Influence branching for learning to solve Mixed Integer Programs online

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Introduction

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Mixed Integer Programs

Mixed integer programs

Mixed integer linear programs are generally defined such as:

$$P : \begin{cases} \min c^T x \\ Ax \leq b \; ; \; x \in \mathbb{N}^{|\mathcal{I}|} \times \mathbb{R}^{n-|\mathcal{I}|} \end{cases}$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

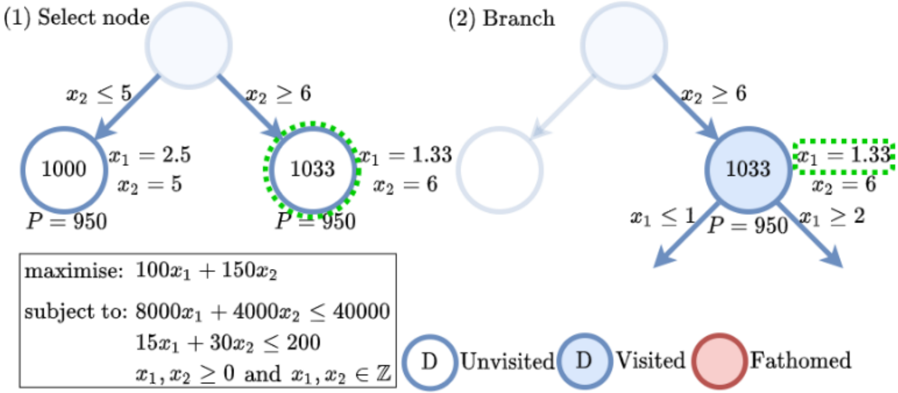
- Linear objective
- Linear constraints
- Integrity constraints makes the problem non-convex
- NP-hard in fact

Branch & bound

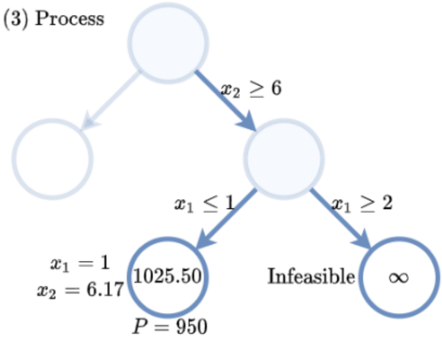
B&B :
Create a partition of the solution space by fixing binary variables at either 0 or 1 (branching).
Such partitioning is built following a tree structure, each node being a sub-problem of the initial MILP.
At each node, we solve the corresponding linear relaxation and hope to find a solution satisfying the binary constraints. A branch is expanded until we prune the leave nodes out of:

- Infeasibility
- Integrality
- Sub-optimality

Branch & bound

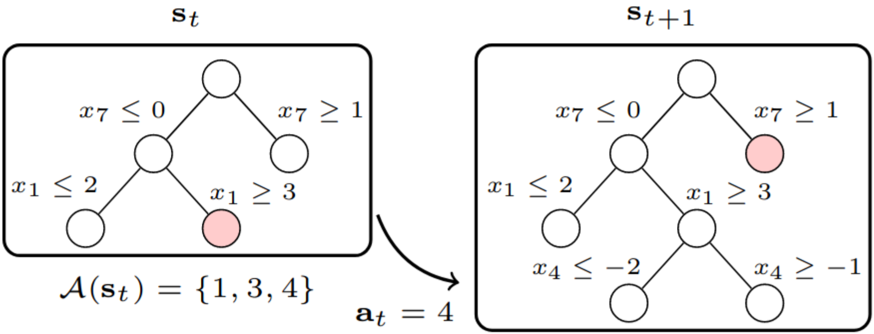


Branch & bound



Strong branching

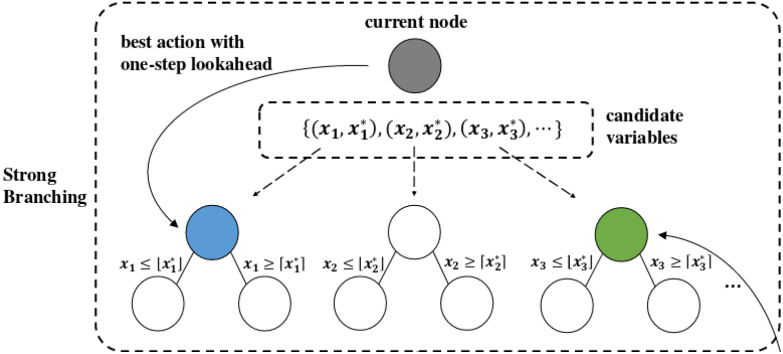
Two major selection strategies to parametrize the B&B solver :



The **node** selection strategy and the **variable** selection strategy !

Strong branching

Strong branching : select the best one-step lookahead branching in terms of **dual gap**



Inconvenient : vast number of LP iterations associated, intractable in most cases

Learning to solve MIPs

Industrial MIP solvers' node and variable selection strategies are based on complex fine-tuned heuristic, designed to perform best (in average) over a vast range of benchmarks.

In the context of real-world applications, in which similar instances with slightly varying inputs are solved on a regular basis, there is a huge incentive to reduce the solving time by learning efficient tailor-made heuristics.

MIP23 Computational competition

Each series $s \in \mathcal{S}$ of the competition if composed of 50 fixed-size MIPs sampled from an unknown distribution $\mathbb{Q}_s$.

$$i \in \mathcal{I}_s \sim \mathbb{Q}_s : \begin{cases} \min c^T x \\ Ax \leq b \ ; \ x \in \mathbb{N}^{|\mathcal{J}|} \times \mathbb{R}^{n-|\mathcal{J}|} \end{cases}$$

For each series, the value of one or several vectors among $\{A, b, c\}$ can change.

MIP23 Computational competition

- Series of 50 instances $i \in \mathcal{I}_s$ to solve **online** (sequentially).
- $t_{max} \sim 300\,s$ per instance
- $f_{s,\,i} = \frac{t}{t_{max}} + dualgap + no\ primal$
- $Total\ score = \sum_{s \in \mathcal{S}} \sum_{i=1}^{50} (1 + 0.1i) \cdot f_{s,\,i}$

MIP23 Computational competition

Trade-off between **learning a model** and **solving an instance** !

$\rightarrow$ Very different framework from the literature of machine learning applied to mixed integer programming.

$\rightarrow$ If you had to learn the absolute minimum about a MIP series, what would it be ?

Our take : for each series, we are going to try to learn graph representations of the instances leading to the best branching decisions near the root node of the B&B tree.

In fact, we introduce a new graph-based branching heuristic, named **influence branching**, and learn to fine tune it across instance series.

Rappel du plan

Definition

Local influence

We define the local influence w_{ij}^l exerted by variable i on variable j through constraint l. w_{ij}^l can be any function on A, b, c, in particular, we say that i has a non-zero influence on j through l if $\mathbb{1}_{A_{li} \neq 0} \mathbb{1}_{A_{lj} \neq 0} \neq 0$.

Direct influence

We define the direct influence w_{ij} exerted by variable i on variable j over P as :

$$w_{ij} = \mathbb{1}_{i \neq j} \sum_{l=1}^{m} w_{ij}^l$$

Definition

We can then derive a definition for influence graphs :

Influence graph

We call influence graph the directed graph $G = (V, E, W)$ where $V = \{1, ..., n\}$, $E = V \times V$ and where $W \in \mathbb{R}^{n \times n}$ the w_{ij} matrix satisfies the definition of direct influence.

Figure: Examples of influence graphs

Influence models

- **Count** $w_{ij}^{l} = \mathbb{1}_{A_{li}}\mathbb{1}_{A_{lj}}$
- **Binary** $w_{ij}^{l} = \frac{\mathbb{1}_{A_{li}}\mathbb{1}_{A_{lj}}}{\sum_{k=1}^{m}\mathbb{1}_{A_{ki}}\mathbb{1}_{A_{kj}}}$
- **Dual** $w_{ij}^{l} = \mathbb{1}_{A_{li}}\mathbb{1}_{A_{lj}}|y_l^*|$
- **Countdual**
  $w_{ij}^{l} = \mathbb{1}_{A_{li}}\mathbb{1}_{A_{lj}}\mathbb{1}_{(y_l{}^*\neq 0)}$
- **Auxiliary**
  $w_{ij}^{l} = \mathbb{1}_{A_{li}}\mathbb{1}_{A_{lj}}s_i|A_{li}y_l|$
- **Adversarial**
  $w_{ij}^{l} = \mathbb{1}_{A_{li}}\mathbb{1}_{A_{lj}}s_i|\frac{A_{li}}{A_{lj}}|\mathbb{1}_{(y_l{}^*\neq 0)}$

Table: Proposed influence models, with y^* the solution of the dual problem at the current node and s_i the minimal distance to a bound for variable i in the primal solution. $\mathbb{1}_{A_{li}\neq 0}$ is noted $\mathbb{1}_{A_{li}}$ to ease the notations.

Influence branching

Influence branching

The influence branching heuristic returns the variable within the graph with the maximal total influence :

$$w^* = \max_i w_i = \max_i \sqrt{1 + c_i} \sum_{j \neq i} w_{ij}(g) \tag{1}$$

as long as the depth of the current node d is inferior or equal to k, the maximum depth.

Influence branching is a variable selection strategy relying on two hyperparameters :

- $g \in \mathcal{G} = \{count, binary, ..., adversarial\}$, the influence model
- $k \in \mathbb{N}$, the maximal depth to apply the heuristic

Speed up potential on MIPcc23

Instance	Influence model	Max depth	Performance $f_{s,i}$	SCIP default	Speed up
1	binary	5	0.64	0.70	-0.06
2	adversarial	5	0.53	1.01	-0.48
3	countdual	5	0.49	0.60	-0.11
4	count	4	0.48	0.76	-0.28
5	countdual	2	0.70	1.03	-0.33
6	count	4	0.26	0.44	-0.18
7	auxiliary	3	0.42	0.53	-0.11
8	countdual	2	0.71	0.98	-0.37
9	countdual	4	0.57	1.39	-0.82
...	...	...	...	...	...
50	binary	5	0.29	0.66	-0.34
Avg			**0.56**	**0.94**	**-0.38**

Multi-armed bandit problem

- Learning which pair (g, k) performs best for any instance of any series would require to shift to a reinforcement learning framework
- We adopt an online bandits framework, as we try to learn which pair (g, k) obtains the best performance in average on a whole series of instances

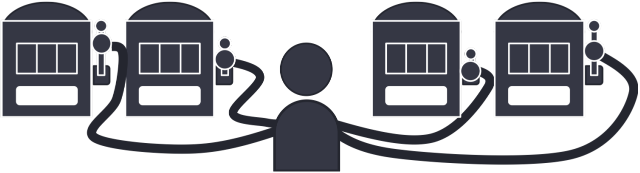
Multi-armed bandit problem

The optimization task can be reformulated as a multi-armed bandit problem on action space $\mathcal{A}$ where

$$\min_{a_i \in \mathcal{A}} \sum_{i=1}^{50} (1 + 0.1i)\, f_{s,\,i}(a_i) \tag{2}$$

is the sum of reward to minimize.

Multi-armed bandit problem



Multi-armed bandit problem

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$$\min_{a_i \in \mathcal{A}} \sum_{i=1}^{50} (1 + 0.1i)\, f_{s,\,i}(a_i) \tag{3}$$

is the sum of reward to minimize.

Multi-armed bandit problem

Influence model	Max depth	Performance	Speed up	Rank
count	5	0.857	**-0.0862**	**1**
base	6	0.865	-0.0783	2
countdual	2	0.874	-0.0691	3
base	5	0.877	-0.0657	4
count	4	0.882	-0.0606	5
...	...	...	...	...
adversarial	3	0.953	0.0520	34
adversarial	2	0.973	0.0721	35
auxiliary	5	1.05	0.148	36

Table: Sorted average performance of influence branching on *obj series 2* for each pair (g, k). The performance column corresponds to the mean of $f_{s,i} = \textit{reltime} + \textit{gap at time limit} + \textit{nofeas}$ over $\mathcal{I}_s$.

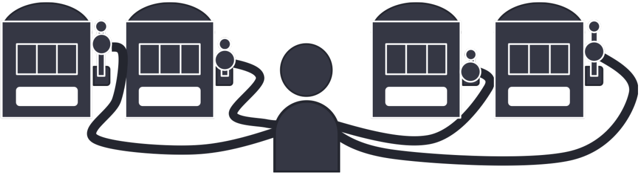
Rappel du plan

1. Mixed Integer Programming
2. Influence branching
3. Online learning
   - Building an action set for MIPcc23
   - Convergence
   - Results

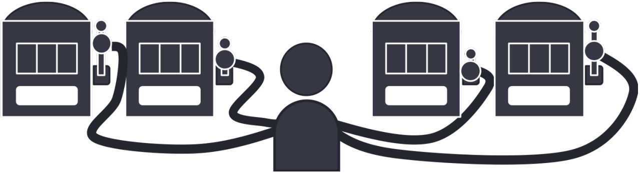
Building an action set

For each series, only 50 samples are available in total:

- Scores $\{f_{s,\,i}(a)\}_{i \in \mathcal{I}_s}$ with $a \in \mathcal{A} = \{(g, k) : g \in \mathcal{G},\ k \in [1, 6]\}$ are assumed to follow an unknown probability distribution $\mathcal{P}_{a,\,s}$.
- In order to minimize (3), the means of $(\mathcal{P}_{a,\,s})_{a \in \mathcal{A}}$, noted $(\mu_{a,\,s})_{a \in \mathcal{A}}$, need to be estimated (or at least ranked) as efficiently as possible for the heuristic to select the action leading to the minimum expected reward.



Building an action set



However :

- The more actions in the action space, the more samples are needed to guarantee the convergence of the bandits algorithm towards optimal actions.
- Moreover, the spreads between $(\mu_{a,\,s})_{a \in \mathcal{A}}$ are rather small, comprised between 0.01 and 0.2, in front of standard deviations of $(\mathcal{P}_{a,\,s})_{a \in \mathcal{A}}$, noted $(\sigma_{a,\,s})_{a \in \mathcal{A}}$, that were measured around $0.1 - 0.3$ across public series.

Action set

After running computationally intensive over test the competition's series, we derive an action set to train our bandit agent :

Action set

$\mathcal{A} = \{(count, 1), (count, 5), (countdual, 2), (binary, 3), (dual, 3)\}$

We use Thompson sampling select to train our agent online :

Thompson sampling

- Hypothesis $(\mathcal{P}_{a,s})_{a\in\mathcal{A}} \sim \mathcal{N}(\mu_{a,s}, \sigma = 0.2)$
- Initialize $(\hat{\mu_a}, \hat{\sigma_a})$ with (μ_0, σ_0)
- Draw samples $x_a \sim \mathcal{N}(\hat{\mu_a}, \hat{\sigma_a})$ for $a \in \mathcal{A}$
- Perform action $a^* = \underset{a}{arg\,min}\, x_a$ and observe reward $f_s(a_i)$
- Perform bayesian update on $\hat{\mu_a}$ and $\hat{\sigma_a}$

TS convergence on MIPcc23 series

$$CS = \frac{\sum_{i=1}^{50} \mu_{i,s}(a_i) - \mu_{i,s}(a_0)}{\sum_{i=1}^{50} \mu_{i,s}(a_s^*) - \mu_{i,s}(a_0)}$$

Series	Convergence score
bnd series 1	72%
bnd series 2	65%
obj series 1	75%
obj series 2	66%
rhs series 1	64%
rhs series 2	72%
rhs obj series 1	74%

Table: Convergence score of Thompson sampling on MIPcc23 public instances series.

Results

Series	Average $f_{s,\,i}$	Speed up
bnd series 1	0.992 ± 0.009	$-\mathbf{0.031 \pm 0.009}$
bnd series 2	0.881 ± 0.020	$-\mathbf{0.037 \pm 0.020}$
obj series 1	0.895 ± 0.006	$-\mathbf{0.022 \pm 0.006}$
obj series 2	0.891 ± 0.022	$-\mathbf{0.052 \pm 0.022}$
rhs series 1	0.875 ± 0.027	$-\mathbf{0.048 \pm 0.027}$
rhs series 2	1.004 ± 0.0001	0.001 ± 0.0001
rhs obj series 1	1.015 ± 0.006	-0.005 ± 0.006
mat series 1	1.050 ± 0.013	-0.009 ± 0.013
mat rhs bnd obj series 1	0.677 ± 0.021	$-\mathbf{0.061 \pm 0.021}$

Table: Averaged speed up obtained across public series. Results are averaged over 2,000 runs, with varying seed.

Conclusion

Criticism

- $\mathcal{A}$ build over public instances, no guarantee that any of the actions will be efficient on the hidden series
- Isn't it just a sophisticated but disguised approach to overfit the competition dataset ?

Strengths

- Sub-optimal actions also achieve significant speed up over public instance series
- We had only 50 instances to train our agent. State of the art methods obtain speed increases of only 4% while training over hundreds of instances.
- With larger dataset, more actions could be added to the action set, thus improving the power of generalization of our method.

Conclusion

Thanks for your attention !
Let’s hear from you :)