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Expected Number of Distinct Sites Visited by a Random Walk with an Infinite Variance

JOSEPH E. GILLIS AND GEORGE H. WEISS*
Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel

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Consider a random walk of n steps on an infinite, simple cubic lattice. Let $p(r)$ be the (symmetric) probability of a vector jump r, and let S_n be the expected number of distinct lattice points visited in the course of the random walk. In the present paper we calculate asymptotic values for S_n for the particular choice of jump probabilities $p(r) = p(-r) = Ar^{-(1+\alpha)}$, where $2 \geq \alpha > 0$, and $p(r_1, r_2) = Ar^{-\beta}$, where $r^2 = r_1^2 + r_2^2$, $2 \geq \beta > 1$, and A denotes the normalizing constant. The results are, in 1D, (1) $S_n \sim An$, $1 > \alpha > 0$, (2) $S_n \sim Bn/\ln n$, $\alpha = 1$, (3) $S_n \sim Cn^{1/\alpha}$, $2 > \alpha > 1$, (4) $S_n \sim D(n \ln n)^{\frac{1}{2}}$, $\alpha = 2$, where A, B, C, and D are calculable constants, and, in 2D, (1) $S_n \sim An$, $2 > \beta > 1$, (2) $S_n \sim Bn/\ln \ln n$, $\beta = 2$.

1. INTRODUCTION

Dvoretzky and Erdos[1] were the first to study the statistics of the distinct number of sites visited in an n-step random walk on a lattice. Their results were subsequently rederived by Vineyard[2] and later extended by Montroll and Weiss[3] to include random walks whose jump probabilities had a finite variance (cf. also Spitzer[4]). The results of Montroll and Weiss were derived through the use of Karamata's Tauberian theorem, a technique first introduced into the study of random walks by Darling and Kac[5] and Kac.[6] Although all of the known results pertain to jump probabilities with finite variance, the use of Tauberian theorems allows one to extend the study to jump probabilities having infinite variance. If $p(n)$ is the probability of changing position by a vector n in a single step, then the variance (in one dimension) is defined by

$$\sigma^2 = \sum_{n=-\infty}^{\infty} n^2 p_n, \tag{1.1}$$

with analogous definitions in higher dimensions. In the present paper, we derive results for certain specific random walks in one and two dimensions for which the variance associated with jump probabilities is infinite. In three and higher dimensions, the problem is of lesser interest since, even in the case of a finite variance, the expected number of distinct sites visited in an n-step random walk is asymptotically An, where A is a calculable constant. Since the result can be at most n, the order of the asymptotic dependence on n cannot depend on whether the variance is finite or infinite, though the value of A, as well as correction terms, will depend on the variance.

For random walks with finite variance, it is known that the expected number of distinct sites visited during an n-step random walk is as follows:

$$\begin{aligned} S_n &= a_1\sqrt{n}, && \text{in 1D,} \\ S_n &= a_2 n/\ln n, && \text{in 2D,} \\ S_n &= a_3 n, && \text{in 3D,} \end{aligned} \tag{1.2}$$

where a_1, a_2, and a_3 are constants. The change of variance from finite to infinite will be shown to lead to an increase in the order of magnitude of S_n in one and two dimensions for the specific models

$$\begin{aligned} p(n) &= An^{-(1+\alpha)}, && \text{in 1D,} \\ p(n_1, n_2) &= B(n_1^2 + n_2^2)^{-\beta}, && \text{in 2D,} \end{aligned} \tag{1.3}$$

and certain extensions of these random walks.

2. A 1-DIMENSIONAL SET OF TRANSITION PROBABILITIES

Let $p(n)$ denote the probability of a jump of n sites at a single step, when $n = (n)$ in one dimension and $n = (n_1, n_2)$ in two dimensions. We define the structure factor of the random walk by

$$\lambda(\boldsymbol{\theta}) = \sum_{\mathbf{n}} p(\mathbf{n}) \exp(i\mathbf{n} \cdot \boldsymbol{\theta}), \tag{2.1}$$

where $\boldsymbol{\theta} = (\theta)$ or $\boldsymbol{\theta} = (\theta_1, \theta_2)$. In k dimensions, we define the integral

$$P(z) = \frac{1}{\pi^k} \int_0^\pi \cdots \int \frac{d^k\boldsymbol{\theta}}{1 - z\lambda(\boldsymbol{\theta})}. \tag{2.2}$$

Then it is known[3] that the asymptotic value of S_n, the expected number of distinct sites visited on an n-step walk, is given by

$$S_n \sim n/P(1 - n^{-1}), \tag{2.3}$$

provided that $P(1-n^{-1})$ has the asymptotic form

$$P(1-n^{-1}) \sim n^{\beta} L(n), \tag{2.4}$$

where $L(n)$ is a slowly varying function, i.e.,

$$\lim L(cn)/L(n) = 1,$$

for all $c > 0$, as $n \to \infty$.

In what follows we assume that $p(\mathbf{n})$ is symmetric in its indices. The specific form of $p(\mathbf{n})$ to be studied in one dimension will be

$$p(n) = p(-n) = B(n^{-(1+\alpha)} + \epsilon_n), \tag{2.5}$$

where it is assumed that ϵ_n is such that

$$\sum_{n=1}^{\infty} n^2 \epsilon_n < \infty \tag{2.6}$$

and B is a normalization constant. The condition in Eq. (2.6) can be weakened somewhat, but we shall not pursue this generalization.

Since we are interested in the case of infinite variance, α must satisfy

$$2 \geq \alpha > 0. \tag{2.7}$$

It follows from Eq. (2.2) that the singular behavior of $P(z)$ near $z = 1$ must come from the behavior of the integrand in the neighborhood of the root of $\lambda(\theta) = 1$. For the cases of present interest, $\theta = 0$ will be the only such root.[7] In order to calculate the behavior of $P(1-n^{-1})$ for large n, we must find an expansion for $\lambda(\theta)$ valid for small $|\theta|$. This requires analysis of the function

$$G_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{1-\cos n\theta}{n^{1+\alpha}} \tag{2.8}$$

for small θ.

We substitute

$$\frac{1}{n^{1+\alpha}} = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty t^\alpha e^{-nt}\, dt \tag{2.9}$$

in (2.8) and perform the resulting summation. This leads to the exact representation

$$G_\alpha(\theta) = \frac{1}{\Gamma(1+\alpha)} \times \int_0^\infty \frac{t^\alpha e^{-t}(1+e^{-t})(1-\cos\theta)\, dt}{(1-e^{-t})[(1-e^{-t})^2 + 2e^{-t}(1-\cos\theta)]}. \tag{2.10}$$

To study the behavior of this integral near $\theta = 0$, we approximate it by

$$g_\alpha(\theta) \sim \frac{\theta^2}{2\Gamma(1+\alpha)} \int_0^\infty \frac{t^\alpha e^{-t}(1+e^{-t})\, dt}{(1-e^{-t})[(1-e^{-t})^2 + \theta^2 e^{-t}]}. \tag{2.11}$$

It is easily verified directly that $|G_\alpha(\theta) - g_\alpha(\theta)|$ is of lower order than the terms retained.

It is clear from (2.11) that, if θ is set equal to zero in the integrand, the resulting integral

$$\int_0^\infty \frac{t^\alpha e^{-t}(1+e^{-t})\, dt}{(1-e^{-t})^3}$$

diverges because of the singularity at $t = 0$. It follows that the behavior of the integral for small θ depends only on the behavior of the integrand in a neighborhood of $t = 0$. It is shown in Appendix A that the limiting behavior of $g_\alpha(\theta)$ can be found from the integral

$$g_\alpha(\theta) \sim h_\alpha(\theta) = \frac{\theta^2}{\Gamma(1+\alpha)} \int_0^\infty \frac{t^{\alpha-1} e^{-t}\, dt}{(t^2+\theta^2)} = \frac{\theta^\alpha}{\Gamma(1+\alpha)} \int_0^\infty \frac{x^{\alpha-1} e^{-\theta x}\, dx}{1+x^2}. \tag{2.12}$$

We can now evaluate the limiting behavior of $g_\alpha(\theta)$ for small θ. For $\alpha < 2$,

$$\frac{h_\alpha(\theta)}{\theta^\alpha} \sim \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{x^{\alpha-1}\, dx}{1+x^2} = \frac{\pi \csc \frac{1}{2}\pi\alpha}{2\Gamma(1+\alpha)}. \tag{2.13}$$

When $\alpha = 2$, we must include the exponential term to insure convergence at $t = \infty$. It follows that

$$\frac{2h_2(\theta)}{\theta^2} \sim \int_0^\infty \frac{x e^{-\theta x}\, dx}{1+x^2}. \tag{2.14}$$

We see that, for small θ,

$$\frac{d}{d\theta}\left(\frac{h_2(\theta)}{\theta^2}\right) \sim -\frac{1}{2\theta} + \tfrac{1}{4}\pi + O(1) \tag{2.15}$$

so that

$$h_2(\theta) \sim -\tfrac{1}{2}\theta^2 \ln\theta = \tfrac{1}{2}\theta^2 \ln^{-1}\theta. \tag{2.16}$$

We now return to the evaluation of $P(z)$ for z close to 1. For $0 < \alpha < 1$, we find

$$P(z) \sim \frac{1}{\pi} \int_0^\pi \frac{d\theta}{1 - z + Bz\theta^\alpha}, \tag{2.17}$$

where B is the normalizing constant in Eq. (2.5). Hence, $P(1)$ is finite and it follows from Eq. (2.2) that

$$S_n \sim n/P(1). \tag{2.18}$$

For $\alpha = 1$, we have

$$P(z) \sim \frac{1}{\pi} \int_0^\pi \frac{d\theta}{1 - z + \frac{1}{2}B\theta} \sim -\frac{2}{B} \ln(1-z) + O(1) \tag{2.19}$$

so that

$$S_n \sim \frac{Bn}{2 \ln n}. \tag{2.20}$$

It is interesting to notice that this resembles the result for a finite variance random walk in two dimensions.[3] When $2 > \alpha > 1$, the integral is of the form shown in (2.17), but $P(1)$ is no longer finite. For that case, we set $\theta^\alpha = (1 - z)x$ so that

$$P(z) \sim \frac{1}{\pi(1-z)^{1-1/\alpha}} \int_0^{\pi/(1-z)^{1/\alpha}} \frac{dx}{(1 + Bx^\alpha)x^{1-1/\alpha}}$$
$$\sim \frac{1}{\pi(1-z)^{1-1/\alpha}} \int_0^{\infty} \frac{dx}{(1 + Bx^\alpha)x^{1-1/\alpha}}. \tag{2.21}$$

This result implies that, for $2 > \alpha > 1$,

$$S_n \sim cn^{1/\alpha} \tag{2.22}$$

where c is found from Eq. (2.21).

Finally, when $\alpha = 2$, we must find the asymptotic behavior of the integral

$$P(z) \sim \frac{1}{\pi} \int_0^{a} \frac{d\theta}{1 - z + Bz\theta^2 \ln \theta^{-1}}, \tag{2.23}$$

where a can be any fixed constant which we choose less than 1 [so that the singularity at $\theta = 1$, which is due to our choice of an approximate form for $\lambda(\theta)$, does not have to be discussed]. For simplicity, let us define the integral

$$F(\epsilon) = \int_0^{a} \frac{d\theta}{\epsilon + \theta^2 \ln \theta^{-1}} \tag{2.24}$$

so that

$$\lim_{z\to 1} P(z) = \frac{1}{\pi B} \lim_{z\to 1} F\left(\frac{1-z}{B}\right). \tag{2.25}$$

In this integral change variables by the transformation

$$\theta^2 \ln \theta^{-1} = y, \tag{2.26}$$

with a solution represented by

$$\theta = f(y). \tag{2.27}$$

Then $F(\epsilon)$ can be formally written

$$F(\epsilon) = \int_0^{b} \frac{f'(y)}{y + \epsilon}\, dy, \tag{2.28}$$

where $b = a^2 \ln a^{-1}$. The integral for $F(\epsilon)$ clearly diverges at $\epsilon = 0$ because of the singularity at $y = 0$ in the integrand. The nature of the singularity can be determined by giving an accurate representation for $f'(y)$ in the neighborhood of $y = 0$. For this purpose set $\theta = \sqrt{y}\, u(y)$ in Eq. (2.26) so that $u(y)$ is the solution to

$$u^2(y)\{\tfrac{1}{2} \ln y^{-1} + \ln [u(y)]^{-1}\} = 1. \tag{2.29}$$

Let us replace this equation by the iterative scheme

$$u_{n+1}^2(y) = \frac{2}{\ln y^{-1} + 2 \ln [u_n(y)]^{-1}}, \quad n = 1, 2, \cdots,$$
$$u_0(y) = 1. \tag{2.30}$$

The first approximation to a solution is

$$u_1(y) = (2/\ln y^{-1})^{\frac{1}{2}} \tag{2.31}$$

and the second approximation is

$$u_2(y) = [2/(\ln y^{-1} + \ln \ln y^{-1} - \ln 2)]^{\frac{1}{2}}. \tag{2.32}$$

This suggests that the general solution to Eq. (2.29) can be expressed in the form

$$u_n(y) = \{2/[\ln y^{-1} + \eta_n(y)]\}^{\frac{1}{2}}, \tag{2.33}$$

where

$$\lim_{y\to 0} \frac{\eta_n(y)}{\ln y^{-1}} = 0. \tag{2.34}$$

An inductive argument serves to establish the validity of Eq. (2.33) and also the approximation $\eta_n(y) \sim \ln \ln y^{-1}$.

With these results we have, finally, that for y in a neighborhood of zero

$$f(y) = \left(\frac{2y}{\ln y^{-1}}\right)^{\frac{1}{2}} \left[1 + O\left(\frac{\ln \ln y^{-1}}{\ln y^{-1}}\right)\right] \tag{2.35}$$

so that $F(\epsilon)$ behaves asymptotically as

$$F(\epsilon) \sim \frac{1}{\sqrt{2}} \int_0^{b'} \frac{dy}{y + \epsilon} \left(\frac{1}{(y \ln y^{-1})^{\frac{1}{2}}} + \frac{1}{(y \ln^3 y^{-1})^{\frac{1}{2}}}\right), \tag{2.36}$$

with the singular behavior of $F(\epsilon)$ still determined by the behavior of the integrand at the origin. The second term in the brackets, $(y \ln^3 y^{-1})^{-\frac{1}{2}}$, can be neglected in comparison with $(y \ln y^{-1})^{-\frac{1}{2}}$, as $y \to 0$, so that

$$F(\epsilon) \sim \frac{1}{\sqrt{2}} \int_0^{b'} \frac{dy}{y + \epsilon} \frac{1}{y \ln y^{-1}}. \tag{2.37}$$

In this integral we make the substitution $y = \epsilon x$ which leads to the representation

$$F(\epsilon) \sim \frac{1}{(2\epsilon)^{\frac{1}{2}}} \int_0^{\infty} \frac{dx}{1 + x} \frac{1}{[x(\ln x^{-1} + \ln \epsilon^{-1})]^{\frac{1}{2}}}, \tag{2.38}$$

where the upper limit of integration has been replaced by ∞ since the resulting integral is convergent. At this point we split the range of integration $(0, \infty)$ into $(0, A\epsilon)$ and $(A\epsilon, \infty)$, where A is chosen so that $A \gg 1$ and $A\epsilon \ll 1$. The first integral can be bounded as follows:

$$\int_0^{A\epsilon} \frac{dx}{(1+x)[x(\ln x^{-1} + \ln \epsilon^{-1})]^{\frac{1}{2}}} < \frac{1}{(\ln \epsilon^{-1})^{\frac{1}{2}}} \int_0^x \frac{dx}{\sqrt{x}} = 2\left(\frac{A\epsilon}{\ln \epsilon^{-1}}\right)^{\frac{1}{2}}. \quad (2.39)$$

In the second range of integration, we can estimate

$$\begin{aligned}\int_{A\epsilon}^{\infty} \frac{dx}{(1+x)[x(\ln x^{-1} + \ln \epsilon^{-1})]^{\frac{1}{2}}} &= \frac{1}{(\ln \epsilon^{-1})^{\frac{1}{2}}} \int_{A\epsilon}^{\infty} \frac{dx}{(1+x)[x(1 + \ln x^{-1}/\ln \epsilon^{-1})]^{\frac{1}{2}}} \\ &\sim \frac{1}{(\ln \epsilon^{-1})^{\frac{1}{2}}} \int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}} + O\left(\frac{1}{\ln \epsilon^{-1}}\right) \\ &= \frac{\pi}{(\ln \epsilon^{-1})^{\frac{1}{2}}}\left[1 + O\left(\frac{1}{(\ln \epsilon^{-1})^{\frac{1}{2}}}\right)\right]. \quad (2.40)\end{aligned}$$

We therefore see, by comparing the last two equations, that the dominant contribution comes from the second range of integration and that

$$F(\epsilon) \sim \pi/(2\epsilon \ln \epsilon^{-1})^{\frac{1}{2}}. \quad (2.41)$$

But, by (2.3), this implies that, for $\alpha = 2$,

$$S_n \sim (2Bn \ln n)^{\frac{1}{2}}. \quad (2.42)$$

When $\alpha > 2$, results in earlier references imply that S_n is asymptotically proportional to $\sqrt{n}$.

3. A 2-DIMENSIONAL SET OF TRANSITION PROBABILITIES

We next consider a 2-dimensional generalization of the jump probabilities given in Eq. (2.5),

$$p(n, m) = B[(n^2 + m^2 + D^2)^{-\beta} + \epsilon_{nm}], \quad (3.1)$$

where we will assume that the ϵ_{nm} satisfy

$$\sum_{n,m} n^2\epsilon_{nm}, \quad \sum_{n,m} nm\epsilon_{nm}, \quad \sum_{n,m} m^2\epsilon_{nm} < \infty. \quad (3.2)$$

The constant β is chosen to satisfy

$$2 \geq \beta > 1, \quad (3.3)$$

which implies that the covariances associated with $p(n, m)$ are infinite. We notice that the inserted constant D^2 implies that $p(0, 0) \neq 0$. This is done to simplify the algebra. No loss of generality follows from the particular form of Eq. (3.1) since the asymptotic dependence on n does not depend on D, although the coefficients may be functions of this parameter.

In order to calculate the function $P(1 - n^{-1})$, we must study the behavior of $\lambda(\theta)$ defined in Eq. (2.1), noting that $\lambda(0) = 1$. To do so, we invoke the 2-dimensional form of the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n, m) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \iint_{-\infty}^{\infty} f(x, y) \exp[2\pi i(rx + sy)]\, dx\, dy, \quad (3.4)$$

where it is assumed that both sides of this equation exist. The contribution to $\lambda(\theta)$ that determines the asymptotic form of $P(1 - n^{-1})$ comes from the sum

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\exp[i(n\theta_1 + m\theta_2)]}{(n^2 + m^2 + D^2)^{\beta}}, \quad (3.5)$$

evaluated near $\theta = 0$. Let us therefore analyze the behavior of the function

$$\frac{\Delta\lambda(\boldsymbol{\theta})}{B} = \sum_{n=-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{1 - \exp[i(n\theta_1 + m\theta_2)]}{(n^2 + m^2 + D^2)^{\beta}} \quad (3.6)$$

by applying the transformation (3.4). It is found that[8]

$$\begin{aligned}\frac{\Delta\lambda(\boldsymbol{\theta})}{B} = \frac{2}{2^{\beta-1}\Gamma(\beta)} \frac{1}{D^{\beta-1}} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \Bigg\{ &[2\pi(r^2 + s^2)^{\frac{1}{2}}]^{\beta-1} \\ &\times K_{\beta-1}(2\pi D(r^2 + s^2)^{\frac{1}{2}}) \\ &- \left\{2\pi\left[\left(r - \frac{\theta_1}{2\pi}\right)^2 + \left(s - \frac{\theta_2}{2\pi}\right)^2\right]^{\frac{1}{2}}\right\}^{\beta-1} \\ &\times K_{\beta-1}\left(2\pi D\left[\left(r - \frac{\theta_1}{2\pi}\right)^2 + \left(s - \frac{\theta_2}{2\pi}\right)^2\right]^{\frac{1}{2}}\right)\Bigg\},\end{aligned} \quad (3.7)$$

where $K_{\beta-1}(x)$ is a modified Bessel function of the second kind. If $\beta < 2$, we see that a possible branch-point singularity occurs in the term $r = s = 0$. The exact nature of such a singularity can be determined from the identity[8]

$$K_\nu(x) = (\pi/\sin \pi\nu)[I_{-\nu}(x) - I_\nu(x)] \quad (3.8)$$

for noninteger ν, where $I_\nu(x)$ is a modified Bessel function of the first kind defined by the series

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu + k + 1)} (\tfrac{1}{2}x)^{\nu+2k}. \quad (3.9)$$

The relation (3.8) requires that ν not be equal to an integer. It follows from (3.8) and (3.9) that

$$\lim_{\epsilon \to 0} \left(\frac{\epsilon}{D}\right)^{\beta-1} K_{\beta-1}(\epsilon) = \frac{2^{\beta-1}}{D^{\beta-1}\Gamma(2 - \beta)}. \quad (3.10)$$

Hence, by considering the term $r = s = 0$ on the right-hand side of Eq. (3.7), we find that

$$\Delta\lambda(\boldsymbol{\theta}) \sim a(\theta_1^2 + \theta_2^2)^{\beta-1}, \quad (3.11)$$

where a is a constant. The asymptotic behavior of $P(z)$ near $z = 1$, therefore, depends on the integral

$$P(z) \sim \frac{1}{\pi^2} \iint_0^\pi \frac{d\theta_1\, d\theta_2}{1 - z + a(\theta_1^2 + \theta_2^2)^{\beta-1}}. \quad (3.12)$$

Transforming to polar coordinates and setting $z = 1$, we see that the integral converges for $\beta < 2$, so that

$$\lim_{n\to\infty} P(1 - n^{-1}) = P(1) \quad (3.13)$$

and, therefore, by Eq. (2.2), when $2 > \beta > 1$, it follows that for large n

$$S_n \sim n/P(1), \quad (3.14)$$

where $P(1)$ must be evaluated numerically.

The case $\beta = 2$ is more difficult. For this case we note that, for small[8] ϵ,

$$K_1(\epsilon) \sim \tfrac{1}{2}\epsilon \ln \epsilon. \quad (3.15)$$

Again we see from Eq. (3.7) that the singular behavior of $\Delta\lambda(\theta)$ near the origin is determined by the term $r = s = 0$. If we use Eq. (3.15) to determine this behavior, we find that

$$\Delta\lambda(\boldsymbol{\theta})/B \sim \tfrac{1}{4}\pi(\theta_1^2 + \theta_2^2) \ln (\theta_1^2 + \theta_2^2)^{-1}. \quad (3.16)$$

Thus, we must study the singular behavior of

$$P(z) \sim \frac{1}{\pi^2} \iint_C \frac{d\theta_1\, d\theta_2}{1 - z + b(\theta_1^2 + \theta_2^2) \ln (\theta_1^2 + \theta_2^2)^{-1}}, \quad (3.17)$$

in which $b = \frac{1}{4}\pi B$ and in which C can be chosen to be a circle in the (θ_1, θ_2) plane without changing the nature of the singularity at $z = 1$ (since only the behavior at $\theta = 0$ determines the singularity). The form of the integral suggests a transformation to polar coordinates in which the angular integration is immediate. We are thus led to consider the integral

$$G(\epsilon) = \int_0^R \frac{r\, dr}{\epsilon + r^2 \ln r^{-1}} = \int_0^{R^2} \frac{dv}{2\epsilon + v \ln v^{-1}}, \quad (3.18)$$

in terms of which we have

$$\lim_{z\to 1} P(z) = \frac{2}{\pi b} \lim_{z\to 1} G\left(\frac{1-z}{b}\right) \quad (3.19)$$

and in which R^2 is chosen less than 1 but is otherwise arbitrary. To determine the limiting behavior of $G(\epsilon)$, we use the same device as for the analysis of the integral appearing in Eq. (2.24). Making the substitution

$$v \ln v^{-1} = x, \quad (3.20)$$

we find that, for x small,

$$v \sim \frac{x}{\ln x^{-1}} \quad (3.21)$$

so that

$$G(\epsilon) \sim \int_0^{R_1} \frac{dx}{2\epsilon + x} \frac{1}{\ln x^{-1}}. \quad (3.22)$$

But we can write

$$\frac{1}{\ln x^{-1}} = \frac{1}{\ln (x + 2\epsilon)^{-1}} + \frac{\ln x(x + 2\epsilon)^{-1}}{\ln x^{-1} \ln (x + 2\epsilon)^{-1}}, \quad (3.23)$$

which, when substituted into Eq. (3.22), yields the estimate

$$G(\epsilon) \sim \ln \ln \epsilon^{-1} + R(\epsilon), \quad (3.24)$$

where

$$R(\epsilon) = \int_0^{R_1} \frac{dx}{2\epsilon + x} \frac{\ln x(x + 2\epsilon)^{-1}}{\ln x^{-1} \ln (x + 2\epsilon)^{-1}}. \quad (3.25)$$

It is shown in Appendix B that $R(\epsilon) = O(1)$ as $\epsilon \to 0$ so that asymptotically $G(\epsilon) \sim \ln \ln \epsilon^{-1}$, which implies that the asymptotic expected number of distinct sites visited is, for $\beta = 2$,

$$S_n \sim \tfrac{1}{8}\pi^2 Bn/\ln \ln n. \quad (3.26)$$

For $\beta > 2$, S_n is asymptotic to $n/\ln n$, as is shown in Ref. 3.

Finally, we note that it has been shown that the asymptotic expected number of points visited exactly once in an n-step random walk is

$$V_n \sim n/P^2(1 - n^{-1}). \quad (3.27)$$

Since we have calculated the function $P(1 - n^{-1})$ for a number of cases, we only list the results. In one dimension, for the jump probabilities given by Eq. (2.5), we have

$$\begin{aligned} &V_n \sim n/P^2(1), && 1 > \alpha > 0, \\ &V_n \sim B^2 n/(4 \ln^2 n), && \alpha = 1, \\ &V_n \sim C^2 n^{(2/\alpha)-1}, && 2 > \alpha > 1, \\ &V_n \sim 2B \ln n, && \alpha = 2. \end{aligned} \quad (3.28)$$

In two dimensions, for jump probabilities given by Eq. (3.1), we have

$$\begin{aligned} &V_n \sim n/P^2(1), && 2 > \beta > 1, \\ &V_n \sim \pi^2 b^2 n/(4 \ln \ln^2 n), && \beta = 2. \end{aligned} \quad (3.29)$$

For $\alpha > 2$ in one dimension and for $\beta > 2$ in two dimensions, results for V_n are given in Ref. 3.

In concluding, we note that the methods of the present paper allow us to analyze results for the 2-dimensional jump probabilities

$$p(n_1, n_2) = A(|n_1| + |n_2|)^{-\alpha},$$

where A and α are constants. It is quite possible that results can also be obtained for jump probabilities of the form that we have considered, but multiplied by a slowly varying function of its parameters. Such investigations would involve analyses of the kind found in Zygmund[9] and appear to be beyond the scope of any method based on Karamata's theorem.

APPENDIX A: LIMITING PROPERTY OF $g_\alpha(\theta)$

Let us write

$$g_\alpha(\theta) = h_\alpha(\theta) + [g_\alpha(\theta) - h_\alpha(\theta)]. \quad \text{(A1)}$$

We will show that

$$\lim_{\theta\to 0} \frac{g_\alpha(\theta) - h_\alpha(\theta)}{h_\alpha(\theta)} = 0, \quad \text{(A2)}$$

for $2 \geq \alpha > 0$. The difference $g_\alpha(\theta) - h_\alpha(\theta)$ can be written explicitly as, say,

$$\begin{aligned} g_\alpha(\theta) - h_\alpha(\theta) &= \frac{\theta^2}{2\Gamma(1+\alpha)} \int_0^\infty t^\alpha e^{-t} \\ &\times \left(\frac{t(t^2+\theta^2)(1+e^{-t}) - 2(1-e^{-t})[(1-e^{-t})^2 + \theta^2 e^{-t}]}{t(1-e^{-t})(t^2+\theta^2)[(1-e^{-t})^2+\theta^2 e^{-t}]} \right) dt \\ &= \frac{\theta^2}{2\Gamma(1+\alpha)} I(\theta). \end{aligned} \quad \text{(A3)}$$

It is clear that, for any θ, the integrand is such that there is no trouble with convergence at the upper limit. The only possible difficulty can arise from the lower limit. If we approximate e^{-t} by $1 - t$ near the origin, then the integrand is approximately $2/(t^2 + \theta^2)$ in the neighborhood of the origin. Hence, if $I(\theta)$ diverges, the divergent behavior will be that of the integral

$$J(\theta) = \int_0^\infty \frac{t^\alpha e^{-t}}{t^2 + \theta^2}\, dt = \theta^{\alpha-1} \int_0^\infty \frac{x^\alpha e^{-\theta x}}{x^2 + 1}\, dx. \quad \text{(A4)}$$

For $2 \geq \alpha > 1$,

$$J(\theta) \to J(0) = \int_0^\infty t^{\alpha-2} e^{-t}\, dt = \text{const}, \quad \text{(A5)}$$

so that $g_\alpha(\theta) - h_\alpha(\theta) = O(\theta^2)$ for $\theta \to 0$ and

$$\frac{g_\alpha(\theta) - h_\alpha(\theta)}{h_\alpha(\theta)} = O(\theta^{2-\alpha}). \quad \text{(A6)}$$

For $\alpha = 1$,

$$J(\theta) = \int_0^\infty \frac{x e^{-\theta x}}{1 + x^2}\, dx \sim \ln \theta^{-1} + O(1) \quad \text{(A7)}$$

by Eq. (A4) so that since $h_1(\theta) \to$ const, the ratio in Eq. (A2) tends to 0 as $\theta^2 \ln(\theta^{-1})$. When $\alpha < 1$, $J(\theta) \sim a\theta^{\alpha-1}$, where a is a constant such that

$$g_\alpha(\theta) - h_\alpha(\theta) \sim b\theta^{1+\alpha}, \quad \text{(A8)}$$

where b is a constant. Since $h_\alpha(\theta)$ is $O(\theta^\alpha)$, the ratio tends to 0 as $\theta \to 0$.

APPENDIX B: PROOF THAT $R(\epsilon)$ IS BOUNDED AS $\epsilon \to 0$

The upper limit R_1 appearing in the definition of R_1 in Eq. (3.25) is strictly less than 1. Hence, ϵ can always be chosen small enough to ensure that $R_1 + 2\epsilon < 1$. Thus, we can write

$$\begin{aligned} |R(\epsilon)| &= \left| \int_0^{R_1} \frac{dx}{2\epsilon + x} \frac{\ln x(x+2\epsilon)^{-1}}{\ln x^{-1} \ln (x+\epsilon)^{-1}} \right| \\ &\leq \frac{1}{\ln R_1^{-1} \ln (R_1 + \epsilon)^{-1}} \left| \int_0^{R_1} \frac{dx}{2\epsilon + x} \ln \frac{x}{x + 2\epsilon} \right| \\ &\leq \frac{1}{\ln R_1^{-1} \ln (R_1 + \epsilon)^{-1}} \left| \int_0^\infty \frac{dy}{y+1} \ln \frac{y}{y+1} \right|. \end{aligned} \quad \text{(B1)}$$

The indicated integral converges, so that $|R(\epsilon)|$ is bounded as $\epsilon \to 0$. Indeed, a more careful analysis reveals that $\lim R(\epsilon) = 0$ as $\epsilon \to 0$.

* Permanent address: Physical Sciences Laboratory, Division of Computer Research and Technology, National Institutes of Health, Department of Health, Education and Welfare, Bethesda, Maryland 20014.

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