ACCQ206

Lecturer: Alex B. Grilo
Lecture # 01 - Basic quantum formalism: qubits, quantum gates, measurements

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1. Show that $\left\{ |+\rangle := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, |-\rangle := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\}$ forms a orthonormal basis for $\mathbb{C}^2$ (i.e. show that the two vectors are orthogonal and that they have norm 1).

2. In this exercise you will show how to convert a state from the computational basis ($\{|0\rangle, |1\rangle\}$) to the Hadamard basis ($\{|+\rangle, |-\rangle\}$).

   (a) Write the vector $|0\rangle$ in the Hadamard basis (i.e. find the values α and β such that $|0\rangle = \alpha |+\rangle + \beta |-\rangle$). Do the same thing for $|1\rangle$.

   (b) Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. Write $|\psi\rangle$ in the Hadamard basis.

3. Let $|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |11\rangle$, $|\phi\rangle = |10\rangle$, $|\rho\rangle = \frac{1}{\sqrt{2}} (|00\rangle - i |11\rangle)$ and $|\gamma\rangle = \frac{1}{\sqrt{2}} (i |00\rangle + |11\rangle)$. Compute

   (a) $\langle\psi|$, $\langle\phi|$, $\langle\rho|$

   (b) $\langle\psi|\phi\rangle$, $\langle\psi|\rho\rangle$, $\langle\gamma|\rho\rangle$, $\langle\rho|\gamma\rangle$

   (c) $|\psi\rangle\langle\psi|$, $|\phi\rangle\langle\rho|$, $|\rho\rangle\langle\rho|$, $|\gamma\rangle\langle\gamma|$

4. Show that for every quantum state $|\psi\rangle$:

   (a) $\langle\psi|\psi\rangle = 1$;

   (b) $|\psi\rangle\langle\psi|$ is a projector (P is a projector if $P^2 = P$).

5. Find the matrix that represents the unitaries

$$Z |b\rangle = (-1)^b |b\rangle$$
$$P |b\rangle = i^b |b\rangle$$
$$c(X) |a\rangle |b\rangle = |a\rangle |a \oplus b\rangle$$

   where $b \in \{0, 1\}$ and $\oplus$ denotes the XOR operation.

6. Show that if U_1 and U_2 are unitaries then U_1U_2, $U_1 \otimes U_2$ and $c(U_1)$ are unitaries.

7. Let $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$. Compute $H |\psi\rangle$ (i.e. find the values of α' and β' such that $H |\psi\rangle = \alpha' |0\rangle + \beta' |1\rangle$).

8. Find the outcomes and the corresponding probabilities and post-measurement states of the measurement of both qubits of the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the computational basis. Do the same for the measurement of both qubits in the Hadamard basis.

9. Let $|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |11\rangle$. We measured the second qubit of $|\psi\rangle$ in the computational basis and the outcome was 1. Then, we measured the first qubit (of the measurement state *after* the first measurement) in the Hadamard basis. Compute the outcomes and corresponding probabilities and post-measurement states of the second measurement.