STA 411 OPERATIONS RESEARCH II

COURSE OUTLINES

1. Explain Queuing Theory
    - 1.1 Define Basic Queuing Terminologies Service time, traffic intensity etc.
    - 1.2 State the distributions of arrival & services time as poison and exponential processes.
    - 1.3 Define and State the assumptions of a simple Queuing (M/M/1).
    - 1.4 State and apply the formulae for M/M/1 to practical problems
    - 1.5 Define and State the assumptions of double channel Queue (M/M/2)
    - 1.6 State and apply the formulae for M/M/2 to practical problems.

2.0 Inventory Theory
- 2.1 Define the classical economic order quantity (EOQ)
- 2.2 Explain the components of EOQ model and (QST)
- 2.3 Obtain the optimal order quantity, optimal time between representing minimizing the cost function apply partial differentials
- 2.4 Solve simple inventory problems
- 2.5 Carry out sensitivity analysis of the classical EOQ
- 2.6 State the cost functions of EOQ with shortages allowed
- 2.7 Explain components in the model stated in 2.6
- 2.8 Determine the optimal order quantity, minimum cost and inventory level just after replenishment from the model 2.6

3.0 Simulation Techniques
- 3.1 Define Simulation
- 3.2 State the various simulation techniques
- 3.3 Explain Monte-Carlo methods
- 3.4 Apply Monte-Carlo methods to simulation
- 3.5 Apply computer packages on simulation techniques

1. Queuing Theory

A queuing situation is basically characterized by a flow of customers arriving at one or more service facilities. On arrival at the facility the customer may be serviced immediately or, if willing, may have to wait until the facility is made available. The service time allocated to each customer may be fixed or random dependency in the type of service.

A typical example occurs in a barbershop. Here the arriving individuals are the customers, and the barbers are the servers. Another example is a machine breakdown situation. A broken machine represents a customer calling for the service of a repairman. These examples show that the term "customer" may be interpreted in a variety of ways. Also, a service may be performed by moving either the server to the customer or the customer to the server.

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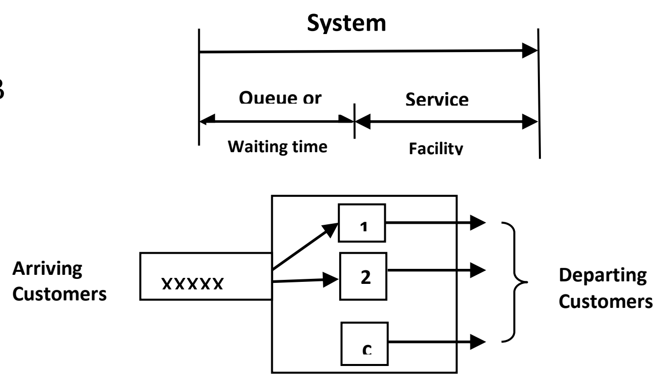


Basic Queuing Terminologies

i. **Customers:** Item, Object, Unit or even a person arriving at station for service.

ii. **Service Station:** This is a place or point where services are being provided.

iii. **Service time:** This is the time taken to serve a customer.

iv. **Waiting time:** This is the time spent by a customer on queue before being served.

v. **Number of customers in the system:** This is the number of customers in the queue together with the number of customers being served.

vi. **Queue System:**This consists of arrival or customers, waiting in queue, service according to discipline and departure of customer.

vii. **Queue Length:** This is the number of customers waiting in the queue.

viii. **Jockeying:** This is the process of leaving the first queue and joining another.

ix. **Reneging:** Leaving a queue without joining another one.

x. **Arrival Rate:** Average number of customers arriving per time. It is normally denoted byλ

xi. **Service Rate:** Average number of customers a server can serve per unit time.
It is denoted by μ.

Traffic Intensity: This is the ratio of arrival rate and service rate i.e. λ/μ which is normally denoted by ρ. The traffic intensity indicates the likelihood and the extent of queuing.

Characteristics of Queuing Models

A queuing system is specified completely by six main characteristics.

1. Input or arrival (interarrival) distribution.
2. Output or departure and service distribution.
3. Service channels
4. Service discipline
5. Maximum number of customers allowed in the system
6. Calling source.

1. **Input or arrival (international) distribution:** The arrival distribution determines the pattern by which the number of customers arrive at the system. Arrivals may also be represented by the interarrival time, which defines the time period between two successive arrivals.

2. **Output or departure (service) distribution:** It determines the pattern by which the number of customers leave from the system. Similarly, departures can be described using the service (inter-departure) time, which defines the time between the commencements of two successive services.

3. **Service Channels:** Service channels may be arranged in parallel or in series or as a more complex combination of both depending on the design of the system's service mechanism. In the case of parallel channels, several customers may be serviced simultaneously. For series channels a customer must pass successively through all the channels before service is completed. A queuing model is called a one-server model when the system has one server only, and a multiple server model when the system has a number of parallel channels each with one server.

4. **Service discipline:** Is a rule for selecting customers from the queue to start service. The most common is the "first come, first served" (i.e FIFO (first in, first out) discipline, where customers are admitted to start service in the strict order of their arrivals. Other discipline include "last come, first served" (sometimes called last-in-first out (LIFO)), "random" and priority.

5. **Maximum number of customers allowed in the system:** it can either finite or infinite depending on the design of the facility. For example, in some facilities only a limited number of customers are allowed to wait in the system. In this case, newly arriving customers are not permitted to join the queue as long as its maximum limit has been reached.

6. **Calling source:** The calling source (or population0 represents an important factor in queuing theory analysis. Since the arrival pattern is dependent in the source from which customers are

generated. The calling source generating the arrivals may be finite or infinite. A finite source exists when an arrival affects the rate of arrival of potential future customers.

Kendall's: Notation for representing Queuing Models

D.G. Kendall and later A. Lee introduced useful notations foe queuing model. The complete notation can be expressed as (a/b/c). (d/e/f) where
a = arrival (or inter-arrival) distribution
b = departure (or service time) distribution
c = number of parallel service channels in the system
d = service discipline
e = maximum number of customers allowed in the system
f = calling source
If we specify the following letters as
M ≡ Poison (Markovian) arrival or departure distribution (or equivalently exponential interarrival or service times distributions).
D ≡ deterministic interarrival or service times
E ≡ Erlangianar Gamma interarrival or service time distribution.
GI = general independent distributions of arrivals (or interarrival time).
G = General service time distribution.
Then (M/ E_k /1): (∞/FIFO) defines a queuing system in which arrivals follow Poisson distribution, service time are Erlangian, single server, infinite capacity and "first in first out" queue discipline.

Model I: (M/M/1): (∞/FIFO):

This model deals with a queuing system having single service channel. Poisson input, exponential service and there is no limit on the system capacity while the customers are served on a "a first in first out (i.e. first come first served)" basis.

The following mathematical notation will be used in connection with queuing models
n = number of customers in the system (waiting line + service facility) at time t.
λ = mean arrival rate (number of arrivals per unit of time).
μ = mean service rate per busy server (number of customers served per unit of time).
λdt= probability that an arrival enters the system between t and $t+dt$ time interval, i.e. within time interval dt.
1-λdt = probability that number arrival enters the system within interval dt plus higher order terms in dt.
μ = mean service rate per channel.
μdt = probability of one service completion between t and $t+dt$ time interval i.e. within time interval dt.
1-μdt = probability of number service reached during the interval dt plus higher order terms in dt.

P_n = Steady state probability of exactly n customers in the system.
$P_n(t)$ = Transient state prob. of exactly n customers in the system at time t, assuming the system started its operation at time zero.
$P_{n+1}(t)$ = Transient state prob. of having n+1 customers in the system at time t.
$P_n(t+dt)$ = Prob. of having n units in the system at time $t+dt$.
L_q = Expected (average) number of customers in the queue.
L_s = Expected number of customers in the system (waiting + being served).
W_q = Expected waiting time per customer in the queue (expected time a customer keeps waiting in line).
W_s = Expected time a customer spends in the system.
L_n = Expected number of customers in a non empty queue
W_n = Expected time a customer waits in line if he has to wait at all, i.e. expected time in the queue for non empty queue.
$\rho = \lambda/\mu$ traffic intensity.

Features of a simple Queue (M/M/1)

a. The arrival pattern is Poisson
b. The service pattern is negative exponential
c. There is only one service channel
d. The traffic intensity (ρ) is less than 1
e. The queue discipline is first-in-first out (FIFO)

Characteristics of Infinite Capacity, Single Server Poisson

Queue Model 1 (M/M/1): (∞/FIFO)

1. Prob. of queue one being greater than or equal to n, the no of customers is given by

$$P(\geq n) = \sum_{k-n}^{\infty} PK$$

$$= (1-\rho)\rho^n \sum_{k-n}^{\infty} p^{k-n} = \frac{(1-\rho)\rho^n}{1-\rho} = \rho^n where \rho = \lambda/\mu < 1$$

2. Average number Ls of customers in the system is given by

$$Ls = E(n) = \frac{\lambda}{\mu-\lambda} or \frac{\rho}{1-\rho}$$

3. Average queue length (number) Lq of customers in the queue is given by

$$Lq = E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} or \frac{\rho^2}{1-\rho}$$

4. Average length of non-empty queue i.e average number Ln of customers in non-empty queue is given by

$$Ln = \frac{\mu}{\mu - \lambda}$$

5. Average waiting time of a customer in the system is

$$E(Ws) = \frac{1}{\mu - \lambda}$$

6. Probability that the waiting tine of a customer in the system exceeds t is

$$P(Ws > t) = \rho^{-(\mu - \lambda)t}$$

7. Average waiting time of customer in the queue is

$$E(Wq) = \frac{\lambda}{\mu(\mu - \lambda)} = Wn \qquad (i.e Wq = \frac{Lq}{\lambda})$$

8. Average waiting time of customer in the queue, if he has to wait.

$$E(Wq / Wq > 0) = \frac{E(Wq)}{P(Wq > 0)} = \frac{1}{\mu - \lambda}$$

Example 1: Cars arrive at a petrol pump, having one petrol unit, in poisson fashion with an average of 10 cars per hour. The service time is distributed exponentially with a mean of 3 minutes. Find

(i) Average number of cars in the system
(ii) Average waiting time in the queue
(iii) Average queue length
(iv) The probability that the number of cars in the system is 2.
(v)

Solution: Mean arrival rate λ=10 per hour
Mean arrival rate μ=1/3 X 60 = 20 per hour

$$\rho = \frac{\lambda}{\mu} = \frac{10}{20} = \frac{1}{2}$$

i. Average number of cars in the system

$$Ls = \frac{\rho}{1-\rho} = \frac{1/2}{1 - 1/2} = 1 car$$

ii. Average waiting time in the queue $= \frac{Lq}{\lambda}$

$$= \frac{0.5}{10} = 0.05 hrs = 3 \min utes$$

iii. Average queue length

$$Lq = \frac{\rho^2}{1-\rho} = \frac{1/4}{1-1/2} = 0.5car$$

iv. Probability of n units in the system $P_n = \rho^n(1-\rho)$

n=2, $P_2 = \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{8}$

Example 2: Association of Statistics students of Nigerian The Polytechnic, Ibadan Chapter employs one clerk at its secretariat. 18 students arrive on an average of every 10 minutes while the clerk can attend to 20 students in 10 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate, calculate.

a. Average number of students in the system

b. Average number of students in the queue.

c. Average time a student spends in the system.

d. Average time a student waits before being attended to

Solution:

The arrival rate $\lambda = \frac{18}{10} = 1.8 students/\min$

The service rate $\mu = \frac{20}{10} = 2 students/\min$

The traffic intensity $\frac{\lambda}{\mu} = \frac{18}{20} = \frac{9}{10}$

a. Average number of students in the system

$$\frac{\lambda}{\mu-\lambda} or \frac{\rho}{1-\rho} = \frac{9/10}{1-9/10} = \frac{9/10}{1/10} = 9students$$

b. Average number of students in the queue

$$\frac{\lambda^2}{\mu-\lambda} or \frac{\rho^2}{1-\rho} = \frac{(9/10)^2}{1-9/10} = \frac{81/100}{1/10} = \frac{81}{100} X10 = 8.1studemts$$

Note: This is also referred to as queue length

c. Average time a student spend in the system

$$\frac{1}{\mu-\lambda} = \frac{1}{2-1.8} = \frac{1}{0.2} = 5\min$$

d. Average time a student waits before being attended to

MULTIPLE CHANNEL QUEUING MODEL (M/M/C) SYSTEM

The only difference in the model component between the (M/M/C) and (M/M/1) systems is that the (M/M/C) system may have more than one service channel.

The letter C stands for the number of parallel service channels

Formulae Relating to Multiple Channel Queuing Model

1. $\rho = \frac{\lambda}{\mu c}$ where λ is the arrival rate,

   μ is the service rate
   C is the no of channel

2. $$P_0 = \frac{C!(1-\rho)}{(C\rho)^c + C!(1-\rho)\left[\sum_{n-0}^{c-1} \frac{1}{n!}(C\rho)^n\right]}$$

   Where P_0 is the probability that is no unit in the system
   n= integers but less than the number of channel

3. $$Wq = \frac{(\rho c)^C}{C!(1-\rho)^2 C\mu} \times P_0$$

4. $$Ws = Wq + \frac{1}{\mu}$$

5. $$Lq = \frac{\rho(\rho c)^c}{C!(1-\rho)^2} \times P_0$$

6. $$L_s = L_q + \rho c$$

7. Probability of queuing on an arrival $= 1 - \frac{(\rho c)^c}{C!(1-\rho)} \times P_0$

Example 3: The arrival rate at Conoil Petrol Station Mokola is 10 vehicles per hr. There are three services points each point can attend to a vehicle at 4 vehicles per hour. Calculate appropriate estimate.

Solution: λ = 10, μ= 4, C= 3 n=C-1 = 2

1. $$\rho = \frac{\lambda}{\mu_c} = \frac{10}{4(3)} = \frac{10}{12} = 0.83$$

2. $$P_0 = \frac{C!(1-\rho)}{(C\rho)^c + C!(1-\rho)\left[\sum_{n=0}^{c-1} \frac{1}{n!}(C\rho)^n\right]}$$

$$= \frac{3!(1-0.83)}{\left(3(0.83)\right)^3 + 6(0.17)\left[\sum_{n=0}^{2} \frac{1}{n!}(0.83x3)^n\right]}$$

$$= \frac{1.02}{(15.4382) + (1.02)\left[\sum_{n=0}^{2} \frac{1}{n!}(2.49)^n\right]}$$

$$= \frac{1.02}{(15.4382) + 1.02\left[\frac{1}{0!}(2.49)^0 + \frac{1}{1!}(2.49) + \frac{1}{2!}(2.49)^2\right]}$$

$$= \frac{1.02}{15.4382 + 1.02[1 + 2.49 + 3.1001]}$$

$$= \frac{1.02}{15.4382 + 6.7219} = \frac{1.02}{22.16}$$

$$= 0.046 \approx 0.05$$

3. $$Wq = \frac{(\rho c)^c}{C!(1-\rho)^2 C\mu} \times P_0$$

$$= \frac{(0.83X3)^3}{3!(0.17)^2 \times 3 \times 4} \times 0.05$$

$$= \frac{0.7719}{2.0808} = 0.37$$

Wq = 0.37 X 60 = 22.26

4. $$Ws = Wq + \frac{1}{\mu}$$

$$= 22.26 + \frac{1}{4} \quad = 22.45$$

5. $$Lq = \frac{P(\rho c)^c}{C!(1-\rho)^2} \times P_0$$

$$= \frac{0.83(0.83X3)^3}{6(0.17)^2} \times 0.05 = \frac{0.6407}{0.1734}$$

= 3.69

6. $L_s = L_q + \rho c$

= 3.69 + 0.83 X 3 = 6.18

7. Prob. of queuing on an arrival $= 1 - \frac{(\rho c)^c}{C!(1-\rho)} \times P_0$

= 0.2432

Exercise:

1. In a super market, the average arrival rate of customer is 10 in every 30minutes following Poisson process. The average time taken by the cashier to list and calculate the customer's purchases is 2.5 minutes, following the exponential distribution. What is the prob. that the queue length exceeds 6?. What is the expected time spent by a customer in the system?
2. A coca-cola centre store at the Polytechnic, Ibadan employ one seller at its counter. Suppose 10 customers arrive on an average of each 6 minutes while cashier can serve 12 customers in 6 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate. Find
   i. Average number of customers in the system.
   ii. Average number of customers in queue
   iii. Average time a customer spends in the system
   iv. Average time a customer waits before being served.
3. Given that the arrival rate at Petro station is 10 vehicles per hour while the service rate is 4 vehicles per hour. If there are three service points at petrol station, calculate
   (i) ρ (ii) Ws (iii) Ls

2. INVENTORY THEORY

Inventory refers to any kind of resource having economic value and is maintained to fulfil the present and future needs of an organization. Fred Hansman, defined inventory as: "An idle resource of any kind provided such a resource has economic value". Such resources may be classified into three categories:

i. Physical Resources such as raw materials, semi finished goods, finished goods, spare parts lubricants etc.
ii. Human Resources such as unused labour (manpower), and
iii. Financial Resources such as working capital etc.

INVENTORY/STOCK CONTROL

Inventory control is an aspect of operation research that deal with management of items stock i.e. result of any items kept to contribute to an organization output.

REASONS FOR CARRYING INVENTORY

1. **Improve customer service:** It provides a level of product or service availability which when located in the proximity of the customer can meet a high customer service level.
2. **Reduce costs:** Holding inventories has a cost associated with it, however, it can indirectly reduce operating costs in other activities and may more than offset the carrying costs.
3. **Maintenance of Operational Capability:** It helps to keep the production going on by acting as a buffer between successive stages of production.
4. **Irregular supply and demand:** It can provide service to customers at various locations by maintaining an adequate supply to meet their immediate and seasonal needs.
5. **Quantity Discounts:** Inventory of items is carried to take advantage of price quantity discount balance many suppliers offer discounts for large orders.

   However, such as advantage must keep a balance between storage cost and higher costs due to spoilage, damaged stock, theft, insurance etc.
6. **Avoid stock outs (shortages):** Labour strikes, natural disasters, variation in demand, and delay in suppliers are the type of contingencies against which inventories can afford some protection as well as to avoid the reputation for constantly being out of stock.

DEFINITION OF TERMS

a. **Lead Time:** This is the time which elapses between the placing of an order for stock and it eventual delivery. Thus, a supply lead time of 2 months means that it will take 2 months from the time an order is placed until the time it is delivered into stores.
b. **Stock-outs:** It refers to a situation where there is a requirement for an item of stock, but the stores of warehouse is temporarily out of stock.
c. **Buffer Stocks:** To avoid stock outs, it may be necessary to hold safety stocks or buffer stocks to meet unexpected high demand. Buffers stocks should only be required from

time to time during the lead time between re-ordering fresh quantities of an item and then re-supply. If there is excessive demand or a delay in re-supply, the buffer stocks might be used. At all other times, buffer stocks represent ‘excess’ stockholding, and give rise to ‘extra’ stockholding costs.

d. **Re-order quantity:** This is the number of units of an item in one order. In OR models a fixed size is usually estimated for the re-order quantity.

e. **Re-order Level:** This is the balance of units remaining in the stock, at which a new order for more units will be placed

ECONOMIC ORDER QUANTITY (EOQ)

The Economic Order Quantity (EOQ) is the number of units that a company should add to inventory with each order to minimize the total cost of inventory such as holding costs, order costs, and shortage cost. The EOQ is used as part of a continous review inventory system, in which the level of inventory is monitored at all times, and a fixed quantity is ordered each time the inventory level reaches a specific reorder point.

ASSUMPTIONS OF EOQ MODEL – (Determine Models)

a. Demand is certain, constant and continuous over time
b. The supply lead time is constant and certain, or else there is instantaneous re-supply.
c. Customers’ orders cannot be held while replenishment stocks are awaited and current stock are down to nothing.
d. No stock outs are permitted.
e. All prices are constant and certain, there are no bulk purchases discounts;
f. The cost of holding stock is proportional to the quantity of stock held.

The notation adapted is:
d = usage in units for one year (i.e demand)
c = cost of making one order (ordering cost)
h = holding cost per unit of stock for one year
Q = re-order quantity

Derivation of the EOQ formula

As demand is constant, average stock $= Q/2$

As h is the holding cost per unit of stock then the cost of holding the average stock for one year $= \frac{Qh}{2}$

The number of orders made in a year $= d/Q$

If the cost of making one order is C then total ordering costs for a year $= cd/Q$

Therefore, the cost of having stock (T) for one year $= \frac{Qh}{2} + \frac{cd}{Q}$

i.e. $T = \frac{Qh}{2} + \frac{cd}{Q}$

The objective is to minimize T

i.e. $\frac{dT}{dQ} = 0$

$$T = \frac{Qh}{2} + cdQ^{-1}$$

Hence $\frac{dT}{dQ} = \frac{h}{2} - \frac{cd}{Q^2}$

$$\frac{dT}{dQ} = 0$$

$$\frac{cd}{Q^2} = \frac{h}{2}$$

$$Q^2 h = 2cd$$

$$Q^2 = \frac{2cd}{h}$$

$$\therefore Q = \sqrt{\frac{2cd}{h}}$$

Example 1:

The demand for a commodity is 40,000 per year, at a steady rate. It cost #20 to place an order, and 40K to hold a unit for a year. Find the batch size to minimize inventory costs, the number of order placed per year and the length of the inventory cycle.

Solution:

$$Q = \sqrt{\frac{2cd}{h}} = \sqrt{\frac{2X\,20X\,40{,}000}{0.4}} = 2{,}000 units$$

This implies that there will be $= d/Q$

$\frac{40{,}000}{2{,}000} = 20$ orders placed one year,

So that the inventory cycle b

$$\frac{52wks}{20orders} = 2.6weeks$$

Example 2:

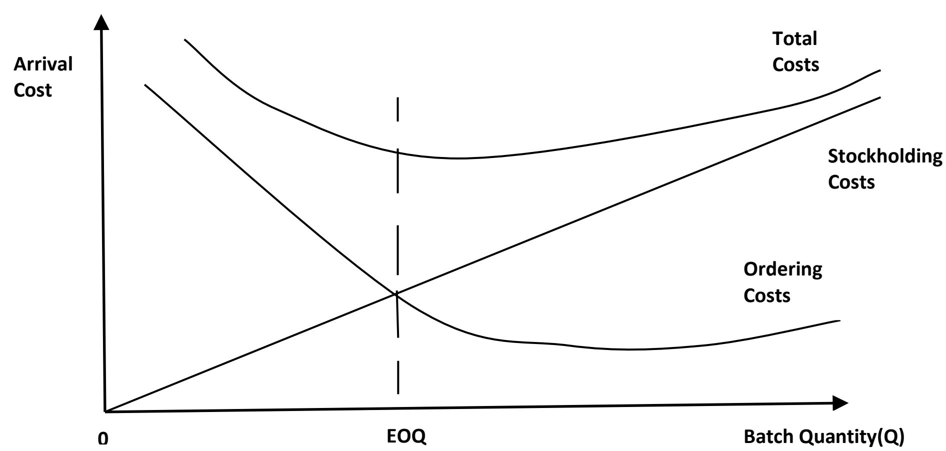
Pam runs a mail order business for gym equipment. Annual demand for the Tricoflexers is 16,000. The annual holding cost per unit is $2.50 and the cost to place an order is $50. What is the economic order quantity?

Solution:

$$Q = \sqrt{\frac{2cd}{h}} = \sqrt{\frac{2X16{,}000X50}{2.50}} = 800 \text{ units per order}$$

A graphical Approach

The variation of costs with batch size can be represented by the following graph



At the EOQ level of order size ordering costs during the period will equal stock holding costs i.e

$$\frac{Qh}{2} = \frac{cd}{Q}$$

Note: Cross multiplying

$$Q^2 h = 2cd$$

$$Q = \sqrt{\frac{2cd}{h}}$$

Other Formulae for Optimum Order Size

There are several other formulae for optimum order size which incorporate the continous flow of products, back ordering issue and the like ones.

We shall discuss some of this

i. When the continuous flow of product is assumed. Instead of receiving the ordered unit all at once, the firm may receive the product continuously at a period of time. If we take d, the rate of using the unit per year, P- the rate of producing units per year.
Q – the number of much produced on each production run.

Therefore, the optimum production cost is expressed as:

$$Q = \sqrt{\frac{2cd}{h} \cdot \frac{P}{P-d}}$$

Example 3: It cost N5,000 to set up production for a product that is used at the rate of 1,000 unit per year. The carrying cost per unit is N500 per year. The rate of production is 2000 units per year. Determine the size of the production run

Solution

C= 5000, d=1000, h=500, p=2000

$$Q = \sqrt{\frac{2cd}{h} \cdot \frac{p}{p-d}} = \sqrt{\frac{2X1000X5000}{500} X \frac{2000}{2000-1000}}$$

= 200 units

ii. Optimal length of each lot size production rum $tp = \frac{Q*}{p}$

iii. Optimal production cycle time, $t = Q/D$

iv. Optimal Number of production cycle $N = D/Q$

where t = D = annual requirement (demand) of an item (units per unit time)

INVENTORY CONTROL MODELS WITH SHORTAGE

The previous model discussed were based on the assumption that shortages and back ordering are not allowed. However, there could be situations in to economic advantage may be gained by allowing shortage is to increase the cycle time, and hence spreading the ordering (or set-up) costs user a longer period. Another advantage of shortage may be seen where the unit value of inventory and hence the inventory carrying cost is high.

Normally, the benefits due to reduce carrying costs or less number of orders in a planning period are less than the increase in the total inventory costs due to a shortage condition.

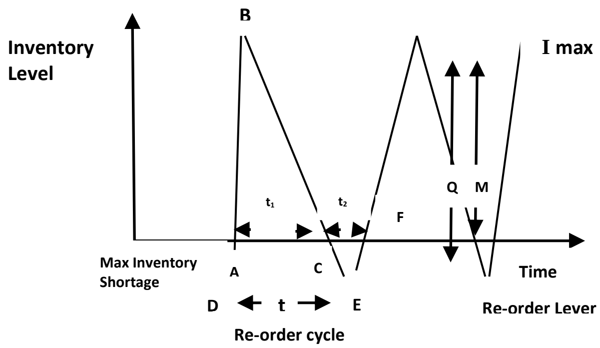
EOQ Model with Constant Rate Of Demand And Variable Order Cycle Time

This model is based on all the assumptions of EOQ model, except that the inventory system runs out of stock for a certain period of time, i.e shortages are allowed and the cost of shortage is assumed to be directly proportional to the average number of units short.

In addition to the notations used previously, let

t_1 = time between the receipt of an order and when the inventory level drops to zero, i.e time when no shortage exist.

t_2 = time during is back order or shortage exists
t = total cycle time: $t = t_1 + t_2$
R = maximum shortage (units), S = shortage cost



The above figure describes the changes in the inventory level with time. Every time the quantity Q (order size) is received, all shortages equal to an amount R are first taken care and the remaining quantity M is placed in inventory as the surplus from to demand during next cycle will be satisfied. Here it may be noted that R units out of Q are always in the shortage list, i.e these are never carried in stock. Thus it yields savings on the inventory carrying cost.

Total variable inventory cost (TVC) = ordering cost + carrying cost + Shortage cost

$t_1 = {}^{M}\!/_{D}$, $\qquad t = {}^{M}\!/_{D}$ $\qquad$ and therefore $t_2 = \dfrac{Q-M}{D}$

Hence carry cost $= \dfrac{M^2}{2Q}.h$

Average shortage level (in units) $= \dfrac{(Q-M)^2}{2Q}$

Shortage cost $= \dfrac{(Q-M)^2}{2Q}.S$

Hence, the total yearly variable inventory cost is given by

$$TVC(Q,M) = \frac{D}{Q}C + \frac{M^2}{2Q}h + \frac{(Q-M)^2}{2Q}S$$

Since TVC is the of functions of two variables Q and M, therefore to determine the optimal order size and the optimal shortage level R, differentiate the total variable C function with respect to Q and M, set the two resulting equations equal to zero and so the simultaneously. Doing so, we get results

$$Q^* = \sqrt{\frac{2DC}{h}\left(\frac{h+s}{s}\right)}, \qquad \text{Economic Order Quantity}$$

$$M^* = \sqrt{\frac{2DC}{h}\left(\frac{s}{h+s}\right)}\ ,\quad \text{Optimal Stock Level}$$

By substituting these values in TVC equation, the minimum so obtained is as follows:

$$TVC = \sqrt{2DCh\left[\frac{s}{h+s}\right]}\ ,\quad \text{Optimal Cost}$$

Other important formulae

Once we have computed values for Q* and M*, the following can be determined as follows;

1. Optimal Shortage Level (in units),

$$R^* = Q^* - M^* = Q^*\left[\frac{h}{h+s}\right]$$

2. Total cycle time, $t = \frac{Q^*}{D} = \sqrt{\frac{2C}{Dh}\left[\frac{h+s}{s}\right]}$

Example 4: A commodity is to be supplied at a constant rate 200 units per day. Suppliers of any amount can be obtained at any required time, but each ordering costs N50, cost of holding the commodity in inventory is N2 per unit per day while the delay in the supply of the item induces a penalty of N10 per unit per day. Find the optimal policy (Q,t), where t is the reorder cycle period and Q is the inventory after reorder. What would be the best policy, if the penalty cost becomes infinite?

Solution:

D = 200 units/day, h=#2 per unit per day

C = #50 per day S = #10 per unit per day

a. Optimal order Quantity

$$Q^* = \sqrt{\frac{2DC}{h}\left[\frac{h+S}{S}\right]} = \sqrt{\frac{2X200X50}{2}\left(\frac{2+10}{10}\right)}$$

= 109.5 units

b. Reorder cycle time, $t^* = \frac{Q^*}{D} = \frac{109.5}{200} = 0.547 day$

Thus optimal order quantity of 109.5 units must be supplied after 0.547 days

If the penalty cost S= α , the expression Q* will be become

$$Q^* = \sqrt{\frac{2DC}{h}\left[\frac{h+S}{S}\right]} = \sqrt{\frac{2DC}{h}} = \sqrt{\frac{2X200X50}{2}} = 100 units$$

And $t^* = \frac{Q^*}{D} = \frac{100}{200} = \frac{1}{2} days$

Example 5: A dealer supplier you the following information with regard to a product dealt in by him: Annual demand = 10,000 units; Ordering cost= N10per order, Price = N20 per unit. Inventory carry cost: 20 per cent of the value of inventory per year. The dealer is considering the possibility of allowing some backorder (stockout) to occur. He has estimated that the annual cost of backordering will be 25 per cent of the value of inventory.

a. What should be the optimum number of units of the product he should buy in one lot?

b. What quantity of the product should be allowed to be back-ordered, if any?

c. What would be the maximum quantity of inventory at any time of the year?

d. Would you recommend to allow back-ordering? If so, what would be the annual cost saving by adopting the policy of back-ordering.

Solution:

D = 10,000 units/year C = N10 per order

P = N20 per units, h = 20% of N20 = N4 per unit per year,

S = 25% of N20 = N5/unit/year

a. Economic order quantity (Q*)

(i) When stock out are not permitted

$$Q^* = \sqrt{\frac{2DC}{h}} = \sqrt{\frac{2X10000X10}{4}} = 223.6 units$$

(ii) When back ordering is permitted

$$Q^* = \sqrt{\frac{2DC}{h}\left[\frac{h+S}{S}\right]} = \sqrt{\frac{2X10000X10}{4}\left[\frac{4+5}{5}\right]} = 300 units$$

b. Optimal quantity of the product o be back ordered.

$$R^* = Q^*\left[\frac{h}{h+S}\right] = 300\left[\frac{4}{4+5}\right] = 133 units$$

c. Maximum Inventory Level

$$M^* = \sqrt{\frac{2DC}{h}\left(\frac{S}{h+S}\right)} = \sqrt{\frac{2X10000X10}{4}\left[\frac{5}{4+5}\right]} = 167 units$$

d. Minimum total variable inventory cost in the cases when stock out are allowed and not allowed is

$$TVC(223.6) = \sqrt{2DCh} = \sqrt{2X10000X10X4} = ₦894.43$$

$$TVC(300) = \sqrt{2DCh\left[\frac{S}{h+S}\right]}$$

$$= \sqrt{2X10,000X10X4\left[\frac{5}{4+5}\right]} = N666.67$$

Since TVC(223.6) > TVC (300), therefore dealer should accept the proposal for back ordering as this will save him N(894.48 – 666.67) = N227.76 per year.

3. Simulation Techniques

Simulation is a descriptive operations research model used to try out several different alternatives while observing how they perform within the system. It is applicable when a problem cannot be expressed by a standard mathematical model because of the complexity of the situation.

Some management problems are so complex that mathematical models, such as linear programming cannot be used or because managers cannot arrive at satisfactory measure of optimality. When this happens, the use of simulation is employed. Applications of simulation in management can be seen in the areas of recording, Queuing theory, inventory control, corporate planning, quality control, capital investment etc.

Note that, simulation enables a manager to evaluate the performance of a proposed system before it is actually installed.

Methods of Simulation

There are two methods of simulation via:

(i) Monte-Carlo method

(ii) System simulation method

It is required in this course to discuss and apply to practical problems the Monte-Carlo method to simulate.

Monte-Carlo method

Monte-Carlo method uses random numbers and experimentation with the help of the probability theory. Monte-Carlo simulation is a statistical technique that uses random sampling to solve mathematical problems and analyze uncertainty. It involves:

(a) Defining a problem or system

(b) Creating a mathematical model

(c) Generating random inputs (e.g. variables, parameters)

(d) Running multiple simulations (iterations)

(e) Analyzing output results.

Advantages of simulation

1. There is less degree of assumptions
2. It is more flexible than the analytical technique because it is an initiation of the reality situation
3. It is useful where time does not one to wait till the real event occurs before action is taken.

Disadvantages (limitation) of simulation

1. It does not produce optimal solution
2. Its results are estimate which are subject to statistical error
3. Simulation tend to be relatively costly, especially simulated models of the complex systems
4. Simulated models may fail to adequately represent important elements of the system
5. It is a slow way of studying a problem when better ones are not available.

Example 1:

A department store manageress is seeking advice on the level of SOYOYO terms to carry on a daily basis so as to maximize her profit. From experience, the behavior of past sales of SOYOYO is given as follows:

Possible Demand	Probability
4500	0.15
6000	0.14
7500	0.40
9000	0.08
10500	0.17
12000	0.06

Simulate for 15 day period and advise the departmental Store on the optimal quantity to carry using the random numbers below:

67, 94, 50, 78, 39, 09, 54, 65, 88, 41, 32, 08, 12, 46and 93

Solution

15 day period simulation

Possible demand	probability	Cumulative probability	Random number allocation
4500	0.15	0.15	00-14
6000	0.14	0.29	15-28
7500	0.40	0.69	29-68
9000	0.08	0.77	69-76
10500	0.17	0.94	77-93
12000	0.06	1.00	94-99

Run	Random numbers	Forecast demand
1	67	7500
2	94	12000
3	50	7500
4	78	10500
5	39	7500
6	09	4500
7	54	7500
8	65	7500
9	88	10500
10	41	7500
11	32	7500
12	08	4500
13	12	4500
14	46	7500
15	93	10500
		117,000

$$\text{Optimal quantity to carry} = \frac{117,000}{15} = 7,800$$

Example 2: A dentist schedules all her patients for 30 minutes appointment. Some patients take more or less than 30 minutes depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and the time needed to complete the work

Category	Time required (min)	Probability of category
Filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Etraction	45	0.10
Checkup	15	0.20

Simulate the dentist's clinic and determine the average waiting time for patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival times, starting at 8 am. Use the following random numbers: 40, 82, 11, 34, 25, 66, 17 and 79.

Solution:

Time	Probability	Cumulative probability	Random number allotted
45 min.	0.40	0.40	00-39
60 min.	0.15	0.55	40-54
15 min.	0.15	0.70	55-69
45 min.	0.10	0.80	70-79
15 min.	0.20	1.00	80-99

Patients No.	Arrival time	Dentist's treatment Starts	ends	service time	waiting time patient	Dentist
1	8.00	8.00	9.00	60	-	-
2	8.30	9.00	9.15	15	30	-
3	9.00	9.15	10.00	45	15	-
4	9.30	10.00	10.45	45	30	-
5	10.00	10.45	11.30	45	45	-
6	10.30	11.30	11.45	15	60	-
7	11.00	11.45	12.30	45	45	-
8	11.30	12.30	1.15	45	60	-
					285	

Average waiting time for the patients $= \frac{285}{8} = 35.625 \text{ min}s.$

Example 3: The time between arrival of students in HOD's office is shown with its probability as follows:

Time between arrivals (minutes)	Probability
5	0.10
10	0.15
15	0.25
20	0.25
25	0.15
30	0.10

The service time is distributed as follows:

Service time (minutes)	Probability
5	0.05
10	0.20
15	0.40
20	0.20
25	0.10
30	0.05

The following random numbers have been provided

Arrivals: 12, 81, 98, 82, 21, 74, 90, 55, 79, 70, 42, and 84

Service: 61, 74, 98, 54, 15, 49, 64, 98, 17, 00, 49, and 58

Assume that HOD resumes at 10.00 am.

Required:

(i) Simulate for twelve (12) students

(ii) Determine (a) Elapsed time (b) estimated average arrival rate (c) estimated average service rate (d) estimated average student time spent in the system (e) estimated average number of students in the system (f) service utilization factors.

(i)

For arrivals

Inter-arrival time(mins)	Probability	Cumulative probability	Random number allocation
5	0.10	0.10	00-09
10	0.15	0.25	10-24
15	0.25	0.50	25-49
20	0.25	0.75	50-74
25	0.15	0.90	75-89
30	0.10	1.00	90-99

For service

Service time (mins)	Probability	Cumulative probability	Random number allocation
5	0.05	0.05	00-04
10	0.20	0.25	05-24
15	0.40	0.65	25-64
20	0.20	0.85	65-84
25	0.10	0.95	85-94
30	0.05	1.00	95-99

Simulation table

Random number	Time till next arrival	Arrival time	Service begins	random number	Service		Waiting time	
					Time	ends	HOD	Students
12	10 mins	10.10am	10.10am	61	15mins	10.25am	10	-
81	25 mins	10.35am	10.35am	74	20mins	10.55am	10	-
98	30 mins	11.05am	11.05am	98	30mins	11.35am	10	-
82	25 mins	11.30am	11.35am	54	15mins	11.50am	-	5
21	10 mins	11.40am	11.50am	15	10mins	12.00pm	-	10
74	20 mins	12.00pm	12.00pm	49	15mins	12.15pm	-	--
90	30 mins	12.30pm	12.30pm	64	15mins	12.45pm	15	-
55	20 mins	12.50pm	12.50pm	98	30mins	1.20pm	5	-
79	25 mins	1.15pm	1.20pm	17	10mins	1.30pm	-	5
70	20 mins	1.35pm	1.35pm	00	5mins	1.40pm	5	-
42	15 mins	1.50pm	1.50pm	49	15mins	2.05pm	10	-
84	25 mins	2.15pm	2.15pm	58	15mins	2.30pm	10	
	255mins				195mins		75mins	20 mins

(ii)

(a) $\quad Elapsed\ time = Time\ last\ student\ left - \mathrm{Re}\,sumption\ time$

$$= 2.30\,pm - 10am = 4hrs\ 30\min s = 270\ \min s$$

(b) $$Estimate\ average\ arrival\ rate = \frac{No.\,of\ students}{Total\ time\,between\ arrivals} = \frac{12}{255} \approx 0.05\min s$$

(c) $$Estimate\ average\ service\ rate = \frac{No.\,of\ students}{Total\ service\ time} = \frac{12}{19.5} \approx 0.06\min s$$

(d) $$Estimate\ average\ student\ time\ spent\ in\ system = \frac{Total\ \ (waiting\ time + service\ time}{Number\ of\ students}$$

$$= \frac{20+195}{12} \approx 17.9\min s$$

(e) $$Estimate\ average\ number\ in\ the\ system = \frac{Total\ (waiting +\ service\ time)}{Elapsed\ time} = \frac{215}{270}$$

$$\approx 1\,student$$

(f) $$Service\ utilization\ factor = \frac{Total\ service\ time}{Elapsed\ time} = \frac{195}{270}$$

$$\approx 72\%$$