

Discrete Mathematics

Week 7

Abeyah Calpatura

5.1

Exercises

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#2, 13, 21, 25, 30, 42, 45, 54, 72, 76

#2 Solution:

$$\begin{aligned}b_1 &= \frac{5-1}{5+1} = \frac{4}{6} = \frac{2}{3} \\b_2 &= \frac{5-2}{5+2} = \frac{3}{7} \\b_3 &= \frac{5-3}{5+3} = \frac{2}{8} = \frac{1}{4} \\b_4 &= \frac{5-4}{5+4} = \frac{1}{9} \\ \text{answer} &= \frac{2}{3}, \frac{3}{7}, \frac{1}{4}, \frac{1}{4}\end{aligned}$$

#13 Solution: $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

n	a_n
1	$a_1 = 1 - \frac{1}{2} = \frac{1}{1} - \frac{1}{1+1}$
2	$a_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{2} - \frac{1}{2+1}$
3	$a_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{3} - \frac{1}{3+1}$
4	$a_4 = \frac{1}{4} - \frac{1}{5} = \frac{1}{4} - \frac{1}{4+1}$
5	$a_5 = \frac{1}{5} - \frac{1}{6} = \frac{1}{5} - \frac{1}{5+1}$
6	$a_6 = \frac{1}{6} - \frac{1}{7} = \frac{1}{6} - \frac{1}{6+1}$

general answer = $a_n = \frac{1}{n} - \frac{1}{n+1}, n \geq 1$

#21 Solution:

$$\begin{aligned}&\sum_{k=1}^3 (k^2 + 1) \\&= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) \\&= 2 + 5 + 10 \\&= 17\end{aligned}$$

#25 Solution:

$$\begin{aligned}&\prod_{k=2}^2 \left(1 - \frac{1}{k}\right) \\&= \left(1 - \frac{1}{2}\right) = \frac{1}{2} \\ \text{answer} &= \frac{1}{2}\end{aligned}$$

#30 Solution:

$$\begin{aligned}&\sum_{j=1}^n j(j+1) \\&= 1(1+1) + 2(2+1) + 3(3+1) + \cdots + n(n+1)\end{aligned}$$

#42 Solution: By separating off the final term, we have

$$\begin{aligned}
 & \sum_{m=1}^{n+1} m(m+1) \\
 &= \sum_{m=1}^n m(m+1) + (n+1)((n+1)+1) \\
 &= \sum_{m=1}^n m(m+1) + (n+1)(n+2)
 \end{aligned}$$

#45 Solution: $(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

$$\prod_{k=2}^4 (k^2 - 1)$$

#54 Solution: $i = k + 1$

$$\begin{aligned}
 & \prod_n^{k=1} \frac{k}{k^2 + 4} \\
 & \prod_n^{k=1} \frac{(k+1) - 1}{((k+1) - 1)^2 + 4} \\
 & \prod_{n+1}^{i=2} \frac{i - 1}{(i - 1)^2 + 4}
 \end{aligned}$$

#72 Solution:

$$\begin{aligned}
 & \binom{7}{4} \\
 &= \frac{7!}{4!(7-4)!} \\
 &= \frac{7!}{4!3!} \\
 &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2 \cdot 1} \\
 &= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \\
 &= 35
 \end{aligned}$$

#76 Solution:

$$\begin{aligned}
 & \binom{n+1}{n-1} \\
 &= \frac{(n+1)!}{(n-1)!(n+1-(n-1))!} \\
 &= \frac{(n+1)!}{(n-1)! \cdot 2!} \\
 &= \frac{(n+1) \cdot n \cdot (n-1)!}{(n-1)!2} \\
 &= \frac{n(n+1)}{2}
 \end{aligned}$$

5.2

Exercises

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#3, 4, 7

#3 Solution: For every positive integer n , $P(n)$ represents the formula:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

a. Is $P(1)$ true?

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

$$1 = \frac{1 \cdot 2 \cdot 3}{6}$$

$$1 = 1$$

Answer: Yes, $P(1)$ is true.

b. $P(k)$

$$\text{Answer: } 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

c. $P(k+1)$

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

d. $P(k+1)$

Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

#4 Solution: For each integer n with $n \geq 1$, $P(n)$ represents the formula:

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

a. Is $P(2)$ true?

$$\sum_{i=1}^{2-1} i(i+1) = \frac{2(2-1)(2+1)}{3}$$

$$1(1+1) = \frac{2(1)(3)}{3}$$

$$2 = \frac{2(3)}{3}$$

$$2 = 2$$

Answer: Yes, $P(2)$ is true.

b. $P(k)$

$$\text{Answer: } \sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

c. $P(k+1)$

$$\sum_{i=1}^k i(i+1) = \frac{(k+1)(k)(k+2)}{3}$$

$$\frac{k(k-1)(k+1)}{3} + k(k+1) = \frac{(k+1)(k)(k+2)}{3}$$

d. $P(k+1)$

Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

#7 Solution: For every integer $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n - 3)}{2}$$

a. Is $P(1)$ true?

$$1 = \frac{1(5 \cdot 1 - 3)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = 1$$

Answer: Yes, $P(1)$ is true.

b. $P(k)$

$$\text{Answer: } 1 + 6 + 11 + 16 + \cdots + (5k - 4) = \frac{k(5k - 3)}{2}$$

c. $P(k + 1)$

$$1 + 6 + 11 + 16 + \cdots + (5k - 4) + (5(k + 1) - 4) = \frac{(k + 1)(5(k + 1) - 3)}{2}$$

$$\frac{k(5k - 3)}{2} + (5(k + 1) - 4) = \frac{(k + 1)(5k + 2)}{2}$$

d. $P(k + 1)$

Answer: $P(k + 1)$ is true assuming that $P(k)$ is true.

5.3

Exercises

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#3, 5, 9, 11

3. Stamps are sold in packages containing either 5 stamps or 8 stamps.

#3a Solution: Show that a person can obtain 5, 8, 10, 13, 15, 16, 20, 21, 24, or 25 stamps by buying a collection of 5-stamp packages and 8-stamp packages.

Number of stamps	How to obtain it
5	5
8	8
10	$5 + 5$
13	$5 + 8$
15	$5 + 5 + 5$
16	$8 + 8$
20	$5 + 5 + 5 + 5$
21	$5 + 8 + 8$
24	$8 + 8 + 8$
25	$5 + 5 + 5 + 5 + 5$

#3b Solution: Use mathematical induction to show that any quantity of at least 28 stamps can be obtained

Show that $P(28)$ is true.

Answer: $28 = 5 + 5 + 5 + 5 + 8$

Show that $P(k)$ is true.

Answer: $k = 5a + 8b$

Show that $P(k+1)$ is true.

Answer: $k + 1 = 5a + 8b$

#5 Solution: For each positive number n , let $P(n)$ be the inequality

$$2^n < (n + 1)!$$

a. Is $P(2)$ true?

$$2^2 < (2 + 1)!$$

$$4 < 3!$$

$$4 < 6$$

Answer: Yes, $P(2)$ is true.

b. $P(k)$

$$\text{Answer: } 2^k < (k + 1)!$$

c. $P(k + 1)$

$$2^{k+1} < (k + 2)!$$

$$2 \cdot 2^k < (k + 2)(k + 1)!$$

$$2 \cdot 2^k < (k + 2)!$$

d. $P(k + 1)$

Answer: $P(k + 1)$ is true assuming that $P(k)$ is true.

#9 Solution: By mathematica induction, prove

$7^n - 1$ is divisible by 6, for each integer $n \geq 0$

a. Is $P(0)$ true?

$$7^0 - 1$$

$$1 - 1$$

$$0$$

Answer: Yes, $P(0)$ is true.

b. $P(k)$

$$7^k - 1 = 6m$$

Answer: $7^k - 1$ is divisible by 6

c. $P(k + 1)$

$$7^{k+1} - 1 = 7 \cdot 7^k - 1$$

$$= 7 \cdot [(7^k - 1) + 1] - 1$$

$$= 7 \cdot [6m + 1] - 1$$

$$= 42m + 7 - 1$$

$$= 42m + 6$$

$$= 6(7m + 1)$$

d. $P(k + 1)$

Answer: $P(k + 1)$ is true assuming that $P(k)$ is true.

#11 Solution: By mathematical induction, prove

$3^{2n} - 1$ is divisible by 8, for each integer $n \geq 0$

a. Is $P(0)$ true?

$$3^{2 \cdot 0} - 1$$

$$3^0 - 1$$

$$0$$

Answer: Yes, $P(0)$ is true.

b. $P(k)$

$$3^{2k} - 1 = 8m$$

Answer: $3^{2k} - 1$ is divisible by 8

c. $P(k + 1)$

$$3^{2(k+1)} - 1 = 3^2 \cdot 3^{2k} - 1$$

$$= 9 \cdot (3^{2k} - 1) + 8$$

$$= 9 \cdot 8m + 8$$

$$= 8(9m + 1)$$

d. $P(k + 1)$

Answer: $P(k + 1)$ is true assuming that $P(k)$ is true.