An essay on irrationality measures of π and other logarithms

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To my teacher and friend A. I. Galochkin
on the occasion of his 60th birthday

Let $a \in \mathbb{Q} \cap (0,2]$, $a \neq 1$. Then the sequence of quantities

$$\int_0^1 \frac{x^n(1-x)^n}{(1-(1-a)x)^{n+1}}\,\mathrm{d}x \in \mathbb{Q}\log a + \mathbb{Q}, \qquad n = 0, 1, 2, \dots, \tag{1}$$

produces 'good' rational approximations to $\log a$. There are several ways of performing integration in (1) in order to show that the integral lies in $\mathbb{Q}\log a + \mathbb{Q}$; we give an exposition of different methods below. The aim of this essay is to demonstrate how suitable generalizations of the integrals in (1) allow to prove the best known results on irrationality measures of the numbers $\log 2$, π and $\log 3$. Although methods presented below work in general situations (e.g., for certain $\mathbb{Q}$-linear forms in logarithms) as well, the three numbers seem to be very nice and important models for our exposition.

Bounds for irrationality measures are presented by means of upper estimates for irrationality exponents. Recall that the *irrationality exponent* of a real irrational number γ is defined by the relation

$$\mu = \mu(\gamma) = \inf\{c \in \mathbb{R} : \text{the inequality } |\gamma - a/b| \leqslant |b|^{-c} \text{ has only finitely many solutions in } a, b \in \mathbb{Z}\}.$$

The estimates for $\mu(\gamma)$ are deduced by constructing sequences of linear forms involving γ and using standard tools of the following shapes.

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Proposition 1 ([10], Lemma 3.1)**.** *Let $\gamma \in \mathbb{R}$ be irrational. Suppose that a sequence of linear forms $b_n x - a_n$, with integer coefficients from the field of rationals or an imaginary quadratic field, satisfies*

$$\limsup_{n\to\infty} \frac{\log|b_n|}{n} \leqslant C_1, \qquad \lim_{n\to\infty} \frac{\log|b_n\gamma - a_n|}{n} = -C_0$$

for some positive real C_0 and C_1. Then $\mu(\gamma) \leqslant 1 + C_1/C_0$.

Proposition 2 ([11], Lemma 2.1)**.** *Let $\omega, \omega' \in \mathbb{R}$ be two irrational numbers. Suppose that sequences of linear forms $b_n x - a_n$ and $b_n x - a'_n$, with integer coefficients from the field of rationals or an imaginary quadratic field, satisfies*

$$\limsup_{n\to\infty} \frac{\log|b_n|}{n} \leqslant C_1, \qquad \lim_{n\to\infty} \frac{\log|b_n\omega - a_n|}{n} = -C_0, \quad \lim_{n\to\infty} \frac{\log|b_n\omega' - a'_n|}{n} = -C'_0$$

for some positive real constants $C_0 < C'_0$ and C_1. Then any nonzero element $\gamma \in \mathbb{Q}\omega + \mathbb{Q}\omega'$ is irrational with the bound $\mu(\gamma) \leqslant 1 + C_1/C_0$ for the irrationality exponent.

Remark. In fact, the statement of Lemma 2.1 in [11] slightly differs from our last claim, but one can easily verify that the proof given there proves our 'modification' as well.

1 Irrationality measure for $\log 2$ (after E. Rukhadze)

1.1 Gauss hypergeometric function

It is worth performing a slightly general integral than (1), namely

$$I(m, n_0, n_1; a) = \int_0^1 \frac{x^{n_0}(1-x)^{n_1}}{(1-(1-a)x)^{m+1}}\,\mathrm{d}x \tag{2}$$

for non-negative integers m, n_0, n_1, provided the condition $\max\{m, n_0\} \leqslant n_1$ holds for further convenience. The integral in (2) is exactly Euler's integral for the Gauss hypergeometric series:

$$\begin{aligned} I(m, n_0, n_1; a) &= \frac{\Gamma(n_0+1)\,\Gamma(n_1+1)}{\Gamma(n_0+n_1+2)}\,{}_2F_1\left(\begin{matrix} m+1,\ n_0+1 \\ n_0+n_1+2 \end{matrix}\,\middle|\, 1-a\right) \\ &= \frac{\Gamma(n_1+1)}{\Gamma(m+1)} \sum_{\nu=0}^{\infty} \frac{\Gamma(m+1+\nu)\,\Gamma(n_0+1+\nu)}{\Gamma(1+\nu)\,\Gamma(n_0+n_1+2+\nu)}(1-a)^{\nu} \end{aligned} \tag{3}$$

(see, e.g., [3], Section 2.2). The latter sum may be written as

$$I(m, n_0, n_1; a) = \sum_{\nu=0}^{\infty} R(\nu)(1-a)^{\nu}, \tag{4}$$

where

$$R(t)=\frac{(t+1)(t+2)\cdots(t+m)}{m!}\cdot\frac{n_1!}{(t+n_0+1)(t+n_0+2)\cdots(t+n_0+n_1+1)} \tag{5}$$

and $R(t)=O(t^{-1})$ as $t\to\infty$ by $m\leqslant n_1$. Denote $m^*=\min\{m,n_0\}$ and $n_0^*=\max\{m,n_0\}$ and decompose the rational function (5) in a sum of partial fractions:

$$R(t)=\sum_{k=n_0^*}^{n_0+n_1}\frac{A_k}{t+k+1}=\sum_{k=n_0^*}^{n_0+n_1}\frac{(-1)^{m+n_0-k}\binom{k}{m}\binom{n_1}{k-n_0}}{t+k+1}. \tag{6}$$

Then by (4) we obtain

$$\begin{aligned} I(m,n_0,n_1;a)&=\sum_{\nu=-m^*}^{\infty}R(\nu)(1-a)^{\nu}=\sum_{k=n_0^*}^{n_0+n_1}A_k(1-a)^{-(k+1)}\sum_{\nu=-m^*}^{\infty}\frac{(1-a)^{\nu+k+1}}{\nu+k+1}\\ &=\sum_{k=n_0^*}^{n_0+n_1}A_k(1-a)^{-(k+1)}\biggl(\sum_{l=1}^{\infty}-\sum_{l=1}^{k-m^*}\biggr)\frac{(1-a)^l}{l}\\ &=-\log a\cdot\sum_{k=n_0^*}^{n_0+n_1}A_k(1-a)^{-(k+1)}-\sum_{k=n_0^*}^{n_0+n_1}\sum_{l=1}^{k-m^*}\frac{A_k(1-a)^{l-(k+1)}}{l}, \end{aligned} \tag{7}$$

hence

$$I(m,n_0,n_1;a)(1-a)^{n_0+n_1+1}\cdot d^{n_0+n_1-m^*}D_{n_0+n_1-m^*}\in\mathbb{Z}\log a+\mathbb{Z}, \tag{8}$$

where d denotes the denominator of a and D_n stands for the least common multiple of the numbers $1,2,\dots,n$. By the prime number theorem, we have the following asymptotic formula:

$$\lim_{n\to\infty}\frac{\log D_n}{n}=1.$$

1.2 Arithmetic valuation

The inclusion (8) may be essentially improved in several cases, and it is the observation that allowed Rukhadze to prove the record irrationality measure for $\log 2$.

The symmetry of the ${}_2F_1$-series in (3) with respect to its upper parameters $m+1$ and n_0+1 gives us a way to write the identity

$$\frac{I(m,n_0,n_1;a)}{\Gamma(n_0+1)\,\Gamma(n_1+1)}=\frac{I(n_0,m,n_0+n_1-m;a)}{\Gamma(m+1)\,\Gamma(n_0+n_1-m+1)} \tag{9}$$

(which is not so evident if one looks on definition (2)). The inclusion (8) written for the I-quantity on the right of (9),

$$I(n_0,m,n_0+n_1-m;a)(1-a)^{n_0+n_1+1}\cdot d^{n_0+n_1-m^*}D_{n_0+n_1-m^*}\in\mathbb{Z}\log a+\mathbb{Z},$$

and the equality

$$\begin{aligned}I(m,n_0,n_1;a)(1-a)^{n_0+n_1+1}\cdot d^{n_0+n_1-m^*}D_{n_0+n_1-m^*}\cdot\frac{m!\,(n_0+n_1-m)!}{n_0!\,n_1!}\\ =I(n_0,m,n_0+n_1-m;a)(1-a)^{n_0+n_1+1}\cdot d^{n_0+n_1-m^*}D_{n_0+n_1-m^*}\end{aligned}$$

imply that if $\Phi(m,n_0,n_1)$ is the denominator of the quotient

$$\frac{m!\,(n_0+n_1-m)!}{n_0!\,n_1!},$$

then

$$I(m,n_0,n_1;a)(1-a)^{n_0+n_1+1}\cdot d^{n_0+n_1-m^*}D_{n_0+n_1-m^*}\cdot\Phi(m,n_0,n_1)^{-1}\in\mathbb{Z}\log a+\mathbb{Z}. \quad (10)$$

By the well-known formula, for each prime p we have $\operatorname{ord}_p N! = \lfloor N/p\rfloor+\lfloor N/p^2\rfloor+\lfloor N/p^3\rfloor+\cdots$, where $\lfloor\,\cdot\,\rfloor$ denotes the integral part of a number. Therefore

$$\Phi(m,n_0,n_1)=\prod_p p^{\phi(p)+\phi(p^2)+\phi(p^3)+\cdots}, \quad (11)$$

where

$$\phi(t)=\max\biggl\{0,\left\lfloor\frac{n_0}{t}\right\rfloor+\left\lfloor\frac{n_1}{t}\right\rfloor-\left\lfloor\frac{m}{t}\right\rfloor-\left\lfloor\frac{n_0+n_1-m}{t}\right\rfloor\biggr\}.$$

The final remark (made by G. Chudnovsky in [7] together with introducing the method of asymptotic evaluation of the factors like (11)) consists in the fact that the divisor

$$\widetilde{\Phi}(m,n_0,n_1)=\prod_{p>\sqrt{n_1}} p^{\phi(p)} \quad (12)$$

of $\Phi(m,n_0,n_1)$ gives the main contribution in the asymptotic of (11) and may be easily controlled.

1.3 Irrationality result

The choice $a=2$ and $n_0=6n$, $m=7n$, $n_1=8n$, where n is the positive integer parameter increasing to ∞, allowed E. Rukhadze in [18] to prove the following result (see also [10], [19] and [6]).

Theorem 1. *The irrationality exponent of* $\log 2$ *satisfies the inequality*

$$\mu(\log 2)\leqslant 3.89139977\ldots.$$

We will briefly indicate required ingredients of the proof. For the above choice of the parameters we set

$$I_n=I(7n,6n,8n;2)=\int_0^1\left(\frac{x^6(1-x)^8}{(1+x)^7}\right)^n\frac{\mathrm{d}x}{1+x}=\bar{A}_n\log 2-\bar{B}_n,$$

where, by (6) and (7),

$$\bar{A}_n = (-1)^n \sum_{k=7n}^{14n} \binom{k}{7n}\binom{8n}{k-6n}.$$

Then

$$\begin{aligned}\lim_{n\to\infty}\frac{\log I_n}{n} &= \log\max_{0<x<1}\frac{x^6(1-x)^8}{(1+x)^7}\\ &= \log\frac{2^5 3^3(7734633\sqrt{393}-153333125)}{7^7} = -11.84497806\ldots \end{aligned}\tag{13}$$

and, thanks to Stirling's asymptotic formula for the factorial,

$$\begin{aligned}\lim_{n\to\infty}\frac{\log|\bar{A}_n|}{n} &= \lim_{n\to\infty}\frac{1}{n}\log\max_{7n\leqslant k\leqslant 14n}\binom{k}{7n}\binom{8n}{k-6n}\\ &= \log\max_{7<y<14}\left(\frac{y^y}{7^7(y-7)^{y-7}}\cdot\frac{8^8}{(y-6)^{y-6}(14-y)^{14-y}}\right)\\ &= \log\frac{2^5 3^3(7734633\sqrt{393}+153333125)}{7^7} = 12.68147230\ldots. \end{aligned}\tag{14}$$

Concerning the asymptotic behaviour of the value $\Phi_n = \widetilde{\Phi}(7n, 6n, 8n)$ in (12), we use the fact $\phi(t) = \varpi_0(n/t)$, where

$$\begin{aligned}\varpi_0(x) &= \max\bigl\{0, \lfloor 6x\rfloor + \lfloor 8x\rfloor - 2\lfloor 7x\rfloor\bigr\}\\ &= \begin{cases}1 & \text{if } x \in \bigl[\frac18,\frac17\bigr)\cup\bigl[\frac14,\frac27\bigr)\cup\bigl[\frac38,\frac37\bigr)\cup\bigl[\frac12,\frac47\bigr)\cup\bigl[\frac23,\frac57\bigr)\cup\bigl[\frac56,\frac67\bigr),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{\log\Phi_n}{n} = \int_0^1 \varpi_0(x)\mathrm{d}\psi(x) = \log\frac{2^{15}3^3}{7^7} + \frac{\pi(3+6\sqrt{2}-4\sqrt{3})}{6} = 2.45775406\ldots, \tag{15}$$

where $\psi(x)$ denotes the logarithmic derivative of the gamma function. Using inclusions (10) and the asymptotics (13)–(15), we obtain

$$\begin{aligned}C_0 &= -\log(7734633\sqrt{393}-153333125) + 10\log 2 - 8 + \frac{\pi(3+6\sqrt{2}-4\sqrt{3})}{6}\\ &= 6.30273213\ldots,\\ C_1 &= \log(7734633\sqrt{393}+153333125) - 10\log 2 + 8 - \frac{\pi(3+6\sqrt{2}-4\sqrt{3})}{6}\\ &= 18.22371823\ldots,\end{aligned}$$

in the notation of Proposition 1 and, finally, conclude with the estimate

$$\mu(\log 2) \leqslant 1 + \frac{C_1}{C_0} = 3.89139977\ldots.$$

The result for the measure of $\log 2$ may be compared with that obtained in simpler settings $n_0 = n_1 = m = n$ (as in (1)):

$$C_0 = -2\log(\sqrt{2}-1) - 1 = 2\log(\sqrt{2}+1) - 1, \qquad C_1 = 2\log(\sqrt{2}+1) + 1,$$

hence

$$\mu(\log 2) \leqslant 1 + \frac{C_1}{C_0} \leqslant 1 + \frac{2\log(\sqrt{2}+1)+1}{2\log(\sqrt{2}+1)-1} = 4.62210083\ldots.$$

2 Irrationality measure for π (after M. Hata)

2.1 Simultaneous approximations to logarithms

The change of variable $z = 1 - (1-a)x$ in (1) transforms the integral (1) into

$$\frac{(-1)^{n+1}}{(1-a)^{2n+1}} \int_1^a \frac{(z-1)^n (z-a)^n}{z^{n+1}} \, \mathrm{d}z. \tag{16}$$

Instead of decomposing the latter integral we will perform a more general *complex* integral

$$I_k(\boldsymbol{a}, m, \boldsymbol{n}; a) = \int_{\Gamma_{1,a}} \frac{(z-1)^{n_0}(z-a_1)^{n_1} \cdots (z-a_k)^{n_k}}{z^{m+1}} \, \mathrm{d}z,$$

where $\Gamma_{1,a}$ denotes a smooth oriented path from 1 to a contained in $\mathbb{C} \setminus \{0\}$; the parameters $a, a_1, \ldots, a_k$ are complex numbers distinct from $0, 1$; the exponents $n_0, n_1, \ldots, n_k, m$ are positive integers. The integral in (16) corresponds to $k = 1$, $a_1 = a$ and $n_0 = n_1 = m = n$. Setting additionally $a_0 = 1$, we may compute, as in [11], Section 3,

$$\begin{aligned} I_k(\boldsymbol{a}, m, \boldsymbol{n}; a) &= \sum_{l_0=0}^{n_0} \sum_{l_1=0}^{n_1} \cdots \sum_{l_k=0}^{n_k} A_{\boldsymbol{l}} \binom{n_0}{l_0} \binom{n_1}{l_1} \cdots \binom{n_k}{l_k} \int_{\Gamma_{1,a}} z^{l_0+l_1+\cdots+l_k-m-1} \, \mathrm{d}z \\ &= \sum \cdots \sum_{l_0+\cdots+l_k \neq m} \frac{A_{\boldsymbol{l}}}{l_0+\cdots+l_k-m} \binom{n_0}{l_0} \cdots \binom{n_k}{l_k} (a^{l_0+\cdots+l_k-m} - 1) \\ &\qquad + \sum \cdots \sum_{l_0+\cdots+l_k=m} A_{\boldsymbol{l}} \binom{n_0}{l_0} \cdots \binom{n_k}{l_k} \cdot \log a, \end{aligned} \tag{17}$$

where

$$A_{\boldsymbol{l}} = A_{l_0,l_1,\ldots,l_k} = (-1)^{l_0+l_1+\cdots+l_k} a_1^{n_1-l_1} \cdots a_k^{n_k-l_k}$$

and we use the formula

$$\int_{\Gamma_{1,a}} z^{l-1}\,\mathrm{d}z = \int_1^a z^{l-1}\,\mathrm{d}z = \begin{cases} a^l/l & \text{if } l \neq 0, \\ \log a & \text{if } l = 0. \end{cases}$$

The main idea is that the coefficient of $\log a$ in the linear form (17) does not depend on the choice of a (but of course the analytic behaviour of the integral does!). The suitable and natural choice of a is from the set $\{a_1, \dots, a_k\}$. Then the above quantities I_k produce simultaneous approximations to $\log a_1, \dots, \log a_k$.

2.2 Analytic and arithmetic ingredients

Our basic consideration will be devoted to the case $k = 2$, which is used in [11] to give the linear independence measure of π and $\log 2$ over $\mathbb{Q}$ (in particular, the irrationality measure of π) and the new irrationality measure of $\pi/\sqrt{3}$.

Thus, Hata [11] takes $k = 2$ (that really gives an extension of (16), and hence of (1)) and substitute $a = a_1$ and $a = a_2$ to get nice simultaneous approximations to $\log a_1$ and $\log a_2$. Hata 'restricts' himself from the beginning to considering the particular case $n_0 = n_1 = n_2 = 2n$ and $m = 3n$, where n is an increasing parameter. However, this simple choice produces the best possible number-theoretic results, and our consideration of the general case

$$n_0 = \alpha_0 n, \quad n_1 = \alpha_1 n, \quad n_2 = \alpha_2 n, \quad m = \alpha n,$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha$ are positive integers, is mostly due to methodological reasons.

Write the integrals in the form

$$J_{j,n} = I_2(a_j) = \int_{\gamma_j} \frac{e^{nf(z)}}{z}\,\mathrm{d}z, \qquad j = 1, 2, \tag{18}$$

where

$$f(z) = \alpha_0 \log(z - a_0) + \alpha_1 \log(z - a_1) + \alpha_2 \log(z - a_2) - \alpha \log z$$

and the path γ_j joints the points 1 and a_j and goes through the corresponding saddle point. The saddle points ξ_0, ξ_1, ξ_2 are solutions of the equation $f'(z) = 0$ becoming the cubic polynomial equation: two of these saddles correspond to the growth of the integrals in (18),

$$\lim_{n\to\infty} \frac{\log |J_{1,n}|}{n} = \operatorname{Re} f(\xi_1), \qquad \lim_{n\to\infty} \frac{\log |J_{2,n}|}{n} = \operatorname{Re} f(\xi_2),$$

while the third saddle ξ_0 determines the asymptotic behaviour of the coefficients of the linear forms.

To compute the arithmetic of the coefficients we should evaluate the true denominators of the products

$$\frac{1}{l_0+l_1+l_2-\alpha n}\binom{\alpha_0 n}{l_0}\binom{\alpha_1 n}{l_1}\binom{\alpha_2 n}{l_2}, \qquad l_0+l_1+l_2\ne\alpha n.$$

Clearly the least common multiple $D_{\beta n}$, where $\beta=\max\{\alpha,\alpha_0+\alpha_1+\alpha_2-\alpha\}$, is required but some primes $p>\sqrt{Cn}$ may be then excluded from this $D_{\beta n}$ by considering the following problem: determine primes p dividing all the integers

$$\binom{\alpha_0 n}{l_0}\binom{\alpha_1 n}{l_1}\binom{\alpha_2 n}{l_2}$$

under the additional condition $l_0+l_1+l_2\equiv\alpha n \pmod p$. Writing $x=\{n/p\}$ and $y_j=\{l_j/p\}$, $j=0,1,2$, for the fractional parts, we reduce the problem to minimizing the 1-periodic integer-valued function

$$\varpi(x,y_0,y_1,y_2)=\sum_{j=0}^{2}\big(\lfloor\alpha_j x\rfloor-\lfloor y_j\rfloor-\lfloor\alpha_j x-y_j\rfloor\big)$$

on the cube $(y_0,y_1,y_2)\in[0,1)^3$ under the additional hypothesis $y_0+y_1+y_2\equiv\alpha x \pmod 1$. (The last condition means that knowledge of x,y_0,y_1 determines the remaining value y_2 uniquely.) Denote by $\varpi_0(x)$ the required minimum. For example, Hata's choice $\alpha_0=\alpha_1=\alpha_2=2$, $\alpha=3$ gives

$$\varpi_0(x)=\begin{cases}1 & \text{if } x\in\big[\frac12,\frac23\big),\\ 0 & \text{otherwise.}\end{cases}$$

There is also a 'problem' of finding the true denominators of $A_{\boldsymbol l}$ and $A_{\boldsymbol l}a^{l_0+l_1+l_2-m}$. For example, in the case $a_1=2$, $a_2=1+i$ (of simultaneous approximations to $\log 2$ and π) we have

$$\begin{aligned}
&(-1)^{l_0+l_1+l_2}A_{\boldsymbol l}a_0^{l_0+l_1+l_2-m}=2^{n_1-l_1}(1+i)^{n_2-l_2}\in\mathbb Z[i],\\
&(-1)^{l_0+l_1+l_2}A_{\boldsymbol l}a_1^{l_0+l_1+l_2-m}=2^{n_1+l_0+l_2-m}(1+i)^{n_2-l_2}\\
&\qquad=2^{n_1+l_0-m}(1+i)^{l_2}(1-i)^{l_2}\cdot(1+i)^{2\lfloor n_2/2\rfloor}(1+i)^{2\{n_2/2\}-l_2}\\
&\qquad=2^{n_1+\lfloor n_2/2\rfloor-m+l_0}i^{\lfloor n_2/2\rfloor}(1+i)^{2\{n_2/2\}}(1-i)^{l_2}\in\mathbb Z[i],\\
&(-1)^{l_0+l_1+l_2}A_{\boldsymbol l}a_2^{l_0+l_1+l_2-m}=2^{n_1-l_1}(1+i)^{n_2+l_0+l_1-m}\\
&\qquad=(1+i)^{n_1-l_1}(1-i)^{n_1-l_1}(1+i)^{n_2+l_0+l_1-m}\\
&\qquad=(1+i)^{n_1+n_2-m+l_0}(1-i)^{n_1-l_1}\in\mathbb Z[i],
\end{aligned}$$

provided that $n_1+\lfloor n_2/2\rfloor-m\geqslant0$ and $n_1+n_2-m\geqslant0$ (i.e., that $\alpha_1+\alpha_2/2\geqslant\alpha$).

2.3 Measure for π

Thus, Hata's choice $a_1 = 2$, $a_2 = 1 + i$ and $n_0 = n_1 = n_2 = 2n$, $m = 3n$ with the help of Proposition 2 gives the following result.

Theorem 2. *The irrationality exponent of any nonzero $\gamma \in \mathbb{Q}\log 2 + \mathbb{Q}\pi$ satisfies the inequality*

$$\mu(\gamma) \leqslant 8.01604539\dots.$$

We would like to refer the interested reader to the notes [5] that could give some feelings of how difficult is evaluating the irrationality measure of π.

2.4 Double hypergeometric series

Here we present a connection of Hata's construction with hypergeometric series (that were a major tool in Section 1).

For simplicity, we will set $a = a_1$, $b = a_2$ and deal with the integrals

$$J = \int_1^a \frac{(z-1)^{n_0}(z-a)^{n_1}(z-b)^{n_2}}{z^{m+1}}\,\mathrm{d}z$$

and

$$J^* = \int_1^b \frac{(z-1)^{n_0}(z-a)^{n_1}(z-b)^{n_2}}{z^{m+1}}\,\mathrm{d}z$$

giving the simultaneous approximations to $\log a$ and $\log b$. Applying the starting change of variable $z = 1 - (1-a)x$ to the first integral we obtain the single integral

$$J = (-1)^{n_0+1}(1-a)^{n_0+n_1+1}(1-b)^{n_2} \int_0^1 \frac{x^{n_0}(1-x)^{n_1}\left(1 - \dfrac{1-a}{1-b}x\right)^{n_2}}{(1-(1-a)x)^{m+1}}\,\mathrm{d}x \qquad (19)$$

that may be identified with the Appell hypergeometric function

$$\begin{aligned} J = (-1)^{n_0+1}(1-a)^{n_0+n_1+1}(1-b)^{n_2}\frac{\Gamma(n_0+1)\,\Gamma(n_1+1)}{\Gamma(n_0+n_1+1)} \\ \times F_1\biggl(n_0+1; m+1, -n_2; n_0+m+2; 1-a, \frac{1-a}{1-b}\biggr) \end{aligned}$$

(see [4], Section 9.3, formula (4)), where the series

$$F_1(A; B, B'; C; X, Y) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} \frac{(A)_{\nu+\mu}(B)_\nu(B')_\mu}{\nu!\,\mu!(C)_{\nu+\mu}} X^\nu Y^\mu$$

is absolutely convergent in the domain $|X| < 1$, $|Y| < 1$.

The next change of variable

$$x = (1-y)\bigg/\biggl(1 - \frac{1-a}{1-b}y\biggr)$$

in (19) gives the integral representation

$$\begin{aligned} J &= (-1)^{n_0+m}(1-a)^{n_0+n_1+1}(1-b)^{n_0+n_2+1}(a-b)^{n_1+n_2+1} \\ &\quad\times \int_0^1 \frac{y^{n_1}(1-y)^{n_0}\,\mathrm{d}y}{\big(a(1-b)-b(1-a)y\big)^{m+1}\big((1-b)-(1-a)y\big)^{n_0+n_1+n_2-m+1}} \\ &= (-1)^{n_0+m}\frac{(1-a)^{n_0+n_1+1}(a-b)^{n_1+n_2+1}}{a^{m+1}(1-b)^{n_1+1}}\,\frac{\Gamma(n_0+1)\,\Gamma(n_1+1)}{\Gamma(n_0+n_1+1)} \\ &\quad\times F_1\biggl(n_1+1; m+1, n_0+n_1+n_2-m+1; n_0+n_1+2; \frac{b(1-a)}{a(1-b)}, \frac{1-a}{1-b}\biggr). \end{aligned} \tag{20}$$

The case $a=2$, $b=1+i$ gives us the following arguments of the last F_1-series:

$$\frac{1-a}{1-b} = -i = e^{-\pi i/2}, \qquad \frac{b(1-a)}{a(1-b)} = \frac{1}{\sqrt{2}}e^{-\pi i/4}.$$

Finally, the above changes of variable applied to the integral J^* produce the same integrals as in (19) and (20) but with integrations over smooth paths from 0 to $(1-b)/(1-a)$ and from ∞ to 1, respectively.

3 Irrationality measure for $\log 3$ (after G. Rhin)

3.1 Preliminary remark

As mentioned, the method of Section 2 have several other applications. For instance, the choice $a=4/3$, $b=3/2$ and $n_0=n_1=n_2=2n$, $m=3n$ (cf. Section 2.3) with the help of Proposition 2 implies that the irrationality exponent of $\gamma \in \mathbb{Q}\log 2 + \mathbb{Q}\log 3$ satisfies the inequality $\mu(\gamma) \leqslant 11.1017577\dots$ (see [12], Corollary 3.1).

3.2 Back to rational approximations to $\log 2$

As we already know from Section 1.1, for our starting integral (1) in the case $a=2$ we have

$$D_n \int_0^1 \left(\frac{x(1-x)}{1+x}\right)^n \frac{\mathrm{d}x}{1+x} \in \mathbb{Z}\log 2 + \mathbb{Z},$$

hence

$$D_n \int_0^1 \left(\frac{x(1-x)}{1+x}\right)^k \frac{\mathrm{d}x}{1+x} \in \mathbb{Z}\log 2 + \mathbb{Z}$$

for any non-negative integer $k \leqslant n$. Considering linear combinations of the latter integrals we arrive at general inclusions

$$D_n \int_0^1 G_n\biggl(\frac{x(1-x)}{1+x}\biggr)\frac{\mathrm{d}x}{1+x} \in \mathbb{Z}\log 2 + \mathbb{Z} \tag{21}$$

valid for all polynomials $G_n(y) \in \mathbb{Z}[y]$ of degree $\deg G_n \leqslant n$. To guess a 'nice' choice for the polynomial G_n, we start with notifying that

$$\int_0^1 \left(\frac{x(1-x)}{1+x}\right)^n \frac{\mathrm{d}x}{1+x} = C \int_0^b \left(\frac{x(1-x)}{1+x}\right)^n \frac{\mathrm{d}x}{1+x},$$

where C is a constant (in our case $C = 2$) and b is the saddle point for the integrand: $b = \sqrt{2} - 1$; therefore,

$$\int_0^1 \left(\frac{x(1-x)}{1+x}\right)^n \frac{\mathrm{d}x}{1+x} = C \int_0^{(\sqrt{2}-1)^2} y^n x(y)\,\mathrm{d}y,$$

where $y = x(1-x)/(1+x)$ and $x(y)\ (0, (\sqrt{2}-1)^2) \to (0, b)$ is the inverse function. Finally,

$$\int_0^1 G_n\left(\frac{x(1-x)}{1+x}\right) \frac{\mathrm{d}x}{1+x} = C \int_0^{(\sqrt{2}-1)^2} G_n(y) x(y)\,\mathrm{d}y;$$

thus, evaluating the required asymptotic, using inclusions (21) and applying Proposition 1 result in the estimate $\mu(\log 2) \leqslant 1 + C_1/C_0$, where

$$\begin{aligned} C_0 &= -1 - \lim_{n\to\infty} \log \max_{0\leqslant y\leqslant(\sqrt{2}-1)^2} \bigl\{|G_n(y)|^{1/n}\bigr\}, \\ C_1 &= 1 + \lim_{n\to\infty} \log \max_{0\leqslant y\leqslant(\sqrt{2}+1)^2} \bigl\{|G_n(y)|^{1/n}\bigr\}. \end{aligned}$$

One might now think to look for a polynomial $G_n \in \mathbb{Z}[y]$ of degree $\leqslant n$ admitting the minimum for the quantity C_1/C_0. Unfortunately, the (non-linear!) problem seems to be very hard for being solved.

The idea of Rhin [15], [16], who introduced the above construction, was to 'linearize' the optimization. He suggested to look for a polynomial $G^* \in \mathbb{Z}[y]$ of degree $\leqslant n^*$, say, which is close enough to the optimal polynomial choice in the problem

$$\min_{\substack{G\in\mathbb{Z}[y] \\ 1\leqslant \deg G\leqslant n^*}} \max_{0\leqslant y\leqslant(\sqrt{2}-1)^2} \bigl\{|G(y)|^{1/n^*}\bigr\}, \tag{22}$$

and then take $G_n(x)$ to be $(G^*(x))^{\lfloor n/n^*\rfloor}$ for n sufficiently greater than n^*. For instance, the fact $(\sqrt{2}-1)^2 \approx 1/6$ gives one the first non-trivial approximation $G^*(y) = y^6(6y-1)$ in the problem.

The problem of minimizing the quantity (22) is deeply related to evaluating the $\mathbb{Z}$-transfinite diameter of the segment $[0, (\sqrt{2}-1)^2]$. (The $\mathbb{Z}$-transfinite diameter of the set $Y \subset \mathbb{R}$ is defined by the formula

$$t_{\mathbb{Z}}(Y) = \inf_{\substack{G\in\mathbb{Z}[x] \\ \deg G\geqslant 1}} \max_{y\in Y} \bigl\{|G(y)|^{1/\deg G}\bigr\},$$

see [1] for problems of computing the quantity.) This relationship is described in [2]; there one can also find the result $\mu(\log 2) < 3.991$, which may be achieved by

the method. The latter estimate looks rather close to the inequality in Theorem 1; however, it seems to be very 'computer dependent'.

The above method may be used in situations à la Section 2 as well. For example, we may go back to simultaneous $\mathbb{Z}[i]$-approximations to $\log a_1$ and $\log a_2$ and write

$$\int_1^{a_1} G_n\biggl(\frac{(z-1)^2(z-a_1)^2(z-a_2)^2}{z^3}\biggr)\frac{\mathrm{d}z}{z} = B_n\log a_1 - B_n',$$
$$\int_1^{a_2} G_n\biggl(\frac{(z-1)^2(z-a_1)^2(z-a_2)^2}{z^3}\biggr)\frac{\mathrm{d}z}{z} = B_n\log a_2 - B_n''$$

for any polynomial $G_n(y)\in\mathbb{Z}[y]$ of degree $\leqslant n$, where

$$d^nB_n,\ d^nD_{3n}B_n',\ d^nD_{3n}B_n''\in\mathbb{Z}[i],$$

the integer $d>0$ emanates from denominators to the numbers $a_1, a_2, a_1^{-1}, a_2^{-1}$. (Using the better inclusions achieved by Hata in [11] is in this case rather problematic.) Unfortunately, this way does not look perspective, again due to the fact that we are required to 'linearize' the appeared optimization problem.

3.3 Another generalization of the integral in (1)

On the other hand, we may perform integration in (1) by putting a general polynomial of degree $\leqslant 2n$ in the numerator of the integrand (in place of $x^n(1-x)^n$). Of course, in this case the polynomial is required to satisfy some additional conditions.

Let $a=c/d\in\mathbb{Q}$ with pairwise coprime c and $d>0$, and let Δ be a common multiple of the numbers c and d. Suppose that a polynomial $H_n(z)\in\mathbb{Z}[z]$ of degree $\leqslant 2n$ may be represented in the form

$$H_n(z)=\sum_{\nu=0}^{n}B_\nu\Delta^{n-\nu}z^\nu+\sum_{\nu=n+1}^{2n}B_\nu z^\nu, \qquad \text{where} \quad B_\nu\in\mathbb{Z},\ \nu=0,1,\dots,2n. \tag{23}$$

(Clearly, for $a=2$ the polynomial $H_n(z)=(z-1)^n(z-2)^n$ has the desired form.) Then for the integral

$$I(n)=(1-a)\int_0^1\frac{H_n(d-d(1-a)x)}{d^n(1-(1-a)x)^{n+1}}\,\mathrm{d}x$$

we deduce

$$\begin{aligned} I(n)&=\sum_{\nu=0}^{n}B_\nu\Delta^{n-\nu}d^{\nu-n}(1-a)\int_0^1(1-(1-a)x)^{\nu-n-1}\,\mathrm{d}x\\ &\qquad+\sum_{\nu=n+1}^{2n}B_\nu d^{\nu-n}(1-a)\int_0^1(1-(1-a)x)^{\nu-n-1}\,\mathrm{d}x\\ &=\sum_{\nu=0}^{n-1}B_\nu\Delta^{n-\nu}d^{\nu-n}\frac{a^{\nu-n}-1}{n-\nu}-B_n\log a-\sum_{\nu=n+1}^{2n}B_\nu d^{\nu-n}\frac{1-a^{\nu-n}}{\nu-n}, \end{aligned}$$

hence

$$I(n) \cdot D_n \in \mathbb{Z} \log a + \mathbb{Z}.$$

In general, having a set of k rational numbers $a_j = c_j/d$ for $j = 1, \dots, k$, we suppose that the polynomial $H_n(z) \in \mathbb{Z}[z]$ of degree $\leqslant 2n$ has representation (23) with Δ being a multiple of the numbers $c_1, \dots, c_k, d$. Then setting

$$I(n; a_j) = (1 - a_j) \int_0^1 \frac{H_n(d - d(1 - a_j)x)}{d^n (1 - (1 - a_j)x)^{n+1}} \, \mathrm{d}x, \qquad j = 1, \dots, k, \tag{24}$$

we obtain

$$I(n; a_j) \cdot D_n = -B_n \log a_j + A_{nj} \in \mathbb{Z} \log a_j + \mathbb{Z}, \qquad j = 1, \dots, k,$$

again simultaneous approximations to $\log a_1, \dots, \log a_k$. (In fact, the choice

$$H_n(z) = \Delta^{2n} (z - 1)^{\lfloor \beta_0 n \rfloor} (z - a_1)^{\lfloor \beta_1 n \rfloor} \cdots (z - a_k)^{\lfloor \beta_k n \rfloor}$$

where $\beta_j = \alpha_j / \alpha$ for $j = 0, 1, \dots, k$, gives us exactly the same approximations as in Section 2. The case $\beta_1 = \dots = \beta_k$ was previously treated in [15] and [17].)

Finding a suitable polynomial $H_n(z)$ for a given set of the numbers $a_1, \dots, a_k$ is very similar to that of Section 3.2. The change of variable $z_j = d - d(1 - a_j)x$ in the integrals (24) (hence, integrating then a simpler expression over the segment $[d, da_j]$) leads to the problem of finding a polynomial $H_n(z) \in \mathbb{Z}[z]$ of degree $\leqslant 2n$ with expansion (23) such that the quantity

$$\max_{z \in Z} \left\{ \left| \frac{H_n(z)}{z} \right|^{1/n} \right\}, \qquad Z = \bigcup_{j=1}^{k} [d, da_j],$$

is as small as possible. The algorithmic solution to this optimization problem by means of the LLL-algorithm was recently proposed by Q. Wu [20]. This gives one a machinery to produce fairly good estimates for linear forms in the logarithms of *rational* numbers.

3.4 Measure for $\log 3$

To derive a nice irrationality measure for $\log 3$, Rhin constructs in [16] simultaneous approximations to the logarithms of $a_1 = 2/3$, $a_2 = 4/3$ and use the following (very complicated) choice of the polynomial (23):

$$\begin{aligned} H_n(z) = 2^{14} \cdot 3^{2n+7} \cdot (z - 1)^{\lfloor 0.704324n \rfloor} \left(z - \frac{2}{3}\right)^{\lfloor 0.552418n \rfloor} \left(z - \frac{4}{3}\right)^{\lfloor 0.447582n \rfloor} \\ \times (5z - 4)^{\lfloor 0.109072n \rfloor} (17z^2 - 34z + 16)^{\lfloor 0.038934n \rfloor} (19z^2 - 36z + 16)^{\lfloor 0.054368n \rfloor} \end{aligned}$$

(a ‘justification’ of the choice is done in [20]). By these means he proves

Theorem 3. *The irrationality exponent of any nonzero* $\gamma \in \mathbb{Q}\log 2+\mathbb{Q}\log 3$ *satisfies the inequality*

$$\mu(\gamma) < 8.616.$$

Further results in this direction (e.g., irrationality measures for $\log 5$, $\log 7$ etc.) may be found in [20].

4 Concluding improvisations

Connections with the hypergeometric subject (indicated in Sections 1.1 and 2.4 above) could play a role in further improvements of the irrationality measures of logarithms and related constants. For instance, Euler's transform (see, e.g., [4], Section 2.4, formula (1))

$${}_2F_1\left(\begin{matrix}A,\, B\\ C\end{matrix}\,\middle|\, z\right) = \frac{1}{(1-z)^A}\cdot {}_2F_1\left(\begin{matrix}A,\, C-B\\ C\end{matrix}\,\middle|\, \frac{-z}{1-z}\right)$$

translates the value $z=1-a=-1$ of Section 1 into $-z/(1-z)=1/2$. This leads to a ${}_2F_1$-series with *positive* terms and makes possible the analytic evaluation of the quantity (3) without using the integral representation (2)—we may get rid of the integral (the idea belongs to K. Ball, cf. [21], the proof of Lemma 4). However, other hypergeometric ingredients are required for real improvements.

We find quite curious that Ramanujan's formulae for π, in particular

$$\begin{aligned}\sum_{\nu=0}^{\infty}\frac{(1/4)_\nu(1/2)_\nu(3/4)_\nu}{\nu!^3}(21460\nu+1123)\cdot\frac{(-1)^\nu}{882^{2\nu+1}} &= \frac{4}{\pi},\\ \sum_{\nu=0}^{\infty}\frac{(1/4)_\nu(1/2)_\nu(3/4)_\nu}{\nu!^3}(26390\nu+1103)\cdot\frac{1}{99^{4\nu+2}} &= \frac{1}{2\pi\sqrt{2}}\end{aligned} \tag{25}$$

(see [14], equations (39) and (44)) and several others, might be used for constructing good rational approximations to π and $\pi\sqrt{d}$, where d is a positive integer. Namely, one can expect reasonable estimates for the corresponding irrationality measures by constructing explicit Padé approximations (of either first or second type) to the functional system 1, $f(z)$, $f'(z)$, $f''(z)$, where

$$f(z)=\sum_{\nu=0}^{\infty}\frac{(1/4)_\nu(1/2)_\nu(3/4)_\nu}{\nu!^3}z^\nu = {}_3F_2\left(\begin{matrix}\frac14,\,\frac12,\,\frac34\\ 1,\,1\end{matrix}\,\middle|\, z\right).$$

The paper [13] provides Padé approximations to the homogeneous system $f(z)$, $f'(z)$, $f''(z)$ (without 1) that are not enough for our purposes. Finally, we should mention that a general result of A. Galochkin in [8] (proved by a proper variation of Siegel's method) yields the qualitative linear independence of the numbers 1, $f(1/b)$, $f'(1/b)$, and $f''(1/b)$ for integers b satisfying $|b|>b_0$, where the value of b_0 is so huge that $b=-882^2$ and $b=99^4$ in (25) do not suit.

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References

[1] F. Amoroso, "Sur le diamètre transfini entier d'un intervalle réel," Ann. Inst. Fourier (Grenoble) **40,** no. 4, 885–911 (1990)

[2] F. Amoroso, "f-transfinite diameter and number theoretic applications," Ann. Inst. Fourier (Grenoble) **43,** no. 4, 1179–1198 (1993)

[3] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications **71** (Cambridge University Press, Cambridge 1999)

[4] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Math. Tracts **32** (Cambridge University Press, Cambridge 1935); 2nd reprinted edition (Stechert-Hafner, New York 1964)

[5] F. Beukers, "A rational approach to π," Notes of a lecture held on the occasion of Pi-day, on July 5, 2000 in Leiden, Nieuw Archief voor Wiskunde, no. 4, 17 pages (2000)

[6] N. Brisebarre, "Irrationality measures of $\log 2$ and $\pi/\sqrt{3}$," Experiment. Math. **10,** no. 1, 35–52 (2001)

[7] G. V. Chudnovsky, "On the method of Thue–Siegel," Ann. of Math. (2) **117,** no. 2, 325–382 (1983)

[8] A. I. Galochkin, "Lower bounds for linear forms of values of G-functions," Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] **51,** no. 3, 23–29 (1996)

[9] J. Guillera, "Some closely related to Ramanujan formulas for π," Unpublished manuscript, 8 pages (December 2003)

[10] M. Hata, "Legendre type polynomials and irrationality measures," J. Reine Angew. Math. **407,** no. 1, 99–125 (1990)

[11] M. Hata, "Rational approximations to π and some other numbers," Acta Arith. **63,** no. 4, 335–349 (1993)

[12] M. Huttner, "On linear independence measures of some abelian integrals," Kyushu J. Math. **57,** no. 1, 129–157 (2003)

[13] Yu. V. Nesterenko, "Padé–Hermite approximants of generalized hypergeometric functions," Mat. Sb. [Russian Acad. Sci. Sb. Math.] **185,** no. 10, 39–72 (1994)

[14] S. Ramanujan, "Modular equations and approximations to π," Quart. J. Math. Oxford Ser. 2 **45,** 350–372 (1914); *Collected Papers of Srinivasa Ramanujan*, eds. G. H. Hardy, P. V. Sechu Aiyar and B. M. Wilson, 23–39 (Cambridge University Press, Cambridge 1927); 2nd reprinted edition (Chelsea Publ., New York 1962)

[15] G. Rhin, "Sur l'approximation diophantienne simultanée de deux logarithmes de nombres rationnels," *Diophantine approximations and transcendental numbers* (Luminy 1982), Progress in Math. **31,** 247–258 (Birkhäuser, Boston 1983)

[16] G. Rhin, "Approximants de Padé et mesures effectives d'irrationalité," *Séminaire de Théorie des Nombres* (Paris 1985–86), ed. C. Goldstein, Progress in Math. **71,** 155–164 (Birkhäuser, Boston 1987)

[17] G. Rhin and P. Toffin, "Approximants de Padé simultanés de logarithmes," J. Number Theory **24,** no. 3, 284–297 (1986)

[18] E. A. Rukhadze, "A lower bound for the approximation of ln 2 by rational numbers," Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] **42,** no. 6, 25–29 (1987)

[19] C. Viola, "Hypergeometric functions and irrationality measures," *Analytic Number Theory* ed. Y. Motohashi, London Math. Soc. Lecture Note Ser. **247,** 353–360 (Cambridge University Press, Cambridge 1997)

[20] Q. Wu, "On the linear independence measure of logarithms of rational numbers," Math. Comput. **72,** no. 242, 901–911 (2002)

[21] W. Zudilin, "An elementary proof of Apéry's theorem," E-print `math.NT/0202159`, 8 pages (2002)

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