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Lecture # 03 - Entanglement + algorithms

Throughout this exercise list, we call $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Entanglement

1. One particular useful application of quantum teleportation is *entanglement swapping*. Suppose that Alice and Bob share an EPR pair, and Bob and Charlie share a second EPR pair (see Figure 1). Show that via quantum teleportation, Bob can help Alice and Charlie share an EPR pair with classical communication only. Does Bob share an EPR pair with any other party after the execution of this protocol?
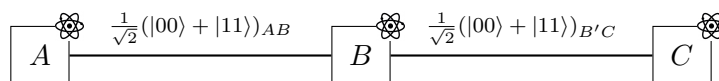

Figure 1: Setup for entanglement swapping

Density matrices

In the following exercise, we denote $((p_1, |\psi_1\rangle), (p_2, |\psi_2\rangle), ..., (p_n, |\psi_n\rangle))$ as the ensemble where we create the state $|\psi_i\rangle$ with probability p_i.

2. Compute the density matrices of:

   (a) $((1, |\psi\rangle))$ and $((1, e^{i\theta}|\psi\rangle))$.
   (b) $((\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle))$ and $((\frac{1}{2}, |+\rangle), (\frac{1}{2}, |-\rangle))$.

   What can you conclude from the previous results?

3. Let $|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|1\rangle_A|00\rangle_B + |1\rangle_A|10\rangle_B + |1\rangle_A|01\rangle_B)$. Compute $Tr_A(|\psi\rangle\langle\psi|_{AB})$.

4. Let $|W_3\rangle_{AB} = \frac{1}{\sqrt{3}}(|1\rangle_A|00\rangle_B + |0\rangle_A|10\rangle_B + |0\rangle_A|01\rangle_B)$. Compute $Tr_A(|W_3\rangle\langle W_3|_{AB})$.

Quantum circuits

5. Show that you can use the CCNOT gate to compute the NAND gate. Since NAND is universal for classical computation, this means that every problem that can be solved with classical circuits can be also solved by quantum circuits.

6. The SWAP gate works as $SWAP|a\rangle|b\rangle = |b\rangle|a\rangle$.

   (a) Show that SWAP gate is unitary.
   (b) Show how to implement the SWAP gate with three CNOT gates.

7. Consider the circuit in Figure 2.

(a) Compute the output of the quantum circuit if the input is $|b\rangle|a\rangle$, for $a, b \in \{0, 1\}$.

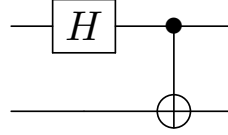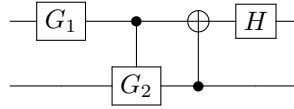(b) Write the 4×4 matrix that describes that circuit.

Figure 2: Circuit for Exercise 7

8. Write the circuit corresponding to the inverse of (where G_1 and G_2 are arbitrary 1-qubit gates)

Quantum algorithms

We say that $f : \{0,1\}^n \to \{0,1\}$ is a linear function if there exists some value $s \in \{0,1\}^n$ such that it can be written as $f(x) = x \cdot s \pmod 2$.

9. Suppose that you have (classical) oracle access to a linear function $f_s(x) = s \cdot x \pmod 2$. Show a classical algorithm that learn s with n queries.

10. Suppose now that you have quantum oracle access to f_s (i.e. you have access to the unitary $O_{f_s}|x\rangle|b\rangle = |x\rangle|b \oplus f_s(x)\rangle$). The Bernstein-Vazirani algorithm (Figure 3) allows us to find s with a single quantum query to this oracle.

(a) What is the state of the algorithm after the first layer of Hadamard gates?

(b) What is the state of the algorithm after the query to the oracle?

(c) What is the state of the algorithm after the last layer of Hadamards?

(d) Compute the probability of measuring s on the first register.

11. Compute every step of the computation of the Bernstein-Vazirani algorithm with oracle access to the linear function $f(x_1, x_2) = x_1 + x_2$. What is the value of s?
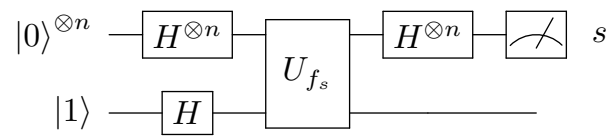
Figure 3: Bernstein-Vazirani algorithm