Finite Difference Scheme for Multi-Asset Black Scholes PDE

Abhas Jain

Contents

Solution using Finite Difference Scheme

- The n-asset Black Scholes equation is:
  - $$\frac{\partial u(\mathbf{s},t)}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} s_i s_j \frac{\partial^2 u(\mathbf{s},t)}{\partial s_i \partial s_j} + r\sum_{i=1}^{n} s_i \frac{\partial u(\mathbf{s},t)}{\partial s_i} = ru(\mathbf{s},t),$$

- Using the following operator, the 3-asset BS equation can be re-written as:
  - $$\mathcal{L}_{BS}u = \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\sigma_z^2 z^2 \frac{\partial^2 u}{\partial z^2} + \rho_{xy}\sigma_x\sigma_y xy \frac{\partial^2 u}{\partial x \partial y} + \rho_{yz}\sigma_y\sigma_z yz \frac{\partial^2 u}{\partial y \partial z} + \rho_{zx}\sigma_z\sigma_x zx \frac{\partial^2 u}{\partial z \partial x} + rx\frac{\partial u}{\partial x} + ry\frac{\partial u}{\partial y} + rz\frac{\partial u}{\partial z} - ru.$$
  - $$\frac{\partial u}{\partial \tau} = \mathcal{L}_{BS}u \quad \text{for } (x,y,z,\tau) \in \Omega \times (0,T], \quad u(x,y,z,0) = u_T(x,y,z)$$

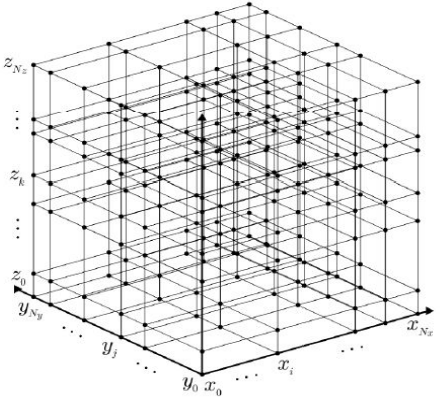
Domain Discretization and Boundary Conditions

❖ The domain is discretized with a non-uniform grid step, i.e.,

➢ $h_i^x = x_{i+1} - x_i$

➢ $h_j^y = y_{j+1} - y_j$

➢ $h_k^z = z_{k+1} - z_k$



❖ At the left end, Dirichlet conditions are used, i.e., $u(0,y,z,\tau) = u(x,0,z,\tau) = u(x,y,0,\tau) = 0$

❖ At the right end, homogeneous Neumann boundary conditions are considered:

$$\frac{\partial u}{\partial x}(L,y,z,\tau) = \frac{\partial u}{\partial y}(x,M,z,\tau) = \frac{\partial u}{\partial z}(x,y,N,\tau) = 0$$

for $0 \le x \le L,\ 0 \le y \le M,\ 0 \le z \le N,\ 0 \le \tau \le T.$

❖ This type of non-uniform discretization is something different from our usual theory content.

Derivatives under non-uniform discretization

- The 1st order derivative is defined as:

$$D_x u_{ijk} = -\frac{h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)} u_{i-1,jk} + \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} u_{ijk} + \frac{h_{i-1}^x}{h_i^x(h_{i-1}^x + h_i^x)} u_{i+1,jk},$$

- The 2nd order derivative is defined as:

$$D_{xx} u_{ijk} = \frac{2}{h_{i-1}^x(h_{i-1}^x + h_i^x)} u_{i-1,jk} - \frac{2}{h_{i-1}^x h_i^x} u_{ijk} + \frac{2}{h_i^x(h_{i-1}^x + h_i^x)} u_{i+1,jk},$$

- The mixed derivative is defined as:

$$D_{xy} u_{ijk} = \frac{u_{i+1,j+1,k} - u_{i-1,j+1,k} - u_{i+1,j-1,k} + u_{i-1,j-1,k}}{h_i^x h_j^y + h_{i-1}^x h_j^y + h_i^x h_{j-1}^y + h_{i-1}^x h_{j-1}^y},$$

Contents

Operator Splitting Method

- A numerical method to compute solutions of a differential equation.
- The method can be understood as a process of three steps:
  - The differential equation is split into multiple parts over a time step
  - Solution to each part is computed separately
  - All solutions are combined to form original solution

Operator Splitting Method

- Consider the option has *m* underlying assets and the price at time level *n*, *u(n)* is known. The Black Scholes PDE is of order *m* in space.

- The basic idea of operator splitting is to split the finite difference equations into *m* such that we get *m* discrete equations solved implicitly one after another to approximate *u(n+1)*.

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\Delta t/2} = \frac{\left(\delta_x^2 u_{ij}^{n+1/2} + \delta_y^2 u_{ij}^{n}\right)}{\Delta x^2}$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\Delta t/2} = \frac{\left(\delta_x^2 u_{ij}^{n+1/2} + \delta_y^2 u_{ij}^{n+1}\right)}{\Delta y^2}$$

$$\frac{u_{ijk}^{n+\frac{1}{3}} - u_{ijk}^{n}}{\Delta\tau} = (\mathcal{L}_{BS}^{x} u)_{ijk}^{n+\frac{1}{3}},$$

$$\frac{u_{ijk}^{n+\frac{2}{3}} - u_{ijk}^{n+\frac{1}{3}}}{\Delta\tau} = \left(\mathcal{L}_{BS}^{y} u\right)_{ijk}^{n+\frac{2}{3}},$$

$$\frac{u_{ijk}^{n+1} - u_{ijk}^{n+\frac{2}{3}}}{\Delta\tau} = (\mathcal{L}_{BS}^{z} u)_{ijk}^{n+1},$$

Discrete Difference Operators

❖ Using operator splitting, the discrete difference operator is defined as:

$$
\begin{aligned}
(\mathcal{L}_{BS}^{x}u)_{ijk}^{n+\frac{1}{3}} &= \frac{(\sigma_x x_i)^2}{2} D_{xx} u_{ijk}^{n+\frac{1}{3}} + r x_i D_x u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3}\sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^{n} \\
&+ \frac{1}{3}\sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^{n} + \frac{1}{3}\sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^{n} - \frac{1}{3} r u_{ijk}^{n+\frac{1}{3}}, \\
\left(\mathcal{L}_{BS}^{y}u\right)_{ijk}^{n+\frac{2}{3}} &= \frac{(\sigma_y y_j)^2}{2} D_{yy} u_{ijk}^{n+\frac{2}{3}} + r y_j D_y u_{ijk}^{n+\frac{2}{3}} + \frac{1}{3}\sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^{n+\frac{1}{3}} \\
&+ \frac{1}{3}\sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3}\sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^{n+\frac{1}{3}} - \frac{1}{3} r u_{ijk}^{n+\frac{2}{3}} \\
(\mathcal{L}_{BS}^{z}u)_{ijk}^{n+1} &= \frac{(\sigma_z z_k)^2}{2} D_{zz} u_{ijk}^{n+1} + r z_k D_z u_{ijk}^{n+1} + \frac{1}{3}\sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^{n+\frac{2}{3}} \\
&+ \frac{1}{3}\sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^{n+\frac{2}{3}} + \frac{1}{3}\sigma_z \sigma_x \rho_{zx} z_k x_i D_{zx} u_{ijk}^{n+\frac{2}{3}} - \frac{1}{3} r u_{ijk}^{n+1},
\end{aligned}
$$

Contents

System of Equations

- Consider this equation: $$\frac{u_{ijk}^{n+\frac{1}{3}} - u_{ijk}^{n}}{\Delta\tau} = (\mathcal{L}_{BS}^{x} u)_{ijk}^{n+\frac{1}{3}}$$
- Re-writing it as: $$\alpha_i u_{i-1,jk}^{n+\frac{1}{3}} + \beta_i u_{ijk}^{n+\frac{1}{3}} + \gamma_i u_{i+1,jk}^{n+\frac{1}{3}} = f_{ijk},$$

where

$$\alpha_i = -\frac{(\sigma_x x_i)^2}{h_{i-1}^x (h_{i-1}^x + h_i^x)} + r x_i \frac{h_i^x}{h_{i-1}^x (h_{i-1}^x + h_i^x)},$$

$$\beta_i = \frac{1}{\Delta\tau} + \frac{(\sigma_x x_i)^2}{h_{i-1}^x h_i^x} - r x_i \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} + \frac{r}{3}, \quad \gamma_i = -\frac{(\sigma_x x_i)^2}{h_i^x (h_{i-1}^x + h_i^x)} - r x_i \frac{h_{i-1}^x}{h_i^x (h_{i-1}^x + h_i^x)}$$

$$f_{ijk}^n = \frac{1}{3}\sigma_x \sigma_y \rho_{xy} x_i y_j D_{xy} u_{ijk}^n + \frac{1}{3}\sigma_y \sigma_z \rho_{yz} y_j z_k D_{yz} u_{ijk}^n + \frac{1}{3}\sigma_x \sigma_z \rho_{zx} x_i z_k D_{zx} u_{ijk}^n - \frac{1}{\Delta\tau} u_{ijk}^n.$$

- The solution to this equation can be found by solving the following tridiagonal system:

$$A_x u_{1:N_x,jk}^{n+\frac{1}{3}} = f_{1:N_x,jk}^{n},$$

System of Equations (*Contd.*)

- Here, the tridiagonal matrix is defined as:

$$A_x = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \cdots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{N_x-1} & \gamma_{N_x-1} \\ 0 & 0 & 0 & \cdots & \alpha_{N_x} & \beta_{N_x} + \gamma_{N_x} \end{pmatrix}.$$

- Similarly forming the system of equations for the next 2 time steps and solving the tridiagonal system in the same way, we can obtain the final solution.

Contents

Cash or Nothing Options

- A cash-or-nothing call option is one which has a binary outcome.
- It pays out either a fixed amount, if the underlying stock exceeds a predetermined threshold or strike price, or pays out nothing.
- The payoff is given by the following expressions based on the number of underlying assets:

$$u_T(x) = \begin{cases} c, & \text{if } x \geq K, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_T(x,y,z) = \begin{cases} c, & \text{if } x \geq K_1,\ y \geq K_2,\ z \geq K_3, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_T(x,y) = \begin{cases} c, & \text{if } x \geq K_1,\ y \geq K_2, \\ 0, & \text{otherwise.} \end{cases}$$

Numerical Results and Analysis

- The experiments are done on following 3 types of non-uniform grids:
  1. Ω_1 = [0, 1.5, 5.5, 9.5, . . . , 77.5, 80.5, 83.5, . . . , 122.5, 126.5, 130.5, . . . , 298.5, 300]
  2. Ω_2 = [0, 1, 4, 7, . . . , 79, 81, 83, . . . , 121, 124, 127, . . . , 298, 300]
  3. Ω_3 = [0, 0.5, 2.5, 4.5, . . . , 80.5, 81.5, 82.5, . . . , 120.5, 122.5, 124.5, . . . , 298.5, 300]
- Also, the error analysis of the numerical solutions is carried out using the respective closed-form solutions and corresponding plots are drawn.
- The Relative Error is defined as:

$$e_{L^2} = \sqrt{\frac{1}{\aleph}\sum_i\sum_j\sum_k\left(\frac{u_{ijk}^{N_\tau} - u(x_i, y_j, z_k, T)}{u(x_i, y_j, z_k, T)}\right)^2}$$

Derivation of Closed Form Solution

The Payoff is given by:

$$u_T(x) = \begin{cases} c, & \text{if } x \geq K \\ 0, & \text{otherwise} \end{cases}$$

Here 'x' follows a Geometric Brownian Motion i.e

$$x(T) = x(t)e^{\sigma(W(T)-W(t)) \; + \; (r-\frac{\sigma^2}{2})(\tau)}$$

where W(t)($0 \leq t \leq T$) is a Brownian Motion and W(t) follows N(0,t) Normal Distribution. Also $x(t)$ is value of the asset at time $T = t$.

Using the above equation the payoff can be rewritten as:

$$u_T(x') = \begin{cases} c, & \text{if } x' \geq ln(K) \\ 0, & \text{otherwise} \end{cases}$$

Here $x' = ln(x)$

Using 'Risk-Neutral Pricing' the price of the option at time T = t can be written as

$$u_t(x') = E[u_T(x')|F_t]$$

where F_t is a filtration w.r.t time.

Using *Independence Lemma* we can say that $u_T(x')|F_t$ has $N(ln(x(t))\ +\ (r - \frac{\sigma^2}{2})t\ ,\ \sigma^2 t)$ distribution.

Now, taking $\frac{x' - ln(x(t))\ +\ (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$ as y and using the above results and PDF of $N(0,1)$ Random Variable:

$$u_t(x') = e^{-r\tau} \times \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

Here $d = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$

The same technique can be followed to derive closed form solutions for 2 and 3 asset Cash or Nothing Options.

One-Asset Cash or Nothing Option

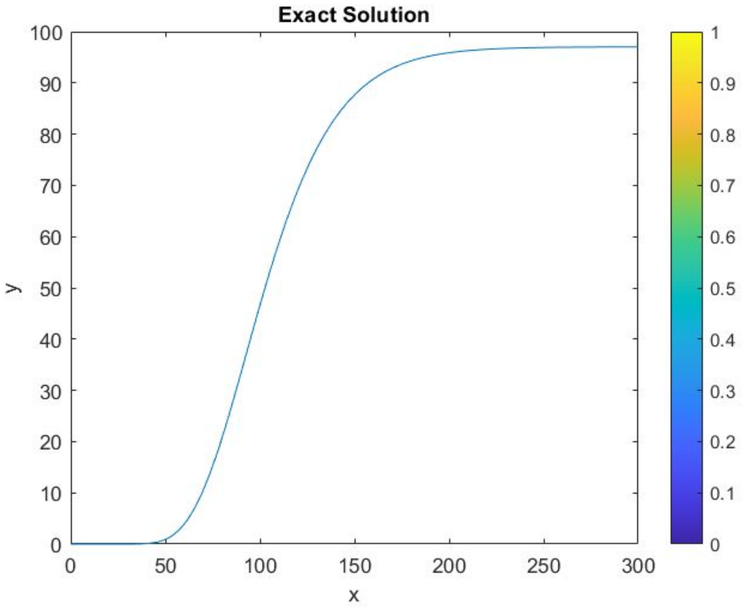
- Error Analysis :

Grid	Numerical Solution	Relative error
Ω_1	4.65790271213551e+01	9.63564812455430e-04
Ω_2	4.65853668211100e+01	4.94269516344677e-04
Ω_3	4.65924828441432e+01	2.48440779469612e-04

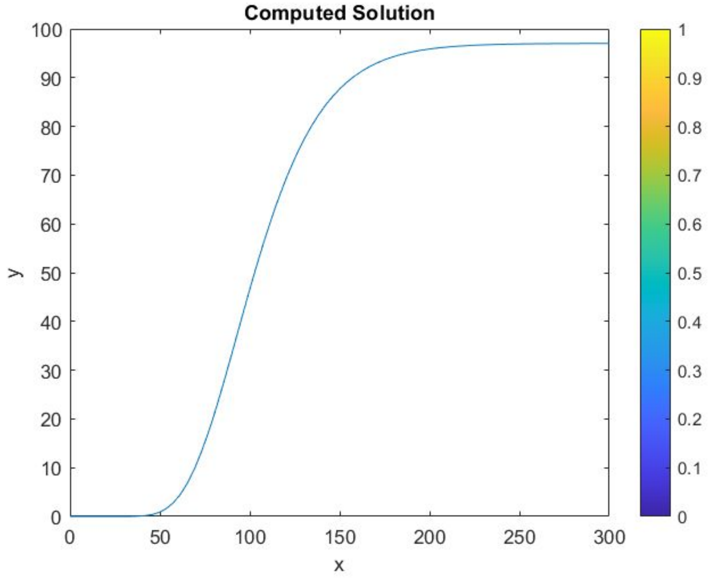
- In above table, the closed form solution at x = 100, that is,

$$u(100, T) = \mathbf{4.658732417041146e{+}01}$$

- Also, **Computed Solution** and **Actual Solution** at t=0 are shown in the following plots:



Exact Solution
100
90
80
70
60
50
40
30
20
10
0
y
0
50
100
150
200
250
300
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0



Computed Solution
100
90
80
70
60
50
40
30
20
10
0
y
0
50
100
150
200
250
300
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Two-Asset Cash or Nothing Option

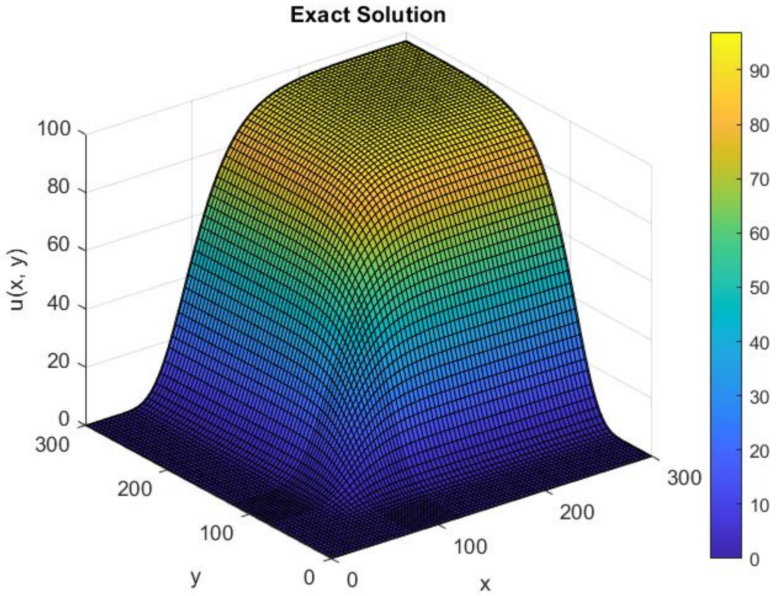
- Error Analysis:

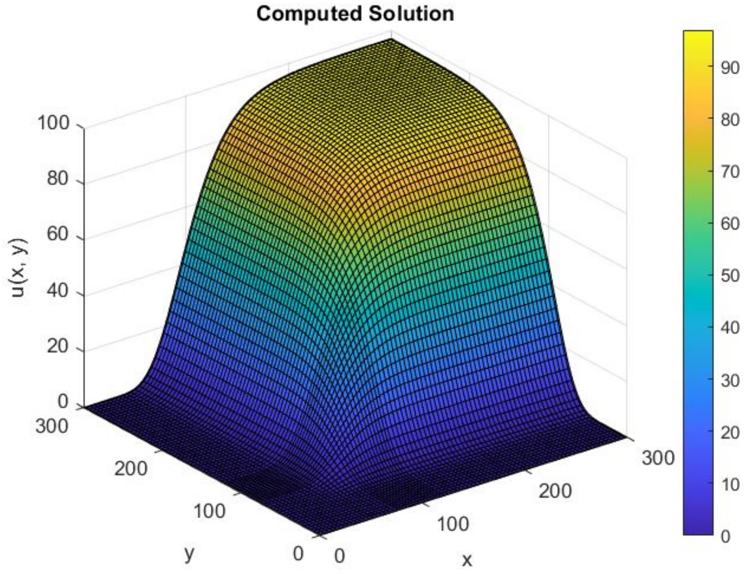
Grid	Numerical Solution	Relative error
Ω_1 x Ω_1	3.04002616369293e+01	1.36875627892725e-03
Ω_2 x Ω_2	3.04241973376953e+01	6.61434543764743e-04
Ω_3 x Ω_3	3.04458994181566e+01	3.41553564224867e-04

- In above table, the closed form solution at x = y = 100, that is,

$$u(100, 100, T) = \mathbf{3.043550958150124e+01}$$

- Also **Computed Solution** and **Actual Solution** at t=0 are shown in the following plots:


Exact Solution
u(x, y)
x
y
0
20
40
60
80
100
100
200
300
0
10
20
30
40
50
60
70
80
90


Computed Solution
u(x, y)
x
y
0
20
40
60
80
100
100
200
300
0
10
20
30
40
50
60
70
80
90

Three-Asset Cash or Nothing Option

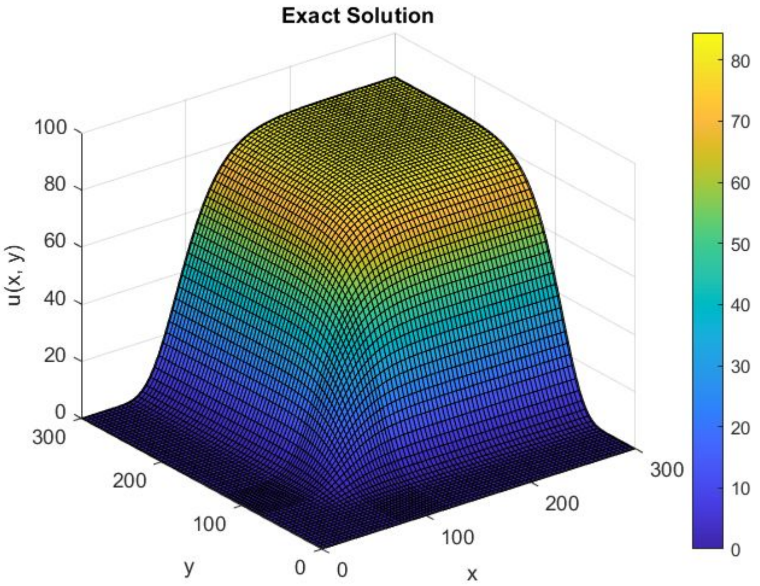
- Error Analysis:

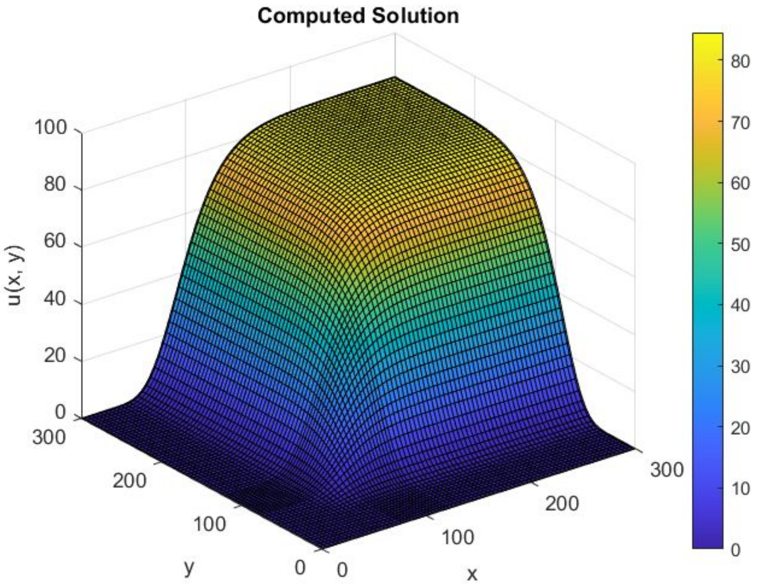
Grid	Numerical Solution	Relative error
Ω_1 x Ω_1 x Ω_1	2.24844278935264e+01	1.70747645283649e-03
Ω_2 x Ω_2 x Ω_2	2.25150423746274e+01	7.41934726348261e-04
Ω_3 x Ω_3 x Ω_3	2.25343435572539e+01	3.11894347568493e-04

- In above table, the closed form solution at x = y = z = 100, that is,

  u(100, 100, 100, T) = **2.252919330866442e+01**

- Also **Computed Solution** and **Actual Solution** at t = 0 *(taking two assets at a time)* are shown in the following plots:





- The above graphs are plotted for a fixed value of asset Z.
- Similarly, other plots (keeping 'X' fixed and 'Y' fixed) show a similar pattern.

Contents

Conclusion

- The paper mainly focused on solving multi-dimensional Black-Scholes Equation using operator splitting method. The BS Equations were discretized non-uniformly in space and implicitly in time (i.e backward).

- In Numerical Analysis, we performed experiment on characteristic examples like ’Cash-or-Nothing Options’.The computational results were in good agreement with the closed form solutions of the Black-Scholes equation.

References

1. *On the use of nonuniform grids in finite-difference equations*
2. *Central-difference schemes on non-uniform grids and their applications in large-eddy simulations of turbulent jets and jet flames*
3. *Operator Splitting - Springer*
4. *Operator Splitting - ResearchGate*
5. *Alternating-direct Implicit Method - Wikipedia*
6. *Cash-or-nothing Call Option - Investopedia*