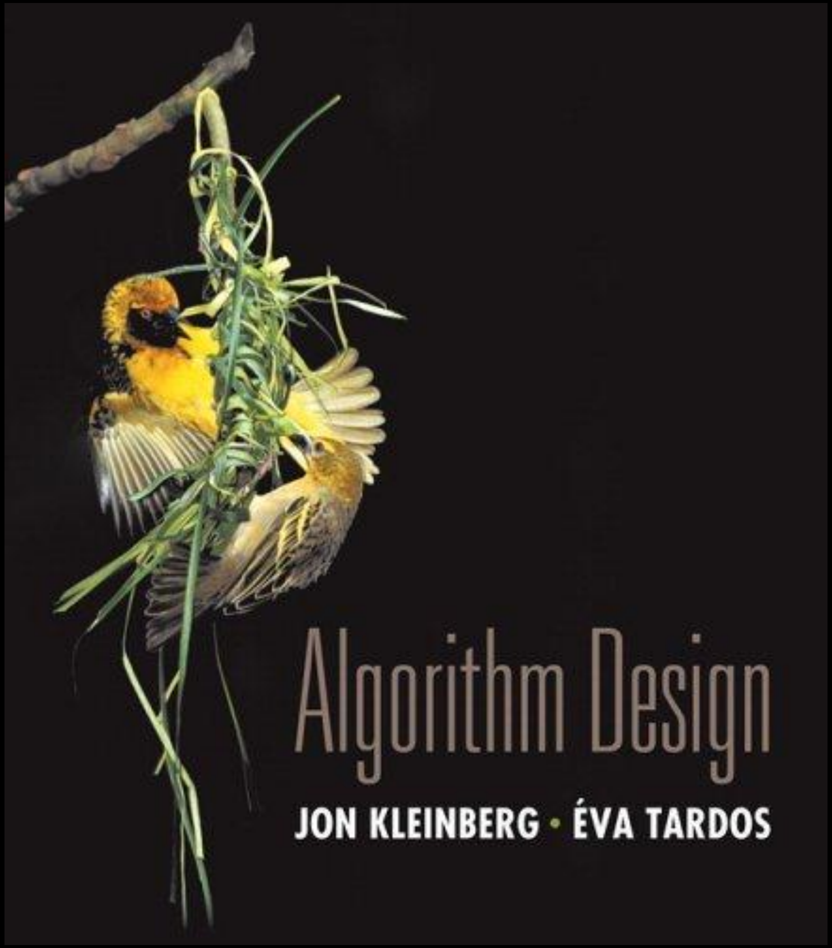


Chapter 4

Greedy Algorithms

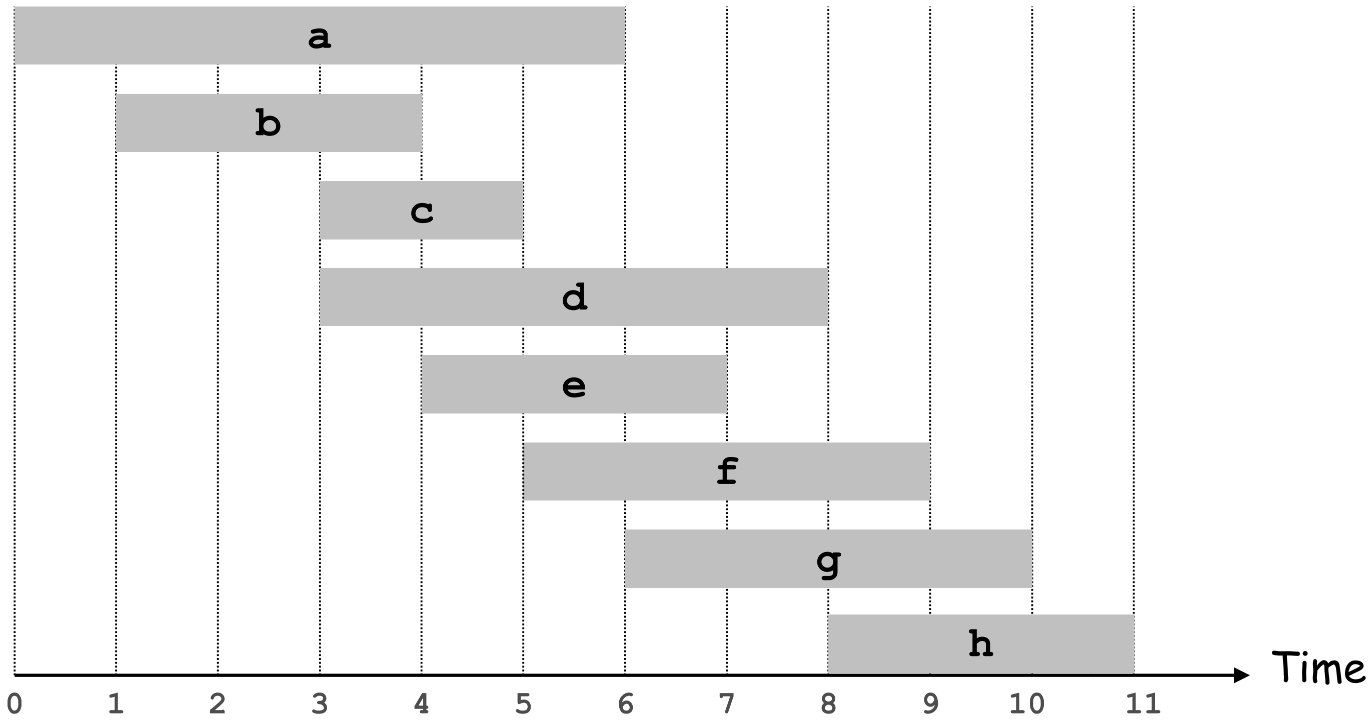
Slides by Kevin Wayne.

4.1 Interval Scheduling

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order. Take each job provided it's compatible with the ones already taken.

- **[Earliest start time]**
  - Consider jobs in ascending order of s_j.
- **[Shortest interval]**
  - Consider jobs in ascending order of $f_j - s_j$.
- **[Fewest conflicts]**
  - For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.
- **[Earliest finish time]**
  - Consider jobs in ascending order of f_j.

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

counterexample for earliest start time

counterexample for shortest interval

counterexample for fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

   set of jobs selected
$A \leftarrow \phi$
for j = 1 to n {
   if (job j compatible with A)
      $A \leftarrow A \cup \{j\}$
}
return A
```

Implementation. O(n log n).

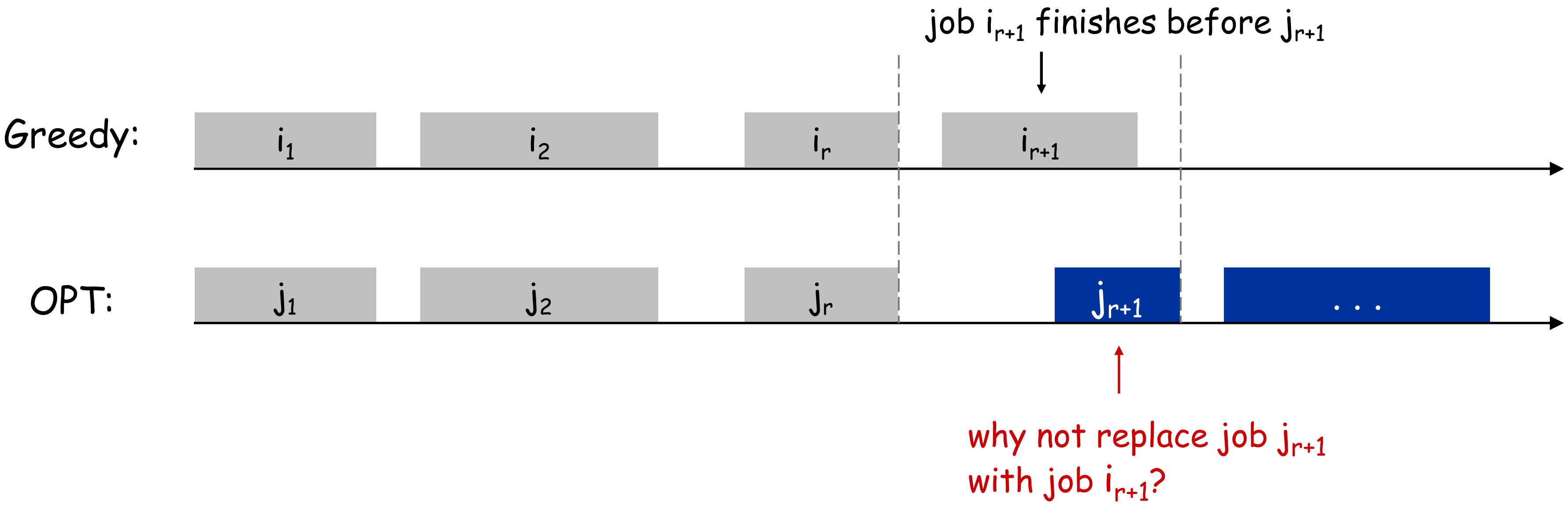
- Remember job j^* that was added last to A.
- Job j is compatible with A if $s_j \ge f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



Interval Scheduling: Analysis

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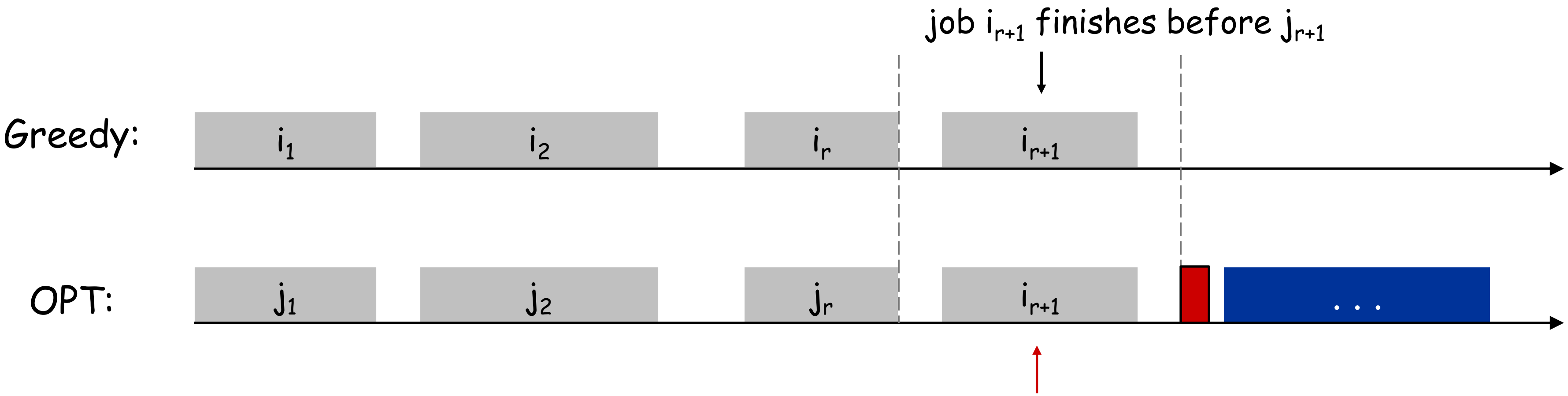
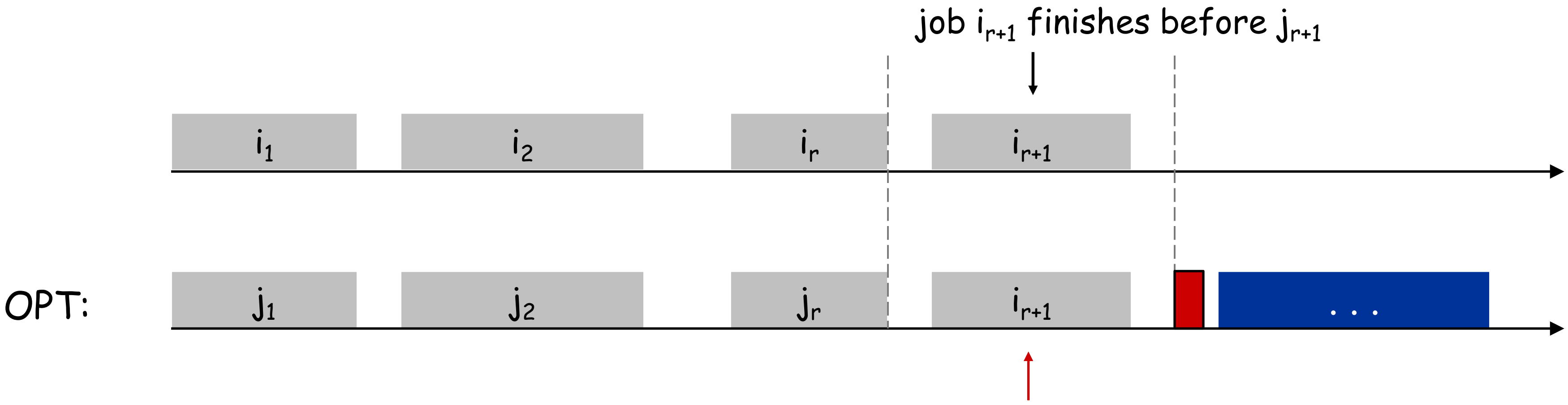
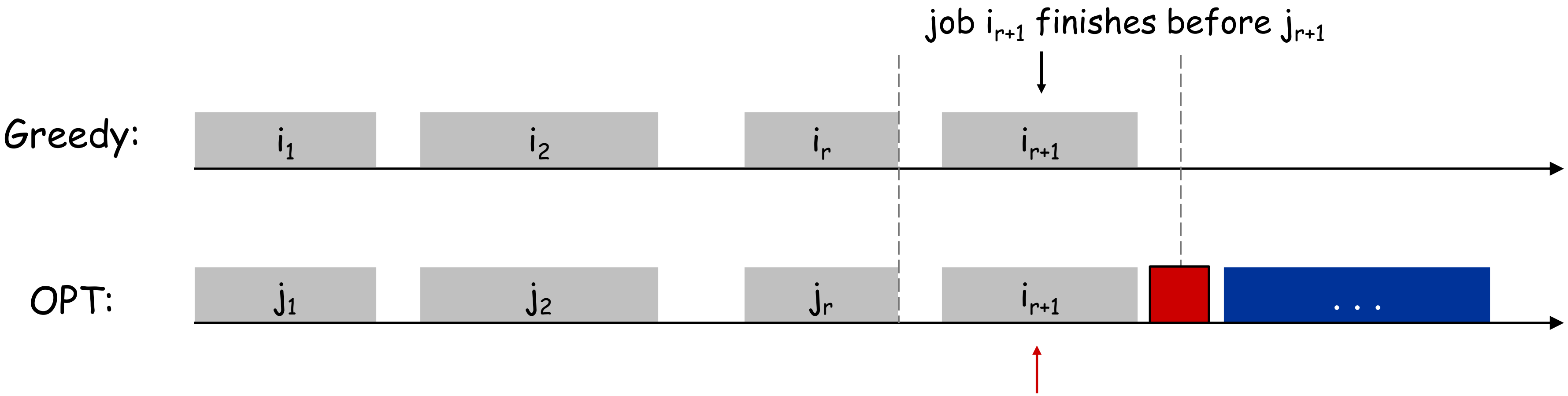


solution still feasible and optimal, but contradicts maximality of r.

Interval Scheduling: Analysis



solution still feasible and optimal, but contradicts maximality of r.



"OPT" solution can be improved by potentially adding another "intervening' task.

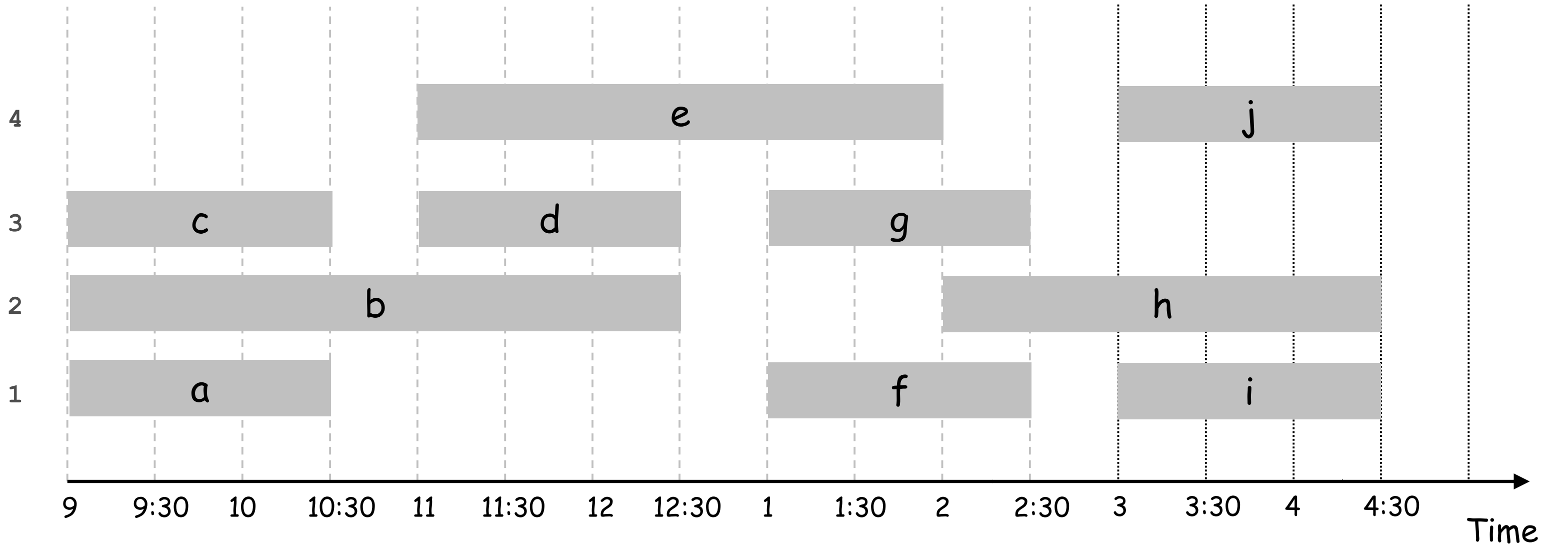
4.1 Interval Partitioning

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- **Goal:** find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

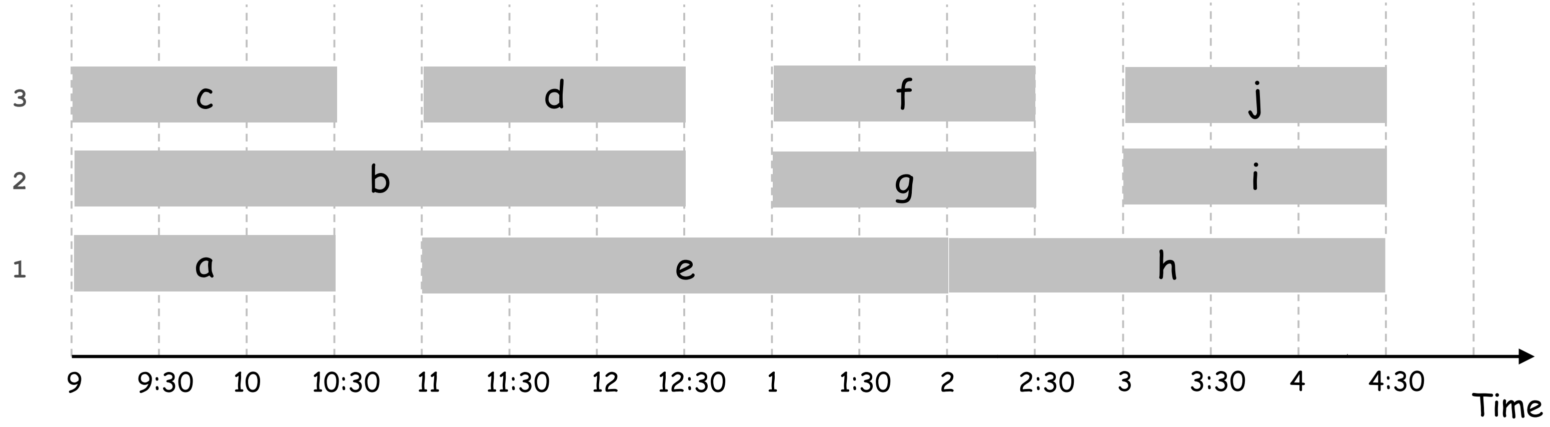


Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- **Goal:** find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3 classrooms.



Interval Partitioning: Lower Bound on Optimal Solution

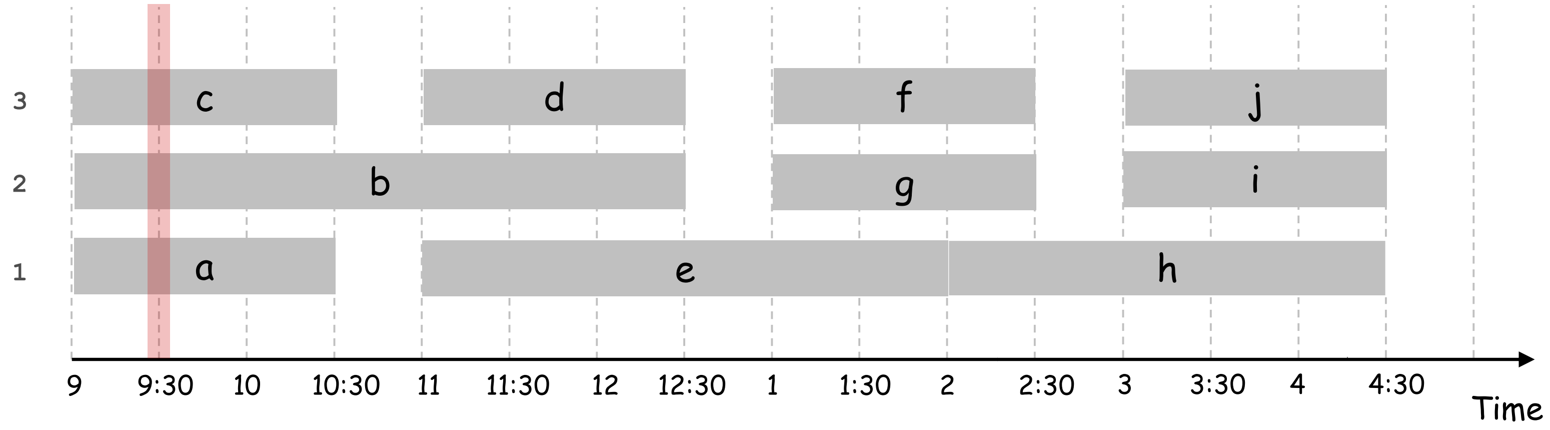
Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed ≥ depth.

Ex: Depth of schedule below = 3 ⇒ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that $s_1 \le s_2 \le ... \le s_n$.
d ← 0 ←  number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).

- For each classroom k, maintain the **finish time** of the last job added.
- Keep the classrooms in a **priority queue**.

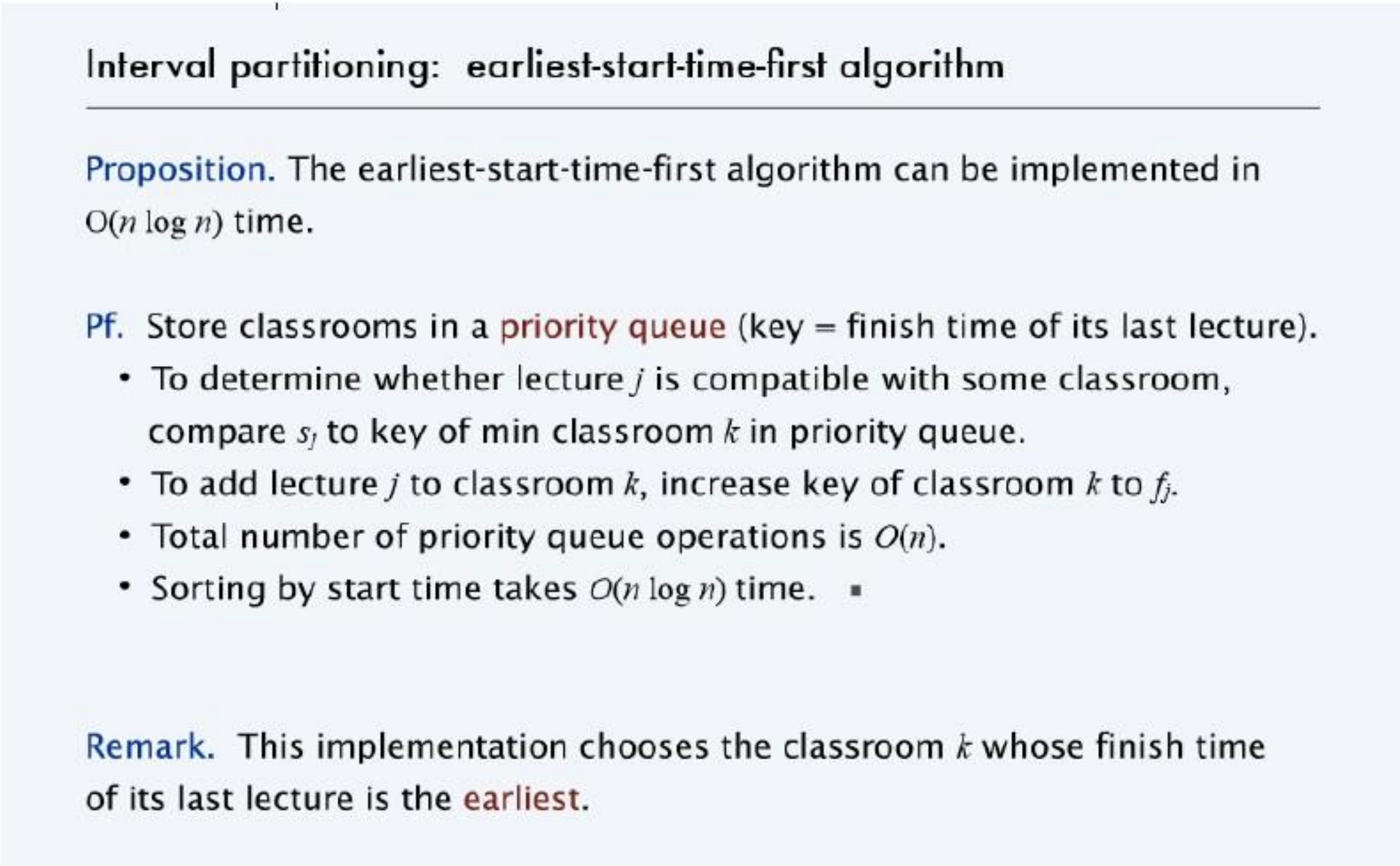
Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.
Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- These d jobs each end after s_j.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- **Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms.** ▪

Interval Partitioning: Greedy Analysis



Interval partitioning: earliest-start-time-first algorithm

Proposition. The earliest-start-time-first algorithm can be implemented in $O(n \log n)$ time.

Pf. Store classrooms in a priority queue (key = finish time of its last lecture).

- To determine whether lecture j is compatible with some classroom, compare s_j to key of min classroom k in priority queue.
- To add lecture j to classroom k, increase key of classroom k to f_j.
- Total number of priority queue operations is $O(n)$.
- Sorting by start time takes $O(n \log n)$ time. ▪

Remark. This implementation chooses the classroom k whose finish time of its last lecture is the earliest.

4.2 Scheduling to Minimize Lateness

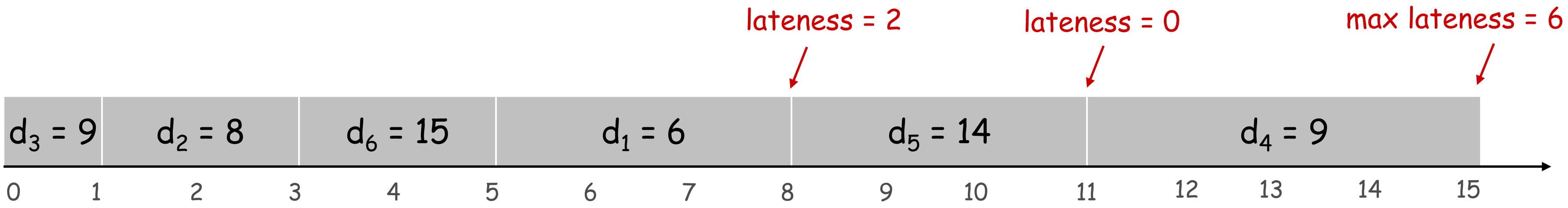
Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max\{0, f_j - d_j\}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- **[Shortest processing time first]**
  - Consider jobs in ascending order of processing time t_j.

- **[Smallest slack]**
  - Consider jobs in ascending order of slack $d_j - t_j$.

- **[Earliest deadline first]**
  - Consider jobs in ascending order of deadline d_j.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

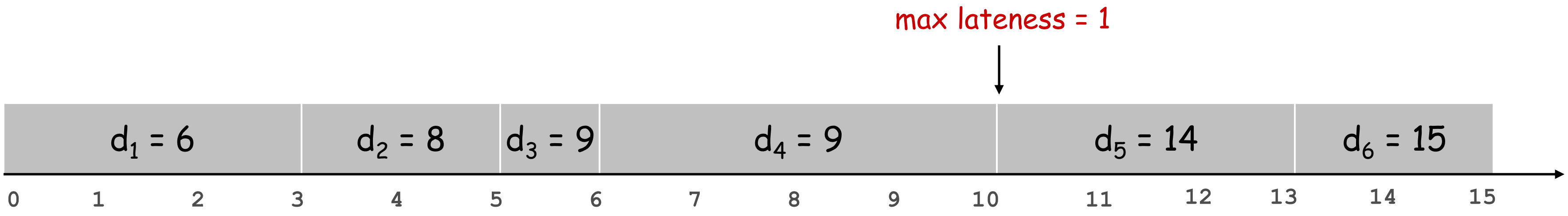
counterexample

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

```
Sort n jobs by deadline so that $d_1 \le d_2 \le \ldots \le d_n$

$t \leftarrow 0$
for j = 1 to n
   Assign job j to interval $[t, t + t_j]$
   $s_j \leftarrow t$, $f_j \leftarrow t + t_j$
   $t \leftarrow t + t_j$
output intervals $[s_j, f_j]$
```



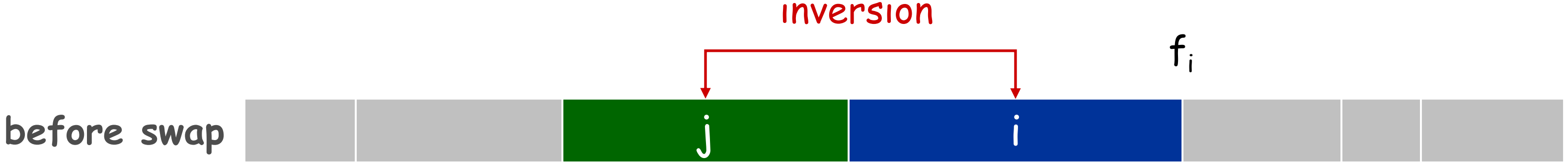
Minimizing Lateness: No Idle Time

Observation. There exists an optimal schedule with no idle time.

Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



[as before, we assume jobs are numbered so that $d_1 \le d_2 \le \ldots \le d_n$]

Observation. Greedy schedule has no inversions.

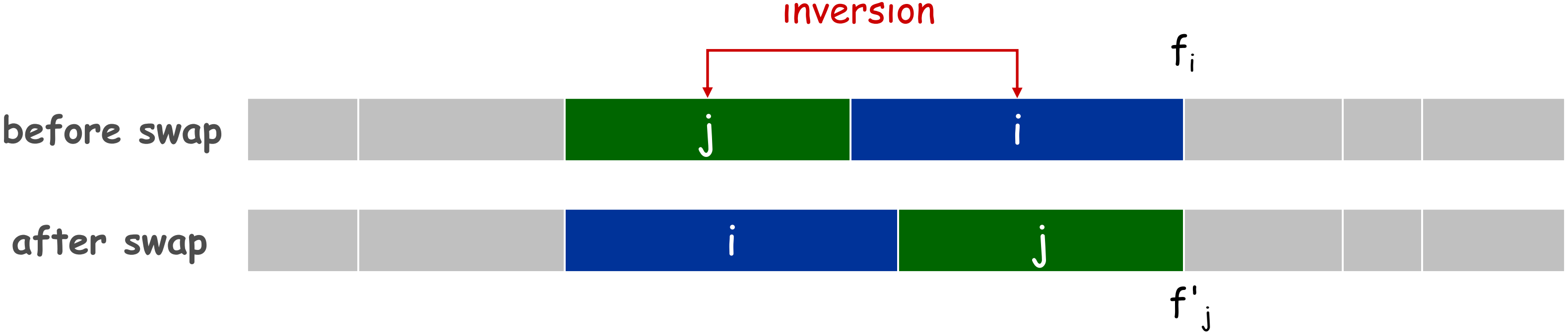
Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Example:

3 … 2
3 4 5 … 8 … 8 2
3 4 4 … 8 6 … 1 2

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is late:

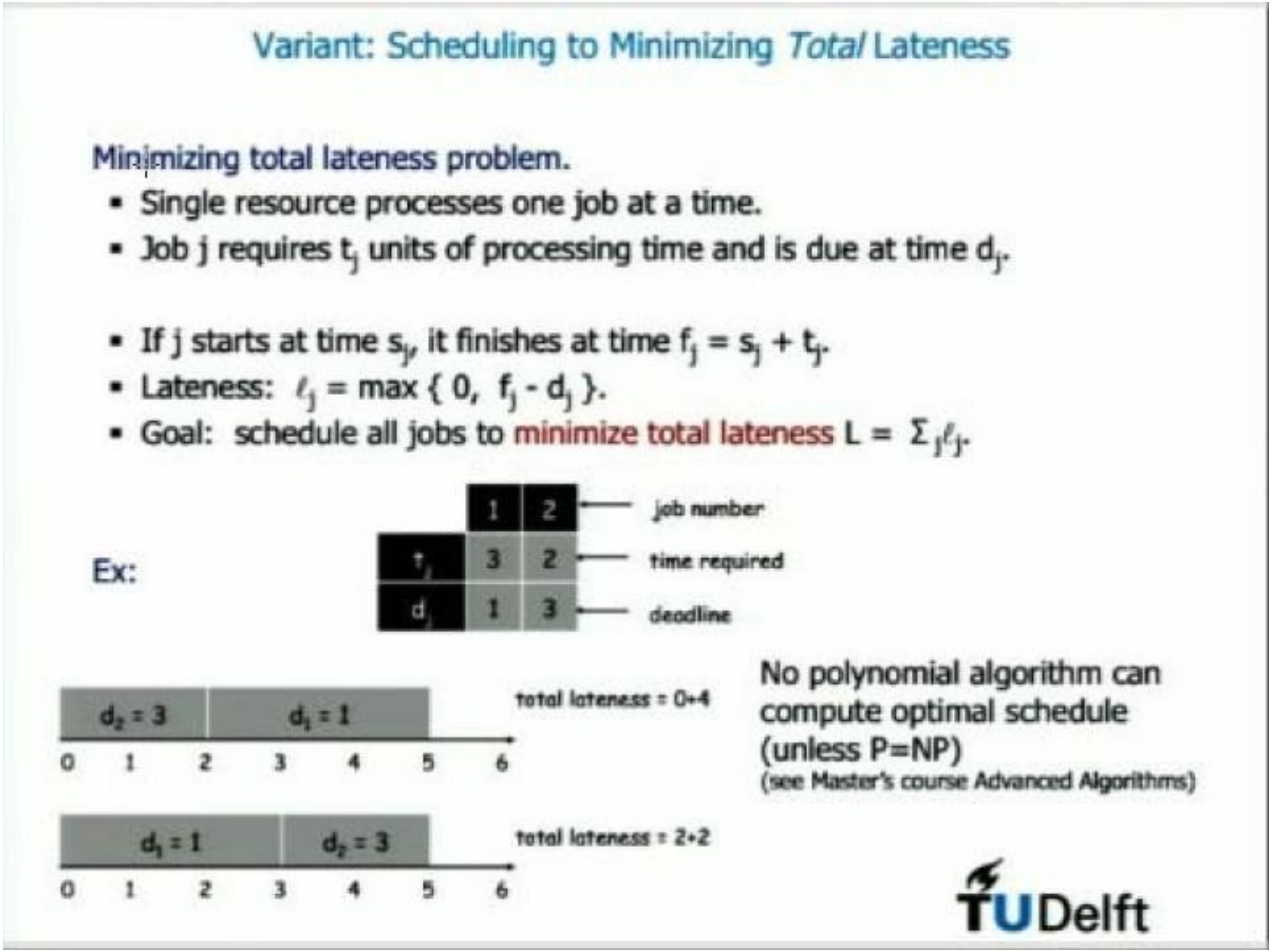
$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*.
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪



Variant: Scheduling to Minimizing *Total* Lateness

Minimizing total lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.

- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize total lateness $L = \Sigma_j \ell_j$.

Ex:

	1	2	← job number
t_j	3	2	← time required
d_j	1	3	← deadline

No polynomial algorithm can compute optimal schedule (unless P=NP)
(see Master's course Advanced Algorithms)

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, …

4.3 Optimal Caching

Optimal Offline Caching

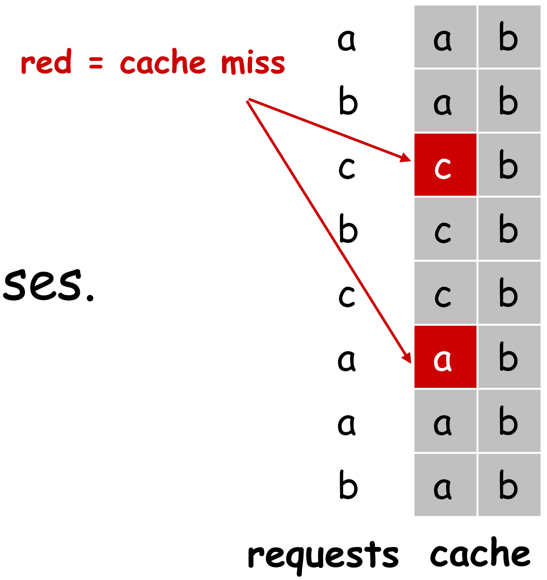
Caching.

- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Ex: k = 2, initial cache = ab,
requests: a, b, c, b, c, a, a, b.

Optimal eviction schedule: 2 cache misses.



red = cache miss

a	a	b
b	a	b
c	c	b
b	c	b
c	c	b
a	a	b
a	a	b
b	a	b
requests	**cache**	

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache:

a	b	c	d	e	f

future queries: `g a b c e d a b b a c d e a f a d e f g h ...`

cache miss (g)

eject this one (f)

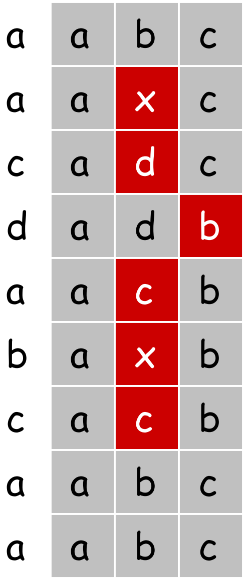
Theorem. [Bellady, 1960s] FF is optimal eviction schedule.

Pf. Algorithm and theorem are intuitive; proof is subtle.

Reduced Eviction Schedules

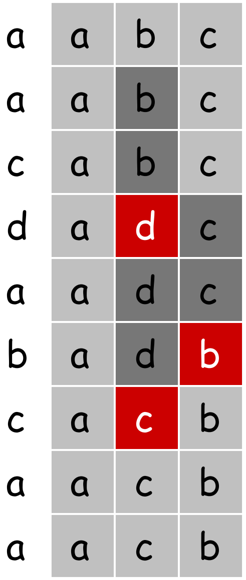
Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.



a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

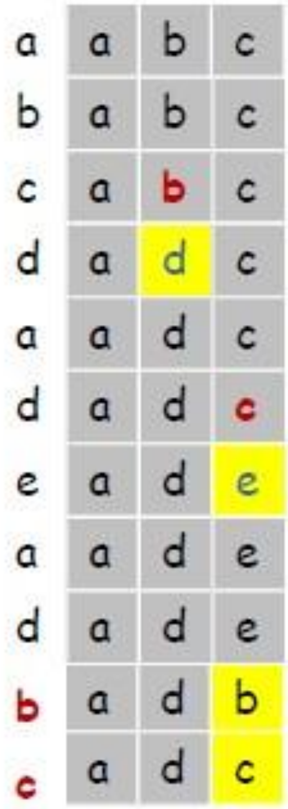
an unreduced schedule



a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

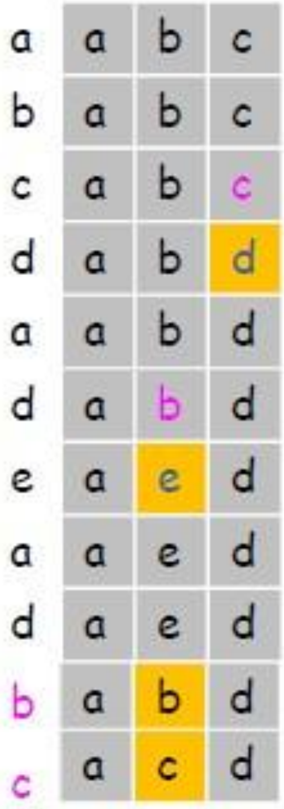
Multiple Optimal Schedules



a	a	b	c
b	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
d	a	d	c
e	a	d	e
a	a	d	e
d	a	d	e
b	a	d	b
c	a	d	c

Another optimal schedule
(2+2 cache misses)

Reduced Eviction Schedules



a	a	b	c
b	a	b	c
c	a	b	c
d	a	b	d
a	a	b	d
d	a	b	d
e	a	e	d
a	a	e	d
d	a	e	d
b	a	b	d
c	a	c	d

Belady Schedule
(Farthest in Future)

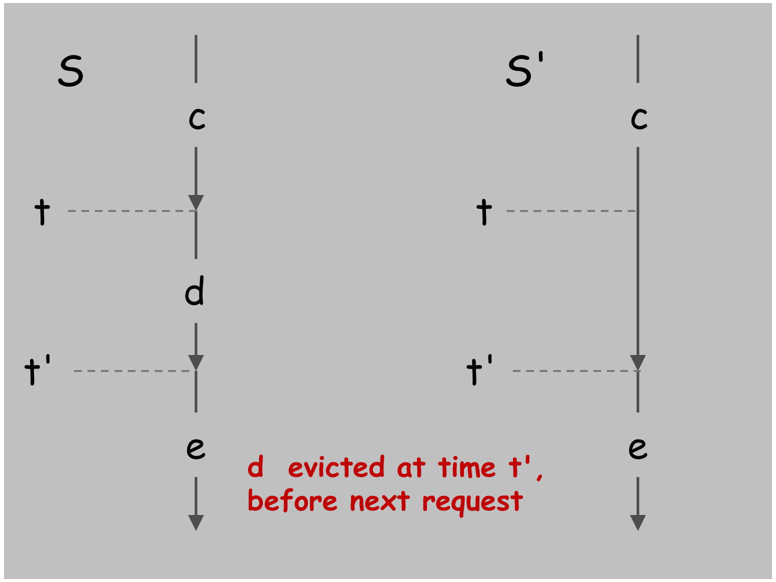
Reduced Eviction Schedules

Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more cache misses.

doesn't enter cache at requested time

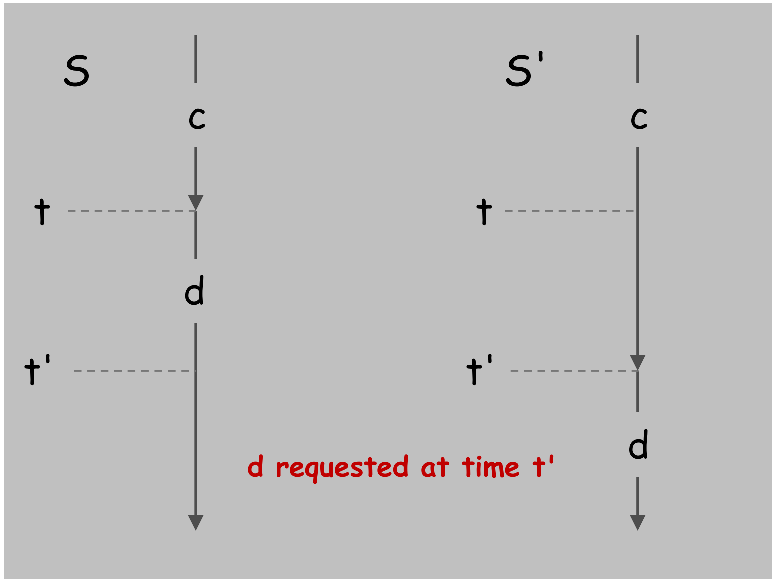
Pf. (by induction on number of **unreduced** items)

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- **Case 1:** d evicted at time t', before next request for d.
- **Case 2:** d requested at time t' before d is evicted. ▪



d evicted at time t', before next request

Case 1



d requested at time t'

Case 2

Farthest-In-Future: Analysis

Theorem. FF is optimal eviction algorithm.
Pf. (by induction on number or requests j)

Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first j+1 requests.

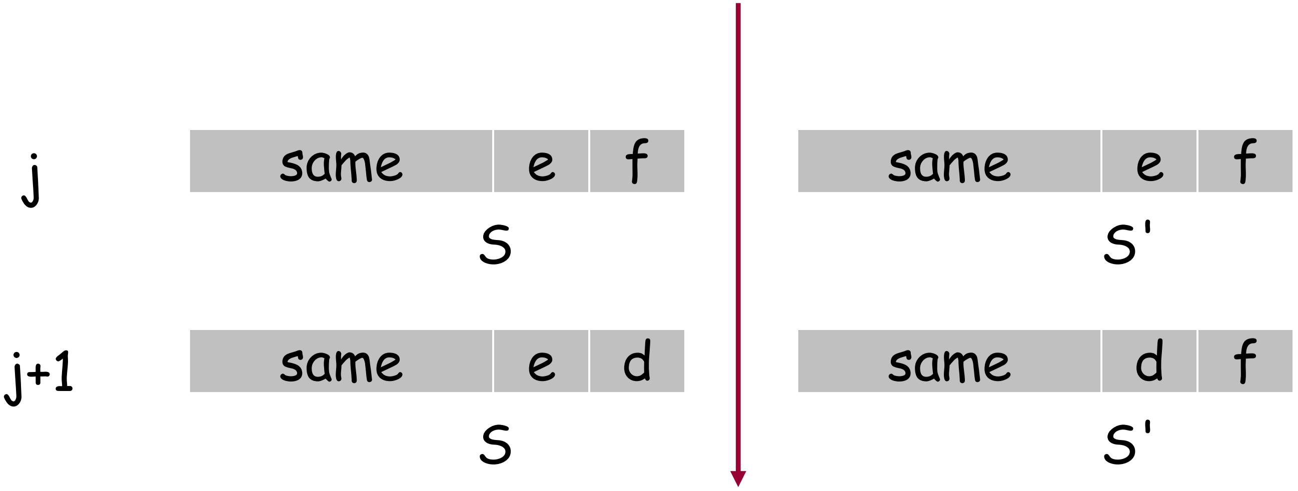
Let S be reduced schedule that satisfies invariant through j requests. We produce S' that satisfies invariant after j+1 requests.

- Consider $(j+1)^{st}$ request $d = d_{j+1}$.
- Since S and S_{FF} have agreed up until now, they have the same cache contents before request j+1.
- **Case 1:** (d is already in the cache). S' = S satisfies invariant.
- **Case 2:** (d is not in the cache and S and S_{FF} evict the same element).
  S' = S satisfies invariant.

Farthest-In-Future: Analysis

Pf. (continued)

- **Case 3:** (d is not in the cache; S_{FF} evicts e; S evicts f $\neq$ e).
  - begin construction of S' from S by evicting e instead of f

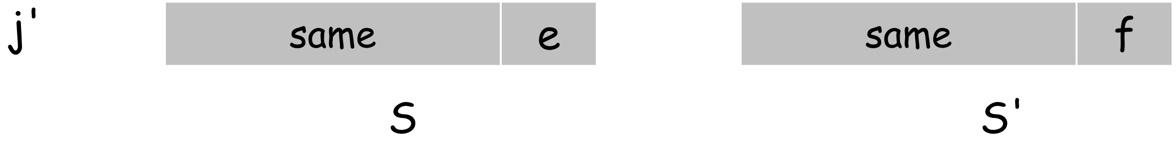


  - now S' agrees with S_{FF} on first j+1 requests; we show that having element f. in cache is no worse than having element e.

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



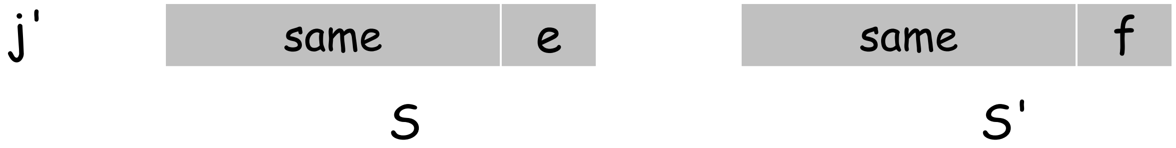
- **Case 3a:** g = e. Can't happen with Farthest-In-Future since there must be a request for f before e. (See S_{FF} definition)
- **Case 3b:** g = f. Element f can't be in cache of S, so let e' be the element that S evicts.
  - if e' = e, S' accesses f from cache; now S and S' have same cache
  - if e' ≠ e, S' evicts e' and brings e into the cache; now S and S' have the same cache

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with S_{FF} through step j+1

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

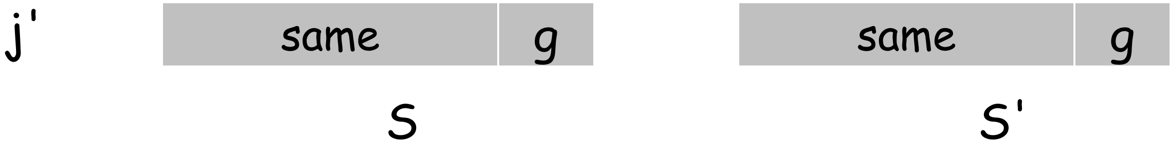
must involve e or f (or both)



otherwise S' would take the same action

- Case 3c: $g \neq e, f$. S must evict e.

  Make S' evict f; now S and S' have the same cache. ▪



Caching Perspective

Online vs. offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

FIFO. Evict page brought in least recently.
LIFO. Evict page brought in most recently.
LRU. Evict page whose most recent access was earliest.

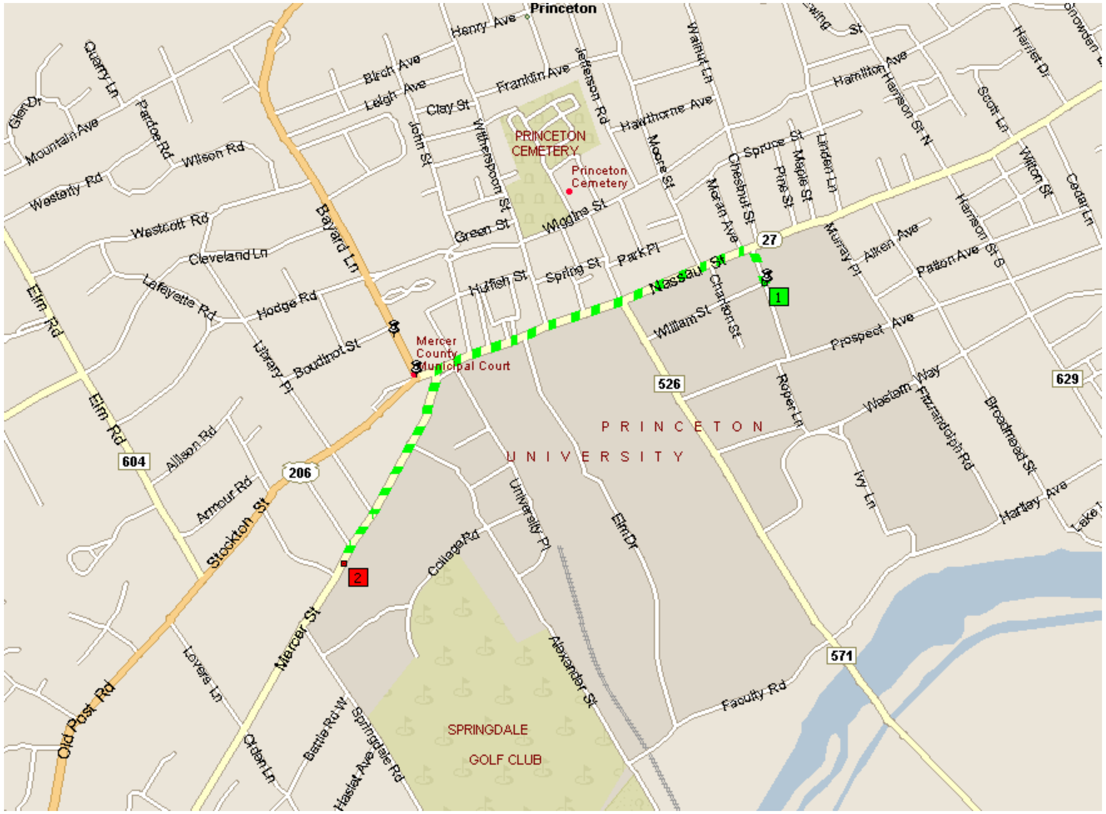
↑

FF with direction of time reversed!

Theorem. FF is optimal offline eviction algorithm.

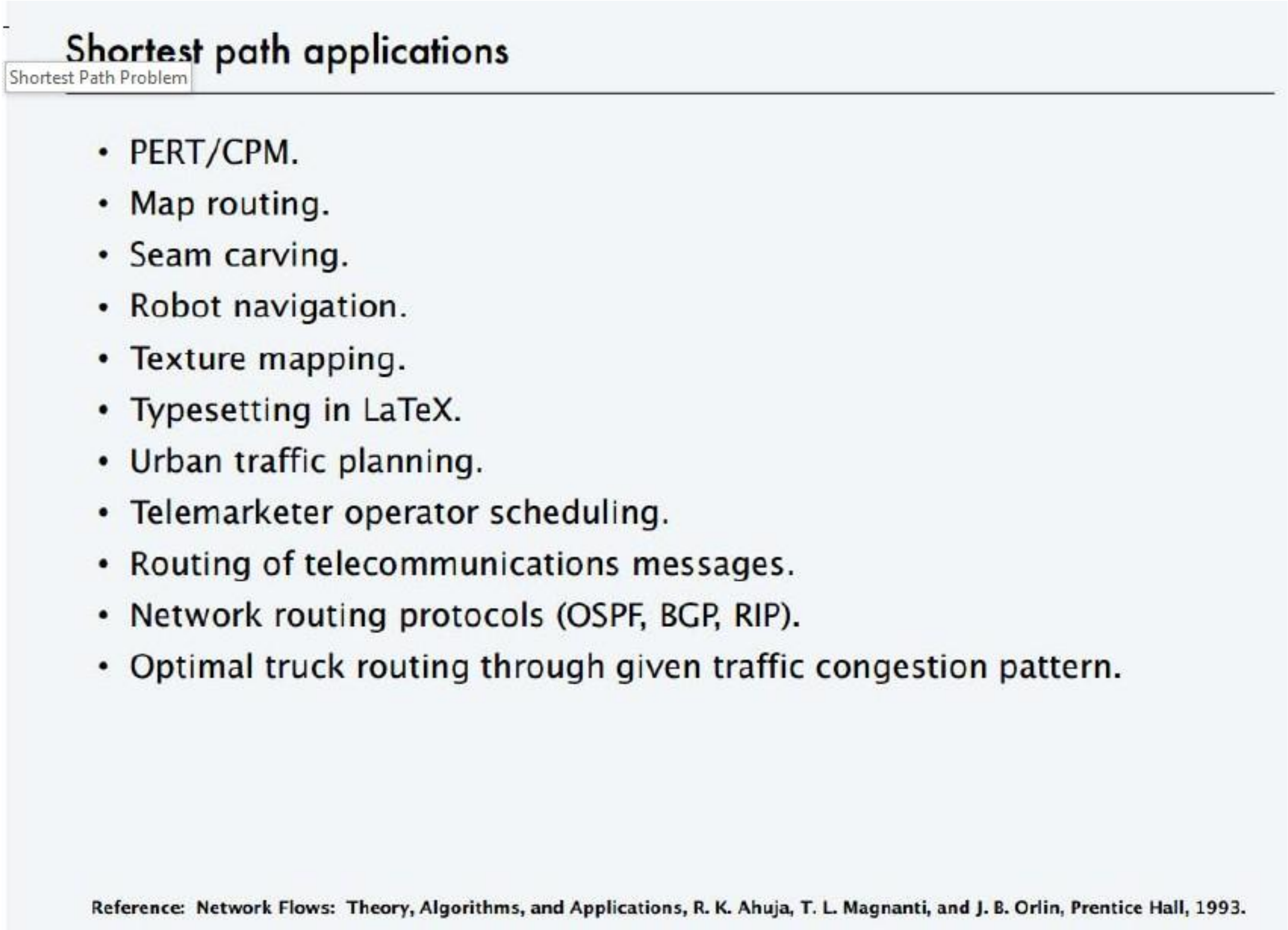
- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.

4.4 Shortest Paths in a Graph



shortest path from Princeton CS department to Einstein's house

Car navigation



Shortest Path Problem

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in LaTeX.
- Urban traffic planning.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

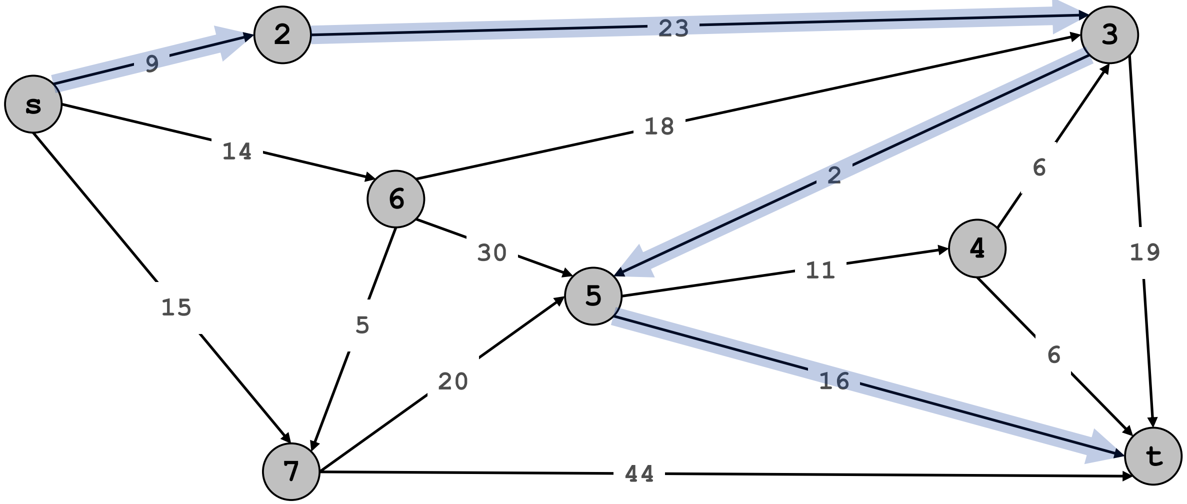
Shortest Path Problem

Shortest path network.

- Directed graph G = (V, E).
- Source s, destination t.
- Length ℓ_e = length of edge e.

Shortest path problem: find **shortest directed path** from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 50.

Dijkstra's Algorithm

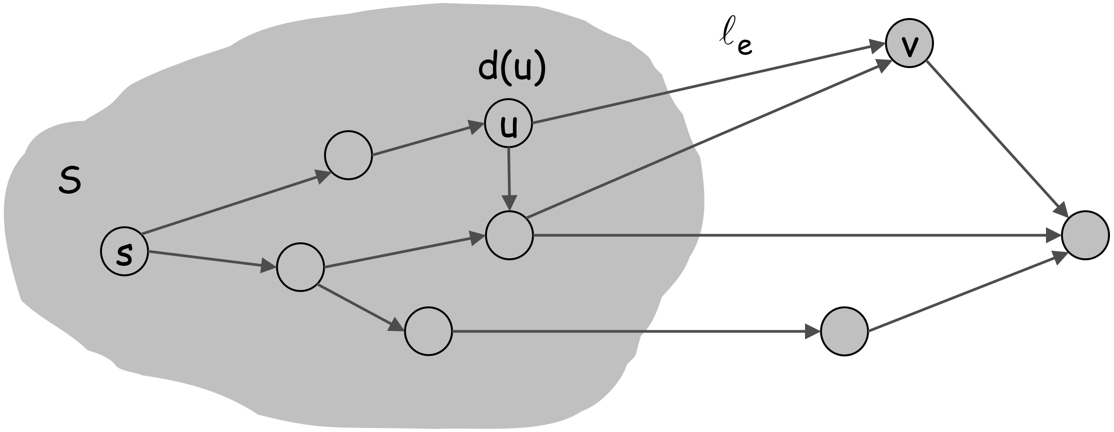
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v (i.e., v ∉ S) that minimizes

$$\pi(v) = \min_{e = (u,v)\,:\, u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = π(v).



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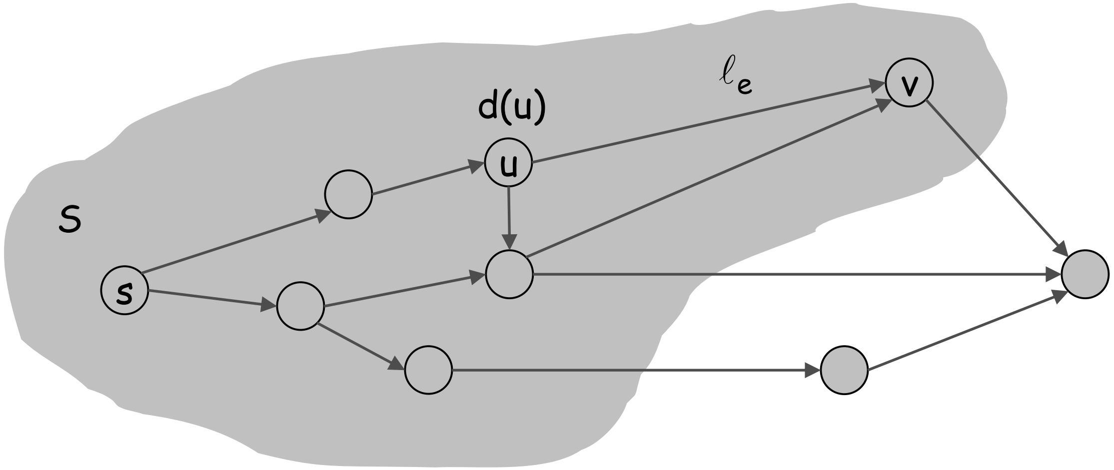


Dijkstra's Algorithm: Proof of Correctness

Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest s-u path.

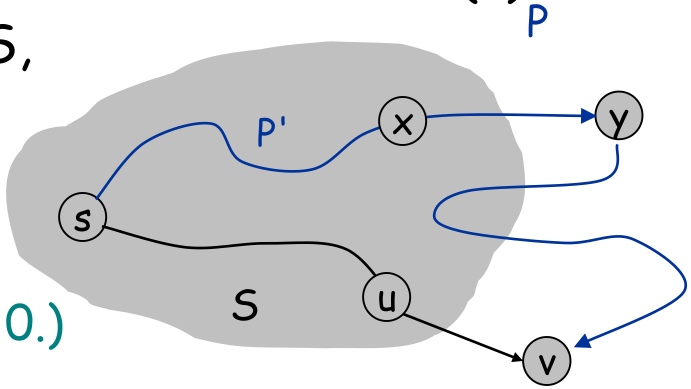
Pf. (by induction on $|S|$)

Base case: $|S| = 1$ is trivial.

Inductive hypothesis: Assume true for $|S| = k \ge 1$.

- Let v be next node added to S, and let u-v be the chosen edge.
- The shortest s-u path plus (u, v) is an s-v path of length $\pi(v)$.
- Consider any s-v path P. We'll see that it's no shorter than $\pi(v)$.
- Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it leaves S and path y-v has positive weight.

(Boundary Case: P has same length with len(y-v) = 0.)



$$\ell(P) \underset{\substack{\uparrow \\ \text{nonnegative} \\ \text{weights}}}{\ge} \ell(P') + \ell(x,y) \underset{\substack{\uparrow \\ \text{inductive} \\ \text{hypothesis}}}{\ge} d(x) + \ell(x, y) \underset{\substack{\uparrow \\ \text{defn of } \pi(y)}}{\ge} \pi(y) \underset{\substack{\uparrow \\ \text{Dijkstra chose v} \\ \text{instead of y}}}{\ge} \pi(v)$$

Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e=(u,v)\,:\,u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{ \ \pi(w), \ \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

PQ Operation	Dijkstra	Array	Binary heap	d-way Heap	Fib heap †
Insert	n	n	log n	$d \log_d n$	1
ExtractMin	n	n	log n	$d \log_d n$	log n
ChangeKey	m	1	log n	$\log_d n$	1
IsEmpty	n	1	1	1	1
Total		n^2	m log n	$m \log_{m/n} n$	m + n log n

† Individual ops are amortized bounds

Dijkstra's algorithm: efficient implementation

Critical optimization 1. For each unexplored node v, explicitly maintain $\pi(v)$ instead of computing directly from formula:

$$\pi(v) = \min_{e = (u,v)\,:\, u \in S} d(u) + \ell_e \ .$$

- For each $v \notin S$, $\pi(v)$ can only decrease (because S only increases).
- More specifically, suppose u is added to S and there is an edge (u, v) leaving u. Then, it suffices to update:

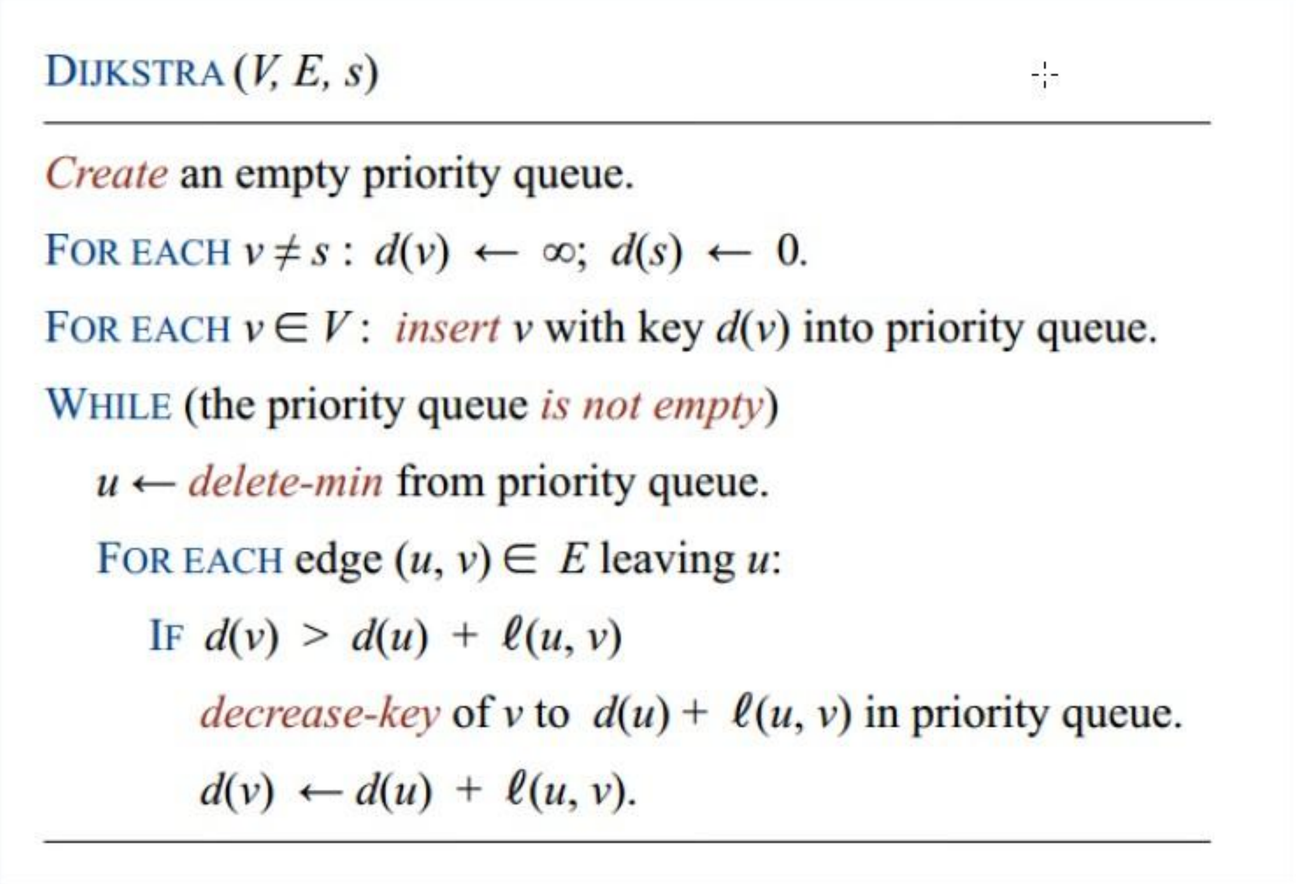
$$\pi(v) = \min \{ \pi(v),\ d(u) + \ell(u, v) \}$$

Critical optimization 2. Use a priority queue to choose the unexplored node that minimizes $\pi(v)$.

Dijkstra's algorithm: efficient implementation

Implementation.

- Algorithm stores $d(v)$ for each explored node v.
- Priority queue stores $\pi(v)$ for each unexplored node v.
- Recall: $d(u) = \pi(u)$ when u is deleted from priority queue.



```
DIJKSTRA ($V, E, s$)
Create an empty priority queue.
FOR EACH $v \neq s$ : $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.
FOR EACH $v \in V$ : insert $v$ with key $d(v)$ into priority queue.
WHILE (the priority queue is not empty)
    $u \leftarrow$ delete-min from priority queue.
    FOR EACH edge $(u, v) \in E$ leaving $u$:
        IF $d(v) > d(u) + \ell(u, v)$
            decrease-key of $v$ to $d(u) + \ell(u, v)$ in priority queue.
            $d(v) \leftarrow d(u) + \ell(u, v)$.
```

Dijkstra's algorithm: which priority queue?

Performance. Depends on PQ: n insert, n delete-min, m decrease-key.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci/Brodal best in theory, but not worth implementing.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
d-way heap (Johnson 1975)	$O(d \log_d n)$	$O(d \log_d n)$	$O(\log_d n)$	$O(m \log_{m/n} n)$
Fibonacci heap (Fredman-Tarjan 1984)	$O(1)$	$O(\log n)$ †	$O(1)$ †	$O(m + n \log n)$
Brodal queue (Brodal 1996)	$O(1)$	$O(\log n)$	$O(1)$	$O(m + n \log n)$

† amortized

Review of Dijkstra
(Non-negative Edge Weights)

Problem: Given a directed graph $G = (V, E)$ with edge-weight function $w : E \rightarrow$ R+, and a node s, find the shortest-path weight $\delta(s, v)$ (and a corresponding shortest path) from s to each v in V.

Greedy iterative approach

1. maintain a set S of vertices whose shortest-path distances from s are known.
2. at each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. update distance estimates of vertices adjacent to v.

Dijkstra's algorithm

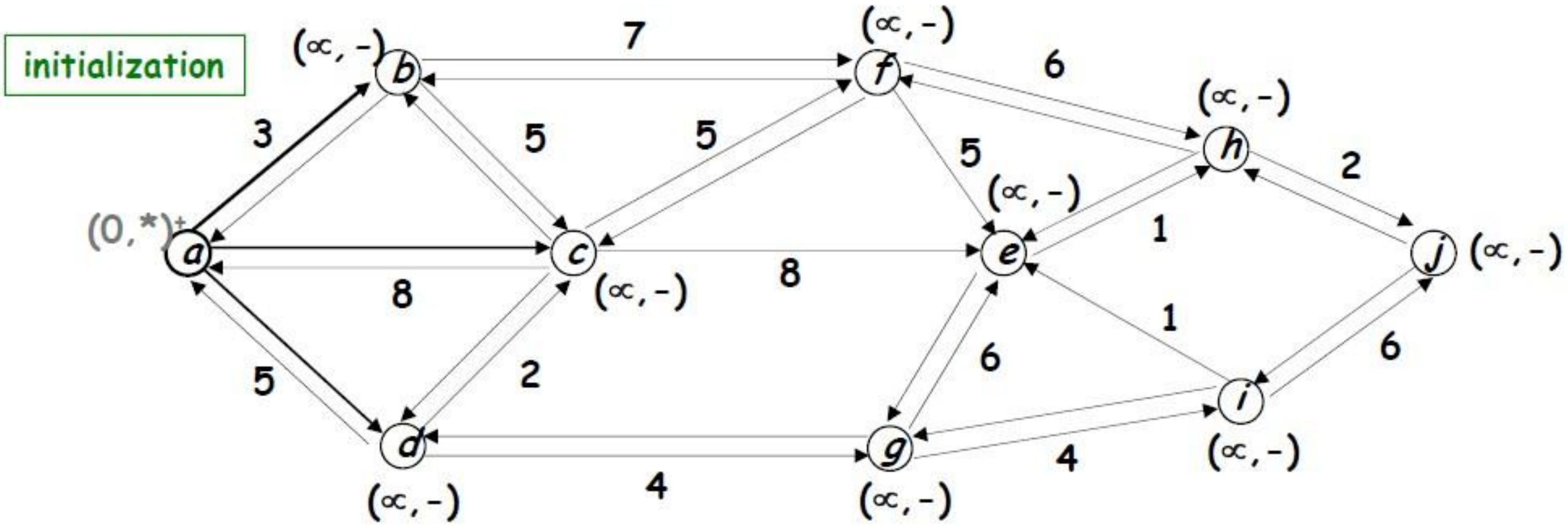
```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \emptyset$
$Q \leftarrow V$
while $Q \neq \emptyset$                ($Q$ min-priority queue for $V - S$)
    do $u \leftarrow$ DELETE-MIN($Q$)
       $S \leftarrow S \cup \{u\}$
       for each $v \in Adj[u]$
           do if $d[v] > d[u] + w(u, v)$
                 then $d[v] \leftarrow d[u] + w(u, v)$
```

initialization

relaxation steps

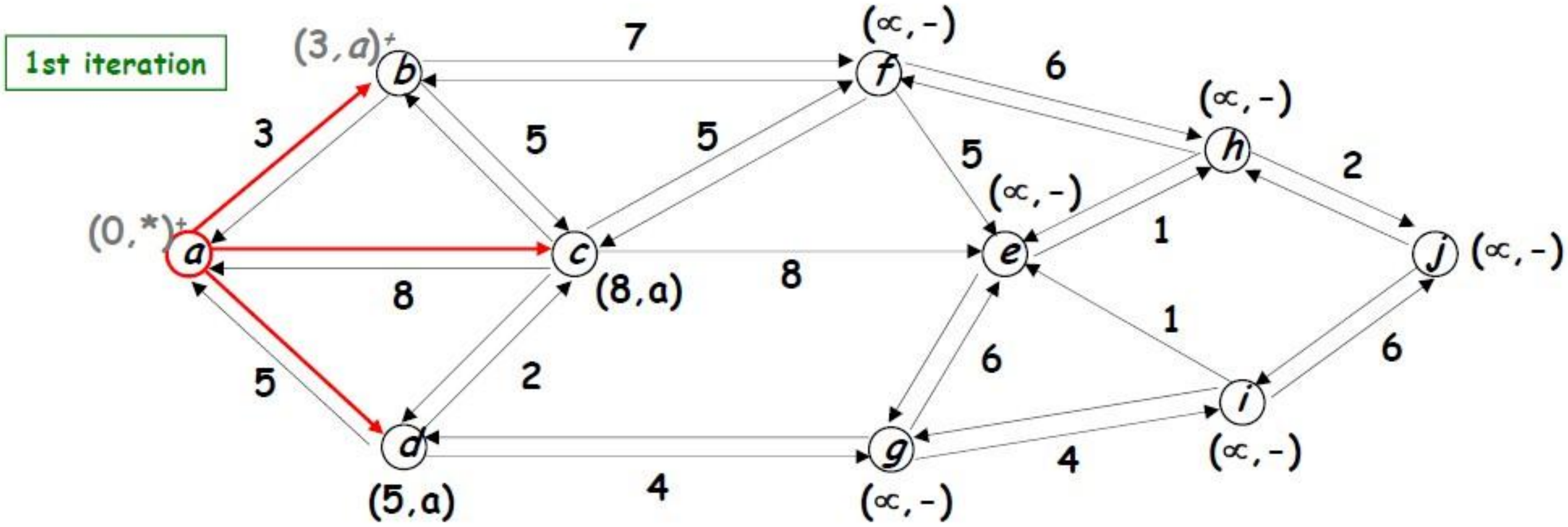
(Implicit DECREASE-KEY)

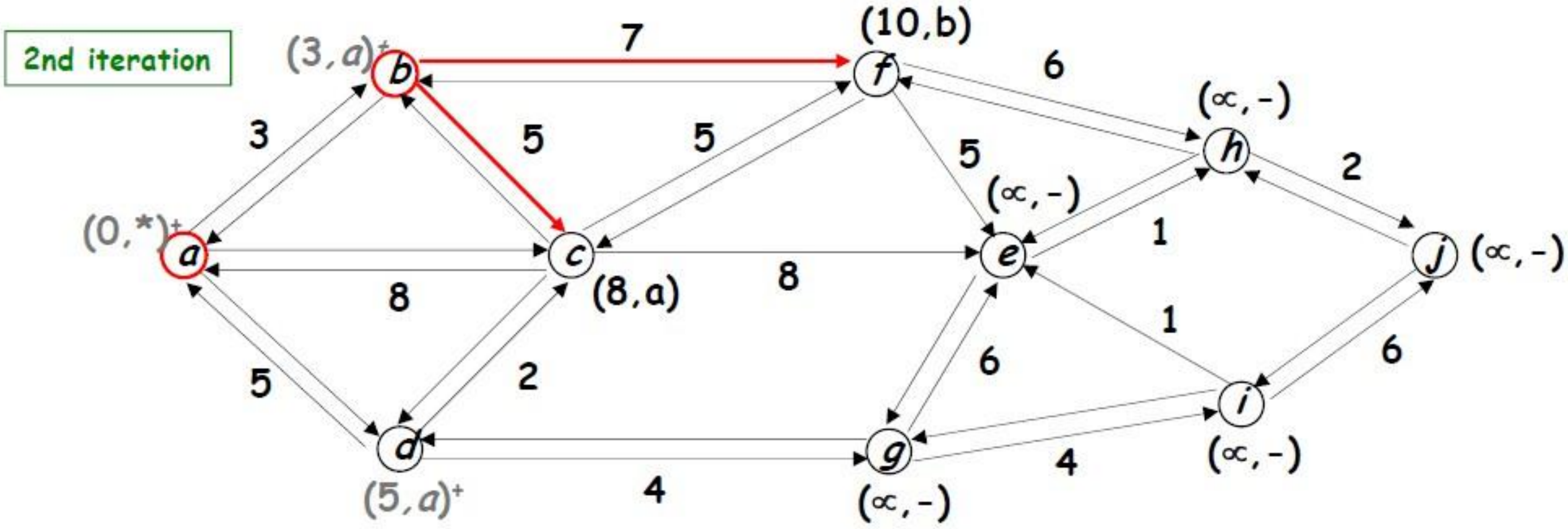
Dijkstra: Example (Min-cost, Min-Path)



$Q = V$, a = DELETE-MIN(Q)

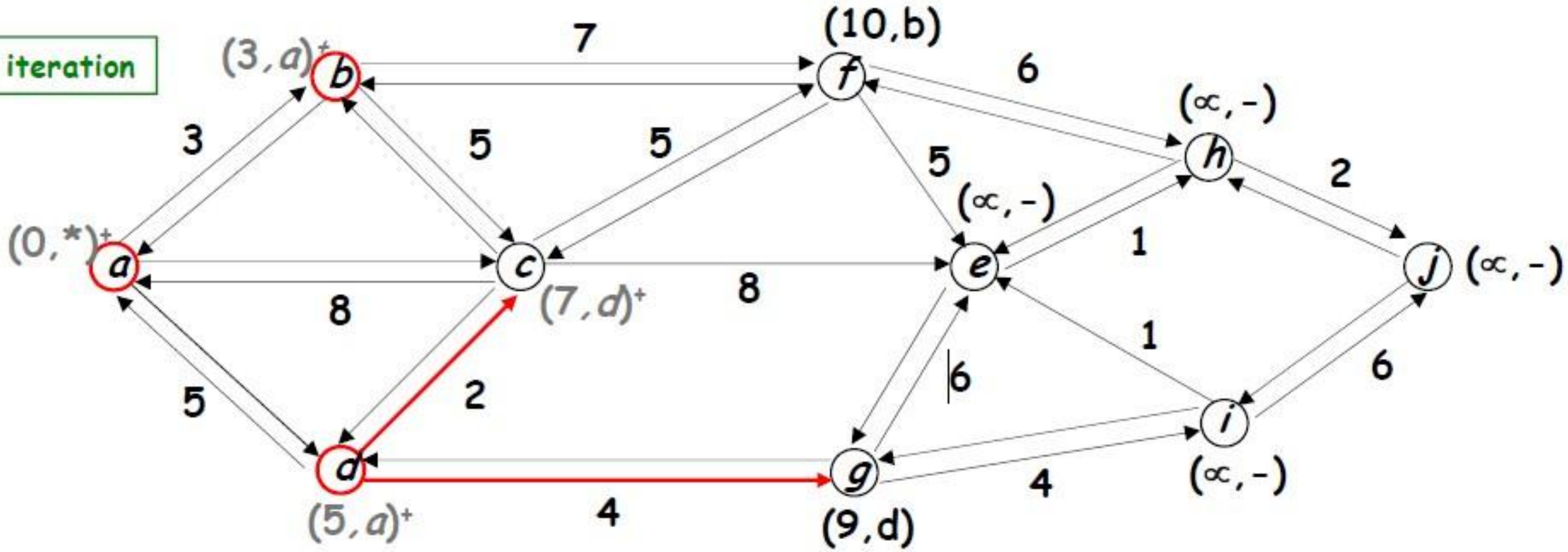
Dijkstra: Example



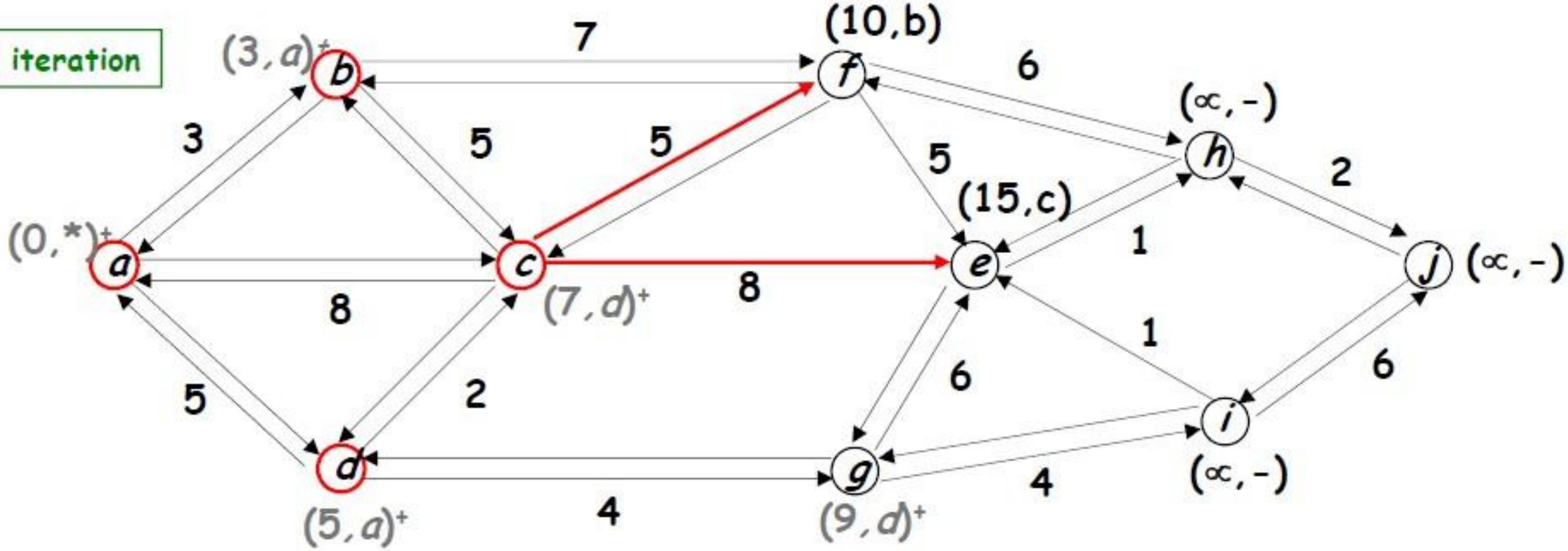


Dijkstra: Example

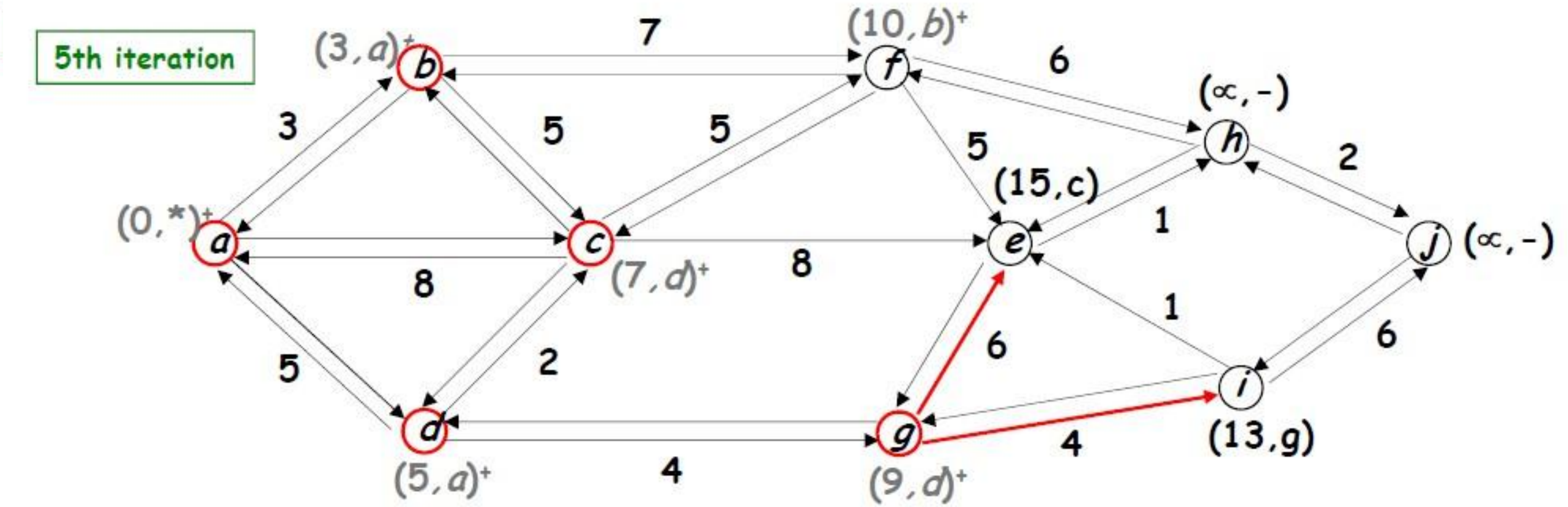
3rd iteration

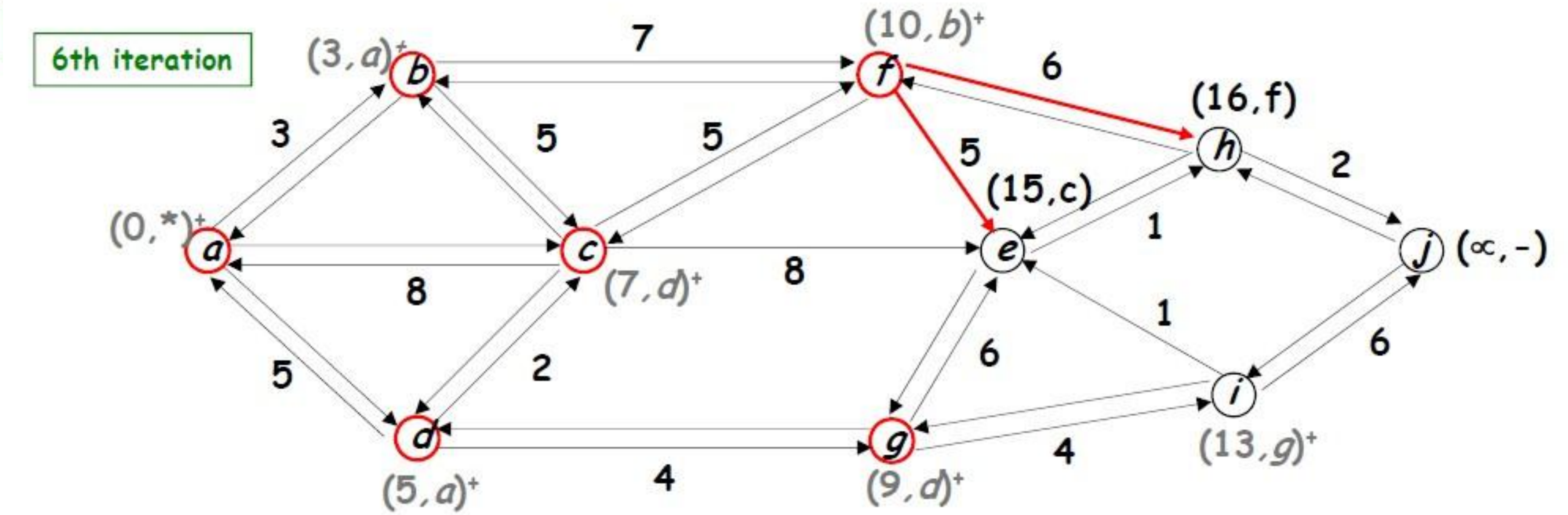


4th iteration

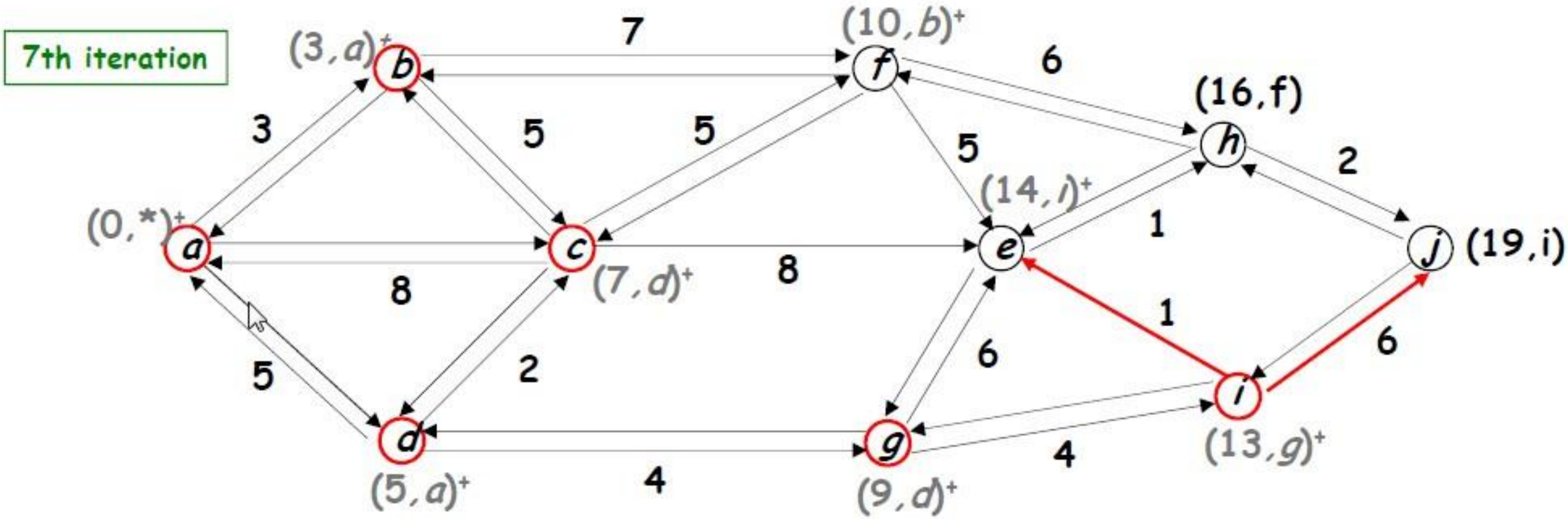


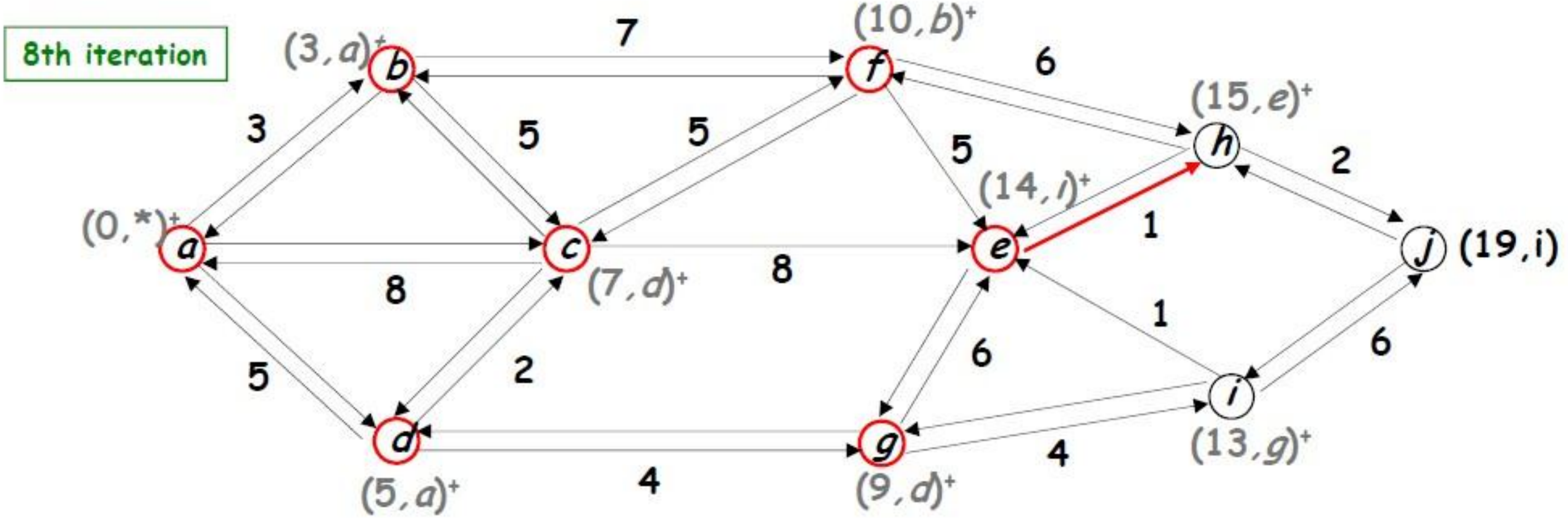
Dijkstra: Example



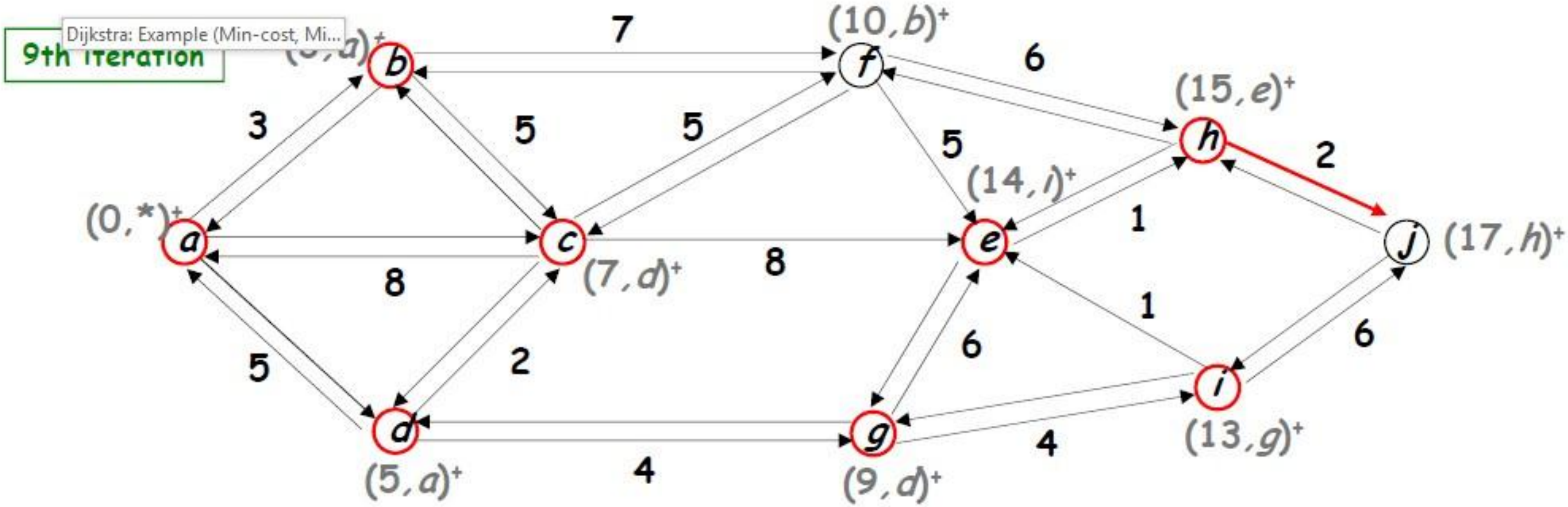


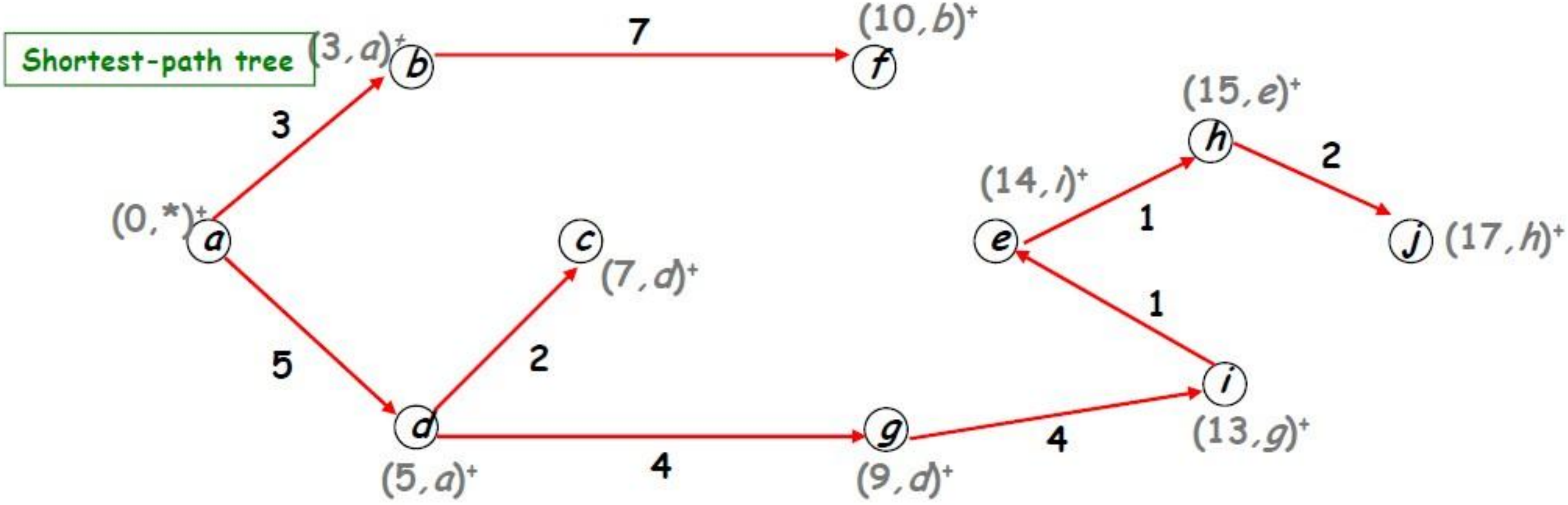
Dijkstra: Example





Dijkstra: Example





Complexity Analysis of Dijkstra

$|V|$ times

```
while $Q \neq \emptyset$
    do $u \leftarrow$ DELETE-MIN($Q$)
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
                then $d[v] \leftarrow d[u] + w(u, v)$
```

degree(u) times

DECREASE-KEY

$$\text{Time} = \Theta(n) \cdot T_{\text{DELETE-MIN}} + \Theta(m) \cdot T_{\text{DECREASE-KEY}}$$

$$\text{Time} = \Theta(n) \cdot T_{\textsc{Delete-Min}} + \Theta(m) \cdot T_{\textsc{Decrease-Key}}$$

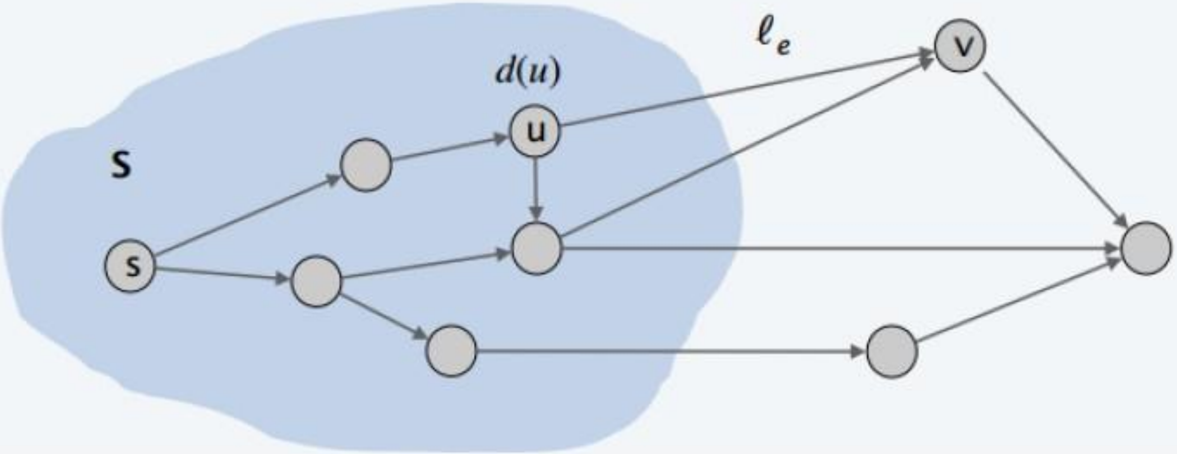
Q	$T_{\textsc{Delete-Min}}$	$T_{\textsc{Decrease-Key}}$	Total
array	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\lg n)$	$O(\lg n)$	$O(m \lg n)$
Fibonacci heap	$O(\lg n)$ amortized	$O(1)$ amortized	$O(m + n \lg n)$ worst case

Extensions of Dijkstra's algorithm

Dijkstra's algorithm and proof extend to several related problems:

- Shortest paths in undirected graphs: $d(v) \le d(u) + \ell(u, v)$.
- Maximum capacity paths: $d(v) \ge \min \{ \pi(u), c(u, v) \}$.
- Maximum reliability paths: $d(v) \ge d(u) \times \gamma(u, v)$.
- ...

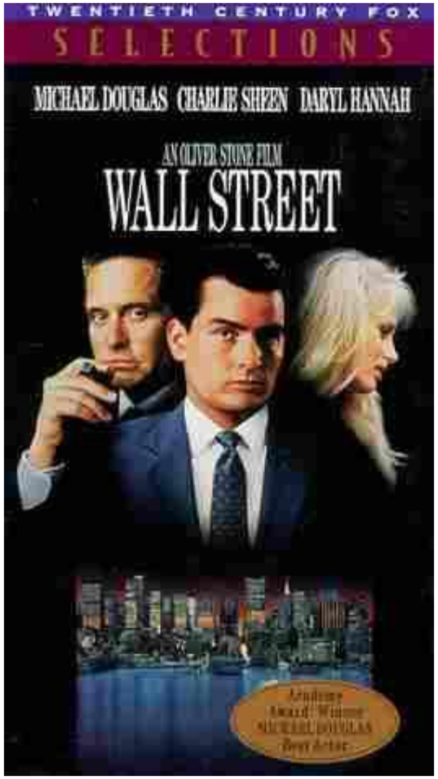
Key algebraic structure. Closed semiring (tropical, bottleneck, Viterbi).



Extra Slides

Coin Changing

> Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.
>
> \- *Gordon Gecko (Michael Douglas)*



Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: \$2.89.

Coin-Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: $c_1 < c_2 < … < c_n$.

  coins selected
S ← $\phi$
while ($x \neq 0$) {
    let k be largest integer such that $c_k \leq x$
    if ($k = 0$)
        return "no solution found"
    $x \leftarrow x - c_k$
    $S \leftarrow S \cup \{k\}$
}
return S
```

Q. Is cashier's algorithm optimal?

Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greed is optimal for U.S. coinage: 1, 5, 10, 25, 100.

Pf. (by induction on x)

- Consider optimal way to change $c_k \leq x < c_{k+1}$: greedy takes coin k.
- We claim that any optimal solution must also take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm. ▪

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, …, k-1 in any OPT
1	1	$P \leq 4$	-
2	5	$N \leq 1$	4
3	10	$N + D \leq 2$	4 + 5 = 9
4	25	$Q \leq 3$	20 + 4 = 24
5	100	no limit	75 + 24 = 99

Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

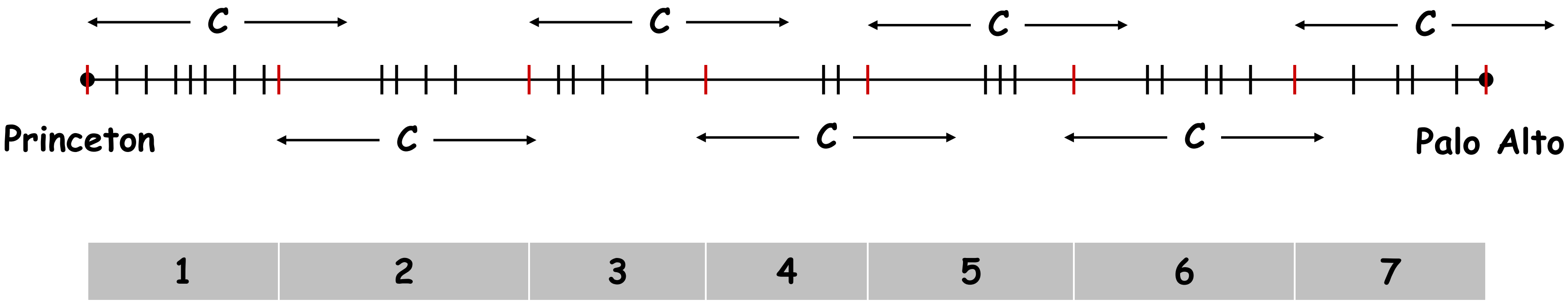
Selecting Breakpoints

Selecting Breakpoints

Selecting breakpoints.

- Road trip from Princeton to Palo Alto along fixed route.
- Refueling stations at certain points along the way.
- Fuel capacity = C.
- Goal: makes as few refueling stops as possible.

Greedy algorithm. Go as far as you can before refueling.



Selecting Breakpoints: Greedy Algorithm

Truck driver's algorithm.

```
Sort breakpoints so that: $0 = b_0 < b_1 < b_2 < ... < b_n$
$= L$

S ← {0} ⟵ breakpoints selected
x ← 0   ⟵ current location

while ($x \neq b_n$)
    let p be largest integer such that $b_p \leq x + C$
    if ($b_p = x$)
        return "no solution"
    $x \leftarrow b_p$
    $S \leftarrow S \cup \{p\}$
return S
```

Implementation. O(n log n)

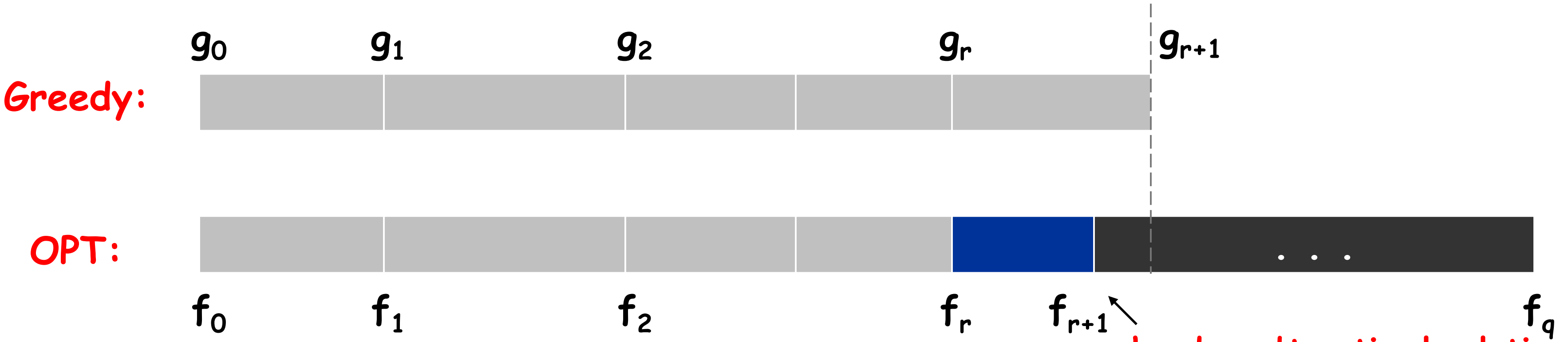
- Use binary search to select each breakpoint p.

Selecting Breakpoints: Correctness

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $0 = g_0 < g_1 < \ldots < g_p = L$ denote set of breakpoints chosen by greedy.
- Let $0 = f_0 < f_1 < \ldots < f_q = L$ denote set of breakpoints in an optimal solution with $f_0 = g_0, f_1 = g_1, \ldots, f_r = g_r$ for largest possible value of r.
- Note: $g_{r+1} > f_{r+1}$ by greedy choice of algorithm.



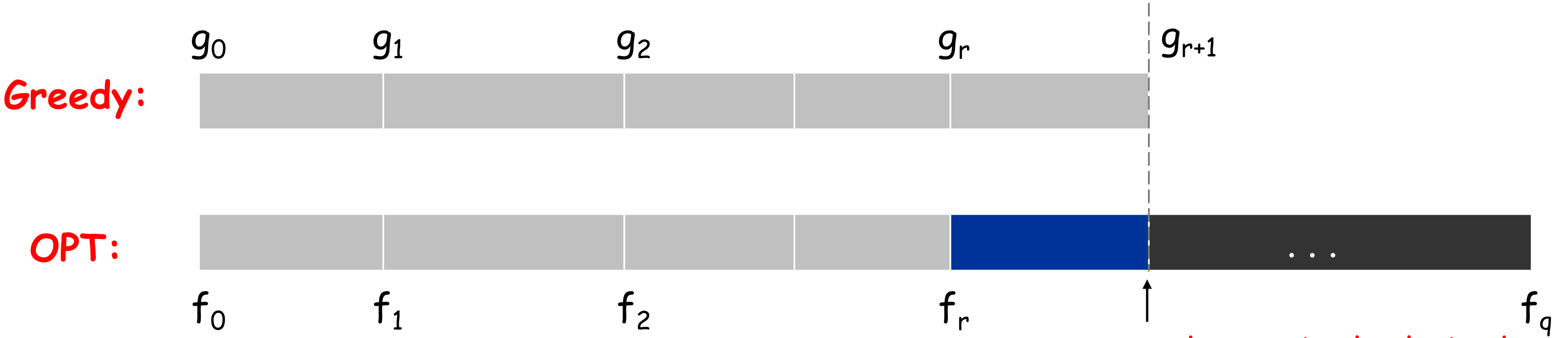
why doesn't optimal solution drive a little further?

Selecting Breakpoints: Correctness

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $0 = g_0 < g_1 < \ldots < g_p = L$ denote set of breakpoints chosen by greedy.
- Let $0 = f_0 < f_1 < \ldots < f_q = L$ denote set of breakpoints in an optimal solution with $f_0 = g_0$, $f_1 = g_1$, . . . , $f_r = g_r$ for largest possible value of r.
- Note: $g_{r+1} > f_{r+1}$ by greedy choice of algorithm.



another optimal solution has one more breakpoint in common ⇒ contradiction

Edsger W. Dijkstra

The question of whether computers can think is like the question of whether submarines can swim.

Do only what only you can do.

In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.

The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.

APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.