Discrete Mathematics
Week 8

Abeyah Calpatura

6.1

Exercises

Abeyah Calpatura
#4ab, 7ab, 8, 9, 12, 15, 20, 31

#4
Let A = $\{n \in \mathbb{Z} \mid n = 5r$ for some integer r$\}$
Let B = $\{m \in \mathbb{Z} \mid m = 20s$ for some integer s$\}$
$A = \{\ldots, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40, \ldots\}$
$B = \{\ldots, -100, -80, -60, -40, -20, 0, 20, 40, 60, 80, 100, \ldots\}$

#4a ***Solution:*** $A \subseteq B$

Not true, prove that we need to find at least one element that belongs to A but not B.

Let's take the element 5.

5 belongs to A because 5 = 5(1).

5 does not belong to B because 5 $\neq$ 20s for any integer s.

Therefore, A $\nsubseteq$ B.

#4b ***Solution:*** $B \subseteq A$

A is the set of all numbers divisble by 5.

B, have all integers that are divisble by 20.

$z \in B$ must be divisble by 4 and 5

Therefore, B $\subseteq$ A. Every element of B also belongs to A

#7
Let A = $\{x \in \mathbb{Z} \mid x = 6a + 4$ for some integer a$\}$
Let B = $\{y \in \mathbb{Z} \mid y = 18b - 2$ for some integer b$\}$
Let C = $\{z \in \mathbb{Z} \mid z = 18c + 16$ for some integer c$\}$
Prove or disprove each of the following statements.

#7a ***Solution:*** $A \subseteq B$

For a = 1, we have that $x = 6 \cdot 1 + 4 = 10 \in A$

Suppose that x = 10. Solve the eqution 10 = 18b - 2

12 = 18b

2 = 3b

$b = \frac{2}{3} \notin \mathbb{Z}$

Therefore, A $\nsubseteq$ B.

#7b ***Solution:*** $B \subseteq A$

A is the set of all numbers that are 4 more than a multiple of 6.
B is the set of all numbers that are 3 less than a multiple of 10.
There is an integer b so that $x = 18b - 2$
$x = 18b - 2 = 6a + 4$
$18b - 2 - 4 = 6a$
$18b - 6 = 6a$
$6(3b - 1) = 6a$
$3b - 1 = a$
Therefore, B ⊆ A. Every element of B also belongs to A.

#8a ***Solution:*** $\{x \in U \mid x \in A \text{ and } x \in B\}$

This is the set of all x from U, such that x is both in A and in B.
Solution: $A \cap B$

#8b ***Solution:*** $\{x \in U \mid x \in A \text{ or } x \in B\}$

This is the set of all x from U, such that x is in A or in B.
Solution: $A \cup B$

#8c ***Solution:*** $\{x \in U \mid x \in A \text{ and } x \notin B\}$

This is the set of all x from U, such that x is in A and not in B.
Solution: $A - B$

#8d ***Solution:*** $\{x \in U \mid x \notin A\}$

This is the set of all x from U that do not belong to A
Solution: A^c

#9a ***Solution:*** $x \notin A \cup B$ if, and only if, $x \notin A$ and $x \notin B$.
#9b ***Solution:*** $x \notin A \cap B$ if, and only if, $x \notin A$ or $x \notin B$.
#9c ***Solution:*** $x \in A - B$ if, and only if, $x \notin A$ and $x \in B$.

#12
Let the universal set be $\mathbb{R}$, the set of all real numbers, and let A = $\{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 0\}$ and B = $\{x \in \mathbb{R} \mid -1 < x < 2\}$, and C = $\{x \in \mathbb{R} \mid 6 < x \leqslant 8\}$
#12a ***Solution:*** $A \cup B$

$$\{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 0 \text{ or } -1 < x < 2\} = \{x \in \mathbb{R} \mid -3 \leqslant x < 2\}$$

#12b ***Solution:*** $A \cap B$

$$\{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 0 \text{ and } -1 < x < 2\} = \{x \in \mathbb{R} \mid -1 < x \leqslant 0\}$$

#12c ***Solution:*** A^c

$$\{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 0\}^c = \{x \in \mathbb{R} \mid x < -3 \text{ or } x > 0\}$$

#12d ***Solution:*** $A \cup C$

$$\{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 0 \text{ or } 6 < x \leqslant 8\} = \{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 8\}$$

#12e ***Solution:*** $A \cap C$

$$\{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 0 \text{ and } 6 < x \leqslant 8\} = \emptyset$$

#12f ***Solution:*** B^c

$$\{x \in \mathbb{R} \mid -1 < x < 2\}^c = \{x \in \mathbb{R} \mid x \leqslant -1 \text{ or } x \geqslant 2\}$$

#12g ***Solution:*** $A^c \cap B^c$

$$\{x \in \mathbb{R} \mid x < -3 \text{ or } x > 0\} \text{ and } \{x \in \mathbb{R} \mid x \leqslant -1 \text{ or } x \geqslant 2\} = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geqslant 2\}$$

#12h ***Solution:*** $A^c \cup B^c$

$$\{x \in \mathbb{R} \mid x < -3 \text{ or } x > 0\} \text{ or } \{x \in \mathbb{R} \mid x \leqslant -1 \text{ or } x \geqslant 2\} = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geqslant 2\}$$

#12i ***Solution:*** $(A \cap B)^c$

$$\{x \in \mathbb{R} \mid -1 < x \leqslant 0\}^c = \{x \in \mathbb{R} \mid x \leqslant -1 \text{ or } x > 0\}$$

#12j ***Solution:*** $(A \cup B)^c$

$$\{x \in \mathbb{R} \mid -3 \leqslant x < 2\}^c = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geqslant 2\}$$

#15
Venn Diagram

#20
Let $B_i = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant i\}$ for each integer i = 1, 2, 3, 4
#20a *Solution:* $B_1 \cup B_2 \cup B_3 \cup B_4 =$

$$\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\} \cup \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 2\} \cup \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 3\} \cup \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 4\}$$
$$= \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 4\}$$

#20b *Solution:* $B_1 \cap B_2 \cap B_3 \cap B_4 =$

$$\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\} \cap \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 2\} \cap \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 3\} \cap \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 4\}$$
$$= \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$$

#20c *Solution:* Are $B_1, B_2, B_3, and B_4$ mutually disjoint?

No, they are not mutually disjoint.
The intersection of any two of them is not empty.

#31
Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following
#31a *Solution:* $\mathcal{P}(A \cap B)$

$$A \cap B = \{2\}$$
$$\mathcal{P}(A \cap B) = \{\emptyset, \{2\}\}$$

#31b *Solution:* $\mathcal{P}(A)$

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

#31c *Solution:* $\mathcal{P}(A \cup B)$

$$A \cup B = \{1, 2, 3\}$$
$$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

#31d *Solution:* $\mathcal{P}(A \times B)$

$$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$
$$\mathcal{P}(A \times B) = \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(2, 2)\}, \{(2, 3)\}, \{(1, 2), (1, 3)\}, \{(1, 2), (2, 2)\}$$
$$\{(1, 2), (2, 3)\}, \{(1, 3), (2, 2)\}, \{(1, 3), (2, 3)\}, \{(2, 2), (2, 3)\}, \{(1, 2), (1, 3), (2, 2)\}, \{(1, 2), (1, 3)$$
$$, (2, 3)\}, \{(1, 2), (2, 2), (2, 3)\}, \{(1, 3), (2, 2), (2, 3)\}, \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$

7.1

Exercises

Abeyah Calpatura
#1, 2, 8cd, 11cd, 12cd, 17, 18
#1
Let X = $\{1, 3, 5\}$ and Y = $\{s, t, u, v\}$. Define f: $X \to Y$

#1a ***Solution:***

$$\text{Domain f} = \{1, 3, 5\}$$
$$\text{Codomain f} = \{s, t, u, v\}$$

#1b ***Solution:*** Find f(1), f(3), f(5)

$$\text{f(1) = v}$$
$$\text{f(3) = s}$$
$$\text{f(5) = v}$$

#1c ***Solution:*** What is the range of f?

$$\text{Range f} = \{s, v\}$$

#1d ***Solution:*** Is 3 an inverse image of s? Is 1 an inverse image of u?

3 is an inverse image of s.
1 is not an inverse image of u.

#1e ***Solution:*** What is the inverse image of s? of u? of v?

$$\text{Inverse image of s} = \{3\}$$
$$\text{Inverse image of u} = \emptyset$$
$$\text{Inverse image of v} = \{1, 5\}$$

#1f ***Solution:*** Represent f as a set of ordered pairs.

$$\text{f} = \{(1, v), (3, s), (5, v)\}$$

#2
Let X = $\{1, 3, 5\}$ and Y = $\{a, b, c, d\}$. Define g: $X \rightarrow Y$

#2a ***Solution:***

$$\text{Domain g} = \{1, 3, 5\}$$
$$\text{Codomain g} = \{a, b, c, d\}$$

#2b ***Solution:*** Find g(1), g(3), g(5)

$$\text{g}(1) = \text{b}$$
$$\text{g}(3) = \text{b}$$
$$\text{g}(5) = \text{b}$$

#2c ***Solution:*** What is the range of g?

$$\text{Range g} = \{b\}$$

#2d ***Solution:*** Is 3 an inverse image of a? Is 1 an inverse image of c?

3 is not an inverse image of a.
1 is the inverse image of c.

#2e ***Solution:*** What is the inverse image of b? of c?

$$\text{Inverse image of b} = \{1, 3, 5\}$$
$$\text{Inverse image of c} = \{\emptyset\}$$

#2f ***Solution:*** Represent g as a set of ordered pairs.

$$\text{g} = \{(1, b), (3, b), (5, b)\}$$

#8 Let $J_5 = \{0, 1, 2, 3, 4\}$, and define a function $F : J_5 \rightarrow J_5$ as follows: For each $x \in J_5, F(x) = (x^3 + 2x + 4)$ mod 5.
#8c ***Solution:*** F(2)

$$\text{F}(2) = (2^3 + 2(2) + 4) \text{ mod } 5$$
$$\text{F}(2) = 16 \text{ mod } 5$$
$$\text{F}(2) = 1$$

#8d *Solution:* F(3)

$$\begin{aligned}
\mathrm{F}(3) &= (3^3 + 2(3) + 4) \text{ mod } 5\\
\mathrm{F}(3) &= 37 \text{ mod } 5\\
\mathrm{F}(3) &= 2
\end{aligned}$$

#11 Let $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ as follows: For every ordered pair (a, b) of integers, $F(a, b) = (2a + 1, 3b - 2)$.
#11c *Solution:* F(3, 2)

$$\begin{aligned}
\mathrm{F}(3,\ 2) &= (2(3) + 1, 3(2) - 2)\\
\mathrm{F}(3,\ 2) &= (7,\ 4)
\end{aligned}$$

#11d *Solution:* F(1, 5)

$$\begin{aligned}
\mathrm{F}(1,\ 5) &= (2(1) + 1, 3(5) - 2)\\
\mathrm{F}(1,\ 5) &= (3,\ 13)
\end{aligned}$$

#12 Let $J_5 = 0, 1, 2, 3, 4$ and define $G : J_5 \times J_5 \to J_5 \times J_5$ as follows: For each (a, b) $\in J_5 \times J_5$, $G(a, b) = ((2a + 1) \bmod 5, (3b - 2) \bmod 5)$ **#12c *Solution:*** G(3, 2)

$$\begin{aligned}
\mathrm{G}(3,\ 2) &= ((2(3) + 1) \bmod 5, (3(2) - 2) \bmod 5)\\
\mathrm{G}(3,\ 2) &= (7 \bmod 5, 4 \bmod 5)\\
\mathrm{G}(3,\ 2) &= (2,\ 4)
\end{aligned}$$

#12d *Solution:* G(1, 5)

$$\begin{aligned}
\mathrm{G}(1,\ 5) &= ((2(1) + 1) \bmod 5, (3(5) - 2) \bmod 5)\\
\mathrm{G}(1,\ 5) &= (3 \bmod 5, 13 \bmod 5)\\
\mathrm{G}(1,\ 5) &= (3,\ 3)
\end{aligned}$$

#17a *Solution:* $\log_2 8$

$$\begin{aligned}
\log_2 8 &= 3\\
2^3 &= 8
\end{aligned}$$

#17b *Solution:* $\log_5(\frac{1}{25})$ = -2

$$\begin{aligned}
\log_5(\frac{1}{25}) &= \text{-}2\\
5^{-2} &= \frac{1}{25}
\end{aligned}$$

#17c *Solution:* $\log_4 4$=1

$$\begin{aligned}
\log_4 4 &= 1\\
4^1 &= 4
\end{aligned}$$

#17d ***Solution:*** $\log_3(3^n)$ = n

$$\log_3(3^n) = \text{n}$$
$$3^n = 3^n$$

#17e ***Solution:*** $\log_4 1 = 0$

$$\log_4 1 = 0$$
$$4^0 = 1$$

#18a ***Solution:*** $\log_3 81$

$$\log_3 81 = 4$$
$$3^4 = 81$$

#18b ***Solution:*** $\log_2 1024$

$$\log_2 1024 = 10$$
$$2^{10} = 1024$$

#18c ***Solution:*** $\log_3(\frac{1}{27})$

$$\log_3(\frac{1}{27}) = \text{-3}$$
$$3^{-3} = \frac{1}{27}$$

#18d ***Solution:*** $\log_2 1$

$$\log_2 1 = 0$$
$$2^0 = 1$$

#18e ***Solution:*** $\log_1 0(\frac{1}{10})$

$$\log_{10}(\frac{1}{10}) = \text{-1}$$
$$10^{-1} = \frac{1}{10}$$

#18f ***Solution:*** $\log_2(2^k)$

$$\log_2(2^k) = \text{k}$$
$$2^k = 2^k$$

7.2

Exercises

Abeyah Calpatura
#1, 7, 8, 10, 17, 24, 25

#1a ***Solution:*** A funtion F is one-to-one if, and only if, each element in the co-domain of F is the image of at **most** one element in the domain of F. **#1b** ***Solution:*** A function F is onto if, and only if, each element in the co-domain of F is the image of **at least** one element in the domain of F. **#7** Let X = $\{a, b, c, d\}$ and Y = $\{e, f, g\}$. Define functions F and G by arrow diagrams

#7a ***Solution:*** Is F one-to-one? Why or why not? Is it onto? Why or why not?

F is not one-to-one because the element b is the image of both a and c.

F is onto because each element in the co-domain is the image of at least one element in the domain.

#7b ***Solution:*** Is G one-to-one? Why or why not? Is it onto? Why or why not?

G is one-to-one because each element in the co-domain is the image of at most one element in the domain.

G is not onto because the element g is not the image of any element in the domain.

#8 Let X = $\{a, b, c\}$ and Y = $\{d, e, f, g\}$. Define functions H and K by arrow diagrams

#8a ***Solution:*** Is H one-to-one? Why or why not? Is it onto? Why or why not?

H is neither one to one nor onto.
H(b) = H(c) = y so H is not one to one
H never takes the value of x, so it is not onto

#8b ***Solution:*** Is K one-to-one? Why or why not? Is it onto? Why or why not?

K is one-to-one but not onto.
K takes three different values on the tree elements of X, so it is one-to-one.
K never takes the value of z, so it is not onto.

#10 Define g: $\mathbb{Z} \to \mathbb{Z}$ by f(n) = 2n. For every integer n.
#10a ***Solution:***
(i) is f one to one? Prove or give a counter example.

Let $x, y \in \mathbb{Z}$ such that x and y have the same image
2x = 2y
x = y
Therefore, f is one-to-one.

(ii) is f onto? Prove or give a counter example.

$3 \in \mathbb{Z}$
3 is not the image of any integer because it's not even.
Therefore, f is not onto if 3 is not the image of any integer while 3 is in the codomain.

#10b ***Solution:***

Let $n \in 2\mathbb{Z}$. Find an element in the domain $\mathbb{Z}$ that has n as its image
Let $n = 2k$ for some integer k
f(k) = 2k = n
Therefore, f is onto.

#17 ***Solution:*** $f(x) = \frac{3x-1}{x}$, for each real number $x \neq 0$. Determine whether or not if f is one-to-one.

Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$

$$\frac{ex_1 - 1}{x_1} = \frac{ex_2 - 1}{x_2}$$

$3x_1x_2 - x_1 = 3x_1x_2 - x_2$

$x_1 = x_2$

Therefore, f is one-to-one.

#24 Let S be the set of all strings of a's and b's, and define $N : S \to \mathbb{Z}$ by
$N(s)$ = the number of a's in the string s, for each $s \in S$

#24a ***Solution:*** Is N one-to-one? Prove or give a counterexample.

Let $s_1, s_2 \in S$ such that $N(s_1) = N(s_2)$

Let $s_1 = aab$ and $s_2 = aba$

N(s_1) = 2 and N(s_2) = 2

Therefore, N is not one-to-one.

#24b ***Solution:*** Is N onto? Prove or give a counterexample.

Not onto.

Any string will either 0 a's or a positve number of a's

N(s) cannot take on negative values. Therefore, N is not onto.

#25 Let S be the set of all strings in a's and b's, and define $C : S \to S$ by
$C(s) = as$, for each $s \in S$

#25a ***Solution:*** Is S one-to-one? Prove or give a counterexample

If two strings are different, adding the letter a at the beginning still means they are different.

Therefore, C is one-to-one.

#25b ***Solution:*** Is S onto? Prove or give a counterexample

Not onto.

Every element in the image of C starts with a

C(s) can never equal b, as the string b does not start with an a.

Therefore, C is not onto.