ACCQ 206 - Entanglement

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Recap

Quantum states

n-qubit state $|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \cdots \\ \alpha_{2^n} \end{pmatrix} \in \mathbb{C}^{2^n}$

$\sum_i |\alpha_i|^2 = 1$

- position : $\{ |x\rangle \}$
- momentum : $\{ |v\rangle \}$

$\underbrace{|0\rangle / |1\rangle}_{\text{position}}$

Evolution

Unitary matrices

$U : UU^\dagger = U(U^*)^T = I$

$|\psi\rangle \to |\phi\rangle$

$\underbrace{|+\rangle / |-\rangle}_{\text{momentum}}$

Measurement in comp. basis

output i w.p. $|\alpha_i|^2$

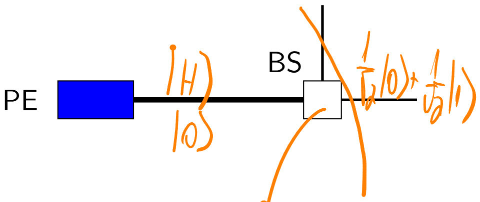
state collapses to $|i\rangle$

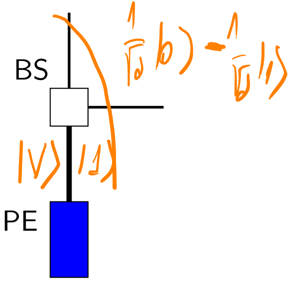
$|\psi\rangle \to$ $\to |i\rangle$

$|+\rangle \to |0\rangle$ w.p. $1/2$

$|+\rangle \to |1\rangle$ w.p. $1/2$

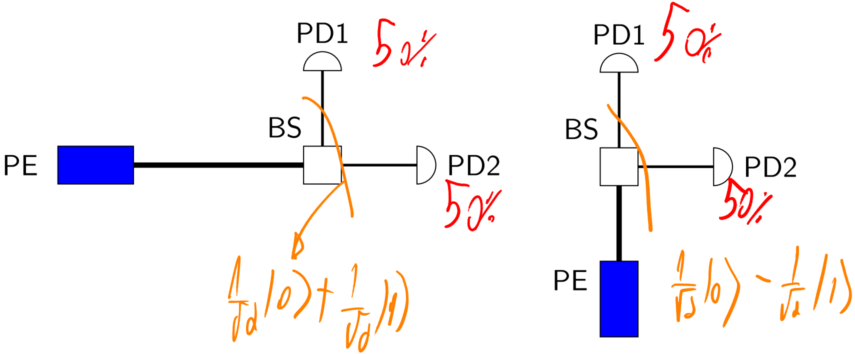
Beam splitter



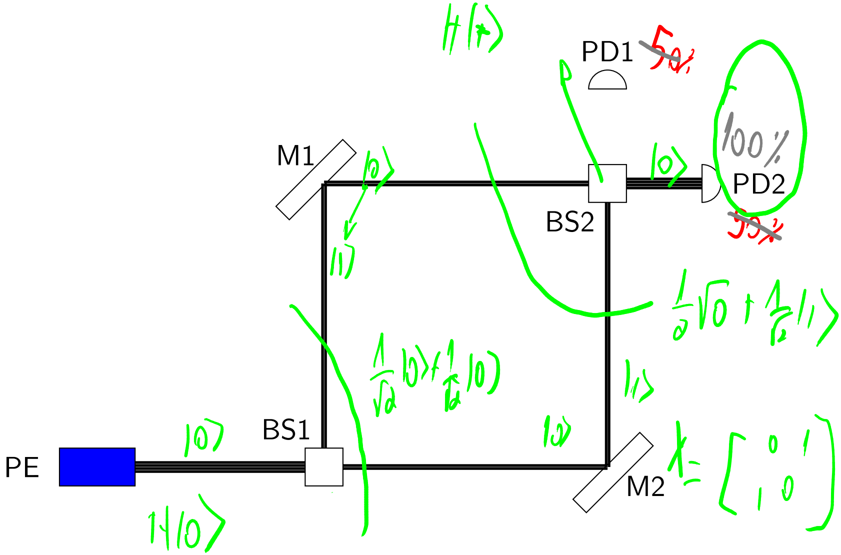


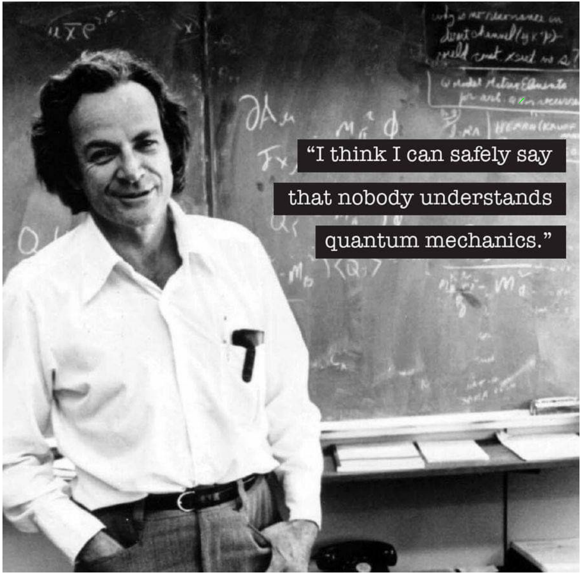
$$H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Beam splitter



Interferometer

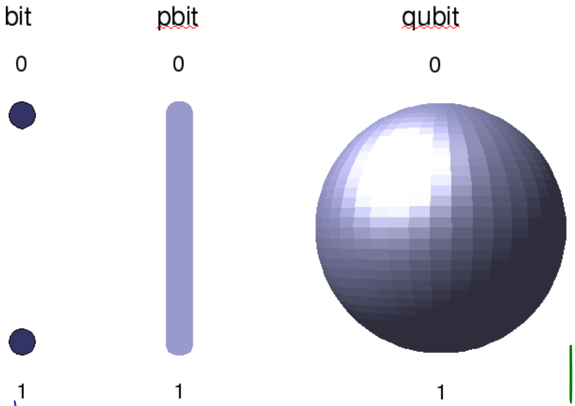




Quantum mechanics is very worthy of regard. But an inner voice tells me that this is not yet the right track. The theory yields much, but it hardly brings us closer to the Old One's secrets. I, in any case, am convinced that He does not play dice.

Albert Einstein

Bit vs. pbit vs qubit



$\{0,1\}$ $[0,1]$

$\alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$

$\alpha = a + ib$

$\beta = c + id$

$|\alpha|^2 + |\beta|^2 = 1$

Observables

P_i is a projector iff $P^2 = P$

- Hermitian operator $A = A^\dagger = (A^T)^*$
  - Real values $\boxed{a_1, \ldots, a_k}$ and corresponding projectors, $\boxed{P_1, \ldots, P_k}$
  - We have $A = \sum_{i\in[k]} a_i P_i$
  - Output a_i with probability $q_i := \langle\psi| P_i |\psi\rangle = \| P_i|\psi\rangle\|^2$
  - State collapses to $|\psi_i\rangle := \boxed{\frac{P_i|\psi\rangle}{\sqrt{q_i}}}$

$\| |\psi_i\rangle \|^2 = 1$

- Average value of the outcome is

$|\psi\rangle \xrightarrow{\text{measure acc to } A} \mathbb{E}[a_i]$

$$\langle A\rangle = \langle\psi| A |\psi\rangle = \sum_i \langle\psi| P_i |\psi\rangle a_i$$

$\to P_i|\psi\rangle$

a_i

Global phases: $|\psi\rangle$ vs. $e^{i\theta} \cdot |\psi\rangle$

$\hookrightarrow$ global phase

Fix obs. $A = \sum a_i P_i$

$|\psi\rangle$

$\rightarrow \boxed{a_i}$ w.p. $q_i = \boxed{\langle\psi|P_i|\psi\rangle}$

$\rightarrow$ post. meas. state is

$P_i |\psi\rangle / \sqrt{q_i}$

$e^{i\theta}|\psi\rangle$ $\leftarrow \cos\theta + i\sin\theta$

$\rightarrow \boxed{a_i}$ w.p. $e^{-i\theta}\langle\psi|P_i e^{i\theta}|\psi\rangle$

$= \cancel{e^{i\theta - i\theta}} \boxed{\langle\psi|P_i|\psi\rangle}$

$\rightarrow$ post. meas state $\underline{e^{i\theta}|\psi\rangle}$
$\sqrt{q_i}$

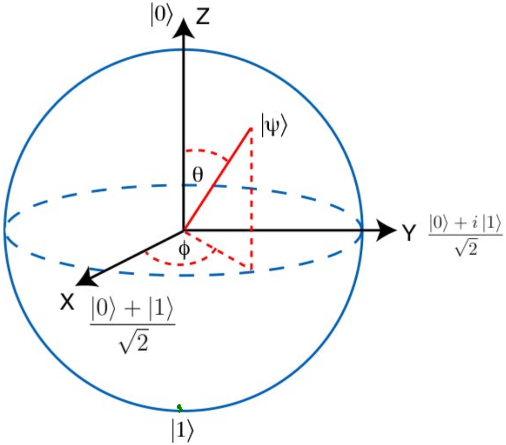
Global phases are not important!

$\alpha|0\rangle + \beta|1\rangle$ $\alpha = \boxed{a} + \ldots$ $|\alpha|^2 + |\beta|^2 = 1$

$\hookrightarrow \in \mathbb{R}$ $\hookrightarrow \mathbb{C}$ $\beta = \boxed{c} + id$

Bloch sphere

Geometrical representation of 1 qubit



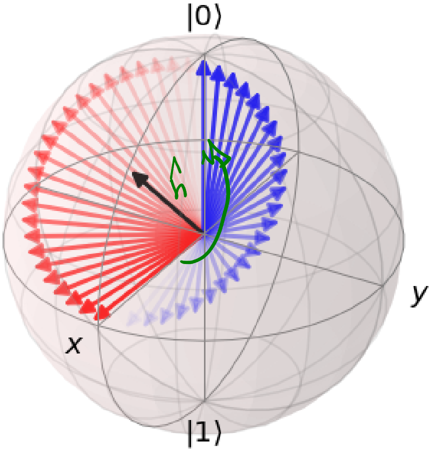
Polarizatio of Photons

$$|\psi\rangle = \cos\theta\,|0\rangle + e^{i\varphi}\sin\theta\,|1\rangle$$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ 180° degrees around the x axis

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ 180° degrees around z axis

Bloch sphere



$$H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$R_{\hat{n}}(\theta)$$

Product states

A two qubit state $|\psi\rangle$ is said to be a product state if there exists $|\psi_1\rangle$ and $|\psi_2\rangle$ such that

$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle\,|\psi_2\rangle$$

$|\psi_1\rangle = \beta|0\rangle + \beta'|1\rangle \qquad |\psi_2\rangle = \gamma|0\rangle + \gamma'|1\rangle$

Not all states are product

$|EPR\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \to EPR$

Proof Let us assume $\exists\ |\psi_1\rangle, |\psi_2\rangle$ st $\cancel{|EPR\rangle = |\psi_1\rangle \otimes |\psi_2\rangle}$

$|EPR\rangle = (\beta|0\rangle + \beta'|1\rangle) \otimes (\gamma|0\rangle + \gamma'|1\rangle) = \beta\gamma|00\rangle + \beta\gamma'|01\rangle + \beta'\gamma|10\rangle + \beta'\gamma'|11\rangle$

(1) $\beta\gamma = \beta'\gamma' = \frac{1}{\sqrt{2}}$

(2) $\beta\gamma' = \beta'\gamma = 0$

(1) $\Rightarrow \beta, \gamma, \beta', \gamma' \neq 0$

(2) $\Rightarrow \beta$ or $\gamma' = 0$, β' or $\gamma = 0$

contradiction!

Entangled states

$|EPR\rangle$

Entangled states: quantum states that are not product

Examples

$\mathrm{GHZ}_n = \frac{1}{\sqrt{2}}(\underbrace{|0...0\rangle}_{n} + \underbrace{|1...1\rangle}_{n})$

$GHZ_3 = \frac{1}{\sqrt{2}}\left(|0\rangle_A|00\rangle_B + |1\rangle_A|11\rangle_B\right)$

$|\psi_1\rangle$ on 1 qubit

$|\psi_2\rangle$ on 2 qubits

$|GHZ_3\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

$\mathrm{W}_n = \frac{1}{\sqrt{n}}(|10...0\rangle + |01...0\rangle + |00...1\rangle)$

$W_3 = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

Bell basis - bases for $\mathbb{C}^4$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

every vector in the basis is entangled

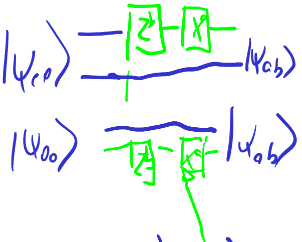
Computational ~~Canonical~~ basis

$$\left\{ \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \right\} \quad \begin{array}{l} |0\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle \\ \ldots \end{array}$$

Bell basis

$|EPR\rangle$

$$\left\{ \begin{array}{l} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi_{00}\rangle \\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\psi_{01}\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\psi_{10}\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\psi_{11}\rangle \end{array} \right\}$$



$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

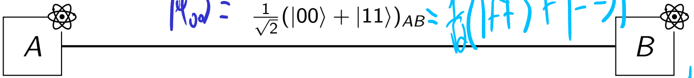
$X^1 = X$ $\quad Z^1 = Z$

$X^0 = I$ $\quad Z^0 = I$

Lem: $|\psi_{ab}\rangle$ is entangled

$$|\psi_{ab}\rangle = (X^a Z^b \otimes I)|\psi_{00}\rangle = (I \otimes X^a Z^b)|\psi_{00}\rangle$$

Source of quantum weirdness



$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB} = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$ exercise

$|00\rangle = |0\rangle\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle|+\rangle + |0\rangle|-\rangle\right)$

- what happens if A measures her share on comp basis? Hadamard basis?

→ outcome is 0 (+) wp $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ $\quad \sqrt{2}\,\frac{1}{\sqrt{2}}|00\rangle = |00\rangle$ ($|++\rangle$) post-m state

Bob measures outcome in comp (Had) basis is 0 (+) w.p 1

$+$ wp $\frac{1}{2}$ $|0\rangle|+\rangle$

$-$ wp $\frac{1}{2}$ $|0\rangle|-\rangle$

$|--\rangle$

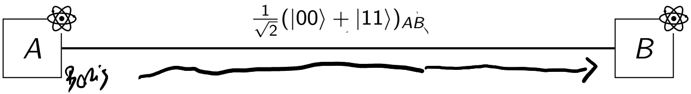
→ outcome is 1 (−) wp $\frac{1}{2}$ $\quad |11\rangle$ p.m.s.o

Bob measure outcome in comp basis is 1 wp 1

$+$ w $\frac{1}{2}$ $|1\rangle|+\rangle$

$-$ wp $\frac{1}{2}$ $|1\rangle|-\rangle$

Source of quantum weirdness



Violation of relativity theory?

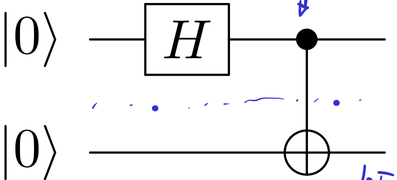
"I cannot seriously believe in it [quantum theory] because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance" Albert Einstein

- non-local games
- Quantum teleportation: how to transmit quantum states via classical channels
- superdense coding: 2 bits of information w/ 1 qubit

Generating entangled states $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Theory

entangling gate



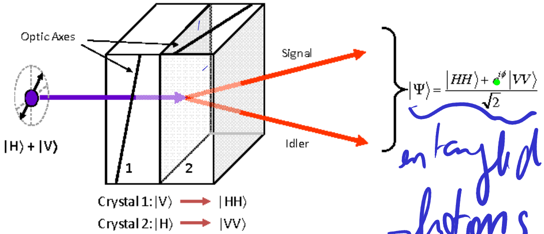
$C(x)$

$\text{CNOT}\,|a\rangle|b\rangle = |a\rangle|b\oplus a\rangle$

$$\text{CNOT}\,(H\otimes I)|00\rangle = \text{CNOT}\,\frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) =$$

$$= \frac{1}{\sqrt{2}}\left(\cancel{\text{CNOT}|00\rangle}^{\,|00\rangle} + \cancel{\text{CNOT}|10\rangle}^{\,|11\rangle}\right) = |\text{EPR}\rangle$$

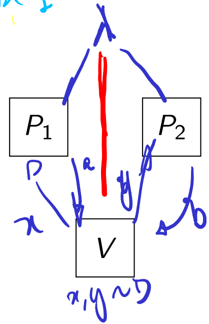
Practice



entangled photons

Non-local games

phase 1



D
V

1) P_1 and P_2 share some strategy
2) Not allowed to communicate after
3) V pick questions to P_1 and P_2

$x, y \sim D$

4) P_1 and P_2 answer
5) $V(a, b \mid x, y) = 0/1$

Classical value $\omega(G)$

λ is a bit string

$$\omega(G) = \max_{\text{classical strategies}} \Pr[V(a,b \mid x,y) = 1]$$

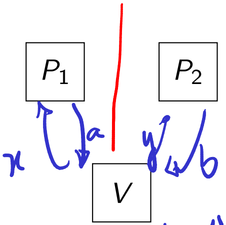
Quantum value $\omega^*(G)$

λ is a quantum state $|\psi\rangle_{AB}$

$$\omega^*(G) = \max_{\text{quantum str.}} \Pr[V(a,b \mid x,y) = 1]$$

Goal: find G s.t. $\omega^*(G) > \omega(G)$

CHSH game



$x, y \in \{0,1\}$

$x \wedge y = a \oplus b$

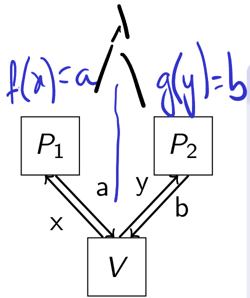
$a \neq b$ iff $x = y = 1$

a, b

D: x, y are random bit

V

CHSH game - classical strategies



$x, y \in_R \{0,1\}$

$x \wedge y = a \oplus b$

$a \neq b$ iff $x = y = 1$

$a, b \in \{0,1\}$

Example Fix f and g

$f(x) = 0$

$g(y) = 0$

x	y	a	b	Win?
0	0	0	0	✓
0	1	0	0	✓
1	0	0	0	✓
1	1	0	0	✗

Win $\frac{3}{4}$

$x \oplus y = 0$

$x \wedge y = 0$

$a \oplus b = 0$

Lem: $\omega(G) = \frac{3}{4}$

$V(a,b \mid x,y) = 1$ iff $x \wedge y = a \oplus b$

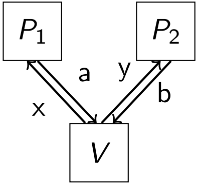
$x = y = 0$

$0 = 0$

CHSH game - quantum strategies

$\omega^*(CHSH) \geq \cos^2 \pi/8 \approx 0.85$

1. quantum state.
2. quantum operations



$x, y \in_R \{0, 1\}$

$x \cdot y = a \oplus b$

$a \neq b$ iff $x = y = 1$

1. $\frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = |\psi_{01}\rangle = Z^b|\psi_{00}\rangle$

2. $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

str. of P_1: if $x=0$ $R(-\pi/16)$

if $x=1$ $R(3\pi/16)$

then measure in comp basis

P_2: if $y=0$ $R(-\pi/16)$

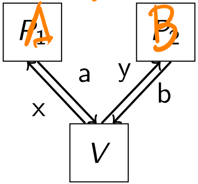
$y=1$ $R(7\pi/16)$

then measure in comp. basis

CHSH game - quantum strategies

$x, y = 0 \quad \theta = -\pi/16$

$x, y = 1 \quad \theta = 3\pi/16$



$$x, y \in_R \{0, 1\}$$
$$x \cdot y = a \oplus b$$
$$a \neq b \text{ iff } x = y = 1$$

if $x \wedge y = 0$

$\Rightarrow \theta_A + \theta_B = \pm \frac{\pi}{8}$

if $x \wedge y = 1$ $\quad \theta_A + \theta_B = \frac{3\pi}{8}$

$\frac{1}{\sqrt{2}} \left(R(\theta_A) \otimes R(\theta_B) \right) \left(|00\rangle - |11\rangle \right) =$

$= \frac{1}{\sqrt{2}} \Big(\left(\cos(\theta_A) |0\rangle - \sin(\theta_A) |1\rangle \right) \left(\cos(\theta_B) |0\rangle - \sin(\theta_B) |1\rangle \right)$

$- \left(\sin(\theta_A) |0\rangle + \cos(\theta_A) |1\rangle \right) \otimes$

$\left(\sin(\theta_B) |0\rangle + \cos(\theta_B) |1\rangle \right) \Big)$

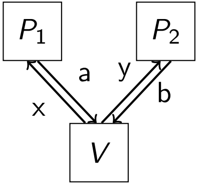
$= \frac{1}{\sqrt{2}} \Big(\left(\cos(\theta_A) \cos(\theta_B) - \sin(\theta_A) \sin(\theta_B) \right) \left(|00\rangle - |11\rangle \right)$

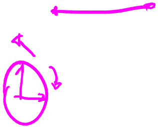
$- \left(\cos(\theta_A) \sin(\theta_B) + \sin(\theta_A) \cos(\theta_B) \right) \left(|01\rangle + |10\rangle \right) \Big)$

$= \frac{1}{\sqrt{2}} \left(\cos(\theta_A + \theta_B) \left(|00\rangle - |11\rangle \right) \right) + \sin(\theta_A + \theta_B) \left(|01\rangle + |10\rangle \right)$

$\cos^2 \pi/8 \qquad \cos^2 \pi/$

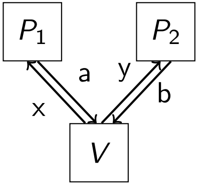
CHSH game - quantum strategies





$$x, y \in_R \{0, 1\}$$
$$x \cdot y = a \oplus b$$
$$a \neq b \text{ iff } x = y = 1$$

CHSH game - quantum strategies



$$x, y \in_R \{0, 1\}$$

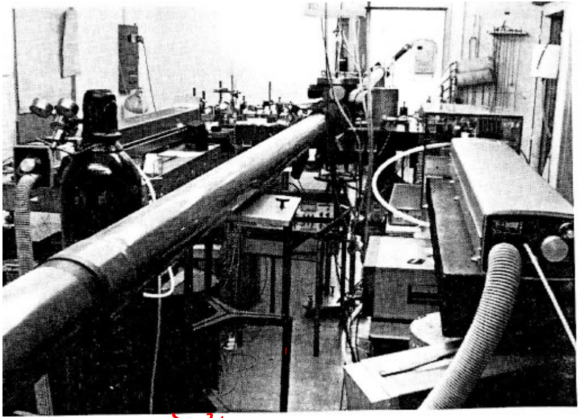
$$x \cdot y = a \oplus b$$

$$a \neq b \text{ iff } x = y = 1$$

Bell inequalities

Bell inequalities give us a way for testing if Nature is not classical $\Rightarrow$ implement the quantum experiments in the lab and if the acceptance probability is strictly larger tha

Alain Aspect (1982, Institut d'Optique)



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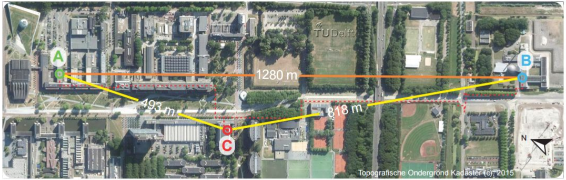
loophole experiment

QuTech group (2015, TU Delft)



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Further notions

We have seen a quantum strategy that achieves $\omega^*(CHSH) = \cos^2 \frac{\pi}{8} \approx 0.85$

Is this strategy optimal? Yes

Tsirelson's inequalities
to bound the quantum
value of games

Is this ~~optimal~~ morally strategy unique? Yes

if achieve $\cos^2 \frac{\pi}{8}$
then one can characterize your
q. devices

→ DI crypto
→ disproof of Connes conj.

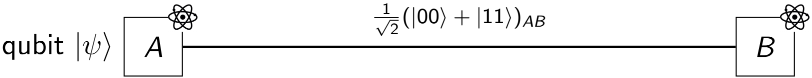
Quantum teleportation

Problem: Alice wants to send a qubit $|\psi\rangle$ to Bob.
If they have a quantum channel to communicate:

If they have a classical channel to communicate:

If they have a classical channel to communicate + pre-shared quantum state:

Quantum teleportation



Super-dense coding

Alice receives two random bits *a* and *b* and she wants Bob to learn both of them.

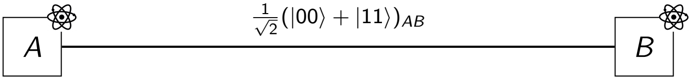
If Alice sends a single classical bit:

If they have shared randomness and Alice sends a single classical bit:

If they have shared quantum state and Alice sends a single classical bit:

If they have shared quantum state and Alice sends a single qubit:

Super-dense coding



Mixed states

- Mixed states: probabilistic distribution of quantum states

Examples

$((\frac{1}{2}, |0\rangle), (\frac{1}{2}, |+\rangle))$, $((\frac{1}{3}, |0\rangle), (\frac{2}{3}, |1\rangle))$

- Density matrices: mathematical representation of mixed states

  $(1, |\psi\rangle)$

  $((p_1, |\psi_1\rangle), (p_2, |\psi_2\rangle), (p_k, |\psi_k\rangle))$

- Properties of density matrices
  - Its trace is 1 (The trace of a square matrix is $\sum_i a_{i,i}$)
  - Positive semi-definitive (all its eigenvalues are non-negative)
- Definition of evolution and measurements can be extended to density matrices

“Parts” of quantum states

- Trace-out: ignore qubits of a larger quantum state