ACCQ206

Lecturer: Alex B. Grilo
Lecture # 02 - Entanglement

Throughout this exercise list, we call $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Obsevables and global phases

1. Write the observable corresponding to the measurement on the computational basis if we assign $|0\rangle$ the outcome $+1$ and $|1\rangle$ the outcome -1. What is the expected value of the measurement outcome of $|+\rangle$ for this observable?

2. Prove that for every qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$, there exists a qubit $|\phi\rangle = \alpha'|0\rangle + \beta'|0\rangle$ where $\alpha' \in \mathbb{R}$ and $\beta' \in \mathbb{C}$ such that $|\psi\rangle = \gamma|\phi\rangle$ for some $\gamma \in \mathbb{C}$ (i.e. $|\psi\rangle$ and $|\phi\rangle$ are indistinguishable)

Bloch sphere

Let $\vec{n} = (n_x, n_y, n_z)$ be a vector with $n_x, n_y, n_z \in \mathbb{R}$ and $||\vec{n}|| = 1$, and $\vec{\sigma} = (X, Y, Z)$ and $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$.

3. Show that $\vec{n} \cdot \vec{\sigma} = I$. [1]

A rotation of θ around the axis $\hat{n}$ can be written as:

$$R_{\vec{n}}(\theta) = e^{i\theta(\vec{n}\cdot\vec{\sigma})}.$$

4. Show that $R_{\vec{n}}(\theta) = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})$.[2]

5. Show that for every $\alpha \in \mathbb{R}$, $e^{i\alpha}R_{\vec{n}}(\theta)$ is a unitary matrix.

6. Show that any unitary U on one qubit can be written as

$$U = e^{i\alpha}R_{\vec{n}}(\theta),$$

for some $\alpha \in \mathbb{R}$.

(Pour aller plus loin...:) Find the values of $\alpha, \vec{n}$ and θ for X, Y, Z and H matrices.

Entanglement

7. Which of the following states are entangled (according to indicated partition):

   (a) $\frac{1}{\sqrt{3}}|0\rangle_A|0\rangle_B + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle_A|1\rangle_B$

   (b) $\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|00\rangle_B$

---

[1] Hint: show that $(\vec{n}\cdot\vec{\sigma})^2 = I$ and use the fact that $XY = -YX$, $XZ = -ZX$,...

[2] Recall that for an $n \times n$ matrix A,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

.

(c) $\frac{1}{\sqrt{2}} |0\rangle_A |1\rangle_B + \frac{1}{\sqrt{2}} |1\rangle_A |0\rangle_B$

8. Show that for every one-qubit basis $\{|b_1\rangle, |b_2\rangle\}$, we have that $|EPR\rangle = \frac{1}{\sqrt{2}}(|b_1\rangle |b_1\rangle + |b_2\rangle |b_2\rangle)$.

9. One particular useful application of quantum teleportation is *entanglement swapping*. Suppose that Alice and Bob share an EPR pair, and Bob and Charlie share a second EPR pair (see Figure 1). Show that via quantum teleportation, Bob can help Alice and Charlie share an EPR pair with classical communication only. Does Bob share an EPR pair with any other party after the execution of this protocol?

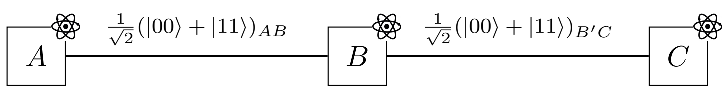

Figure 1: Setup for entanglement swapping

Density matrices

10. Compute the density matrices of:

    (a) $(1, |\psi\rangle)$ and $(1, e^{i\theta} |\psi\rangle))$.

    (b) $((\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle))$ and $((\frac{1}{2}, |+\rangle), (\frac{1}{2}, |+\rangle))$.

    What can you conclude from the previous results?

11. Let $|W_3\rangle_{AB} = \frac{1}{\sqrt{3}}(|1\rangle_A |00\rangle_B + |0\rangle_A |10\rangle_B + |0\rangle_A |01\rangle_B)$. Compute $Tr_A(|W_3\rangle\langle W_3|_{AB})$.

Pour aller (beaucoup) plus loin... [3]

We will now prove the Tsirelson's bound for CHSH (i.e., we will show that the quantum strategy that we saw is optimal).

For that, we will consider a more convenient notation for the CHSH game: the answers from P_1 and P_2 are $a, b \in \{\pm 1\}$, and they win the game if $(-1)^{xy} = ab$.[4]

We can consider a generic quantum strategy for P_1 and P_2 in the CHSH game as follows:

(A) P_1 and P_2 share an arbitrary quantum state $|\psi\rangle_{P_1,P_2}$ (notice that we make no assumption on the size of such a quantum state)

(B) For each question x, P_1 chooses an observable M^b that she will use to measure her share of $|\psi\rangle$ and will answer with the outcome of the this measurement. In other words, for each x, P_1 chooses two values M^x_{+1} and M^x_{-1} such that $M^x_{+1} + M^x_{-1} = I$ and $M^x = M^x_{+1} - M^x_{-1}$.

(C) Likewise, for each y, P_2 chooses an observable N^y that she will use to measure her share of $|\psi\rangle$ and will answer with the outcome of the this measurement.

[3] This exercise was based on Exercise 6 of chapter 16 of [1]

[4] Notice that we are just considering the map $b \leftrightarrow (-1)^b$ for their answers.

Show that:

12. Show that for a fixed x and y, the expected value of ab is $\langle\psi| M^x \otimes N^y |\psi\rangle$.

13. Let $C = (M^0 \otimes N^0 + M^0 \otimes N^1 + M^1 \otimes N^0 - M^1 \otimes N^1)$. Show that the winning probability of P_1 and P_2 in the game is $\frac{1}{2} + \frac{1}{8} \langle\psi| C |\psi\rangle$. (Hint: argue that $\langle\psi| C |\psi\rangle = Pr[\text{win}] - Pr[\text{lose}]$)

14. Show that
$$C^2 = 4I + (M^0M^1 - M^1M^0) \otimes (N^0N^1 - N^1N^0)$$
(Hint: Show (and use) the fact that $(M^x)^2 = (N^y)^2 = I$ for all $x, y \in \{0, 1\}$)

15. Show that $\langle\psi| C |\psi\rangle \leq 2\sqrt{2}$. (Hint: Use Cauchy-Schwartz inequality: $(\langle\psi| C |\psi\rangle)^2 \leq \langle\psi| C^2 |\psi\rangle$)

16. What can you say about maximum the quantum value of CHSH?

References

[1] Ronald de Wolf. Quantum computing: Lecture notes. `http://arxiv.org/abs/1907.09415`, 2019.