1 FIRST ORDER ORDINARY DIFFERENTIAL EQUATION

Exact Equations

A first order differential equation of the form

$$M\,dx + N\,dy = 0 \qquad \text{.... (1)}$$

where both M and N are functions of x, y is said to be exact, if there exist a function $u\,(x, y)$ such that

$$M\,dx + N\,dy = du \qquad \text{.... (2)}$$

Then equation (1) becomes $du = 0$, which on integration gives

$$u(x, y) = c, \; c \text{ being a constant.}$$

Therefore, $u\,(x, y) = c$ is a solution of (1).

For example $\log x\,dy + \dfrac{y}{x}\,dx = d\,(y \log x)$.

Hence $y \log x = c$, c being a constant, is a general solution of the equation.

Theorem. The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Proof. Let the equation $M\,dx + N\,dy = 0$ be exact.

Then there exist a function $u\,(x, y)$ such that $Mdx + Ndy = du$.

But
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

So
$$Mdx + Ndy = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$$

Therefore
$$\frac{\partial u}{\partial x} = M, \; \frac{\partial u}{\partial y} = N$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}, \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

But, if both $\frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial y}$ are continuous, then $\frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y \partial x}$.

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Thus the condition is necessary.

Conversely, let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

We are to show that $Mdx + Nidy = 0$ is exact.

Let $$\int M dx = F(x, y)$$

$$\therefore \quad \frac{\partial F}{\partial x} = M$$

So, $$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$\therefore \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right).$$

$$\therefore \quad N = \frac{\partial F}{\partial y} + f(y) \qquad \text{... (3)}$$

$$\therefore \quad M dx + N dy = \frac{\partial F}{\partial x} dx + \left\{\frac{\partial F}{\partial y} + f(y)\right\} dy.$$

$$= \left(\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy\right) + f(y)\, dy$$

$$= dF + f(y)\, dy$$

$$= d\left[\mathrm{F} + \int f(y)\,dy\right]$$

$$= du, \qquad \text{.... (4)}$$

where $$u = \mathrm{F} + \int f(y)\,dy. \qquad \text{.... (5)}$$

which shows that $\mathrm{M}dx + \mathrm{N}dy = 0$ is exact.

Note. By (5), the solution of $\mathrm{M}dx + \mathrm{N}dy = 0$ is $u\,(x, y) =$ constant.

i.e., $$\mathrm{F} + \int f(y)\,dy = \text{constant}$$

i.e., $$\int \mathrm{M}dx + \int f(y)\,dy = \text{constant, by (5)}$$

i.e., $$\int \mathrm{M}dx + \int (\text{terms of N not containing } x)\ dy = \text{constant.} \qquad \text{... (6)}$$

Example 1 : Solve $(x^2 - y)\,dx + (y^2 - x)dy = 0$.

Sol^n : Here $M = x^2 - y$ and $N = y^2 - x$

Now $\dfrac{\partial M}{\partial y} = -1$ and $\dfrac{\partial N}{\partial x} = -1$

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, hence the differential equation is exact.

Now integrating M w.r.t. keeping y as constant, we get

$$\int M\,dx = \int (x^2 - y)\,dx = \frac{x^3}{3} - yx \qquad \frac{x^3}{3} - yx$$

Integrating N w.r.t. y keeping x as constant, we get

$$\int N\,dy = \int (y^2 - x)\,dy = \frac{y^3}{3} - xy$$

Then the general solution (or complete primitive) is $\dfrac{x^3}{3} - xy + \dfrac{y^3}{3} = c$

where c is an arbitrary constant.

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Example 2. Solve the following equations :

(i) $ydx + (x + \cos y)\, dy = 0$

(ii) $(4x^3 y^3 - 2xy)dx + (3x^4y^2 - x^2)\, dy = 0$

$\int Y \cdot dx$

$y \cdot x + \sin y$

Sol^n : (i) Here $M = y$, $N = (x + \cos y)$

Now $\dfrac{\partial M}{\partial y} = 1$ and $\dfrac{\partial N}{\partial x} = 1$,

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, hence the differential equation is exact.

Now integrating M w.r.t. x keeping y as constant, we get

$$\int M\, dx = \int y\, dy = xy$$

Again integrating N w.r.t. keeping x constant, we get

$$\int N\, dx = \int y\, dx = xy$$

$\therefore$ $\boxed{\text{The general solution is } xy + \sin y = c}$ (c = arbitrary constant)

(ii) Here $M = 4x^3y^3 - 2xy$, $N = 3x^4y^2 - x^2$

$$\frac{\partial M}{\partial y} = 12x^3y^2 - 2x$$

and $\dfrac{\partial N}{\partial x} = 12x^3y^2 - 2x$

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, hence the given differential equation is exact.

Now integrating M w.r.t. x keeping y as constant, we get

$$\int M\, dx = \int (4x^3y^3 - 2xy)\, dx = x^4y^3 - x^2y$$

Again integrating N w.r.t. y keeping x as constant, we get

$$\int N\,dy = \int (3x^3y^2 - x^2)\,dy = x^4y^3 - x^2y$$

$$\boxed{\text{Hence the general solution is } x^4y^3 - x^2y = c}$$

(where c is an the arbitrary constant)

Example 1. Solve : $(xy^2 - e^{\frac{1}{x^3}})\,dx - x^2y\,dy = 0.$

Sol^n : Here, $M = xy^2 - e^{\frac{1}{x^3}}, N = -x^2y$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy, \frac{\partial N}{\partial x} = -2xy$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

So the equation is not exact but

$$\boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)} = \frac{4xy}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

Hence, I.F. $= e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log x^{-4}} = \frac{1}{x^4}.$

Multiplying the equation by $\frac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx - \frac{y}{x^2}dy = 0, \text{ which is an exact equation.}$$

Therefore the solution is

$$\int\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx + 0 = \frac{c}{6}, \text{ since there is no term which does not}$$

contain x in N.

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i.e., $y^2\int\frac{dx}{x^3}+\frac{1}{3}\int e^{\frac{1}{x^3}}\,d\left(\frac{1}{x^3}\right)=\frac{c}{6}$

i.e., $-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=\frac{c}{6}$

$\therefore\ 2x^2\,e^{\frac{1}{x^3}}-3y^2=cx^2$ which is the required solution.

Example 2. Solve : $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$.

Sol^n : Here $M=3x^2y^4+2xy$, $N=2x^3y^3-x^2$

$\therefore\ \frac{\partial M}{\partial y}=12x^2y^3+2x,\ \frac{\partial N}{\partial x}=6x^2y^3-2x$

$\frac{1}{M}\left|\frac{dM}{}\right.$

$\therefore\ \boxed{\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}}$

So the equation is not exact but

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{6x^2y^3+4x}{3x^2y^4+2xy}=\frac{2}{y}$ which is a function of y only.

Hence I.F. $=e^{-\int\frac{2}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=\frac{1}{y^2}$. $=Y_L$

Multiplying the equation to $\frac{1}{y^2}$, we get

$(3x^2y^6+2xy^3)\,dx$

$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$ which is an exact equation.

Therefore the solution is

$\cancel{\int\left(3x^2y^2+\frac{2x}{y}\right)dx}+0=c$, since there is no term which does not contain x in N.

$\int(3x^2y^6+2xy^3)\,dx$

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$\frac{3x^3}{2}y^6 + \frac{2x^2y^2}{2}$

$\boxed{2(3x^3y^6 + x^2y^2)}$

$\therefore \quad x^3y^2 + \frac{x^2}{y}$ c, which is the required solution.

Example 3. Solve : $(x^4 + y^4)\,dx - xy^3dy = 0$.

Sol^n : Here, $M = x^4 + y^4$, $N = -xy^3$.

So M, N are both homogeneous functions in x, y of degree 4.

Now, $\frac{\partial M}{\partial y} = 4y^3, \frac{\partial N}{\partial x} = -y^3$.

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

So the equation is not exact, but $Mx + Ny = x^5 \neq 0$.

Hence, $\frac{1}{x^5}$ is an integrating factor.

Multiplying the equation by $\frac{1}{x^5}$, we get $\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx - \frac{y^3}{x^4}dy = 0$

which is an exact equation.

Therefore the solution is

$$\int\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx = \frac{c}{4},$$

since there no term which does not contain x in N.

or, $\log x - \frac{y^4}{4x^4} = \frac{c}{4}$.

$\therefore \quad 4x^4 \log x - y^4 = cx^4$ which is the required solution.

Example 4. Solve : $y(xy + 2x^2y^2)\,dx + (xy - x^2y^2)\,xdy = 0$.

Sol^n : Here $M = y\,(xy + 2x^2y^2)$, $N = (xy - x^2y^2)\,x$.

Also the equation is of the form $yf(xy)\,dx + xg\,(xy)\,dy = 0$.

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Moreover $Mx - Ny = 3x^3y^3$.

Hence $\dfrac{1}{3x^3y^3}$ is an integrating factor.

Multiplying the equation by $\dfrac{1}{3x^3y^3}$, we get

$$\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$$ which is an exact equati

Therefore the solution is given by

$$\frac{1}{3}\int\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\int\left(-\frac{1}{y}\right)dy=\frac{c}{3},$$ c being a constant.

or, $-\dfrac{1}{xy}+2\log x-\log y=c.$

$\log\dfrac{x^2}{y}=\dfrac{1}{xy}+c,$ which is the required solution.

Example 5. Solve : $3ydx - 2xdy + x^2y^{-1}(10ydx - 6xdy) = 0.$

Comparing the given equation with the equation.

$$x^ay^b(mydx + nxdy) + x^{a'}y^{b'}(m'ydx + n'xdy) = 0$$

we have $a = b = 0$, $m = 3$, $n = -2$, $a' = 2$, $b' = -1$, $m' = 10$, $n' = -6$.

Sol^n : Let $x^h y^k$ be an integrating factor.

Then $\dfrac{a+h+1}{m}=\dfrac{b+k+1}{n}$

i.e., $\dfrac{0+h+1}{3}=\dfrac{0+k+1}{-2}$

i.e., $2h + 3k = -5.$

and $\dfrac{a'+h+1}{m'}=\dfrac{b'+k+1}{n'}$

$$\therefore \quad \frac{2+h+1}{m'} = \frac{b'+k+1}{n'}$$

$\therefore \quad 3h + 5k = -9.$ .... (ii)

Solving (i) and (ii), we get $h = 2$, $k = -3$.

Therefore, x^2y^{-3} is an integrating factor.

Multiplying the equation by x^2y^{-3}, we get

$(3x^2y^{-2} + 10x^4y^{-3})\, dx - (2x^3y^{-3} + 3x^5y^{-4})\, dy = 0$ which is an exact equation.

Therefore the solution is given by

$\int (3x^2y^{-2} + 10x^4y^{-3})dx = c,$ since there is no term which does not contain x in N.

$\therefore \quad x^3y^{-2} + 2x^5y^{-3} = c$, which is the required solution.

Example 6. Solve $\left(x^2y - 2xy^2\right) dx + \left(3x^2y - x^3\right)dy = 0.$

Sol^n : Here, $M = x^2y - 2xy^2$, $N = 3x^2y - x^3$ are both homogeneous functions in x, y of same degree 3.

Now, $Mx + Ny = (x^2y - 2xy^2)\, x + (3x^2y - x^3)\, y = x^2y^2$.

$$\therefore \quad \text{I.F.} = \frac{1}{x^2y^2}$$

Multiplying the given differential equation by $\frac{1}{x^2y^2}$, we get,

$$\frac{1}{x^2y^2}\left(x^2y - 2xy^2\right)dx + \frac{1}{x^2y^2}\left(3x^2y - x^3\right)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$$ which is an exact eqauation.

Hence the solution of the equation is

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$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

or, $\dfrac{x}{y}-2\log x+3\log y=c$

$\therefore\ \dfrac{x}{y}+\log\dfrac{y^3}{x^2}=c,$ c being a constant.

Example 7 : Solve, $\left(2xy+e^x\right)y\,dx-e^x dy=0$

Sol^n : Hence $M=(2xy+e^x)\,y,\ N=-e^x$

$m = 2x \cdot y + e^x$
$N = -e^x$
$\dfrac{dm}{dy} = 2x + 0$
$\dfrac{dN}{dx} = -e^x$

$\therefore\ \dfrac{\partial M}{\partial y}=4xy+e^x,\ \dfrac{\partial N}{\partial x}=-e^x$

$\therefore\ \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ $\quad \dfrac{4xy+e^x}{2x\cdot y+e^x}$

$\therefore\ \dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}=2\left(2xy+e^x\right)$

$\dfrac{1}{M}\left(\dfrac{dN}{dx}-\dfrac{dM}{dy}\right)$ $\quad \dfrac{1}{N}\left(\dfrac{dm}{dy}-\dfrac{dN}{dx}\right)$

$\therefore\ \left(\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}\right)=\dfrac{2}{y}$ $\quad \dfrac{1}{2x\cdot y+e^x}$ $\quad \dfrac{1}{-e^x}\left(2x+e^x\right)$

$\therefore\ \text{I.F.}=e^{-\int\frac{2}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=\dfrac{1}{y^2}$; $\dfrac{1}{2x\cdot y+e^x}\left(-e^x+4xy\right)$

Multiplying the given equation by $\dfrac{1}{y^2}$, we get $\dfrac{1}{2x\cdot y+e^x}$

$$\frac{1}{y}\left(2xy+e^x\right)dx-\frac{e^x}{y^2}dy=0$$

i.e., $\left(2x+\frac{e^x}{y}\right)dx-\frac{e^x}{y^2}dy=0$ which is an exact equation.

Hence the required solution is

$$\int\left(2x+\frac{e^x}{y}\right)dx+0=c$$

or, $2\cdot\frac{x^2}{2}+\frac{e^x}{y}=c$

$\therefore\quad x^2+\frac{e^x}{y}=c$

Example 8. Show that the I.F. of the equation

$(x^2+y^2+2x)\,dx-+2y\,dy=0$

is e^x and its particular solution $x^2+y^2=2e^{1-x}$ when $x=y=1$.

Sol^n : Multiplying the given equation by e^x we get,

$$e^x\left(x^2+y^2+2x\right)dx+2ye^x\,dy=0 \qquad \text{.... (i)}$$

which is of the form $Mdx+Ndy=0$

$\therefore\quad M=e^x(x^2+y^2+2x),\ N=2ye^x$

$\therefore\quad \frac{\partial M}{\partial y}=e^x.2y.\frac{\partial N}{\partial x}=2ye^x$

$\therefore\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

So the equation (i) is exact.

Hence e^x is an integrating factor.

Therefore, the solution of the given equation is

$$\int e^x\left(x^2+y^2+2x\right)dx+0=c$$

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$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

or, $\dfrac{x}{y}-2\log x+3\log y=c$

$\therefore\ \dfrac{x}{y}+\log\dfrac{y^3}{x^2}=c,$ c being a constant.

Example 7 : Solve, $(2xy+e^x)y\,dx-e^x dy=0$

Sol^n : Hence $M=(2xy+e^x)y$, $N=-e^x$

$\therefore\ \dfrac{\partial M}{\partial y}=4xy+e^x,\ \dfrac{\partial N}{\partial x}=-e^x$

$\therefore\ \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

$\dfrac{4xy+e^x}{2x\cdot y+e^x}$

$M=2x\cdot y+e^x$

$N=-e^x$

$\dfrac{dM}{dy}=2x+0$

$\dfrac{dN}{dx}=-e^x$

$\therefore\ \dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}=2(2xy+e^x)$

$\therefore\ \left(\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}\right)=\dfrac{2}{y}$

$\dfrac{1}{M}\left(\dfrac{dN}{dx}-\dfrac{dM}{dy}\right)\ \dfrac{1}{N}\left(\dfrac{dM}{dy}-\dfrac{dN}{dx}\right)$

$\dfrac{1}{2x\cdot y+e^x}\left(\dfrac{1}{-e^x}(2x+e^x)\right)$

$\therefore$ I.F. $=e^{-\int\frac{2}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=\dfrac{1}{y^2}$; $\dfrac{1}{2x\cdot y+e^x}\left(-e^x+4xy\right)$

Multiplying the given equation by $\dfrac{1}{y^2}$, we get $\dfrac{1}{2x\cdot y+e^x}$

$$\frac{1}{y}(2xy+e^x)dx-\frac{e^x}{y^2}dy=0$$

i.e., $\left(2x+\frac{e^x}{y}\right)dx-\frac{e^x}{y^2}dy=0$ which is an exact equation.

Hence the required solution is

$$\int\left(2x+\frac{e^x}{y}\right)dx+0=c$$

or, $2\cdot\frac{x^2}{2}+\frac{e^x}{y}=c$

$$\therefore\quad x^2+\frac{e^x}{y}=c$$

Example 8. Show that the I.F. of the equation

$(x^2+y^2+2x)\,dx-+2y\,dy=0$

is e^x and its particular solution $x^2+y^2=2e^{1-x}$ when $x=y=1$.

Sol^n : Multiplying the given equation by e^x we get,

$$e^x\left(x^2+y^2+2x\right)dx+2ye^x\,dy=0 \qquad \text{.... (i)}$$

which is of the form $Mdx+Ndy=0$

$\therefore\quad M=e^x(x^2+y^2+2x),\ N=2ye^x$

$$\therefore\quad \frac{\partial M}{\partial y}=e^x.2y.\frac{\partial N}{\partial x}=2ye^x$$

$$\therefore\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

So the equation (i) is exact.

Hence e^x is an integrating factor.

Therefore, the solution of the given equation is

$$\int e^x\left(x^2+y^2+2x\right)dx+0=c$$

or, $\int e^x x^2 dx + y^2 \int e^x dx + 2\int xe^x dx = c$

or, $x^2 e^x - \int 2xe^x + y^2 e^x + \int xe^x dx = c$

$$x^2 + y^2 = ce^{-x}$$

When $x = y = 1$, we have

$$1 + 1 = c.e^{-1}$$

$\therefore \quad c = 2e$

Thus the particular solution is

$$x^2 + y^2 = 2ce^{-x}$$

i.e., $x^2 + y^2 = 2e^{1-x}$

Example 9 : Prove that $(x + y + 1)^{-4}$ is integrating factor of the equa $(2xy - y^2 - y)\, dx + (2xy - x^2 - x)\, dy = 0$ and hence solve it.

Sol^n : Multiplying the given equation by $(x + y + 1)^{-4}$ we get

$$\frac{2xy - y^2 - y}{(x+y+1)} dx + \frac{2xy - x^2 - x}{(x+y+1)^4} dy = 0.$$

which is of the form $Mdx + Ndy = 0$

$$\therefore \quad M = \frac{2xy - y^2 - y}{(x+y+1)^4}, \quad N = \frac{2xy - x^2 - x}{(x+y+1)^4}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{(2x-2y-1)(x+y+1)^4 - 4(x+y+1)^3(2xy - y^2 - y)}{(x+y+1)^8}$$

$$(2x^2 + 2y^2 - 8xy + x + y + 1)/(x + y + 1)^5$$

and $$\frac{\partial N}{\partial x} = \frac{(2y-2x-1)(x+y+1)^4 - 4(2xy - x^2 - x)(x+y+1)^3}{(x+y+1)^8}$$

$$= (2x^2 + 2y^2 - 8xy + x + y + 1)/(x + y + 1)^5$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So the equation (i) is exact. Hence $(x + y + 1)^{-1}$ is an intengrating factor. Therefore, the solution of the equation is given by

$$\int \frac{2xy - y^2 - y}{(x+y+1)^4} dx + 0 = e,$$

since there is no term which does not contain x.

$$\text{or,} \quad 2y\int \frac{(x+y+1)-(y+1)}{(x+y+1)^4} dx - \int \frac{y(y+1)}{(x+y+1)^4} dx = c$$

$$\text{or,} \quad 2y\int \frac{dx}{(x+y+1)^3} - 3y(y+1)\int \frac{dx}{(x+y+1)^4} = c$$

$$\text{or,} \quad -\frac{x}{(x+y+1)^2} + \frac{y(y+1)}{(x+y+1)^3} = c$$

or, $xy = c\,(x + y + 1)^3$ which is the required solution.

Example 10. Solve $2 \sin y^2 dx + xy \cos y^2 dy = 0$, $y(2) = \sqrt{\frac{\pi}{2}}$.

Sol^n : Here $M = 2 \sin y^2$, $N = xy \cos y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 4y \cos y^2, \quad \frac{\partial N}{\partial x} = y \cos y^2$$

$$\therefore \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y\cos y^2}{xy\cos y^2} = \frac{3}{x}$$

which is a function of x only.

$$\therefore \quad \text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3.$$

Multiplying the equation by x^3 we get

$2x^3 \sin y^2 dx + x^4 y \cos y^2 dy = 0$ which is an exact equation.

Therefore, the solution is

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$\int 2x^3 \sin y^2 dx + 0 = \frac{c}{2}$, since there is no term which does not contain in N.

i.e., $x^4 \sin y^2 = c$.

As we are given $y(2) = \sqrt{\frac{\pi}{2}}$, so $2^4 \cdot \sin\frac{\pi}{2} = c$

$\therefore \quad c = 16.$

So the required solution is $x^4 \sin y^2 = 16$.

Linear Queations

A first order differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{.... (1)}$$

where P and Q are functions of x alone or constants, is known as first order linear equations.

To solve such type equations, we use the integrating factor as $e^{\int P dx}$.

Multiplying both side of (1) by $e^{\int P dx}$ we get

$\frac{dx}{dy} + P(y)$

$$\frac{dy}{dx} e^{\int P dx} + Pye^{\int P dx} = Qe^{\int P dx}$$

$\frac{dy}{dx} + P(y) = 0$

$e^{\int P \cdot dx}$

$e^{\int P \cdot dx}$

or, $\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$

or, $d\left(ye^{\int P dx}\right) = Qe^{\int P dx} dx.$

Integrating both si des, we have

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + c, \text{ } c \text{ being a constant.}$$

1. Firs
2. Orc
3. Cor
4. Cor
5. Ap
6. Pa
7. Pa

$$\therefore \qquad y = e^{-\int P dx}\left[\int Q e^{\int P dx} dx + c\right] \qquad \text{.... (2)}$$

which is the required solution of (1).

Note : Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. Then the equation takes the form $\dfrac{dx}{dy} + Px = Q$ .... (3)

where P, Q are functions of y alone or constants.

In this case I.F. is $e^{\int P dy}$ and the solution is

$$x = e^{-\int P dy}\left[\int Q e^{-\int P dy} dy + c\right] \qquad \text{... (4)}$$

Example 1. Solve : $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

Sol^n : The given equation can be written as

$$\frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x,$$

which is a linear equation in y.

Here I.F. = $e^{\int \sec^2 x . dx = e^{\tan x}}$.

Multiplying both sides of (i) by $e^{\tan x}$, we get

$$\frac{d}{dx}\left(ye^{\tan x}\right) = \tan x \sec^2 x \, e^{\tan x}$$

or, $d\left(ye^{\tan x}\right) = \tan x \sec^2 x \, e^{\tan x} \, dx$

which on integration gives

$$ye^{\tan x} = \int \tan x \sec^2 x \, e^{\tan x} dx$$

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$= \int ze^z\, dz$, by putting $\tan x = z$ i.e. $\sec^2 x dx = dz$

$= ze^z - e^z + c$

$= e^{\tan x}(\tan x - 1) + c$, c being a constant.

$\therefore \quad y = \tan x - 1 + ce^{-\tan x}$, which is the required solution.

Eample 2. Solve : $(1 + y^2)dx = (\tan^{-1} y - x)\, dy$.

The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is a linear equation in x.

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Sol^n : Here I. F. $= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$.

Multiplying both sides of (i) by $e^{\tan^{-1} y}$, we get

$$\frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = e^{\tan^{-1} y} \cdot \frac{\tan^{-1} y}{1+y^2}$$

or, $d\left(xe^{\tan^{-1} y}\right) = \dfrac{\tan^{-1} y\, e^{\tan^{-1} y}}{1+y^2} dy$

which on integration gives

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y\, e^{\tan^{-1} y}}{1+y^2} dy$$

1. Firs
2. Orc
3. Con
4. Cor
5. Ap
6. Pa
7. Pa

$= \int ze^z\, dz$, by putting $\tan^{-1} y = z$ i.e. $\dfrac{1}{1+y^2} dy = dz$

$= ze^z - e^z + c$, c being a constant.

$= e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) + c.$

$\therefore \quad x = \tan^{-1} y - 1 + c\, e^{-\tan^{-1} y}$, which is the required solution

Examplwe 3. Solve : $(x + y + 1)\dfrac{dy}{dx} = 1$.

Sol^n : The given equation can be written

$$\frac{dx}{dy} - x = y + 1 \quad \text{.... (i)}$$

$\frac{dx}{dy} = x+y+1$

which is a linear equation in x.

$\frac{dx}{dy} + x = y+1$

Here I.F. = $e^{\int(-1)dy} = e^{-y}$

$e^{\int 1 \cdot dy}$

e^{-}

Multiplying both sides of (i) by e^{-y}, we get

$$\frac{d}{dx}\left(xe^{-y}\right) = (y+1)e^{-y}$$

or, $d\left(xe^{-y}\right) = (y + 1)\, e^{-y}\, dy$

which on integration gives

$$xe^{-y} = \int (y+1)e^{-y}dy$$

$= -ye^{-y} - 2e^{-y} + c$, c being a constant

$\therefore \quad x = -y - 2 + c\, e^{y}$

$\therefore \quad x + y + 2 = c\, e^{y}$, which is the required solution.

Example 4. Show that the equation of the curve whose slope at any point is equal to $(y + 2x)$ and which passes through the origin is $y = 2(e^x - x - 1)$.

Sol^n : The slope of the curve at any point (x, y) is $\dfrac{dy}{dx}$

$\therefore$ By the given condition,

$$\frac{dy}{dx} = y + 2x$$

$$\therefore \quad \frac{dy}{dx} - y = 2x$$

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which is a linear equation in y.

$$\therefore \quad \text{I.F.} = e^{\int -dx} = e^{-x}$$

Multiplying both side of (i) by e^{-x} we get,

$$e^{-x}\frac{dy}{dx} - ye^{-x} = 2xe^{-x}$$

$$\text{or,} \quad \frac{d}{dx}\left(ye^{-x}\right) = 2xe^{-x}$$

$$\therefore \quad d\left(ye^{-x}\right) = 2xe^{-x}\,dx$$

which on integration gives

$$ye^{-x} = 2\int xe^{-x}dx + c = 2\left[x\frac{e^{-x}}{-1} - \int 1.\frac{e^{-x}}{-1}dx\right] = 2\left[-xe^{-x} - e^{-x}\right] + c$$

$$\therefore \quad y = -2(x+1) + ce^{x}$$

which passes through the origin

$$\therefore \quad 0 = -2(0+1) + c.1$$

$$\therefore \quad c = 2$$

Thus the required equation of the curve is

$$y = -2(x+1) + 2e^{x}$$

i.e., $y = 2\,(e^{x} - x - 1)$

$\frac{dy}{dx} + Py = Qy^n$

BERNOULLI'S EQUATION

1. Firs
2. Ord
3. Con
4. Con
5. Apl
6. Par
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The equation $\boxed{\frac{dy}{dx} + Py = Qy^{n}}$

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation*.

To solve (1), divide both sides by y^{n}, so that $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

Put $y^{1-n} = z$ so that $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dz}$.

i.e. $y^{-n}\dfrac{dy}{dx}=\dfrac{1}{1-n}\dfrac{dz}{dx}$; we get

$\therefore$ **(2)** become $\dfrac{1}{1-n}\dfrac{dz}{dx}+Pz=Q$

$$\Rightarrow \quad \frac{dz}{dx}+P(1-n)z=Q(1-n).$$

which is a linear equation in z and can be solved by the method of the previous article.

Remarks : (1) Putting $z=\int y^{-n}dy$, (6) can be reduced to a linear form $\dfrac{dz}{dx}+\mathrm{P}(1-z)z=\mathrm{Q}$ also.

(2) Another type Bernoulli's equation of the form

$$\frac{dx}{dy}+Px=Qx^{n}$$

where P, Q are function of y only or constants, can be reduced to linear equation by putting $x^{1-n}=z$.

$y^{-2}\dfrac{dy}{dx}+\dfrac{1}{x}y^{-1}=\dfrac{1}{x^2}$

$e^{\int \frac{1}{x}dx}$

$e^{\log x}$

x

Illustrative Examples.

Example 1. Solve : $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^2}{x^2}$.

Dividing both sides by y^2, the given equation becomes

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=\frac{1}{x^2} \qquad \text{.... (i)}$$

$y^{-1}=t$,

$-1y^{-2}=\dfrac{dt}{dx}$

$-y^{-2}\cdot dx=dt$

Sol^n : Put $y^{-1}=z$ so that $-y^{-2}\dfrac{dy}{dx}=\dfrac{dz}{dx}$.

i.e., $y^{-2}\dfrac{dy}{dx}=-\dfrac{dz}{dx}$

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5. App
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(student can solve by putting)

$$z = \int y^{-2} dy = -y^{-1} \text{ also)}$$

$\therefore$ Equation (i) becomes $-\dfrac{dz}{dx} + \dfrac{1}{x} z = \dfrac{1}{x^2}$

i.e., $\dfrac{dz}{dx} - \dfrac{1}{x} z = -\dfrac{1}{x^2}$

which is a linear equation in z.

So, I.F. $= e^{-\int \frac{1}{x} dx} = e^{-\log x} = x^{-1} = \dfrac{1}{x}$

Multiplying both sides of (ii) by $\dfrac{1}{x}$, we get

$$\frac{d}{dx}\left(z, \frac{1}{x}\right) = -\frac{1}{x^3} \qquad \text{or, } d\left(z, \frac{1}{x}\right) = -\frac{1}{x^3} dx$$

which on integratioin gives

$$z.\frac{1}{x} = -\int \frac{1}{x^3} dx = \frac{1}{2x^2} + c \quad \text{or, } y^{-1} \frac{1}{x} = \frac{1}{2x^2} + c$$

$\therefore$ $\dfrac{1}{xy} = \dfrac{1}{2x^2} + c$, which is the required solution.

Example 2. Solve $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\, e^x$

Putting $\sin y = z$ so that $\cos y \dfrac{dy}{dx} = \dfrac{dz}{dx}$ in (i), we get

$$\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x$$

which is a linear equation in z.

$\therefore$ I.F. $= e^{-\int \frac{1}{1+x}dx} = e^{-\log(1+x)} = (1+x)^{-1}$

Multiplying both sides of (2) by $(1 + x)^{-1}$ we get

$$(1+x)^{-1}\frac{dz}{dx} - z(1-1x)^{-2} = e^x$$

$\frac{dv}{dx} + \frac{1}{x}\sin 2y = x^2\cos^2 y.$

or, $d\{z(1+x)^{-1}\} = e^x\,dx$

which on integration gives

$\sec^2 y\frac{dy}{dx} + \frac{1}{x}$

$$z(1+x)^{-1} = e^x + c$$

or, $d\{z(1+x)^{-1}\} = e^x dx$

which on integration gives

$$z(1+x)^{-1} = e^x + c$$

or, $\sin y(1+x)^{-1} = e^x + c$

$\therefore \sin y = (1+x)(e^x + c)$

which is the required general solution.

Example 3. Solve : $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$.

Dividing by $\cos^2 y$, the given equation becomes

$$\sec^2 y\frac{dy}{dx} + \frac{1}{x}2\tan y = x^3 \qquad \text{.... (i)}$$

Sol^n : Put $\tan y = z$, so that $\sec^2 y\,\frac{dy}{dx} = \frac{dz}{dx}$

Therefore the equation (i) becomes

$$\frac{dz}{dx} + \frac{1}{x}2z = x^3 \qquad \text{.... (ii)}$$

which is a linear equation in z.

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so I.F. $= e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2$.

Multiplying both sides of (ii) by x^2, we get

$$\frac{d}{dx}(z.x^2) = x^5$$

or, $d(z.x^2) = x^5 dx$

which on integration gives

$$zx^2 = \frac{x^6}{6} + \frac{c}{6}, \text{ } c \text{ being a constant.}$$

$\therefore$ $6x^2 \tan y = x^6 + c$, which is the required solution.

Example 4. Solve, $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)^2}{x^2}$

The given equation can be written as

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{\log y} = \frac{1}{x^2}$$

Sol^n : Putting $z = \frac{1}{\log y}$ so that

$$\frac{dz}{dx} = \frac{1}{(\log y)^2}.\frac{1}{y}.\frac{dy}{dx}$$

i.e., $\frac{1}{y(\log y)^2}\frac{dy}{dx} = -\frac{dz}{dx}$

in (i) we get,

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$$

1. Firs
2. Ord
3. Con
4. Con
5. Apf
6. Par
7. Par

$$\therefore \quad \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

which is a linear equation in z

$$\therefore \quad \text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = x^{-1}$$

Multiplying both sides of (i) by x^{-1}, we get,

$$x^{-1}\frac{dz}{dx} - zx^{-2} = -x^{-3}$$

or, $\frac{d}{dx}\left(zx^{-1}\right) = -x^{-3}$ $\quad \therefore \; d\left(zx^{-1}\right) = -x^3 dx$

which on integration gives

$$zx^{-1} = -\frac{x^{-2}}{-2} + c$$

or, $\frac{z}{x} = \frac{1}{2x^2} + c$ $\quad$ or, $\frac{1}{x \log y} = \frac{1}{2x^2} + c$

or, $\frac{1}{\log y} = \frac{1 + 2cx^2}{2x}$

$\therefore \quad \log y(1 + 2cx^2) = 2x$

which is the required general solution.

EQUATION SOLVABLE FOR p

$1 - \cos x = 2\sin^2\frac{x}{2}$

$\sin x = 2\sin\frac{x}{2} \cdot \cos\frac{x}{2}$

Let the equation (1) can be put in the form

$$\{p - f_1(x, y)\}\ \{p - f_2(x, y)\}\ \ldots\ldots\ \{p - f_n(x, y)\} = 0$$

which is equivalent to

$$p - f_1(x, y) = 0,\ p - f_2(x, y) = 0,\ \ldots\ldots,\ p - f_n(x, y) = 0 \qquad \ldots (2)$$

i.e., $\frac{dy}{dx} - f_1(x,y) = 0, \frac{dy}{dx} - f_2(x,y) = 0, \ldots\ldots, \frac{dy}{dx} - f_n(x,y) = 0.$

$(\cos x + 1) = (2\cos^2\frac{x}{2})$

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Each of these equations is of first order and first degree and can be solved by the methods discussed previous chapter. Let the solution be

$$F_1(x, y, c_1) = 0, F_2(x, y, c_2) = 0, \ldots\ldots, F_n(x, y, c_n) = 0 \qquad \ldots (3)$$

where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants..

Hence the general solution of (1) is given by

$$F_1(x, y, c)\, F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0 \qquad \ldots (4)$$

where $c_1 = c_2 = \ldots\ldots = c_n = c$, is any arbitrary constant. (We should have one arbitrary constant because this is a first order different equation).

Example 1. Solve $p^2 + px + py + xy = 0$.

Sol^n : Here $p^2 + px + py + xy + 0$ where $p = \dfrac{dy}{dx}$

$$\Rightarrow p(p + x) + y(p + x) = 0$$

$$\Rightarrow (p + x)(p + y) = 0$$

Now, $p + x = 0$ gives $dy + x\,dx = 0$

$$\Rightarrow y + \frac{x^2}{2} = c_1$$

and $p + y = 0$ gives $dy + y\,dx = 0$

$$\Rightarrow \frac{dy}{y} + dx = 0 \text{ or, } \log y + x = c_2.$$

Hence the required general solution is

$$\left(y + \frac{x^2}{2} - c\right)(x + \log y - c) = 0$$

where c is an arbitrary constant.

Example 2. Solve $p^2 + 2py \cot x = y^2$

Sol^n : Here $p^2 + 2py \cot x = y^2$

$$\Rightarrow p^2 + 2py \cot x$$

$$= y^2 (\operatorname{cosec}^2 x - \cot^2 x)$$

1. Fir
2. Or
3. Co
4. Co
5. Ap
6. Pa
7. Pa

$\Rightarrow p^2 + 2py \cot x + y^2 \cot^2 x = y^2 \operatorname{cosec}^2 x$

$\Rightarrow (p + y \cot x)^2 - (y \operatorname{cosec} x)^2 = 0$

$\Rightarrow (p + y \cot x + y \operatorname{cosec} x)(p + y \cot x - y \operatorname{cosec} x) = 0$

$\therefore \quad p + y \cot x + y \operatorname{cosec} x = 0$ gives $\dfrac{dy}{dx} + y\,(\operatorname{cosec} x + \cot x) = 0$

$\Rightarrow \dfrac{dy}{y} + (\operatorname{cosec} x + \cot x)\,dx = 0$

$\Rightarrow \dfrac{dy}{y} + \dfrac{1+\cos x}{\sin x}dx = 0$

$\Rightarrow \dfrac{dy}{y} = -\cot\dfrac{x}{2}dx$

Integrating, we get $\log y = -2 \log \sin\dfrac{x}{2} + c_1$

$\Rightarrow \log y + \log \sin^2\dfrac{x}{2} = \log c'$

$\Rightarrow y \sin^2 \dfrac{x}{2} = c'$ where $c_1 = \log c'$

Again $p + y \cot x - y \operatorname{cosec} x = 0$ gives $\dfrac{dy}{y} = \dfrac{1-\cos x}{\sin x} = \tan\dfrac{x}{2}$

Integrating, we get $\log y = 2 \log \sec \dfrac{x}{2} + \log c_2$

$\Rightarrow \log y - \log \sec^2 x/2 = \log c_2$

$\Rightarrow y \cos^2 \dfrac{x}{2} = c_2$

The general solution is

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$$\boxed{\left(y\sin^2\frac{x}{2}-c\right)\left(y\cos^2\frac{x}{2}-c\right)=0}$$

where c is an arbitrary constant.

Example 1. Solve : $p^2 - p(e^x + e^{-x}) + 1 = 0$.

Sol^n : The equation can be written as $(p - e^x)(p - e^{-x}) = 0$.

i.e., $p - e^x = 0$ or, $p - e^{-x} = 0$

$(p - e^x) = 0$ gives $\dfrac{dy}{dx} = e^x$.

or, $dy = e^x dx$

which on integration gives $y = e^x + c_1$.

Also $p - e^{-x} = 0$ gives $\dfrac{dy}{dx} = e^{-x}$

or, $dy = e^{-x}\, dx$

which on integration gives $y = -e^{-x} + c$

Also, $p - e^{-x} = 0$ gives $\dfrac{dy}{dx} = e^{-x}$

or, $dy = e^{-x}\, dx$

which on integration gives $y = -e^{-x} + c_2$

Hence the general solution of the equation is

$(y - e^x - c)(y + e^{-x} - c) = 0$, c is an arbitrary constant.

Example 2. Solve : $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

Sol^n : The given equation can be written as $(px - 2y)(py + 3x) = 0$.

i.e., $px - 2y = 0$

or, $py + 3x = 0$.

Now $px - 2y = 0$ gives $\dfrac{dy}{dx} = \dfrac{2y}{x}$

or, $\dfrac{dy}{y} - \dfrac{2dx}{x} = 0$

which on integration gives

$\log y - 2\log x = \log c_1$

$\therefore \dfrac{y}{x^2} = c_1$

$\therefore y = c_1x^2$.

Again $py + 3x = 0$ gives $p = -$

or, $\dfrac{dy}{dx} = -\dfrac{3x}{y}$

or, $ydy = -3xdy$

which on integration gives

$$\frac{y^2}{2} = -\frac{3x^2}{2} + \frac{c_2}{2}$$

$\therefore y^2 + 3x^2 = c_2$.

Hence the general solution

$(y - cx^2)(y^2 + 3x^2 - c)$

Example 3. Solve : $xy\{($

Sol^n : The given equation

or, $(xp + y)(yp - x) = 0$

$\therefore xp + y = 0$

or, $yp - x = 0$.

Now $xp + y = 0$ gives $x\dfrac{d}{d}$

or, $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$

1. Firs
2. Ord
3. Con
4. Con
5. App
6. Par
7. Par

which on integration gives

$$\log y - 2\log x = \log c_1$$

$$\therefore \quad \frac{y}{x^2} = c_1$$

$$\therefore \quad y = c_1 x^2.$$

Again $py + 3x = 0$ gives $p = -\dfrac{3x}{y}$

or, $\dfrac{dy}{dx} = -\dfrac{3x}{y}$

or, $ydy = -3xdy$

which on integration gives

$$\frac{y^2}{2} = -\frac{3x^2}{2} + \frac{c_2}{2}$$

$\therefore \quad y^2 + 3x^2 = c_2.$

Hence the general solution of the equation is

$(y - cx^2)(y^2 + 3x^2 - c) = 0$, c is an arbitrary constant.

Example 3. Solve : $xy\left\{\left(\dfrac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\dfrac{dy}{dx}.$

Sol^n : The given equation can be written as $xy(p^2 - 1) = (x^2 - y^2)p.$

or, $(xp + y)(yp - x) = 0$

$\therefore \quad xp + y = 0$

or, $yp - x = 0.$

Now $xp + y = 0$ gives $x\dfrac{dy}{dx} = -y$

or, $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$

which on integration gives

$\log y + \log x = \log c_1$

$\therefore \quad xy = c_1$

Again $yp - x = 0$ gives $y\dfrac{dy}{dx} - x = 0$

or, $ydy - xdx = 0$

which on integration gives

$$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c_2}{2}$$

$\therefore \quad y^2 - x^2 = c_2.$

Hence the general solution of the equation is

$(xy - c)(x^2 - y^2 + c) = 0$, c is an arbitrary constant.

Example 4. Solve the following differential equation

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

Sol^n : The gives equation can be written as

$$p - \frac{1}{p} = \frac{x^2 - y^2}{xy} \quad \text{where } p = \frac{dy}{dx}$$

or, $(p^2 - 1)\,xy = p\,(x^2 - y^2)$

or, $p^2xy - xy - px^2 + py^2 = 0$

or, $(py - x)\,(px + y) = 0$

$\therefore \quad py - x = 0 \qquad$ or, $px + y = 0$

Now, $py - x = 0$ gives

$$p = \frac{x}{y}$$

or, $y\,dy - x\,dx = 0$

which on integration gives

$$\frac{y^2}{2}-\frac{x^2}{2}=\frac{c_1}{2}$$

$\therefore \quad y^2 - x^2 - c_1 = 0$

Again $px + y = 0$ gives

$$\frac{dy}{y}+\frac{dx}{x}=0$$

which on integration gives

$$\log y + \log x = \log c_2$$

$\therefore \quad xy - c_2 = 0$

Thus the required general solution is

$$\left(y^2 - x^2 - c\right)\left(xy - c\right) = 0$$

where c is an arbitrary constant.

EQUATION SOLVABLE FOR y

In this case, the given differential equation can be written in the form

$$y = f(x, p) \text{ where } p = \frac{dy}{dx}$$

then differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x}.\frac{dx}{dx}+\frac{\partial f}{\partial p}\frac{dp}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}.\frac{dp}{dx}$$

$$= \phi\left(x, p, \frac{dp}{dx}\right) \text{ (say)}$$

Then we solve $p = \phi\left(x, p, \frac{dp}{dx}\right)$ to obtain its solution $\psi(x, p, c) = 0$, say.

The general solution is obtained by eliminating p between

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$y = f(x, p)$ and $\psi(x, p, c) = 0$.

Sometimes, the solutions is stated in the parametric form

$x = u(p)$ and $y = v(p)$, p being treated as a parameter.

Example 1. Solve $y = a\sqrt{1+p^2}$.

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Sol^n : Here $y = a\sqrt{1+p^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = p = a\,\frac{2p}{2\sqrt{1+p^2}}\frac{dp}{dx}$$

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$$\Rightarrow \quad p = \frac{ap}{\sqrt{1+p^2}}\frac{dp}{dx}$$

$$\Rightarrow \quad dx = \frac{a\,dp}{\sqrt{1+p^2}}$$

$\int \frac{dx}{\sqrt{1+x^2}} = \log(x+\sqrt{1+x^2}) + C$

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Integrating, we get

$$\int dx = \int \frac{a\,dp}{\sqrt{1+p^2}} + c \text{ or } x = a\log\left(p + \sqrt{1+p^2}\right) + c \quad \ldots(2)$$

From equation (1), $p = \sqrt{\frac{y^2-a^2}{a^2}} = \frac{1}{a}\sqrt{y^2-a^2}$

Eliminating p, between equations (2) and (3), we get

$$x = a\log\left(\frac{\sqrt{y^2-a^2}+y}{a}\right) + c$$

which is the required solution and c is an arbitrary constant.

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$y = px + p^2)$

Example 2. Solve $y = px + p^2x$.

Sol^n : Here $y = px + p^2x$

$\frac{dy}{dx} =$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + p^2 + 2xp\frac{dp}{dx}$$

$$\Rightarrow \quad p = p + x\frac{dp}{dx}(1 + 2p) + p^2$$

$$\Rightarrow \quad -p^2 = x\frac{dp}{dx}(1 + 2p)$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{1+2p}{p^2}dp = 0$$

Integrating, we get $\log x + \log c = \frac{1}{p} - 2\log p$.

$$\Rightarrow \quad c\,x\,p^2 = e^{1/p}$$

$$\Rightarrow \quad x = \frac{e^{1/p}}{cp^2}$$

and $$y = x\,(p + p^2) = \frac{e^{1/p}}{cp^2}(p + p^2)$$

which is the parametric solution of the given equation (p is the parameter).

Example 3. Solve $y = 2px - p^2$.

Sol^n : The given equation is $y = 2px - p^2$ .... (i)

Differentiating (i) w.r.t. x,

$$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

or, $$p + 2\,(x - p)\,\frac{dp}{dx} = 0$$

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or, $p\dfrac{dx}{dp}+2(x-p)=0$

$$\therefore \quad \frac{dx}{dp}+\frac{2}{p}x=2$$

which is a linear equation in x.

$$\therefore \quad \text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

Multiplying both sides of (ii) by p^2, we get

$$\frac{d}{dp}(x.p^2)=2p^2$$

or, $d(xp^2) = 2p^2dp$

which on integration gives $xp^2 = \dfrac{2p^3}{3}+c.$

$$\therefore \quad x=\frac{2}{3}p+cp^{-2}$$

Putting this value of x in (i), we get

$$y=2p\left(\frac{2}{3}p+cp^{-2}\right)-p^2=\frac{1}{3}p^2+2cp^{-1}$$

So the required general solution is

$$\left.\begin{aligned} x&=\frac{2}{3}p+cp^{-2}\\ y&=\frac{1}{3}p^2+2cp^{-1}\end{aligned}\right\}, \text{ where } p \text{ is the parameter.}$$

Example 4. Solve : $y = 2px + p^4x^2$.

Sol^n : The given equation is $y = 2\,px + p^4x^2$

Differentiating (i) w.r.t. x,

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$$p = 2p + 2x\frac{dp}{dx} + 2p^4x + 4p^3x^2\frac{dp}{dx}$$

or, $$4p^3x^2\frac{dp}{dx} + 2x\frac{dp}{dx} + 2p^4x + p = 0$$

or, $(p + 2x\dfrac{dp}{dx})(1 + 2p^3x) = 0.$

Discarding the factor $(1 + 2p^3x)$, we have

$$p + 2x\frac{dp}{dx} = 0$$

$$\int \frac{1}{\sqrt{1+p^2}} = \log|$$

or, $$2\frac{dp}{p} + \frac{dx}{x} = 0$$

which on integration gives

$2 \log p + \log x = \log c.$

$\therefore\ p^2x = c.$

$\therefore\ p^2 = \dfrac{c}{x}.$ .... (ii)

Eliminating p from (i) and (ii) we get

$$y = 2\left(\pm\sqrt{\frac{c}{x}}\right)x + c^2$$

or, $(y - c^2)^2 = 4\dfrac{c}{x}.x^2$

$\therefore\ (y - c^2)^2 = 4cx$, which is the required general solution.

Example 5. Solve : $e^y - p^3 - p = 0$.

Sol^n : The given equation can be written as

$y = \log(p^3 + p)$ .... (i)

Differentiating (i) w.r.t. x,

$$p = \frac{1}{p^3+p}.(3p^2+1)\frac{dp}{dx}$$

or, $$\frac{dp}{dx} = \frac{p^2(p^2+1)}{3p^2+1}$$

or, $$dx = \frac{3p^2+1}{p^2(p^2+1)}dp$$

or, $$dx = \left(\frac{2}{1+p^2} + \frac{1}{p^2}\right)dp.$$

which on integration gives

$$x = 2\tan^{-1} p - \frac{1}{p} + c.$$

Thus the required solution is

$$\left.\begin{aligned} x &= 2\tan^{-1} p - \frac{1}{p} + c \\ y &= \log(p^3+p) \end{aligned}\right\}, \text{ where } p \text{ is the parameter.}$$

Example 6. Solve : $y = (p + p^2)\,x + p^{-1}$

Sol^n : We have $y = (p + p^2)\,x + p^{-1}$

Differentiating (i) w.r.t. x,

$$p = p = p^2 + x\left(\frac{dp}{dx} + 2p\frac{dp}{dx}\right) - \frac{1}{p^2}\frac{dp}{dx}$$

or, $$\frac{dx}{dp} + x\left(\frac{1+2p}{p^2}\right) = \frac{1}{p^4}$$

which is a linear equation in x

$$\therefore \text{ I.F} = e^{\int\left(\frac{1+2p}{p^2}\right)dp} = e^{-\frac{1}{p}+2\log p} = p^2 e^{-\frac{1}{p}}$$

Multiplying both sides of (ii) by I.F. and then integrating we get

$$xp^2 e^{-\frac{1}{p}} = \int \frac{1}{p^4}.p^2.e^{-\frac{1}{p}}dp = \int \frac{1}{p^2}e^{-\frac{1}{p}}dp$$

$$= \int e^{-\frac{1}{p}}d\left(-\frac{1}{p}\right) = e^{-\frac{1}{p}} + c$$

$p = \frac{dy}{dx}$ $\frac{dy}{dx} = p$ $\frac{dy}{dx} = \frac{1}{p}$

$$\therefore \quad xp^2 = 1 + ce^{\frac{1}{p}}$$

$$\therefore \quad x = \frac{1+ce^{\frac{1}{p}}}{p^2} \qquad \text{.... (iii)}$$

$$\therefore \quad \text{From (i), } y = (p+p^2)\cdot\frac{1+ce^{\frac{1}{p}}}{p^2} + \frac{1}{p}$$

$p = \frac{dy}{dx}$

$$\therefore \quad y = \left(1+\frac{1}{p}\right)\left(1+ce^{\frac{1}{p}}\right) + \frac{1}{p} \qquad \text{.... (iv)}$$

Then (iii) and (iv) together gives the required solution.

EQUATION SOLVABLE FOR x

In this case, the given differential equation can be written in the form $x = f(y, p)$ where $p = \frac{dy}{dx}$. Then differentiating with respect to y, we get

$$\boxed{\frac{dy}{dx} = \frac{\partial f}{\partial y}.\frac{dy}{dy} + \frac{\partial f}{\partial p}\frac{dp}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}}$$

x

$$\Rightarrow \qquad \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \text{ (say).}$$

Then we solve this equation and obtain the solution $\psi = (y, p, c) = 0$. Th general solution is obtained by eliminating p from the relation $x = f(y, p)$ an $\psi(y, p, c) = 0$. The complete primitive or general solution can be obtained i the parameteric form $x = u(p)$ and $y = v(p)$ where p is the parameter.

Example 1 : Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Sol^n : We have $p = \tan\left(x - \frac{p}{1+p^2}\right)$

$$\frac{(1+p^2)\cdot\frac{dp}{dx} - p\,d(1+p^2)}{\left(\frac{u}{v}\right)}$$

$$\Rightarrow \qquad x = \frac{p}{1+p^2} + \tan^{-1} p$$

Differentiating w.r.t. y, we get $\frac{dx}{dy}$

$$= \left[\frac{1+p^2 - p.2p}{(1+p^2)^2}\right]\frac{dp}{dy} + \frac{1}{1+p^2}\frac{dp}{dy}$$

$$\Rightarrow \qquad \frac{1}{p} = \left[\frac{1-p^2}{(1+p^2)^2} + \frac{1}{1+p^2}\right]\frac{dp}{dy} = \frac{2}{(1+p^2)}\frac{dp}{dy}$$

$$\Rightarrow \qquad dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating, we get $y + \frac{1}{1+p^2} = c$ where c is an arbitrary constant.

$\therefore \quad x = \frac{p}{1+p^2} + \tan^{-1} p$ and $y = c - \frac{1}{1+p^2}$ givs the parametric solution of the given differential equation.

Example 2. Solve $x = py - p^2$.

Sol^n : Now $x = py - p^2$

Differentiating w.r.t. y, we get $\frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$

$$\Rightarrow \left(\frac{1}{p} - p\right) = (y - 2p)\frac{dp}{dy}$$

$(1+p^2)\cdot\frac{dp}{dy} - p\frac{d(1+p^2)}{dy} + \frac{1}{1+p^2}$

$(1+p^2)\cdot\frac{dp}{dy} -$

$$\Rightarrow \frac{dy}{dp}\left(\frac{1}{p} - p\right) = y - 2p$$

$$\Rightarrow \frac{dy}{dp} + \frac{y}{p - \frac{1}{p}} = \frac{2p}{p - \frac{1}{p}}$$

Which is a first order linear differential equation in y with p as its independent variable. This integrating factor (I.F.)

$$= e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log(p^2-1)} = \sqrt{p^2-1}$$

Multiplying equation (2) by I.F. and integrating w.r.t. p, we get

$$y\sqrt{p^2-1} = \int \frac{2p^2}{(p^2-1)}\sqrt{p^2-1}\,dp + c = \int \frac{2p^2}{\sqrt{p^2-1}}dp + c$$

$$= 2\int \frac{p^2-1+1}{\sqrt{p^2-1}}dp + c = 2\int \sqrt{p^2-1}\,dp + 2\int \frac{dp}{\sqrt{p^2-1}} + c$$

$$= 2\left[\frac{p\sqrt{p^2-1}}{2} - \frac{1}{2}\log(p + \sqrt{p^2-1})\right]$$

$$+2\log(p + \sqrt{p^2-1}) + c$$

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$$= p\sqrt{p^2-1} + \log\,(p+\sqrt{p^2-1}) + c$$

where c is an arbitrary constant.

$$\Rightarrow\ y = p + [\log(p+\sqrt{p^2-1}) + e]/\sqrt{p^2-1}$$

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$$\therefore\ x = py - p^2 = (p^2-1)^{-\frac{1}{2}}\,[\log\,(p+\sqrt{p^2-1} + c]$$

This gives the parameteric solution of the given equation.

Example 3. Solve : $y^2 \log y = xyp + p^2$

Sol^n : The given equation can be written as

$$x = \frac{y\log y}{p} - \frac{p}{y}$$

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Differentiating (i) w.r.t. y,

$$\frac{dx}{dy} = \frac{p(1+\log y) - y\log y\dfrac{dp}{dy}}{p^2} - \frac{y\dfrac{dp}{dy} - p}{y^2}$$

or, $$\frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log y - \frac{y}{p^2}\log y\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

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or, $$\frac{1}{y}\frac{dp}{dy}\left(1+\frac{y^2}{p^2}\log y\right) = \frac{p}{y^2}\left(1+\frac{y^2}{p^2}\log y\right)$$

or, $$\frac{dp}{dy} = \frac{p}{y}\ \text{[discharding the factor}\ \left(1+\frac{y^2}{p^2}\log y\right)$$

or, $$\frac{dp}{p} = \frac{dy}{y}$$

which on inegration gives

$\log p = \log y + \log c$

or, $p = cy.$ .... (ii)

Eliminating p from (i) and (ii) we get

$\log y = cx + c^2$, which is the required solution.

Example 4. Solve : $ayp^2 + (2x - b)\,p - y = 0,\ a > 0$

Sol^n. : The given equation can be written as

$$x = \frac{y}{2p} - \frac{ayp}{2} + \frac{1}{2}b. \qquad \text{.... (i)}$$

Differentiating both sides of (i) w.r.t. y, we get

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{ap}{2} - \frac{ay}{2}\frac{dp}{dy}$$

or, $$\frac{1}{p} + ap + \left(\frac{y}{p^2} + ay\right)\frac{dp}{dy} = 0$$

or, $$a\left(p + y\frac{dp}{dy}\right) + \frac{1}{p^2}\left(p + y\frac{dp}{dy}\right) = 0$$

or, $$\left(a + \frac{1}{p^2}\right)\left(p + y\frac{dp}{dy}\right) = 0$$

$\therefore\ p + y\dfrac{dp}{dy} = 0.$ $\left[\because\ a + \dfrac{1}{p^2} \neq 0, \text{ since } a > 0.\right]$

or, $$\frac{dy}{y} + \frac{dp}{p} = 0$$

which on integration gives

$\log y + \log p = \log c$

$\therefore\ yp = c.$

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Eliminating p from (i) and (ii), we get

$$\frac{ac^2}{y} + (2x - b)\frac{c}{y} - y = 0$$

$\therefore$ $ac^2 + (2x - b)\,c - y^2 = 0$, which is the required solution.

Example 5. Solve : $\tan^{-1} p = x - \dfrac{p}{1+p^2}$.

Sol^n : The given equation can be written as

$$x = \tan^{-1} p + \frac{p}{1+p^2}.$$

Differentiating (i) w.r.t. y, we get

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2) - 2p.p}{(1+p^2)^2}\cdot\frac{dp}{dy}$$

or, $$\frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

or, $$dy = \frac{2p}{(1+p^2)^2}dp$$

which on integration gives

$$y = c - \frac{1}{1+p^2}$$

So the required solution is

$$\left.\begin{aligned} x &= \tan^{-1} p + \frac{p}{1+p^2} \\ y &= c - \frac{1}{1+p^2} \end{aligned}\right\}, \text{ where } p \text{ is the parameter.}$$

Example 6. Solve : $y = 2px + yp^2$

Sol^n : The given equation can be written as

$$x = \frac{y}{2p} - \frac{py}{2} \qquad \text{.... (i)}$$

Differentiting (i) w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy} = 0$$

or, $\left(\frac{1}{p} + p\right) + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + p\right) = 0$ or, $\left(\frac{1+p^2}{p}\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$

$\therefore \quad 1 + \frac{y}{p}\frac{dp}{dy} = 0 \qquad [\because p^2 + 1 \neq 0]$

or, $\frac{dy}{y} + \frac{dp}{p} = 0$

which on integration gives

$$\log y + \log p = \log c$$

or, $py = c \qquad \therefore y^2 = 2cx + c^2$

$\boxed{y = px + f(p)}$

which is the required general solution.

CLAIRAUT'S FORM

A differential equation of the form $y = px + f(p)$ .... (1)

is known as Clairaut's equation.

To solve the equation (11), we differentiate both sides of (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or, $\frac{dp}{dx}\{x + f'(p)\} = 0$

Example 6. Solve : $y = 2px + yp^2$

Sol^n : The given equation can be written as

$$x = \frac{y}{2p} - \frac{py}{2} \qquad \text{.... (i)}$$

Differentiting (i) w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy} = 0$$

or, $\left(\frac{1}{p} + p\right) + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + p\right) = 0$ or, $\left(\frac{1+p^2}{p}\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$

$\therefore \quad 1 + \frac{y}{p}\frac{dp}{dy} = 0 \qquad [\because p^2 + 1 \neq 0]$

or, $\frac{dy}{y} + \frac{dp}{p} = 0$

which on integration gives

$$\log y + \log p = \log c$$

or, $py = c \qquad \therefore y^2 = 2cx + c^2$

$\boxed{y = px + f(p)}$

which is the required general solution.

CLAIRAUT'S FORM

A differential equation of the form $y = px + f(p)$ .... (1)

is known as Clairaut's equation.

To solve the equation (11), we differentiate both sides of (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or, $\frac{dp}{dx}\{x + f'(p)\} = 0$

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Either $\dfrac{dp}{dx} = 0$ i.e., $p = c$, a constant.

or, $x + f'(p) = 0$.

Eliminating p between (1) and (2), we get

$$y = cx + f(c)$$

which is the required general solution of (11).

Again eliminating p between (12) and (13) we get a relation betw and y wherer no arbitrary constant are involved. This is called *singular tion* of the differential equation (11).

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Example 1. Solve the following Clairaut's equation

$$y = px + \sin^{-1} p.$$

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Sol^n : Here $y = px + \sin^{-1} p$

Differentiating w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left(x + \frac{1}{\sqrt{1-p^2}}\right) = 0.$$

Now $\dfrac{dp}{dx} = 0$ gives $p = c$ and hence the general solution is

$$y = cx + \sin^{-1} c.$$

The singular solution is obtained by eliminating p between the gi equation and $x + \dfrac{1}{\sqrt{1-p^2}} = 0$.

Now $x + \dfrac{1}{\sqrt{1-p^2}} = 0$ gives $\sqrt{1-p^2} = -\dfrac{1}{x}$

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$$\Rightarrow \quad 1 - p^2 = \frac{1}{x^2}$$

$$\Rightarrow \quad p = \pm \frac{\sqrt{x^2 - 1}}{x} \qquad \text{.... (2)}$$

From equations (1) and (2), we get the singular solution as

$$y = \pm \sqrt{x^2 - 1} + \sin^{-1}\left(\pm \frac{\sqrt{x^2 - 1}}{x}\right).$$

Example 2. Solve $p = \sin(y - xp)$.

Also find its singular solutions.

Sol^n : Given equation can be written as

$\sin^{-1} p = y - xp \Rightarrow y = px + \sin^{-1} p$ which is the Clairaut's equation.

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (i) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1 - c^2}}$$

To eliminate c from (i) and (ii), we rewrite (ii) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (i), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Example 3. Solve $(px - y)(py + x) = a^2 p$.

Sol^n : Put $x = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$$p = \frac{dy}{dx} = \frac{dv}{y} \Big/ \frac{du}{x} = \frac{x}{y} P, \text{ where } P = \frac{dv}{du}$$

Then the given equation becomes $\left(\frac{xP}{y}.x-y\right)\left(\frac{xP}{y}.y+x\right)=a^2\frac{xP}{y}$

$\Rightarrow$ $(uP - v)(P + 1) = a^2P \Rightarrow uP - v = \frac{a^2P}{P+1}$.

$\Rightarrow$ $u = uP - a^2P/(P + 1)$, which is Clairaut's form.

$\therefore$ its solution is $v = uc - a^2c/(c + 1)$, i.e., $y^2 = cx^2 - a^2c/(c + 1)$.

Example 4. Solve : $(y - px)(p - 1) = p$ and obtain the singular solut

The given equation can be written as $y - px = \frac{p}{p-1}$

i.e. $y = px + \frac{p}{p-1}$ ....

Differentiating (i) w.r.t. x, we get

$$p = p.1 + x\frac{dp}{dx} - \frac{1}{(p-1)^2}\frac{dp}{dx}$$

or, $\frac{dp}{dx}\left(x - \frac{1}{(p-1)^2}\right) = 0.$

Either $\frac{dp}{dx} = 0$ i.e. $p = c$, a constant ... (

or, $x - \frac{1}{(p-1)^2} = 0$ i.e., $p = \frac{1+\sqrt{x}}{\sqrt{x}}$ .... (i

Eliminating p between (i) and (ii), we get

$$y = cx + \frac{c}{c-1}.$$

which is the required general solution.

Again eliminating p between (i) and (ii), we get

$$y = (1+\sqrt{x})\sqrt{x} + 1 + \sqrt{x}$$

$$\therefore \quad (x - y + 1)^2 = 4x$$

which is the required singular solution.

Example 5. Obtain the general solution and singular solution of the equation $y = px + \sqrt{a^2p^2 + b^2}$.

Sol^n : The given equation is $y = px + \sqrt{a^2p^2 + b^2}$ .... (i)

Differentiating (i) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{a^2p^2+b^2}} \times a^2 p$$

$$\text{or,} \quad \frac{dp}{dx}\left(x + \frac{a^2p}{\sqrt{a^2p^2+b^2}}\right) = 0$$

$$\therefore \quad x = \frac{a^2p}{\sqrt{a^2p^2+b^2}} \qquad \text{.... (ii)}$$

Eliminating p between (i) and (ii), we get

$y = cx + \sqrt{a^2c^2 + b^2}$, which is the required general solution.

Again from (i) and (iii), we have

$$y = -\frac{a^2p^2}{\sqrt{a^2p^2+b^2}} + \sqrt{a^2p^2+b^2} = \frac{b^2}{\sqrt{a^2p^2+b^2}}$$

$$\therefore \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(-\frac{ap}{\sqrt{a^2p^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2p^2+b^2}}\right)^2$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which is the required singular solution.

Example 6. Find the general and singular solution of the equation

$$py = p^2(x-b)+a.$$

Sol^n : The given equation can be written as

$$y = px+\frac{a}{p}-bp$$

Differentiating (i) w.r.t. x, we get

$$p = p+x\frac{dp}{dx}-\frac{a}{p^2}\frac{dp}{dx}-b\frac{dp}{dx}$$

$$\text{or,} \quad \frac{dp}{dx}\left(x-\frac{a}{p^2}-b\right)=0.$$

Either $\dfrac{dp}{dx}=0$ i.e., $p=c$, a constant

$$\text{or,} \quad c-\frac{a}{p^2}-b=0.$$

$$\text{i.e.} \quad p=\pm\sqrt{\frac{a}{x-b}}$$

Eliminating p from (i) and (ii), we get

$$cy = c^2(x-b)+a,$$

which is the required general solution.

Again eliminating p between (i) and (iii), we get

$$\pm y\sqrt{\frac{a}{x-b}}=\frac{a}{x-b}(x-b)+a$$

or, $y^2a = 4a^2(x-b)$.

$\therefore\ y^2 = 4a(x-b)$,

which is the required singular solution.

Example 7. Findt he general and singular solution of

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

Sol^n : The given equation can be written as

$$y = xp + p^2 \text{ where } p = \frac{dy}{dx}$$

which is in Clairut's form

Differentiating (i) w.r.t. x, we get,

$$p = 1, p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or, $\frac{dp}{dx}(x+2p) = 0$

$\therefore$ Either $\frac{dp}{dx} = 0$ i.e., p

or, $x + 2p = 0$.

$\therefore\ p = -\frac{x}{2}$

Putting $p = c$ in (i) we get,

$y = cx + c^2$

which is the required general solution

Again putting $p = -\frac{x}{2}$ in (i) we get,

$$y = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 = -\frac{x^2}{4}$$

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$\therefore \quad x^2 + 4y = 0$

which is the required singular solution.

Example 8. Find the general solution of $p = \cos(y - px)$ wherer

$$p = \frac{dy}{dx}$$

Sol^n : The given equation can be written as

$$y - px = \cos^{-1} p$$

$$\therefore \quad y = px + \cos^{-1} p$$

which is in Clairaut's form

Differentiating (i) w.r.t. x, we get,

$$\frac{dy}{dx} = p.1 + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or, $$p = p + \frac{dp}{dx}\left(x - \frac{1}{\sqrt{1-p^2}}\right)$$

$$\therefore \quad \frac{dp}{dx}\left(x - \frac{1}{\sqrt{1-p^2}}\right) = 0$$

$\therefore$ Either $\dfrac{dp}{dx} = 0$ i.e., $p = c$

or, $$x - \frac{1}{\sqrt{1-p^2}} = 0$$

or, $$x = \frac{1}{\sqrt{1-p^2}}$$

Putting $p = c$ in (i) we get,

1. Fi
2. O
3. C
4. C
5. A
6. P
7. P

$$y = cx + \cos^{-1} c$$

which is the required general solution.

Note : We can obtain the singular solution as follows :

$$c\,x = \frac{1}{\sqrt{1-p^2}}$$

$$\therefore\ 1-p^2 = \frac{1}{x^2}$$

$$\therefore\ p = \pm\frac{\sqrt{x^2-1}}{x}$$

Putting these values in (i), we get

$$y = \pm\frac{\sqrt{x^2-1}}{x}.x + \cos^{-1}\left(\pm\frac{\sqrt{x^2-1}}{x}\right)$$

$$\text{or,}\quad y \mp \sqrt{x^2-1} = \cos^{-1}\left(\pm\frac{\sqrt{x^2-1}}{x}\right)$$

$$\text{or,}\quad \pm\frac{\sqrt{x^2-1}}{x} = \cos\left(y \mp \sqrt{x^2-1}\right)$$

$$\text{or,}\quad x^2 - 1 = x^2\cos^2\left(y \pm \sqrt{x^2-1}\right)$$

$$x^2\sin^2\left(y \pm \sqrt{x^2-1}\right) = 1$$

which is the singular solution.

□□□

2 ORDINARY DIFFERENTIAL EQUATION OF HIGHER ORDERS

Second order linear differential equation

The 2nd order linear equation

$D^2+D+=0$

$$\boxed{\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=0}$$

where P_1, P_2 are constants and X is either constant or a function of x only.

Using the symbol D for the different operaor $\frac{d}{dx}$, the above equation can be written in symbolic formas

$$(D^2+P_1D+P_2)\,y=X \qquad \text{.... (1)}$$

or, $f(D)\,y=X$ .... (2)

where $f(D)=D^2+P_1D+P_2$ is a polynomial in D.

When $X=0$ then $f(D)\,y=0$ .... (2)

is called the homogeneous equation.

Theorem 1. If y_1, y_2 are two linearly independent solutions of the differerntial equation $(D^2+P_1D+P_2)\,y=0$, ... (i)

then $u=c_1y_1+c_2y_2$ is also its solution, wherer c_1c_2 are arbitrary constants.

Proof : Left as an exercise.

Note : Since this soluton contains two arbitrary constants, it is the *general or complete solution* of the equation (i).

Theorem 2. If $y=u$ is the general solution of the equation $f(D)\,y=0$ and $y=v$ is a particular solution (contain no arbitrary constants) of the equation $f(D)\,y=X$, then the general solution of the equation $f(D)\,y=X$ is $y=u+v$.

Proof : Since u is the general solution oft he equation $f(D)y = 0$, so

$$f(D)\, u = 0. \qquad \text{... (i)}$$

Also, since $y = v$ is a particular solution of the equation $f(D)\, y = X$,

so $f(D)\, v = X$. ... (ii)

Adding (i) and (ii), we have $f(D)\,(u + v) = X$

which shows that $y = u + v$ satisfies the equation $f(D)\, y = X$.

Hence $y = u + v$ is the general solution of the equation $f(D)\, y = X$.

Here $y = u$ is called the **complementary function (C.F.)** and $y = v$ is called the **particular integral (P.I.)** of the equation $f(D)\, y = X$.

Thus the complete or general solution of the equation (3) i.e. of

$$\frac{d^3y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X \text{ is } y = \text{C.F.} + \text{P.I.}$$

Note. Sometimes the symbols y_c and y_p are used to denote the C.F. and P.I. of the differential equation.

Rules for finding C.F.

Consider the differential equation

$$(D^2 + P_1 D + P_2)\, y = 0. \qquad \text{.... (5)}$$

The general solution of this is nothing but the C.F. of (2) or (3).

Let $y = e^{mx}$ be a trial solution of (5).

Then $Dy = me^{mx}$, $D^2y = m^2e^{mx}$.

Hence from (5), we get

$$(m^2 + P_1 m + P_2)\, e^{mx} = 0$$

i.e., $m^2 + P_1 m + P_2 = 0$, since $e^{mx} \neq 0$

which is called the *auxiliary equation* (A.E.) for the differential equation (5).

Let m_1, m_2 be the root of the auxiliary equation (6). The solution of the equation (5) depends upon the nature of roots of the A.E. (6).

The following cases arise :

Case (i) Let m_1, m_2 be real and distinct. Then the general solution of (5) is $y = c_1e^{m_1x} + c_2e^{m_2x}$, where c_1, c_2 are arbitrary constants.

Case (ii) Let the roots of the a.E. are real and equal (m_1). Then the general solution of the equation (5) is

$$y = (c_1 + c_2 x)\ e^{m_1 x}$$

where c_1, c_2 are arbitrary constants.

Case (iii) Let the roots of A.E. be imaginary which are

$$m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta.$$

Then the general solution of the equation (5) is

$y = e^{\alpha x}\ (c_1 \cos \beta x + c_2 \sin \beta x)$ where c_1, c_2 are arbitrary constants.

Example 1. $(D^3 - 2D^2 - 5D + 6)\ y = (e^{2x} + 3)^2 + e^{3x} \cosh x.$

Sol^n : The reduced equation is

$$(D^3 - 2D^2 - 5D + 6)\ y = 0$$

Let $y = Ae^{mx}$ be a trial solution of equation (1). Then the auxiliary equation is

$$m^3 - 2m^2 - 5m + 6 = 0$$

$$\Rightarrow\ m^3 - m^2 + m - 6m + 6 = 0$$

$$\Rightarrow\ m^2 (m - 1) - m (m - 1) - 6 (m - 1) = 0$$

$$\Rightarrow\ (m - 1) (m^2 - M - 6) = 0$$

$$\Rightarrow\ (m - 1) (m^2 - 3m + 2m - 6) = 0$$

$$\Rightarrow\ (m - 1) (m - 3) (m + 2) = 0$$

$$\Rightarrow\ M = 1, 3, -2.$$

$\therefore$ The complementary function is

$y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$ where c_1, c_2, c_3 are arbitrary constants.

Again, $(e^{2x} + 3)^2 + e^{3x} \cosh x = e^{4x} + 6\ e^{2x} + 9 + e^{3x} \left(\dfrac{e^x + e^{-x}}{2}\right)$

$$= e^{4x} + 6\ e^{2x} + 9e^{0.x} + \frac{e^{4x}}{2} + \frac{e^{2x}}{2}$$

$$= \frac{3}{2} e^{4x} + \frac{13}{2} e^{2x} + 9e^{0.x}$$

$\therefore$ The particular integral is

$$y = \frac{1}{D^3 - 2D^2 - 5D + 6}\left\{\frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}\right\}$$

$$= \frac{1}{(D-1)(D-3)(D+2)}\left\{\frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}\right\}$$

$$= \frac{3}{2}\frac{1}{(D-1)(D-3)(D+2)}e^{4x} + \frac{13}{2}\frac{1}{(D-1)(D+2)(D-3)}e^{2x}$$

$$+9\frac{1}{(D-1)(D-3)(D+2)}e^{0.x}$$

$$= \frac{3}{2}\frac{e^{4x}}{(4-1)(4-3)(4+2)} + \frac{13}{2}\frac{e^{2x}}{(2-1)(2+2)(2-3)}$$

$$+9\frac{e^{0.x}}{(0-1)(0-3)(0+2)}$$

$$= \frac{3}{2}\frac{e^{4x}}{3.1.6} + \frac{13}{2}\frac{e^{2x}}{1.4.(-1)} + 9\frac{e^{0.x}}{(-1)(-3).2} = \frac{e^{4x}}{12}\frac{13}{8}e^{2x} + \frac{3}{2}.$$

Hence the general solution is

y = C.F. + P.I.

$$\boxed{y = c_1e^x + c_2e^{3x} + c_3e^{-2x} + \frac{e^{4x}}{12} + \frac{13}{8}e^{2x} + \frac{3}{2}.}$$

Example 2. Solve $(D^4 + 2D^2 + 1)\,y = \cos x$.

Sol^n : The reduced equation is $(D^4 + 2D^2 + 1)\,y = 0$.

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equation is

$\therefore m^4 + 2m^2 + 1 = 0$ or, $[(m^2 + 1)]^2 = 0$ or, $m = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x$ where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\therefore \text{ P.I.} = \frac{1}{D^4+2D^2+1}\{\cos x\}$$

$$= x\frac{1}{4D^3+4D}\{\cos x\}$$

$[\because \phi(D^2) = D^4 + 2D^2 + 1, \phi(-1^2) = 1 - 2 + 1 = 0$

then $\frac{1}{F(D)}\{f(x)\} = x\frac{1}{F'(D)}\{f(xx)\}]$

$$= \frac{x}{4}\frac{1}{D^3+D}\{\cos x\} = \frac{x}{4}\cdot\frac{x}{3D^2+1}\{\cos x\}$$

$$= \frac{x^2}{4}\frac{1}{3D^2+1}\{\cos x\} = \frac{x^2}{4}\cdot\frac{\cos x}{-3+1} = -\frac{x^2}{8}\cos x$$

Hence the general solution is

y = C.F. + P.I.

$$\boxed{y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{x^2}{8}\cos x.}$$

Example 3. Solve $(D^2 - 4)y = \sin 2x$.

Sol^n : The reduced equation is

$(D^2 - 4)y = 0$

Let $y = Ae^{mx}$ be a trial solution and then auxiliary equation is

$m^2 - 4 = 0 \Rightarrow m = \pm 2$

The complementary function is

$y = c_1 e^{2x} + c_2 e^{-2x}$ where c_1, c_2 are arbitrary constants.

The particular integral is

$$y = \frac{1}{D^2 - 4}\{\sin 2x\} = \frac{1}{-2^2 - 4}\sin 2x \text{ [Replace } D^2 \text{ by } -2^2]$$

$$= -\frac{1}{8}\sin 2x$$

The general solution is

y = C.F. + P.I.

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}\sin 2x.}$$

Example 4. Solve $\frac{d^2y}{dx^2} + a^2 y = \sin ax$.

Sol^n : The reduced equation is $(D^2 + a^2)\, y = 0$. .... (1)

Let $y = Ae^{mx}$ be a trial solution of equation (1). Then the auxiliary equation is

$m^2 + a^2 = 0$ or, $m = \pm\, ai$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\therefore \text{ P.I. } = \frac{1}{D^2 + a^2}\{\sin ax\}$$

$$= \frac{1}{F(D)}\{\sin ax\} \text{ where } F(D)$$

$= D^2 + a^2$ and $F(-a^2)$

$= -a^2 + a^2 = 0$

$$= x\frac{1}{F'(D)}\{\sin ax\}$$

$$= x\frac{1}{2D}\{\sin ax\}$$

$$= \frac{x}{2}\int \sin ax\, dx = -\frac{x}{2a}\cos ax$$

Hence the general solution is

$$\boxed{y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax.}$$

Example 5. Solve : $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 144y = 0.$

Solution : Let $y = e^{mx}$ be a trial solution.

Then the auxiliary equation is $m^2 - 24m + 144 = 0$

or, $(m - 12)^2 = 0$

$\therefore \quad m = 12, 12.$

Hence the general solution is $y = (c_1 + c_2x)\, e^{12x}$ where c_1, c_2 are arbitra constants.

Example 6. Solve : $(D^2 - 3D + 2)\, y = 0.$

Solution : Let $y = e^{mx}$ be a trial solution.

Then the auxiliary equation is $m^2 - 3m + 2 = 0$

or, $(m - 1)(m - 2) = 0$

$\therefore \quad m = 1, 2$

Hence the general solution is

$y = c_1e^x + c_2e^{2x}$ where c_1, c_2 are arbitary constants.

Example 3. Solve : $\dfrac{d^2y}{dx^2} + 4y = 0.$

Solution : Let $y = e^{mx}$ be a trial solution.

Then the auxiliary equation is $m^2 + 4 = 0$

$\therefore \quad m = 0 \pm 2i.$

Hence the general solution is

$$y = e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$$

$$= c_1 \cos 2x + c_2 \sin 2x, \text{ wherre } c_1, c_2 \text{ are arbitrary constants.}$$

Example 7. Solve the equation $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$ and find the particular solution if $x = 3$ when $t = 0$ and $x = 8$ when $t = \log 2$.

Solution : Let $x = e^{mt}$ be a trial solution.

Then the auxiliary equation is $m^2 - 5x + 6 = 0$.

$\therefore \quad m = 2, 3.$

So the general solution is

$x = c_1e^{2t} + c_2e^{3t}$ where c_1, c_2 are arbitary constants.

Now $x = 3$ when $t = 0$.

$\therefore \quad 3 = c_1 + c_2$ .... (i)

Also it is given that $x = 8$ when $t = \log 2$.

$\therefore \quad 8 = c_1e^{2\log 2} + c_2e^{\log 2}$

$= c_1e^{\log 4} + c_2e^{\log 8} = 4c_1 + 8c_2$

$\therefore \quad c_1 + 2c_2 = 2.$ .... (ii)

Solving (i) and (ii), we get $c_1 = 4$, $c_2 = -1$.

Hence the particular solution is $x = 4e^{2t} - e^{3t}$.

Example 8. Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0, y(0) = 1, \left(\frac{dy}{dt}\right) = 0$

Solution : Let $y = e^{mt}$ be a trial solution.

Then the auxiliary equation is

$$m^2 + 4m + 4 = 0$$

or, $(m + 2)^2 = 0$

$\therefore \quad m = -2, -2$

So the general solution is

$$y = (c_1 + c_2t)\, e^{-2t}$$

$$\therefore \quad \frac{dy}{dt} = c_2 e^{-2t} - 2(c_1 + c_2 t)e^{-2t}$$

Given $y(0) = 1$ and $\left(\frac{dy}{dt}\right) = 0$

$\therefore \quad 1 = (c_1 + c_2.0).1 \qquad \therefore c_1 = 1$

Also, $0 = c_2.1 - 2\,(c_1 + c_2.0).1$

$= c_2 - 2c_1$

$\therefore \quad c_2 = 2c_1 = 2.1 = 2$

Thus the particular solution is $y = (1 + 2t)\,e^{-2t}$

Example 9. Solve : $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = x^2$.

The given equation can be written as $(D^2 + 5D + 4)\,y = x^2$.

Solution : Let $y = e^{mx}$ be a trial solution of the homogeneous equati

$(D^2 + 5D + 4)y = 0$.

Then the auxiliary equation is $m^2 + 5m + 4 = 0$

or, $(m + 4)\,(m + 1) = 0$.

$\therefore \quad m = -4, -1$.

$\therefore$ C.F. $= c_1 e^{-4x} + c_2 e^{-x}$, where c_1, c_2 are arbitary constants.

Now, $\quad$ P.I. $= \frac{1}{D^2 + 5D + 4} x^2$

$$= \frac{1}{4}\left(1 + \frac{D^2 + 5D}{4}\right)^{-1} x^2$$

$$= \frac{1}{4}\left\{1 - \left(\frac{D^2 + 5D}{4}\right) + \left(\frac{D^2 + 5D}{4}\right)^2 - \ldots\right\} x^2$$

$$= \frac{1}{4}\left\{1 - \frac{D^2 + 5D}{4} + \frac{25}{16}D^2 - \ldots\right\} x^2$$

[we write upto D^2 term since x^2 is a 2 degree polynomial.]

$$= \frac{1}{4}\left(x^2 - \frac{5}{2}x + \frac{21}{8}\right).$$

So the general solution is

$$y = c_1 e^{-4x} + c_2 e^{-x} + \frac{1}{4}\left(x^2 - \frac{5}{2}x + \frac{21}{8}\right)$$

Example 10. Solve : $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 e^{3x}$.

The given equation can be written as $(D^2 - 5D + 6)\, y = x^2 e^{3x}$.

Solution : Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$(D^2 - 5D + 6)\, y = 0.$

Then the auxiliary equation is $m^2 - 5x + 6 = 0$

or, $(m-3)(m-2) = 0.$ $\therefore m = 2, 3.$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{3x}$, where $c_1 c_2$ are arbitrary constants.

Now. P.I. $= \frac{1}{D^2 - 5D + 6}(x^2 e^{3x})$

$D^2 + 9 + 6D - 5D + 15 + 6$

$$= e^{3x} \frac{1}{(D+3)^2 - 5(D+3) + 6} x^2 \text{ by (2v)}$$

$\frac{1}{D^2 + 9 + 6D - 5D - 15 + 6}$

$$= e^{3x} \frac{1}{D^2 + D} x^2 = e^{3x} \frac{1}{D}(1+D)^{-1} x^2$$

$\frac{1}{D^2 + D}$

$$= e^{3x} \frac{1}{D}(1 - D + D^2 - \ldots.)\, x^2 = e^{3x} \frac{1}{D}(x^2 - 2x + 2)$$

$$= e^{3x}\left(\frac{1}{3}x^3 - x^2 + 2x\right) \left[\frac{1}{D}x^2 = \int x^2 dx \text{ etc.}\right]$$

So the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + e^{3x}\left(\frac{1}{3}x^3 - x^2 + 2x\right)$$

Example 11. Solve : $\dfrac{d^2y}{dx^2}+4y = \text{cosec } 2x.$

Solution : The given equation can be written as $(D^2 + 4)\, y = \text{cosec } 2x$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$(D^2 + 4)\, y = 0.$

Then the auxiliary equation is $m^2 + 4 = 0$.

$\therefore \quad m = \pm 2i$

$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$

$$\text{Now, P.I.} = \frac{1}{D^2+4}\ \text{cosec } 2x = \frac{1}{4i}\left(\frac{1}{D-2i}-\frac{1}{D+2i}\right)\ \text{cosec } 2x$$

$$\therefore \quad \frac{1}{D-2i}\ \text{cosec } 3x = \int e^{-2ix}\,\text{cosec}\,2x\,dx$$

$$= e^{2xi}\int(\cos 2x - i\sin 2x)\ \text{cosec } 2x\ dx$$

$$= e^{2xi}\int(\cos 2x - i)\,dx$$

$$= (\cos 2x + i \sin 2x)\left(\frac{1}{2}\log\sin 2x - ix\right)$$

$$= \frac{1}{2}\cos 2x \log \sin 2x + x \sin 2x + i\left(\frac{1}{2}\sin 2x\log\sin 2x - x\cos 2x\right)$$

Similarly $\dfrac{1}{D+2i}\ \text{cosec } 2x$

$$= \frac{1}{2}\cos 2x \log \sin 2x + x \sin 2x - i\left(\frac{1}{2}\sin 2x\log\sin 2x - x\cos 2x\right)$$

$$\therefore \quad \text{P.I.} = \frac{1}{4}\,(\sin 2x \log \sin 2x - 2x \cos 2x).$$

So the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\,(\sin 2x \log \sin 2x - 2x \cos 2x).$$

Example 12. Solve : $(D^2 - 4)\, y = x \sinh x$.

Solution : Let $y = e^{mx}$ be a trial solution of the homogeneous equation $(D^2 - 4)\, y = 0$.

Then the auxiliary equation is $m^2 - 4 = 0$.

$\therefore \quad m = \pm 2.$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$, where c_1, c_2 are arbitrary constants.

$$\therefore \text{ P.I.} = \frac{1}{D^2 - 4}\, x \sinh x$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} x e^{x} - \frac{1}{D^2-4} x e^{-x}\right] \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[e^{x} \frac{1}{(D+1)^2 - 4} x - e^{-x} \frac{1}{D^2 - 2D - 3} x\right]$$

$$= \frac{1}{2}\left[e^{x} \frac{1}{D^2 + 2D - 3} x - e^{-x} \frac{1}{D^2 - 2D - 3} x\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{3}\left(1 - \frac{D^2 + 2D}{3}\right)^{-1} x + \frac{e^{-x}}{3}\left(1 - \frac{D^2 - 2D}{3}\right)^{-1} x\right]$$

$$= -\frac{1}{6} e^{x}\left(1 + \frac{D^2 + 2D}{3} + \ldots\right) x + \frac{1}{6} e^{-x}\left(1 + \frac{D^2 - 2D}{3} + \ldots\right) x$$

$$= -\frac{1}{6} e^{x}\left(x + \frac{2}{3}\right) + \frac{1}{6} e^{-x}\left(x - \frac{2}{3}\right)$$

$$= -\frac{1}{6} x\left(e^{x} - e^{-x}\right) - \frac{1}{9}\left(e^{x} + e^{-x}\right)$$

$$= -\frac{1}{3} x \sinh x - \frac{2}{9} \cosh x$$

So the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3}\, x\sinh x - \frac{2}{9}\cosh x.$$

Example 13. Solve : $(D^2 + 1)\, y = x \cos x$.

Solution : Let $y = e^{mx}$ be a trial solution of the homogeneous equatio $(D^2 + 1)\, y = 0$.

Then the auxiliary equation is $m^2 + 1 = 0$.

$\therefore \quad m = \pm i$

$\therefore \quad$ C.F. $= c_1 \cos x + c_2 \sin x$, where c_1, c_2 where are arbitrary constant

$$\text{Now, P.I.} = \frac{1}{D^2+1}\, x\cos x = \left\{x - \frac{1}{D^2+1}.2D\right\}\frac{1}{D^2+1}\cos x$$

$$= \left\{x - \frac{1}{D^2+1}.2D\right\}\frac{x}{2}\sin x$$

$$\left[\because \frac{1}{D^2+1}\cos x = x\frac{1}{2D}\cos x = \frac{x}{2}\int \cos x\, dx = \frac{x}{2}\sin\right.$$

$$= \frac{1}{2}x^2 \sin x - \frac{1}{D^2+1}D(x\sin x)$$

$$= \frac{1}{2}x^2 \sin x - \frac{1}{D^2+1}(\sin x + x\cos x)$$

$$= \frac{1}{2}x^2 \sin x - \frac{1}{D^2+1}\sin x - \frac{1}{D^2+1}x\cos x$$

$$= \frac{1}{2}x^2 \sin x - \frac{1}{D^2+1}\sin x - \text{P.I.}$$

$$\therefore \quad 2 \times \text{P.I.} = \frac{1}{2}x^2 \sin x - \frac{x}{2D}\sin x = \frac{1}{2}x^2 \sin x + \frac{1}{2}x\cos x$$

$$\therefore \quad \text{P.I.} = \frac{1}{4}x^2 \sin x + \frac{1}{4}x\cos x$$

So the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} x \sinh x - \frac{2}{9} \cosh x.$$

Example 13. Solve : $(D^2 + 1) y = x \cos x$.

Solution : Let $y = e^{mx}$ be a trial solution of the homogeneous equation $(D^2 + 1) y = 0$.

Then the auxiliary equation is $m^2 + 1 = 0$.

$\therefore \quad m = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$, where c_1, c_2 where are arbitrary constants.

$$\text{Now, P.I.} = \frac{1}{D^2+1} x \cos x = \left\{x - \frac{1}{D^2+1}.2D\right\} \frac{1}{D^2+1} \cos x$$

$$= \left\{x - \frac{1}{D^2+1}.2D\right\} \frac{x}{2} \sin x$$

$$\left[\because \frac{1}{D^2+1} \cos x = x \frac{1}{2D} \cos x = \frac{x}{2} \int \cos x \, dx = \frac{x}{2} \sin x\right]$$

$$= \frac{1}{2} x^2 \sin x - \frac{1}{D^2+1} D(x \sin x)$$

$$= \frac{1}{2} x^2 \sin x - \frac{1}{D^2+1} (\sin x + x \cos x)$$

$$= \frac{1}{2} x^2 \sin x - \frac{1}{D^2+1} \sin x - \frac{1}{D^2+1} x \cos x$$

$$= \frac{1}{2} x^2 \sin x - \frac{1}{D^2+1} \sin x - \text{P.I.}$$

$$\therefore \quad 2 \times \text{P.I.} = \frac{1}{2} x^2 \sin x - \frac{x}{2D} \sin x = \frac{1}{2} x^2 \sin x + \frac{1}{2} x \cos x$$

$$\therefore \quad \text{P.I.} = \frac{1}{4} x^2 \sin x + \frac{1}{4} x \cos x$$

$\sinh h = \frac{e^{x}(1+e^{-2})}{2}$

[Alternatively. P.I. $= \frac{1}{D^2+1} x\cos x = Rl\left\{e^{ix}\frac{1}{D^2+1}xe^{ix}\right\}$

$$= Rl\left\{e^{ix}\frac{1}{(D+i)^2+1}x\right\} = Rl\left\{e^{ix}\frac{1}{D^2+2Di}x\right\}$$

$$= Rl\left\{e^{ix}\frac{1}{2Di}\left(1+\frac{D}{2i}\right)^{-1}x\right\} = Rl\left\{e^{ix}\frac{1}{2Di}\left(1-\frac{D}{2i}+\ldots\right)x\right\}$$

$$= Rl\left\{e^{ix}\frac{1}{2Di}\left(x-\frac{1}{2i}\right)\right\} = Rl\left\{\frac{1}{2i}(\cos x+i\sin x)\left(\frac{x^2}{2}-\frac{x}{2i}\right)\right\}$$

$$= Rl\left\{-\frac{1}{4}(\cos x+i\sin x)(x^2 i-x)\right\}$$

$$= \frac{1}{4}(x^2\sin x + x\cos x)]$$

Hence the general solution is

$$y = c_1\cos x + c_2\sin x + \frac{1}{4}x^2\sin x + \frac{1}{4}x\cos x.$$

METHOD OF VARIATION OF PARAMETERS

In this chapter we shall discuss the alternative method of finding the particular integral (P.I.) of the linear non-homogeneous equation whose complementary function (C.F.) is known. In this method, the P.I. is obtained by replacing the arbitrary constants of the C.F. with functions of x and so the method is known as variation of parameter.

Consider the linear equation of second order with constant coefficient.

$$\boxed{\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = X.} \qquad \ldots (1)$$

Let its C.F. be $\boxed{y = c_1y_1 + c_2y_2}$ where y_1 and y_2 arer L. I. solutions of the equation $\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = 0.$

$$\therefore \quad \frac{d^2y_1}{dx^2} + P_1\frac{dy_1}{dx} + P_2y_1 = 0$$

$$\frac{d^2y_2}{dx^2} + P_1\frac{dy_2}{dx} + P_2y_2 = 0, \qquad \text{.... (2)}$$

Now, let us assume that P.I. of (1) be

$$y_p = uy_1 + vy_2 \qquad \text{.... (3)}$$

where u and v are unknown functions of x.

Differentiating (3) w.r.t. x, we have

$$y'_p = uy'_1 + vy'_2 + u'y_1 + v'y_2$$

We shall choose u, v in such a manner that

$$u'y_1 = v'y_2 = 0 \qquad \text{.... (4)}$$

Then $y'_p = uy'_1 + vy'_2$ .... (5)

$$\therefore \quad y''_p = uy''_1 + vy''_2 + u'y'_1 + u'y'_1 + v'y'_2 \qquad \text{.... (6)}$$

Substituting the values of y_p, y'_p, y''_p in (1) and rearranging we get

$$u(y''_1 + P_1y'_1 + P_2y_1) + v(y''_2 + P_1y'_2 + P_2y_2) + u'y'_1 + v'y'_2 = X.$$

or, $u'y'_1 + v'y'_2 = X$, by (2) .... (7)

Solving (4) & (7), we get

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ X & y'_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}} = \frac{-y_2X}{W}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & X \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}} = \frac{y_1X}{W}$$

where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Integrating we get, $\quad u = -\int \frac{y_2X}{W}dx$ ....

$$v = \int \frac{y_1 X}{W} dx \qquad \text{.... (9)}$$

Putting these values of u and v in (3) we obtain a particular integral of (1), namely $y_p = uy_1 + vy_2$.

Example 1. Apply method of variation of parameter to solve

$$\frac{d^2y}{dx^2} + 9y = \text{Sec } 3x.$$

Sol^n : The reduced equation is $\frac{d^2y}{dx^2} + 9y = 0$

Let $y = Ae^{mx}$ be a trial solution and then the auxiliary equation is

$$m^2 + 9 = 0, \; m = \pm 3i$$

$\therefore$ The complementary function is $y = c_1 \cos 3x + c_2 \sin 3x$

where c_1 and c_2 are arbitrary constants.

In the method of variation of parameter, we replace the parameters c_1 and c_2 in C.F. by the two unknown functions $u(x)$ and $v(x)$, then

$$\text{P.I.} = u \cos 3x + v \sin 3x$$

where u and v are determined from the relation

$$u = -\int \frac{y_2 \sec 3x}{W} dx \text{ and } v = -\int \frac{y_1 \sec 3x}{W} dx$$

where $y_1 = \cos 3x$, $y_2 = \sin 3x$ and $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_1 \end{vmatrix}$

$$\therefore \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 2\cos 3x \end{vmatrix} = 3\,(\cos^2 3x + \sin^2 3x) = 3$$

$$\therefore \; u = -\int \frac{\sin 3x}{3\cos 3x} dx = \frac{1}{9} \log \cos 3x$$

$$v = \int \frac{\sin 3x}{3\cos 3x}\,dx = \frac{1}{9}\log\cos 3x$$

$$\therefore \quad \text{P.I.} = \frac{1}{9}\cos 3x \log\cos 3x + \frac{1}{3}x\sin 3x$$

$\therefore$ The general solution is

$$\boxed{y = c_1\cos 3x + c_2\sin 3x + \frac{1}{9}\cos 3x\log\cos 3x + \frac{1}{3}x\sin 3x.}$$

Example 2. Solve by the method of variation of parameters :

$$(D^3 - 3D^2 + 3D - 1)\,y = \frac{e^x}{x}.$$

Sol^n : Now $(D^3 - 3D^2 + 3D - 1)\,y = \dfrac{e^x}{x}$ ....

The reduced equation (1) is $(D-1)^3\,y = 0$ ....

Let $y = Ae^{mx}$ be a trial solution of equation (2) and then the auxil equation is

$$(m-1)^3 = 0,\ m = 1, 1, 1$$

The complementary function is $y = (c_1 + c_2x + c_3x^2)\,e^x$ where c_1, c_2, c_3 arbitrary constants.

In the method of variation of parameter, we replace the parameter c_2, c_3 in C.F. by the unknown functions $u(x)$, $v(x)$ and $p(x)$ respectively,

P.I. $= uy_1 + vy_2 + p\,y_3$ where $y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2e^x$ and u, v and determined from the relations

$$u = \int \frac{\begin{vmatrix} y_2 & y_3 \\ y'_2 & y'_3 \end{vmatrix} X}{W}\,dx,$$

$$v = \int \frac{\begin{vmatrix} y_1 & y_3 \\ y'_1 & y'_3 \end{vmatrix} X}{W},$$

$$v = \int \frac{\sin 3x}{3\cos 3x}\,dx = \frac{1}{9}\log\cos 3x$$

$$\therefore \quad \text{P.I.} = \frac{1}{9}\cos 3x \log\cos 3x + \frac{1}{3}x \sin 3x$$

$\therefore$ The general solution is

$$\boxed{y = c_1\cos 3x + c_2\sin 3x + \frac{1}{9}\cos 3x\log\cos 3x + \frac{1}{3}x\sin 3x.}$$

Example 2. Solve by the method of variation of parameters :

$$(D^3 - 3D^2 + 3D - 1)\,y = \frac{e^x}{x}.$$

Sol^n : Now $(D^3 - 3D^2 + 3D - 1)\,y = \dfrac{e^x}{x}$ .... (1)

The reduced equation (1) is $(D-1)^3\,y = 0$ .... (2)

Let $y = Ae^{mx}$ be a trial solution of equation (2) and then the auxiliary equation is

$$(m-1)^3 = 0,\ m = 1, 1, 1$$

The complementary function is $y = (c_1 + c_2x + c_3x^2)\,e^x$ where c_1, c_2, c_3 are arbitrary constants.

In the method of variation of parameter, we replace the parameters c_1, c_2, c_3 in C.F. by the unknown functions $u(x)$, $v(x)$ and $p(x)$ respectively, then

P.I. $= uy_1 + vy_2 + p\,y_3$ where $y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2e^x$ and u, v and p are determined from the relations

$$u = \int \frac{\begin{vmatrix} y_2 & y_3 \\ y'_2 & y'_3 \end{vmatrix} X}{W}\,dx,$$

$$v = \int \frac{\begin{vmatrix} y_1 & y_3 \\ y'_1 & y'_3 \end{vmatrix} X}{W},$$

$$p = \int \frac{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} X}{W} dx \text{ and } X = \frac{e^x}{x}$$

$$\text{where } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & 2xe^e + x^2e^x \\ e^x & xe^x + 2e^x & 2e^x + 4xe^x + x^2 + e^x \end{vmatrix}$$

$$= e^x\, e^x\, e^x \begin{vmatrix} 1 & x & x^2 \\ 1 & x+1 & 2x+x^2 \\ 1 & x+2 & x^2+4x+2 \end{vmatrix}$$

$$= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 4x+2 \end{vmatrix} \quad \begin{matrix} R'_2 = R_2 - R_1 \\ R'_3 = R_3 - R_1 \end{matrix} \quad = e^{3x}\,[1.\,(4x + 2 - 4x)] = 2e^{3x}$$

$$\therefore \quad u = \int \frac{\begin{vmatrix} y_2 & y_3 \\ y'_2 & y'_3 \end{vmatrix}}{W} dx = \int \frac{\begin{vmatrix} xe^x & x^2e^x \\ xe^x + e^x & 2xe^x + x^2e^x \end{vmatrix}}{2e^{3x}} \frac{e^x}{x} dx$$

$$= \int \frac{xe^x.e^x \begin{vmatrix} 1 & x \\ 1+x & 2x+x^2 \end{vmatrix}}{2e^{3x}} \frac{e^x}{x} dx$$

$$= \frac{1}{2}\int (x^2 + 2x - x - x^2)\, dx = \frac{1}{2}\int x\, dx = \frac{x^2}{4}$$

$$v = -\int \frac{\begin{vmatrix} y_1 & y_3 \\ y'_1 & y'_3 \end{vmatrix}}{W} \frac{e^x}{x} dx = -\int \frac{\begin{vmatrix} e^x & x^2e^x \\ e^x & 2xe^2 + x^2e^x \end{vmatrix}}{2e^{3x}} \frac{e^x}{x} dx$$

$$= -\int \frac{e.e^x \begin{vmatrix} 1 & x^2 \\ 1 & 2x+x^2 \end{vmatrix}}{2e^{3x}} \frac{e^x}{x} dx = -\frac{1}{2}\int \frac{(x^2+2x-x^2)}{x} dx = -\int dx = -$$

$$p = \int \frac{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}{W} \frac{e^x}{x} dx = \int \frac{\begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}}{2e^{3x}} \frac{e^x x}{x} dx$$

$$= \int \frac{e^x e^x \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} e^x}{2e^{3x}.x} dx$$

$$= \int \frac{(x+1-x)}{2x} dx = \frac{1}{2}\int \frac{1}{2} dx = \frac{1}{2}\log x$$

$$\therefore \quad \text{P.I.} = \frac{x^2}{4}.e^x - x.xe^x + x^2 e^x \frac{1}{2}\log x$$

$\therefore$ The general solution is

$$\boxed{y = (c_1 + c_2 x + c_3 x^2)e^x + \frac{x^2}{4}e^x - x^2 e^x + \frac{1}{2}(\log x)x^2 e^x.}$$

Example 3. Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^{-x} + x.$$

Sol^n : $(D^2 + 3D + 2)y = 3e^{-x} + x$ ....

The reduced equation (1) is $(D^2 + 3D + 2)y = 0$ ....

Let $y = Ae^{mx}$ be a trial solution of equation (2) and then the auxil equation is

$$m^2 + 3m + 2 = 0 \text{ or, } m^2 + 2m + m + 2 = 0 \Rightarrow (m+2) + 1\,(m+2)$$

$$\Rightarrow \quad (m+2)(m+1) = 0 \qquad \therefore m = -2, -1$$

The complementary function is $y = c_1 e^{-x} + c_2 e^{-2x}$

In the method of variation of parameters, we replace the arbitrary constants c_1 and c_2 by the unknown functions $u(x)$ and $v(x)$, then

$$\text{P.I.} = u\,y_1 + v\,y_2 \text{ where } y_1 = e^{-x},\ y_2 = e^{-2x}$$

and u and v are determined from the relations

$$u = -\int \frac{y_2 X}{W} dx \text{ and } v = -\int \frac{y_1 X}{W} dx \text{ where } X = 3e^{-x} + x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = e^{-x}.e^{-2x}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}$$

$$= e^{-3x}(-2+1) = -e^{-3x}$$

$$\therefore\ u = -\int \frac{e^{-2x}}{-e^{-3x}}(3e^{-x}+x)\,dx = \int (3 + xe^{x})\,dx$$

$$= 3x + xe^{x} - e^{x} = 3x + e^{x}(x-1)$$

$$v = \int \frac{e^{-x}}{-e^{-3x}}(3e^{-x}+x)\,dx = -\int (3e^{x} + xe^{2x})\,dx$$

$$= -\left(3e^{x} + \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right)$$

$$\therefore\ \text{P.I.} = [3x + e^{x}(x-1)]\,e^{-x} + \left[-\left(3e^{x} + \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right)\right]e^{-2x}$$

$\therefore$ The general solution is

$$\boxed{y = c_1 e^{x} + c_2 e^{-x}[3x + e^{x}(x-1)] - e^{2x}\left[3e^{x} + \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]}$$

Example 4. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + 4y = \sin 2x$.

Solution : Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} + 4y = 0$.

Then the auxiliary equation is $m^2 + 4 = 0$.

$\therefore \quad m = \pm 2i.$

$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$

Let the particular solution be

$$y_p = u(x) \cos 2x + v(x) \sin 2x.$$

Then $Dy_p = 2(-u \sin 2x + v \cos 2x) + (u' \cos 2x + v' \sin 2x)$

Choose u, v such that

$$u' \cos 2x + v' \sin 2x = 0 \quad \text{....(}$$

so that $Dy_p = 2(-u \sin 2x + v \cos 2x)$.

$\therefore \quad D^2 y_p = 2(-u' \sin 2x + v' \cos 2x) - 4(u \cos 2x + v \sin 2x).$

Substituting the value of $D^2 y_p$, y_p in the given equation

$$\frac{d^2 y}{dx^2} + 4y = \sin 2x,$$

we get $2(-u' \sin 2x + v' \cos 2x) = \sin 2x$

$$\therefore \quad -u' \sin 2x + v' \cos 2x = \frac{1}{2} \sin 2x \quad \text{....}$$

Solving (i) and (ii), we get $u' = -\frac{1}{2} \sin^2 2x = -\frac{1}{4}(1 - \cos 4x)$

$$v' = \frac{1}{2} \sin 2x \cos 2x = \frac{1}{4} \sin 4x.$$

Integrating we get,

$$\therefore \quad u = -\frac{1}{4}\left(x - \frac{\sin 4x}{4}\right) \text{ and } v = -\frac{\cos 4x}{16}$$

$$y_p = -\frac{1}{4} x \cos 2x + \frac{1}{16}(\cos 2x \sin 4x - \sin 2x \cos 4x)$$

$$= -\frac{1}{4} x \cos 2x + \frac{1}{16} \sin 2x.$$

So the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x \cos 2x + \frac{1}{16} \sin 2x$$

$$= c_1 \cos 2x + c_3 \sin 2x - \frac{1}{4}x \cos 2x \text{ where } c_3 = c_2 + \frac{1}{16}$$

where c_1, c_3 are arbitrary constants.

Alternatively. Let the particular solution be

$$y_p = u(x) \cos 2x + v(x) \sin 2x.$$

Then, $u(x) = -\int \frac{y_2 X}{W} dx$ by (8),

$$v(x) = \int \frac{y_1 X}{W} dx \text{ by (9)}$$

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$, $X = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(\cos^2 2x + \sin^2 x) = 2$$

$$\therefore \quad u = -\int \frac{\sin 2x \times \sin 2x}{2} dx = -\frac{1}{4}\int (1 - \cos 4x)\, dx$$

$$= -\frac{1}{4}\left(x - \frac{\sin 4x}{4}\right).$$

$$\therefore \quad v = \int \frac{\cos 2x . \sin 2x}{2} dx = \frac{1}{4}\int \sin 4x dx = -\frac{1}{16} \cos 4x.$$

$$\therefore \quad y_p = -\frac{1}{4}x \cos 2x + \frac{1}{16} (\cos 2x \sin 4x - \sin 2x \cos 4x)$$

$$= -\frac{1}{4}x \cos 2x + \frac{1}{16} \sin 2x.$$

Example 5. Solve by the method of variation of parameters the equation

$$\frac{d^2 y}{dx^2} + y = \sec^3 x \tan x.$$

Solution : Let $y = e^{mx}$ be a trial solution of

$$\frac{d^2y}{dx^2} + y = 0$$

$\therefore$ The auxiliary equation is $m^2 + 1 = 0$

$\therefore$ $m = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

Let the particular solution be

$$y_p = u(x) \cos x + v(x) \sin x$$

Then $u(x) = -\int \frac{y_2 X}{W} dx,$

$$v(x) = \int \frac{y_1 X}{W} dx$$

Here, $y_1 = \cos x$, $y_2 = \sin x$, $x = \sec^3 x \tan x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - 1 \sin^2 x = 1$$

$$\therefore \quad u(x) = -\int \frac{\sin x \sec^3 x \tan x}{1} dx = -\int \tan^2 x \sec^2 x \, dx$$

$$= -\int z^2 dz \text{ [Putting } \tan x = z]$$

$$= -\frac{z^3}{3} = -\frac{\tan^3 x}{3}$$

$$v(x) = \int \frac{\cos x \sec^3 x \tan x}{1} dx = \int \tan x \sec^2 x \, dx$$

$$= \int z dz$$ [putting $\tan x =$

$$= \frac{z^2}{2} = \frac{\tan^2 x}{2}$$

$$\therefore \quad y_p = -\frac{\tan^3 x}{3} . \cos x + \frac{\tan^2 x}{2} . \sin x$$

$$= -\frac{1}{3}\tan^2 x.\sin x + \frac{1}{2}\tan^2 x \sin x = -\frac{1}{6}\tan^2 x \sin x$$

Thus the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{6}\tan^2 x \sin x$$

Example 6. Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax. \ (a \neq 0)$$

Solution : Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} + a^2 y = 0$.

Then the auxiliary equation is $m^2 + a^2 = 0$.

$\therefore \quad m = \pm ai,$

$\therefore \quad$ C. F. $= c_1 \cos ax + c_2 \sin ax.$

Let the particular solution be

$$y_p = u(x) \cos ax + v(x) \sin ax.$$

Then, $Dy_p = a(-u \sin ax + v \cos ax) + (u' \cos ax + v' \sin ax)$ .... (i)

Choose u, v such that

$$u' \cos ax + v' \sin ax = 0.$$

so that $Dy_p = a(-u \sin ax + v \cos ax)$

$\therefore \quad D^2 y_p = a(-u' \sin ax + v' \cos ax) - a^2 (u \cos ax + v \sin ax).$

Substiuting the value of $D^2 y_p$, y_p in the given equation

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax,$$

we get $a(-u' \sin ax + v' \cos ax) = \sec ax.$

$$\therefore \quad -u' \sin ax + v' \cos ax = \frac{1}{a} \sec ax. \qquad \text{.... (ii)}$$

Solving (i) and (ii), we get $u' = -\frac{1}{a} \tan ax$, $v' = \frac{1}{a}$.

Integrating, we get $u = \frac{1}{a^2} \log \cos ax$, $v = \frac{x}{a}$.

Alternatively

Let the particular solution be

$$y = u(x)\cos ax + v(x)\sin ax$$

where $u(x) = -\int \frac{y_2 X}{W} dx$

$$v(x) = -\int \frac{y_1 X}{W} dx$$

Here $y_1 = \cos ax$, $y_2 = \sin ax$, $X = \sec ax$

$$\therefore \quad W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a$$

$$\therefore \quad u(x) = -\int \frac{\sin ax \sec ax}{a} dx = -\frac{1}{a}\int \tan ax\, dx$$

$$= -\frac{1}{a^2}\log\sec ax = \frac{1}{a^2}\log\cos ax$$

and $v(x) = \int \frac{\cos ax \sec ax}{a} dx = \frac{1}{a}\int dx = \frac{x}{a}$

$$\therefore \quad y_p = \frac{1}{a^2}\cos ax \log\cos ax = \frac{x}{a}\sin ax.$$

So the general solution is

$$y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log\cos ax,$$

where c_1, c_2a re arbitrary constants.

EULER-CAUCHY EQUATION

An equation of the form

$$x^2\frac{d^2y}{dx^2} + P_1 x\frac{dy}{dx} + P_2 y = X$$

i.e., $(x^2 D^2 + P_1 x D + P_2)\, y = X$

where P_1, P_2 arre constants and X is a function of x alone, is called Cauchy's homogeneous linear equation.

The above equation can be reduced to linear differential equation with constant coefficients by the substitution $x = e^z$.

or,
$$z = \log x$$

$$\therefore \quad \frac{dz}{dx} = \frac{1}{x}.$$

Now
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\therefore \quad x\frac{dy}{dx} = \frac{dy}{dz}.$$

$$\therefore \quad xDy = D'y \text{ where } D' \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}.$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}.$$

$$\left(\because D' \equiv \frac{d}{dz}\right)$$

$$\therefore \quad x^2D^2y = D'(D'-1)\,y.$$

Substituting these values in equation (1) or (2), we get a linear differential equation with constant coefficient which can be solved by the methods already discussed.

Note : An equation of the form

$$(a+bx)^2\frac{d^2y}{dx} + P_1(a+bx)\frac{dy}{dx} + P_2y = X$$

where P_1, P_2 are constants and X is a function of x alone, is called Legendre's homogeneous linear equation. The equation can be reduced to the

homogeneous linear form by putting $a + bx = e^z$ and then can be solved by the method indicated in the above article.

Example 1. Solve $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65 \cos(\log x)$.

Sol^n : Let $\log x = z$.

Then $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}$

$$\Rightarrow \quad x\frac{dy}{dx} = \frac{dy}{dz} = Qy \text{ where } Q = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (Q^2 - Q)y = Q(Q-1)y$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}\right)$$

$$= -\frac{2}{x^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{x^2}\left(\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right)\frac{dz}{dx}$$

$$= -\frac{2}{x^3}\left(\frac{d^2y}{dx^2} - \frac{dy}{dz}\right) + \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right)$$

$$\Rightarrow \quad x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + \frac{2dy}{dz}$$

$$= Q(Q^2 - 3Q + 2)y$$

$$= Q(Q-1)(Q-2)y$$

Then the given equation becomes

$$(Q^3 - 3Q^2 + 2Q)y + 3(Q^2 - Q)y + Qy + 8y = 65 \cos z$$

$$\Rightarrow \quad (Q^3 + 8)y = 65 \cos z \qquad \text{.... (1)}$$

The reduced equation of equation (1) is $(Q^3 + 8)y = 0$ .... (2)

Let $y = Ae^{mz}$ be a trial solution of equation (2) and then the auxiliary equation is

$$m^3 + 8 = 0 \text{ or, } (m+2)(m^2 - 2m + 4) = 0$$

$$\therefore \quad m = -2,\ 1 \pm \sqrt{3}\,i$$

The complementary function is $c_1 e^{-2z} + e^z (c_2 \cos \sqrt{3}\,z + c_3 \sin \sqrt{3}\,z)$ where c_1, c_2 and c_3 are arbitrary constants.

$$\text{P.I.} = \frac{1}{Q^3+8}\, 65 \cos z = 65\, \frac{1}{Q^2(Q)+8} \cos z$$

$$= 65\, \frac{1}{8-Q} \cos z \qquad (\because Q^2 \text{ is replace by } -1^2)$$

$$= 65 \frac{8+Q}{64-Q^2} \cos z = \frac{65}{64-(-1)} (8+Q) \cos z = 8 \cos z - \sin z$$

$\therefore$ The general solution is

$$y = c_1 e^{-2z} + e^z (c_2 \cos \sqrt{3}\, z + c_3 \sin \sqrt{3}\, z) + 8 \cos z - \sin z$$

$$y = c_1 x^{-2} + x\,[c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x)]$$
$$+ 8 \cos(\log x) - \sin(\log x).$$

Example 2. Solve $(3x+2)^2 \dfrac{d^2y}{dx^2} + 5(3x+2)\dfrac{dy}{dx} - 3y = x^2 + x + 1.$

Sol^n : Let $(3x+2) = e^z$ or, $z = \log(3x+2)$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{3}{(3x+2)}$$

$$\Rightarrow \quad (3x+2) \frac{dy}{dx} = 3 \frac{dy}{dz}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3}{3X+2} \cdot \frac{dy}{dz}\right) = \frac{9}{(3x+2)^2} \frac{dy}{dz} + \frac{3}{3x+2} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{9}{(3x+2)^2} \frac{dy}{dz} + \frac{3}{3x+2} \cdot \frac{d^2y}{dz^2} \cdot \frac{3}{3x+2}$$

$$\Rightarrow (3x+2)\frac{d^2y}{dx^2}=9\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

Then the given equation becomes

$$9\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+5.3\frac{dy}{dz}-3y=\left(\frac{e^z-z}{3}\right)^3+\frac{e^z-2}{3}+1$$

$$\Rightarrow 9\frac{d^2y}{dz^2}+6\frac{dy}{dz}-3y$$

$$=\frac{e^{2z}-4e^z+4}{9}+\frac{e^z-2}{3}+1=\frac{1}{9}\left(e^{2z}-e^z+7\right) \quad \text{.... (1}$$

Let $y = Ae^{mz}$ be a trial solution of the reduced equation (1) and then the auxiliary equation is $9m^2 + 6m - 3 = 0$ or, $9m^2 + 9m - 3m = 0$.

$\Rightarrow 9m(m+1) - 3(m+1) = 0$

$\Rightarrow 3(m+1)(3m-1) = 0$

$\therefore m = 1/3, -1$

$\therefore$ C.F. $= c_1e^{z/3} + c_2e^{-z}$ where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{9Q^2+6Q-3}\{e^{2z}-ez+7\}\frac{1}{9} \text{ where } \frac{d}{dz}=Q$$

$$=\frac{1}{9}\left[\frac{e^{2z}}{9.4+6.2-3}-\frac{e^z}{9.1+6.1-3}+\left(-\frac{1}{3}\right)(1-2Q-3Q^2)^{-1}\{7\}\right]$$

$$=\frac{1}{9}\left[\frac{e^{2z}}{45}-\frac{e^z}{12}-\frac{7}{3}\right]$$

Hence the general solution is

$$y = c_1e^{z/3} = c_2e^{-z} + \frac{1}{9}\left[\frac{e^{2z}}{45}-\frac{e^z}{12}-\frac{7}{3}\right]$$

$$\boxed{y=c_1(3x+2)^{1/3}+c_2(3x+2)^{-1}+\frac{1}{9}\left[\frac{(3x+2)^2}{45}-\frac{(3x+2)^2}{45}-\frac{(3x+2)}{12}-\frac{7}{13}\right]}$$

Example 3. Solve $(x^2D^2 - x\,D + 4)y = x \sin(\log x)$.

Sol^n : Let $\log x = z$ or, $x = e^z$ and $Q = \dfrac{d}{dz}$

$\therefore\ x\,Dy = Qy$ and $x^2D^2y = Q(Q-1)y$

Then given equation becomes

$$(Q^2 - Q)y - Qy + 4y = e^z \sin z$$

$$\Rightarrow\ (Q^2 - 2Q + 4)\,y = e^z \sin z \qquad \text{.... (1)}$$

The reduced equation is $(Q^2 - 2Q + 4)\,y = 0$ .... (2)

Let $y = e^{mz}$ be a trial solution of equation (1) and then the auxiliary equation is

$$m^2 - 2m + 4 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = i \pm \sqrt{3}i$$

$\therefore$ C.F. $= e^z\,(c_1 \cos \sqrt{3}\,z + c_2 \sin \sqrt{3}\,z)$

where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{Q^2-Q+4}\,\{e^z \sin z\} = e^z\,\frac{1}{(Q+1)^2-2(Q+1)+4}\,\{\sin z\}$$

$$= e^z\,\frac{1}{Q^2+2Q+1-2Q-2+4}\,\{\sin z\}$$

$$= e^z\,\frac{1}{Q^2+3}\{\sin z\} = \frac{e^z}{-1^2+3}\sin z = \frac{e^z \sin z}{2}$$

$\therefore$ The gneral solution is $y = e^z\,(c_1 \cos \sqrt{3}\,z + c_2 \sin \sqrt{3}z + \dfrac{e^z \sin z}{2}$

$$= x\,[c_1 \cos(\sqrt{3}\,\log x) + c_2 \sin(\sqrt{3}\,\log x)] + \frac{x \sin(\log x)}{2}$$

Example 4. Solve : $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

Solution : The given equation can be written as

$$(x^2D^2 + xD + 1)\,y = \log x \sin(\log x) \qquad \text{.... (i)}$$

Put $x = e^z$, i.e., $z = \log x$

so that $xDy = D'y$, $D' \equiv \dfrac{d}{dz}$

$$x^2D^2y = D'(D'-1)\,y.$$

Thus the equation (i) reduces to $[D'(D'-1) + D' + 1]\,y = z \sin z$

i.e., $(D'^2 + 1)\,y = z \sin z$..... (ii)

Let $y = e^{mz}$ be a trial soluton of $(D'^2 + 1)\,y = 0$.

Then the auxiliary equation is $m^2 + 1 = 0$.

$\therefore \quad m = \pm i$

$\therefore \quad$ C.F. $= c_1 \cos z + c_2 \sin z$,

Now, P.I. $= \dfrac{1}{D'^2+1}\, z \sin z$

$$= \left\{z - \frac{1}{D'^2+1}.2D'\right\}\frac{1}{D'^2+1}\sin z,$$

$$= \left\{z - \frac{1}{D'^2+1}.2D'\right\}(-1)\frac{z}{2}\cos z$$

$$\left[\because \frac{1}{D'^2+1}\sin z = z\frac{1}{2D'}\sin z = -\frac{z}{2}\cos z\right]$$

$$= -\frac{1}{2}z^2\cos z + \frac{1}{D'^2+1}D'(z\cos z)$$

$$= -\frac{1}{2}z^2 \cos z + \frac{1}{D'^2+1}(\cos z - z \sin z)$$

$$= -\frac{1}{2}z^2 \cos z + \frac{1}{D'^2+1}\cos z - \frac{1}{D'^2+1}(z \sin z)$$

$$= -\frac{1}{2}z^2 \cos z + z.\frac{1}{2D'}\cos z - \text{P.I.}$$

$$\therefore \quad 2 \times \text{P.I.} = -\frac{1}{2}z^2 \cos z + \frac{1}{2}z \sin z$$

$\therefore$ P.I. $= -\frac{1}{4}z^2 \cos z + \frac{1}{4} z \sin z.$

So the general solution of (ii) is

$$y = c_1 \cos z + c_2 \sin z - \frac{1}{4}z^2 \cos z + \frac{1}{4} z \sin z.$$

Therefore the general solution of the given equation is

$$y = c_1 \cos (\log x) + c_2 \sin (\log x) - \frac{1}{4} (\log x)^2 \cos (\log x) + \frac{1}{4} \log x \sin (\log x)$$

where c_1, c_2 arer arbitrary constants.

Example 5. Solve : $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \log x$.

Solution : The given equation can be written as

$(x^2D^2 + 4xD + 2)\, y = \log x.$

Put $x = e^z$, i.e., $z = \log x$.

Then $xD = D'$, $x^2D^2 = D'(D' - 1)$, $D' \equiv \frac{d}{dz}$

So (i) reduces to $[D'(D' - 1) + 4D' + 2]\, y = z$

or, $(D'^2 + 3D' + 2)\, y = z$..... (iii)

Let $y = e^{mz}$ be a trial solution of $(D'^2 + 3D' + 2)\, y = 0$.

Then the auxiliary equation is $m^2 + 3m + 2 = 0$.

or, $(m + 2)(m + 1) = 0$.

$\therefore$ $m = -1, -2$

$\therefore$ C.F. $= c_1 e^{-z} + c_2 e^{-2z}$.

Now, P.I. $= \frac{1}{D'^2 + 3D' + 2} z = \frac{1}{2}\left(1 + \frac{D'^2 + 3D'}{2}\right)^{-1} z$

$$= \frac{1}{2}\left(1 - \frac{D'^2 + 3D'}{2}\right) z = \frac{1}{2}\left(z - \frac{3}{2}\right).$$

So the general solution of (ii) is

$$y = c_1 e^{-z} + c_2 e^{-2z} + \frac{1}{2}\left(z - \frac{3}{2}\right).$$

Therefore the general solution of the given equation is

$$y = c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2}\left(\log x - \frac{3}{2}\right).$$

where c_1, c_2 are arbitrary constants.

Example 6. Solve : $(x^2D^2 - xD - 3)\, y = x^2 \log x$.

Solution : Let $x = e^z$, i.e., $z = \log x$.

Then $xD = D'$, $x^2D^2 = D'(D' - 1)$, $D' \equiv \dfrac{d}{dz}$

So the given equation reduces to

$$[D'(D' - 1) - D' - 3]\, y = ze^{2z}$$

or, $(D'^2 - 2D' - 3)\, y = ze^{2z}$..... (i)

Let $y = e^{mz}$ be a trial solution of $(D'^2 - 2D' - 3)\, y = 0$.

Then the auxiliary equation in $m^2 - 2m - 3 = 0$.

or, $(m - 3)(m + 1) = 0$.

$\therefore$ $m = 1, 3$

$\therefore$ C.F. $= c_1 e^{-z} + c_2 e^{3z}$.

$$\text{Now, P.I.} = \frac{1}{D'^2 - 2D' - 3} ze^{2z} = e^{2z} \frac{1}{(D'+2)^2 - 2(D'+2) - 3} . z$$

$$= e^{2z} \frac{1}{D'^2 + 2D' - 3} . z = -\frac{1}{3} e^{2z} \left(1 - \frac{D'^2 + 2D'}{3}\right)^{-1} z$$

$$= -\frac{1}{3} e^{2z} \left(1 + \frac{D'^2 + 2D'}{3} + ...\right) z = -\frac{1}{3} e^{2z} \left(z + \frac{2}{3}\right) = -\frac{1}{9} e^{2z} (3z + 2).$$

So the general solution of (i) is

$$y = c_1 e^{-z} + c_2 e^{3z} - \frac{1}{9} e^2 z (3z + 2).$$

Therefore the general solution of the given equation is

$$y = c_1x^{-1} + c_2x^3 - \frac{1}{9}\, x^2\, 93 \log x + 2),$$

where c_1, c_2 are arbitrary constants.

LEGENDRE'S EQUATION

The differential equation $(1 - x^2)\ \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)\,y = 0$

.... (1)

where n is real number, is called Legendre's differential equation. This equation is of considerable importance in applied mathematics, particularly in boundary value problems involving spherical configurations.

Though n is a real number, in most physical applications only integral values of n are required. Also, equation (1) can be solved in series of ascending or decending powers of x. The solution in descending powers of x is more important than the one in ascending powers.

Let $y = \sum_{k=0}^{\infty} a_k x^{m-k}$

then $\dfrac{dy}{dx} \sum_{k=0}^{\infty} (m-k)\, a_k x^{m-k-1}$

and $\dfrac{d^2y}{dx^2} \sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2}$

Substituting for y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$(1 - x^2) \sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2} - 2x \sum_{k=0}^{\infty} (m-k)$$

$$a_k x^{m-k-1} + n(n+1) \sum_{k=0}^{\infty} a_k x^{m-k} = 0$$

or, $\sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2} - \sum_{k=0}^{\infty}$

$$[(m-k)(m-k-1)+2(m-k)-n(n+1)]\,a_k x^{m-k}=0$$

or, $$\sum_{k=0}^{\infty}(m-k)(m-k-1)a_k x^{m-k-2}$$

$$-\sum_{k=0}^{\infty}(m-k)^2-n^2+(m-k)-n]a_k x^{m-k}=0$$

or, $$\sum_{k=0}^{\infty}(m-k)(m-k-1)a_k x^{m-k-2}$$

$$\sum_{k=0}^{\infty}[(m-k-n)(m-k+n+1)a_k x^{m-k}=0.$$

Equating to zero the co-efficient of highest power of x, i.e., x^m, we get the indical equation.

$$(m-n)(m+n+1)\,a_0=0$$

whence $m=n$ or, $m=-(n+1)$ since $a_0\neq 0$.

Equating to zero the co-efficient of the next lower power of x, i.e., x^{m-1}, we get

$$(m+n)(m-n-1)\,a_1=0 \quad \text{or, } a_1=0$$

since $(m+n)$ and $(m-n-1)$ are not zero for $m=n$ or, $-(n+1)$.

Equating to zero the co-efficient of x^{m-k}, we get the recurrence relation

$$[m-(k-2)]\,[m-(k-2)-1]\,a_{k-2}-(m-k-n)(m-k+n+1)\,a_k=0$$

or, $$a_k=-\frac{(m-k+2)(m-k+1)}{(n-k+k)(n+m-k+1)}a_k \qquad \dots (2)$$

Since $a_1=0$ therefore, from (2), we get $a_3=a_5=a_7=\dots\dots 0$.

When m = m, the recurrence relation (2) reduces to a_k

$$=-\frac{(n-k+2)(n-k+1)}{k(2n-k+1)}a_{k-2}$$

putting $k=2, 4, 6, \dots\dots$, we get $a_2=\dfrac{n(n-1)}{2(2n-1)}a_0$,

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}a_0 \text{, etc.}$$

Therefore, one solution of Legendre's equation is given by

$$y_1 = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-3)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

When $m = -(m+1)$, the recurrence relation (2) reduce to a_k

$$= \frac{(n+k-1)(n+k)}{k(2n+k+1)}a_{k-2}$$

Putting $k = 2, 4, 6, \ldots\ldots$ we get

$$a_2 = -\frac{(n+1)(n+2)}{2(2n+3)}a_0,$$

$$a_4 = -\frac{(n+3)(n+4)}{4(2n+5)}a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}a_0, \text{ etc.}$$

Therefore, the second solution of Lengendre's equation is given by

$$= y_2 = a_0$$

$$\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5}x^{-n-5} + \ldots\right]$$

When n is a positive integer and $a_0 = \dfrac{1.3.5\ldots\ldots(2n-1)}{n!}$

the first solution is denoted by $P_n(x)$ and is called the **Legendre's function of the first kind.**

Thus

$$P_n(x) = \frac{1.3.5\ldots\ldots(2n-1)}{n!}$$

$$\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

$P_n(x)$ is a terminating series and gives what are called **Legendre's Polynomials** for different values of n such that $P_n(1) = 1$.

We can write

$$P_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2n-2k)!}{2^n . k!(n-k)!(n-2k)!} x^{n-2k}$$

$$\text{where, } N = \begin{cases} \frac{n}{2}, \text{if } n \text{ is even} \\ \frac{n-1}{2}, \text{if } n \text{ is odd} \end{cases}$$

Again, when n is a positive integer and $a_0 = \dfrac{n!}{1.3.5.....(2n+1)}$

the second solution is denoted $Q_n(x)$ and is called the **Legendre's function of the second kind.** Thus

$$Q_n(x) = \frac{n!}{1.3.5....(2n+1)}$$

$$\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + ... \right]$$

which is an infinite series.

Hence the solution of Lengendre's equation is given by $y = c_1 P_n(x) + c_2 Q_n(x)$ where c_1 and c_2 are arbitrary constants.

Example 1. Explain in terms of Legendre polynomials the expression :

$x^4 + x^3 + x^2 + x + 1$

Sol^n : Let, $x^4 + x^3 + x^2 + x = 1$ $a P_4(x) + b P_3(x) + c P_2(x) + d P_1(x) + e P_2(x)$ **[2012]**

$$\text{Then } = \frac{a}{8}(35x^4 - 30x^2 + 3) + \frac{b}{2}(5x^3 - 3x) + \frac{c}{2}(3x^2 - 1) + dx + e$$

$$= \frac{35}{8}ax^4 - \frac{30}{8}ax^2 + \frac{3a}{8} + \frac{5b}{2}x^2 - \frac{3c}{2}x^2 - \frac{c}{2} + dx + e$$

$$= \frac{35a}{8}x^4 + x^3\frac{5b}{2} + x^2\left(\frac{3c}{2} - \frac{30a}{8}\right) + x\left(d - \frac{3b}{2}\right) + \left(\frac{3a}{8} - \frac{c}{2} + e\right)$$

Now, equating the coefficients we get

$$a = \frac{8}{25}, b = \frac{2}{5}, c = \left(1 - \frac{30a}{8}\right)\frac{2}{3} = \frac{2}{21}, d = \frac{8}{5}, e = \frac{101}{105}$$

Now, putting the value of a, b, c, d & e we get,

$$x^4 + x^3 + x^2 + 1 = \frac{8}{35}P_4(x) + \frac{2}{5}P_3(x) + \frac{2}{21}P_2(x) + \frac{8}{5}P_1(x) + \frac{101}{105}$$

Now, putting the value of a, b, c, d & e we get,

$$x^4 + x^3 + x^2 + x + 1$$

$$= \frac{8}{35}P_4(x) + \frac{2}{5}P_3(x) + \frac{2}{21}P_2(x) + \frac{8}{5}P_1(x) + \frac{101}{105}.$$

Example 2. Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre polynomials. **[2012A]**

Sol^n : Let, $4x^3 + 6x^2 + 7x + 2 = a\,P_3(x) + b\,P_2(x) + c\,P_1(x) + d\,P_0(x)$

$$= a\left(\frac{5x^2}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + c(x) + d$$

$$= \frac{5ax^2}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$4x^3 + 6x + 7x + 2 = \frac{5ax^2}{2} + \frac{3bx^2}{2} + \left(\frac{-3a}{2} + c\right)x - \frac{b}{2} + d$$

Equating the coefficients of like powers of x, we have

$$a = \frac{8}{5}, b = 4, c = \frac{47}{5} \text{ and } d = 4$$

putting the values of a, b, c and d, we get

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$$

Example 3. Express $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of Legendre's Polynomia's. |2018|

Sol^n : We know that

$$P_0(x) = 1, \qquad P_1(x) = x.$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$\Rightarrow \quad 3x^2 = 2P_2(x) + P_2(x)$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$2x^2 = \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x)$$

Again, $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$

$$\Rightarrow \quad 2P_3(x) = 5x^3 - 3x$$

$$5x^3 = 2P_3(x) + 3P_1(x)$$

$$\Rightarrow \quad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$4x^3 = \frac{8}{5}P_3(x) + \frac{12}{5}P_1(x)$$

Putting the above value in given $f(x)$

$$f(x) = 4x^3 - 2x^2 - 3x + 8$$

$$= \frac{8}{5}P_3(x) + \frac{12}{5}P_1(x) - \frac{4}{3}P_2(x) - \frac{2}{3}P_0(x) - 3P_1(x) + 8P_0(x)$$

$$= \frac{8}{5}P_3(x) \quad \frac{4}{3}P_2(x) - \frac{3}{5}P_1(x) + \frac{22}{3}P_0(x)$$

Example 4. Show that $x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$.

Solution : We know that

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_0(x) = 1$$

$$\therefore \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)] = \frac{1}{35}[35x^4 - 30x^2 + 3) + 10(3x^2 - 1) + 7] = x^4.$$

Example 5. Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Lendre's polynomials.

Solution : We know that

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\therefore x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$$

$$\therefore f(x) = \left[\frac{2}{3}P_3(x) + \frac{3}{5}(x)\right] - 5x^2 + x + 2$$

$$= \frac{2}{5}P_3(x) - 5x^2 + \frac{8}{5}x + 2 = \frac{2}{5}P_3(x) - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + \frac{8}{5}x + 2$$

$$[\because P_2(x) = \frac{1}{2}(3x^2 - 1) \therefore x^2 = \frac{2}{3}P_2(x) - \frac{1}{3}]$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}x + \frac{1}{3}$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$

$$[\because x = P_1(x) \text{ and } 1 = P_0(x)]$$

Example 6. Express $x^4 - 3x^2 + x$ as a series of Legendre's polynomi

Sol^n : We know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

and $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$\therefore \quad 2P_2(x) = 3x^2 - 1 \text{ or, } x^2 = \frac{1}{3}[2P_2(x) + 1] = \frac{1}{3}[2P_2(x) + P_0(x)]$$

$$x = P_1(x)$$

and $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$ [By Ex. (1)]

$$x^4 - 3x^2 + x = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x) - [2P_2(x) + P_0(x)] + P_1($$

$$= \frac{8}{35}P_4(x) + \left(\frac{4}{7} - 2\right)P_2(x) + P_1(x) + \left(\frac{1}{5} - 1\right)P_0(x)$$

$$= \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x).$$

ESSEL'S EQUATION

One of the most important differential equations in applied mathemat is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \ldots$$

which is known as *Bessel's equation of order n*. Its particular soluti are called *Bessel functions or cylinderical functions of order n*. Many phy cal problems involving vibrations or heat conduction in cylindrical regi give rise to this equation.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots\ldots$

(1) takes the form

$$a_0(m^2 - n^2)x^m + a_1\{(m+1)^2 - n\}x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\}x^{m+2} + \ldots\ldots =$$

Example 6. Express $x^4 - 3x^2 + x$ as a series of Legendre's polynomia

Sol^n : We know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

and $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$\therefore \quad 2P_2(x) = 3x^2 - 1$ or, $x^2 = \frac{1}{3}[2P_2(x) + 1] = \frac{1}{3}[2P_2(x) + P_0(x)]$

$x = P_1(x)$

and $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$ [By Ex. (1)] $\therefore$

$$x^4 - 3x^2 + x = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x) - [2P_2(x) + P_0(x)] + P_1(x$$

$$= \frac{8}{35}P_4(x) + \left(\frac{4}{7} - 2\right)P_2(x) + P_1(x) + \left(\frac{1}{5} - 1\right)P_0(x)$$

$$= \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x).$$

BESSEL'S EQUATION

One of the most important differential equations in applied mathemati is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad .....(1)$$

which is known as *Bessel's equation of order n*. Its particular solutio are called *Bessel functions or cylinderical functions of order n*. Many phys cal problems involving vibrations or heat conduction in cylindrical regio give rise to this equation.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + ...$

(1) takes the form

$$a_0(m^2 - n^2)x^m + a_1\{(m+1)^2 - n\}x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\}x^{m+2} + ... = 0$$

Equating to zero the coefficient of x^m, we obtain the indical equation $m^2 - n^2 = 0$ (as $a_0 \neq 0$) where $m = n$ or $-n$.

$$a_1 = a_3 = a_5 = a_7 = \ldots\ldots = 0$$

and $a_2 = -\dfrac{a_0}{(m+2)^2 - n^2}$, $a_4 = -\dfrac{a_2}{(m+4)^2 - n^2}$ etc.

These give

$$y = a_0 x^m \left(1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^4 - \ldots\right)$$

For $m = n$, we get

$$y_1 = a_0 x^n \left\{1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2.2!(n+1)(n+2)} x^4 - \frac{1}{4^3.3!(n+1)(n+2)(n+3)} x^6 + \ldots\right\} \quad \ldots (2)$$

and for $m = -n$, we have

$$y_2 = a_0 x^{-n} \left\{1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2.2!(-n+1)(-n+2)} x^4 - \frac{1}{4^3.3!(-n+1)(-n+2)(-n+3)} x^6 + \ldots\right\} \quad \ldots (3)$$

Case I. When n is not integral or zero, the complete solution of (1) is $y = c_1 y_1 + c_2 y_2$.

If we take $a_0 = 1/2^n \Gamma(n+1)$, then the solution gives by (2) is called the *Bessel function of the first kind of order n* and is denoted by $J_n(x)$. Thus

$$J_n(x) = \left(\frac{x}{2}\right)^n$$

$$\left\{\frac{1}{\Gamma(n+1)} - \frac{1}{1!\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \ldots\right\} \quad (n > 0)$$

i.e., $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\Gamma(n+r+1)} \quad \ldots (4)$$

and corresponding to (3), we have

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\Gamma(n+r+1)} \quad \text{... (5)}$$

which is called the *Bessel function of the first kind of order –n.*

Hence complete solution of the Bessel's equation (1) may be expressed in the form.

$$y = AJ_n(x) + BJ_{-n}(x).$$

Example 1. Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Sol^n : From the recurrence relation (vi), we get

$$J_{n+1}(x) = \left(\frac{2n}{x}\right)J_n(x) - J_{n-1}(x) \quad \text{.... (1)}$$

Replacing n by 3 in (1), we get

$$J_4(x) = (6/x)\, J_3(x) - J_2(x) \quad \text{.... (2)}$$

Now replacing n by 2 in (1), we get

$$J_3(x) = (4/x)\, J_2(x) - J_1(x) \quad \text{.... (3)}$$

Using (3), (2) become

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x) = \left(\frac{24}{x^2} - 1\right)J_2(x) - J_1(x)\frac{6}{x} \quad \text{.... (4)}$$

Now replacing n by 1 in (1), we get

$$J_2(x) = \left(\frac{2}{x}\right) J_1(x) - J_0(x) \quad \text{.... (5)}$$

Using (5), (4) become

$$J_4 = \left(\frac{24}{x^2} - 1\right)\left(\frac{2}{x}J_1(x) - J_0(x)\right) - \frac{6}{x}J_1(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x).$$

❑❑❑

3 COMPLEX VARIABLES DIFFERNTIATION

DIFFERENTIABILITY

A single-valued function $f(z)$ is said to be differentiable at a point z_0 if

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

Note Derivative of $f(z)$ at a point z_0 can also be written as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example 1 :

Find the derivative of $f(z) = \dfrac{1+z}{1-z}$ *by using the definition at* $z = 2$.

Solution

$$f(z) = \frac{1+z}{1-z}$$

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

At $z = 2$,

$$f'(2) = \lim_{\Delta z \to 0} \frac{f(2+\Delta z) - f(2)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{1+(2+\Delta z)}{1-(2+\Delta z)} - \dfrac{1+2}{1-2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{3+\Delta z}{-1-\Delta z} + 3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3+\Delta z - 3 - 3\Delta z}{-(1+\Delta z)\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-2\Delta z}{-(1+\Delta z)\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{1+\Delta z}$$

$$= 2$$

Example 2 :

Show that $f(z) = i\,z^3$ is differentiable for all z.

Solution

$$f(z) = iz^3$$

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{i(z+\Delta z)^3 - iz^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{i\left[z^3 + (\Delta z)^3 + 3z^2\Delta z + 3z(\Delta z)^2\right] - iz^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} i\left[(\Delta z)^2 + 3z^2 + 3z\Delta z\right]$$

$$= 3iz^2$$

$f'(z)$ exists for all values of z.
Hence, $f(z)$ is differentiable.

Example 3 :

Show that $f(z) = z\,\mathrm{Im}\,(z)$ is differentiable only at $z = 0$ and $f'(0) = 0$.

Solution

$$f(z) = z\,\mathrm{Im}\,(z)$$

$$= (x+iy)\,\mathrm{Im}\,(x+iy)$$

$$= (x+iy)\,y$$

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\,\mathrm{Im}(z+\Delta z) - z\,\mathrm{Im}(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(x+iy+\Delta x+i\Delta y)(y+\Delta y) - (x+iy)\,y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{(x+iy)y + (x+iy)\Delta y + (\Delta x + i\Delta y)(y+\Delta y) - (x+iy)y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{(x+iy)\Delta y + (\Delta x + i\Delta y)(y+\Delta y)}{\Delta x + i\Delta y}$$

For $f'(z)$ to exist, Δz may approach zero along any path.

(i) Let the path be parallel to the x-axis, i.e., $\Delta y = 0$,

$$f'(z) = \lim_{\Delta x \to 0} \frac{y\Delta x}{\Delta x}$$

$$= y$$

(ii) Let the path be parallel to the y-axis, i.e., $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{(x+iy)\Delta y + i\Delta y(y+\Delta y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x+iy+i(y+\Delta y)}{i}$$

$$= \frac{x+2iy}{i}$$

$$= -ix+2y$$

Since $f'(z)$ is different along different paths, $f(z)$ is not differentiable.

At $z=0$, $x=0, y=0$

$\therefore$ $f'(z)=0$ at $z=0$ along all paths.

Hence, $f(z)$ is differentiable only at $z=0$.

Example 4 :

Show that $f(z) = |z|^2$ is continuous but not differentiable, at each point in the plane.

Solution

Let $z = x+iy$

$f(z) = |z|^2 = x^2+y^2$

$|z|^2$, being a polynomial function, is continuous everywhere in the z-plane.

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x + y^2 + (\Delta y)^2 + 2y\Delta y - x^2 - y^2}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{(\Delta x)^2 + 2x\Delta x + (\Delta y)^2 + 2y\Delta y}{\Delta x + i\Delta y}$$

For $f'(z)$ to exist, Δz may approach zero along any path.

(i) Let the path be parallel to the x-axis, i.e., $\Delta y = 0$,

$$f'(z) = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (\Delta x + 2x)$$
$$= 2x$$

(ii) Let the path be parallel to the y-axis, i.e., $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{(\Delta y)^2 + 2y\Delta y}{i\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{\Delta y + 2y}{i}$$
$$= \frac{2y}{i}$$
$$= -2iy$$

Since $f'(z)$ is different along different paths, $f(z)$ is not differentiable.

EXERCISE

Limits

1. Use the definition of limit to prove that

(i) $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$ (ii) $\lim_{z \to i} \frac{z - i}{z^2 + 1} = \frac{1}{2i}$

2. Show that the limit of the function $f(z)$ as z tends to zero does not exist.

(i) $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ (ii) $f(z) = (\bar{z})^2, \quad z \neq 0$
$= 0, \quad z = 0$

3. Prove that $\lim_{z \to i} \left(z + \frac{1}{z}\right) = 0$.

Continuity

1. Show that the function $f(z)$ is continuous of $z = i$.

$$f(z) = \frac{z^2 + 1}{z + i}, \quad z \neq i$$
$$= 0, \quad z = i$$

2. Discuss the continuity of the function $f(z) = (x + y^2) + i\,xy$ in the complex plane.

For $f'(z)$ to exist, Δz may approach zero along any path.

(i) Let the path be parallel to the x-axis, i.e., $\Delta y = 0$,

$$f'(z) = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (\Delta x + 2x)$$
$$= 2x$$

(ii) Let the path be parallel to the y-axis, i.e., $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{(\Delta y)^2 + 2y\Delta y}{i\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{\Delta y + 2y}{i}$$
$$= \frac{2y}{i}$$
$$= -2iy$$

Since $f'(z)$ is different along different paths, $f(z)$ is not differentiable.

EXERCISE

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$$= 0, \quad z = i$$

2. Discuss the continuity of the function $f(z) = (x + y^2) + i\,xy$ in the complex plane.

[Ans.: Continuous everwhere]

3. Show that the function $f(z)$ is not continuous at $z = i$.

$$f(z) = z^2, \quad z \neq i$$
$$= 0, \quad z = i$$

4. Determine whether the function $f(z)$ is continuous at the origin. If not then redefine it to make it continuous.

$$f(z) = \frac{z\,\mathrm{Re}(z)}{|z|}, \quad z \neq 0$$
$$= 2, \quad z = 0$$

[Ans.: Not continuous, define $f(0) = 0$]

Differentiability

1. Show that the functions $\bar{z}$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are not differentiable in the complex plane.

2. Find the derivative of $f(z) = \dfrac{2z - i}{z + zi}$ at $z = -i$ using definition.

$\left[\text{Ans.: } \dfrac{1+i}{2}\right]$

3. Show that the function $f(z) = z^2\,\bar{z}$ is not differentiable in the complex plane.

4. Determine whether the function $f(z)$ is differentiable at the origin.

$$f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

[Ans.: No]

ANALYTIC FUNCTIONS

A function $f(z)$ is said to be an analytic function if it is defined and differentiable at each point of a region R. This function is also known as a *regular* or *holomorphic function*.

Entire Function A function $f(z)$ is said to be an entire function if it is analytic everywhere in the finite plane (complex plane).

An entire function is not analytic at $z = \infty$, e.g., $\cos z$, $\sin z$, e^z and polynomial functions, etc.

CAUCHY-RIEMANN EQUATIONS IN CARTESIAN COORDINATES

Cauchy-Riemann equations for an analytic function are

(i) $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

These equations are necessary conditions for a function $f(z)=u+iv$ to be an analytic function at all the points in a region R.

Proof Let $f(z)=u+iv$ be an analytic function and Δu, Δv be the increments in u and v corresponding to increments Δx, Δy in x and y respectively.

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{[(u+\Delta u)-i(v+\Delta v)]-(u+iv)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\Delta u+i\Delta v}{\Delta z} \qquad ...(2.1)$$

Since $f(z)$ is differentiable at each point of the region R, Δz may approach zero along any path. Consider two paths as follows:

(i) Parallel to x-axis $\Delta y=0$

$$\Delta z=\Delta x+i\Delta y$$
$$=\Delta x$$

Substituting in Eq. (2.1),

$$f'(z)=\lim_{\Delta x\to 0}\frac{\Delta u+i\Delta v}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+i\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \qquad ...(2.2)$$

(ii) Parallel to y-axis $\Delta x=0$

$$\Delta z=\Delta x+i\Delta y$$
$$=i\Delta y$$

Substituting in Eq. (2.1),

$$f'(z)=\lim_{\Delta y\to 0}\frac{\Delta u+i\Delta v}{i\Delta y}$$

$$=\lim_{\Delta y\to 0}\frac{\Delta u}{i\Delta y}+\lim_{\Delta y\to 0}\frac{\Delta v}{\Delta y}$$

$$f'(z)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \qquad ...(2.3)$$

Since $f(z)$ is differentiable, $f'(z)$ must be unique.

Equating Eqs (2.2) and (2.3),

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

Comparing real and imaginary parts,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

These equations are known as *Cauchy–Riemann* (*C–R*) equations.

Note Sufficient Conditions for function to be Analytic

For a function $f(z)=u+iv$, if the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous in the region R and $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ then the function of $f(z)$ is analytic.

Proof Let $f(z) = u + iv$ be a single valued function possessing continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in the region R satisfying C–R equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$f(z+\delta z)=u(x+\delta x, y+\delta y)+iv(x+\delta x, y+\delta y)$$

$$=\left[u(x, y)+\left(\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y\right)+\cdots\right]+i\left[v(x, y)+\left(\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y\right)+\cdots\right]$$

[By Taylor's theorem]

$$=u(x, y)+iv(x, y)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta y$$

[Neglecting terms with power more than one]

$$=f(z)+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y \quad \text{[Using C-R equations]}$$

$$f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(i^2\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\delta y)$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z$$

$$\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial v}{\partial y} \quad \text{[Using C-R equations]}$$

Hence, using Eqs (2.2) and (2.3), $f(z)$ is analytic at all points in the region R.

CAUCHY–RIEMANN EQUATIONS IN POLAR COORDINATES

If $f(z) = u + iv$ is an analytic function where u and v are functions of r, θ and $z = re^{i\theta}$ then

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Proof $f(z)=u+iv, z=re^{i\theta}$

$$u+iv=f(re^{i\theta}) \qquad ...(2.4)$$

Differentiating Eq. (2.4) partially w.r.t. r,

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta})\cdot e^{i\theta}$$

$$=f'(z)\cdot e^{i\theta}$$

$$f'(z)=\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \qquad ...(2.5)$$

Differentiating Eq. (2.4) partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta})\cdot ire^{i\theta}$$

$$=f'(z)\cdot ire^{i\theta}$$

$$f'(z)=\frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}\right) \qquad ...(2.6)$$

Since $f(z)$ is differentiable, $f'(z)$ must be unique.
Equating Eqs (2.5) and (2.6),

$$\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)=\frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}\right)$$

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=-\frac{i}{r}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \theta}$$

Comparing real and imaginary parts,

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Example 5 :

Find the real and imaginary part of $f(z) = z^2 + 3z$.

Solution

$$\begin{aligned} f(z) &= z^2+3z \\ &= (x+iy)^2+3(x+iy) \\ &= (x^2+i^2y^2+2ixy)+3(x+iy) \\ &= (x^2-y^2+3x)+i(2xy+3y) \end{aligned}$$

Real part $= x^2 - y^2 + 3x$

Imaginary part $= 2xy + 3y$

Example 6 :

Find the real and imaginary parts of $f(z)=2z^3-3z$.

Solution

$$\begin{aligned} f(z) &= 2z^3-3z \\ &= 2(x+iy)^3-3(x+iy) \\ &= 2[x^3+i^3y^3+3x^2\cdot iy+3x(iy)^2]-3(x+iy) \\ &= 2[x^3-iy^3+i\cdot 3x^2y-3xy^2]-3(x+iy) \\ &= (2x^3-6xy^2-3x)+i(6x^2y-2y^3-3y) \end{aligned}$$

Real part $= 2x^3-6xy^2-3x$

Imaginary part $= 6x^2y-2y^3-3y$

Example 7 :

Write the function $f(z)=z+\frac{1}{z}$ *in* $f(z) = u(r, \theta) + iv(r, \theta)$ *form.*

Solution

$$f(z)=z+\frac{1}{z}$$

Let $z=re^{i\theta}$

$$f(z)=re^{i\theta}+\frac{1}{re^{i\theta}}$$

$$=re^{i\theta}+\frac{1}{r}e^{-i\theta}$$

$$=r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)$$

$$=\left(r+\frac{1}{r}\right)\cos\theta+i\left(r-\frac{1}{r}\right)\sin\theta$$

Example 8 :

Determine whether the function $2xy+i(x^2-y^2)$ *is analytic or not.*

Solution

Let

$$f(z)=2xy+i(x^2-y^2)$$

$$u+iv=2xy+i(x^2-y^2)$$

Comparing real and imaginary parts,

$$u=2xy, \qquad v=x^2-y^2$$

$$\frac{\partial u}{\partial x}=2y, \qquad \frac{\partial v}{\partial x}=2x$$

$$\frac{\partial u}{\partial y}=2x, \qquad \frac{\partial v}{\partial y}=-2y$$

$$\therefore \frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.

Hence, $f(z)$ is not analytic.

Example 9 :

Show that $f(z) = z^3$ is analytic everywhere.

Solution

$$f(z) = z^3$$

$$u + iv = (x+iy)^3$$

$$= x^3 + (iy)^3 + 3x^2 iy + 3x(iy)^2$$

$$= x^3 - 3xy^2 + i(3x^2 y - y^3)$$

Comparing real and imaginary parts,

$$u = x^3 - 3xy^2, \qquad v = 3x^2 y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy, \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R equations are satisfied.
Hence, z^3 is analytic everywhere.

Example 10 :

Test the analyticity of the function $w = \sin z$.

Solution

$$w = \sin z$$

$$u + iv = \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y, \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R equations are satisfied.
Hence, $\sin z$ is analytic.

Example 11 :

Are $|z|$, Re (z), $Im(z)$ *analytic? Give reasons.*

Solution

Let $$z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\text{Re}(z) = x$$

$$\text{Im}(z) = y$$

Since all the above functions are real, $v = 0$.

$$\therefore \quad \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0 \text{ for all functions.}$$

But $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not zero. Thus, C–R equations are not satisfied for all the functions.

Hence, $|z|$, Re(z), $Im(z)$ are not analytic.

Example 12 :

State the basic difference between the limit of a function of a real variable and that of a complex variable.

Solution

In real calculus, x approaches x_0 only along the line called real line; whereas in case of the complex variable z, z approaches z_0 from any direction in the z-plane.

Example 13 :

Show that the function $f(z) = \bar{z}$ *is nowhere differentiable.*

Solution

$$f(z) = \bar{z}$$

$$u + iv = x - iy$$

Comparing real and imaginary parts,

$$u = x, \qquad v = -y$$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = -1$$

$$\therefore \quad \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial y}$$

Example 11 :

Are $|z|$, Re (z), $Im(z)$ *analytic? Give reasons.*

Solution

Let $$z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\text{Re}(z) = x$$

$$\text{Im}(z) = y$$

Since all the above functions are real, $v = 0$.

$$\therefore \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0 \text{ for all functions.}$$

But $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not zero. Thus, C–R equations are not satisfied for all the functions.

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Show that the function $f(z) = \bar{z}$ *is nowhere differentiable.*

Solution

$$f(z) = \bar{z}$$

$$u + iv = x - iy$$

Comparing real and imaginary parts,

$$u = x, \qquad v = -y$$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = -1$$

$$\therefore \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial y}$$

C–R equations are not satisfied.
Hence, $f(z)$ is nowhere differentiable.

Example 14 :

Discuss the differentiability of $f(z) = x^2 + iy^2$.

Solution

$$f(z) = x^2 + iy^2$$
$$u + iv = x^2 + iy^2$$

Comparing real and imaginary parts,

$$u = x^2, \qquad v = y^2$$
$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = 2y$$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ except at the origin, i.e., at $x = 0, y = 0$.

Since C–R equations are satisfied only at the origin, $f(z)$ is differentiable only at $z = 0$.

Example 15 :

Show that $f(z) = z|z|$ is not analytic anywhere.

Solution

$$f(z) = z|z|$$
$$u + iv = (x + iy)\sqrt{x^2 + y^2}$$
$$= x\sqrt{x^2 + y^2} + iy\sqrt{x^2 + y^2}$$

Comparing real and imaginary parts,

$$u = x\sqrt{x^2 + y^2}, \qquad v = y\sqrt{x^2 + y^2}$$
$$\frac{\partial u}{\partial x} = \sqrt{x^2 + y^2} + x \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x, \qquad \frac{\partial v}{\partial x} = y \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x$$
$$= \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}} \qquad = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y, \qquad \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + y \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y$$

$$= \frac{xy}{\sqrt{x^2+y^2}} \qquad = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.

Hence, $f(z)$ is not analytic anywhere.

Example 16 :

Find the analytic region of $f(z) = (x-y)^2 + 2i(x+y)$.

Solution

$$f(z) = (x-y)^2 + 2i(x+y)$$
$$u + iv = (x-y)^2 + 2i(x+y)$$

Comparing real and imaginary parts,

$$u = (x-y)^2, \qquad v = 2(x+y)$$
$$\frac{\partial u}{\partial x} = 2(x-y), \qquad \frac{\partial v}{\partial x} = 2$$
$$\frac{\partial u}{\partial y} = -2(x-y), \qquad \frac{\partial v}{\partial y} = 2$$

All the partial derivatives are continuous.

For $f(z)$ to be analytic, u and v should satisfy C–R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$2(x-y) = 2, \qquad -2(x-y) = -2$$
$$x - y = 1, \qquad x - y = 1$$

$f(z)$ is analytic at all points satisfying $x - y = 1$.

Hence, the analytic region is the line $x - y = 1$.

Example 17 :

Examine the analyticity of $\sinh z$.

Solution

Let
$$w = \sinh z$$
$$u + iv = -i \sin iz$$
$$= -i \sin i(x+iy)$$
$$= -i \sin(ix - y)$$
$$= -i(\sin ix \cos y - \cos ix \sin y)$$
$$= -i(i \sinh x \cos y - \cosh x \sin y)$$
$$\sinh x \cos y + i \cosh x \sin y$$

Comparing real and imaginary parts,

$$u = \sinh x \cos y, \qquad v = \cosh x \sin y$$
$$\frac{\partial u}{\partial x} = \cosh x \cos y, \qquad \frac{\partial v}{\partial x} = \sinh x \sin y$$
$$\frac{\partial u}{\partial y} = -\sinh x \sin y, \qquad \frac{\partial v}{\partial y} = \cosh x \cos y$$
$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R equations are satisfied.
Hence, $\sinh z$ is analytic.

Example 18 :

Show that $w = e^z$ *is analytic everywhere in the complex plane. Hence, find* $\frac{dw}{dz}$.

Solution

$$w = e^z$$
$$u + iv = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x \cos y + ie^x \sin y$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C–R equations are satisfied. Also e^x, $\cos y$ and $\sin y$ are continuous for all values of x and y.

Hence, e^z is analytic everywhere in the complex plane.

Since $w = u + iv$ is analytic everywhere,

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\\ &= e^x \cos y + ie^x \sin y\\ &= e^x(\cos y + i\sin y)\\ &= e^x e^{iy}\\ &= e^{x+iy}\\ &= e^z\end{aligned}$$

Example 19 :

Prove that $\cosh z$ *is an analytic function and, hence, find its derivative.*

Solution

$$\begin{aligned}f(z) &= \cosh z\\ &= \cos iz\\ u + iv &= \cos i(x+iy)\\ &= \cos(ix - y)\\ &= \cos ix \cos y + \sin ix \sin y\\ &= \cosh x \cos y + i \sinh x \sin y\end{aligned}$$

Comparing real and imaginary parts,

$$u = \cosh x \cos y, \qquad v = \sinh x \sin y$$

$$\frac{\partial u}{\partial x} = \sinh x \cos y, \qquad \frac{\partial v}{\partial x} = \cosh x \sin y$$

$$\frac{\partial u}{\partial y} = -\cosh x \sin y, \qquad \frac{\partial v}{\partial y} = \sinh x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

6. Show that $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}, z \neq 0$

$= 0, \qquad z = 0$

is not analytic at the origin although C–R equations are satisfied at the origin.

HARMONIC FUNCTIONS

A real function ϕ of two variables x and y is said to be a harmonic function in a region R if it has continuous second-order partial derivatives and satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\nabla^2\phi = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

PROPERTIES OF ANALYTIC FUNCTIONS

(1) If $f(z) = u + iv$ is an analytic function, u and v are harmonic functions.

Proof

$f(z) = u + iv$ is an analytic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \cdots(2.7)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \cdots(2.8)$$

Differentiating Eq. (2.7) w.r.t. x and Eq. (2.8) w.r.t. y,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \cdots(2.9)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \cdots(2.10)$$

Adding Eqs (2.9) and (2.10),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$= 0 \qquad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right]$$

Hence, u is a harmonic function.

Differentiating Eq. (2.7) w.r.t. y and Eq. (2.8) w.r.t. x,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \cdots(2.11)$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad \cdots(2.12)$$

Subtracting Eq. (2.12) from Eq. (2.11),

$$\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$$

Hence, v is a harmonic function.

Note u and v of an analytic function $u + iv$ are called *conjugate harmonic functions* of each other.

(2) If $f(z) = u + iv$ is an analytic function then the family of curves $u(x, y) = c_1$ and the family of curves $v(x, y) = c_2$ cut orthogonally.

Proof

$f(z) = u + iv$ is an analytic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots(2.13)$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots(2.14)$$

$$u(x, y) = c_1$$

$$\frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

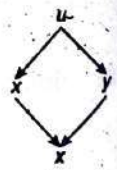


Fig.

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1, \text{ say}$$

which represents the slope of the family of curves $u(x, y) = c_1$

$$v(x, y) = c_2$$

$$\frac{\partial v}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}=m_2, \text{ say}$$

which represents slope of family of curves $v(x, y) = c_2$

$$m_1m_2=\left[-\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right]\left[-\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}\right]$$

$$=\left[\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right]\left[\frac{-\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}\right] \qquad \text{[Using Eqs (2.7) and (2.8)]}$$

$$=-1$$

Hence, both the families of curves cut orthogonally.

Note In polar form, the condition of orthogonality for the families of curves $u(r, \theta) = c_1$ and $v(r, \theta) = c_2$ is

$$\left(r\frac{d\theta}{dr}\right)_{u=c_1}\cdot\left(r\frac{d\theta}{dr}\right)_{v=c_2}=-1$$

Example 20 :

Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

Solution

$$u=2x-x^3+3xy^2$$

$$\frac{\partial u}{\partial x}=2-3x^2+3y^2, \quad \frac{\partial u}{\partial y}=6xy$$

$$\frac{\partial^2 u}{\partial x^2}=-6x \qquad \frac{\partial^2 u}{\partial y^2}=6x$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-6x+6x$$

$$=0$$

Hence

Example 21 :

Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ *is harmonic.*

Solution

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x, \qquad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6, \qquad \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x+6) + (-6x-6)$$

$$= 0$$

Hence, u is harmonic.

Example 22 :

If $f(z) = e^z$ *then show that* u *and* v *are harmonic functions.*

Solution

$$f(z) = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \qquad \frac{\partial^2 v}{\partial x^2} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y, \qquad \frac{\partial^2 v}{\partial y^2} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y + (-e^x \cos y)$$

$$= 0$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=e^x \sin y+(-e^x \sin y)$$
$$=0$$

Hence, u and v are harmonic functions.

Example 23 :

Prove that $u=e^{-y}\cos x$ and $v=e^{-x}\sin y$ the satisfy the Laplace equation, but $u+iv$ is not an analytic function of z.

Solution

$$u=e^{-y}\cos x, \qquad v=e^{-x}\sin y$$

$$\frac{\partial u}{\partial x}=-e^{-y}\sin x, \qquad \frac{\partial v}{\partial x}=-e^{-x}\sin y$$

$$\frac{\partial^2 u}{\partial x^2}=-e^{-y}\cos x, \qquad \frac{\partial^2 v}{\partial x^2}=e^{-x}\sin y$$

$$\frac{\partial u}{\partial y}=-e^{-y}\cos x, \qquad \frac{\partial v}{\partial y}=e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2}=e^{-y}\cos x, \qquad \frac{\partial^2 v}{\partial y^2}=-e^{-x}\sin y$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-e^{-y}\cos x+e^{-y}\cos x$$
$$=0$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=e^{-x}\sin y-e^{-x}\sin y$$
$$=0$$

Hence, u and v satisfy the Laplace equation.

$$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$$

C-R equations are not satisfied.
Hence, $u+iv$ is not an analytic function.

Example 24 :

Prove that $u=x^2-y^2$ and $v=-\dfrac{y}{x^2+y^2}$ are harmonic but $u+iv$ is not regular.

Solution

$$u = x^2 - y^2, \qquad v = -\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = \frac{y}{(x^2+y^2)^2}(2x) = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 v}{\partial x^2} = 2y\left[\frac{(x^2+y^2)^2 - x\cdot 2(x^2+y^2)^2\cdot 2x}{(x^2+y^2)^4}\right]$$

$$\frac{\partial u}{\partial y} = -2y \qquad = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \qquad \frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\right]$$

$$= -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y(x^2+y^2)^2 - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2-2y^2+2x^2)}{(x^2+y^2)^3}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2$$

$$= 0$$

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} + \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$= 0$$

Hence, u and v are harmonic.

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.

Hence, $u + iv$ is not regular.

Example 25 :

If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ *is an analytic function of $z = x + iy$.*

Solution

Let $$U+iV=\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Comparing real and imaginary parts,

$$U=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}, \qquad V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

$$\frac{\partial U}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}, \qquad \frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}$$

$$\frac{\partial U}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial y\partial x}, \qquad \frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}$$

$$=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\partial y}, \qquad =\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}$$

Since u and v are harmonic functions,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 v}{\partial y^2}$$

$$\therefore \frac{\partial U}{\partial y}=-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 v}{\partial x\partial y}$$

and $$\frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}, \qquad \frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}$$

Hence, $U + iV$, i.e., the given function, is analytic.

Example 26 :

Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally when $u + iv = z^3$.

Solution

$$u + iv = z^3$$
$$= (x+iy)^3$$
$$= x^3 + i^3y^3 + 3x^2 \cdot iy + 3x \cdot i^2y^2$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Comparing real and imaginary parts,

$$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$$

Slope of families of curves $u = c_1$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}$$
$$= -\frac{3x^2 - 3y^2}{-6xy}$$
$$= \frac{x^2 - y^2}{2xy} = m_1, \text{ say}$$

Slope of families of curves $v = c_2$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$
$$= -\frac{6xy}{3x^2 - 3y^2}$$
$$= -\frac{2xy}{x^2 - y^2} = m_2, \text{ say}$$
$$m_1m_2 = \left(\frac{x^2 - y^2}{2xy}\right)\left(-\frac{2xy}{x^2 - y^2}\right)$$
$$= -1$$

Hence, families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

Example 27 :

Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally when $w = e^{z^2}$.

Solution

$$w = e^{z^2}$$

$$u + iv = e^{(x+iy)^2}$$

$$= e^{x^2-y^2+2ixy}$$

$$= e^{x^2-y^2} e^{i2xy}$$

$$= e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$$

Comparing real and imaginary parts,

$$u = e^{x^2-y^2}\cos 2xy, \qquad v = e^{x^2-y^2}\sin 2xy$$

Slope of families of curves $u = c_1$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}$$

$$= -\frac{e^{x^2-y^2}\cdot 2x\cos 2xy + e^{x^2-y^2}(-2y\sin 2xy)}{e^{x^2-y^2}(-2y)\cos 2xy + e^{x^2-y^2}(-2x\sin 2xy)}$$

$$= \frac{x\cos 2xy - y\sin 2xy}{y\cos 2xy + x\sin 2xy}$$

$$= m_1, \text{ say}$$

Slope of families of curves $v = c_2$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= -\frac{e^{x^2-y^2}\cdot 2x\sin 2xy + e^{x^2-y^2}(2y\cos 2xy)}{e^{x^2-y^2}(-2y)\sin 2xy + e^{x^2-y^2}(2x\cos 2xy)}$$

$$= -\frac{x\sin 2xy + y\cos 2xy}{-y\sin 2xy + x\cos 2xy}$$

$$= m_2, \text{say}$$

$$m_1 m_2 = \left(\frac{x\cos 2xy - y\sin 2xy}{x\sin 2xy + y\cos 2xy}\right)\left(-\frac{x\sin 2xy + y\cos 2xy}{x\cos 2xy - y\sin 2xy}\right)$$

$$= -1$$

Hence, families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

Example 28 :

Show that the families of the curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ cut orthogonally.

Solution

Let $u = r^n - a\sec n\theta = 0$ ...(1)

and $v = r^n - b\operatorname{cosec} n\theta = 0$ ...(2)

Differentiating Eq. (1) w.r.t. r,

$$nr^{n-1} - a(\sec n\theta \tan n\theta)\cdot n\frac{d\theta}{dr} = 0$$

$$\frac{d\theta}{dr} = \frac{r^{n-1}}{a\sec n\theta \tan n\theta}$$

$$r\frac{d\theta}{dr} = \frac{r^n}{a\sec n\theta \tan n\theta}$$

$$= \frac{a\sec n\theta}{a\sec n\theta \tan n\theta}$$

$$= \cot n\theta$$

$$= m_1, \text{say}$$

Differentiating Eq. (2) w.r.t. r,

$$nr^{n-1} - b(-\operatorname{cosec} n\theta \cot n\theta)n\frac{d\theta}{dr} = 0$$

$$\frac{d\theta}{dr} = \frac{-r^{n-1}}{b\operatorname{cosec} n\theta \cot n\theta}$$

$$r\frac{d\theta}{dr} = -\frac{r^n}{b\operatorname{cosec} n\theta \cot n\theta}$$

$$= -\frac{r^n}{r^n \cot n\theta}$$

$$= -\frac{1}{\cot n\theta}$$

$$= m_2, \text{say}$$

$$m_1 m_2 = \cot n\theta\left(-\frac{1}{\cot n\theta}\right)$$

$$= -1$$

Hence, families of the given curves cut orthogonally.

Example 29 :

Show that $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}=4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}$.

Solution

Let $f(z)$ be a function of x and y

where $z = x+iy$

$$\bar{z} = x-iy$$

$$\therefore \quad x=\frac{1}{2}(z+\bar{z})$$

$$y=\frac{1}{2i}(z-\bar{z})$$

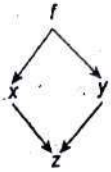


Fig.

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$=\frac{\partial f}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial f}{\partial y}\left(\frac{1}{2i}\right)$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}+\frac{1}{i}\frac{\partial f}{\partial y}\right)$$

$$2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y} \qquad \cdots(1)$$

$$\frac{\partial f}{\partial \bar{z}}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$=\frac{\partial f}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}-\frac{1}{i}\frac{\partial f}{\partial y}\right)$$

$$2\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y} \qquad \cdots(2)$$

From Eqs (1) and (2),

$$2\frac{\partial}{\partial z}\left(2\frac{\partial}{\partial \bar{z}}\right)=\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$$

$$4\frac{\partial^2}{\partial z\,\partial \bar{z}}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

This is also known as the complex form of the Laplace equation.

Example 30 :

If $f(z) = u + iv$ is analytic in the domain D then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2|f'(z)|^2.$$

Solution

Let $f(z) = u + iv$

Since $f(z)$ is analytic, u and v are harmonic functions

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(1)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad ...(2)$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$|f'(z)|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2$$

$$=\frac{\partial^2}{\partial x^2}u^2+\frac{\partial^2}{\partial y^2}u^2$$

$$=\frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)$$

$$=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] \qquad \text{[Using Eq. (1)]}$$

$$=2|f'(z)|^2$$

Example 31 :

If $f(z)$ is an analytic function of z in any domain, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f'(z)|^2|f(z)|^{p-2}$$

Solution

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\frac{\partial^2}{\partial z\partial \bar{z}}\left[|f(z)|^2\right]^{\frac{p}{2}}$$

$$=4\frac{\partial^2}{\partial z\partial \bar{z}}\left[f(z)\cdot\overline{f(z)}\right]^{\frac{p}{2}}$$

$$=4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left\{f(z)\cdot f(\bar{z})\right\}^{\frac{p}{2}}\right] \quad \left[\because \overline{f(z)}=f(\bar{z})\right]$$

$$=4\frac{\partial}{\partial z}\left[\left\{f(z)\right\}^{\frac{p}{2}}\cdot\frac{p}{2}\left\{f(\bar{z})\right\}^{\frac{p}{2}-1}f'(\bar{z})\right] \left[\because f(z)\text{ is a function of } z \text{ alone}\right]$$

$$=4\cdot\frac{p}{2}\left[f(\bar{z})\right]^{\frac{p}{2}-1}f'(\bar{z})\cdot\frac{\partial}{\partial z}\left[f(z)\right]^{\frac{p}{2}} \quad \left[\because f(\bar{z})\text{ is a function of } \bar{z} \text{ alone}\right]$$

$$=4\cdot\frac{p}{2}\left[f(\bar{z})\right]^{\frac{p}{2}-1}f'(\bar{z})\cdot\frac{p}{2}\left[f(z)\right]^{\frac{p}{2}-1}f'(z)$$

$$=p^2\left[f(\bar{z})\cdot f(z)\right]^{\frac{p}{2}-1}\left[f'(\bar{z})\cdot f'(z)\right]$$

$$=p^2\left[\overline{f(z)}\cdot f(z)\right]^{\frac{p-2}{2}}\cdot\left[\overline{f'(z)}\cdot f'(z)\right]$$

$$=p^2\left[|f(z)|^2\right]^{\frac{p-2}{2}}\left[|f'(z)|^2\right]$$

$$=p^2|f'(z)|^2|f(z)|^{p-2}$$

Example 32 :

1. Prove that $\tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.
2. Prove that $\log\sqrt{x^2+y^2}$ is harmonic.

3. If $f(z)$ is analytic, prove that

(i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re}\, f(z)|^2 = 2|f'(z)|^2$

(ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Im}\, f(z)|^2 = 2|f'(z)|^2$

(iii) $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

4. Prove that $\log\{(x-1)^2+(y-2)^2\}$ is harmonic in every region which do not include the point (1, 2).

CONJUGATE HARMONIC FUNCTIONS – MILNE-THOMSON METHOD

u and v of analytic function $u + iv$ are called conjugate harmonic functions of ea other. If one of the function is given, other can be obtained using Milne-Thomso method.

Working rule for Milne-Thomson method is as follows:

Case (i) If the real part u of $f(z)$ is given

Step 1: Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Step 2: Find $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

Step 3: Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4: Integrate $f'(z)$ to obtain $f(z)$.

Case (ii) If the imaginary part v of $f(z)$ is given

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Step 2: Find $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$$=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]}$$

Step 3: Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4: Integrate $f'(z)$ to obtain $f(z)$.

Example 33 :

Find the analytic function $f(z) = u + iv$ if $u = x^3 - 3xy$.

Solution

$$u = x^3 - 3xy$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y, \qquad \frac{\partial u}{\partial y} = -3x$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= (3x^2 - 3y) - i(-3x)$$

$$= 3x^2 - 3y + 3ix$$

Putting $x = z$, $y = 0$,

$$f'(z) = 3z^2 + 3iz$$

Integrating w.r.t. z,

$$f(z) = \int (3z^2 + 3iz)\,dz$$

$$= z^3 + i\frac{3z^2}{2} + c$$

Example 34 :

Construct the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x, \qquad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= (3x^2 - 3y^2 + 6x) - i(-6xy - 6y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = (3z^2 + 6z) + i(0)$$

Integrating w.r.t. z,

$$f(z) = \int (3z^2 + 6z)dz$$

$$= 3\left(\frac{z^3}{3}\right) + 6\left(\frac{z^2}{2}\right) + c$$

$$= z^3 + 3z^2 + c$$

Example 35 :

Find the analytic function $w = u + iv$ whose imaginary part is given by $v = e^x (x \sin y + y \cos y)$.

Solution

$$v = e^x(x\sin y + y\cos y)$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y)$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]}$$

$$= e^x(x\cos y + \cos y - y\sin y) + i\,e^x(x\sin y + y\cos y + \sin y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = e^z(z\cos 0 + \cos 0 - 0) + i\,e^z(z\sin 0 + 0 + \sin 0)$$

$$= e^z(z+1)$$

Integrating w.r.t. z,

$$f(z) = \int (z+1)e^z\,dz$$

$$= (z+1)e^z - (1)e^z + c$$

$$= ze^z + c$$

Example 36 :

Find the analytic function $f(z) = u + iv$, whose real part u is $\frac{x}{x^2+y^2}$.

Solution

$$u = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{(x^2+y^2)^2} \cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} - i\left[-\frac{2xy}{(x^2+y^2)^2}\right]$$

Putting $x = z, y = 0$,

$$f'(z) = \frac{-z^2}{z^4} + i(0)$$

$$= -\frac{1}{z^2}$$

Integrating w.r.t. z,

$$f(z) = \int -\frac{1}{z^2}\,dz$$

$$= -\left(\frac{z^{-1}}{-1}\right) + c$$

$$= \frac{1}{z} + c$$

Example 37 :

Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Solution

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \sin 2x\left[-\frac{1}{(\cosh 2y - \cos 2x)^2}\cdot(2\sinh 2y)\right]$$

$$= -\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Putting $x = z$, $y = 0$,

$$f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} \qquad [\because \cosh 0 = 1, \sinh 0 = 0]$$

$$= -\frac{2}{1 - \cos 2z}$$

$$= -\frac{2}{2\sin^2 z}$$

$$= -\operatorname{cosec}^2 z$$

Integrating w.r.t. z,

$$f(z) = \int -\operatorname{cosec}^2 z \, dz$$

$$= \cot z + c$$

$$u+iv=i\left[(x+iy)^2+\frac{1}{x+iy}\right]+c$$

$$=i\left[x^2+i^2y^2+2ixy+\frac{x-iy}{x^2+y^2}\right]+c$$

$$=ix^2-iy^2+2i^2xy+\frac{ix}{x^2+y^2}-\frac{i^2y}{x^2+y^2}+c$$

$$=i\left(x^2-y^2+\frac{x}{x^2+y^2}\right)-2xy+\frac{y}{x^2+y^2}+c$$

$$=\left(\frac{y}{x^2+y^2}-2xy\right)+i\left(x^2-y^2+\frac{x}{x^2+y^2}\right)+c$$

Comparing real parts,

$$u=\frac{y}{x^2+y^2}-2xy+c$$

Example 38 :

Find the analytic function $w = u + iv$ when $v=e^{-2y}(y\cos 2x+x\sin 2x)$ and find u.

Solution

$$v=e^{-2y}(y\cos 2x+x\sin 2x)$$

$$\frac{\partial v}{\partial x}=e^{-2y}(-2y\sin 2x+\sin 2x+2x\cos 2x)$$

$$\frac{\partial v}{\partial y}=-2e^{-2y}(y\cos 2x+x\sin 2x)+e^{-2y}(\cos 2x)$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]}$$

$$=-2e^{-2y}(y\cos 2x+x\sin 2x)+e^{-2y}\cos 2x$$
$$+ie^{-2y}(-2y\sin 2x+\sin 2x+2x\cos 2x)$$

Putting $x=z,\ y=0$,

$$f'(z)=-2e^0(0+z\sin 2z)+e^0\cos 2z+ie^0(-0+\sin 2z+2z\cos 2z)$$
$$=-2z\sin 2z+\cos 2z+i(\sin 2z+2z\cos 2z)$$

Integrating w.r.t z,

$$f(z)=\int(\cos 2z-2z\sin 2z)dz+i\int(\sin 2z+2z\cos 2z)dz$$

$$=\frac{\sin 2z}{2}-2\left[z\left(-\frac{\cos 2z}{2}\right)-(1)\left(-\frac{\sin 2z}{4}\right)\right]$$

$$+i\left[-\frac{\cos 2z}{2}+2\left\{z\frac{\sin 2z}{2}-(1)\left(-\frac{\cos 2z}{4}\right)\right\}\right]+c$$

$$=\frac{\sin 2z}{2}+z\cos 2z-\frac{\sin 2z}{2}+i\left[-\frac{\cos 2z}{2}+z\sin 2z+\frac{\cos 2z}{2}\right]+c$$

$$=z\cos 2z+iz\sin 2z+c$$

$$=z(\cos 2z+i\sin 2z)+c$$

$$=ze^{i2z}+c$$

$$u+iv=(x+iy)e^{i2(x+iy)}+c$$

$$=(x+iy)e^{2ix-2y}+c$$

$$=(x+iy)e^{-2y}e^{2ix}+c$$

$$=e^{-2y}(x+iy)(\cos 2x+i\sin 2x)+c$$

$$=e^{-2y}\left[(x\cos 2x-y\sin 2x)+i(y\cos 2x+x\sin 2x)\right]+c$$

Comparing real parts,

$$u=e^{-2y}(x\cos 2x-y\sin 2x)$$

Example 39 :

Show that $u(x, y)=x^2-y^2$ is harmonic. Find the corresponding analytic function $f(z)=u+iv$.

Solution

(i) $u(x, y)=x^2-y^2$

$$\frac{\partial u}{\partial x}=2x, \qquad \frac{\partial u}{\partial y}=-2y$$

$$\frac{\partial^2 u}{\partial x^2}=2, \qquad \frac{\partial^2 u}{\partial y^2}=-2$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2-2$$

$$=0$$

Hence, u is harmonic.

(ii) $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= 2x - i(-2y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = 2z$$

Integrating w.r.t. z

$$\int f'(z)\,dz = \int 2z\,dz$$

$$f(z) = z^2 + c$$

Example 40 :

Show that $u(x, y) = x^2 - y^2 + x$ is harmonic. Find the corresponding analytic function $f(z) = u + iv$.

Solution

(i) $u(x, y) = x^2 - y^2 + x$

$$\frac{\partial u}{\partial x} = 2x + 1, \qquad \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2$$

$$= 0$$

Hence u is harmonic.

(ii) $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= (2x + 1) - i(-2y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = 2z + 1$$

Integrating w.r.t. z,

$$\int f'(z)dz = \int (2z+1)dz$$

$$= z^2 + z + c$$

Example 41 :

Prove that $u = e^x (x \cos y - y \sin y)$ is harmonic and, hence, find the analytic function $f(z) = u + iv$.

Solution

$$u = e^x (x\cos y - y\sin y)$$

$$\frac{\partial u}{\partial x} = e^x (x\cos y - y\sin y) + e^x (\cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x (x\cos y - y\sin y + \cos y) + e^x (\cos y)$$

$$= e^x (x\cos y - y\sin y + 2\cos y)$$

$$\frac{\partial u}{\partial y} = e^x (-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^x (-x\cos y - \cos y - \cos y + y\sin y)$$

$$= e^x (-x\cos y - 2\cos y + y\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x (x\cos y - y\sin y + 2\cos y) + e^x (-x\cos y - 2\cos y + y\sin y)$$

$$= 0$$

Hence, u is a harmonic function.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= e^x (x\cos y - y\sin y + \cos y) - ie^x (-x\sin y - \sin y - y\cos y)$$

Putting $x = z, y = 0,$

$$f'(z) = e^z (z\cos 0 - 0 + \cos 0) - ie^z (0 - 0 - 0)$$

$$= e^z (z+1)$$

CONFORMAL MAPPINGS

INTRODUCTION

A *conformal mapping*, also called a *conformal map* or *conformal transformation*, is a transformation that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative. Conformal mappings are useful for solving problems in engineering and physics that can be expressed in terms of functions of a complex variable but exhibit inconvenient geometrics. By choosing an appropriate mapping, the inconvenient geometry can be transformed into a much more convenient geometry.

CONFORMAL MAPPINGS

Conformal mapping transforms curves from one complex plane to another with respect to size and orientation. If the point z (x, y) describes some curve C in the z-plane, the point w (u, v) describes a corresponding curve C' in the w-plane. Hence, for each point (x, y) in the z-plane, there corresponds a point (u, v) in the w-plane. Thus, a curve C in the z-plane is mapped into the corresponding curve C' in the w-plane by the mapping $w = f(z)$.

A mapping or transformation $w = f(z)$ is said to be conformal at a point z_0 if it preserves the angle between any two curves passing through z_0 in magnitude and sense.

Isogonal Transformation A transformation which preserves the angle between any two curves passing through a point only in magnitude and not in sense is known as the isogonal transformation at that point.

Conformal Property of Analytic Function If $w = f(z)$ is an analytic function and $f'(z) \neq 0$ in a region R of the z-plane then the mapping $w = f(z)$ is conformal at all points of the region R.

Critical Points A point of the transformation $w = f(z)$ at which $f'(z) = 0$ is known as the critical point. At this point, mapping is not conformal.

Example 42 :

At what points is the mapping $w = z^2 + \frac{1}{z^2}$ *not conformal?*

Solution

The mapping is not conformal at critical points, i.e., where $f'(z) = 0$.

$$w = f(z) = z^2 + \frac{1}{z^2}$$

$$f'(z) = 2z - \frac{2}{z^3}$$

$$f'(z) = 0$$

$$2z - \frac{2}{z^3} = 0$$

$$2z^4 - 2 = 0$$

$$z^4 = 1$$

$$z^2 = \pm 1$$

$$z^2 = 1, z^2 = -1 = i^2$$

$$z = \pm 1, z = \pm i$$

Hence, the mapping is not conformal at $z = \pm 1$, $z = \pm i$.

SOME STANDARD TRANSFORMATIONS

Translation

The transformation $w = z + c$, where c is a complex constant, represents translation.

Let $z = x + iy, w = u + iv, c = a + ib$

$$w = z + c$$
$$u + iv = (x + iy) + (a + ib)$$
$$= (x + a) + i(y + b)$$

Comparing real and imaginary parts,

$$u = x + a, \qquad v = y + b$$

Therefore, the image of the point (x, y) in the z-plane is the point $(x+a, y+b)$ in the w-plane.

Hence, $w = z + c$ is simply a translation of the axes and preserves shape and size of the region of the z-plane.

Example 43 :

Find the image of $2x + y - 3 = 0$ under the transformation $w = z + 2i$.

Solution

$$w = z + 2i$$
$$u + iv = x + iy + 2i$$
$$= x + i(y + 2)$$

Comparing real and imaginary parts,

$$u = x, \quad v = y + 2$$
$$x = u, \quad y = v - 2$$

Given

$$2x + y - 3 = 0$$
$$2u + (v - 2) - 3 = 0$$
$$2u + v - 5 = 0$$

Hence, the line $2x + y - 3 = 0$ in the z-plane is mapped onto a line $2u + v - 5 = 0$ in the w-plane (Fig. 3.1).

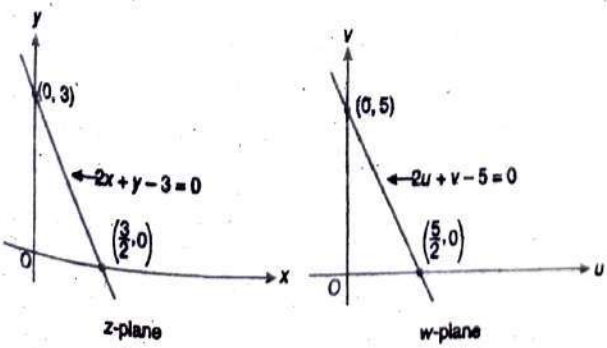


Fig.

Example 44 :

Determine and sketch the image of $|z| = 1$ under the transformation $w = z + i$.

Solution

$$w = z + i$$
$$u + iv = x + iy + i$$
$$= x + i(y+1)$$

Comparing real and imaginary parts,

$$u = x, \quad v = y + 1$$
$$x = u, \quad y = v - 1$$

Given $$|z| = 1$$
$$\sqrt{x^2 + y^2} = 1$$
$$x^2 + y^2 = 1$$
$$u^2 + (v-1)^2 = 1$$

Hence, the image of the circle $|z| = 1$ in the z-plane is the circle $u^2 + (v-1)^2 = 1$ with the centre at (0, 1) and a radius of 1 in the w-plane (Fig. 3.2).

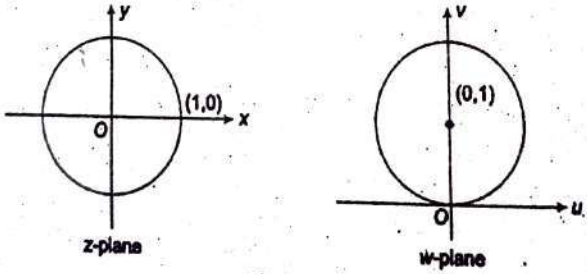


Fig. 3.2

Example 45 :

Find the image of $|z| = 2$ under the mapping $w = z + 3 + 2i$.

Solution

$$w = z + 3 + 2i$$
$$u + iv = x + iy + 3 + 2i$$
$$= (x+3) + i(y+2)$$

Comparing real and imaginary parts,

$$u = x + 3, \quad v = y + 2$$
$$x = u - 3, \quad y = v - 2$$

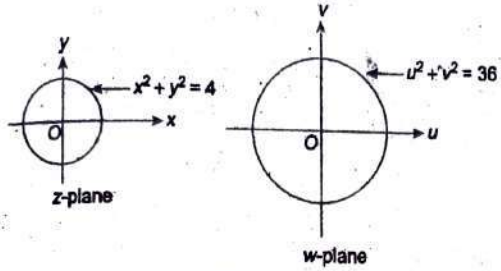


Fig.

Example 46 :

Find and sketch the image of the region $|z| > 1$ under the transformation $w = 4z$.

Solution

$$w = 4z$$

$$u + iv = 4(x + iy)$$

Comparing real and imaginary parts,

$$u = 4x, \quad v = 4y$$

$$x = \frac{u}{4}, \quad y = \frac{v}{4}$$

Given

$$|z| > 1$$

$$\sqrt{x^2 + y^2} > 1$$

$$x^2 + y^2 > 1$$

$$\frac{u^2}{16} + \frac{v^2}{16} > 1$$

$$u^2 + v^2 > 16$$

Hence, the image of $|z| > 1$ (Exterior of the circle $|z| = 1$) in the z-plane is the exterior of the circle $u^2 + v^2 = 16$ in the w-plane

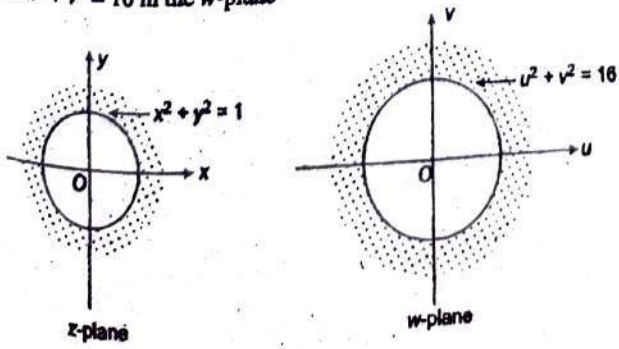


Fig.

Example 47 :

Determine the region in the w-plane into which the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the z-plane is mapped under the transformation $w = 4z$.

Solution

$$w = 4z$$

$$u + iv = 4(x + iy)$$

Comparing real and imaginary parts,

$$u = 4x, \quad v = 4y$$

$$x = \frac{u}{4}, \quad y = \frac{v}{4}$$

(i) When $x = 0, u = 0$

Hence, the line $x = 0$ is mapped onto the line $u = 0$.

(ii) When $y = 0, v = 0$

Hence, the line $y = 0$ is mapped onto the line $v = 0$.

(iii) When $x + y = 1$,

$$\frac{u}{4} + \frac{v}{4} = 1$$

$$u + v = 4$$

Hence, the line $x + y = 1$ is mapped onto the line $u + v = 4$.

Hence, the triangle OAB in the z-plane is mapped onto the triangular region $OA'B'$ in the w-plane

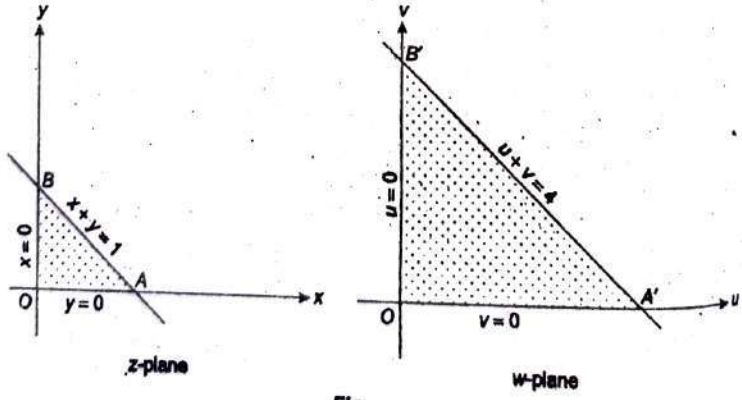


Fig.

Example 48 :

Draw the image of the square whose vertices are at (0, 0), (1, 0), (1, 1), (0, 1) in the z-plane, under the transformation $w = (1 + i)z$. What has this transformation done to the original square?

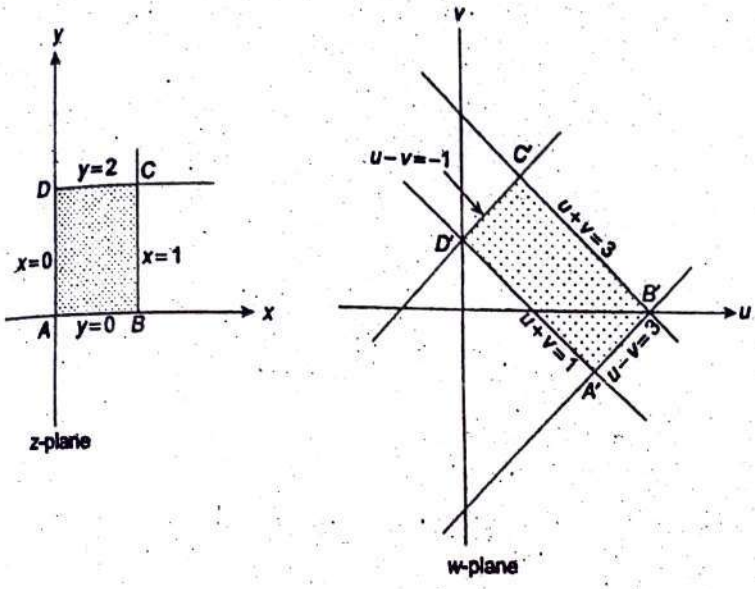


Fig.

Example 49 :

Find the image of the infinite strip $0 \le x \le 1$ under the transformation $w = iz + 1$. Sketch the region in the w-plane.

Solution

$$w = iz + 1$$
$$u + iv = i(x + iy) + 1$$
$$= ix - y + 1$$
$$= (1 - y) + ix$$

Comparing real and imaginary parts,

$$u = 1 - y, \quad v = x$$
$$y = 1 - u, \quad x = v$$

Given

$$0 \le x \le 1$$
$$0 \le v \le 1$$

Hence, the image of the infinite strip $0 \le x \le 1$ in z-plane is the infinite strip $0 \le v \le 1$ in the w-plane

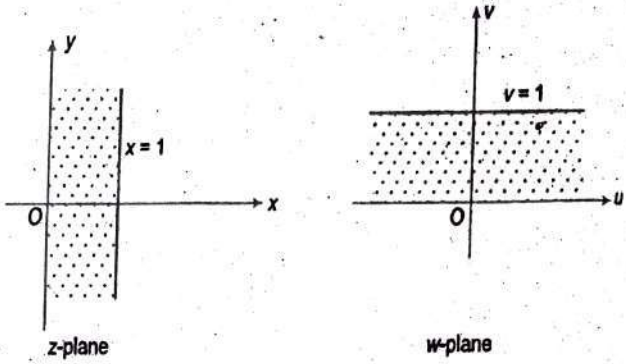


Fig.

Example 50 :

Find the image of the semi-infinite strip $x > 0, 0 < y < 2$ when $w = iz$. Sketch the strip and its image.

Solution

$$w = iz + 1$$
$$u + iv = i(x + iy) + 1$$
$$= ix - y + 1$$
$$= (1 - y) + ix$$

Comparing real and imaginary parts,

$$u = 1 - y, \quad v = x$$
$$y = 1 - u, \quad x = v$$

Given $\quad x > 0$

$v > 0$

and $\quad 0 < y < 2$

$0 < 1 - u < 2$

$-1 < -u < 1$

$1 > u > -1$

or $\quad -1 < u < 1$

Hence, the image of the semi-infinite strip $x > 0, 0 < y < 2$ in the z-plane is the semi-infinite strip $v > 0, -1 < u < 1$ in the w-plane

$$1 \le r^2 \le 4$$
$$1 \le R \le 4$$

and $$\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$
$$\frac{\pi}{6} \le \frac{\phi}{2} \le \frac{\pi}{3}$$
$$\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$$

Hence, the image of the region bounded by $1 \le r \le 2$ and $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ in the z-plane is the region $1 \le R \le 4$ and $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ in the w-plane

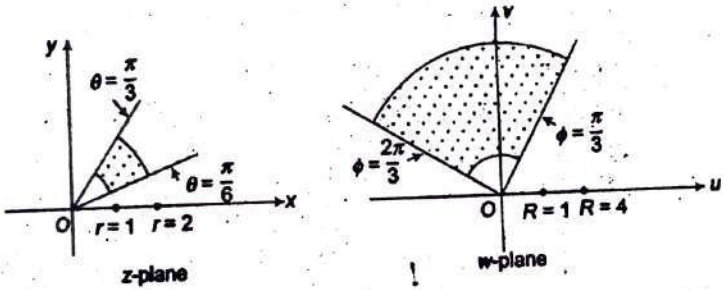


Fig.

$$W = e^z$$

$$u + iv = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x(\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

(i) Let x = constant = a, say

Substituting $x = a$ in u and v,

$$u = e^a \cos y \qquad \cdots(3.1)$$

$$v = e^a \sin y \qquad \cdots(3.2)$$

Squaring and adding Eqs (3.1) and (3.2),

$$u^2 + v^2 = e^{2a}$$

which represents a circle with the centre at (0, 0) and a radius of e^a.

For $a > 0$, $e^a > 1$, the radius of the circle is greater than 1.

For $a < 0$, $e^a < 1$, the radius of the circle is less than 1.

For $a = 0$, $e^a = 1$, the radius of the circle is 1.

Hence, the lines parallel to the y-axis in the z-plane are mapped onto the circles in the w-plane with the centre at (0, 0) and radius, $r > 1$ or $r < 1$ or $r = 1$ according to whether $a > 0$ or $a < 0$ or $a = 0$ respectively.

(ii) Let y = constant = b, say

Substituting $y = b$ in u and v,

$$u = e^x \cos b \quad \cdots(3.3)$$

$$v = e^x \sin b \quad \cdots(3.4)$$

Dividing Eq. (3.3) by Eq. (3.4),

$$\frac{v}{u} = \frac{e^x \sin b}{e^x \cos b}$$

$$= \tan b$$

$$v = u \tan b$$

which represents a radial line.

Hence, the lines parallel to the x-axis in the z-plane are mapped onto radial lines in the w-plane.

Example 51 :

Prove that the transformation $w = e^z$ transforms the region between the real axis and a line parallel to the real axis $y = \pi$ into the upper half of the w-plane.

Solution

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

Given region lies between real axis (i.e., $y = 0$) and the line $y = \pi$.

In the region $0 < y < \pi$,

(i) $\sin y > 0$

$$e^x \sin y > 0 \qquad \left[\because\ e^x \text{ is always positive}\right]$$

$$v > 0$$

(ii) $\cos y > 0, 0 < y < \dfrac{\pi}{2}$

and $\cos y < 0, \dfrac{\pi}{2} < y < \pi$

$\therefore$ u may be negative or positive.

Hence, the region between the real axis and the line $y = \pi$ in the z-plane is transformed into the upper half of the w-plane (i.e., $v > 0$).

Example 52 :

Determine and sketch the image of the region $0 \le x \le 1, 0 \le y \le \pi$ under the transformation $w = e^z$.

Solution

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \quad v = e^x \sin y$$

The given region is $0 \le x \le 1$ and $0 \le y \le \pi$.

In the region $0 \le y \le \pi$,

(i) $\sin y \ge 0$

and $0 \le \sin y \le 1$ ... (1)

Also, $0 \le x \le 1$

$$e^0 \le e^x \le e$$

$$1 \le e^x \le e$$ ... (2)

From Eqs (1) and (2),

$$0 \le e^x \sin y \le e$$

$$0 \le v \le e$$

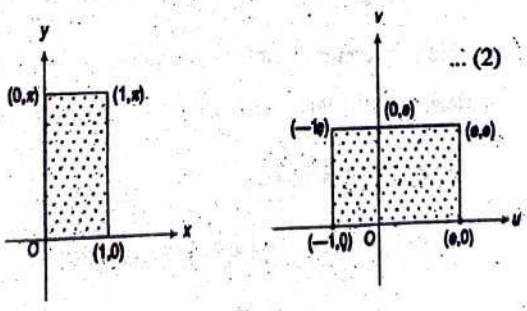


Fig.

(ii) $\cos y \ge 0, \quad 0 \le y \le \dfrac{\pi}{2}$

$\cos y \le 0, \quad \dfrac{\pi}{2} \le y \le \pi$

$-1 \le \cos y \le 1$ ... (3)

From Eqs (2) and (3),

$$-1 \le e^x \cos y \le e$$

$$-1 \le u \le e$$

Hence, the image of the region $0 \le x \le 1$, $0 \le y \le \pi$, in the z-plane is the region $-1 \le u \le e$, $0 \le v \le e$, in the w-plane (Fig. 3.26).

$\therefore$ u may be negative or positive.

Hence, the region between the real axis and the line $y = \pi$ in the z-plane is transformed into the upper half of the w-plane (i.e., $v > 0$).

Example 52 :

Determine and sketch the image of the region $0 \le x \le 1, 0 \le y \le \pi$ under the transformation $w = e^z$.

Solution

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \quad v = e^x \sin y$$

The given region is $0 \le x \le 1$ and $0 \le y \le \pi$.

In the region $0 \le y \le \pi$,

(i) $\sin y \ge 0$

and $0 \le \sin y \le 1$

Also, $0 \le x \le 1$ ... (1)

$$e^0 \le e^x \le e$$

$$1 \le e^x \le e$$ ... (2)

From Eqs (1) and (2),

$$0 \le e^x \sin y \le e$$

$$0 \le v \le e$$

(ii) $\cos y \ge 0, \quad 0 \le y \le \frac{\pi}{2}$

$\cos y \le 0, \quad \frac{\pi}{2} \le y \le \pi$

$$-1 \le \cos y \le 1$$ ... (3)

From Eqs (2) and (3),

$$-1 \le e^x \cos y \le e$$

$$-1 \le u \le e$$

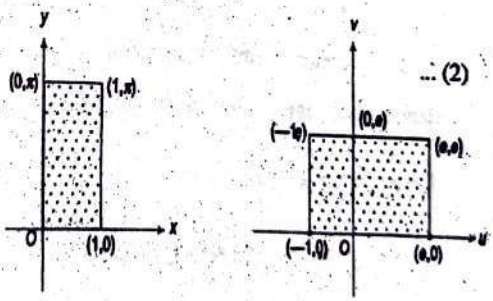


Fig.

Hence, the image of the region $0 \le x \le 1, 0 \le y \le \pi$, in the z-plane is the region $-1 \le u \le e, 0 \le v \le e$, in the w-plane (Fig. 3.26).

Solution

$$w = e^z$$

$$Re^{i\phi} = e^{x+iy} \qquad \text{[Considering the polar form of } w\text{]}$$

$$= e^x e^{iy}$$

$$R = e^x, \phi = y$$

$$x = \log R$$

Given $y = mx$.

Substituting x and y,

$$\phi = m \log R$$

$$\log R = \frac{\phi}{m}$$

$$R = e^{\frac{\phi}{m}}$$

which represents an equiangular spiral.

Hence, the image of the straight line $y = mx$ in the z-plane is an equiangular spiral $R = e^{\frac{\phi}{m}}$ in the w-plane.

MOBIUS TRANSFORMATIONS

The transformation $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are complex or real constants such that $ad - bc \neq 0$ is known as Mobius transformation. Mobius transformation is also known as *bilinear transformation.*

Note

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2}$$

$$\neq 0 \qquad [\because ad - bc \neq 0].$$

Example 53 :

Prove that the image of the straight line $y = mx$ is an equiangular spiral under the transformation $w = e^z$.

Solution

$$w = e^z$$

$$Re^{i\phi} = e^{x+iy} \qquad \text{[Considering the polar form of } w\text{]}$$

$$= e^x e^{iy}$$

$$R = e^x, \phi = y$$

$$x = \log R$$

Given $y = mx$.

Substituting x and y,

$$\phi = m \log R$$

$$\log R = \frac{\phi}{m}$$

$$R = e^{\frac{\phi}{m}}$$

which represents an equiangular spiral.

Hence, the image of the straight line $y = mx$ in the z-plane is an equiangular spiral $R = e^{\frac{\phi}{m}}$ in the w-plane.

MOBIUS TRANSFORMATIONS

The transformation $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are complex or real constants such that $ad - bc \neq 0$ is known as Mobius transformation. Mobius transformation is also known as *bilinear transformation.*

Note $$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2}$$

$$\neq 0 \qquad [\because ad - bc \neq 0].$$

Thus, bilinear transformation is conformal.

Special Cases

(i) If $c=0, a=d\neq 0$,

$$w=z+\frac{b}{d}$$

which represents translation.

(ii) If $c=0,\ b=0, d\neq 0$,

$$w=\frac{a}{d}z$$

which represents rotation and magnification.

(iii) If $a=0, d=0, b=c\neq 0$,

$$w=\frac{1}{z}$$

which represents inversion and reflection.

Fixed (Invariant) Points

A fixed point is a point $z = x + iy$ which maps into itself in the w-plane (i.e., $w = z$) under the bilinear transformation $w=\frac{az+b}{cz+d}$.

$$z=\frac{az+b}{cz+d}$$

$$cz^2+z(d-a)-b=0$$

This is a quadratic equation if $c\neq 0$. The roots of this equation are the fixed point of the bilinear transformation.

Notes

(i) If both the roots are equal then the bilinear transformation is said to be parabolic.

(ii) A bilinear transformation with two finite fixed points α, β can be written as $\frac{w-\alpha}{w-\beta}=\lambda\left(\frac{z-\alpha}{z-\beta}\right)$, where λ is a complex constant.

(iii) If a bilinear transformation has only one fixed point α then it can be written as $\frac{1}{w-\alpha}=k+\frac{1}{z-\alpha}$, where $k=\frac{c}{a-\alpha c}\neq 0$

This form is known as *normal form* or *canonical form* of a bilinear transformation.

The cross ratio of four points z_1, z_2, z_3, z_4 is defined as $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ and is denoted by (z_1, z_2, z_3, z_4).

Theorem

A bilinear transformation preserves the cross ratio of four points, i.e., the cross ratio of four points is invariant under a bilinear transformation.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Proof

Let $w=\frac{az+b}{cz+d}$ be the bilinear transformation.

Let z, z_1, z_2, z_3 are four points in the z-plane which are mapped to the points w, w_1, w_2, w_3 in the w-plane respectively.

$$w_1=\frac{az_1+b}{cz_1+d}$$

$$w-w_1=\frac{az+b}{cz+d}-\frac{az_1+b}{cz_1+d}$$

$$=\frac{(az+b)(cz_1+d)-(cz+d)(az_1+b)}{(cz+d)(cz_1+d)}$$

$$=\frac{aczz_1+adz+bcz_1+bd-cazz_1-cbz-daz_1-db}{(cz+d)(cz_1+d)}$$

$$=\frac{z(ad-bc)-z_1(ad-bc)}{(cz+d)(cz_1+d)}$$

$$=\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

Similarly,

$$w_2-w_3=\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$(w-w_1)(w_2-w_3)=\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\cdot\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

Similarly,

$$(w-w_3)(w_2-w_1)=\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\cdot\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}$$

$$\therefore\ \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Thus, bilinear transformation preserves the cross ratio.

Example 54 :

Find the fixed points of the transformation $w = \dfrac{6z-9}{z}$.

Solution

The fixed points are obtained by putting $w = z$.

$$z = \frac{6z-9}{z}$$
$$z^2 = 6z-9$$
$$z^2 - 6z + 9 = 0$$
$$(z-3)^2 = 0$$
$$z = 3, 3$$

Example 55 :

Find the invariant points of the transformation $w = \dfrac{1+z}{1-z}$.

Solution

The invariant points are obtained by putting $w = z$.

$$z = \frac{1+z}{1-z}$$
$$z - z^2 = 1 + z$$
$$z^2 = -1$$
$$z = \pm i$$

Example 56 :

Find the invariant points of the transformation $w = \dfrac{2z+6}{z+7}$.

Solution

The invariant points are obtained by putting $w = z$.

$$z = \frac{2z+6}{z+7}$$
$$z^2 + 7z = 2z + 6$$
$$z^2 + 5z - 6 = 0$$
$$(z+6)(z-1) = 0$$
$$z = -6, 1$$

Example 57 :

Find the fixed points of the mapping $w = \dfrac{1}{z+2i}$.

Solution

The fixed points are obtained by putting $w = z$.

$$z = \frac{1}{z+2i}$$

$$z^2 + 2zi = 1$$

$$z^2 + 2zi - 1 = 0$$

$$z = \frac{-2i \pm \sqrt{-4+4}}{2}$$

$$= -i$$

Example 58 :

Obtain the invariant points of the transformation $w = 2 - \dfrac{2}{z}$.

Solution

The invariant points are obtained by putting $w = z$.

$$z = 2 - \frac{2}{z}$$

$$= \frac{2z-2}{z}$$

$$z^2 = 2z - 2$$

$$z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

Example 59 :

Find the fixed point of $w = \frac{z+1}{z}$. *Verify your result.*

Solution

The fixed points are obtained by putting $w = z$.

$$z = \frac{z+1}{z}$$

$$z^2 = z+1$$

$$z^2 - z - 1 = 0$$

$$z = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Verification

$$w = \frac{z+1}{z}$$

Putting $z = \frac{1+\sqrt{5}}{2}$,

$$w = \frac{\frac{1+\sqrt{5}}{2}+1}{\frac{1+\sqrt{5}}{2}}$$

$$= \frac{3+\sqrt{5}}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{3-3\sqrt{5}+\sqrt{5}-5}{1-5}$$

$$= \frac{-2-2\sqrt{5}}{-4}$$

$$= \frac{1+\sqrt{5}}{2}$$

Putting $z = \frac{1-\sqrt{5}}{2}$,

$$w = \frac{\frac{1-\sqrt{5}}{2}+1}{\frac{1-\sqrt{5}}{2}}$$

Example 60 :

Determine the bilinear transformation which maps the points $z = 0, i, 1$ into $w = i, -1, \infty$.

Solution

Let $z_1 = 0, z_2 = i, z_3 = 1$ and $w_1 = i, w_2 = -1, w_3 = \infty$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0-1)}{(0-1)(-1-i)} = \frac{(z-0)(i-1)}{(z-1)(i-0)} \qquad \left[\because w_3 = \infty, \frac{w_2}{w_3} = 0, \frac{w}{w_3} = 0\right]$$

$$w - i = \frac{(1+i)(1-i)z}{i(z-1)}$$

$$w = \frac{2z(-i)}{z-1} + i$$

$$= \frac{-2iz + iz - i}{z-1}$$

$$= \frac{-i(z+1)}{z-1}$$

Example 61 :

Determine the linear fractional transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$, on to $w_1 = -1, w_2 = -i, w_3 = 1$ respectively.

Solution

Let $z_1 = 0, z_2 = 1, z_3 = \infty$ and $w_1 = -1, w_2 = -i, w_3 = 1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$\frac{w+1}{w-1} = -\frac{z(1-i)}{1+i}$$

$$= -\frac{z(1-i)(1-i)}{(1+i)(1-i)}$$

$$= -\frac{z(-2i)}{2}$$

$$= \frac{z}{-i}$$

Applying componendo-dividendo,

$$\frac{w+1+w-1}{w+1-w+1} = \frac{z-i}{z+i}$$

$$\frac{2w}{2} = \frac{z-i}{z+i}$$

$$w = \frac{z-i}{z+i}$$

Example 62 :

Determine the bilinear transformation which maps the points $0, \infty, i$ *into* $\infty, 1, 0$.

Solution

Let $z_1 = 0, z_2 = \infty, z_3 = i$ and $w_1 = \infty, w_2 = 1, w_3 = 0$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{\left(\frac{w}{w_1}-1\right)(w_2-w_3)}{(w-w_3)\left(\frac{w_2}{w_1}-1\right)} = \frac{(z-z_1)\left(1-\frac{z_3}{z_2}\right)}{(z-z_3)\left(1-\frac{z_1}{z_2}\right)}$$

$$\left[\begin{array}{l} \because z_2 = \infty, \frac{z_3}{z_2} = 0, \frac{z_1}{z_2} = 0 \\ \because w_1 = \infty, \frac{w}{w_1} = 0, \frac{w_2}{w_1} = 0 \end{array}\right]$$

$$\frac{(0-1)(1-0)}{(w-0)(0-1)} = \frac{(z-0)(1-0)}{(z-i)(1-0)}$$

$$\frac{1}{w} = \frac{z}{z-i}$$

$$w = \frac{z-i}{z}$$

Example 63 :

Find the bilinear transformation that maps the points $z_1 = 1, z = i, z_3 = -1$, onto $w_1 = -1, w_2 = 0, w_3 = 1$ *respectively. Find the image of* $|z| < 1$ *under this transformation.*

Solution

(i) Let $z_1 = 1, z_2 = i, z_3 = -1$ onto $w_1 = -1, w_2 = 0, w_3 = 1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(0-1)}{(w-1)(0+1)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w+1)(-1)}{(w-1)} = \frac{(z-1)}{(z+1)}\frac{(i+1)^2}{(i-1)(i+1)}$$

$$= \frac{(z-1)(2i)}{(z+1)(-2)}$$

$$\frac{w+1}{w-1} = \frac{iz-i}{z+1}$$

Applying componendo-dividendo,

$$\frac{w+1+w-1}{w+1-w+1} = \frac{iz-i+z+1}{iz-i-z-1}$$

$$\frac{2w}{2} = \left[\frac{z(i+1)-(i-1)}{z(i-1)-(i+1)}\right]\left(\frac{i+1}{i+1}\right)$$

$$w = \frac{z(i+1)^2-(i-1)(i+1)}{z(i-1)(i+1)-(i+1)^2}$$

$$= \frac{2iz+2}{-2z-2i}$$

$$= -\frac{iz+1}{z+i}$$

(ii) $$w = -\frac{iz+1}{z+i}$$

$$(z+i)w = -(iz+1)$$

$$zw + iw = -iz - 1$$

$$z(w+i) = -1 - iw = i^2 - iw = i(i-w)$$

$$z = \frac{i(i-w)}{(i+w)}$$

$$|z|<1$$

$$\left|\frac{i(i-w)}{(i+w)}\right|<1$$

$$|i|\,|i-w|<|i+w|$$

$$1\cdot|i-(u+iv)|<|i+u+iv|$$

$$|-u+i(1-v)|<|u+i(1+v)|$$

$$\sqrt{u^2+(1-v)^2}<\sqrt{u^2+(1+v)^2}$$

$$u^2+(1-v)^2<u^2+(1+v)^2$$

$$u^2+1-2v+v^2<u^2+1+2v+v^2$$

$$4v>0$$

$$v>0$$

Hence, the image of $|z|<1$ in the z-plane is the upper half of the w-plane.

Example 64 :

Find the bilinear transformation which maps the points 1, i, −1 onto the points 0, 1, ∞. Also show that the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane.

Solution

(i) Let $z_1=1, z_2=i, z_3=-1$ and $w_1=0, w_2=1, w_3=\infty$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)}=\frac{(z-1)(i+1)}{(z+1)(i-1)}\qquad\left[\because w_3=\infty,\frac{w_2}{w_3}=0,\frac{w}{w_3}=0\right]$$

$$\frac{-w}{-1}=-\frac{(z-1)(1+i)}{(z+1)(1-i)}$$

$$=-\frac{(z-1)(1+i)}{(z+1)(1-i)}$$

$$=-\frac{(z-1)(1+i)(1+i)}{(z+1)(1-i)(1+i)}$$

Applying componendo-dividendo,

$$\frac{w+i+w-i}{w+i-w+i}=\frac{iz+i-z+1}{iz+i+z-1}$$

$$\frac{2w}{2i}=\frac{z(i-1)+(i+1)}{z(i+1)+(i-1)}$$

$$w=\left[\frac{z(-1-i)+(-1+i)}{z(i+1)+(i-1)}\right]\left(\frac{i-1}{i-1}\right)$$

$$=\frac{-z(i+1)(i-1)+(i-1)^2}{z(i+1)(i-1)+(i-1)^2}$$

$$=\frac{2z-2i}{-2z-2i}$$

$$=-\frac{z-i}{z+i}$$

(ii) $$w=-\frac{z-i}{z+i}$$

$$=-\frac{1-\frac{i}{z}}{1+\frac{i}{z}}$$

At $z=\infty$,

$w=-1$

EXERCISE

1. Find the invariant point of the transformation $w=\dfrac{1}{z-2i}$.

[Ans.: i]

2. Find the invariant points of the bilinear transformation $\dfrac{z-1}{z+1}$.

[Ans.: $\pm i$]

3. Find the fixed points of $w=\dfrac{3z-4}{z-1}$.

[Ans.: 2, 2]

4. Find the fixed points under the transformation $w=\dfrac{2z-5}{z+4}$.

[Ans.: $-1\pm 2i$]

5. Find the bilinear transformation that maps the points -1, 0, 1 in the z-plane onto the points 0, i, $3i$ in the w-plane.

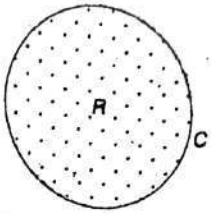
$\left[\text{Ans.: } -3i\dfrac{(z+1)}{(z-3)}\right]$

4 COMPLEX VARIABLES INTEGRATION

SIMPLY CONNECTED AND MULTIPLY CONNECTED REGIONS

Simply Connected Region

A simply connected region R is the region enclosed by a simple curve, e.g., interior of a circle, rectangle, triangle, ellipse, etc.,



Multiply Connected Region

A multiply connected region is the region enclosed by more than one simple curve, e.g., annulus region, regions with holes, etc.,

A multiply connected region can be converted into a simply connected region by introducing cross-cuts

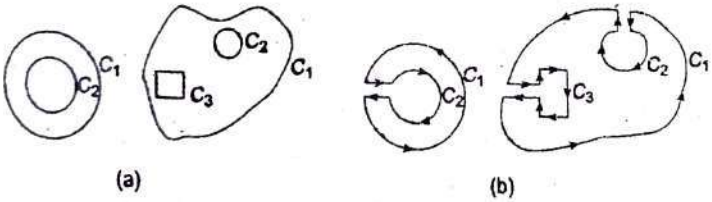


Fig.

Independence of Path

If $f(z)$ is analytic in a simply connected region then the line integral is independent of the path

CAUCHY'S INTEGRAL THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point inside and on a closed curve C then

$$\oint f(z)\,dz = 0$$

Proof : Let $f(z) = u + iv$, $z = x + iy$ $\qquad dz = dx + idy$

$$\oint f(z)\,dz = \oint (u+iv)(dx + i\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{[Using Green's theorem]}$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy \quad \text{[Using C-R equations]}$$

$$= 0$$

Cauchy–Goursat Theorem

If $f(z)$ is analytic at all points inside and on a simple closed curve C contained in a simply connected domain D then

$$\oint_C f(z)\,dz = 0$$

Extension of Cauchy's Integral Theorem to a Multiply Connected Region

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

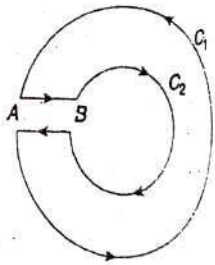


Fig.

Proof The multiply connected region is made simply connected region by introducing a cross-cut AB

The path of integration along AB and C_2 is in the clockwise direction, and along BA and C_1 is in the anticlockwise direction.

By Cauchy's integral theorem, in a simply connected region C_1ABC_2,

$$\oint f(z)\,dz = 0$$

$$\int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz + \int_{C_1} f(z)\,dz = 0 \quad \left[\because \int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz\right]$$

Reversing the direction of the integral around C_2,

$$-\int_{C_2} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Note If $C_1, C_2, C_3, \ldots, C_n$ be n closed curves within C (Fig. 4.17) then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

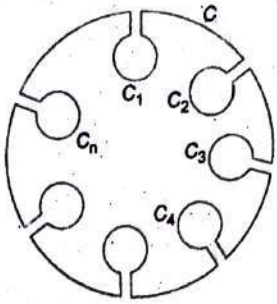


Fig.

Example 1

Evaluate $\int_C e^z\,dz$, *where* C *is* $|z| = 1$.

Solution

(i) Let $f(z) = e^z$

(ii) Since $f(z)$ is differentiable, it is analytic inside and on C.
$f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C e^z\,dz = 0$$

Example 2

Evaluate $\oint_C e^{z^2}\,dz$, *where* C *is any closed contour. Justify your answer.*

Solution

(i) Let $f(z) = e^{z^2}$.

$$f'(z) = e^{z^2} \cdot 2z$$

(ii) Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.
$f'(z)$ is continuous in the z-plane.

(iii) By Cauchy's integral theorem,

$$\oint_C e^{z^2}\,dz = 0$$

Example 3

If C is any simple closed contour, in either direction then show th

$$\int_C \exp(z^3)\,dz = 0.$$

Solution

(i) Let $f(z) = \exp(z^3) = e^{z^3}$

$$f'(z) = e^{z^3}(3z^2)$$

(ii) Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane. $f'(z)$ is also continuous in the z-plane.

(iii) By Cauchy's integral theorem,

$$\int_C \exp(z^3)\,dz = 0$$

Example 4

Evaluate $\int_C e^{\sin z^2}\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = e^{\sin z^2}$

(ii) Since $f(z)$ is differentiable, it is analytic inside and on C.

$f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C e^{\sin z^2}\,dz = 0$$

Example 5

Evaluate $\oint_C (z^2+3)\,dz$, *where C is any closed contour. Justify your answer.*

Solution

(i) Let $f(z) = z^2 + 3$

$f'(z) = 2z$

(ii) Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.

$f'(z)$ is also continuous in the z-plane.

(iii) By Cauchy's integral theorem,

$$\oint_C (z^2 + 3)\,dz = 0$$

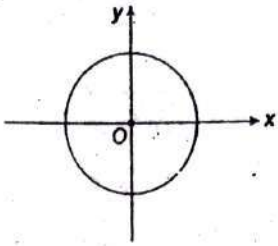


Fig.

Example 6

Evaluate $\int_C (z^2 + 2z)\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = z^2 + 2z$

(ii) Since $f(z)$ is a polynomial, it is analytic inside and on C.

$f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C (z^2 + 2z)\,dz = 0$$

Example 7

Evaluate $\int_C \frac{dz}{z+4}$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{1}{z+4}$

$f(z)$ is not analytic at $z = -4$.

(ii) C is the circle $|z| = 2$ with the centre at O and a radius of 2

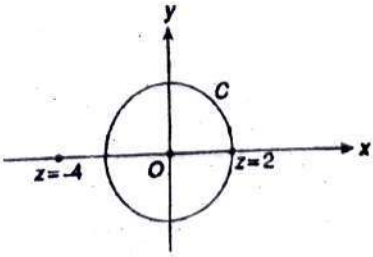


Fig.

(v) By Cauchy's integral theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)} = 0$$

Example 8 :

Evaluate $\oint_C \frac{e^z}{z+i} dz$, *where* $C : |z-1| = 1$.

Solution

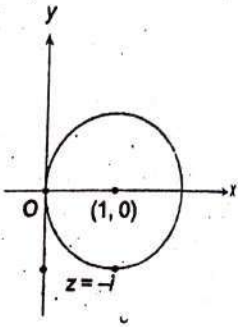


Fig.

(i) Let $f(z) = \frac{e^z}{z+i}$.

(ii) C is a circle $|z-1| = 1$ with the centre at (1, 0) and a radius of 1

(iii) For $z = -i, |z-1| = |-i-1| = \sqrt{2} > 1$
Hence, $z = -i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.
$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\oint_C \frac{e^z}{z+i} dz = 0$$

Example 9 :

Evaluate $\int_C \frac{e^{2z}}{z^2+1} dz$, *where* C *is* $|z| = \frac{1}{2}$.

Solution

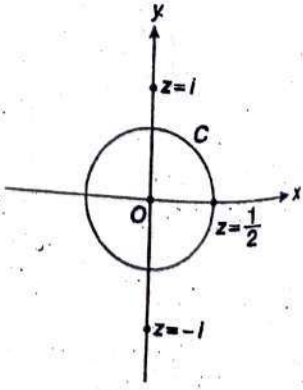


Fig.

(i) Let $f(z) = \frac{e^{2z}}{z^2+1} = \frac{e^{2z}}{(z+i)(z-i)}$

$f(z)$ is not analytic at $z = \pm 1$.

(ii) C is the circle $|z| = \frac{1}{2}$ with the centre at O and a radius of $\frac{1}{2}$

(iii) For $z = -i, |z| = |-i| = 1 > \frac{1}{2}$

Hence, $z = -i$ lies outside C.

For $z = i, |z| = |i| = 1 > \frac{1}{2}$

Hence, $z = i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{e^{2z}}{z^2+1} dz = 0$$

Example 10 :

Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2} dz$, *where C is* $|z-1+i| = 2$.

Solution

(i) Let $f(z) = \frac{z+1}{(z^2+2z+4)^2}$

$$= \frac{z+1}{\left[(z+1-\sqrt{3}i)(z+1+\sqrt{3}i)\right]^2}$$

$f(z)$ is not analytic at $z = -1+\sqrt{3}i$ and $z = -1-\sqrt{3}i$.

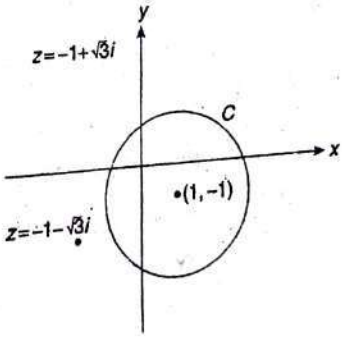


Fig.

(ii) C is the circle $|z-1+i| = 2$ with the centre at $(1,-1)$ and a radius of 2

(iii) For $z = -1+\sqrt{3}i, |z-1+i| = |-1+\sqrt{3}i-1+i|$

$= 3.39 > 2$

Hence, $z = -1+\sqrt{3}i$ lies outside C.

For $z = -1-\sqrt{3}i, |z-1+i| = |-1-\sqrt{3}i-1+i| = 2.13 > 2$

Hence, $z = -1-\sqrt{3}i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{z+1}{(z^2+2z+4)^2} dz = 0$$

LINE INTEGRALS

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at every point of a curve C whose end points are A and B (Fig. 4.1). Divide the curve C into n parts at the points.

$A = P_0(z_0), P_1(z_1), P_2(z_2), \cdots P_i(z_i) \cdots P_n(z_n) = B$

Let $\delta z_i = z_i - z_{i-1}$ and let ξ_i be a point on the arc $P_{i-1} P_i$. The limit of the sum $\sum_{i=1}^{n} f(\xi_i)\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the line integral of $f(z)$ along the path C and is denoted by $\int_C f(z)\,dz$. If C is a closed curve, i.e., if P_o and P_n coincide, the integral is called the contour integral and is denoted by $\oint_C f(z)\,dz$.

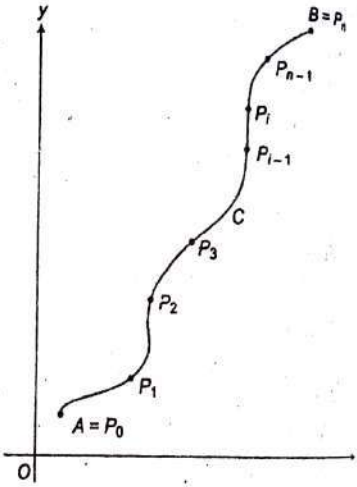


Fig.

Properties of Line Integrals

If $f(z)$ and $g(z)$ are integrable along a curve C then the following properties hold.

1. Linearity

$$\int_C [k_1 f(z) + k_2 g(z)]\,dz = k_1 \int_C f(z)\,dz + k_2 \int_C g(z)\,dz$$

2. Sense Reversal

$$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

3. Partitioning of Path

If the curve C consists of the curves C_1 and C_2 then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

4. Integral Inequality

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)||dz|$$

5. ML Inequality

If $f(z)$ is continuous on the curve C of length L and $|f(z)| \le M$ then

$$\left|\int_C f(z)\,dz\right| \le ML$$

Example 11 :

Find an upper bound for the absolute value of the integral $\int_C z^2\,dz$, *where* C *is the straight-line segment from* 0 *to* $1 + i$.

Solution

The length of the line segment OA from 0 to $1 + i$

$$L = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$$

Let $\quad f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

Along the line OA, $y = x$

$$f(z) = x^2 - x^2 + 2ix^2$$
$$= 2ix^2$$
$$|f(z)| = |2ix^2|$$
$$= 2x^2$$
$$\le 2 \qquad [\because\ 0 \le x \le 1]$$
$$M = 2$$

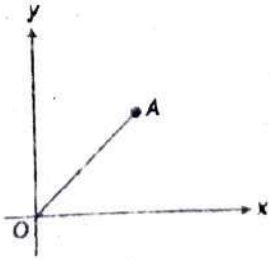


Fig.

By ML inequality,

$$\left|\int_C f(z)\,dz\right| \le ML$$

$$\left|\int_C z^2\,dz\right| \le 2\sqrt{2}$$

Example 12 :

Find an upper bound for the absolute value of the integral $\oint_C \frac{e^z}{z+1}dz$

where C is $|z| = 4$.

Solution

The length of the circular arc $|z| = 4$ (Fig. 4.3) is

$$L = 2\pi r$$
$$= 2\pi(4)$$
$$= 8\pi$$

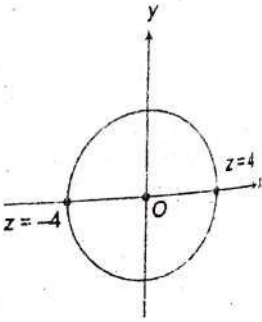


Fig.

Let $$f(z) = \frac{e^z}{z+1}$$

$$|f(z)| = \left|\frac{e^z}{z+1}\right|$$

$$\le \frac{|e^z|}{|z+1|}$$

$$\le \frac{e^4}{3} \qquad \left[\because \begin{array}{l} \max(z) = 4, \quad e^z \le e^4 \\ \min(z) = -4, |z+1| \ge 3, \frac{1}{|z+1|} \le \frac{1}{3} \end{array}\right]$$

$$M = \frac{e^4}{3}$$

By ML inequality,

$$\left|\int_C f(z)\,dz\right| \le ML$$

$$\left|\oint_C \frac{e^z}{z+1}dz\right| \le \frac{8\pi e^4}{3}$$

Example 13 :

Find an upper bound for the integral $\int_C \frac{e^z}{z^2} dz$, *where C is the line segment from* $z = i$ *to* $z = 2$.

Solution

Let $z = i$ is the point $A(0, 1)$ and $z = 2$ is the point $B(2, 0)$.

The length of the line segment AB

$$L = \sqrt{(0-2)^2 + (1-0)^2}$$
$$= \sqrt{5}$$

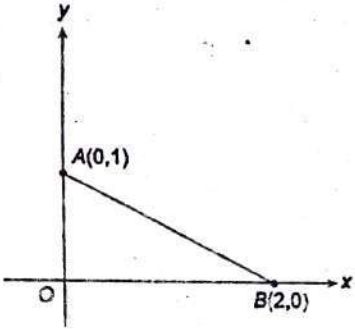


Fig.

The equation of the line AB is

$$y - 0 = \frac{1-0}{0-2}(x-2)$$
$$-2y = x - 2$$
$$x + 2y = 2$$

Let $$f(z) = \frac{e^z}{z^2}$$

$$= \frac{e^{x+iy}}{(x+iy)^2}$$

$$= \frac{e^x e^{iy}}{(x+iy)^2}$$

$$|f(z)| = \left|\frac{e^x e^{iy}}{(x+iy)^2}\right|$$

$$= \frac{|e^x||e^{iy}|}{|x+iy|^2}$$

$$= \frac{e^x}{x^2+y^2} \qquad [\because e^x \geq 0 \text{ for all values of } x]$$

Along the line AB, $y = \frac{2-x}{2}$

$$|f(z)| = \frac{e^x}{x^2 + \left(\frac{2-x}{2}\right)^2}$$

$$= \frac{4e^x}{4x^2 + 4 + x^2 - 4x}$$

$$= \frac{4e^x}{5x^2 - 4x + 4}$$

For maximum value,

$$\frac{d}{dx}|f(z)| = 0$$

$$\frac{4e^x(5x^2 - 4x + 4) - 4e^x(10x - 4)}{(5x^2 - 4x + 4)^2} = 0$$

$$5x^2 - 14x + 8 = 0$$

$$x = 2, \frac{4}{5}$$

But $|f(z)|$ is maximum at $x = \frac{4}{5}$.

$$|f(z)| = \frac{4e^x}{5x^2 - 4x + 4}$$

$$\leq \frac{4e^{\frac{4}{5}}}{5\left(\frac{4}{5}\right)^2 - 4\left(\frac{4}{5}\right) + 4}$$

$$\left|\frac{e^z}{z^2}\right| \leq e^{\frac{4}{5}}$$

$$M = e^{\frac{4}{5}}$$

By ML inequality,

$$\left|\int_C f(z)\,dz\right| \leq ML$$

$$\left|\int_C \frac{e^z}{z^2}\,dz\right| \leq \sqrt{5}\, e^{\frac{4}{5}}$$

Evaluation of Line Integrals

Let $f(z) = u + iv$

$z = x + iy, \quad dz = dx + idy$

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Here, u and v are functions of x and y. To evaluate the line integral, it is converted to a single variable x or y using the equation of the curve C.

Note If $f(z)$ is analytic in a simply connected domain D then the line integral is independent of the path, i.e.,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

where $F'(z) = f(z)$

Example 14 :

Evaluate $\int_0^{2+i} z^2\,dz$ *along the line* $y = \frac{x}{2}$.

Solution

$$f(z) = z^2$$
$$f'(z) = 2z$$

Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.

Hence, $\int_0^{2+i} z^2\,dz$ will be independent of the path.

$$\int_0^{2+i} z^2\,dz = \left|\frac{z^3}{3}\right|_0^{2+i}$$
$$= \frac{1}{3}(2+i)^3$$
$$= \frac{1}{3}(8 + i^3 + 6i^2 + 12i)$$
$$= \frac{1}{3}(8 - i - 6 + 12i)$$
$$= \frac{1}{3}(2 + 11i)$$

Example 15 :

Evaluate $\int_0^{1+i} (x - y + ix^2)\,dz$ *along the line from* $z = 0$ *to* $z = 1 + i$.

Solution

Let

$$z = x + iy$$
$$dz = dx + i\,dy$$

The equation of the line OA

$$y-0=\frac{1-0}{1-0}(x-0)$$

$$y=x$$

$$\int_0^{1+i}(x-y+ix^2)\,dz=\int_0^{1+i}(x-y+ix^2)\,(dx+idy)$$

$$=\int_{OA}(x-y+ix^2)\,(dx+idy)$$

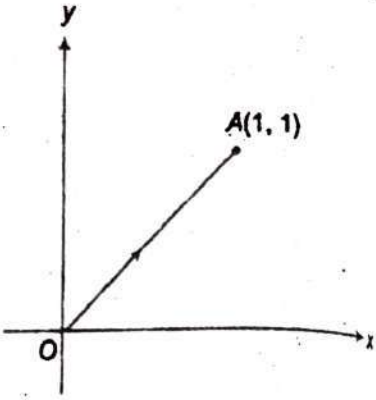


Fig.

Along OA, $y=x$, $dx=dy$, x varies from 0 to 1.

$$\int_0^{1+i}(x-y+ix^2)\,dz=\int_0^1 ix^2(dx+idx)$$

$$=\int_0^1 ix^2(1+i)\,dx$$

$$=(-1+i)\int_0^1 x^2\,dx$$

$$=(-1+i)\left|\frac{x^3}{3}\right|_0^1$$

$$=\frac{-1+i}{3}$$

$$=-\frac{1}{3}+\frac{1}{3}i$$

Example 16 :

Evaluate $\int_C \operatorname{Re} z\,dz$, *where C is the shortest path from* $1+i$ *to* $3+2i$.

Solution

Let

$$z=x+iy$$

$$\operatorname{Re} z=x$$

$$dz=dx+idy$$

The shortest distance from $A(1+i)$ to $B(3+2i)$ is the line joining these points

The equation of the line AB is

$$y-1=\frac{2-1}{3-1}(x-1)$$

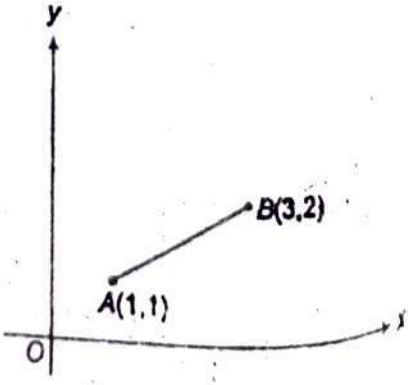


Fig.

$$2(y-1)=x-1$$
$$x-2y=-1$$
$$\int_C \text{Re}\, z\, dz = \int_C x(dx+i\,dy)$$

Along AB, $y=\frac{x+1}{2}$, $dy=\frac{1}{2}dx$, x varies from 1 to 3.

$$\int_C \text{Re}\, z\, dz = \int_1^3 x\left(dx+i\frac{1}{2}dx\right)$$
$$=\left(1+\frac{i}{2}\right)\int_1^3 x\,dx$$
$$=\left(1+\frac{i}{2}\right)\left|\frac{x^2}{2}\right|_1^3$$
$$=\left(1+\frac{i}{2}\right)\left(\frac{9}{2}-\frac{1}{2}\right)$$
$$=\left(1+\frac{i}{2}\right)4$$
$$=4+2i$$

Example 17 :

Evaluate $\int_C z^2\, dz$, *where C is the line joining the points* (0, 0) *and* (4, 2).

Solution

Let
$$z=x+iy$$
$$dz=dx+i\,dy$$

The equation of the line joining $O(0, 0)$ and $A(4, 2)$

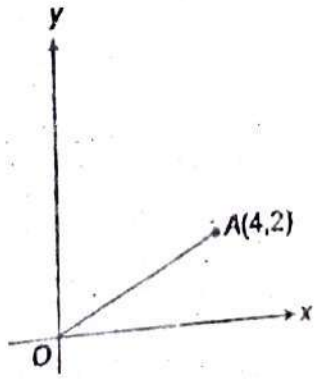


Fig.

$$y-0=\frac{2-0}{4-0}(x-0)$$
$$2y=x$$

Along OA, $x=2y$, $dx=2dy$, y varies from 0 to 2.

$$\int_0^{4+2i} z^2\, dz = \int_C (x+iy)^2\,(dx+i\,dy)$$
$$=\int_C (x^2-y^2+2ixy)\,(dx+i\,dy)$$

$$= \int_0^2 (4y^2 - y^2 + 4iy^2)(2dy + idy)$$

$$= \int_0^2 (3+4i)\, y^2 (2+i)\, dy$$

$$= (6+3i+8i+4i^2)\left|\frac{y^3}{3}\right|_0^2$$

$$= (2+11i)\left(\frac{2^3}{3}\right)$$

$$= \frac{8}{3}(2+11i)$$

Aliter:

Let $f(z) = z^2$

$f'(z) = 2z$

Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.

Hence, $\int_0^{4+2i} z^2\, dz$ will be independent of the path.

$$\int_0^{4+2i} z^2\, dz = \left|\frac{z^3}{3}\right|_0^{4+2i}$$

$$= \frac{1}{3}(4+2i)^3$$

$$= \frac{2^3}{3}(2+i)^3$$

$$= \frac{8}{3}(8+i^3+6i^2+12i)$$

$$= \frac{8}{3}(8-i-6+12i)$$

$$= \frac{8}{3}(2+11i)$$

Example 18 :

Evaluate $\int_C \bar{z}\, dz$, *where C is along the sides of the triangle havin*

vertices $z = 0, 1, i$.

Solution

Let $z = x + iy, \bar{z} = x - iy$

$$dz = dx + i\,dy$$

$$\int_C \bar{z}\,dz = \int_C (x - iy)(dx + i\,dy)$$

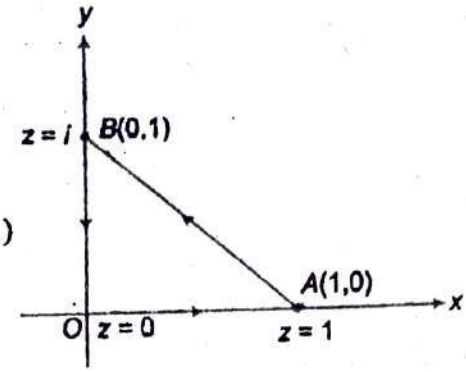


Fig.

The equation of the line joining $A(1, 0)$ and $B(0, 1)$

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$y = -x + 1$$

$$x + y = 1$$

Along OA, $y = 0$, $dy = 0$, x varies from 0 to 1.

$$\int_C \bar{z}\,dz = \int_0^1 x\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{2}. \qquad \text{...(1)}$$

Along AB, $y = -x + 1$, $dy = -dx$, x varies from 1 to 0.

$$\int_C \bar{z}\,dz = \int_1^0 [x - i(-x+1)](dx - i\,dx)$$

$$= \int_1^0 [(1+i)x - i](1-i)\,dx$$

$$= (1-i)\left|(1+i)\frac{x^2}{2} - ix\right|_1^0$$

$$= (1-i)\left[-\frac{(1+i)}{2} + i\right]$$

$$= (1-i)\left[\frac{-1-i+2i}{2}\right]$$

$$= \frac{(1-i)(i-1)}{2} \qquad \text{...(2)}$$

$$= i$$

Along BO, $x = 0$, y varies from 1 to 0.

$$\int_C \bar{z}\,dz = \int_{1}^{0} (-y)\,i\,dy$$

$$= \int_{1}^{0} y\,dy$$

$$= \left|\frac{y^2}{2}\right|_1^0$$

$$= -\frac{1}{2} \qquad \ldots(3)$$

Adding Eqs (1), (2), and (3),

$$\int_C \bar{z}\,dz = \frac{1}{2} + i - \frac{1}{2}$$

$$= i$$

Example 19 :

Find the value of the integral $\int_C \bar{z}\,dz$, *where C is the right-hand half* $z = 2e^{i\theta}\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ *of the circle* $|z| = 2$, *from* $z = -2i$ *to* $z = 2i$.

Solution

$$z = 2e^{i\theta}$$

$$dz = 2ie^{i\theta}\,d\theta$$

$$\bar{z} = 2e^{-i\theta}$$

$$\int_C \bar{z}\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta} \cdot 2ie^{i\theta}\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4i\,d\theta$$

$$= 4i\,|\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4i\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 4i\pi$$

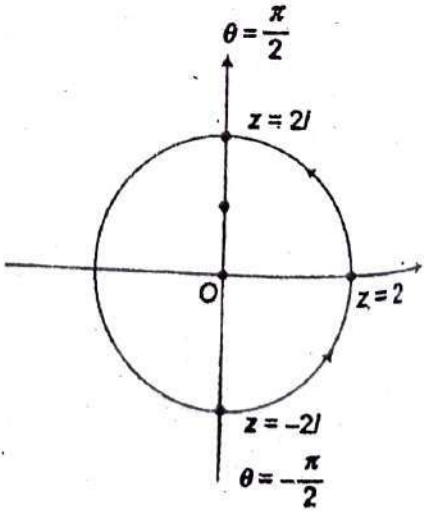


Fig.

If $f(z)$ is analytic inside and on a closed curve C and if a is any point inside C then

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

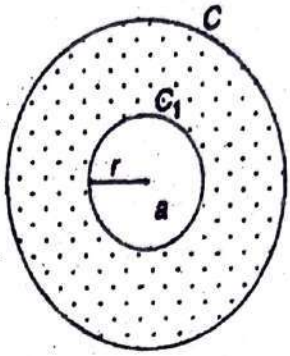


Fig.

Proof Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-a}$ is also analytic inside and on C except at $z=a$.

Draw a small circle C_1 with the centre at $z=a$ and the radius r lying entirely inside C

Now, $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

By Cauchy's integral theorem for a multiply connected region,

$$\oint_C \frac{f(z)}{z-a}\,dz=\oint_{C_1}\frac{f(z)}{z-a}\,dz$$

For the circle C_1, let $z-a=re^{i\theta}, dz=rie^{i\theta}\,d\theta$

$$\oint_C \frac{f(z)}{z-a}\,dz=\oint_0^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}\,ire^{i\theta}\,d\theta$$

$$=i\oint_0^{2\pi} f(a+re^{i\theta})\,d\theta \qquad ...(4.1)$$

In the limiting position, as $r\to 0$, the circle C_1 shrinks to the point a.
Hence, Eq. (4.1) reduces to

$$\oint_0 \frac{f(z)}{z-a}\,dz=i\oint_0^{2\pi} f(a)\,d\theta$$

$$=i\,f(a)\left|\theta\right|_0^{2\pi}$$

$$=i\,f(a)(2\pi-0)$$

$$=2\pi i\,f(a)$$

$$\therefore \quad f(a)=\frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

GENERALIZED CAUCHY'S INTEGRAL FORMULA

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i} \oint_{0} \frac{f(z)}{(z-a)^2} dz$$

where C is any closed curve in R surrounding the point $z = a$.

Proof By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{z-a} dz$$

Differentiating w.r.t. a using Differentiation Under the Integral Sign (DUIS),

$$f'(a) = \frac{1}{2\pi i} \oint_{C} \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right] dz$$

$$= \frac{1}{2\pi i} \oint_{C} f(z)\left[-\frac{1}{(z-a)^2}(-1)\right] dz$$

$$= \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{(z-a)^2} dz$$

Differentiating again w.r.t. a,

$$f''(a) = \frac{2!}{2\pi i} \oint_{C} \frac{f(z)}{(z-a)^3} dz$$

Similarly,

$$f'''(a) = \frac{3!}{2\pi i} \oint_{C} \frac{f(z)}{(z-a)^4} dz$$

In general,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C} \frac{f(z)}{(z-a)^{n+1}} dz$$

LIOUVILLE THEOREM

If $f(z)$ is analytic and bounded in the entire z-plane for all z then $f(z)$ is constant.

MAXIMUM MODULUS THEOREM

If $f(z)$ is nonconstant and analytic inside and on a simple closed contour C then the maximum value of $|f(z)|$ occurs on C.

Example 20 :

Prove that $\int_C \frac{dz}{z-a} = 2\pi i$ *and* $\int_C (z-a)^n \, dz = 0$ *(n is an integer and* $n \neq -1$*) where C is the circle* $|z-a| = r$.

Solution

(a) (i) Let $I = \int_C \frac{dz}{z-a}$

(ii) C is the circle $|z-a| = r$ with the centre at $(a, 0)$ and a radius of r (Fig. 4.29).

(iii) For $z = a$, $|z-a| = |a-a| = 0 < r$

Hence, $z = a$ lies inside C.

(iv) Let $f(z) = 1$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

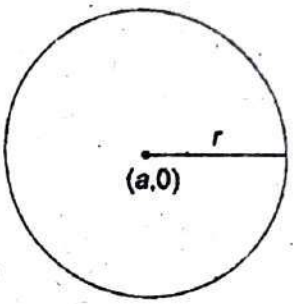


Fig. 4.29

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{dz}{z-a} = 2\pi i \, (1)$$

$$= 2\pi i$$

(b) (i) Let $f(z) = (z-a)^n$

(ii) Since $f(z)$ is a polynomial, it is analytic inside and on C. $f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C (z-a)^n dz = 0$$

Example 21 :

Evaluate $\int_C \frac{z}{z-2} dz$, *where C is the circle* $|z-2| = \frac{3}{2}$.

Solution

(i) Let $I = \int_C \frac{z}{z-2} dz$

(ii) C is the circle $|z-2| = \frac{3}{2}$ with the centre at $(2, 0)$ and a radius of $\frac{3}{2}$ (Fig. 4.30).

(iii) For $z = 2$, $|z-2| = |2-2| = 0 < \frac{3}{2}$

Hence, $z = 2$ lies inside C.

(iv) Let $f(z) = z$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

$$\int_C \frac{z}{z-2} dz = 2\pi i [z]_{z=2}$$

$$= 2\pi i(2)$$

$$= 4\pi i$$

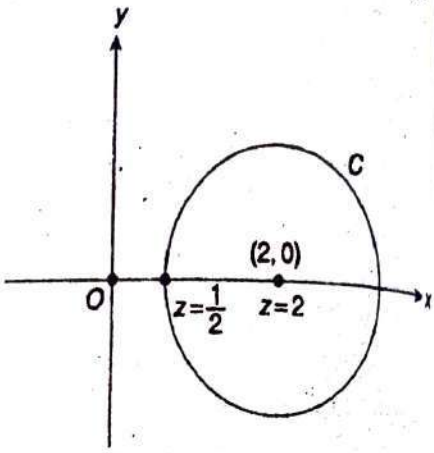


Fig.

Example 22 :

Evaluate $\int_C \frac{e^z}{z-1} dz$ *if* C *is* $|z| = 2$.

Solution

(i) Let $I = \int_C \frac{e^z}{z-1} dz$

(ii) C is the circle $|z| = 2$ with the centre at O and a radius of 2

(iii) For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

(iv) Let $f(z) = e^z$

$f(z)$ is analytic inside and on C.

(v) By Cauchy s integral formula,

$$\int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

$$\int_C \frac{e^z}{z-1} dz = 2\pi i \left[e^z\right]_{z=1}$$

$$= 2\pi i e$$

$$= 2\pi e i$$

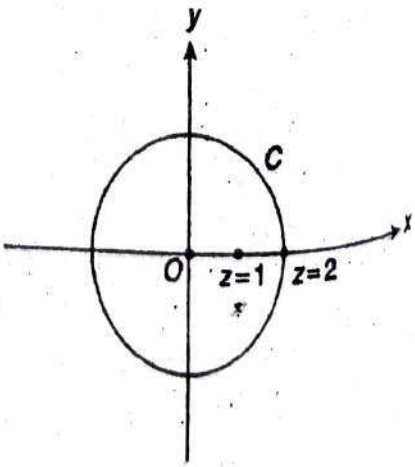


Fig. 4.31

Example 23 :

Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $I = \int_C \frac{e^{2z}}{(z+1)^4} dz$

(ii) C is the circle $|z| = 2$ with the centre at O and a radius of 2

(iii) For $z = -1, |z| = |-1| = 1 < 2$

Hence, $z = -1$ lies inside C.

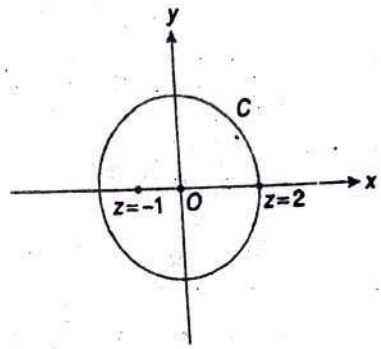


Fig.

(iv) Let $f(z) = e^{2z}$

$f(z)$ is analytic inside and on C.

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

(v) By generalized Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^4} dz = \frac{2\pi i}{3!} f'''(a)$$

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{3!}\left[8e^{2z}\right]_{z=-1}$$

$$= \frac{2\pi i}{6}\left[8e^{-2}\right]$$

$$= \frac{8\pi i}{3e^2}$$

Example 24 :

Evaluate $\int_C \left[\frac{3}{z-i} - \frac{6}{(z-i)^2}\right] dz$, *where* $C : |z| = 2$.

Solution

(i) Let $I = \int_C \left[\frac{3}{z-i} - \frac{6}{(z-i)^2}\right] dz$

(ii) C is the circle $|z| = 2$ with the centre at O and a radius of 2

(iii) For $z = i$, $|z| = |i| = 1 < 2$.

Hence, $z = i$ lies inside C.

(iv) Let $f_1(z) = 3, f_2(z) = 6$

$f_1(z)$ and $f_2(z)$ are analytic inside and on C.

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

and $$\int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

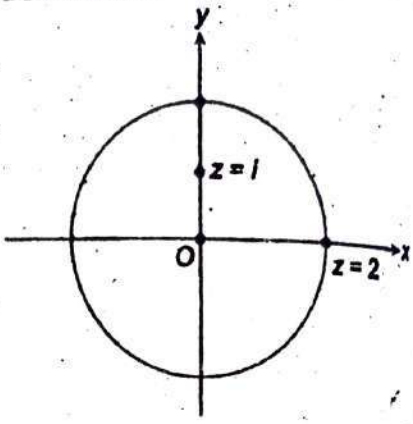


Fig.

$$\int_C \left[\frac{3}{z-i} - \frac{6}{(z-i)^2}\right] dz = 2\pi i \left[f_1(z)\right]_{z=i} - 2\pi i \left[f_2'(z)\right]_{z=i}$$

$$= 2\pi i\,(3) - 2\pi i\,(0)$$

$$= 6\pi i$$

Example 25 :

Evaluate $\int_C \frac{dz}{z^4 e^z}$, *where C is* $|z| = 1$.

Solution

(i) Let $I = \int_C \frac{dz}{z^4 e^z}$

$$= \int_C \frac{e^{-z}}{z^4} dz$$

(ii) C is the circle $|z| = 1$ with the centre at O and a radius of 1

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

(iv) Let $f(z) = e^{-z}$

$f(z)$ is analytic inside and on C.

$f'(z) = -e^{-z}$

$f''(z) = e^{-z}$

$f'''(z) = -e^{-z}$

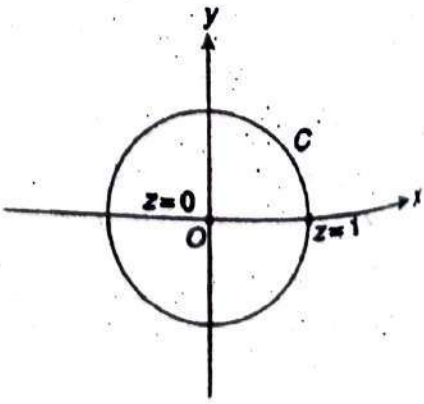


Fig.

(v) By generalized Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^4}\,dz = \frac{2\pi i}{3!} f''(a)$$

$$\int_C \frac{dz}{z^4 e^4} = \frac{2\pi i}{3!}\left[-e^{-z}\right]_{z=0} = \frac{2\pi i}{6}(-1) = -\frac{\pi i}{3}$$

TAYLOR'S SERIES

If $f(z)$ is analytic inside a circle C with centre at $z = a$ then for each z inside C, $f(z)$ can be expanded as a power series about $z = a$ as

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \cdots$$

Proof Let z be any point inside the circle C. Draw a circle C_1 inside C with the centre at $z = a$, enclosing the point z

Let w be any point on C_1.

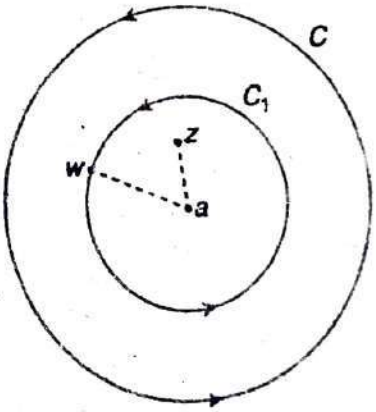


Fig.

$$|z-a| < |w-a|$$

$$\frac{|z-a|}{|w-a|} < 1$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw \qquad \text{...(5.1)}$$

Consider $\dfrac{1}{w-z} = \dfrac{1}{(w-a)-(z-a)}$

$$= \frac{1}{(w-a)\left(1-\dfrac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

$$\left[\because \left|\frac{z-a}{w-a}\right| < 1 \quad \text{Using binomial expansion}\right]$$

$$= \frac{1}{(w-a)} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

Subsituting $\frac{1}{w-z}$ in Eq. (5.1),

$$f(z) = \frac{1}{2\pi i} \int_{C_1} f(w) \left[\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} \right] dw$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left[\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \right]$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left[2\pi i \frac{f^{(n)}(a)}{n!} \right] \left[\begin{array}{l} \text{Using Cauchy's integral formula} \\ \text{for derivatives} \end{array} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

$$= f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Note: Taylor's series of $f(z)$ about $z = 0$ is known as Maclaurin's series.

Example 26 :

Expand $f(z) = e^z$ in a Taylor series about $z = 0$.

Solution

$$f(z) = e^z, \quad a = 0$$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Putting $a = 0$,

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots \quad \ldots(1)$$

$f(z) = e^z$ $\qquad f(0) = e^0 = 1$

$f'(z) = e^z$ $\qquad f'(0) = e^0 = 1$

$f''(z) = e^z$ $\qquad f''(0) = e^0 = 1$

$f'''(z) = e^z$ $\qquad f'''(0) = e^0 = 1$

Substituting in Eq. (1),

$$f(z)=1+z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\ldots$$

$$e^z=1+z+\frac{z^2}{2}+\frac{z^3}{6}+\ldots$$

Example 27 :

Expand $\frac{1}{z+2}$ at $z=1$ *in Taylor's series.*

Solution

Let $f(z)=\frac{1}{z+2}$, $a=1$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\cdots$$

Putting $a=1$,

$$f(z)=f(1)+f'(1)(z-1)+\frac{f''(1)}{2!}(z-1)^2+\frac{f'''(1)}{3!}(z-1)^3+\cdots \quad \ldots(1)$$

$$f(z)=\frac{1}{z+2} \qquad f(1)=\frac{1}{1+2}=\frac{1}{3}$$

$$f'(z)=-\frac{1}{(z+2)^2} \qquad f'(1)=-\frac{1}{(1+2)^2}=-\frac{1}{9}$$

$$f''(z)=\frac{2}{(z+2)^3} \qquad f''(1)=\frac{2}{(1+2)^3}=\frac{2}{27}$$

$$f'''(z)=-\frac{6}{(z+2)^4} \qquad f'''(1)=-\frac{6}{(1+2)^4}=-\frac{6}{81}=-\frac{2}{27}$$

Substituting in Eq. (1),

$$f(z)=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{2!}\frac{2}{27}(z-1)^2+\frac{1}{3!}\left(-\frac{2}{27}\right)(z-1)^3+\cdots$$

$$\frac{1}{z+2}=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{27}(z-1)^2-\frac{1}{81}(z-1)^3+\cdots$$

Aliter:

Let $f(z)=\frac{1}{z+2}$

$z-1=t$ then $z=t+1$.

$$f(z)=\frac{1}{t+1+2}$$

$$\frac{1}{z+2}=\frac{1}{3+t}$$

$$=\frac{1}{3\left(1+\frac{t}{3}\right)}$$

$$=\frac{1}{3}\left(1+\frac{t}{3}\right)^{-1}$$

$$=\frac{1}{3}\left(1-\frac{t}{3}+\frac{t^2}{9}-\frac{t^3}{27}+\cdots\right)$$

$$=\frac{1}{3}-\frac{1}{9}t+\frac{1}{27}t^2-\frac{1}{81}t^3+\cdots$$

$$=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{27}(z-1)^2-\frac{1}{81}(z-1)^3+\ldots$$

Example 28 :

Obtain the Taylor series of $f(z)=\frac{1-z}{z^2}$ *in powers of* $z-1$.

Solution

$$f(z)=\frac{1-z}{z^2}=\frac{1}{z^2}-\frac{1}{z},\quad a=1$$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots$$

Putting $a=1$,

$$f(z)=f(1)+f'(1)(z-1)+\frac{f''(1)}{2!}(z-1)^2+\frac{f'''(1)}{3!}(z-1)^3+\cdots$$

$$f(z)=\frac{1}{z^2}-\frac{1}{z} \qquad f(1)=\frac{1}{1}-\frac{1}{1}=0$$

$$f'(z)=-\frac{2}{z^3}+\frac{1}{z^2} \qquad f'(1)=-2+1=-1$$

$$f''(z)=\frac{6}{z^4}-\frac{2}{z^3} \qquad f''(1)=6-2=4$$

$$f'''(z)=-\frac{24}{z^5}+\frac{6}{z^4} \qquad f'''(1)=-24+6=-18$$

Substituting in Eq. (1),

$$f(z)=0-1(z-1)+\frac{4}{2!}(z-1)^2-\frac{18}{3!}(z-1)^3+\cdots$$

$$\frac{1-z}{z^2}=-(z-1)+2(z-1)^2-3(z-1)^3+\cdots$$

Example 29 :

Expand $\frac{z-1}{z+1}$ *in a Taylor series about the point* $z = 1$.

Solution

Let $f(z)=\frac{z-1}{z+1}, \quad a=1$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots$$

Putting $a = 1$,

$$f(z)=f(1)+f'(1)(z-1)+\frac{f''(1)}{2!}(z-1)^2+\cdots \qquad \ldots(1)$$

$$f(z)=\frac{z-1}{z+1} \qquad f(1)=0$$

$$f'(z)=\frac{(z+1)(1)-(z-1)(1)}{(z+1)^2} \qquad f'(1)=\frac{1}{2}$$

$$=\frac{2}{(z+1)^2}$$

$$f''(z)=\frac{0-2\{2(z+1)\}}{(z+1)^4} \qquad f''(1)=-\frac{4}{2^3}=-\frac{1}{2}$$

$$=-\frac{4}{(z+1)^3}$$

Substituting in Eq. (1),

$$f(z)=0+\frac{1}{2}(z-1)-\frac{1}{2!}\frac{1}{2}(z-1)^2+\cdots$$

$$\frac{z-1}{z+1}=\frac{1}{2}(z-1)-\frac{1}{4}(z-1)^2+\cdots$$

Example 30 :

Expand $\log(1+z)$ *as a Taylor series about* $z=0$ *when* $|z|<1$.

Solution

Let. $f(z)=\log(1+z),\ a=0$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots$$

Putting $a=0$,

$$f(z)=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+\ldots \quad \ldots(1)$$

$$f(z)=\log(1+z) \qquad f(0)=\log 1=0$$

$$f'(z)=\frac{1}{1+z} \qquad f'(0)=\frac{1}{1+0}=1$$

$$f''(z)=-\frac{1}{(1+z)^2} \qquad f''(0)=-\frac{1}{1}=-1$$

$$f'''(z)=\frac{2}{(1+z)^3} \qquad f'''(0)=2$$

Substituting in Eq. (1),

$$f(z)=0+z-\frac{z^2}{2!}+\frac{2}{3!}z^3+\ldots$$

$$\log(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}+\ldots$$

Example 31 :

Expand $f(z)=\sin z$ *about* $z=\frac{\pi}{4}$.

Solution

$$f(z)=\sin z, \qquad a=\frac{\pi}{4}$$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots$$

Putting $a=\frac{\pi}{4}$,

$$f(z)=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)\left(z-\frac{\pi}{4}\right)+\frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z-\frac{\pi}{4}\right)^2+\frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z-\frac{\pi}{4}\right)^3+\cdots \quad ...(1)$$

$f(z)=\sin z$ $\qquad f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$f'(z)=\cos z$ $\qquad f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$f''(z)=-\sin z$ $\qquad f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$

$f'''(z)=-\cos z$ $\qquad f'''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$

Substituting in Eq. (1),

$$f(z)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2-\frac{1}{3!}\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3+\cdots$$

$$\sin z=\frac{1}{\sqrt{2}}\left[1+\left(z-\frac{\pi}{4}\right)-\frac{1}{2}\left(z-\frac{\pi}{4}\right)^2-\frac{1}{6}\left(z-\frac{\pi}{4}\right)^3+\cdots\right]$$

LAURENT'S SERIES

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with the centre at $z=a$ and radii $r_1, r_2 (r_2<r_1)$ and in the annular region R between C_1 and C_2 then for all z in R,

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\cdots+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\cdots$$

$$=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where

$$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

$$b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

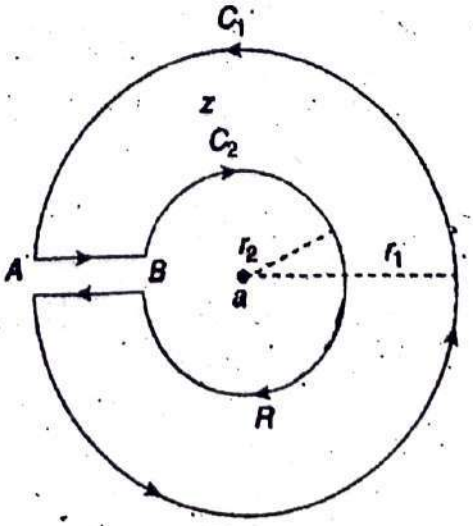


Fig.

Proof

Let C_1, C_2 be two concentric circles with centre at $z = a$ and radii r_1 and $r_2 (r_2 < r_1)$ respectively

The annular region R between C_1 and C_2 is a multiply connected region. This multiply connected region is converted to a simply connected region by introducing a cross-cut AB.

$f(z)$ is analytic in this simply connected region.

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{AB}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{BA}\frac{f(w)}{w-z}dw$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw$$

[$\because$ integrals around AB and BA cancel each other]

$$= f_1(z) - f_2(z)$$

(i) $$f_1(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw \qquad ...(5.2)$$

z is any point in R and for this integral, w lies on C_1.

$$\therefore \ |z-a| < |w-a|$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

Consider $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

$$= \frac{1}{(w-a)}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\frac{1}{w-z}$ in Eq. (5.2),

$$f_1(z)=\frac{1}{2\pi i}\int_{C_1} f(w)\left[\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right]dw$$

$$=\sum_{n=0}^{\infty}\left[(z-a)^n\left\{\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw\right\}\right]$$

$$=\sum_{n=0}^{\infty}(z-a)^n a_n \qquad \cdots(5.3)$$

where $$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

(ii) $$f_2(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw \qquad \cdots(5.4)$$

z is any point in R and for this integral, w lies on C_2.

$$\therefore \quad |z-a|>|w-a|$$

$$1>\left|\frac{w-a}{z-a}\right|$$

$$\left|\frac{w-a}{z-a}\right|<1$$

Consider $$\frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}$$

$$=\frac{1}{(z-a)\left(\frac{w-a}{z-a}-1\right)}$$

$$=-\frac{1}{(z-a)}\cdot\frac{1}{\left(1-\frac{w-a}{z-a}\right)}$$

$$=-\frac{1}{(z-a)}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$=-\frac{1}{(z-a)}\left[1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+\cdots+\left(\frac{w-a}{z-a}\right)^{n-1}+\cdots\right]$$

$$=-\frac{1}{(z-a)}\sum_{n=1}^{\infty}\left(\frac{w-a}{z-a}\right)^{n-1}$$

$$=\sum_{n=1}^{\infty}-\frac{(w-a)^{n-1}}{(z-a)^n}$$

Substituting $\frac{1}{w-z}$ in Eq. (5.4),

$$f_2(z)=\frac{1}{2\pi i}\int_{C_2} f(w)\left[\sum_{n=1}^{\infty}-\frac{(w-a)^{n-1}}{(z-a)^n}\right]dw$$

$$=\sum_{n=1}^{\infty}\left[\frac{-1}{(z-a)^n}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right\}\right]$$

$$=\sum_{n=1}^{\infty}-\frac{1}{(z-a)^n}b_n$$

where $$b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

$$\therefore f(z)=f_1(z)-f_2(z)$$

$$=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where $$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

$$b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

The second term $\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ is known as the principal part of Laurent's series.

Example 32 :

Find the Laurent's series of $f(z)=z^2e^{\frac{1}{z}}$ about $z=0$.

Solution

$$f(z)=z^2e^{\frac{1}{z}}$$

$f(z)$ is analytic for $|z|>0$.

$$f(z)=z^2\left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\cdots\right)$$

$$=z^2+z+\frac{1}{2}+\frac{1}{6z}+\cdots$$

Example 33 :

Find the Laurent's series of $f(z)=\dfrac{e^{2z}}{(z-1)^3}$ *about* $z=1$.

Solution

$$f(z)=\frac{e^{2z}}{(z-1)^3}$$

$f(z)$ is not analytic at $z=1$. But it is analytic in the region $|z-1|>0$ about $z=1$.
Let $z-1=t$ then $z=t+1$.

$$f(z)=\frac{e^{2(t+1)}}{t^3}$$

$$=\frac{e^2}{t^3}e^{2t}$$

$$=\frac{e^2}{t^3}\left[1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots\right]$$

$$=e^2\left[\frac{1}{t^3}+\frac{2}{t^2}+\frac{2}{t}+\frac{4}{3}+\cdots\right]$$

$$=e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)}+\frac{4}{3}+\cdots\right]$$

Example 34 :

Find Laurent's series expansion in powers of z that represent $f(z)=\dfrac{1}{z^2(1-z)}$ *for the following domains: (i)* $|z|<1$ *(ii)* $|z|>1$.

Solution

$$f(z)=\frac{1}{z^2(1-z)}$$

$$=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{1-z}$$

$$1=Az(1-z)+B(1-z)+Cz^2$$

Putting $z=0$,

$$1=B$$

Putting $z = 1$,

$$1 = C$$

Putting $z = -1$,

$$1 = -2A + 2B + C$$

$$= -2A + 2 + 1$$

$$A = 1$$

$$\therefore \quad f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$$

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + (1-z)^{-1}$$

$$= \frac{1}{z} + \frac{1}{z^2} + (1 + z + z^2 + \cdots)$$

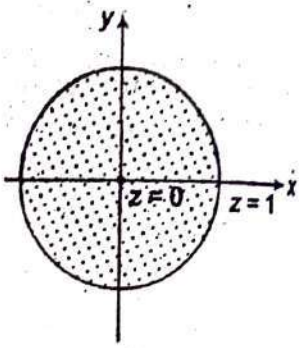


Fig.

(ii) $|z| > 1$

$f(z)$ is analytic in the region $|z| > 1$ about $z = 0$ (Fig. 5.5).

$$\because \quad |z| > 1, \quad \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{-z\left(-\frac{1}{z} + 1\right)}$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\right)$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

$$= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

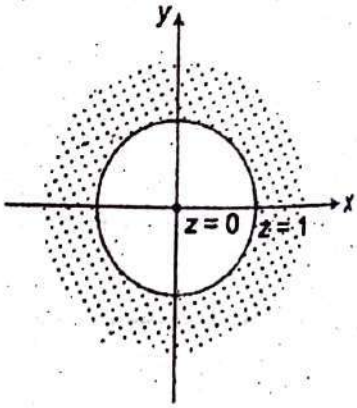


Fig.

Example 35 :

Expand $f(z) = \dfrac{1}{(z+1)(z-2)}$ *in Laurent's series in the following regions:*

(i) $|z| < 1$ *(ii)* $1 < |z| < 2$ *(iii)* $|z| > 2$.

Solution

$$f(z)=\frac{1}{(z+1)(z+2)}$$
$$=\frac{1}{3}\frac{(z+1)-(z-2)}{(z+1)(z-2)}$$
$$=\frac{1}{3}\left[\frac{1}{z-2}-\frac{1}{z+1}\right]$$

$f(z)$ is not analytic at $z=2$ and $z=-1$.

(i) $|z|<1$

$f(z)$ is analytic in the region $|z|<1$ about $z=0$

$\because\ |z|<1<2,\quad \left|\frac{z}{2}\right|<1$

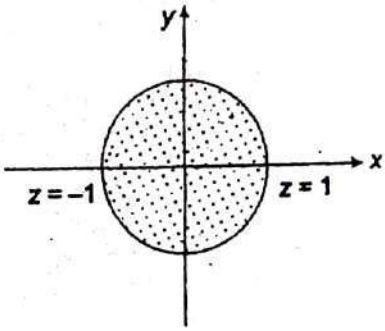


Fig.

$$f(z)=\frac{1}{3}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z+1}\right]$$
$$=-\frac{1}{3}\left[\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1+z)^{-1}\right]$$
$$=-\frac{1}{3}\left[\frac{1}{2}\left\{1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right\}+(1-z+z^2-\cdots)\right]$$
$$=-\frac{1}{3}\left(\frac{3}{2}-\frac{3}{4}z+\frac{9}{8}z^2-\cdots\right)$$
$$=-\frac{1}{2}+\frac{1}{4}z-\frac{3}{8}z^2+\cdots$$

(ii) $1<|z|<2$

$f(z)$ is analytic in the annular region $1<|z|<2$ about $z=0$ (Fig. 5.7).

$1<|z|,\ \left|\frac{1}{z}\right|<1$

$|z|<2,\ \left|\frac{z}{2}\right|<1$

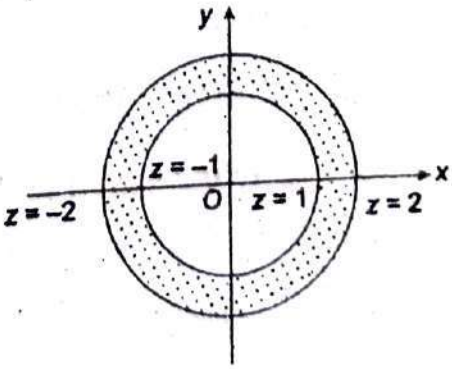


Fig.

$$f(z)=\frac{1}{3}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1+\frac{1}{z}\right)}\right]$$
$$=\frac{1}{3}\left[-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}\right]$$
$$=\frac{1}{3}\left[-\frac{1}{2}\left\{1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right\}-\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right)\right]$$

$$=-\frac{1}{3}\left[\left(\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\cdots\right)+\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\right]$$

(iii) $|z|>2$

$f(z)$ is analytic in the region $|z|>2$ about $z=0$

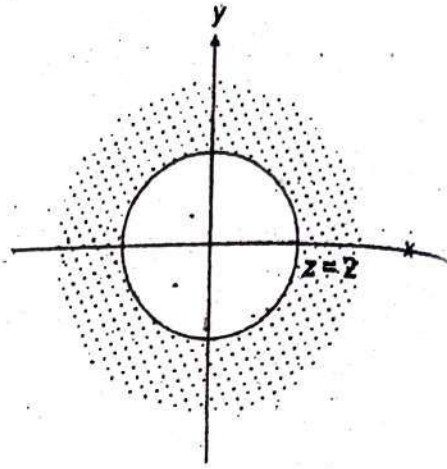


Fig.

$\because\ |z|>2>1,\ \left|\frac{2}{z}\right|<1,\ \left|\frac{1}{z}\right|<1$

$$f(z)=\frac{1}{3}\left[\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1+\frac{1}{z}\right)}\right]$$

$$=\frac{1}{3}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}\right]$$

$$=\frac{1}{3}\left[\frac{1}{z}\left\{1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right\}-\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right)\right]$$

$$=\frac{1}{3}\left[\left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right)-\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\right]$$

$$=\frac{1}{3}\left(\frac{3}{z^2}+\frac{3}{z^3}+\cdots\right)$$

$$=\frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

Example 36 :

Find all possible Laurent's expansion of $f(z)=\dfrac{4-3z}{z(1-z)(2-z)}$ ***abou*** $z=0$ ***Indicate the region of convergence in each case.***

Solution

$$f(z)=\frac{4-3z}{z(1-z)(2-z)}$$

$$=\frac{A}{z}+\frac{B}{1-z}+\frac{C}{2-z}$$

$$4-3z=A(1-z)(2-z)+Bz(2-z)+Cz(1-z)$$

Putting $z=0$,

$$4=2A$$

$$A=2$$

Putting $z = 1$,

$$B = 1$$

Putting $z = 2$,

$$4 - 3(2) = -2C$$

$$C = 1$$

$$\therefore \quad f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = 2$.

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 5.9).

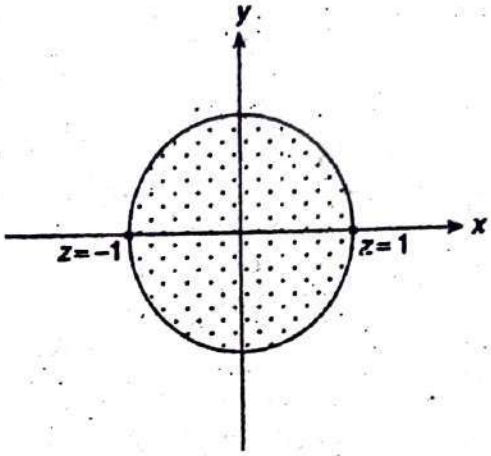


Fig.

$$\because \quad |z| < 1 < 2, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} + \left(1 + z + z^2 + \cdots\right) + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots\right) + \frac{2}{z}$$

(ii) $1 < |z| < 2$

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 5.10)

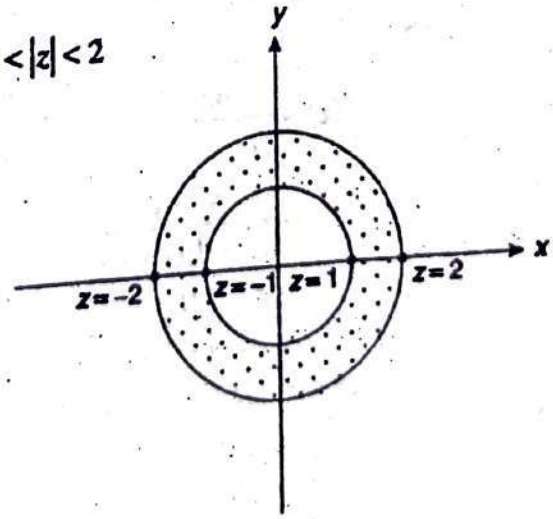


Fig.

$$1 < |z|, \quad \left|\frac{1}{z}\right| < 1$$

$$|z| < 2, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{2}{z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right) + \left(\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right)$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$

$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{2}{z} - \frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$-\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$$= -\frac{3}{z^2} - \frac{5}{z^3} - \cdots$$

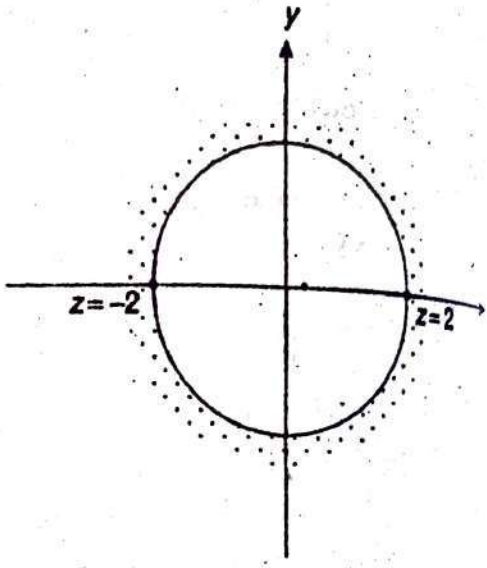


Fig.

Example 37 :

Find the Laurent's series expansion of

$$f(z) = \frac{4z+1}{z(z^2+z-2)} \text{ in } |z| > 2.$$

Solution

$$f(z) = \frac{4z+1}{z(z^2+z-2)}$$

$$= \frac{4z+1}{z(z-1)(z+2)}$$

$$= \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$4z+1 = A(z+2)(z-1) + Bz(z+2) + Cz(z-1)$$

Putting $z = 0$,

$$1 = (2)(-1)A$$

$$A = -\frac{1}{2}$$

Putting $z = 1$,

$$5 = 3B$$

$$B = \frac{5}{3}$$

Putting $z = -2$,

$$-7 = (-2)(-3)C$$

$$C = -\frac{7}{6}$$

$$\therefore\ f(z) = -\frac{1}{2}\frac{1}{z} + \frac{5}{3}\frac{1}{z-1} - \frac{7}{6}\frac{1}{z+2}$$

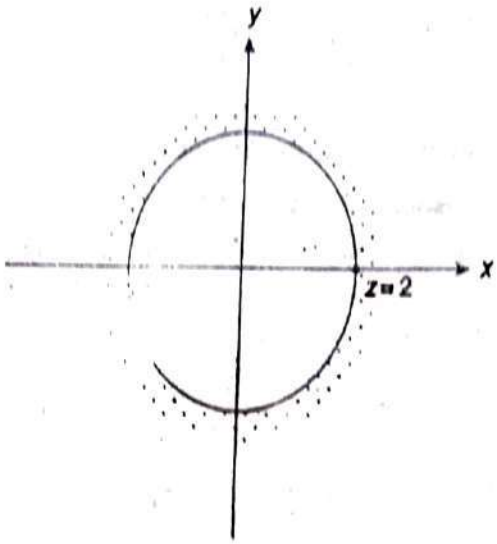


Fig.

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 5.12).

$$\because\ |z| > 2 > 1,\quad \left|\frac{2}{z}\right| < 1,\quad \left|\frac{1}{z}\right| < 1$$

$$f(z) = -\frac{1}{2z} + \frac{5}{3z\left(1-\frac{1}{z}\right)} - \frac{7}{6z\left(1+\frac{2}{z}\right)}$$

$$= -\frac{1}{2z} + \frac{5}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{7}{6z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{5}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{7}{6z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\cdots\right]$$

$$= -\frac{1}{2z} + \left(\frac{5}{3z}+\frac{5}{3z^2}+\frac{5}{3z^3}+\cdots\right) - \left(\frac{7}{6z}-\frac{7}{3z^2}+\frac{14}{3z^3}-\cdots\right)$$

$$= \frac{4}{z^2} - \frac{3}{z^3} + \cdots$$

A point at which the function $f(z)$ is not analytic is known as a singular point or singularity of the function.

Types of Singularities

1. Isolated Singularity

A singular point $z = a$ is known as an isolated singularity if there is no other singularity within a small circle surrounding the point $z = a$; otherwise it is called a non-isolated singularity.

For example, $f(z) = \dfrac{z^2}{z-2}$ has isolated singularity at $z = 2$.

2. Removable Singularity

An isolated singular point $z = a$ is known as a removable singularity if $\lim\limits_{z \to a} f(z)$ exists or there is no negative power term of $(z - a)$ in Laurent's series of $f(z)$, i.e., $b_n = 0$.

For example, $f(z) = \dfrac{\sin z}{z}$

$z = 0$ is a singularity of $f(z)$.

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

$\therefore$ $f(z)$ has a removable singularity at $z = 0$.

3. Essential Singularity

A singular point $z = a$ is known as an essential singularity if the number of negative power terms of $(z - a)$ in Laurent's series is infinite.

For example, $f(z) = e^{\frac{1}{z-2}}$

$$= 1 + \frac{1}{z-2} + \frac{1}{2!}\frac{1}{(z-2)^2} + \cdots$$

Since the number of negative power terms of $(z - 2)$ is infinite, $z = 2$ is an essential singularity.

4. Pole of Order m

An isolated singularity $z = a$ is known as a pole of order m if in Laurent's series, all the negative powers of $(z - a)$ after the mth power are zero, i.e., highest power of $\dfrac{1}{z-a}$ is m.

For example, $$f(z)=\frac{1}{(z-1)(z-3)^2}$$

$z=1$ is a pole of order 1 and $z=3$ is a pole of order 2.

Note A pole of order one is also known as a simple pole.

Meromorphic Function A function $f(z)$ which is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example 38 :

Define the singular point of the function and state whether the function $f(z)=\dfrac{z^2+1}{(z+2)(z^2+2z+2)}$ *is analytic?*

Solution

A point at which the function $f(z)$ is not analytic is known as a singular point or a singularity of the function.

$f(z)$ is analytic everywhere except at the points given by

$$(z+2)(z^2+2z+2)=0$$

$$z+2=0, \quad z^2+2z+2=0$$

$$z=-2, \quad z=\frac{-2\pm\sqrt{4-8}}{2}$$

$$=\frac{-2\pm 2i}{2}$$

$$=-1\pm i$$

The singular points are $z=-2$ and $z=-1\pm i$.

Hence, z is analytic everywhere except at $z=-2$ and $z=-1\pm i$.

Example 39 :

Identify the type of singularities of the function $f(z)=e^{\frac{1}{z-1}}$.

Solution

$$f(z)=e^{\frac{1}{z-1}}$$

$$=1+\frac{1}{(z-1)}+\frac{1}{2!(z-1)^2}+\cdots\infty$$

Since the number of terms of negative powers of $(z-1)$ are infinite, $z=1$ is an essenti singularity.

Example 40 :

Classify the poles of $f(z)=\dfrac{1}{z^2-z^6}$.

Solution

$$f(z)=\frac{1}{z^2(1-z^4)}$$

The poles are given by

$$z^2(1-z^4)=0$$

$$z^2(1+z^2)(1-z^2)=0$$

$$z=0,0,\pm1,\pm i$$

$z=0$ is a pole of order 2 and $z=\pm1,\pm i$ are poles of order one or simple poles.

Example 41 :

Find the residue at $z=0$ *of* $f(z)=z\cos\dfrac{1}{z}$.

Solution

$$f(z)=z\cos\frac{1}{z}$$

$$=z\left[1-\frac{\left(\frac{1}{z}\right)^2}{2!}+\frac{\left(\frac{1}{z}\right)^4}{4!}-\cdots\right]$$

$$=z-\frac{1}{2z}+\frac{1}{24z^3}-\cdots$$

Res $[f(z);z=0]$ = coefficient of $\dfrac{1}{z}$

$$=-\frac{1}{2}$$

Example 42 :

Find the pole and its order of the function $f(z)=\dfrac{\sin z}{z^4}$.

Solution

$$f(z)=\frac{\sin z}{z^4}$$

$$=\frac{1}{z^4}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)$$

$$=\frac{1}{z^3}-\frac{1}{6z}+\frac{z}{120}-\frac{z^3}{5040}+\cdots$$

Since the highest power of $\frac{1}{z}$ is 3, $z=0$ is a pole of order 3.

Example 43 :

Expand $f(z)=\frac{z-\sin z}{z^2}$ *at* $z=0$, *classify the singular point* $z=0$.

Solution

$$f(z)=\frac{z-\sin z}{z^2}$$

$$=\frac{1}{z^2}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\right]$$

$$=\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}+\cdots$$

$$\lim_{z\to 0} f(z)=\lim_{z\to 0}\left(\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-\cdots\right)$$

$$=0$$

Since the limit is finite, $z=0$ is a removable singularity.

Example 44 :

What is the nature of the singularity at $z=0$ *of the function* $f(z)=\frac{\sin z-z}{z^3}$?

Solution

$$f(z)=\frac{\sin z-z}{z^3}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z - z}{z^3} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z\to 0} \frac{\cos z - 1}{3z^2} \qquad [\text{Using L'Hospital's rule}]$$

$$= \lim_{z\to 0} \frac{-\sin z}{6z} \qquad [\text{Using L'Hospital's rule}]$$

$$= -\frac{1}{6} \qquad \left[\because \lim_{z\to 0} \frac{\sin z}{z} = 1\right]$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 45 :

Identify the type of singularity of $f(z) = \dfrac{2 - e^z}{z^3}$.

Solution

$$f(z) = \frac{2 - e^z}{z^3}$$

$$= \frac{1}{z^3}\left[2 - \left(1 + z + \frac{z^2}{2!} + \frac{z}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots\right)\right]$$

$$= \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{6} - \frac{z}{24} - \frac{z^2}{120} - \cdots$$

Since the highest power of $\dfrac{1}{z}$ is 3, $z = 0$ is a pole of order 3.

RESIDUES

The coefficient of $\frac{1}{z-a}$ in the Laurent series expansion of $f(z)$ is known as the residue of $f(z)$ at $z=a$.

The Laurent series expansion about $z=a$ is

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

$$\text{Residue (at } z=a) = \text{Coefficient of } \frac{1}{z-a}$$

$$=b_1$$

$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-1+1}}dz \quad \left[\because \quad b_n=\frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^{-n+1}}dz\right]$$

$$=\frac{1}{2\pi i}\int_C f(z)dz$$

Evaluation of Residues

1. Residue at a Simple Pole

(i) If $f(z)$ has a simple pole at $z=a$ then

$$\text{Res}[f(z); z=a]=\lim_{z\to a}(z-a)f(z)$$

(ii) If $f(z)$ is of the form $f(z)=\frac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic at $z=a$.

If $h(a)=0$ but $H(a)\neq 0$ then $z=a$ is a simple pole.

If $h(a)=0$ but $g(a)\neq 0$ then

$$\text{Res}[f(z); z=a]=\lim_{z\to a}\frac{g(z)}{h'(z)}$$

2. Residue at a Pole of Order m

If $f(z)$ has a pole of order m at $z=a$ then

$$\text{Res}[f(z); z=a]=\frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]$$

Example 46 :

If $f(z)=\frac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+\cdots\right]$, *find the residue of* $f(z)$ *at* $z=1$.

Solution

$$\text{Res}\left[f(z); z=1\right] = \text{Coefficient of } \frac{1}{z-1} \text{ in the Laurent's series of } f(z) \text{ about } z=1$$

$$=-1$$

Example 47 :

Find the residue of $f(z)=z^2\sin\left(\frac{1}{z}\right)$ at $z=0$.

Solution

$$f(z)=z^2\sin\left(\frac{1}{z}\right)$$

$$=z^2\left[\frac{1}{z}-\frac{\left(\frac{1}{z}\right)^3}{3!}+\cdots\right]$$

$$=z-\frac{1}{6z}+\cdots$$

$$\text{Res}\left[f(z); z=0\right]=\text{Coefficient of } \frac{1}{z}$$

$$=-\frac{1}{6}$$

Example 48 :

Obtain the residue of $f(z)=\dfrac{z-3}{(z+1)(z+2)}$ *at its poles.*

Solution

$$f(z)=\frac{z-3}{(z+1)(z+2)}$$

The poles are given by

$$(z+1)(z+2)=0$$

$$z=-1,-2$$

$z=-1$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z);z=-1\right]&=\lim_{z\to-1}(z+1)f(z)\\&=\lim_{z\to-1}\frac{z-3}{z+2}\\&=\frac{-1-3}{-1+2}\\&=-4\end{aligned}$$

$z=-2$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z);z=-2\right]&=\lim_{z\to-2}(z+2)f(z)\\&=\lim_{z\to-2}\frac{z-3}{z+1}\\&=\frac{-2-3}{-2+1}\\&=5\end{aligned}$$

Example 49 :

Test for singularity of $\dfrac{1}{z^2+1}$ *and, hence, find the corresponding residues.*

Solution

Let $$f(z)=\frac{1}{z^2+1}$$

$$=\frac{1}{(z+i)(z-i)}$$

$z=-i$ and $z=i$ are simple poles.

$$\text{Res}\left[f(z); z=-i\right]=\lim_{z\to -i}(z+i)f(z)$$

$$=\lim_{z\to -i}\frac{1}{z-i}$$

$$=\frac{1}{-i-i}$$

$$=-\frac{1}{2i}$$

$$\text{Res}\left[f(z); z=i\right]=\lim_{z\to i}(z-i)f(z)$$

$$=\lim_{z\to i}\frac{1}{z+i}$$

$$=\frac{1}{i+i}$$

$$=\frac{1}{2i}$$

Example 50 :

Find the residue of $\dfrac{1-e^{2z}}{z^4}$ at $z=0$.

Solution

$$f(z)=\frac{1-e^{2z}}{z^4}$$

$z=0$ is a pole of order 4.

$$\text{Res}\left[f(z);z=0\right]=\frac{1}{(4-1)!}\lim_{z\to 0}\frac{d^3}{dz^3}\left[(z-0)^4 f(z)\right]$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$=\frac{1}{6}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$$

$$=\frac{1}{6}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$=\frac{1}{6}\lim_{z\to 0}(-8e^{2z})$$

$$=\frac{1}{6}(-8e^{0})$$

$$=-\frac{4}{3}$$

Example 51 :

Find the residue of $f(z)=\dfrac{50z}{(z+4)(z-1)^2}$ *at* $z=1$.

Solution

$$f(z)=\frac{50z}{(z+4)(z-1)^2}$$

$z=1$ is a pole of order 2.

$$\text{Res}\left[f(z);z=1\right]=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$\frac{d}{dz}\left[\frac{50z}{}\right]$$

Example 50 :

Find the residue of $\dfrac{1-e^{2z}}{z^4}$ at $z=0$.

Solution

$$f(z)=\frac{1-e^{2z}}{z^4}$$

$z=0$ is a pole of order 4.

$$\text{Res}\left[f(z);z=0\right]=\frac{1}{(4-1)!}\lim_{z\to 0}\frac{d^3}{dz^3}\left[(z-0)^4 f(z)\right]$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}(1-e^{2z})$$

$$=\frac{1}{6}\lim_{z\to 0}\frac{d^2}{dz^2}(-2e^{2z})$$

$$=\frac{1}{6}\lim_{z\to 0}\frac{d}{dz}(-4e^{2z})$$

$$=\frac{1}{6}\lim_{z\to 0}(-8e^{2z})$$

$$=\frac{1}{6}(-8e^{0})$$

$$=-\frac{4}{3}$$

Example 51 :

Find the residue of $f(z)=\dfrac{50z}{(z+4)(z-1)^2}$ *at* $z=1$.

Solution

$$f(z)=\frac{50z}{(z+4)(z-1)^2}$$

$z=1$ is a pole of order 2.

$$\text{Res}\left[f(z);z=1\right]=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}\left[\frac{50z}{z+4}\right]$$

$$= \lim_{z \to 1} \left[\frac{(z+4)50 - 50z}{(z+4)^2} \right]$$

$$= \frac{5(50) - 50}{(5)^2}$$

$$= 8$$

Example 52 :

Find the residues of the function $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$ *at each of poles.*

Solution

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

The poles are given by

$$(z+1)^2(z^2+4) = 0$$

$$z = -1, -1, \pm 2i$$

$$\therefore \qquad f(z) = \frac{z^2 - 2z}{(z+1)^2(z+2i)(z-2i)}$$

$z = -1$ is a pole of order 2.

$$\text{Res}\left[f(z); z = -1\right] = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z^2 - 2z}{z^2 + 4}\right]$$

$$= \lim_{z \to -1} \left[\frac{(2z-2)(z^2+4) - (z^2-2z)\cdot 2z}{(z^2+4)^2}\right]$$

$$= \lim_{z \to -1} \left[\frac{2z^3 + 8z - 2z^2 - 8 - 2z^3 + 4z^2}{(z^2+4)^2}\right]$$

$$= \lim_{z \to -1} \left[\frac{2z^2 + 8z - 8}{(z^2+4)^2}\right]$$

$$= \frac{2 - 8 - 8}{25}$$

$$= -\frac{14}{25}$$

$z = 2i$ is a simple pole,

$$\text{Res}\left[f(z); z = 2i\right] = \lim_{z \to 2i} \left[(z - 2i)\, f(z)\right]$$

$$= \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2 (z+2i)}$$

$$= \frac{(2i)^2 - 2(2i)}{(2i+1)^2\, 4i}$$

$$= \frac{-4 - 4i}{(-4+1+4i)\, 4i}$$

$$= \frac{-(1+i)}{i(4i-3)}$$

$$= \frac{(i-1)}{(4i-3)} \times \frac{(4i+3)}{(4i+3)} \qquad \left[\because \ \frac{1}{i} = -i\right]$$

$$= \frac{(4i^2 + 3i - 4i - 3)}{-25}$$

$$= \frac{(7+i)}{25}$$

$z = -2i$ is a simple pole.

$$\text{Res}\left[f(z); z = -2i\right] = \lim_{z \to -2i} \left[(z + 2i) f(z)\right]$$

$$= \lim_{z \to -2i} \frac{z^2 - 2z}{(z+1)^2 (z-2i)}$$

$$= \frac{(-2i)^2 + 4i}{(-2i+1)^2 (-4i)}$$

$$= \frac{-4 + 4i}{(-4+1-4i)(-4i)}$$

$$= \frac{-(1+i)}{-(3+4i)} \times \frac{(3-4i)}{(3-4i)}$$

$$= \frac{3 - 4i + 3i - 4i^2}{25}$$

$$= \frac{7-i}{25}$$

CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of singular points inside C then

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

where the integral is taken in the anticlockwise direction around C.

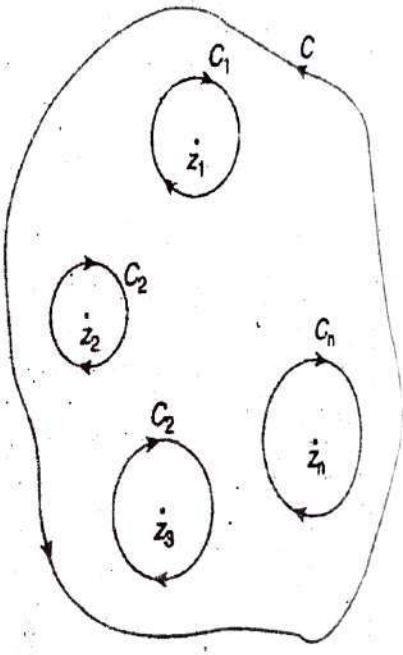


Fig.

Proof

Let $z_1, z_2, \ldots, z_n$ be finite numbers of singular points inside C Enclose each of the singular point in a small circle such that no other singular point lies inside this circle. The curve C along with these small circles $C_1, C_2, C_3, \ldots, C_n$ form a multiply connected region and $f(z)$ is analytic in this region.

By Cauchy's integral theorem for a multiply connected region,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i\left[\text{Res. at } z_1\right] + 2\pi i\left[\text{Res. at } z_2\right] + \cdots + 2\pi i\left[\text{Res. at } z_n\right]$$

$$= 2\pi i \text{ (sum of residues)}$$

Example 53 :

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$ *where C is* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{z^2-4z+4}{z+i}$

The poles are given by $z = -i$.

(ii) C is the circle $|z| = 2$ with the centre at (0, 0) and a radius of 2

(iii) For $z = -i$, $|z| = |-i| = 1 < 2$

Hence, $z = -i$ lies inside C.

(iv) $z = -i$ is a simple pole.

$$\text{Res}[f(z); z = -i] = \lim_{z\to -i}(z+i)f(z)$$

$$= \lim_{z\to -i}(z^2-4z+4)$$

$$= i^2+4i+4$$

$$= 3+4i.$$

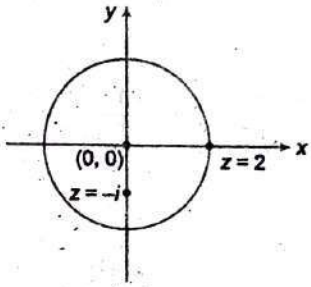


Fig.

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^2-4z+4}{z+i} = 2\pi i\,(3+4i)$$

$$= 2\pi(3i-4)$$

Example 54 :

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$ *where C is* $|z| = 2$.

Solution

(i) $f(z) = \frac{1}{(z-1)^2(z-3)}$

The poles are given by $z = 1$, $z = 3$.

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 5.38).

(iii) For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3$, $|z| = |3| = 3 > 2$

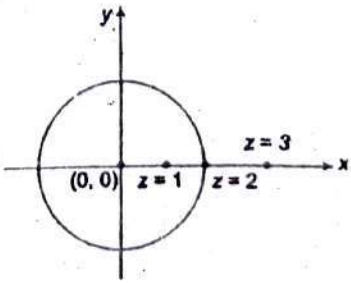


Fig.

Hence, $z = 3$ lies outside C.

(iv) $z = 1$ is a pole of order 2.

$$\text{Res}\left[f(z); z=1\right] = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{1}{z-3}\right]$$

$$= \lim_{z\to 1}\left[-\frac{1}{(z-3)^2}\right]$$

$$= -\frac{1}{4}$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i\left(-\frac{1}{4}\right)$$

$$= -\frac{\pi i}{2}$$

Example 55 :

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, *where* C *is* $|z| = 3$.

Solution

(i) Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The poles are given by $z = 1, z = 2$.

(ii) C is a circle $|z| = 3$ with the centre at (0, 0) and a radius of 3

(iii) For $z = 1$; $|z| = |1| = 1 < 3$

Hence, $z = 1$ lies inside C.

For $z = 2$, $|z| = |2| = 2 < 3$

Hence, $z = 2$ lies inside C.

(iv) $z = 1$ is a simple pole.

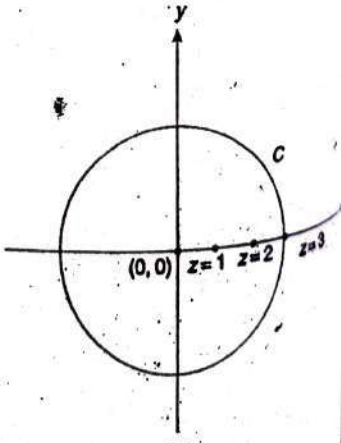


Fig.

$$\text{Res}[f(z); z=1] = \lim_{z\to 1}(z-1)\, f(z)$$

$$= \lim_{z\to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$$

$$= \frac{\sin \pi + \cos \pi}{-1}$$

$$= 1$$

$z = 2$ is a simple pole.

$$\text{Res}[f(z); z=2] = \lim_{z\to 2}(z-2)\, f(z)$$

$$= \lim_{z\to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1}$$

$$= 1$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i(1+1)$$

$$= 4\pi i$$

Example 56 :

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3} dz$, *where C is* $|z-2| = 2$.

Solution

(i) Let $f(z) = \frac{5z+7}{z^2+2z-3}$

$$= \frac{5z+7}{(z+3)(z-1)}$$

The poles are given by $z = -3$, $z = 1$.

(ii) C is a circle $|z-2| = 2$ with the centre at $(2, 0)$ and a radius of 2

(iii) For $z = -3$, $|z-2| = |-3-2| = 5 > 2$

Hence, $z = -3$ lies outside C.

For $z = 1$, $|z-2| = |1-2| = 1 < 2$

Hence, $z = 1$ lies inside C.

(iv) $z = 1$ is a simple pole.

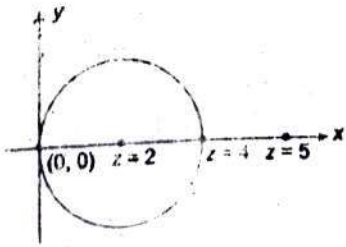


Fig.

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\,f(z)$$

$$= \lim_{z\to 1}\frac{5z+7}{z+3}$$

$$= \frac{5+7}{1+3}$$

$$= 3$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 2\pi i\,(3)$$

$$= 6\pi i$$

Example 57 :

Using the residue theorem, evaluate $\int_C \frac{e^z+z}{z^3-z}\,dz$, *where* $C: |z| = \frac{\pi}{2}$.

Solution

(i) Let $f(z) = \frac{e^z+z}{z^3-z} = \frac{e^z+z}{z(z+1)(z-1)}$

The poles are given by $z=0$, $z=-1$, $z=1$.

(ii) C is the circle $|z| = \frac{\pi}{2} = 1.57$ with the centre (0, 0) and a radius of $\frac{\pi}{2}$

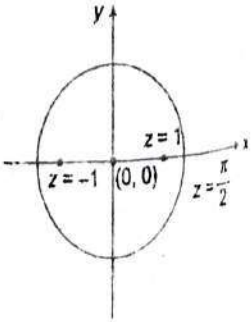


Fig.

(iii) For $z=0$, $|z| = 0 < \frac{\pi}{2}$

Hence, $z=0$ lies inside C.

For $z=-1$, $|z| = |-1| = 1 < \frac{\pi}{2}$

Hence, $z = -1$ lies inside C.

For $z = 1$, $|z| = |1| = 1 < \frac{\pi}{2}$

Hence, $z = 1$ lies inside C.

(iv) $z = 0$ is a simple pole.

$$\text{Res}\left[f(z); z = 0\right] = \lim_{z \to 0} z f(z)$$

$$= \lim_{z \to 0} \frac{e^z + z}{(z+1)(z-1)}$$

$$= \frac{e^0}{-1}$$

$$= -1$$

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z = -1\right] = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} \frac{e^z + z}{z(z-1)}$$

$$= \frac{e^{-1} - 1}{-1(-2)}$$

$$= \frac{1-e}{2e}$$

$z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z = 1\right] = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{e^z + z}{z(z+1)}$$

$$= \frac{e+1}{2}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^z + z}{z^3 - z}\,dz = 2\pi i\left(-1 + \frac{1-e}{2e} + \frac{e+1}{2}\right)$$

$$= 2\pi i\left(\frac{-2e + 1 - e + e^2 + e}{2e}\right)$$

$$= \frac{\pi i}{e}(e^2 - 2e + 1)$$

$$= \frac{\pi i}{e}(e-1)^2$$

Example 58 :

Use residues to evaluate the integrals of the function $\frac{\exp(-z)}{z^2}$ *around the circle* $|z| = 3$ *in the positive sense.*

Solution

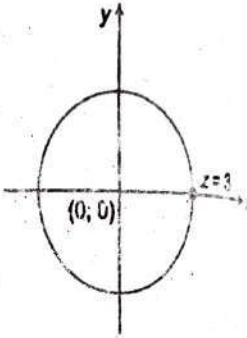


Fig.

(i) Let $f(z) = \frac{\exp(-z)}{z^2} = \frac{e^{-z}}{z^2}$

The poles are given by $z = 0$.

(ii) C is the circle $|z| = 3$ with the centre at (0, 0) and a radius of 3

(iii) For $z = 0$, $|z| = 0 < 3$

Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a pole of order 2.

$$\text{Res}\left[f(z); z = 0\right] = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left[z^2 f(z)\right]$$

$$= \lim_{z \to 0} e^{-z}$$

$$= e^0$$

$$= 1$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i\,(1)$$

$$= 2\pi i$$

Example 59 :

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$ *where C is* $|z-i| = 2$.

Solution

(i) Let $f(z) = \frac{2z+6}{z^2+4}$

$$= \frac{2z+6}{(z+2i)(z-2i)}$$

The poles are given by $z = -2i$, $z = 2i$.

(ii) C is a circle $|z-i| = 2$ with the centre at (0, 1) and a radius of 2

(iii) For $z = 2i$, $|z-i| = |2i-i| = |i| = 1 < 2$

Hence, $z = 2i$ lies inside C.

For $z = -2i$, $|z-i| = |-2i-i| = |-3i| = 3 > 2$

Hence, $z = -2i$ lies outside C.

(iv) $z = 2i$ is a simple pole.

$$\text{Res}\left[f(z); z = 2i\right] = \lim_{z\to 2i}\left[(z-2i)\,f(z)\right]$$

$$= \lim_{z\to 2i}\frac{2z+6}{z+2i}$$

$$= \frac{2(2i)+6}{2i+2i}$$

$$= \frac{2i+3}{2i}$$

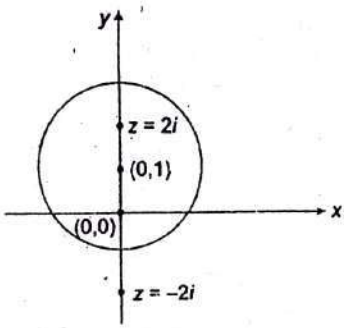


Fig.

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\left(\frac{2i+3}{2i}\right)$$

$$= \pi(2i+3)$$

Example 60 :

Find the value of the integral $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$ *taken counterclockwise around the circle* $C : |z-2| = 2$.

Solution

(i) Let $f(z) = \frac{3z^2+2}{(z-1)(z^2+9)} = \frac{3z^2+2}{(z-1)(z+3i)(z-3i)}$

The poles are given by $z = 1, z = -3i, z = 3i$.

(ii) C is the circle $|z-2| = 2$ with the centre at $(2, 0)$ and a radius of 2

(iii) For $z = 1$, $|z-2| = |1-2| = |-1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3i$,

$$|z-2| = |3i-2| = \sqrt{9+4} = \sqrt{13} > 2$$

Hence, $z = 3i$ lies outside C.

For $z = -3i$,

$$|z-2| = |-3i-2| = \sqrt{9+4} = \sqrt{13} > 2$$

Hence, $z = -3i$ lies outside C.

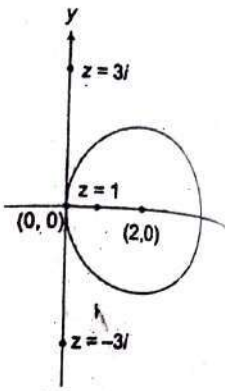


Fig.

(iv) $z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\,f(z)$$

$$= \lim_{z\to 1}\frac{3z^2+2}{z^2+9}$$

$$= \frac{3+2}{1+9}$$

$$= \frac{1}{2}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\frac{1}{2}\right)$$

$$= \pi i$$

Example 61 :

Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$, *where C is the circle* $|z| = 2.5$.

Solution

(i) Let $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

The poles are given by $z = 1, z = -2$.

(ii) C is a circle $|z| = 2.5$ with the centre at (0, 0) and a radius of 2.5

(iii) For $z = 1, |z| = |1| = 1 < 2.5$

Hence, $z = 1$ lies inside C.

For $z = -2, |z| = |-2| = 2 < 2.5$

Hence, $z = -2$ lies inside C.

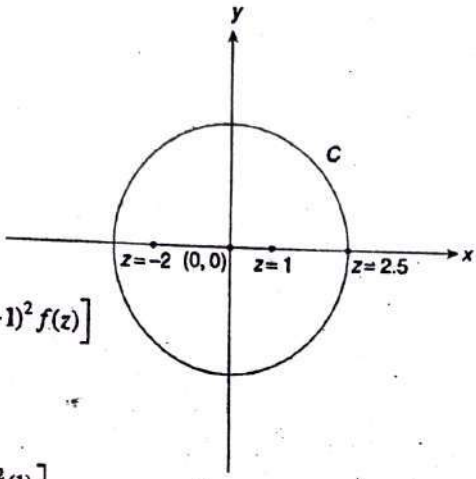


Fig.

(iv) $z = 1$ is a pole of order 2.

$$\text{Res}[f(z); z=1] = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+2}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+2)(2z)-z^2(1)}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\frac{2z^2+4z-z^2}{(z+2)^2}$$

$$= \frac{2+4-1}{9}$$

$$= \frac{5}{9}$$

$z = -2$ is a simple pole.

$$\text{Res}[f(z); z=-2] = \lim_{z\to -2}(z+2)\,f(z)$$

$$= \lim_{z\to -2}\frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2}$$

$$= \frac{4}{9}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz = 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right)$$

$$= 2\pi i(1)$$

$$= 2\pi i$$

Example 62 :

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\,dz$, *where C is* $|z-2|=2$.

Solution

(i) Let $f(z)=\frac{5z+7}{z^2+2z-3}=\frac{5z+7}{(z+3)(z-1)}$

The poles are given by $z=1,\ \ z=-3$.

(ii) C is the circle $|z-2|=2$ with the centre at (2, 0) and a radius of 2

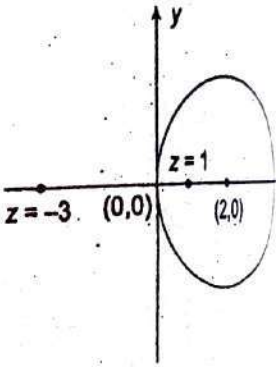


Fig.

(iii) For $z=1, |z-2|=|-1|=1<2$

Hence, $z=1$ lies inside C.

For $z=-3, |z-2|=|-3-2|=|-5|=5>2$

Hence, $z=-3$ lies outside C.

(iv) $z=1$ is a simple pole

$$\text{Res}\left[f(z); z=1\right]=\lim_{z\to 1}(z-1)f(z)$$

$$=\lim_{z\to 1}\frac{5z+7}{z+3}$$

$$=\frac{12}{4}$$

$$=3$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz=2\pi i\ (\text{sum of residues})$$

$$=2\pi i(3)$$

$$=6\pi i$$

Solution

(i) Let $f(z)=\dfrac{z^4}{(z+1)(z-i)^2}$

The poles are given by $z=-1$, $z=i$.

(ii) C is an ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

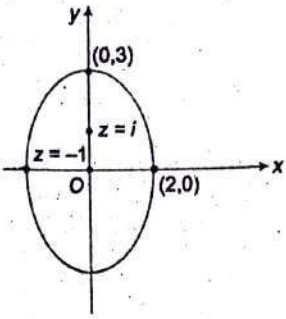


Fig.

(iii) For $z=-1, x=-1, y=0$

$$\therefore \frac{x^2}{4}+\frac{y^2}{9}=\frac{(-1)^2}{4}+0=\frac{1}{4}<1$$

Hence, $z=-1$ lies inside C.

For $z=i, x=0, y=1$

$$\therefore \frac{x^2}{4}+\frac{y^2}{9}=0+\frac{1}{9}<1$$

Hence, $z=i$ lies inside C.

(iv) $z=-1$ is a simple pole.

$$\text{Res}\left[f(z); z=-1\right]=\lim_{z\to-1}(z+1)f(z)$$

$$=\lim_{z\to-1}\frac{z^4}{(z-i)^2}=\frac{(-1)^4}{(-1-i)^2}=\frac{1}{2i}$$

$z=i$ is a pole of order 2.

$$\text{Res}\,[f(z);\ z=i]=\frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2 f(z)\right]$$

$$=\lim_{z\to i}\frac{d}{dz}\left(\frac{z^4}{z+1}\right)=\lim_{z\to i}\frac{4z^3(z+1)-z^4}{(z+1)^2}=\lim_{z\to i}\frac{3z^4+4z^3}{(z+1)^2}$$

$$=\frac{3i^4+4i^3}{(i+1)^2}=\frac{3-4i}{2i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz=2\pi i\ (\text{sum of residues})$$

$$\int_C\frac{z^4}{(z+1)(z-i)^2}dz=2\pi i\left(\frac{1}{2i}+\frac{3-4i}{2i}\right)$$

$$=4\pi(1-i)$$

Example 64 :

Evaluate $\int_C \tan z \, dz$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

The poles are given by

$$\cos z = 0$$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} + \cdots$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2

(iii) For $z = \pm\dfrac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.57 < 2$$

Hence, $z = \pm\dfrac{\pi}{2}$, lies inside C.

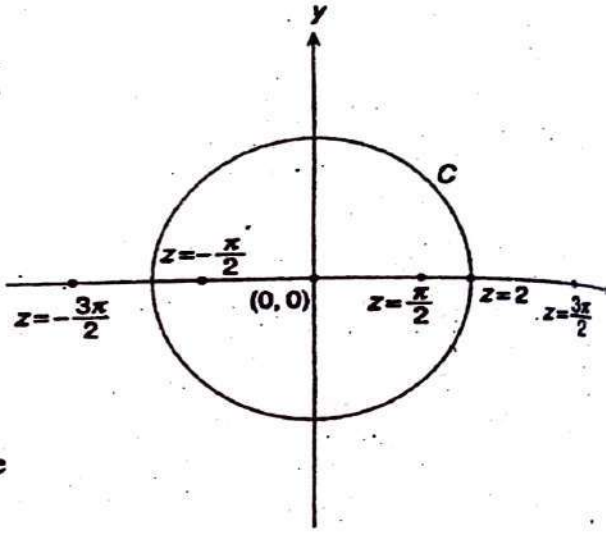


Fig.

(iv) $z = \dfrac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = \frac{\pi}{2}\right] = \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right) f(z) = \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)\tan z$$

$$= \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)\tan z$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)}{\cot z} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{1}{-\text{cosec}^2 z} \qquad \text{[Using L'Hospital's rule]}$$

$$= -\frac{1}{\text{cosec}^2 \frac{\pi}{2}}$$

$$= -1$$

$\therefore -\dfrac{\pi}{2}$ is a simple pole.

$$f(z); z = -\frac{\pi}{2}\Big] = \lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right)\tan z$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{z+\frac{\pi}{2}}{\cot z} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{1}{-\operatorname{cosec}^2 z} \quad [\text{Using L'Hospital's rule}]$$

$$= -\frac{1}{\operatorname{cosec}^2\left(-\frac{\pi}{2}\right)}$$

$$= -1$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \tan z\,dz = 2\pi i(-1-1)$$

$$= -4\pi i$$

Example 65 :

Evaluate $\int_C \frac{dz}{z \sin z}$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \frac{1}{z \sin z}$

The poles are given by $z \sin z = 0$

$z = 0, \quad \sin z = 0$

$z = 0, \pm\pi, \pm 2\pi, \ldots$

(ii) C is a circle $|z| = 1$ with the centre at (0, 0) and a radius of 1

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

Other poles $z = \pm\pi, \pm 2\pi, \ldots$ lies outside C.

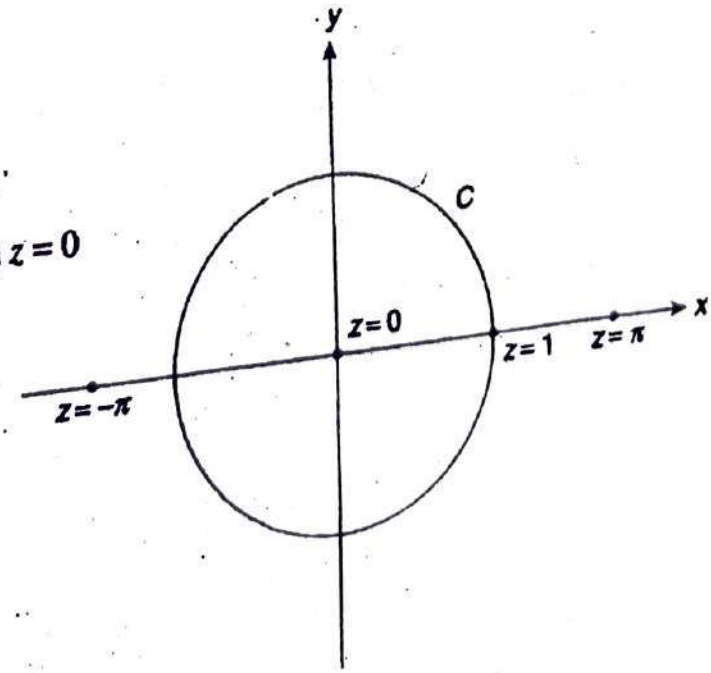


Fig.

(iv) $z = 0$ is a pole of order 2.

$$\text{Res}\left[f(z); z=0\right] = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{\sin z}\right)$$

$$= \lim_{z\to 0}\frac{(1)\sin z - z(\cos z)}{(\sin z)^2} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z\to 0}\frac{\cos z - (\cos z - z\sin z)}{2\sin z\cos z} \qquad [\text{Using L'Hospital's rul}$$

$$= \lim_{z\to 0}\frac{z\sin z}{2\sin z\cos z}$$

$$= \lim_{z\to 0}\frac{z}{2\cos z}$$

$$= 0$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{dz}{z\sin z} = 2\pi i(0)$$

$$= 0$$

5 APPLICATIONS OF COMPLEX INTEGRATION BY RESIDUES

EVALUATION OF IMPROPER REAL INTEGRAL OF A RATIONAL FUNCTION INCLUDING TRIGONOMETRIC FUNCTIONS

Let $I = \int_{-\infty}^{\infty} \frac{\cos mx}{Q(x)} dx$ or $\int_{-\infty}^{\infty} \frac{\sin mx}{Q(x)} dx$

where $Q(x)$ is a polynomial in x.

To evaluate the integral,

consider $\int_C f(z)\,dz = \int e^{imz} g(z)\,dz$

where $g(z) = \frac{1}{Q(z)}$ and C is the contour consisting of the upper semicircle C_1 of radius R with centre at the origin and the part of the real axis from $-R$ to R

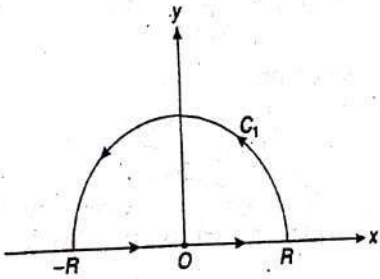


Fig.

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues at poles in the upper half of z-plane)}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

(Taking $R \to \infty$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_1} f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty}\int_{C_1} e^{imz}g(z)\,dz = 2\pi i\text{(sum of residues)}$$

By Jordan's lemma, $\lim_{R\to\infty}\int_{C_1} e^{imz}g(z)dz = 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

Example 1

Evaluate $\int_0^{\infty}\frac{\cos mx}{x^2+a^2}dx, a>0, m>0$ *using contour integration.*

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{imz}}{z^2+a^2}dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R

(ii) Let $f(z) = \frac{e^{imz}}{z^2+a^2}$

The poles are given by

$$z^2+a^2=0$$

$$z=\pm ai$$

$$\therefore f(z) = \frac{e^{imz}}{(z+ai)(z-ai)}$$

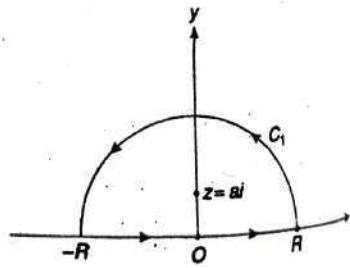


Fig.

(iii) The pole $z = ai$ lies inside C.

(iv) $\text{Res}[f(z); z=ai] = \lim_{z\to ai}(z-ai)f(z)$

$$= \lim_{z\to ai}\frac{e^{imz}}{(z+ai)}$$

$$= \frac{e^{im(ai)}}{(ai+ai)}$$

$$= \frac{e^{-ma}}{2ai}$$

(v) $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$ (sum of residues)

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = 2\pi i\left(\frac{e^{-ma}}{2ai}\right)$$

$$= \frac{\pi e^{-ma}}{a}$$

$$\int_{-\infty}^{\infty} \frac{(\cos mx + i\sin mx)}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{2a}$$

Example 2 :

Evaluate a real integral $\int_{0}^{\infty} \frac{x\sin x}{x^2+9}\,dx$ *using residue.*

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{ze^{iz}}{z^2+9}\,dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R

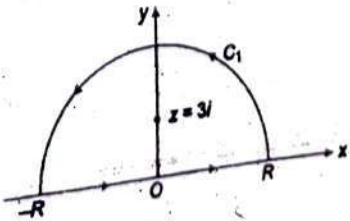


Fig.

(ii) Let $f(z) = \frac{ze^{iz}}{z^2+9}$

The poles are given by

$$z^2+9=0$$

$$z = \pm 3i$$

$$\therefore\ f(z) = \frac{ze^{iz}}{(z+3i)(z-3i)}$$

(iii) The pole $z = 3i$ lies inside C.

(iv) $\text{Res}[f(z); z = 3i] = \lim_{z \to 3i} (z - 3i) f(z)$

$$= \lim_{z \to 3i} \frac{ze^{iz}}{z + 3i}$$

$$= \frac{3ie^{3i^2}}{3i + 3i}$$

$$= \frac{3ie^{-3}}{6i}$$

$$= \frac{e^{-3}}{2}$$

$$= \frac{1}{2e^3}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 9}\,dx = 2\pi i\left(\frac{1}{2e^3}\right)$$

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)}{x^2 + 9}\,dx = \frac{\pi i}{e^3}$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9}\,dx = \frac{\pi}{e^3}$$

$$2\int_{0}^{\infty} \frac{x \sin x}{x^2 + 9}\,dx = \frac{\pi}{e^3} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + 9}\,dx = \frac{\pi}{2e^3}$$

Example 3 :

Evaluate $\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2}\,dx, a > 0, m > 0.$

Solution

(i) Let $\int_C f(z)\,dz = \frac{ze^{imz}}{z^2 + a^2}\,dz$

Consider the contour C consisting of upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R

(ii) Let $f(z)=\dfrac{ze^{imz}}{z^2+a^2}$

The poles are given by

$$z^2+a^2=0$$

$$z=\pm ai$$

$$\therefore f(z)=\frac{ze^{imz}}{(z+ai)(z-ai)}$$

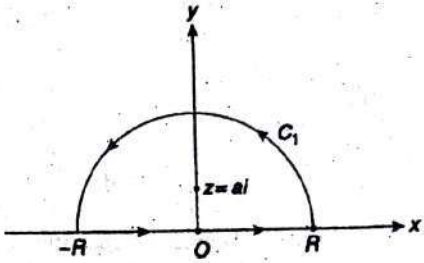


Fig.

(iii) The pole $z=ai$ lies inside C.

(iv) $\text{Res}[f(z);z=ai]=\lim\limits_{z\to ai}(z-ai)f(z)$

$$=\lim_{z\to ai}\frac{ze^{imz}}{z+ai}$$

$$=\frac{aie^{i^2ma}}{ai+ai}$$

$$=\frac{aie^{-ma}}{2ai}$$

$$=\frac{e^{-ma}}{2}$$

(v) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx=2\pi i\,(\text{sum of residues})$

$$\int_{-\infty}^{\infty}\frac{xe^{imx}}{x^2+a^2}dx=2\pi i\left(\frac{e^{-ma}}{2}\right)$$

$$=\pi ie^{-ma}$$

$$\int_{-\infty}^{\infty}\frac{x(\cos mx+i\sin mx)}{x^2+a^2}dx=\pi ie^{-ma}$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^2+a^2}dx=\pi e^{-ma}$$

$$2\int_{0}^{\infty}\frac{x\sin mx}{x^2+a^2}dx=\pi e^{-ma}\quad\left[\because\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)\text{ if }f(-x)=f(x)\right]$$

$$\int_{0}^{\infty}\frac{x\sin mx}{x^2+a^2}dx=\frac{\pi e^{-ma}}{2}$$

Example 4 :

Evaluate $\int_0^\infty \frac{\cos mx}{(1+x^2)^2}\,dx,\ m>0$ *using contour integration.*

Solution

(i) Let $f(z)dz = \int_C \frac{e^{imz}}{(1+z^2)^2}dz$

Consider the contour C consisting of the upper semicircle of radius R and the part of the real axis from $-R$ to R

(ii) Let $f(z) = \frac{e^{imz}}{(1+z^2)^2}$

The poles are given by

$$(1+z^2)^2 = 0$$

$$1+z^2 = 0$$

$$z = \pm i$$

$$\therefore f(z) = \frac{e^{imz}}{(z+i)^2(z-i)^2}$$

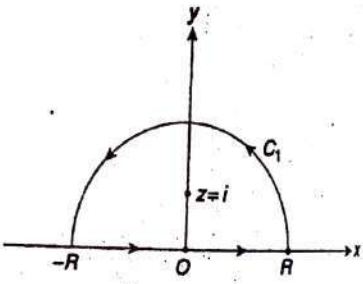


Fig.

(iii) The pole $z = i$ of order 2 lies inside C.

(iv) $\text{Res}[f(z); z=i] = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)]$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{e^{imz}}{(z+i)^2}\right]$$

$$= \lim_{z\to i}\frac{(z+i)^2 e^{imz}\cdot im - e^{imz}2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{[im(z+i)-2]e^{imz}}{(z+i)^3}$$

$$= \frac{[(i+i)im-2]e^{i^2m}}{(i+i)^3}$$

$$= \frac{(-2m-2)e^{-m}}{-8i}$$

$$= \frac{(m+1)e^{-m}}{4i}$$

(v) $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i(\text{sum of residues})$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(1+x^2)^2}\,dx = 2\pi i\left[\frac{(m+1)e^{-m}}{4i}\right]$$

$$= \frac{\pi(m+1)e^{-m}}{2}$$

$$\int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{4}$$

EVALUATION OF IMPROPER REAL INTEGRAL WHEN SIMPLE POLES LIE ON THE REAL AXIS

Let $I = \int_{-\infty}^{\infty} f(x)\,dx$

To evaluate the integral, consider $\int_C f(z)\,dz$.

If $f(z)$ has a simple pole on the real axis then this pole is deleted by indenting the contour. The contour is indented by drawing a small semi-circle C_1 of radius r having the pole as its centre

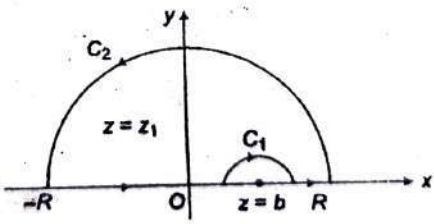


Fig.

If $f(z)$ has a simple pole at $z = b$ on the real axis then

$$\lim_{r\to 0} \int_{C_1} f(z)\,dz = \pi i \,\text{Res}\left[f(z); z = b\right]$$

If $f(z)$ has a simple pole $z = b$ on the real axis and another simple pole $z = z_1$ in the upper half of the z-plane then

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i \,\text{Res}\left[f(z); z = b\right] + 2\pi i \,\text{Res}\left[f(z); z = z_1\right]$$

Example 5 :

Find the principal value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{dz}{(z^2-3z+2)(z^2+1)}$

(ii) Let $f(z) = \frac{1}{(z^2-3z+2)(z^2+1)}$

The poles are given by

$$(z^2-3z+2)(z^2+1)=0$$

$$(z-1)(z-2)(z+i)(z-i)=0$$

$$z = 1, 2, \pm i$$

$$\therefore \quad f(z) = \frac{1}{(z-1)(z-2)(z+i)(z-i)}$$

(iii) The poles $z = 1$, $z = 2$ lie on the real axis and the pole $z = i$ lies in the upper hal of the z-plane

(iv) $\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1} \frac{1}{(z-2)(z^2+1)}$$

$$= -\frac{1}{2}$$

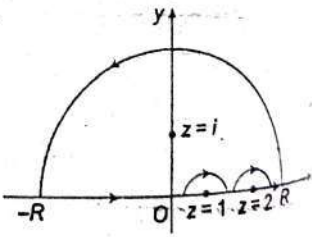


Fig.

$\text{Res}\left[f(z); z=2\right] = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2} \frac{1}{(z-1)(z^2+1)}$$

$$= \frac{1}{5}$$

$\text{Res}\left[f(z); z=i\right] = \lim_{z\to i}(z-i)f(z)$

$$= \lim_{z\to i} \frac{1}{(z^2-3z+2)(z+i)}$$

$$= \frac{1}{(-1-3i+2)2i}$$

$$= \frac{1}{(1-3i)2i}$$

$$= \frac{1}{2(i+3)} \times \frac{(-i+3)}{(-i+3)}$$

$$= \frac{-i+3}{2(1+9)}$$

$$= \frac{3-i}{20}$$

(v) PV of $\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = \pi i\{\text{Res}[f(z); z=1]+[f(z); z=2]\}$

$$+ 2\pi i \text{ Res}[f(z); z=i]$$

$$= \pi i\left(-\frac{1}{2}+\frac{1}{5}\right)+2\pi i\left(\frac{3-i}{20}\right)$$

$$= \pi i\left(-\frac{3}{10}\right)+\pi i\left(\frac{3-i}{10}\right)$$

$$= -\frac{3\pi i}{10}+\frac{3\pi i}{10}-\frac{\pi i^2}{10}$$

$$= \frac{\pi}{10}$$

Example 6 :

Show that the principal value of $\int_{-\infty}^{\infty} \frac{\sin x\, dx}{(x^2+4)(x-1)} = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{iz}}{(z^2+4)(z-1)} dz$

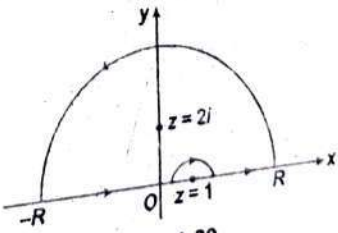


Fig. 6.29

(ii) Let $f(z) = \frac{e^{iz}}{(z^2+4)(z-1)}$

The poles are given by

$$(z^2+4)(z-1) = 0$$

$$z = \pm 2i, 1$$

$$\therefore \quad f(z) = \frac{e^{iz}}{(z+2i)(z-2i)(z-1)}$$

(iii) The pole $z = 1$ lies on the real axis and the pole $z = 2i$ lies in the upper half of the z-plane

(iv) $\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}\frac{e^{iz}}{(z^2+4)}$$

$$= \frac{e^{i}}{5}$$

$$\text{Res}\left[f(z); z=2i\right] = \lim_{z\to 2i}(z-2i)\,f(z)$$

$$= \lim_{z\to 2i}\frac{e^{iz}}{(z+2i)(z-1)}$$

$$= \frac{e^{i(2i)}}{4i(2i-1)}$$

$$= \frac{e^{-2}}{-8-4i}$$

$$= \frac{e^{-2}}{-4(2+i)}\times\frac{(2-i)}{(2-i)}$$

$$= \frac{e^{-2}(2-i)}{-4(5)}$$

$$= \frac{e^{-2}(i-2)}{-20}$$

$$= \frac{i-2}{20e^{2}}$$

(v) $\text{PV of}\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x-1)}dz = \pi i\,\text{Res}\left[f(z); z=1\right] + 2\pi i\,\text{Res}\left[f(z); z=2i\right]$

$$\text{PV of}\int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{(x^2+4)(x-1)}dx = \pi i\left(\frac{e^{i}}{5}\right) + 2\pi i\left(\frac{i-2}{20e^{2}}\right)$$

$$= \frac{\pi i}{5}(\cos 1 + i\sin 1) + \frac{\pi}{10e^{2}}(-1-2i)$$

$$= \frac{\pi}{5}(i\cos 1 - \sin 1) - \frac{\pi}{10e^{2}}(1+2i)$$

$$= -\frac{\pi}{5}\left(\sin 1 + \frac{1}{2e^{2}}\right) + i\left(\frac{\pi}{5}\cos 1 - \frac{2\pi}{10e^{2}}\right)$$

$$= -\frac{\pi}{5}\left(\sin 1 + \frac{1}{2e^{2}}\right) + i\frac{\pi}{5}\left(\cos 1 - \frac{1}{e^{2}}\right)$$

Comparing imaginary parts,

$$\text{PV of } \int_{-\infty}^{\infty} \frac{\sin x \, dx}{(x^2+4)(x-1)} = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$$

Example 7 :

Using indentation along a branch cut, show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z} dz$

(ii) Let $f(z) = \frac{e^{iz}}{z}$

The pole is given by

$z = 0$

(iii) The pole $z = 0$ lies on the real axis

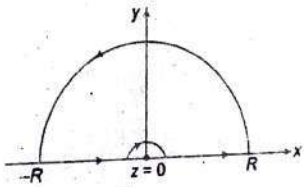


Fig.

(iv) $\text{Res}\left[f(z); z=0\right] = \lim_{z \to 0} z f(z)$

$$= \lim_{z \to 0} e^{iz}$$

$$= e^0$$

$$= 1$$

(v) $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i \, \text{Res}\left[f(z); z=0\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx = \pi i \, (1)$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$2\int_0^{\infty} \frac{\sin x}{x} dx = \pi \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(iv) $\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}\frac{e^{iz}}{(z^2+4)}$$

$$= \frac{e^{i}}{5}$$

$$\text{Res}\left[f(z); z=2i\right] = \lim_{z\to 2i}(z-2i)f(z)$$

$$= \lim_{z\to 2i}\frac{e^{iz}}{(z+2i)(z-1)}$$

$$= \frac{e^{i(2i)}}{4i(2i-1)}$$

$$= \frac{e^{-2}}{-8-4i}$$

$$= \frac{e^{-2}}{-4(2+i)}\times\frac{(2-i)}{(2-i)}$$

$$= \frac{e^{-2}(2-i)}{-4(5)}$$

$$= \frac{e^{-2}(i-2)}{20}$$

$$= \frac{i-2}{20e^{2}}$$

(v) PV of $\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x-1)}dz = \pi i\,\text{Res}\left[f(z); z=1\right] + 2\pi i\,\text{Res}\left[f(z); z=2i\right]$

$$\text{PV of }\int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{(x^2+4)(x-1)}dx = \pi i\left(\frac{e^{i}}{5}\right) + 2\pi i\left(\frac{i-2}{20e^{2}}\right)$$

$$= \frac{\pi i}{5}(\cos 1 + i\sin 1) + \frac{\pi}{10e^{2}}(-1-2i)$$

$$= \frac{\pi}{5}(i\cos 1 - \sin 1) - \frac{\pi}{10e^{2}}(1+2i)$$

$$= -\frac{\pi}{5}\left(\sin 1 + \frac{1}{2e^{2}}\right) + i\left(\frac{\pi}{5}\cos 1 - \frac{2\pi}{10e^{2}}\right)$$

$$= -\frac{\pi}{5}\left(\sin 1 + \frac{1}{2e^{2}}\right) + i\frac{\pi}{5}\left(\cos 1 - \frac{1}{e^{2}}\right)$$

Comparing imaginary parts,

$$\text{PV of } \int_{-\infty}^{\infty} \frac{\sin x\, dx}{(x^2+4)(x-1)} = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$$

Example 8 :

Using indentation along a branch cut, show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z} dz$

(ii) Let $f(z) = \frac{e^{iz}}{z}$

The pole is given by

$z = 0$

(iii) The pole $z = 0$ lies on the real axis

(iv) $\text{Res}\left[f(z); z=0\right] = \lim_{z\to 0} z f(z)$

$= \lim_{z\to 0} e^{iz}$

$= e^0$

$= 1$

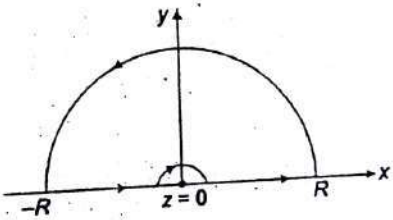


Fig.

(v) $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i \text{ Res}\left[f(z); z=0\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx = \pi i\,(1)$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$2\int_0^{\infty} \frac{\sin x}{x} dx = \pi \quad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Example 9 :

Determine the residues at the poles of $\dfrac{z+1}{(z^2-16)(z+2)}$

Solution :

Let $$f(z) = \frac{z+1}{(z^2-16)(Z+2)} = \frac{z+1}{(z^2-16)(z+2)}.$$

Hence poles are given by :

$$z_0 = -4, -2 \text{ and } 4.$$

Now,

Residue at $z_0 = -4$

$$r_1 = \lim_{z\to -4} (z+4)$$

$$= \frac{(z+1)}{(z+4)(z-4)(z+4)}$$

$$= \lim_{z\to -4} \frac{(z+1)}{(z-4)(z+2)} = \frac{-3}{16}$$

Residue at $z_0 = -2$

$$r_2 = \lim_{z\to -2} (z+2)\frac{(z+1)}{(z+4)(z-4)(z+2)}$$

$$= \lim_{z\to -2} \frac{(z+1)}{(z-4)(z+4)} = \frac{1}{12}$$

Residue at $z_0 = 4$

$$r_2 = \lim_{z\to 4} (z-4)\frac{(z+1)}{(z+4)(z-4)(z+2)}$$

$$= \lim_{z\to 4} \frac{(z+1)}{(z+2)(z+4)} = \frac{5}{18}$$

6 MULTIVARIABLE CALCULUS (INTEGRATION)

Double Integrals (Cartesian Co-ordinates)

Evaluation of Double Integrals

Let A be a region bounded by a curves $y = y_1$ and $y = y_2$ and the straight line $x = x_1$ and $x = x_2$ then

$$\boxed{\int_A f(x, y)\, dx\, dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\, dx\, dy} \quad \ldots (1)$$

Case I : When x_1 and x_2 are constant and y_1, y_2 are function of x. Then the function $f(x, y)$ keeping x is fixed between the limit x_1 and x_2 and then integrating with respect to y in the limit between y_1 and y_2. After that result is integrated with respect to x between the limit x_1 and x_2.

$$\boxed{\int_A f(x, y) = \int_{x_1}^{x_2}\left\{\int_{y_1}^{y_2} f(x, y)\, dy\right\} dx}$$

Case II : When y_1 and y_2 are constant and x_1 and x_2 are function of y then first integrated with respect to x with the limt x_1 and x_2 keeping y as a constant. After result integrating with respect to y between the limt y_1 and y_2

$$\boxed{\int_A f(x, y)\, dx\, dy = \int_{y_1}^{y_2}\left\{\int_{x_1}^{x_2} f(x, y)\, dx\right\} dy}$$

Case III : If x_1, x_2 y_1 and y_2 are constant then case I integrated vertical and Case Ii integrated along horizontal.

Note : If both are constant then first integrated with respect to x and integrate.

Change of Order of Integration in Double Integrals

When evaluating a double integrals

$$\boxed{\int_A f(x, y)\, dx\, dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\, dx\, dy}$$

Then change of order of integrals $= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\, dy\, dx$

the limit of integration must cover the boundary of the region.

Double Integration in Polar Co-ordinates

$$\boxed{\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta = \int_{r_1}^{r_2}\left\{\int_{\theta_1}^{\theta_2} f(r, \theta)\, d\theta\right\} dr}$$

Ist integration is to performed with respect to right hand variable.

Objective Question Answer :

Q.1. Value of the integral $\int_0^{\pi/4} \cos^2 x\, dx$ is

(i) $\frac{\pi}{4}+\frac{1}{4}$ (ii) $\frac{\pi}{8}-\frac{1}{2}$ (iii) $-\frac{\pi}{8}-\frac{1}{4}$ (iv) $-\frac{\pi}{8}$

Sol^n.: (i)

Case III : If x_1; x_2 y_1 and y_2 are constant then case I integrated al vertical and Case Ii integrated along horizontal.

Note : If both are constant then first integrated with respect to x and integrate.

Change of Order of Integration in Double Integrals

When evaluating a double integrals

$$\boxed{\int_A f(x, y)\, dx\, dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\, dx\, dy}$$

Then change of order of integrals $= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\, dy\, dx$

the limit of integration must cover the boundary of the region.

Double Integration in Polar Co-ordinates

$$\boxed{\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta = \int_{r_1}^{r_2}\left\{\int_{\theta_1}^{\theta_2} f(r, \theta)\, d\theta\right\} dr}$$

Ist integration is to performed with respect to right hand variable. i.e.

Objective Question Answer :

Q.1. Value of the integral $\int_0^{\pi/4} \cos^2 x\, dx$ is

(i) $\frac{\pi}{4}+\frac{1}{4}$ (ii) $\frac{\pi}{8}-\frac{1}{2}$ (iii) $-\frac{\pi}{8}-\frac{1}{4}$ (iv) $-\frac{\pi}{8}+\frac{1}{4}$

Sol^n.: (i)

Q.2 : The value of the following integral $\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{1+\cos x} dx$

(i) $-2 \log 2$ (ii) 2 (iii) 0 (iv) $(\log 2)^2$

Sol^n.: (iv)

Q.3. The double integral $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\, dx\, dy$ is |A.K.U. 2015|

(i) 0 (ii) π (iii) $\frac{\pi}{2}$ (iv) 2

Sol^n.: (i)

Q.4 : Fill in the blanks |A.K.U. 2016|

(i) $\int_0^2 \int_0^x (x+y)\, dx\, dy = \ldots\ldots$

Ans. $\frac{8}{2}$

(ii) $\int_0^{\pi/2} \int_0^{a\sin\theta} r\, dr\, d\theta = \ldots\ldots\ldots$

Ans. $\frac{a^2\pi}{8}$

(iii) On changing to polar co-ordinate $\int_0^{2a} \int_0^{2ax-x^2} dx\, dy$ becomes ...

Ans. $\int^{\pi a/2} \int^{2a\cos\theta} r\, dr\, d\theta.$

Q.5. : The length of the curve $y = \frac{(x^2+2)^{2/3}}{3}$ from $x = 0$ to $x =$

(i) 10 (ii) 12 (iii) 3π (iv) 6π

Sol^n.: (ii)

Q.6 : The value of the integral $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} x\, dx\, dy$ is [A.K.U. 2

(i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{3}$

(iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{6}$

Sol^n.: (iii)

Hints : $I = \int_0^\infty \left\{ \int_0^\infty e^{-x^2(1+y^2)} x\, dx \right\} dy. = \int_0^\infty I_1\, dy$ (let)

Where $I_1 = \int_0^\infty e^{-x^2(1+y^2)} x\, dx.$

Let $a + 1 + y^2$ treat as a constant.

Therefore

$$I_1 = \int_0^\infty e^{-ax^2} x\, dx - \frac{1}{2a} \int_0^\infty e^{-ax^2} (-2ax)\, dx$$

$$= -\frac{1}{2a}\left[e^{-ax^2}\right]_0^\infty$$

$$= -\frac{1}{2a}\left[e^\infty - e^0\right] = \frac{1}{2a} = \frac{1}{2(1+y^2)}$$

Then from equation (1)

$$I = \int_0^\infty \frac{1}{2(1+y^2)}\,dy = \frac{1}{2}\left[\tan^{-1} y\right]_0^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1} 0\right] = \frac{1}{2}\left(\frac{\pi}{2-0}\right) = \frac{\pi}{4}$$

Q.6(a) :The value of the integral $\int_0^\infty\int_0^\infty e^{-x^2(1+y^2)}\, x\,dx\,dy$ is **[A.K.U. 2017]**

(i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{6}$

Sol^n.: (iii)

Hints : Let $I = \int_0^\infty\int_0^\infty e^{-x^2(1+y^2)}\, x\,dx\,dy$

Putting $e^{-x^2(1+y^2)} = t$

and the limit t varies from i to 0.

$$\Rightarrow \quad e^{-x^2(1+y^2)}\left[-2x(1+y^2)\right]dx = dt$$

$$\Rightarrow \quad e^{-x^2(1+y^2)}\, x\,dx = \frac{-dt}{2(1+y^2)}$$

$$\Rightarrow I = \int_0^\infty\int_1^0 -\frac{dt}{2(1+y^2)}\,dy = \frac{1}{2}\int_0^\infty\int_0^1 dt\cdot\frac{dy}{1+y^2}$$

$$= \frac{1}{2}\int_0^\infty \frac{dy}{1+y^2}\left[t\right]_0^1 = \frac{1}{2}\int_0^\infty \frac{dy}{1+y^2}$$

$$= \frac{1}{2}\left[\tan^{-1} y\right]_0^\infty = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4} = \frac{\pi}{4}$$

Q.7. Evaluate the double integral $\int_1^2\int_1^3 xy^2\,dx\,dy$

Sol^n. Here both limits are constant

Then $\int_1^3\left\{\int_1^2 (xy^2)\,dx\right\}dy = \int_1^3\left[\frac{x^2y^2}{2}\right]_1^2 dy$

$$= \int_1^3\left(2y^2 - \frac{y^2}{2}\right)dy = \int_1^3 \frac{3y^2}{2}\,dy = \left[\frac{3y^3}{3x^2}\right]_1^3 = \left[\frac{y^3}{2}\right]_1^3$$

$$= \frac{27}{2} - \frac{1}{2} = \frac{26}{2} = 13 \text{ Ans.}$$

Q.8. Evaluate $\iint_R xy\,dx\,dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ and $x \geq 0,\ y \geq 0.$ [A.K.U. 2014]

Sol^n.: According to given condition limit lies between 0 to a. for both axis.

$$\iint_R xy\,dx\,dy = \frac{1}{2}\int_0^a\int_0^a xy\,dx\,dy$$

$$= \frac{1}{2}\int_0^a\left[\frac{x^2}{2}y\right]_0^a dy = \frac{1}{2}\int_0^a\left(\frac{a^2}{2}y\right)dy$$

$$= \frac{a^2}{4}\int_0^a y\,dy = \frac{a^2}{4}\cdot\left[\frac{y^2}{z}\right]_0^a = \frac{a^2}{4}\cdot\left[\frac{y^2}{z}\right]_0^a$$

Q.9. Evaluate $\iint_R (x+y)^2\, dx\, dy$ where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$. **[A.K.U. 2016]**

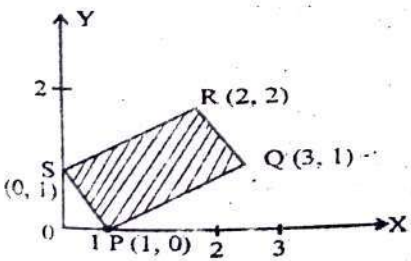
Sol^n.: **A parallelogram PQRS in the xy-plane becomes the another region P′Q′R′S′ in the uv plane as shown in figure.**

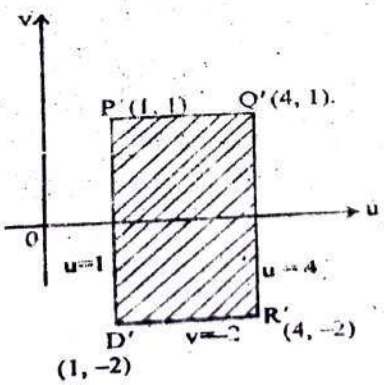
From equation (1)

$$x = \frac{1}{3}(2u+v), \quad y = \frac{1}{3}(u-v)$$

Then from Jacobian Transformation

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \frac{-2}{9} - \frac{1}{9} = \frac{-3}{9} = \frac{-1}{3}$$





Hence the given integral

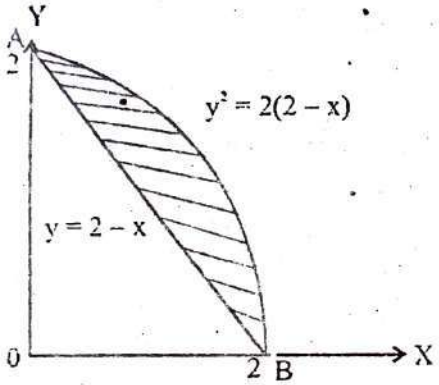
$$\iint_{P'Q'R'S'} u^2 |J|\, du\, dv = \int_1^4 \int_{-2}^1 u^2 \cdot \frac{1}{3}\, du\, dv$$

$$= \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 [v]_{-2}^1 = \frac{1}{3}\left[\frac{64}{3} - \frac{1}{3}\right][1+2] = \frac{63}{3} = 21 = 21 \text{ Ans.}$$

Q.10. Find the double integration the area enclosed by the pair of cu
$y = 2 - x$ and $y^2 = 2(2 - x)$. [A.K.U. 201

Sol^n. Given two curve $y = 2 - x$ and $y^2 = 2(2 - x)$ then x varies fro
0 to 2 and y varies from $2 - x$ to $\sqrt{2(2-x)}$

Area enclosed by two curve is



$$A = \int_0^2 \left[\int_{2-x}^{\sqrt{4-2x}} dy \right] dx = \int_0^2 [y]_{2-x}^{\sqrt{4-2x}} dx = \int_0^2 \left[\sqrt{4-2x} - 2 + x \right] dx$$

$$= \left[\frac{2}{3} \frac{(4-2x)^{3/2}}{(-2)} - 2x + \frac{x^2}{2} \right]_0^2$$

$$= \left[-\frac{1}{3}(4-2x)^{3/2} - 2x + \frac{x^2}{2} \right]_0^2 = \left[-4 + 2 + \frac{8}{3} \right]$$

$$= \frac{8}{3} - 2 = \frac{8-6}{3} = \frac{2}{3} = \frac{8-6}{3} = \frac{2}{3} = \frac{2}{3}$$

....Ans

Q.11. By changing the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

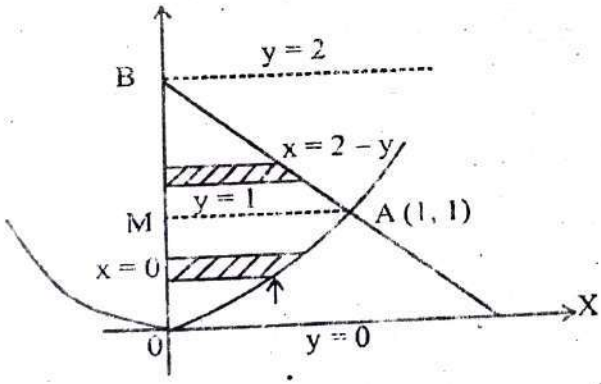
Evaluate the integral. [A.K.U. 2016]

Sol^n.: On taking f (x, y) = xy and region of integration is bounded by following curve

$y = x^2, \quad y = 2 - x, \; x = 0, \; x = 1$

$x^2 = y$ is a parabola whose vertex is at the origin and the axis is the positive reaction of y-axis and the curve $y = 2 - x \Rightarrow x + y = 2$ is a straight line which cuts intercept 2, 2 on the co-ordinate axes.

Hence we draw a line AM parallel to x-axis which divides the region OAB into two regions OAM and MAB.



For the region OAM : x varies from $x = 0$ to $x = \sqrt{y}$

y varies from $y = 0$ to $y = 1$

For the region MAB : x varies from $x = 0$, $x = 2 - y$

and y varies $y = 1$ to $y = 2$

Hence changing the order of integration

$$= \int_0^1 \int_0^{\sqrt{y}} xy\,dy\,dx + \int_0^2 \int_0^{2-y} xy\,dy\,dx$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_0^2 y\cdot\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y\cdot y\,dy + \frac{1}{2}\int_0^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_0^2 (4y-4y^2+y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{4y^2}{2}-\frac{4y^3}{3}+\frac{y^4}{4}\right]_0^2$$

$$= \frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\left[\frac{4.4}{2}-\frac{4}{3}\cdot 8+\frac{16}{4}\right] = \frac{1}{6}+\frac{1}{2}\left[8-\frac{32}{3}+4\right]$$

$$= \frac{1}{6}+\frac{1}{2}\times\frac{4}{3} = \frac{1}{6}+\frac{2}{3}=\frac{5}{6}$$ **...Ans.**

Q.12. Using the transformation $x + y = u$ and $y = uv$ show tha

$$\int_0^1\int_0^{1-x} \frac{y}{e^{x+y}}\,dy\,dx = \frac{1}{2}(e-1)$$ [A.K.U. 2017]

Sol^n.: Given $x + y = u$, $y = uv$

Then $x = u(1 - v)$ $y = uv$

Then from Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

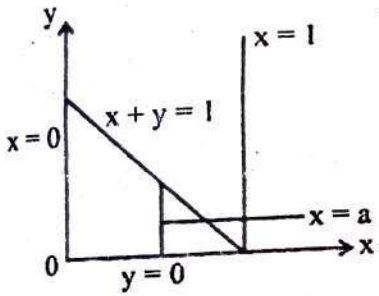
$\Rightarrow \quad J = u - uv + uv = u$

$\Rightarrow \quad dx\,dy = |J|\,du\,dv = u\,du\,dv$

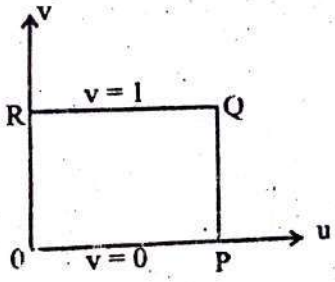
Also $x = 0$

$\Rightarrow \quad u(1-v) = 0 \qquad \Rightarrow u = 0,\ v = 1$

if $y = 0 \qquad \Rightarrow uv = 0 \qquad \Rightarrow u = 0,\ v = 0$



xy-plane



uv-plane

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$\Rightarrow \quad J = u - uv + uv = u$

$\Rightarrow \quad dx\,dy = |J|\,du\,dv = u\,du\,dv$

Also $x = 0$

$\Rightarrow \quad u\,(1-v) = 0$

$\Rightarrow \quad u = 0,\ v = 1$

if $y = 0 \Rightarrow uv = 0 \Rightarrow u = 0,\ v = 0$

$\Rightarrow \quad x + y = u = 1$

Hence the limits of u are 0 to 1 and the limits of v are 0 to 1

The area of integration is OPQR is uv-plane.

$$\int_0^1 \int_0^{1-x} e^{y/x+y}\,.dy\,dx = \int_0^1 \int_0^1 e^{\frac{uv}{u}}\,|J|\,du\,dv$$

$$\Rightarrow \quad \int_0^1 \int_0^{1-x} e^{y/x+y}\, dy\, dx = \int_0^1 \int_0^1 u\, e^v\, du\, dv$$

$$= \left[\frac{v^2}{2}\right]_0^1 \left[e^u\right]_0^1 = \frac{1}{2}\cdot[e-1]$$

Q.13. Show that

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx.$$ [A.K.U. 2013, 2

Sol^n. **Taking L.H.S.**

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy = \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\, dy$$

$$= \int_0^1 dx \int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] dy$$

$$= \int_0^1 dx \left[2x \cdot \frac{(x+y)^{-2}}{-2} - \frac{(x+y)^{-1}}{-1}\right]_0^1$$

$$= \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{(x+y)}\right]_0^1 dx = \int_0^1 \left(\frac{-x}{(x+2)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x}\right) d$$

$$= \int_0^1 \frac{-x+x+1}{(x+1)^2}\, dx = \int_0^1 \frac{1}{(x+1)^2}\, dx = \left[-\frac{1}{x+1}\right]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}$$

R.H.S. $$\int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx = \int_0^1 dy \left[\int_0^1 \frac{x+y-2y}{(x+y)^3}\, dx\right]$$

$$= \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx$$

$$= \int_0^1 \left[\frac{-1}{x+y} \frac{-2y}{-2} \frac{1}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{1}{y} \right] dy$$

$$= \int_0^1 \frac{-1-y+y}{(1+y)^2} dy = -\int_0^1 \frac{1}{(1+y)^2} dy = \left[\frac{1}{1+y} \right]_0^1 = \frac{1}{2} - 1 = \frac{-1}{2}$$

This shows that L.H.S. $\neq$ R.H.S.

$$\frac{1}{2} \neq \frac{-1}{2}$$

Q.14 : Evaluate $\iint r \sin\theta \, dr \, d\theta$ over the cardioid, $r = a(1 - \cos\theta)$. above the initial line. **[A.K.U. 2016]**

Solⁿ.: Here 1st integrate with respect to r and r varies from 0 to a (1−cosθ) and θ varies from 0 to π

$$\iint_R r \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\int_0^{a(1-\cos\theta)} r \, dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)}$$

$$= \frac{a^2}{2}\int_0^{\pi} (1-\cos\theta)^2 \cdot \sin\theta \, d\theta.$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi} = \frac{a^2}{2}\cdot\frac{8}{3} = \frac{4a^2}{3} \quad = \frac{4a^2}{3}$$

Q.15 : Evaluate $\iint r^4 \cos^3\theta \, dr \, d\theta$ over the interior of the circle $r = 2a\cos\theta$. [A.K.U.

Sol^n.: Here radius of the circle $r = 2a\cos\theta$ is a. The pole lies o circumference of the circle.

It is clear that the region of integration is bounded by the follo curve $r = 0$, $r = 2a\cos\theta$, and $\theta = \frac{-\pi}{2}$; $\theta = \frac{\pi}{2}$.

$$\iint_R r^4 \cos^3\theta \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^4 \cos^3\theta \cdot dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\int_0^{2a\cos\theta} r^4 \, dr\right] \cos^3\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^5}{5}\right]_0^{2a\cos\theta} \cos^3\theta \, d\theta$$

$$= \frac{32}{5}a^5 \int_{-\pi/2}^{\pi/2} \cos^8\theta \, d\theta.$$

$$= \frac{32}{5}a^5\left[\frac{2}{8}-1\right] \qquad \text{(from Walli's formula)}$$

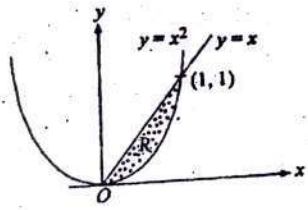
$$= \frac{32}{5}a^5\left(\frac{-6}{8}\right) = \frac{-24a^5}{5}$$

$$= \frac{-24a^5}{5}$$

...Ans.

Q. 16. Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region bounded by the curves $y = x^2$ and $y = x$.

Sol^n.



The curves $y = x^2$ and $y = x$ meet at the points (1, 1) and (0, 0). The region of integration is shaded in the figure.

$$\therefore \iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left\{\int_{y=x^2}^{x} xy(x+y)\,dy\right\}dx$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{x} x(xy+y^2)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[x\left(\frac{y^2}{2}+\frac{y^3}{3}\right)\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[x\left(\frac{x.x^2}{2}+\frac{x^3}{3}\right) - x\left(\frac{x.x^4}{2}+\frac{x^6}{3}\right)\right] dx$$

$$= \int_{x=0}^{1} x\left(\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^5}{2}-\frac{x^6}{3}\right) dx$$

$$= \int_{x=0}^{1} \left[\frac{5}{6}x^4 - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

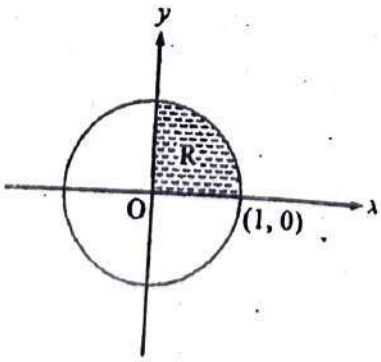
$$= \left[\frac{5}{6}.\frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{4-1}{24} - \frac{1}{14} = \frac{1}{8} - \frac{1}{14} = \frac{7-4}{56} = \frac{3}{56}.$$

Q. 17. Find the value of the double integral $\iint x^2 y^2 \, dx\, dy$ over the regio $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$.

Sol^n. : As $x^2 + y^2 = 1$ $\quad \therefore y = \pm\sqrt{1-x^2}$

$\because$ $y \geq 0$, then $y = +\sqrt{1-x^2}$



Here, the region of integration R is bounded by $x = 0$, $y = 0$and $y = \sqrt{1-x^2}$.

$$\therefore \iint_R x^2 y^2 dx\, dy = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x^2 y^2 dx\, dy$$

$$= \int_{x=0}^{1} \left\{ \int_{y=0}^{\sqrt{1-x^2}} x^2 y^2 dy \right\} = \int_{x=0}^{1} \left[x^2 \frac{y^3}{3} \right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \int_{x=0}^{1} \frac{1}{3} x^2 \ (1 - x^2)^{3/2} \ dx, \text{ let } x = \sin\theta, \ dx$$

$= \cos\theta \ d\theta$ and

x	0	1
θ	0	$\pi/2$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^2\theta \ (1 - \sin^2\theta)^{3/2} \cos \ d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^2\theta \cos^3\theta \cos\theta \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^2\theta \cos^2\theta \cos^2\theta \, d\theta$$

$$= \frac{1}{3} \cdot \frac{1}{8} \int_0^{\pi/2} \sin^2 2\theta (1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{24} \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4\theta)(1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{48} \int_0^{\pi/2} [1 + \cos 2\theta - \cos 4\theta - \cos 4\theta \cos 2\theta] \, d\theta$$

$$= \frac{1}{48} \int_0^{\pi/2} \frac{1}{2} [2 + 2\cos 2\theta - 2\cos 2\theta - 2\cos 4\theta \cos 2\theta] \, d\theta$$

$$= \frac{1}{96} \int_0^{\pi/2} [2 + 2\cos 2\theta - 2\cos 4\theta - \cos 6\theta - \cos 2\theta] \, d\theta$$

$$= \frac{1}{96} \int_0^{\pi/2} [2 + \cos 2\theta - 2\cos 4\theta - \cos 6\theta] \, d\theta$$

$$= \frac{1}{96} \left[2\theta + \frac{\sin 2\theta}{2} - \frac{1}{2} \sin i4\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/2}$$

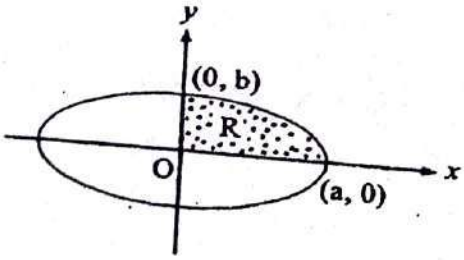
$$= \frac{1}{96}[\pi + 0 - 0 - 0] \qquad [\because \sin n\pi = 0 \text{ for } n \in$$

$$= \frac{\pi}{96}$$

Q. 18. Compute the value of $\iint_R y\,dx\,dy$ where R is the region in the fi quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol^n. : Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

$\because$ $y > 0$, then $y = +\frac{b}{a}\sqrt{a^2 - x^2}$



Here the region of integration R is bounded by $x = 0$, $y = 0$ and $y = \frac{b}{a}\sqrt{a^2 - x^2}$

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dx\,dy = \int_0^a \left\{\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\right\}dx$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2 - x^2)dx$$

$$= \frac{b^2}{2a^2}\left[a^2x - \frac{x^3}{3}\right]\frac{b^2}{2a^2}\left[a^3 - \frac{a^3}{3}\right] = \frac{ab^2}{3}.$$

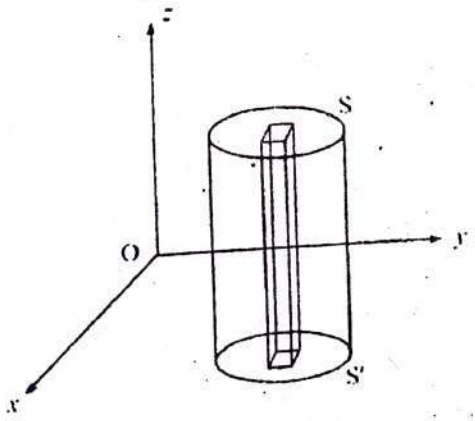
APPLICATIONS : EVALUATION OF AREA AND VOLUME, CENTER OF MASS AND GRAVITY

Double and triple integrals can be effectively used in many problems such as evaluation of area and volume, calculation of centroid. moment of inertia etc.

EVALUATION OF VOLUME

The following are the two popular methods :

(i) If a region is bounded by a bounded surface S given by $z = f(x, y)$, the xy-plane and the orthogonal projection of the boundary of S on the xy-plane, the volume of the region is found out by taking an elemental volume $z\ \delta x\ \delta y$ and then region is found out by taking an elemental volume $z\ \delta x\ \delta y$ and then summing over the ranges of x and y. That is, volume = $\iint z\,dx\,dy$



(ii) By starting with an elemental volume $\delta x\ \delta y\ \delta z$ and then summing over the ranges of x, y and z, volume can also be evaluated.

The following examples illustrate the methods.

DETERMINATION OF MASS

For a plane if $\rho(x, y)$ denotes density at (x, y), then the elementary mass at $P(x, y)$ is $\rho\ dx\ dy$.

Hence the total mass = $\iint \rho(x,y)\,dx\,dy$.

In polar coordinate system, the elementary mass at $P(r, \theta)$ is $\rho r\ dr\ d\theta$ and hence the total mass = $\iint \rho(r,\theta)\,r\,d\theta\,dr$.

For a solid of density $\rho(x, y, z)$ at $P(x, y, z)$.

$$\boxed{\text{the total mass} = \iiint \rho\,dx\,dy\,dz}$$

Triple Integrals

Let $f(x, y, z)$ be a function of three independednt variables x, y defined for all points in a finite closed three dimensional region v. Div v into n elementary volumes $\Delta v_1, \Delta v_2, \ldots\ldots \Delta v_n$. Let us select an arbitra point $P_k\ (x_k, y_k, z_k)$ in each Δv_k and the form of sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta v_k.$$

Taking a limt $n \to \infty$ and $\Delta v_k \to 0$ is called the triple integral of $f(x, y, z)$ over the region v denoted by

$$\boxed{\iiint f(x, y, z)\, dv = \lim_{n \to \infty} \sum_{k=1}^{\infty} f(x_k, y_k, z_k)\, \Delta v_k}$$

Evaluation of Triple Integral

Evaluate the triple integral 1st keep x and y constatnt and integrate with respect z between the limts in general depending upon x and y. Then we have to reduce $f(x, y, z)$ to a function of x and y.

After that keep x constant and integrate with respect to y between limit in depending upon x.

Finally integrated with respect to x between remeining the integrals

$$\boxed{\int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right\} dy \right] dz}$$

Objective Question Answer

Q.1 : The value of $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ is **[A.K.U. 2014]**

(i) $\frac{4}{35}$ (ii) $\frac{3}{35}$ (iii) $\frac{8}{35}$ (iv) $\frac{6}{35}$

Sol^n.: (i)

Hints : $\int_0^1 \int_{y^2}^1 \left[\int_0^{1-x} x\,dz \right] dx\,dy$

$$= \int_0^1 \int_{y^2}^1 [xz]_0^{1-x}\,dx\,dy = \int_0^1 \left[\int_{y^2}^1 x(1-x)\,dx \right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 dy = \int_0^1 \left[\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3} \right] dy$$

$$= \int_0^1 \left[\frac{y^6}{3} - \frac{y^4}{2} + \frac{1}{6} \right] dy = \left[\frac{y^7}{21} - \frac{y^5}{10} + \frac{y}{6} \right]_0^1$$

$$= \frac{1}{21} - \frac{1}{10} + \frac{1}{6} = \frac{10-21+35}{210} = \frac{24}{210} = \frac{8}{70} = \frac{4}{35}$$ **...Ans.**

Q.2 : Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$ **[A.K.U. 2012]**

Sol^n.: We have

$$I = \int_0^1 x \left[\int_0^{\sqrt{1-x^2}} y \left\{ \int_0^{\sqrt{1-x^2-y^2}} z\,dz \right\} dy \right] dx$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{1-x^2}} y \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy \right\} dx.$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{1-x^2}} y \cdot (1-x^2-y^2)\,dy \right\} dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_0^1 x\left[(1-x^2)^2\cdot 2x-(1-x^2)^4\cdot x\right]dx$$

$$= \frac{1}{8}\int_0^1 (x-2x^3+x^5)\,dx = \frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right]=\frac{1}{48}$$ **...Ans.**

Q.3. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\cdot dx\cdot dy\, dz$. **[A.K.U. 2015]**

Sol^n.: Let the given triple integral be denoted by I. Then

$$I = \int_1^e \int_1^{\log y}\left\{\int_1^{e^x} \log z\, dz\right\} dy\, dx.$$

$$= \int_1^e \int_1^{\log y} [z\log z - z]_1^{e^x}\, dy\, dx.$$

$$= \int_1^e \left\{\int_0^{\log y}\left[(x-1)e^x+1\right]dx\right\}dy$$

$$= \int_1^e \left\{\left[(x-1)e^x\right]_0^{\log y}-\left[e^x\right]_0^{\log y}+[x]_0^{\log y}\right\}dy$$

$$= \int_1^e \{(\log y-1)y+1-y+1+\log y\}\,dy$$

$$= \int_1^e \{(y+1)\log y - 2y + 2\}\,dy$$

$$= \int_1^e (y+1)\log y\,dy + 2\int_1^e (1-y)\,dy$$

$$= \left[\log y\left(\frac{y^2}{2}+y\right)\right]_1^e - \int_1^e \frac{1}{y}\left(\frac{y^2}{2}+y\right)dy + \left[2y+y^2\right]_1^e$$

$$= \frac{e^2}{2}+e-\int_1^e\left(\frac{y}{2}+1\right)dy + 2e - e^2 - 1$$

$$= \frac{-e^2}{2}+3e-1-\left[\frac{y^4}{4}+y\right]_1^e$$

$$= \frac{-e^2}{2}+3e-1-\frac{e^4}{4}-e+\frac{1}{4}+1 = \frac{-e^2}{4}-\frac{e^2}{2}+2e+\frac{1}{4} \quad \text{.....Ans.}$$

Q.4 : Evaluate $\iiint x^2\,yz\,dx\,dy\,dz$ over the region bounded by the planes $x = 0,\ y = 0,\ z = 0$ and $x + y + z = 1$. **[A.K.U. 2017]**

Sol^n.: Let

$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\,yz\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-x} x^2 y\left[\frac{z^2}{2}\right]_0^{1-x-y} dx\,dy$$

$$= \int_0^1\int_0^{1-x} \frac{1}{2}x^2 y(1-x-y)^2\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} x^2 y\left[(1-x)^2 - 2(1-x)y + y^2\right] dx\, dy$$

$$= \frac{1}{2}\int_0^1 x^2 \left[(1-x)^2 \frac{y^2}{2} - 2(1-x)\cdot\frac{y^3}{3} + \frac{y^4}{4}\right]_0^{1-x} \cdot dx$$

$$= \frac{1}{2}\int_0^1 x^2 \left[\frac{(1-x)^4}{2} - \frac{2(1-x)^4}{3} + \frac{(1-x)^4}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 \frac{x^2(1-x)^4}{12} dx = \frac{1}{24}\int_0^1 x^2 (1-x)^4 dx$$

$$= \frac{1}{24}\int_0^1 x^{3-1}(1-x)^{5-1} dx = \frac{1}{24}\beta(3,5)$$

$$= \frac{1}{24}\frac{\Gamma(3)\,\Gamma(5)}{\Gamma(8)} = \frac{1}{24}\cdot\frac{\lfloor 2\cdot\lfloor 4}{\lfloor 7}$$

$$= \frac{1}{24}\cdot\frac{1\cdot2\cdot1\cdot2\cdot3\cdot4}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} = \frac{1}{2520} = \frac{1}{2520}$$ ...Ans.

CURVILINEAR COORDINATES

Of immense use is the rectangular coordinate system in expressing analytically various physical plenomena. The basis of the 3-dimensional rectangular coordinate system is three mutually perpendicular lines meeting at a point usually called the a-axis, y-axis and z-axis. There are many occasions where the rectangular coordinate system is not the best system for analytical treatment. This limitation calls for a new system.

There are many after systems but the following two are the simples

(a) Cylindrical coordinate system.

(b) Spherical coordinate system.

The Two Popular Systems

(a) Cylindrical Coordinates System : In this system the coordinates of a point P are ρ, ϕ and z where ρ is the distance of projection of P on the xy-plane, ϕ is the angle made by ON with the positive x-axis and z is the z-coordinate of the point P ?

Thus $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$.

where $\rho \geq 0$, $0 \leq \phi < 2\pi$, $-\infty < z < \infty$

Clearly, the scale factors are

$$h_\rho = 1,\ h_\phi = \rho,\ h_z = 1$$

The coordinate surface are

$\rho = c_1$, i.e., a cylinder coaxial with the z-axis.

$\phi = c_2$, i.e., planes through the z-axis.

$z = c_3$, i.e., planes perpendicular to the z-axis.

The coordinate curves are :

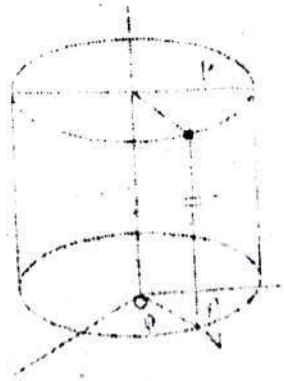
Intersection of $\rho = c_1$ and $\varphi = c_2$ is a straight line.

Intersection of $\rho = c_1$ and $z = c_3$ is a circle (or point)

Intersection of $\phi = c_2$ and $z = c_3$ is a straight line

The transformation from cylindrical to rectangular coordinates is given

by $\rho = \sqrt{x^2 + y^2}$, $\varphi = \tan^{-1} \dfrac{y}{x}, z = z$.



Note that for points on the z-axis, i.e., $x = 0$, $y = 0$, ϕ is indeterminate. Such points are called singular points of the transformation.

...ment d^2s of arc length, we have

The Two Popular Systems

(a) Cylindrical Coordinates System : In this system the coordinates of a point P are ρ, ϕ and z where ρ is the distance of projection of P on the xy-plane, ϕ is the angle made by ON with the positive x-axis and z is the z-coordinate of the point P ?

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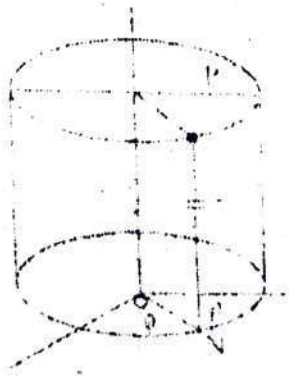


Note that for points on the z-axis, i.e., $x = 0$, $y = 0$, ϕ is indeterminate. Such points are called singular points of the transformation.

For the square element d^2s of arc length, we have

$$x = \rho \cos\phi,\ y = \rho \sin\phi,\ z = z$$

$$dx = -\rho \sin\varphi\ d\phi + \cos\phi\ d\rho,$$

$$dy = \rho \cos\varphi\ d\varphi + \sin\varphi\ d\rho$$

$$dz = dz$$

Hence

$$d^2s = d^2x + d^2y + d^2z$$

$$= (-\rho \sin\phi\ d\varphi + \cos\phi\ d\rho)^2 + (\rho \cos\varphi\ d\varphi + \sin\varphi\ d\rho)^2 + (dz)^2$$

$$= (d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2$$

$$= h_1^2 (d\rho)^2 + h_2^2 (d\phi)^2 + h_3^2 (dz)^2$$

where $h_1 = h_\rho = 1$, $h_2 = h_\phi = \rho$, $h_3 = h_z = 1$ are the scale factors.

For volume element dv, we recall that

$$dv = h_1 h_2 h_3\ du\ dv\ dw.$$

In cylindrical coordinates

$$u = \rho,\ v = \phi,\ w = z,\ h_1 = 1,\ h_2 = \rho,\ h_3 = 1$$

$$\therefore\ dv = (1)\ (\rho)\ (1)\ d\rho\ d\phi\ dz$$

$$= \rho\ d\rho\ d\phi\ dz = \rho\ d\rho\ d\phi\ dz.$$

(b) Spherical Coordinates : In this system the coordinates of a point P are γ, θ and ϕ where γ is the distance P from the origin O, θ is the angle made by OP with the positive z-axis and ϕ is the angle made by ON, with the positive x-axis. N being the projection of P on the xy-plane. Note O is the centre of the sphere.

Thus $x = \gamma \sin\phi \cos\phi$, $y = \gamma \sin\theta \sin\phi$, $z = \gamma \cos\theta$

where $\gamma \geq 0$, $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$

Clearly the scale factors are

$$h\gamma = 1,\ h_\theta = \gamma,\ h_\phi = \gamma \sin\theta.$$

The coordinate surfaces are

$\gamma = C_1$, i.e., spheres having centre at the origin

$\theta = C_2$, i.e., cones having vertex at the origin

$\varphi = C_3$ i.e., planes through the z-axis.

The coordionate curves are :

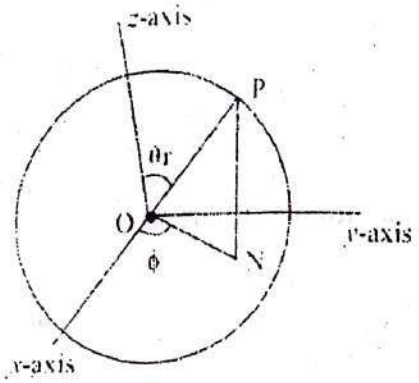
Intersection of $\gamma = C_1$ and $\theta = C_2$ is a circle (or point)

Intersection of $\gamma = C_1$ and $\phi = C_3$ is a semicircle ($C_1 \neq 0$)

Intersection of $\theta = C_2$ and $\phi = C_3$ is a line

The transformation from spherical to rectnagular coordinates is given by

$$\gamma = \sqrt{x^2+y^2+z^2},\ \theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z}},\ \phi = \tan^{-1}\frac{y}{x}$$



The square element of arc length in spherical coordinates can be found out as follows.

$x = \gamma \sin\theta \cos\phi,\ y = \gamma \sin\theta \sin\varphi,\ z = \gamma \cos\theta$

Hence, $dx = -\gamma \sin\theta \cos\phi\, d\phi + \gamma \cos\theta \cos\varphi\, d\theta + \sin\theta \cos\phi\, d\gamma$

$dy = \gamma \sin\theta \cos\phi\, d\phi + \gamma \cos\theta \sin\varphi\, d\theta + \sin\theta \sin\phi\, d\gamma$

$dz = -\gamma \sin\theta\, d\theta + \cos\theta\, d\gamma.$

Therefore, $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

$= (d\gamma)^2 + \gamma^2(d\theta)^2 + \gamma^2 \sin^2\theta\, (d\phi)^2.$

The scale factors are

$h_1 = h\gamma = 1,\ h_2 = h\theta = \gamma,\ h_3 = h\phi = \gamma \sin\theta.$

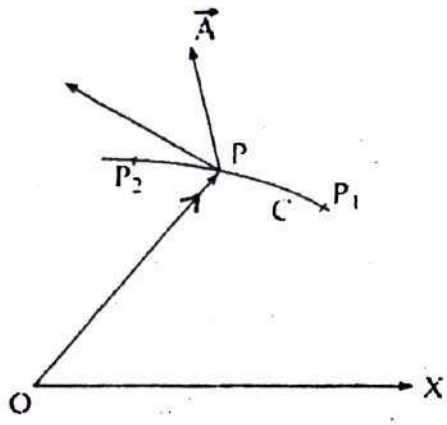
The volume element dv in spherical coordiinate system is

$dv = (1)\ (\gamma)\ (\gamma \sin\theta)\ d\gamma\ d\theta\ d\varphi = \gamma^2 \sin^2\theta\ d\gamma\ d\theta\ d\phi.$

where $u = \gamma,\ v = \theta,\ w = \phi.$

LINE INTEGRAL

Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, the position vector of a point P w.r.t. the origin O, define a curve C joining points, P_1 and P_2 where $t = t_1$ and $t = t_2$ respectively.



Let us assume that C is composed of a finite number of segments fo each of which $\vec{r}(t)$ has continuous derivatives.

Let $\vec{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector function of position defined and continuous over C. Then the integral of the tangential component of $\vec{A}$ along C from P_1 to P_2, given by

$\int_{P_1}^{P_2} \vec{A}.d\vec{r} = \int_C \vec{A}.d\vec{r} = \int_C A_2 dy + \int_C A_3 dz$ is called the *line integral* of $\vec{A}$ over C.

Example 1. Find the total work done in moving a particle in a force field given to $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ form $t = 1$ to $t = 2$.

Sol^n : Total work done

$$= \int_C \vec{F}.d\vec{r} = \int_C (3xy\hat{i} - 5z\hat{j} + 10x\hat{k}).(dx\hat{i} + dy\hat{j} + dx\hat{k})$$

$$= \int_C 3xydx - 5zdy + 10xdz.$$

$$= \int_{t=1}^{2} 3(t^2+1)2t^2 d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)$$

$$= \int_{t=1}^{2} 3(t^2+1)2t^2 d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)$$

$$= \int_1^2 [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]dt$$

$$= \int_1^3 [12t^5 + 10t^4 + 12t^3 + 30t^2]dt$$

$$= [2t^6 + 2t^5 + 8t^4 + 10t^3]_1^2$$

$$= 128 + 64 + 48 + 80 - 2 - 2 - 3 - 10$$

$$= 303$$ ...Ans.

Example 2. If $f = xy\ \hat{i} + (x^2 + y^2)j$, obtain $\int_c \vec{f}.d\vec{r}$ where C is the arc of the parabola $y = x^2 - 4$ from (2, 0) to (4, 12).

Sol^n : $\int_C \vec{f}.d\vec{r} = \int_C xydx + (x^2 + y^2)dy$

$$= \int_{x=2}^{d} x(x^2 - 4)\,dx + \int_{y=0}^{12} (y^2 + y + 4)\,dy$$

$$= \int_{x=2}^{4} (x^3 - 4x)\,dx + \int_{y=0}^{12} (y^2 + y + 4)dy$$

$$= \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_2^4 + \left[\frac{y^3}{3} + \frac{y^2}{2} + 4y\right]_0^{12}$$

$$= 732.$$

Example 3 : If C is a simple closed curve in the xy-plane not enclosing the origin, show that $\int_C \vec{A}.d\vec{r} = 0$ where $\vec{A} = \frac{y\hat{i} + x\hat{j}}{x^2 + y^2}$.

Solution : Here $\int_C \vec{A}.d\vec{r} = \int_C \left(\frac{y\hat{i} + x\hat{j}}{x^2 + y^2}\right)(dx\hat{i} + dy\hat{j})$

$$= \int_C \frac{ydx + xdy}{x^2 + y^2}$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and θ varies from 0 to 2π

$$= \int_0^{2\pi} \frac{r\sin\theta\, d(r\cos\theta) + r\cos\theta\, d(r\sin\theta)}{r^2}$$

$$= \int_0^{2\pi} -\sin^2\theta\, d\theta + \cos^2\theta\, d\theta = \int_0^{2\pi} \cos 2\theta\, d\theta$$

$$= \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{2}[\sin i 4\pi - \sin 0] = \frac{1}{2}\ (0 - 0)$$

Example 5 : Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 3 and if the force field is given by

$$\vec{F} = (2x - y + z)\ \hat{i} + (x + y - z^2)\ \hat{j} + (3x - 2y + 4z)\ \hat{k}.$$

Sol^n : In the plane $z = 0$, $\vec{F} = (2x - y)\ \hat{i} + (x + y)\ \hat{j} + (3x - 2y)\ \hat{k}$ and $d\vec{r} = dx\ \hat{i} + dy\ \hat{j}$.

$\therefore$ Work done $= W = \int_C \vec{F}.d\vec{r}$

$$= [(2x - y)\ \hat{i} + (x + y)\ \hat{j} + (3x - 2y)\ \hat{k}].\ [\hat{i}\ dx + \hat{j}\ dy]$$

$$= \int_C (2x - y)\, dx + (x + y)\, dy$$

The parametric equation of circle $x^2 = y^2 = 3^2$ is $x = 3\cos\theta$ $y = 3\sin\theta$ where $0 \le \theta \le 2\pi$, then

$$W = \int_0^{2\pi} (2.3\cos\theta - 3\sin\theta)\, d(3\cos\theta)$$

$$+ (3\cos\theta + 3\sin\theta)\ d\,(3\sin\theta)$$

$$= \int_0^{2\pi} 9[-2\sin\theta\cos\theta + \sin^2\theta]\, d\theta + 9\ [\cos^2\theta + \sin\theta\cos\theta]\ d\theta$$

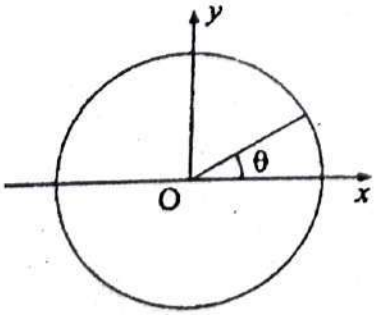
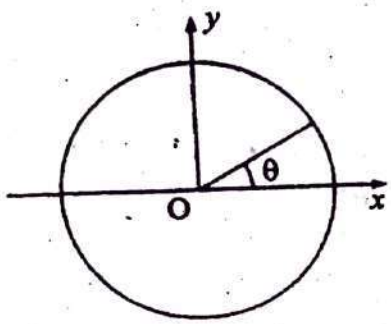


Fig.

$$= 9\int_0^{2\pi} [-\sin\theta\cos\theta + (\sin^2\theta + \cos^2\theta)]\,d\theta$$

$$= 9\int_0^{2\pi} \left(1 - \frac{1}{2}\sin 2\theta\right) d\theta = 9\left[\theta + \frac{1}{4}\cos 2\theta\right]_0^{2\pi}$$

$$= 9\left[2\pi + \frac{1}{4}(1-1)\right] = 18\pi.$$

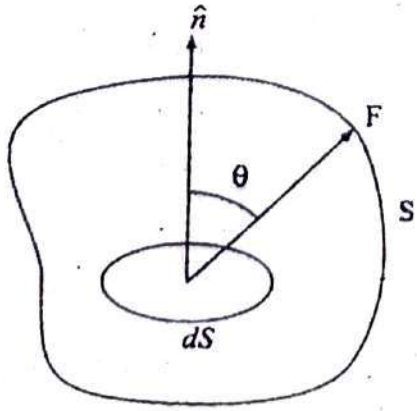


SURFACE INTEGRALS

Let $\vec{F}(x, y, z)$ be a vector function and S be a given surface.

The surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the component of $\vec{F}$ along the normal to the surface. As the component of $\vec{F}$ along the normal is $\vec{F}.\hat{n}$ where $\hat{n}$ is the unit normal vector to an element dS, the surface integral of $\vec{F}$ over S is

$$\iint_S (\vec{F}.\vec{n})\, dS$$



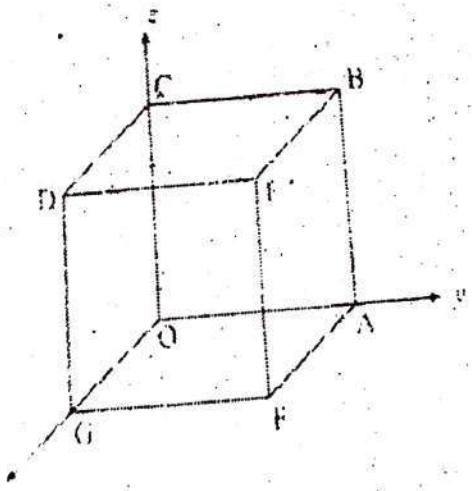
(i) If $\vec{F}$ represents the velocity of a liquid, then $\iint_S (\vec{F}.\vec{n})\,dS$ is called the flux of $\vec{F}$ over S.

(ii) If $\iint_S (\vec{F}.\vec{n})\,dS = 0$, then $\vec{F}$ is said to be *solenoidal*.

Example 1. If $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$.

Evaluate $\iint_S (\vec{F}.\vec{n})\,dS$ where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 1$.

Solution : For face DEFG : $\hat{n} = \hat{i}$ and $x = 1$



$$= \int_0^1 \int_0^1 4xz\,dy\,dz$$

$$= 4\int_0^1 \int_0^1 dz\,dy\,dz = 4\int_0^1 \left[\frac{z^2}{2}\right]_0^1 dy = 4\int_0^1 \frac{1}{2}\,dy = 2[y]_0^1 = 2$$

For Face ABDO; $\hat{n} = -\hat{i}, x = 0$.

$$\int_{ABCD} \vec{F}.\vec{n}\,ds = \int_0^1 \int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(-\hat{i})\,dy\,dz$$

$$\int_0^1 \int_0^1 (4xz)\,dy\,dz = 0 \text{ (Since } x = 0)$$

For Face ABEF : $\hat{n}=\hat{j}, y=1$, then

$$\iint_{ABEF} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1}\int_{z=0}^{1} (4xzi - y^2 j + yzk).\,j\,dx\,dz = \int_0^1\int_0^1 -y^2\,dx\,dz$$

$$= \int_0^1\int_0^1 (-1)\,dx\,dz = (-1)\int_0^1 [x]_0^1\,dz = -1\int_0^1 dz = -[z]_0^1 = -1.$$

For Face OGDC : $\hat{n}=-\hat{j}, y=0$,

$$\int_{OGDC} F.n\,dS = \int_{x=0}^{1}\int_{z=0}^{1} (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(-j)\,dxdz$$

$$= \int_0^1\int_0^1 (y^2)\,dxdz = 0. \qquad (\because y=0)$$

For Face BCDE : $\hat{n}=\hat{k}$

and $z = 1$

$$\int_{BCDE} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1} (4x\hat{i} - y^2\hat{j} + y\hat{k}).\ \hat{k}dxdy = \int_0^1\int_0^1 y\,dxdy$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

For Face AFGO: $\hat{n}=-\hat{k}, z=0$ then

$$\iint_{AFGO} \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 (-y^2\hat{j}),(-\hat{k})\,dxdy = 0.$$

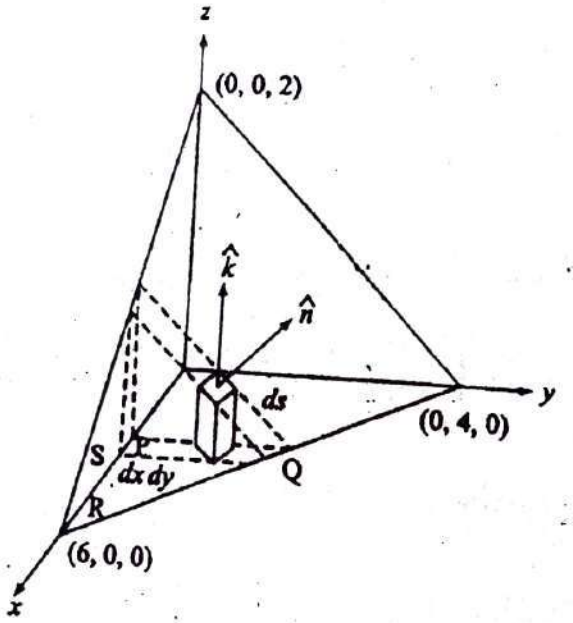
Adding : $\iint_S \vec{F}.\hat{n}\,ds = \left(2+0+(-1)+0+\frac{1}{2}+0\right) = \frac{3}{2}$

Example 2 : Evaluate $\iint_S \vec{A}.\hat{n}\,ds$ where $\vec{A} = 18z\,i - 12\,j + 3y\,k$ and S is the part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

Sol^n : Let R be the projection of S on the xy-plane, then

$$\iint_S \vec{A}.\hat{n}\,ds = \iint_R \vec{A}.\hat{n}\frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

Now $\bar{\nabla}\phi = \bar{\nabla}(2x+3y+6z) = 2\hat{i}+6\hat{k}$



$\therefore$ Then a unit normal to the surface S at any point is

$$\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2\hat{j}+3\hat{j}+6\hat{k}}{\sqrt{49}} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$\therefore \quad \hat{n}.\hat{k} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}).\hat{k} = \frac{6}{7}$$

$$\bar{A}.\hat{n} = (18z\hat{i}-12\hat{j}+3y\hat{k}).(2\hat{i}+3\hat{j}+6\hat{k}) = \frac{36z-36+18y}{7}$$

$$= \frac{6(12-2x-3y)+18y-36}{7} \qquad [\because 2x+3y+6z=12]$$

$$= \frac{36-12x}{7}$$

$$\therefore \quad \iint_S \bar{A}.\hat{n}\,dS = \iint_R A.\mathrm{n}\frac{dx\,dy}{|\hat{n}.\hat{k}|} = \iint_R \frac{36-12x}{7} \times \frac{dydx}{\frac{6}{7}}$$

$$= \iint_R (6-2x)\,dx\,dy$$

In R, keeping x fixed and integrating w.r.t. y from $y = 0$ to $y = \frac{12-2x}{3}$ and then integrating w.r.t. x from $x = 0$ to $x = 6$, we have the double integral (1) as

$$\iint_R (6-2x)\,dx\,dy = \int_{x=0}^{6}\left[\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dy\right]dx$$

$$= \int_{x=0}^{6}\left[(6-2x)y\right]_{y=0}^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 (6-2x)\frac{(12-2x)}{3}dx$$

$$= \frac{1}{3}\int_0^6 (72-36x+4x^2)\,dx$$

$$= \frac{1}{3}\left[72x-18x^2+\frac{4}{3}x^3\right]_0^6$$

$$= \frac{1}{3}\left[72.6-18\times 36+4\times\frac{6^3}{3}\right] = 24$$

GREEN'S THEOREM IN THE PLANE

Statement : Let R be a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous partial derivatives in R, then

$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy}$$

where C is traversed in the anti-clockwise direction.

Example 1. Using Green's theorem, evaluate $\oint_C (x^2 y\,dx + x^2 dy)$ where C is the boundary described clockwise of the triangle with vertices (0, 0), (1, 0), (1, 1).

Sol^n : By the Green's theorem, we get

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy .$$

Here $M = x^2 y$ and $N = x^2$.

$$\oint_C (x^2 y dx + x^2 dy) = \iint_R \left[\frac{\partial x^2}{\partial x} - \frac{\partial}{\partial y} \right] dx\, dy$$

$$= \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (2x - x^2)\, dx\, dy$$

$$= \int_{x=0}^{1} [2x - x^2][y]_0^x\, dx$$

$$= \int_0^1 x(2x - x^2)\, dx = \int_0^1 (2x^2 - x^3)\, dx$$

$$= \left[\frac{2}{3} x^3 - \frac{x^4}{4} \right]_0^1 = \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5}{12}.$$

Example 2. Verify Green's theorem in the plane for

$$\oint_C (xy + y^2)\, dx + x^2 dy$$

where C is the closed curved of the region bounded by $y = x$ and $y = x^2$.

Sol^n : Here $y = x$ and $y = x^2$ intersect at (0, 0) and (1, 1). The positive direction in tranversing C is as shown in the figure.

Along $y = x^2$, the line integral

$$= \int_0^1 (x.x^2 + x^4)\, dx + x^2\, d(x^2)$$

$$= \int_0^1 (x^3 + x^4)\, dx + 2x^3 dx = \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3}{4} x^4 + \frac{x^5}{5} \right]_0^1 = \left(\frac{3}{4} + \frac{1}{5} \right) = \frac{19}{20}.$$

Along $y = x$ from (1, 1) to (0, 0), the line integral

$$= \int_1^0 (x, x + x^2)\, dx + x^2 dx = \int_1^0 3x^2\, dx = \left[x^3\right]_1^0 = -1$$

$\therefore$ The required line integral $= \dfrac{19}{20} - 1 = -\dfrac{1}{20}$

Again $\iint_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\, dy\, dx = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx = \int_0^1 (x^4 - x^3) dx = \left(\frac{x^5}{5} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= \left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{1}{20}.$$

Example 3 : Apply Green's theorem to evaluate

$$\oint_C (y - \sin x)\, dx + \cos x\, dy$$

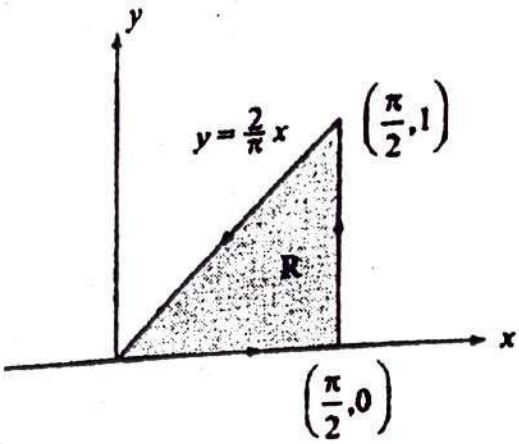
where C is the plane triangle enclosed by the lines $y = 0$, $x = \dfrac{\pi}{2}$ and $y = x$.

Sol^n. : From Green's theorem, we know that

$$\oint_C M dx = N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Here $M = y - \sin x$, $N = \cos x$

$\therefore \quad \dfrac{\partial M}{\partial y} = -1, \qquad \dfrac{\partial N}{\partial x} = -\sin x$



$$\oint_C (y-\sin x)\,dx+\cos x\,dy = \iint_C \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \int_{y=0}^{1}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}y}(-\sin x-1)\,dx\,dy = \int_{y=0}^{1}\left[\cos x - x\right]_{x=\frac{\pi}{2}}^{\frac{\pi}{2}y} dy$$

$$= \int_{y=0}^{1}\left[\cos\frac{\pi y}{2}-\frac{\pi y}{2}+\frac{\pi}{2}\right]dy$$

$$= \left[\frac{2}{\pi}\sin\frac{\pi y}{y}-\frac{\pi y^2}{4}+\frac{\pi}{2}y\right]_0^1 = \frac{2}{\pi}-\frac{\pi}{4}+\frac{\pi}{2} = \frac{2}{\pi}+\frac{\pi}{4} = \frac{\pi^2+8}{4\pi}.$$

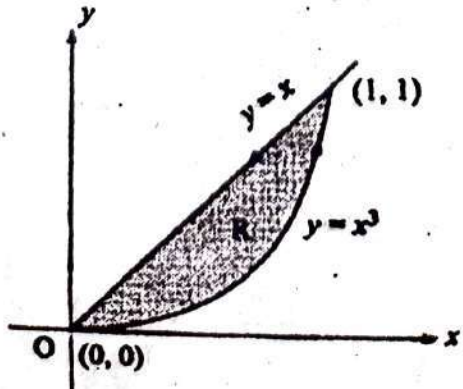
Example 4. Use Green's theorem to evaluate

$$\oint_C (x^2 y\,dx + y^3\,dy)$$

where C is the closed path formed by $y = x$ and $y = x^3$ from $(0, 0)$ to $(1, 1)$.

Sol^n : From Green's theorem in the xy-plane, we know that

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$



Here $M = x^2y$ and $N = y^3$.

$$\therefore \quad \frac{\partial M}{\partial y} = x^2 \text{ and } \frac{\partial N}{\partial x} = 0$$

$\therefore$ By Green's theorem

$$\oint_C (x^2y + dx + y^3dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \iint_R (-x^2)dx\,dy \text{ where } R \text{ is the region bounded by } C.$$

$$= \int_{x=0}^{1}\int_{y=x^3}^{x} (-x^2)\,dx\,dy = \int_0^1 [-x^2y]_{x^3}^{x}\,dx = \int_0^1 -x^2(x - x^3)dx$$

$$= \int_0^1 (x^5 - x^3)dx = \left[\frac{x^6}{6} - \frac{x^4}{4}\right]_0^1$$

STOKE'S THEOREM

Statement : Let S be an open, two-sided surface bounded by a closed, non-intersecting simple curve C and $\vec{A}$ be a vector valued function having continuous first order partial derivatives. Then

$$\boxed{\oint \vec{A}.d\vec{r} = \iint_S (\vec{\nabla} \times A).\hat{n}\,dS}$$

where C is traversed in the anti-clockwise direction.

Example 1. Evalulate $\oint_C e^x dx + 2y\,dy - dz$ by using S stokes theorem, where C is the curve

$x^2 + y^2 = 4,\ z = 2.$

Sol^n : The Stoke's theorem is $\oint_C \vec{A}.d\vec{r} = \iint_S \nabla \times \vec{A}.\hat{n}\,ds$

Here $e^x dx + 2y\,dy - dz = (e^x\hat{i} + 2y\hat{j} - \hat{k}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$= \vec{A}.d\vec{r}$ where $\vec{A} = e^x\hat{i} + 2y\hat{j} - \hat{k}$ and $d\vec{r} = dx\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial k} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial(-1)}{\partial y} - \frac{\partial}{\partial z}(2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial z}e^x\right] + \hat{k}\left[\frac{\partial}{\partial x}2y - \frac{\partial}{\partial y}e^x\right]$$

$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0] = \hat{i}.0 + \hat{j}.0 + \hat{k}.0.$$

Hence by the Stokes's theorem

$$\oint_C e^x dx + 2y\,dy - dz = \oint_C \vec{A}.d\vec{r}$$

$$= \oint_S \nabla \times \vec{A}.\hat{n}\,ds = 0. \qquad [\because \nabla \times \vec{A} = 0]$$

Example 2 : Verify Stokes theorem for $\vec{A} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2\,\hat{z}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Sol^n : In the xy-plane the boundary C of S is a circle of radius one and with centre at the origin. Then the parametric equation of C is $y = \sin t$, $x = \cos t$ where $0 \le t \le 2\pi$.

$$\therefore \quad \oint_C \vec{A}.d\vec{r} = \oint_C [(2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}].[\hat{i}dx + \hat{j}dy + \hat{k}dz]$$

$$= \oint_C (2x - y)dx - yz^2dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\,d(\cos t) - \cos t.0.d(\sin t) - 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \int_0^{2\pi} (\sin^2 t - 2\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2}2\sin^2 t - \sin 2t\right)dt = \int_0^{2\pi} \left[\frac{1}{2}(1 - \cos 2t) - \sin 2t\right]dt$$

$$= \left[\frac{1}{2}\left(t - \frac{\sin 2t}{2}\right) + \frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi - \frac{\sin 4\pi}{2} + \cos 4\pi - \cos 0\right] = \frac{1}{2} \times 2\pi = \pi.$$

Again $\bar{\nabla} \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-y^2 z) - \frac{\partial}{\partial z}(yz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-y^2 z) - \frac{\partial}{\partial z}(2x - y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x - y)\right]$$

$$= \hat{i}[-2yz + 2yz] - \hat{j}[0 - 0] + \hat{k}[0 + 1]$$

$$= \hat{i}.0 + \hat{j}.0 + \hat{k} = \hat{k}$$

Let R be the projection of S on the xy-plane. Then

$$\iint_S \nabla \times \bar{A}.n\, ds = \iint_S k.\hat{n} \frac{dx\, dy}{|\hat{k}.\hat{n}|} = \iint_R dx\, dy$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2}\, dx\, dy = \int_{x=-1}^{1} [y]_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} dx$$

$$= \int_{x=-1}^{1} 2\sqrt{1 - x^2}\, dx = 2\left[\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_{-1}^{1}$$

$$= 2\left[0 + \frac{1}{2}(\sin^{-1} 1 - \sin^{-1}(-1))\right]$$

$$= 2\left[\frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right] = 2\times\frac{1}{2}\pi = \pi$$

Hence $\oint_C \vec{A}.d\vec{r} = \iint_S \nabla\times\vec{A}.\hat{n}\,ds = \pi$

So Stokes theorem is verified.

Example 2 : Evaluate $\oint_C e^x\,dx + 2y\,dy - dz$ by using S stokes' theorem, where C is the curve $x^2 + y^2 = 4$, $z = 2$.

Sol^n : The Stoke's theorem is $\oint_C \vec{A}.d\vec{r} = \iint_S \vec{\nabla}\times\vec{A}.\hat{n}\,ds$

Here $e^x\,dx + 2y\,dy - dz = (e^x\hat{i}+2y\hat{j}-k).(\hat{i}\,dx+\hat{j}\,dy+\vec{k}\,dz)$

$= \vec{A}.d\vec{r}$ where $\vec{A} = e^x\hat{i}+2y\hat{j}-\hat{k}$ and $d\vec{r} = dx\hat{i}+dy\hat{j}+dz\hat{k}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial(-1)}{\partial y}-\frac{\partial}{\partial z}(2y)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-1)-\frac{\partial}{\partial z}e^x\right] + k\left[\frac{\partial}{\partial x}2y-\frac{\partial}{\partial y}e^x\right]$$

$$= \hat{i}[0-0]-\hat{j}(0-0)+\hat{k}[0-0] = \hat{i}.0+\hat{j}.0+\hat{k}.0$$

Hence by the Stoke's theorem

$$\oint_C e^x\,dx + 2y\,dy - dz = \oint_C \vec{A}.d\vec{r}$$

$$= \iint_S \nabla\times\vec{A}.\hat{n}\,ds = 0. \qquad [\because \nabla\times\vec{A}=0]$$

THEOREM OF GAUSS

Statement : Let V be the volume bounded by a closed surface S and $\vec{A}$ vector fucnction having continuous partial derivatives, then

$$\boxed{\oiint_S \vec{A}.d\vec{S} = \iiint_V \vec{\nabla}.\vec{A}\,dV}$$

where $\hat{n}$ is the unit normal (outward drawn) to S.

Example 1. Use the divergence theorem to evaluate

$\iint_S \vec{A}.\vec{ds}$ where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Sol^n : From the divergence theorem, we get

$$\iint_S \vec{A}.\hat{n}.ds$$

$$= \iiint_V \vec{\nabla}.\vec{A}\,dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^3\hat{i} + y^3\hat{j} + z^2\hat{k})\,dV$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\,dV$$

Let $x = r\sin\theta\cos\phi$, y
$= r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\therefore\quad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,[r^2\sin^2\theta\cos\phi]\cos^2\phi - \sin\theta\sin\phi\,[-r^2\sin^2\theta\sin\phi]$$
$$+ \cos\theta\,[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\phi\sin^2\phi]$$
$$= r^2\,[\sin^3\theta\cos^2\phi + \sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta\,(\cos^2\phi + \sin^2\phi)]$$
$$= r^2\,[\sin^3\theta\,(\cos^2\phi + \sin^2\phi + \sin^2\phi) + \cos^2\theta.\sin\theta]$$
$$= r^2\,[\sin^3\theta + \cos^2\theta.\sin\theta] = r^2\sin\theta$$

$$\therefore \quad 3\iiint_V (x^2+y^2+z^2)\,dV = \iiint_V r^2.J\,dr\,d\theta\,d\phi$$

$$= 3\iiint_V r^2.\sin\theta\,dr\,d\theta\,d\phi = 3\iiint_V r^2.r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3.\,8\int_{r=0}^{a_r}\int_{\theta=0}^{\pi/2} r^4\sin\theta[\phi]_0^{\pi/2}\,dr\,d\theta = 24\int_0^a\int_{\theta=\theta}^{\pi/2} r^4\sin\theta[\phi]_0^{\pi/2}\,dr\,d\theta$$

$$= 12\pi\int_0^a r^4[-\cos\theta]_0^{\pi/2}\,dr = 12\pi\int_0^a r^4\,dr = \frac{12}{5}\pi r^5\Big|_0^a = \frac{12\pi a^5}{5}$$

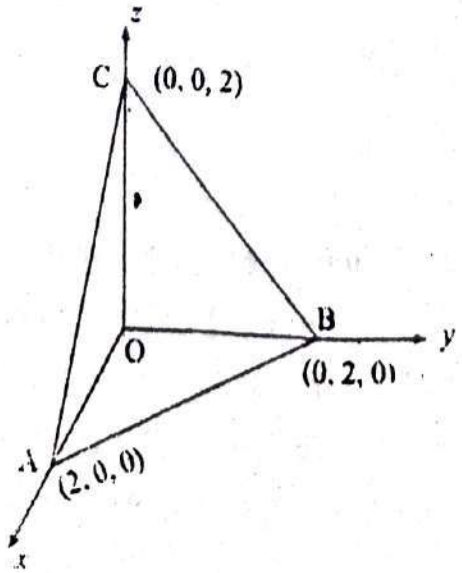
$$= 12\pi\int_0^a r^4\,dr = \frac{12}{5}\pi r^5\Big|_0^a = \frac{12\pi a^5}{5}.$$

Example 2. Evaluate the surface integral $\iint_S \vec{F}.\hat{n}\,ds$

where $\vec{F} = (x^2+y^2+z^2)\,(\hat{i}+\hat{j}+\hat{k})$, S being the surface of the tetrahedron $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$ and $\hat{n}$ being the unit normal in the outward direction to the closed surface S.

Sol^n : By the divergence theorem, we get

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \vec{\nabla}.\vec{F}\,dV$$



$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . [(x^2 + y^2 + z^2)\ (\hat{i} + \hat{j} + \hat{k})] dV$$

$$= \iiint_V (2x + 2y + 2z)\, dV = 2\iiint_V (x + y + z)\, dV$$

$$= 2\iiint_V 2.dx\, dy\, dz = 4\iiint_V dx\, dy\, dz$$

= 4 Volume of tetrahedron OABC

$$= 4 \times \left[\frac{1}{3} \text{area of the base OAB} \times \text{Height OC} \right]$$

$$= 4 \times \left[\frac{1}{3}\frac{1}{2}(2 \times 2) \times 2 \right] = \frac{16}{3}.$$

$$= \frac{16}{3}.$$

Example 3. Verify Gauss's divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ for the cylindrical region S given by $x^2 + y^2 = a^2$, $z = 0$ and $z = h$.

Sol^n : Now $\vec{\nabla}.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right) \cdot (y\hat{i} + x\hat{j} + z^2\hat{k}) = 2z$

$$\therefore \iiint_V \vec{\nabla}.\vec{F}\, dV = \iiint_V 2z\, dx\, dy\, dz = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h 2z\, dx\, dy\, dz$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\left[\frac{z^2}{2} \right]_0^h dx\, dy$$

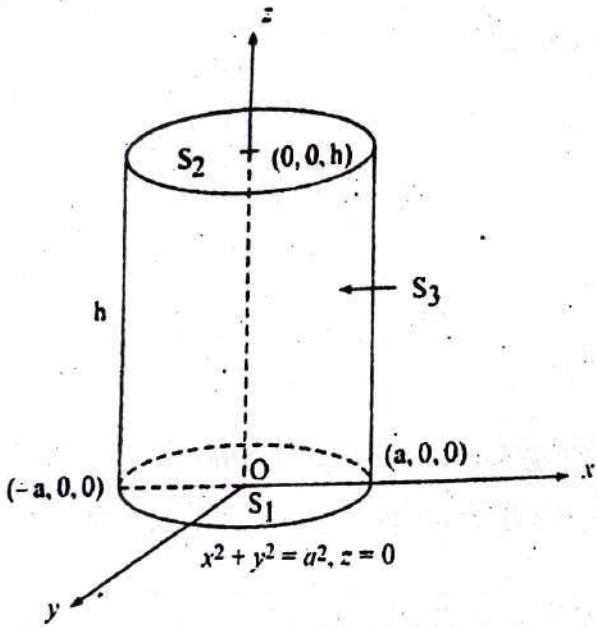
$$= \int_{-a}^{a} h^2 [y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\left[\frac{z^2}{2} \right]_0^h dx\, dy = h^2 \int_{-a}^{a} 2\sqrt{a^2 - x^2}\, dx$$

$$= 2h^2 \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_{-a}^{a}$$

$$= 2h^2\left[\frac{a^2}{2}.\left(\sin^{-1}1-\sin^{-1}(-1)\right)\right]$$

$$= h^2a^2\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \pi a^2h^2.$$



The surface S of the cylinder has a base S_1, the top S_2 and the curved portion S_3.

For S_1, $z = 0$, $\hat{n}=-\hat{k}$ and $\vec{F}.\hat{n} = -z^2 = 0$

So $\iint\limits_{S_1} \vec{F}.\hat{n}\,ds = 0$

For S_2, $z = h$ and $\hat{n}=\hat{k}$ and $\vec{F}.\hat{n} = z^2 = h^2$

$\therefore \iint\limits_{S_2} \vec{F}.\hat{n}\,ds = \iint\limits_{R} h^2\,dx\,dy$

where R is the region bounded by the circle $x^2 + y^2 = a^2$.

$$= h^2 \iint_R dx\,dy$$

$$= h^2 \times \text{area of the circle}$$

$$= h^2 \pi a^2$$

For S_3, $\hat{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \dfrac{\tilde{N}(x^2+y^2-a^2)}{|\vec{\nabla}\phi|} = \dfrac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}} = \dfrac{x\hat{i}+y\hat{j}}{a}$

$$\therefore \quad \hat{n}.\hat{j} = \frac{y}{a} \text{ and } \hat{F}.\hat{n} = \frac{2xy}{a}$$

$$\iint_{S_3} \vec{F}.\hat{n}\,ds = \iint_R \frac{2xy}{a}\frac{dy\,dz}{|\hat{n}.\hat{j}.|} = \iint_R \frac{2xy}{a}\frac{d}{y}dy\,dz = \iint_R 2x\,dy\,dz$$

$$= \int_{z=0}^{h}\int_0^{2\pi} a^2 2\cos\theta\,d\theta\,dz \quad [\because x = a,\ d(a\sin\theta) - a\cos\theta]$$

$$= 2a^2 \int_0^h [\sin\theta]_0^{2\pi}\,dz$$

$$= 2a^2\,[0-0] = 0.$$

$$\therefore \quad \iint_S \vec{F}.\hat{n}\,ds = \iint_{S_1} \vec{F}.\hat{n}\,ds + \iint_{S_2} \vec{F}.\hat{n}\,ds + \iint_{S_3} \vec{F}.\hat{n}\,ds$$

$$= \pi a^2 h^2$$

$$= \iiint_V \vec{\nabla}.\vec{F}.dV$$

Hence Gauss divergence theorem is verified.

Examle 4. Use Stokes' theorem to prove that (i) curl grad $\phi = 0$.

(ii) div. cur $\vec{F} = 0$ where ϕ is a scalar function and $\vec{F}$ is a vector function.

Sol^n : (i) By Stokes theorem, we get $\iint_S \text{curl grad}\,\phi.\hat{n}\,ds$

$= \int_C \text{grad}\,\phi . d\vec{r}$ where C is closed curve bonding of S.

$$= \oint_C \vec{\nabla}\phi . d\vec{r} = \oint_C \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$= \oint_C d\phi = 0$, since C is closed curve.

Since the result holds for arbitrary S, so curl grad $\phi = 0$.

(ii) Let V be the volume bounded by a closed surface S. Then by the divergence theorem, we get

$$\iiint_V \vec{\nabla}.curl\,\vec{F}\,dV = \iint_C 0 \times \vec{F}.\hat{n}\,dS$$

We divided the close surface S into two parts S_1 and S_2 by the curve C which is closed. Then

$$\iint_S \vec{\nabla} \times \vec{F}.\hat{n}\,dS = \iint_{S_1} \vec{\nabla} \times \vec{F}.\hat{n}\,dS + \iint_{S_1} \vec{\nabla} \times \vec{F}.\hat{n}\,dS$$

$= \oint \vec{F}.\vec{dr} - \oint_C \vec{F}.d\vec{r}$ (by the Stockes theorem) $= 0$.

[Negative sign in the 2nd integral because the positive senses of the enclosing C for the two surfaces area opposite in character.]

$$\therefore \quad \iiint_V \vec{\nabla}.\text{curl}\,\vec{F}.d\,V = 0$$

This is true for all V; so $\vec{\nabla}$. curl $\vec{F} = 0$ i.e., div curl $F = 0$.

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