ACCQ206

Lecturer: Alex B. Grilo
Lecture # 02 - Entanglement

Throughout this exercise list, we call $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Obsevables and global phases

1. Write the observable corresponding to the measurement on the computational basis if we assign $|0\rangle$ the outcome $+1$ and $|1\rangle$ the outcome -1. What is the expected value of the measurement outcome of $|+\rangle$ for this observable?
   Solution:
   We have the projector $|0\rangle\langle 0|$ associated with the eigenvalue $a_0 = +1$ and the projector $|1\rangle\langle 1|$ associated with the eigenvalue $a_1 = -1$.

   The observable is then $Z = +1 \cdot |0\rangle\langle 0| + (-1) \cdot |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   We have that the expected outcome measuring $|+\rangle$ according to Z is

   $$\langle +| Z |+\rangle = \langle +| -\rangle = 0.$$

2. Prove that for every qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$, there exists a qubit $|\phi\rangle = \alpha'|0\rangle + \beta'|1\rangle$ where $\alpha' \in \mathbb{R}$ and $\beta' \in \mathbb{C}$ such that $|\psi\rangle = \gamma|\phi\rangle$ for some $\gamma \in \mathbb{C}$ (i.e. $|\psi\rangle$ and $|\phi\rangle$ are indistinguishable)
   Solution:
   Let $\alpha = r_0 e^{i\phi_0}$ and $\beta = r_1 e^{i\phi_1}$ for $r_0, r_1 \in \mathcal{R}$ and $\phi_0, \phi_1 \in (-\pi, \pi]$ such that $r_0^2 + r_1^2 = 1$.

   We have that

   $$\begin{aligned} |\psi\rangle &= r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle \\ &= e^{i\phi_0}(r_0|0\rangle + r_1 e^{i-\phi_0} e^{i\phi_1}|1\rangle) \\ &= e^{i\phi_0}(r_0|0\rangle + r_1 e^{i(\phi_1-\phi_0)}|1\rangle). \end{aligned}$$

Bloch sphere

Let $\vec{n} = (n_x, n_y, n_z)$ be a vector with $n_x, n_y, n_z \in \mathbb{R}$ and $||\vec{n}|| = 1$, and $\vec{\sigma} = (X, Y, Z)$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. Show that $(\vec{n} \cdot \vec{\sigma})^2 = I$. [1]
   Solution:
   The I, X, Y and Z matrices are known as the Pauli matrices. We have that for $P, P' \in \{X, Y, Z\}$, $PP' = -P'P$ if $P \neq P'$ or $PP' = P'P = I$ if $P = P'$.

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[1] Hint: use the fact that $XY = -YX$, $XZ = -ZX$,...

Therefore we have that

$$
\begin{aligned}
&(\vec{n}\cdot\vec{\sigma})^2\\
&= n_x^2X^2 + n_xn_yXY + n_xn_zXZ + n_yn_xYX + n_y^2Y^2 + n_yn_zYZ + n_zn_xZX + n_zn_yZY + n_z^2Z^2\\
&= n_x^2I + n_xn_yXY + n_xn_zXZ + -n_yn_xXY + n_y^2I + n_yn_zYZ + -n_zn_xXZ - n_zn_yYZ + n_z^2I\\
&= (n_x^2 + n_y^2 + n_z^2)I + (n_xn_y - n_xn_y)XY + (n_xn_z - n_zn_x)XZ + (n_yn_z - n_yn_z)\\
&= I.
\end{aligned}
$$

A rotation of θ around the axis $\hat{n}$ can be written as:

$$R_{\vec{n}}(\theta) = e^{i\theta/2(\vec{n}\cdot\vec{\sigma})}.$$

4. Show that $R_{\vec{n}}(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})$.[2]
Solution:

$$
\begin{aligned}
&e^{i\theta/2(\vec{n}\cdot\vec{\sigma})}\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(i^k\left(\frac{\theta}{2}\right)^k(\vec{n}\cdot\vec{\sigma})^k\right)\\
&= \left(\sum_{k=2j}^{\infty}\frac{1}{2j!}\left(i^{2j}\left(\frac{\theta}{2}\right)^{2j}(\vec{n}\cdot\vec{\sigma})^{2j}\right)\right) + \left(\sum_{k=2j+1}^{\infty}\frac{1}{2j+1!}\left(i^{2j+1}\left(\frac{\theta}{2}\right)^{2j+1}(\vec{n}\cdot\vec{\sigma})^{2j+1}\right)\right)\\
&= \left(\sum_{j=0}^{\infty}\frac{1}{2j!}(-1)^j\left(\frac{\theta}{2}\right)^{2j}\right)I + i\left(\sum_{j=0}^{\infty}\frac{1}{2j+1!}(-1)^j\left(\frac{\theta}{2}\right)^{2j+1}\right)(\vec{n}\cdot\vec{\sigma})\\
&= \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma}),
\end{aligned}
$$

where in the third equality we use the result of exercise 3 and in the fourth equality we use the Maclaurin series of sin and cos [3]

5. Show that for every $\alpha\in\mathbb{R}$, $e^{i\alpha}R_{\vec{n}}(\theta)$ is a unitary matrix.
Solution:

$$
\begin{aligned}
&e^{i\alpha}R_{\vec{n}}(\theta)\left(e^{i\alpha}R_{\vec{n}}(\theta)\right)^\dagger && (1)\\
&= e^{i\alpha}\overline{e^{i\alpha}}R_{\vec{n}}(\theta)R_{\vec{n}}(\theta)^\dagger && (2)\\
&= \left(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})\right)\left(\cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})\right) && (3)\\
&= \cos^2\frac{\theta}{2}I + i\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma}) - i\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma}) + \sin^2\frac{\theta}{2}I && (4)\\
&= I, && (5)
\end{aligned}
$$

[2] Recall that for an $n\times n$ matrix A,

$$e^A = \sum_{k=0}^{\infty}\frac{1}{k!}A^k$$

.

[3] https://en.wikipedia.org/wiki/Taylor_series#Trigonometric_functions

where in the second equality we use Exercise 4 and in the last equality we use exercise 3.

6. Show that any unitary U on one qubit can be written as

$$U = e^{i\alpha} R_{\vec{n}}(\theta),$$

for some $\alpha \in \mathbb{R}$.

Solution:

Any matrix $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be written as $U = \beta_0 I + \beta_1 X + \beta_2 Y + \beta_3 Z$ with

$$\beta_0 = \frac{a+d}{2}, \quad \beta_1 = \frac{b+c}{2}, \quad \beta_2 = \frac{c-b}{2i}, \quad \beta_3 = \frac{a-d}{2},$$

We have that

$$\begin{aligned}
&U^\dagger U \\
&\quad = (\beta_0 I + \beta_1 X + \beta_2 Y + \beta_3 Z)(\beta_0 I + \beta_1 X + \beta_2 Y + \beta_3 Z)^\dagger \\
&\quad = (|\beta_0|^2 + |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2)I + (\overline{\beta_0}\beta_1 + \overline{\beta_1}\beta_0 + i\overline{\beta_2}\beta_3 - i\overline{\beta_3}\beta_2)X \\
&\qquad + (\overline{\beta_0}\beta_2 + i\overline{\beta_1}\beta_3 + \overline{\beta_2}\beta_0 - i\overline{\beta_3}\beta_1)Y + (\overline{\beta_0}\beta_3 + i\overline{\beta_1}\beta_2 + -i\overline{\beta_2}\beta_1 + \overline{\beta_3}\beta_0)Y.
\end{aligned}$$

Since U is unitary (and therefore $U^\dagger U = I$, we have that

$$|\beta_0|^2 + |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 = 1 \tag{6}$$

$$\overline{\beta_0}\beta_1 + \overline{\beta_1}\beta_0 + i\overline{\beta_2}\beta_3 - i\overline{\beta_3}\beta_2 = 0 \tag{7}$$

$$\overline{\beta_0}\beta_2 + i\overline{\beta_1}\beta_3 + \overline{\beta_2}\beta_0 - i\overline{\beta_3}\beta_1 = 0 \tag{8}$$

$$\overline{\beta_0}\beta_3 + i\overline{\beta_1}\beta_2 + -i\overline{\beta_2}\beta_1 + \overline{\beta_3}\beta_0 = 0 \tag{9}$$

Define

- θ such that $|\beta_0| = \cos\frac{\theta}{2}$
- $n_x = \frac{|\beta_1|}{|\sin\frac{\theta}{2}|}$
- $n_y = \frac{|\beta_2|}{|\sin\frac{\theta}{2}|}$
- $n_z = \frac{|\beta_3|}{|\sin\frac{\theta}{2}|}$
- α such that $e^{i\alpha} = \frac{\beta_0}{\cos\theta 2}$

Notice that using Equation 6, we have that $n_x^2 + n_y^2 + n_z^2 = 1$.

Let us denote α_1, α_2 and α_3 the phases of β_1, β_2 and β_3 (i.e. $\alpha_i = \frac{\beta_i}{|\beta_i|}$).

We have that

$$\begin{aligned}
0 &= \overline{\beta_0}\beta_1 + \overline{\beta_1}\beta_0 + i\overline{\beta_2}\beta_3 - i\overline{\beta_3}\beta_2 \\
&= e^{i(\alpha_1-\alpha)}\cos\frac{\theta}{2}\sin\frac{\theta}{2}n_x + e^{i(\alpha-\alpha_1)}\cos\frac{\theta}{2}\sin\frac{\theta}{2}n_x + ie^{i(\alpha_3-\alpha_2)}\sin^2\frac{\theta}{2}n_y n_z - ie^{i(\alpha_2-\alpha_3)}\sin^2\frac{\theta}{2}n_y n_z \\
&= 2\cos(\alpha-\alpha_1)\cos\frac{\theta}{2}\sin\frac{\theta}{2}n_x + 2i\sin(\alpha_3-\alpha_2)\sin^2\frac{\theta}{2}n_y n_z.
\end{aligned}$$

From this, we have that $\cos(\alpha - \alpha_1) = \sin(\alpha_3 - \alpha_2) = 0$, and therefore $\alpha_1 = \alpha - \pi/2$ and $\alpha_2 = \alpha_1$. Using Equations 8 and 9, we can similarly achieve $\alpha_1 = \alpha_2 = \alpha_3 = \alpha - \pi/2$.

Therefore, we have that

$$\beta_0 = e^{i\alpha} \cos \frac{\theta}{2}$$
$$\beta_1 = -e^{i\alpha} \sin \frac{\theta}{2} n_x$$
$$\beta_2 = -e^{i\alpha} \sin \frac{\theta}{2} n_y$$
$$\beta_3 = -e^{i\alpha} \sin \frac{\theta}{2} n_z$$

This implies that we can write

$$U = e^{i\alpha}(\cos \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2}(n_x X + x_y Y + n_z Z)) = e^{i\alpha} R_{\vec{n}}(\theta).$$

(**Pour aller plus loin...:**) Find the values of $\alpha, \vec{n}$ and θ for X, Y, Z and H matrices.

Entanglement

7. Which of the following states are entangled (according to indicated partition):

   (a) $\frac{1}{\sqrt{3}}|0\rangle_A|0\rangle_B + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle_A|1\rangle_B$
   Solution:
   Suppose $\exists|\psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\psi_2\rangle = c|0\rangle + d|1\rangle$ such that

   $$\frac{1}{\sqrt{3}}|0\rangle_A|0\rangle_B + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle_A|1\rangle_B = |\psi_1\rangle|\psi_2\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle.$$

   $ac = \frac{1}{\sqrt{3}}$ and $bd = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow a, b, c, d \neq 0$
   $ad = bc = 0 \Rightarrow a$ or d is 0; b or c is 0 **Contradiction!**
   $\frac{1}{\sqrt{3}}|0\rangle_A|0\rangle_B + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle_A|1\rangle_B$ is entangled.

   (b) $\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|00\rangle_B$
   Solution:
   $\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|00\rangle_B = |+\rangle \otimes |00\rangle$.
   $\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|00\rangle_B$ is not entangled

   (c) $\frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B$
   Solution:
   Suppose $\exists|\psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\psi_2\rangle = c|0\rangle + d|1\rangle$ such that

   $$\frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B = |\psi_1\rangle|\psi_2\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle.$$

   $ac = 0$ and $bd = 0 \Rightarrow a$ or c is 0; b or d is 0
   $ad = bc = \frac{1}{\sqrt{2}} \Rightarrow a, b, c, d \neq 0$ **Contradiction!**
   $\frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B$ is entangled.

8. Show that for every one-qubit basis $\{|b_1\rangle, |b_2\rangle\}$ such that $|0\rangle = \alpha_0|b_0\rangle + \alpha_1|b_1\rangle$ and $|1\rangle = \alpha_1|b_0\rangle - \alpha_0|b_1\rangle$, we have that $|EPR\rangle = \frac{1}{\sqrt{2}}(|b_1\rangle|b_1\rangle + |b_2\rangle|b_2\rangle)$.
Solution:
It follows that

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \\
&= \frac{1}{\sqrt{2}}\left((\alpha_0|b_1\rangle + \alpha_1|b_2\rangle)(\alpha_0|b_1\rangle + \alpha_1|b_2\rangle) + (\alpha_1|b_1\rangle - \alpha_0|b_2\rangle)(\alpha_1|b_1\rangle - \alpha_0|b_2\rangle)\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha_0^2 + \alpha_1^2\right)|b_1\rangle|b_1\rangle \frac{1}{\sqrt{2}}\left(\alpha_0\alpha_1 - \alpha_0\alpha_1\right)|b_1\rangle|b_2\rangle \frac{1}{\sqrt{2}}\left(\alpha_1\alpha_0 - \alpha_1\alpha_0\right)|b_2\rangle|b_1\rangle \frac{1}{\sqrt{2}}\left(\alpha_0^2 + \alpha_1^2\right)|b_2\rangle|b_2\rangle \\
&= \frac{1}{\sqrt{2}}|b_1\rangle|b_1\rangle + \frac{1}{\sqrt{2}}|b_2\rangle|b_2\rangle.
\end{aligned}$$

We can actually prove an stronger statement that

$$U \otimes I(|EPR\rangle) = I \otimes U^T(|EPR\rangle),$$

which implies that $U \otimes U|EPR\rangle = |EPR\rangle$ for real unitaries.

Pour aller (beaucoup) plus loin... [4]

We will now prove the Tsirelson's bound for CHSH (i.e., we will show that the quantum strategy that we saw is optimal).

For that, we will consider a more convenient notation for the CHSH game: the answers from P_1 and P_2 are $a, b \in \{\pm 1\}$, and they win the game if $(-1)^{xy} = ab$.[5]

We can consider a generic quantum strategy for P_1 and P_2 in the CHSH game as follows:

(A) P_1 and P_2 share an arbitrary quantum state $|\psi\rangle_{P_1,P_2}$ (notice that we make no assumption on the size of such a quantum state)

(B) For each question x, P_1 chooses an observable M^b that she will use to measure her share of $|\psi\rangle$ and will answer with the outcome of the this measurement. In other words, for each x, P_1 chooses two values M^x_{+1} and M^x_{-1} such that $M^x_{+1} + M^x_{-1} = I$ and $M^x = M^x_{+1} - M^x_{-1}$.

(C) Likewise, for each y, P_2 chooses an observable N^y that she will use to measure her share of $|\psi\rangle$ and will answer with the outcome of the this measurement.

Show that:

9. Show that for a fixed x and y, the expected value of ab is $\langle\psi|\, M^x \otimes N^y |\psi\rangle$.

10. Let $C = (M^0 \otimes N^0 + M^0 \otimes N^1 + M^1 \otimes N^0 - M^1 \otimes N^1)$. Show that the winning probability of P_1 and P_2 in the game is $\frac{1}{2} + \frac{1}{8}\langle\psi|\, C|\psi\rangle$. (Hint: argue that $\langle\psi|\, C|\psi\rangle = Pr[\text{win}] - Pr[\text{lose}]$)

11. Show that
$$C^2 = 4I + (M^0M^1 - M^1M^0) \otimes (N^0N^1 - N^1N^0)$$
(Hint: Show (and use) the fact that $(M^x)^2 = (N^y)^2 = I$ for all $x, y \in \{0, 1\}$)

[4] This exercise was based on Exercise 6 of chapter 16 of [1]

[5] Notice that we are just considering the map $b \leftrightarrow (-1)^b$ for their answers.

12. Show that $\langle\psi|\, C|\psi\rangle \leq 2\sqrt{2}$. (Hint: Use Cauchy-Schwartz inequality: $(\langle\psi|\, C|\psi\rangle)^2 \leq \langle\psi|\, C^2|\psi\rangle$)

13. What can you say about maximum the quantum value of CHSH?

References

[1] Ronald de Wolf. Quantum computing: Lecture notes. `http://arxiv.org/abs/1907.09415`, 2019.