

Discrete Mathematics

Week 8

Abeyah Calpatura

6.1

Exercises

Abeyah Calpatura

#4ab, 7ab, 8, 9, 12, 15, 20, 31

#4

Let $A = \{n \in \mathbb{Z} \mid n = 5r \text{ for some integer } r\}$

Let $B = \{m \in \mathbb{Z} \mid m = 20s \text{ for some integer } s\}$

$A = \{\dots, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40, \dots\}$

$B = \{\dots, -100, -80, -60, -40, -20, 0, 20, 40, 60, 80, 100, \dots\}$

#4a Solution: $A \subseteq B$

Not true, prove that we need to find at least one element that belongs to A but not B.

Let's take the element 5.

5 belongs to A because $5 = 5(1)$.

5 does not belong to B because $5 \neq 20s$ for any integer s.

Therefore, $A \not\subseteq B$.

#4b Solution: $B \subseteq A$

A is the set of all numbers divisible by 5.

B, have all integers that are divisible by 20.

$z \in B$ must be divisible by 4 and 5

Therefore, $B \subseteq A$. Every element of B also belongs to A

#7

Let $A = \{x \in \mathbb{Z} \mid x = 6a + 4 \text{ for some integer } a\}$

Let $B = \{y \in \mathbb{Z} \mid y = 18b - 2 \text{ for some integer } b\}$

Let $C = \{z \in \mathbb{Z} \mid z = 18c + 16 \text{ for some integer } c\}$

Prove or disprove each of the following statements.

#7a Solution: $A \subseteq B$

For $a = 1$, we have that $x = 6 \cdot 1 + 4 = 10 \in A$

Suppose that $x = 10$. Solve the equation $10 = 18b - 2$

$$12 = 18b$$

$$2 = 3b$$

$$b = \frac{2}{3} \notin \mathbb{Z}$$

Therefore, $A \not\subseteq B$.

#7b Solution: $B \subseteq A$

A is the set of all numbers that are 4 more than a multiple of 6.

B is the set of all numbers that are 3 less than a multiple of 10.

There is an integer b so that $x = 18b - 2$

$$x = 18b - 2 = 6a + 4$$

$$18b - 2 - 4 = 6a$$

$$18b - 6 = 6a$$

$$6(3b - 1) = 6a$$

$$3b - 1 = a$$

Therefore, $B \subseteq A$. Every element of B also belongs to A.

#8a Solution: $\{x \in U \mid x \in A \text{ and } x \in B\}$

This is the set of all x from U , such that x is both in A and in B.

Solution: $A \cap B$

#8b Solution: $\{x \in U \mid x \in A \text{ or } x \in B\}$

This is the set of all x from U , such that x is in A or in B.

Solution: $A \cup B$

#8c Solution: $\{x \in U \mid x \in A \text{ and } x \notin B\}$

This is the set of all x from U , such that x is in A and not in B.

Solution: $A - B$

#8d Solution: $\{x \in U \mid x \notin A\}$

This is the set of all x from U that do not belong to A

Solution: A^c

#9a Solution: $x \notin A \cup B$ if, and only if, $x \notin A$ and $x \notin B$.

#9b Solution: $x \notin A \cap B$ if, and only if, $x \notin A$ or $x \notin B$.

#9c Solution: $x \in A - B$ if, and only if, $x \notin A$ and $x \in B$.

#12

Let the universal set be \mathbb{R} , the set of all real numbers, and let $A = \{x \in \mathbb{R} \mid -3 \leq x \leq 0\}$ and $B = \{x \in \mathbb{R} \mid -1 < x < 2\}$, and $C = \{x \in \mathbb{R} \mid 6 < x \leq 8\}$

#12a Solution: $A \cup B$

$$\{x \in \mathbb{R} \mid -3 \leq x \leq 0 \text{ or } -1 < x < 2\} = \{x \in \mathbb{R} \mid -3 \leq x < 2\}$$

#12b Solution: $A \cap B$

$$\{x \in \mathbb{R} \mid -3 \leq x \leq 0 \text{ and } -1 < x < 2\} = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$$

#12c Solution: A^c

$$\{x \in \mathbb{R} \mid -3 \leq x \leq 0\}^c = \{x \in \mathbb{R} \mid x < -3 \text{ or } x > 0\}$$

#12d Solution: $A \cup C$

$$\{x \in \mathbb{R} \mid -3 \leq x \leq 0 \text{ or } 6 < x \leq 8\} = \{x \in \mathbb{R} \mid -3 \leq x \leq 8\}$$

#12e Solution: $A \cap C$

$$\{x \in \mathbb{R} \mid -3 \leq x \leq 0 \text{ and } 6 < x \leq 8\} = \emptyset$$

#12f Solution: B^c

$$\{x \in \mathbb{R} \mid -1 < x < 2\}^c = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 2\}$$

#12g Solution: $A^c \cap B^c$

$$\{x \in \mathbb{R} \mid x < -3 \text{ or } x > 0\} \text{ and } \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 2\} = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geq 2\}$$

#12h Solution: $A^c \cup B^c$

$$\{x \in \mathbb{R} \mid x < -3 \text{ or } x > 0\} \text{ or } \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 2\} = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geq 2\}$$

#12i Solution: $(A \cap B)^c$

$$\{x \in \mathbb{R} \mid -1 < x \leq 0\}^c = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x > 0\}$$

#12j Solution: $(A \cup B)^c$

$$\{x \in \mathbb{R} \mid -3 \leq x < 2\}^c = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geq 2\}$$

#15

Venn Diagram

#20

Let $B_i = \{x \in \mathbb{R} \mid 0 \leq x \leq i\}$ for each integer $i = 1, 2, 3, 4$

#20a Solution: $B_1 \cup B_2 \cup B_3 \cup B_4 =$

$$\begin{aligned} & \{x \in \mathbb{R} \mid 0 \leq x \leq 1\} \cup \{x \in \mathbb{R} \mid 0 \leq x \leq 2\} \cup \{x \in \mathbb{R} \mid 0 \leq x \leq 3\} \cup \{x \in \mathbb{R} \mid 0 \leq x \leq 4\} \\ &= \{x \in \mathbb{R} \mid 0 \leq x \leq 4\} \end{aligned}$$

#20b Solution: $B_1 \cap B_2 \cap B_3 \cap B_4 =$

$$\begin{aligned} & \{x \in \mathbb{R} \mid 0 \leq x \leq 1\} \cap \{x \in \mathbb{R} \mid 0 \leq x \leq 2\} \cap \{x \in \mathbb{R} \mid 0 \leq x \leq 3\} \cap \{x \in \mathbb{R} \mid 0 \leq x \leq 4\} \\ &= \{x \in \mathbb{R} \mid 0 \leq x \leq 1\} \end{aligned}$$

#20c Solution: Are $B_1, B_2, B_3,$ and B_4 mutually disjoint?

No, they are not mutually disjoint.

The intersection of any two of them is not empty.

#31

Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following

#31a Solution: $\mathcal{P}(A \cap B)$

$$\begin{aligned} A \cap B &= \{2\} \\ \mathcal{P}(A \cap B) &= \{\emptyset, \{2\}\} \end{aligned}$$

#31b Solution: $\mathcal{P}(A)$

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

#31c Solution: $\mathcal{P}(A \cup B)$

$$\begin{aligned} A \cup B &= \{1, 2, 3\} \\ \mathcal{P}(A \cup B) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \end{aligned}$$

#31d Solution: $\mathcal{P}(A \times B)$

$$\begin{aligned} A \times B &= \{(1, 2), (1, 3), (2, 2), (2, 3)\} \\ \mathcal{P}(A \times B) &= \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(2, 2)\}, \{(2, 3)\}, \{(1, 2), (1, 3)\}, \{(1, 2), (2, 2)\}, \\ & \{(1, 2), (2, 3)\}, \{(1, 3), (2, 2)\}, \{(1, 3), (2, 3)\}, \{(2, 2), (2, 3)\}, \{(1, 2), (1, 3), (2, 2)\}, \{(1, 2), (1, 3), \\ & (2, 3)\}, \{(1, 2), (2, 2), (2, 3)\}, \{(1, 3), (2, 2), (2, 3)\}, \{(1, 2), (1, 3), (2, 2), (2, 3)\}\} \end{aligned}$$

7.1

Exercises

Abeyah Calpatura

#1, 2, 8cd, 11cd, 12cd, 17, 18

#1

Let $X = \{1, 3, 5\}$ and $Y = \{s, t, u, v\}$. Define $f: X \rightarrow Y$

#1a Solution:

Domain $f = \{1, 3, 5\}$

Codomain $f = \{s, t, u, v\}$

#1b Solution: Find $f(1)$, $f(3)$, $f(5)$

$f(1) = v$

$f(3) = s$

$f(5) = v$

#1c Solution: What is the range of f ?

Range $f = \{s, v\}$

#1d Solution: Is 3 an inverse image of s ? Is 1 an inverse image of u ?

3 is an inverse image of s .

1 is not an inverse image of u .

#1e Solution: What is the inverse image of s ? of u ? of v ?

Inverse image of $s = \{3\}$

Inverse image of $u = \emptyset$

Inverse image of $v = \{1, 5\}$

#1f Solution: Represent f as a set of ordered pairs.

$f = \{(1, v), (3, s), (5, v)\}$

#2

Let $X = \{1, 3, 5\}$ and $Y = \{a, b, c, d\}$. Define $g: X \rightarrow Y$

#2a Solution:

Domain $g = \{1, 3, 5\}$

Codomain $g = \{a, b, c, d\}$

#2b Solution: Find $g(1)$, $g(3)$, $g(5)$

$$g(1) = b$$

$$g(3) = b$$

$$g(5) = b$$

#2c Solution: What is the range of g ?

$$\text{Range } g = \{b\}$$

#2d Solution: Is 3 an inverse image of a ? Is 1 an inverse image of c ?

3 is not an inverse image of a .

1 is the inverse image of c .

#2e Solution: What is the inverse image of b ? of c ?

$$\text{Inverse image of } b = \{1, 3, 5\}$$

$$\text{Inverse image of } c = \{\emptyset\}$$

#2f Solution: Represent g as a set of ordered pairs.

$$g = \{(1, b), (3, b), (5, b)\}$$

#8 Let $J_5 = \{0, 1, 2, 3, 4\}$, and define a function $F: J_5 \rightarrow J_5$ as follows: For each $x \in J_5$, $F(x) = (x^3 + 2x + 4) \bmod 5$.

#8c Solution: $F(2)$

$$F(2) = (2^3 + 2(2) + 4) \bmod 5$$

$$F(2) = 16 \bmod 5$$

$$F(2) = 1$$

#8d Solution: $F(3)$

$$F(3) = (3^3 + 2(3) + 4) \bmod 5$$

$$F(3) = 37 \bmod 5$$

$$F(3) = 2$$

#11 Let $F : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ as follows: For every ordered pair (a, b) of integers, $F(a, b) = (2a + 1, 3b - 2)$.

#11c Solution: $F(3, 2)$

$$F(3, 2) = (2(3) + 1, 3(2) - 2)$$

$$F(3, 2) = (7, 4)$$

#11d Solution: $F(1, 5)$

$$F(1, 5) = (2(1) + 1, 3(5) - 2)$$

$$F(1, 5) = (3, 13)$$

#12 Let $J_5 = 0, 1, 2, 3, 4$ and define $G : J_5 \times J_5 \rightarrow J_5 \times J_5$ as follows: For each $(a, b) \in J_5 \times J_5$, $G(a, b) = ((2a + 1) \bmod 5, (3b - 2) \bmod 5)$

#12c Solution: $G(3, 2)$

$$G(3, 2) = ((2(3) + 1) \bmod 5, (3(2) - 2) \bmod 5)$$

$$G(3, 2) = (7 \bmod 5, 4 \bmod 5)$$

$$G(3, 2) = (2, 4)$$

#12d Solution: $G(1, 5)$

$$G(1, 5) = ((2(1) + 1) \bmod 5, (3(5) - 2) \bmod 5)$$

$$G(1, 5) = (3 \bmod 5, 13 \bmod 5)$$

$$G(1, 5) = (3, 3)$$

#17a Solution: $\log_2 8$

$$\log_2 8 = 3$$

$$2^3 = 8$$

#17b Solution: $\log_5(\frac{1}{25}) = -2$

$$\log_5(\frac{1}{25}) = -2$$

$$5^{-2} = \frac{1}{25}$$

#17c Solution: $\log_4 4 = 1$

$$\log_4 4 = 1$$

$$4^1 = 4$$

#17d Solution: $\log_3(3^n) = n$

$$\log_3(3^n) = n$$
$$3^n = 3^n$$

#17e Solution: $\log_4 1 = 0$

$$\log_4 1 = 0$$
$$4^0 = 1$$

#18a Solution: $\log_3 81$

$$\log_3 81 = 4$$
$$3^4 = 81$$

#18b Solution: $\log_2 1024$

$$\log_2 1024 = 10$$
$$2^{10} = 1024$$

#18c Solution: $\log_3(\frac{1}{27})$

$$\log_3(\frac{1}{27}) = -3$$
$$3^{-3} = \frac{1}{27}$$

#18d Solution: $\log_2 1$

$$\log_2 1 = 0$$
$$2^0 = 1$$

#18e Solution: $\log_{10}(\frac{1}{10})$

$$\log_{10}(\frac{1}{10}) = -1$$
$$10^{-1} = \frac{1}{10}$$

#18f Solution: $\log_2(2^k)$

$$\log_2(2^k) = k$$
$$2^k = 2^k$$

7.2

Exercises

Abeyah Calpatura

#1, 7, 8, 10, 17, 24, 25

#1a Solution: A function F is one-to-one if, and only if, each element in the co-domain of F is the image of at **most** one element in the domain of F . **#1b Solution:** A function F is onto if, and only if, each element in the co-domain of F is the image of **at least** one element in the domain of F . **#7** Let $X = \{a, b, c, d\}$ and $Y = \{e, f, g\}$. Define functions F and G by arrow diagrams

#7a Solution: Is F one-to-one? Why or why not? Is it onto? Why or why not?

F is not one-to-one because the element b is the image of both a and c .

F is onto because each element in the co-domain is the image of at least one element in the domain.

#7b Solution: Is G one-to-one? Why or why not? Is it onto? Why or why not?

G is one-to-one because each element in the co-domain is the image of at most one element in the domain.

G is not onto because the element g is not the image of any element in the domain.

#8 Let $X = \{a, b, c\}$ and $Y = \{d, e, f, g\}$. Define functions H and K by arrow diagrams

#8a Solution: Is H one-to-one? Why or why not? Is it onto? Why or why not?

H is neither one to one nor onto.

$H(b) = H(c) = y$ so H is not one to one

H never takes the value of x , so it is not onto

#8b Solution: Is K one-to-one? Why or why not? Is it onto? Why or why not?

K is one-to-one but not onto.

K takes three different values on the three elements of X , so it is one-to-one.

K never takes the value of z , so it is not onto.

#10