# Continuous Probability Distributions

5CS037

# Contents

# Continuous Random Variables

- Continuous random variables are used to represent outcomes of events numerically
- A set of all possible outcomes represented by a continuous random variable, is always infinite
- **Note:** *It is never able to be counted.*

For example, a *discrete random variable* can be used to represent the number of coin flips necessary before getting the first 'tails' result.

It is theoretically possible for someone to have to flip the coin continuously for a endless amount of time before ever seeing a tails result.

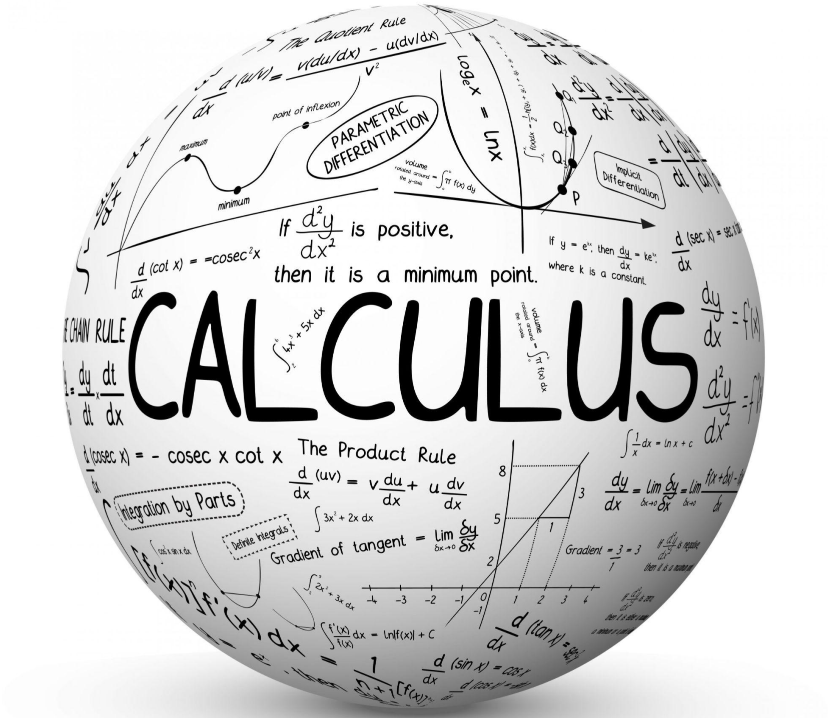
In that sense, the sample space is infinite. However, each flip of the coin can be counted (1, 2, 3, 4, . . ., infinite), so you must consider this to be a ***discrete random variable*.**

*As a counter example*, the sample space of the random variable which represents the exact amount of time one spends waiting for a taxi cab **cannot be counted**.

That is, one could wait 1 minute, or 1.1 minutes, or 1.11 minutes, or 1.111 minutes, etc . . .

Being that one could just continue to add additional precision in the form of additional decimal digits, this sample space is not countable, and therefore this random variable must be considered as **continuous**.

- Calculating probabilities, expected values, and the variance or standard deviation of a continuous random variable typically requires the application of Calculus.
- The exception is that some specific distributions have distinct formulas for calculating probabilities, mean or expected value, and variation or standard deviation.

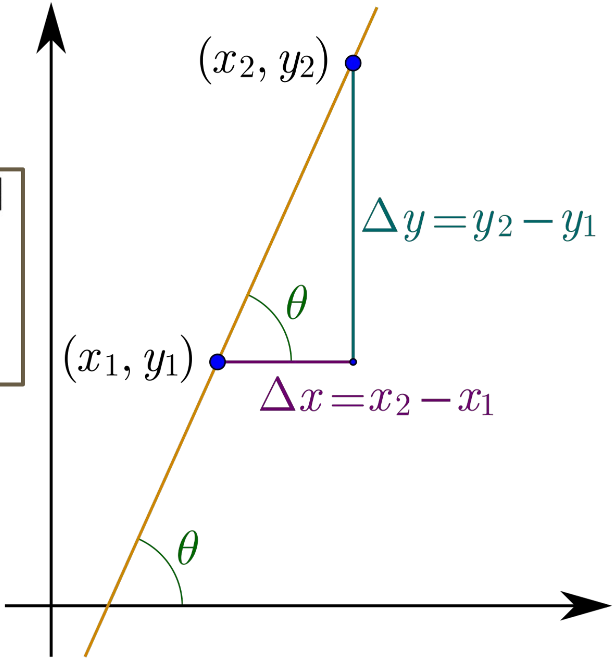

CALCULUS
The Quotient Rule
PARAMETRIC DIFFERENTIATION
point of inflexion
maximum
minimum
If $\frac{d^2y}{dx^2}$ is positive,
then it is a minimum point.
Implicit Differentiation
THE CHAIN RULE
The Product Rule
$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
Integration by Parts
Definite Integrals
Gradient of tangent = $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
$\frac{d}{dx}(\mathrm{cosec}\ x) = -\mathrm{cosec}\ x \cot x$
where k is a constant.

# Slope-The concept

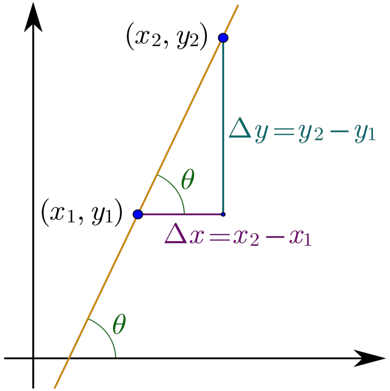
- Any continuous function defined in an interval can possess a quality called slope

Mathematically, the slope between two points (x1,y1) and (x2,y2) is defined as

- $m = \frac{y_2 - y_1}{x_2 - x_1}$

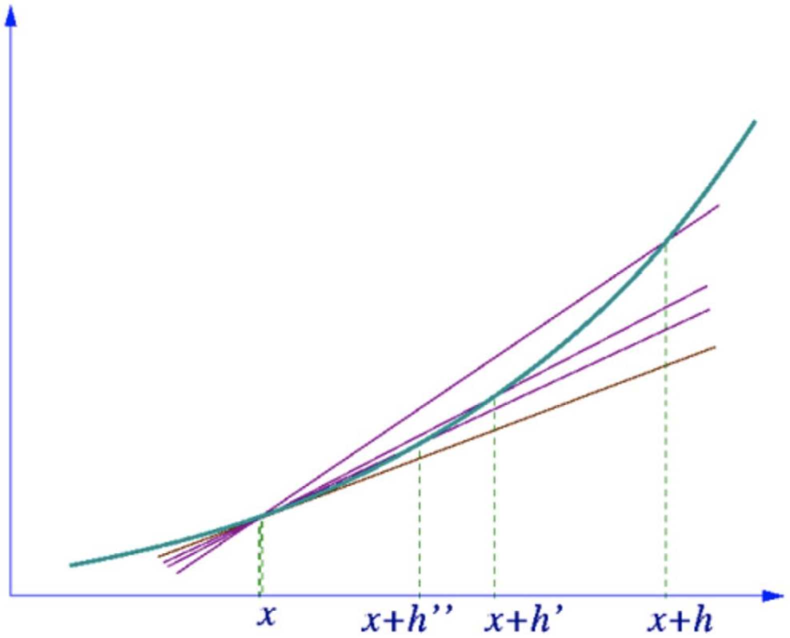


- In simple words, it can be thought of as "rise-over-run". It refers to the change of the function's value when moving from one x value to another.
- all straight lines have a constant slope, but for curves, the above approach only gives an AVERAGE slope.
- This is because if you take small segments of a curve at different places, the slope you find will be different.



# Slope of a curve

Look at this curve. You will notice that for different sizes of the x intervals, the average slopes are all different.
(The inclinations of the different lines are different)



# Derivative-The Concept

- As we saw, the slope can be very ambiguous if applied to most functions in general

Here, we modify the idea of a slope. Using the idea of a limit, we rewrite the slope as:

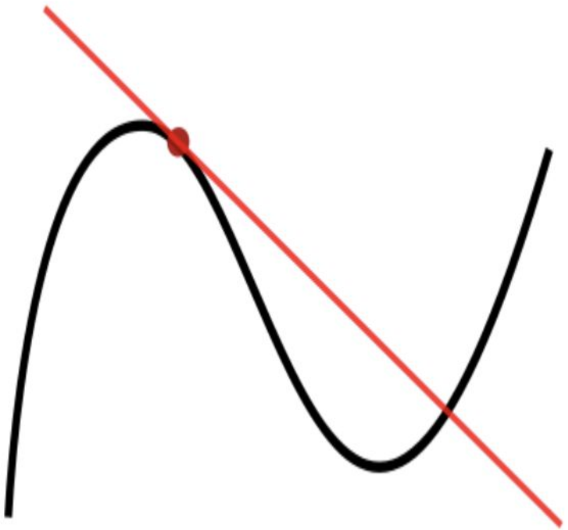
- $m = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- This is defined as the derivative.

It may seem absurd to do this, since intuition says that as $\Delta x \to 0$, then $\Delta y \to 0$. This would imply we are dividing 0 by 0, which is meaningless?!

- Let's work with the function
  - $f(x) = x^2$
  - For the sake of argument, let's pick a fixed point at x=2. then f(2)=4.
  - now, pick intervals of different sizes, say 0.1,0.01, and 0.001. find the changes in y and tabulate them.

| Δx | Δy | Δy/Δx |
|---|---|---|
| 0.1 | 0.41 | 4.1 |
| 0.01 | 0.0401 | 4.01 |
| 0.001 | 0.004001 | 4.001 |

- As we saw, as the change in x is made smaller and smaller, the value of the quotient – often called the Difference Quotient – comes closer and closer to 4.
- The formal way of writing it is
- $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 4$
- Think of the variable h as a “slider”. You could slide along the x-axis to get it as close to 2 as possible.
- Look at the picture, this is what you end up with geometrically.



The derivative is the slope at a single point, so it is also called the tangent line

- Given $f(x) = 3x^2 + 1$, find the value of the derivative at x=4.
- $f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$,
- Simply substitute 4+h for x in the function and find the limit.

$$f'(4) = \lim_{h \to 0} \frac{3(4+h)^2 + 1 - [3(4)^2 + 1]}{h}$$

$$= \lim_{h \to 0} \frac{3(16 + h^2 + 8h) + 1 - 48 - 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{48} + 3h^2 + 24h - \cancel{48} - \cancel{1} + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3h + 24)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3h + 24$$

$$f'(4) = 24$$

- This idea can be extended and used to find the general derivative for any function.
- The definition is changed slightly and written
- $f'(x) = \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$
- Here, a is an arbitrary point and x acts as the “slider”.
- It is interesting to note that if the substitution x=a+h is made, then we get the previously mentioned difference quotient.
- The next slide works out the derivative for f(x) using this idea.

- $f(x) = 3x^2 + 1$
- Using the definition,
  - $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
  - We substitute 'a' into the function and simplify the difference quotient.
  - $f'(a) = \lim_{x \to a} \frac{3x^2 + 1 - [3a^2 + 1]}{x - a}$
  - Distributing the negative sign,
  - $f'(a) = \lim_{x \to a} \frac{3x^2 + 1 - 3a^2 - 1}{x - a}$
  - Canceling out the 1 and factoring a 3 from both terms,
  - $f'(a) = \lim_{x \to a} \frac{3(x^2 - a^2)}{x - a}$

- Now, observe that the numerator is a difference of squares, we can expand them,
- $f'(a) = \lim_{x \to a} \frac{3(x-a)(x+a)}{x-a}$
- Here, we have the same factor in the numerator and denominator. Cancelling them out,
- $f'(a) = \lim_{x \to a} 3(x + a)$
- Now, just substitute the limit and you get,
- $\boldsymbol{f'(a) = 6a}$
- Since the variable a is arbitrary, you can rewrite this expression as

$$\underline{\boldsymbol{f'(x) = 6x}}$$

# Derivative as a function

- As we saw in the answer in the previous slide, the derivative of a function is, in general, also a function
- This derivative function can be thought of as a function that gives the value of the slope at any value of x
- This method of using the limit of the difference quotient is also called "ab-initio differentiation" or "differentiation by first principle"

*Note: there are many ways of writing the derivative symbol. Some of them are* $y'$, $f'(x)$, $\dot{y}$, $\frac{dy}{dx}$ *and D.*

# Rules of differentiation

- It can easily be verified by first principle that
- $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$
  - The derivative of the sum of two functions is the sums of their individual derivatives.
- $\left(f(x) - g(x)\right)' = f'(x) - g'(x)$
  - The derivative of the difference of two functions is the difference of their individual derivatives.
- $\left(cf(x)\right)' = c \times f'\,(x)$
- The derivative of a function multiplied by a constant is the constant multiplied by the derivative.
- (c)’=0
  - The derivative of a constant is zero.

# The Power Rule

- For any function of the form
- $f(x) = x^n$
  - A function where a variable is raised to a real power
- The derivative is given by :
- $f'(x) = nx^{n-1}$

- REMEMBER: the index n is a constant real number.
- Examples of such functions are : $x^2, x^7, x^{\frac{2}{3}}, x^{-5}$.

In Differential Calculus, we are given functions of $x$ and asked to obtain their derivatives.

In Integral Calculus, we are given functions of $x$ and asked what they are the derivatives of.

The process of answering this question is called **"integration"**.

**Integration is the reverse of differentiation.**

**DEFINITION**

Given a function $f(x)$, another function ,$z$, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x)$$

is called an integral of $f(x)$ with respect to $x$.

**Notes:**

(i) having found $z$, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x),$$

$z + C$ is also an integral for any constant value, $C$.

(ii) We call $z + C$ the **"indefinite integral of $f(x)$ with respect to $x$"** and we write

$$\int f(x)\mathrm{d}x = z + C.$$

(iii) $C$ is an **arbitrary constant** called the **"constant of integration"**.

**Result:**
Two functions $z_1$ and $z_2$ are both integrals of the same function $f(x)$ if and only if they differ by a constant.

1.

$$\int 3x^2 \mathrm{d}x = x^3 + C.$$

2.

$$\int x^2 \mathrm{d}x = \frac{x^3}{3} + C.$$

3.

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{Provided} \quad n \neq -1.$$

4.

$$\int \frac{1}{x} \mathrm{d}x \quad \text{i.e.} \quad \int x^{-1} \mathrm{d}x = \ln x + C.$$

5.

$$\int e^x \mathrm{d}x = e^x + C.$$

6.

$$\int \cos x \mathrm{d}x = \sin x + C.$$

7.

$$\int \sin x \mathrm{d}x = -\cos x + C.$$

## ILLUSTRATIONS

1.

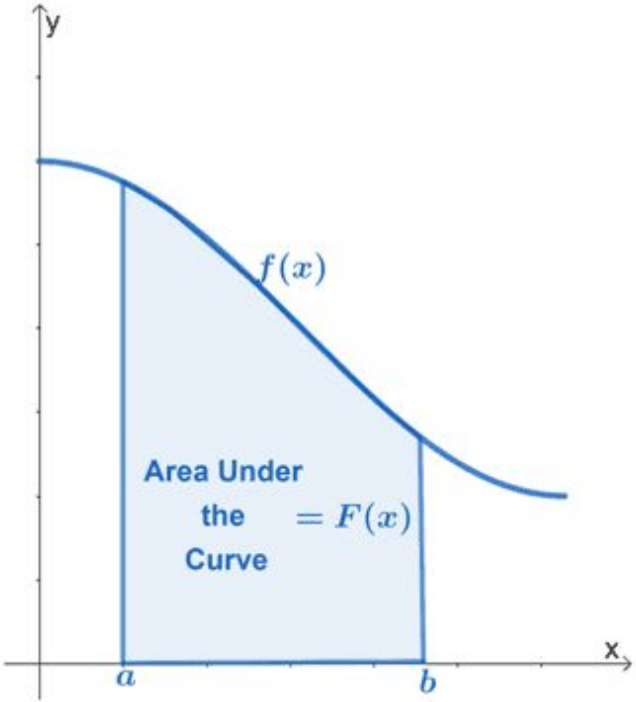
$$\int(x^2+3x-7)\mathrm{d}x=\frac{x^3}{3}+3\frac{x^2}{2}-7x+C.$$

2.

$$\int(3\cos x+4\mathrm{sec}^2 x)\mathrm{d}x=3\sin x+4\tan x+C.$$

# Definite Integrals - Area under the curve

$$\int_a^b f(x)dx$$



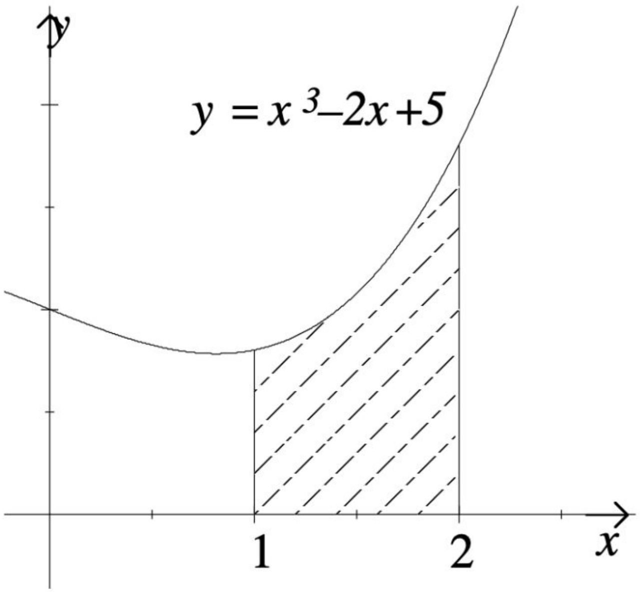
Find $\int_0^1 x^2 dx$.

An anti-derivative of $x^2$ is $\frac{1}{3}x^3$, so we write

$$\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}.$$

If the required area is $A$ square units, then

$$
\begin{aligned}
A &= \int_1^2 \left(x^3 - 2x + 5\right) dx \\
&= \left[\frac{x^4}{4} - x^2 + 5x\right]_1^2 \\
&= (4 - 4 + 10) - \left(\frac{1}{4} - 1 + 5\right) \\
&= 5\frac{3}{4}.
\end{aligned}
$$



# Continuous Random Variables

- Continuous random variables are used to represent outcomes of events numerically
- A set of all possible outcomes represented by a continuous random variable, is always infinite
- **Note:** *It is never able to be counted.*

# Expected Value of a Continuous Random Variable

$$E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

**Example #1:**

Let $X$ be a continuous uniform random variable on the interval $[0, 1]$, with the probability density function $f_X(x) = 1$. Find the $E(X)$.

$$E(X) = \int_0^1 x\ f(x)\ dx$$

$$E(X) = \int_0^1 x * 1\ dx$$

$$E(X) = \int_0^1 x\ dx$$

$$\boxed{E(X) = \int_0^1 x dx = \frac{x^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}}$$

Our expected value for this random variable is $\frac{1}{2}$. Notice this is intuitively what would be expected from a random variable which follows a uniform distribution . . . the expected value lies at the midpoint of the range.

# Variance of a Continuous Random Variable

$$V(X) = E(X^2) - (E(X))^2$$

Let $Y$ be a continuous random variable which has a probability density function of $f_Y(y) = \frac{3}{8}y^2$ on the interval $(0, 2)$, and $f_Y(y) = 0$ otherwise. Find the expected value of the random variable $Y$.

$$E(Y) = \int_0^2 y * f_Y(y)\, dy$$

$$E(Y) = \int_0^2 y * \frac{3}{8}y^2\, dy$$

$$E(Y) = \int_0^2 \frac{3}{8}y^3\, dy$$

$$\boxed{E(Y) = \int_0^2 \frac{3}{8}y^3\, dx = \frac{3y^4}{32}\Big|_0^2 = \frac{48}{32} - 0 = \frac{3}{2}}$$

$\boxed{\text{Compute the variance of the random variable } Y}$

- Setup the appropriate integral

$$E(h(Y)) = \int_0^2 h(y) * f_Y(y)\, dy$$

$$E(Y^2) = \int_0^2 y^2 * \frac{3}{8} y^2\, dy$$

$$E(Y^2) = \int_0^2 \frac{3}{8} y^4\, dy$$

- Compute the definite integral

$$\int_0^2 \frac{3}{8} y^4\, dy = \frac{3y^5}{40}\Big|_0^2 = \frac{96}{40} = \frac{12}{5}$$

- From above: $E(Y) = \frac{3}{2}$

$$E(Y)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\boxed{\begin{gathered} V(Y) = E(Y^2) - E(Y)^2 \\ V(Y) = \frac{12}{5} - \frac{9}{4} = \frac{48}{20} - \frac{45}{20} = \frac{3}{20} \end{gathered}}$$

# Probability Density Function

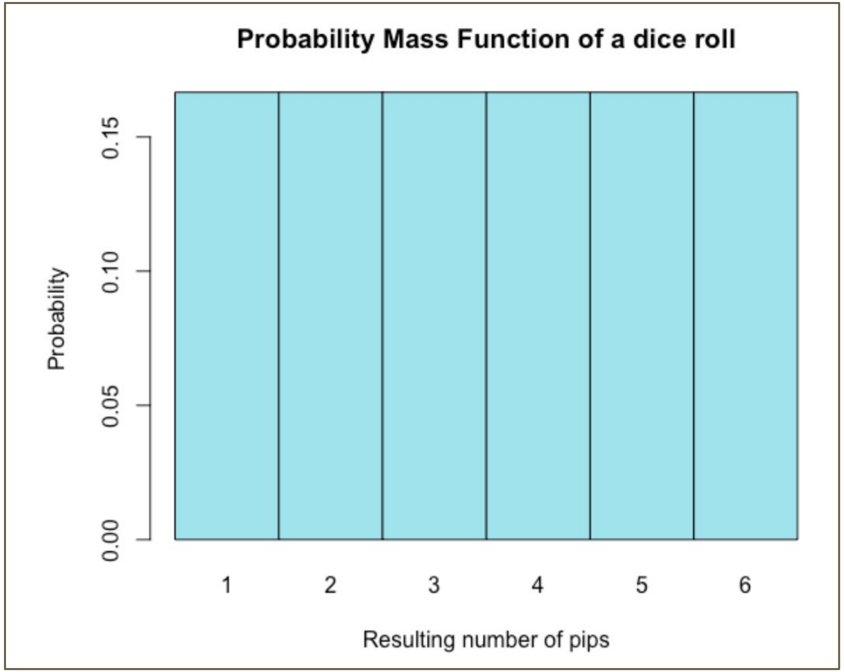
- The probability distribution function(or PDF) of a continuous random variable, in many ways, is similar to the probability mass function(or PMF) of a discrete random variable

**Aside: Notation**

Similar to the *PMF* a common notation for the *PDF* of a random variable ($X$ in this case) is:

$$f_X(x)$$

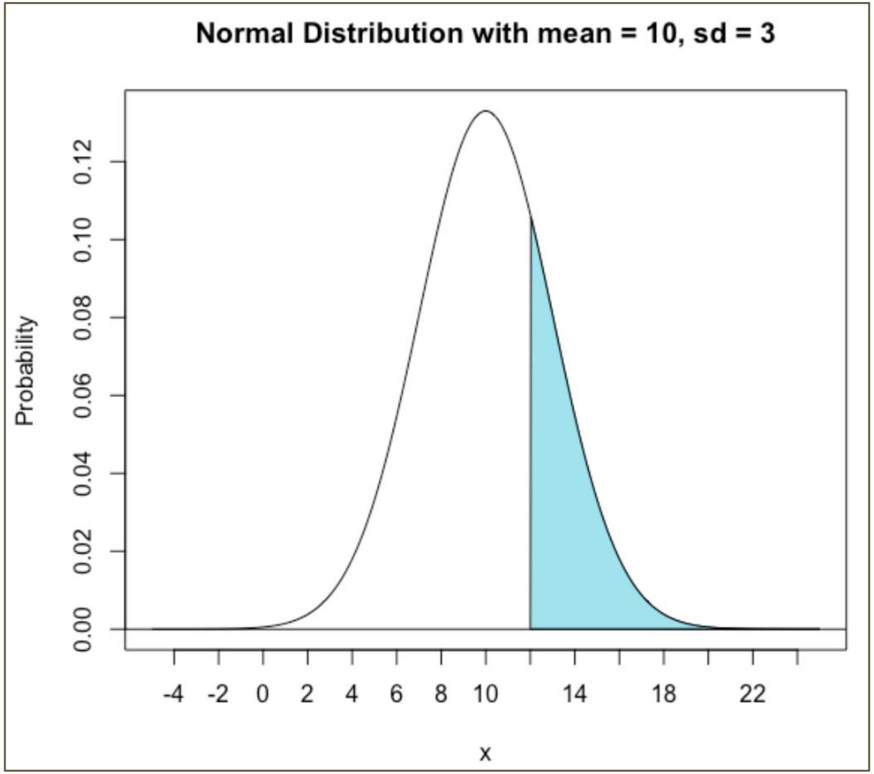
While dealing with *PMF*s for discrete random variables, probabilities can typically be interpreted as the total 'area' of the bars, in a histogram. See below for an example:

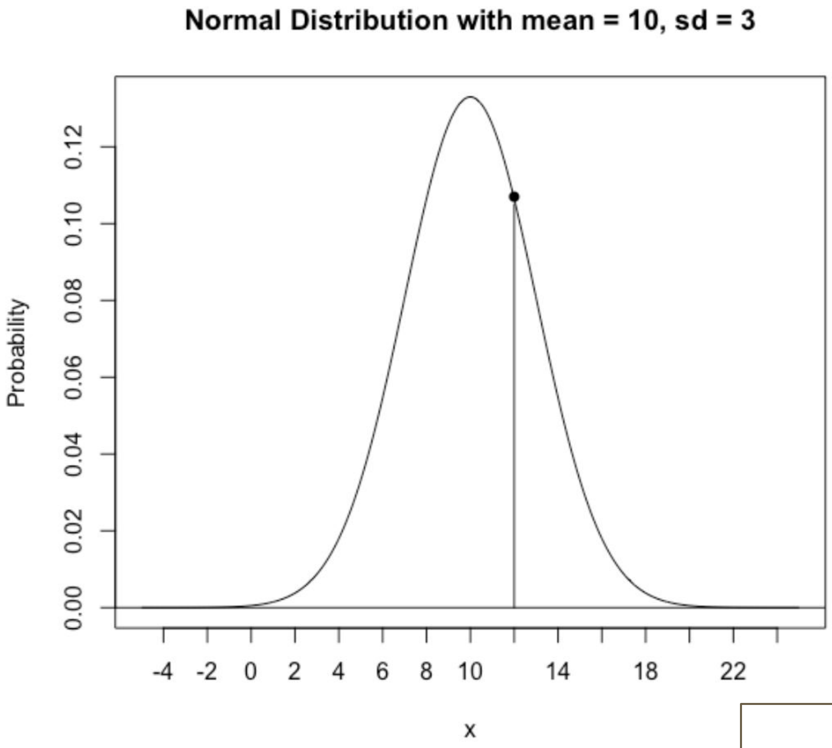



$$\text{area} = \text{height} * \text{width}$$

$$\frac{1}{6} * 1 = 1/6$$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$


Normal Distribution with mean = 10, sd = 3
Probability
0.00
0.02
0.04
0.06
0.08
0.10
0.12
-4
-2
0
2
4
6
8
10
14
18
22
x




$$P(A) = \frac{|A|}{|S|}$$

$$P(A) = \frac{1}{\infty}$$

$$P(A) = \lim_{x \to \infty} \frac{1}{x} = 0$$

$$\boxed{P(a \leq X \leq b) = \int_a^b f_X(x)\,dx}$$

Let the random variable $X$ represent a continuous random variable which has the probability distribution function defined below on the interval $(0, 1)$. There is no probability outside of the given interval.

$$f_X(x) = 3x^2$$

What is the probability that $X$ falls between $0$ and $\frac{1}{2}$?

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

$$P(0 \leq X \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} 3x^2\, dx$$

$$P(0 \leq X \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} 3x^2\, dx = x^3\Big|_0^{\frac{1}{2}} = \frac{1}{8} - 0 = \frac{1}{8} = 0.125$$

# Checking the validity of a PDF

The total probability for any given random variable (or random experiment) must always equal one. Integration can be used to determine whether or not a given *PDF* is legitimate by integrating over the entire interval where the probability distribution function is greater than zero, and checking whether the result is 1.

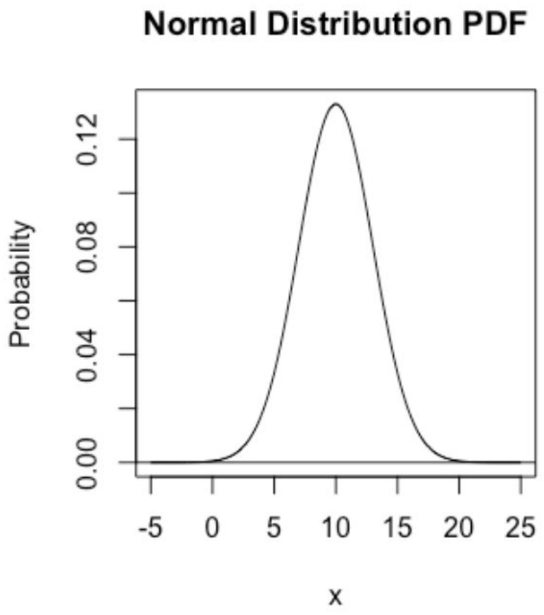
$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

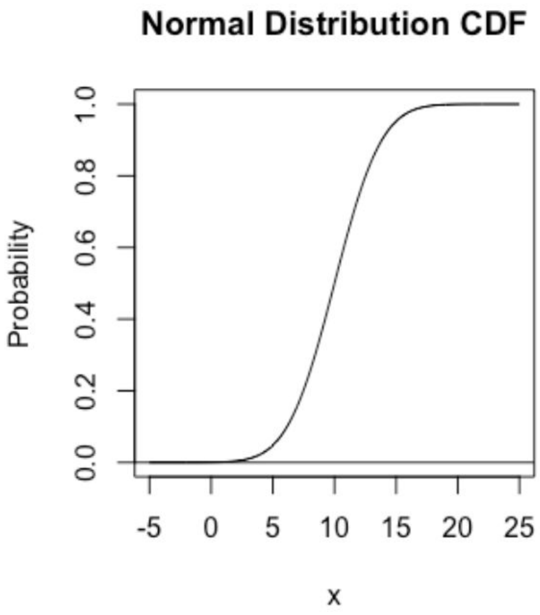
$$P(0 \le X \le 1) = \int_0^1 3x^2\,dx$$

$$P(0 \le X \le 1) = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1^3 - 0 = 1$$

# Cumulative Distribution Function

Much like a *PDF* for a random variable $X$ is expressed as $f_X(x)$, the *CDF* is expressed as $F_X(x)$. Notice the only difference is that a capital 'F' is used for the *CDF* where the lower cased 'f' is used for a *PDF*.




Whereas the *CDF* of a discrete distribution looks like a step-wise function, the *CDF* of a normal distribution is continuous. In both cases, once the *CDF* reaches a value of 1, it has hit its maximum, and continues on at that maximum out until infinity.

Let the random variable $X$ be continuous with the probability density function defined below:

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq X \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the *CDF* to find $P(X \leq \frac{1}{2})$

- Step 1: Setup the indefinite integral for the defined interval $(0, 1)$

$$F_X(x) = \int f_X(x)\, dx$$

$$F_X(x) = \int 3x^2\, dx$$

- Step 2: Compute the indefinite integral

$$F_X(x) = \int 3x^2\, dx = x^3$$

- Step 3: Interpret the result and express as a cumulative distribution function, be careful to mind the appropriate intervals

$$F_X(x) = \begin{cases} 0 & X < 0 \\ x^3 & 0 \leq X \leq 1 \\ 1 & X > 1 \end{cases}$$

- Step 4: Find $P(X \leq \frac{1}{2})$
  - First, be sure to choose the correct interval to apply the CDF. In this case $F_X(x) = x^3$.
  - Second, plug in the given value to the ***CDF*** to find the result

$$P(X \leq \frac{1}{2}) = \frac{1}{2}^3 = \frac{1}{8}$$

# The Normal Distribution - Gaussian Distribution

Common notations are used for the normal distribution. A random variable  which is told to follow a normal distribution is expressed as below:

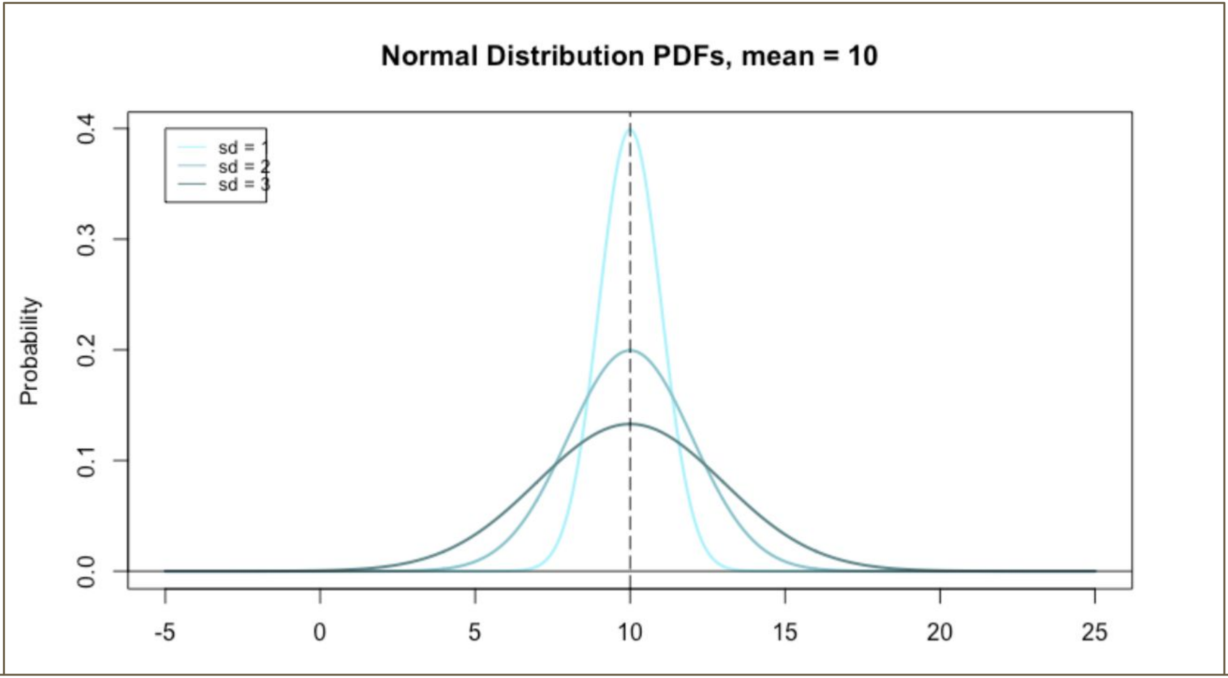
$$X \sim N(\mu, \sigma^2) \quad \text{or} \quad X \sim N(\mu, \sigma)$$

## Probability Distribution Function

The *PDF* of the normal distribution is a relatively complex function. The *PDF* is not used by itself very often, and conventional wisdom dictates that the integral of this *PDF* cannot be computed with conventional integration techniques, rather one must employ the use of accurate estimations. Because of this, tables and/or software is typically employed to calculate probabilites from a Normal Distributed random variable. That being said the definition of the *PDF* for a Normal Distribution is below.

The *PDF* of a Normal distribution with unknown mean, $\mu$, and variance, $\sigma^2$, is:

$$f_X(\mu, \sigma^2) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$




In the plot above, it can be plainly seen that the normal distribution has the common "bell-shaped" curve, it turns out that this type of curve (and the normal distribution) is used ubiquitously to represent both natural and man-made phenomena. Also, notice that the functions with larger standard deviations have a greater "spread" of the distribution.

How many standard deviations above or below the mean a given value lies?

$$Z = \frac{X - \mu}{\sigma}; \text{ Where } X \text{ is a normally distributed random variable}$$

Now, having a numerical representation of how many standard deviations above or below the mean a given value lies, the standard normal distribution tables can be utilized to find a probability.

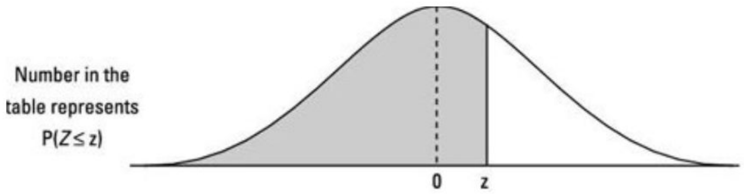
The response time of a driver seeing brake lights on the vehicle in front of them is critical for drivers to avoid collisions. In the journal *Ergonomics*, a study of response times to brake lights was found to follow a normal distribution with a mean value of 1.25 seconds, and a standard deviation of .46 seconds. What is the probability that a driver's response time is less than 1.5 seconds?

- Step 1: Write down all given information in the problem statement
  - Mean, $\mu = 1.25 \ seconds$
  - Standard Deviation, $\sigma = 0.46 \ seconds$
  - Random Variable is normally distributed

- Step 2: Determine the z-Score for the given value
  - $z = \frac{X-\mu}{\sigma}$
  - $z = \frac{1.5-1.25}{0.46} = 0.543$

- Step 3: Interpret the z-Score
  - With a z-Score of $0.543$, it can be said that the value we are testing, 1.5 seconds, is 0.543 standard deviations above the population mean of 1.25. If the z-Score were negative, that would indicate the value being tested lies below the population mean.

- Step 4: Use a Z-table to find the probability corresponding to the z-Score




| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | $\boxed{.7054}$ | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | | | | | |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | | | | | |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | | | | | |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | | | | | |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | | | | | |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | | | | | |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | | | | | |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | | | | | |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | | | | | |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | | | | | |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | | | | | |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | | | | | |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | | | | | |

- Step 5: Interpret the result from the z-Table
  - The z-Table gives the probability of $0.7054$ for the z-Score of $0.543$. Notice that the probability given by the table is the area under the curve which is less than or equal to the given value of $X$. This is the same type of result that is typically used when using a *CDF* ($P(X \leq x)$). This can interpreted as there being a 70.54% chance that a randomly chosen driver will respond to brake lights in 1.5 seconds or less.

# Exponential Distribution

- The exponential random variable is typically used to model the time in between events of a Poisson Process
- The exponential distribution is commonly used to model waiting times, typically the amount of time for a single event to occur
- he Exponential Distribution is the so-called "memoryless" property.
- The memoryless property means that a given probability distribution operates independently of its history.

# Example

For example, if there is prior knowledge that one available taxi typically drives by a NYC office every ten minutes (or 0.1 cabs/minute)

we can calculate expected wait times (Expected Value),

we could calculate the probability that it would take less than ten, or more than twenty minutes for an available taxi to arrive.

## Required Assumption

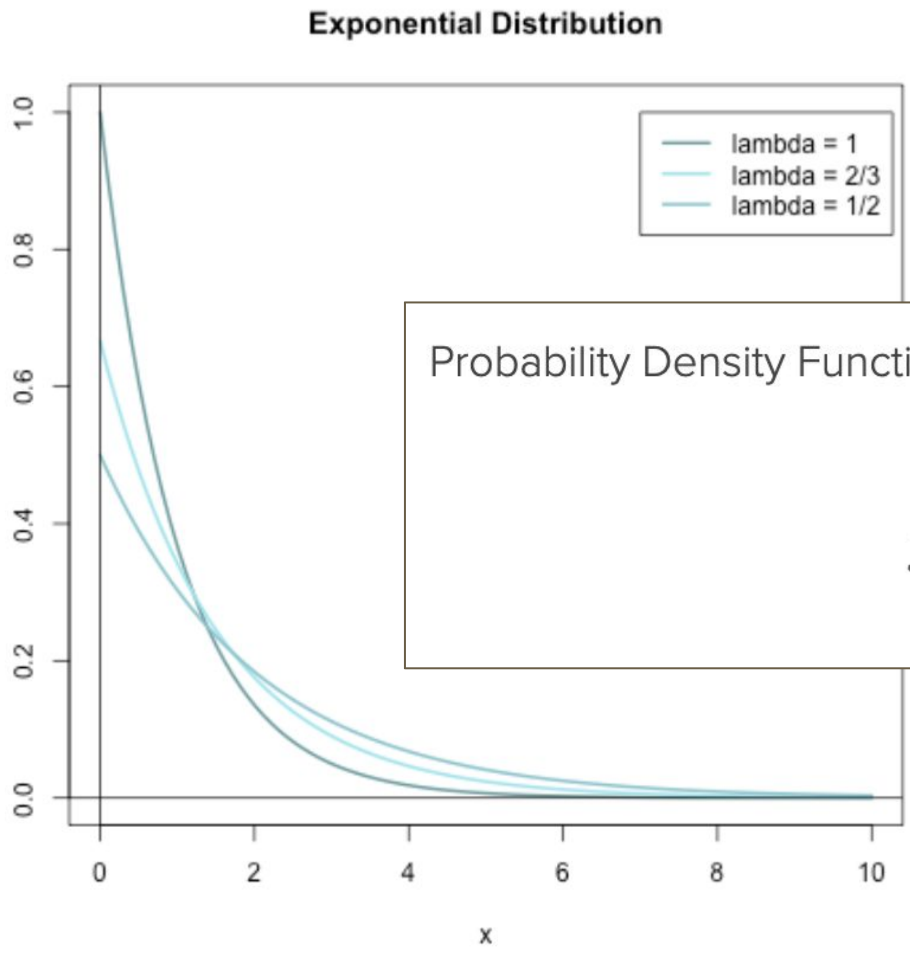
Much like the Poisson random variable, there is an important assumption that must be made when applying the exponential distribution. To apply the exponential distribution, the parameter which describes a rate $(\lambda)$ must be constant. In more complicated real world models which implement the exponential distribution, it is common to see the parameter $\lambda$ expressed as a function which models a rate in different intervals of time.

### Aside: Notation

A random variable follows an exponential distribution is typically notated by the expression:

$$X \sim \mathrm{Exp}(\lambda)$$

Notice that there is only one parameter for this distribution, and much like the Poisson Distribution, it is represented by a lower case lambda, $\lambda$, and it represents a rate of events. Where $\lambda$ represented "events per time/space" in the Poisson Distribution, the $\lambda$ operates similarly in the Exponential Distribution. In the example above, one would likely set $\lambda = 0.1$, representing the rate of taxi cab arrivals per minute; as minutes are a more commonly used measure of time than ten minute intervals.




Probability Density Function of an Exponential Random Variable $X$:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & : \ x \geq 0 \\ 0 & : \ \text{otherwise} \end{cases}$$

Notice that much like a Poisson Distribution, the Exponential Distribution operates on the interval $[0, \infty)$. Intuitively makes sense, as there cannot be a negative amount of time or space inbetween events (or a negative waiting time before a first event). It is worth noting that the function does not have an upper bound, that is, there could theoretically be an infinite amount of time inbetween events (or before a first event), though it is usually quite improbable.

# Mean, Variance & Standard Deviation

$$\mu = \frac{1}{\lambda}$$

$$\sigma^2 = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

## Cumulative Distribution Function for the Exponential Distribution

Like all other continuous distributions, the *PDF* of an exponential distribution cannot be used to directly calculate probabilities; rather, one would have to integrate to find any probabilities related to the distribution. However, the Exponential Distribution does have a well-defined *Cumulative Distribution Function*, or *CDF* which can be used to calculate probabilities. The *CDF* of an Exponential Random Variable $X$ is defined below:

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & : \ x \geq 0 \\ 0 & : \ \text{otherwise} \end{cases}$$

As always with any continuous distribution, the *CDF* can be used to directly calculuate probabilities. See an example problem below:

The amount of time spent choosing breakfast cereal types can be modeled by an exponential distribution with the average amount of time equal to five minutes ($\lambda = \frac{1}{5} = .2$). What is the probability that a randomly chosen shopper will choose their breakfast cereal in fewer than three minutes?

- Step 1: Set up the problem using the *CDF*
  - $P(X \leq x) = 1 - e^{-\lambda x}$
  - $\lambda = 0.2$
  - $P(X \leq 3) = 1 - e^{-0.2 \cdot 3}$

- Step 2: Make the calculation
  - $P(X \leq 3) = 1 - e^{-0.2 \cdot 3} = 0.451$
- Step 3: Interpret the result
  - It is shown that the probability of a shopper taking fewer than three minutes to choose their breakfast cereal is $0.451$ or $45.1$%

# Memoryless Property of an Exponential Distribution

In our previous example, recall that the amount of time spent choosing breakfast cereals is exponentially distributed. Suppose that a customer has already spent four minutes in the cereal aisle making their decision, intuition may lead one to believe that since the customer has already spent four minutes in the aisle, it would seem more likely that they are to make a decision within the next minute. However, when using the Exponential Distribution, this is not the case, the additional time spent choosing their cereal does not depend on how much time the customer has already spent looking at the options. This is an example of the ***memoryless property***. See below for an example:

In the previous example, we find the distribution of a grocery store customer shopping for breakfast cereal. Given that a customer has already spent two minutes in the cereal aisle observing their choices, what is the probability that it will take the customer between two and seven additional minutes to make their decision?

- Step 1: Set up the problem using the *CDF*
  - $P(2 \leq X \leq 7) = F_X(7) - F_X(2)$
  - $P(2 \leq X \leq 7) = (1 - e^{-7\lambda}) - (1 - e^{-2\lambda})$
  - $\lambda = 0.2$

- Step 2: Make the calculation
  - $P(2 \leq X \leq 7) = (1 - e^{-7 \cdot 0.2}) - (1 - e^{-2 \cdot 0.2}) = 0.424$

- Step 3: Interpret the solution
  - The probability that a customer will take an additional 2-7 minutes is $0.424$ or $42.4\%$
  - Note: Because of the 'memoryless' property held by the exponential distribution, any and all prior time spent in the aisle choosing is irrelevant.

> Intuitively, the memoryless property tells us that, just because a customer has already spent some time in the cereal aisle; we still lack any evidence that they are closer to making a choice than when they began. In the taxi cab example, the intuition is similar; when a taxi driver arrives in front of the office building, they have no knowledge of how long the client has been waiting, these two events (the customer waiting, and the cab arriving), are independent (assuming that the customer had not called the cab company to order a cab).

# Thank You

Happy Mathematics

**Luck is the probability taken personally**

*-Chip Denman*