Aperiodic Tilings on the Computer

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Summary. We briefly review the standard methods used to construct quasiperiodic tilings, such as the projection, the inflation, and the grid method. A number of sample *Mathematica* programs, implementing the different approaches for one-and two-dimensional examples, are discussed. Apart from small examples, the corresponding programs are not contained in the text, but are archived on the WWW.

1. Introduction

Structure models of quasicrystals are based on tilings of space, either on perfect quasiperiodic patterns or on random tilings. However, besides their physical applications, many of these tilings are also aesthetically appealing – not by chance Penrose's original article [29] is entitled "The Rôle of Aesthetics in Pure and Applied Mathematical Research"! Thus, even a reader who does not plan to enter the subject of quasicrystals on a deeper level might be interested to know how such tilings can be produced on the computer.

The computer programs discussed here were provided for two afternoon sessions of the summer school. Because no knowledge of particular programming languages could be assumed, and since good graphic tools were essential for this purpose, we decided to use an algebraic computer package that provides all one needs. Which of the commercially available packages one chooses is mainly a matter of taste, and we decided in favour of $Mathematica^1$ [36]. Note that the actual Mathematica programs, apart from a small example, are not reproduced in this text – instead, the routines are archived on the WWW [16] and can be downloaded free of charge. At present, this directory contains a total of six files:

- Read.Me
- ChairTiling.m
- FibonacciChain.m
- OctagonalTiling.m
- GridMethod.m
- PenrosePuzzle.m

including some documentation, but we may choose to update or supplement the content in the future.

During the summer school, the participants worked with *Mathematica* 'notebooks'. The corresponding front end of *Mathematica* provides a rather

 $^{^{1}}$ $Mathematica^{\circledR}$ is a registered trademark of Wolfram Research.

nice programming environment. Nevertheless, there are several reasons why we prefer to use a standard package format [27] for the archived programs: first of all, these are much smaller in size; secondly, they are less version-dependent (the notebook format in *Mathematica* has changed recently, and notebooks prepared with version 3.0 of *Mathematica* cannot be used with earlier versions); and finally, as these are simple ASCII files, file transfer should pose no problems. Furthermore, if desired, our programs can of course be included into the notebook environment by the user himself easily.

After a short survey on the different approaches employed to construct quasiperiodic tilings, we first present a simple example – a *Mathematica* program for the inflation of the so-called chair tiling [17]. This is followed by concise descriptions of our *Mathematica* programs. We also briefly sketch the theory behind the constructions, for a more detailed account we refer the reader to the literature compiled in [14], and to the other chapters of this volume.

2. How to Construct Quasiperiodic Tilings

Whether one wants to describe the structure of real (albeit idealized) quasicrystals, or whether one is just after a more elaborate way to tile the bathroom floor, one faces the problem of constructing quasiperiodic tilings. There exists a variety of mathematical procedures [2, 18, 32], the most popular being the projection (or cut-and-projection) method [7, 10, 20, 23, 25]. Here, one starts from a higher-dimensional periodic lattice, cuts a certain slice out of it, and projects this slice onto a lower-dimensional (the 'physical' or 'parallel') space (the remaining directions are usually referred to as 'orthogonal', 'perpendicular', or 'internal' coordinates). In fact, there are several slightly different, albeit equivalent, formulations of this method, for details see [2].

The simplest example – and in fact the only one that can easily be visualized on the computer – is given by a projection from a periodic two-dimensional (2D) lattice to a 1D quasiperiodic structure, the usual toy model being the Fibonacci chain which is related to the golden mean $\tau = (1+\sqrt{5})/2$. This is part of one of the programs which is described in detail in Sect. 4. It also contains the description of the Fibonacci chain as a substitution sequence, which is a common method to construct aperiodic (in general not quasiperiodic) self-similar sequences. These serve as toy models of 1D aperiodic order in many areas of physics [3].

Inflation rules [17] provide a concept to construct self-similar, and in particular quasiperiodic tilings in the same spirit. The rules describe in which way each tile of a given set is decomposed into scaled copies of the tiles, such that, after applying the rules to a patch of a tiling, one obtains a patch consisting of the same tiles, but with all lengths scaled by a common factor. Iterating this procedure on some initial patch, rescaling after each step such

that the tiles stay the same, one generates larger and larger patches approaching an infinite self-similar tiling. In Sect. 5, an example for both the projection and the inflation approach is given for the 2D octagonal (Ammann–Beenker) tiling [1, 4, 8, 21].

Another approach, the dualization or grid method, is due to de Bruijn [6], see also [22]: Rotating equidistant parallel lines by angles $2\pi k/n$, $k = 0, 1, \ldots, n-1$, yields an n-fold grid. Its dualization gives an n-fold tiling in which each p-gon of the grid corresponds to a vertex with p neighbours in the tiling. This method can very easily be translated into a computer program, see Sect. 6, and it is also straightforward to generalize it to higher dimensions.

Finally, quasiperiodic tilings may also be constructed via so-called matching rules [17, 29, 30]. Starting with a fixed set of proto-tiles, these give an atlas of the allowed local configurations. If these are *perfect* matching rules, the quasiperiodic tiling is essentially determined; more precisely, it is the corresponding 'local indistinguishability' or 'local isomorphism' (LI) class that is fixed [2]. On first view, this looks like a constructive prescription that should allow to model the growth of such structures, but this turns out to be a fallacy because it contains an inherent non-locality, compare [15]. In practice, this means that an erroneous tile addition during construction may only show up much later and at a completely different place, see e.g. [31] for a nice example. In Sect. 7, we present a program which allows the reader to try it himself for the rhombic Penrose tiling.

3. A Small Example Program: Inflating the Chair Tiling

Before we address our main subject, the generation of quasiperiodic tilings, we first present a small program to give the reader a taste of the syntax and the structure of the programs. Clearly, we cannot give an extensive introduction into the *Mathematica* language, for this the reader is referred to [27, 34, 36] and to the vast literature on *Mathematica* (including a regularly published journal with electronic supplements) and its applications in science, particularly in mathematics and theoretical physics – it is beyond the scope of this article to list all of these. Let us only mention a few features that we use frequently, so that even the non-incriminated reader gets some feeling how to change the provided *Mathematica* programs if he wants to interfere and produce his own tilings accordings to his own aesthetic pretensions.

The main ingredient in our programs is the manipulation of lists of objects, as we represent a tiling in a natural way as a list of its tiles (and the tiles, occasionally, as a list of its vertices, which in turn are lists of their coordinates). For this, functions are used, which are either explicitly defined (like, for instance, $f[x_{-}]:=x^2$ which defines the function $f(x)=x^2$), or are used as so-called 'pure functions'. The latter are expressions delimited by an ampersand (&) involving arguments #1 (or just #), #2, #3, and so on (for instance, to compute $x^2 + y^2$ one may use $(\#1^2+\#2^2)\&[x,y]$ or,

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synonymously, Function[#1^2+#2^2][x,y]). Such functions can easily be applied to the elements of a list by using the command Map, for example, Map[#^3&,{1,2,3,4,5}] or, shorter, #^3&/0{1,2,3,4,5} results in the output {1,8,27,64,125}. There is also a functional operation Nest that allows one to apply a function iteratively, as we are going to do with the inflation rule. For instance, Nest[f,x,3] is equivalent to f[f[f[x]]].

For simplicity, we consider a tiling that consists of a single tile only. We choose the so-called chair tiling, and generate it by iterated application of the inflation rule given in Fig. 3.1. Note that the chair tiling, albeit aperiodic, is not quasiperiodic – it is an example of a limit-periodic tiling, which means that its Fourier module, though discrete, is not finitely generated [12]. The 'chair-formed' tile, which occurs in four different orientations, can be thought of as a combination of three congruent squares, hence the tiling belongs to the class of so-called triomino (a term that is deduced from the word 'domino', the general case is referred to as 'polyomino') tilings [13].



Fig. 3.1. Inflation rule of the chair tiling.

An implementation of the inflation rule of Fig. 3.1 is shown in Fig. 3.2. ScaleFactor gives the linear scaling factor of the inflation rule, which is 2 in this case. In an inflation step, we rescale all lengths by this factor such that, provided one starts with a patch with integer coordinates of the vertices, the coordinates of the inflated patch are integers. A single tile is represented in the form $\{or, \{x0,y0\}\}\$, where or denotes the angular orientation of the tile, with values 0, 1, 2, and 3 (mod4), and {x0,y0} denote the coordinates of the reference point. The function TileCoordinates computes the actual coordinates of the six vertices of a tile, which is used in the graphics function PlotTiling. All displacements are expressed in terms of four 2D vectors, namely TwoVector [0] = $\{1,1\}$ and its m-times 90° -rotated copies TwoVector[m]. The inflation rule is encoded in the function TileInflation which produces a list of the four tiles into which a single tile is dissected. Since tiles in all orientations are dissected in the same way, it is sufficient to use a single definition by employing the rotated vectors TwoVector[m]. To perform an n-fold inflation of a patch, consisting of a list of one or more tiles, the function Inflation is used, with n as its second argument. Finally, PlotTiling enables one to plot the tiling, with optional arguments that allow to give different colours to the four orientation of the tile and to change the colour and width of the lines outlining the tiles.

```
{\tt Clear[ScaleFactor,TwoVector,TileCoordinates,TileInflation,}
      Inflation,PlotTiling];
ScaleFactor = 2;
TwoVector[num_Integer] :=
  TwoVector[num] =
 Dot[MatrixPower[{{0,-1}},
                   {1, 0}},num],{1,1}];
TileCoordinates[{tile_Integer,refpoint_List}] :=
  Map[(refpoint+#)&,
      {0,(#3+#4)/2,#4,#1,#2,(#2+#3)/2}]&[TwoVector[tile],
                                          TwoVector[tile+1],
                                          TwoVector[tile+2],
                                          TwoVector[tile+3]];
TileInflation[{tile_Integer,refpoint_List}] :=
  {{#1,#4},
   {#2,#4+TwoVector[#2]},
   {#1,#4+TwoVector[#1]},
   {#3,#4+TwoVector[#3]}}&[Mod[tile,4],
                           Mod[tile-1,4],
                           Mod[tile+1,4],
                           ScaleFactor*refpoint];
Inflation[tiling_List,
          num_Integer:1] /; num>=0 :=
 Nest[Flatten[Map[TileInflation,#],1]&,tiling,num];
PlotTiling[tiling_List,
           tilecol_List:Table[GrayLevel[1],{4}],
           linecol_:GrayLevel[0],
           linewidth_:1/200] :=
  Graphics[Table[{tilecol[[i+1]],
                  Map[Polygon,#],
                  linecol,Thickness[linewidth],
                  Map[Line[Join[#,Take[#,2]]]&,#]}&[
                    Map[TileCoordinates,
                        Select[tiling,Mod[First[#],4]==i&]]],
                 {i,0,3}],
           AspectRatio->1];
```

Fig. 3.2. The *Mathematica* program ChairTiling.m [16] for the inflation of the chair tiling. It is also available in electronic form [16].

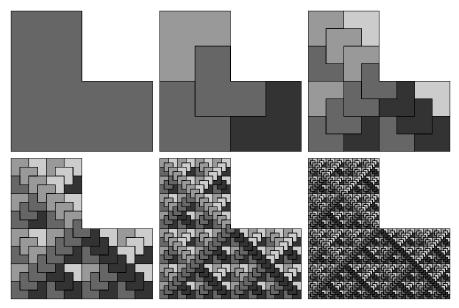


Fig. 3.3. Inflation of the chair tiling. The four orientations of the tile are distinguished by different shadings.

The result can be seen in Fig. 3.3, where the first five inflation steps of a single tile are shown, giving different shadings to the four orientations of the tile. This figure has been produced by the following input

within a Show or Display command to display the result on the screen or write it to a PostScript file, respectively. Note that our rescaling by ScaleFactor has effectively been reversed by *Mathematica*, since the different subgraphs in a GraphicsArray arrangement are scaled to the same total size. For completeness, we mention that Fig. 3.1 was produced by

where the second part of the argument of GraphicsArray produces the arrow.

This completes our little excursion into the mysteries of *Mathematica*. Of course, the preceding remarks can only offer a glimpse at the possibilities of this algebraic computer package, although our example program of Fig. 3.2 – on purpose – is not written in the simplest possible way, but already contains some little tricks to show at least some of the ingredients of the programs discussed below.

4. Once Again: The Ubiquitous Fibonacci Chain

The toy model of a 1D quasicrystal is the Fibonacci chain – one will hardly find any introductory text on quasicrystals where this example is not discussed. The usual way to introduce the Fibonacci sequence employs an inflation-like procedure, a so-called substitution rule

$$\varrho: \begin{cases} S \to L \\ L \to LS \end{cases} \tag{4.1}$$

on two letters S and L, generating a semi-infinite word w_{∞} by iterated application $w_{n+1} = \varrho(w_n)$ on some initial word, say $w_0 = S$,

$$\begin{aligned}
 w_0 &= S & w_4 &= LSLLS \\
 w_1 &= L & w_5 &= LSLLSLSL \\
 w_2 &= LS & w_6 &= LSLLSLSLLSLLS \\
 w_3 &= LSL & w_7 &= LSLLSLSLLSLLSLLSLSLS
 \end{aligned} \tag{4.2}$$

and so forth. Obviously, in place of the substitution rule (4.1), one can use concatenation of subsequent words to generate the same sequence

$$w_0 = S, w_1 = L, w_{n+1} = w_n w_{n-1}, (4.3)$$

where $w_n w_{n-1}$ denotes the word obtained by appending w_{n-1} to w_n . From this, it is easy to see that the length of the word w_n (i.e., the number of its letters) is given by the Fibonacci number $|w_n| = f_{n+1}$ defined by the recursion

$$f_0 = 0$$
, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$, (4.4)

and that w_n contains precisely f_n letters L and f_{n-1} letters S [for n > 0, otherwise one has to extend the definition (4.4) to negative values of n, which amounts to setting $f_{-n} = (-1)^{n+1} f_n$ for n > 0]. Thus, the frequency of the predominant letter L in the limit word w_{∞} is given by

$$\nu_{L} = \lim_{n \to \infty} \frac{f_{n}}{f_{n+1}} = \lim_{n \to \infty} \frac{f_{n}}{f_{n} + f_{n-1}}$$

$$= 1 + \left(\lim_{n \to \infty} \frac{f_{n-1}}{f_{n}}\right)^{-1} = 1 + \nu_{L}^{-1}$$

$$\Rightarrow \nu_{L} = \frac{\sqrt{5} - 1}{2} = \tau - 1 = \tau^{-1}$$
(4.5)

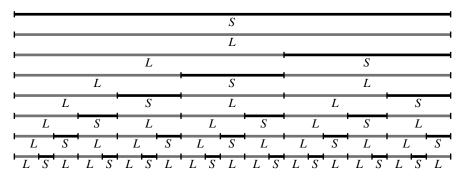


Fig. 4.1. Inflation of the Fibonacci chain.

where $\tau = (1 + \sqrt{5})/2 = \lim_{n \to \infty} (f_{n+1}/f_n)$ is the golden ratio. As this is an irrational number, we thus proved that the Fibonacci sequence is aperiodic. Such statistical properties of the sequence can also be derived from the associated substitution matrix

$$M_{\varrho} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \lambda^{\pm} = \pm \tau^{\pm 1}, \qquad v^{+} = \begin{pmatrix} \tau^{-2} \\ \tau^{-1} \end{pmatrix},$$
 (4.6)

whose elements just count the number of letters S and L appearing in $\varrho(S)$ and $\varrho(L)$, respectively. Its largest eigenvalue λ^+ , related to the exponential growth of the word length in a substitution step, is just τ ; and the elements of the corresponding eigenvector v^+ , when properly normalized, encode the frequencies of the two letters, cf. (4.6). Obviously, substitution rules with irrational maximum eigenvalue of the corresponding substitution matrix generically give aperiodic (but, in general, not quasiperiodic) chains. However, a rational or integer maximum eigenvalue does not imply that the sequence one obtains is periodic; the Thue–Morse sequence with substitution $a \to ab$, $b \to ba$, is a prominent example: it is aperiodic though the two eigenvalues of the substitution matrix (whose elements are all equal to 1) are 2 and 0.

In the file FibonacciChain.m [16], a program SubstitutionSystem is contained that defines several functions (SubstitutionRule, Substitution, SubstitutionMatrix, SubstitutionSequence, SubstitutionWord) to generate arbitrary substitution sequences. The substitution rule and an initial word have to be given as parameters, the defaults (which are {"S"->{"L"}, "L"->{"L", "S"}} for the rule and {"S"} as initial word) yield the Fibonacci sequence as in Eqs. (4.1) and (4.2). The recursive definition (4.3) is also implemented (FibonacciRecursionSequence, FibonacciRecursionWord). Here, the Fibonacci (or other substitution) words may be represented in two ways; for instance, $w_4 = LSLLS$ may either be represented as a list of single letters {"L", "S", "L", "L", "S"} (e.g., FibonacciRecursionSequence[4]) or as a string {"LSLLS"} (e.g., FibonacciRecursionWord[4]). There are a number of functions to analyze the statistical properties of the sequence which are written such that they can simultaneously deal with both representa-

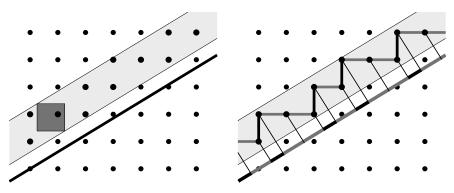


Fig. 4.2. Projection of the Fibonacci chain from the square lattice in the 'strip projection'. The slope of the 'physical space' is τ^{-1} . For details see text.

tions of the words. For example, SubWordCount[w1,w2] counts the number of occurrences of the word w2 in the word w1. Furthermore, the Fibonacci numbers are defined (FibonacciNumber[n] gives f_n), and a small program PlotLinearChain for a geometric representation of n-letter sequences as linear arrangements of n intervals is included. This we used to prepare Fig. 4.1 by

where the length ratio of the two intervals is the golden mean τ , with L corresponding to the 'long' and S to the 'short' interval, respectively. In this way, the substitution rule may be interpreted as an inflation procedure for the geometric object; and, upon suitable rescaling of lengths (i.e., by a factor of τ), the semi-infinite line representing the limit word w_{∞} is a fixed point under this procedure.

This 'geometric Fibonacci chain', however, can also be obtained by a suitable projection from the 2D square lattice. As mentioned previously, there are several variants of projection schemes, which are essentially equivalent, see Figs. 4.2, 4.3, and 4.4. The file FibonacciChain.m [16] contains the three functions StripProjection, AtomicSurfaceProjection, and KlotzConstruction, which show, in a sequence of pictures, three variations of the projection to a line. Among others, the slope can be given as a parameter, the default value $1/\tau$ again yields the Fibonacci chain. As commonly, the 'window' or 'acceptance domain', determining which part of the square lattice is projected, is chosen as the projection of a unit cell (e.g. the dark grey square in the left part of Fig. 4.2) of the square lattice onto 'perpendicular space', i.e., into the direction orthogonal to the cut line which corresponds to the 'physical space' and is indicated by the thick line in the left part of Fig. 4.2. In

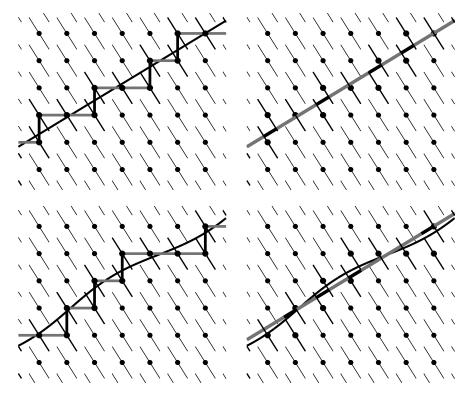


Fig. 4.3. Projection of the Fibonacci chain from the square lattice in the 'atomic hypersurface' scheme. Below, small 'phasonic' fluctuations in the cut space lead to 'phason jumps', creating subwords (LSLSLS and LLL) that are not present in a perfect Fibonacci sequence.

the 'strip projection' shown in Fig. 4.2, this determines the width of the strip, its vertical position can be chosen as a parameter. The actual sequence obtained by projection depends on this parameter, but one always stays within the same LI class [2], except for some 'singular' cases which are obtained if one lattice point falls precisely on the boundary of the strip. In the method of the 'atomic (hyper)surfaces' [5, 19] shown in Fig. 4.3, the cut line is dissected by 'atomic surfaces' attached to the vertices of the square lattice. The function AtomicSurfaceProjection also allows to introduce fluctuations of the cut space (mimicking 'phason' degrees of freedom, see [35]), an example is shown in the lower part of Fig. 4.3. Finally, the 'Klotz construction' or 'dualization' scheme [24, 26, 28, 33] shown in Fig. 4.4 employs a different unit cell of the square lattice (e.g. consisting of the two dark grey squares in the left part of Fig. 4.4), chosen such that all its boundaries are parallel or perpendicular to the 'physical space'. In addition to the pictures shown in Fig. 4.4, the function KlotzConstruction also gives a detailed description of the construction of this unit cell.

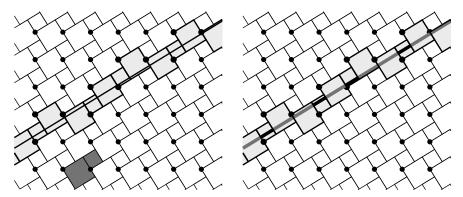


Fig. 4.4. Projection of the Fibonacci chain from the square lattice in the 'Klotz construction' or 'dualization' scheme.

We now turn our attention to arguably more interesting examples of 2D quasiperiodic tilings, both from an aesthetic and a physical point of view. Nevertheless, the methods we are going to use are very similar to the 1D case considered above, particularly the projection scheme is a straightforward generalization of the method shown in Figs. 4.2, 4.3, and 4.4.

5. The Octagonal Tiling: Projection and Inflation

We now consider the canonical octagonal (also known as Ammann–Beenker) tiling [1, 4, 8, 21]. The corresponding programs are contained in the file OctagonalTiling.m [16]. The tiling can be obtained by projection from the 4D hypercubic lattice, choosing the orientation of the 'physical space' as one of the two unique invariant subspaces with respect to the action of the dihedral group D_8 (the symmetry group of the regular octagon), which is a subgroup of the point group of the 4D hypercubic lattice. The acceptance domain is the orthogonal projection of the hypercube, it is a regular octagon as shown in Fig. 5.1, which was produced with the function PlotCubeOrthogonalProjection.



Fig. 5.1. Projection of the 4D hypercube.

The projection of a small portion of the hypercubic lattice, together with the orthogonal projection of the vertices, is shown in Fig. 5.2. The patch

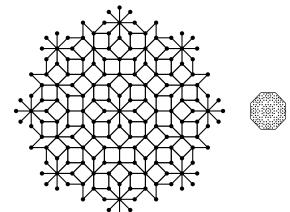


Fig. 5.2. Parallel (vertices and edges, left side) and perpendicular (vertices only, right side) projection of part of the hypercubic lattice, drawn at the same scale. The octagon is the projection of the 4D hypercube, cf. Fig. 5.1.

comprises 177 vertices and 536 edges; it was obtained by considering all points that can be reached by at most eight steps along the hypercubic edges from the origin which corresponds to the center of this eightfold symmetric patch. Here, we used the commands PlotParallelProjection and PlotOrthogonalProjection defined in OctagonalTiling.m [16].

The octagonal tiling can also be obtained by inflation. The inflation rule, i.e., the description of how to cut tiles into smaller pieces, can be read off from Fig. 5.3, which shows three subsequent inflation steps. Apparently, the inflation rule is rather complicated: first of all, the parts obtained by inflating

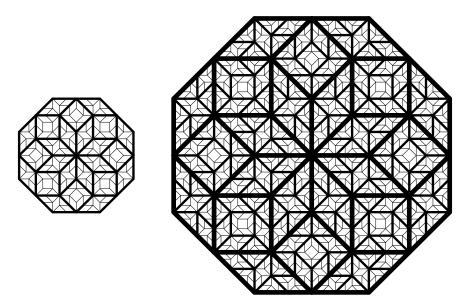


Fig. 5.3. Two (left) and three (right) inflation steps of the octagonal tiling.

a tile cover sometimes only half a tile at the smaller scale, and secondly, the square tile is dissected in an asymmetric way. The best way to represent the inflation rule is therefore in terms of two triangular tiles obtained by cutting the rhomb along the short diagonal, and the square such that the two halves, when oriented properly, are dissected in the same way. This inflation procedure has been implemented in OctagonalInflation, and only in the end, when using PlotOctagonalTiling to display the tiling, one may choose to recombine the triangles into squares and rhombs as it has been done in Fig. 5.3.

In the programs, the tilings are always treated in terms of integer coordinates, only for the purpose of the graphical presentation it is necessary to translate these into real 2D coordinates. In the projection approach, it is natural to use the coordinates of the 4D hypercubic lattice point, whereas in the inflation approach, we instead performed all calculations in the module $\mathbb{Z} + \sqrt{2}\mathbb{Z}$.

6. De Bruijn's Ingenuity: The Dualization Method

Another method that is particularly easy to program is de Bruijn's famous grid (or dualization) method [6, 22]. Here, one starts from an n-fold grid that is created rotating a set of equidistant parallel lines by angles $2\pi k/n$, $k=0,1,\ldots,n-1$, and shifting them by certain amounts. The sum of these shifts, Γ , plays an important role – in general, the tilings constructed with different values of Γ (modulo 1) belong to different LI classes. A grid is called 'regular' if there are no intersection points where more than two lines meet. From this, an n-fold tiling is obtained by dualization, so that each p-gon of

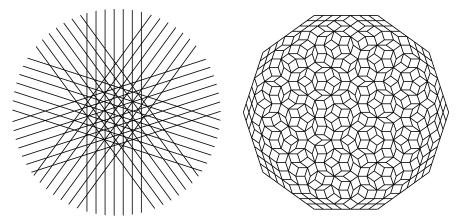


Fig. 6.1. A pentagrid (left) and the corresponding part of a ten-fold tiling (right, disregarding the boundary region), which belongs to the LI class of the Penrose tiling ($\Gamma = 0$).

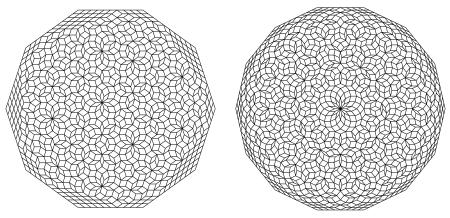


Fig. 6.2. Two tilings obtained by the grid method: a ten-fold tiling (left) in the LI class of the 'anti-Penrose' tiling ($\Gamma = 1/2$) and an example for n = 7 (right), again disregarding the boundary region which is due to the finite grid.

the grid yields a vertex with p neighbours in the tiling. Its coordinates can be calculated easily. For a regular grid, each intersection point of the grid thus corresponds to a rhombic tile. Note that for odd values of n one could as well have considered the case 2n since a rotation by an angle of π is irrelevant.

In Fig. 6.1, we present a grid and the corresponding rhombic tiling for the Penrose case, i.e., for n=5 where the shifts (grid parameters) satisfy the condition $\Gamma = 0$. The portion of a regular lattice on the boundary of the patch is due to the finiteness of the grid (because we used only seven grid lines for each direction). The picture has been produced with the commands PlotGrid and PlotDualTiling defined in GridMethod.m [16]. Note that in the dualization, performed in DualizeGrid, the regularity of the grid is not verified, although it is assumed in the construction, hence results are unpredictable in case the grid is not regular. Fig. 6.2 shows, on the left, another example for n=5, but now with $\Gamma=1/2$ which is 'maximally remote' from the Penrose case $\Gamma = 0$ wherefore it is occasionally referred to as an 'anti-Penrose tiling'. Though it is built from the same two rhombic tiles as the Penrose tiling, it obviously belongs to a different LI class because it contains vertex configurations forbidden in a perfect Penrose tiling, as for instance the star of ten 'thin' rhombs. On the right of Fig. 6.2, an example for n=7 is shown. We note that the grid method is equivalent to the projection scheme, see e.g. [33] for details.

7. Giving it a Trial: A Penrose-Tiling Puzzle

Finally, we like to invite the reader to take the part of the pitiable craftsman who, unsuspectingly, accepted the honourable commission to tile a hypocrite

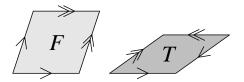


Fig. 7.1. The two Penrose rhombs ('fat' and 'thin') with arrow decorations encoding the matching rules.

quasicrystallographer's bathroom floor with a portion of a rhombic Penrose tiling. Prudently, the quasicrystallographer provided two sets of rhombic tiles with edges that only fit in certain ways, thus encoding the matching rules of the Penrose tiling as in Fig. 7.1.

To understand the desperate situation of the tiler, the reader should use the program PenrosePuzzle contained in the file PenrosePuzzle.m [16]. One may give an arbitrary initial patch as input, default is a single 'fat' rhomb. Once started, the program asks whether to add or to remove tiles. An addition is specified by the letter A followed by the type of the new tile (F for the 'fat' and T for the 'thin' rhomb, respectively) and by the number of the surface edge at which the new tile should be attached. This leaves only one possibility to attach the tile without creating an overlap with the tile that it is attached to; and for each addition the program checks whether it leads to an arrow mismatch in the resulting tiling, or whether an overlap between the new tile and a tile of the previous patch occurs. If that happens, an error message is produced and one may choose whether to continue with the addition. A tile

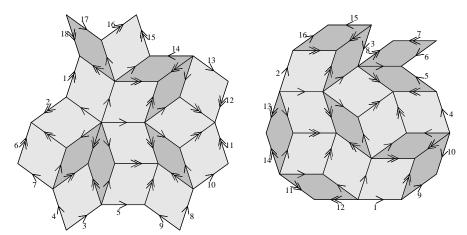


Fig. 7.2. Two patches consisting of Penrose rhombs, grown with PenrosePuzzle. In the right example, the two crevices with surfaces edges (5,6) and and (3,8) cannot be filled with the tiles of Fig. 7.1 – any attempted addition of a tile will produce an error due to a mismatch of the arrows or an overlap of two tiles. Note that it does not suffice to remove the tile with surface edges 6, 7 and 8 – one also has to remove the thin rhomb with surface edge 5 in order to obtain a patch that occurs in a perfect Penrose tiling.

removal is simply specified by the letter R together with the number of one of the surface edges of the tile to remove (thus only tiles that are on the surface of the patch can be removed). The edges are numbered consecutively, the patch with the edge labels is shown after each successful addition or removal of a tile, see Fig. 7.2.

Of course, matching rules just state that one constructed a perfect Penrose tiling when one succeeded to tile the entire plane without violating the rules. However, since both tiles and bathrooms are usually of finite size, the tiler might in the end turn up with a finite 'legal' patch (i.e., the matching rules are not violated) that nevertheless is not 'correct', i.e., it does not occur in a perfect Penrose tiling. In other words, it is impossible to continue the addition of further tiles without violating the matching rules at some point, see the right part of Fig. 7.2 for a simple example and [15, 31] for more details.

8. Concluding Remarks

Clearly, the programs introduced here form by no means a complete package that allows one to create all those tilings that quasicrystallographers play around with – but this was not what we had intended. Instead, it was our purpose to show with a few comparatively simple examples how quasiperiodic tilings can be produced, and thereby give insight into the different approaches that one may choose. In particular, we restricted ourselves to one- and 2D examples, not because it would be substantially more difficult to generate higher-dimensional tilings, but mainly because it requires much more effort to visualize the result on a 2D screen. For the 3D case, commercially available balls-and-sticks kits, e.g. [37], which allow to build icosahedral structures, may still prove more useful than computer-generated pictures – however, there is also a *Mathematica* implementation of the balls-and-sticks tools [11].

Nevertheless, our sample programs [16] may serve as a starting point for the layman who aims to generate some beautiful pictures of tilings, his interest be of mathematical, physical, or purely aesthetical nature. Whether experts may still profit from our small example programs, they have to judge by themselves – we did not spend too much effort to find the most elegant or most concise (but probably also quite unreadable) form of the program code, neither did we try to optimize our programs with respect to CPU time or memory consumption (which sooner or later – more likely sooner – becomes the limiting factor in using *Mathematica*). Still, it should be rather easy to modify our programs in order to treat most of the quasiperiodic tilings one may encounter in the literature.

Finally, we like to note that computer programs generating quasiperiodic tilings can also be found in other places. For example, some *Mathematica* programs are contained in the book [32], and there even exists an interactive WWW site [9] devoted to quasiperiodic tilings.

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