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SINGLE-DEGREE-OF-FREEDOM RIGIDLY FOLDABLE ORIGAMI FLASHERS

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ABSTRACT

We present the design for a family of deployable structures based on the origami flasher that are rigidly foldable, i.e., foldable with revolute joints at the hinges and planar rigid faces, and that exhibit a single degree of freedom in their motion. These structures may be used to realize highly compact deployable mechanisms.

1 Introduction

Many patterns from the world of origami have application in the world of engineering, particularly in the area of deployable structures. Whenever there is a need for a mechanism to transform between a large, flat, sheet-like state (the "deployed" state), and a much smaller state ("stowed"), origami-based mechanisms can provide efficient solutions.

One origami mechanism that has received considerable attention over the years is the pattern called a "flasher," which was introduced and explored in the origami world by Palmer and Shafer [1, pp. 149–151]; they developed their concept from the twist-fold forms of Toshikazu Kawasaki (see, e.g., [2]). However, the concept had a long existence outside of the origami world. As noted by Guest and Pellegrino (see [3] and references therein), several authors have discovered and explored similar structures since the early 1960s and it is regularly rediscovered. Nojima, for example, [4] shows a variety of similar forms with varying degrees of helicity and rotational symmetry. For purposes of this work, we will refer to all of these patterns that are rotationally symmetric, roughly flat in the deployed state, roughly cylindrical in the stowed state, and deploy in a spiral pattern, as (generalized) *flashers*.

Most work has focused on the use of the flasher mechanism in concert with membranes, or at least, structures that have distributed flexibility [5, 6]. While many flashers have been demonstrated from relatively stiff materials, the basic flasher mechanism and, to our knowledge, all versions proposed and demonstrated to date are not *rigidly foldable*; they cannot be folded continuously from the stowed to the deployed state with rigid panels and pure revolute joints.

To get around this problem, several alternatives have been proposed and/or demonstrated that work with rigid panels. Guest and Pellegrino have proposed a structure composed of separate panels joined by struts [7]. Zirbel et al. demonstrated a prototype solar array using a flasher structure with rigid panels and finite-width membrane hinges between the panels, in which the flexible membrane hinges provide the necessary additional compliance needed for deployment [8], albeit at the cost of introducing potentially undesirable additional degrees of freedom into the motion.

An additional challenge with using idealized flasher patterns in real-world engineering is the problem of thickness: idealized patterns assume zero (or negligible) thickness, but in real-world applications, the thickness of each panel is usually non-negligible. Finite thickness matters in two ways. First, it affects *metric foldability*: offsets and displacements of hinges from their idealized zero-thickness positions can affect the mechanics of folding or even prevent folding by turning a flexible mechanism into a locked structure. Second, it affects *injectivity*, or self-intersection: the layers of a thick structure can collide with each other even if the corresponding zero-thickness model does not self-intersect. Fortunately, recent work by Tachi [9] and Edmonson et al. [10] have demonstrated effective techniques for adapting zero-thickness structures with non-negligible thickness panels, and such techniques are applicable here.

Self-intersection aside, the problem of metric foldability remains. In particular, for applications with rigid panels, it would be desirable to have a folding pattern that is rigidly foldable. Even better, it would be desirable for the folding motion to have a single degree of freedom (DOF), so that there is one and only one path between the stowed and deployed states.

In this paper, we propose, describe, and analyze a member of the flasher family of mechanisms that meets both criteria: it is rigidly foldable with planar panels and pure revolute joints and transforms continuously from a fully flat state to a compact, cylindrical configuration with a single DOF. Like another well-known deployable structure, the Miura-ori [11], it is overconstrained according to the Kutzbach criteria, but by careful choice of angles in the design, we can realize a single DOF in the deployment motion. Furthermore, there is a tuning parameter for the spacing between layers in the stowed state, so that nonzero-thickness panels of varying thickness may be accommodated in the folding pattern. Thus, the thickness-accommodating techniques of Tachi and/or Edmonson may be applied to realize a full thick-panel mechanism while preserving the single-DOF deployment motion.

2 Preliminaries

Figures 1 and 2 show implementations of the flasher structure by Scheel [12] and Palmer/Shafer [1] that illustrate the fundamental structure. There is a central planar region (henceforth, the *central polygon*), surrounded by a series of mountain and valley folds that emanate roughly radially, but are offset somewhat from being center-directed.

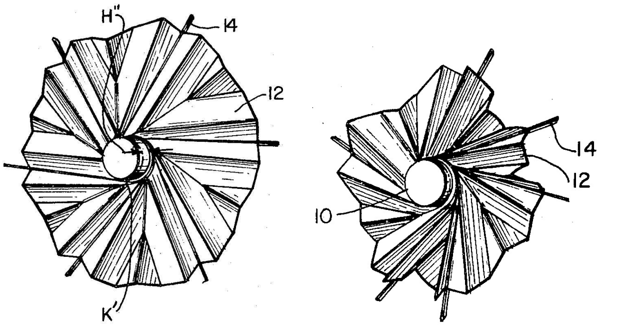

FIGURE 1: Scheel's wind-up membrane. Left: nearly open. Right: starting to close. From [12].

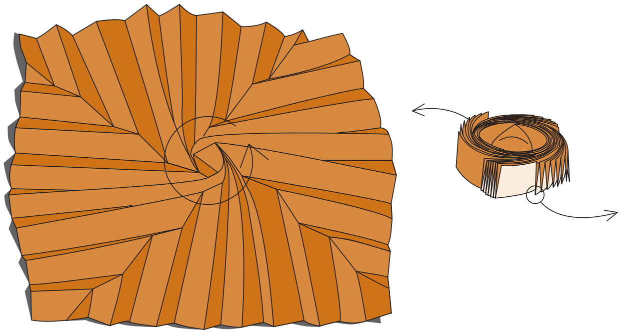

FIGURE 2: The Palmer-Shafer origami flasher. Left: nearly fully deployed. Right: stowed. From [1].

These images show curved and bent facets, but it is possible to create polyhedral (planar facet) versions of the flasher [3, 8]. Figure 3 shows three versions of a polyhedral flasher with a square central polygon and fourfold rotational symmetry, constructed according to the algorithm of [8].

There are four distinct types of folds in a flasher, illustrated in Figure 4. The *diagonal folds* emanate from the corners of the central polygon and, in the folded form, propagate helically around the axis of symmetry. Next, there are *reverse* folds, which propagate axially around the structure, each fold forming a spiral that lies (nearly) in the same plane. Both reverse and diagonal folds are sharp folds, being folded nearly to $\pm\pi$. Next, there are the *bend* folds, which are (nearly) axis-parallel folds at the corners of the central polygon; they are the folds used for the layers to wrap around the central polygon. And last, there are the *central polygon* folds, which appear to be continuations of the reverse folds but, unlike the latter, have a fold angle of about $\pi/2$, rather than $\pm\pi$.

For a flasher to be rigidly foldable with a single DOF, the fold angles around each vertex must flex continously in such a way that the fold angles along each fold are compatible when they meet up at every vertex. In general with flashers, this is not possible; for all of the patterns shown to date, at the very least, facets must bend along their diagonals. A general flasher pattern consists of degree-6, degree-5, and degree-4 vertices, with the last usually being the most numerous. Degree-4 vertices individually have a single DOF in their motion, and it will be these vertices that are the key to realizing single-DOF motion for a flasher. We now review briefly the important and relevant properties of degree-4 vertices.

3 Degree-4 Vertices

Figure 5 shows a generic degree-4 vertex, with four *sector angles* $\alpha_i, i = 1, \ldots, 4$, and four *dihedral angles*, $\gamma_i, i = 1, \ldots, 4$, with $\gamma_i \in [-\pi, \pi]$. For mountain folds, $\gamma_i < 0$; for valley folds,

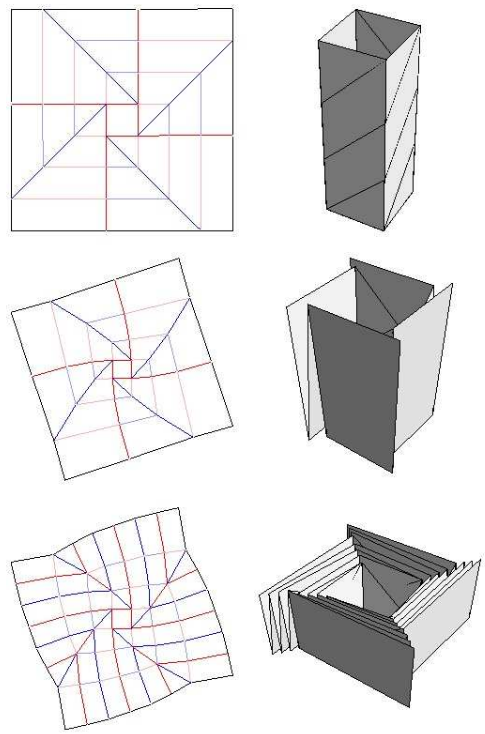

FIGURE 3: A polyhedral flasher with fourfold rotational symmetry. Top row: an ideal polyhedral flasher (left: crease pattern; right: folded form). Middle row: the same structure, modified to spread the layers to accommodate nonzero thickness. Bottom row: the same structure, but with additional reverse folds added to reduce the height. Note that the crease patterns and folded forms are shown at different scales; the diameter of the folded form is approximately the diameter of the central polygon of the crease pattern in each case.

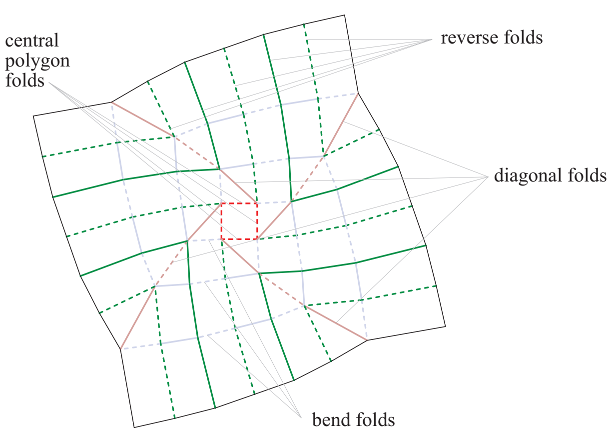

FIGURE 4: The four distinct types of fold in a flasher. Here we use the origami convention of drawing mountain folds as solid lines, valleys as dashed, with different tones for the four families of fold.

$\gamma_i > 0$. If $|\gamma_i| = \pi$, the fold is *fully folded*; if $|\gamma_i| \in (0, \pm\pi)$, it is *partially folded*; and if $\gamma_i = 0$ it is *unfolded*.

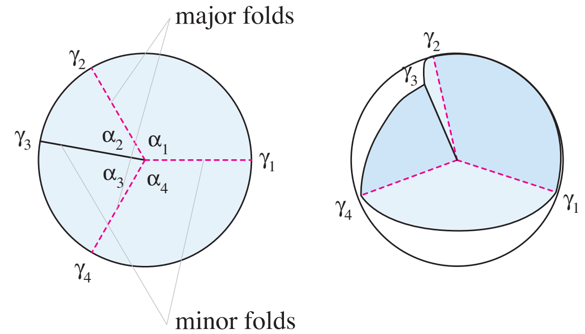

FIGURE 5: A degree-4 vertex. Left: crease pattern. Right: folded form.

If all four creases are partially or fully folded, then there must be three mountains and one valley or three valleys and one mountain. If we look at alternating (not consecutive) folds around the vertex, two will be of the same type and the other two will be of opposite type. We call the two folds of the same type (γ_2 and γ_4 in Figure 5) the *major folds* (or major pair) of the vertex. The two folds of opposite type (γ_1 and γ_3 in Figure 5) are the *minor folds* of the vertex.

We also recognize two special cases: if $\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 = \pi$, the vertex is *straight-major*, because the major folds are collinear. Similarly, if $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = \pi$, the vertex is *straight-minor*. Straight-minor vertices are perfectly well behaved, but straight-major vertices are a special case; it is not possible for all four folds of a straight-major vertex to be partially folded at the same time. In order for a straight-major vertex to fold, first, γ_2 and γ_4 must fold from 0 to $\pm\pi$, and only then can γ_1 and γ_3 fold (and then only if $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$).

For non-straight-major vertices, if one fold angle is chosen, the other three are fully determined from trigonometric relationships between the sector angles and the fold angles. For the generic case, these relationships can be rather complex (as we show in the Appendix). However, for a flat-foldable vertex—one that can be pressed flat with all layers in a common plane—the relations collapse to simple forms.

First, as is well known from Kawasaki's Theorem [13], a degree-4 vertex is flat-foldable if and only if

$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \pi. \tag{1}$$

For a flat-foldable vertex, the major fold angles are equal [14]:

$$\gamma_2 = \gamma_4, \tag{2}$$

and the minor fold angles are equal and opposite [14]:

$$\gamma_1 = -\gamma_3. \tag{3}$$

The relationship between adjacent fold angles has been described by several authors [14–16]; a particularly simple and useful expression (derived in Appendix A) is

$$\frac{\tan\frac{1}{2}\gamma_2}{\tan\frac{1}{2}\gamma_1} = -\frac{\tan\frac{1}{2}\gamma_2}{\tan\frac{1}{2}\gamma_3} = \frac{\tan\frac{1}{2}\gamma_4}{\tan\frac{1}{2}\gamma_1} = -\frac{\tan\frac{1}{2}\gamma_4}{\tan\frac{1}{2}\gamma_3} = \frac{\sin\frac{1}{2}(\alpha_1+\alpha_2)}{\sin\frac{1}{2}(\alpha_1-\alpha_2)}. \tag{4}$$

Equation (4) also gives some justification for the names “major” and “minor”; a consequence of Equation (4) is that

$$|\gamma_{2,4}| \geq |\gamma_{1,3}| \tag{5}$$

with strict inequality at all partially folded angles and equality only at the flat (fully unfolded and fully folded) states.

We define the ratio in Equation (4) as the *fold angle multiplier* μ for the vertex. In general, $\mu > 1$. The fold angle multiplier is a measure of the *geometric advantage* between a major and minor fold of the vertex. If we denote either major fold angle by γ_+ and either minor fold angle by γ_-, we have that

$$\lim_{\gamma_-\to 0}\frac{d\gamma_+}{d\gamma_-} = \pm\mu,\ \lim_{\gamma_-\to\pi}\frac{d\gamma_+}{d\gamma_-} = \pm 1/\mu,\ \text{and}$$
$$\left|\frac{d\gamma_+}{d\gamma_-}\right| \in (1/\mu,\mu)\ \text{at angles in between.} \tag{6}$$

There is a remarkable property implicit in Equations (2)–(4), noted by Tachi [16]. For any two angles γ_i, γ_j at a flat-foldable degree-4 vertex, $\tan\frac{1}{2}\gamma_i/\tan\frac{1}{2}\gamma_j = constant$, independent of the state of foldedness. This property extends to any mesh of degree-4 vertices: if γ_i and γ_j are connected by a path containing only non-straight-major flat-foldable degree-4 vertices, their half-angle-tangents are proportion by some fixed value that depends on the sector angles around all of the vertices along the path. Because of this constant of proportionality, it is guaranteed that γ_i, γ_j, and all folds in between can fold smoothly all the way from flat to fully folded, at least, if we ignore all vertices outside the path. And, more broadly, for a mesh whose interior vertices are all flat-foldable degree-4 vertices, if we can find a partially-folded state involving all folds for a single fold angle, then it is guaranteed that the entire pattern folds smoothly over the full range of fold angles from fully flat to fully folded, with a single DOF.

This property is extremely powerful and useful, and we can employ it when we are seeking single-DOF folding mechanisms. If we can construct a fold pattern consisting of flat-foldable degree-4 vertices and find a single consistent partially folded state, then we have a single-DOF mechanism. In principle, such a pattern could fold from flat unfolded to flat fully folded. In practice, self-intersection may limit the full range of motion, but even if we don’t need full flat-foldability, we can still use this technique to achieve single-DOF mechanisms. And we will now do this with flashers.

4 Simple Flashers

We now turn our attention back to the flasher. For simplicity, we will consider first a flasher that has no reverse folds at all, such as the one illustrated in Figure 6. There are only diagonal, bend, and central polygon folds. This choice ensures that all of the interior vertices of the pattern are degree-4.

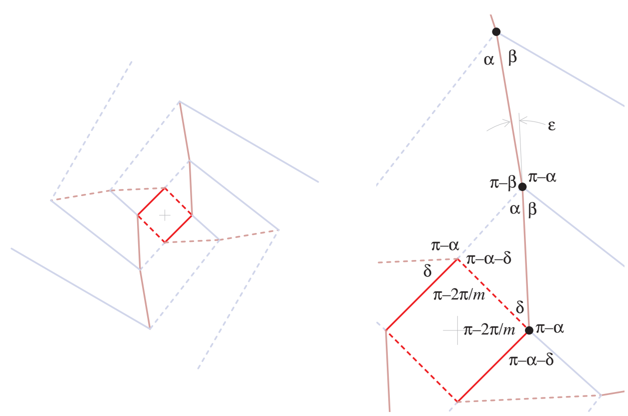

FIGURE 6: A simplified flasher, containing only diagonal, bend, and central polygon folds. Left: the full crease pattern. Right: a close-up, with labeled sector angles.

We will first look at the constraints on the angles in the pattern, assuming m-fold rotational symmetry on the positions of the fold lines, m-fold rotational symmetry on the magnitudes of the fold angles, and $m/2$-fold rotational symmetry on the sign of the fold angles (mountain/valley assignment). We also assume $m \geq 3$. The central polygon is a regular m-gon, and so the interior angles at its corners are $(\pi - 2\pi/m)$. We take δ to be the angle between the incident diagonal fold and the side of the central polygon. In Figure 6, we have $m = 4$ and the central polygon is a square.

Moving out along the diagonal folds, we have a sequence of degree-4 vertices. Denote the two angles to left and right of the diagonal fold as α and β, respectively. If we are going to attempt to create a single-DOF mechanism using the property described in the previous section, then this vertex must be flat-foldable, with opposite angles summing to π. This lets us complete the sector angle assignment around this vertex—and, as well, around all of the other degree-4 vertices. Since, according to Equations (2) and (3), the major and minor folds around each of the degree-4 vertices are equal, this means that the fold angles of all of the diagonal folds are equal to one another in magnitude, and the fold angles of all of the bend folds are equal to one another in magnitude. We denote the magnitude of the diagonal fold angles by γ_{diag} and the magnitude of the bend fold angles by γ_{bend}.

All of the degree-4 vertices are similar to one another (similar in the geometric sense), and because of that, the diagonal and bend fold angles at each degree-4 vertex are related to each other by Equation (4). So, up to now, the sector angles and fold angles are consistent with one another at every degree-4 vertex. Since each degree-4 vertex is a single-DOF mechanism, the entire array of degree-4 vertices (apart from those of the central polygon) must be, itself, a single-DOF mechanism—if, that is, it is not locked by other interactions.

We have not yet considered the degree-4 vertices around the central polygon. If we force each central polygon vertex to be developable (its sector angles sum to 2π), so that the crease pattern is a flat sheet, that condition lets us solve for α:

$$\alpha = \frac{\pi}{2} - \frac{\pi}{m}. \tag{7}$$

While α is given by Equation (7), angles β and δ may be chosen independently. But there is not complete freedom to choose. Looking at one of the vertices along the diagonal fold, if we choose $\beta = \alpha$, then the major folds at that vertex become collinear, resulting in the straight-major condition. As noted above, in a straight-major vertex, the major and minor folds become uncoupled; such a vertex must fold entirely from flat to fully folded along the major crease before the minor crease can fold at all (if even possible). So $\beta = \alpha$ is forbidden if we want a single-DOF mechanism where all the folds happen together.

It is convenient to introduce a new angle, ε, as the deviation from straightness of the major folds, as illustrated in Figure 6. We then have

$$\beta = \alpha + \varepsilon. \tag{8}$$

And because we now know the sector angles around these vertices, we can compute the fold angle multiplier between the major folds (diagonal folds) and minor folds (bend folds) at each vertex. We denote this fold angle multipler by μ_{db}. It is given by

$$\mu_{db} = \cos\alpha + \cot\tfrac{\varepsilon}{2}\sin\alpha. \tag{9}$$

If we choose any bend fold angle γ_{bend}, then every other bend fold has the fold angle $\pm\gamma_{bend}$, with the sign depending on its mountain/valley assignment, and every diagonal fold has the fold angle

$$\pm\gamma_{diag} = \pm 2\tan^{-1}\left(\mu_{db}\tan\tfrac{1}{2}\gamma_{bend}\right), \tag{10}$$

with, again, the sign determined from the mountain/valley assignment.

Now we consider fold angles around the central polygon, and here a problem arises. Consider the black-dotted central polygon vertex in Figure 6. It is clearly not flat-foldable, because opposite angles sum to $(\pi/2+\pi/m)$, not π. It is still a single-DOF mechanism, and so if we choose a generic value of the diagonal fold angle, γ_{diag}, we can compute the fold angles of the two incident central polygon folds (which will, in general, be different from one another in magnitude, as well as sign).

Let us denote the dihedral angle of the valley fold of the central polygon by $\gamma_{cp,v}$ and that of the mountain fold by $\gamma_{cp,m}$. Using the general expressions for opposite and adjacent dihedral angles from Appendix A and the angles from Figure 6, we can solve for both $\gamma_{cp,v}$ and $\gamma_{cp,m}$ in terms of γ_{diag}. The expressions are both rather large, and we omit them for brevity, but the important thing is this: they are quite *different*.

The problem is that if we move to the next central polygon vertex and compute the fold angles for the two incident central polygon folds, we will get the same two values, but with opposite sign. Consistency from one vertex to the next therefore requires that

$$\gamma_{cp,m} = -\gamma_{cp,v}, \tag{11}$$

and this is not the case; they have fundamentally different functional dependence upon γ_{diag}, no matter what values of α and δ might be chosen. So *it is not possible to find an assignment of fold angles around the central polygon consistent with the single-DOF mechanism surrounding it.*

Well, then: how about if we simply cut out the central polygon entirely, so that its edges now become edges of the fold pattern? Then there would no longer be a consistency condition on the fold angles around each vertex of the central polygon because there are no central polygon fold angles to contend with.

But there is still a consistency condition to consider. If we cut out the central polygon, at each of its vertices, we have three fixed sector angles and two fold angles whose values are linked by the single-DOF mechanism, and so we can solve for the angle between two adjacent sides of the central polygon—what would have been the interior angle of the central polygon.

We compute this angle by making use of 3D rotation matrices. Define the usual rotation matrices about the x, y, and z axes as

$$\begin{aligned}
\mathbf{R_x}(\phi) &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \\
\mathbf{R_y}(\phi) &\equiv \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \\
\mathbf{R_z}(\phi) &\equiv \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{12}$$

Left-multiplying a vector by matrix $\mathbf{R_x}(\phi)$ rotates the vector through angle ϕ about the global x-axis; similarly for $\mathbf{R_y}(\phi)$ and $\mathbf{R_z}(\phi)$. If we have a local coordinate system defined by a 3×3 matrix, that we can rotate that coordinate system about its own local axes by *right-multiplying* by the transpose of these same matrices. We set up a coordinate system centered on the central polygon vertex whose local x-axis runs along $\gamma_{cp,m}$ and whose local x-y plane contains folds $\gamma_{cp,m}$ and γ_{bend} and describe this coordinate system by some matrix $\mathbf{I}$. Then we can find the direction vector for fold $\gamma_{cp,v}$ by successively rotating the coordinate system about the local z axis for each sector angle and about the local x axis for each dihedral angle as we work our way around the the vertex. The transformed coordinate system is thus given by

$$\begin{aligned}
\mathbf{I}' = \mathbf{I} \cdot \mathbf{R_z}^T(\pi - \alpha - \delta) \cdot \mathbf{R_x}^T(\gamma_{bend}) \cdot \\
\mathbf{R_z}^T(\pi - \alpha) \cdot \mathbf{R_x}^T(\gamma_{diag}) \cdot \mathbf{R_z}^T(\delta).
\end{aligned} \tag{13}$$

If the first component of $\mathbf{I}$ was the direction vector of fold $\gamma_{cp,m}$, then the first component of $\mathbf{I}'$ should be the direction vector of fold $\gamma_{cp,v}$, and their dot product must be the cosine of the angle between the two folds in 3D. We denote that angle by α_{3D}. Its angle cosine is:

$$\begin{aligned}
\cos\alpha_{3D} = \cos\alpha\,(\cos\delta\cos(\alpha+\delta) + \\
\sin\delta\sin(\alpha+\delta)\cos\gamma_{bend}\cos\gamma_{diag}) \\
+ \sin(\alpha+\delta)\left(\sin\delta\sin\gamma_{bend}\sin\gamma_{diag} - \sin\alpha\cos\delta\cos\gamma_{bend}\right) \\
+ \sin\alpha\sin\delta\cos(\alpha+\delta)\cos\gamma_{diag}.
\end{aligned} \tag{14}$$

Unfortunately, this quantity *varies* as the rest of the mechanism changes its folded state (i.e., as γ_{bend} and γ_{diag} vary, which they do together). In the flat, unfolded state, the edges of the central polygon form a closed polygon. But, it turns out, for nearly all other partially folded states, the corner angles change, and so the central polygon no longer closes up.

And so, there is no consistent assignment of sector and fold angles that makes this pattern an isometric mechanism. We're almost there: we can achieve consistency, isometry, and single-DOF motion at every other interior vertex, but the fold pattern fails when we require consistency going around the loop of the central polygon.

The solution, therefore, is to break the loop; we cut the pattern from the outside edge in to the center, so that there is no longer a loop condition around the central polygon to be satisfied. The cut path can be anywhere from the outside in, but for simplicity, we will cut along one of the diagonal folds, as illustrated in Figure 7, and then remove the central polygon.

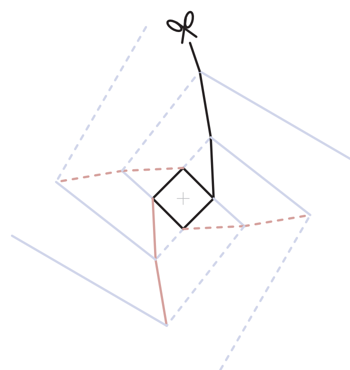

FIGURE 7: Cut lines on the simple flasher. We cut out the central polygon and cut in from the edges along a diagonal fold.

And that does the trick! We now have a pattern consisting entirely of degree-4 vertices in which the fold angles and sector angles at each interior vertex are mutually consistent at every folded state, from unfolded to fully flat (or as close to fully flat as we can get without self-intersection).

We note that the idea of cutting a flasher from edge to center is not without precedent. Tibbalds and Pellegrino introduced the notion of cutting multiple panels of a disk-like form apart in order to produce a single-DOF mechanism [17] at the cost of introducing multiple sets of struts. If we create a single-DOF flasher, though, then we only need to create a single cut, and we end up with a single connected single-DOF mechanism.

So what does this pattern look like when folded? For that, we need to compute the vertex coordinates in 2D and 3D.

We denote each vertex of the crease pattern by $\mathbf{p}_{i,j}$, where $i = 0, \ldots$ indexes the vertices heading out along the diagonal folds from the central polygon, and $j = 0,\ldots,m$ denotes the rotational position around the origin, as illustrated in Figure 8. Rotational indices "wrap around": $\mathbf{p}_{i,m} = \mathbf{p}_{i,0}$, and so forth.

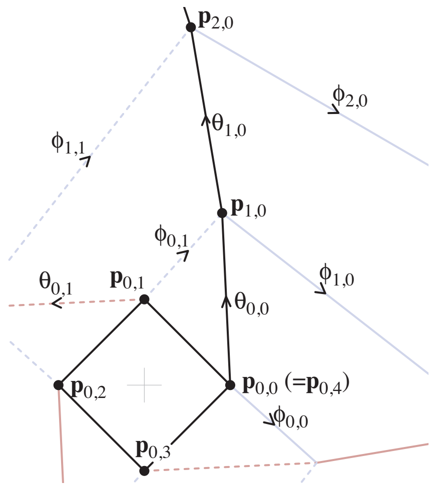

FIGURE 8: Vertex and angle indexing in the simple flasher.

We assume for simplicity that $\mathbf{p}_{0,0} = (1,0)$. Define the 2D rotational matrix

$$\mathbf{R}_m(k) \equiv \begin{pmatrix} \cos\frac{2\pi k}{m} & -\sin\frac{2\pi k}{m} \\ \sin\frac{2\pi k}{m} & \cos\frac{2\pi k}{m} \end{pmatrix}. \tag{15}$$

Then we have that

$$\mathbf{p}_{i,j} = \mathbf{R}_m(j)\cdot\mathbf{p}_{i,0}. \tag{16}$$

We further define angles $\phi_{i,j}$ and $\theta_{i,j}$ as the absolute angles (measured in a global coordinate system as a rotation from the x axis) of the fold lines emanating from $\mathbf{p}_{i,j}$, as illustrated in Figure 8. From consideration of the angles in Figure 6, we have that

$$\theta_{i,j} = (\pi-\delta-\alpha)+i\varepsilon+2\pi j/m, \tag{17}$$
$$\phi_{i,j} = (-\delta)+i\varepsilon+2\pi j/m. \tag{18}$$

Now we can compute the position of $\mathbf{p}_{i,j}$ for $i>0$ as the intersection of lines emanating from lower-i vertices. We introduce the vector-valued function

$$\mathbf{u}(\xi) \equiv (\cos\xi, \sin\xi), \tag{19}$$

the matrix determinant

$$\det(\mathbf{x},\mathbf{y}) \equiv \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - y_1x_2, \tag{20}$$

and the line intersection function $\textsc{lineint}(\mathbf{a}_1,\mathbf{d}_1,\mathbf{a}_2,\mathbf{d}_2)$ that returns the intersection between two lines emanating from points $\mathbf{a}_1$ and $\mathbf{a}_2$ with direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, given by

$$\textsc{lineint}(\mathbf{a}_1,\mathbf{d}_1,\mathbf{a}_2,\mathbf{d}_2) = \mathbf{a}_1 + \mathbf{d}_1\frac{\det((\mathbf{a}_2-\mathbf{a}_1),\mathbf{d}_2)}{\det(\mathbf{d}_1,\mathbf{d}_2)}. \tag{21}$$

Then

$$\mathbf{p}_{i+1,0} = \textsc{lineint}(\mathbf{p}_{i,0},\mathbf{u}(\theta_{i,0}),\mathbf{p}_{i,1},\mathbf{u}(\phi_{i,1})). \tag{22}$$

This relation, plus Equation (16), allows us to recursively compute all of the points of the crease pattern, for as far out as we wish to go.

An open question is what to do for the outer boundary of the pattern. For simplicity, we have chosen to simply close the pattern by connecting the points $\mathbf{p}_{n,j}$ for some value n.

What about the folded form? Since we know the crease pattern, and we know the angle of all of the folds, we can compute the folded form by rotating the facets of the crease pattern relative to their neighbors about the known fold angles of their shared creases. This is done efficiently by constructing a spanning tree on the facets linked by their adjacency, then traversing the tree and composing rotations along the way.

Using these formulas, we have computed representative examples of single-DOF flasher mechanisms. An example is shown in Figure 9 for parameters $m=4$, $n=3$, $\varepsilon=3°$, $\delta=43.5°$, and γ_{bend} taking on various values from 0 (fully flat) to (nearly) fully folded.

The 3D plots verify the impossibility of achieving a closed central polygon. As can be seen in the first few images, the central polygon opens up quite widely, then (in the middle image) it begins to curl up on itself. The maximum value of γ_{bend} in the figure was chosen to be the point at which the central polygon closes back up on itself—and, not incidentally, the originally cut edges of the ring meet up with each other once more.

It is possible to take the pattern all the way to a bend fold angle of $\gamma_{bend} = 180°$, i.e., fully flatly folded, at which point the entire pattern lies in a common plane. However, to get there, the panels of the pattern must intersect each other in numerous places and ways. No practical application would take this structure all the way to flatly folded (at least none that we can envision at present). The "stowed" state for this structure stops when the

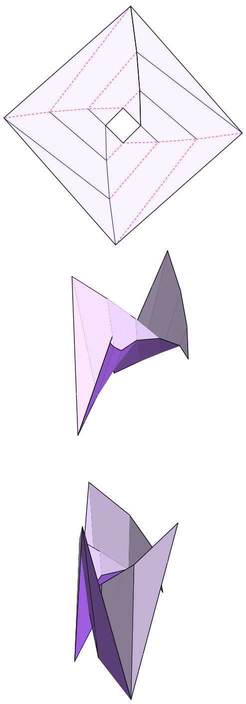

FIGURE 9: A single-DOF flasher for various values of γ_{bend}. From upper left to lower right: $\gamma_{bend} = 0°, 3°, 5°, 10°, 20°, 30°, 40°, 65°, 87°$. Note that the scale varies from one subfigure to the next.

central polygon re-closes and the panels have some small dihedral angle between them. Such a model may be used in a thickening algorithm, such as that of Tachi [9], which requires slightly angularly separated panels.

There are two noticeably different motions in going from the flat to the curled-up state. First, for small γ_{bend}, most of the motion happens on the diagonal folds, and as they fold up, the flat disc forms into a slightly curved vertical stack of layers. There then comes a point where the diagonal folds are mostly folded, and the remaining motion comes from the bend folds curling the layers around until they meet up again.

In general, the vertices of the central polygon are non-planar in the partially folded state. However we have found that a particular value of δ gives rise to a central polygon that is planar through the full range of motion:

$$\delta_{planar} = \frac{\pi}{m} - \frac{\varepsilon}{2}. \tag{23}$$

We chose $\delta = \delta_{planar}$ in Figure 9.

As noted already, as the mechanism proceeds away from the flattened state, the central polygon opens up, and then it recloses. The "fully stowed" state would be that where the central polygon has closed to its original state and the edges of the cut have come back together. We would like to know what bend angle γ_{bend} gives rise to this state: this would define the full range of useful motion of the mechanism.

Recall that Equation (14) gave the cosine of the angle between folds $\gamma_{cp,m}$ and $\gamma_{cp,v}$, which was α_{3D}. The mechanism has re-closed when that angle takes on the value of the interior angle of the original planar central polygon, whose angle cosine is

$$\cos\alpha_0 = \cos(\pi - \frac{2\pi}{m}) = -\cos(\frac{2\pi}{m}). \tag{24}$$

So we can equate $\cos\alpha_{3D}$ from Equation (14) and $\cos\alpha_0$ from Equation (24) and solve for the bend angle (or equivalently, the diagonal angle) that satisfies the equality.

To keep the algebra tractable, we introduce the Weirstrauss substitution

$$x \equiv \tan\frac{\gamma_{bend}}{2}, \tag{25}$$

which gives rise to the following simplifying substitutions:

$$\begin{aligned} \sin\gamma_{bend} &= \frac{2x}{1+x^2}, \\ \cos\gamma_{bend} &= \frac{1-x^2}{1+x^2}, \\ \sin\gamma_{diag} &= \frac{2(\mu_{db}x)}{1+(\mu_{db}x)^2}, \\ \cos\gamma_{diag} &= \frac{1-(\mu_{db}x)^2}{1+(\mu_{db}x)^2}. \end{aligned} \tag{26}$$

Substituting these into Equation (14) along with taking $\delta = \delta_{planar} = \frac{\pi}{m} - \frac{\varepsilon}{2}$, $\mu_{db} = \cos\alpha + \cot\frac{\varepsilon}{2}\sin\alpha$, and $\alpha = \frac{\pi}{2} - \frac{\pi}{m}$, gives

$$\begin{aligned} &\cos\alpha_{3D} = \\ &\quad \left(x^2 \left[\left(x^2+1\right)\sin\left(\frac{2\pi}{m}\right)\sin(\varepsilon) + x^2 - \cos(\varepsilon) \right] + \right. \\ &\quad \left. \cos\left(\frac{2\pi}{m}\right)\left[-2x^2 + \left(x^4+x^2+1\right)\cos(\varepsilon) - 1\right] \right) / \\ &\quad \left(\left(x^2+1\right)\left[x^2\cos\left(\frac{2\pi}{m} - \varepsilon\right) + x^2 - \cos(\varepsilon) + 1 \right] \right). \end{aligned} \tag{27}$$

This angle cosine is parameterized on the variable x, which is the transformed version of γ_{bend}. We would like to know the value of x (and thus, by extension, γ_{bend}), that makes this value equal that of the planar central polygon. Setting the two values equal and solving for x gives two solutions: first, $x = 0$, which is the unfolded state. The second solution is the desired fully folded state. We find

$$x = \frac{\sec\left(\frac{\pi}{m}\right)}{\mu_{db}\sqrt{2}} \times \left[\mu_{db}^2\left(\cos\varepsilon - \cos\frac{2\pi}{m}\right) - 4\mu_{db}\cos\frac{\varepsilon}{2}\sin\left(\frac{\pi}{m} - \frac{\varepsilon}{2}\right) + \cos\left(\frac{2\pi}{m} - \varepsilon\right) - \cos\frac{2\pi}{m}\right]^{1/2}, \quad (28)$$

which sets the maximum bend angle to be

$$\gamma_{bend,max} = 2\tan^{-1}\left[\sec\left(\frac{\pi}{m} - \frac{\varepsilon}{2}\right)\sqrt{\sin\left(\frac{\pi}{m}\right)\sin\left(\frac{\pi}{m} - \varepsilon\right)}\right] \approx \frac{2\pi}{m} - \varepsilon - \frac{1}{4}\cot\left(\frac{\pi}{m}\right)\varepsilon^2 + O(\varepsilon^3). \quad (29)$$

As we fold the pattern beyond $\gamma_{bend,max}$ the pattern self-intersects. There is also a small amount of self-intersection even before closure. This can be seen in Figure 10, which is a view "looking down the barrel" of the tubular form just before the central polygon closes.

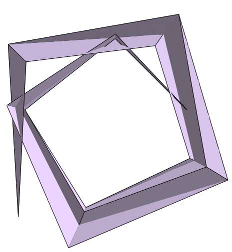

FIGURE 10: A view from the top of the flasher of Figure 9 for $\gamma_{bend} = 80°$, showing collisions with the inner layers.

So, this is potentially a problem with practical applications of this structure. However, this collision happens because of where we chose our cut, which results in one portion of the pattern wrapping around another as we approach the stowed state. In the stowed state, the cut edges re-align with one another. We can therefore begin with the curled-up state and make our cut somewhere else in a way that prevents this wrap-around issue. Two possible alternate locations for a cut are illustrated in Figure 11.

Judicious choice of cut line can potentially eliminate self-intersection throughout the full range of motion, from stowed to fully deployed. We will show one such choice and its effects in the next section.

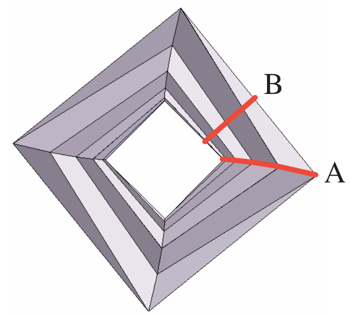

FIGURE 11: A top view of the flasher with $m = 4$, $n = 3$, $\varepsilon = 3°$, $\delta = 43.5°$, and $\gamma_{bend} = 87°$. Lines A and B indicate possible alternate cut lines.

Another issue getting in the way of practical application is the fact that the stowed form is long and tubular, which arises from the fact that this is the simplest possible flasher structure. That length can be reduced by choosing a larger rotational order, but that strategy gives small slivers of triangles near the inner rim, which are undesirable in applications.

Scheel [12] and subsequent investigators show that one can reduce the height of a flasher by incorporating what in the origami world are called *reverse folds* into the pattern; such were included in the constructive algorithms of Guest et al. [3] and Zirbel et al. [8]. We can incorporate such folds into this flasher while preserving the single-DOF motion, as we now show.

5 Reduced-Height Flashers

We now consider adding a pair of reverse folds that emanates from some point along one of the diagonal folds, as illustrated in Figure 12. We denote by η the angle that the reverse fold makes relative to the diagonal fold. In order to avoid disturbing the single-DOF mechanism that already exists, we will assume that the diagonal and bend folds are unchanged, except for the addition of vertices and selective inversion of the fold type (sign change of the fold angle). The name "reverse fold" comes from the origami world; such a set of folds inverts the parity of all of the diagonal and bend folds that lie within the V of the reverse fold.

What freedom do we have in the choice of η? If we are to leave the magnitude of the fold angles unchanged along the diagonal and bend folds, then all of the new vertices must be flat-foldable. That means that both sides of the base of the V (where the reverse folds hit the diagonal) must make the same angle η with the diagonal fold, as shown in Figure 12.

It also means that each of the vertices where a reverse fold meets a bend fold must be geometrically similar to one another, as well as flat-foldable. Considering what happens at two successive vertices along a reverse fold reveals that there is only one

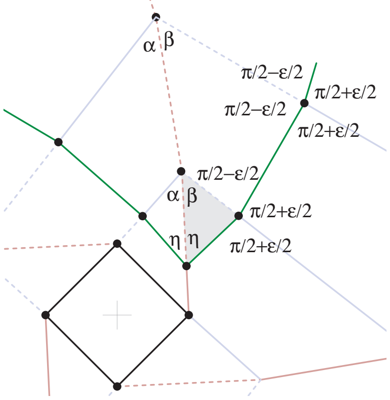

FIGURE 12: Crease pattern with the addition of a pair of reverse folds along one of the diagonals.

possible set of sector angles at those vertices that make them all geometrically similar, namely, sector angles of $(\pi/2-\varepsilon/2)$ and $(\pi/2+\varepsilon/2)$ (two of each), as illustrated in Figure 12.

With the four sector angles at each reverse/bend vertex assigned, the value of η is fully defined, and can be worked out from the interior angles of the shaded triangle in Figure 12. We find that

$$\eta = \frac{\pi}{2} - \alpha - \frac{\varepsilon}{2} = \frac{\pi}{m} - \frac{\varepsilon}{2} = \delta_{planar}. \tag{30}$$

This is a nice result; it tells us that if we choose $\delta = \delta_{planar}$ and place the tip of the reverse fold V along the first diagonal fold, the left side of the V will be parallel to the edge of the central polygon. If we place the tip of the reverse fold V on a vertex of the central polygon, then the left side will be coincident with the edge of the central polygon.

There is one thing still to check, however. We have three types of folds with distinct fold angle magnitudes: diagonal (γ_{diag}), bend (γ_{bend}), and now reverse (γ_{rvrs}). We also have three types of interior vertices that enforce proportionality between the half-angle-tangents of the fold angles at each vertex. Each vertex can be labeled by the two types of fold incident on the vertex: diagonal/bend (which we have already met, characterized by fold angle multiplier μ_{db}), and now two new ones: reverse/bend, which will be characterized by a fold angle multiplier μ_{rb}, and reverse/diagonal, with a fold angle multiplier μ_{rd}.

The shaded triangle in Figure 12 has one of each type of vertex. There is a self-consistency condition that must be satisfied going around this triangle. In particular, we must have:

$$\mu_{rb} = \mu_{rd}\mu_{db}. \tag{31}$$

For a general triangle of three flat-foldable vertices, this relationship is not guaranteed to hold. What about this particular case?

The value of μ_{db} was given by Equation (9). For the other two, we find that

$$\mu_{rb} = \csc\tfrac{\varepsilon}{2}, \tag{32}$$
$$\mu_{rd} = \csc(\alpha + \tfrac{\varepsilon}{2}). \tag{33}$$

Substituting these into Equation (31) reveals that the latter is satisfied for all values of α and ε. So, no matter what simple flasher we start with, we can add one or more reverse folds anywhere along the diagonal folds, and the resulting pattern is guaranteed to be a single-DOF mechanism. The spacing between successive reverse folds can be chosen arbitrarily. The farther apart successive reverse folds are placed, the taller the resulting mechanism.

When placing the reverse folds, one can think of each reverse fold as "sliding" along the diagonal, creating triangular and quadrilateral panels along each diagonal; each reverse fold breaks a diagonal fold into two segments whose lengths depend upon the position of the tip of the V of the reverse fold. There is a special case, where the tip of the V of the reverse fold coincides with an existing vertex along the diagonal. This choice gives rise to a particularly elegant crease pattern consisting of triangular facets along the diagonals and near-rectangular panels everywhere else.

It also creates degree-6 vertices along the diagonal, each created by the effective merging of two degree-4 vertices. This merging potentially increases the number of degrees of freedom of the mechanism—an issue we will come back to presently—but it does not alter the consistency between the values of γ_{bend}, γ_{diag}, and γ_{rvrs} given above.

We now compute 2D and 3D representations of this structure. We introduce triply-subscripted points for the vertices of the reverse folds, illustrated in Figure 13.

In each rotational section, we define $\mathbf{r}_{i,j,k}$ for vertices on the left side of the diagonal fold and $\mathbf{s}_{i,j,k}$ for vertices on the right side (as viewed from the central polygon). For vertices on the diagonal folds, we define both

$$\mathbf{r}_{i,j,0} = \mathbf{s}_{i,j,0} \equiv \mathbf{p}_{i,j}, \tag{34}$$

where $\{\mathbf{p}_{i,j}\}$ are the original vertices of the pattern as defined in the previous section. As we move out from $\mathbf{r}_{i,j,0}$ along a reverse fold, the k index of $\mathbf{r}_{i,j,k}$ increments each time we hit a bend fold, and similarly with $\mathbf{s}_{i,j,k}$.

We note that this gives multiple names to the same point; not only does $\mathbf{r}_{i,j,0} = \mathbf{s}_{i,j,0}$, but also $\mathbf{r}_{i,j,1} = \mathbf{r}_{i,j-1,0}$ and $\mathbf{s}_{0,j,k} = \mathbf{r}_{0,j-1,k+1}$. This requires a bit of care in the bookkeeping of distinct vertices, but otherwise causes no problems.

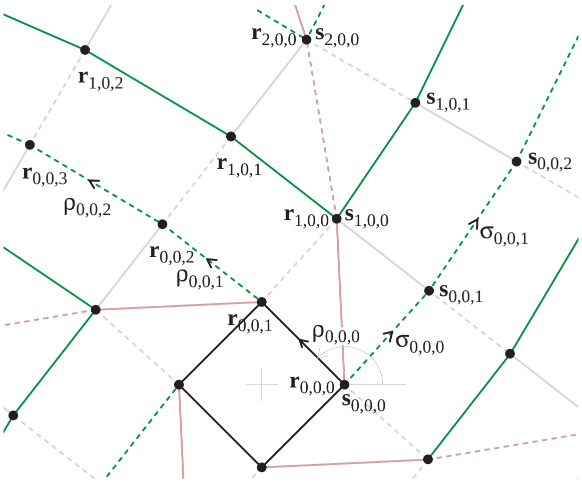

FIGURE 13: A portion of the crease pattern of a reverse-folded flasher, with reverse folds emanating from each of the diagonal vertices.

We further define $\rho_{i,j,k}$ as the absolute angle of the reverse fold emanating outward from $\mathbf{r}_{i,j,k}$, and $\sigma_{i,j,k}$ as the absolute angle of the reverse fold emanating outward from $\mathbf{s}_{i,j,k}$ (like $\theta_{i,j}$ and $\phi_{i,j}$, measured as a rotation relative to the x axis in a global coordinate system).

With these definitions, the vertex coordinates and angles are as follows.

$$\rho_{i,j,k} = \theta_{i,j} + \eta + k\varepsilon, \tag{35}$$

$$\sigma_{i,j,k} = \theta_{i,j} - \eta + k\varepsilon, \tag{36}$$

$$\mathbf{r}_{i,j,k+1} = \text{LINEINT}(\mathbf{r}_{i,j,k}, \mathbf{u}(\rho_{i,j,k}), \mathbf{p}_{i+k,j+1}, \mathbf{u}(\phi_{i+k,j+1})), \tag{37}$$

$$\mathbf{s}_{i,j,k+1} = \text{LINEINT}(\mathbf{s}_{i,j,k}, \mathbf{u}(\sigma_{i,j,k}), \mathbf{p}_{i+k+1,j}, \mathbf{u}(\phi_{i+k+1,j})), \tag{38}$$

where $\theta_{i,j}$ and $\phi_{i,j}$ were given by Equation (17)

The outermost vertices are a special case, and their position depends on how we choose to finish the pattern. For the tubular flasher, we simply connected points $\{p_{n,j}\}$ for some n. Doing that with this reverse-folded case will slice some of the outermost panels, giving trapezoidal and/or triangular facets. For simplicity and elegance, we have chosen to terminate the pattern along what would be bend folds, which gives roughly rectangular panels for all panels not along the diagonal folds. We have also chosen the cut line along one of the reverse folds (specifically, along the $\mathbf{s}_{0,0,k}$ chain of folds), rather than along a diagonal, as in the previous example.

Figure 14 shows this new flasher design from unfolded to fully folded for the same parameters as Figure 9, with $m = 4$, $n = 3$, and $\varepsilon = 3°$. Once again, we have a rigidly foldable single-DOF mechanism.

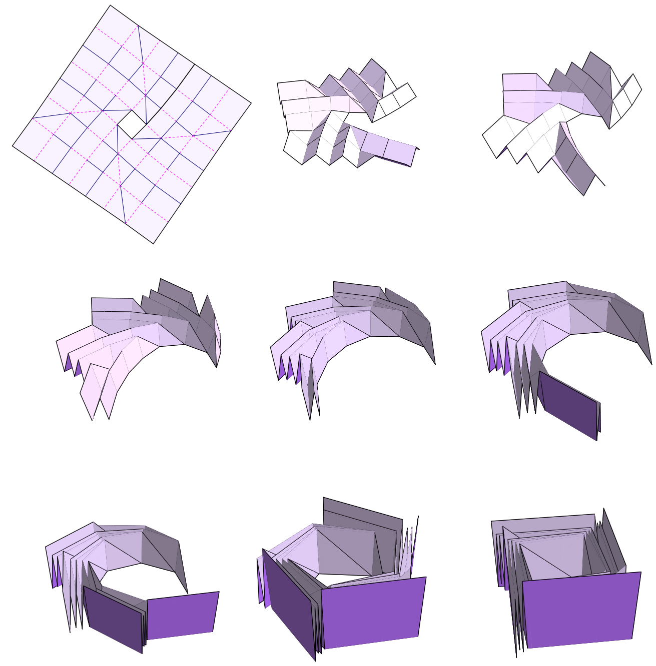

FIGURE 14: A single-DOF reverse-folded flasher for various values of γ_{bend}. From upper left to lower right: $\gamma_{bend} = 0°, 3°, 5°, 10°, 20°, 30°, 40°, 65°, 87°$.

6 Further Developments and Discussion

The tip of any given reverse fold may be positioned anywhere along the diagonal fold without perturbing the mechanical action. In the example developed here, we have chosen the reverse fold vertices to coincide with bend/diagonal vertices. This choice has an important side effect: it creates several degree-6 vertices. The number of degrees of freedom of a degree-n vertex is $n-3$, so a degree-6 vertex has three, not one, degrees of freedom. This could, in principle, give the mechanism extra degrees of freedom.

However, those degree-6 vertices do not exist in isolation. The facets surrounding each degree-6 vertex are themselves connected to degree-4 vertices, and those vertices are constrained to single-DOF paths in phase space. If enough facets surrounding a high-degree vertex are constrained to single-DOF paths, then the remaining facets will be clamped as well to the single-DOF motion, which happens in this mechanism.

There is still the question of layer collisions. For the simulation of Figure 14, we have cut the crease pattern along a reverse fold, rather than along a diagonal fold as in the simpler flasher. This cut still gives a layer collision as we approach the folded state, as can be seen in Figure 15.

The presence of layer collisions depend upon the position and orientation of the cut. A straight cut in the flat state—either

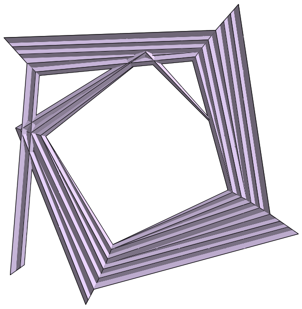

FIGURE 15: A view from the top of the flasher of Figure 14 for $\gamma_{bend} = 80^\circ$, showing collisions with the inner layers.

along a diagonal fold, as in the simple flasher, or along a reverse fold, as in the previous example—assumes a spiral form in the 3D state, and it is the overlap of that spiral that gives rise to the collision in the nearly-stowed state. We can avoid such a collision by choosing a cut that is nearly straight in the stowed state, as was noted earlier in Figure 11. Such a cut will give rise to a spiral cut in the flat state, but with appropriate choice of cut, can give rise to a collision-free motion as the fold angle approaches $\gamma_{bend,max}$.

In Figure 16 we show a different cut position with otherwise the same parameters as Figure 14. As before, this pattern moves rigidly between the deployed (flat) and stowed (cylindrical) shape.

By making the cut follow along a bend, we get the cut edges to more cleanly line up in the stowed state. As $\gamma \to \gamma_{bend,max}$, the cut edges realign and form a perfect butt joint.

However, this particular introduces a very slight layer intersection near the deployed state. Figure 17 shows top views in the nearly-deployed and nearly-stowed states.

This collision is slight and only results in a slight overlap; complete elimination of the intersection could be achieved with a slight removal of material from a few of the panels.

Contributing factors to the presence of collisions in the near-stowed state are the sharp corners in the bend that arises from low rotational order. In higher-rotational order, the residual overlap is reduced. Figure 18 shows a hexagonal flasher with the same cut pattern; in this pattern, there is still a very slight overlap in the near-stowed state, but it is extremely small, and only a tiny amount of material would need to be removed to eliminate the overlap.

It seems likely that an appropriately chosen path for the cut could entirely eliminate layer collisions; we leave that exploration as a topic for future work.

Another potential drawback of this particular path is that it creates facets that are connected by only a single fold; clearly, these possess their own degree of freedom independent of the rest of the mechanism. It also creates one degree-6 vertex that is not

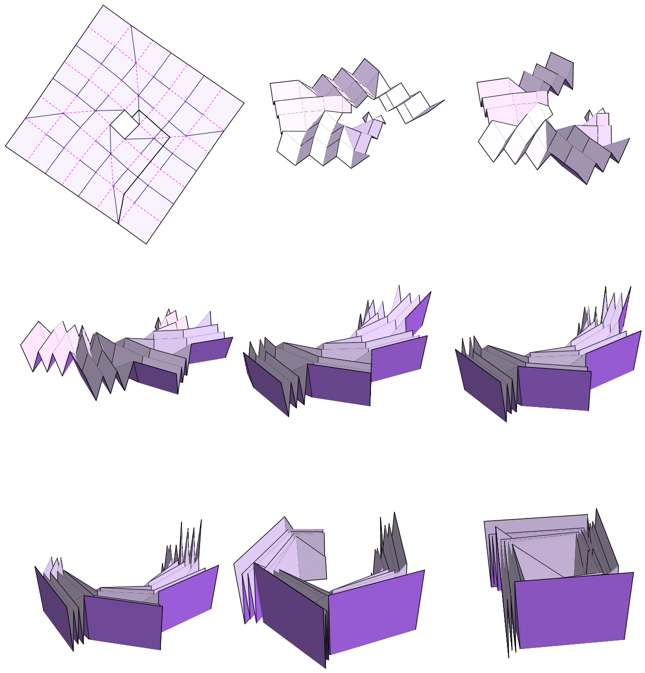

FIGURE 16: A rigidly foldable flasher for various values of γ_{bend}. From upper left to lower right: $\gamma_{bend} = 0^\circ, 3^\circ, 5^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 65^\circ, 87^\circ$.

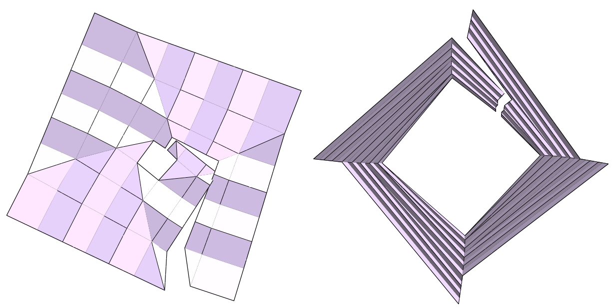

FIGURE 17: A rigidly foldable flasher near the endpoints of the motion. Left: $\gamma_{bend} = 1^\circ$. Right: $\gamma_{bend} = 85^\circ$ (different scale).

surrounded by degree-4 vertices, and so also possesses multiple degrees of freedom.

Clearly, then, there are considerable further avenues for development of this family of structures. The degree-6 vertices were created by positioning the reverse folds at existing vertices along the diagonals; they could be eliminated simply by choosing reverse-fold positions *anywhere else* along the diagonals. The positions of the reverse folds are another axis of parameter variation in the design process. One could, for example, choose their positions in such a way that each of the concentric

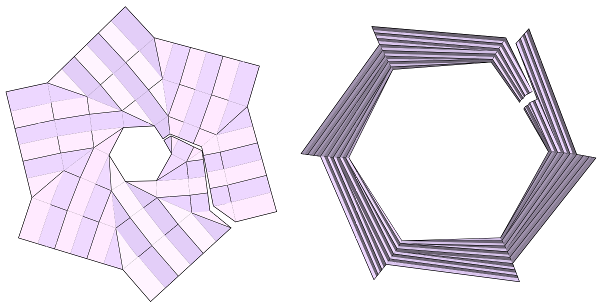

FIGURE 18: A rigidly foldable hexagonal flasher near the endpoints of the motion. Left: $\gamma_{bend} = 0.5°$. Right: $\gamma_{bend} = 56°$ (different scale).

spirals possesses exactly the same height in the stowed configuration. This choice would give the most efficient possible stowed form, and by displacing the reverse folds from the existing vertices along the diagonals, would give rise to all degree-4 vertices and a greater ease of obtaining a single-DOF mechanism.

In conclusion, we have presented a family of flasher mechanisms that possess rigid foldability, thereby making them suitable for the implementation of deployable mechanisms with rigid hinged panels. Many of these mechanisms exhibit single-DOF motion, which is another desirable trait. We presented constructive algorithms for the 2D and 3D forms and the relevant fold angles. We presented specific examples with fourfold rotational symmetry, but the analysis is fully parameterized on the rotational order m, so that other rotational orders may be readily similarly constructed.

More broadly, we have shown an approach for constructing large-scale single-DOF mechanisms by making use of the unique properties of flat-foldable degree-4 vertices—specifically, the proportionality relationship of Equation (4). This behavior was noted by Tachi [16], who demonstrated flat-foldable "generalized Miura-ori." As shown here, this property can be used to construct large single-DOF networks even when the mechanism is never folded to the flat state; individual vertex flat-foldability is used simply as a means to attain constancy of $\tan(\gamma_i)/\tan(\gamma_j)$ for every pair $\{i, j\}$ of vertices in the network. We expect that this property can be used to construct many more complex, single-DOF, origami-based mechanisms in the future.

ACKNOWLEDGMENT

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Appendix A
Sector-dihedral relations

Here we derive new formulas relating the dihedral angles of a general degree-4 vertex to the values of the surrounding sector angles. A portion of this derivation is also presented in [18].

We build off of the work of Huffman [15], following his approach and using several of his results. We consider the trace of the vertex on the Gaussian sphere, as shown in Figure 19 (analogous to Figure 3 of [15]). The Gaussian sphere is a dual-space representation of the vertex: sector angles of the vertex map to corners of the trace, while dihedral angles (fold angles) map to arcs on the Gaussian sphere.

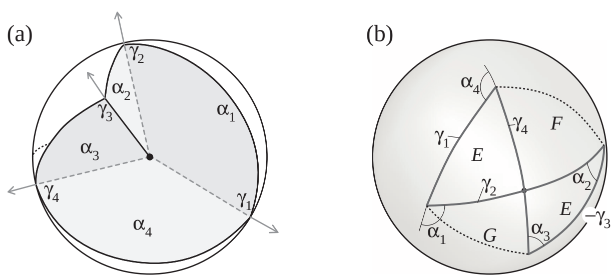

FIGURE 19: Schematic of a degree-4 vertex. (a) The vertex embedded in a unit sphere. Dashed lines are valley folds, dotted lines are mountain folds. (b) The trace of the vertex on the Gaussian sphere. Since γ_3 is a mountain fold, its sign is negative.

As noted by Huffman, the trace of a developable degree-4 vertex always takes the general form of a (not necessarily symmetric) bow-tie, and the solid angles subtended by the two triangles of the bow are equal (this follows directly from developability of the vertex).

Huffman derived relations between opposite dihedral angles for a general vertex (Equations (2a–c) in [15]). Using our notation, they are:

$$\frac{\sin^2\left(\frac{1}{2}\gamma_2\right)}{\sin^2\left(\frac{1}{2}\gamma_4\right)} = \frac{\sin\alpha_3\sin\alpha_4}{\sin\alpha_1\sin\alpha_2}, \tag{39}$$

and

$$\frac{\sin^2\left(\frac{1}{2}\gamma_1\right)}{\sin^2\left(\frac{1}{2}\gamma_3\right)} = \frac{\sin\alpha_2\sin\alpha_3}{\sin\alpha_1\sin\alpha_4}. \tag{40}$$

For a flat-foldable vertex ($\alpha_1+\alpha_3=\pi$, $\alpha_2+\alpha_4=\pi$) with crease assignment as shown in Figure 19, these simplify to

$$\gamma_2 = \gamma_4, \gamma_1 = -\gamma_3. \tag{41}$$

Huffman also derived a relationship between adjacent dihedral angles (Equation (3) in [15]); here, we derive one that is somewhat simpler.

We consider the four triangles

$$\triangle(\alpha_4,\alpha_1,\alpha_2), \triangle(\alpha_4,\alpha_3,\alpha_2), \triangle(\alpha_1,\alpha_4,\alpha_3), \text{ and } \triangle(\alpha_1,\alpha_2,\alpha_3),$$

each composed of two of the smaller lettered triangles in Figure 19.

Because the first two of the four share the triangle F and the two halves of the bow-tie of the trace have equal area, the first pair of triangles have equal area, as do the second pair:

$$\text{area}(\triangle(\alpha_4,\alpha_1,\alpha_2)) = \text{area}(\triangle(\alpha_4,\alpha_3,\alpha_2)) = E+F, \tag{42}$$
$$\text{area}(\triangle(\alpha_1,\alpha_4,\alpha_3)) = \text{area}(\triangle(\alpha_1,\alpha_2,\alpha_3)) = E+G. \tag{43}$$

Using a cotangent formula for triangle areas from [15] (see unmarked equation preceding Equation (1) in [15]), we can establish an equality for each pair of triangles:

$$\cot(\tfrac{1}{2}\gamma_1)\cot(\tfrac{1}{2}\gamma_2)\csc(\pi-\alpha_1)+\cot(\pi-\alpha_1) = \cot(\tfrac{1}{2}\gamma_4)\cot(-\tfrac{1}{2}\gamma_3)\csc\alpha_3+\cot\alpha_3, \tag{44}$$
$$\cot(\tfrac{1}{2}\gamma_1)\cot(\tfrac{1}{2}\gamma_4)\csc(\pi-\alpha_4)+\cot(\pi-\alpha_4) = \cot(\tfrac{1}{2}\gamma_2)\cot(-\tfrac{1}{2}\gamma_3)\csc\alpha_2+\cot\alpha_2. \tag{45}$$

Now, we can eliminate one of the half-angle cotangents from these two equations and solve for any one of the four in terms of the two remaining. For example,

$$\cot(\tfrac{1}{2}\gamma_2) = \frac{\cot(\tfrac{1}{2}\gamma_3)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_3-\cot(\tfrac{1}{2}\gamma_1)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_4}{\cot^2(\tfrac{1}{2}\gamma_3)\csc\alpha_2\csc\alpha_3-\cot^2(\tfrac{1}{2}\gamma_1)\csc\alpha_1\csc\alpha_4}. \tag{46}$$

But this can be simplified; by using Equation (40), the denominator simplifies to

$$\cot^2(\tfrac{1}{2}\gamma_3)\csc\alpha_2\csc\alpha_3-\cot^2(\tfrac{1}{2}\gamma_1)\csc\alpha_1\csc\alpha_4 = \csc\alpha_1\csc\alpha_4-\csc\alpha_2\csc\alpha_3. \tag{47}$$

.

Applying this and corresponding relations for all four half-angle cotangents, we find linear relationships among the four:

$$\cot(\tfrac{1}{2}\gamma_1) = \frac{\cot(\tfrac{1}{2}\gamma_2)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_2-\cot(\tfrac{1}{2}\gamma_4)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_3}{\csc\alpha_3\csc\alpha_4-\csc\alpha_1\csc\alpha_2}, \tag{48}$$

$$\cot(\tfrac{1}{2}\gamma_2) = \frac{\cot(\tfrac{1}{2}\gamma_3)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_3-\cot(\tfrac{1}{2}\gamma_1)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_4}{\csc\alpha_1\csc\alpha_4-\csc\alpha_2\csc\alpha_3}, \tag{49}$$

$$\cot(\tfrac{1}{2}\gamma_3) = \frac{\cot(\tfrac{1}{2}\gamma_2)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_1-\cot(\tfrac{1}{2}\gamma_4)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_4}{\csc\alpha_3\csc\alpha_4-\csc\alpha_1\csc\alpha_2}. \tag{50}$$

$$\cot(\tfrac{1}{2}\gamma_4) = \frac{\cot(\tfrac{1}{2}\gamma_3)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_2-\cot(\tfrac{1}{2}\gamma_1)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_1}{\csc\alpha_1\csc\alpha_4-\csc\alpha_2\csc\alpha_3}, \tag{51}$$

Now, each of Equations (39) and (40) may be rewritten in terms of cotangents:

$$\cot^2(\tfrac{1}{2}\gamma_2) = \csc^2(\tfrac{1}{2}\gamma_4)\frac{\sin\alpha_1\sin\alpha_2}{\sin\alpha_3\sin\alpha_4} - 1 = (1+\cot^2(\tfrac{1}{2}\gamma_4))\frac{\sin\alpha_1\sin\alpha_2}{\sin\alpha_3\sin\alpha_4} - 1, \tag{52}$$

$$\cot^2(\tfrac{1}{2}\gamma_4) = \csc^2(\tfrac{1}{2}\gamma_2)\frac{\sin\alpha_3\sin\alpha_4}{\sin\alpha_1\sin\alpha_2} - 1 = (1+\cot^2(\tfrac{1}{2}\gamma_2))\frac{\sin\alpha_3\sin\alpha_4}{\sin\alpha_1\sin\alpha_2} - 1, \tag{53}$$

$$\cot^2(\tfrac{1}{2}\gamma_1) = \csc^2(\tfrac{1}{2}\gamma_3)\frac{\sin\alpha_1\sin\alpha_4}{\sin\alpha_2\sin\alpha_3} - 1 = (1+\cot^2(\tfrac{1}{2}\gamma_3))\frac{\sin\alpha_1\sin\alpha_4}{\sin\alpha_2\sin\alpha_3} - 1, \tag{54}$$

$$\cot^2(\tfrac{1}{2}\gamma_3) = \csc^2(\tfrac{1}{2}\gamma_1)\frac{\sin\alpha_2\sin\alpha_3}{\sin\alpha_1\sin\alpha_4} - 1 = (1+\cot^2(\tfrac{1}{2}\gamma_1))\frac{\sin\alpha_2\sin\alpha_3}{\sin\alpha_1\sin\alpha_4} - 1. \tag{55}$$

Recall that for the two pairs of opposite angles, one pair must have the same sign and the other must have the opposite sign. When we take square roots of Equations (52) and (54), we must choose signs that respect this convention. We can denote the possible choices by introducing operations $\pm_i$, where the sign of each is chosen to match the desired crease directions. Then we have that

$$\cot(\tfrac{1}{2}\gamma_2) = \pm_2\sqrt{(1+\cot^2(\tfrac{1}{2}\gamma_4))\frac{\sin\alpha_1\sin\alpha_2}{\sin\alpha_3\sin\alpha_4} - 1}, \tag{56}$$

$$\cot(\tfrac{1}{2}\gamma_4) = \pm_4\sqrt{(1+\cot^2(\tfrac{1}{2}\gamma_2))\frac{\sin\alpha_3\sin\alpha_4}{\sin\alpha_1\sin\alpha_2} - 1}, \tag{57}$$

$$\cot(\tfrac{1}{2}\gamma_1) = \pm_1\sqrt{(1+\cot^2(\tfrac{1}{2}\gamma_3))\frac{\sin\alpha_1\sin\alpha_4}{\sin\alpha_2\sin\alpha_3} - 1}, \tag{58}$$

$$\cot(\tfrac{1}{2}\gamma_3) = \pm_3\sqrt{(1+\cot^2(\tfrac{1}{2}\gamma_1))\frac{\sin\alpha_2\sin\alpha_3}{\sin\alpha_1\sin\alpha_4} - 1}, \tag{59}$$

and each of these may be substituted into the proceeding four equations to give a one-to-one relationship between any dihedral angle and either of its immediately adjacent angles. We summarize these for the general case and for the specific cases of straight-major, straight-minor, and flat-foldable, where the relationships take on somewhat simpler forms.

From Minor Dihedral

Consider first a minor pair, where γ_1 is the known angle. Then, from Equation (40), the other minor angle is given by:

$$\gamma_3 = -2\sin^{-1}\left[\sin(\tfrac{1}{2}\gamma_1)\sqrt{\frac{\sin\alpha_1\sin\alpha_4}{\sin\alpha_2\sin\alpha_3}}\right], \tag{60}$$

which is valid for either sign of γ_1.

The two major angles are then given by

$$\gamma_2 = 2\cot^{-1}\left[\frac{\cot(\tfrac{1}{2}\gamma_3)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_3 - \cot(\tfrac{1}{2}\gamma_1)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_2}{\csc\alpha_1\csc\alpha_4 - \csc\alpha_2\csc\alpha_3}\right], \tag{61}$$

$$\gamma_4 = 2\cot^{-1}\left[\frac{\cot(\tfrac{1}{2}\gamma_3)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_2 - \cot(\tfrac{1}{2}\gamma_1)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_1}{\csc\alpha_1\csc\alpha_4 - \csc\alpha_2\csc\alpha_3}\right]. \tag{62}$$

These two expressions give the proper signs for γ_2 and γ_4 for all possible sets of valid sector angles and given angle γ_1, except for the special cases where the two expressions are undetermined: flat-foldable, straight-minor, and straight-major.

For the flat-foldable special case, where $\alpha_1+\alpha_3=\pi$, $\alpha_2+\alpha_4=\pi$, we have

$$\gamma_3=-\gamma_1, \tag{63}$$

$$\gamma_2=\gamma_4=2\cot^{-1}\left[\cot(\tfrac{1}{2}\gamma_1)\frac{\sin\frac{1}{2}(\alpha_1-\alpha_2)}{\sin\frac{1}{2}(\alpha_1+\alpha_2)}\right]=2\cot^{-1}\left[\mu^{-1}\cot(\tfrac{1}{2}\gamma_1)\right], \tag{64}$$

where we have included the fold angle multiplier μ that we introduced above. These equations, too, give the proper sign for all possible sets of sector angles and γ_1.

For the straight-minor special case, where $\alpha_1+\alpha_2=\pi$, $\alpha_3+\alpha_4=\pi$, we have

$$\gamma_3=-\gamma_1, \tag{65}$$

$$\gamma_2=2\cot^{-1}\left[\frac{(1-\cot^2(\frac{1}{2}\gamma_1))\cot\alpha_1+(1+\cot^2(\frac{1}{2}\gamma_1))\cot\alpha_3}{2\cot(\frac{1}{2}\gamma_1)\csc\alpha_1}\right], \tag{66}$$

$$\gamma_4=-2\cot^{-1}\left[\frac{(1-\cot^2(\frac{1}{2}\gamma_1))\cot\alpha_3+(1+\cot^2(\frac{1}{2}\gamma_1))\cot\alpha_1}{2\cot(\frac{1}{2}\gamma_1)\csc\alpha_3}\right]. \tag{67}$$

And finally, for the straight-major special case, where $\alpha_1+\alpha_4=\pi$, $\alpha_2+\alpha_3=\pi$, we have

$$\gamma_3=-\gamma_1, \tag{68}$$

$$\gamma_2=\gamma_4=\begin{cases}\pm\pi & \text{if } \gamma_1=0,\\ \text{unspecified} & \text{if } \gamma_1=0.\end{cases} \tag{69}$$

From Major Dihedral

Instead of being given a minor dihedral angle, we might instead be given a major dihedral angle, e.g., γ_4. From this we can calculate the other three dihedrals. For the general case, the other major angle comes from Equation (39), given by

$$\gamma_2=2\sin^{-1}\left[\sin(\tfrac{1}{2}\gamma_4)\sqrt{\frac{\sin\alpha_3\sin\alpha_4}{\sin\alpha_1\sin\alpha_2}}\right]. \tag{70}$$

The two minor angles are then given by

$$\gamma_1 = 2\cot^{-1}\left[\frac{\cot(\frac{1}{2}\gamma_2)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_2-\cot(\frac{1}{2}\gamma_4)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_3}{\csc\alpha_3\csc\alpha_4-\csc\alpha_1\csc\alpha_2}\right], \tag{71}$$

$$\gamma_3 = 2\cot^{-1}\left[\frac{\cot(\frac{1}{2}\gamma_2)(\cot\alpha_2+\cot\alpha_4)\csc\alpha_1-\cot(\frac{1}{2}\gamma_4)(\cot\alpha_1+\cot\alpha_3)\csc\alpha_4}{\csc\alpha_3\csc\alpha_4-\csc\alpha_1\csc\alpha_2}\right]. \tag{72}$$

For the flat-foldable special case, where $\alpha_1+\alpha_3=\pi$, $\alpha_2+\alpha_4=\pi$, we have

$$\gamma_2 = \gamma_4, \tag{73}$$

$$\gamma_1 = -\gamma_3 = 2\cot^{-1}\left[\cot(\tfrac{1}{2}\gamma_4)\frac{\sin\frac{1}{2}(\alpha_1+\alpha_2)}{\sin\frac{1}{2}(\alpha_1-\alpha_2)}\right] = 2\cot^{-1}\left[\mu\cot(\tfrac{1}{2}\gamma_4)\right]. \tag{74}$$

For the straight-minor special case, where $\alpha_1+\alpha_2=\pi$, $\alpha_3+\alpha_4=\pi$, we have

$$\gamma_2 = 2\sin^{-1}\left[\sin(\tfrac{1}{2}\gamma_4)\frac{\sin\alpha_3}{\sin\alpha_1}\right], \tag{75}$$

$$\gamma_1 = -\gamma_3 = 2\cot^{-1}\left[\frac{\left(\cot(\frac{1}{2}\gamma_2)\csc\alpha_1+\cot(\frac{1}{2}\gamma_4)\csc\alpha_4\right)\left(\cot\alpha_1+\cot\alpha_3\right)}{\csc^2\alpha_3-\csc^2\alpha_1}\right]. \tag{76}$$

And last, for the straight-major special case, where $\alpha_1+\alpha_4=\pi$, $\alpha_2+\alpha_3=\pi$, we have

$$\gamma_2 = \gamma_4, \tag{77}$$

$$\gamma_1 = -\gamma_3 = \begin{cases} 0 & \text{if } \gamma_4 \neq 0, \\ \text{unspecified} & \text{if } \gamma_4 = \pm\pi. \end{cases} \tag{78}$$