ACCQ 206 - Basic quantum formalism: qubits, quantum gates, measurements

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Postulates of quantum mechanics

1. Quantum states: how to represent physical objects (at the atomic level)?
2. Evolution: how do quantum states change in time?
3. Measurements: how can we “read” the properties of quantum states?

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2. Evolution: how do quantum states change in time?
3. Measurements: how can we “read” the properties of quantum states?

Computational route: definition of these concepts tailored to their use in quantum computing

Quantum states

Postulate

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The choice of basis $\{|0\rangle\,, |1\rangle\}$ is arbitrary. We could choose the basis $\left\{|+\rangle := \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, |-\rangle := \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$, and write

$$|\psi\rangle = \alpha'\,|+\rangle = \beta'\,|-\rangle\,,\ \ \alpha', \beta' \in \mathbb{C} \text{ and } |\alpha'|^2 + |\beta'|^2 = 1$$

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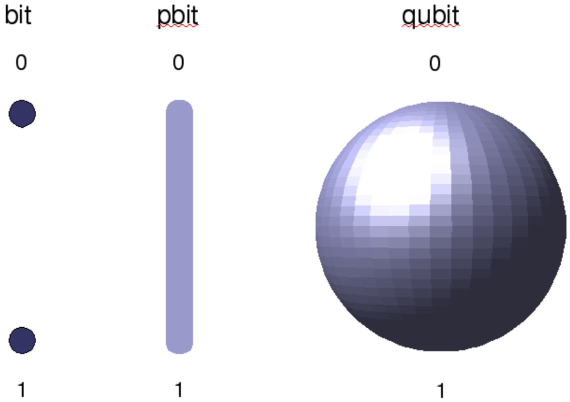
Example

- Energy levels of a trapped ion
- Photon polarization
- ...

Bit vs. pbit vs qubit



- A (deterministic) bit has values in $\{0, 1\}$
- A probabilistic bit has values in $[0, 1]$
- A quantum bit has values $\{\alpha\,|0\rangle + \beta\,|1\rangle : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}$

Examples

$|0\rangle$

$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$

Examples

$$|0\rangle = 1\,|0\rangle + 0\,|1\rangle\,;\, |1|^2 + |0|^2 = 1$$

$$|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \;\; = |1/\sqrt{2}|^2 + |1/\sqrt{2}|^2 = 2 \cdot \frac{1}{2} = 1$$

Examples

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$$|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \;\; = |1/\sqrt{2}|^2 + |1/\sqrt{2}|^2 = 2 \cdot \frac{1}{2} = 1$$

Exercise

Which of these vectors correspond to valid quantum states?

$a) \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$

$b) \frac{1}{\sqrt{2}}\,|0\rangle + i\,\frac{1}{\sqrt{2}}\,|+\rangle$

$c) \frac{1}{\sqrt{3}}\,|0\rangle + \left(\frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)|+\rangle$

$d) cos\theta\,|0\rangle + e^{i\phi} sin\theta\,|1\rangle$

Math background: tensor product

Definition

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n'} \\ \dots \\ b_{m',1} & b_{m',2} & \dots & b_{m',n'} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ \dots \\ a_{m,1}B & a_{m,2}B & \dots & a_{m,n}B \end{pmatrix}$$
$$= \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & \dots & a_{1,1}b_{1,n'} & a_{1,2}b_{1,1} & \dots & a_{1,2}b_{1,n'} & \dots & a_{1,n}b_{1,n'} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & \dots & a_{1,1}b_{2,n'} & a_{1,2}b_{2,1} & \dots & a_{1,2}b_{2,n'} & \dots & a_{1,n}b_{2,n'} \\ \dots \\ a_{m,1}b_{m',1} & a_{m,1}b_{m',2} & \dots & a_{m,1}b_{m',n'} & a_{m,2}b_{m',1} & \dots & a_{m,2}b_{m',n'} & \dots & a_{1,n}b_{m',n'} \end{pmatrix}$$

Tensor product

Examples

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{\otimes 2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Quantum states - multiple qubits

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- Register with n qubits: n qubits "side-by-side"
  - Hilbert space: $\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2 \cong \mathbb{C}^{2^n}$
  - Computational basis: $|i\rangle, i \in \{0,1\}^n$
  - Quantum state: $|\phi\rangle = \sum_{i\in\{0,1\}^n} \alpha_i |i\rangle$, $\alpha_i \in \mathbb{C}$ and $\sum |\alpha_i|^2 = 1$

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Example

$$|000\ldots000\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$$

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{3}}|100\rangle + \frac{1}{\sqrt{6}}|111\rangle$$

Math background: transpose, conjugate, inverse

For a complex number $w = a + bi$, its conjugate is $\overline{w} = a - bi$.

For a matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$, we have

- Conjugate: $A^* = \begin{pmatrix} \overline{a_{1,1}} & \overline{a_{1,2}} & \dots & \overline{a_{1,n}} \\ \overline{a_{2,1}} & \overline{a_{2,2}} & \dots & \overline{a_{2,n}} \\ \dots \\ \overline{a_{m,1}} & \overline{a_{m,2}} & \dots & \overline{a_{m,n}} \end{pmatrix}$
- Transpose: $A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{m,1} \\ a_{1,2} & a_{2,2} & \dots & a_{m,2} \\ \dots \\ a_{1,n} & a_{2,n} & \dots & a_{m,n} \end{pmatrix}$
- Conjugate transpose: $A^\dagger = \begin{pmatrix} \overline{a_{1,1}} & \overline{a_{2,1}} & \dots & \overline{a_{m,1}} \\ \overline{a_{1,2}} & \overline{a_{2,2}} & \dots & \overline{a_{m,2}} \\ \dots \\ \overline{a_{1,n}} & \overline{a_{2,n}} & \dots & \overline{a_{m,n}} \end{pmatrix}$
- Inverse (if exists): A^{-1} s.t. $AA^{-1} = A^{-1}A = I$

Dirac notation

ket: column vector

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix}$$

$$|\phi\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

bra: row vector

$$\langle\psi| = (|\psi\rangle)^\dagger = \begin{pmatrix} \overline{a_1} & \overline{a_2} & \dots & \overline{a_m} \end{pmatrix}$$

$$\langle\phi| = \begin{pmatrix} \overline{b_1} & \overline{b_2} & \dots & \overline{b_n} \end{pmatrix}$$

bra.ket: inner product

complex number

$$\langle\phi|\,\psi\rangle = \begin{pmatrix} \overline{b_1} & \overline{b_2} & \dots & \overline{b_n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix}$$

$$= \sum_{i\in[m]} \overline{b_i} a_i$$

bra.ket: outer product

complex matrix

$$|\phi\rangle\langle\psi| = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} \begin{pmatrix} \overline{a_1} & \overline{a_2} & \dots & \overline{a_m} \end{pmatrix}$$

$$= \begin{pmatrix} b_1\overline{a_1} & b_1\overline{a_2} & \dots & b_1\overline{a_m} \\ b_2\overline{a_1} & b_2\overline{a_2} & \dots & b_2\overline{a_m} \\ & & \dots & \\ b_m\overline{a_1} & b_m\overline{a_2} & \dots & b_m\overline{a_m} \end{pmatrix}$$

Quantum operations

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    In particular for every valid quantum state $|\psi\rangle$, $U|\psi\rangle$ is also a valid quantum state
  - Reversible: no information loss

Quantum unitaries - examples

Example (X)

Definition: $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
Unitary:

Operational:

Quantum unitaries - examples

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Definition: $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

Unitary:

$X^\dagger = X^T = X$

$X^2 = I$

Operational:

$X\left|0\right\rangle = \left|1\right\rangle$

$X\left|1\right\rangle = \left|0\right\rangle$

$X\left|+\right\rangle = X\left(\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(X\left|0\right\rangle + X\left|1\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|1\right\rangle + \left|0\right\rangle\right) = \left|+\right\rangle$

Quantum gates - examples

Example (Hadamard)

Definition: $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

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Unitary:

$H^\dagger = H^T = H$

$H^2 = I$

Operational:

$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$

$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$

$H|+\rangle = H(H|0\rangle) = I|0\rangle = |0\rangle$

$H|0\rangle = |1\rangle$

Quantum gates - examples

Example (Other gates)

$$Z\left|b\right\rangle = (-1)^b \left|b\right\rangle$$
$$P\left|b\right\rangle = i^b \left|b\right\rangle$$
$$T\left|b\right\rangle = e^{b(i\pi/4)} \left|b\right\rangle$$

Quantum circuits

Unitary composition

For unitaries U_1 and U_2:

- $U_1 U_2$ is unitary
- $U_1 \otimes U_2$ is unitary

$$(U_1 \otimes U_2)(|\psi\rangle \otimes |\phi\rangle) = U_1 |\psi\rangle \otimes U_2 |\phi\rangle$$

- Controlled unitary $c(U_1)$ is a unitary

$$c(U_1)\,|0\rangle\,|\psi\rangle \to |0\rangle\,|\psi\rangle \text{ and } c(U_1)\,|1\rangle\,|\psi\rangle \to |1\rangle\,U_1\,|\psi\rangle$$
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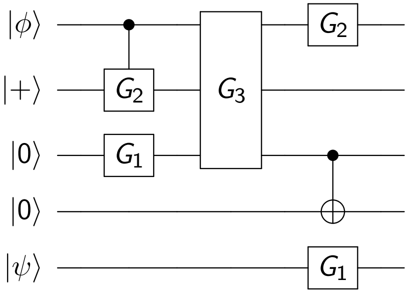
$$c(U_1) = \begin{pmatrix} I & 0 \\ 0 & U_1 \end{pmatrix}$$

- Quantum circuits: composition of smaller building blocks (gates)

Quantum circuits

Fix a *gateset* $\mathcal{G} = \{G_1, G_2, \ldots\}$



$$(G_2 \otimes I \otimes c(X) \otimes G_1)(G_3 \otimes I_4)(c(G_2) \otimes G_1 \otimes I_4) \, |\phi\rangle \, |+\rangle \, |00\rangle \, |\psi\rangle$$

Measurement in the computational basis

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Example

- Measure $|0\rangle$
  - $|0\rangle$ with probability 1, post measurement state is $|0\rangle$
  - $|1\rangle$ with probability 0

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- Measure $|0\rangle$
  - $|0\rangle$ with probability 1, post measurement state is $|0\rangle$
  - $|1\rangle$ with probability 0
- Measure $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
  - $|0\rangle$ with probability $\frac{1}{2}$, post measurement state is $|0\rangle$
  - $|1\rangle$ with probability $\frac{1}{2}$, post measurement state is $|1\rangle$

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- Measure $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
  - $|0\rangle$ with probability $\frac{1}{2}$, post measurement state is $|0\rangle$
  - $|1\rangle$ with probability $\frac{1}{2}$, post measurement state is $|1\rangle$
- Measure $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$
  - $|0\rangle$ with probability $\frac{1}{2}$, post measurement state is $|0\rangle$
  - $|1\rangle$ with probability $\frac{1}{2}$, post measurement state is $|1\rangle$

More on measurements

- Measuring $|\psi\rangle = \sum_{i\in\{0,1\}^n} \alpha_i |i\rangle$ in the basis $\{|\phi_1\rangle, ..., |\phi_{2^n}\rangle\}$

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  - Output i with probability $|\beta_i|^2$
  - Quantum state collapses to $|\phi_i\rangle$
- Measure the first qubit of $|\psi\rangle = \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle$ in the computational basis
  - Output is b with probability $p_b = \sum_{i=bj, j=\{0,1\}^{n-1}} |\alpha_i|^2$
  - Quantum state collapses to $\frac{1}{\sqrt{p_b}} \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle$

Example

- Measure first qubit of $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ in the computational basis

More on measurements

- Measuring $|\psi\rangle = \sum_{i\in\{0,1\}^n} \alpha_i |i\rangle$ in the basis $\{|\phi_1\rangle, \ldots, |\phi_{2^n}\rangle\}$
  - Write $|\psi\rangle = \sum_{i\in\{0,1\}^n} \beta_i |\phi_i\rangle$
  - Output i with probability $|\beta_i|^2$
  - Quantum state collapses to $|\phi_i\rangle$
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  - Output is b with probability $p_b = \sum_{i=bj, j=\{0,1\}^{n-1}} |\alpha_i|^2$
  - Quantum state collapses to $\frac{1}{\sqrt{p_b}} \sum_{i\in\{0,1\}^n} \alpha_i |i\rangle$

Example

- Measure first qubit of $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ in the computational basis
  - Output is 0 with probability $\frac{2}{3}$, post measurement state is $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$
  - Output is 1 with probability $\frac{1}{3}$, post measurement state is $|10\rangle$

General measurements

- Hermitian operator $A = A^\dagger$
  - For some vectors $|\phi_1\rangle, \ldots, |\phi_{2^n}\rangle$ and real values $a_1, \ldots, a_{2^n}$, we have $A = \sum_{i \in [2^n]} a_i |\phi_i\rangle\langle\phi_i|$
  - Output v with probability $p_v := \sum_{i : a_i = v} |\langle\psi|\phi_i\rangle|^2$
  - State collapse to

$$\frac{1}{\sqrt{p_v}} \left(\sum_{i : a_i = v} |\phi_i\rangle\langle\phi_i| \right) |\psi\rangle = \frac{1}{\sqrt{p_v}} \sum_{i : a_i = v} \langle\phi_i|\psi\rangle \phi_i$$