ACCQ206

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Lecture # 01 - Basic quantum formalism: qubits, quantum gates, measurements

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1. Show that $\left\{|+\rangle := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, |-\rangle := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right\}$ forms a orthonormal basis for $\mathbb{C}^2$ (i.e. show that the two vectors are orthogonal and that they have norm 1).
   Solution:
   $|| \, |+\rangle \, ||^2 = | \, \langle +| +\rangle \, | = (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$
   $|| \, |-\rangle \, ||^2 = | \, \langle -| -\rangle \, | = (\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$
   $\langle +| -\rangle = \frac{1}{2} - \frac{1}{2} = 0$

2. In this exercise you will show how to convert a state from the computational basis ($\{|0\rangle, |1\rangle\}$) to the Hadamard basis ($\{|+\rangle, |-\rangle\}$).
   (a) Write the vector $|0\rangle$ in the Hadamard basis (i.e. find the values α and β such that $|0\rangle = \alpha |+\rangle + \beta |-\rangle$). Do the same thing for $|1\rangle$.
   Solution:
   $|0\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$
   $|1\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle$

   (b) Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. Write $|\psi\rangle$ in the Hadamard basis.
   Solution:
   $$\begin{aligned} |\psi\rangle &= \alpha_0 |0\rangle + \alpha_1 |1\rangle \\ &= \alpha_0 \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right) + \alpha_1 \left(|1\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \right) \\ &= \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |+\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |-\rangle . \end{aligned}$$

3. Let $|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |11\rangle$, $|\phi\rangle = |10\rangle$, $|\rho\rangle = \frac{1}{\sqrt{2}} (|00\rangle - i |11\rangle)$ and $|\gamma\rangle = \frac{1}{\sqrt{2}} (i |00\rangle + |11\rangle)$. Compute
   (a) $\langle \psi|$, $\langle \phi|$, $\langle \rho|$
   Solution:
   $\langle \psi| = \frac{1}{\sqrt{2}} \langle 00| + \frac{1}{\sqrt{3}} \langle 01| + \frac{1}{\sqrt{6}} \langle 11| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}$
   $\langle \phi| = \langle 10| = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$
   $\langle \rho| = \frac{1}{\sqrt{2}} (\langle 00| + i \langle 11|) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{+i}{\sqrt{2}} \end{pmatrix}$
   (b) $\langle \psi| \phi\rangle$, $\langle \psi| \rho\rangle$, $\langle \gamma| \rho\rangle$, $\langle \rho| \gamma\rangle$
   Solution:
   $\langle \psi| \phi\rangle = 0$
   $\langle \psi| \rho\rangle = \frac{1}{2} - \frac{i}{\sqrt{12}}$
   $\langle \gamma| \rho\rangle = \frac{-i}{2} + \frac{-i}{2} = -i$
   $\langle \rho| \gamma\rangle = \frac{i}{2} + \frac{i}{2} = i$
   Remark: For every $|\psi\rangle, |\phi\rangle$, $\langle \psi| \phi\rangle = \overline{\langle \phi| \psi\rangle}$ (exercise!)

(c) $|\psi\rangle\langle\psi|$, $|\phi\rangle\langle\rho|$, $|\rho\rangle\langle\rho|$, $|\gamma\rangle\langle\gamma|$

Solution:

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{6}} & \frac{1}{3} & 0 & \frac{1}{\sqrt{18}} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & 0 & \frac{1}{6} \end{pmatrix}$$

$$|\phi\rangle\langle\rho| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|\rho\rangle\langle\rho| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-i}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$|\gamma\rangle\langle\gamma| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-i}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Remark: For every $|\psi\rangle$ and $|\phi\rangle = e^{i\theta}|\psi\rangle$, $|\psi\rangle\langle\psi| = |\phi\rangle\langle\phi|$ (exercise!)

4. Show that for every quantum state $|\psi\rangle$:

(a) $\langle\psi|\psi\rangle = 1$;

Solution:

Let $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_d \end{pmatrix}$. We have that $\langle\psi| = \begin{pmatrix} \overline{\alpha_1} & \ldots & \overline{\alpha_d} \end{pmatrix}$.

$\langle\psi|\psi\rangle = \sum_i \overline{\alpha_i}\alpha_i = \sum_i |\alpha_i|^2 = 1$, where the last equality holds since $|\psi\rangle$ is a quantum state.

(b) $|\psi\rangle\langle\psi|$ is a projector (P is a projector if $P^2 = P$).

Solution:

$|\psi\rangle\langle\psi|\,|\psi\rangle\langle\psi| = \langle\psi|\psi\rangle\,|\psi\rangle\langle\psi| = 1 \cdot |\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$

5. Find the matrix that represents the unitaries

$$Z|b\rangle = (-1)^b|b\rangle$$
$$P|b\rangle = i^b|b\rangle$$
$$c(X)|a\rangle|b\rangle = |a\rangle|a \oplus b\rangle$$

where $b \in \{0,1\}$ and $\oplus$ denotes the XOR operation.

Solution:

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$c(X) = CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

6. Show that if U_1 and U_2 are unitaries then U_1U_2, $U_1 \otimes U_2$ and $c(U_1)$ are unitaries.

Solution:

$U_1U_2(U_1U_2)^\dagger = U_1U_2U_2^\dagger U_1^\dagger = U_1IU_1^\dagger = I$

$(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger = (U_1 \otimes U_2)(U_1^\dagger \otimes U_2^\dagger) = (U_1U^\dagger \otimes U_2U_2^\dagger) = I$

$$\begin{aligned}
&(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U)^\dagger \\
&= (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U^\dagger) \\
&= |0\rangle\langle 0|\,|0\rangle\langle 0| \otimes I + |0\rangle\langle 0|\,|1\rangle\langle 1| \otimes U + |1\rangle\langle 1|\,|0\rangle\langle 0| \otimes U + |1\rangle\langle 1|\,|1\rangle\langle 1| \otimes UU^\dagger \\
&= |0\rangle\langle 0| \otimes I + 0 + 0 + |1\rangle\langle 1| \otimes I = I \otimes I
\end{aligned}$$

7. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Compute $H|\psi\rangle$ (i.e. find the values of α' and β' such that $H|\psi\rangle = \alpha'|0\rangle + \beta'|1\rangle$).
   Solution:
   $H|\psi\rangle = \alpha H|0\rangle + \beta|1\rangle = \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$

8. Find the outcomes and the corresponding probabilities and post-measurement states of the measurement of both qubits of the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the computational basis. Do the same for the measurement of both qubits in the Hadamard basis.
   Solution:
   Computational basis:

   00 w.p. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and post-measurement state is $\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}|00\rangle\right) = |00\rangle$

   01 and 10 w.p. 0

   11 w.p. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and post-measurement state is $\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}|11\rangle\right) = |11\rangle$

   Hadamard basis: The state written in the Hadamard basis is

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right)\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right)\right) \\
&= \frac{1}{2\sqrt{2}}|++\rangle + \frac{1}{2\sqrt{2}}|+-\rangle + \frac{1}{2\sqrt{2}}|-+\rangle + \frac{1}{2\sqrt{2}}|--\rangle \\
&\quad + \frac{1}{2\sqrt{2}}|++\rangle + \frac{1}{2\sqrt{2}}|+-\rangle - \frac{1}{2\sqrt{2}}|-+\rangle - \frac{1}{2\sqrt{2}}|--\rangle \\
&= \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)
\end{aligned}$$

   The measurement outcomes are:
   $++$ w.p. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and post-measurement state is $\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}|++\rangle\right) = |++\rangle$

   $+-$ and $+-$ w.p. 0

   $--$ w.p. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and post-measurement state is $\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}|--\rangle\right) = |--\rangle$

9. Let $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|11\rangle$. We measured the second qubit of $|\psi\rangle$ in the computational basis and the outcome was 1. Then, we measured the first qubit (of the state *after* the first measurement) in the Hadamard basis. Compute the outcomes and corresponding probabilities and post-measurement states of the second measurement.

Solution:

The first measurement has outcomes:

0 w.p. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and the post-measurement state is $|00\rangle$

1 w.p. $|\frac{1}{\sqrt{3}}|^2 + \frac{1}{\sqrt{6}}|^2 = \frac{1}{2}$ and the post-measurement state is $\sqrt{2} \cdot (\frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|11\rangle) = \frac{\sqrt{2}}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|11\rangle$

Since we assume that the outcome of the first measurement is 1, we have the state $\frac{\sqrt{2}}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|11\rangle$. If we write the first qubit in the Hadamard basis, we have

$$\begin{aligned}
\frac{\sqrt{2}}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|11\rangle &= \frac{1}{\sqrt{3}}(|+,1\rangle + |-,1\rangle) + \frac{1}{\sqrt{6}}(|+,1\rangle - |-,1\rangle) \\
&= \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|+,1\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|-,1\rangle
\end{aligned}$$

The outcome of measuring the first qubit in the Hadamard basis is:

$+$ with probability $|\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|^2 = \frac{1}{2} + \frac{1}{3\sqrt{2}}$ and the post-measurement state is $|+1\rangle$

$-$ with probability $|\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|^2 = \frac{1}{2} - \frac{1}{3\sqrt{2}}$ and the post-measurement state is $|-1\rangle$