\Big) \ . \tag{1.16}$$

Moreover, if $k(N)$ is the number of Theorem 1.7, we have

$$P(|G_I - \mu_I| \geq t) \leq K\Big(\frac{Kk(N)N^{1+2/k(N)}}{t^2}\Big)^{k(N)/2} \ . \tag{1.17}$$

Finally, we have

$$E(|G_I - \mu_I|^2) \leq KN^{-1} \ . \tag{1.18}$$

Neither of these results is really satisfactory. In fact (1.18) shows that $P(|G_I - \mu_I| \geq t)$ becomes small for t of order $N^{-1/2}$. On the other hand, the right hand side of (1.16) becomes small for t of order $N^{-1/4}$, a loss of accuracy explained by the use of the crude inequality (1.15). The right hand side of (1.17) becomes small for t of order $k(N)^{1/2}N^{1/2+1/k(N)}$, but (1.17) is worse than (1.16) for $t \gg N^{-1/4}$. My only excuse for presenting results that are so far from the optimal is that Theorem 1.9 requires several new and arguably nontrivial arguments, and that these arguments would yield a much better inequality if Problem 1.6 had a positive answer.

The proof of Theorem 1.9 is also the first time we meet the main theme of the entire model, namely the fact that it is important to understand the random variables

$$\left\langle \exp \frac{t}{\sqrt{N}} \sum_{i \leq N} g_i \epsilon_i \right\rangle$$

conditionally in $(g_{ij})_{i,j \leq N}$, where $(g_i)_{i \leq N}$ is a fresh standard Gaussian sequence. (By "fresh", we mean that this sequence is independent of all the previous random quantities).

In order to discuss further results, we need to explain what is the SK "solution", or, more accurately, the free energy function associated to this solution.

Throughout the paper, we consider the function

$$\Phi(x,h) = \mathrm{E}\mathrm{th}^2(xg+h)$$

where g is standard normal and where th is the hyperbolic tangent. (We assume $x \geq 0, h \geq 0$.) If $h > 0$, the equation

$$x^2 = \Phi(\beta x, \beta h) \tag{1.19}$$

has a unique root $x(\beta,h)$. If $h = 0$, $x = 0$ is the unique root if $\beta \leq 1$, and there is another root if $\beta > 1$. In the case $h = 0$, we denote by $x(\beta,h)$ the largest root of this equation.

We consider the function $\mathrm{SK}(\beta,h)$ defined as follows

$$\mathrm{SK}(\beta,h) = \tfrac{\beta^2}{4}\big(1 - x^2(\beta,h)\big)^2 + E\log\mathrm{ch}(\beta g x(\beta,h) + \beta h) \tag{1.20}$$

Sherrington and Kirkpatrick predicted that

$$\lim_{N\to\infty} \tfrac{1}{N}\, EF_N(\beta,h) = \mathrm{SK}(\beta,h) \tag{1.21}$$

It is now believed [A-T] that this holds provided

$$\beta^2 E\left(\frac{1}{\mathrm{ch}(\beta x(\beta,h)g + \beta h)^4}\right) < 1 \tag{1.22}$$

but fails provided

$$(1.23) \qquad \beta^2 E\left(\frac{1}{\operatorname{ch}(\beta x(\beta,h)g+\beta h)^4}\right) > 1$$

(In Section 6, we will provide very strong support for this). The region where (1.23) holds, the so called "spin glass region" is where the SK model gets **really** challenging. (Unfortunately, it is at present not clear to the author how to even formulate mathematically most of the phenomenon described in the physics literature.)

In Section 3 we turn to the case $h \neq 0$, and we will prove the following.

Theorem 1.10. *There exists* β_0 *such that* (1.21) *holds if* $\beta < \beta_0$.

It must be mentioned here that the case $h \neq 0$ is very much harder than the case $h = 0$; this is because a key argument (relation (2.3) below) breaks down for $h \neq 0$. Thereby a more indirect approach is needed. We will use here (and at many other places) the "cavity method", i.e. induction upon N.

When $h \neq 0$ one has to replace τ_N by

$$\overline{\tau}_N = \tfrac{1}{N^2}\Big\langle \big((\epsilon - \langle\epsilon\rangle)\cdot(\epsilon' - \langle\epsilon'\rangle)\big)^2 \Big\rangle_2 \ .$$

We do know that $E\overline{\tau}_N(\beta,h) \leq K(\beta,h)/N$ (to be compared with (1.12)), a key step in the proof of Theorem 1.10. This fact was actually proved long ago (using delicate high-power expansions) by J. Fröhlich and B. Zegarlinski in a much more general setting. We give a simpler proof using the cavity method.

One could naively expect some kind of law of large numbers to come into play, and thus expect that, as N becomes large, about every quantity associated to the Gibbs measure becomes essentially independent upon the particular realization of the quenched variables. An essential feature of the prediction of the physicists about the SK model is that this is not the case for large β. (It is expected that "large β" means (1.23).) The following results goes in this direction.

Theorem 1.11. *Assume* $h = 0$, *and consider* $\beta_1 > 1$. *If*

$$(1.24) \qquad \lim_{N\to\infty} \int_1^{\beta_1} \operatorname{Var} \tau_N(\beta)\, d\beta = 0$$

then

$$(1.25) \qquad \forall \beta \leq \beta_1, \liminf_{N\to\infty} N^{-1} E F_N(\beta) \geq \mathrm{SK}(\beta) \ .$$

The point of the theorem is to show that (1.24) fails. Since F. Comets showed [C] that $\limsup_{N\to\infty} N^{-1}EF_N(\beta) < \mathrm{SK}(\beta)$ for β large enough, we have proved that (1.24) fails for β_1 large enough. It is extremely likely that this is the case as soon as $\beta_1 > 1$.

It would be nice to show that (1.25) can be strengthened into

$$\lim_{N\to\infty} N^{-1}EF_N(\beta) = \mathrm{SK}(\beta) \ .$$

We have arguments to show that this would be the case if we could do the following.

Problem 1.12. *Prove that $E\tau_N(\beta)$ increases with β, or at least prove that lim $sup_{N\to\infty}EF''_N(\beta)/N \le 1/2$.*

We will prove Theorem 1.11 in Section 5. The proof of Theorem 1.11 is plagued by an intrinsic difficulty (the symmetry of G around zero) and this obscures the fundamentally simple underlying idea. Thereby, before proving Theorem 1.11, we will prove a result of the same nature that involves the parameter

$$\sigma_N = \sigma_N(\beta, h) = \frac{1}{N}\sum_{i\le N}\langle\epsilon_i\rangle^2 = \frac{1}{N}\|\langle\boldsymbol{\epsilon}\rangle^2\|^2$$

(which is also a measure of the order of the system). Unfortunately, we do not know how to do that with the original Hamiltonian (1.2). Rather, we will use the Hamiltonian

$$H(\boldsymbol{\epsilon}) = -\left(\frac{1}{\sqrt{N}}\sum_{i<j} g_{ij}\epsilon_i\epsilon_j + \sum_{i\le N}\epsilon_i h + \gamma\varphi(N)\sum_{i\le N}\epsilon_i u_i\right) \ . \tag{1.26}$$

There, γ is a parameter, $\varphi(N)$ is a certain function of N, and $(u_i)_{i\le N}$ is a fresh standard gaussian sequence. In the case $h = 0$, there is compelling motivation to change the Hamiltonian (1.2), for then σ_N is zero by symmetry, and the extra term in (1.26) can be justified as a "symmetry breaking" term. In Section 4, we will prove the following.

Theorem 1.13. *We consider the Hamiltonian* (1.26), *and we assume*

$$\lim_{N\to\infty}\varphi(N) = 0, \quad \lim_{N\to\infty} N\varphi^2(N) = \infty, \quad \lim_{N\to\infty}\frac{\varphi(N+1)}{\varphi(N)} = 1 \ . \tag{1.27}$$

Consider $0 < \beta_1,\ \gamma_0 >\ 0$. *Assume that*

$$\lim_{N\to\infty}\int_0^{\beta_1}\int_{-\gamma_0}^{\gamma_0}\mathrm{Var}\,\sigma_N(\beta,\gamma,h)\,d\beta\,d\gamma = 0 \tag{1.28}$$

Then, if $h \ne 0$, *we have*

$$\forall\, 0 < \beta < \beta_1, \forall\gamma < \gamma_0, \lim_{N\to\infty}\tfrac{1}{N}EF_N(\beta,\gamma,h) = \mathrm{SK}(\beta,h) \ . \tag{1.29}$$

In the case $h = 0$,

$$\forall\beta \le \beta_1, \forall\gamma < \gamma_0, \liminf_{N\to\infty}\tfrac{1}{N}EF_N(\beta,\gamma) \ge \mathrm{SK}(\beta) \ .$$

In technical jargon, a result such as (1.29) is expressed by "If σ_N is self-averaging, the SK solution holds". A result of a similar nature was proved by L. Pastur and M. Shcherbina [P-S]. Our hypothesis are however not identical. The one weakness of Theorem 1.12 compared to [P-S] is that we do require information for a whole interval of values of γ. On the other hand, [P-S] does require a symmetry breaking term $N^{-1/4}\sum_{i\le N} u_i\epsilon_i$ for a very special sequence u_i. Moreover, our approach is quite straightforward (and we hope, transparent!)

Problem 1.14. *When* $h \neq 0$, *prove Theorem* 1.13 *for the original Hamiltonian* (1.2).

In Section 6, we will provide arguments towards (1.22), (1.23) (conditions first identified in [A-T]). We will make a series of conjectures, each of them asserting that certain pathological behavior does not occur. These are extremely likely to hold, even though we could not see how to approach them. A complete proof that the SK solution does not hold under (1.21) is conceivably not too far away. Proving that the SK solution holds under (1.22) seems however an entirely different matter. In particular my understanding is that the physicists have arguments only towards a type of "stability" of the SK solution (a stability we have been able to prove under the validity of the conjectures mentioned above), but do not have arguments showing that the SK solution is the true solution.

To avoid cultural misunderstandings, a few comments are in order about the present paper. Both mathematicians and physicists have written about the present topic, and they have rather different ideas of rigor. This is a mathematical paper, which intends to present fully rigorous arguments, even when this means going through unpleasant and tiresome technical considerations (such as those contained at the end of Sections 3 and 4) while these could be avoided simply by considering that some slightly wrong statements (such as the one mentioned after Proposition 4.2) are correct. (In fact, it should be said that some of the technical difficulties solved in the present paper were not even recognized in previous – supposedly rigorous – literature.) Full rigor does not however mean complete details. We have felt appropriate to give these for each important argument or calculation, but neither for calculations that require no idea, or that are similar to previous computations, nor for results that are somewhat secondary issues.

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2. High temperature, no external field

Throughout this section, $h = 0$.

On the space $\Sigma_N = \{-1,1\}^N$ there is a natural probability, namely the uniform probability. For a function A on Σ_N, we set $E_\epsilon A = 2^{-N}\sum_{\boldsymbol{\epsilon}} A(\boldsymbol{\epsilon})$, the expectation with respect to the uniform probability. Thus

$$Z_N = 2^N E_\epsilon a(\boldsymbol{\epsilon}) \ ,$$

where we have set $a(\boldsymbol{\epsilon}) = \exp(-\beta H(\boldsymbol{\epsilon}))$.

The following simple lemma plays a crucial role.

Lemma 2.1. *We have*

$$EZ_N = 2^N \exp\frac{\beta^2(N-1)}{4} \ . \tag{2.1}$$

If $\gamma + \beta^2 < 1$, *we have*

$$E\Big(\sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')\exp\frac{\gamma}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\Big) \le \frac{1}{\sqrt{1-\beta^2-\gamma}}(EZ_N)^2 \ . \tag{2.2}$$

(*The summation is of course over all* $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in \Sigma_N$).

Comment. When $\gamma = 0$, (2.2) implies in particular that

$$E(Z_N^2) \le K(\beta)(EZ_N)^2 \ . \tag{2.3}$$

This crucial fact is no longer true when $h \neq 0$, even for small β. This makes the case $h \neq 0$ much harder.

Proof. If g is a standard normal r.v. and $a \in \mathbb{R}$, we have

$$E\exp ag = \exp\left(\frac{a^2}{2}\right) \ , \tag{2.4}$$

a fact that will be used many times. This, and independence, imply (2.1).

To prove (2.2), we want to estimate

$$\begin{aligned} EE_{\epsilon,\epsilon'}\exp&\left[\frac{\beta}{\sqrt{N}}\sum_{i<j}(\epsilon_i\epsilon_j + \epsilon'_i\epsilon'_j)g_{ij} + \frac{\gamma}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right] \\ &= E_{\epsilon,\epsilon'}\exp\left[\frac{\beta^2}{2N}\sum_{i<j}(\epsilon_i\epsilon_j + \epsilon'_i\epsilon'_j)^2 + \frac{\gamma}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right] \end{aligned} \tag{2.5}$$

Now,

$$\begin{aligned} \sum_{i<j}(\epsilon_i\epsilon_j + \epsilon'_i\epsilon'_j)^2 &= N(N-1) + \sum_{i<j}2\epsilon_i\epsilon'_i\epsilon_j\epsilon'_j \\ &= N(N-2) + \left(\sum_i \epsilon_i\epsilon'_i\right)^2 \\ &= N(N-2) + (\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 \ . \end{aligned}$$

Thus the quantity (2.5) is bounded by

$$\exp\frac{\beta^2(N-2)}{2}E_{\epsilon,\epsilon'}\exp\frac{\beta^2+\gamma}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 \quad .$$

Now

$$E_{\epsilon,\epsilon'}\exp\frac{\beta^2+\gamma}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 = E_\epsilon\exp\frac{\beta^2+\gamma}{2N}\left(\sum_{i\le N}\epsilon_i\right)^2 \tag{2.6}$$

because $\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'$ and $\sum_{i\le N}\epsilon_i$ have the same distribution. Thus using (2.4)

$$\exp\frac{\beta^2+\gamma}{2N}\left(\sum_{i\le N}\epsilon_i\right)^2 = E\exp\sqrt{\frac{\beta^2+\gamma}{N}}g\left(\sum_{i\le N}\epsilon_i\right)$$

where g is standard normal, so that the quantity (2.6) is bounded by

$$\begin{aligned}EE_\epsilon\exp g\sqrt{\frac{\beta^2+\gamma}{N}}\sum_{i\le N}\epsilon_i &= E\left(\mathrm{ch}\left(g\sqrt{\frac{\beta^2+\gamma}{N}}\right)\right)^N\\ &\le E\exp\frac{\beta^2+\gamma}{2}g^2 = \frac{1}{\sqrt{1-\beta^2-\gamma}}\end{aligned}$$

using the elementary inequality $\mathrm{ch}x\le\exp\frac{x^2}{2}$. The proof is complete. □

We note the following consequence of (2.3) (when $\gamma=0$)

$$E(Z_N^2)\le\frac{1}{\sqrt{1-\beta^2}}(EZ_N)^2 \quad . \tag{2.7}$$

Thus, by Chebyshev inequality

$$P(Z_N\ge tEZ_N)\le\frac{1}{t^2\sqrt{1-\beta^2}}$$

and thus, using (2.1)

$$P\left(F_N\ge u+\frac{\beta^2(N-1)}{4}\right)\le\frac{1}{\sqrt{1-\beta^2}}\mathrm{e}^{-2u} \tag{2.8}$$

which implies (1.9).

The proof of (1.8) requires more work. We start with some preparation.

Lemma 2.2. We have

$$P(Z_N\ge\frac{EZ_N}{2})\ge\frac{\sqrt{1-\beta^2}}{4}$$

Proof. This is a consequence of (2.7), and of the Paley-Zygmund inequality that states that, for any r.v. $Y\ge 0$, we have

$$P\left(Y\ge\frac{EY}{2}\right)\ge\frac{1}{4}\frac{(EY)^2}{EY^2} \quad .$$

Indeed,

$$\tfrac{1}{2}\, EY \le E(Y1_{\{Y \ge EY/2\}}) \le (EY^2)^{1/2} P(Y \ge EY/2)^{1/2}$$

by Cauchy-Schwarz. □

Lemma 2.3. *We have*

$$P\left(Z_N \ge \frac{EZ_N}{2}\,;\, \langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle_2 \le KN(1-\beta^2)^{-2}\right) \ge \frac{\sqrt{1-\beta^2}}{8} \ .$$

Proof. Since $e^x \ge x$, we get from (2.2) that, for $\gamma < 1-\beta^2$

$$E\left(\sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')\frac{(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2}{N}\right) \le \frac{2}{\gamma\sqrt{1-\beta^2-\gamma}}(EZ_N)^2 \ .$$

Taking $\gamma = (1-\beta^2)/2$, and using Chebyshev inequality, we get

$$P\left(\sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')\frac{(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2}{N} \ge u\right) \le \frac{4\sqrt{2}}{u(1-\beta^2)^{3/2}}(EZ_N)^2 \ .$$

Taking $u = 32\sqrt{2}(1-\beta^2)^{-2}(EZ_N)^2$, and combining with Lemma 2.2, we get

$$P\left(Z_N \ge \frac{EZ_N}{2}\,;\, \sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 \le KN(EZ_N)^2(1-\beta^2)^{-2}\right) \ge \frac{\sqrt{1-\beta^2}}{8} \ .$$

To conclude, we simply observe that

$$\left\langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right\rangle_2 = \frac{1}{Z_N^2}\sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 \ . \qquad \square$$

The proof of (1.8) relies upon the Gaussian isoperimetric inequality. In order to apply this inequality, it is convenient to think of Z_N as a function of $(g_{ij}) \in \mathbb{R}^M$ (where $M = \frac{N(N-1)}{2}$). We provide $\mathbb{R}^M$ with the canonical gaussian measure γ_M. Thus, in this language, the content of Lemma 2.3 is that there is a set A of $\mathbb{R}^M$ such that, when $(g_{ij}) \in A$, we have

$$Z_N(g_{ij}) \ge \frac{EZ_N}{2}\,;\, \langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle_2 \le KN(1-\beta^2)^{-2} \tag{2.8}$$

and moreover such that

$$\gamma_M(A) \ge \frac{\sqrt{1-\beta^2}}{8} \ . \tag{2.9}$$

Let us denote by $d(x,A)$ the Euclidean distance in $\mathbb{R}^M$ of a point x and of a set A. Let us denote by γ_1 the standard one-dimensional gaussian measure. The Gaussian isoperimetric inequality states that (for any $A \subset \mathbb{R}^M$)

(2.10) $$\gamma_M(d(\cdot,A) \ge u) \le \gamma_1([a+u,\infty))$$

where $\gamma_1((-\infty,a]) = \gamma_M(A)$.

It is elementary to see that, if $t > 0$, we have

$$\gamma_1([t,\infty)) \le \tfrac{1}{2}\exp\left(-\tfrac{t^2}{2}\right) .$$

Thus, if $\gamma_M(A) \le \frac{1}{2}$,

$$\gamma_M(A) = \gamma_1((-\infty,a]) \le \tfrac{1}{2}\exp\left(-\tfrac{a^2}{2}\right)$$

so that

$$|a| \le \sqrt{2\log\frac{1}{2\gamma_M(A)}}$$

and thus, for $u \ge \sqrt{2\log(1/2\gamma_M(A))}$, we have

(2.11) $$\begin{aligned}\gamma_M(d(\cdot,A) \ge u) &\le \frac{1}{2}\exp -\frac{1}{2}(u-|a|)^2 \\ &\le \frac{1}{2}\exp\left(-\frac{1}{2}\left(u-\sqrt{2\log\frac{1}{2\gamma_M(A)}}\right)^2\right) .\end{aligned}$$

In order to use (2.11), given $(g_{ij}) \in A$, and $(g'_{ij}) \in \mathbb{R}^M$, we must know how to find a lower bound for $Z'_N = Z_N(g'_{ij})$ in function of $Z_N = Z_N(g_{ij})$ and $u = (\sum(g_{ij}-g'_{ij})^2)^{1/2}$, a task to which we turn now. Setting $u_{ij} = g'_{ij} - g_{ij}$, we note that

$$\begin{aligned}Z'_N &= \sum_{\epsilon} \exp\frac{\beta}{\sqrt{N}}\sum_{i<j}\epsilon_i\epsilon_j g'_{ij} \\ &= \sum_{\epsilon} a(\epsilon)\exp\frac{\beta}{\sqrt{N}}\sum_{i<j}\epsilon_i\epsilon_j u_{ij} \\ &= Z_N\left\langle \exp\frac{\beta}{\sqrt{N}}\sum_{i<j}\epsilon_i\epsilon_j u_{ij}\right\rangle\end{aligned}$$

where the Gibbs measure implicit in $\langle\ \rangle$ is associated to (g_{ij}).

Using Jensen's inequality, Cauchy-Schwarz and (1.4) we get

$$\begin{aligned}Z'_N &\ge Z_N\exp\frac{\beta}{\sqrt{N}}\sum_{i<j}u_{ij}\langle\epsilon_i\epsilon_j\rangle \\ &\ge Z_N\exp -\frac{\beta u}{\sqrt{N}}\left\langle(\epsilon\cdot\epsilon')^2\right\rangle_2^{1/2} .\end{aligned}$$

Combining with Lemma 2.4 we see that

$$Z_N \ge \frac{EZ_N}{4}\exp\left(-K\beta(1-\beta^2)^{-1}u\right)$$

so that

$$Z_N(g'_{ij}) \geq \frac{EZ_N}{4} \exp\Big(-K\beta(1-\beta^2)^{-1} d((g_{ij}), A)\Big) \ .$$

Combining with (2.11), we see that

$$\begin{aligned} P(F_N(\beta) &\leq \frac{\beta^2}{4}(N-1) - \log 4 - K\beta(1-\beta^2)^{-1}u) \\ &\leq \frac{1}{2} \exp - \frac{1}{2}\left(\sqrt{2\log\frac{4}{(1-\beta^2)^{1/2}}}\right)^2 \end{aligned}$$

when $u \geq \sqrt{2\log\frac{4}{(1-\beta^2)^{1/2}}}$, so that, in particular

$$P\left(F_N(\beta) \leq \frac{\beta^2}{4}(N-1) - t\right) \leq \frac{1}{2}\exp\left(-\frac{(1-\beta^2)^2 t^2}{K\beta^2}\right)$$

provided $t \geq K\beta(1-\beta^2)^{-1}\left(1 + \sqrt{\log(1/(1-\beta^2)}\right)$. Theorem 1.2 is proved. □

Since the proof of Theorem 1.3 (and of the related Theorem 1.7) is considerably more elaborate, these proofs are better presented in the last section of the paper, and we now start the proof of Theorem 1.4. Most of the work has already been done.

Using Cauchy-Schwarz for $G \otimes G$, we have

$$\begin{aligned} \left\langle \exp\left(\frac{1-\beta^2}{4N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right)\right\rangle_2 &\leq \left\langle \exp\frac{1-\beta^2}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right\rangle_2^{1/2} \\ &= \frac{1}{Z}\left(\sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')\exp\frac{1-\beta^2}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right)^{1/2} \ . \end{aligned}$$

Using Cauchy-Schwarz again, and then (2.2)

$$\begin{aligned} &E\left\langle \exp\frac{1-\beta^2}{4N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right\rangle \\ &\quad\leq \left(E\frac{1}{Z_N^2}\right)^{1/2}\left(E\sum a(\boldsymbol{\epsilon})a(\boldsymbol{\epsilon}')\exp\frac{1-\beta^2}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right)^{1/2} \\ &\quad\leq \left(E\frac{1}{Z_N^2}\right)^{1/2}\left((EZ_N)^2\frac{2}{\sqrt{1-\beta^2}}\right)^{1/2} \ . \end{aligned}$$

Now, it is an elementary consequence of (1.8) that $EZ_N^{-2} \leq K(\beta)(EZ_N)^{-2}$. (There $K(\beta)$ denotes of course a number depending on β only, that may vary at each occurrence.)

Thus we have shown that

$$E\left\langle \exp\frac{1-\beta^2}{4N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\right\rangle_2 \leq K(\beta) \ , \tag{2.12}$$

and, by Jensen's inequality, that

$$E\langle \exp \frac{1-\beta^2}{4N} \langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \rangle_2 \le K(\beta) \ . \tag{2.13}$$ □

We now turn to the proof of Theorem 1.8.

For a subset J of $\{1, \ldots, N\}$, we set $r_J = \langle \prod_{i \in J} \epsilon_i \rangle$. Consider $\xi \in \{-1, 1\}^I$, and observe that

$$1_{\{\xi\}}(\epsilon) = 2^{-n} \prod_{i \in I} (1 + \xi_i \epsilon_i) \tag{2.14}$$

for all $\epsilon \in \{-1, 1\}^I$. Thus, setting $\xi_J = \prod_{i \in J} \xi_i$ for a subset J of $\{1, \ldots, N\}$, we have

$$G_I(\{\xi\}) = 2^{-n} \sum_{J \subset I} r_J \xi_J \tag{2.15}$$

and

$$G_I(\{\xi\}) - 2^{-n} = 2^{-n} \sum_{\emptyset \ne J \subset I} r_J \xi_J \ . \tag{2.16}$$

Now, using Cauchy-Schwarz and (2.16)

$$\begin{aligned} E|G_I - \mu_I|^2 &= E\left(\sum_{\xi} |G_I(\xi) - 2^{-n}| \right)^2 \\ &\le 2^n E \sum_{\xi} (G_I(\xi) - 2^{-n})^2 \\ &= 2^{-n} E \sum_{\xi} \sum_{\emptyset \ne J, J' \subset I} r_J \xi_J r_{J'} \xi_{J'} \\ &= \sum_{\emptyset \ne J \subset I} E r_J^2 \end{aligned} \tag{2.17}$$

because $\sum_{\xi} \xi_J \xi_{J'} = 0$ if $J \ne J'$, and $= 2^n$ if $J = J'$.

Now

$$\begin{aligned} r_J^2 &= \left\langle \prod_{i \in J} \epsilon_i \right\rangle \left\langle \prod_{i \in J} \epsilon'_i \right\rangle \\ &= \left\langle \prod_{i \in J} \epsilon_i \epsilon'_i \right\rangle_2 \ . \end{aligned} \tag{2.18}$$

Thus

$$\sum_{\text{card } J = 2k} r_J^2 = \left\langle \frac{1}{(2k)!} \sum_{i_1, \ldots, i_{2k}} \prod_{\ell \le 2k} \epsilon_{i_\ell} \epsilon'_{i_\ell} \right\rangle_2$$

where the sum is over all possible choices of *distinct* integers $i_1, \ldots, i_{2k} \le N$. Now, by the computation above

$$\left\langle \prod_{\ell \le 2k} \epsilon_{i_\ell} \epsilon'_{i_\ell} \right\rangle = \left\langle \prod_{\ell \le 2k} \epsilon_{i_\ell} \right\rangle^2 \ge 0$$

whether the indexes $i_1, \ldots, i_{2k}$ are distinct or not. Thus

$$\sum_{\operatorname{card} J = 2k} r_J^2 \le \frac{1}{(2k)!} \left\langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^{2k} \right\rangle_2 \ .$$

Now, $x^{2k} \le k! \exp x^2$, so that, for all $a > 0$

$$\left\langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^{2k} \right\rangle_2 \ \le \ k! N^k a^{-k} \langle \exp \tfrac{a}{N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \rangle_2$$

and finally

$$\sum_{\operatorname{card} J = 2k} r_J^2 \le N^k a^{-k} U \ ,$$

where $U = \langle \exp \frac{a}{N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \rangle_2$.

Since Er_J^2 depends only upon $\operatorname{card} J$, we have

$$\begin{aligned} \sum_{J \subset I, \operatorname{card} J = 2k} Er_J^2 &\le \binom{n}{2k} \binom{N}{2k}^{-1} \sum_{\operatorname{card} J = 2k} Er_J^2 \\ &\le \left(\frac{n}{N}\right)^{2k} \left(\frac{N}{a}\right)^k EU = \left(\frac{n^2}{aN}\right)^k EU \ , \end{aligned}$$

where we have used that

$$\binom{n}{2k} \binom{N}{2k}^{-1} = \frac{n!(N-2k)!}{(n-2k)!N!} \le \left(\frac{n}{N}\right)^{2k} \ .$$

Using (2.18) we have, since $r_J = 0$ when $\operatorname{card} J$ is odd,

$$\begin{aligned} E|G_I - \mu_I|^2 &\le \sum_{0 < 2k \le n} \sum_{J \subset I, \operatorname{card} J = 2k} Er_J^2 \\ &\le \frac{2n^2}{aN} EU \end{aligned} \tag{2.19}$$

provided $2n^2 \le aN$. We then take $a = \sqrt{1 - \beta^2}/4$. The result follows. □

We now turn to the proof of Theorem 1.9.

For $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k) \in \{-1, 1\}^k$, we set

$$c(\boldsymbol{\eta}) = \sum \{ a(\boldsymbol{\epsilon}); \epsilon_1 = \eta_1, \ldots, \epsilon_k = \eta_k \} \ .$$

Thus

$$G_I(\{\boldsymbol{\eta}\}) = \frac{c(\boldsymbol{\eta})}{\sum_{\boldsymbol{\eta}'} c(\boldsymbol{\eta}')} \ . \tag{2.20}$$

Now, if $\epsilon_1 = \eta_1, \ldots, \epsilon_k = \eta_k$ we have

$$a(\boldsymbol{\epsilon}) = \exp -\beta H(\boldsymbol{\epsilon}) \tag{2.21}$$
$$= \exp \frac{\beta}{\sqrt{N}} \left(\sum_{i,j \le k} \eta_i \eta_j g_{ij} + \sum_{i \le k} \sum_{j > k} g_{ij} \eta_i \epsilon_j \right)$$
$$\times \exp \frac{\beta}{\sqrt{N}} \left(\sum_{k < i < j \le N} g_{ij} \epsilon_i \epsilon_j \right) .$$

Consider on $\{-1,1\}^{N-k}$ the Hamiltonian given by

$$H'(\xi) = -\frac{1}{\sqrt{N}} \sum_{k < i < j \le N} \xi_i \xi_j g_{ij} . \tag{2.22}$$

We set $\boldsymbol{a}'(\xi) = \exp(-\beta H'(\xi))$, $Z' = \sum_\xi a'(\xi)$, and for a function $f(\xi)$ on $\{-1,1\}^{N-k}$, we set

$$\langle f(\xi) \rangle' = (Z')^{-1} \sum_\xi a'(\xi) f(\xi)$$

We then see from (2.21) that

$$c(\boldsymbol{\eta}) = Z' \langle f_{\boldsymbol{\eta}}(\xi) \rangle'$$

where

$$f_{\boldsymbol{\eta}}(\xi) = \exp \frac{\beta}{\sqrt{N}} \left(\sum_{i,j \le k} \eta_i \eta_j g_{ij} + \sum_{i \le k} \sum_{j > k} g_{ij} \eta_i \xi_j \right) .$$

so that, by (2.20)

$$G_I(\{\boldsymbol{\eta}\}) = \frac{\langle f_{\boldsymbol{\eta}}(\xi) \rangle'}{\sum \langle f_{\boldsymbol{\eta}'}(\xi) \rangle'} .$$

It should be obvious that $E\langle f_{\boldsymbol{\eta}}(\xi) \rangle'$ does not depend upon $\boldsymbol{\eta}$. Moreover, using the fact that $\langle f_{\boldsymbol{\eta}}(\xi) \rangle' \ge \exp \frac{\beta}{\sqrt{N}} \sum_{i,j \le k} \eta_i \eta_j g_{ij}$, it is elementary to see that to prove (1.16) it suffices to show that there is a constant $K = K(k, \beta)$ such that for each $\boldsymbol{\eta}$ we have, for each $t > 0$

$$P\big(|\langle f_{\boldsymbol{\eta}}(\xi) \rangle' - E\langle f_{\boldsymbol{\eta}}(\xi) \rangle'| \ge Kt\big) \tag{2.23}$$
$$\le K \exp\Big(-t^{4/3} N^{1/3}\Big) + K \exp\left(-\frac{N^{1/4}}{K}\right) .$$

Now, $\langle f_{\boldsymbol{\eta}}(\xi) \rangle'$ is distributed like

$$\left\langle \exp \frac{1}{\sqrt{N}} \left(g + \sum_{j > k} g_j \xi_j \right) \right\rangle' \tag{2.24}$$

where the r.v. g, g_j are normal, independent, independent of the $(g_{ij})_{i>k}$, and $Eg^2 = \beta^2 k(k-1)/2, Eg_j^2 = k\beta^2$.

Since

$$P\Big(g \ge N^{1/4}\Big) \le \exp\Big(-N^{1/2}/K(\beta,k)\Big) \ ,$$

we leave the reader to check that the contribution of g to (2.24) is of smaller order; then we are reduced to the study of

$$\left\langle \exp \frac{1}{\sqrt{N}} \sum_{j>k} g_j \xi_j \right\rangle' \ .$$

Now the Hamiltonian H' of (2.22) that we study is of the type (1.2) (where N has been replaced by $N-k$), except that we have a coefficient $\beta/\sqrt{N}$ rather than $\beta/\sqrt{N-k}$; but this simply means that we have replaced β by $\beta' = \beta\sqrt{1-k/N} < \beta$.

We are reduced to prove the following:

Proposition 2.4. *Given* $\beta_0 < 1,\ \alpha_0 > 0$ *there exists a number* $K = K(\beta_0, \alpha_0)$ *with the following property. Consider a fresh independent standard normal sequence* $(g_i)_{i \le N}$. *Then if* $\beta \le \beta_0, \alpha \le \alpha_0$, *the r.v.*

$$X_\alpha = \left\langle \exp \frac{\alpha}{\sqrt{N}} \sum_{i \le N} g_i \epsilon_i \right\rangle$$

satisfies, for $t > 0$:

$$P(|X_\alpha - EX_\alpha| \ge t) \le K \exp\left(-\frac{t^{4/3} N^{1/3}}{K}\right) + K \exp\left(-\frac{N^{1/4}}{K}\right) \ .$$

We will denote by E_g and P_g respectively the conditional expectation and probability at (g_{ij}) fixed. Thus

$$E_g X_\alpha = \exp \tfrac{\alpha^2}{2} \ .$$

We start with a preparatory lemma.

Lemma 2.5. *There exists an exceptional set of quenched variables of probability at most* $K(\beta_0) \exp -(N^{1/4} \alpha_0 K(\beta_0))$ *such that, outside this set, for* $v \ge 2$ *and* $\alpha \le 2\alpha_0$ *we have*

$$P_g\Big(X_\alpha \ge v \exp \tfrac{\alpha^2}{2}\Big) \le v^{-N^{1/4}/\alpha_0^2 K(\beta_0)} \tag{2.25}$$

Proof. It follows from (2.12) that for a certain number $C_0(\beta_0)$, we have

$$EA \le C_0(\beta_0) \tag{2.26}$$

where

$$A = \left\langle \exp \frac{\epsilon \cdot \epsilon'}{C_0(\beta_0)\sqrt{N}} \right\rangle \ .$$

We write, for any integer q

$$X_\alpha^q = \left\langle \exp \frac{\alpha}{\sqrt{N}} \sum_{i \le N} g_i \left(\sum_{\ell \le q} \epsilon_i^\ell \right) \right\rangle_q .$$

There the bracket represents an integral over Σ_N^q with respect to the measure $G^{\otimes q}$, and the generic point of Σ_N^q is denoted by $(\boldsymbol{\epsilon}^\ell)_{\ell \le q}$.

Thus

$$\begin{aligned} E_g X_\alpha^q &= \left\langle \exp \frac{\alpha^2}{2N} \sum_{i \le N} \left(\sum_{\ell \le q} \epsilon_i^\ell \right)^2 \right\rangle_q \\ &= \exp \frac{q\alpha^2}{2} \left\langle \exp \frac{\alpha^2}{2N} \sum_{1 \le \ell < \ell' \le q} 2\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \right\rangle_q . \end{aligned}$$

Using Holder's inequality

$$\begin{aligned} E_g X_\alpha^q &\le \exp \frac{q\alpha^2}{2} \prod_{1 \le \ell < \ell' \le q} \left\langle \exp \frac{\alpha^2}{2N} q(q-1) \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \right\rangle_2^{\frac{2}{q(q-1)}} \\ &= \exp \frac{q\alpha^2}{2} \left\langle \exp \frac{\alpha^2}{2N} q(q-1) \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \right\rangle_2 . \end{aligned}$$

Now, for

$$q(q-1)\frac{\alpha^2}{2\sqrt{N}} \le \frac{1}{C_0(\beta_0)} , \tag{2.27}$$

Holder's inequality and (2.26) show that

$$\left\langle \exp \frac{\alpha^2}{2N} q(q-1) \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \right\rangle \le A$$

so that

$$E_g X_\alpha^q \le A \exp \frac{q\alpha^2}{2}$$

and

$$P_g\left(X_\alpha \ge v \exp \tfrac{\alpha^2}{2} \right) \le A v^{-q} .$$

We can take $q = N^{1/4}/\alpha_0^2 C_1(\beta_0)$ to satisfy (2.27) for $\alpha \le 2\alpha_0$. Then

$$P_g\left(X_\alpha \ge v \exp \tfrac{\alpha^2}{2} \right) \le v^{-\left(N^{1/4}/2\alpha_0^2 C_1(\beta_0)\right)}$$

for $v \ge 2, A \le 2^{N^{1/4}/2\alpha_0^2 C_1(\beta_0)}$.

The result then follows from (2.26). □

The main idea is simply to use concentration properties for the gaussian measure such as (1.10) for the function $X_\alpha(\boldsymbol{g})$ defined for $\boldsymbol{g} \in \mathbb{R}^N$ by

$$X_\alpha(\boldsymbol{g}) = \left\langle \exp \frac{\alpha}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \right\rangle .$$

We use the notation $Q = \sup_{\|\boldsymbol{x}\| \le 1} \langle (\boldsymbol{x} \cdot \boldsymbol{\epsilon})^2 \rangle$.

Lemma 2.6. *Consider* $\boldsymbol{g}, \boldsymbol{g}' \in \mathbb{R}^N$, *and* $u = \|\boldsymbol{g} - \boldsymbol{g}'\|$. *Then if we have*

$$X_{2\alpha_0}(\boldsymbol{g}) \le 2 \exp 2\alpha_0^2, X_{2\alpha_0}(\boldsymbol{g}') \le 2 \exp 2\alpha_0^2 \ , \tag{2.28}$$

we have

$$|X_\alpha(\boldsymbol{g}) - X_\alpha(\boldsymbol{g}')| \le K(\alpha_0) \frac{u}{\sqrt{N}} Q^{1/2} \ . \tag{2.29}$$

Proof. We start with the easy inequality

$$|\mathrm{e}^x - \mathrm{e}^y| \le |x - y| \mathrm{e}^{\max(x,y)} \le |x - y|(\mathrm{e}^x + \mathrm{e}^y) \ .$$

Thus, using Cauchy-Schwarz and (2.28),

$$\begin{aligned} |X_\alpha(\boldsymbol{g}) - X_\alpha(\boldsymbol{g}')| &\le \frac{\alpha}{\sqrt{N}} \left\langle \left| ((\boldsymbol{g} - \boldsymbol{g}') \cdot \boldsymbol{\epsilon}) \left(\exp \frac{\alpha}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} + \exp \frac{\alpha}{\sqrt{N}} \boldsymbol{g}' \cdot \boldsymbol{\epsilon} \right) \right| \right\rangle \\ &\le \frac{\alpha}{\sqrt{N}} K(\alpha_0) \left\langle ((\boldsymbol{g} - \boldsymbol{g}') \cdot \boldsymbol{\epsilon})^2 \right\rangle^{1/2} \ . \qquad \square \end{aligned}$$

Lemma 2.7. *If the realization of the quenched variables is such that* (2.25) *holds, then for* $t > 0$

$$\begin{aligned} \gamma_N \left(\left| X_\alpha(\boldsymbol{g}) - \exp \frac{\alpha^2}{2} \right| \ge t \right) &\le 4 \exp - \frac{t^2 N}{Q K(\alpha_0, \beta_0)} \\ &\quad + K(\alpha_0, \beta_0) 2^{-N^{1/4}/K(\alpha_0,\beta_0)} \end{aligned} \tag{2.30}$$

Since the proof of (2.30) from (2.29) is tedious routine, we first conclude the proof of Proposition 2.4. We recall that from (1.13) and (1.15) that we have

$$E \exp \frac{Q^2}{N K(\beta_0)} \le K(\beta_0) \ . \tag{2.31}$$

Thus, for $s > 0$, we have

$$P(Q \ge s) \le K(\beta_0) \exp\left(- \frac{s^2}{N K(\beta_0)} \right) \ .$$

Thus, combining with (2.30), (and taking into account the exceptional set involved in (2.25)), we have for each s

$$\begin{aligned} P\left(\left| X_\alpha - \exp \frac{\alpha^2}{2} \right| \ge t \right) &\le K(\alpha_0, \beta_0) 2^{-N^{1/4}/K(\alpha_0,\beta_0)} \\ &\quad + 4 \exp\left(- \frac{t^2 N}{s K(\alpha_0, \beta_0)} \right) + K(\beta_0) \exp - \frac{s^2}{N K(\beta_0)} \end{aligned}$$

and to prove Proposition 2.7 it suffices to take $s = (t^2N^2)^{1/3}$.

We now turn to the proof of (2.30). Consider the set

$$B = \left\{ \boldsymbol{g} \in \mathbb{R}^N ; X_{2\alpha_0}(\boldsymbol{g}) \leq 2\exp 2\alpha_0^2 \right\} \ .$$

Thus by (2.25) we have

$$\gamma_N(B) \geq 1 - \exp\left(- \frac{N^{1/4}}{K(\alpha_0, \beta_0)} \right) \ .$$

The content of (2.29) is that the function X_α has a Lipschitz constant $L = K(\alpha_0, \beta_0) u N^{-1/2} Q^{1/2}$ when restricted to B. Consider a function Y from $\mathbb{R}^N$ to $\mathbb{R}$, that coincides with X_α on B and with the same Lipschitz constant as X. Thus, by [I-S-T]

$$\forall t > 0, \gamma_N\left(\left| Y - \int Y \, d\gamma_N \right| \geq t \right) \leq 2\exp - \frac{t^2}{2L^2}$$

so that

$$\gamma_N\left(\left| X - \int Y \, d\gamma_N \right| \geq t \right) \leq 2\exp - \frac{t^2}{2L^2} + \gamma_N(B^c) \ .$$

By Cauchy-Schwarz, we have $X_\alpha(\boldsymbol{g}) \leq 2\exp \alpha_0^2$ for $\boldsymbol{g} \in A$; so we can assume $Y \leq 2\exp\alpha_0^2$ everywhere.

Thus

$$\begin{aligned} \text{(2.32)} \qquad \int |X - Y| \, d\gamma_N &= \int_{B^c} |X - Y| \, d\gamma_N \\ &\leq \int_{B^c} X' \, d\gamma_N + 4\exp\alpha_0^2 \gamma_N(B^c) \end{aligned}$$

where $X' = \min(X, 2\exp\alpha_0^2)$. We then see from (2.25) that

$$\left| \int X \, d\gamma_N - \int Y \, d\gamma_N \right| \leq K(\alpha_0, \beta_0) \exp - \frac{N^{1/4}}{K(\alpha_0, \beta_0)} \ .$$

Combining with (2.32) concludes the proof. □

We have proved (1.16). The proof of (1.17) is entirely similar, except that instead of (2.31) we use that, according to Theorem 1.7,

$$EQ^{k(N)} \leq N^2$$

so that, for $s > 0$,

$$P\left(Q \geq sN^{2/k(N)} \right) \leq s^{-k(N)}$$

and we take

$$s = \frac{x}{k(N)} \Big/ \log \frac{x}{k(N)}$$

for $x = t^2 N^{1-2/k(N)}/K(\alpha_0, \beta_0)$, when $x \geq 2$.

As for (1.18), it is a special case of Theorem 1.8. A simpler direct argument is as follows. As in Lemma 2.5, we see that the variable X_α of Proposition 2.4 satisfies

$$E_g X_\alpha^2 = (E_g X_\alpha)^2 \left\langle \exp \frac{\alpha^2}{2N} \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \right\rangle_2$$

so that

$$E_g X_\alpha^2 - (E_g X_\alpha)^2 \leq \frac{K(\alpha, \beta)}{N} \left\langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \right\rangle_2$$

and

$$E\left(E_g X_\alpha^2 - (E_g X_\alpha)^2 \right) \leq \frac{K(\alpha, \beta)}{N} \quad .$$

Since $E_g X_\alpha$ is constant, this implies $E(X_\alpha - EX_\alpha)^2 \leq K(\alpha, \beta)/N$. The conclusion follows easily. □

Another topic of interest related to the material of the present section is the case $\beta = 1$. F. Guerra [G1] proved in that case that

$$E\left\langle \left(\frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'}{N} \right)^2 \right\rangle \leq \frac{K}{N^{1/4}} \quad . \tag{2.33}$$

We have tried to adapt the proof of Theorem 1.4 to this case. This requires a number of careful, but mostly standard estimates. For example, (2.3) has to be replaced by $EZ_N^2 \leq K(\beta) N^{1/4} (EZ_N)^2$. The best we could obtain is the inequality

$$\forall v > 0, P\left(\left\langle \left(\frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'}{N} \right)^2 \right\rangle \geq K \frac{v + \log N}{N^{1/4}} \right) \leq \exp\left(-\frac{v^2}{\log N} \right) \quad . \tag{2.34}$$

Unfortunately, this fails to recover (2.33).

3. High temperature, non-zero external field

This section is devoted to the proof of Theorem 1.10. The proof has two rather different parts, that are based on the so called "cavity method", that is, induction on N. Each of this part contains one of the crucial ideas of the paper.

Before starting the work, let us settle a secondary issue. The definition (1.2) is justified of physical grounds. From the mathematical point of view, it is however a nuisance that h occurs everywhere with coefficient β. Thus, through the rest of the paper we make the change of variables $\hbar = \beta h$, and we think to $\beta, \hbar$ as two independent parameters. The parameter $\hbar$ will remain fixed through this section, so we do not indicate it in the notation.

Throughout the section, we define

$$\overline{\tau}_N = \overline{\tau}_N(\beta) = \tfrac{1}{N^2}\big\langle((\boldsymbol{\epsilon} - \langle\boldsymbol{\epsilon}\rangle)\cdot(\boldsymbol{\epsilon}' - \langle\boldsymbol{\epsilon}'\rangle))^2\big\rangle_2 \ .$$

It might help the reader to mention at this stage that the fact that $E\overline{\tau}_N(\beta) = o(1)$ should be seen as the central feature of the "high temperature" region. We will describe an iteration technique that allows to approach results of this type. Once we know that $E\overline{\tau}_N(\beta) = o(1)$, there is a simple method (Proposition 3.4 below) that allows to handle the crucial functions $\langle\exp\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot\boldsymbol{\epsilon}\rangle$ (where $\boldsymbol{g}$ is a fresh sequence of Gaussian variables). This proposition is central. It allows considerable simplifications over previous methods. While based on a schoolboy idea (a second moment computation) quite amazingly it does not seem to have been observed before.

Theorem 3.1. *There exists $\beta_0 > 0$ such that if $\beta < \beta_0$ we have*

$$E\overline{\tau}_N(\beta) \le \frac{K}{N} \ . \tag{3.1}$$

The proof of this result given in [F-Z] uses high power expansions. These appear powerless to prove (3.1) in the correct region (1.20). The reason we give a new proof is that our approach appears to have a better potential at this (beside the fact that it is technically simpler).

Let us start by a simple observation. As in Lemma 1.1 we have

$$\langle((\boldsymbol{\epsilon} - \langle\boldsymbol{\epsilon}\rangle)\cdot(\boldsymbol{\epsilon}' - \langle\boldsymbol{\epsilon}'\rangle))^2\rangle = \sum_{i,j\le N}\big\langle(\epsilon_i - \langle\epsilon_i\rangle)(\epsilon_j - \langle\epsilon_j\rangle)\big\rangle^2 \ .$$

Using the symmetry between the variables ϵ_i, we get

$$E\overline{\tau}_N = 2\left(\frac{N-1}{N}\right)E\langle(\epsilon_1 - \langle\epsilon_1\rangle)(\epsilon_2 - \langle\epsilon_2\rangle)\rangle^2 + \tfrac{1}{N}\,E\big\langle(\epsilon_1 - \langle\epsilon_1\rangle)^2\big\rangle^2 \ . \tag{3.2}$$

Now

$$\big\langle(\epsilon_1 - \langle\epsilon_1\rangle)^2\big\rangle^2 = (1 - \langle\epsilon_1\rangle^2)^2 \le 1$$

so that

$$E\overline{\tau}_N \le \tfrac{1}{N} + 2E\langle(\epsilon_1 - \langle\epsilon_1\rangle)(\epsilon_2 - \langle\epsilon_2\rangle)\rangle^2 \ , \tag{3.3}$$

and also

$$E\overline{\tau}_N \ge 2\big(1 - \tfrac{1}{N}\big)E\langle(\epsilon_1 - \langle\epsilon_1\rangle)(\epsilon_2 - \langle\epsilon_2\rangle)\rangle^2 \ . \tag{3.4}$$

Thus, the issue to compute $E\overline{\tau}_N$ (with accuracy of order $1/N$) is to estimate the correlation coefficient

$$C_N(\beta) = E\langle(\epsilon_1 - \langle\epsilon_1\rangle)\cdot(\epsilon_2 - \langle\epsilon_2\rangle)\rangle^2 \ .$$

This is done by the following proposition.

Proposition 3.2. *Consider fresh independent standard normal r.v. $(g_i)_{i\le N}$, $(g_i')_{i\le N}, g$, and set $\boldsymbol{g} = (g_i)_{i\le N}, \boldsymbol{g}' = (g_i')_{i\le N}$. For $\eta, \eta' \in \{-1, 1\}$, consider*

$$A(\eta,\eta') = \left\langle \exp\left(\frac{\beta}{\sqrt{N}}((\eta \boldsymbol{g} + \eta' \boldsymbol{g}') \cdot \boldsymbol{\epsilon} + \eta\eta' g) + \hbar(\eta + \eta')\right)\right\rangle \ .$$

Then

$$(3.5) \quad C_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) = 16E\left(\frac{(A(1,1)A(-1,-1) - A(-1,1)A(1,-1))^2}{\left(\sum_{\eta,\eta'=\pm 1} A(\eta,\eta')\right)^4}\right) \ .$$

Proof. Consider $Z = \sum_{\eta,\eta'=\pm 1} A(\eta,\eta')$. Looking back at the proof of Theorem 1.8, we see that the random measure $\overline{G}$ on $\{-1,1\}^2$ that gives weight $Z^{-1}A(\eta,\eta')$ to (η,η') is distributed like the projection of $G_{N+2}(\beta\sqrt{1+\frac{2}{N}})$ onto, any two coordinates. Thus

$$C_{N+2}\left(\beta\sqrt{1+\tfrac{2}{N}}\right) = E\left((\mathrm{cov}_{\overline{G}}(\eta,\eta'))^2\right) \ .$$

Now, simple algebra shows that

$$Z^2 \mathrm{cov}_{\overline{G}}(\eta,\eta') = 4(A(1,1)A(-1,-1) - A(1,-1)A(-1,1)) \ . \qquad \square$$

It remains now to compute the right-hand side of (3.5). In this section, we present a crude computation that is sufficient to prove Theorem 3.1, but we make no attempt to obtain a good value for β_0. In Section 6, we will make a considerably more accurate computation.

Proposition 3.3. *There exists $\beta_0 > 0$ and $C_0 < 1, C_1 < \infty$ such that if $\beta < \beta_0$, for N large enough we have*

$$(3.6) \qquad E\overline{\tau}_{N+2}\left(\beta\sqrt{1+\tfrac{2}{N}}\right) \le \tfrac{C_1}{N} + C_0 E\overline{\tau}_N(\beta) \ .$$

Once this is proved, we proceed as follows. We fix N_0 large enough that $\frac{N_0}{N_0+2} > C_0$. We then fix C_2 large enough that

$$N \ge N_0 \Rightarrow -\frac{C_2}{N+2} + \frac{C_1}{N} \le -\frac{C_0 C_2}{N} \ .$$

Then (3.6) implies that for $N \ge N_0$ we have

$$E\overline{\tau}_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) - \frac{C_2}{N+2} \le C_0\left(E\overline{\tau}_N(\beta) - \frac{C_2}{N}\right) \ .$$

If $\beta\sqrt{1+\frac{2k}{N}} < \beta_0$, we can iterate k times the above to get

$$\begin{aligned} E\overline{\tau}_{N+2k}\left(\beta\sqrt{1+\frac{2k}{N}}\right) &\le C_0^k(E\overline{\tau}_N(\beta)) + \frac{C_2}{N} \\ &\le \frac{C_2}{N} + C_0^k \end{aligned}$$

and taking k of order $\log N$ finishes the proof. $\square$

We now start to prove Proposition 3.3.

First, we notice that, since $e^x \geq 1 + x$, we have $Z \geq 4$, so that (3.5) yields

$$C_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) \leq EX^2 \tag{3.7}$$

where $X = X_1 - X_2$,

$$X_1 = A(1,1)A(-1,-1); \qquad X_2 = A(1,-1)A(-1,1) \ .$$

We now observe that

$$X_1 = \exp\frac{2g\beta}{\sqrt{N}}\left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}')\right\rangle_2 =: \exp\frac{2g\beta}{\sqrt{N}}Y_1$$

$$X_2 = \exp-\frac{2g\beta}{\sqrt{N}}\left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}')\right\rangle_2 =: \exp\frac{-2g\beta}{\sqrt{N}}Y_2 \ .$$

where we use again the trick $\langle A(\boldsymbol{\epsilon})\rangle\langle B(\boldsymbol{\epsilon})\rangle = \langle A(\boldsymbol{\epsilon})B(\boldsymbol{\epsilon}')\rangle_2$. It then turns out quite pleasantly that Y_1, Y_2 are independent conditionally in the quenched variables. Thus, if we denote by E_g conditional expectation at given value of the quenched variables, we have

$$\begin{aligned} E_gX^2 &= E_gX_1^2 + E_gX_2^2 - 2E_gX_1X_2 \\ &= 2\exp\frac{8\beta^2}{N}E_gY_2^2 - 2(E_gY_2)^2 \\ &= 2\left(\exp\frac{8\beta^2}{N} - 1\right)E_gY_2^2 + 2\left(E_gY_2^2 - (E_gY_2)^2\right) \ . \end{aligned} \tag{3.8}$$

Throughout the proof we assume $\beta \leq 1$, without loss of generality; we recall that K denotes a constant that may change at each occurrence. Thus from (3.8) we get

$$E_gX^2 \leq \frac{K}{N} + 2\left(E_gY_2^2 - (E_gY_2)^2\right) \ . \tag{3.9}$$

To simplify notation, we set $\boldsymbol{\theta} = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}'$ and we consider an independent copy $\boldsymbol{\theta}^*$ of $\boldsymbol{\theta}$. Then

$$Y_2 = \left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot\boldsymbol{\theta}\right\rangle_2$$

$$Y_2^2 = \left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\right\rangle_4$$

and

$$E_gY_2^2 = \left\langle \exp\frac{\beta^2}{N}\|\boldsymbol{\theta}+\boldsymbol{\theta}^*\|^2\right\rangle_4 \tag{3.10}$$

$$E_gY_2 = \left\langle \exp\frac{\beta^2}{N}\|\boldsymbol{\theta}\|^2\right\rangle_2 \ . \tag{3.11}$$

Thus

$$E_g Y_2^2 - (E_g Y_2)^2 = \left\langle \exp \frac{\beta^2}{N} \|\boldsymbol{\theta} + \boldsymbol{\theta}^*\|^2 \right\rangle_4 - \left\langle \exp \frac{\beta^2}{N} \left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}^*\|^2 \right) \right\rangle_4$$

$$= \left\langle \exp \frac{\beta^2}{N} \left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}^*\|^2 \right) \left(\exp \frac{2\beta^2}{N} \boldsymbol{\theta} \cdot \boldsymbol{\theta}^* - 1 \right) \right\rangle_4 . \tag{3.12}$$

We now observe that exchanging $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ changes $\boldsymbol{\theta}$ in $-\boldsymbol{\theta}$, so that

$$E_g Y_2^2 - (E_g Y_2)^2 = \left\langle \exp \frac{\beta^2}{N} \left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}^*\|^2 \right) \left(\operatorname{ch} \frac{2\beta^2}{N} \boldsymbol{\theta} \cdot \boldsymbol{\theta}^* - 1 \right) \right\rangle_4 \tag{3.13}$$

$$\leq K\beta^4 \left\langle \left(\frac{\boldsymbol{\theta} \cdot \boldsymbol{\theta}^*}{N} \right)^2 \right\rangle_4 .$$

We leave to the reader to check the elementary fact that

$$N^{-2} \left\langle (\boldsymbol{\theta} \cdot \boldsymbol{\theta}^*)^2 \right\rangle_4 = 4\overline{\tau}_N . \tag{3.14}$$

Combining with (3.9), (3.13), we have

$$E_g X^2 \leq K\left(\tfrac{1}{N} + \beta^2 \overline{\tau}_N\right) .$$

Taking expectations and recalling (3.7) finish the proof. □

The importance of $\overline{\tau}_N$ being small is demonstrated by the following, where we consider a fresh sequence $\boldsymbol{g}$ of standard normal r.v.

Proposition 3.4. *We have*

$$E_g \left(\left| \left\langle \exp \frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \right\rangle - \exp \left(\frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \langle \boldsymbol{\epsilon} \rangle \right) \left\langle \exp \frac{\beta'^2}{2N} \|\boldsymbol{\epsilon} - \langle \boldsymbol{\epsilon} \rangle\|^2 \right\rangle \right| \right) \leq K(\beta') \overline{\tau}_N^{1/4} .$$

Comment. We have not tried to get a sharp bound (that would involve a better dependence in $\overline{\tau}_N$). What matters here is that we have a formula to compute $\langle \exp \frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \rangle$, namely

$$\left\langle \exp \frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \right\rangle \simeq \exp \left(\frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \langle \boldsymbol{\epsilon} \rangle \right) C$$

where C does not depend on $\boldsymbol{g}$. This opens the way to all kind of computations.

Proof. Setting $X = \langle \exp \frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot (\boldsymbol{\epsilon} - \boldsymbol{a}') \rangle$, where $\boldsymbol{a} = \langle \boldsymbol{\epsilon} \rangle$, and using Cauchy-Schwarz, it suffices to show that $E_g((X - E_g X)^2) \leq K(\beta') \overline{\tau}_N^{1/2}$. Now

$$\begin{aligned} E_g X^2 - (E_g X)^2 &= \left\langle \exp \frac{\beta'^2}{2N} \|\boldsymbol{\epsilon} - \boldsymbol{a}' + \boldsymbol{\epsilon}' - \boldsymbol{a}'\| \right\rangle_2^2 \\ &\quad - \left\langle \exp \frac{\beta'^2}{2N} \|\boldsymbol{\epsilon} - \boldsymbol{a}'\|^2 + \|\boldsymbol{\epsilon}' - \boldsymbol{a}'\|^2 \right\rangle_2 \\ &= \left\langle \exp \frac{\beta'^2}{2N} (\|\boldsymbol{\epsilon} - \boldsymbol{a}'\|^2 + \|\boldsymbol{\epsilon}' - \boldsymbol{a}'\|^2) \right. \\ &\quad \left. \times \left(\exp \frac{\beta'^2}{N} (\boldsymbol{\epsilon} - \boldsymbol{a}') \cdot (\boldsymbol{\epsilon}' - \boldsymbol{a}') - 1 \right) \right\rangle_2 \\ &\leq K(\beta') \langle ((\boldsymbol{\epsilon} - \boldsymbol{a}') \cdot (\boldsymbol{\epsilon}' - \boldsymbol{a}'))^2 \rangle^{1/2} \ . \end{aligned}$$

□

Throughout the rest of the section, we set

$$\sigma_N = \sigma_N(\beta) = \frac{1}{N} \sum_{i \leq N} \langle \epsilon_i \rangle^2 = \frac{1}{N} \|\langle \boldsymbol{\epsilon} \rangle\|^2 = \frac{1}{N} \|\boldsymbol{a}'\|^2 \ .$$

Proposition 3.5. *Consider* $\beta_0 > 0$, *and assume that*

$$\lim_{N \to \infty} \int_0^{\beta_0} E(\overline{\tau}_N(\beta))\, d\beta = 0 \ . \tag{3.15}$$

Then

$$\lim_{N \to \infty} \int_0^{\beta_0} \left| E\sigma_N \left(\beta \sqrt{\frac{N+1}{N}} \right) - E\Phi\Big(\beta \sigma_N(\beta)^{1/2}, \hbar \Big) \right| d\beta = 0 \ . \tag{3.16}$$

Comment. The hypothesis (3.15) is weaker than what we know here (namely $\lim_{N \to \infty} E\big(\overline{\tau}_N(\beta)\big) = 0$ for each $\beta < \beta_0$). The point of this weaker assumption is to avoid repetition later on.

Proof. Setting $A(\eta) = \langle \exp \eta (\frac{\beta}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} + \hbar) \rangle$ for $\eta \in \{-1, 1\}$ we see that

$$E\sigma_{N+1} \left(\beta \sqrt{1 + \frac{1}{N}} \right) = E \left(\frac{A(1) - A(-1)}{A(1) + A(-1)} \right)^2 \ .$$

Consider, for $\eta \in \{-1, 1\}$,

$$B(\eta) = \exp \eta \left(\frac{\beta}{\sqrt{N}} g \cdot \langle \boldsymbol{\epsilon} \rangle + \hbar \right) \left\langle \exp \frac{\beta^2}{2N} \|\boldsymbol{\epsilon} - \langle \boldsymbol{\epsilon} \rangle\|^2 \right\rangle \ .$$

It follows from Proposition 3.4 and elementary inequalities that

$$\left| E_g \left(\frac{A(1) - A(-1)}{A(-1) + A(1)} \right)^2 - E_g \left(\frac{B(1) - B(-1)}{B(1) + B(-1)} \right)^2 \right| \leq K(\beta_0) (\overline{\tau}_N)^{1/4} \ . \tag{3.17}$$

Now

$$\left(\frac{B(1)-B(-1)}{B(1)+B(-1)}\right)^2 = \mathrm{th}^2\left(\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot\langle\boldsymbol{\epsilon}\rangle + \hbar\right)$$

and $\boldsymbol{g}\cdot\langle\boldsymbol{\epsilon}\rangle/\sqrt{N}$ has variance σ_N. Then (3.17) implies

$$\left|E_g\left(\frac{A(1)-A(-1)}{A(1)+A(-1)}\right)^2 - \Phi\Big(\beta\sigma_N(\beta)^{1/2},\hbar\Big)\right| \le K(\beta)(\overline{\tau}_N(\beta))^{1/4} \ .$$

Taking expectation and integrating yield the result. □

Proposition 3.6. *Assume that* (3.15) *holds, and that moreover*

$$\lim_{N\to\infty}\int_0^{\beta_0} \mathrm{Var}\,\sigma_N(\beta)\,d\beta = 0 \ . \tag{3.18}$$

Then

$$\lim_{N\to\infty}\int_0^{\beta_0}\left|E\sigma_{N+1}\left(\beta\sqrt{\frac{N+1}{N}}\right) - \Phi\Big(\beta(E\sigma_N(\beta))^{1/2},\hbar\Big)\right| d\beta = 0 \ . \tag{3.19}$$

Proof. This is rather obvious, since $0\le\sigma_N(\beta)\le 1$ and by continuity of Φ. □

Since (3.19) is quite more interesting than (3.16), we try to prove (3.18). Unfortunately I see no very simple way to do this, although at least two different types of argument work. The most direct goes as follows. We observe that

$$\begin{aligned}\mathrm{Var}\,\sigma_N(\beta) &= \frac{1}{N^2}\sum_{i,j\le N} E\Big(\langle\epsilon_i\rangle^2\langle\epsilon_j\rangle^2\Big) - E(\langle\epsilon_i\rangle^2)E(\langle\epsilon_j\rangle^2)\\ &\le \frac{1}{N} + E(\langle\epsilon_1\rangle^2\langle\epsilon_2\rangle^2) - E(\langle\epsilon_1\rangle^2)^2 \ .\end{aligned}$$

Now, if we proceed as in the proof of Proposition 3.5, we find that

$$\limsup_{N\to\infty}\mathrm{Var}\,\sigma_N\left(\beta\sqrt{\frac{1+2}{N}}\right) \le \limsup_{N\to\infty}\mathrm{Var}\Big(\Phi\Big(\beta\sigma_N^{\frac{1}{2}}(\beta),\hbar\Big)\Big) \ . \tag{3.20}$$

Lemma 3.7. *We have* $\frac{d}{dx}\Phi(\beta\sqrt{x},\hbar)\le\beta^2$

Proof. Let us start with a general principle. If g is standard normal, we have

$$\frac{d}{dx}Ef(g\sqrt{x}) = \frac{1}{2}\ Ef''(g\sqrt{x}) \ . \tag{3.21}$$

This follows from the integration by parts formula

$$Egh(g) = Eh'(g)$$

since

$$\frac{d}{dx}Ef(g\sqrt{x}) = \frac{1}{2\sqrt{x}}Egf'(g\sqrt{x}) \ .$$

The lemma then follows from the fact that $(\mathrm{th}^2)'' \leq 2$. □

If f is a Lipschitz function, and X as r.v., it is easy to see that $\mathrm{Var}\, f(X) \leq \|f\|_{\mathrm{Lip}}^2 \mathrm{Var}\, X$. This is quite obvious if one observes that $2\mathrm{Var}\, X = E(X-Y)^2$ where Y is an independent copy of X. Thus we see for (3.20) and Lemma 3.7 that

$$\limsup_{N\to\infty} \mathrm{Var}\, \sigma_N \left(\beta\sqrt{1+\frac{2}{N}}\right) \leq \beta^4 \limsup_{N\to\infty} \mathrm{Var}\, \sigma_N(\beta) \tag{3.22}$$

from which (3.18) follows by iteration.

We now continue the argument; we assume that $\hbar \neq 0$. Consider the function $f_\beta(x) = \Phi(\beta\sqrt{x}, \hbar)$ and for $k \geq 0$, consider the set $C_k(\beta) = f_\beta\big(C_{k-1}(\beta)\big)$, where $C_0(\beta) = [0,1]$. Then the sequence of intervals $C_k(\beta)$ converges to the unique root $x^2(\beta, h)$ of the equation $x = \Phi(\beta\sqrt{x}, \hbar)$. Also, induction on k using (3.19) show that

$$\lim_{N\to\infty} \int_0^{\beta_0} d(E\sigma_N(\beta), C_k(\beta))\, d\beta = 0$$

and thus

$$\lim_{N\to\infty} \int_0^{\beta_0} \left|E\sigma_N(\beta) - x^2(\beta, h)\right| d\beta = 0 \ . \tag{3.23}$$

Lemma 3.8. *We have*

$$\lim_{N\to\infty} \int_0^{\beta_0} |E\tau_N(\beta) - E\sigma_N^2(\beta)| d\beta = 0$$

Proof. This is rather obvious, since $0 \leq \sigma_N(\beta) \leq 1$, and

$$\begin{aligned} \left|\tau_N - \sigma_N^2\right| &\leq \frac{1}{N^2}\langle |(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 - \|\boldsymbol{a}\|^4|\rangle_2 \\ &\leq \frac{2}{N}\langle |\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' - \|\boldsymbol{a}\|^2|\rangle_2 \\ &\leq \frac{2}{N}\langle |(\boldsymbol{\epsilon}-\boldsymbol{a})\cdot(\boldsymbol{\epsilon}'-\boldsymbol{a})|\rangle_2 + \frac{2}{N}(\langle|(\boldsymbol{\epsilon}-\boldsymbol{a})\cdot\boldsymbol{a}|\rangle \\ &\quad + \langle|(\boldsymbol{\epsilon}'-\boldsymbol{a})\cdot\boldsymbol{a}|\rangle) \\ &\leq \frac{2}{N}\left(\overline{\tau}_N^{1/2} + 2\overline{\tau}_N^{1/4}\right) \end{aligned} \tag{3.24}$$

using Cauchy-Schwarz and (1.15). The result follows by integration. □

Thus, we have now proved that

$$\lim_{N\to\infty}\int_0^{\beta_0} |E\tau_N(\beta) - x^4(\beta,h)\,|d\beta = 0 \ . \tag{3.25}$$

We now make the convention to write $F_N(\beta,\hbar)$ (or $\mathrm{SK}(\beta,\hbar)$) when we think to $F_N(\beta,h)$ (or $\mathrm{SK}(\beta,h)$) as a function of the two independent variables $\beta,\hbar$.

We observe that

$$\frac{\partial F_N}{\partial\beta}(\hbar) = \frac{1}{\sqrt{N}}\sum_{i<j} g_{ij}\langle\epsilon_i\epsilon_j\rangle \ . \tag{3.26}$$

It will be shown in Section 4, using integration by parts that this yields

$$E\left(\frac{1}{N}\frac{\partial F_N}{\partial\beta}(\beta,\hbar)\right) = E\frac{\beta}{N^2}\sum_{i<j}\left(1-\langle\epsilon_i\epsilon_j\rangle^2\right) \ . \tag{3.27}$$

Now $N + 2\sum_{i<j}\langle\epsilon_i\epsilon_j\rangle^2 = N^2\tau_N$, so that

$$E\left(\frac{1}{N}\frac{\partial F_N}{\partial\beta}(\beta,\hbar)\right) = \frac{\beta}{2}(1-E\tau_N) \ . \tag{3.28}$$

One of the magical facts about (1.19) is that

$$\frac{\partial}{\partial\beta}\mathrm{SK}(\beta,\hbar) = \frac{\beta}{2}\left(1-x^4(\beta,h)\right) \ , \tag{3.29}$$

a fact that is easily checked after one observes that (1.18) means that if one thinks to $\mathrm{SK}(\beta,h)$ as being also a function of the independent parameter $x = x(\beta,h)$, then $\partial\mathrm{SK}/\partial x = 0$.

Combining (3.25), (3.28), (3.29), we get

$$\int_0^{\beta_0}\left|E\left(\frac{1}{N}\frac{\partial F_N}{\partial\beta}(\beta,\hbar)\right) - \frac{\partial}{\partial\beta}\mathrm{SK}(\beta,\hbar)\,\right|d\beta = 0 \ .$$

It remains only to check that $\lim_{N\to\infty}\frac{1}{N}F_N(0,\hbar) = \mathrm{SK}(0,\hbar)$ to finish the proof. This is very easy, and left to the reader. □

4. Lack of self-averaging, I

In this section, we prove Theorem 1.13, or rather, since it is somewhat more natural, a similar result where $\hbar = \beta h$ rather than h remain fixed. (Only a few details need to be changed to get the statement of Theorem 1.13 itself.) Since $\hbar$ will remain fixed, we will not indicate it in the notation.

The main part of the proof is the following. (A few days before submitting this paper I learned of related work by F. Guerra [G2].)

Proposition 4.1. *Under the conditions of Theorem* 1.13, *we have*

$$(4.1) \qquad \lim_{N\to\infty} \int_0^{\beta_1} \int_{-\gamma_0}^{\gamma_0} E(\overline{\tau}_N(\beta,\gamma))\, d\beta\, d\gamma = 0 \ .$$

(Once this is obtained one then proceeds as in Section 3.)

The proof of Proposition 4.1 uses an idea that goes back to [A-L-R]. It significantly simplifies the proof to assume, as we do, that the variables are Gaussian, but this is not needed for this type of argument to apply.

We start with the relation

$$(4.2) \qquad \frac{\partial F}{\partial \gamma} = \beta\varphi(N) \sum_{i\le N} u_i \langle \epsilon_i \rangle \ .$$

Integration by part show that, for a smooth function A of u_i, we have

$$(4.3) \qquad E(u_i A) = E\left(\frac{\partial A}{\partial u_i}\right) \ .$$

A straightforward calculation shows that

$$(4.4) \qquad \frac{\partial \langle \epsilon_i \rangle}{\partial u_i} = \beta\varphi(N)(1 - \langle \epsilon_i \rangle^2)$$

so that we can rewrite (4.2) as

$$(4.5) \qquad \beta\varphi(N)\left(\sum_{i\le N} u_i\langle\epsilon_i\rangle - \frac{\partial\langle\epsilon_i\rangle}{\partial u_i}\right) = \frac{\partial F}{\partial\gamma} - \beta^2\varphi(N)^2 N(1-\sigma_N(\beta,\gamma)) \ .$$

We now observe the inequality

$$\mathrm{Var}(X+Y) \le 2(\mathrm{Var}\, X + \mathrm{Var}\, Y) \ .$$

Since the left-hand side of (4.5) has zero expectation, we have

$$(4.6) \qquad \beta^2 \frac{\varphi(N)^2}{N^2} \sum_{i,j\le N} E_u\left(\left(u_i\langle\epsilon_i\rangle - \frac{\partial\langle\epsilon_i\rangle}{\partial u_i}\right)\left(u_j\langle\epsilon_j\rangle - \frac{\partial\langle\epsilon_j\rangle}{\partial u_j}\right)\right)$$
$$\le 2\mathrm{Var}_u \frac{1}{N}\frac{\partial F}{\partial\gamma} + 2\beta^4\varphi(N)^4 \mathrm{Var}_u \sigma_N(\beta,\gamma)$$

where the subscript u means that we work conditionally in the quenched variables.

To evaluate the generic term of the summation in (4.6), we expend it, and use (4.3) again to get rid of terms containing either u_i or u_j. When $i \ne j$, this term is

$$E_u\left(\frac{\partial^2}{\partial u_i \partial u_j}(\langle\epsilon_i\rangle\langle\epsilon_j\rangle) - \frac{\partial}{\partial u_j}\left(\frac{\partial\langle\epsilon_i\rangle}{\partial u_i}\langle\epsilon_j\rangle\right) - \frac{\partial}{\partial u_i}\left(\langle\epsilon_i\rangle\frac{\partial\langle\epsilon_j\rangle}{\partial u_j}\right) + \frac{\partial\langle\epsilon_i\rangle}{\partial u_j}\frac{\partial\langle\epsilon_j\rangle}{\partial u_j}\right)$$
$$= E_u\left(\frac{\partial}{\partial u_j}\langle\epsilon_i\rangle \frac{\partial}{\partial u_i}\langle\epsilon_j\rangle\right) = \beta^2\varphi(N)^2 E_u\left((\langle\epsilon_i\epsilon_j\rangle - \langle\epsilon_i\rangle\langle\epsilon_j\rangle)^2\right)$$

after some tedious computations. Thus (4.6) gives

$$
\begin{aligned}
2\beta^4 E_u \frac{1}{N^2}\Big(\sum_{i<j}\Big((\langle\epsilon_i\epsilon_j\rangle - \langle\epsilon_i\rangle\langle\epsilon_j\rangle)^2\Big)\Big) \\
\leq \frac{2}{\varphi(N)^4}\mathrm{Var}_u \frac{1}{N}\frac{\partial F}{\partial\gamma} + 2\beta^4 \mathrm{Var}_u \sigma_N(\beta,\gamma) \ .
\end{aligned}
$$

Thereby we have shown the following.

Proposition 4.2. *To prove Proposition* 4.1, *it suffices to show that for each* β,

$$
\lim_{N\to\infty}\frac{1}{\varphi(N)^4}\int_{-\gamma_0}^{\gamma_0}\mathrm{Var}_u\Big(\frac{1}{N}\frac{\partial F}{\partial\gamma}(\beta,\gamma)\Big)\,d\gamma = 0 \ . \tag{4.7}
$$

The proof of (4.7) relies upon a general principle. Given a random convex function, that, at each point, does not fluctuate much, it is NOT true (despite what one reads in the "rigorous" physics literature) that at each point, the derivative does not fluctuate much. This is true when one can bound uniformly the second derivative of the expected function. When this information is not a priori available, one can use instead the fact, formalized in the next proposition, that this is true "in average".

Proposition 4.3. *Consider a number* $\gamma_0 > 0$, *and a random convex function* W *defined on* $[-3\gamma_0, 3\gamma_0]$. *We assume* $EW(0) = 0, EW(x) \geq 0$ *for* $|x| \leq 3\gamma_0$. *We set* $V = EW$,

$$
\sigma^2 = \sup_{|x|\leq 2\gamma_0} \mathrm{Var}\, W(x)
$$

Then for $v > 0, v \leq \gamma_0$, *we have*

$$
\int_{|x|\leq\gamma_0}\mathrm{Var}(W'(x))\,dx \leq 12\frac{\gamma_0\sigma^2}{v} + 12v\Big(\frac{V(3\gamma_0)+V(-3\gamma_0)}{\gamma_0}\Big)^2 \ . \tag{4.8}
$$

Proof. This proof gives a new meaning to the word "tedious". We have by convexity that for $v > 0$,

$$
\begin{aligned}
v^{-1}(V(x+v) - V(x)) \leq V'(x+v) \\
W'(x) \leq v^{-1}(W(x+v) - W(x))
\end{aligned}
$$

so that

$$
\begin{aligned}
W'(x) - V'(x) \leq v^{-1}(W(x+v) - V(x+v)) - v^{-1}(W(x) - V(x)) \\
+ V'(x+v) - V'(x)
\end{aligned}
$$

Using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we get

$$
E((\max(W'(x) - V'(x), 0))^2) \leq \frac{6\sigma^2}{v^2} + 3(V'(x+v) - V'(x))^2 \ .
$$

By a similar argument,

$$\text{(4.9)}\qquad \operatorname{Var}(W'(x)) \le \frac{12\sigma^2}{v^2} + 3(V'(x+v) - V'(x))^2 + 3(V'(x) - V'(x-v))^2$$
$$\le \frac{12\sigma^2}{v^2} + 3(V'(x+v) - V'(x-v))^2 \ .$$

We now observe that, since $V \ge 0$, we have

$$-\frac{V(3\gamma_0)}{\gamma_0} \le V'(x) \le \frac{V(3\gamma_0)}{\gamma_0}$$

for $|x| \le 2\gamma_0$. Thus

$$V'(x+v) - V'(x-v) \le \frac{V(3\gamma_0) + V(-3\gamma_0)}{\gamma_0}$$

if $|x| \le \gamma_0, |v| \le \gamma_0$. Also,

$$\text{(4.10)}\qquad \int_{-\gamma_0}^{\gamma_0} (V'(x+v) - V'(x-v))\, dx$$
$$= V(\gamma_0 + v) - V(\gamma_0 - v) - (V(-\gamma_0 + v) - V(-\gamma_0 - v))$$
$$\le 2v(V'(\gamma_0 + v) - V'(-\gamma_0 - v))$$
$$\le \frac{4v}{\gamma_0}(V(3\gamma_0) + V(-3\gamma_0)) \ .$$

Combining with (4.9), (4.10) completes the proof. □

We now prove (4.7). Fixing any value of the quenched variables, we consider the random convex function $W(\gamma) = \frac{1}{N}(F_N(\beta, \gamma) - F_N(\beta, 0))$, where naturally the randomness occurs through the variables $(u_i)_{i \le N}$. The use of Jensen's inequality (integration inside the log rather than outside) show that $V(\gamma) \le \beta^2 \varphi(N)^2 \gamma^2 / 2$. Moreover use of [I-S-T] show that

$$\operatorname{Var}_u W(\gamma) \le K\beta^2 \frac{\varphi(N)^2}{N} \ .$$

Thus from (4.8), we see that for each v,

$$\limsup_{N \to \infty} \frac{1}{\varphi(N)^4} \int_{-\gamma_0}^{\gamma_0} \operatorname{Var}_u\left(\frac{1}{N}\frac{\partial F}{\partial \gamma}(\beta, \gamma)\right) d\gamma \le K(\beta, \gamma_0) v$$

and this concludes the proof since v is arbitrary. □

In the case $h \ne 0$, most of the work that remains to deduce Theorem 1.13 from Proposition 4.1 has been done in Section 3. In particular Proposition 3.5 and 3.6 remain valid, when the integrals are replaced by $\int_0^{\beta_1} \int_{-\gamma_0}^{\gamma_0} \cdots \, d\beta \, d\gamma$. The term $\gamma \varphi(N) \sum_{i \le N} u_i \epsilon_i$ in the Hamiltonian has a vanishing influence as $N \to 0$ since $\varphi(N) \to 0$. The routine details are left to the reader.

In the case $h = 0$, the extra difficulty is that there are two roots to the equation $x = \Phi(\beta\sqrt{x})$, where, for simplicity we write $\Phi(\beta\sqrt{x}) = \Phi(\beta\sqrt{x}, 0)$. Thereby it is harder to use information such as contained in (3.19) because, if we have a sequence x_n with $x_{n+1} \simeq \Phi(\beta\sqrt{x_n})$, we cannot guarantee that (x_n)

converges to the largest root $x(\beta)$ of the equation $x = \Phi(\beta\sqrt{x})$. Nonetheless, the sequence of intervals $C_k(\beta)$ defined just after (3.22) converges to $[0, x(\beta)]$, and rather than (3.23) we have

$$\lim_{N\to\infty}\int_0^{\beta_1}\int_{-\gamma_0}^{\gamma_0}\left(E\sigma_N(\beta,\gamma) - x^2(\beta)\right)^+ d\beta\, d\gamma = 0$$

from which the proof proceeds as in Section 3.

We now want to show more subtle arguments that prove a bit more, that will explain the difficulty involved in proving (1.25) and that will motivate Problem 1.12. It seems much better to do this at a somewhat informal level. For simplicity, we will ignore all terms containing γ, and we start by discussing the main issue. By convexity of the free energy, the right-hand side of (3.28) is an increasing function of β, so that by differentiating in β we get

$$0 \le \frac{1}{2}(1 - E\tau_N(\beta)) - \frac{\beta}{2}\frac{\partial}{\partial\beta}E\tau_N(\beta)$$

and thus

$$\frac{\partial}{\partial\beta}E\tau_N(\beta) \le 1 \tag{4.11}$$

so that, if $\beta' < \beta$ we have

$$E\tau_N(\beta') \ge E\tau_N(\beta) - (\beta - \beta')\ . \tag{4.12}$$

Thus if we know that $E\tau_N(\beta)$ is not too small, this remains true in a small interval to the left of β. The main source of complication is that I do not see how to prove that this is true in a little interval to the right of β (unless one solves Problem 1.12.)

Consider $L(\beta) = \limsup_{N\to\infty} E\sigma_N(\beta)$, and define $\beta_2 = \sup\{\beta < \beta_1;\ L(\beta) > 0\}$. As strange as it may seem, I see no way to guarantee that $\beta_2 = \beta_1$. If $\beta_2 < \beta_1$, for $\beta_2 < \beta < \beta_1$ we have $\lim_{N\to\infty} E\sigma_N(\beta) = 0$, and thus

$$\lim_{N\to\infty}\int_{\beta_2}^{\beta_1}\left|E\frac{1}{N}\frac{\partial F_N}{\partial\beta}(\beta) - \frac{\beta}{2}\right| d\beta = 0\ .$$

Let us fix $\beta_3 < \beta_2$. Then we can find $\eta > 0$ and N such that the set $A_N = \{E\sigma_N(\beta) > \eta\}$ has positive measure in $[\beta_3, \beta_2]$. Let us define by induction $C_0(\beta) = [\eta, 1]$ and

$$C_k(\beta) = \{\Phi(\beta\sqrt{y}) : y \in C_{k-1}(\beta)\}$$

so that $x(\beta) = \bigcap_k C_k(\beta)$. Rather than (3.10), we now have

$$\lim_{N\to\infty}\int_0^{\beta_1}\left|E\sigma_{N+1}\left(\beta\sqrt{\frac{N+1}{N}}\right) - \Phi(\beta E\sigma_N(\beta))\right| d\beta = 0\ . \tag{4.13}$$

For simplicity, we will ignore the influence of the factor $\sqrt{(N+1)/N}$. It follows from (4.13) that for all but a small proportion of the points β of

$A_N, E\sigma_{N+1}(\beta)$ is very close to $C_2(\beta)$. After a small number L of iterations (the number of which does not depend upon N) for most of the points β of $A_N, E\sigma_{N+L}(\beta)$ is close to $x(\beta)$, so that by (3.24) $E\tau_{N+L}(\beta)$ is close to $x^2(\beta)$. Now (4.12) tells us that there is a little interval to the left of each such β on which $E\tau_{N+L}(\beta')$ is not too small. Using (3.24) again, we see we have gone from the information that $\sigma_N(\beta)$ is not small on A_N to the information that $\sigma_{N+L}(\beta)$ is not small (on most of) $A_N+]-\tau, 0]$ for some $\tau > 0$ (not depending on N). After a few iterations, we can find N' so that $\sigma_{N'}(\beta) \simeq x(\beta)$ on (almost) all of $[0, \beta_3]$ and so

$$\liminf_{N\to\infty} \int_0^{\beta_2} \left| E\frac{1}{N}\frac{\partial F_N}{\partial \beta}(\beta) - \frac{\beta}{2}(1 - x^4(\beta))\right| d\beta = 0 \ .$$

In particular there is a subsequence N_p such that if $\beta < \beta_2$

$$\lim_{N_p\to\infty} E\frac{1}{N_p}F_{N_p}(\beta) = \mathrm{SK}(\beta) \ .$$

The problem however is that even if we know that $E\sigma_N(\beta) \simeq x(\beta)$ on $[0, \beta_2]$, there is no obvious way to guarantee that this will remain true for larger N, because the small errors permitted by (4.13) might creep in from the right and accumulate over many iterations. □

5. Lack of self-averaging, II

In this section we prove Theorem 1.11. The proof starts with the following counterpart of Proposition 4.1.

Proposition 5.1. *Under* (1.21) *we have*

$$\lim_{N\to\infty} \int_{\beta_0}^{\beta_1} \rho_N(\beta)\, d\beta = 0 \tag{5.1}$$

where

$$\rho_N(\beta) = E\frac{1}{N^4} \sum_{i,j,k,\ell\le N} (\langle \epsilon_i\epsilon_j\epsilon_k\epsilon_\ell\rangle - \langle \epsilon_i\epsilon_j\rangle\langle \epsilon_k\epsilon_\ell\rangle)^2 \tag{5.2}$$

Proof. The proof of this result is similar (but simpler) than that of Proposition 4.1 except that we now start with the relation

$$\frac{\partial F_N}{\partial \beta} = \frac{1}{\sqrt{N}} \sum_{i<j} g_{ij}\langle \epsilon_i\epsilon_j\rangle \ .$$

The details are left to the reader.

The leading idea of the deduction of Theorem 1.13 from Proposition 4.1 is that (following Proposition 3.3), when $\overline{\tau}_N$ is small we have an explicit formula to calculate $\langle \exp \frac{\beta}{\sqrt{N}} \boldsymbol{g}\cdot\boldsymbol{\epsilon}\rangle$, namely

$$\left\langle \exp \frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \right\rangle \simeq \mathrm{Const} \times \exp\left(\frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \langle \boldsymbol{\epsilon} \rangle \right) \ . \tag{5.3}$$

Behind Theorem 1.13 is also a similar formula, but now we cannot expect something as simple as (5.3). What we will show is that when ρ_N is small, we can (approximately) split Σ_N in two pieces A and $-A$, of Gibbs measure about $1/2$, such that

$$\int_A \exp \frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \, dG(\boldsymbol{\epsilon}) \simeq \mathrm{Const} \times \exp\left(\frac{\beta'}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{a} \right)$$

where $\boldsymbol{a}$ is the barycenter of $\boldsymbol{\epsilon}$ over A. In physical terms, this means that the system is a mixture of two phases, (symmetric to each other) such that (5.3) holds for each phase. (The presence of an external field, or even the extra term in the Hamiltonian (1.23) breaks the symmetry, and the system has only one phase, which makes it much simpler to study.) This feature is possibly the main interest of the present section.

One problem in the proof will be how to identify the set A. The answer is natural. For most vectors $\boldsymbol{x}$, the distribution of $\boldsymbol{x} \cdot \boldsymbol{\epsilon}$ takes essentially only 2 values, so the set $A = \{\boldsymbol{\epsilon}; \boldsymbol{x} \cdot \boldsymbol{\epsilon} > 0\}$ will do (of course, one also must take care not to choose one of the few vectors $\boldsymbol{x}$ that possibly would not work).

For a function f on Σ_N, we write $\mathrm{Var}_G f$ its variance with respect to G, i.e.

$$\mathrm{Var}_G f = \langle f^2 \rangle - \langle f \rangle^2 \ .$$

Lemma 5.2. *We have*

$$\mathrm{Var}_G\left(\tfrac{1}{N^2} (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \right) \le K \rho_N^{1/2} \ . \tag{5.4}$$

Proof. We have $\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' = \sum_{i \le N} \epsilon_i \epsilon'_i$ so that

$$\left(\frac{1}{N} \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \right)^2 = \frac{1}{N^2} \sum_{i,j \le N} \epsilon_i \epsilon_j \epsilon'_i \epsilon'_j$$

and

$$\left(\frac{1}{N} \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \right)^4 = \frac{1}{N^4} \sum_{i,j,k,\ell \le N} \epsilon_i \epsilon_j \epsilon_k \epsilon_\ell \epsilon'_i \epsilon'_j \epsilon'_k \epsilon'_\ell \ .$$

Thus

$$\mathrm{Var}_G\left(\left(\frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'}{N} \right)^2 \right) = \frac{1}{N^4} \sum_{i,j,k,\ell} \left(\langle \epsilon_i \epsilon_j \epsilon_k \epsilon_\ell \rangle^2 - \langle \epsilon_i \epsilon_j \rangle^2 \langle \epsilon_k \epsilon_\ell \rangle^2 \right) \ .$$

Using Cauchy-Schwarz and the fact that $|A^2 - B^2| \le 2|A - B|$ for $A, B \le 1$, we get

$$\mathrm{Var}_G\left(\left(\frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'}{N} \right)^2 \right) \le 2 \left(\frac{1}{N^4} \sum_{i,j,k,\ell} \left(\langle \epsilon_i \epsilon_j \epsilon_k \epsilon_\ell \rangle - \langle \epsilon_i \epsilon_j \rangle \langle \epsilon_k \epsilon_\ell \rangle \right)^2 \right)^{1/2} \le K \rho_N^{1/2} \ . \qquad \square$$

Since $\tau_N = \langle (\frac{\epsilon\cdot\epsilon'}{N})^2 \rangle_2$, we rewrite (5.4) as

$$\iint \left(\left(\frac{1}{N} \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \right)^2 - \tau_N \right)^2 dG(\boldsymbol{\epsilon})\, dG(\boldsymbol{\epsilon}') \leq K \rho_N^{1/2} \ .$$

Using Fubini theorem, we pick $\boldsymbol{\eta} \in \Sigma_N$ such that

$$\left\langle \left(\frac{1}{N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\eta})^2 - \tau_N \right)^2 \right\rangle \leq K \rho_N^{1/2} \ . \tag{5.5}$$

This vector $\boldsymbol{\eta}$ will remain fixed throughout the proof. We set

$$A = \{\boldsymbol{\epsilon}; \boldsymbol{\epsilon} \cdot \boldsymbol{\eta} > 0\}$$
$$B = -A = \{\boldsymbol{\epsilon}; \boldsymbol{\epsilon} \cdot \boldsymbol{\eta} < 0\}$$
$$C = \{\boldsymbol{\epsilon}; \boldsymbol{\epsilon} \cdot \boldsymbol{\eta} = 0\}$$

Thus by (5.5) we have

$$G(C) \leq \frac{K}{\tau_N} \rho_N^{1/2} \ . \tag{5.6}$$

Since $B = -A$, we have $G(B) = G(A) = \frac{1}{2}(1 - G(C))$. We set

$$\boldsymbol{a} = \frac{1}{G(A)} \int_A \boldsymbol{\epsilon}\, dG(\boldsymbol{\epsilon}) \ , \tag{5.7}$$

so that

$$-\boldsymbol{a} = \frac{1}{G(A)} \int_B \boldsymbol{\epsilon}\, dG(\boldsymbol{\epsilon}) \ .$$

Before we continue with the argument, we need one more observation.

Lemma 5.3. *Consider numbers $(\alpha_{ij})_{i,j \leq N}$, and $f(\boldsymbol{\epsilon}) = \sum_{i,j \leq N} \alpha_{ij} \epsilon_i \epsilon_j$. Then*

$$\mathrm{Var}_G f \leq N^2 \left(\sum_{i,j \leq N} \alpha_{ij}^2 \right) \rho_N^{1/2} \ .$$

Proof. It is straightforward that

$$\mathrm{Var}_G f = \sum_{i,j,k,\ell} \alpha_{ij} \alpha_{k\ell} (\langle \epsilon_i \epsilon_j \epsilon_k \epsilon_\ell \rangle - \langle \epsilon_i \epsilon_j \rangle \langle \epsilon_k \epsilon_\ell \rangle)$$

from which the result follows from Cauchy-Schwarz. □

Corollary 5.4. *If $f(\boldsymbol{\epsilon}) = (\sum_{i \leq N} \alpha_i \epsilon_i)(\sum_{i \leq N} \beta_i \epsilon_i)$ then we have*

$$Var_G f \leq N^2 \left(\sum_{i \leq N} \alpha_i^2 \right) \left(\sum_{i \leq N} \beta_i^2 \right) \rho_N^{1/2} \ .$$

We go back to the main argument.

To simplify notation, we denote by R_N a quantity of the type $K(\beta) \rho_N(\beta)^{\alpha} \tau_N(\beta)^{-\alpha'}$ where $\alpha > 0, \alpha' < \infty$, and where $K(\beta)$ remains bounded with β. This quantity may vary at each occurrence.

Lemma 5.5. *Consider* $\boldsymbol{x}$ with $\|\boldsymbol{x}\|^2 \le 4N$. *Then*

$$\int_A \frac{1}{N}|\boldsymbol{x}\cdot(\boldsymbol{\epsilon}-\boldsymbol{a})|\, dG(\boldsymbol{\epsilon}) \le R_N \ .$$

Proof. From (5.5) we have

$$\int_A \left(\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} - \tau_N^{1/2}\right)^2 \left(\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} + \tau_N^{1/2}\right)^2 dG(\boldsymbol{\epsilon}) \le R_N \ .$$

Now, for $\boldsymbol{\epsilon}\in A$, we have $\boldsymbol{\epsilon}\cdot\boldsymbol{\eta}\ge 0$, so that

$$\tfrac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} + \tau_N^{1/2} \ge \tau_N^{1/2} \ ,$$

and hence

$$\int_A \left(\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} - \tau_N^{1/2}\right)^2 dG(\boldsymbol{\epsilon}) \le R_N$$

and thus

$$\int_A \left|\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} - \tau_N^{1/2}\right| dG(\boldsymbol{\epsilon}) \le R_N \ .$$

Since $|N^{-1}\boldsymbol{x}\cdot\boldsymbol{\epsilon}|\le 2$, we have

$$\int_A \left|\frac{1}{N^2}(\boldsymbol{x}\cdot\boldsymbol{\epsilon})(\boldsymbol{\eta}\cdot\boldsymbol{\epsilon}) - \tau_N^{1/2}\frac{\boldsymbol{x}\cdot\boldsymbol{\epsilon}}{N}\right| dG(\boldsymbol{\epsilon}) \le R_N \ . \tag{5.8}$$

Using Corollary 5.4 and Cauchy-Schwarz we see that for some constant c,

$$\int_A \left|c - \tau_N^{1/2}\frac{\boldsymbol{x}\cdot\boldsymbol{\epsilon}}{N}\right| dG(\boldsymbol{\epsilon}) \le R_N$$

so that

$$\left|c - \tau_N^{1/2}\frac{\boldsymbol{x}\cdot\boldsymbol{a}}{N}\right| \le R_N \ .$$

The result follows combining these two inequalities. □

Lemma 5.6. *We have*

$$\left|\tau_N^{1/2} - \tfrac{1}{N}\|\boldsymbol{a}\|^2\right| \le R_N \ .$$

Proof. Lemma 5.5 shows that for each $\boldsymbol{\epsilon}'$ in Σ_N we have

$$\int_A \left|\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' - \frac{1}{N}\boldsymbol{a}\cdot\boldsymbol{\epsilon}'\right| dG(\boldsymbol{\epsilon}) \le R_N$$

so that

$$\int_{A\times A} \left|\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' - \frac{1}{N}\boldsymbol{a}\cdot\boldsymbol{\epsilon}'\right| dG(\boldsymbol{\epsilon})\, dG(\boldsymbol{\epsilon}') \le R_N \ .$$

Now, by Lemma 5.5 again,

$$\int_{A\times A}\left|\frac{1}{N}\boldsymbol{a}\cdot\boldsymbol{\epsilon}'-\frac{1}{N}\boldsymbol{a}\cdot\boldsymbol{a}'\right|dG(\boldsymbol{\epsilon})\,dG(\boldsymbol{\epsilon}')\le R_N \tag{5.9}$$

so that

$$\int_{A\times A}\left|\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'-\frac{1}{N}\boldsymbol{a}\cdot\boldsymbol{a}\right|dG(\boldsymbol{\epsilon})\,dG(\boldsymbol{\epsilon}')\le R_N\ . \tag{5.10}$$

Since $|x^2-y^2|\le 2|x-y|$ for $|x|,|y|\le 1$, we have

$$\int_{A\times A}\left|\left(\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'\right)^2-\frac{1}{N^2}\|\boldsymbol{a}\|^4\right|dG(\boldsymbol{\epsilon})\,dG(\boldsymbol{\epsilon}')\le R_N\ .$$

Using symmetry, we get

$$\int_{(A\cup B)\times(A\cup B)}\left|\left(\frac{1}{N}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'\right)^2-\frac{1}{N^2}\|\boldsymbol{a}\|^4\right|dG(\boldsymbol{\epsilon})\,dG(\boldsymbol{\epsilon}')\le R_N\ .$$

Now (5.6) shows that this remains true if we now take the integral over Σ_N^2. It follows that $|\tau_N-\frac{1}{N^2}\|\boldsymbol{a}'\|^4|\le R_N$. The result follows, using $|x-y|\le(x^2-y^2)/x$ for $x,y>0$. □

Lemma 5.7. *We have*

$$\int_{A\times A}\left|\frac{1}{N}(\boldsymbol{\epsilon}-\boldsymbol{a})\cdot(\boldsymbol{\epsilon}'-\boldsymbol{a})\right|dG(\boldsymbol{\epsilon})\,dG(\boldsymbol{\epsilon}')\le R_N\ .$$

Proof. We have $\int|f(\boldsymbol{\epsilon},\boldsymbol{\epsilon})-\frac{1}{N}\|\boldsymbol{a}\|^2|\le R_N$ whenever $f(\boldsymbol{\epsilon},\boldsymbol{\epsilon}')=\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'/N$ (by (5.10)) or $f(\boldsymbol{\epsilon},\boldsymbol{\epsilon}')=\boldsymbol{a}\cdot\boldsymbol{\epsilon}'/N$ or $\boldsymbol{a}\cdot\boldsymbol{\epsilon}/N$ (by (5.9)). □

At last we come within sight of accomplishing the program outlined at the beginning of the section.

Lemma 5.8. *Consider a fresh sequence $\boldsymbol{g}=(\boldsymbol{g}_i)_{i\le N}$ of standard Gaussian r.v., and*

$$U=\int_A\exp\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot\boldsymbol{\epsilon}\,dG(\boldsymbol{\epsilon})$$
$$V=G(A)\exp\left(\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot\boldsymbol{a}+\frac{\beta^2}{2}\left(1-\frac{1}{N}\|\boldsymbol{a}\|^2\right)\right)\ .$$

Then we have

$$E_g|U-V|\le R_N\ .$$

Proof. We write

$$U_1=\int_A\exp\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot(\boldsymbol{\epsilon}-\boldsymbol{a})\,dG(\boldsymbol{\epsilon})$$
$$V_1=G(A)\exp\frac{\beta^2}{2}\left(1-\frac{\|\boldsymbol{a}\|^2}{N}\right)$$

so that, using Cauchy-Schwarz, we are reduced to show that

$$E_g(U_1 - V_1)^2 \le R_N \ .$$

Now

$$E_g(U_1 - V_1)^2 = E_g(U_1 - E_g U_1)^2 + (E_g U_1 - E_g V_1)^2 \ .$$

The control of the first term is done as usual, and relies upon Lemma 5.7. The control of the second term relies upon

$$E_g U_1 - E_g V_1 = \int_A \left(\exp \frac{\beta^2}{2N} \|\boldsymbol{\epsilon} - \boldsymbol{a}\|^2 - \exp \frac{\beta^2}{2} \left(1 - \frac{\|\boldsymbol{a}\|^2}{N} \right) \right) dG(\boldsymbol{\epsilon}) \ .$$

The integrand is at most

$$K(\beta) \left| \frac{\boldsymbol{\epsilon} \cdot \boldsymbol{a}}{N} - \frac{\|\boldsymbol{a}\|^2}{N} \right|$$

and we now use Lemma 5.5, for $\boldsymbol{x} = \boldsymbol{a}$. □

Proposition 5.9. *We have*

$$(5.11) \qquad E_g \left| \left\langle \exp \frac{\beta}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{\epsilon} \right\rangle - \left(\mathrm{ch} \frac{\beta}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{a} \right) \exp \frac{\beta^2}{2} \left(1 - \frac{\|\boldsymbol{a}\|^2}{N} \right) \right| \le R_N \ .$$

Proof. We decompose the integral $\langle \cdot \rangle$ as the sum of the integral over $A, -A$, and C, and we use Lemma 5.8 and (5.6). The routine details are left to the reader. □

We observe that Proposition 5.9 remains true if we replace in (5.11) β by $\beta\sqrt{2}$ (which is what we actually need).

Consider a fresh sequence $\boldsymbol{g}' = (g'_i)_{i \le N}$ of standard normal r.v. For $\eta, \eta' \in \{-1, 1\}$, we set

$$A(\eta, \eta') = \left\langle \exp \frac{\beta}{\sqrt{N}} (\eta \boldsymbol{g} \cdot \boldsymbol{\epsilon} + \eta' \boldsymbol{g}' \cdot \boldsymbol{\epsilon}) \right\rangle$$

and

$$U = \frac{\sum_{\eta, \eta'} \eta \eta' A(\eta, \eta')}{\sum_{\eta, \eta'} A(\eta, \eta')} \ .$$

Consider

$$B(\eta, \eta') = \mathrm{ch} \left(\frac{\beta}{\sqrt{N}} (\eta \boldsymbol{g} + \eta' \boldsymbol{g}') \cdot \boldsymbol{a} \right) \exp \frac{\beta}{2} \left(1 - \frac{\|\boldsymbol{a}\|^2}{N} \right)$$

and

$$V = \frac{\sum_{\eta,\eta'} \eta\eta' B(\eta,\eta')}{\sum_{\eta,\eta'} B(\eta,\eta')} \quad .$$

Thus $V = \mathrm{th}(\frac{\beta}{\sqrt{N}} \boldsymbol{g} \cdot \boldsymbol{a})\ \mathrm{th}(\frac{\beta}{\sqrt{N}} \boldsymbol{g}' \cdot \boldsymbol{a})$. Thus, by independence of $\boldsymbol{g}, \boldsymbol{g}'$,

$$E_g V^2 = \Phi\left(\beta \frac{\|\boldsymbol{a}\|}{\sqrt{N}}\right)^2 \quad .$$

On the other hand, if $|x| \le y, |x'| \le y', y, y' \ge 1$, we have

$$\left|\frac{x}{y} - \frac{x'}{y'}\right| \le |x - x'| + |y - y'| \quad .$$

Thus, using (5.11), we have

$$E_g(|U - V|) \le R_N$$

so that

$$\left|E_g U^2 - E_g V^2\right| = \left|E_g U^2 - \Phi^2\left(\beta \frac{\|\boldsymbol{a}\|}{\sqrt{N}}\right)\right| \le R_N \quad ,$$

and using Lemma 5.6

$$|E_g U^2 - \Phi^2\big(\beta \tau_N^{1/4}(\beta)\big)| \le R_N \quad . \tag{5.12}$$

Now, since

$$\tau_N = \frac{1}{N} + \frac{2}{N^2} \sum_{i<j} \langle \epsilon_i \epsilon_j \rangle^2$$

it should be clear that (looking back at Proposition 3.2, and since the influence of the term $\eta\eta' g$ there was shown in Proposition 3.3 to be of order K/N)

$$\left|EU^2 - E\tau_{N+2}\big(\beta\sqrt{(N+2)/N}\big)\right| \le \tfrac{K}{N}$$

so that, from (5.12), and using the form of R_N, we have

$$\left|E\Phi^2(\beta\tau_N^{1/4}(\beta)) - E\tau_{N+2}(\beta\sqrt{(N+2)/N})\right| \le \frac{K}{N} E \min\left(\frac{\rho_N^\alpha}{\tau_N^{\alpha'}}, 2\right) \quad . \tag{5.13}$$

It is rather obvious, but tedious and elementary to show, that there is a function $\varphi\colon (\mathbb{R}^+)^3 \to \mathbb{R}$, depending on β only, with a limit zero at zero, such that (5.13) implies

$$\begin{aligned} &\left|\Phi(\beta(E\tau_N^{1/2}(\beta))^{1/2}) - E\tau_{N+2}^{1/2}(\beta\sqrt{(N+2)/N})\right| \\ &\le \frac{K}{N} + \varphi(\rho_N(\beta), \operatorname{Var}\tau_N(\beta), \operatorname{Var}\tau_{N+2}(\beta\sqrt{(N+2)/N})) \quad . \end{aligned} \tag{5.14}$$

Thereby it follows that

$$\lim_{N\to\infty}\int_1^{\beta_1} |x_{N+2}(\beta\sqrt{(N+2)/N}) - \Phi\big(\beta x_N^{1/2}(\beta)\big)|\, d\beta = 0 \tag{5.15}$$

where we have set $x_N(\beta) = E\tau_N^{1/2}(\beta)$. The purpose of this change of notation is to make apparent the relationship between (5.15) and (4.13). Once (5.15) is obtained the conclusion is reached by the arguments of Section 4, and is left to the reader. □

6. Toward the Almeida-Thouless line

The purpose of this section is to evaluate the right-hand side of (3.5) more carefully than was done in Section 3. Throughout this section, $\boldsymbol{a} = \langle \boldsymbol{\epsilon} \rangle, T = \langle \exp \frac{\beta^2}{N} \|\boldsymbol{\epsilon} - \boldsymbol{a}\|^2 \rangle$. Consider (with the notation of (3.5)) the quantity

$$B(\eta, \eta') = T \exp\left(\frac{\beta}{\sqrt{N}}(\eta \boldsymbol{g} + \eta' \boldsymbol{g}') \cdot \boldsymbol{a} + \beta h(\eta + \eta')\right)$$

and

$$Z = \sum_{\eta,\eta'=\pm 1} A(\eta, \eta'); Z' = \sum_{\eta,\eta'=\pm 1} B(\eta, \eta')$$

$$X = (A(1,1)A(-1,-1) - A(-1,1)A(1,-1))^2$$

so that, as shown in Section 3, $EX^2 \le K(\beta)(C_N(\beta) + 1/N)$.

Thus

$$\begin{aligned}\left|E\left(\frac{X^2}{Z^4}\right) - E\left(\frac{X^2}{Z'^4}\right)\right| &\le E\left|X^2\left|\frac{Z^4 - Z'^4}{Z^4 Z'^4}\right|\right| \\ &\le KE\left|X^2(Z - Z')\right| \ .\end{aligned}$$

Conjecture 6.1. $E(X^2|Z - Z'|) = o(C_N(\beta))$.

This is very likely in view of $E(|Z - Z'|^2) \le K(\beta)C_N(\beta), E(X^2) \le K(\beta)C_N(\beta)$. Using Cauchy-Schwarz, it even suffices to prove that

Conjecture 6.2. $EX^4 = o(C_N(\beta))$.

Thus, under Conjecture 6.2, to calculate the right hand side of (3.5), it suffices to calculate $16E(\frac{X^2}{Z'^4})$ and we have

$$Z' = 4T \operatorname{ch}\left(\frac{\beta}{\sqrt{N}}\boldsymbol{g} \cdot \boldsymbol{a} + \beta h\right) \operatorname{ch}\left(\frac{\beta}{\sqrt{N}}\boldsymbol{g}' \cdot \boldsymbol{a} + \beta h\right) \ .$$

Thus, to evaluate $C_{N+2}\left(\beta\sqrt{1 + \frac{2}{N}}\right)$ we are led to evaluate

$$E\left(\frac{X^2}{16T^4 \operatorname{ch}^4\left(\frac{\beta}{\sqrt{N}}\boldsymbol{g} \cdot \boldsymbol{a} + \beta h\right) \operatorname{ch}^4\left(\frac{\beta}{\sqrt{N}}\boldsymbol{g}' \cdot \boldsymbol{a} + \beta h\right)}\right) \ . \tag{6.2}$$

The only apparent way to do this is to compute the expectation of X^2 conditionally upon the quenched variables, and upon $\boldsymbol{g}\cdot\boldsymbol{a}, \boldsymbol{g}'\cdot\boldsymbol{a}$. (Such expectation is denoted by E_0). This computation is extremely elementary and dull, and we strongly advise the reader, at first reading (and indeed at all subsequent readings) to jump directly to Proposition 6.3 that summarizes its conclusions. We denote by $S(\boldsymbol{x})$ the projection of $\boldsymbol{x}$ along the direction of $\boldsymbol{a}$, given by

$$S(\boldsymbol{x}) = (\boldsymbol{x}\cdot\boldsymbol{a})\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2}$$

and we set

$$R(\boldsymbol{x}) = \boldsymbol{x} - S(\boldsymbol{x}) \ .$$

Thus $R(\boldsymbol{g})$ is Gaussian and independent of $\boldsymbol{g}\cdot\boldsymbol{x}$. Moreover for $\boldsymbol{x}$ in $\mathbb{R}^N$ we have

$$E_0(\exp\boldsymbol{x}\cdot\boldsymbol{g}) = \exp\Big(\tfrac{1}{2}\|R(\boldsymbol{x})\|^2 + \boldsymbol{g}\cdot S(\boldsymbol{x})\Big) \tag{6.3}$$

Now, using the notation of Section 3, we have

$$\begin{aligned} X^2 = {} & \left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\right\rangle_4 + \left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\right\rangle_4 \\ & - \left\langle \exp\frac{\beta}{\sqrt{N}}\big((\boldsymbol{g}+\boldsymbol{g}')\cdot\boldsymbol{\theta} + (\boldsymbol{g}-\boldsymbol{g}')\cdot\boldsymbol{\theta}^*\big)\right\rangle_4 \\ & - \left\langle \exp\frac{\beta}{\sqrt{N}}\big((\boldsymbol{g}+\boldsymbol{g}')\cdot\boldsymbol{\theta} + (\boldsymbol{g}-\boldsymbol{g}')\cdot\boldsymbol{\theta}^*\big)\right\rangle_4 \end{aligned}$$

and thus

$$\begin{aligned} E_0X^2 = {} & \left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot S\big(\boldsymbol{\theta}+\boldsymbol{\theta}^*\big)\exp\frac{\beta^2}{N}\|R(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\|^2\right\rangle_4 \\ & + \left\langle \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\exp\frac{\beta^2}{N}\|R(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\|^2\right\rangle_4 \\ & - \Big\langle \exp\frac{\beta}{\sqrt{N}}\big((\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}) + (\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*)\big) \\ & \qquad \times\exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2)\Big\rangle_4 \\ & - \Big\langle \exp\frac{\beta}{\sqrt{N}}\big((\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}) + (\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*)\big) \\ & \qquad \times\exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2)\Big\rangle_4 \\ = {} & \left\langle \exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2)H\right\rangle_4 \end{aligned}$$

where

$$H = \exp\frac{2\beta^2}{N}R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*)\Big[\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}+\boldsymbol{\theta}^*)$$
$$+\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}+\boldsymbol{\theta}^*)\Big]$$
$$-\exp\frac{\beta}{\sqrt{N}}((\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta})+(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*))$$
$$-\exp\frac{\beta}{\sqrt{N}}((\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta})+(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*))\ .$$

Now we write

$$\exp\frac{2\beta^2}{N}R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*) = 1+\frac{2\beta^2}{N}R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*)$$
$$+\left(\exp\frac{2\beta^2}{N}R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*)-1-\frac{2\beta^2}{N}R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*)\right)$$

Thus

$$E_0X^2 = L_1+L_2+L_3$$

where

$$L_1 = \left\langle \exp\frac{\beta^2}{N}\left(\|R(\boldsymbol{\theta})\|^2+\|R(\boldsymbol{\theta}^*)\|^2\right)H_1\right\rangle_4 \tag{6.4}$$

for

$$H_1 = \exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}+\boldsymbol{\theta}^*)+\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}+\boldsymbol{\theta}^*)$$
$$-\exp\frac{\beta}{\sqrt{N}}((\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta})+(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*))$$
$$-\exp\frac{\beta}{\sqrt{N}}((\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta})+(\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*))$$
$$=\left(\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta})-\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta})\right)$$
$$\times\left(\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*)-\exp\frac{\beta}{\sqrt{N}}(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*)\right)$$

and

$$L_2 = \frac{2\beta^2}{N}\Big\langle R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*)\exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2+\|R(\boldsymbol{\theta}^*)\|^2)$$
$$\times\exp\frac{\beta}{\sqrt{N}}((\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta})+(\boldsymbol{g}+\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*))\Big\rangle_4$$
$$+\frac{2\beta^2}{N}\Big\langle R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*)\exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2+\|R(\boldsymbol{\theta}^*)\|^2)$$
$$\times\exp\frac{\beta}{\sqrt{N}}((\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta})+(\boldsymbol{g}-\boldsymbol{g}')\cdot S(\boldsymbol{\theta}^*))\Big\rangle_4 \tag{6.5}$$

and

$$L_3 = \left\langle \left(\exp \frac{2\beta^2}{N} R(\boldsymbol{\theta}) \cdot R(\boldsymbol{\theta}^*) - 1 - \frac{2\beta^2}{N} R(\boldsymbol{\theta}) \cdot R(\boldsymbol{\theta}^*) \right) \right.$$
$$\left. \times \exp \frac{\beta^2}{N} \left(\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2 \right) \right\rangle_4 .$$

We now observe that L_1, L_2, L_3 are all nonnegative. Indeed

$$L_1 = \left\langle \exp \frac{\beta^2}{N} \|R(\boldsymbol{\theta})\|^2 \left(\exp \frac{\beta}{\sqrt{N}} (\boldsymbol{g} + \boldsymbol{g}') \cdot S(\boldsymbol{\theta}) - \exp \frac{\beta}{\sqrt{N}} (\boldsymbol{g} - \boldsymbol{g}') \cdot S(\boldsymbol{\theta}) \right) \right\rangle_2^2$$

$$L_2 = \frac{2\beta^2}{N} \left\| \left\langle R(\boldsymbol{\theta}) \exp \left(\frac{\beta^2}{N} \|R(\boldsymbol{\theta})\|^2 + \frac{\beta}{\sqrt{N}} (\boldsymbol{g} + \boldsymbol{g}') \cdot S(\boldsymbol{\theta}) \right) \right\rangle_2 \right\|^2$$
$$+ \frac{2\beta^2}{N} \left\| \left\langle R(\boldsymbol{\theta}) \exp \left(\frac{\beta^2}{N} \|R(\boldsymbol{\theta})\|^2 + \frac{\beta}{\sqrt{N}} (\boldsymbol{g} - \boldsymbol{g}') \cdot S(\boldsymbol{\theta}) \right) \right\rangle_2 \right\|^2 .$$

We observe that, since exchanging $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ changes $\boldsymbol{\theta}$ in $-\boldsymbol{\theta}$, we have

$$L_3 = \left\langle \exp \frac{\beta^2}{N} (\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2) \left(\operatorname{ch} \frac{2\beta^2}{N} R(\boldsymbol{\theta}) \cdot R(\boldsymbol{\theta}^*) - 1 \right) \right\rangle_4 .$$

Also, straight forward computation from (6.4), (6.5) gives

$$E_0 L_1 = 4 \left\langle \exp \frac{\beta^2}{N} (\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}^*\|^2) \operatorname{sh}^2 \left(\frac{\beta}{N} \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \left(\boldsymbol{\theta}^* \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \right) \right\rangle_4$$
$$\leq K(\beta) \frac{1}{N^2} \left\langle \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right)^2 \right\rangle_2^2$$

$$E_0 L_2 = \frac{2\beta^2}{N} \left\langle R(\boldsymbol{\theta}) \cdot R(\boldsymbol{\theta}^*) \exp \frac{\beta}{N} \left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}^*\|^2 + 2 \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \left(\boldsymbol{\theta}^* \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \right) \right\rangle_4$$
$$= \frac{2\beta^2}{N} \left\langle R(\boldsymbol{\theta}) \cdot R(\boldsymbol{\theta}^*) \exp \frac{\beta}{N} (\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}^*\|^2) \operatorname{sh} \frac{2\beta}{N} \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \left(\boldsymbol{\theta}^* \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \right\rangle_4$$

using symmetry. We observe that

$$R(\boldsymbol{\theta}) \cdot R(\boldsymbol{\theta}^*) = \boldsymbol{\theta} \cdot \boldsymbol{\theta}^* - \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \left(\boldsymbol{\theta}^* \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) . \tag{6.6}$$

Thus, using Cauchy Schwarz,

$$E_0 L_2 = \frac{K(\beta)}{N^2} \left\langle |\boldsymbol{\theta} \cdot \boldsymbol{\theta}^*| \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \left(\boldsymbol{\theta}^* \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right) \right\rangle_4 + \frac{K(\beta)}{N^2} \left\langle \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right)^2 \right\rangle_2^2$$
$$\leq K(\beta) \left\langle \left(\frac{\boldsymbol{\theta} \cdot \boldsymbol{\theta}^*}{N} \right)^2 \right\rangle_4^{1/2} \left\langle \frac{1}{N} \left(\boldsymbol{\theta}^* \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right)^2 \right\rangle_4 + \frac{K(\beta)}{N^2} \left\langle \left(\boldsymbol{\theta} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right)^2 \right\rangle_2^2 .$$

We now summarize the result of this computation. We recall that $T = \langle \exp \frac{\beta^2}{N} \|\boldsymbol{\epsilon} - \boldsymbol{a}\|^2 \rangle$.

Proposition 6.3. *Consider a standard normal r.v. z. Then we have*

$$\frac{L_3}{16T^4}\left(E_z\left(\frac{1}{\mathrm{ch}^4\left(\beta\frac{\|\boldsymbol{a}\|}{\sqrt{N}}z+\beta h\right)}\right)\right)^2 \le E_0\frac{X^2}{Z'^4}$$

$$\le \frac{L_3}{16T^4}\left(E_z\left(\frac{1}{\mathrm{ch}^4\left(\beta\frac{\|\boldsymbol{a}\|}{\sqrt{N}}z+\beta h\right)}\right)\right)^2 + K(\beta)\frac{1}{N^2}\left\langle\left(\boldsymbol{\theta}\cdot\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right)^2\right\rangle_2^2$$

$$+K(\beta)\left\langle\left(\frac{\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*}{N}\right)^2\right\rangle_4^{1/2}\left\langle\frac{1}{N}\left(\boldsymbol{\theta}\cdot\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right)^2\right\rangle_2 .$$

There, by E_z we mean the expectation in z, conditionally in the quenched variables.

We now make some overwhelmingly convincing conjectures that will allow to pursue the computations.

Conjecture 6.4. *Given $\alpha > 0$, there is $\gamma > 0$ with the following property. For each function $f\colon \Sigma_N^4 \to [0,1]$ (that depends upon the quenched variables) we have (for each N)*

$$E\langle f\rangle \le \gamma \Rightarrow E\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2 f\rangle_4 \le \alpha E\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\rangle_4$$

Comment. What this means is that very small sets for the Gibbs measure, and exceptional events in the quenched variables do not contribute much to $E\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\rangle_4$.

Conjecture 6.5. *We have*

$$E\left\langle\frac{1}{N}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{\theta}}{\|\boldsymbol{a}\|}\right)^2\right\rangle_2^2 = o\left(E\left\langle\left(\frac{\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*}{N}\right)^2\right\rangle_4\right) = o(E\overline{\tau}_N(\beta)) .$$

Comment. Here, and in what follows, o denotes a function such that $\lim_{x\to 0} o(x)/x = 0$, that may vary at each occurrence. As we have seen, the statement that

$$\sup_{\|\boldsymbol{x}\|=1}\left\langle\left(\frac{\boldsymbol{x}\cdot\boldsymbol{\theta}}{N}\right)^2\right\rangle_2 \le \left\langle\left(\frac{\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*}{N}\right)^2\right\rangle_4^{1/2}$$

simply expresses the domination of the operator norm by the Hilbert-Schmidt norm of a certain operator. If Conjecture 6.5 were wrong, this would mean not only that these norms are of the same order, but also that $\boldsymbol{a}$ is a very special direction in relation to the symmetrized vector $\boldsymbol{\epsilon}-\boldsymbol{\epsilon}'$, a most unlikely situation.

Consider the function

$$\Psi_\beta(x) = \left(E\frac{1}{\operatorname{ch}^4(\beta\sqrt{x}z + \beta h)}\right)^2$$

Proposition 6.6. *Under Conjectures* 6.1, 6.4, 6.5 *we have*

$$C_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) = \frac{\beta^4}{2}\Psi_\beta(E\sigma_N(\beta))E\overline{\tau}_N(\beta) + o(E\overline{\tau}_N(\beta))$$
$$+ E\overline{\tau}_N(\beta)\varphi(\operatorname{Var}\sigma_N(\beta)) \ .$$

where φ denotes a function such that $\lim_{x\to 0}\varphi(x) = 0$.

Proof. It should be obvious from Proposition 6.3 that

$$C_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) = E\left(\frac{L_3}{16T^4}\Psi_\beta(\sigma_N(\beta))\right) + o(E\overline{\tau}_N(\beta)) \ .$$

We observe that $L_3 \le K(\beta)N^{-2}\langle(R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*))^2\rangle_4$. We observe that by (6.6)

$$E\langle|(R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*))^2 - (\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2|\rangle_4 \le 2E\left\langle\left|\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*\right|\left|\boldsymbol{\theta}\cdot\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right|\left|\boldsymbol{\theta}^*\cdot\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right|\right\rangle_4$$

$$+E\left\langle\left(\boldsymbol{\theta}\cdot\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right)^2\left(\boldsymbol{\theta}^*\cdot\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\right)^2\right\rangle_4 \le N^2 o(E\overline{\tau}_N(\beta)) \tag{6.7}$$

where we have used Cauchy-Schwarz and Conjecture 6.5. It then follows from Conjecture 6.4 that

$$E(|L_3(\Psi_\beta(\sigma_N(\beta)) - \Psi_\beta(E\sigma_N(\beta)))|) \le E\overline{\tau}_N(\beta)\varphi(\operatorname{Var}\sigma_N(\beta)) + o(E\overline{\tau}_N(\beta)) \ .$$

Consider

$$L_4 = \frac{2\beta^4}{N^2}\left\langle(R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*))^2\exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2)\right\rangle_4$$
$$L_5 = \frac{2\beta^4}{N^2}\left\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\exp\frac{\beta^2}{N}(\|R(\boldsymbol{\theta})\|^2 + \|R(\boldsymbol{\theta}^*)\|^2)\right\rangle_4$$

Thus

$$E|L_3 - L_4| \le K(\beta)E\left\langle\frac{(R(\boldsymbol{\theta})\cdot R(\boldsymbol{\theta}^*))^4}{N^4}\right\rangle_4 \ .$$

Using (6.6) and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we see from Conjectures 6.4, 6.5 that this is $o(E\overline{\tau}_N(\beta))$. The same holds true for $E|L_4 - L_5|$ from (6.7).

Next, we recall that

$$\begin{aligned}\|R(\boldsymbol{\theta})\|^2 &= \|\boldsymbol{\theta}\|^2 - \frac{(\boldsymbol{a}\cdot\boldsymbol{\theta})^2}{\|\boldsymbol{a}\|^2} \\ &= 2N\left(1-\frac{\|\boldsymbol{a}\|^2}{N}\right) - 2(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' - \|\boldsymbol{a}\|^2) - \frac{(\boldsymbol{a}\cdot\boldsymbol{\theta})^2}{\|\boldsymbol{a}\|^2} \quad .\end{aligned}$$

Thus

$$L_5 = \frac{2\beta^4}{N^2}\exp 4\beta^2\left(1-\frac{\|\boldsymbol{a}\|^2}{N}\right)\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\xi\rangle_4$$

where ξ is a function on Σ_N^4 such that $E\langle|\xi-1|\rangle_4$ goes to zero with $E\left\langle\left(\frac{\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*}{N}\right)^2\right\rangle_4$.

Now

$$T = \left\langle\exp\frac{\beta^2}{N}\|\boldsymbol{\epsilon}-\boldsymbol{a}\|^2\right\rangle = \exp\beta^2\left(1-\frac{\|\boldsymbol{a}\|^2}{N}\right)\left\langle\exp\frac{2\beta^2}{N}(\boldsymbol{\epsilon}-\boldsymbol{a})\cdot\boldsymbol{a}\right\rangle$$

and thus

$$\frac{L_5}{T^4} = \frac{2\beta^4}{N^2}\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\xi'\rangle_4$$

where $E\langle|\xi'-1|\rangle_4$ goes to zero with $E\left\langle\left(\frac{\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*}{N}\right)^2\right\rangle_4$.

Thus, by Conjecture 6.4, we have

$$E\frac{L_5}{T^4} = \frac{2\beta^2}{N^2}E\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\rangle_4 + o(E\overline{\tau}_N(\beta)) \quad .$$

Finally, it remains only to use that $N^{-2}\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^2\rangle_4 = 4\overline{\tau}_N(\beta)$. □

Proposition 6.7. *If*

(6.8)
$$\frac{1}{\beta^2} > E\left(\frac{1}{\mathrm{ch}^4(\beta x(\beta,h)z+\beta h)}\right)$$

and if Conjectures 6.1, 6.4, 6.7 *are true*, "*the* SK *solution is stable*" *that is*, *under the conditions of Theorem* 1.12, *for* N *large enough*, $E\big(\sigma_N(\beta)-x^2(\beta,h)\big)^2$ and $C_N(\beta)$ *small enough*, *we have*

(6.9)
$$C_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) < A\left(\frac{1}{N}+C_N(\beta)\right)$$

where $A<1$.

Moreover, *if* (6.8) *fails*, *under the same other conditions as above*, "*the* SK *solution is unstable*", *that is*

(6.10)
$$C_{N+2}\left(\beta\sqrt{1+\frac{2}{N}}\right) > AC_N(\beta)$$

where $A>1$.

Proof. We combine Proposition 6.6 with the fact that

$$2\left(1-\tfrac{1}{N}\right)C_N(\beta) \le E\overline{\tau}_N(\beta) \le \tfrac{1}{N} + 2C_N(\beta) \qquad \square$$

A true notion of stability would contain the fact that the quantity $E\big(\sigma_N(\beta) - x^2(\beta,h)\big)^2$ satisfies a relation of the type (6.9). However, due to space limitations, I will not pursue this idea here. In a different (but more subtle) setting, the rigorous idea of stability will be developed in the forthcoming paper [T2].

The most natural way to attack Conjectures 6.1, 6.4, 6.5 is to try to carry information by induction upon quantities such as $\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^4\rangle_4, \langle(\boldsymbol{a}\cdot\boldsymbol{\theta})^2\rangle_2$, etc.This requires computations of the same nature as those presented here, but on a vaster scale.

The real challenge however is that computation of $\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^4\rangle_4$ by induction requires control $\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^6\rangle_4$, etc., so the only chance of success lies incontrolling *all* the moments $\langle(\boldsymbol{\theta}\cdot\boldsymbol{\theta}^*)^k\rangle_4$ together, a task better left as a topic for further research.

7. Proof of Theorems 1.3 and 1.7

The present paper intends to be both an introduction to the topic and to present the latest results. These are somewhat contradictory aims. Even though the material of the present section pertains to Section 2, it is of a somewhat higher level of sophistication and is better presented separately. While more elaborate, the proofs of the present section contain a number of ideas that should provide ample reward to the reader.

The proof of Theorem 1.3 relies upon moment estimates. The basic formula, which is closely related to the replica trick, is that for $n \ge 1$

$$Z_N^n(\beta) = \sum_\epsilon \exp\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}\left(\sum_{\ell\le n}\epsilon_i^\ell\epsilon_j^\ell\right) \tag{7.1}$$

where the summation is over all n-tuples $\boldsymbol{\epsilon}^1,\ldots,\boldsymbol{\epsilon}^n$ of Σ_N. To simplify notation, we will write $\sum_\epsilon$ for summations over (sequences of) configurations $\boldsymbol{\epsilon}^1,\ldots,\boldsymbol{\epsilon}^n$ or $\boldsymbol{\epsilon}^1,\ldots,\boldsymbol{\epsilon}^r$; it should be clear from the context what is meant. We will also write

$$e_n(\boldsymbol{\epsilon}) = \exp\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}\left(\sum_{\ell\le n}\epsilon_i^\ell\epsilon_j^\ell\right) \tag{7.2}$$

(keeping the value of β, N implicit) so that (7.1) becomes

$$Z_N^n = \sum_\epsilon e_n(\boldsymbol{\epsilon}) \ .$$

Now from (7.1) we have

$$EZ_N^n = \sum_{\epsilon} \exp \frac{\beta^2}{2N} \sum_{i<j} \left(\sum_{\ell \le n} \epsilon_i^\ell \epsilon_j^\ell \right)^2 \tag{7.3}$$

$$= \exp \frac{\beta^2}{4} n(N-n) \sum_{\epsilon} \exp \frac{\beta^2}{2N} \sum_{\ell < \ell'} (\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'})^2$$

using simple algebra as in Lemma 2.1.

If we consider the term of the summation corresponding to the case where $\boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^n$ have all their components equal to one, we get

$$E\left(\left(2^{-N} Z_N \exp\left(-\frac{N\beta^2}{4} \right) \right)^n \right) \ge 2^{-nN} \exp \frac{\beta^2}{4} n^2 (N-1) \tag{7.4}$$

and this proves (1.11). Thus we cannot directly use the moments of Z_N. What saves the situation is that the moment explosion of (7.4) is created by a few configurations, and that these have a small influence in probability. The situation is described rather precisely by Proposition 7.1 below. Throughout the section, we set

$$B(x) = \left\{ \boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^n \in \Sigma_N^n ; \sum_{\ell < \ell'} (\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'})^2 \le x N^2 \right\} \tag{7.5}$$

and we denote by $B^c(x)$ the complement of $B(x)$.

Proposition 7.1. *There exists a constant* $c(\beta) > 0$ *and a constant* $K(\beta) < \infty$ *such that if* $n^2 \le N/K(\beta)$ and $B = B(c(\beta))$ *we have*

$$E\Big(\sum_{\epsilon} 1_B(\boldsymbol{\epsilon}) e_n(\boldsymbol{\epsilon}) \Big) \le (K(\beta))^{n^2} (EZ_N)^n \tag{7.6}$$

$$P\left(\sum_{\epsilon} 1_{B^c}(\boldsymbol{\epsilon}) e_n(\boldsymbol{\epsilon}) \ge \exp\left(-\frac{N}{K(\beta)} \right) \max(Z_N^n, (EZ_N)^n) \right) \tag{7.7}$$

$$\le \exp\left(-\frac{N}{K(\beta)} \right) .$$

Let us first show why this implies Theorem 1.3. Consider $n \ge 1$, and write

$$U = \sum_{\epsilon} 1_B(\boldsymbol{\epsilon}) e_n(\boldsymbol{\epsilon}) ; U^c = \sum_{\epsilon} 1_{B^c}(\boldsymbol{\epsilon}) e_n(\boldsymbol{\epsilon}) \;.$$

For $u > 0$, we have

$$P(Z_N \ge \mathrm{e}^u EZ_N) = P(Z_N^n \ge \mathrm{e}^{nu} (EZ_N)^n) \tag{7.8}$$

$$\le P\left(U \ge \frac{\mathrm{e}^{nu}}{2} (EZ_N)^n \right) + P\big(U^c \ge \tfrac{1}{2} \max(Z_N^n, (EZ_N)^n) \big) \;.$$

Using (7.6) and Chebyshev inequality to control the first term, and (7.7) to control the second, we see that provided $n^2 \le N/K(\beta)$ we have

$$P(Z_N \geq e^u EZ_N) \leq \frac{2(K(\beta))^{n^2}}{e^{nu}} + \exp\left(-\frac{N}{K(\beta)}\right)$$

and the result follows by taking $n \simeq u/K'(\beta)$ for $K'(\beta)$ large enough.

We now start the proof of (7.6). We consider a fresh sequence of independent standard normal r.v. $(g_{\ell\ell'})_{1\leq\ell<\ell'\leq n}$. Following the physicist's idea, we observe, using (2.4), that

$$\exp\frac{\beta^2}{2N}\sum_{\ell<\ell'}(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'})^2 = E_g \exp\frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'} g_{\ell\ell'}\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \quad . \tag{7.9}$$

There, and in the sequel, E_g denotes expectation in the variables $g_{\ell\ell'}$ only.

Even though the connection between (7.6) and the following lemma will become clear only gradually, this seems to be the key step.

Lemma 7.2. *There exists constants $K_1(\beta) > 0$, $\eta(\beta) > 1$, such that if we define*

$$D = \left\{\sum_{\ell<\ell'} g_{\ell\ell'}^2 \leq \frac{N}{K_1(\beta)}\right\} \tag{7.10}$$

the function $\varphi(\boldsymbol{\epsilon})$ given by

$$\varphi(\boldsymbol{\epsilon}) = E_g 1_D(\boldsymbol{g}) \exp\frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'} g_{\ell\ell'}\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \tag{7.11}$$

satisfies

$$E_\epsilon \varphi^{\eta(\beta)} \leq K_1(\beta)^{n^2} \quad . \tag{7.12}$$

There, of course, $E_\epsilon = 2^{-nN}\sum_\epsilon$, and $\boldsymbol{g} = (g_{\ell\ell'})_{\ell<\ell'}$.

Proof. First, we observe that, since $(E_g Y)^\eta \leq E_g Y^\eta$, choosing $\eta(\beta)$ with $\beta\eta(\beta) = (\beta+1)/2$, it suffices to prove that $E_\epsilon\varphi \leq K_1(\beta)^{n^2}$.

Since $\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} = \sum_{i\leq N} \epsilon_i^\ell \epsilon_i^{\ell'}$, we have by independence,

$$E_\epsilon\varphi = E_g 1_D(\boldsymbol{g})\left(E_\epsilon \exp\frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'} g_{\ell\ell'}\epsilon^\ell\epsilon^{\ell'}\right)^N \quad . \tag{7.13}$$

In the last expectation, $(\epsilon^\ell)_{\ell\leq n}$ are independent Bernoulli random variables and E_ϵ denotes expectation in these variables. To compute this expectation, let

$$X = \frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'} g_{\ell\ell'}\epsilon^\ell\epsilon^{\ell'}$$

so that $E_\epsilon X = 0$,

$$E_\epsilon X^2 = \frac{\beta^2}{N}\sum_{\ell<\ell'} g_{\ell\ell'}^2 \tag{7.14}$$

and

$$E_\epsilon \exp X = 1 + \frac{E_\epsilon X^2}{2} + \sum_{p \geq 3} E_\epsilon \frac{X^p}{p!} \tag{7.15}$$

Conditionally in the variables $g_{\ell\ell'}, X$ is an "order two Bernoulli chaos", and, by a very useful result of C. Borell [B] for $p \geq 2$ it satisfies

$$(E_\epsilon |X|^p)^{1/p} \leq (p-1)\big(E_\epsilon X^2\big)^{p/2} \tag{7.16}$$

so that by (7.15) we have

$$E_\epsilon \exp X \leq 1 + \frac{E_\epsilon X^2}{2}\left(1 + 2\sum_{p \geq 3} \frac{p^p}{p!}\big(E_\epsilon X^2\big)^{\frac{p-2}{2}}\right) \tag{7.17}$$

Thus, by (7.14), if $K_1(\beta)$ is large enough, we have

$$1_D(\boldsymbol{g}) E_\epsilon \exp X \leq 1 + \frac{\beta'^2}{2N}\sum_{\ell<\ell'} g_{\ell\ell'}^2 \leq \exp \frac{\beta'^2}{2N}\sum_{\ell<\ell'} g_{\ell\ell'}^2$$

where $\beta' = (\beta+1)/2$. Thus, by (7.13), we have

$$E_\epsilon \varphi \leq E_g \exp \frac{\beta'^2}{2}\sum_{\ell<\ell'} g_{\ell\ell'}^2 = \left(\frac{1}{1-\beta'^2}\right)^{n(n-1)/4}$$

□

We cannot apply (7.9) to (7.11) because the integration there is restricted to D. The following lemma shows that such a restriction need not matter.

Lemma 7.3. *Consider i.i.d. standard normal r.v.* $(g_\ell)_{\ell \leq m}$ *and numbers* $(\alpha_\ell)_{\ell \leq m}$. *Then, if*

$$y \geq 16\left(m + \sum_{\ell \leq m} \alpha_\ell^2\right) \tag{7.18}$$

we have

$$E\left(1_{\{\sum g_\ell^2 \geq y\}} \exp \sum_{\ell \leq m} \alpha_\ell g_\ell\right) \leq \exp\left(-\frac{y}{16}\right) . \tag{7.19}$$

Proof. We first use Cauchy-Schwarz to see that the left-hand side of (7.19) is at most

$$P\left(\sum_{\ell \leq m} g_\ell^2 \geq y\right)^{1/2} \exp\left(\sum_{\ell \leq m} \alpha_\ell^2\right) . \tag{7.20}$$

Next, since $E \exp g_\ell^2/4 \leq 2$, we have $E \exp \frac{1}{4}\sum_{\ell \leq m} g_\ell^2 \leq 2^m \leq e^m$, so that

$$P\left(\sum_{\ell \leq m} g_\ell^2 \geq y\right) \leq \exp\left(m - \frac{y}{4}\right)$$

and (7.20) is bounded by

$$\exp\left(\sum_{\ell\le m}\alpha_\ell^2+\frac{m}{2}-\frac{y}{8}\right) \qquad \square$$

Lemma 7.4. *There exists constants $\eta'(\beta), c(\beta) > 0, K_2(\beta) < \infty$ such that for $B = B(c(\beta))$, and for each subset A of Σ_N^n, if $n^2 \le N/K_2(\beta)$ we have*

$$E_\epsilon\left(1_A 1_B \exp\frac{\beta^2}{2N}\sum_{\ell<\ell'}\left(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\right)^2\right) \le P_\epsilon(A)^{\eta'(\beta)}(K_2(\beta))^{n^2}+\exp\left(-\frac{N}{K_2(\beta)}\right) . \tag{7.21}$$

Proof. We define $c(\beta) = (32K_1(\beta))^{-1}$, where $K_1(\beta)$ is the constant of Lemma 7.2. Using (7.9), (7.19), we see that if $n^2 \le 1/K(\beta)$ we have

$$1_B \exp\frac{\beta^2}{2N}\sum_{\ell<\ell'}\left(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\right)^2 \le \varphi(\boldsymbol{\epsilon}) + \exp\left(-\frac{N}{K(\beta)}\right)$$

where φ is defined in (7.11). The conclusion then follows from (7.12) and Hölder's inequality, if we define $\eta'(\beta)+\eta^{-1}(\beta) = 1$. $\square$

We have actually proved a bit more than (7.6). The following is a consequence of the computation of (7.3) and (7.21).

Corollary 7.5. *With the previous notation, we have*

$$E\sum_\epsilon 1_A 1_B e_n(\boldsymbol{\epsilon}) \le (EZ_N)^n\left[P_\epsilon(A)^{\eta'(\beta)}(K(\beta))^{n^2}+\exp\left(-\frac{N}{K(\beta)}\right)\right] \tag{7.22}$$

provided $n^2 \le N/K(\beta)$.

We now turn to the proof of (7.7). We will consider various values of n, so we write $B(n,x)$ rather than $B(x)$. We set

$$W(n,x) = \sum_\epsilon 1_{B(n,x)}(\boldsymbol{\epsilon})e_n(\boldsymbol{\epsilon}) . \tag{7.23}$$

Given $a > 0$, we will study $P(W(n,x) \ge a\overline{Z}_N^n)$ where we set $\overline{Z}_N = \max(Z_N, EZ_N)$. We are interested only in the case $x = c(\beta), a = 1/2$. Considering other values is required by the proof. The proof is based on iterative reductions of the value of n. We consider an integer $r \le n, r \ge 2$.

Lemma 7.6. *If $\sum_{1\le\ell<\ell'\le n}(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'})^2 > xN^2$, we can find a subset I of $\{1,\dots,n\}$ of cardinal r, with*

$$\sum_{\ell<\ell',\ell,\ell'\in I}\left(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\right)^2 > \frac{xr^2}{2n^2}N . \tag{7.24}$$

Proof. We simply average the left-hand side of (7.24) over all values of I and observe that

$$\frac{\binom{n}{r-2}}{\binom{n}{r}} \geq \frac{1}{2}\frac{r^2}{n^2}$$ □

Lemma 7.7. *We have*

$$P\big(W(n,x) \geq a\overline{Z}_N^n\big) \leq \left(\frac{en}{r}\right)^r P\left(W\left(r, \frac{xr^2}{2n^2}\right) \geq a\left(\frac{r}{en}\right)^r \overline{Z}_N^r\right) . \tag{7.25}$$

Proof. For a subset I of $\{1, \dots, n\}$ of cardinality r, we write

$$B_I = \left\{\boldsymbol{\epsilon}^1, \dots, \boldsymbol{\epsilon}^n; \sum_{\ell<\ell', \ell,\ell' \in I} \left(\boldsymbol{\epsilon}^{\ell} \cdot \boldsymbol{\epsilon}^{\ell'}\right)^2 > \frac{xr^2}{2n^2} N^2\right\} .$$

Thus Lemma 7.6 shows that $B^c(n,x) \subset \bigcup_I B_I$, where the union is over all subsets I of $\{1, \dots, n\}$ of cardinal r. Now, if we define
$W_I = E_\epsilon 1_{B_I}(\boldsymbol{\epsilon}) e_n(\boldsymbol{\epsilon})$, we have

$$W(n,x) \leq \sum_I W_I$$

so that

$$P\big(W(n,x) \geq a\overline{Z}_N^n\big) \leq \sum_I P\left(W_I \geq a\binom{n}{r}^{-1} \overline{Z}_N^n\right) . \tag{7.26}$$

The condition that $(\boldsymbol{\epsilon}^1, \dots, \boldsymbol{\epsilon}^n)$ belongs to B_I imposes no condition on $(\boldsymbol{\epsilon}^\ell)_{\ell \notin I}$. Observing that $\overline{Z}_N^n \geq \overline{Z}_N^r Z_N^{n-r}$, it should be clear that each term of the right hand side of (7.26) is at most

$$P\left(W\left(r, \frac{xr^2}{2n^2}\right) \geq a\binom{n}{r}^{-1} \overline{Z}_N^r\right) .$$

The result then follows from the inequality $\binom{n}{r} \leq (en/r)^r$. □

Lemma 7.8. *If* $n^2 \leq N/K(\beta)$, *and* $r \geq n/3$ *we have*

$$\begin{aligned} &P\big(W(n, c(\beta)) \geq a\overline{Z}_N^n\big) \\ &\quad \leq (3e)^r P\left(W(r, c(\beta)) \geq \frac{a}{2}\left(\frac{1}{3e}\right)^r \overline{Z}_N^r\right) + \frac{2}{a}(3e)^r \exp\left(-\frac{N}{K(\beta)}\right) . \end{aligned} \tag{7.27}$$

Proof. We note that $(en/r)^r \leq (3e)^r, r^2/2n^2 \geq 1/18$. We use (7.25) with $x = c(\beta)$, so that

$$P\big(W(n, c(\beta)) \geq a\overline{Z}_N^r\big) \leq (3e)^r P\left(W\left(r, \frac{c(\beta)}{18}\right) \geq a\left(\frac{1}{3e}\right)^r \overline{Z}_N^r\right) . \tag{7.28}$$

Next we observe that

$$W\left(r,\frac{c(\beta)}{18}\right)\leq W(r,c(\beta))+\sum_{\epsilon}1_A(\boldsymbol{\epsilon})e_r(\boldsymbol{\epsilon}) \tag{7.29}$$

for

$$A=\left\{\boldsymbol{\epsilon}^1,\ldots,\boldsymbol{\epsilon}^r;\frac{c(\beta)}{18}N^2<\sum_{\ell<\ell'}\left(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\right)^2\leq c(\beta)N^2\right\} .$$

Now, since $A\subset B(r,c(\beta))$, using (7.21) for r rather than n we have

$$E_\epsilon\left(1_A(\boldsymbol{\epsilon})\exp\frac{\beta^2}{2N}\sum_{\ell<\ell'}\left(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\right)^2\right)\leq K(\beta)^{r^2} . \tag{7.30}$$

Since $\sum_{\ell<\ell'}(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'})^2\geq c(\beta)N^2/18$ on A, it follows that for $n^2\leq N/K(\beta)$ we have $P_\epsilon(A)\leq\exp(-N/K(\beta))$. Thus (7.22) implies

$$E\sum_{\epsilon}1_A(\boldsymbol{\epsilon})e_r(\boldsymbol{\epsilon})\leq(EZ_N)^r\exp\left(-\frac{N}{K(\beta)}\right) \tag{7.31}$$

and

$$P\left(\sum_{\epsilon}1_A(\boldsymbol{\epsilon})e_r(\boldsymbol{\epsilon})\geq b(EZ_N)^r\right)\leq\frac{1}{b}\exp\left(-\frac{N}{K(\beta)}\right) .$$

The conclusion follows from the inequality

$$P(X+Y\geq b)\leq P\big(X\geq\tfrac{b}{2}\big)+P\big(Y\geq\tfrac{b}{2}\big) . \qquad\square$$

We now finish the proof of (7.7). Denoting by $p(n,a)$ the left-hand side of (7.27), we rewrite this inequality as

$$p(n,a)\leq(3e)^r p(r,a_1)+\frac{2}{a}(3e)^r\exp\left(-\frac{N}{K(\beta)}\right) , \tag{7.32}$$

where $a_1=(a/2)(3e)^{-r}$. We consider now a sequence $n_0=2\leq n_1\leq\cdots\leq n_p=n$, where $2\leq n_{i+1}/n_i\leq 3$. It is a simple matter using induction over p to see from (7.32) that

$$p(n,a)\leq K^n p(2,K^{-n}a)+\frac{K^n}{a}\exp\left(-\frac{N}{K(\beta)}\right) . \tag{7.33}$$

The conditions $n_{i+1}/n_i\geq 2$ are used to ensure $\sum_{i<p}n_i\leq n$. Now, using (2.2) and Chebyshev inequality we have

$$p(2,K^{-n}a)\leq\frac{K^n}{a}\exp\left(-\frac{N}{K(\beta)}\right)$$

so that from (7.33)

$$p(n,a)\leq\frac{K^n}{a}\exp\left(-\frac{N}{K(\beta)}\right)$$

and this finishes the proof. $\square$

We now turn to the proof of Theorem 1.7. We are interested in the quantity

$$Q = \sup_{\|\boldsymbol{x}\| \le 1} \langle (\boldsymbol{x} \cdot \boldsymbol{\epsilon})^2 \rangle \ . \tag{7.34}$$

Let us recall that the identity

$$\langle (\boldsymbol{x} \cdot \boldsymbol{\epsilon})^2 \rangle = \sum_{i,j \le N} x_i x_j \langle \epsilon_i \epsilon_j \rangle$$

shows that Q is the norm of the symmetric matrix $M = (\langle \epsilon_i \epsilon_j \rangle)$. Denoting by M^k_{ij} the entries of the k-th power M^k of M, we see that, since M is symmetric

$$Q^k = \|M^k\| \le \|M^k\|_{\mathrm{HS}} = \left(\sum_{i,j} \left(M^k_{ij} \right)^2 \right)^{1/2}$$

and thus

$$Q^{2k} \le \sum_{i,j \le N} \left(M^k_{ij} \right)^2 \ . \tag{7.35}$$

These quantities are easily computed using replicas, as the following lemma (with a proof immediate by induction over k) shows.

Lemma 7.9. $M^k_{ij} = \langle \epsilon^1_i (\boldsymbol{\epsilon}^1 \cdot \boldsymbol{\epsilon}^2) \cdots (\boldsymbol{\epsilon}^{k-1} \cdot \boldsymbol{\epsilon}^k) \epsilon^k_j \rangle_k$

In this notation, $\boldsymbol{\epsilon}^1, \dots, \boldsymbol{\epsilon}^k$ are of course thermally independent copies of $\boldsymbol{\epsilon}^1$.

Moreover, the right-hand side itself of (7.35) can be expressed simply using $2k$ replicas $\boldsymbol{\epsilon}^1, \dots, \boldsymbol{\epsilon}^{2k}$. Throughout the proof, we set

$$S(\boldsymbol{\epsilon}) = \left(\boldsymbol{\epsilon}^1 \cdot \boldsymbol{\epsilon}^2 \right) \left(\boldsymbol{\epsilon}^2 \cdot \boldsymbol{\epsilon}^3 \right) \cdots \left(\boldsymbol{\epsilon}^{2k} \cdot \boldsymbol{\epsilon}^1 \right) \tag{7.36}$$

and we note right away that $|S(\boldsymbol{\epsilon})| \le N^{2k}$.

Lemma 7.10. *We have*

$$Q^{2k} \le \langle S(\boldsymbol{\epsilon}) \rangle_{2k} \ . \tag{7.37}$$

Proof. We observe that, by symmetry

$$\left\langle \epsilon^1_i \left(\boldsymbol{\epsilon}^1 \cdot \boldsymbol{\epsilon}^2 \right) \cdots \left(\boldsymbol{\epsilon}^{k-1} \cdot \boldsymbol{\epsilon}^k \right) \epsilon^k_j \right\rangle_k = \left\langle \epsilon^{2k}_i \left(\boldsymbol{\epsilon}^{2k} \cdot \boldsymbol{\epsilon}^{2k-1} \right) \cdots \left(\boldsymbol{\epsilon}^{k+2} \cdot \boldsymbol{\epsilon}^{k+1} \right) \epsilon^{k+1}_j \right\rangle_k$$

so that by Lemma 7.9

$$\begin{aligned} M^2_{ij} &= \left\langle \epsilon^1_i \left(\boldsymbol{\epsilon}^1 \cdot \boldsymbol{\epsilon}^2 \right) \cdots \left(\boldsymbol{\epsilon}^{k-1} \cdot \boldsymbol{\epsilon}^k \right) \epsilon^k_j \right\rangle_k \left\langle \epsilon^{2k}_i \left(\boldsymbol{\epsilon}^{2k} \cdot \boldsymbol{\epsilon}^{2k-1} \right) \cdots \left(\boldsymbol{\epsilon}^{k+2} \cdots \boldsymbol{\epsilon}^{k+1} \right) \epsilon^{k+1}_j \right\rangle_k \\ &= \left\langle \epsilon^1_i \epsilon^{2k}_i \left(\boldsymbol{\epsilon}^1 \cdot \boldsymbol{\epsilon}^2 \right) \cdots \left(\boldsymbol{\epsilon}^{k-1} \cdot \boldsymbol{\epsilon}^k \right) \left(\epsilon^k_j \epsilon^{k+1}_j \right) \left(\boldsymbol{\epsilon}^{k+1} \cdot \boldsymbol{\epsilon}^{k+2} \right) \cdots \left(\boldsymbol{\epsilon}^{2k-1} \cdot \boldsymbol{\epsilon}^{2k} \right) \right\rangle_{2k} \end{aligned}$$

and the result follows by summation. □

What (7.37) means is that

$$Q^{2k} \le \frac{U}{Z_N^{2k}} \tag{7.38}$$

where $U = E_\epsilon S(\boldsymbol{\epsilon}) e_{2k}(\boldsymbol{\epsilon})$. Since, by Theorem 1.2, we control Z_N from below, it is natural to try to control U from above. This is where the trouble starts; we cannot even control EU, for the same reason that we cannot control EZ_N^{2k}. We have seen in the proof of Theorem 1.3 that the trouble comes from a few configurations, that, in probability, do not matter. Recalling Definition 7.5, it is natural to consider

$$U(x) = \sum_\epsilon S(\boldsymbol{\epsilon}) 1_{B(x)}(\boldsymbol{\epsilon}) e_{2k}(\boldsymbol{\epsilon}) \ . \tag{7.39}$$

This gives us some hope of controlling $E(U(x))$. Much of the trouble arises from the fact that $S(\boldsymbol{\epsilon})$ is not always greater or equal than zero, and, while we know by construction that $U \geq 0$, the same does not seem necessarily true for $U(x)$. The first step is to prove the following:

Proposition 7.11. *There is a constant $K(\beta)$ such that if $x \leq 1/K(\beta)$ and $k^2 \leq N/K(\beta)$ we have*

$$E(Q^k) \leq K(\beta)^{k^2} + K(\beta)^{k^2} N^{2k} \exp\left(-\frac{Nx}{K(\beta)}\right) + \frac{E(U(x))}{(EZ_N)^{2k}} \ . \tag{7.40}$$

This reduces the task of bounding $E(Q^k)$ to the task of bounding $E(U(x))$, so we have succeeded in getting rid of the denominator. On the other hand, the term $K(\beta)^{k^2}$ means that we cannot take k of order larger than $\sqrt{\log N}$, while to solve Problem 1.6 we would need to take k of order $\log N$, so a rather different approach has to be invented.

Proof. **Step 1.** We set $H_1 = \{Z_N \leq \mathrm{e}^{\sqrt{N}} EZ_N\}$ so that from (1.12) we have that for $u \geq 0$

$$P(1_{H_1} Z_N \geq \mathrm{e}^u EZ_N) \leq K(\beta) \exp\left(-\frac{u^2}{K(\beta)}\right) \tag{7.41}$$

and also

$$P\left(H_1^c\right) \leq K(\beta) \exp\left(-\frac{N}{K(\beta)}\right) \ . \tag{7.42}$$

From (7.41) and a simple calculation, we have

$$E\left(1_{H_1} Z_N^{2k}\right) \leq K(\beta)^{k^2} (EZ_N)^{2k} \ . \tag{7.43}$$

Step 2. From (7.7) we see that if $k^2 \leq N/K(\beta)$ there is a set H_2 of quenched variables with $P(H_2^c) \leq \exp(-N/K(\beta))$ and

$$1_{H_2} \sum_\epsilon 1_{B^c}(\boldsymbol{\epsilon}) e_{2k}(\boldsymbol{\epsilon}) \leq \exp\left(-\frac{N}{K(\beta)}\right) \max\left(Z_N^{2k}, (EZ_N)^{2k}\right) \tag{7.44}$$

where $B = B(c(\beta))$. Thus, setting $H = H_2 \cap H_1$, we have

$$1_H U \leq 1_H U(c(\beta)) + N^{2k} \exp\left(-\frac{N}{K(\beta)}\right) \max\left(1_H Z_N^{2k}, (EZ_N)^{2k}\right) \ . \tag{7.45}$$

Step 3. Since $U \leq N^{2k} Z_N^{2k}$, we have from (7.38) and (7.45) that

$$E(Q^k) \leq E\left(1_H \frac{\sqrt{U}}{Z_N^k}\right) + N^k P(H^c)$$
$$\leq E\left(1_H Z_N^{-k}\left[U(c(\beta)) + N^{2k}\exp\left(-\frac{N}{K(\beta)}\right)\max\left(Z_N^{2k}, (EZ_N)^{2k}\right)\right]^{1/2}\right)$$
$$+ K(\beta)N^k \exp(-N/K(\beta)) \ .$$

Using that $ab \leq a^2 + b^2$, we have

$$(7.46)\ E(Q^k) \leq K(\beta)N^k \exp\left(-\frac{N}{K(\beta)}\right) + E\left(\left(\frac{EZ_N}{Z_N}\right)^{2k}\right)$$
$$+ (EZ_N)^{-2k} E\left(1_H\left[U(c(\beta)) + N^{2k}\exp\left(-\frac{N}{K(\beta)}\right)\max\left(Z_N^{2k}, (EZ_N)^{2k}\right)\right]\right) \ .$$

Using (1.8), by the same calculation as in (7.43) we have $E(Z_N^{-2k}) \leq K(\beta)^{k^2}(EZ_N)^{-2k}$. Using (7.43) to control the last term of (7.46), we then get

$$(7.47)\ \ E\left(Q^k\right) \leq K(\beta)^{k^2} N^k \exp\left(-\frac{N}{K(\beta)}\right) + K(\beta)^{k^2} + (EZ_N)^{-2k} E 1_H U(c(\beta)) \ .$$

Step 4. We would be done if we had a last term $E(U(c(\beta)))$ rather than $E(1_H U(c(\beta)))$. The restriction to H is a nuisance, because it prevents us from computing the expectation. The difficulty to remove it is that $U(c(\beta))$ need not be ≥ 0.

It will be easier to prove that we can replace $E(1_H U(x))$ by $E(U(x))$ when x is somewhat smaller than $c(\beta)$. Thus we first show that

$$(7.48) \qquad E(Q^k) \leq N^{2k} K(\beta)^{k^2} \exp\left(-\frac{xN}{K(\beta)}\right) + K(\beta)^{k^2}$$
$$+ (EZ_N)^{-2k} E(1_H U(x)) \ .$$

It suffices to observe that

$$|U(c(\beta)) - U(x)| \leq N^{2k} E_\epsilon 1_{B(c(\beta))\backslash B(x)} e_{2k}(\boldsymbol{\epsilon})$$

and that

$$E E_\epsilon 1_{B(c(\beta))\backslash B(x)} e_{2k}(\boldsymbol{\epsilon}) \leq K(\beta)^{k^2} \exp\left(-\frac{xN}{K(\beta)}\right) \ .$$

The argument to prove this inequality (based on Corollary 7.5) has been given in the proof of Lemma 7.8 (see (7.40)).

Step 5. We write

$$E(1_H U(x)) \leq E(U(x)) + 2E(1_{H^c}|U(x)|)$$

and we show that the last term is small using Cauchy-Schwarz:

$$E(1_{H^c}|U(x)|) \le P(H^c)^{1/2} E\Big(U(x)^2\Big)^{1/2} \le \exp\Big(-\frac{N}{K(\beta)}\Big) E\Big(U(x)^2\Big)^{1/2} . \tag{7.49}$$

Unfortunately, the control of $E(U(x)^2)$ is a significant task. Introducing new replicas $\boldsymbol{\epsilon}^{2k+1}, \ldots, \boldsymbol{\epsilon}^{4k}$, and computing the expectation in the quenched variables as usual, we have

$$E\Big(U(x)^2\Big) \le (EZ_N)^{4k} E_\epsilon 1_{B'(x)} \exp \frac{\beta^2}{2N} \sum_{\ell<\ell'} (\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'})^2$$

where

$$B'(x) = \left\{ \boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^{4k}; \sum_{1 \le \ell < \ell' \le 2k} \Big(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\Big)^2 \le xN^2, \sum_{2k < \ell < \ell' \le 4k} \Big(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\Big)^2 \le xN^2 \right\} .$$

Thus we have

$$(EZ_N)^{-4k} E\Big(U(x)^2\Big) \le \exp \beta^2 xN E_\epsilon 1_{B''(x)} \exp \frac{\beta^2}{2N} \sum_{1 \le \ell \le 2k < \ell' \le 4k} \Big(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\Big)^2$$

where

$$B''(x) = \left\{ \boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^{4k}; \sum_{1 \le \ell < \ell' \le 2k} \Big(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\Big)^2 \le xN^2 \right\} .$$

There is now independence in the replicas $\boldsymbol{\epsilon}^{\ell'}, \ell' > 2k$. Thus, introducing a new replica $\boldsymbol{\eta}$, we have

$$E_\epsilon 1_{B''(x)} \exp \frac{\beta^2}{2N} \sum_{1 \le \ell \le 2k < \ell' \le 4k} \Big(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\Big)^2 = E_\epsilon 1_{B(x)} V^{2k}(\boldsymbol{\epsilon}) \tag{7.50}$$

for

$$V(\boldsymbol{\epsilon}) = E_\eta \exp \frac{\beta^2}{2N} \sum_{1 \le \ell \le 2k} (\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\eta})^2 .$$

(We hope that the fact that E_ϵ on the right of (7.50) represents an average over $\boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^{4k}$ and on the left an average over $\boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^{2k}$ is not confusing.) To bound $V(\boldsymbol{\epsilon})$, we introduce auxiliary r.v. $(g_\ell)_{\ell \le 2k}$ and we write

$$V(\boldsymbol{\epsilon}) = E_g E_\eta \exp \frac{\beta}{\sqrt{N}} \sum_{1 \le \ell \le 2k} g_\ell \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\eta}$$

Since $\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\eta} = \sum_{i \le N} \epsilon_i^\ell \eta_i$ we can compute E_η to get

$$V(\boldsymbol{\epsilon}) = E_g \left(\prod_{i \le N} \mathrm{ch}\left(\frac{\beta}{\sqrt{N}} \sum_{1 \le \ell \le 2k} \epsilon_i^\ell g_\ell \right) \right) \le E_g \exp \frac{\beta^2}{2N} \sum_{i \le N} \left(\sum_{1 \le \ell \le 2k} \epsilon_i^\ell g_\ell \right)^2 .$$

Now

$$\left(\sum_{1\le\ell\le 2k}\epsilon_i^\ell g_\ell\right)^2=\sum_{1\le\ell\le 2k}g_\ell^2+2\sum_{1\le\ell<\ell'\le 2k}g_\ell g_{\ell'}\epsilon_i^\ell\epsilon_i^{\ell'}$$

so that

$$\begin{aligned}\sum_{i\le N}\left(\sum_{1\le\ell\le 2k}\epsilon_i^\ell g_\ell\right)^2&=N\sum_{1\le\ell\le 2k}g_\ell^2+2\sum_{1\le\ell\le\ell'\le 2k}g_\ell g_{\ell'}\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\\&\le N(1+2\sqrt{x})\sum_{1\le\ell\le 2k}g_\ell^2\ ,\end{aligned}$$

using Cauchy-Schwarz and the fact that $\boldsymbol{\epsilon}\in B(x)$.

Thus, if we fix x small enough that $x\le 1/K(\beta)$ and $\beta^2(1+2\sqrt{x})\le(1+\beta)^2/4$, we get that $V(\boldsymbol{\epsilon})\le K(\beta)^{2k}$ and thus

$$E\left(U(x)^2\right)\le(EZ_N)^{4k}K(\beta)^{k^2}\exp x\beta^2N$$

which, when combined with (7.49) finishes the proof of Proposition 7.11.

□

The progress we have made is that we can integrate in the quenched variables. We have

$$(7.51)\qquad (EZ_N)^{-2k}E(U(x))=\exp\left(-k\beta^2\right)E_\epsilon 1_{B(x)}S(\boldsymbol{\epsilon})\exp\frac{\beta^2}{2N}\sum_{\ell<\ell'}\left(\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\right)^2\ .$$

As in the proof of Theorem 1.3, we will linearize the quadratic form in order to be able to perform the integration in $\boldsymbol{\epsilon}$. We introduce i.i.d. $N(0,1)$ variables $g_{\ell\ell'}, 1\le\ell<\ell'\le 2k$. We set

$$D(x)=\left\{\sum_{\ell<\ell'}g_{\ell\ell'}^2\le 32xN\right\}\ .$$

Lemma 7.12. *If* $Nx\ge k^2$, *we have*

$$\begin{aligned}(7.52)\qquad (EZ_N)^{-2k}E(U(x))&\le N^{2k}\exp(-xN/K(\beta))\\&+\exp\left(-k\beta^2\right)E_g 1_{D(x)}(\boldsymbol{g})E_\epsilon S(\boldsymbol{\epsilon})\exp\frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'}g_{\ell\ell'}\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\ .\end{aligned}$$

Proof. From (7.51) we have

$$(EZ_N)^{-2k}E(U(x))=\exp\left(-k\beta^2\right)E_\epsilon E_g 1_{B(x)}(\boldsymbol{\epsilon})S(\boldsymbol{\epsilon})\exp\frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'}g_{\ell\ell'}\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}$$

and it follows from Lemma 7.3 that if $k^2\le Nx$ we have

$$\begin{aligned}(EZ_N)^{-2k}E(U(x))&\le N^{2k}\exp(-Nx/K(\beta))\\&+E_\epsilon E_g 1_{D(x)}(\boldsymbol{g})1_{B(x)}(\boldsymbol{\epsilon})S(\boldsymbol{\epsilon})\exp\frac{\beta}{\sqrt{N}}\sum_{\ell<\ell'}g_{\ell\ell'}\boldsymbol{\epsilon}^\ell\cdot\boldsymbol{\epsilon}^{\ell'}\ .\end{aligned}$$

We now get rid of the term $1_{B(x)}(\boldsymbol{\epsilon})$ (that is an obstacle to the computation of E_ϵ). It follows from Holder's inequality and Lemma 7.2 that if $x \leq 1/K(\beta)$ we have

$$E_\epsilon 1_{B(x)^c} E_g 1_{D(x)} \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \leq K(\beta)^{k^2} P_\epsilon(B(x)^c)^{\eta'(\beta)}$$

where $\eta'(\beta) > 0$. The proof will be finished if we can show that

$$P_\epsilon(B(x)^c) \leq K(\beta)^{k^2} \exp(-kN/K(\beta)) \ . \tag{7.53}$$

We cannot appeal to the argument used in the proof of Lemma 7.8 because we do not know that (7.53) holds for $x = c(\beta)$, so we have to give a separate argument, that will occupy the end of this proof.

We write

$$\sum_{\ell<\ell'} \left(\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\right)^2 = \left(\sup \sum_{\ell<\ell'} \alpha_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'}\right)^2 \tag{7.54}$$

where the supremum is over all sequences $\alpha_{\ell\ell'}$ for which $\sum_{\ell<\ell'} \alpha_{\ell\ell'}^2 = 1$. We have

$$\sum_{\ell<\ell'} \alpha_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} = \sum_{i \leq N} Y_i$$

where $Y_i = \sum_{\ell<\ell'} \alpha_{\ell\ell'} \epsilon_i^\ell \epsilon_i^{\ell'}$. The variables $(Y_i)_{i \leq N}$ are i.i.d, $EY_i = 0$, $EY_i^p = 1$, $EY_i^p \leq p^p$ by the result of C. Borell that was crucial in the proof of Proposition 7.1. One of the many forms of Bernstein's inequality then implies that

$$P\left(\sum_{i \leq N} Y_i \geq t\right) \leq \exp\left(-\frac{1}{K} \min\left(\frac{t^2}{N}, t\right)\right) \leq \exp\left(-\frac{t^2}{KN}\right)$$

for $t \leq N$, which reads as

$$P\left(\sum_{\ell<\ell'} \alpha_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \geq t\right) \leq \exp\left(-\frac{t^2}{KN}\right) \ .$$

A classical argument [L-T, Lemma 15-1] then shows that

$$P\left(\sup \sum_{\ell<\ell'} \alpha_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \geq t\right) \leq 5^{k^2} \exp\left(-\frac{t^2}{KN}\right)$$

where the supremum is as (7.54). The result follows. □

Combining Proposition 7.11 and Lemma 7.12, we have shown that, provided $k^2 \leq Nx$ and $x \leq 1/K(\beta)$, we have

$$\begin{aligned} E(Q^k) \leq &\left(N^{2k} \exp\left(-\frac{Nx}{K(\beta)}\right) + 1\right) K(\beta)^{k^2} \\ &+ E_g 1_{D(x)} E_\epsilon S(\boldsymbol{\epsilon}) \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \ . \end{aligned} \tag{7.55}$$

It remains to estimate the last term.

Proposition 7.13. *If* $x \le 1/K(\beta)$, $k^2\sqrt{x} \le 1/12$, $xN \ge k^2$, *we have*

$$E_g 1_{D(x)} E_\epsilon S(\boldsymbol{\epsilon}) \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \le N\left(Kk^3\sqrt{xN}\right)^{2k} \exp e^2 k^2 \sqrt{xN} \ . \tag{7.56}$$

For the typical choice of $\boldsymbol{\epsilon}, S(\boldsymbol{\epsilon})$ is of order N^k, and Proposition 7.13 expresses that a lot of cancellation occurs. (Think of the case $xN = k^2$). As the proof of this fact is delicate we provide motivation by showing how to conclude the proof of Theorem 1.7. For this, we simply take $Nx = kK \log N$ and $k = k(N) = (\log N)^{1/5}/K$ to get $E(Q^{k(N)}) \le K(\beta)N^2$.

Proof of Proposition 7.13. Unfortunately the only way I see to prove this is by expanding everything and estimating the number and the size of the non zero terms. We write $D = D(x)$ for simplicity.

First, we expand $S(\boldsymbol{\epsilon})$ and we get

$$\begin{aligned} &E_g 1_D E_\epsilon S(\boldsymbol{\epsilon}) \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \\ &= \sum_{i_1,\ldots,i_{2k}} E_g 1_D E_\epsilon \left(\epsilon^1_{i_1}\epsilon^2_{i_1}\right) \ldots \left(\epsilon^{2k}_{i_{2k}}\epsilon^1_{i_{2k}}\right) \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \end{aligned} \tag{7.57}$$

where the sum is over all possible choices of indexes $i_1, \ldots, i_{2k}$ between 1 and N. It will be necessary to distinguish how many of the indexes $i_1, \ldots, i_{2k}$ are different. To simplify notation, we make the convention that $\boldsymbol{\epsilon}^{2k+1} = \boldsymbol{\epsilon}^1$.

Lemma 7.14. *If there are exactly* $p \le 2k$ *different indexes* $i_1, \ldots, i_{2k}$, *we have*

$$\begin{aligned} &E_g 1_D E_\epsilon \prod_{\ell \le 2k} \epsilon^\ell_{i_\ell} \epsilon^{\ell+1}_{i_\ell} \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \\ &\le 2N^{1-p}\left(e^2 k^2 \sqrt{xN}\right)^{2p} \exp e^2 k^2 \sqrt{xN} \end{aligned} \tag{7.58}$$

This implies (7.56) because (very crudely) there are at most $N^p p^{2k}$ choices of indexes $i_1, \ldots, i_{2k} \le N$ such that at most p indexes are different.

Using the symmetry between the indexes $i_1, \ldots, i_N$, to prove (7.58), it suffices to prove the following. Given a partition $H_1, \ldots, H_p$ of $\{1, \ldots, 2k\}$, such that none of the sets $H_1, \ldots, H_p$ is empty, the quantity

$$E_g 1_D E_\epsilon \prod_{i \le p} \left(\prod_{\ell \in H_i} \epsilon^\ell_i \epsilon^{\ell+1}_i \right) \exp \frac{\beta}{\sqrt{N}} \sum_{\ell<\ell'} g_{\ell\ell'} \boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} \tag{7.59}$$

is bounded by the right-hand side of (7.58). Writing $\boldsymbol{\epsilon}^\ell \cdot \boldsymbol{\epsilon}^{\ell'} = \sum_{i \le N} \epsilon^\ell_i \epsilon^{\ell'}_i$ and using independence over i, we see that (7.59) is equal to

$$E_g 1_D \prod_{i \le N} R_i \tag{7.60}$$

where

$$R_i = E_\epsilon \prod_{\ell \in H_i} \epsilon^\ell \epsilon^{\ell+1} \exp \frac{\beta}{\sqrt{N}} \sum_{\ell < \ell'} g_{\ell\ell'} \epsilon^\ell \epsilon^{\ell'}$$

for $i \leq p$, and, for $i > p$,

$$R_i = E_\epsilon \exp \frac{\beta}{\sqrt{N}} \sum_{\ell < \ell'} g_{\ell\ell'} \epsilon^\ell \epsilon^{\ell'} \quad .$$

We now expend the exponential as $e^x = \sum_{r \geq 0} x^r / r!$.

Given $\boldsymbol{r} = (r_i)_{i \leq N} (r_i \in \mathbb{N})$ we write $|\boldsymbol{r}| = \sum_{i \leq N} r_i$, and we define

$$Y(\boldsymbol{r}) = \frac{\beta^{|\boldsymbol{r}|}}{N^{|\boldsymbol{r}|/2}} \frac{1}{\prod_{i \leq N} r_i!} E_g 1_D \prod_{i \leq N} T_i \tag{7.61}$$

where

$$T_i = E_\epsilon \left(\prod_{\ell \in H_i} \epsilon^\ell \epsilon^{\ell+1} \left(\sum_{\ell < \ell'} g_{\ell\ell'} \epsilon^\ell \epsilon^{\ell'} \right)^{r_i} \right)$$

for $i \leq p$, while

$$T_i = E_\epsilon \left(\left(\sum_{\ell < \ell'} g_{\ell\ell'} \epsilon^\ell \epsilon^{\ell'} \right)^{r_i} \right)$$

for $p < i \leq N$. The quantity (7.60) is equal to $\sum Y(\boldsymbol{r})$, where the summation is over all possible values of $\boldsymbol{r}$. The central point of the proof is the following fact.

Lemma 7.15. *If* $Y(\boldsymbol{r}) \neq 0$, *we have*

$$|\boldsymbol{r}| \geq 2p - 2 + 2M \tag{7.62}$$

where $M = \operatorname{card}\{i > p; r_i \geq 1\}$.

Moreover

$$|Y(\boldsymbol{r})| \leq \left(k^2 \sqrt{x}\right)^{|\boldsymbol{r}|} \quad . \tag{7.63}$$

Let us first show why this finishes the proof of Lemma 7.14 (and of Theorem 1.7). We define

$$I(\boldsymbol{r}) = \{i > p; r_i \geq 1\}$$

and we write

$$\sum Y(\boldsymbol{r}) = \sum_I \sum_s \sum_{I(\boldsymbol{r}) = I, |\boldsymbol{r}| = s} Y(\boldsymbol{r}) \quad . \tag{7.64}$$

The first summation is over all subsets I of $\{p+1, \ldots, N\}$, the second summation over $s \geq 0$, and the third over $\boldsymbol{r}$. By (7.62), the second summation is in fact over $s \geq 2p - 2 + 2M$, where $M = \operatorname{card} I$. The number of ways to write s as a sum of $M + p$ integers ≥ 1 is

$$\binom{s+M+p-1}{M+p-1} \le \left(e\frac{(s+M+p-1)}{M+p-1}\right)^{M+p-1}$$
$$\le \mathrm{e}^{M+p-1}\left(1+\frac{s}{M+p-1}\right)^{M+p-1} \le \mathrm{e}^{M+p-1+s} \ .$$

Thus the third summation is at most $\mathrm{e}^{M+p-1}(ek^2\sqrt{x})^s$, using (7.63). The second summation is then at most $2\mathrm{e}^{M+p-1}(ek^2\sqrt{x})^{2p-2+2M}$ since $ek^2\sqrt{x} \le 1/2$. Finally

$$\sum Y(\boldsymbol{r}) \le \sum_{M \le N-p} 2\binom{N-p}{M}\mathrm{e}^{M+p-1}(ek^2\sqrt{x})^{2p-2+2M} \ .$$

Since $\binom{N-p}{M} \le (\frac{eN}{M})^M$, we have

$$\sum Y(\boldsymbol{r}) \le \frac{2}{N^{p-1}}(e^2k^2\sqrt{xN})^{2p-2}\sum_M \frac{(e^2k^2\sqrt{xN})^M}{M^M}$$

and the conclusion follows since $M^M \ge M!$

It remains only to prove Lemma 7.15. For this, we will expand each term $(\sum_{\ell<\ell'} g_{\ell\ell'}\epsilon^\ell\epsilon^{\ell'})^{r_i}$. At that stage it is convenient to think to each pair $(\ell,\ell')(\ell<\ell')$ as an edge of the complete graph G_{2k} on $\{1,\ldots,2k\}$. Given an edge $e=(\ell,\ell')$, we write $f(e)=\epsilon^\ell\epsilon^{\ell'}, g_e = g_{\ell\ell'}$. We then have

$$Y(\boldsymbol{r}) = \frac{\beta^{|\boldsymbol{r}|}}{N^{|\boldsymbol{r}|/2}}\frac{1}{\prod_{i\le N} r_i!}\sum_{\boldsymbol{e}} E_g 1_{D_i} g_{\boldsymbol{e}}\prod_{i\le N} R_i(\boldsymbol{e}) \tag{7.64}$$

where the summation is over all sequences $\boldsymbol{e}=(e(r,i))_{r\le r_i, i\le N}$ of edges of G, where $g_{\boldsymbol{e}} = \prod_{r\le r_i, i\le N} g_{e(r,i)}$, and where, for $i \le p$

$$R_i(\boldsymbol{e}) = E_\epsilon\left(\prod_{\ell\in H_i}\epsilon^\ell\epsilon^{\ell+1}\prod_{r\le r_i} f(e(r,i))\right)$$

while for $i > p$,

$$R_i(\boldsymbol{e}) = E_\epsilon\left(\prod_{r\le r_i} f(e(r,i))\right) \ .$$

(When $r_i = 0$, the product $\prod_{r\le r_i} f(e(r,i))$ is defined as equal to one.)

For each $\boldsymbol{e}$, we have $|R_i(\boldsymbol{e})| \le 1$; there are at most $(k(k-1)/2)^{|\boldsymbol{r}|}$ choices for the sequence $\boldsymbol{e}$, and any $|g_{\ell\ell'}|$ is at most $\sqrt{xN}$ on $D(x)$, so that (7.63) is obvious.

It is more delicate to prove (7.62). In order that a term of the summation of (7.64) be non zero, we must have $E_g 1_D g_{\boldsymbol{e}} \ne 0$ and $R_i(\boldsymbol{e}) \ne 0$ for $i \le N$. The first condition implies

(7.65) For each $e \in G$, there is an even number of values of $(r,i), r \le r_i$, $i \le N$ for which $e = e(r,i)$.

This is simply because the set D is invariant by change of sign of the $g_{\ell\ell'}$. Next, each $R_i(\boldsymbol{e})$ is $E_\epsilon \prod$, where $\prod$ is a product of terms ϵ^ℓ; $R_i(\boldsymbol{e})$ is not zero exactly if each such term occurs an even number of times.

To prove (7.62), we find it convenient to prove the following statement of graph theory.

Lemma 7.16. *Consider $q \geq 2$, and the complete graph G_q on $\{1, \ldots, q\}$. Consider two disjoint sets J_1, J_2, and their union J. We assume that J_1 is not empty. For $i \in J, e \in G$, consider an integer $m_i(e) \geq 0$, and set $r_i = p \sum_{e \in G} m_i(e)$. Consider a partition $(H_i)_{i \in J_1}$ of the set of edges $(1,2), (2,3), \ldots, (q,1)$, such that no set H_i of the partition is empty.*

For $i \in J_1$, we define $n_i(e) = 1$ if $e \in H_i, n_i(e) = 0$ otherwise. For $i \in J_2$, we set $n_i(e) = 0$ for all e. We set $m_i'(e) = m_i(e) + n_i(e)$ for $i \in J, e \in G_q$. We assume the following

$$\forall i \in J, \forall \ell \leq q, \text{the sum of the integers } m_i'(e) \text{ over the edges } e \text{ that have } \ell \text{ as an endpoint is even.} \tag{7.66}$$

$$\forall e \in G, \sum_{i \in J} m_i(e) \text{ is even} \tag{7.67}$$

$$\forall i \in J_2, r_i \geq 1 \ . \tag{7.68}$$

Then we have

$$\sum_{i \in J} r_i \geq 2(\operatorname{card} J - 1) \ . \tag{7.69}$$

To relate this statement to (7.62), we take $q = 2k, J_1 = \{1, \ldots, p\}$, $J_2 = \{i; p < i \leq N; r_i \geq 1\}$. Given a sequence $\boldsymbol{e} = (e(r,i))_{r \leq r_i, i \leq N}$, we define $m_i(e)$ as the number of integers $r \leq r_i$ for which $e(r,i) = e$. Then (7.69) is (7.62). Condition (7.68) is obvious; condition (7.67) follows from (7.65) and (7.66) from the fact that $R_i(\boldsymbol{e}) \neq 0$.

Proof of Lemma 7.16. This proof is by induction over $q \geq 2$. A first observation is that (by (7.66)), we have $r_i \geq 2$ if $i \in J_2$. Assume first $q = 2$. Then either $\operatorname{card} J_1 = 1$ or $\operatorname{card} J_1 = 2$. If $\operatorname{card} J_1 = 1$, we have

$$\sum_{i \in J} r_i \geq \sum_{i \in J_2} r_i \geq 2\operatorname{card} J_2 \geq 2(\operatorname{card} J - 1) \ .$$

If $\operatorname{card} J_1 = 2$, one of the sets H_i consists of the edge $(1,2)$ and the other one of the edge $(2,1)$. Thus 1 is the endpoint of exactly one edge of H_i. Using (7.66) we then see that $r_i \geq 1$ for $i \in J_1$. Thus

$$\sum_{i \in J} r_i \geq \sum_{i \in J_1} r_i = \operatorname{card} J_1 = 2 = 2(\operatorname{card} J - 1) \ .$$

For the induction step from $q-1$ to q, let us first consider the case where $r_i \geq 2$ for all $i \in J_1$. In that case we have $\sum_{i \in J} r_i \geq 2\operatorname{card} J$, and there is nothing to prove. So, let us assume that for some $i_0 \in J_1$ we have $r_{i_0} = 1$. We know that H_{i_0} is not empty. For the simplicity of notation we assume without loss of generality that $(q-1, q) \in H_{i_0}$. The basic procedure is then the identification of the vertices $q-1$ and q. The edge $(q-1,q)$ is removed. For $\ell \leq q-1$, the edges $(\ell, q-1), (\ell, q)$ are identified. We denote the quantities related to the "contracted graph" by the same letter as for G_q, but with a bar on top, e.g. $\overline{G}_q = G_{q-1}$. For an edge $e = (\ell, \ell')$ of $G_q, (\ell' < q-1)$, we define $\overline{m}_i(e) = m_i(e)$. For an edge $e = (\ell, q-1)$ of $\overline{G}_q$, we define

$$\overline{m}_i(e) = m_i((\ell, q-1)) + m_i((\ell, q)) \ .$$

We define the partition $(\overline{H}_i)_{i \in \overline{J}_1}$ of the edges $(1,2), \ldots, (q-1,1)$ in the obvious manner. Specifically, if $H_{i_0} = \{(q-1,q)\}$, then $\overline{J}_1 = J_1 \backslash \{i_0\}$;otherwise $\overline{J}_1 = J_1$. For $i \neq i_0$, we define $\overline{H}_i$ as the set of edges $(\ell, \ell+1)(\ell+1 \leq q-1)$ that belong to H_i. We define $\overline{r}_i = \sum \overline{m}_i(e)$, where the sum is over $e \in \overline{G}_q$. Thus

$$r_i - \overline{r}_i = m_i((q-1,q)) \ .$$

Consider

$$\overline{J}_2 = \{i \in J_2; \overline{r}_i \geq 1\} \ .$$

For $i \in J_2 \backslash \overline{J}_2$, we have $\overline{r}_i = 0$. This means that $(q-1,q)$ is the only edge e for which $m_i(e) > 0$. Using (7.66) for $\ell = q-1$, we then see that $m_i((q-1,q))$ is even, and thus $m_i((q-1,q)) \geq 2$.

Thus

$$\forall i \in J_2 \backslash \overline{J}_2, \quad r_i - \overline{r}_i \geq 2 \ . \tag{7.70}$$

If we are in the case $J_1 = \overline{J}_1$, we then have

$$\sum_{i \in J} r_i \geq \sum_{i \in \overline{J}} \overline{r}_i + 2\mathrm{card}(J \backslash \overline{J}) \ . \tag{7.71}$$

If we are in the case $\overline{J}_1 = J_1 \backslash \{i_0\}$, we have $H_{i_0} = \{(q-1,q)\}$. Since $r_{i_0} = 1$, condition (7.66) forces that $m_{i_0}((q-1,q)) = 1$ and thus

$$\sum_{i \in J} m_i((q-1,q)) \geq 1 + 2\mathrm{card}\, J_2 \backslash \overline{J}_2 \ .$$

But the left-hand side is even by (7.67), and thus

$$\sum_{i \in J} m_i((q-1,q)) \geq 2(1 + \mathrm{card}(J_2 \backslash \overline{J}_2)) = 2\mathrm{card}(J \backslash \overline{J})$$

so that (7.71) holds again . To complete the proof, we then check that (7.66) and (7.67) hold for $\overline{G}_q$ (which is easy) and we use (7.71) and the induction hypothesis. □

Note added in proof. I received from M.V. Shcherbina an unpublished 1991 paper (CARR preprint N3/91) containing material closely related to our Section 4. I also received from her a fascinating preprint called "On the replica symmetric solution for the Sherrington-Kirkpatrick model". The model she studies is not the same as the standard one I consider here. The last term of the Hamiltonian (1.1) is replaced by $\Sigma_{i \leq N} h_i \varepsilon_i$ where $(h_i)_{i \leq N}$ is an i.i.d. gaussian sequence. However it is natural to believe that the behavior of this model is closely related to the behavior of the model we consider. The preprint claims the proof of the validity of (1.21), suitably modified of course, in a region that at $h = 0$ contains all values $\beta < 1$. The methods of this paper are rather different from ours, but the proof does say that C_N is small in this region. What is extremely interesting is that a closer look at the cavity method seems to show that for β close to 1 and h very small, it simply cannot be true that $C_N \approx 0$ unless, with the notation of Section 6, $N^{-4}E\langle(\theta \cdot \theta^*)^4\rangle_4$ is much smaller than $N^{-2}E\langle(\theta \cdot \theta^*)^2\rangle_2$. Thus, if correct, the arguments of Shcherbina are very likely to contain a proof of this fact in the region she considers. This indeed would be a major progress, and one can only hope that the author will produce a version of her work with sufficiently many details to make it acessible to others, which, unfortunately, is not the case of the current version.

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