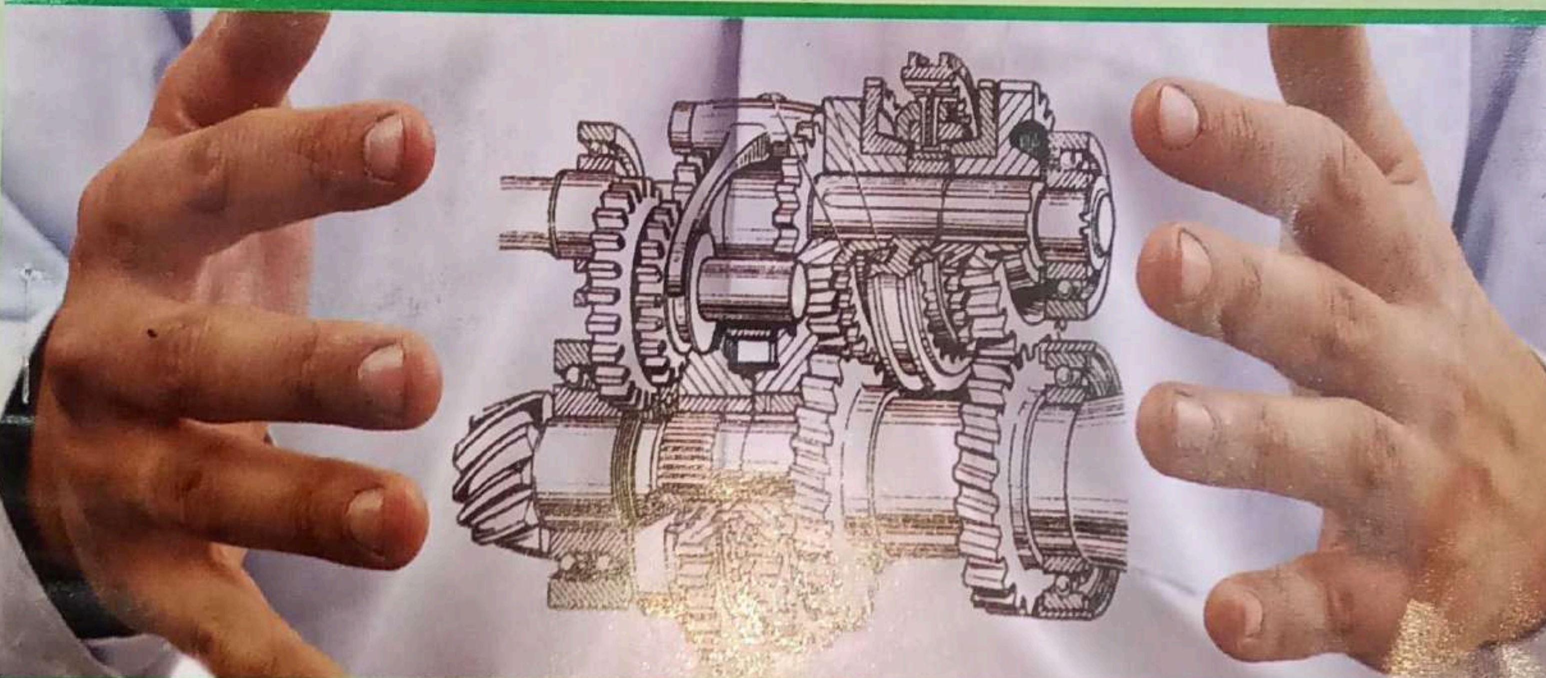
ELECTRICAL BRANCH

For 3rd Semester

Engineering Mechanics

Strictly according to the
NEW SYLLABUS



ORGANIZER

UNIT 1 INTRODUCTION TO VECTORS AND TENSORS

Chapter at a glance

- *Scalar* $\Rightarrow$ A physical quantity which is completely described by a single real number (magnitude).
- *Example of scalar quantities* $\Rightarrow$ temperature, density, mass, etc.
- *Vector* $\Rightarrow$ A physical quantity which possesses (i) a certain **magnitude**, including a null value (ii) a certain orientation in space, called its **direction** and (iii) a **sense**, which qualifies its nature of action (moving toward or away from a given point).
- Vector Notations $\Rightarrow$ **ab**, $\overline{ab}$, $\overrightarrow{ab}$, $\widehat{ab}$ etc.
- Magnitude of a vector $\Rightarrow$ represented by $|ab|$ or $|\overline{ab}|$
- Types of Vectors
  Equality Vectors $\Rightarrow$ have same magnitude, same direction and same sense.
  Equivalence Vectors $\Rightarrow$ produce the same effect in a certain respect.
  Free Vector $\Rightarrow$ action is not confined to a unique line in space
  Bound or Fixed Vector $\Rightarrow$ must remain at the same point of application
  Position Vector $\Rightarrow$ directed line segment from the origin of a co-ordinate system to a point
  Displacement Vector $\Rightarrow$ directed line segment connecting any two points on the path of motion
- Parallelogram Law of Forces $\Rightarrow$ If two forces, acting at a point be represented in magnitude, direction and sense by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude, direction and sense by the diagonal of the parallelogram passing through that point.

  $\Rightarrow$Magnitude of resultant: $R = (P^2 + Q^2 + 2PQ\cos\alpha)^{1/2}$

  $\Rightarrow$Direction of resultant: $\phi = \tan^{-1}[(Q\sin\alpha)/(P+Q\cos\alpha)]$
- Direction Cosines of Vector

  $\Rightarrow l = \cos\alpha = \cos(v, x)$; $m = \cos\beta = \cos(v, y)$; $n = \cos\gamma = \cos(v, z)$ $l^2 + m^2 + n^2 = 1$
- Unit Vector $\Rightarrow$ unit magnitude and a particular direction and sense along the vector;

  $e_v = \dfrac{\vec{v}}{v}$
- Unit Co-ordinate Vector $\Rightarrow$ Unit vectors lying along the axes of reference
- Component form of Vectors $\Rightarrow \vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}$
- Vector algebra

$\Rightarrow \vec{A}+\vec{B}=(A_x+B_x)\,\vec{i}+(A_y+B_y)\,\vec{j}+(A_z+B_z)\,\vec{k}$

$\Rightarrow \vec{A}-\vec{B}=(A_x-B_x)\,\vec{i}+(A_y-B_y)\,\vec{j}+(A_z-B_z)\,\vec{k}$

$\Rightarrow$Scalar product or dot product: $\vec{A}.\vec{B}=AB\cos\theta=AB\cos(\vec{A},\vec{B})$;

$\vec{i}.\vec{i}=\vec{j}.\vec{j}=\vec{k}.\vec{k}=\cos 0°=1$;

$\vec{i}.\vec{j}=\vec{j}.\vec{k}=\vec{k}.\vec{i}=\cos 90°=0$; $\vec{A}.\vec{B}=A_xB_x+A_yB_y+A_zB_z$

$\Rightarrow \vec{A}.\vec{B}=\vec{B}.\vec{A}$ [*Commutative Law*]

$\Rightarrow \vec{A}.(\vec{B}+\vec{C})=\vec{A}.\vec{B}+\vec{A}.\vec{C}$ [*Distributive Law*]

$\Rightarrow$ Cross product: magnitude is $AB\sin\theta$; Direction is perpendicular to plane containing $\vec{A}$ *and* $\vec{B}$

$\Rightarrow \vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$; $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{i}=-\vec{k}$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

- Couple $\Rightarrow$Two non-collinear parallel and equal forces acting in opposite sense
- Principle of transmissibility $\Rightarrow$ The effect of a force upon a body is the same at every point in its line of action
- Varignon's theorem $\Rightarrow$ The net moment of a system of forces about a point is equal to the moment of the resultant about the same point

Tensors

- Tensors $\Rightarrow$ zeroth order tensors, first-order tensors, second-order tensors, and so on. Apart from the zeroth.
- Applications $\Rightarrow$continuum mechanics, relativity, electromagnetism, quantum theory, etc.
- Zeroth Order Tensors $\Rightarrow$ simply another name for a scalar (magnitude only – 1 component)
- First Order Tensors $\Rightarrow$ simply another name for a vector (magnitude and one direction – 3 components)
- Second Order Tensors $\Rightarrow$ also called as Dyad (magnitude and two directions – $3^2 = 9$ components)
- Third Order Tensors $\Rightarrow$ also called as Triad (magnitude and two directions – $3^3 = 81$ components) and so on
- Vector cross product gives the direction perpendicular to the plane. However, tensors (2nd order onwards) cross product gives the direction which is in any orientation.
- A tensor of order n in three-dimensional space has 3^n components

- Tensors can be multiplied by other tensors to form new tensors
- The product of a tensor and a scalar (tensor of rank 0) is commutative
- The pre-multiplication of a given tensor by another tensor produces a different result from post-multiplication; i.e., tensor multiplication in general is not commutative.
- The order of a new tensor formed by the product of two other tensors is the sum of their individual order.
- Second order Tensor Notations ⇒ uppercase bold-face Latin letters

  $u.v$ is a scalar (a zeroth order tensor)
  $u \times v$ is a vector (a first order tensor)
  $u \otimes v$ is a dyad(a second order tensor)

- $\boxed{T_{ij} = e_i T e_j}$ Components of a Tensor

- α, β, γ … 0^{th}-order tensors ("scalars")
  a, b, c. … 1^{st}-order tensors ("vectors")
  A, B, C … 2^{nd}-order tensors ("dyadic")
  A, B, C … 3^{rd}-order tensors ("triadics")
  A, B, C … 4^{th}-order tensors ("tetradics")

- Index notation ⇒ $A_{ijm} B_{mk} = C_{ijk}$
- Matrix notation ⇒ $A_{ijk} B_{jk} = c_i$
- $$Tu = [T][u] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11}u_1 & T_{12}u_2 & T_{13}u_3 \\ T_{21}u_1 & T_{22}u_2 & T_{23}u_3 \\ T_{31}u_1 & T_{32}u_2 & T_{33}u_3 \end{bmatrix}$$

  Symbolic notation "short" matrix notation "full" matrix notation

- Identity Tensor ⇒ The linear transformation which transforms every tensor into itself
- Transpose of a Tensor ⇒second order tensor A with components A_{ij} is the tensor A^T with components A_{ji}.
- Trace of a Tensor (trA) ⇒ scalar equal to the sum of the diagonal elements of its matrix representation
- Norm of a Tensor ⇒ $|A| = \sqrt{A : A}$
- Determinant of a Tensor (detA) ⇒ determinant of the matrix [A] of components of the tensor
- Inverse of a Tensor ⇒ The inverse of a second order tensor A, denoted by A^{-1}
- Orthogonal Tensors ⇒ linear vector transformation satisfying the condition **Qu.Qv = u.v**
- Rotation Tensors ⇒ Proper orthogonal tensors
- Symmetric and Skew Tensors ⇒ A tensor T is said to be symmetric if it is identical to the transposed tensor, $T = T^T$, and skew (antisymmetric) if $T = -T^T$.
- Transpose of a Tensor ⇒second order tensor A with components A_{ij} is the tensor A^T with components A_{ji}

Short Answer Type Questions

1. Explain Varignon's theorem of resultant force.

Answer:

The net moment of a system of forces about a point is equal to the moment of the resultant about the same point. It can also be stated, as "the moment of a vector about a point is same as that of the sum of the moments of its components about same point."

2. State and prove Varignon's theorem.

Answer:

The net moment of a system of forces about a point is equal to the moment of the resultant about the same point. It can also be stated, as "the moment of a vector about a point is same as that of the sum of the moments of its components about same point."

For concurrent forces: Let $\vec{F}_1, \vec{F}_2, \ldots\ldots, \vec{F}_n$ are concurrent forces at a point A.

$\vec{F}_1 + \vec{F}_2 + \ldots\ldots + \vec{F}_n = \vec{R}$ = Resultant force.

Consider moment of forces about O, which is origin of a reference system.

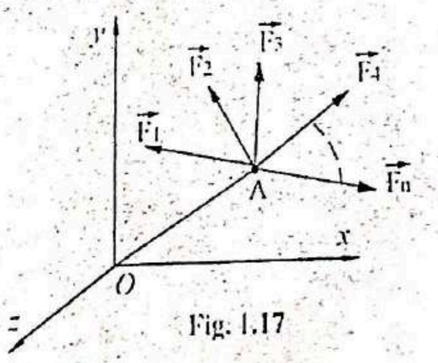
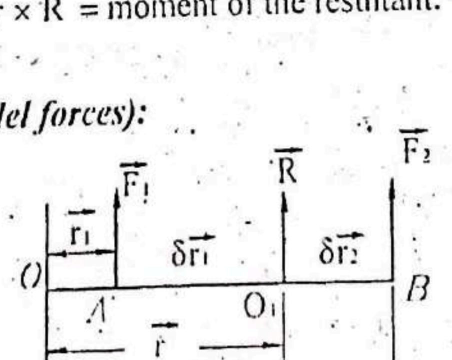


Fig. 1.17

$$\vec{M}_0 = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \ldots\ldots + \vec{r} \times \vec{F}_n$$

$$= \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \ldots\ldots + \vec{F}_n) = \vec{r} \times \vec{R} = \text{moment of the resultant.}$$

For Coplanar forces (Parallel forces):



Let us assume,

$\vec{F}_1$ and $\vec{F}_2$ acting at A and B having a resultant R acting at O_1.

Assume 'O' origin of the reference system.

$$\text{Hence } \vec{M}_0 = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = (\vec{r}_1 - \delta\vec{r}_1) \times \vec{F}_1 + (\vec{r} + \delta\vec{r}_2) \times \vec{F}_2$$

$$= \vec{r}_1 \times (\vec{F}_1 + \vec{F}_2) + (\delta\vec{r}_2 \times \vec{F}_2 - \delta\vec{r}_1 \times \vec{F}_1)$$

For all parallel forces in equilibrium the two moments are equal in magnitude but opposite in sense and hence,

$$(\delta\vec{r}_2 \times \vec{F}_2 = \delta\vec{r}_1 \times \vec{F}_1)$$

$$\vec{M}_0 = \vec{r} \times (\vec{F}_1 + \vec{F}_2) = \vec{r} \times \vec{R} = \text{moment of the resultant about O.}$$

3. State the principle of Transmissibility of forces.

Answer:

The condition of a rigid body whether in equilibrium or in motion remains unchanged if a force acting at a given point on the rigid body is transmitted along its line of action to another point of the rigid body without changing its sense. It is similar to sliding vector.

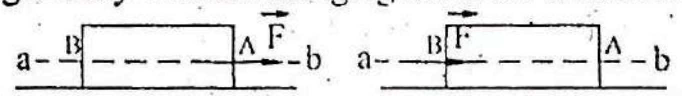


Fig. 1.16

Fig. 1.16(a) shows a body on which a force is applied at point A along the line of action ab. This force can also be applied at any other point such as B Fig. 1.16(b) along the line of action ab and the condition of equilibrium or of motion will remain unchanged.

Long Answer Type Questions

1. A force given by $F = 3i + 2j - 4k$ is applied at the point $P(1, -1, 2)$. Find the moment of the force F about the point $O(2, -1, 3)$.

Answer:

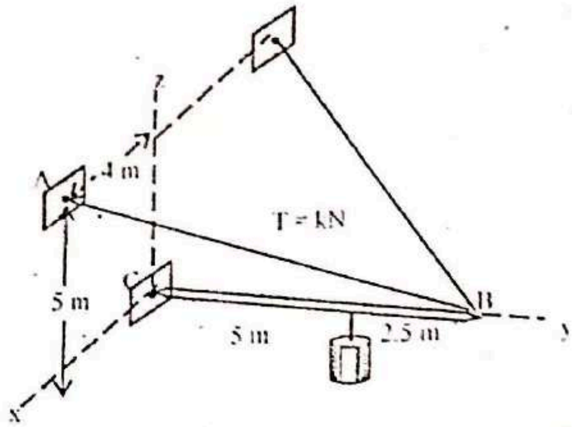
Displacement vector $\overline{OP} = (1-2)\vec{i} + (-1+1)\vec{j} + (2-3)\vec{k} = -\vec{i} - \vec{k}$

$\therefore$ Moment of force $\overline{M} = \overline{r_{op}} \times \overline{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2\vec{i} - 7\vec{j} - 2\vec{k}$$

$\therefore M = \sqrt{2^2 + (-7)^2 + (-2)^2}$ unit $= 7.549$ unit

2. The tension in the supporting cable AB (shown in Fig.) is 10 kN. Write the force which the cable exerts on the boom BC as a vector T. Determine the angle θ_x, θ_y and θ_z which the line of action of T forms with the positive x, y and z – axes.



Answer:

Co-ordinate of the points A(4, 0, 4); B(0, 7.5, 0)

Tension 10 kN is acting on BC at B along $\vec{BA}$.

$\overrightarrow{BA} = 4\hat{i} - 7.5\hat{j} + 5\hat{k}$

$$\therefore \text{Unit vector along } \vec{BA} = \frac{4\hat{i} - 7.5\hat{j} + 5\hat{k}}{\sqrt{4^2 + (-7.5)^2 + 5^2}} = \frac{4\hat{i} - 7.5\hat{j} + 5\hat{k}}{\sqrt{97.25}}$$

$$\therefore \text{Tension as a vector } (\vec{T}) = 10 \times \left(\frac{4\hat{i} - 7.5\hat{j} + 5\hat{k}}{\sqrt{97.25}}\right) kN$$

$$= 4.056\hat{i} - 7.60\hat{j} + 5.07\hat{k}\ kN$$

Now if θ_x is the angle made by the tension vector with positive x axis, then

$\vec{T}\cdot\hat{i} = |\vec{T}|(1)\cos\theta_x$

$$\therefore \cos\theta_x = \frac{(4.056\hat{i} - 7.60\hat{j} + 5.07\hat{k})\cdot\hat{i}}{9.993} = \frac{4.056}{9.993} = 0.4058$$

$\therefore \theta_x = \cos^{-1}(0.4058) = 66.05°$

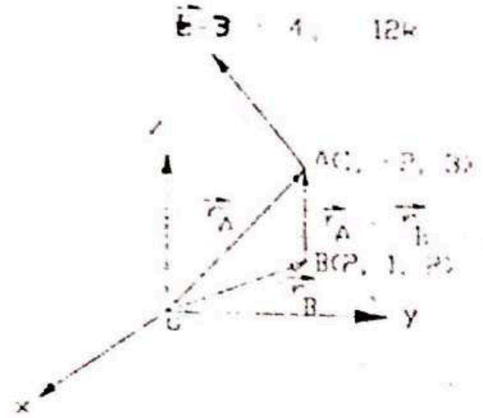
$$\text{Similarly} \quad \theta_y = \cos^{-1}\left(\frac{-7.60}{9.993}\right) = \cos^{-1}\left(-\frac{7.60}{9.993}\right)$$

$$= \cos^{-1}(-0.7605) = 139.5°$$

and $\theta_z = 59.51°$

3. A force $\vec{F} = (3\hat{i} - 4\hat{j} + 12\hat{k})N$ acts at a point A whose coordinates are (1, -2, 3) m. Compute (a) moment of force about origin (b) moment of force about the point B (2, 1, 2) m.

Answer:



Position vector of the point A w.r.t. O as origin is $\vec{r}_A = \vec{i} - 2\vec{j} + 3\vec{k}$

Position vector of the point B w.r.t. O as origin is $\vec{r}_B = 2\vec{i} + \vec{j} + 2\vec{k}$

(a) Moment of force about origin O is

$$\vec{M}_O = \vec{r}_A \times \vec{F} = (\vec{i} - 2\vec{j} + 3\vec{k}) \times (3\vec{i} - 4\vec{j} + 12\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & -4 & 12 \end{vmatrix} = (-12\vec{i} - 3\vec{j} + 2\vec{k})\,Nm$$

(b) Moment of force about origin B is

$$\vec{M}_B = (\vec{r}_A - \vec{r}_B) \times \vec{F} = \{(\vec{i} - 2\vec{j} + 3\vec{k}) - (2\vec{i} + \vec{j} + 2\vec{k})\} \times (3\vec{i} - 4\vec{j} + 12\vec{k})$$

$$= (-\vec{i} - 3\vec{j} + \vec{k}) \times (3\vec{i} - 4\vec{j} + 12\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 1 \\ 3 & -4 & 12 \end{vmatrix} = (-32\vec{i} + 15\vec{j} + 13\vec{k})\,Nm$$

4. **The vector $\vec{A}$ and $\vec{B}$ are given as**

$$\vec{A} = 2i + 3j \text{ and } \vec{B} = 3i - j$$

Determine : (a) the dot product and cross product of the vectors, (b) the angle between vectors $\vec{A}$ and $\vec{B}$ (c) the included angle between vector $\vec{A}$ a nd the vector resulting from the cross product.

Answer :

The dot product is given as

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y$$

$$= 2 \times 3 + 3 \times (-1) = 6 - 3 = 3 \text{ units}$$

The cross product is expressed as

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= [3 \times 0 - (-1) \times 0]\, i - (2 \times 0 - 2 \times 0)\, j + [2 \times (-1) - 3 \times 3]\, k$$

$$= 0 - 0 - 11k = -11k$$

(b) From the definition of dot product : $\vec{A}.\vec{B} = AB\cos\theta$

$$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{3}{\sqrt{2^2+3^2}\times\sqrt{3^2+(-1)^2}}$$

$$= \frac{3}{\sqrt{13}\times\sqrt{10}} = 0.263$$

$$\theta = \cos^{-1}(0.263) = \mathbf{74.75°}$$

Check : From the definition of cross product : $|\vec{A}\times\vec{B}| = AB\sin\theta$

$$\sin\theta = \frac{\vec{A}\times\vec{B}}{AB} = \frac{11}{\sqrt{13}\times\sqrt{10}} = 0.965$$

$$\theta = \sin^{-1}(0.965) = \mathbf{74.79°}$$

(c) $\vec{A}\times(\vec{A}\times\vec{B}) = \begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 0 & 0 & -11 \end{vmatrix} = -33i + 22j$

The above vector has a magnitude of $\sqrt{(-33)^2+(-22)^2} = 39.66$

$$\sin\theta = \frac{|\vec{A}\times(\vec{A}\times\vec{B})|}{|\vec{A}|\times|\vec{A}\times\vec{B}|} = \frac{39.6}{\sqrt{13}\times 11} = 1.00$$

Thus the angle between $\vec{A}$ and $(\vec{A}\times\vec{B})$ is 90°. The included angle between vector $\vec{B}$ and the vector $(\vec{A}\times\vec{B})$ would also workout to be 90°.

Remarks : The included angle between the vector resulting from the cross product and either of the constituent vectors must be 90°.

5. **Force of 100 N is directed along the line that is drawn from the ponts A(5, 2, –1) m to the point B(3, –4, 7) m. Make calculations for the moment of this force about the z-axis.**

Answer :

Positon vector

$$\vec{AB} = (3-5)\,i + (-4-2)\,j + (7+1)\,k$$

$$= (-2i - 6j + 8k)\text{ m}$$

Unit vector along $\vec{AB} = \dfrac{-2i-6j+8k}{\sqrt{(-2)^2+(-6)^2+8^2}}$

$$= \frac{-2i-6j+8k}{\sqrt{104}} = -0.196\,i - 0.294\,j + 0.784\,k$$

Then force vector along $\vec{AB}$ = F × unit vector along $\vec{AB}$

$$= 100\times(-0.196\,i - 0.294\,j + 0.784\,k)$$

$$= -19.6\,i - 29.4\,j + 78.4\,k$$

Moment of the force about the origin O = $\vec{OA}\times\vec{F}$

where $\vec{OA} = 5i + 2j - k$

$$\therefore \quad M_0 = \begin{vmatrix} i & j & k \\ 5 & 2 & -1 \\ -19.6 & -29.4 & 78.4 \end{vmatrix}$$

Moment about z-axis $= (-147 + 29.3)\,k = -107.8$ Nm

6. A force $\vec{F} = 3i - 4j + 12k$ acts at a pont A whose coordinates are (1, –2, 3) m. Compute :

(a) moment of force about origin.

(b) moments of force about the point B (2, 1, 2) m.

(c) vector component of force F along line AB and the moment of this force about the origin.

Answer :

Position vector $\vec{OA}$ = Position of O with respect to A

$$= (x_a - x_0)\, i + (y_n - y_0)\, j + (z_a - z_0)\, k$$
$$= (1-0)\, i + (-2-0)\, j + (3-0)\, k = i - 2j + 3k$$

Moment of force about the origin

$$= \vec{OA} \times \vec{F} = (i - 2j + 3k) \times (3i - 4j + 12k)$$

$$= \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 3 & -4 & 12 \end{vmatrix}$$

$$= [(-2 \times 12) - (-4 \times 3)]\, i - [(1 \times 12) - (3 \times 3)\, j + [(-4 \times 1) - (-2 \times 3)]k$$

(b) Position vector $\vec{BA}$ = Position of B with respect to A

$$= (x_a - x_b)\, i + (y_a - y_b)\, j + (z_a - z_b)\, k$$
$$= (1-2)\, i + (-2-1)\, j + (3-2)\, k = -i - 3j + k$$

Moment of force about point B

$$= \vec{BA} \times \vec{F} = (-i - 3j + k) \times (3i - 4j + 12k)$$

$$= \begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 3 & -4 & 12 \end{vmatrix}$$

$$= [-3 \times 12 - (-3 \times 1)]\, i - [(-1 \times 12) - (3 \times 1)]\, j + [-1 \times (-4) - 3 \times (-3)]\, k$$

$$= -32i + 15j + 13k.$$

(c) Position vector $\vec{AB} = (2-1)\,i + [1-(-2)]\,j + (2-3)\,k = i + 3j - k$

Unit vector along $\vec{AB}$ (along the direction of force)

$$= \frac{i+3j-k}{\sqrt{1^2+3^2+(-1)^2}} = \frac{i+3j-3}{\sqrt{11}} = 0.3\,i + 0.9j - 0.3k$$

Projection of $\vec{F}$ on AB = $\vec{F}$ · unit vector along AB

$$= (3i - 4j + 12k)\cdot(0.3i + 0.9j - 0.3k)$$

$$= 3 \times 0.3 - 4 \times 0.9 + 12 \times (0.3) = \mathbf{-6.3\ m}$$

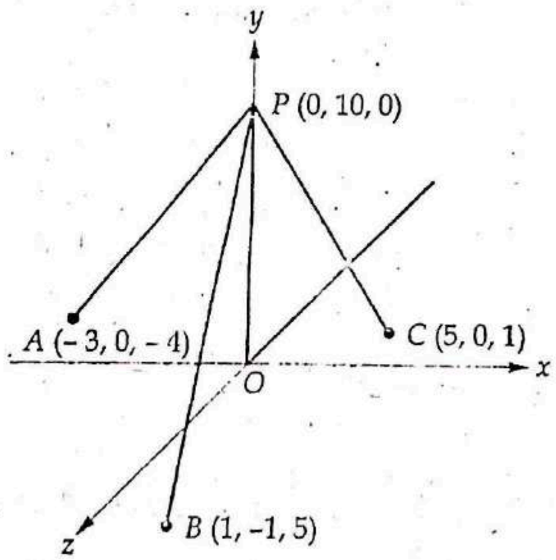
7. A vertical pole is guyed by three cables PA, PB and PC tied at a common point P, 10 m above the ground. The base points of the cable are

A(–3, 0, –4), B (1, –1, 5) and C (5, 0, –1)

If the tensile forces in the cable are adjusted to be 15, 18 and 20 kN, make calculations for the resultant force on the pole at P.

Answer :

Refer Fig. for the arrangement of the three cables and the pole.



The pole P is 10 m above the ground at O and as such the forces in the cables would be directed along PA, PB and PC.

Distance $\quad PA = \sqrt{(x_a - x_p)^2 + (y_a - y_b)^2 + (z_a - z_p)^2}$

$$= \sqrt{(-3-0)^2 + (0-10)^2 + (-4-0)^2}$$

$$= \sqrt{125} = 11\cdot18$$

$$PB = \sqrt{(1-0)^2+(-1-10)^2+(5-0)^2}$$

$$= \sqrt{147} = 12.12$$

$$PC = \sqrt{(5-0)^2+(0-10)^2+(-1-0)^2}$$

$$= \sqrt{126} = 11.22$$

Unit vectors along these directions are :

$$n_1 = \frac{1}{11.18}(-3i-10j-4k)$$

$$n_2 = \frac{11}{12.12}(i-11j+5k)$$

$$n_3 = \frac{1}{11.22}(5j-10j-1k)$$

Force vector $\vec{F} = |\vec{F}| \times \vec{n}$

= magnitude of force × unit vector in the direction of force

$$\vec{F_1} = 15 \times \frac{(3i-10j-4k)}{11.18} = -4.02j - 13.4j - 5.36k$$

$$\vec{F_2} = 18 \times \frac{(i-11j+5k)}{12.12} = 1.48\,i - 16.34j + 7.42k$$

$$\vec{F_3} = 20 \times \frac{(5i-10j+k)}{11.22} = 8.91i\,/0\;17.8j - 1.78k$$

Resultant $\vec{R} = \vec{F_1}+\vec{F_2}+\vec{F_3} = 6.37i - 47.54\,j + 0.28\,k$

Magnitude of resultant force = $\sqrt{(6.37)^2+(-47.54)^2+(0.28)^2} = 47.96$ kN

The resultant force on the pole is 47.96 kN in magnitude and it acts predominantly downwards to hold the pole in position.

8. A vertical tower AB shown in Fig. is subjected to a horizontal force F = 50 kN at its t and it is anchored by two guy wires AC and AD. Compute the tensions in the guy wi and thrust in the pole.

Assume following coordinates for the points :

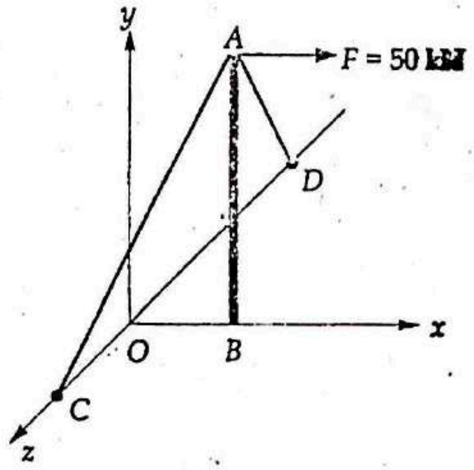
O (0, 0, 0); A (3, 20, 0);

B (3, 0, 0); C (0, 0, 4)

and D (0, 0, –4).

Answer :

The cvector expression for tension in a guy wire is = magnitude of tension × unit vector along the direction of that wire



Position vector $\vec{AC} = (0-3)\,i + (0-20)\,j + (4-0)\,k$

$$= -3i - 20j + 4k$$

Unit vector along $\vec{AC} = \dfrac{-3i-20j+4k}{\sqrt{(-3)^2+(-20)^2+4^2}}$

$$= \frac{-3i-20j+4k}{20.61}$$

$\therefore$ $\vec{T}_{ac} = T_{ac} \times \dfrac{-3i-20j+4k}{20.61}$

$$= T_{ac}\,[-0.146i - 0.97j + 0.194k]$$

Similarly $\vec{T}_{ad} = T_{ad} \times \dfrac{-3i-20j-4k}{\sqrt{(-3)^2+(-20)^2+(-4)^2}}$

$$= T_{ad}\,[-0.146\,i + 0.097\,j - 0.195\,k]$$

$$\vec{T}_{ab} = T_{ab} \times \frac{0i-20j-0k}{\sqrt{0^2+(-20)^2+(0)^2}} = T_{ab}\,(-j)$$

The force F = 50 kN can be expressed as a force vector

$$\vec{F} = 50i$$

Resultant of all forces meeting at pont A

$$= \vec{T}_{ac} + \vec{T}_{ad} + \vec{T}_{ab} + \vec{F}$$

$$= (-0.146\,T_{ac} - 0.146\,T_{ad} + 50)i + (-0.97\,T_{ac} - T_{ab})\,j + (0.194\,T_{ac} - 0\,0.194\,T_{ad})$$

Applying the equilibrium conditions to point A : $\vec{R} = 0$

Then equating the coefficients of *i*, *j* and *k* to zero, we get

$$-0.146\,T_{ac} - 0.146\,T_{ad} + 50 = 0 \quad \text{....(i)}$$

$$-0.97\,T_{ac} - 0.97\,T_{ad} - T_{ab} = 0 \quad \text{...(ii)}$$

$$0.194\,T_{ac} - 0.194\,T_{ad} = 0;\ T_{ac} = T_{ad} \quad \text{...(iii)}$$

From identities (i) and (iii) we obtain

$$-0.146\,T_{ac} - 0.194\,T_{ac} + 50 = 0$$

$$\therefore \quad T_{ac} = \frac{50}{0.146+0.146} = 171.23 \text{ N}$$

$$T_{ad} = T_{ac} = 171.23 \text{ N}$$

Substituting these values of T_{ad} and T_{ac} in identify (ii), we get

$$-0.97 \times 171.23 - 0.97 \times 171.23 - T_{ab} = 0$$

$$\therefore \quad T_{ab} = -2 \times 0.97 \times 171.23 = -332.18 \text{ N}$$

The negative value for T_{ad} indicates that the tower would experience thrust rather than tension.

9. In the system shown in Fig. a 5 m long beam is held in vertical position AO by three guy wires AB, AC and AD. If a tension equivalent to 600 N is induced in AD and the resultant force at A is to be vertical, compute the magnitude of forces in AC and AB.

Answer :

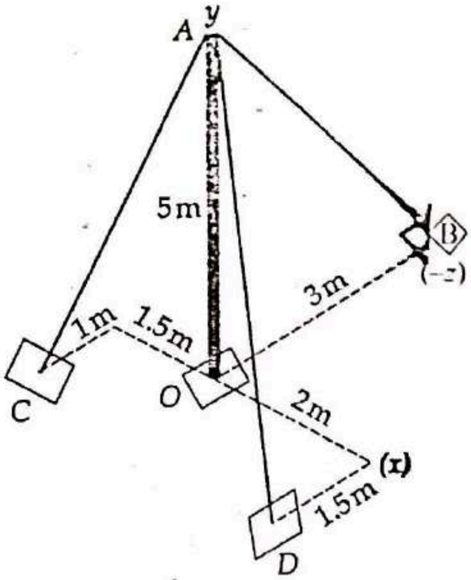
Taking *y* and *z* axis along A and B respectively, the coordinates of various points with respect to O as origin are :

A(0, 5, 0)

B(0, 0, –3);

C(–1.5, 0, 1)

and D(2, 0, 1.5)



The vector expression for tension induced in guy wire is = magnitude of tension × unit vector along the direction of that wire

Position vector $\vec{AD} = (2-0)\,i + (0-5)\,j + (1.5-0)\,k = 2i - 5j + 1.5k$

Unit vector along $\vec{AD} = \dfrac{2i-5j+1.5k}{\sqrt{1^2+(-5)^2+1.5^2}}$

$$= \frac{2i-5j+1.5k}{5.59}$$

$$\therefore \quad \vec{T}_{ad} = T_{ad} \times \frac{2i-5j+1.5k}{5.59}$$

$$= 600 \times \frac{2j-5j+1.5k}{5.59}$$

$$= 214.67i - 536.67j + 161k$$

Similarly, $\vec{T}_{ad} = T_{ab} \times \dfrac{(-5j-3k)}{\sqrt{(-5)^2+(-3)^2}}$

$$= T_{ab} \times \frac{(-5j-3k)}{5.83}$$

$$= -T_{ab}\,(0.857j + 0.514k)$$

$$\vec{T}_{ac} = T_{ac} \times \frac{1.5i-5j+k}{\sqrt{(-1.5)^2+(-5)^2+1^2}} = T_{ac} \times \frac{(-1.5i-5j+k)}{5.31}$$

$$= -T_{ac}\,(0.282\,i + 0.942j - 0.188\,k)$$

Resultant of all force meeting at point A

$$= \vec{T}_{ad} + \vec{T}_{ab} + \vec{T}_{ac}$$

$$= (214.67 - 0.282\,T_{ac})\,i - (536.67 + 0.857\,T_{ab} + 0.942\,T_{ac})\,j + (161 - 0.514\,T_{ab} + 0.188\,T_{ac})\,k$$

As resultant force at A is to be vertical

$$\Sigma F_x = 0 \text{ and } \Sigma F_z = 0$$

That gives : $214.67 - 0.282\,T_{ac} = 0$

∴ Tension in cable AC,

$$T_{ac} = \frac{214.67}{0.282} = 761.24\text{ N}.$$

Also $161 - 0.514\,T_{ab} + 0.188\,T_{ac} = 0$

∴ Tension in cable AB,

$$T_{ab} = \frac{161 + 0.188 \times 761.24}{0.514} = 591.66\text{ N}$$

10. A tripod supports a load of 2.5 kN at point P as shown in Fig. The end points A, B and C of the three legs lie in the *x-z* plane. Make calculations for the force developed in each leg.

Answer :

With point O as origin, the coordinates of the end points are :

A(1.5, 0, 0); B (0, 0, 1.5) m and C (−1.25, 0, −1).

The coordinates of the point P which supports the load are P(0, 2, 0).

Let T_1, T_2 and T_3 be the tension in the legs AP, BP and CP respectively.

The vector expression for tension induced in a leg is

= magnitude of tension × unit vector along the direction of that leg

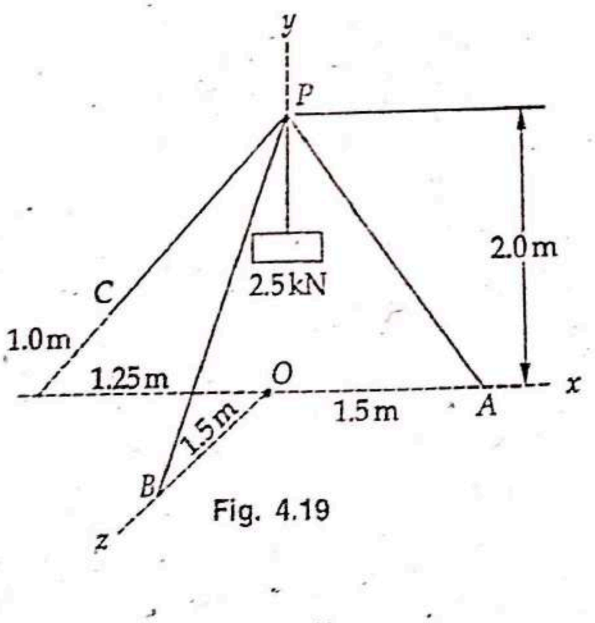


Fig. 4.19

$$\vec{AP} = (0-1.5)\,i + (2-0)\,j + (0-0)\,k = -1.5i + 2j$$

$$\therefore \qquad \vec{T_1} = T_1 \times \frac{-1.5i + 2j}{\sqrt{(-1.5)^2 + (2)^2}} = T_1\,(-0.6i + 0.8j)$$

Similarly :

$$\vec{T_2} = T_2 \times \frac{2j - 1.5k}{\sqrt{(2)^2 + (-1.5)^2}} = T_2\,(0.8j - 0.6k)$$

$$\vec{T_3} = T_3 \times \frac{1.25i + 2j - k}{\sqrt{(1.25)^2 + 2^2 + 1^2}} = T_3\,(0.49\,i + 0.78\,j + 0.39\,k)$$

Also : $\vec{W} = -2.5j$

Resultant of all forces meeting at point P

$$= \vec{T_1} + \vec{T_2} + \vec{T_3} + \vec{W}$$
$$= (-0.6T_1 + 0.49T_3)\,i + (0.8T_1 + 0.8T_2 - 2.5)\,j + (-0.6\,T_2 + 0.39T_3)\,k$$

Applying the equilibrium conditions to point P

$$\vec{R} = \Sigma\,\vec{F} = 0$$

Then equating the coefficients of i, j and k to zero, we get

$$-0.6\,T_1 + 0.49T_3 = 0 \qquad \text{.... (i)}$$
$$0.8T_1 + 0.8T_2 + 0.78T_3 - 2.5 = 0 \qquad \text{... (ii)}$$
$$-0.6T_2 + 0.39T_3 = 0 \qquad \text{.... (iii)}$$

From identities (i) and (iii) :

$$T_1 = \frac{0.49}{0.6}\,T_3 = 0.816T_3$$

and

$$T_2 = \frac{0.39}{0.6}\,T_3 = 0.65\,T_3$$

Substituting these values for T_1 and T_2 in identity (ii), we get

$$0.8 \times 0.816\,T_3 + 0.8 \times 0.65\,T_3 + 0.78T_3 = 2.5$$

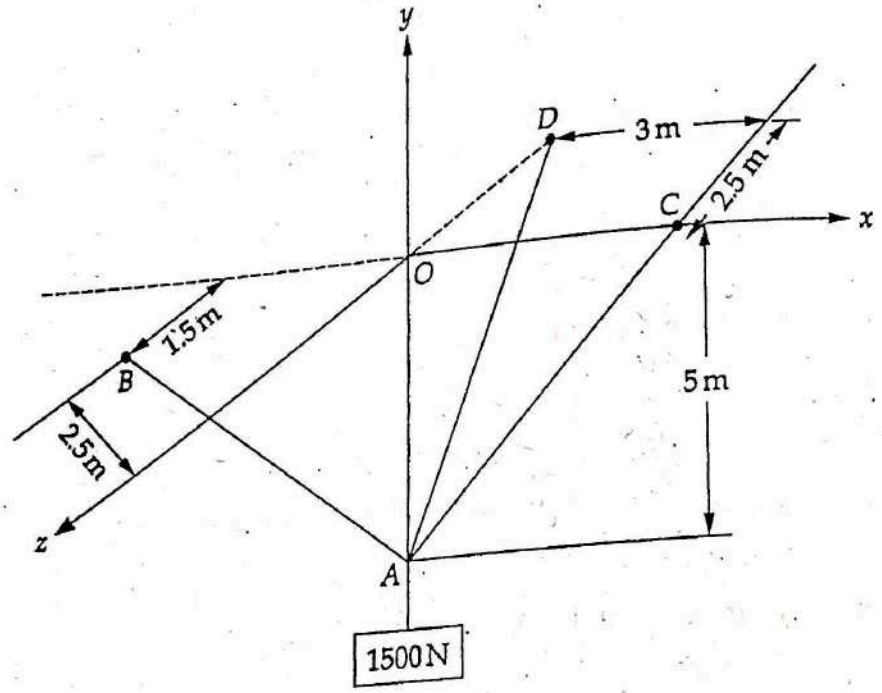
or, $(0.653 + 0.52 + 0.78)\,T_3 = 2.5$

$\therefore$ Force developed in leg CP : $T_3 = \frac{2.5}{1.953} = \mathbf{1.28\ kN}$

Force developed in leg BP : $T_2 = 0.65 \times 1.28 = \mathbf{0.832\ kN}$

$\therefore$ Force developed in leg AP : $T_1 = 0.816 \times 1.28 = \mathbf{1.04\ kN}$

11. A load of 1500 N is supported at point A by three cables AB, AC and AD as shown in Fig. Make calculations for the tensions induced in each cable.



Answer :

Let T_1, T_2 and T_3 bew the tensions induced in the cable AB, AC and AD respectively.

The co-ordinates of various points with O as origin are

A(0, –5, 0); B(–2.5, 0, 1.5); C(3, 0, 0) and D(0, 0, –2.5)

The vector expression for tension induced in a cable is

= magnitude of tension × unit vector along the direction of that cable

$$\vec{AB} = (-2.5 - 0)\,i + [0 - (-5)]\,j + (1.5 - 0)\,k$$

$$= -2.5i + 5j + 1.5k$$

$$\therefore \quad \vec{T_1} = T_1 \times \frac{-2.5i + 5j + 1.5k}{\sqrt{(-2.5)^2 + (5)^2 + (1.5)^2}}$$

$$= (-0.432i + 0.864\,j + 0.259\,k)\,T_1$$

Similarly :

$$\vec{T_2} = T_2 \times \frac{3i+5j}{\sqrt{3^2+5^2}} = (0.514\, i + 0.857\, j)\, T_2$$

$$\vec{T_3} = T_3 \times \frac{5i-2.5k}{\sqrt{5^2+(2.5)^2}} = (.894\, j - 0.447\, k)\, T_3$$

Also : $\vec{W} = -1500j$

Resultant of all the forces meeting at pont A

$= \vec{T_1} + \vec{T_2} + \vec{T_3} + \vec{W}$

$= (-0.432\, T_1 - 0.514\, T_2)\, i + (0.864\, T_2 + 0.894\, T_3 - 1500)\, j + (0.259\, T_1 + 0.447\, T_3)\, k$

Since the state of equilibrium exists, $\vec{R} = 0$. That is

$-0.432\, T_1 + 0.514\, T_2 = 0;\ T_2 = 0.84\, T_1$ ... (i)

$-0.864\, T_1 + 0.857\, T_2 + 0.894\, T_3 - 1500 = 0$ ... (ii)

$-0.259\, T_1 - 0.447\, T_3 = 0;\ T_3 = 0.579\, T_1$ .... (iii)

Substituting the values of T_2 and T_3 in expression (ii) we get

$0.864T_1 + 0.857 \times 0.84T_1 + 0.894 \times 0.579T_1 = 1500$

or, $(0.864 + 0.72 + 0.517)\, T_1 = 1500$

$\therefore$ Tension in cable AB, $T_1 = 713.94$ N (Tension)

Tension in cable AC, $T_2 = 0.84T_1 = 599.71$ N (Tension)

Tension in cable AD, $T_3 = 0.579T_1 = 413.37$ N (Tension

12. A force $\vec{F}$ with a magnitude of 100 N is applied at the origin O of the axes x, y, z as shown in Fig. The line of action of $\vec{F}$ passes through a point A whose co-ordinates are 3 m, 4 m and 5 m. Determine :

(a) the x, y, z scalare compo-nents of $\vec{F}$

(b) the projection of $\vec{F}$ on the x-y plane, and

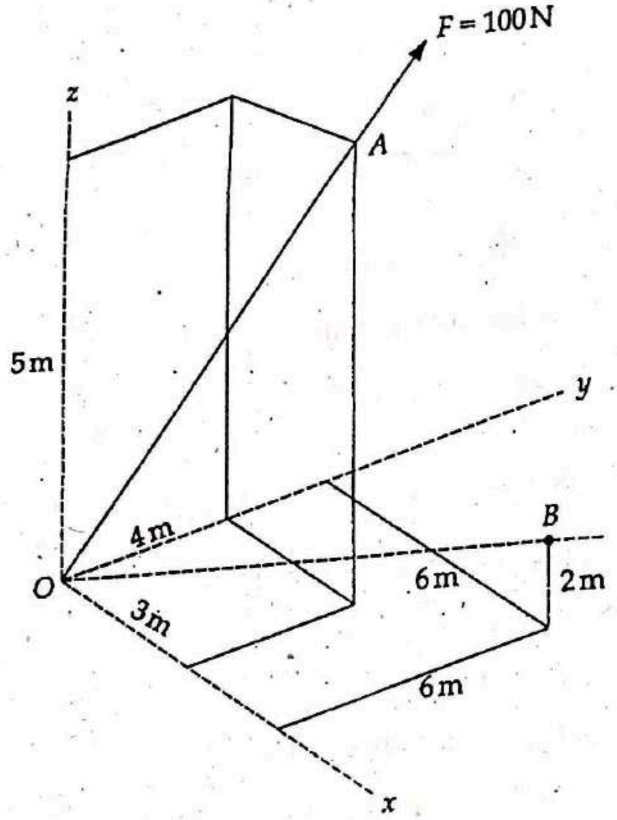
(c) the projection of F along of the line OB

Answer :

Coordinates of point A

$= (3, -4, 5)$

Vector $\vec{OA} = (x_2 - x_1) + (y_2 - y_1)\,j + (z_2 - z_1)\,k$



$= 3i + 4j + 5k$

Unit vector along $\vec{OA} = \dfrac{3i + 4j + 5k}{\sqrt{3^2 + 4^2 + 5^2}}$

$= 0.424i + 0.566\,j + 0.707\,k$

Force vector $\vec{F}$ = magnitude of force × unit vector along the direction of force

$= 100\,(0.424\,i + 0.566\,j + 0.707\,k)$

$= 42.4i + 56.6j + 70.7k$

The scale components of force vector $\vec{F}$ are

$F_x = 42.4$ N; $F_y = 56.6$ N and $F_z = 70.7$ N

(b) Projection of $\vec{F}$ on x - y plane

Projection $= 3i + 4j$

Unit vector $= \dfrac{3i + 4j}{\sqrt{3^2 + 4^2}} = 0.6i + 0.8j$

Then $F_{xy} = \vec{F} \times$ unit vector along projecion on x-y plane

$$= (42.4i + 56.6j + 70.7k) \cdot (0.6i + 0.8j)$$

$$= 42.4 \times 0.6 = 56.6 \times 0.8 = \mathbf{70.72\ N}$$

Alternatively : The cosine of angle θ between $\vec{F}$ and x - y plane is given by

$$\cos\theta = \frac{\sqrt{3^2+4^2}}{\sqrt{3^2+4^2+5^2}} = 0.707;\ \theta = 45^\circ$$

$$F_{xy} = F\cos\theta = 100 \times \cos 45^\circ = \mathbf{70.7\ N}$$

(c) Vector $\vec{OB} = 6i + 6j + 2k$

Unit vector along $\vec{OB} = \frac{6i+6j+2k}{\sqrt{6^2+6^2+2^2}} = 0.688\ i + 0.688\ j + 0.229\ k$

Then $F_{OB} = \vec{F} \times$ unit vector OB

$$= (42.4i + 56.6j + 70.7k) \cdot (0.688i + 0.688j + 0.229k)$$

$$= 42.4 \times 0.688 + 56.6 \times 0.688 + 70.7 \times 0.229$$

$$= 29.17 + 38.94 + 16.19 = \mathbf{84.3\ N}$$

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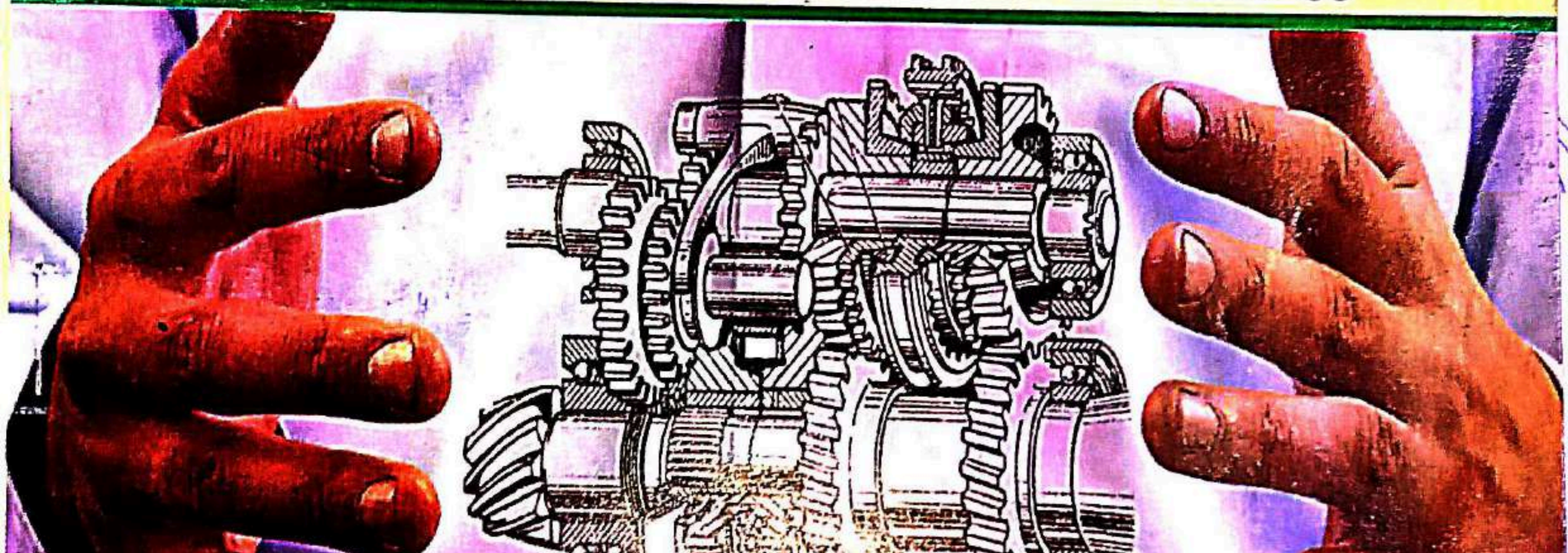
ELECTRICAL BRANCH

For 3rd Semester

ENGINEERING MECHANICS

Strictly according to the
NEW SYLLABUS



ORGANIZER

UNIT 2

THREE-DIMENSIONAL ROTATION IN SPACE

Chapter at a glance

- Euler's rotation theorem ⇒ Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O
- *Euler angles* ⇒ angles of rotation of a three-dimensional coordinate frame. A rotation Euler angles is represented as a matrix of trigonometric functions of the angles.
- Rodrigues' formula ⇒ $\mathbf{R} = \mathbf{I}\cos\theta - \mathbf{u}\mathbf{u}^T(1 - \cos\theta) - \mathbf{u}\sin\theta$
- Chasles' Theorem ⇒ The most general rigid body displacement can be produced by translation along a line followed (or preceded) by a rotation about that line.

UNIT 3

MOMENT OF INERTIA

Chapter at a glance

- Moment of Inertia of the area A about the x-axis is $I_x = \int y^2 dA = \iint y^2 dx\, dy$
- Moment of Inertia of the area A about the y-axis is $I_y = \int x^2 dA = \iint x^2 dx\, dy$
- *Radius of gyration* ⇒ The area may be assumed to be concentrated at a point such that the magnitude of the second moment remains unaltered. The distance of the said point from the axis concerned is called the radius of gyration of the area.
- *Perpendicular Axis theorem* ⇒ The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point O is equal to the sum of moments about any two mutually perpendicular axis through the same point.

  $J_o = \int r^2 dA$ $\qquad J_o = I_x + I_y$
- *Theorem of Parallel Axes for Moment of Inertia* ⇒ The moment of inertia of an area about an axis is equal to the moment of inertia of the area about a parallel axis passing through the centroid together with the area multiplied by the square of the distance of the centroid from the axis.

  $\therefore I_{xx} = I_{Gx} + b^2 A$
- *Mass moment of Inertia* ⇒ $I = \sum_{i=1}^{n} r_i^2 \Delta m_i$ becomes $\int r^2 dm$ ie. $I = \int r^2 dm$.
- The moment of inertia of a circular cylinder of radius R, length L and weight density ρ about its axis, $\therefore I = MK^2 = \frac{W}{g}\left(\frac{R}{\sqrt{2}}\right)^2$
- Moment of inertia of a sphere of homogeneous material of radius about any diameter Hence, $I_x = I_y = I_z = I = \frac{2}{5}\frac{WR^2}{g}$ and $K = \sqrt{\frac{I}{W/g}} = \sqrt{\frac{2}{5}\frac{WR^2}{g}\cdot\frac{1}{W/g}} = \sqrt{\frac{2}{5}}R$
- M. I. of a thin circular ring of radius r and mass M about an axis perpendicular to plane of ring is Mr^2
- M. I. of a rectangular area of base 'b' and height 'd' about x axis is given by $bd^3/3$
- M. I. Of circular area whose diameter is 'd' about an axis perpendicular to the area passing through its centre is given by $\pi d^4/32$
- M.I of a hollow circular section about a central axis perpendicular to the section is double its M.I. about horizontal axis.
- M. I. of a triangle of base 'a' and height 'h' about the base is given by $ah^3/12$
- Moment of inertia of a quadrant about its X-X axis is given by $0.055r^4$

Multiple Choice Type Questions

1. M. I. of a rectangular area of base 'b' and height 'd' about x axis is given by

a) $bd^3/3$ b) $bd^3/4$ c) $bd^3/6$ d) $bd^3/12$ e) $bd^3/8$

Answer: (a)

2. Mass moment of inertia of body is
a) moments of its inertia
b) rotational analogue of mass
c) inertial moment about the centroidal axis
d) none of these

Answer: (b)

3. Moment of inertia of a triangle of base *b* and height *h* about the centroidal axis parallel to base is

a) $bh^3\frac{1}{36}$ b) $\frac{bh^3}{12}$ c) $bh\frac{3}{3}$ d) none of these

Answer: (a)

4. Moment of inertia of a semicircle of radius R about its centroidal axis $x-x$ is

a) $0.22R^4$ b) $0.055R^4$ c) $0.11R^4$ d) none of these

Answer: (a) & (c)

5. Moment of inertia of a circle with its centroidal *x*-axis is

a) $\pi d^4/32$ b) $\pi d^4/256$ c) $\pi d^4/64$ d) $\pi d^4/128$

Answer: (c)

Short Answer Type Questions

1. State the Parallel axes theorem of Moment of Inertia (MI) of lamina.

Answer:

The theorem states that the moment of inertia of an area about an axis is equal to the moment of inertia of the area about a parallel axis passing through the centroid together with the area multiplied by the square of the distance of the centroid from the axis.

2. a) State parallel axis and perpendicular axis theorem for moment of inertia.
b) Define radius of gyration. How is it related to mass moment of inertia?

Answer:

a) *Theorem of Parallel Axes for Moment of Inertia of lamina*

The theorem states that the moment of inertia of a lamina about an axis is equal to moment of inertia of the lamina about a parallel axis passing through the centr together with the area multiplied by the square of the distance of the centroid from the axis.

Perpendicular axes theorem (or plane figure theorem) can be used to determine the moment of inertia of a rigid object which are laminar in nature and therefore may be assumed to lie entirely within a plane, about an axis perpendicular to that plane, given the moments of inertia of the object about two perpendicular axes lying within the plane. The three axes must all pass through a single point in the plane.
Define perpendicular axes x, y, and z (which meet at origin O) so that the body lies in the xy plane, and the z axis is perpendicular to the plane of the body. Let I_x, I_y and I_z be moments of inertia about axis x, y, z respectively, the perpendicular axis theorem states that $I_z = I_x + I_y$

b) In case of moment of inertia the area may be assumed to be concentrated at a point such that the magnitude of the second moment remains unaltered. The distance of the said point from the axis concerned is called the radius of gyration of the area and is denoted frequently by the letter K.

But, $I_x = AK_x^2 = \int y^2 dA$ or, $K_x = \sqrt{\frac{I_x}{A}}$ ---(1)

Similarly, $K_y = \sqrt{\frac{I_y}{A}}$ and $K_0 = \sqrt{\frac{J_0}{A}}$ ......(2)

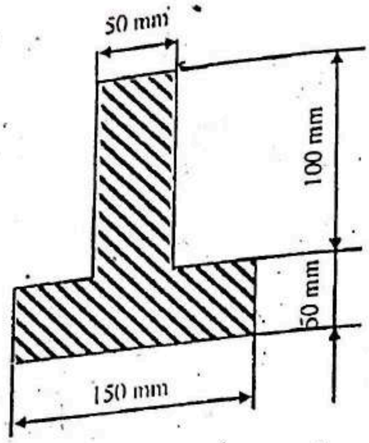
If the whole mass M of the body is assumed to be concentrated at a point at a distance K from the axis of reference, such that the moment of inertia of the body remains the same, then this distance K is called the radius of gyration of the body about the said axis of reference.

According to the new concept $I = MK^2$, where M is the mass of the body concentrated at a distance K from the axis.

$\therefore$ Radius of gyration, $K = \sqrt{\frac{I}{M}}$ ....(3)

Long Answer Type Questions

1. Determine MI of the inverted T section as shown in the figure below about its centroidal axis parallel to the base.



Answer:

Let X-X is the centroidal axis parallel to the base.

$$\therefore \bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{150 \times 50 \times 25 + 100 \times 50 \times 100}{150 \times 50 + 100 \times 50} mm = 55mm$$

For the (1) Area, moment of inertia about its centroidal axis, parallel to base.

$$= \frac{1}{12} bh^3 = \frac{1}{12} \times 150 \times 50^3 mm^4 = 1562500 mm^4$$

Applying parallel axis theorem, moment of Inertia of Area (1) about X-X axis,

$$I_{x_1} = 1562500 + 150 \times 50 \times 30^2 mm^4 = 8312500 mm^4$$

For the (2) Area, moment of inertia about its centroidal axis, parallel to base,

$$= \frac{1}{12} \times 50 \times 100^3 = 4166666.67 mm^4$$

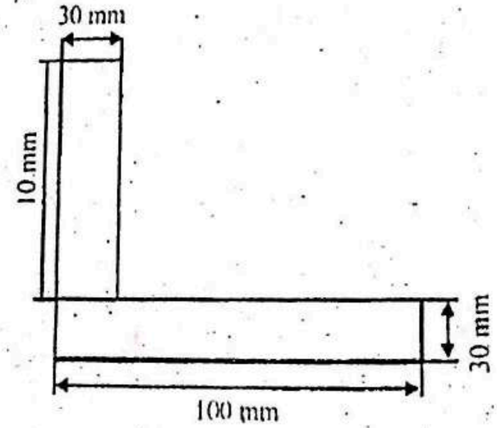
Applying parallel axis theorem, moment of inertia of Area (2) about X-X axis,

$$I_{x_2} = 4166666.67 + 50 \times 100 \times 45^2 mm^4 = 14291666.67 mm^4$$

Therefore, moment of inertia of the inverted T section about X-X axis,

$$I_x = I_{x_1} + I_{x_2} = (8312500 + 14291666.67) mm^4 = 22.6 \times 10^6 mm^4$$

2. Find out the moment of inertia about centroidal axes of an area as shown i



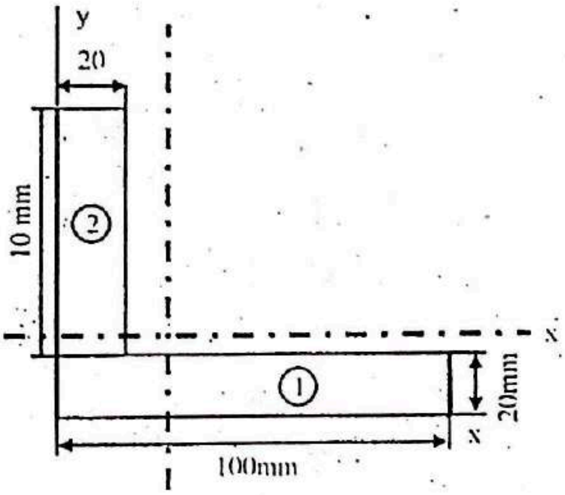
Answer:

From composite figure principle, we can write, centroidal distance from y axis

$$\bar{x} = \frac{A_1\bar{x}_1 + A_2\bar{x}_2}{A_1 + A_2} = \frac{50\times(100\times 20) + 10\times(60\times 20)}{100\times 20 + 60\times 20}\text{mm} = 35\text{mm}$$

and centroidal distance from x axis

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = \frac{10\times(100\times 20) + (20+30)\times(60\times 20)}{100\times 20 + 60\times 20}\text{mm} = 25\text{mm}$$



Moment of inertia w.r.t. centroidal X axis

$$I_X = I_{cix1} + Ah_1^2 + I_{cix2} + Ah_2^2$$

$$= \left[\frac{1}{12}\times 100\times 20^3 + (100\times 20)(5+10)^2\right] + \left[\frac{1}{12}\times 20\times 60^3 + (20\times 60)(30-5)^2\right]\text{mm}^4$$

$$= 1626666.67\text{mm}^4$$

Moment of inertia w.r.t. centroidal Y axis

$$I_Y = I_{ciy1} + Ad_1^2 + I_{ciy2} + Ad_2^2$$

$$= \left[\frac{1}{12}\times 20\times 100^3 + (100\times 20)(50-35)^2\right] + \left[\frac{1}{12}\times 60\times 20^3 + (60\times 20)(30-10)^2\right]\text{mm}^4$$

$$= 2906666.67\text{mm}^4$$

3. **Find the moment of inertia of a rolled steel joist girder of symmetrical I section shown in Fig.**

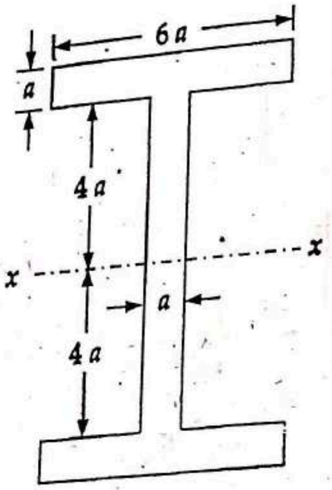
Answer :

The areas of the three reactangles comprising the I-section are :

$$\text{upper flange } A_1 = 6a \times a = 6a^2$$

$$\text{web } A_2 = 8a \times a = 8a^2$$

$$\text{lower flange } A_3 = 6a \times a = 6a^2$$



MOI of upper flange about x-axis (using parallel axis theorem)

$$= \frac{6a \times a^3}{12} + 6a^2 \times \left(4a + \frac{a}{2}\right)^2$$

$$= \frac{a^4}{2} + \frac{243a^4}{2} = 122a^4$$

$$\text{MOI of web about } x\text{-axis} = \frac{a \times (8a)^3}{12} = \frac{128a^4}{3}$$

MOI of lower flange about x-axis (using parallel axis theorem)

$$= \frac{6a \times a^3}{12} + 4a^2\left(4a + \frac{a}{2}\right)^2$$

$$= \frac{a^4}{2} + \frac{243a^4}{2} = 122a^4$$

Moment of inertia of the I-section about centroidal axis is

= MOI of area a_1 about centroidal axis + MOI of area a_2 about centroidal axis + MOI of area a_3 about centroidal axis.

$$= (I_{G1} + A_1 h_1^2) + (I_{G2} + A_2 h_2^2) + (I_{G3} + A_3 h_3^2)$$

$$= \left[\frac{10\times1^3}{12} + 10\times(5.07)^2\right] + \left[\frac{1\times12^3}{12} + 12\times(1.43)^2\right] + \left[\frac{8\times1^3}{12} + 8\times(7.93)^2\right]$$

$$= (0.833 + 257.05) + (144 + 24.54) + (0.67 + 503.08) = 930.17 \text{ cm}^4$$

and $I_{yy} = \frac{1\times10^3}{12} + \frac{12\times1^3}{12} + \frac{1\times8^3}{12} = 83.33 + 1 + 42.64 = 127 \text{ cm}^4$

Polar moment of inertia $= I_{xx} + I_{yy} = 930.17 + 127 =$ **1057.17 cm⁴**

(b) The radius of gyration is given by :

$$k = \sqrt{\frac{I}{A}}$$

$$\therefore \quad k_{xx} = \sqrt{\frac{930.17}{30}} = \mathbf{5.567\ cm}$$

$$k_{yyy} = \sqrt{\frac{127}{30}} = \mathbf{2.057\ cm}$$

Determine the moment of inertia about centroidal axes *x-x* and *y-y* of the channel section shown in Fig.

Soln. : The section is divided into three rectangles with areas

$$A_1 = 10 \times 1.5 = 15 \text{ cm}^2$$

$$A_2 = (40 - 1.5 - 1.5) \times 1 = 37 \text{ cm}^2$$

$$A_3 = 10 \times 1.5 = 15 \text{ cm}^2$$

$$\Sigma A = A_1 + A_2 + A_3$$

$$= 15 + 37 + 15 = 67 \text{ cm}^2$$

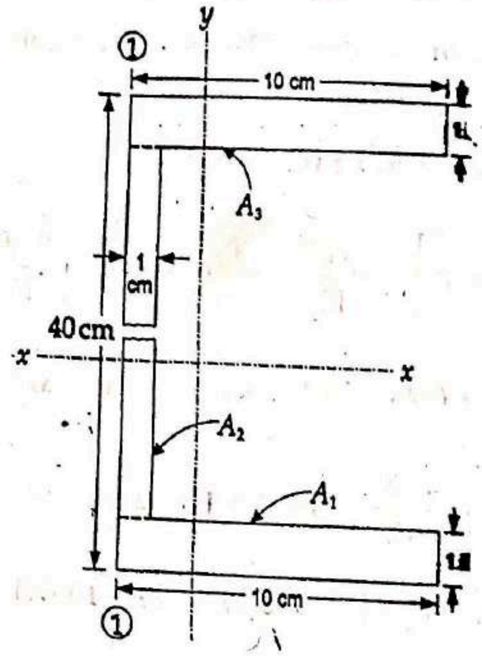


Fig. 8.20

The given section is symmetrical about the horizontal axis passing through the centroid rectangle A_2.

The distance of the centroid of the section with reference to section 1 – 1 is

$$\frac{\Sigma Ax}{\Sigma A} = \frac{(15\times5)+(37\times1/2)+(15\times5)}{67} = 2.51 \text{ cm}$$

With reference to the centroidal axes x-x and y-y, the centroids of the rectangles are :

[(5 – 2.51), (40/2 – 1.5/2)]

or, (2.49, 19.25) for rectangle A_1

[(2.51 – 1/2), 0.0] or, (2.01, 0.0) for rectangle A_2

[(5 – 2.51), (40/2 – 1.5/2] or, (2.49, 19.25) for rectangle A_3

Then invoking parallel axis theorem, the moment of inertia of areas A_1, A_2 and A_3 about

$$I_{xx} = \left[\frac{10\times1.5^3}{12}+15\times19.25^2\right]+\left[\frac{1\times37^3}{12}\right]+\left[\frac{10\times1.5^3}{12}+15\times19.25^2\right]$$

$$= (2.812 + 5558.437) + (42221.083) + (2.812 + 558.437) = \mathbf{155343.58\ cm^4}$$

$$\text{Similarly, } I_{yy} = \left[\frac{1.5\times10^3}{12}+15\times2.49^2\right]+\left[\frac{37\times1^3}{12}\right]+\left[\frac{1.5\times10^3}{12}+15\times2.49^2\right]$$

8. Determine I_{xx} and I_{yy} of the cross-section of a cast iron beam shown in Fig.

Answer :

The MOI of the given sections can be worked out by looking it as a rectangle minus two semi-circles.

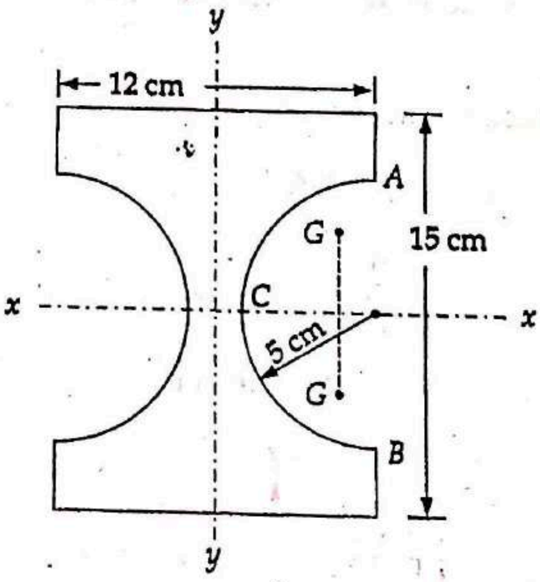
$$\therefore \quad I_{xx} = I_{xx} \text{ of rectangle} - I_{xx} \text{ of circular part}$$

$$= \frac{bd^3}{12} - \frac{\pi r^4}{4}$$

$$= \frac{12\times 15^3}{12} - \frac{\pi \times 50^3}{4}$$

$$= 33.75 - 490.87$$

$$= \mathbf{2884.13\ cm^4}$$



Likewise : $I_{yy}' = I_{yy}$ of rectnagle – I_{yy} of semi-circular parts

$$I_{yy} \text{ of rectangle} = \frac{15\times 12^3}{12} = 2160\ \text{cm}^4$$

For the semi-circular part ACB;

MOI about its diameter, $I_{AB} = \frac{1}{2}\times\frac{\pi\times 5^4}{4} = 245.43\ \text{cm}^4$

Distance of its CG from the diameter,

$$h = \frac{4r}{3\pi} = \frac{4\times 5}{3\pi} = 2.12\ \text{cm}$$

$$\text{Area A} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 5^2 = 39.27 \text{ cm}^2$$

From the correlation, $I_{AB} = I_{GG} + Ah^2$, the moment of inertia of semi-circular part about ... centroidal axis

$$I_{GG} = 245.43 - 39.27 \times (2.12)^2 = 68.94 \text{ cm}^4$$

Again from the parallel axis theorem,

$$I_{yy} = I_{GG} + Ah_1^{\,2}$$

where h_1 = distance between axis and G-axis, $= 6 - 2.12 = 3.88$ cm

$$\therefore \quad I_{yy} = 68.94 + 39.27 \times 3.88^2 = 660.13 \text{ cm}^4$$

Since there are two semi-circular parts,

I_{yy} for two semi-circular parts $= 2 \times 660.13 = 1320.26 \text{ cm}^4$

$\therefore$ I_{yy} for the section $= 2160 - 1320.26 =$ **839.74 cm⁴**

9. Determine the moments of inertia about the x and y centroidal axis of a beam wh... cross sectional area is as shown in Fig. All dimensions are in cm.

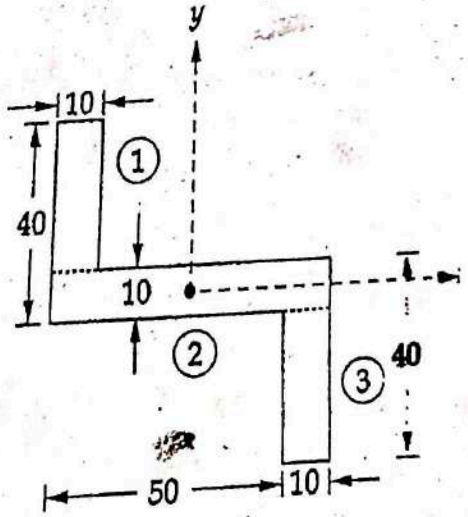
Answer :

The given section has been divided into three segments marked 1, 2 and 3

$$(I_{xx})_1 = I_{G_1} + A_1 h_1^{\,2} = I_{G_1} + A_1\,(\bar{y} - y_1)^2$$

$$= \frac{1}{12} \times 10 \times 30^3 + (30 \times 10)\,(35 - 15)^2$$

$$= 1.425 \times 10^5 \text{ cm}^4$$

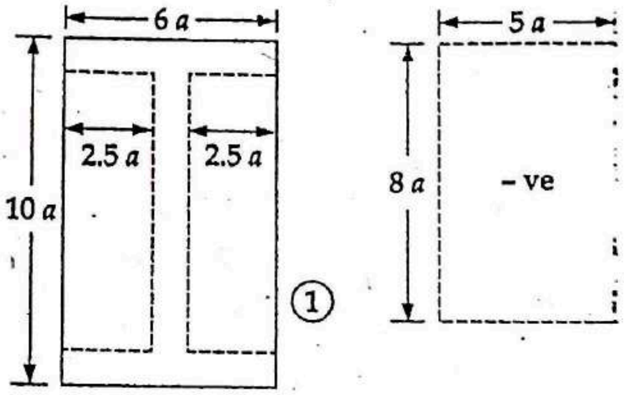


$$(I_{xx})_2 = I_{G_2} + A_2 h_2^{\,2} = I_{G_2} + A_2\,(\bar{y} - y_2)^2$$

$$= \frac{1}{12} \times 10 \times 30^3 + (30 \times 10)\,(35 - 15)^2 = 1.425 \times 10^5 \text{ c...}$$

∴ Total MOI of the given I-section about x-axis

$$= 122a^4 + \frac{128a^4}{3} + 122a^4 = \frac{860}{3}a^4$$



The MOI of the given I-section could also be worked out with reference to Fig.

$$I_{xx} = I_{x1} - I_{x2}$$

$$= \frac{6a\times(10a)^3}{12} - \frac{5a\times(8a)^3}{12}$$

$$= 500a^4 - \frac{640}{3}a^4 = \frac{860}{3}a^4$$

4. Determine the moment of inertia of the T-section shown in Fig. about an axis passing through the centroid and parallel to top most fibre of the section. Proceed to determine the moment of inertia about axis of symmetry and hence find out the radii of gyration.

Answer :

From the calculations made in Example the CG of the given T-section lies on the y-axis and at distance 43.71 mm from the top face of its flange

$\bar{x} = 0$ and $\bar{y} = 43.71$ mm.

Refering to this centroidal axis, the centroid of a_1 is (0.0, 38.71 mm) and that of a_2 is (0.0, 41.29 mm).

Moment of inertia of the section about centroid axis is

I_{xx} = MOI of area a_1 about centroidal axis

+ MOI of area a_2 about centroidal axi

$$= \left[\frac{160\times10^3}{12}+1600\times(38.71)^2\right]+\left[\frac{10\times150^3}{12}+1500\times(41.29)^2\right]$$

$$= 7780672 \text{ mm}^4$$

Similarly, $$I_{yy} = \frac{10\times160^3}{12}+\frac{150\times10^3}{12} = 342833 \text{ mm}^4$$

The radius of gyration is given by $k = \sqrt{\frac{I}{A}}$

$$\therefore \quad k_{xx} = \sqrt{\frac{7780672}{3100}} = 50.1 \text{ mm}$$

$$k_{yy} = \sqrt{\frac{3425833}{3100}} = 34.24 \text{ mm}$$

5. Determine the moment of inertia of the area shown sbhaded in Fig. about axis xx which coincides with the base edge AB.

Answer :

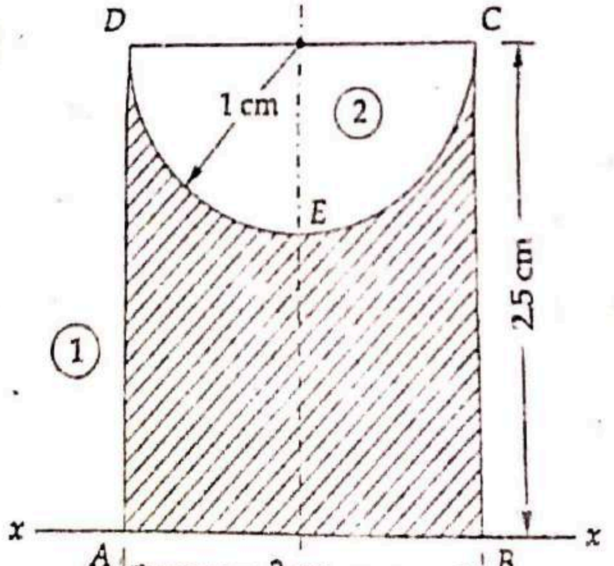
The given section comprises the full rectangle ABCD minus the semi-circle DEC.

Moment of cnertia of rectangle ABCD about AB

$$I_1 = I_{G1} + A_1 h_1^2$$

$$= \frac{2\times2.5^3}{12} + (2\times2.5)\times1.25^2$$

$$= 2.604 + 7.812 = 10.416 \text{ cm}^4$$



Moment of inertia of semi-circle about AB

$$I_2 = I_{G2} + A_2 h_2^2$$

$$= 0.11\, r^2 + \frac{1}{2}\pi r^2 \times \left(2.5 - \frac{4r}{4\pi}\right)^2$$

The parameter $\frac{4r}{3\pi}$ is the distance of centroid of semi-circle from DC.

$$\therefore \quad I_2 = 0.11 \times 1^2 + \frac{1}{2}\pi \times (1)^2 \times \left(2.5 - \frac{4\times 1}{3\pi}\right)^2$$

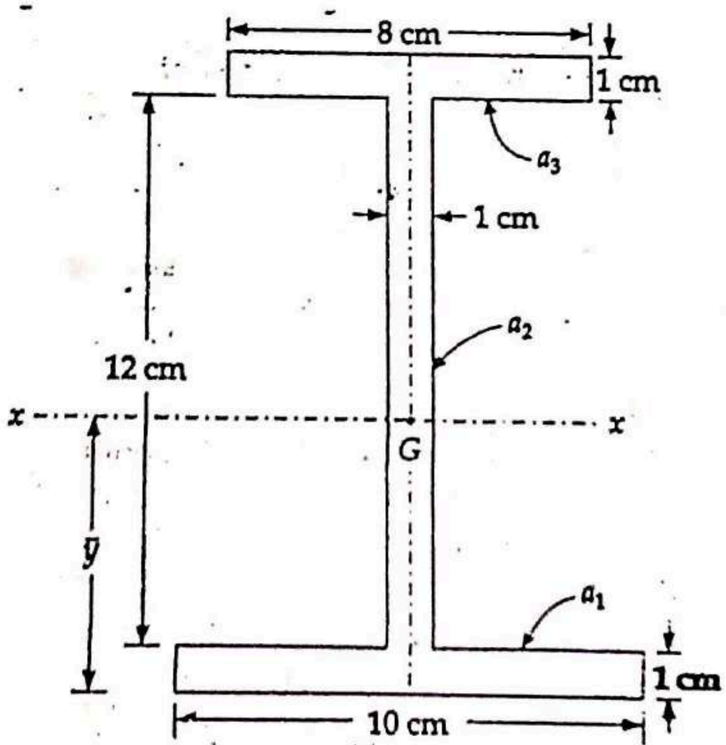
$$= 0.11 + 6.76 = 6.87 \text{ cm}^4$$

$\therefore$ Moment of inertia of shaded area about AB = $10.416 - 6.87 = 3.546$ cm^4

6. Determine the polar moment of inertia of the I-section shown in Fig. Also make calculations for the radius of gyration with respect to *x*-axis and *y*-axis.

Answer :

The I-section is symmetrical about y-axis and accordingly its CG lies at point G on the y-axis, i.e., $x = 0$. Further, the bottom fibre of lower flange has been chosen as reference axis to locate the centroid $\bar{y}$.



(a)

The areas and co-ordinates of centroids of the three rectangles comprising the given section are :

Lower flange :

$$a_1 = 10 \times 1 = 10 \text{ cm}^2$$

$$y_1 = \frac{1}{2} = 0.5 \text{ cm}$$

Web :

$$a_2 = 12 \times 1 = 12 \text{ cm}^2$$

$$y_2 = 1 + \frac{12}{2} = 7 \text{ cm}$$

Upper flange :

$$a_3 = 8 \times 1.8 \text{ cm}^2$$

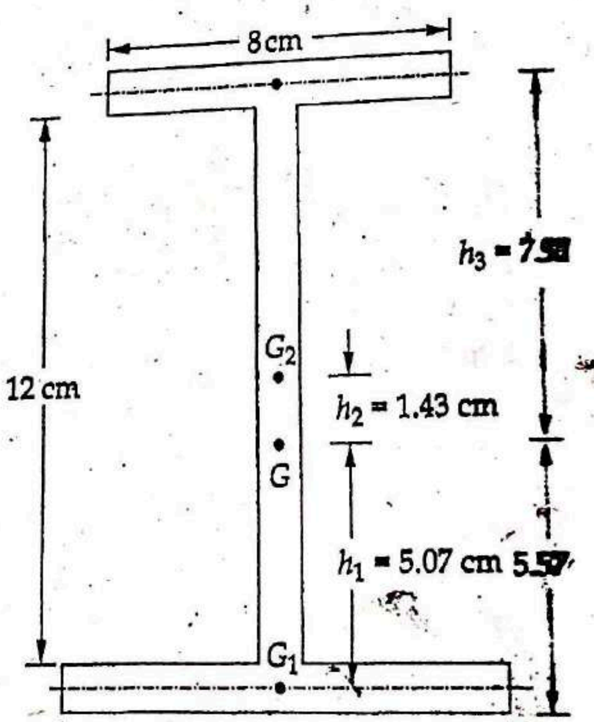
$$y_3 = 1 + 12 + \frac{1}{2} = 13.5 \text{ cm}$$

Then :

$$\overline{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{10 \times 0.5 + 12 \times 7 + 8 \times 13.5}{10 + 12 + 8}$$

$$= \frac{5 + 84 + 108}{30} = 5.57 \text{ cm}$$



With reference to the centroidal axes, the centroid of the lower flange, web and upper fla are (0, 5.07), (0, 1.43) and (0, 7.93) respectively.

$\therefore \quad I_{xx} = 1.425 \times 10^5 + 0.05 \times 10^5 + 1.425 \times 10^5 = \mathbf{2.90 \times 10^5\ cm^4}$

$$(I_{yy})_1 = I_{G_1} + A_1 h_1^2 = I_{G_1} = A_1\ (\bar{x} - x_1)^2$$

$$= \frac{1}{12} \times 30 \times 10^3 + (30 \times 10) \times (30 - 5)^2 = 1.9 \times 10^5\ cm^4$$

$$(I_{yy})_2 = I_{G_2} + A_2 h_2^2 = I_{G_2} + A_2\ (\bar{x} - x_2)^2$$

$$= \frac{1}{12} \times 60^3 \times 10 + (60 \times 10) \times 0 = 1.8 \times 10^5\ cm^4$$

$$(I_{yy})_3 = I_{G_3} + A_3 h_3^2 = I_{G_3} + A_3\ (\bar{x} - x_3)^2$$

$$= \frac{1}{12} \times 30 \times 10^3 + (30 \times 10) \times (30 - 5)^2$$

$$= 1.9 \times 10^5\ cm^4$$

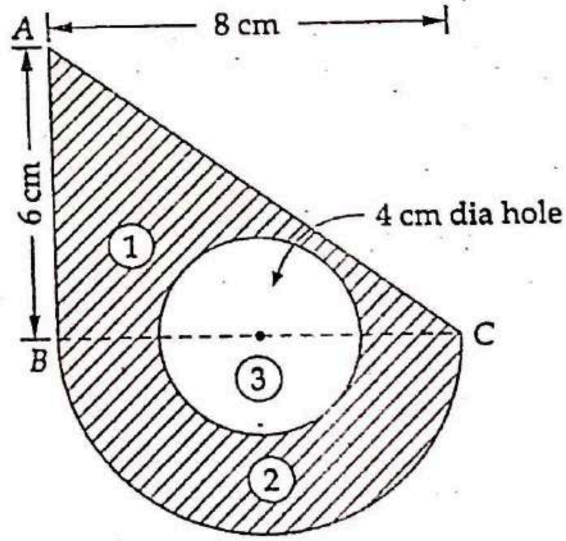
$\therefore \quad I_{yy} = 1.9 \times 10^5 + 1.8 \times 10^5 + 1.9 \times 10^5 = \mathbf{5.6 \times 10^5\ cm^4}$

10. Find the moment of inertia about the centroi horizontal axis of the area shown shaded in fig. The section consists of triangle ABC, semi-circle on BC as diameter, and a circular hole of diameter 4 cm with its central on BC.

Soln. : The shaded area can be considered as a triangle (1), semicircle (2) and a circular hole (3)

Location of Centroid : For the triangular element,

$$a_1 = \frac{1}{2} \times 6 \times 8 = 24\ cm^2$$



y_1 (distance of centroid from BC) $= \dfrac{6}{3} = 2$ cm

For the semi-circular element,

$$a_2 = \frac{1}{2}\pi r^2$$

$$= \frac{1}{2}\pi \times 4^2 = 25.12 \text{ cm}^2$$

$$y_2 \text{ (distance of centroid from BC)} = \frac{-4r}{3\pi} = \frac{-4\times 4}{3\pi} = -1.7 \text{ cm}$$

The negative sign stems from the fact that it lies below BC.

For the circular hole

$$a_3 = \pi r^2 = \pi \times 2^2 = 12.56 \text{ cm}^2 \text{ (this area is removed)}$$

$$y_3 = 0 \text{ (centroid lies on BC)}$$

$\therefore$ Distance of the centroid of the shaded area from BC

$$= \frac{\Sigma ay}{\Sigma a} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3} = \frac{24\times 2 + 25.12\times(-1.7) - 12.56\times 0}{24 + 25.12 - 12.56} = 0.145 \text{ cm}$$

Moment of Inertia

I_1 = moment of inertia of triangle ABC about base BC

$$= \frac{1}{12}bh^3 = \frac{1}{12}\times 8\times 6^3 = 144 \text{ cm}^4$$

I_2 = moment of inertia of semi-circle about BC

$$= \frac{1}{128}\pi d^4 = \frac{1}{128}\times\pi\times 8^4 = 100.48 \text{ cm}^4$$

I_3 = moment of inertia of circular hole about BC

$$= \frac{\pi}{64}d^4 = \frac{\pi}{64}\times 4^4 = 12.56 \text{ cm}^2$$

$\therefore$ Moment of inertia of the shaded area about BC

$= 144 + 100.48 - 12.56 = 231.92 \text{ cm}^4$

Area of the shaded portion $= 24 + 25.12 - 12.56 = 36.56$ cm

Invoking parallel axis theorem,

Moment of inertia of shaded area about centroidal axis

$$I_G = I_{BC} - Ah^2$$

$$= 231.92 - 36.56 \times 0.145^2$$

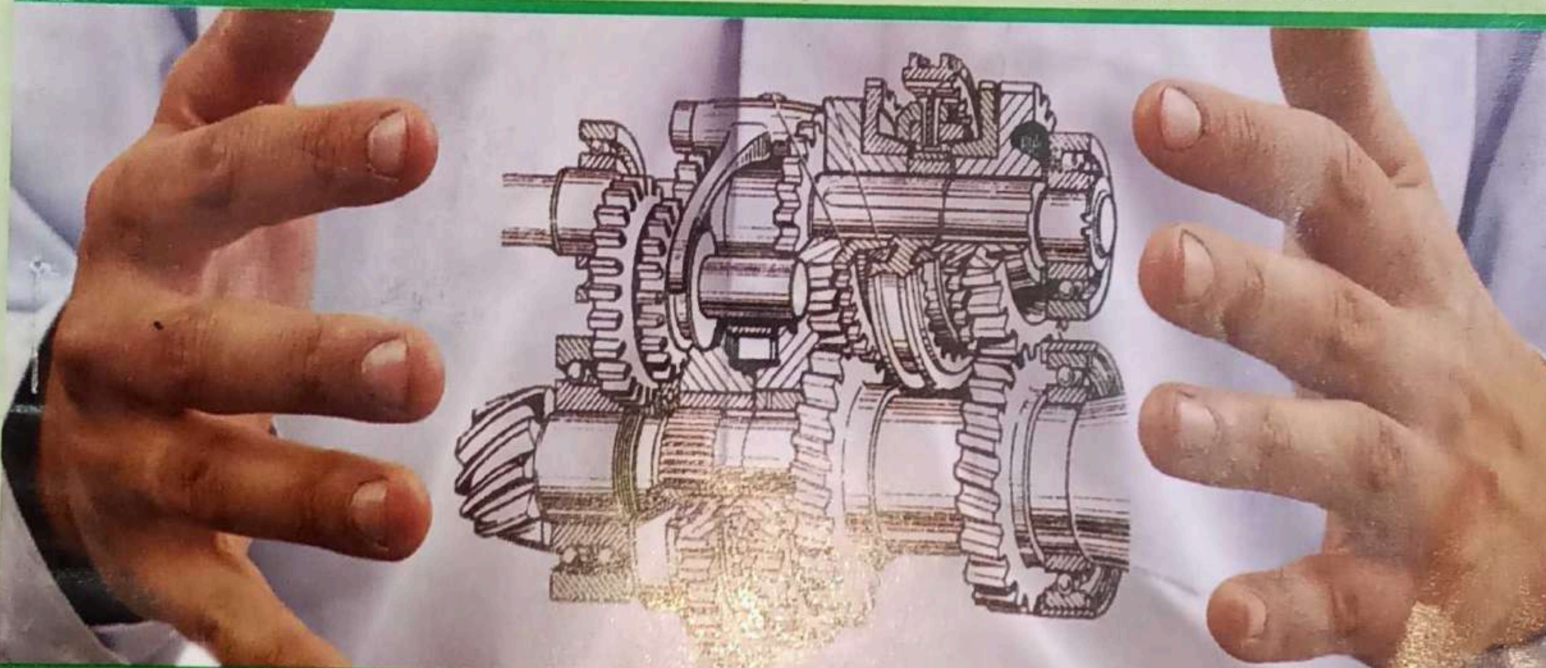
$$= \mathbf{231.15 \text{ cm}^4}$$

ELECTRICAL BRANCH

For 3rd Semester

Engineering Mechanics

Strictly according to the
NEW SYLLABUS



ORGANIZER

UNIT 4

INTRODUCTION TO KINETICS OF RIGID BODIES

Chapter at a glance

- *Translation* ⇒ Any straight line in the body always retains an orientation parallel to its original direction during such a motion.
- *Rectilinear Translation* ⇒ The path of motion of points is parallel straight lines.
- *Curvilinear Translation* ⇒ Points move parallel curve lines.
- *Rotation* ⇒ The motion of a rigid body, in which all the particles move in concentric circular path. The common centre of circles may be located in the body or outside the body.
- *Important Relations* ⇒ When motion with uniform angular velocity.
  (i) Angular velocity = ω
  (ii) Linear or tangential velocity = v = ωr
  (iii) Linear or tangential acceleration = a = 0
  (iv) Angular acceleration = α = 0
  (v) Radial acceleration = $\omega^2 r$
- *Important Relations* ⇒ When motion with uniform angular acceleration.
  (i) $\omega = \omega_0 + \alpha t$, (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, (iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$,
- *Inertia Force* ⇒ When an effective force acts on a particle and produces acceleration, there is a force of equal magnitude in same direction but oppositely sensed, which is known as Inertia force.
- *Dynamic Equilibrium* ⇒ If the inertia force is considered together with the effective force, the net force acting on the particle will be zero. Hence the particle will be in equilibrium, known as dynamic equilibrium.
- *D'Alembert's Principle* ⇒ Converted a problem of dynamics into an equivalent problem of static equilibrium, by invoking an inertia force opposite to the direction of acceleration. This is known as ***D'Alembert's principle.***
- *General plane motion* ⇒ A rigid body undergoes translation and rotation simultaneously.
- When path of motion of different points are not parallel to each other but they are moving parallel planes, we call it **general plane motion.**
- *Equations for General plane motion:*
  $F_x = m\ddot{X}; F_y = m\ddot{Y}; F_z = m\ddot{Z}; M = I\alpha \}$

Multiple Choice Type Questions

1. D'.Alembert's principle is applied to solve problems related to
a) statics b) stress of a structure c) dynamics d) none of these

Answer: (c)

2. The D' Alembert's principle
   a) is based upon the presence of inertia force
   b) provides advantage over Newton's law
   c) is purely a hypothetical principle
   d) allows a dynamic problems to be treated as a static one

Answer: (d)

3. D'Alembert's principle
   a) is based upon the presence of inertia force
   b) provides advantage over Newton's law
   c) is purely a hypotietical law
   d) allowa a dynamic problem to be treated as a static one.

Answer: (d)

Short Answer Type Questions

1. What is D' Alembert's principle? What is the advantage of using the principle? How does it differ from Newton's second law of motion?

Answer:

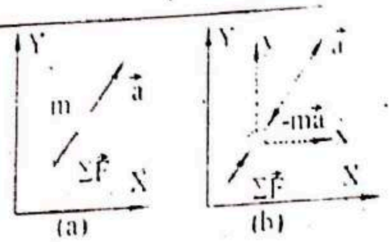


Fig: 1

When a particle is observed from a fixed set of axes X-Y-Z, its absolute acceleration $\vec{a}$ i measured and the familiar relation $\sum\vec{F} = m\vec{a}$ is applied.

When the particle is observed from a moving system x-y-z to which it is attached at th origin, the particle necessarily appears to be at rest or in equilibrium in x-y-z. Thus th observer who is accelerating with x-y-z concludes that a force $-m\vec{a}$ acts on the particle t balance $\sum\vec{F}$.

This point of view, which permits the treatment of a dynamics problem by the method o statics, was an outgrowth of the work of D'Alembert contained in his 'Traite d Dynamique' published in 1743.

This approach merely amounts to rewriting the equation of motion as $\sum\vec{F} - m\vec{a} = 0$, whic assumes the form of zero force summation if $-m\vec{a}$ is treated as a force. This fictitio force is known as the ***inertia force or D'Alembert's force or reversed effective force.*** an the artificial state of equilibrium created is known as ***dynamic equilibrium.***

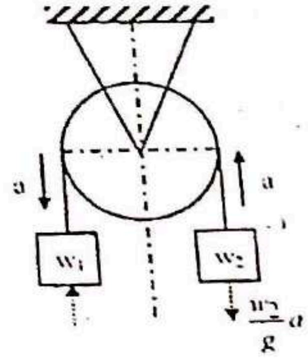
Thus we have converted a problem of dynamics into an equivalent problem of sta equilibrium, by invoking an inertia force opposite to the direction of acceleration. This known as ***D'Alembert's principle.***

Analysis of the motion of two bodies connected by a string

Two bodies of weights W_1 and W_2 are connected to the two ends of a light inextensible string passing over a smooth pulley as shown in Figure 2.

Consider $W_1 > W_2$

Let a = acceleration of the system. Now net external force acting on the system in the direction of motion = $(W_1 - W_2)$.



, Fig: 2

Inertia force or reversed effective force on weight $W_1 = -\frac{W_1}{g}a$ and

Inertia force or reversed effective force on weight $W_2 = -\frac{W_2}{g}a$.

$\therefore$ Resultant reversed effective force or inertia force $= -\left(\frac{W_1}{g}a + \frac{W_2}{g}a\right) = -\frac{a}{g}(W_1 + W_2)$.

But according to D'Alembert's Principle the net external force acting on the system and the resultant reversed effective force should be in equilibrium.

$\therefore$ Net external force + resultant reversed effective force = 0

or. $(W_1 + W_2) - \frac{a}{g}(W_1 + W_2) = 0$ $\qquad \therefore \quad a = \frac{g(W_1 - W_2)}{(W_1 + W_2)}$

2. Two blocks of weights *P* and *Q* are connected by a flexible but inextensible cord and supported as shown in Fig. C. If the co-efficient of friction between the block *P* and the horizontal surface is μ (mu) and all other friction is negligible, find (a) the acceleration of the system and (b) the tensile force S in the cord. The following numerical data are given *P* = 53.4N; Q = 26.7N, $\mu = \frac{1}{3}$.

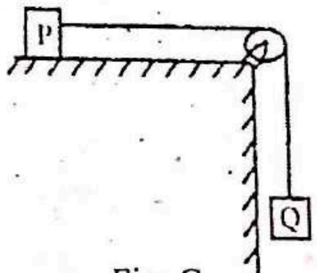
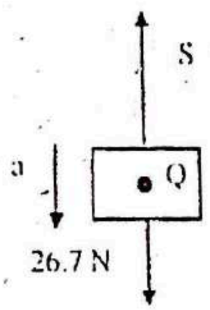


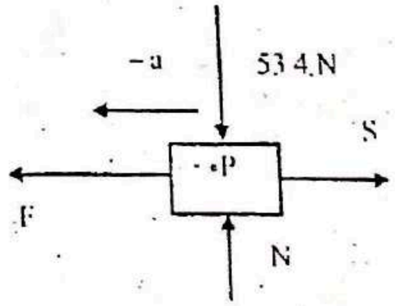
Fig: C

Answer:

The free body diagrams of the two bodies are shown below.



FBD of Q



FBD of P

From the FBD of Q

$$\sum F_y = m_Q a \text{ or, } W - S = m_Q a \quad \text{or, } 26.7 - S = \frac{26.7}{9.81} \times a$$

$$\Rightarrow S = 26.7 - 2.7217 \times a \quad \text{---(i)}$$

From the FBD of P

$$\sum F_x = m_P a \quad \text{or, } S - F = m_P a \quad \text{or, } S - \mu N = \frac{53.4}{9.81} \times a$$

$$\text{or, } S - \frac{1}{3} \times 53.4 = 5.4434a \quad \text{or, } S = 17.8 + 5.4434a \quad \text{...(ii)}$$

From equation (i) & (ii). we get

$$26.7 - 2.7217 \times a = 17.8 + 5.4434a$$

$$a = 1.09$$

$$\therefore a = 1.09\ m/s^2 \text{ and } T = 26.7 - 2.7217 \times 1.09 = 23.733$$

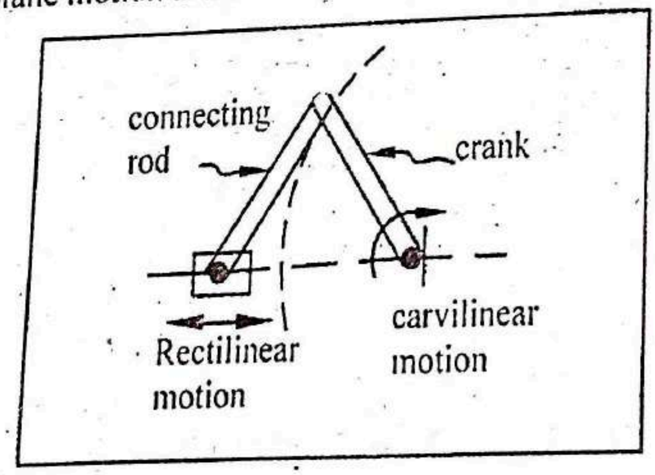
$$\therefore T = 23.733\ N$$

3. Derive the expressions of the Equation of motion for General plane motion.

Answer:

When path of motion of different points are not parallel to each other but they are moving parallel planes, we call it **general plane motion.**

In fact the general plane motion is a combination of rectilinear and curvilinear translation.



Let C be the centre of gravity of a body that moves parallel to the XY plane under the action of applied forces and let A be any particle of mass dm at the distance r from an axis through C normal to the plane of motion.

Point A may now be regarded as having two motions:

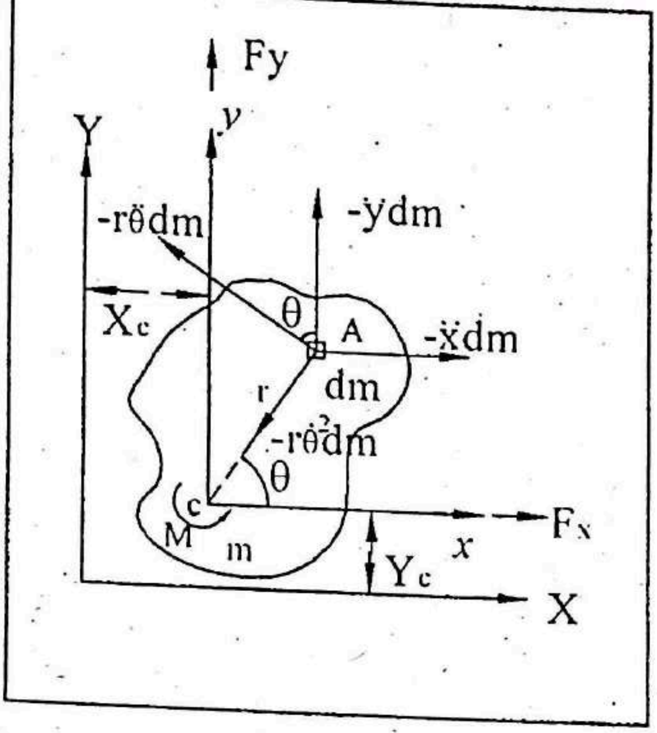
(1) Translation together with the pole C and

(2) Rotation about the axis through C normal to the plane of motion.

As a result of translation, the particle A has two components of inertia force $-\ddot{X}$ dm and $-\ddot{Y}$ dm

As a result of rotation about the normal axis through C, it has tangential and normal components of inertia force.

$-r\ddot{\theta}dm$ and $-r\dot{\theta}^2dm$.



$$\left[\begin{aligned} \text{Note: } a_t &= \dot{v} = \frac{d}{dt}(v) = \frac{d}{dt}(r\omega) = \frac{d}{dt}(r\dot{\theta}) = r\ddot{\theta} \\ a_n &= \frac{v^2}{\rho} = \frac{v^2}{r} = \frac{(r\dot{\theta})^2}{r} = r\dot{\theta}^2 \end{aligned}\right]$$

In accordance with D'Alembert's principle, this system of inertia forces for all particle in the body is in equilibrium with the external applied forces and we may write the following equations of dynamic equilibrium.

$$\sum F_x = 0 \text{ gives, } F_x - \int \ddot{X}\,dm + \int (r\dot{\theta}^2 dm)\text{Cos}\theta + \int (r\ddot{\theta}\,dm)\text{Sin}\theta = 0 \;......(9.9)$$

$$\sum F_y = 0 \text{ gives, } F_y - \int \ddot{Y}\,dm + \int (r\dot{\theta}^2 dm)\text{Sin}\theta - \int (r\ddot{\theta}\,dm)\text{Cos}\theta = 0 \;......(9.10)$$

$$M_c = 0 \text{ gives, } M + \int (\ddot{X}\,dm)r\text{Sin}\theta - \int (\ddot{Y}\,dm)r\text{Cos}\theta - \int (r\ddot{\theta}\,dm)r = 0 \;......(9.11)$$

where F_x, F_y and M represent the applied external forces.

Since C is the centre of gravity, the statical moments $\int r\,\text{Sin}\theta\,dm = 0$ and $\int r\,\text{Cos}\theta\,dm = 0$

Therefore, from equation (9.9), we get, $F_x = m\ddot{X}$

from equation (9.10) we get, $F_y = m\ddot{Y}$

and from equation (9.11) we get,

$$M = \ddot{\theta}\int r^2 dm$$

$i.e., M = I\ddot{\theta} = I\alpha$, where I = moment of inertia about the centroidal axis normal to the plane of motion.

$$\therefore \boxed{\left.\begin{aligned} F_x &= m\ddot{X} \\ F_y &= m\ddot{Y} \\ M &= I\alpha \end{aligned}\right\} \text{......(9.12)}}$$

We can say that the moment M of external forces with respect to the centre of gravity has no effect on the motion of that point.

Equation (9.12) can be used to solve two kinds of problems involving the motion of a rigid body parallel to a fixed plane (i) the motion of the body is given, and the forces required to produce the motion are to be determine; (ii) the acting forces are given, and it is required to determine the kind of motion of the body that they will produce.

Long Answer Type Questions

1. Determine the tension in the strings and accelerations of two blocks of masses 150 kg and 50 kg connected by a string and a frictionless, weightless pulley as shown in fig: (i).

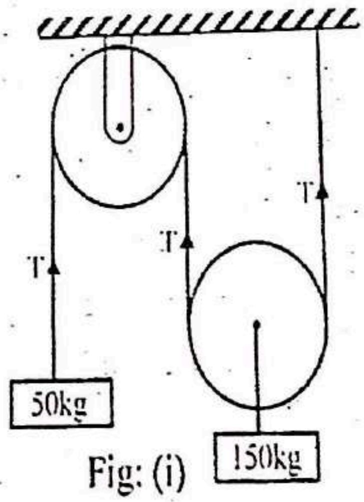
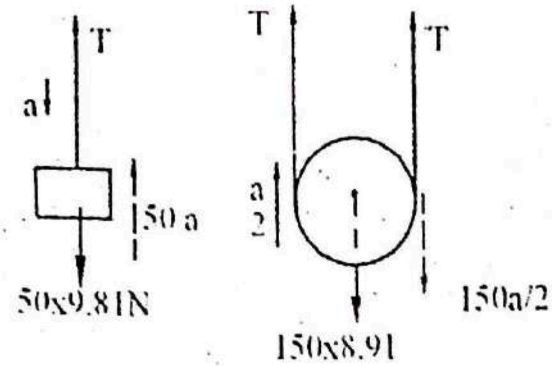


Fig: (i)

Answer:

Let us consider the acceleration of 50 kg block is 'a' and downward.

(By D'Alembert's principle) FBD of weights are shown below.



Equations of motions are for left figure:

$$T + (50)a = 50 \times 9.81 \text{ ......(1)}$$

and for right figure: $2T = 150 \times 9.81 + 150\left(\dfrac{a}{2}\right)$ ......(2)

From equation (1) we can write $T = 490.5 - 50a$

Putting the value of T in equation (2) we get

$$2 \times (490.5 - 50a) = 150 \times 9.81 + 150\left(\frac{a}{2}\right)$$

or, $981 - 100a = 1471.5 + 75a$

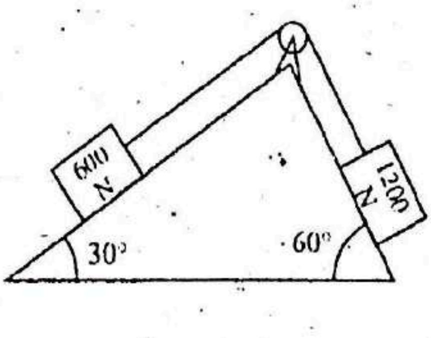
or, $-175a = 490.5 \;\therefore a = -2.803\, m/s^2$

$\therefore T = 50 \times 9.81 - (50)a = (490.5 + 140.14)N = 630.64N$ **(Ans.)**

Therefore the acceleration of 50 kg block is 2.803 m/s^2 (upward) and for 150 kg block is 1.40 m/s^2 (downward). **(Ans.)**

2. a) State D'Alembert principle.

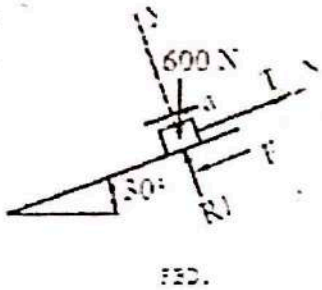
b) Two blocks weighing 600 N and 1200 N are placed on 30° and 60° planes respectively as shown in Fig. The blocks are connected by an extensible string which is passing over a friction pulley. If μ = 0.25 for both the plane, find the tension in the string and the acceleration of blocks.

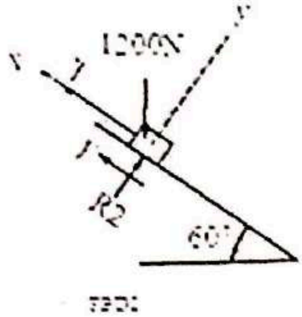


Answer:

a) *Refer to Question No. 2.1.*

b)





From $FBD1$, We can write, $R1 = 600\sin 60° N = 300\sqrt{3}\, N$

and $T - 600\cos 60° - 0.25\times 300\sqrt{3} = \frac{600}{g}\times a$ ----(1)

From $FBD2$, We can write, $R2 = 1200\sin 30° N = 600N$

and $1200\cos 30° - T - 0.25\times 600 = \frac{1200}{g}\times a$ ----(2)

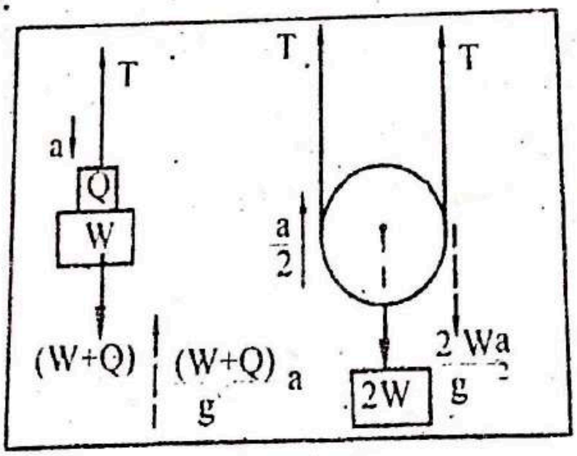
Adding equation (1) and (2) we get, $a = \frac{525\sqrt{3} - 450}{1800}\times g = 2.5 \text{ m/s}^2$

and $T = \left(600\times\frac{1}{2} + 75\sqrt{3} + \frac{600}{9.8}\times 2.5\right)\text{N} = 583\,\text{N}$

3. Weight W and 2W are supported in a vertical plane by a string and pulleys arranged as shown in Fig. E. Find the magnitude of an additional weight Q applied on the left which will give a downward acceleration $a = 0.1g$ to the weight W. Neglect friction and inertia of pulleys.

Answer:

(By D'Alembert's principle) FBD of weight 2W and (W + Q) as shown below.



Equations of motions are for left figure:

$$T + \left(\frac{W+Q}{g}\right)a = W + Q \qquad \text{....(1)}$$

And for right figure: $2T = 2W + \frac{2W}{g}(\frac{a}{2})$ ..... (2)

From equation (1) we can write $T = W + Q - \left(\frac{W+Q}{g}\right)a$

Putting the value of T in equation (2) we get:

$$2\left\{W + Q - (\frac{W+Q}{g})a\right\} = 2W + \frac{2W}{g}(\frac{a}{2})$$

$$or,\quad 2\left\{W + Q - (\frac{W+Q}{10})\right\} = 2W + \frac{W}{10}\ [\because a = 0.1g]$$

$$or,\quad 2Q - \frac{W}{5} - \frac{Q}{5} = \frac{W}{10}\ or, \frac{9}{5}Q = \frac{3}{10}W \quad \therefore Q = \frac{W}{6}$$

4. Determine velocity V of the falling weight W of the system as shown in Fig: 1.1 as a function of its displacement from the initial position of rest. Assume weight of the cylinder as 2W.

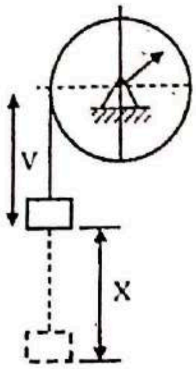


Fig: 1.1

Answer:

Let ω be angular velocity of the cylinder. Then $K.E = \frac{1}{2}I_c\omega^2$ where $I_c = \frac{1}{2}\left(\frac{2W}{g}\right)r^2$

For the weight W, $K.E = \frac{1}{2}\left(\frac{W}{g}\right)v^2$

Thus total K.E of the system $= \frac{1}{4}\left(\frac{2W}{g}\right)r^2\omega^2 + \frac{1}{2}\left(\frac{W}{g}\right)v^2$ ......(1)

From Kinematics, $v = r\omega$; (c is centre of rotation)

Therefore, the equation (1) becomes

$$K.E = \frac{1}{2}\frac{W}{g}v^2 + \frac{1}{2}\frac{W}{g}v^2$$

$$= \left(\frac{W}{g}\right)v^2 \text{ ......(2)}$$

Therefore, work done

$$U_{1-2} = (K.E)_2 - (K.E)_1$$

$$= \left(\frac{W}{g}\right)v^2 - 0 = \frac{W}{g}v^2 \quad ......(3)$$

Also work done = W.x ......(4) [since the force 2W does not do any work]

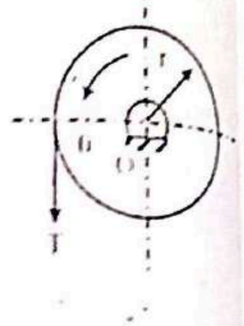
Therefore, we can write $\left(\frac{W}{g}\right)v^2 = W.x$

$$\boxed{\therefore v = \sqrt{gx} \quad ......(5)}$$

which is the required relation.

Alternative method:

Moment of the forces about point O



$$Tr = I\alpha$$

$$\Rightarrow Tr = \left(\frac{1}{2} \times \frac{2W}{g} \times r^2\right)\ddot{\theta}$$

$$\Rightarrow T = \left(\frac{W}{g} \times r\right)\ddot{\theta} \qquad ....(i)$$

Again we can write.

$$r\theta = x \quad \Rightarrow r\dot{\theta} = \dot{x}$$

$$\therefore r\ddot{\theta} = \ddot{x} \Rightarrow \ddot{\theta} = \frac{\ddot{x}}{r} \qquad ....(ii)$$

From D'Alembert's principle we get.

$$W - T = \frac{W}{g}\ddot{x}$$

$$\Rightarrow W - \frac{W}{g}\ddot{x} = \frac{W}{g}\ddot{x}$$

$$\Rightarrow W = \frac{2W}{g}\ddot{x} \qquad \therefore \frac{g}{2} = \ddot{x}$$

$$\Rightarrow \frac{g}{2} = v\frac{dv}{dx} \qquad \therefore \int_0^v v\,dv = \frac{g}{2}\int_0^x dx \Rightarrow \frac{v^2}{2} = \frac{g}{2}x$$

$$\therefore v = \sqrt{gx}$$

which is the required relation.

5. The angular displacement of a point on a plate cam rotating about an axis is given by $\theta = 0.3t^2 + 1$ radians. Determine the angular velocity and angular acceleration of the point when t = 0 and t = 10 seconds.

Answer:

Given $\theta = 0.3t^2 + 1$

$$\omega = \frac{d\theta}{dt} = 0.6t$$

$$\alpha = \frac{d\omega}{dt} = 0.6$$

$$\therefore \omega_{t=0} = 0 \text{ and } \omega_{t=10} = 0.6 \times 10 = 6\text{rad / sec}$$

$$\alpha_{t=0} = \text{constant} = 0.6\text{rad / sec}^2 \text{ and } \alpha_{t=10} = 0.6 = 6\text{rad / sec}^2$$

6. The angular velocity $\vec{\omega} = 4\vec{i} + 6.28\vec{k}$ rad/sec., angular acceleration $\alpha = 25.1j$ rad/sec² and position vector of A is given by $\vec{r} = 0.693\vec{j} + 0.4\vec{k}m$. Find velocity and acceleration vector of A.

Answer:

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & 6.28 \\ 0 & 0.693 & 0.4 \end{vmatrix} = -4.35\vec{i} - 1.60\vec{j} + 2.77\vec{k} \text{ m / s}$$

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 25.1 & 0 \\ 0 & 0.693 & 0.4 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & 6.28 \\ -4.35 & -1.6 & 2.77 \end{vmatrix}$$

$$= 20.1\vec{i} - 38.4\vec{j} - 6.4\vec{k} \text{ m / s}^2$$

7. A grinding wheel is rotating at 3000 rpm. Due to some reason the power goes off and the wheel stops in 10 seconds. Determine the deceleration of the wheel, assumed constant, and find the number of revolutions made by the wheel before coming to a stop.

Answer:

Initial angular speed

$$= \omega_0 = \frac{2\pi \times 3000}{60} \text{rad/s} = 314.16 \text{rad/sec.}$$

Final angular speed = 0

$\therefore \omega = \omega_0 + \alpha t$ gives

$$\alpha = -\frac{314.16}{10} \text{rad/s}^2 = -31.416 \text{rad/s}^2$$

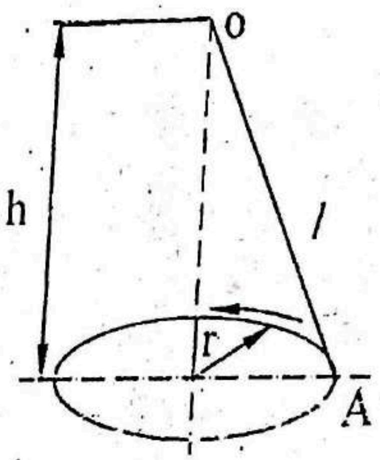
Therefore, angular deceleration = 31.4 rad/s^2

Also, $\omega^2 = \omega_0^2 + 2\alpha(\theta)$ gives

$0 = (314.16)^2 - 2 \times 31.4 \times \theta \quad \therefore \theta = 1571.6$ radians.

$\therefore$ Number of revolutions performed $= \frac{1571.6}{2\pi} = 250.$

8. A particle A of mass m is attached to a fixed point O by a string of length l whirls in a horizontal circular path of radius r with a uniform speed v as shown in fig. Determine the tensile force in the string during such motion.



Solution:

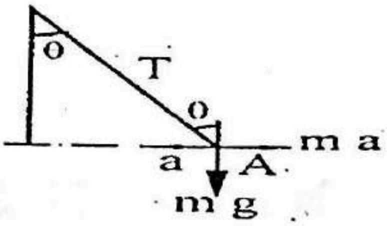
Let us draw the FBD of the particle, as shown in fig.

For dynamic equilibrium, using D'Alembert's principle

$\sum F_x = 0$ gives, $ma - T \sin\theta = 0$ ......9.8(1)

$\sum F_y = 0$ gives, $T \cos\theta - mg = 0$ ......9.8(2)

From equation 9.8(1) and 9.8(2), we get,



$$\tan\theta = \frac{a}{g} \quad ......9.8(3)$$

Now, for circular motion $a = a_n = \frac{v^2}{\rho} = \frac{v^2}{r}$ ......9.8(4)

Therefore, from equation 9.8(3), we get

$$\tan\theta = \frac{v^2}{rg}$$

$$\therefore \sin\theta = \frac{v^2}{\sqrt{v^4 + r^2g^2}} \quad ......9.8(5)$$

From equation 9.8(1), we get,

$$T = \frac{ma}{\sin\theta} = \frac{ma\sqrt{v^4 + r^2g^2}}{v^2}$$

$$= \frac{mv^2}{rv^2}\sqrt{v^4 + r^2g^2} = mg\sqrt{1 + \frac{v^4}{r^2g^2}}$$

$$\boxed{\therefore T = mg\sqrt{1 + \frac{v^4}{r^2g^2}} \quad ......9.8(6)}$$

9. A pile hammer of 250 kg mass is made to fall freely on a pile from a height of 6 m. If the hammer comes to rest in 0.012 second, determine the change in momentum, impulse and average force.

Answer :

Since the initial velocity $u = 0$, the velocity with which the hammer falls on the pile is

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 6} = 10.85 \text{ m/s}$$

Change in momentum = mass × change in velocity = 250 × (10.85 – 0) = **2712.5 Ns**

Impulse equals the change in moment

$\therefore$ Impulse = **2712.5 Ns**

Further, impulse represents the product of force and time during which it acts. Then

$$F \times t = 2712.5$$

$$\therefore \quad \text{Average force } F = \frac{2712.5}{0.012} = 226042 \text{ N} \simeq \mathbf{226.04 \text{ kN}}$$

❊ ❊ ❊

UNIT 5 KINETICS OF RIGID BODIES

Chapter at a glance

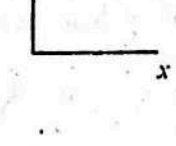
- *Angular momentum about a point* $\Rightarrow$
  $H_G = \int_m r' \times (\omega \times r')dm = \int_m [(r'.r')\omega - (r'.\omega)r']dm$
- *Angular momentum in matrix form* $\Rightarrow$
- $H_G = [I_G]\omega$ or $\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$
- *Tensor of inertia in matrix form* $\Rightarrow [I_G]$
- *Principal axes of inertia* $\Rightarrow$ axes are such that the tensor of inertia is diagonal
- *Parallel axes theorems* $\Rightarrow$
  $(I_{xx})o = I_{xx} + m(y_G^2 + z_G^2)$; $(I_{yy})o = I_{yy} + m(x_G^2 + z_G^2)$; $(I_{zz})o = I_{zz} + m(x_G^2 + y_G^2)$;
  $(I_{xy})o = (I_{yx})o = I_{xy} + mxGyG$;
  $(I_{xz})o = (I_{zx})o = I_{xz} + mxGzG$; $(I_{yz})o = (I_{zy})o = I_{yz} + myGzG$;

Long Answer Type Questions

1. Compute the moment of inertia of a thin squareplate of size a and mass M about its centre of mass.

Answer:

We choose the xy axis shown in Figure



Using the formulas of MI

$$I_{xx} = \sigma \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy\left(y^2\right) = \sigma a \frac{a^3}{12} = M\frac{a^2}{12}$$

where σ is the two-dimensional mass density of the plate. and

$M = \sigma a^2$. Similarly.

$$I_{yy} = \sigma \int_{-a/2}^{a/2} dx\left(x^2\right) \int_{-a/2}^{a/2} dy = I_{xx};$$

$$I_{zz} = \sigma \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy\left(x^2 + y^2\right) = I_{xx} + I_{yy} = M\frac{a^2}{6};$$

Due to symmetry we can conclude that $I_{xy} = I_{yx} = 0$, hence, x and y are the principal axes of the plate. Also, since $z = 0$ for all the points on the thin plate.

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0;$$

It turns out that any rotated axes $(x'y')$ on the xy plane yields the same moment of inertia. You can obtain this result using somewhat complex integration. This result shows that any orthogonal $x'y'z$ axes are principal axes for the square plate. In matrix form,

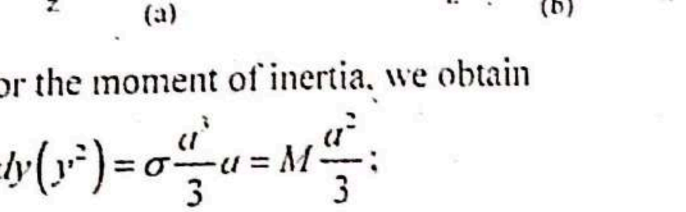
$$I = \begin{pmatrix} \frac{Ma^2}{12} & 0 & 0 \\ 0 & \frac{Ma^2}{12} & 0 \\ 0 & 0 & \frac{Ma^2}{6} \end{pmatrix}$$

In the above example, the principal axes pass through the CM. However, it is not necessary for the principal axes to pass through the CM of the rigid body, as we illustrate in the following example.

2. Compute the moment of inertia of a thin squareplate of size a and mass M about one of its corners. Find theprincipal axes of the plate passing through the aforementioned corner.

Answer:

We compute the moment of inertia of the square aboutthe corner point P shown in Figure.



Using the formulas for the moment of inertia, we obtain

$$I_{xx} = \sigma \int_0^a dx \int_0^a dy\left(y^2\right) = \sigma\frac{a^3}{3}a = M\frac{a^2}{3};$$

$$I_{yy} = \sigma \int_0^a dx\left(x^2\right) \int_0^a dy = M\frac{a^2}{3} = I_{xx};$$

$$I_{zz} = \sigma \int_0^a dx \int_0^a dy \left(x^2 + y^2\right) = I_{xx} + I_{yy} = 2M\frac{a^2}{3};$$

$$I_{xy} = I_{yx} = -\sigma \int_0^a dx \int_0^a dy \left(xy\right) = -\sigma \frac{a^2}{2}\frac{a^2}{2} = -M\frac{a^2}{4};$$

$$I_{yz} = I_{zx} = I_{xz} = I_{zy} = 0;$$

where σ is the constant mass density of the thin plate. The above values yield the following moment of inertia matrix:

$$\tilde{I} = \begin{pmatrix} \frac{Ma^2}{3} & -\frac{Ma^2}{4} & 0 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} & 0 \\ 0 & 0 & \frac{2Ma^2}{3} \end{pmatrix}$$

.... (1)

Since the matrix is not in a diagonal form, the axes xyz are not principal axes.

We diagonalize the above matrix, which is

$$I = \begin{pmatrix} \frac{Ma^2}{12} & 0 & 0 \\ 0 & \frac{7Ma^2}{12} & 0 \\ 0 & 0 & \frac{2Ma^2}{3} \end{pmatrix}$$

with the eigenvectors being (1, 1, 0), (1, −1, 0), (0, 0, 1). The first two vectors are along the x' and y' axes respectively shown in Figure.

This is an example where the principal axes do not pass through the CM of the rigid body.

3. **Describe kinetics of rigid bodies.**

Answer :

Kinetics of a Rigid Body

Kinetics of rigid body is the study of motion of rigid body in a plane. Motions such as translation and rotation are involved in plane motion. It uses the relation between the forces exerted on the body externally and the respective translation and rotational motions. The problems under kinetics of rigid bodies are solved by using any of the two methods namely,

1) Force acceleration method

2) Work energy method.

Various aspects of kinetics of rigid bodies are as follows.

Kinetic Energy (KE) of a Rigid Body

The sum of kinetic energies of every individual particle contained in the body is known as the kinetic energy of a rigid body. But, every particle in the body has both rotational and translation motions. Thus, the kinetic energy is the sum of the both energy in rotational and energy in translational motion of all individual particles of the body.

KE of a rigid body = KE in rotational + KE in translation.

(i) Kinetic Energy of Rigid Body in Translation

Kinetic energy of a particle or body represents the form of energy that arises from motion. Kinetic energy of a particle in translation is given by the expression,

$$k_i = \frac{1}{2} m_i v_i^2$$

Kinetic energy of rigid body,

$$k = \frac{I}{2} \Sigma m_i v_i^2$$

(ii) Kinetic Energy of Rigid Body in Rotation

In pure rotation, the rigid body has no "overall" translation of the body. However, the body in rotation must have kinetic energy as it involves certain motion. A closer look on the rotation of rigid body reveals that, though we may not be able to assign translation to the rigid body as a whole, but we can recognize translation of individual particles, as each of them rotate about the axis in circular motion with different linear speeds. The speed of the particle is given by,

$$V_i = \omega r_i$$

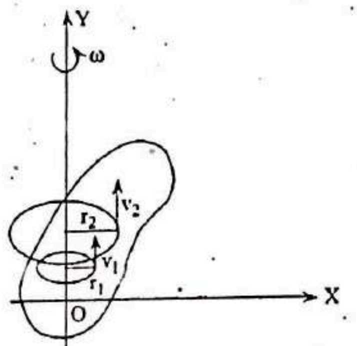


Figure: Rigid Body Rotating about Fixed Axis

Thus, kinetic energy of the individual particle is,

$$k_i = \frac{1}{2} m_i v_i^2$$

Where, "k_i" is the kinetic energy of "i" the particle having a speed "v_i". In terms of angular speed, the kinetic energy of an individual particle is,

$$k_i = \frac{1}{2} m_i (\omega r_i)^2$$

Kinetic energy of the rigid body is sum of the kinetic energies of the particles constituting the rigid body.

$$k = \Sigma k_i = \Sigma \frac{1}{2} m_i \omega^2 r_i^2$$

Note

The angular speeds of all particles constituting the body are same. Hence, constants 1/2 and ω^2 are taken out,

$$k = \frac{1}{2} \omega^2 \Sigma m_i r_i^2$$

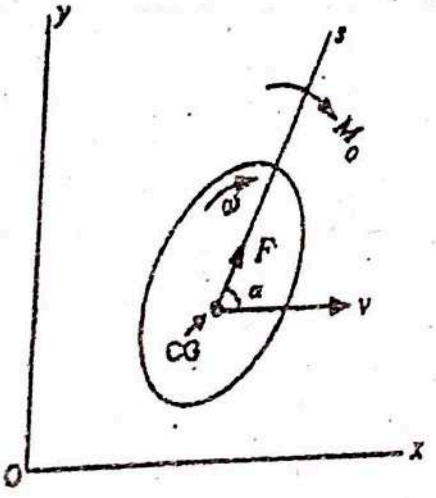
But, $\Sigma m_i r_i^2 = I$

$$\therefore k = \frac{1}{2}\omega^2 I$$

Hence the kinetic energy of a rigid body rotating about a fixed axis i.e., in pure rotational motion.

2. Workdone on a Rigid Body

Consider, a rigid body moving parallel to xy plane with plane motion. The mass of the body be 'M', resultant couple of moment 'M_o' and resultant force 'f'.



Figure

Work done on the body,

$$W = \int_{s_1}^{s_2} F \cos\alpha \, ds + \int_{\theta_1}^{\theta_2} M \, d\theta$$

$$W = F_x + M_o \theta.$$

Where,

α – Angle made by resultant (F) with horizontal

s – Direction of displacements in linear motion

ds – Length of elementary path

θ – Angular displacement of the body

$d\theta$ – Angle of elementary path.

4. List out the principles of dynamics and state their applications.

Answer :

Principles of Dynamics

1. Newton's I law of motion (or) Law of inertia.

   Application : It is used for solving static and dynamic problems.

2. Newton's II law of motion (or) Law of force and acceleration.

   Application : It is applicable for solving dynamic problems in which forces and acceleration are known.

3. Newton's III law of motion (or) Law of action and reaction.

   Application : It is useful in solving statics and dynamic problems.

4. Principle of work and energy.

   Application : It is applied in case of solving dynamic problems where forces, velocities and displacements known.

5. Principle of linear momentum and impulse.

   Application : It can be used when forces, velocities time of the dynamic problem are given.

6. Conservation of linear momentum principle.

   Application : This principle is applicable for solving problem of particles explosion and collision.

7. Conservation of energy principle.

8. D'Alembert's principle.

   Application : This principle is used for solving dynamic problems where inertia forces and forces that are applied externally are known.

5. Explain motion analysis of a rigid body in translation.

Answer :

Consider a rigid body in translation, as shown in translates either rectilinearly or curvilinearly in the xy On rigid body, consider two points 'A' and 'B'. Motion of 'A' is known and motion of point 'B' is unknown

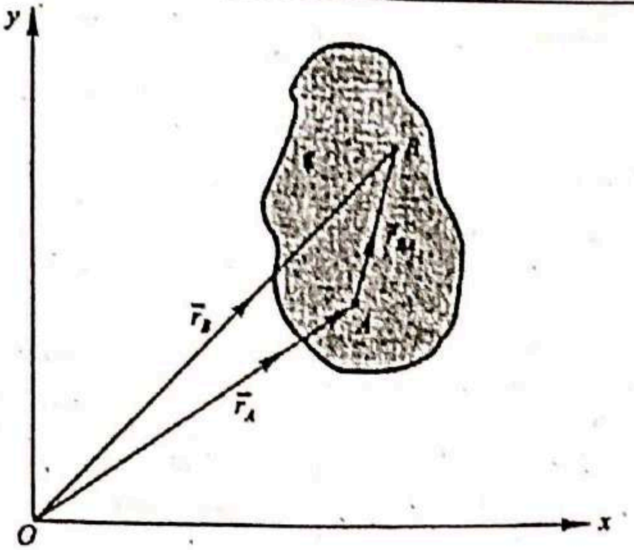


Figure : Rigid Body in Translation

Relation between known motion of point 'A' and unmotion of point '*B*' is obtained as follows.

When measured from origin of fixed reference frame, position vectors of '*A*' and '*B*' are given by $\vec{r}_A$ and $\vec{r}_B$ respectively. When measured from '*A*' to '*B*' position vector is given by $\vec{r}_{BA}$ which represents position of '*B*' relative to '*A*'. Relation between these position vectors is given by,

$$\vec{r}_B = \vec{r}_A + \vec{r}_{BA} \quad \text{... (1)}$$

To obtain relation between the velocity, vectors equation (1) is differentiated with respect to '*t*'.

$$\frac{d}{dt}\left(\vec{r}_B\right) = \frac{d}{dt}\left(\vec{r}_A + \vec{r}_{BA}\right)$$

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{BA}}{dt}$$

$$\vec{v}_B = \vec{v}_A + \vec{v}_{BA} \quad \text{... (2)}$$

To obtain relation between the acceleration vectors, equation (2) is differentiated with respect to '*t*'.

$$\frac{d}{dt}\left(\vec{v}_B\right) = \frac{d}{dt}\left(\vec{v}_A + \vec{r}_{BA}\right)$$

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{BA}}{dt}$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{BA} \quad \text{... (3)}$$

From rigid body definition it is evident that distance between the points '*A*' and '*B*' remains constant.

Therefore,

$$\vec{r}_{BA} = \text{Constant}$$

Thereby, terms $\vec{v}_{BA}$ and $\vec{a}_{BA}$ becomes zero in equations (2) and (3) respectively.

Thus,

$$\vec{v}_B = \vec{v}_A \quad \text{... (4)}$$

$$\vec{a}_B = \vec{a}_A. \quad \text{... (5)}$$

From equations (4) and (5) it is evident that, if one point located on the rigid body is defined completely, then motion of rigid body in translation is fully known. That is, when rigid body undergoes translation, motion of each and every point or particle on rigid body are same or identical.

6. Discuss about rigid body rotation about a fixed axis.

Answer :

Fixed Axis Rotation of a Rigid Body

A body is said to be rigid, in which after the application of external loads the distance between the two points remains constant (i.e., invariable). The rotation of motion of a rigid body with reference to the particle moving in a circular path with centres on fixed straight lines are known of "fixed axis rotation of a rigid body". In this motion of rotation, the plane of circular path remains parallel to the axis of rotation.

In fixed axis rotation, the motion produced from unbalanced couple acting on rigid body is given as,

$$M = I \times \alpha$$

Where,

M – Unbalanced couple

I – Mass moment of inertia

α – Angular acceleration $= \dfrac{d\omega}{dt}$ rad/s²

ω – Angular velocity $= \dfrac{d\theta}{dt}$ rad/s

Where,

θ – Angular displacement

7. A flywheel which is at rest attains a constant speed of 300 r.p.m after accelerating uniformly for 10 seconds. Determine the number of revolutions made by the flywheel during the period.

Answer :

Given that,

Speed, $N = 300$ r.p.m

Time, $t = 10$ secs

$$\text{Angular velocity, } \omega = \frac{2\pi N}{60} = \frac{2\pi 300}{60}$$

$$\omega = 31.416 \text{ rad/s}$$

But,

$$\omega = \omega_0 + \alpha t$$

$$31.416 = 0 + \alpha \times 10$$

$$\therefore \quad \alpha = 3.142 \text{ rad/s}^2$$

Angular displacement,

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

$$= 0 \times 10 + \frac{1}{2} \times 3.142 \times 10^2$$

$$= 157.1 \text{ radians}$$

$$= \frac{157.1}{2\pi} = 25.003$$

$$\therefore \ \theta \simeq 25 \text{ revolutions.}$$

8. A 600 mm diameter flywheel is brought uniformly from rest to a speed of 350 r.p.m in 20 seconds. Determine the velocity and acceleration of a point on the rim 2 seconds after starting from rest.

Answer :

Given that,

Diameter of flywheel, $d = 600$ mm $= 0.6$ m

$r = d/2 = 0.3$ m

Speed, $N = 350$ r.p.m

Time, $t = 20$ s

Initial angular velocity, $\omega_1 = 0$

$$\text{Final angular velocity, } \omega_2 = \frac{2\pi \times 350}{60} = 36.65 \text{ rad/s} \quad \left(\because \omega = \frac{2\pi N}{60}\right)$$

Angular acceleration,

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{36.65 - 0}{20} = 1.83 \text{ rad/s}^2$$

Velocity at a point on the rim 2 seconds after start from rest is,

$$\omega = \omega_0 + \alpha t$$

Where,

$$\omega_0 = 0, \ \alpha = 1.83 \text{ rad/s}^2 \text{ and } t = 2$$

Then,

$$\omega = 0 + 1.83 \times 2$$

$$\therefore \quad \omega = 3.66 \text{ rad/s}$$

Linear velocity,

$$v = r \times \omega$$

$$= 0.3 \times 3.66$$

$$v = 1.098 \text{ m/s.}$$

Tangential acceleration,

$$a_t = \alpha . r$$

$$= 1.83 \times 0.3 \quad (\because \alpha = 1.83 \text{ rad/s}^2)$$

$$a_t = 0.549 \text{ m/s}^2$$

Normal acceleration,

$$a_n = r . \omega^2$$

$$= 0.3 \times 3.66^2$$

$$a_n = 4.019 \text{ m/s}^2$$

$\therefore$ Resultant acceleration $= \sqrt{a_t^2 + a_n^2}$

$$= \sqrt{0.549^2 + 4.019}$$

$$= 4.056 \text{ m/s}^2$$

---

9. A wheel rotating about a fixed axis at 20 r.p.m is uniformly accelerated for 70 seconds during this time it makes 50 revolutions. Find,

(a) Angular velocity at the end of this interval

(b) Time required for the speed to reach 100 rev/min.

Answer :

Given that,

Speed of wheel, $N = 20$ r.p.m

Number of revolutions in 70 seconds = 50

Time, $t = 70$ s

Angular displacement,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad \text{... (1)}$$

Also, angular displacement, θ = Angular displacement in one revolution × Number of revolutions.

$$\theta = 2\pi \times 50$$

$$= 100\,\pi$$

$$\therefore \quad \theta = 314.159 \text{ radians}$$

Initial angular velocity, $\omega_0 = \frac{2\pi N}{60}$

$$= \frac{2\pi \times 20}{60}$$

$$\omega_0 = 2.094 \text{ rad/s}$$

Substituting above values in equation (1),

$$314.159 = (2.094 \times 70) + \frac{1}{2} \times \alpha \times (70)^2$$

$$314.159 = 146.580 + 2450.\alpha$$

$$167.579 = 2450.\alpha$$

$$\therefore \ \alpha = 0.068 \text{ rad/s}^2$$

(a) Angular Velocity at the End of Period of 70 secends

Final angular velocity,

$$\omega = \omega_0 + \alpha t$$

$$= 2.094 + 0.068 \times 70$$

$$= 2.094 + 4.76$$

$$\therefore \ \omega = 6.854 \text{ rad/s}$$

$\therefore$ Angular velocity at the end of period of 70 secconds, $\omega = 6.854$ rad/s

(b) Time Required for the Speed to Reach 100 rev/min

Final angular velocity,

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 100}{60}$$

$$\omega = 10.472 \text{ rad/s}$$

Using relation,

$$\omega = \omega_0 + \alpha t$$

$$10.472 = 2.094 + 0.068 \times t$$

$$8.378 = 0.068 \times t$$

$$\therefore \ t = 123.206 \text{ s}$$

$\therefore$ The time required for the speed to reach 100 rev/min, $t = 123.206$ s.

10. A flywheel rotates at a speed of 3600 rpm and its speed uniformly decreases to 800 rpm in 1500 revolutions. Find,

(i) Time interval required to reach 800 rpm.

(ii) Number of revolutions the wheel can perform before it is brought to rest from 800 rpm.

Model Paper-II, Q11

Answer :

Given that,

Initial speed of flywheel, $N_o = 3600$ rpm

Initial angular velocity of flywheel,

$$\omega_o = \frac{2\pi N_o}{60}$$

$$= \frac{2\pi \times 3600}{60} = 376.8 \text{ rad/s}$$

Final speed of flywheel, $N = 800$ rpm

Final angular velocity of flywheel,

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 800}{60} = 83.74 \text{ rad/s}$$

Angular displacement,

$$\theta = 1500 \times 2\pi = 3000\pi \text{ rad}$$

From equation, $\omega^2 - \omega_o^2 = 2\alpha\theta$.

Where, α – Angular acceleration of flywheel

$$(83.74)^2 - (376.8)^2 = 2 \times \alpha \times 3000\pi$$

$$\alpha = \frac{(83.74)^2 - (376.8)^2}{2 \times 3000\pi} = -7.16$$

$\therefore$ Angular acceleration, $\alpha = -7.16$ rad/s^2

($\because$ '–ve' sign indicates the decrease in velocity of body)

(i) From equation, $\omega = \omega_o + \alpha t$

$$\text{Time interval, } t = \frac{\omega - \omega_o}{\alpha}$$

$$= \frac{83.74 - 376.8}{-7.16}$$

$$\therefore \ t = 40.93 \approx 41 \text{ s}$$

(ii) At rest, $\omega = 0$, & $\omega_o = 8374$ rad/s

From equation, $\omega^2 = \omega_o^2 + 2\alpha\theta'$

Where, θ' is the angular displacement made by the flywheel before it comes to rest

$$0 = (8374)^2 + 2 \times -7.16 \times \theta'$$

$$\theta' = \frac{-(83.74)^2}{2 \times -7.16} = 489.7 \text{ radians}$$

$\therefore$ Number of revolutions made by the flywheel before it comes to rest,

$$N' = \frac{\theta'}{2\pi} = \frac{489.7}{11}$$

$$\therefore \ N' = 77.97 \approx 78 \text{ revolutions}$$

UNIT
6

SHEAR FORCE AND BENDING MOMENT

Chapter at a glance

- *Beam* ⇒a bar subjected to forces or couples that lie in a plane containing the longitudinal axis of the bar.
- *Types of load* ⇒Point load, Uniformly distributed load, Uniformly varying load
- *Type of beam* ⇒Simply supported beam, Cantilever beam, Overhanging beam, Built-in beam, Continuous beam
- *Shear force* ⇒the algebric sum of the forces taken on the either side of the section.
- *Bending moment* ⇒the algebric sum of the moments of the forces about the section, taken on either side of the section.
- *Shear forces Sign Conventions* ⇒upwards to the left of a section or downwards to the right of the section are positive. Downwards to the left of a section or upwards to the right of the section are negative.
- *Bending Moment Sign Conventions* ⇒Clockwise to the left of the section or counterclockwise to the right of the section are positive. Counterclockwise to the left of the section or clockwise to the right of the section are negative
- *For auniformly distributed load of intensity* w ⇒ bending moment. $M = \frac{wlx}{2} - wx \times \frac{x}{2}$

  or, $M = \frac{1}{2}wlx - \frac{1}{2}wx^2$

  Shear force, $F = \frac{1}{2}wl - wx$, which is equal to $\frac{dM}{dx}$
- *Point of Contraflexure* ⇒ point where BM will change sign from positive to negative or vice versa.

Long Answer Type Questions

1. Draw the SF and BM diagrams for simply supported beams with Point load

Answer:

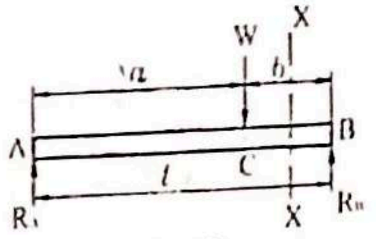


Fig (i)

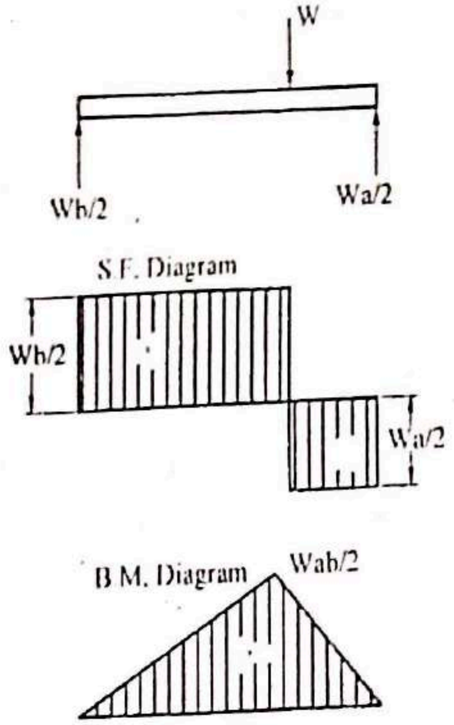


Fig (ii)

Consider the simply supported beam AB as shown in Fig. (i).
The beam can be assumed to rest on knife-edges or roller supports and is free to bend at the supports without restraint.
A concentrated load W is applied at point C on the beam. The reactions at the end supports of the beam may be calculated by applying normal equilibrium conditions, i.e. by taking moment about A.

$$R_B = l = W \times a$$

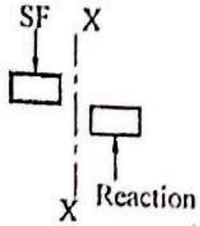
or, $$R_B = \frac{Wa}{l}$$

Again, $R_B + R_A = W$

$$\therefore \quad R_A = W\left(\frac{l-a}{l}\right) = \frac{Wb}{l}$$

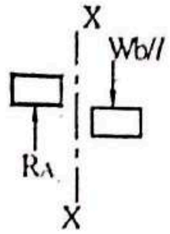
SF Calculations

Now SF at section X–X, taken in between B and C,



i.e., negative and the value is $= -\frac{Wa}{l} = -R_B$

SF in between A and C



i.e., positive and the value is $= \frac{Wb}{l}$

BM Calculations

BM at points A and $B = 0$

BM at point $C = R_A \times a$, which is clockwise to the left of A so, positive

$$R_A \times a = \frac{Wba}{l}$$

At point C, SF changes sign from negative to positive through zero. Hence BM is maximum at point C. The SF and BM diagrams are shown in Fig. (ii).

2. Draw the SF and BM diagram of Simply Supported beam with UDL.

Answer:

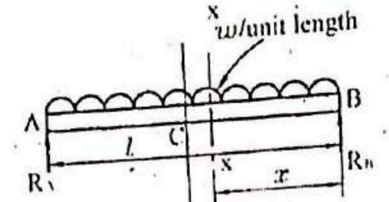


Fig (i)

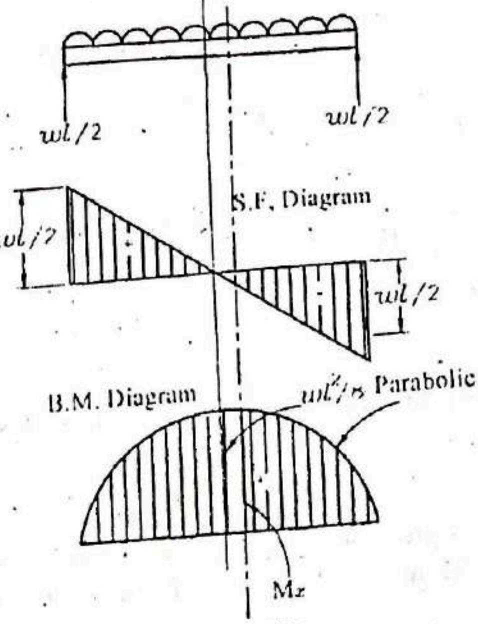


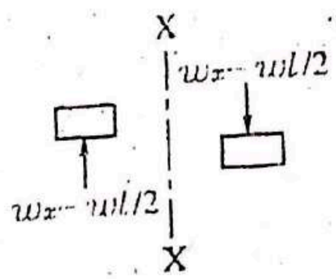
Fig (ii)

Consider the simply supported beam AB, carrying a UDL w per unit length as shown Fig. (i). Here again it is necessary to evaluate the reactions, but in this case the problem simplified by the symmetry of the beam. Each reaction will therefore take half of applied load.

i.e., $R_A = R_B = \dfrac{wl}{2}$

Now consider a section x-x at a distance x from the right end of the beam.

Now, SF at that section

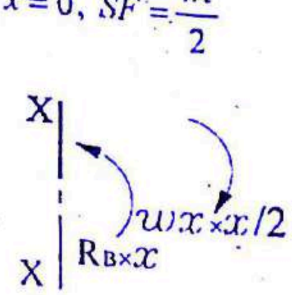


i.e., positive and the value is, $F_x = wx - \dfrac{wl}{2}$

Now, when $x = \dfrac{l}{2}$, $SF = 0$

when, $x = l$, $SF = \frac{wl}{2}$,and when, $x = 0$, $SF = \frac{wl}{2}$

BM Calculations



∴ Net BM at $x - x$

$$M_x = R_B x - wx^2 = \frac{wlx}{2} - \frac{wx^2}{2}$$

At $x = 0$, $M = 0$; At $x = l$, $M = 0$

$$\text{At } x = \frac{l}{2},\ M = \frac{wl^2}{4} - \frac{wl^2}{8} = \frac{wl^2}{8}$$

The SF and BM diagrams are shown in Fig.(ii).

3. 6 Simply Supported beam with UVL from zero at one end to w/unit length at other end.

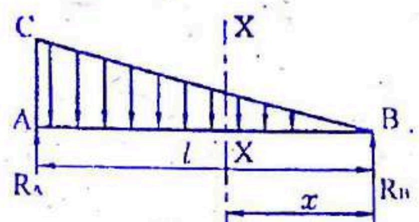


Fig (i)

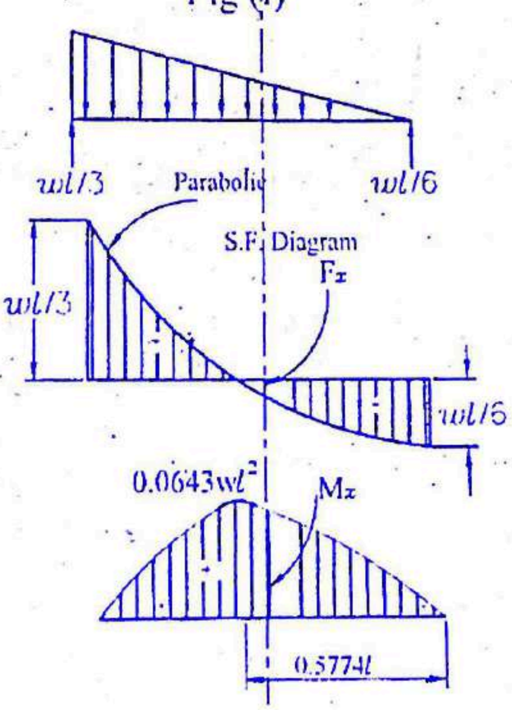


Fig (ii)

To calculate, reactions at A and B,
Taking moment about A

$$R_B \times l = \left(\frac{0+w}{l}\right) \times l \times \frac{1}{3}$$

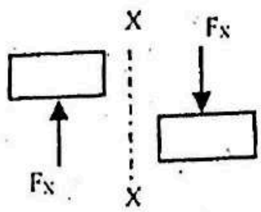
[Average load/unit length $= \frac{w+0}{2} = \frac{w}{2}$.

The total load $\frac{w}{2} \times l$ can be assumed to be acted at the centroid of the triangle ABC]

$\therefore \quad R_B = \frac{wl}{6}$

$\therefore \quad R_A = \text{Total load} - \frac{wl}{6} = \frac{w}{2} - \frac{wl}{6} = \frac{wl}{3}$

$\therefore$ SF at $X - X$



So, positive and $F_X = w\left(\frac{x}{l}\right) \times \frac{1}{2} x - R_h = \frac{wx^2}{2l} - \frac{wl}{6}$

Thus, at $x = 0$, $F = -\frac{wl}{6}$

at $\quad x = l,\ F = \frac{wl}{2} - \frac{wl}{6} = \frac{wl}{3}$

This is to be noted that as $\frac{dF_X}{dx} = \frac{wx}{l}$, which is positive, the slope will be increasing type

Now, BM at section X – X

$$M_X = R_B x - \frac{w}{l}\frac{x}{2} \times \frac{x}{3}, \text{ Counter-clockwise, i.e., positive}$$

or, $\quad M_X = \frac{wlx}{6} - \frac{wx^3}{6l}$

$\therefore$ at $x = 0;\ M = 0$

at $x = l,\ M = 0$

BM is maximum, where SF = 0

Now making SF = 0.

$$\frac{wx^2}{2l} = \frac{wl}{6}.$$

or, $\quad x = 0.5774l$

$$\therefore \qquad (BM)_{max} = \frac{wl \times 0.5774l.}{6} - \frac{w \times (0.5774l)^3}{6} = 0.0643wl^2$$

The SF and BM diagrams are shown in Fig. (ii).

4. Draw the SF and BM Diagram for Cantilever beam with a point load at the free end.

Answer:

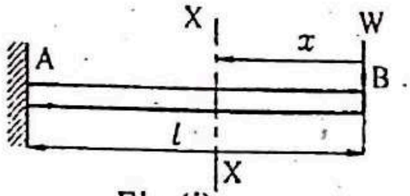


Fig (i)

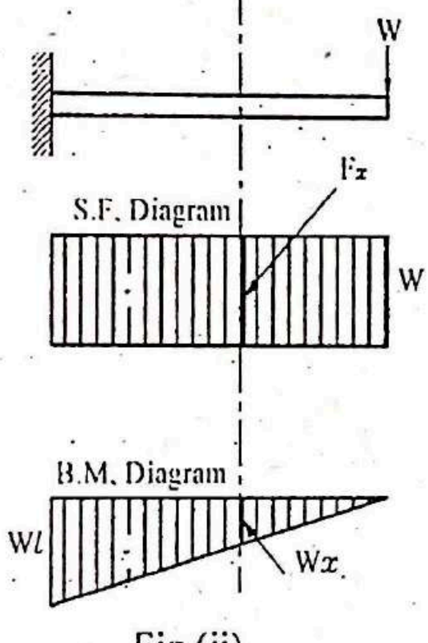
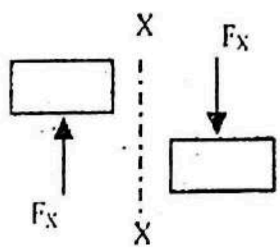


Fig (ii)

Consider the beam AB with a point load W at B, which is free end as shown in Fig. (i).
Consider a section X – X at a distance x from the free end.
Now SF at X – X



i.e., positive and the value
$F_x = W$

Now BM at section X – X

i.e., negative and $M_x = -Wx$

The SF and BM diagrams are shown in Fig. (ii).

5. Draw the SF and BM Diagram for Cantilever beam with UDL.

Answer:

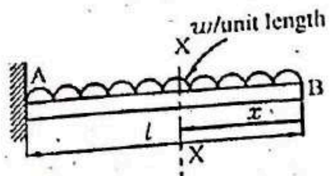


Fig (i)

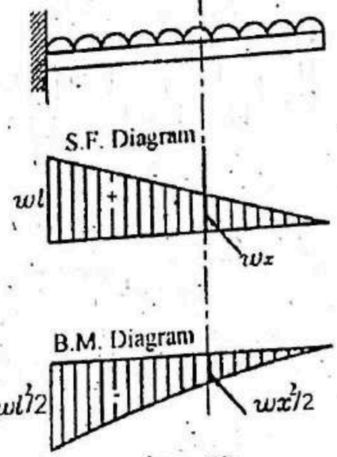
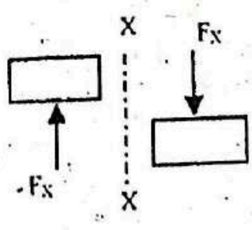


Fig (ii)

The Fig.(i) shows a cantilever beam AB with uniformly distributed load w per unit length.

Let take a section X – X at a distance x from the free end.

SF at X – X



i.e., positive and

$F_x = wx$

Therefore at x = 0, SF = 0 and at $x = l$, SF = wl which is maximum.

BM at section X – X is negative and $M_x = -(wx)x/2 = -wx^2/2$
at $x = 0$, BM = 0 and at $x = l$, BM = $wl^2/2$.
The SF and BM diagrams are shown in Fig.(ii).

6. Draw the SF and BM Diagram for Cantilever beam with UVL.

Answer:

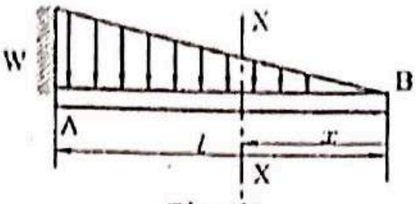


Fig (i)

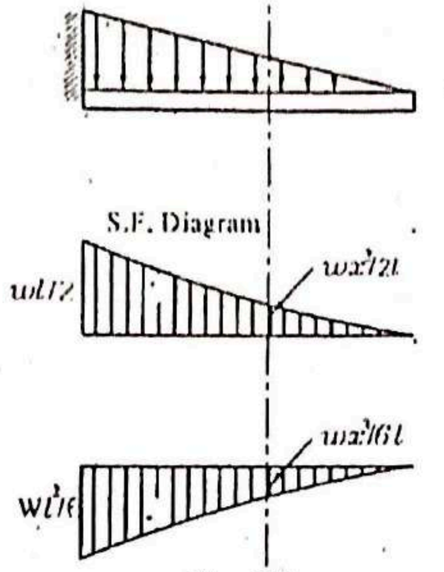


Fig (ii)

Fig. (i) shows a cantilever beam AB with uniformly varying load, varying from zero to w per unit length.
Consider a section X – X at a distance x from the free end.
SF at X – X

$$F_x = \frac{wx}{l} \times \frac{x}{2} = \frac{wx^2}{2l}$$

which is at right hand side to the section and downward. So, positive. Therefore at x = 0, SF = 0 and at $x = l$, SF = $wl/2$.
Now, BM at section X X,

$$M_x = -\frac{wx^2}{2l} \times \frac{x}{3} = \frac{wx^3}{6l}$$

[Taken negative because the moment isacting

at right hand side to the section and in clockwise direction]
Therefore, at x = 0, M = 0 and at $x = l$, BM = $wl^2/6$. The SF and BM diagrams are shown in Fig.(ii).

7. How to find out the Point of Contraflexure.

Answer:

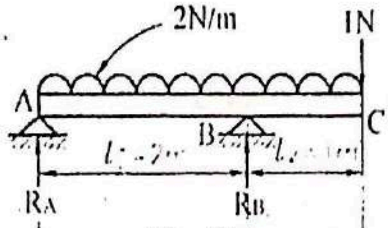


Fig (i)

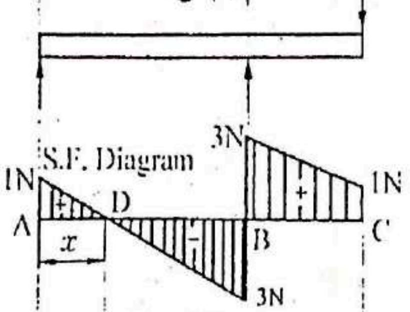


Fig (ii)

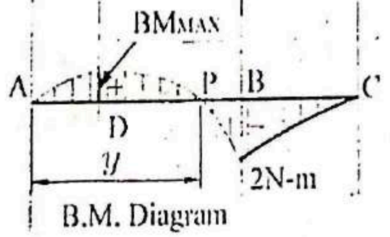


Fig (iii)

Consider the overhanging beam shown in Fig.(i).

First step is to find out reactions R_A and R_B.

Taking moment about A.

$R_B \times 2 = 1 \times 3 + 2 \times 3 \times 3/2$

or, $R_B = 6N$

Again, $R_A + R_B = 2 \times 3 + 1$.

or, $R_A = 7 - 6 = 1N$

Next step is to draw the SF and BM diagram.

Now, SF at C and at A is equal to zero.

$\therefore F_C = F_A = 0$.. Now when moving from C to B the shear force will increase for UDL.

$SF = 1 + w \times 1 = 1 + 2 \times 1 = 3N$.

Again when moving A to B, SF will decrease as the left downward forces to the section are increasing.

Therefore, SF at B = $R_A - 2 \times w = 1 - 2 \times 2 = -3N$

At B, SF changes sign due to the effect of R_B.

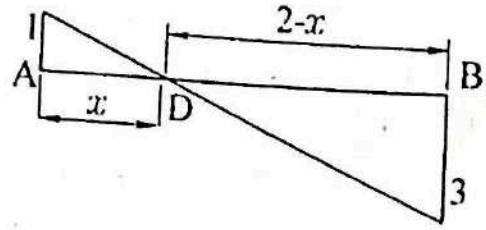
Now BM at A and C are equal to zero, as point A is a support of a simply supported beam and point C is the extreme end of the cantilever.

Now BM at B.

$$M_B = R_A \times l_1 - l_1 \times w \times \frac{l_1}{2} = 1 \times 2 - 2 \times 2 \times \frac{2}{2} = -2N - m$$

Also from Right hand side, $M_B = -1 \times l_2 - l_2 \times w \times \frac{l_2}{2} = -1 \times 1 - 1 \times 2 \times \frac{1}{2} = -2N - m$

The SF diagram is shown in Fig.(ii). From this diagram this can be concluded that, maximum BM will occur either at B or D where SF changes sign.
Let the point D is at a distance x from A as shown in figure.



$$\therefore \quad \frac{x}{1} = \frac{2-x}{3}$$

or, $\quad x = 0.5$ m

$\therefore$ BM at D

$$= R_A \times x - w \times x \times \frac{x}{2}$$

$$= 1 \times 0.5 - 2 \times 0.5 \times \frac{0.5}{2}$$

$$= 0.25 \text{ N-m}$$

To find out the point of contraflexure

Let the point of contraflexure is P, which is a point where BM = 0 and say P is at a distance y from A.

$$\therefore BM \text{ at P} = R_A \times y - w \times y \times \frac{y}{2}$$

$$= 1 \times y - 2 \times \frac{y^2}{2} = y - y^2$$

y definition, $y - y^2 = 0$ or, $y(1 - y) = 0$

$\therefore y = 1$ m [since $y \neq 0$]

ie BM diagram is shown in Fig.(ii).

How to find out Load and BM diagrams from SF diagram.

swer:

this case, first load diagram is drawn from SF diagram and then BM diagram is npleted.

: following points are to be kept in mind during making load diagram:

1) If there is sudden increase or decrease (i.e., vertical line of SF diagram). it indicates that there is either a point load or reaction (i.e., support) at that point.

2) When SF line is horizontal and forms a rectangle, then there is no load between the points.
3) If SF line is an inclined straight line between any two points, then there is uniformly distributed load between the points.
4) If SF line is parabolic between any two points, then there is uniformly varying load between the points.

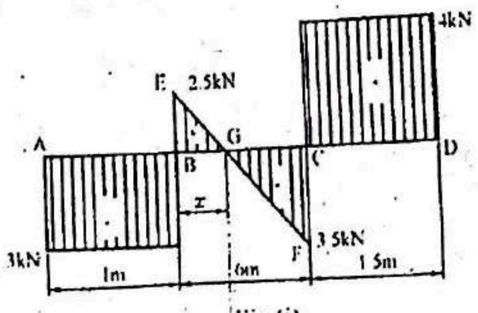


Fig (i)

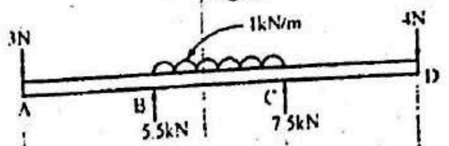


Fig (ii) Load Diagram

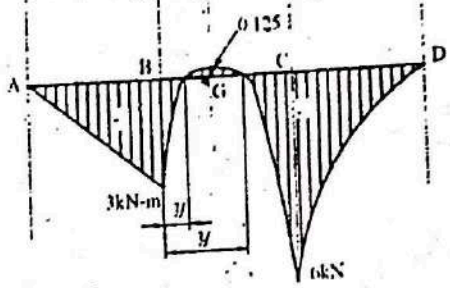


Fig (iii) BM Diagram

Consider the SF diagram as shown in Fig.(i).
The following conclusions can be made based on the SF diagram given in Fig.(i)
(1) As SF decreases suddenly from 0 to -3kN at point A, therefore, at point A there is point load of 3kN.
(2) As from A to B there is no change in SF, so there is no loading between A and B.
(3) As SF increases suddenly from –3 to 2.5kN at point B, therefore, at point B there support reaction of 3 + 2.5 = 5.5kN.
(4) SF diagram shows that the line EF is a inclined line and SF is decreasing from 2.5k to –3.5kN. So, there is a UDL from B to C and this decrease (2.5 + 3.5 = 6kN) occurring in 6 m length, so w = 6/6 kN/m = 1kN/m
(5) As SF is increasing suddenly from –3.5 to 4kN at point C, therefore, at point C the is a support reaction of 3.5 + 4 = 7.5kN
(6) From C to D there is no change in SF, so there is no intermediate load in between and D.
(7) At D SF diagram shows a sudden decrease in SF from 4kN to 0. So there is a p load of 4kN at D.

Now from the above conclusions, the load diagram is drawn as shown in Fig.(ii).

BM Calculations

BM at A and C = 0, because these are free extreme ends of cantilever.
BM at B = 3 x 1 = 3kN-m (calculated from left end)
BM at C = 4 x 1.5 = -6kN-m (calculated from right end)
Maximum BM will occur either at B, or G or C where SF changes sign.

From geometry, $\frac{x}{2.5} = \frac{6-x}{3.5}$ or, x = 2.5 m

Now BM at G = 3 x (1 + 2.5) + 5.5 x 2.5 – w x 2.5 x 2.5/2 = 0.125kN-m
Hence maximum positive BM is at G, i.e., 0.125kN-m and maximum negative BM is at C, i.e., –6kN-m.

To find out the point of contra flexure
At this point BM = 0.
Say the point P is at a distance y from B and toward right.
Therefore BM at P=3(1 + y) + 5.5y – w x y x y/2
Now by definition BM at P = 0

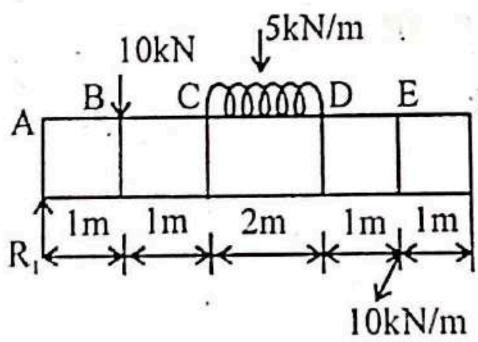
$\therefore \frac{y^2}{2} - 2.5y + 3 = 0,\ or,\ y^2 - 5y + 6 = 0$

$or, (y-3)(y-2) = 0$

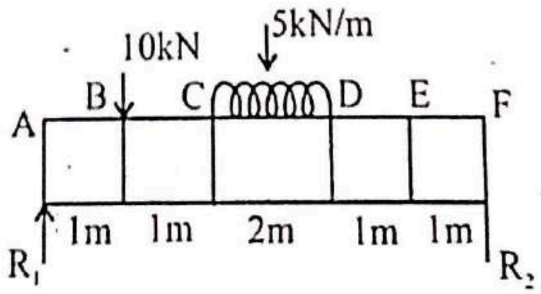
$\therefore y = 2$ m and 3 m

The BM diagram is shown in Fig.(iii).

9. Draw S.F.D and B.M.D for the beam shown in Fig.



Answer :



Taking moments about F

$$R_1 \times 6 = 10 \times 5 + 5 \times 2 \times 3$$

$$R_1 = \frac{80}{6} = 13.33\text{kN}$$

$$R_2 = 20 + 5 \times 2 - 13.33 = 6.67\text{ kN}$$

S.F. Diagram :– Shear force between A and B is equal to R_1 consider a section at a distance x from A. For x to lie between B and C,

$$F = R_1 - 10$$
$$= 13.33 - 10 = 3.33\text{ kN}$$

For x to lie between C and D,

$$F_1 = 3.33 - 5(x-2)$$
$$= 13.33 - 5x$$

For x = 2m

$$F_1 = 3.33\text{ kN}$$

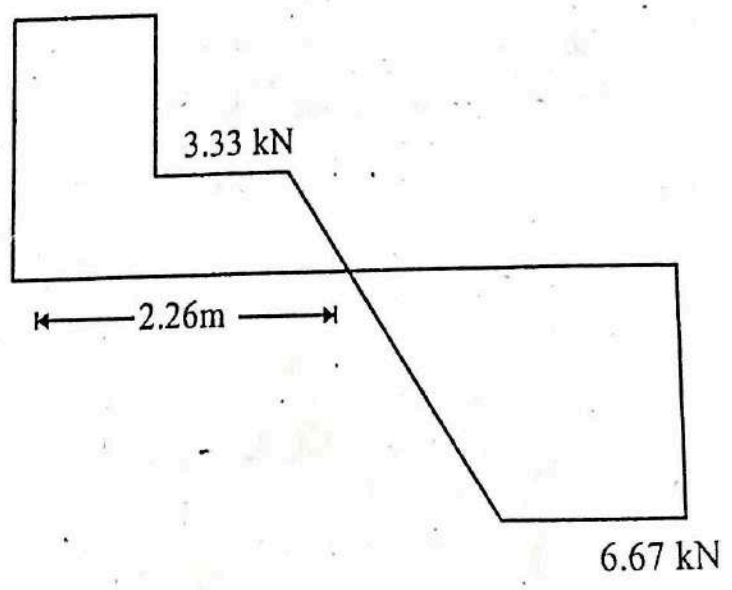
For x = 4m

$$F_1 = 13.33 - 5 \times 4$$
$$= -6.67\text{ kN}$$

$$F = 0$$

$$x = \frac{13.33}{5} = 2.66\text{m}$$

The S.F diagram is shown in the fig.



B.M diagram between A and B,

$$M = R_1 \times x = 13.33x$$

At $x = 0, M = 0$

and at $x = 1m, M = 13.33$ kN.m

The variation of moment is linear

Between B and C

$$M = R_1 \times x - 10 \times (x - 1)$$

$$= 13.33x - 10x + 10 = 3.33x + 10$$

At $x = 1, M = 13.33$ kN.m

$x = 2, M = 16.66$ kN.m

The variation of moment is linear

Between C and D,

$$M = R_1 \times x - 10\,(x - 1) - 5\,(x - 2)\,\frac{(x-2)}{2}$$

$$= R_1 \times x - 10x - 10 - \frac{5}{2}(x-2)^2$$

At $x = 2m, M = 16.66$ kNm

At $x = 4m$

$$M = 13.33 \times 4 - 10 \times 4 - 10 - \frac{5}{2}(4-2)^2$$

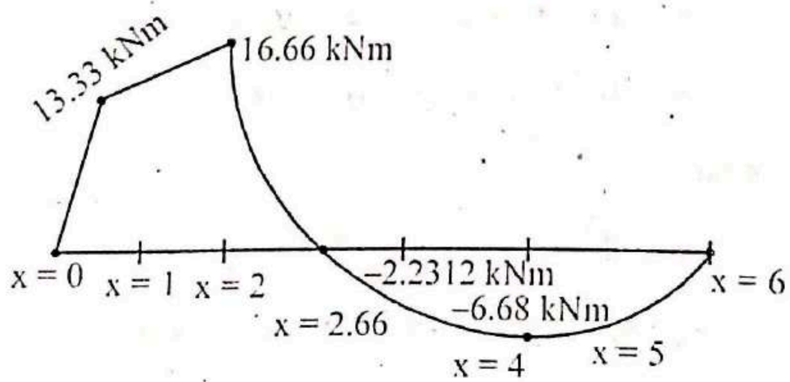
$$= 53.32 - 60 = -6.68 \text{ kNm}$$

At $x = 2.66$ m

$$M = 13.33 \times 2.66 - 10 \times 2.66 - 10 - \frac{5}{2}(2.66-2)^2$$

$$= -2.2312 \text{ kNm}$$

The B. M diagram is shown in the fig.



10. A beam ABCDE of total length 18 meters is supported and loaded as shown in fig. (i) Draw S. F. D and B. M. D and locate points of contra flexure if any :

Answer :

Let R_B and R_D are reactions at B and D respectively

$$R_D + R_B = 3\times7+10 = 31 \quad ...(i)$$

Now, taking moment about B, and equating the same.

$$-10\times15+R_D\times11-3\times4\times2+3\times31.5=0$$

$$\Rightarrow \quad -150+R_D\times11-24+13.5=0$$

$$\therefore \quad R_D = 14.6\ kN$$

Substituting the value of R_D in eqn(i),

$$\therefore \quad R_B = 16.4\ kN$$

Shear force at every point $(\uparrow|\downarrow +)$

Part E - D $(0\le x\le4)$

$$F_x = 10\ kN$$

$$F_E = 10 kN$$

$$F_D = 10\ kN$$

Part D - C $(4\le x\le11)$

$$F_X = 10-R_D$$

$$F_D = 10-14.6 = -4.6\ kN$$

$$F_C = 10-14.6 = -4.6\ kN$$

Part C - B $(11\le x\le15)$

$$F_X = 10-R_D+3\times(x-11)$$

$$F_C = 10-14.6+3(11-11) = -4.6\ kN$$

$$F_B = 10-14.6+3x(15-11) = 7.4\ kN$$

Part B - A $(15\le x\le18)$

$$F_X = 10-R_D+3\times4-R_B+3(x-15)$$

$$F_B = 10-14.6+12-16.4+3\,(15-15) \;=-9\ kN$$

$$F_A = 10 - 14.6 + 12 - 16.4 + 3(18 - 15) \ = 0$$

Bending moment at every point ($\uparrow | \uparrow +$)

Part E - D $(0 \leq x \leq 4)$

$$M_X = -\ 10 \times 2 \qquad \therefore\ M_E = -10 \times 0 = 0$$

$$M_D = -10 \times 4 - 40\ kN - m$$

Part D - C $(4 \leq x \leq 11)$

$$M_X = -10 \times x + R_D \times (x - 4)$$

$$M_D = -10 \times 4 + 14.6(4 - 4)$$

$$= -40\ kN - m$$

$$M_C = -10 \times 11 + 14.6(11 - 4)$$

$$= -7.8 \text{ kN m.}$$

Part C - B $(11 \leq x \leq 1 - 5)$

$$M_X = -10 \times x + R_D \times (x - 4) - \frac{3(x - 11)^2}{2}$$

$$M_C = -10 \times 11 + 14.6 \times (11 - 4) - \frac{3 \times (11 - 11)^2}{2} \quad = -7.8\ kN - m$$

$$M_B = -10 \times 15 + 14.6 \times (15 - 4) - 3 \times \frac{(15 - 11)^2}{2} = -13.4\ kN\ m$$

Part B - A $(15 \leq x \leq 18)$

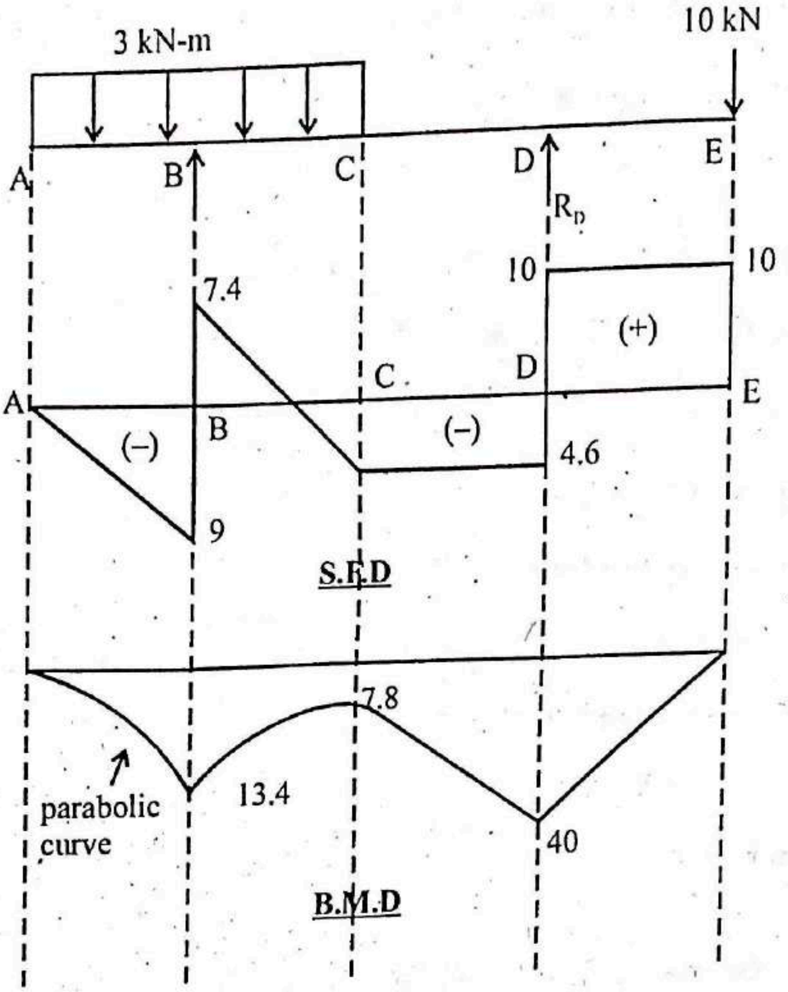
$$M_X = -10x + R_D \times (x - 4) - \frac{3 \times (x - 11)^2}{2} + R_B \times (x - 15)$$

$$M_B = -10 \times 15 + 14.6(15 - 4) - \frac{3 \times (15 - 11)^2}{2} + 16.4 \times (15 - 15)$$

$$= -13.4\ kN - m$$

$$M_A = -10 \times 18 + 14.6\ (18 - 4) - \frac{3 \times (18 - 11)^2}{2} + 16.4(18 - 15)$$

$$= 0\ .$$



Point of contraflexture occurs at point D which is 4m from E.

$$B.M_{max} = 40 \text{ kN-m}.$$

11. A beam ABCDE is simply supported over B and E. AB = CD = DE = 1m, BC = 3m. The beam is subjected to a concentrated load of 60 kN at A, an UDL of intensity 40 kN/m over the portion BC and a clockwise moment of 120 kN-m at D. Draw S.F. and B.M. diagrams. Locate the point of contraflexures if any.

Answer :

For equilibrium,

$$f_y = 0$$

$$R_B + R_E - 60 - 40 \times 3 = 0$$

$$\Rightarrow \quad R_B + R_E = 180$$

Taking moment about B, and equating the same

$$60 \times 1 - 40 \times 3 \times 1.5 - 120 + R_E \times 5 = 0$$

$$\Rightarrow 60 - 180 - 120 + 5R_E = 0$$

$$R_E = 48\ kN$$

Substituting the value of $R_E = 48\,kN$ in equation (1), we obtain

$$R_B = 180 - 48 = 132\ kN$$

we find shear force at every point ($\downarrow | \uparrow +$)

Part D - E $(0 \le x \le 1)$

$$F_x = +R_E$$

$$F_E = 48\,kN$$

$$F_D = 48\ kN$$

Part D - C $(1 \le x \le 2)$

$$F_D = 48\,kN$$

$$F_C = 48\ kN$$

Part C - B $(2 \le x \le 5)$

$$F_x = 48 - 48(x-2)$$

$$F_C = 48 - 40(2-2) = 48\,kN$$

$$F_B = 48 - 40(5-2) = -70\,kN$$

Part B - A $(5 \le x \le 6)$

$$F_x = 48 - 40 \times 3 + 132$$

$$F_B = 48 - 120 + 132 = 60\ kN$$

$$F_A = 60\,kN$$

Now, we find bending moment at every point ($\circlearrowleft | \circlearrowright +$)

Part E - D $(0 \le x \le 1)$

$$M_x = +R_E \times x$$

$$M_E = 48 \times 0 = 0$$

$$M_D = 48 \times 1 = 48\ kN - m$$

Part D - C $(1 \le x \le 2)$

$$M_x = 48 \times x - 120\ kN - m$$

$$M_D = 48 \times 1 - 120 = -72\ kN - m$$

$$M_C = 48 \times 2 - 120 = -24\ kN - m$$

Part C - B $(2 \le x \le 5)$

$$M_x = 48\times x - 120 - 40\times\frac{(x-2)^2}{2}$$

$$M_C = 48\times 2 - 120 - 40\times\left(\frac{2-2}{2}\right)^2 = -24\ kN-m$$

$$M_B = 48\times 5 - 120 - 40\times\left[\frac{5-2}{2}\right]^2 = 240-120-40\times\frac{9}{2} = -60\ kN-m$$

Part B - A $(5 \le x \le 6)$

$$M_x = 48\times x - 120 - 40\times 3\times(x-3.5) + 132\times(x-5)$$

$$M_B = 48\times 5 - 120 - 40\times 3\times(5-3.5) + 132(5-5)$$

$$= -60\ kN-m$$

$$M_A = 48\times 6 - 120 - 40\times 3\times(6-3.5) + 132(6-5) = 0.$$

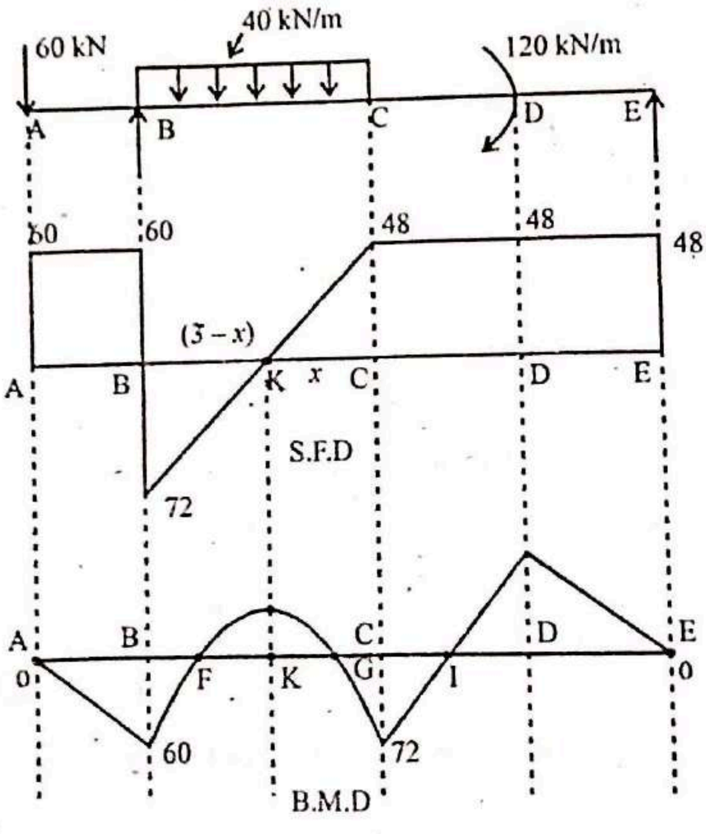
We know that the point where shear force changes sign, at that point bending in moment becomes maximum. Bending moment at point k;

$$\frac{x}{48} = \frac{(3-x)}{72}$$

$\Rightarrow$ $72x = 144 - 48x \Rightarrow 120x = 144$

$\therefore$ $x = 1.2$

$EK = 211.2 = 3.2$



$$M_K = 48\times 3.2 - 120 - 40\frac{(3.2-2)^2}{2} = 4.8\ kN-m$$

Here at three point bending moment changes sign, Here there is 3 point of contraflexture which is denoted by F, G and I in bending moment diagram.

12. A beam is simply supported at A and F. AB = 1m, BC = 2m, CD = 4m, DE = 1m, EF = 2m. Concentrated load of 40 kN at B, UDL of 5 kN/m over CD and a clockwise moment 120 kN-m at E are applied. Determine the slope at F and deflection at B and mid-point of beam.

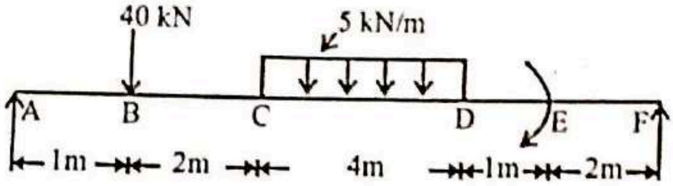
Answer :

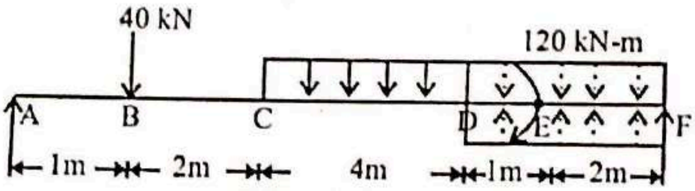
Taking moment about A, we get

$$R_F \times 10 = 5\times 4\times 5 + 120 + 40\times 1$$

$$\therefore\ R_F = 26\ kN$$

$$\therefore\ R_A = \text{Total load - } R_F = 40 + 5\times 4 - 26\ = 34\ kN$$





Taking moment about A, we get

$$R_F \times 10 = 5\times 4\times 5 + 120 + 40\times 1$$

$$\therefore\ R_F = 26\ kN$$

$$\therefore\ R_A = \text{Total load} - R_F = 40 + 5\times 4 - 26$$

$$= 34\ kN$$

Now, we apply macaulay's method to solve this question :

The B.M. at any section at a distance x from end A is given by

$$\text{E.I. } \frac{d^2y}{dx^2} = R_A x \,\vdots\, - 40(x-1) \,\vdots\, - 5\times(x-3)\times\frac{(x-3)}{2} + 5x(x-7)\frac{(x-7)}{2} \,\vdots\, - 120$$

$$= 34x \,\vdots\, - 40(x-1) \,\vdots\, - \frac{5}{2}(x-3)^2 \,\vdots\, + \frac{5}{2}(x-7)^2 \,\vdots\, - 120$$

Integrating the above equation, we get

$$E.I\frac{dy}{dx}=34\times\frac{x^2}{2}+C_1 \vdots -40\frac{(x-1)^2}{2} \vdots -\frac{5}{2}\frac{(x-3)^3}{3} \vdots +\frac{5}{2}\frac{(x-7)^3}{3} \vdots -120x \quad \text{.........(1}$$

Integrating again weget

$$E.I \quad y=\frac{34\times x^3}{2\times 3}+C_1x+C_2 \vdots -\frac{40\times(x-1)}{2\times 3}-\frac{5}{2}\frac{(x-3)^4}{3\times 4} \vdots +\frac{5}{2}\frac{(x-7)^4}{3\times 4} \vdots -120\frac{x^2}{2} \vdots$$

$$=\frac{17}{3}x^3+C_1x+C_2 \vdots -\frac{20}{3}\times(x-1)^3 \vdots$$

$$-\frac{5}{24}(x-3)^4 \vdots +\frac{5}{24}(x-7)^4 \vdots -60\,x^2 \quad \text{.........(2}$$

where C_1 and C_2 are constants of integration. Their values are obtained from bounda conditions which are:

(i) at $x=0,\ y=0$

(ii) at $x=10,\ y=0$

(i) Substituting $x=0$ and $y=0$ in eqn, (2) upto first dotted line (as $x=0$ lies in the first p AC of the beam, we get

$$0=0+C_1\times 0+C_2 \Rightarrow C_2=0$$

(ii) Substituting $x=10$ and $y=0$ in complete equation (2)

$$0=\frac{17}{3}\times(10)^3+C_1\times 10+0+\frac{5}{24}(10-7)^4$$

$$-\frac{5}{24}\times(10-3)^4-\frac{20}{3}\times(10-1)^3-60\times 1($$

$$0=5666.67+10C_1+16.875-500.2-4860-6000$$

$$\therefore \Rightarrow C_1=-567.67$$

Substituting the value of C_1 and C_2 in eqn, (2), we get

$$E.I_y=\frac{17}{3}x^3-567.67x \vdots -\frac{20}{3}(x-1).$$

$$-\frac{5}{24}(x-3)^4 \vdots +\frac{5}{24}(x-7)^4 \vdots -60\,x^2 \quad \text{......}$$

(a) Slope at point F

substituting $x = 10$ and $C_1 = -567.67$ in eqn. (1)

$$E.I\ \frac{dy}{dx} = 34 \times \frac{(10)^2}{2} - 567.67 - 40 \times \frac{(10-1)^2}{2}$$

$$-\frac{5}{2}\frac{(10-3)^3}{3} + \frac{5}{2}\frac{(10-7)^3}{3} - 120 \times 10$$

$$\Rightarrow E.I\ \theta_F = 1700 - 567.67 - 1620 - 285.83 + 22.5 - 1200$$

$$E.I\ \theta_F = -1951$$

$$\therefore \quad \theta_F = -\frac{1951}{E.I}\text{rad.}$$

(b) Deflection at B

substituting $x = 1$ in equation (3) we obtain

$$EI\ y_B = \frac{17}{3} \times (1)^3 - 567.67 \times 1 - 60 \times 1^2$$

$$\Rightarrow \quad EI\ y_B = 5.67 - 567.67 - 60 = -622.003$$

$$\therefore\ y_B = -\frac{622.003}{EI} mm$$

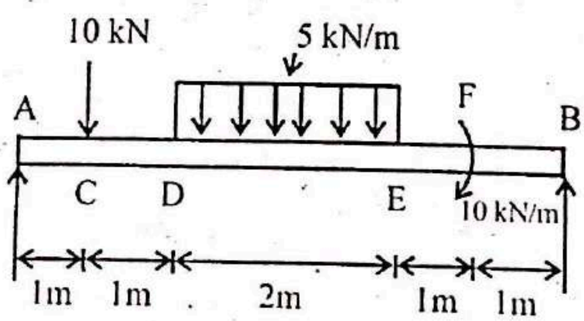
(c) Deflection at mid-point of beam (i.e. $x = 5m$)

$$EI\ \ y_B = \frac{17}{3} \times 5^3 - 567.67 \times 5 - \frac{20}{3} \times (5-1)^3 - \frac{5}{24}(5-3)^4 - 60 \times 5^2$$

$$= 708.33 - 2838.35 - 426.67 - 3.33 - 1500 = -4060.02$$

$$\therefore \quad y_B = -\frac{4060.02}{EI} mm.$$

13. A beam supported simply at two ends as shown in fig. Using singularity function, write the equation for the shear force and bending moment at any section of the beam and plot the S.P.D. and B.M.D. and locate points of inflection.



Answer :

Let R_A and R_B = Reaction at point A and B respectively.

$$\sum V = 0$$

$\therefore \quad R_A + R_B - 10 - 5 \times 2 = 0$

$\Rightarrow \quad R_A + R_B = 20$ ....(i

Now, Taking moment about A and equating the same.

$$R_B \times 6 - 10 - 5 \times 2 \times 3 - 10 \times 1 = 0$$

$\Rightarrow 6 . R_B = 10 + 30 + 10$

$\therefore R_B = 8.33\ kN$

and, $R_A = 20 - 8.33 = 11.67\ kN.$

Now, we find S.F at all point $(\uparrow | \downarrow +)$

Plane B-F $(0 \le x \le 1)$

$$F_x = -R_B$$

$$F_B = -8.33\ kN$$

$$F_F = -8.33\ kN$$

Plane F.E. $(1 \le x \le 2)$

$$F_x = -R_B$$

$$F_F = -8.33\ kN$$

$$F_E = -8.33\ kN$$

Plane E-D $(2 \le x \le 4)$

$$R_x = -R_B + 5x\ (x - 2)$$

$$F_E = -8.33 + 5\ (2 - 2) = -8.33\ kN$$

$$F_D = -8.33 + 5\ (4 - 2) = 1.67\ kN$$

Plane D-C $(4 \le x \le 5)$

$$F_x = -R_B + 10$$

$$F_D = -8.33 + 10 = 1.67\ kN$$

$$F_C = 11.67\ kN$$

Plane C - A $(5 \le x \le 6)$

$$F_x = -R_B + 10 + 10$$

$$F_C = 8.33 + 10 + 10 = 11.67\ kN$$

$$F_A = 11.67\ kN$$

Now, we find B.M at every point $(\uparrow | \uparrow +)$

Plane B-F $(0 \le x \le 1)$

$$M_x = +R_B . x$$

$$M_B = +8.33 \times 0 = 0$$

$$M_F = +8.33 \times 1 = +8.33 \text{kN} - \text{m}$$

Plane F-E $(1 \le x \le 2)$

$$M_x = +R_B \times x - 10$$

$$M_F = 8.33 \times 1 - 10 = -1.67$$

$$M_E = 8.33 \times 2 - 10 = 6.67 \text{ KN} - \text{m}$$

Plane E-D $(2 \le x \le 4)$

$$M_x = R_B \times x - 10 - 5 \times (x-2) \times \frac{(x-2)}{2}$$

$$M_E = 8.33 \times 2 - 10 - 5(2-2)\frac{(2-2)}{2} = +6.67\ kN$$

$$M_D = 8.33 \times 4 - 10 - 5(4-2)\frac{(4-2)}{2}$$

$$= 33.32 - 10 - 10 = 13.32\ kN\ m.$$

Plane D-C $(4 \le x \le 5)$

$$M_x = R_B \times x - 10 - 5 \times 2 \times (x-3)$$

$$M_D = 8.33 \times 4 - 10 - 5 \times 2 \times (4-3) = 13.32\ kN - m$$

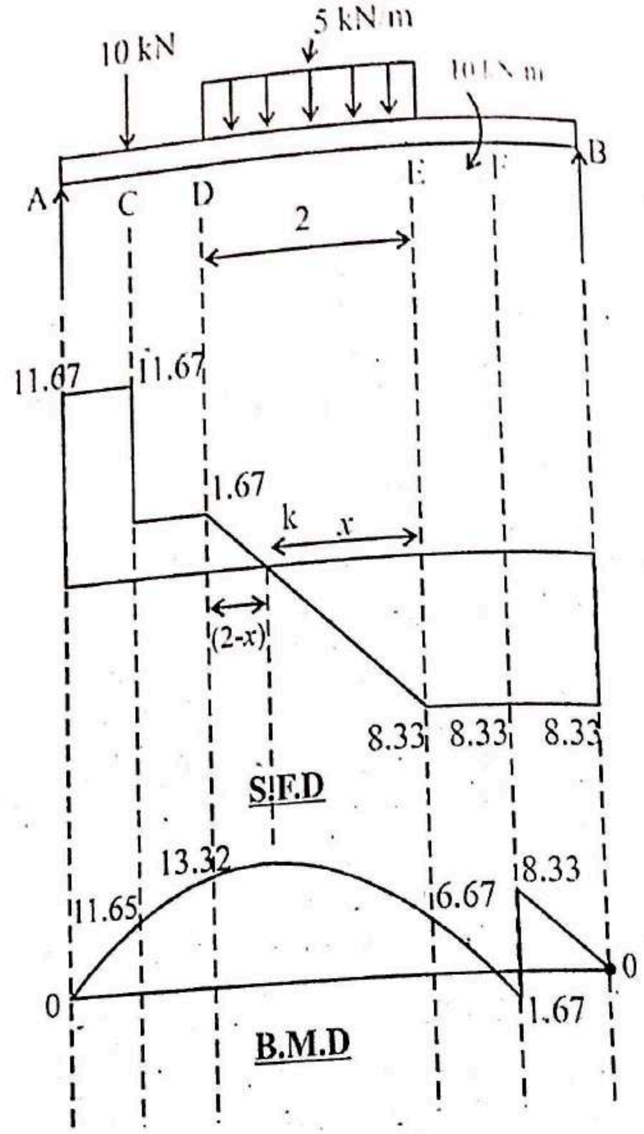
$$M_C = 8.33 \times 5 - 10 - 5 \times 2 \times (5-3) = 11.65\ kN\ m.$$

Plane C - A $(5 \le x \le 6)$

$$M_x = R_B \times x - 10 - 5 \times 2 \times (x-3) - 4(x-5)$$

$$M_C = 8.33 \times 5 - 10 - 5 \times 2 \times (5-3) - 10(5-5) = 11.65\ kN\ m.$$

$M_A = 8.33\times 6 - 10 - 5\times 2(6-3) - 10\,(6-5) = 0$ kN...



from similar triangle

$$\frac{1.67}{2-x} = \frac{8.33}{x}$$

$\Rightarrow \quad 1.67x = 16.67 - 8.33x$

$\Rightarrow \quad x = 1.67\,m$

Put $\quad x = 1.67 + 2 = 3.67$ in equation of plane E-D.

$$B.M_{\max} = 8.33\times 3.67 - 10 - 5\times(67-2)\,\frac{(3.67-2)}{2} = 13.59$$

❊ ❊ ❊

UNIT 7

TORSIONAL MOTION

Chapter at a glance

- *Torsion* ⇒twisting of a structural member, when it is loaded by couples that produce rotation about its longitudinal axis.
- *Torsional Deformation of a Circular Bar*⇒shear strain inside the bar $\gamma = \frac{\rho}{r}\gamma_{max}$
- *Torsional Deformation of a Circular tube*⇒shear strain $\gamma_{min} = \frac{r_1}{r_2}\gamma_{max}$
- *Torsional flexibility*⇒ $f = \frac{L}{GI_p}$
- *Torsional stiffness*⇒ $k = \frac{GI_p}{L}$
- *Shear stress*⇒ $\tau = \frac{T\rho}{I_p}$
- *For a circular tube* ⇒ $I_p = \frac{\pi(r_2^4 - r_1^4)}{2} = \frac{\pi(d_2^4 - d_1^4)}{32}$

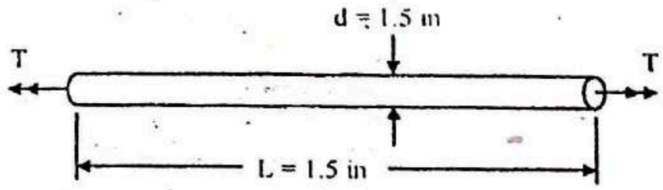
Long Answer Type Questions

1. A solid bar of circular cross-section

$d = 40\,\text{mm},\ L = 1.3\,\text{m},\ G = 80\,\text{GPa}$

a) $T = 340$ **N-m,** $\tau_{max},\ \phi = ?$

b) $\tau_{all} = 42$ **MPa,** $\phi_{all} = 2.5°,\ T = ?$



Answer:

(a) $\tau_{max} = \frac{16T}{\pi d^3} = \frac{16 \times 340\,\text{N-m}}{\pi(0.04m)^3} = 27.1\,\text{MPa}$

$$I_p = \frac{\pi d^4}{32} = 2.51 \times 10^{-7} m^4$$

$$\phi = \frac{TL}{GI_p} = \frac{340\,\text{N-m} \times 1.3m}{80\,\text{GPa} \times 2.51 \times 10^{-7} m^4} = 0.02198\,\text{rad} = 1.26°$$

(b) Dut to $\tau_{all} = 42\,\text{MPa}$

$$T_1 = \pi d^3 \frac{\tau_{all}}{16} = \pi(0.04m)^3 \times \frac{42\,\text{MPa}}{16} = 528\,\text{N-m}$$

Due to $\phi_{all} = 2.5^\circ = \dfrac{2.5 \times \pi \text{ rad}}{180^\circ} = 0.04363 \text{ rad}$

$$T_2 = \frac{GI_p\phi_{all}}{L} = \frac{80\text{GPa} \times 2.51 \times 10^{-7} m^4 \times 0.04363}{1.3m} = 674 \text{ N-m}$$

Thus $T_{all} = \min\left[T_1, T_2\right] = 528 \text{ N-m}$

2. A steel shaft of either solid bar or circular tube

$$T = 1200 \text{ N-m}, \quad \tau_{all} = 40 \text{ MPa}$$

$$\theta_{all} = \frac{0.75^\circ}{m}, \qquad G = 78 \text{ GPa}$$

a) determine d_0 of the solid bar

b) for the hollow shaft, $t = \dfrac{d_2}{10}$, determine d_2

c) determine $\dfrac{d_2}{d_0}$, $\dfrac{W_{hollow}}{W_{solid}}$

(a) (b)

Answer:

(a) For the solid shaft, due to $\tau_{all} = 40 \text{ MPa}$

$$d_0^3 = \frac{16T}{\pi\tau_{all}} = \frac{16 \times 1200}{\pi 40} = 152.8 \times 10^{-6} \text{ m}^3$$

$$d_0 = 0.0535 \text{ m} = 53.5 \text{ mm}$$

Due to $\theta_{all} = 0.75^\circ/\text{m} = \dfrac{0.75 \times \pi \text{ rad}}{180^\circ/\text{m}} = 0.01309 \text{ rad/m}$

$$I_p = \frac{T}{G\theta_{all}} = \frac{1200}{78 \times 10^9 \times 0.01309} = 117.5 \times 10^{-8} \text{ m}^4$$

$$d_0^4 = \frac{32I_p}{\pi} = \frac{32 \times 117.5 \times 10^{-8}}{\pi} = 1197 \times 10^{-8} \text{ m}^4$$

$$d_0 = 0.0588 \text{ m} = 58.8 \text{ mm}$$

Thus, we choose $d_0 = 58.8 \text{ mm}$ [in practical design, $d_0 = 60 \text{ mm}$]

(b) For the hollow shaft

$$d_1 = d_2 - 2t = d_2 - 0.2d_2 = 0.8d_2$$

$$I_p = \frac{\pi\left(d_2^4 - d_1^4\right)}{32} = \frac{\pi\left[d_2^4 - (0.8d_2)^4\right]}{32} = 0.05796d_2^4$$

Due to $\tau_{all} = 40 \text{ MPa}$

$$I_p = 0.05796 d_2^4 = \frac{Tr}{\tau_{all}} = \frac{1200\left(\frac{d_2}{2}\right)}{40}$$

$$d_2^3 = 258.8 \times 10^{-6}\ \text{m}^3$$

$$d_2 = 0.0637\ \text{m} = 63.7\ \text{mm}$$

Due to $\theta_{all} = 0.75°/\text{m} = 0.01309\ \text{rad/m}$

$$\theta_{all} = 0.01309 = \frac{T}{GI_p} = \frac{1200}{78 \times 10^9 \times 0.05796 d_2^4}$$

$$d_2^4 = 2028 \times 10^{-8}\ \text{m}^4$$

$$d_2 = 0.0671\ \text{m} = 67.1\ \text{mm}$$

Thus, we choose $d_0 = 67.1\ \text{mm}$ [in practical design. $d_0 = 70\ \text{mm}$]

(c) The ratios of hollow and solid bar are

$$\frac{d_2}{d_0} = \frac{67.1}{58.8} = 1.14$$

$$\frac{W_{hollow}}{W_{solid}} = \frac{A_{hollow}}{A_{solid}} = \frac{\frac{\pi\left(d_2^2 - d_1^2\right)}{4}}{\frac{\pi d_0^2}{4}} = 0.47$$

The hollow shaft has 14% greater in diameter but 53% less in weight.

3. Draw and explain the shear stress distribution through the cross-section of a circular shaft under pure torsion for hollow shaft.

Answer :

Expression for shear stress distribution :

When a circular shaft is subjected to torsion, shear stresses are set up in the material of the shaft. To determine the magnitude of shear stress at any point on the shaft, consider a shaft fixed at one end *AA* and free at the end *BB* as shown in fig. Let *CD* is any line on the outer surface of the shaft. Now let the shaft is subjected to a torque *T* at the end *BB* as shown in fig. As a result of this torque *T*, the shaft at the end *BB* will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses. The point *D* will shift to *D'* and hence line *CD* will be deflected to *CD'* as shown in fig.(a). The line *OD* will be shifted to *OD'* as shown in fig.(b).

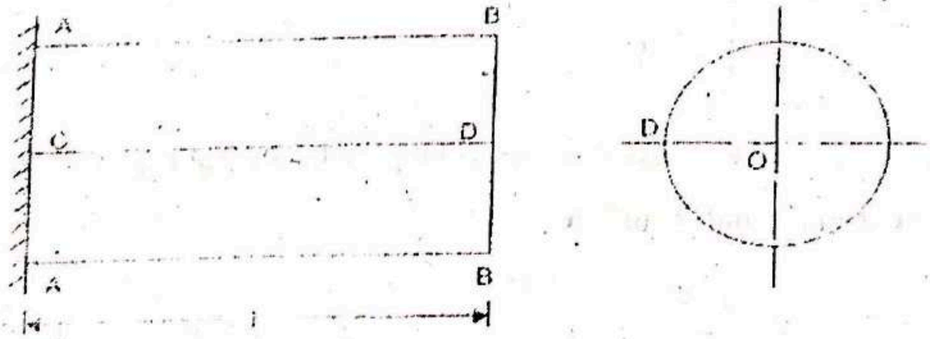


Fig. Shaft fixed at one end *AA* before torque *T* is applied.

Let R = Radius of shaft

L = Length of shaft

T = Torque applied at the end BB

τ = Shear stress induced at the surface of the shaft due to torque T

C = Modulus of rigidity of the material of the shaft.

$\phi = \angle DCD'$ also equal to shear strain

$\theta = \angle DOD'$ and is also called angle of twist

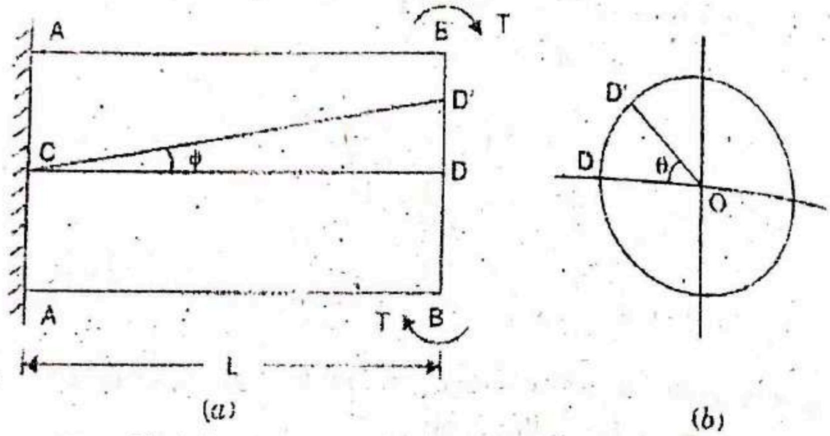


Fig. Shaft fixed at AA and subjected to torque T at BB.

Now distortion at the outer surface due to torque T

$$= DD'$$

$\therefore$ Shear strain at outer surface

$$= \text{Distortion per unit length}$$

$$= \frac{\text{Distortion at the outer surface}}{\text{Length of shaft}} = \frac{DD'}{L}$$

$$= \frac{DD'}{CD} = \tan\phi$$

$$= \phi \qquad (\text{If } \phi \text{ is very small then } \tan\phi = \phi)$$

$\therefore$ Shear strain at outer surface,

$$\phi = \frac{DD'}{L}$$

Now from Fig.(b)

Arc $DD' = OD \times \theta = R\theta$ $(\because OD = R = \text{Radius of shaft})$

Substituting the value of DD' m equation (1), we get

Shear strain at outer surface

$$\phi = \frac{R \times \theta}{L}$$

Now the modulus of rigidity (C) of the material of the shaft is given as

$$C = \frac{\text{Shear stress induced}}{\text{Shear strain produced}} = \frac{\text{Shear stress at the outer surface}}{\text{Shear strain at outer surface}}$$

$$= \frac{\tau}{\left(\frac{R\theta}{L}\right)} \qquad \left(\because \text{From equation (ii), shear strain} = \frac{R\theta}{L}\right)$$

$$= \frac{\tau \times L}{R\theta}$$

$$\therefore \quad \frac{C\theta}{L} = \frac{\tau}{R}$$

$$\therefore \quad \tau = \frac{R \times C \times \theta}{L}$$

Now for a given shaft subjected to a given torque (T), the values of C, θ and L are constant. Hence shear stress produced is proportional to the radius R.

$$\therefore \quad \tau \propto R \quad \text{or} \quad \frac{\tau}{R} = \text{constant}$$

If q is the shear stress induced at a radius 'r' from the centre of the shaft then

$$\frac{\tau}{R} = \frac{q}{r}$$

But $\quad \frac{\tau}{R} = \frac{C\theta}{L}$ from equation

$$\therefore \quad \frac{\tau}{R} = \frac{C\theta}{L} = \frac{q}{r}$$

From eqn., (iii), it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

4. **What are the assumptions made in the derivation of torsional formula ?**

Answer :

The following assumptions are made in the derivation of the torsion equation for twisting of a shaft of solid circular section :

(i) The bar is straight and has uniform circular cross-section over its length.

(ii) The shaft is subjected to a uniform twisting moment.

(iii) The stress induced in the shaft remains within the proportional limit.

(iv) Radial lines remain straight and radial.

(v) Plane sections remain plane and distance between any two sections remains constant.

(vi) Generators parallel to longitudinal axis are assumed to remain straight during torsion (although they deform into helices during torsion).

5. (a) **What do you mean by core of the section. Find the core of a hollow circular section having external diameter D and internal diameter d.**

(b) **A solid circular shaft is subjected to a bending moment M and torque T. Find the equivalent torque Tc. Which will produce the same maximum sheer stress as produced by the bending moment and torque acting simultaneously.**

Answer :

(a) Core of the section :

For a hollow circular section, having external diameter D and internal diameter d,

$$I = \frac{\pi}{64}(D^4 - d^4) \text{ and } A = \frac{\pi}{4}(D^2 - d^2)$$

$$\therefore \quad K^2 = \frac{I}{A} = \frac{D^4 - d^4}{16(D^2 - d^2)} = \frac{D^2 + d^2}{16}$$

For no tension to develop

$$e \leq \frac{2k^2}{d} \quad \text{Hence} \leq \frac{2}{d}\left(\frac{D^2 + d^2}{16}\right) \leq \frac{D^2 + d^2}{8D}$$

$$\therefore \quad e_{max} = \frac{D^2 + d^2}{8D}$$

Hence the core for a hollow circular section is a concentric circle of diameter $\frac{D^2 + d^2}{4D}$

Answer :

(b) Considering any point on the cross section of a shaft

Let T = Torque at the section ; D = Diameter of the shaft

M = B.M at the section

The torque T will produce shear stress at the point where as the B.M will produce bending stress

Let q = shear stress at the point produced by torque T

σ = Bending stress at the point produced by B.M(M)

The shear stress at any point due to torque (τ) is given by

$$q/r = \tau/J \qquad \left[\because \frac{q}{r} = \frac{\tau}{R} = \frac{T}{J}\right]$$

or
$$q = \frac{T}{J} \times r$$

The bending stress at any point due to bending moment (M) is given by

$$\frac{M}{I} = \frac{\sigma}{Y} \quad or \quad \sigma = \frac{M \times y}{I}$$

We know that the angle θ made by the plane of maximum shear with the normal cross - section is given by

$$\tan 2\theta = \frac{2I}{\sigma}$$

The bending stress and shear stress is maximum at a point on the surface of the shaft , Where

$$r = R = D/2 \text{ and } Y = D/2$$

σ_b = Maximum bending stress i.e on the surface of the shaft

$$\frac{M}{I} \times \frac{D}{2} = \frac{M}{\pi/64 \times D^4} \times D/2 = 32M/\pi D^3$$

τ = Maximum shear stress i.e on the surface of the shaft

$$\frac{T}{J} \times R = \frac{T}{\pi/32 \times D^4} \times D/2 = 16T/\pi D^3$$

$$\therefore \quad \tan\theta = \frac{2\tau}{\sigma} = \frac{2\tau_c}{\sigma_b} = \frac{2 \times 16\, T/\pi D^3}{32M/\pi D^3} = \frac{T}{M}$$

Major Principle Stress

$$\frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{\sigma_b}{2} + \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_s^2}$$

$$= \frac{32M}{2\times\pi D^3} + \sqrt{\left(\frac{32M}{2\times\pi D^3}\right)^2 + \frac{16T}{\pi D^3}}$$

$$= \frac{16}{\pi D^3}(M + \sqrt{M^2+T^2}$$

and Minor Principle Stress

$$= \frac{16}{\pi D^3}(M - \sqrt{M^2+T^2})$$

∴ Maximum Shear Stress

= Major principle stress - Minor principle stress / 2

$$= \frac{16}{\pi D^3}\sqrt{M^2+T^2}) = \frac{16}{\pi D^3}.T_e \quad \text{Where } T_e = \sqrt{M^2+T^2}$$

∴ Equivalent torque Te which will produce the same maximum shear stress as produced by the bending moments and torque acting simultaneously.

6. A shaft of the same material and same length are subjected to the same torque. The first shaft is of a solid circular section and the second shaft is of hollow circular whose internal diameter is ¾ th of the outside diameter. Compare the weights of two shafts.

Answer :

Inner Dia. of hollow shaft

Dia. = $3/4$ *Dia.at outside*

$= 3/4\, D_0 = 0.75 D_0$

Torque transmitted by a solid shaft

$$T = \frac{\pi}{16}\tau.D^3 \quad \text{.................(i)}$$

Torque transmitted by hollow shaft

$$T = \frac{\pi}{16}\tau.\left[\frac{D_0^{\,4} - D_i^{\,4}}{D_0}\right] = \frac{\pi}{16}\tau.\left[\frac{D_0^{\,4} - (0.75D_0)^4}{D_0}\right]$$

$$= \frac{\pi}{16}\tau D_0^{\,3} \times [1 - 0.3164]$$

$$= \frac{\pi}{16}\tau \times D_0^{\,3} \times 0.6836 \quad \text{......(ii)}$$

But torque transmitted by solid shaft = torque transmitted by hollow shaft

from eq. (i) and (ii)

$$=\frac{\pi}{16}\times\tau\times D^3=\frac{\pi}{16}\times\tau\times D_0^{\,3}\times 0.6836$$

$$\Rightarrow \qquad D^3 = 0.6836\ D_0^{\,3}$$

$$\therefore \qquad D = 0.8809\ D_0 \qquad \text{.... (iii)}$$

Now weight of solid shaft

$$W_S = \rho\times g\times volume$$

$$=\rho\times g\times\left(\frac{\pi}{4}D^2\times L\right) \qquad \text{...... (iv)}$$

similarly weight of hollow shaft

$$W_h=\rho\times g\times\left(\frac{\pi}{4}(D_0^{\,2}-D_1^{\,2}\right)\times L$$

$$=\rho\times g\times\frac{\pi}{4}\times\left((D_0^{\,2}-0.5625D_0^{\,2}\right)\times L$$

$$=\rho\times g\times\frac{\pi}{4}\times D_0^{\,2}\left((1-0.5625\right)\times L$$

$$=\rho\times g\times\frac{\pi}{4}\times D_0^{\,2}\times 0.4375\times L \qquad \text{.... (v)}$$

Dividing equation (iv) by (v)

$$\frac{W_S}{W_H}=\frac{\rho\times g\times\frac{\pi}{4}\times D^2\times L}{\rho\times g\times\frac{\pi}{4}\times D_0^{\,2}\times 0.4375\times L}$$

$$=\frac{D^2}{0.4375\ D_0^{\,2}}=\frac{(0.8809D_0)^2}{0.437D_0^{\,2}}=\frac{(0.776D_0^{\,2}}{0.4375D_0^{\,2}}=1.7737$$

7. Determine the diameter of a solid shaft which will transit 337.5 KW at 300 rpm. The maximum shear stress should not exceed 35 N/mm² and twist should not more than 1° in a shaft lenght of 2.5 m. Take modulus of rigidity = 9 ×10⁴ N/mm².

Answer :

Power transmitted,

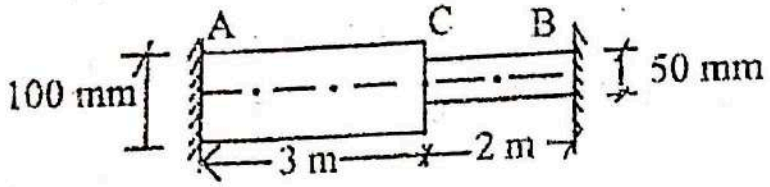
$$P = \frac{2\pi NT}{60000};\ T = \frac{60000P}{2\pi N} = \frac{60000 \times 337.5}{2\pi \times 300} Nm = 10742.95\ Nm$$

$$T = \frac{f_s \pi d^3}{16};\ d^3 = \frac{16T}{f_s \pi} = \frac{16 \times 10742.95 \times 10^3}{35\pi}$$

$$= 1563.24 \times 10^3\ mm^3\ ;\ d = 116\ mm$$

8. **Two solid shafts AC and CB of brass and copper are rigidly fastened together at C as shown in figure below. A torque of 100 Nm is applied at the junction C. Compute the maximum shear stress in each material and angle of twist at the junction.**

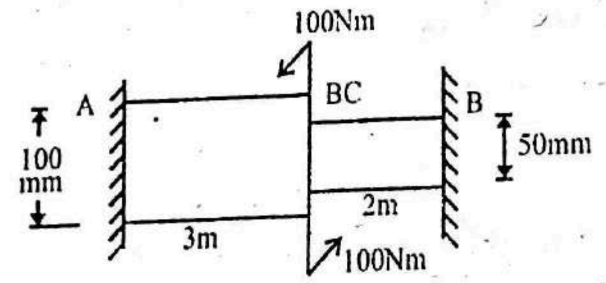
Take $G_b = 8 \times 10^4\ MPa$ **and** $G_c = 3 \times 10^4\ MPa$



Answer :

Solid shaft AC

Material → Brass



Length, $L_1 = 3m = 3000\,mm$

$d_1 = 100\,mm$

$G_b = 8 \times 10^4\ MPa$

$= 8 \times 10^4\ N/mm^2$

Solid shaft CB

Material → Copper

$$L_2 = 2m = 2000\,mm$$

$$d_2 = 50\,mm$$

$$G_C = 3\times10^4\ MPa$$

$$= 3\times10^4\ N/mm^2$$

$$T = 100\,Nm = 100000\,Nmm$$

The torque is applied at the junction B, hence the angle of twist in shaft AC and in shaft CB will be same (i.e., $\theta_1 = \theta_2 = \theta$)

where, θ_1 = Angle of twist in shaft AB

θ_2 = Angle of twist in shaft BC

Let T_1 = Torque transmitted to shaft AC

T_2 = Torque transmitted to shaft OCB

$$T_1 + T_2 = 100000\,\text{Nmm} \qquad \text{......... (i)}$$

Using eqn., we get

$$\frac{T}{J} = \frac{C\times\theta}{L}$$

For the shaft AC, the above equation becomes as

$$\frac{T_1}{J_1} = \frac{C_1\times\theta_1}{L_1}$$

$$\theta_1 = \frac{T_1\times L_1}{J_1\times G_1} \quad \text{where, } J_1 = \frac{\pi}{32}\times d^2$$

$$= \frac{\pi}{32}\times(100)^4$$

$$= \frac{T_1\times3000}{\frac{\pi}{32}\times(100)^4\times8\times10^4}$$

$$= \frac{T_1\times3000\times32}{\pi\times(100)^4\times8\times10^4} \qquad \text{....... (ii)}$$

For shaft CB, the value of θ_2 is given by,

$$\theta_2 = \frac{T_2 \times L_2}{J_2 \times G_2} \text{ where, } J_2 = \frac{\pi}{32} \times (50)^4$$

$$= \frac{T_2 \times 2000}{\frac{\pi}{32} \times (50)^4 \times 3 \times 10^4}$$

$$= \frac{T_2 \times 2000 \times 32}{\pi \times (50)^4 \times 3 \times 10^4}$$

But $\theta_1 = \theta_2$

$$\therefore \quad \frac{T_1 \times 3000 \times 32}{\pi \times (100)^4 \times 8 \times 10^4} = \frac{T_2 \times 2000 \times 32}{\pi \times (50)^4 \times 3 \times 10^4}$$

$$\Rightarrow \quad T_1 \times 3 \times (50)^4 \times 3 = T_2 \times 2 \times (100)^4 \times 8$$

$$\Rightarrow \quad T_1 = \frac{T_2 \times 2 \times (100)^4 \times 8}{3 \times (50)^4 \times 3}$$

$$T_1 = 28.44\ T_2$$

Substituting this value in eqn., (1)

$$28.44 T_2 + T_2 = 100000$$

$$\Rightarrow \quad 29.44\ T_2 = 100000$$

$$\therefore\ T_2 = 3396.74\ N-mm$$

$$\therefore\ T_2 = 28.44 \times 3396.74$$

$$= 96603.26\ N-mm$$

Using eqn, $\frac{T}{J} = \frac{\tau}{R}$

for shaft AC, $\frac{T_1}{J_1} = \frac{\tau_1}{R_1}$

$$\Rightarrow \quad \tau_1 = \frac{T_1 \times R_1}{J_1} = \frac{96603.26 \times 50 \times 32}{\pi \times (100)^4}$$

). **A solid shaft is required totransmit 750 kW at 60 r.p.m. If the maximum value of shear stess is not to exceed 50 N/mm², calculate the dimeter of shaft. If this shaft is replaced by a hollow shaft. If this shaft is replaced by a hollow shaft of diameter ratio 0.6, then what will be the percentage saving? The torque, maximum shear stress, material and length of shafts are same in either case.**

Ans. Here, given

$$P = 750 \text{ kN}, N = 60 \text{ r.p.m.},$$

$$\tau = 50\, N/mm^2$$

Since Torque
$$\tau = \frac{\pi}{16} \times \Sigma \times D^3$$

$\therefore$
$$P = \frac{2\pi NT}{60} kW$$

Here
$$T = \frac{\pi}{16} \times \tau \times D^3$$

Le
$$T = N\,mm = 10^{-6}\, kNm$$

Now
$$P = \frac{2\pi NT}{60} kN$$

$\Rightarrow$
$$750 = \frac{2\pi \times 60 \times \pi \times 50 \times D^3 \times 10^{-6}}{60 \times 16}$$

$\Rightarrow$
$$3\sqrt{\frac{750 \times 16}{2\pi^2 \times 50 \times 10^{-6}}} = D$$

$$D = 229.95\, mm$$

Diameter of the shaft = 229.95mm.

% saving of material :

$$\% \text{ saving} = \frac{T_1 - T_2}{T_1} \times 100$$

where
$$T_1 = \frac{\pi}{IC} \times \tau \times D^3$$

$$T_2 = \frac{\pi}{IC} \times \tau \times \left(\frac{D^4 - d^4}{D}\right)$$

$$= \frac{D^3 - \frac{D^4 - d^4}{D}}{D^2} \times 100$$

$$= \frac{D^4 - D^4 + d^4}{D^4} \times 100$$

$$= \frac{1}{(1 \div (0.6)^4)} \times 100 = 12.96\%.$$

10. Define the following :–

(i) Torsional rigidity **(ii) Strength of a shaft**

Answer :

(i) Torsional rigidity : Torsional rigidity or stiffness of the shaft is defined as the produc modulus of rigidity (C) and polar moment of inertia of the shaft (J). Mathematically, The sional rigidity is given as,

Torsional rigidity = C × J

Torsional rigidity is also defined as the torque required to produce a twist of one radian unit length of the shaft.

(ii) Strength of a shaft : The strength of shaft means the maximum torque or maxim power of the shaft can transmit.

11. A hollow steel shaft, 200 mm internal diameter and 300 mm external diameter is to replaced by a solid alloy shaft. If the polar modulus has the same value for both, cal late the diameter of the solid shaft and the ratio of torsional rigidities. [G for steel = × G for alloy.]

Answer :

Given, internal dia. of hollow shaft, $d_i = 200$ mm external dia of hollow shaft, $d_0 = 300$

Dia of solid shaft = d m

From the question,

Polar modulus of hollow shaft = polar modulus of solid shaft

$$\frac{J_{\text{hollow shaft}}}{R_{\text{hollow shaft}}} = \frac{J_{\text{solid shaft}}}{R_{\text{solid shaft}}}$$

$$= \frac{\frac{\pi}{32} \times \left(D_0^4 - D_i^4\right)}{\frac{D_0}{2}} = \frac{\frac{\pi}{32} \times D^4}{D/2}$$

$$= \frac{D_0^4 - D_2^4}{D_0} = D^3$$

$$= \frac{300^4 - 200^4}{300} = D^3$$

$\therefore$ $$D = \sqrt[3]{21666666.67} = 278.782 \text{ mm}$$

Torsional rigidity for hollow steel shaft,

$$= G \times J = G \times \frac{\pi}{32} \times \left(D_0^4 - D_2^4\right)$$

Similarly, torsional rigidity for solid alloy shaft

$$= 2.5 \times G \times \frac{\pi}{32} \times D^4$$

$\therefore$ ratio of torsional rigidities

$$= \frac{\text{Torsional rigidities of hollow shaft}}{\text{Torsional rigidities of solid shaft}}$$

$$= \frac{G \times \pi/32 \times \left(300^4 - 200^4\right)}{2.5 \times G \times \pi/32 \times (278.782)^4}$$

$$= 0.43 = \frac{43}{100}$$

12. Find the maximum shear stress induced in a solid circular shaft of diameter 15 cm when the shaft transmits 150 kW power at 180 r.p.m.

Answer :

Given :

Diameter of shaft, $D = 15 \text{ cm} = 150 \text{ mm}$

Power transmitted, $P = 150 \text{ kW} = 150 \times 10^3 \text{ W}$

Speed of shaft, $N = 180 \text{ r.p.m.}$

Let τ = Maximum shear stress induced in the shaft

Power transmitted is given by equation (16.7) as

$$P = \frac{2\pi NT}{60}$$

$$150 \times 10^3 = \frac{2\pi \times 180 \times T}{60}$$

$$\therefore \quad T = \frac{150 \times 10^3 \times 60}{2\pi \times 180}$$

$$= 7957.7 \text{ Nm} = 7957700 \text{ Nmm}$$

Now using equation (16.4) as.

$$T = \frac{\pi}{16} \tau D^3$$

$$7957700 = \frac{\pi}{16} \times \tau \times 150^3$$

$$\therefore \quad \tau = \frac{16 \times 7957700}{\pi \times 150^3}$$

$$= 12 \text{ N/mm}^2 \text{ Ans.}$$

13. A solid cylindrical shaft is to transmit 300 kW power at 100 r.p.m.

(a) If the shear stress is not to exceed 80 N/mm², find its diameter.

(b) What percent saving in weight would be obtained if this shaft is replaced by a hollo[w] one whose internal diameter equals to 0.6 of the external diameter, the length, t[he] material and maximum shear stress being the same ?

Answer :

Given :

Power, $P = 300 \text{ kW} = 300 \times 10^3 \text{ W}$

Speed, $N = 100$

Max. shear stress, $\tau = 80 \text{ N/mm}^2$

(a) Let D = Dia. of solid shaft

Using eqation (16.7),

$$P = \frac{2\pi NT}{60}$$

$$300 \times 10^3 = \frac{2\pi \times 100 \times T}{60}$$

$$\therefore \quad T = \frac{300 \times 10^3 \times 60}{2\pi \times 100}$$

$= 28647.8$ N-m

$= 28647800$ N-mm

Now using equation (16.4),

$$T = \frac{\pi}{16} \times \tau \times D^3$$

or,
$$28647800 = \frac{\pi}{16} \times 80 \times D^3$$

$$D = \left(\frac{16 \times 28647800}{\pi \times 80}\right)^{1/3} = 121.8 \text{ mm}$$

= **say 122.0 mm Ans.**

(b) Percent saving in weight

Let D_0 = External dia. of hollow shaft

D_i = Internal dia. of hollow shaft

$= 0.6 \times D_0$ (given)

The length material and maximum shear stress in solid and hollow shafts are given the same. Hence torque transmitted by solid shaft is equal to the torque transmitted by hollow shaft. But the torque transmitted by hollow shaft is given by equation (16.6).

∴ Using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$= \frac{\pi}{16} \times 800 \times \frac{[D_0^4 - (0.6 D_0)^4]}{D_0}$$

$(\because D_i = 0.6 D_0)$

$$= \pi \times 50 \times \frac{[D_0^4 - (0.6 D_0)^4]}{D_0}$$

But torque transmitted by solid shaft = 28647800 N-mm

∴ Equating the two torques, we get

$$28647800 = \pi \times 50 \times \left(\frac{0.8704 D_0^4}{D_0}\right) = \pi \times 50 \times 0.8704 D_0^3$$

$$\therefore \quad D_o = \left(\frac{28647800}{\pi \times 50 \times 0.8704}\right)^{1/3}$$

$$= 127.6 \text{ mm} = \text{say } 128 \text{ mm}$$

$\therefore$ Internal dia.

$$D_i = 0.6 \times D_0 = 0.6 \times 128 = 76.8 \text{ mm}$$

Now let

W_s = Weight of solid shaft.

and

W_h = Weight of hollow shaft.

Then

W_s = Weight density × Area of solid shaft × Length

$$= w \times \frac{\pi}{4} D^2 \times L \text{ (where } w = \text{weight density)}$$

Similarly, W_h = Weight density × Area of hollow shaft × Length

$$= w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L$$

($\because$ Both shafts are of same length and of same material)

Now percent saving in weight

$$= \frac{W_s - W_h}{W_s} \times 100$$

$$= \frac{w \times \frac{\pi}{4} D^2 \times L - w \times \frac{\pi}{4}[D_0^2 - D_i^2] \times L}{w \times \frac{\pi}{4} D^2 \times L}$$

$$= \frac{D^2 - (D_0^2 - D_i^2)}{D^2} \times 100 \qquad \left(\text{Cancelling } w \times \frac{\pi}{4} \times L\right.$$

$$= \frac{122^2 - (128^2 - 76.8^2)}{122^2} \times 100$$

$$= \frac{14884 - (16384 - 5898)}{14884} \times 100$$

$$= \frac{1484 - 10486}{14884} \times 100 = \mathbf{29.55\%} \textbf{ Ans.}$$

14. A solid steel shaft has to transmit 75 kW at 200 r.p.m. Taking allowable shear stress as 70 Nmm², find suitable diameter for the shaft, if the maximum torque transmitted at each revolution exceeds the mean by 30%.

Answer :

Given :

Power of transmitted, $P = 75 \text{ kW} = 75 \times 10^3 \text{ W}$

R.P.M. of the shaft, $N = 200$

Shear stress, $\tau = 70 \text{ N/mm}^2$

Let T = Mean torque transmitted

T_{max} = Maximum torque transmitted

$= 1.3\ T$

D = Suitable diameter of the shaft

Power is given by the relation,

$$= \frac{2\pi NT}{60}$$

or,
$$75 \times 10^3 = \frac{2\pi \times 200 \times T}{60}$$

$$\therefore \quad T = \frac{75 \times 10^3 \times 60}{2\pi \times 200}$$

$$= 3580.98 \text{ N-m}$$

$$= 3580980 \text{ N-mm}$$

$$\therefore \quad T_{max} = 1.3\ T = 1.3 \times 3580980$$

$$= 4655274 \text{ N-mm.}$$

Maximum torque transmitted by a solid shaft is given by equation (16.4) as,

$$T_{max} = \frac{\pi}{16} \times \tau \times D^3$$

or,
$$4655274 = \frac{\pi}{16} \times 70 \times D^3$$

$$\therefore \quad D = \left(\frac{16 \times 4655274}{\pi \times 70}\right)^{1/3}$$

$$= \mathbf{69.57\ mm = 70\ mm\ Ans.}$$

15. **A hollow shaft is to transmit 300 kW power at 80 r.p.m. If the shear stress is not to exceed 60 N/mm² and the internal diameter is 0.6 of the external diameter, find the external and internal diameters assuming that the maximum torque is 1.4 times the mean.**

Answer :

Given :

Power transmitted, $P = 300\ \text{kW} = 300 \times 10^3\ \text{W}$

Speed of the shaft, $N = 80$ r.p.m.

Maximum shear stress, $\tau = 60\ \text{N/mm}^2$

Internal diameter, $D_i = 0.6 \times$ External diameter

$= 0.6\ D_0$

Maximum torque, $T_{max} = 1.4$ times the mean torque

$= 1.4 \times T$

Power is given by the relation,

$$P = \frac{2\pi NT}{60}$$

or,

$$T = \frac{60 \times P}{2\pi N}$$

$$= \frac{60 \times 300 \times 10^3}{2\pi \times 80}$$

$$= 35809.8\ \text{N-m}$$

$\therefore$

$$T_{max} = 1.4\,T = 1.4 \times 35809.8\ \text{N-m}$$

Now maximum troque transmitted by a haollow shaft is given by equation (16.6) as,

$$T_{max} = \frac{\pi}{16} \times \tau \times \left[\frac{D_0^4 - D_i^4}{D_0}\right]$$

or,

$$50133700 = \frac{\pi}{16} \times 60 \times \left[\frac{D_0^4 - (0.5D_0)^4}{D_0}\right]$$

$$= \frac{\pi}{16} \times 60 \left[\frac{D_0^4 - 1296\,D_0^4}{D_0}\right]$$

$$= \frac{\pi}{16} \times 60 \times .8704\,D_0^3$$

$$\therefore \qquad D_0 = \left(\frac{16 \times 50133700}{\pi \times 60 \times .8704}\right)^{1/3} = \mathbf{169.2}$$

$$= \mathbf{170\ mm\ Ans.}$$

and

$$D_i = 0.6 \times D_0 = 0.6 \times 170$$

$$= \mathbf{102\ mm\ Ans.}$$

16. A steel shaft ABCD having a total length of 2.4 m consists of three lengths having different sections as follows :

AB is hollow having outside and inside diameters of 80 mm and 50 mm respectively and BC and CD are solid, BC having a diameter of 80 mm and CD a diameter of 70 mm. If the angle of twist is the same for each section, determine the length of each section and the total angle of twist if the maximum shear stress in the hollow portion is 50 N/mm². Take C = 8.2 × 10⁴ N/mm².

Answer :

Given :

Total length of shaft, $L = 2.4\ m = 2400\ mm$

Shaft AB : Length $= L_1$

Outer dia., $D_1 = 80\ mm$

Inner dia., $d_1 = 50\ mm$

Shaft BC : Length $= L_2$

Diameter, $D_2 = 80\ mm$

Shaft CD : Length $= L_3$

Diameter, $D_3 = 70\ mm$

Angle of twist is same for each section.

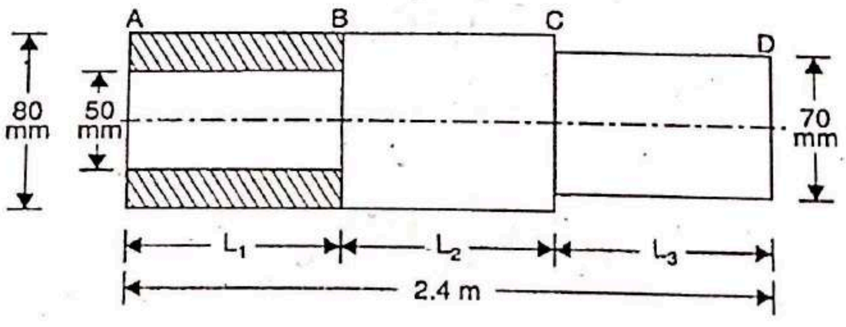
Hence $\theta_1 = \theta_2 = \theta_3$

Max. shear stress in hollow portion,

$$\tau_1 = 60\ N/mm^2$$

$$= 8.2 \times 10^4\ N/mm^2$$

Polar moment of inertia of each shaft is given as :



For shaft AB,

$$J_1 = \frac{\pi}{32}(D_1^4 - d_1^4)$$

$$= \frac{\pi}{32}(80^4 - 50^4)$$

$$= 340.9 \times 10^4 \text{ mm}^4$$

For shaft BC,

$$J_2 = \frac{\pi}{32} D_2^4$$

$$= \frac{\pi}{32} \times 80^4 = 402.4 \times 10^4 \text{ mm}^4$$

For shaft CD,

$$J_3 = \frac{\pi}{32} D_3^4$$

$$= \frac{\pi}{32} \times 70^4$$

$$= 235.8 \times 10^4 \text{ mm}^4$$

Now using equation,

$$\frac{T}{J} = \frac{C \times \theta}{L}$$

or,

$$\theta = \frac{T.L}{J.C}$$

Hence

$$\theta_1 = \frac{T \times L_1}{J_1 \times C},$$

$$\theta_2 = \frac{T \times L_2}{J_2 \times C}$$

and

$$\theta_3 = \frac{T \times L_3}{J_3 \times C}$$

But

$$\theta_1 = \theta_2 = \theta_3$$

$\therefore$

$$\frac{T.L_1}{J_1.C} = \frac{T.L_2}{J_2.C} = \frac{T.L_3}{J_3.C}$$

[Torque T and C are same for each portion]

or,
$$\frac{L_1}{J_1} = \frac{L_2}{J_2} = \frac{L_3}{J_3}$$

or,
$$\frac{L_1}{340.9 \times 10^4} = \frac{L_2}{402.4 \times 10^4} = \frac{L_3}{235.8 \times 10^4}$$

or,
$$\frac{L_1}{340.9} = \frac{L_2}{402.4} = \frac{L_3}{235.8}$$

$\therefore$
$$L_1 = \frac{340.9}{235.8} L_3 = 1.44 L_3$$

$$L_2 = \frac{402.4}{235.8} L_3 = 1.71 L_3$$

But $L_1 + L_2 + L_3 = 2.4 \text{ m} = 2400 \text{ mm}$

or, $1.44 L_3 + 1.71 L_3 + L_3 = 2400$

or, $4.15 L_3 = 2400$

$\therefore$
$$L_3 = \frac{2400}{4.15} = 578.35 \text{ mm.}$$

Substituting the value of L_3 in equations (i) and (ii), we get

$$L_1 = 1.44 \times 578.3 = 832.75 \text{ mm}$$
$$L_2 = 1.71 \times 578.3 = 988.80 \text{ mm}$$

As the shear stress is given in shaft AB. The angle of twist of shaft AB can be obtained by using equation

$$\frac{\tau}{R} = \frac{C \times \theta}{L}$$

$\therefore$ For shaft AB,

$$\frac{\tau_1}{\left(\frac{D_1}{2}\right)} = \frac{C \times \theta_1}{L_1}$$

$\therefore$
$$\theta_1 = \frac{\tau \times L_1}{\left(\frac{D_1}{2}\right) \times C} = \frac{50 \times 832.75}{\left(\frac{80}{2}\right) \times 8.2 \times 10^4}$$

$= 0.01269$ radians

$= 0.7273°$

∴ Total angle of twist of the whole shaft

$$= \theta_1 + \theta_2 + \theta_3$$

$$= 0.7273 \times 3 = \mathbf{2.1819° \text{ Ans.}}$$

17. Determine the diameter of a solid shaft which will transmit 300 kW at 250 r.p.m. The maximum shear stress should not exceed 30 N/mm² and twist should not be more than 1° in a shaft length of 2 m. Take modulus of rigidity = 1 × 10⁵ N/mm².

Answer :

Given :

Power transmitted, $P = 300 \text{ kW} = 300 \times 10^3 \text{ W}$

Speed of the shaft, $N = 250$ r.p.m.

Maximum shear stress, $\tau = 30 \text{ N/mm}^2$

Twist in shaft, $\theta = 1° = \dfrac{\pi}{180}$

$= 0.01745$ radian

Length of shaft, $L = 2 \text{ m} = 2000 \text{ mm}$

Modulus of rigidity, $C = 1 \times 10^5 \text{ N/mm}^2$

Let D = Diameter of the shaft.

Power is given by the relation,

$$P = \frac{2\pi NT}{60}$$

or,
$$300 \times 10^3 = \frac{2\pi \times 250 \times T}{60}$$

∴
$$T = \frac{300 \times 10^3 \times 60}{2\pi \times 250}$$

$$= 11459.1 \text{ N-m}$$

$$= 11459.1 \times 10^3 \text{ N-mm}$$

(i) Diameter of the shaft when maximum shear stress,

$$\tau = 30 \text{ N/mm}^2$$

Maximum torque transmitted by a solid shaft is given by equation (16.4) as

$$T = \frac{\pi}{16} \times \tau \times D^3$$

$$\therefore \quad 11459100 = \frac{\pi}{16} \times 30 \times D^3$$

$$\therefore \quad D = \left(\frac{16 \times 11459100}{\pi \times 30}\right)^{1/3} = 124.5 \text{ mm} \quad \text{... (i)}$$

(ii) Diameter of shaft when twist should not be more than 1°.

Using equation (16.9),

$$\frac{T}{J} = \frac{C\theta}{L}$$

where J = Polar moment of inertia of solid shaft

$$= \frac{\pi}{32} D^4$$

$$\therefore \quad \frac{11459100}{\frac{\pi}{32} D^4} = \frac{10^5 \times 0.01745}{2000}$$

$$\therefore \quad D^4 = \frac{32 \times 2000 \times 11459100}{10^5 \times \pi \times 0.01745}$$

$$= 13377.81 \times 10^4$$

The suitable diameter of the shaft is the greater* of the two values given by equation (i) and ii).

$\therefore$ Diameter of the shaft = **124.5 mm say 125 mm Ans.**

*(If diameter is taken smaller of the two values say 107.5 mm, then from equation $T = \frac{\pi}{16}$ D^3, the value of shear stress will be

$$11459100 = \frac{\pi}{16} \tau \times (107.5)^3$$

or, $$11459100 = 243920 \tau$$

or, $$\tau = \frac{11459100}{243920}$$

$$= 46.978 \text{ N/mm}^2$$

which is more than the given value of 30 N/mm²).

18. A hollow shaft of diameter ratio $\frac{3}{8}$ (internal dia. to outer dia.) is to transmit 375 kW power at 100 r.p.m. The maximum torque being 20% greater than the mean. The shear stress is not to exceed 60 N/mm^2 and twist in a length of 4 m not to exceed 2°. Calculate its external and internal diameters which would satisfy both the above condition. Assume modulus of rigidity, C = 0.85 × 10^5 N/mm^2.

Answer :

Given :

Diameter ratio, $\frac{D_i}{D_0} = \frac{3}{8}$

$\therefore$ $D_i = \frac{3}{8} D_0$

Power, $P = 375$ kW
$= 375000$ W

Speed, $N = 100$ r.p.m.

Max. torque, $T_{max} = 1.2\ T_{mean}$

Length, $L = 4$ m $= 4000$ mm

Max. twist, $\theta = 2° = 2 \times \frac{\pi}{180}$ radians
$= 0.0349$ radians

Modulus of rigidity, $C = 0.85 \times 10^5$ N/mm^2

Power is given by, $P = \frac{2\pi NT}{60}$

Here torque is T_{mean}

or, $T = \frac{P \times 60}{2\pi N}$

$$= \frac{375000 \times 60}{2\pi \times 100} = 35810 \text{ N-m}$$

or, $T_{mean} = 35810$ N-m

$\therefore$ $T_{max} = 1.2 \times T_{mean} = 1.2 \times 35810$
$= 42972$ N-m $= 42972 \times 1000$ N-mm.

(i) Diameters of the shaft when shear stress is not to exceed 60 N/mm^2.

For the hollow shaft, the torque transmitted is given by

$$T_{max} = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

or, $$42972 \times 1000 = \frac{\pi}{16} \times 60 \times \frac{\left[D_0^4 - \left(\frac{3}{8} D_0\right)^4\right]}{D_0}$$

or, $$\frac{42972000 \times 16}{\pi \times 60} = \frac{D_0^4}{D_0}$$

$$\left(1 - \frac{81}{4096}\right) = D_0^3 \times \frac{4015}{4096}$$

or, $$D_0^3 = \frac{42972000 \times 16 \times 4096}{\pi \times 60 \times 4015}$$

$\therefore$ $$D_0 = \left(\frac{42972000 \times 16 \times 4096}{\pi \times 60 \times 4015}\right)^{1/3}$$

$$= 154.97 \text{ mm say } 155 \text{ mm}$$

$\therefore$ $$D_i = \frac{3}{8} D_0 = \frac{3}{8} \times 155 = 58.1 \text{ mm.}$$

(ii) Diameter of the shaft when the twist is not to exceed 2 degrees.

Using equation (16.9) in terms of torque and θ, we get

$$\frac{T}{J} = \frac{C \times \theta}{L}$$

or, $$\frac{42972000}{\frac{\pi}{32}\left[D_0^4 - D_i^4\right]} = \frac{(0.85 \times 10^5) \times 0.0349}{4000}$$

or, $$\frac{42972000 \times 4000 \times 32}{\pi \times 0.85 \times 10^5 \times 0.0349} = D_0^4 - D_i^4 = D_0^4$$

$$-\left(\frac{3}{8}D_0\right)^4 = D_0^4 - \frac{81}{4096}D_0^4$$

$$= D_0^4\left[1-\frac{81}{4096}\right] = \frac{4015}{4096}D_0^4$$

$\therefore$

$$D_0^4 = \frac{42972000 \times 4000 \times 32 \times 4096}{\pi \times 0.85 \times 10^5 \times 0.0349 \times 4015}$$

$\therefore$

$$D_0 = 156.65 \text{ mm say } 157 \text{ mm}$$

and

$$D_i = \frac{3}{8} \times 156.65$$

$$= 58.74 \text{ mm say } 59 \text{ mm.}$$

The diameters of the shaft, which would satisfy both the conditions are the greater of the two values.

$\therefore$ External dia., D_0 = **157 mm. Ans.**

Internal dia., D_i = **59 mm. Ans.**

❊ ❊ ❊

UNIT 8

FRICTION

Chapter at a glance

- *Friction* ⇒ When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces.
- *Sliding friction* ⇒ Encountered commonly in the case of two solid surfaces having a tendency of relative motion.
- *Rolling Friction* ⇒ Resistance between two surfaces is experienced when they roll over each other.
- *Fluid friction* ⇒ Resistive force is offered by a fluid to body which moves over or through it.
- *Limiting friction* ⇒ Maximum value of frictional force, which comes into play, when the motion is impending.
- *Static friction* ⇒ Applied force is less than the limiting friction, the body remains at rest.
- *Dynamic friction* ⇒ Applied force exceeds the limiting friction, the body starts moving over the other body.
- *Laws of Coulomb's friction* ⇒ (i) The direction of friction force always acts in a direction opposite to which the body moves or tends to move, (ii) The frictional force developed is independent of the area of contact, (iii) The total frictional force depends on the roughness of contact surfaces, (iv) The frictional force is equal to the force applied to the body, so long as the body is at rest, (v) The coefficient of Kinetic friction is less than the coefficient of Static friction, (vi) The total frictional force is proportional to the normal Reactive force ($F \propto N$).
- *Coefficient of friction* ⇒ Ratio between the limiting friction and the normal reaction.
- *Angle of friction* ⇒ The angle between the resultant reaction and the normal to the surface.
- *Angle of repose* ⇒ The maximum inclination of the plane on which a body, free from external forces, can slip.

Multiple Choice Type Questions

1. A body is resting on a plane inclined at angle of 30° to horizontal. What force would be required to slide it down, if the coefficient of friction between body and plane is 0.3

a) zero b) 1 kg c) 5 kg
d) would depend on weight of body e) none of the above

Answer: (a)

2. The ratio of limiting friction and reaction is known as
a) coefficient friction
b) angle of friction
c) angle of repose
d) sliding friction
e) friction resistance

Answer: (a)

3. Column friction is between
a) Solids and liquids
b) dry surfaces
c) between bodies having relative motion
d) none of these

Answer: (b). (c)

4. A particle inside a hollow sphere of radius r, having co-efficient of friction $1/\sqrt{3}$, can be in rest up to a height of
a) $r/2$
b) $r/4$
c) $3r/8$
d) none of these

Answer: (d)

5. Coulomb friction is
a) the friction between solids and liquids
b) the friction between dry surfaces
c) the friction between bodies having reactive motion
d) none of these

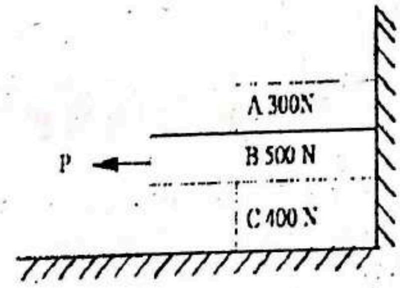
Answer: (b)

6. Frictional force has the following relation with the normal reaction between the two connecting surfaces
a) $F = \mu N$
b) $F = \mu^2 N$
c) $F = \mu / N$
d) None of these

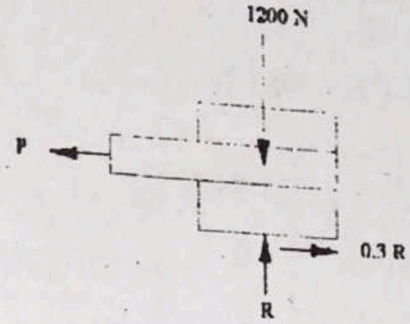
Answer: (a) for maximum friction

Short Answer Type Questions

1. Determine the force P required to intend the motion of the block b shown in the figure below. Take μ = 0.3 for all surfaces of contact, where μ = coefficient of friction.



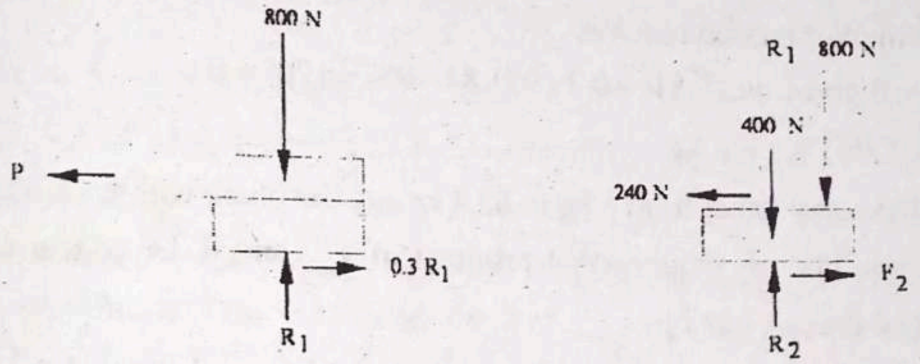
Answer:
Case-I: We consider all blocks are moving altogether



$\sum F_y = 0$ gives, $R = 1200$ N

$\sum F_x = 0$ gives. $0.3R - P = 0 \Rightarrow P = 0.3 \times 1200\ \text{N} = 360\ \text{N}$

Case-II: We consider blocks A and B are moving together as a unit w.r.t. block C



FBD of A and B　　　　FBD of C

Again, $\sum F_y = 0$ gives, $R_1 = 800$ N

$\sum F_x = 0$ gives, $0.3R_1 - P = 0 \Rightarrow P = 0.3 \times 800\ \text{N} = 240\ \text{N}$

It is clear from FBD of block C. $F_2 = 240$ N which is less than $F_{max} = 360$ N.
Thus, the required value of P in order to move the block B is = 240 N.

2. Refer to the Fig: 1, determine the range of values of mass m_0 so that the 100 kg block will neither move up nor slip down the inclined plane. The coefficient of static friction for the surfaces in contact is 0.3. [WBUT 2009]

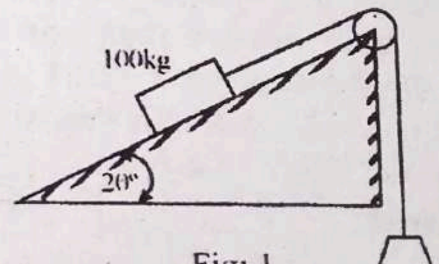
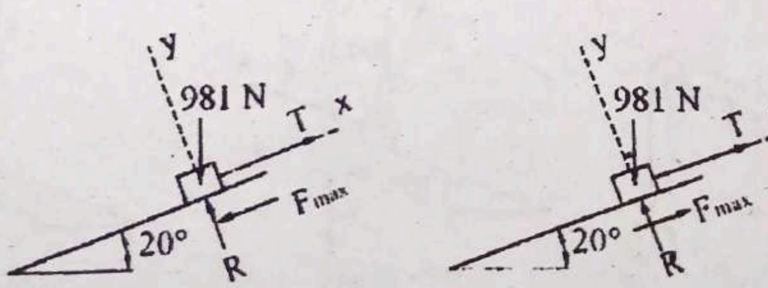


Fig: 1

Answer:



CASE I　　　　CASE II

Case-I: The maximum value of m_0 is determined when motion is impending up the plane.

With the weight mg = 100 × 9.81 N = 981 N.

The equations of equilibrium give

$\sum F_y = 0$ gives, $R = 981\cos 20^\circ N = 921.84$ N

$\sum F_x = 0$ gives, $T - F_{max} - 981\sin 20^\circ = 0$

i.e. $m_0 g - \mu R - 335.52 = 0$

or, $m_0 \times 9.81 - 0.3 \times 921.84 - 335.52 = 0$

$\therefore m_0 = 62.4$ kg

Case-II: The minimum value of m_0 is determined when motion is impending down the plane.

Equilibrium in the x-direction requires

$\sum F_x = 0$ gives, $m_0(9.81) + 0.3 \times 921.84 - 981\sin 20^\circ = 0$

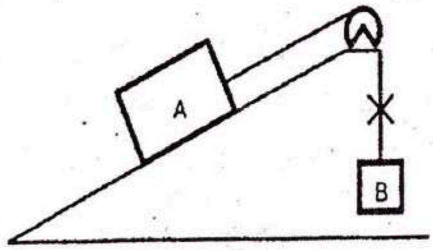
$\therefore m_0 = 5.965$ Kg ≈ 6 kg

Thus, m_0 may have any value from 6 kg to 62.4 kg and the block will remain at rest.

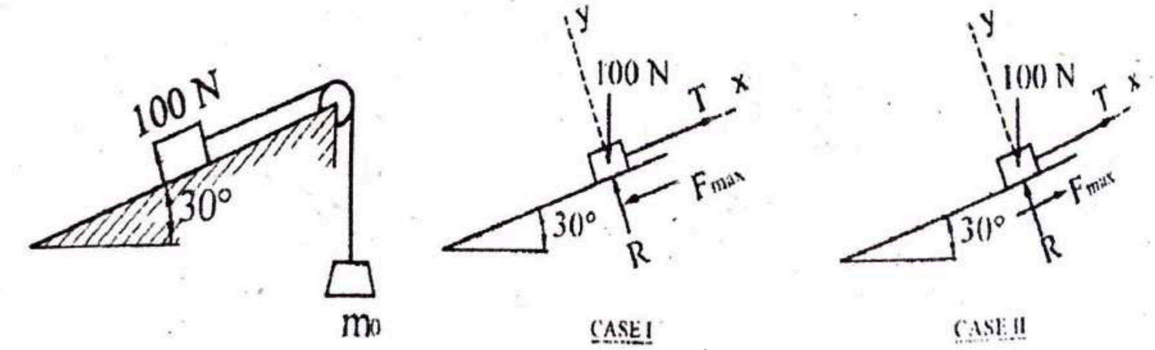
[In both cases equilibrium requires that resultant of F_{max} and R be concurrent with the 981 N weight and the tension T].

3. A block A of weight 100 N is placed on an inclined plane which makes an angle 30° to the horizontal, an extensible string is connected to block A and is passed over a smooth pulley. Another block B is hung freely at the other end of the string as shown in fig. Determine the range of weight of block B, such that the block A has motion neither up the plane nor down the plane. Take $\mu = 0.3$ for all contact surfaces.

e



Answer:



Case-I: The maximum value of m_0 is determined when motion is impending up the plane.
With the weight mg = 100 N
The equations of equilibrium give

$\sum F_y = 0$ gives, $R = 100\cos 30° N = 86.60$ N

$\sum F_x = 0$ gives, $T - F_{max} - 100\sin 30° = 0$

i.e. $m_0 g - \mu R - 50 = 0$

or. $m_0 \times 9.81 - 0.3 \times 86.60 - 50 = 0$

$\therefore m_0 = 7.745$ kg

Case-II: The minimum value of m_0 is determined when motion is impending down the plane.
Equilibrium in the x-direction requires

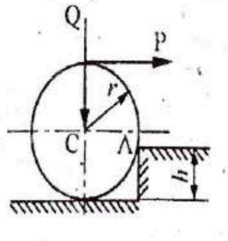
$$\sum F_x = 0 \text{ gives, } m_0(9.81) + 0.3 \times 86.6 - 100\sin 30° = 0$$

$$\therefore m_0 = 2.448 \text{ kg}$$

Thus, m_0 may have any value from 2.448 kg to 7.745 kg and the block will remain at rest. [In both cases equilibrium requires that resultant of F_{max} and R be concurrent with the 100 N weight and the tension T].

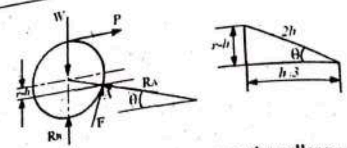
Long Answer Type Questions

1. A roller of radius $r = 12$ cm and weight $Q = 5$ kN is to be rolled over a curb of height $h = 6$ cm by a horizontal force P applied to the end of a string wound around the circumference of the roller as shown in the next Figure. Find the magnitude of P required to start the roller over the curb. There is sufficient friction between the roller surface and the edge of the curb to prevent slip at A.



Answer:

First, we draw the free body diagram of the roller, as shown in figure
Note that there is some frictional force at A = F (say).



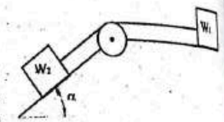
When the roller starts rolling over the curb, contact between the roller and the ground at B is lost, which means that the ground will not exert any force on the roller under this condition, i.e., $R_B = 0$

For rolling the roller over the curb, net clockwise moment of all forces about A must be greater than zero.

i.e., $P\times(0.12+0.06) - 5000\times 0.06\sqrt{3} > 0$

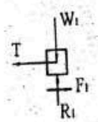
i.e., $P\times(0.18) > 5000\times 0.06\sqrt{3}$, *i.e.*, $P > \frac{5000}{\sqrt{3}}$

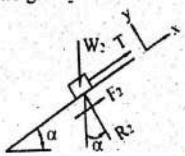
2. Two rectangular blocks of weights W_1 and W_2 are connected by a flexible cord and rest upon a horizontal and an inclined plane, respectively, with the cord passing over a pulley as shown in the next Figure. In the particular case where $W_1 = W_2$ and the coefficient of static friction μ is the same for all contiguous surfaces, then find the angle α of inclination of the inclined plane at which motion of the system will impend. Neglect friction in the pulley. **[WBUT 2007]**



Answer:

First, we draw the FBD of weight W_1 and weight W_2 as shown in figure.





For Block 1

For equilibrium

$\sum F_x = 0$ gives, $F_1 = T$ ......(1)

$\sum F_y = 0$ gives, $R_1 = W_1$ ......(2)

For limiting equilibrium, $F_1 = \mu R_1$ ......(3)

From above equations, we can write, $T = \mu W_1$ ......(4)

For Block 2

For equilibrium

$\sum F_x = 0$ gives, $T + F_2 - W_2 \sin\alpha = 0$ ......(5)

$\sum F_y = 0$ gives, $R_2 - W_2 \cos\alpha = 0$ ......(6)

For limiting equilibrium, $F_2 = \mu R_2$ ......(7)

Combining above equations $T = W_2 \sin\alpha - \mu W_2 \cos\alpha$ ......(8)

Now, from equation (4) and (8), we get,

$\mu W = W \sin\alpha - \mu W \cos\alpha \quad [\because W_1 = W_2 = W]$

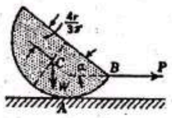
or, $\mu = \sin\alpha - \mu\cos\alpha$

or, $\mu(1+\cos\alpha) = \sin\alpha$

or, $\mu = \dfrac{\sin\alpha}{1+\cos\alpha} = \tan\dfrac{\alpha}{2}$

$\boxed{\text{i.e. } \alpha = 2\tan^{-1}\mu}$

3. A short semi-circular right cylinder of radius r and weight W rests on a horizontal surface and is pulled at right angles to its geometric axis by a horizontal force P applied at the middle B of the front edge as shown. Find the angle α that the flat face will make with the horizontal plane just before sliding begins if the coefficient of friction at the line of contact A is μ. The gravity force W must be considered as acting at the centre of gravity C as shown in the figure.

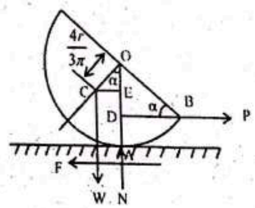


Answer:

From figure we can write, OA = OB = r

In triangle ODB, $OD = OB\sin\alpha = r\sin\alpha$

$DA = OA - OD = r - r\sin\alpha$, $OC = \dfrac{4r}{3\pi}$



In triangle OCE, $CE = OC\sin\alpha = \dfrac{4r}{3\pi}\sin\alpha$

$\sum F_x = 0$ gives, $P = F = \mu N$; $\sum F_y = 0$ gives, $N = W$

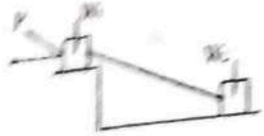
$\therefore P = F = \mu W$

$\sum M_A = 0$ gives, $W \times CE = P \times DA$ or, $W \times \dfrac{4r}{3\pi}\sin\alpha = \mu W \times (r - r\sin\alpha)$

or, $\left(\dfrac{4}{3\pi} + \mu\right)\sin\alpha = \mu \quad [\because W \neq 0, r \neq 0]$

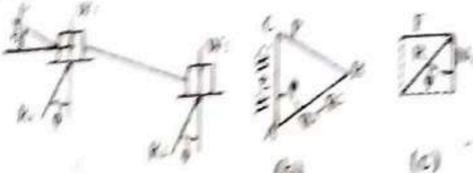
$\therefore \sin\alpha = \frac{3\pi\mu}{4+3\pi\mu}$ and hence $\alpha = \sin^{-1}\left(\frac{3\pi\mu}{4+3\pi\mu}\right)$

4. Two blocks having weights W_1 and W_2 are connected by a string and rest on horizontal planes as shown in figure. If the angle of friction for each block is ϕ, find the magnitude and direction of the least force P applied to the upper block that will induce sliding.



Answer:

FBD of the two blocks connected together, as shown in figure.



(a) (b) (c)

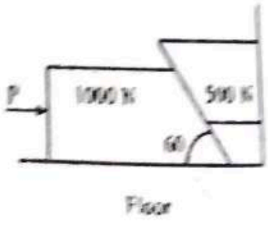
Observe that tension in the string connecting the two blocks does not appear in the FBD. This is simply because of the fact that the blocks are not disconnected from the string. Here, we show resultant of friction and normal reaction on each block. For instance, for block 1, the friction force F_1 acts towards the right and the resultant R_1 makes an angle with the direction N_1.

Three forces $\vec{W}_1 + \vec{W}_2, \vec{R}_1 + \vec{R}_2$ and $\vec{P}$ form a closed vector triangle. Now for the force P to be minimum (i.e., the length BC to be minimum) i.e., the point B should be so located that BC becomes perpendicular to the line AB.

The above consideration complete construction of ΔABC.

$$\text{Since } \angle ABC = 90^\circ, \therefore P_{min} = (W_1 + W_2)\sin\phi$$

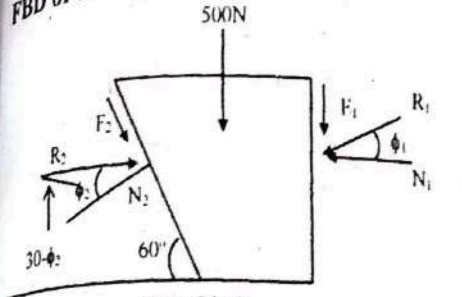
6. As shown in the figure, find the minimum value of horizontal force P applied to the lower block that will keep the system in equilibrium, given the coefficient of friction between the lower block and floor = 0.25, between the upper block and vertical wall = 0.30 & between the blocks = 0.20.



Floor

Answer:

FBD of the *Upper Block*:



Upper Block

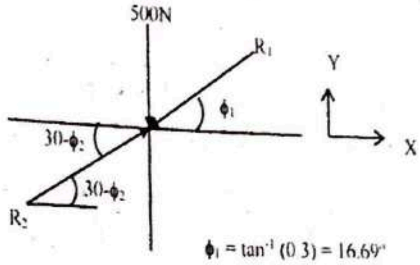


Fig. (i)

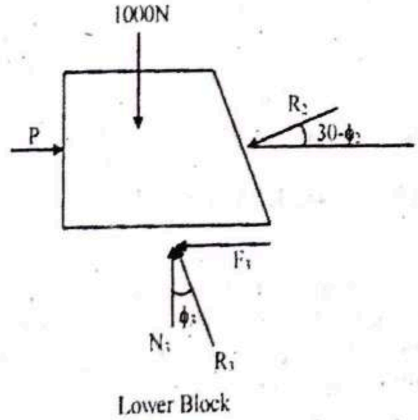
$\phi_1 = \tan^{-1}(0.3) = 16.69°$
$\phi_2 = \tan^{-1}(0.2) = 11.31°$

From Fig (i), Applying Lami's theorem,

$$\frac{500}{\sin(90+\phi_1+90-30+\phi_2)} = \frac{R_1}{\sin(90+30-\phi_2)} = \frac{R_2}{\sin(90-\phi_1)}$$

$$\therefore R_2 = \frac{500 \times \sin 73.3°}{\sin(178°)} \text{N} = 13722.64\text{N}$$

FBD of the *Lower Block*:



Lower Block

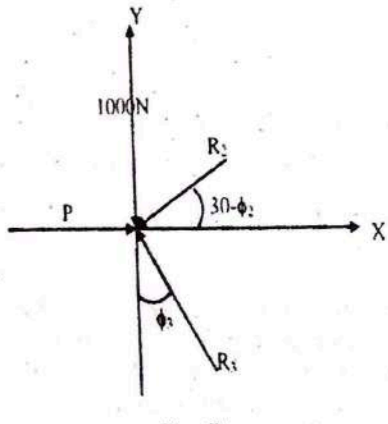


Fig: (ii)

Now, $\phi_3 = \tan^{-1}(0.25) = 14.036°$

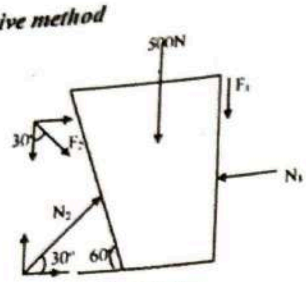
From Fig (ii)

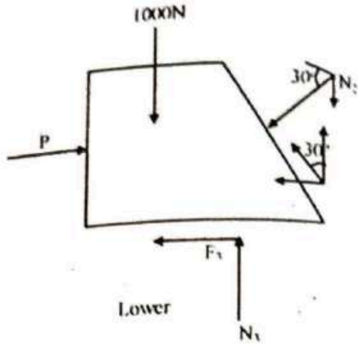
$\sum F_y = 0 \Rightarrow R_3 \cos\phi_3 = 1000 + R_2 \sin(30-\phi_2)$

$\therefore R_3 = 5563.52\text{N}$

$\sum F_x = 0 \Rightarrow P = R_2 \cos(30-\phi_2) + R_3 \sin\phi_3 = 14348.34\text{N}$

Alternative method





For equilibrium of *Upper Block*

$\sum F_x = 0$ gives, $N_2 \cos 30° + F_2 \sin 30° - N_1 = 0$

$\Rightarrow N_2 \cos 30° + 0.2N_2 \sin 30° - N_1 = 0$

$\Rightarrow N_2 = 1.035N_1 ......(1)$

$\sum F_y = 0$ gives, $N_2 \sin 30° - F_2 \cos 30° - N_1 - 500 = 0......(2)$

From equation (1) and (2), we get,

$N_2 = 13535.5N$

$F_2 = 2707N$

For equilibrium of *Lower Block*

$\sum F_y = 0$ gives. $N_3 + F_2 \cos 30° - N_2 \sin 30° = 1000 \quad \Rightarrow N_3 = 5423.33N$

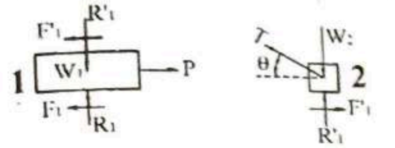
$\sum F_x = 0$ gives. $P = 0.25N_3 + F_2 \sin 30° + N_2 \cos 30° = 14431.5N = 14.43kN$

6. A block of weight W_1 = 200 N rests on a horizontal surface and supports on top of it, another block of weight W_2 = 50 N. The block W_2 is attached to a vertical wall by the inclined string AB. Find the magnitude of the horizontal force P applied to the lower block as shown in Fig. (a), that will be necessary to cause slipping to impend. The coefficient of static friction for all contiguous surfaces is $\mu = 0.3$.

Fig: (a)

Answer:

Let us first draw FBD of two blocks as shown in Fig.



$\tan\theta = \frac{3}{4}$

$\therefore \theta = \tan^{-1}\frac{3}{4} = 36.87$

For Block 1

For equilibrium

$$\sum F_x = 0 \text{ gives, } P = F_1 + F_1' \quad ......(1)$$

$$\sum F_y = 0 \text{ gives, } R_1 = W_1 + R_1' \quad ......(2)$$

For limiting equilibrium,

$$F_1 = \mu R_1 \quad ......(3)$$

and $\quad F_1' = \mu R_1' \quad ......(4)$

From equation (2), (3) and (4), we, get, $\frac{F_1}{\mu} = W_1 + \frac{F_1'}{\mu} \quad ......(5)$

For Block 2

For equilibrium

$$\sum F_x = 0 \text{ gives, } F_1' = T\cos\theta \quad ......(6)$$

$$\sum F_y = 0 \text{ gives, } R_1' + T\sin\theta = W_2 \quad ......(7)$$

From equation (4), (6) and (7), we get,

$$\frac{F_1'}{\mu} + \frac{F_1'}{\cos\theta} \times \sin\theta = W_2$$

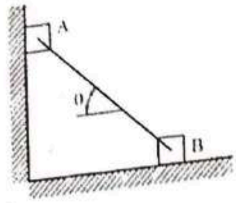
$$\therefore F_1' = \frac{W_2}{\frac{1}{\mu} + \tan\theta} = \frac{50}{\frac{1}{0.3} + \frac{3}{4}} N = 12.244 \text{ N}$$

Now, from equation (5), we get,

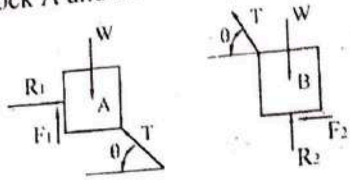
$$F_1 = 0.3(200 + \frac{12.244}{0.3})N = 72.244 \text{ N}.$$

$$\therefore P = F_1 + F_1' = (72.244 + 12.244)N = 84.49 \text{ N}.$$

7. Two identical blocks *A* and *B* each having weight W are connected by rigid link and supported by a vertical wall and a horizontal plane having same co-efficient of friction (μ) as shown in figure. If sliding impends for θ = 45°, calculate μ.



Answer:
Step-I: Draw the FBD of block A and B.



Block A

For equilibrium

$\sum F_x = 0$ gives, $R_1 - T\cos\theta = 0$ ......(1)

$\sum F_y = 0$ gives, $F_1 + T\sin\theta - W = 0$ ......(2)

For limiting equilibrium, $F_1 = \mu R_1$ ......(3)

Block B

For equilibrium

$\sum F_x = 0$ gives, $T\cos\theta - F_2 = 0$ ......(4)

$\sum F_y = 0$ gives, $R_2 - T\sin\theta - W = 0$ ......(5)

For limiting equilibrium, $F_2 = \mu R_2$ ......(6)

From equation (1) and (3), we get, $F_1 = \mu T\,Cos\theta$

Now from equation (2), we get, $\mu T\cos\theta + T\sin\theta = W$

$$\therefore T = \frac{W}{\mu\cos\theta + \sin\theta} \quad ......(7)$$

From equation (4) and (6), we, get, $T\cos\theta = \mu R_2$

Therefore from equation (5), we get, $T\cos\theta = \mu(T\sin\theta + W)$

$$\therefore T = \frac{\mu W}{\cos\theta - \mu\sin\theta} \quad ......(8)$$

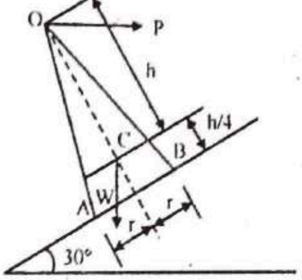
From equation (7) and (8), we get, $\dfrac{W}{\mu\cos\theta+\sin\theta}=\dfrac{\mu W}{\cos\theta-\mu\sin\theta}$

i.e. $\cos 45^0-\mu\sin 45^0=\mu^2\cos 45^0+\mu\sin 45^0$

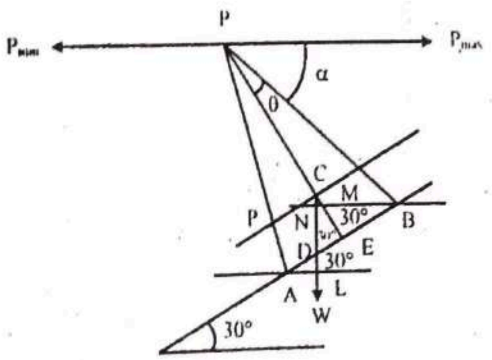
$\therefore \mu^2+2\mu-1=0$ ......(9)

Since $\mu > 0$ (physically), positive root of equation (9) is only acceptable, which yields μ = 0.414.

8. A solid right circular cone of altitude $h=12\,\text{cm}$ and radius $r=3\,\text{cm}$ has its cg C on its geometric axis at a distance $h/4$ above the base. This cone rests on the inclined plane AB which makes an angle of 30° with the horizontal and for which the angle of friction is 0.5. A horizontal force P is applied to the vertex O of the cone and acts in the vertical plane of the figure. Find the maximum and minimum values of P consistent with equilibrium of the cone of weight $W=10\,\text{kgf}$.



nswer:



From ΔCDE, $\frac{DE}{CE} = \tan 30° \Rightarrow DE = \frac{3}{\sqrt{3}}\text{cm} = \sqrt{3}\text{cm}$

$AD = r - DE = (3-\sqrt{3})\text{cm} = 1.27\text{cm}$

From ΔADL, $\frac{AL}{AD} = \cos 30° \Rightarrow AL = \frac{\sqrt{3}}{2} \times 1.27\text{cm} = 1.1\text{cm}$

$\text{DB} = (2\times 3 - 1.27)\text{cm} = 4.73\text{cm}$

From ΔDNB, $\frac{NB}{BD} = \cos 30° \Rightarrow NB = 4.73 \times \frac{\sqrt{3}}{2}\text{cm} = 4.1\text{cm}$

$CE = \frac{h}{4} = \frac{12}{4}\text{cm} = 3\text{cm}$

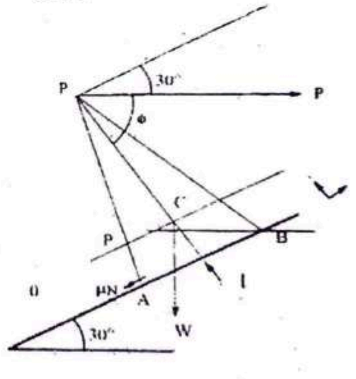
Toppling about B, $M_B = 0 \Rightarrow W \times NB = P \times BS \; ---(i)$

$\tan\theta = \frac{r}{h} = \frac{3}{12} \Rightarrow \theta = 14°$

$\alpha = 90° - 30° - 14° = 46°$

$OB = \sqrt{3^2 + 12^2}\text{cm} = 12.4\text{cm} \qquad BS = OB\sin\alpha = 8.92\text{cm}$

From equation (i), we get, $P = \frac{10kgf \times 4.1}{8.92}\text{N} = 4.6\text{kgf}$



$\phi = 60°$

$\sum F_x = 0 \Rightarrow P\cos 30° = W\sin 30° + \mu N \; ---(ii)$

$\sum F_y = 0 \Rightarrow N = P\sin 30° + W\cos 30° = \frac{P}{2} + \frac{\sqrt{3}\times 10}{2}kgf \; ---(iii)$

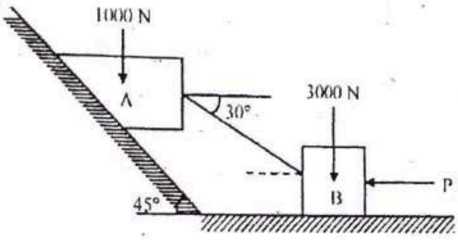
From equation (i) & (ii) we can write,

$$\frac{P\sqrt{3}}{2}=\frac{10\,\text{kgf}\times 1}{2}+0.5\left[\frac{P}{2}+\frac{10\times\sqrt{3}\,\text{kgf}}{2}\right]$$

$\therefore P = 15.1\,\text{kgf}$

Therefore, $P_{max} = 15.1\,\text{kgf}$ and $P_{min} = 4.6\,\text{kgf}$

9. A block A weighing 1000 N rests on a rough inclined plane whose inclination to the horizontal is 45°. The block is connected to another block B weighing 3000 N resting on a rough horizontal plane, by a weightless rigid bar inclined at an angle 30° to the horizontal as shown in figure. Find the horizontal force that has to be applied on the block B to just move the block A up the slope. Assume coefficients of friction for all contact surfaces is 0.26.



Answer:

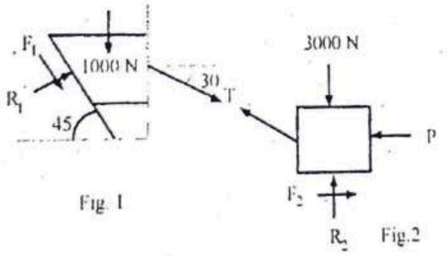


Fig. 1 Fig.2

FBD

For equilibrium of *Upper Block (fig1)*

$\sum F_x = 0$ gives,

$F_1\cos 45° + R_1\cos 45° + T\cos 30° = 0$

$\Rightarrow \mu R_1\cos 45° + R_1\cos 45° + T\cos 30° = 0$

$\Rightarrow 0.26R_1\times\cos 45° + R_1\cos 45° + T\cos 30° = 0$

$\Rightarrow 0.891R_1 + T\times 0.866 = 0 \quad \Rightarrow R_1 = -0.972T \;---(i)$

$\sum F_y = 0$ gives,

$-F_1\sin 45° + R_1\sin 45° - T\sin 30° - 1000 = 0$

Btech

$\Rightarrow -0.26R_1 \times \sin 45° + R_1 \sin 45° - T\sin 30° - 1000 = 0$

$\Rightarrow 0.523R_1 - 0.5T - 1000 = 0 \Rightarrow 0.523 \times (-0.972T) - 0.5T - 1000 = 0$

$\Rightarrow -1.0083T = 1000 \Rightarrow T = -991.8N$

$\therefore R_1 = -0.972 \times -991.8N = 964N$

For equilibrium of Lower Block (fig2)

$\sum F_y = 0$ gives. $R_2 + T\sin 30° = 3000$

$\Rightarrow R_2 = 3000 - T\sin 30° = 3000 - (-991.8)0.5\,N = 3495.9\,N$

$\Sigma F_x = 0$ gives.

$P = F_2 - T\cos 30° = 0.26R_2 - (-991.8 \times 0.866)$

$= 0.26 \times 3495.9 - (-991.8 \times 0.866)N = 1767.83\,N$

10. Three flat blocks are positioned on the 30° incline as shown in Fig. J and a force P parallel to the incline is applied to the middle block. The upper block is prevented from moving by a wire which attaches is to the fixed support. The coefficient of static friction for each of the three pairs of mating surfaces is shown. Determine the maximum value of P which may have before any slipping takes place.

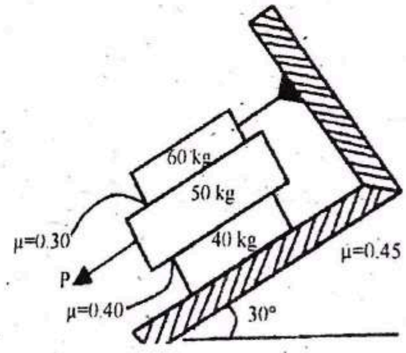
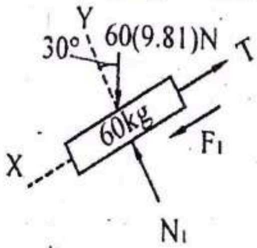


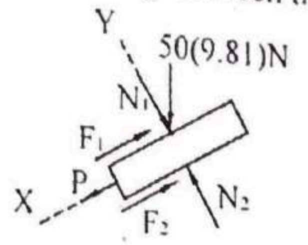
Fig: J

Answer:

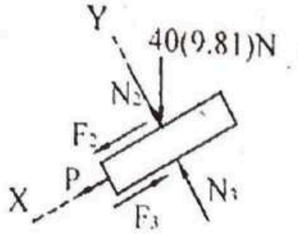
The free body diagram of each block is drawn. The friction forces are assigned in the directions to oppose the relative motion, which would occur if no friction were present. There are two possible conditions for impending motion.



Either the 50 kg block slips and the 40 kg block remains in place, or the 50 and 40 kg blocks move together with slipping occurring between the 40 kg block and the incline.



The normal forces, which are in the y-direction, may be determined without reference to the friction forces, which are all in the x-direction.



Thus,

$\left[\Sigma F_Y = 0\right]$

(60 kg) $\rightarrow$ $N_1 - 60(9.81)\cos 30° = 0$ $\quad N_1 = 509.74\ N$

(50 kg) $\rightarrow$ $N_2 - 50(9.81)\cos 30° - 509.74 = 0$ $\quad N_2 = 934.52\ N$

(40 kg) $\rightarrow$ $N_3 - 40(9.81)\cos 30° - 934.52 = 0$ $\quad N_3 = 1274.34\ N$

We shall assume arbitrarily that only the 50 kg block slips, so that the 40 kg block remains in place. Thus, for impending slippage at both surfaces of the 50 kg block, we have

$[F_{max} = \mu_s N]$ $\quad F_1 = 0.30(509.74)\text{N} = 152.92\ \text{N}$

$F_2 = 0.40(934.52)\text{N} = 373.808\ \text{N}$

The assumed equilibrium of forces at impending motion for the 50 kg block gives

$\left[\Sigma F_X = 0\right]$ $\quad P - 152.92 - 373.808 + 50(9.81)\sin 30° = 0 \quad \therefore\ P = 281.478\ \text{N}$

We now check on the validity of our initial assumption. For the 40 kg block with $F_2 = 373.808$ N the friction force F_3 would be given by

$\left[\Sigma F_X = 0\right] \Rightarrow 373.808 + 40(9.81)\sin 30° - F_3 = 0 \quad F_3 = 570\ \text{N}$

But the maximum possible value of

F_3 is $F_3 = \mu_s N_3 = 0.45(1274.34)\ \text{N} = 573.45\ \text{N}$.

Thus, 570 N cannot be supported and our initial assumption was wrong. We conclude, therefore, that slipping occurs first between the 40 kg block and the incline. With the corrected value $F_3 = 573.45$ N, equilibrium of the 40 kg block for its impending motion requires

$$\left[\Sigma F_x = 0\right] \quad F_2 + 40(9.81)\sin 30° - 573.45 = 0 \quad \therefore \; F_2 = 377.25 \text{ N}$$

Equilibrium of the 50 kg block gives, finally,

$$\left[\Sigma F_x = 0\right] \quad P + 50(9.81)\sin 30° - 377.25 - 152.92 = 0$$

$$P = 284.92 \text{ N}$$

Thus, with P = 284.92 N, motion impends for the 50 kg and 40 kg blocks as a unit.

11. **A body of weight 2000 N rests in a horizontal plane. If the coefficient of friction is 0.4, find the horizontal force required to be applied parallel to the plane to move the body.**

Answers :

For limiting equilibrium,

$$\Sigma F_y = 0 \; (\uparrow +, \downarrow -)$$

$$R - 2000 = 0$$

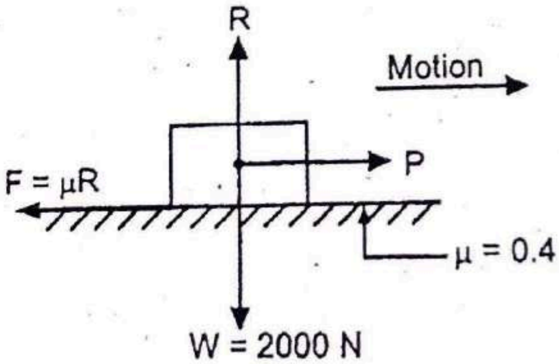
$$R = 2000 \text{ N} \qquad \text{.... (i)}$$

$$\Sigma F_x = 0 \; (\uparrow +, \downarrow -)$$

$$P - F = 0$$

$$P - \mu R = 0$$

$$P - 0.4\, R = 0 \qquad \text{.... } (\because \mu = 0.4)$$



Substituting the value of R from equation (i) in equation (ii),

$$P - 0.4 \times 200 = 0$$

$$\therefore \qquad P - 800 = 0$$

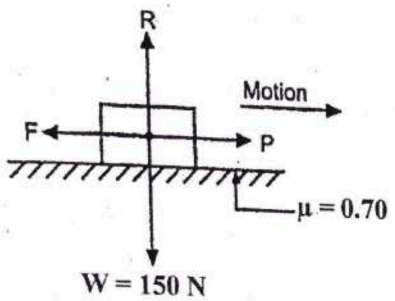
$$P = 800 \text{ N} \; \textbf{Ans.}$$

12. **A body of weight 150 N lies on a horizontal plane for which Co-efficient of friction is 0.70.**

Determine the following :

(i) Limiting force of friction (ii) Angle of friction (iii) Horizontal force required to move it.

Answer :



Given :

$$\mu = 0.70;$$
$$R_N = 150 \text{ N};$$

(i) Limiting force of friction

$$F = \mu R_N$$
$$= 0.70 \times 150$$
$$= 150 \text{ N } \textbf{Ans.}$$

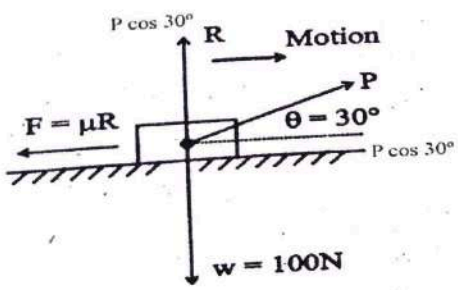
(ii) Angle of friction

$$\tan\theta = \mu = 0.70$$
$$\theta = \tan^{-1}(0.70)$$
$$= 34.99^\circ \textbf{ Ans.}$$

(iii) More than 150 N force required.

3. **Define friction. A box weighing 100 N is resting in a horizontal plane, the Co-efficient of friction is 0.25. Find the least force acting at an angle of 30°, with the horizontal which move the box.**

Answer :



We Resolve the force in horizontal and vertical directions :

Now,

$$\Sigma f_x = 0$$
$$F = P\cos 30^\circ = \mu R$$

Again,

$$\Sigma f_y = 0$$

DON'T

$$R = W - p \sin 30^\circ$$

$$p \cos 30^\circ = \mu (W - P \sin 30^\circ)$$

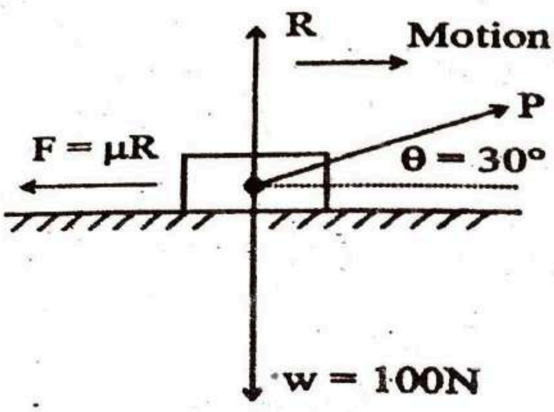
$$\Rightarrow \quad p = \frac{\mu w}{\cos 30^\circ + \mu \sin 30^\circ}$$

$$\Rightarrow \quad = \frac{0.25 \times 100}{\frac{\sqrt{3}}{2} + 0.25 \times \frac{1}{2}}$$

$$\boxed{P = 25.25\text{N}} \ \textbf{Ans.}$$

14. A box of weight 100N is resting on a horizontal plane, the co-efficient of frition Find the least force acting at an angle of 30° with the horizontal which would the box.

Answers :



That is

$$\Sigma f_x = 0$$

or,

$$F = P \cos\theta$$

$$\Sigma f_y = 0;$$

$$R = w - P \sin\theta$$

Also,

$$F = \mu R$$

$\therefore$

$$\mu (w - P \sin\theta) = P \cos\theta$$

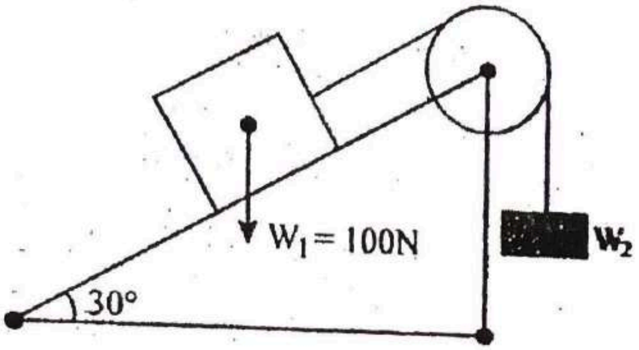
or,

$$P = \frac{\mu w}{\cos\theta + \mu \sin\theta}$$

$$= \frac{0.4 \times 100}{\cos 30^\circ + 0.4 \sin 30^\circ}$$

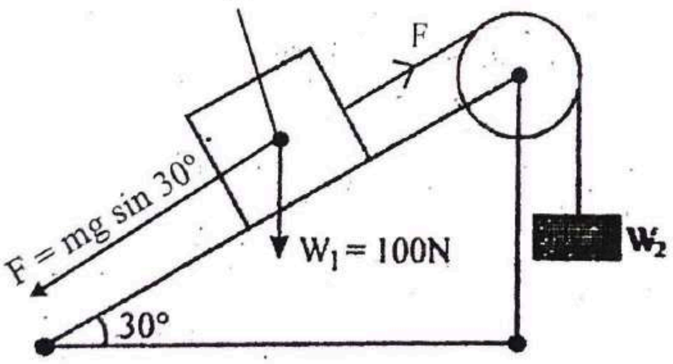
$$= \frac{40}{0.866 + 0.2} = 37.52 \text{ N } \textbf{Ans.}$$

15. **A block of weight W_1 =100N rests on an inclined plane and another weight W_2 is attached to the first weight through a string as shown in figure. If co-efficient of friction between block and plane 0.2. Find maximum value of W_2 so that equilibrium can exist.**



Answer :

Since the system is in equilibrium there must be no acceleration in any direction. So all you need to do is balance the force along the incline.



So, let the t ension in the string is T. Then,

$$T - 100 \sin 30^\circ = 0$$

$$T = 50 \text{ N}$$

For Block of weight $\quad w_2 = w_2 = T = 50$ N **Ans.**

A weight of 25 kN is pulled up on a rough inclined plane. Whose inclination to the horizontal is 30° by a force of 18 kN acting parallel to the plane. Find the Co-efficient of friction.

Answer :

We get

$$F - \mu R_N - W \sin\alpha = 0$$

or, $$18 = \mu R + 25 \sin 30^\circ$$

$$\mu R_N = 18 - 25 \sin 30^\circ \quad \text{.... (i)}$$

or,

$$= 5.5$$

and

$$R_N - W \cos 30^\circ = 0$$

or,

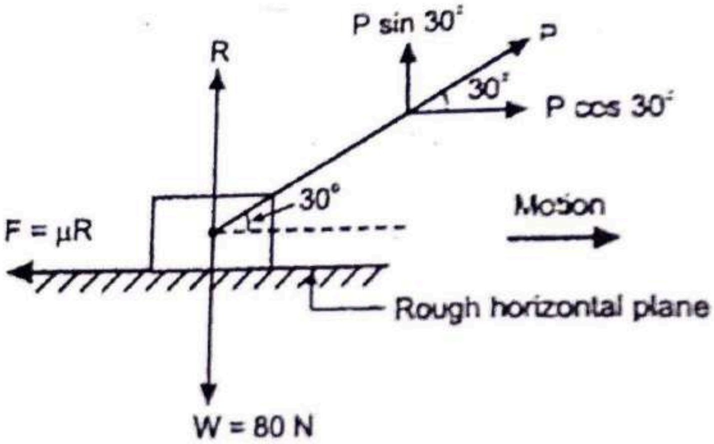
$$R_N = W \cos 30^\circ = 25 \cos 30^\circ$$

$$= 21.65 \text{ kN}$$

from equation (i),

$$\mu = \frac{5.5}{21.65} = 0.25 \textbf{ Ans.}$$

17. A block of 80 N is placed on a horizontal plane where the coefficient of friction is 0.25. Find the force at 30° upto the horizontal to just move the block.



Answer :

For limiting equilibrium,

$$\Sigma F_x = 0 \, (\uparrow +, \downarrow -)$$

$$P \cos 30^\circ - F = 0$$

$$\therefore \quad P \cos 30^\circ - \mu R = 0$$

$$\therefore \quad 0.866 \, P - 0.25 \, R = 0$$

$$\therefore \quad 0.25 \, R = 0.866 \, P$$

$$R = 3.464 \, P \quad \text{.... (i)}$$

$$\Sigma F_y = 0 \, (\uparrow +, \downarrow -)$$

$$\therefore \quad R + P \sin 30^\circ - W = 0$$

$$\therefore \quad 3.464 \, P + 0.5 \, P - 80 = 0$$

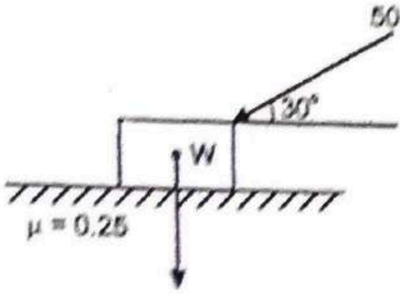
$$\therefore \quad (3.464 + 0.5) \, P = 80$$

$$\therefore \quad 3.964 \, P = 80$$

$$\therefore \quad P = 20.18 \text{ N}$$

$$\boxed{P = 20.18 \text{ N}} \textbf{ Ans.}$$

18. Find the value of W if the body is in the limiting equilibrium. See Fig



Answer :

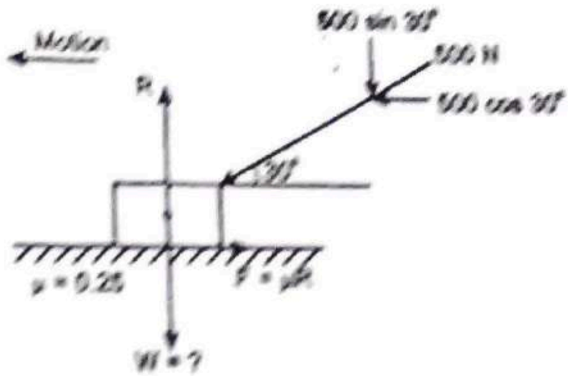
Fig. (ii) shows the F.B.D. (free body diagram) of given Fig.

Resolve 500 N force along the plane and normal to the plane.

Components of 500 N are :

(i) 500 cos 30° (←)

(ii) 500 sin 30° (↓)



For limiting equilibrium, $\Sigma F_x = 0$ (↑ +, ↓ –)

$$\therefore \quad \mu R - 500 \cos 30^\circ = 0$$

$$\therefore \quad 0.25\,R = 0 \qquad \text{.....}(\because F = \mu R)$$

$$\therefore \quad 0.25\,R = 433.01$$

$$R = \frac{433.01}{0.25} = 1732.04 \text{ N}$$

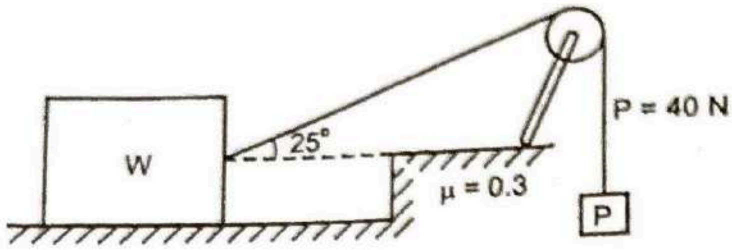
$$\Sigma F_y = 0 \; (\uparrow +, \downarrow -)$$

$$R - 500 \sin 30^\circ = 0$$

$$\therefore \quad 1732.04 - 250 = W$$

$$\boxed{W = 1482.04 \text{ N}} \text{ Ans.}$$

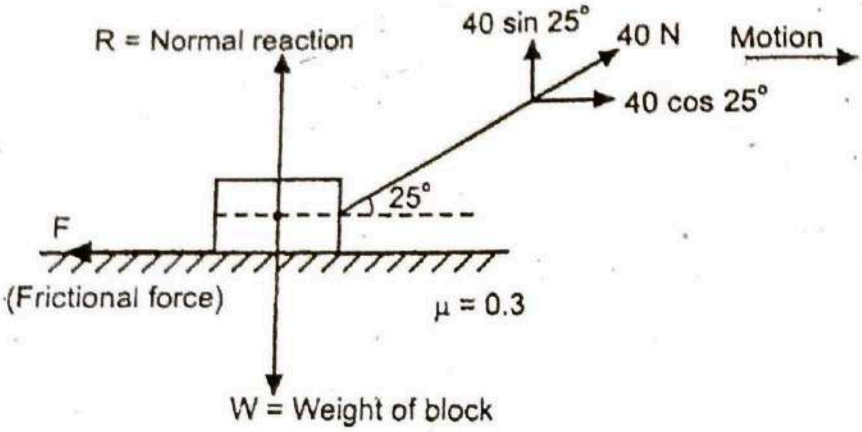
19. **Fig. 1.14 (i) shows an arrangement for just moving the block of weight 'W' horizontally. An effort of 40 N just moves the block. If the coefficient of static friction is 0.30, determine the weight of the block.**



Answer :

Component of 40 N force are : 40 cos 25° (→), 40 sin 25° (↑).

Shows all the forces acting on the block.



For limiting equilibrium,

$$\Sigma F_x = 0\ (\uparrow +, \downarrow -) \qquad\qquad \Sigma F_y = 0\ (\uparrow +, \downarrow -)$$

$$40 \cos 25^\circ - F = 0 \qquad\qquad R + 40 \sin 25^\circ - W = 0$$

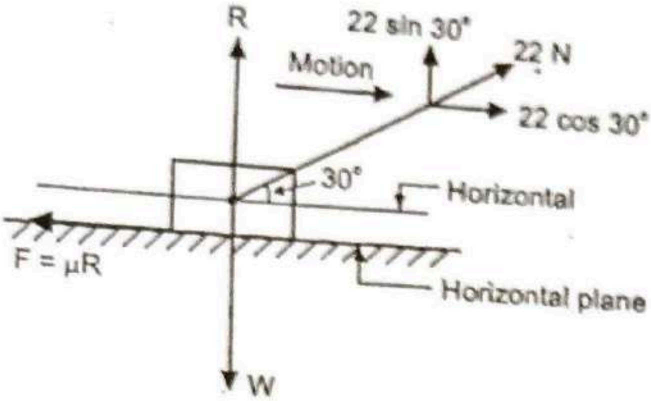
$$36.25 - \mu R = 0 \qquad\qquad \therefore \quad W = R + 40 \sin 25^\circ$$

$$= 120.84 + 16.9$$

A body resting on a rough horizontal plane is on the point of moving by a pull of 22 N acting 30° inclined to horizontal. It is pushed by a force of 28 N acting 30° inclined to horizontal. Find the weight of body and coefficient of friction.

Answer :

Case I : When a pull of 22 N is applied at 30°



For limiting equilibrium,

$$\Sigma F_x = 0\ (\uparrow +, \downarrow -)$$

$$22 \cos 30^\circ - \mu R = 0$$

$\therefore$

$$\mu R = 19.05 \qquad \text{.... (i)}$$

$$\Sigma F_y = 0\ (\uparrow +, \downarrow -)$$

$$R + 22 \sin 30^\circ - W = 0$$

$$R + 11 - W = 0$$

$\therefore$

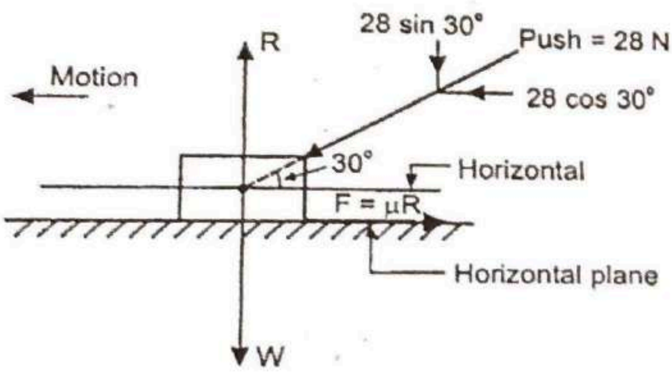
$$R = W - 11 \qquad \text{.... (ii)}$$

$\therefore$ From equations (i) and (ii),

$$\mu \times (W - 11) = 19.05$$

$$\mu = \frac{19.05}{W-11} \qquad \text{.... (A)}$$

Case II : When a push of 28 N is applied at 30°



For limiting equilibrium,

$$\Sigma F_x = 0\ (\uparrow +, \downarrow -)$$

$$\mu R - 28 \cos 30^\circ = 0$$

$$\therefore \quad \mu R = 24.25 \qquad \text{...(iii)}$$

$$\Sigma F_y = 0 \; (\uparrow +, \downarrow -)$$

$$R - 28 \sin 30^\circ - W = 0$$

$$\therefore \quad R \; W + 14 \qquad \text{...(iv)}$$

∴ From equations (iii) and (iv),

$$\mu \times (W + 14) = 24.25$$

$$\therefore \quad \mu = \frac{24.25}{W + 14} \qquad \text{...(B)}$$

Equating two values of μ from (A) and (B),

$$\frac{19.05}{W-11} = \frac{24.25}{W+14}$$

$$19.05\,(W + 14) = 24.25\,(W - 11)$$

$$\therefore \quad 19.05\,W + 266.7 = 24.25\,W - 266.75$$

$$266.7 + 266.75 = 24.25\,W - 19.05\,W$$

$$\therefore \quad 533.45 = 5.2\,W$$

$$W = 102.58\text{ N}$$

To find μ : From equation (A), $\mu = \frac{11.05}{W-11} = \frac{19.05}{102.58-11} = 0.208 \text{ » } 0.21$

Check : From equation (B), $\mu = \frac{24.25}{W+14} = \frac{24.25}{102.58+14} = 0.208 \text{ » } 0.21$

$\boxed{W = 102.58\text{ N}, \mu = 0.21}$ **Ans.**

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