Discrete Mathematics
Week 7

Abeyah Calpatura

5.1

Exercises

Abeyah Calpatura
#2, 13, 21, 25, 30, 42, 45, 54, 72, 76
#2 Solution:

$$
\begin{aligned}
b_1 &= \frac{5-1}{5+1} = \frac{4}{6} = \frac{2}{3} \\
b_2 &= \frac{5-2}{5+2} = \frac{3}{7} \\
b_3 &= \frac{5-3}{5+3} = \frac{2}{8} = \frac{1}{4} \\
b_4 &= \frac{5-4}{5+4} = \frac{1}{9} \\
\text{answer } &= \frac{2}{3}, \frac{3}{7}, \frac{1}{4}, \frac{1}{4}
\end{aligned}
$$

#13 Solution: $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \frac{1}{5}-\frac{1}{6}, \frac{1}{6}-\frac{1}{7}$

n	a_n
1	$a_1 = 1-\frac{1}{2} = \frac{1}{1}-\frac{1}{1+1}$
2	$a_2 = \frac{1}{2}-\frac{1}{3} = \frac{1}{2}-\frac{1}{2+1}$
3	$a_3 = \frac{1}{3}-\frac{1}{4} = \frac{1}{3}-\frac{1}{3+1}$
4	$a_4 = \frac{1}{4}-\frac{1}{5} = \frac{1}{4}-\frac{1}{4+1}$
5	$a_5 = \frac{1}{5}-\frac{1}{6} = \frac{1}{5}-\frac{1}{5+1}$
6	$a_6 = \frac{1}{6}-\frac{1}{7} = \frac{1}{6}-\frac{1}{6+1}$

$$\text{general answer } = a_n = \frac{1}{n} - \frac{1}{n+1}, n \geq 1$$

#21 Solution:

$$
\begin{aligned}
&\sum_{k=1}^{3}(k^2+1) \\
&= (1^2+1)+(2^2+1)+(3^2+1) \\
&= 2+5+10 \\
&= 17
\end{aligned}
$$

#25 Solution:

$$
\begin{aligned}
&\prod_{k=2}^{2}(1-\frac{1}{k}) \\
&= (1-\frac{1}{2}) = \frac{1}{2} \\
&\text{answer } = \frac{1}{2}
\end{aligned}
$$

#30 Solution:

$$
\begin{aligned}
&\sum_{j=1}^{n} j(j+1) \\
&= 1(1+1)+2(2+1)+3(3+1)+\cdots+n(n+1)
\end{aligned}
$$

#42 ***Solution:*** By separating off the final term, we have

$$\sum_{m=1}^{n+1} m(m+1)$$
$$= \sum_{m=1}^{n} m(m+1) + (n+1)((n+1)+1)$$
$$= \sum_{m=1}^{n} m(m+1) + (n+1)(n+2)$$

#45 ***Solution:*** $(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

$$\prod_{k=2}^{4} (k^2 - 1)$$

#54 ***Solution:*** $i = k + 1$

$$\prod_{n}^{k=1} \frac{k}{k^2 + 4}$$
$$\prod_{n}^{k=1} \frac{(k+1) - 1}{((k+1) - 1)^2 + 4}$$
$$\prod_{n+1}^{i=2} \frac{i - 1}{(i-1)^2 + 4}$$

#72 ***Solution:***

$$\binom{7}{4}$$
$$= \frac{7!}{4!(7-4)!}$$
$$= \frac{7!}{4!3!}$$
$$= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2 \cdot 1}$$
$$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$$
$$= 35$$

#76 ***Solution:***

$$\binom{n+1}{n-1}$$
$$= \frac{(n+1)!}{(n-1)!(n+1-(n-1))!}$$
$$= \frac{(n+1)!}{(n-1)! \cdot 2!}$$
$$= \frac{(n+1) \cdot n \cdot (n-1)!}{(n-1)!2}$$
$$= \frac{n(n+1)}{2}$$

5.2

Exercises

Abeyah Calpatura
#3, 4, 7

#3 Solution: For every positive integer n, $P(n)$ represents the formula:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

a. Is P(1) true?

$1^2 = \dfrac{1(1+1)(2 \cdot 1 + 1)}{6}$

$1 = \dfrac{1 \cdot 2 \cdot 3}{6}$

$1 = 1$

Answer: Yes, $P(1)$ is true.

b. $P(k)$

Answer: $1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

c. $P(k+1)$

$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \dfrac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$\dfrac{k(k+1)(2k+1)}{6} + (k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$

d. $P(k+1)$

Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

#4 *Solution:* For each integer n with $n \geq 1$, $P(n)$ represents the formula:

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

a. Is P(2) true?

$$\sum_{i=1}^{2-1} i(i+1) = \frac{2(2-1)(2+1)}{3}$$

$$1(1+1) = \frac{2(1)(3)}{3}$$

$$2 = \frac{2(3)}{3}$$

$$2 = 2$$

Answer: Yes, $P(2)$ is true.

b. $P(k)$

Answer: $\displaystyle\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$

c. $P(k+1)$

$$\sum_{i=1}^{k} i(i+1) = \frac{(k+1)(k)(k+2)}{3}$$

$$\frac{k(k-1)(k+1)}{3} + k(k+1) = \frac{(k+1)(k)(k+2)}{3}$$

d. $P(k+1)$

Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

#7 Solution: For every integer $n \geqslant 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n-3)}{2}$$

a. Is P(1) true?

$1 = \dfrac{1(5 \cdot 1 - 3)}{2}$

$1 = \dfrac{1(2)}{2}$

$1 = 1$

Answer: Yes, $P(1)$ is true.

b. $P(k)$

Answer: $1 + 6 + 11 + 16 + \cdots + (5k - 4) = \dfrac{k(5k-3)}{2}$

c. $P(k+1)$

$1 + 6 + 11 + 16 + \cdots + (5k - 4) + (5(k+1) - 4) = \dfrac{(k+1)(5(k+1)-3)}{2}$

$\dfrac{k(5k-3)}{2} + (5(k+1) - 4) = \dfrac{(k+1)(5k+2)}{2}$

d. $P(k+1)$

Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

5.3

Exercises

Abeyah Calpatura
#3, 5, 9, 11

3. Stamps are sold in packages containing either 5 stamps or 8 stamps.

#3a ***Solution:*** Show that a person can obtain 5, 8, 10, 13, 15, 16, 20, 21, 24, or 25 stamps by buying a collectino of 5-stamp packages and 8-stamp packages.

Number of stamps	How to obtain it
5	5
8	8
10	5 + 5
13	5 + 8
15	5 + 5 + 5
16	8 + 8
20	5 + 5 + 5 + 5
21	5 + 8 + 8
24	8 + 8 + 8
25	5 + 5 + 5 + 5 + 5

#3b ***Solution:*** Use mathematical induction to show that any quantity of at least 28 stamps can be obtained

Show that P(28) is true.

Answer: $28 = 5 + 5 + 5 + 5 + 8$

Show that P(k) is true.

Answer: $k = 5a + 8b$

Show that P(k+1) is true.

Answer: $k + 1 = 5a + 8b$

#5 ***Solution:*** For each positive number n, let $P(n)$ be the inequality

$2^n < (n+1)!$

a. Is P(2) true?
$2^2 < (2+1)!$
$4 < 3!$
$4 < 6$
Answer: Yes, $P(2)$ is true.

b. $P(k)$
Answer: $2^k < (k+1)!$

c. $P(k+1)$
$2^{k+1} < (k+2)!$
$2 \cdot 2^k < (k+2)(k+1)!$
$2 \cdot 2^k < (k+2)!$

d. $P(k+1)$
Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

#9 ***Solution:*** By mathematica induction, prove

$7^n - 1$ is divisible by 6, for each integer $n \geq 0$

a. Is P(0) true?
$7^0 - 1$
$1 - 1$
0
Answer: Yes, $P(0)$ is true.

b. $P(k)$
$7^k - 1 = 6m$
Answer: $7^k - 1$ is divisible by 6

c. $P(k+1)$
$7^{k+1} - 1 = 7 \cdot 7^k - 1$
$= 7 \cdot [(7^k - 1) + 1] - 1$
$= 7 \cdot [6m + 1] - 1$
$= 42m + 7 - 1$
$= 42m + 6$
$= 6(7m + 1)$

d. $P(k+1)$
Answer: $P(k+1)$ is true assuming that $P(k)$ is true.

#11 ***Solution:*** By mathematical induction, prove

$3^{2n} - 1$ is disivible by 8, for each integer $n \geq 0$

a. Is P(0) true?

$3^{2\cdot 0} - 1$

$3^0 - 1$

0

Answer: Yes, $P(0)$ is true.

b. $P(k)$

$3^{2k} - 1 = 8m$

Answer: $3^{2k} - 1$ is divisible by 8

c. $P(k+1)$

$3^{2(k+1)} - 1 = 3^2 \cdot 3^{2k} - 1$

$= 9 \cdot (3^{2k} - 1) + 8$

$= 9 \cdot 8m + 8$

$= 8(9m + 1)$

d. $P(k+1)$

Answer: $P(k+1)$ is true assuming that $P(k)$ is true.