Grover's Algorithm

Quantum Search Algorithm in $\mathcal{O}(\sqrt{N})$ complexity

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Outline

Classical Search

Motivation

- "Imagine a phone directory containing N names arranged in completely random order. In order to find someone's phone number with a probability of 0.5, any classical algorithm (whether deterministic or probabilistic) will need to look at a minimum of $N/2$ names."

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- Consider the function $F : \{0,1\}^3 \to \{0,1\}$,

$$\begin{aligned} F(x,y,z) = &(x \vee y \vee z) \wedge \\ &(\neg x \vee \neg y \vee z) \wedge \\ &(x \vee \neg y \vee \neg z) \wedge \\ &(\neg x \vee y \vee \neg z) \end{aligned}$$

  - Question: For what values of the input does $F(x,y,z) = 1$?

Classical Search

Complexity

- Problems are $\mathcal{O}(N)$ on a classical computer
- Lov Grover published a *quantum algorithm* in 1996 with complexity $\mathcal{O}(\sqrt{N})$.
  - For large N, this represents a significant improvement over the classical case
  - Quantum algorithm cannot guarantee the correct answer, only returns the solution with high probability

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- $\langle\psi|$ is *Dirac notation* for a row vector (*bra* vector)

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  - An example is $\langle\psi| = |\psi\rangle^\dagger$
- The quantity $\langle\phi|\psi\rangle$ is called the *inner product*
  - The quantity is a scalar with value $\sum_{i=1}^{n} \phi_i^* \psi_i$

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Vector Norm

- The norm/length/magnitude squared of $|\psi\rangle \in \mathbf{C}^n$ is defined as

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  - This will always be 1 for a valid quantum mechanical state

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$$(A|\psi\rangle)^\dagger = (|\psi\rangle)^\dagger A^\dagger = \langle\psi|A^\dagger$$

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- Unitary operators are the way in which quantum states are altered or evolved

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\ \sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \sigma_y = \begin{bmatrix}0 & i\\ -i & 0\end{bmatrix},\ \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

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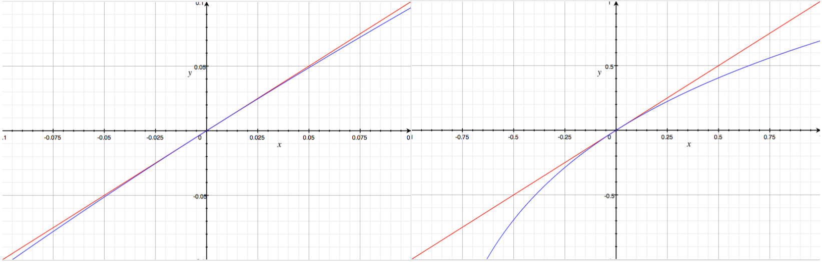
- Common modeling approximation in engineering
  - Example: $\log(1 + x)$ is very nearly x for small x



Linearity & QM

- Linearity is a *fundamental* property of QM
  - *"With classical fields, we often use linear equations, such as the differential equations that allow us to solve for small oscillatory motion of, say, a pendulum. In such a classical case, the linear equation is an approximation; a pendulum with twice the amplitude of oscillation will not oscillate at exactly the same frequency, for example. Hence we cannot take the solution derived at one amplitude of oscillation of the pendulum and merely scale it up for larger amplitudes of oscillation, except as a first approximation. We should emphasize right away, however, that, in quantum mechanics, this linearity of the equations with respect to the quantum mechanical amplitude is not an approximation of any kind; it is apparently an absolute property."-David Miller (Prof. Stanford University)*

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  - Classical analogy is Schrodinger's cat that is both alive and dead

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  - Its state will be $|\psi\rangle = \psi_0|0\dots00\rangle + \psi_1|0\dots01\rangle + \cdots + \psi_{N-1}|1\dots11\rangle$ where $\psi_i \in \mathbf{C}$ and $\sum_{i=0}^{N-1} |\psi_i|^2 = 1$

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  - Copenhagen Interpretation
  - Many World's Hypothesis
  - Quantum Decoherence

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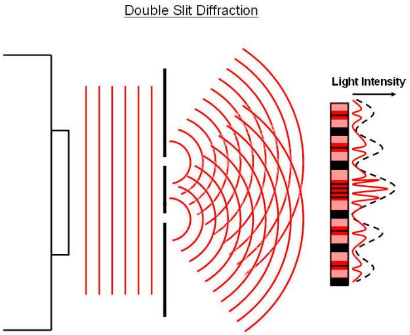

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  - A diffraction pattern! The electron is interfering with itself
- A measurement device is placed on one of the slits to see which one the electron went through
  - Diffraction pattern is gone.
  - Electron behaves like a particle that goes through one slit or the other when measured
  - The act of measurement has altered the outcome

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- If there is no solution or multiple solutions, algorithm does not work out of the box

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  - Define the oracle operator $U_\omega \in \mathbf{R}^{n\times n}$ as

$$U_\omega = I - 2|\omega\rangle\langle\omega| = \mathbf{diag}(\underbrace{1, 1, \ldots, -1, \ldots, 1, 1}_{0,1,\ldots,\omega,\ldots,N-2,N-1})$$

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- Measure qubits

Inversion About the Mean

- The operator U_s can be written as

$$U_s = \begin{bmatrix} \frac{2}{N} - 1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} - 1 & \cdots & \frac{2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} - 1 \end{bmatrix}$$

- Applying the operator to $|x\rangle$ gives the update

$$x_i \leftarrow -x_i + \frac{2}{N} \sum_{j=1}^{N} x_j$$

- Adds twice the mean of the coefficients to negation of each state
- State x_ω was already negated which boosts its value.

Grover's Example

N=4

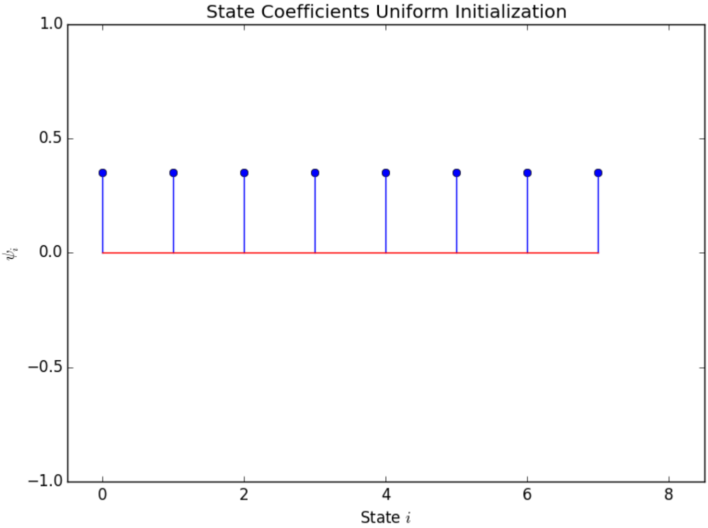
- For the case of $N = 4$ we need only compute $r = \left\lfloor \frac{\pi}{4}\sqrt{4} \right\rfloor = 1$ iteration

$$U_\omega|s\rangle = (I - 2|\omega\rangle\langle\omega|)|s\rangle = |s\rangle - 2|\omega\rangle\langle\omega|s\rangle = |s\rangle - 2|\omega\rangle(1/\sqrt{4})$$

$$\begin{aligned} U_s\left(|s\rangle - \frac{2}{\sqrt{4}}|\omega\rangle\right) &= (2|s\rangle\langle s| - I)\,(|s\rangle - |\omega\rangle) \\ &= 2|s\rangle\langle s|s\rangle - |s\rangle - |s\rangle\langle s|\omega\rangle + |\omega\rangle \\ &= 2|s\rangle - |s\rangle - |s\rangle - |\omega\rangle \\ &= |\omega\rangle \end{aligned}$$

- Measure the system to get the answer

Grover's Illustration (N=8, ω=3)



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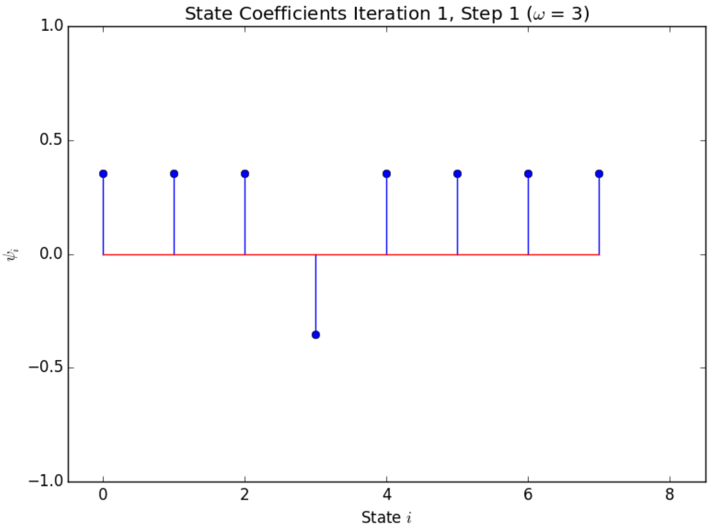

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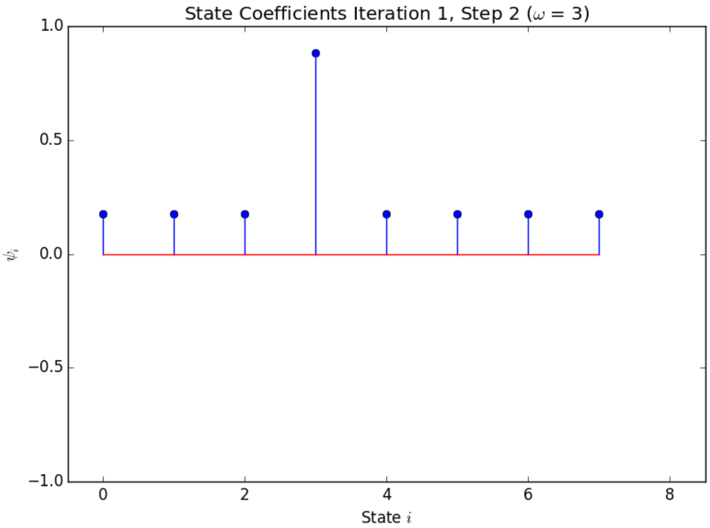

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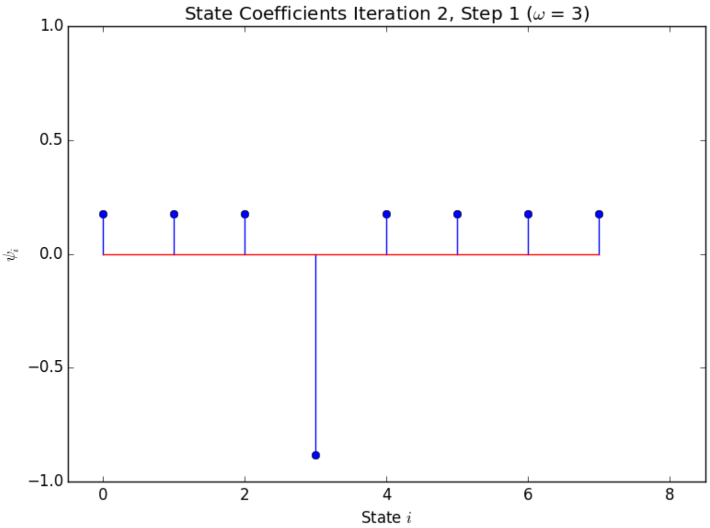
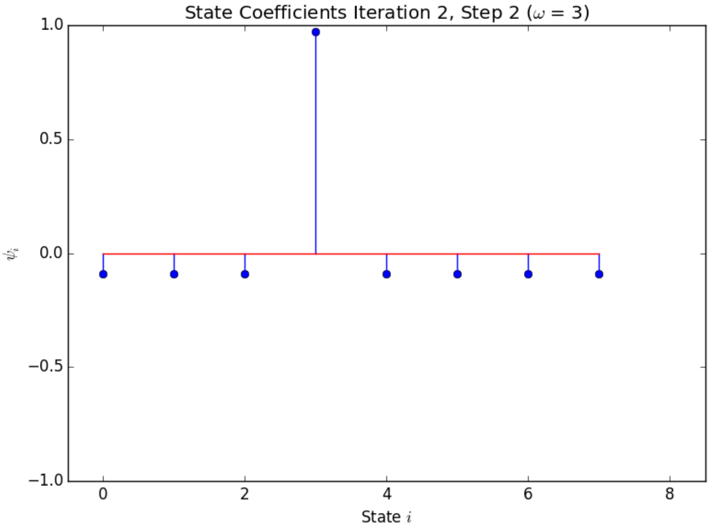


Grover's Illustration (N=8, ω=3)

Graphical



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  - Gates such as Hadamard, Pauli Spin, Phase shift, CNOT etc (all are unitary operators) used to build oracles
  - Programming a quantum computer is more like programming an FPGA than writing software

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- The $\mathcal{O}(\sqrt{N})$ bound assumes predicate can be evaluated on superposition of all states
- $f(\cdot)$ must be implemented in quantum hardware using *quantum gates*
  - May be difficult to find compact gate representation
  - Problems like database search must first convert to an *implicit* list
  - Cost of evaluating $f(\cdot)$ may dominate

Current State of Quantum Computing

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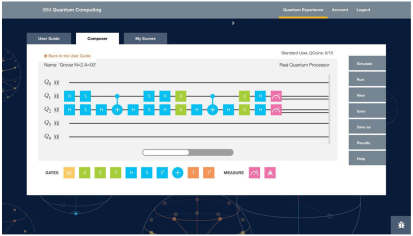

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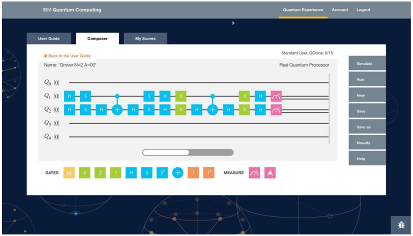

- *D Wave* solves optimization problems by exploiting QM effects
- Much of quantum computing is still theoretical and uses simulation
  - Q#, Libquantum, IBM Q etc
  - Simulation requires *lots* of memory (e.g. 32 qubits implies $2 \cdot 2^{32}$ real numbers)

Outline

References

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