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An Inequality Involving sin(n)

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Abstract. We prove an inequality involving sine by using an estimate of the irrationality measure of π.

At a meeting in Jacksonville, Florida in 2010 the author was asked about proving the inequality

$$|\sin(n)| > 2^{-n}$$

for all positive integers n. A more interesting problem is to find the smallest α for which $|\sin(n)| > \alpha^{-n}$ for all positive integers n. Note that to find such a number is equivalent to finding $sup_{n\in\mathbb{N}} \frac{1}{\sqrt[n]{|\sin(n)|}}$. It seems likely that this could be shown to be $\frac{1}{\sqrt[3]{\sin(3)}}$ without all the "heavy machinery" used in the proof that follows. The author was unable to do this, but it would be interesting to see a self-contained proof of the following result.

In this note, we prove the following.

Proposition. $|\sin(n)| > \alpha^{-n}$ *for all positive integers* n *if and only if* $\alpha > \frac{1}{\sqrt[3]{\sin(3)}} \approx 1.92$.

Proof. ($\Longrightarrow$) If $\alpha = \frac{1}{\sqrt[3]{\sin(3)}}$ then $\sin(3) = (\frac{1}{\sqrt[3]{\sin(3)}})^{-3} = \alpha^{-3}$, so α must be greater than $\frac{1}{\sqrt[3]{\sin(3)}}$.

($\Longleftarrow$) To begin with, note that $\frac{1}{2}|x| \le |\sin(x)| \le |x|$ for $-1 < x < 1$ by the tangent line approximation for $\sin(x)$ at $x = 0$. It is clear that we need to concern ourselves with values of n that make $\sin(n)$ small, which are those that are close to a multiple of π. So $\frac{1}{2}|n - m\pi| \le |\sin(n - m\pi)| \le |n - m\pi|$ for $n \approx m\pi$. From this it is easy to see that $\frac{1}{2}|n - m\pi| \le |\sin(n)| \le |n - m\pi|$ for $n \approx m\pi$. We need to show that $\frac{1}{2}|n - m\pi| > \alpha^{-n}$ or $|n - m\pi| > 2 \cdot \alpha^{-n}$ for all positive integers n.

By a result of Mahler ([1, Theorem 1, p. 33]),

$$\left|\pi - \frac{n}{m}\right| \ge m^{-42} \text{ for all integers } m, n \ge 2.$$

There are more accurate estimates of the irrationality measure of π, but this one holds for virtually all m and n. We can assume that $3m < n < 4m$ since $\frac{n}{m} \approx \pi$, so $|n - m\pi| \ge m^{-41} \ge (\frac{n}{3})^{-41}$. We need to find N so that $2 \cdot \alpha^{-n} < (\frac{n}{3})^{-41}$ for all $n > N$. Using a calculator (with α equal to $\frac{1}{\sqrt[3]{\sin(3)}}$) we determined that this inequality is true for $n > 290$. We then checked (again by calculator) that the integers less than or equal to 290 work in the inequality $|\sin(n)| > \alpha^{-n}$ for any $\alpha > \frac{1}{\sqrt[3]{\sin(3)}}$. So we have that the inequality holds for all positive integers n. ■

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