1) a. Give the derivation of what L converges to in probability?

$f_Y(y) = \frac{1}{2b} \exp^{-}\left(\frac{|y-\mu|}{b}\right)$ $\quad -\infty < y < \infty \quad -\infty < \mu < \infty \quad b > 0 \quad \mu = 0, b = 5$

⇒ plug in $\mu = 0$, $b = 5$, rewrite

$f_Y(y) = \frac{1}{10} \exp^{-}\left(\frac{|y|}{5}\right)$

(must stay negative in exp) ⇒ two cases.

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} \frac{y}{10} e^{-|y|/5}\, dy = \frac{1}{10}\int_{-\infty}^{0} y e^{y/5}\, dy + \frac{1}{10}\int_{0}^{\infty} y e^{-y/5}\, dy$$

for first case:

integrate by parts: recall: $\int u\, dv = uv - \int v\, du$

$u = y \quad dv = e^{y/5}\, dy$

$du = dy \quad v = 5e^{y/5}$

$$= \frac{1}{10}\left(5ye^{y/5} - \int_{-\infty}^{0} 5e^{y/5}\, dy\right)$$

$$\int_{-\infty}^{0} 5e^{y/5}\, dy = 5\int_{-\infty}^{0} e^{y/5}\, dy \qquad u = y/5 \quad du = \frac{1}{5}dy$$

$$5ye^{y/5}\Big|_{-\infty}^{0} = 0 - 0 = \underline{0}$$

$$= 25\int_{-\infty}^{0} e^{u}\, du$$

$$= 25e^{y/5}\Big|_{-\infty}^{0} = 25 - 0 = \underline{25}$$

$$= \frac{1}{10}(0 - 25) = -2.5$$

for second case: $\frac{1}{10}\int_0^{\infty} ye^{-y/5}\, dy$

integrate by parts: $u = -y/5 \quad du = -\frac{1}{5}dy$

$u = y \quad dv = e^{-y/5}$ $\Rightarrow -5ye^{-y/5}\Big|_0^{\infty} + 5\int_0^{\infty} e^{-y/5}\, dy$

$du = dy \quad v = -5e^{-y/5}$ $\qquad = \int_0^{\infty} ye^{-y/5} =$

$$= 0 - 25\left(e^{-y/5}\right)\Big|_0^{\infty} = -25(0-1) = 25$$

$$\frac{1}{10}(25) = 2.5$$

Combining both cases we get:

$$E(Y) = -2.5 + 2.5 = 0.$$

Now find Var(y).

$\Rightarrow Var(Y) = E(Y^2) - [\cancel{E(Y)}]^2 \quad \Rightarrow E(Y)=0$

$= E(Y^2)$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \frac{1}{10}\int_{-\infty}^{0} y^2 e^{y/5}\,dy + \frac{1}{10}\int_{0}^{\infty} y^2 e^{-y/5}\,dy.$$

First case:

$$\frac{1}{10}\int_{-\infty}^{0} y^2 e^{y/5}\,dy$$

$u = y^2 \qquad dv = e^{y/5}\,dy \Rightarrow \frac{1}{10}\left(\underbrace{5y^2 e^{y/5}\Big|_{-\infty}^{0}}_{=0} - 10\underbrace{\int_{-\infty}^{0} y e^{y/5}\,dy}_{=-25}\right)$

$du = 2y\,dy \qquad v = 5e^{y/5}$

$$= \frac{1}{10}\left(0 - 10(-25)\right) = 25$$

Second case:

$$\frac{1}{10}\int_{0}^{\infty} y^2 e^{-y/5}\,dy$$

$u = y^2 \qquad dv = e^{-y/5}\,dy \Rightarrow \frac{1}{10}\left(\underbrace{-5y^2 e^{-y/5}\Big|_{0}^{\infty}}_{=0} + 10\underbrace{\int_{0}^{\infty} y e^{-y/5}\,dy}_{=25}\right)$

$du = 2y\,dy \qquad v = -5e^{-y/5}$

$$= \frac{1}{10}\left(0 + 10(25)\right) = 25$$

Adding both cases:

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = 25 + 25 = 50 = Var(Y)$$

$L = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 \xrightarrow{P} E(Y_i^2)$ by Law of Large Numbers.

Therefore:

$$\boxed{L \xrightarrow{P} E(Y_i^2) \xrightarrow{P} 50}$$

b) $K = \sqrt{L} \Rightarrow$ meaning: $K \xrightarrow{P} \sqrt{50} = 5\sqrt{2}$: (by continuity theorem)

$$\boxed{K \xrightarrow{P} 5\sqrt{2}}$$

c) $F_Y(y) = \begin{cases} \int_{-\infty}^{t} f_Y'(y)\,dt & \text{for } y \geq \mu \\ \int_{-\infty}^{t} f_Y(y)\,dt & \text{for } y \leq \mu \end{cases}$

$\Rightarrow$ for $y \leq \mu$

$$\int_{-\infty}^{Y} \frac{1}{2b} \cdot e^{-\left(\frac{t-\mu}{b}\right)}\,dt = \frac{1}{2b}\int_{-\infty}^{Y} e^{-\left(\frac{t-\mu}{b}\right)} \overset{\text{positive exponent}}{} = \frac{1}{2b}\left[be^{\left(\frac{t-\mu}{b}\right)}\right]_{-\infty}^{Y}$$

$$= \frac{1}{2}\left[\exp\left(\frac{t-\mu}{b}\right)\right]_{-\infty}^{Y} = \underline{\frac{1}{2}\exp\left(\frac{Y-\mu}{b}\right) \text{ for } y \leq \mu}$$

$\Rightarrow$ for $y \geq \mu$ $\Rightarrow$ split into two cases $\Rightarrow$ exponent negative.

$$= \frac{1}{2b}\left(\int_{-\infty}^{\mu} \exp-\left(\frac{t-\mu}{b}\right)dt + \int_{\mu}^{Y} \exp-\left(\frac{t-\mu}{b}\right)dt.\right)$$

$$= -\frac{1}{2b}\cdot be^{-\left(\frac{t-\mu}{b}\right)}\Big|_{-\infty}^{\mu} + \frac{1}{2b}\cdot -be^{-\left(\frac{t-\mu}{b}\right)}\Big|_{\mu}^{Y}$$

$$= -\frac{1}{2}(-1) - \frac{1}{2}\left(e^{-\left(\frac{Y-\mu}{b}\right)} - 1\right) = \frac{1}{2} - \frac{1}{2}e^{\left(\frac{\mu-Y}{b}\right)} + \frac{1}{2}$$

$$= \underline{1 - \frac{1}{2}e^{\left(\frac{\mu-Y}{b}\right)} \quad \text{if } X \geq \mu}$$

$\therefore$ Therefore:

$$F_Y(y) = \begin{cases} \frac{1}{2}\exp\left(\frac{X-\mu}{b}\right) & \text{if } X \leq \mu \\ 1 - \frac{1}{2}\exp\left(\frac{\mu-Y}{b}\right) & \text{if } X \geq \mu \end{cases}$$

d) Generating RVs → need inverse fxns.

Recall CDF

$$\begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right) & x \le \mu \\ 1-\frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right) & x \ge \mu \end{cases}$$

for $x \le \mu$

$\rightarrow u = \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right)$

$\rightarrow 2u = \exp\left(\frac{x-\mu}{b}\right) \rightarrow \ln(2u) = \frac{x-\mu}{b} = b\ln(2u) + \mu = x$

for $x \ge \mu$

$\rightarrow u = 1-\frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right)$

$\ln(2-2u) = -\frac{x-\mu}{b} \rightarrow -b\ln(2-2u) + \mu = x$

$$\begin{cases} \mu - b\ln(2-2u) & x \ge \mu \\ \mu + b\ln(2u) & x \le \mu \end{cases}$$

$x-\mu < 0$

for $x \ge \mu \Rightarrow$
$0 \le -b\ln(2-2u)$
$0 \ge \ln(2-2u)$
$1 \ge 2-2u$
$-1 \ge -2u$
$u \ge \frac{1}{2}$

for $x \le \mu$
$0 \ge x-\mu = b\ln(2u)$
$0 \ge \ln(2u)$
$1 \ge 2u$
$u \le \frac{1}{2}$

3) Kurtosis is the 4^{th} standardized moment, defined as:

$$Kurt[Y] = E\left[\left(\frac{Y-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma_4} = \frac{E[(Y-\mu)^4]}{(E[(Y-\mu)^2])^2} = 6 \text{ (given)}$$

$\mu = 0$ (given)

$$\Rightarrow \frac{E(y)^4}{E[(y)^2]^2} = 6 \Rightarrow [E(y)^4] = 6E[(y)^2]^2$$

must rewrite

substitute

Recall: $Var(L) = Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right)$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i^2) = \frac{1}{n^2}\sum_{i=1}^{n}\left(E(Y_i^2)^2 - [E(Y_i^2)]^2\right) = \frac{1}{n^2}\sum_{i=1}^{n}\left(E(Y_i^4) - [E(Y_i^2)]^2\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(6E[(Y_i^2)]^2 - E[(Y_i^2)]^2\right) = \frac{1}{n^2}\sum_{i=1}^{n} 5E[(Y_i^2)]^2 = Var(L)$$

By CLT: $\frac{L-\mu}{\sqrt{Var(L)}} \xrightarrow{d} N(0,1)$

$$Var(L) = \frac{1}{n^2}\sum_{i=1}^{n} 5E[(Y_i^2)]^2$$

General form for $E[(Y_i^2)]\big|_{\mu=0} = 2b^2$

$$\Rightarrow = \frac{1}{n^2}\sum_{i=1}^{n} 5(2b^2)^2$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} 5(4b^4) = \frac{1}{n^2}\sum_{i=1}^{n}(20b^4)$$

$$= \frac{1}{n^2}(20b^4 n) = 20b^4/n$$

$$\Rightarrow \frac{L-0}{\sqrt{\frac{20b^4}{n}}} = \frac{L}{\frac{2b^2\sqrt{5}}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

Derivation for general form is on the next page.

Derivation of $E(Y_i^2)$ with arbitrary b. and $\mu = 0$.

$$f_Y(y) = \frac{1}{2b}\exp\left(\frac{-|y|}{b}\right) \qquad \mu = 0, \; -\infty < y < \infty,$$

$$= \int_{-\infty}^{\infty} \frac{1}{2b} y^2 \exp\left(\frac{-|y|}{b}\right) = \frac{1}{2b}\int_{-\infty}^{0} y^2 \exp\left(\frac{y}{b}\right) + \frac{1}{2b}\int_{0}^{\infty} y^2 \exp\left(\frac{-y}{b}\right)$$

First case: $\frac{1}{2b}\int_{-\infty}^{0} y^2 \exp\left(\frac{y}{b}\right) \Rightarrow \frac{1}{2b}\left(\underbrace{by^2 e^{y/b}\Big|_{-\infty}^{0}}_{0} - 2b\underbrace{\int_{-\infty}^{0} ye^{y/b}}_{-b^2 \,\to\, \text{Inferred from 1a)}}\right)$

$$= \frac{1}{2b}(-2b)(-b^2) = b^2$$

Second case: $\frac{1}{2b}\int_{0}^{\infty} y^2 \exp\left(\frac{-y}{b}\right) \Rightarrow \frac{1}{2b}\left(\underbrace{-by^2 e^{-y/b}\Big|_{0}^{\infty}}_{0} + 2b\underbrace{\int_{0}^{\infty} ye^{-y/b}}_{b^2 \,\to\, 1a)}\right)$

$$= \frac{1}{2b}2b(b^2) = b^2$$

$$= E(Y_i)^2 = 2b^2$$

Plug into Var(L) equation in previous page