ACCQ206

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Lecture # 02 - Entanglement

Throughout this exercise list, we call $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Obsevables and global phases

1. Write the observable corresponding to the measurement on the computational basis if we assign $|0\rangle$ the outcome $+1$ and $|1\rangle$ the outcome -1. What is the expected value of the measurement outcome of $|+\rangle$ for this observable?

2. Prove that for every qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$, there exists a qubit $|\phi\rangle = \alpha'|0\rangle + \beta'|1\rangle$ where $\alpha' \in \mathbb{R}$ and $\beta' \in \mathbb{C}$ such that $|\psi\rangle = \gamma|\phi\rangle$ for some $\gamma \in \mathbb{C}$ (i.e. $|\psi\rangle$ and $|\phi\rangle$ are indistinguishable)

Bloch sphere

Let $\vec{n} = (n_x, n_y, n_z)$ be a vector with $n_x, n_y, n_z \in \mathbb{R}$ and $||\vec{n}|| = 1$, and $\vec{\sigma} = (X, Y, Z)$ and $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

3. Show that $(\vec{n} \cdot \vec{\sigma})^2 = I$. [1]

A rotation of θ around the axis $\hat{n}$ can be written as:

$$R_{\vec{n}}(\theta) = e^{i\theta/2(\vec{n}\cdot\vec{\sigma})}.$$

4. Show that $R_{\vec{n}}(\theta) = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})$.[2]

5. Show that for every $\alpha \in \mathbb{R}$, $e^{i\alpha}R_{\vec{n}}(\theta)$ is a unitary matrix.

6. Show that any unitary U on one qubit can be written as

$$U = e^{i\alpha}R_{\vec{n}}(\theta),$$

for some $\alpha \in \mathbb{R}$.

(**Pour aller plus loin...:**) Find the values of $\alpha, \vec{n}$ and θ for X, Y, Z and H matrices.

Entanglement

7. Which of the following states are entangled (according to indicated partition):

   (a) $\frac{1}{\sqrt{3}}|0\rangle_A|0\rangle_B + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle_A|1\rangle_B$
   (b) $\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|00\rangle_B$
   (c) $\frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B$

8. Show that for every one-qubit basis $\{|b_1\rangle, |b_2\rangle\}$ such that $|0\rangle = \alpha_0|b_0\rangle + \alpha_1|b_1\rangle$ and $|1\rangle = \alpha_1|b_0\rangle - \alpha_0|b_1\rangle$, we have that $|EPR\rangle = \frac{1}{\sqrt{2}}(|b_1\rangle|b_1\rangle + |b_2\rangle|b_2\rangle)$.

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[1] Hint: use the fact that $XY = -YX$, $XZ = -ZX$,...

[2] Recall that for an $n \times n$ matrix A,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

.

Pour aller (beaucoup) plus loin... [3]

We will now prove the Tsirelson's bound for CHSH (i.e., we will show that the quantum strategy that we saw is optimal).

For that, we will consider a more convenient notation for the CHSH game: the answers from P_1 and P_2 are $a, b \in \{\pm 1\}$, and they win the game if $(-1)^{xy} = ab$.[4]

We can consider a generic quantum strategy for P_1 and P_2 in the CHSH game as follows:

(A) P_1 and P_2 share an arbitrary quantum state $|\psi\rangle_{P_1,P_2}$ (notice that we make no assumption on the size of such a quantum state)

(B) For each question x, P_1 chooses an observable M^b that she will use to measure her share of $|\psi\rangle$ and will answer with the outcome of the this measurement. In other words, for each x, P_1 chooses two values M^x_{+1} and M^x_{-1} such that $M^x_{+1} + M^x_{-1} = I$ and $M^x = M^x_{+1} - M^x_{-1}$.

(C) Likewise, for each y, P_2 chooses an observable N^y that she will use to measure her share of $|\psi\rangle$ and will answer with the outcome of the this measurement.

Show that:

9. Show that for a fixed x and y, the expected value of ab is $\langle\psi| M^x \otimes N^y |\psi\rangle$.

10. Let $C = (M^0 \otimes N^0 + M^0 \otimes N^1 + M^1 \otimes N^0 - M^1 \otimes N^1)$. Show that the winning probability of P_1 and P_2 in the game is $\frac{1}{2} + \frac{1}{8} \langle\psi| C |\psi\rangle$. (Hint: argue that $\langle\psi| C |\psi\rangle = Pr[\text{win}] - Pr[\text{lose}]$)

11. Show that
$$C^2 = 4I + (M^0 M^1 - M^1 M^0) \otimes (N^0 N^1 - N^1 N^0)$$
(Hint: Show (and use) the fact that $(M^x)^2 = (N^y)^2 = I$ for all $x, y \in \{0, 1\}$)

12. Show that $\langle\psi| C |\psi\rangle \leq 2\sqrt{2}$. (Hint: Use Cauchy-Schwartz inequality: $(\langle\psi| C |\psi\rangle)^2 \leq \langle\psi| C^2 |\psi\rangle$)

13. What can you say about maximum the quantum value of CHSH?

References

[1] Ronald de Wolf. Quantum computing: Lecture notes. `http://arxiv.org/abs/1907.09415`, 2019.

[3] This exercise was based on Exercise 6 of chapter 16 of [1]

[4] Notice that we are just considering the map $b \leftrightarrow (-1)^b$ for their answers.