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ELECTRICAL BRANCH

For 3rd Semester

ELECTRO MAGNETIC Field

Strictly according to the
NEW SYLLABUS

ORGANIZER

SYLLABUS

Module 1 : Review of Vector Calculus & Static Electric Field (6 hours) :

Vectoralgebra-addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus-differentiation, partial differentiation, integration, vector operator del, gradient, divergence and curl; integraltheorems of vectors. Conversion of a vector from one coordinate system to an other.

Module 2 : Static Electric Field (6 hours) :

Coulomb's law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

Module 3 : Conductors, Dielectrics and Capacitance (6 hours) :

Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson's equation, Laplace's equation, Solution of Laplace and Poisson's equation, Application of Laplace's and Poisson's equations.

Module 4 : Static Magnetic Fields Static Magnetic Fields (6 hours) :

Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

Module 5 : Magnetic Forces, Materials and Inductance (6 hours) :

Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

Module 6 : Time Varying Fields and Maxwell's Equations (6 hours) :

Faraday's law for Electromagntic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions.

Module 7 : Electromagnetic Waves (6 hours) :

Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogeneous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

UNIT 1

Review of Vector Calculus & Static Electric Field

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Define vector and state parallelogram law of vectors.

Ans:

Vector

The physical quantity that has both magnitude and direction is known as 'Vector quantity'. Force, field strength, acceleration, displacement are some of the examples of vector.

Parallelogram Law of Vectors

Parallelogram law of vectors states that, if the adjacent sides of a parallelogram are represented by two vectors (both in magnitude and direction) drawn from a point, then the diagonal drawn from that point gives the resultant of two vectors (both in magnitude and direction).

Q2. Write the different coordinate systems.

Ans:

The following are the different coordinate systems used to represent field vectors.

1. Cartesian (or) Rectangular coordinate system
2. Cylindrical coordinate system
3. Spherical coordinate system.

Q3. How are the unit vectors defined in cylindrical coordinate systems?

Ans:

The vectors having unity magnitude and directed along the coordinates axes are known as unit vectors.

A cylindrical co-ordinate system has three unit vectors a_r, a_ϕ a_z and any point P in a cylindrical coordinate system is represented as $P(r, \phi, z)$.

Hence, the vector of point 'P' can be given as,

$$\bar{P} = P_r\bar{a}_r + P_\phi\bar{a}_\phi + P_z\bar{a}_z$$

Where,

a_r is a unit vector parallel to the xy -plane and directed radially outwards perpendicular to the cylindrical surface.

a_ϕ is a unit vector parallel to the xy -plane and pointing in the direction of increasing ϕ.

a_z is a unit vector parallel to z-axis and pointing in the direction of increasing z.

Q4. What is electrostatic field. Give its applications.

Ans:

Electrostatic Field

The electric field produced by a static charge distribution is said to be electrostatic field. Basically, this field is conservative in nature i.e., it obeys conservative property. It means that, the work done in carrying a unit charge around any closed path within the field is zero. These fields are governed by two basic laws namely, Coulomb's law and Gauss's law.

Applications

Electrostatic fields are having a very wide range of applications, some of them are listed.

- To get the electron beam deflection, electrostatic field is applied in CRO
- For the mobility of charges
- To develop potential.
- In X-ray machines
- In electric power transmission.

Q5. State Coulomb's law. Write the expression in SI form.

Ans:

Coulomb's Law

It states that the force between two charged bodies is directly proportional to the product of their charges and also inversely proportional to the square of the distance between charges. Force is a vector quantity and its units are Newton.

Coulomb's Law in SI Form

Let Q_1 and Q_2 be the two charges separated by a distance R, then the force between the charged bodies is given as,

$$F = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{R^2}a_R$$

Where, a_R is a unit vector.

Q6. State Coulomb's law. Give its applications.

Ans:

Coulomb's Law

For answer refer Unit-I, Q2.

Applications

- It is employed to determine the force between two charges.
- It is employed to determine the electric field at a point due to a fixed charge
- Potential and electric field due to any type of charge distribution can be found with the help of Coulomb's law.

Q7. State and list the applications of Gauss's law.

Ans:

Gauss Law

It states that the net flux (ψ) coming out of any closed surface is equal to the total charge (Q) enclosed by that surface.

$$\therefore \psi = Q$$

Applications of Gauss's Law

1. Electric field due to long infinite wire
2. Electric field at any point on a uniform charged sphere.
3. Electric field at any point due to infinite charge surface.
4. Application of Gauss's law to point charge.
5. Application of Gauss's law to differential volume element.

Q8. What are the limitations of Gauss's Law?

Ans: (Dec.-14, (R13), Q1(b) | Model Paper-II, Q1(a))

(1) Gauss law cannot be used to determine E or D, when the charge distribution is not symmetric.

(2) It cannot be used on non-Gaussian surfaces.

(3) It cannot be used if the surface includes the complete volume.

Q9. List out the properties of a potential function.

Ans:

Properties of a Potential Function

The different properties of potential function are as follows,

1. Potential function has a single value at any point in any electrostatic field.
2. Potential function is a continuous function.
3. The difference in potential between any two points does not depends on the path of integration.
4. An electrostatic field is conservative in nature since $\Delta \times \Delta V = 0$ that is the work done in moving a point charge around any closed path in the field is zero.

Q10. Define electric field intensity. Mention any two sources of electromagnetic field.

Ans:

Electric Field Intensity

Electric Field Intensity is defined as the force exerted per unit charge at a point in the vicinity of field.

i.e., $\overline{E} = \frac{\overline{F}}{Q}$

Where,

$\overline{F}$ = Force exerted between the charges

Q = Charge present at the point of consideration.

$$\overline{E} = \frac{\frac{Q_1Q_2}{4\pi\epsilon_0 R^2}\overline{a}_R}{Q_2} = \frac{Q_1}{4\pi\epsilon_0 R^2}\overline{a}_R$$

Where,

R = Distance between the charges Q_1 and Q_2.

EFI due to a point charge distribution is given by,

$$E = \frac{1}{4\pi\epsilon_0}\sum_{K=1}^{n}\frac{Q_K}{R_K^2}\overline{a}_K$$

Sources of Electromagnetic Field

- The sources of electromagnetic field includes,

Natural Sources: Atmospheric thunderstorms, precipitation static, ratio stars and cosmic rays etc.

Artificial Sources: Electronic communication devices, Nuclear, electric power sources, Ignition systems etc.

Q11. Justify the electric field is conservative.

Ans: (Nov./Dec.-17, (R16), Q1(c) | Model Paper-I, Q1(b))

A filed is said to be conservative, if it is independent of path. Electric field is conservative as it's force is radially symmetric. In vector calculus , conservative vector field is gradient of some function i.e., scalar potential. Line integral of independent path is equivalent to vector field for being conservative.

Q12. Define electro static field and mention any two sources.

Ans: (Nov./Dec.-18, R16, Q1(a) | Model Paper-II, Q1(b))

Electrostatic Field

For answer refer Unit-1, Q1.

(i) Positive and negative electric charges and links are the sources of electric field.

(ii) Time varying magnetic fields are also the sources of electric field.

Q13. What is dipole moment?

Ans:

The dipole moment is defined as the product of charge 'Q' and the distance between the charges.

Mathematically it is given as,

$$P = Q\overline{d}$$

Since the displacement $\overline{d}$ is a vector directed from negative charge to positive charge, the direction of dipole moment is same as that of displacement vector.

Q14. What is meant by a pure electric dipole?

Ans:

An electric dipole, or a dipole, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance 'd' as shown in figure.

Figure: Dipole

An important characteristic of dipole is the dipole moment defined as, the product of magnitude of charges and distance between them.

$$P = Qd$$

Where, 'P' denotes dipole moment.

A dipole is said to be pure (or ideal) when the distance 'd' approaches zero and charge Q tends to infinity, but the dipole moment P which is the product of Qd remains constant.

Q15. Describe what are the source of electric field and magnetic fields?

Ans: Dec.-14, (R13), Q1(a)

Electric and magnetic fields are produced by all the power sources including conductors, wires, tools, home appliances etc.

The most common source is moving charge.

The different sources of electric field include,

(i) Thunderclouds

(ii) Friction

(iii) Sparks

(iv) Direct current

(v) Television and computer screens with CRT's.

The sources of magnetic field are,

(i) Transformers

(ii) Inductors

(iii) Current carrying conductors

(iv) Permanent magnets etc.

Q16. Find the maximum charge that can be held on the isolated sphere 2 m in diameter, the sphere being in air with dielectric strength 40 kV/cm.

Ans:

Given that,

Diameter of sphere, $D = 2$ m

Dielectric strength, $E = 40$ kV/cm $= 40 \times 10^3$ V/cm

Radius of sphere, $R = \frac{D}{2}$

$= \frac{2}{2} = 1$ m

$= 1 \times 10^2$ cm

$R^2 = (1 \times 10^2)^2$

$= 1 \times 10^4$ cm

Maximum Charge when the Sphere is in Air

Electric field intensity is given by,

$$E = \frac{Q}{4\pi\epsilon_0\epsilon_r R^2}$$

$\therefore$ Charge,

$$Q = (E)(4\pi\epsilon_0\epsilon_r)(R)^2 \quad \text{... (1)}$$

The dielectric medium is air thus $\epsilon_r = 1$.

Substituting the values in equation (1), we get,

$$Q = (40 \times 10^3)(4\pi \times 8.854 \times 10^{-12} \times 1)(1 \times 10^3)$$

$$= 0.044 \text{ C}$$

Q17. Verify that the potential field given below satisfies the Laplace's equation,

$V = 4x^2 - 6y^2 + 2z^2$.

Nov.-10, Set-1, Q6(

Ans:

Given that,

$$V = 4x^2 - 6y^2 + 2z^2 \quad \text{... (1)}$$

Now,

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}(V) + \frac{\partial^2}{\partial y^2}(V) + \frac{\partial^2}{\partial z^2}(V) \quad \text{... (2)}$$

Substituting equation (1) in equation (2) we have,

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}[4x^2 - 6y^2 + 2z^2] + \frac{\partial^2}{\partial y^2}[4x^2 - 6y^2 + 2z^2] + \frac{\partial^2}{\partial z^2}[4x^2 - 6y^2 + 2z^2]$$

$$= \frac{\partial}{\partial x}[8x] + \frac{\partial}{\partial y}[-12y] + \frac{\partial}{\partial z}[4z]$$

$$= 8 - 12 + 4$$

$$= 0$$

$$\therefore \boxed{\nabla^2 V = 0}$$

Hence, the given potential field (V) satisfies Laplace's equation.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

1.1 REVIEW OF CONVERSION OF A VECTOR FROM ONE COORDINATE SYSTEM TO ANOTHER COORDINATE SYSTEM

Q18. Explain Cartesian co-ordinate system and differential elements in Cartesian co-ordinate system.

Model Paper-II, Q3(a)

Ans:

Cartesian Co-ordinate System

The other name for Cartesian co-ordinate system is the rectangular co-ordinate system. This system consists of 3 co-ordinate axes intersecting at the origin. The angle between these co-ordinate axis is 90° to each other and these axis are denoted as x, y, and z.

Types

Based on axis rotation the system is divided into two types,

(i) Right hand system and

(ii) Left hand system.

(i) Right Hand System

If the x-y plane is made to rotate in anti-clock wise direction then due to this rotation the screw will move upward in the direction of positive z-axis as shown in figure (a).

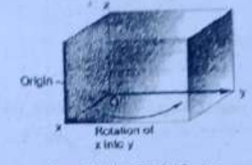


Figure (1): Right Handed System

With respect to right hand rule, thumb indicates the x-axis, fore finger indicates the y-axis and the middle finger indicates the z-axis (direction of moment).

(ii) Left Hand System

If the x-y plane is made to rotate in clock wise direction then due to this rotation the screw will move downwards in the direction of negative z-axis as shown in figure (b).

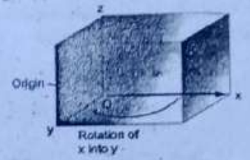


Figure (b): Left Hand System

With respect to left hand rule, thumb indicates the x-axis, fore finger indicates the y-axis (direction of moment).

A plane is said to be a y-z plane if $x = 0$

A plane is said to be a x-z plane if $y = 0$

A plane is said to be a x-y plane if $z = 0$.

Differential Elements in Cartesian Co-ordinate System

The differential elements of Cartesian co-ordinates is obtained by assuming a point $P(x, y, z)$ in the space. If the point 'P' is moved by a small incremental distance in all the directions then another point 'P'' is obtained at $(x + dx, y + dy, z + dz)$.

The differential length in the direction of x-axis is 'dx'.

The differential length in the direction of y-axis is 'dy'.

The differential length in the direction of z-axis is 'dz'.

Then the total differential vector length 'dl' is obtained by the sum of all individual lengths.

Therefore, $\overline{dl} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$ ... (1)

This is called as the differential vector joining both points 'P' and 'P''.

Because of increment in all directions at P', six planes come into existence and this results in a differential volume, known as rectangular parallelopiped. The diagonal formed between point 'P' and 'P'' is called as differential vector length as shown in figure (a).

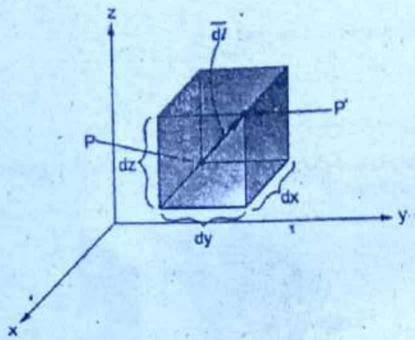


Figure (a): Differential Elements and Different Length in Cartesian System

The magnitude of differential vector can be calculated as,

$$\boxed{|\overline{dl}| = \sqrt{dx^2 + dy^2 + dz^2}}$$

And, the differential volume is given as,

$$dV = dxdydz$$

Depending on the direction of the axis, three differential surface areas are defined. The notation used for differential surface element is $\overline{ds}$ given as,

$$\overline{ds} = ds\,\bar{a}_n$$

In Cartesian co-ordinate system, the differential surface elements are as shown in figure (b).

$\overline{ds}_x = dy\,dz\,\bar{a}_x$ differential surface area perpendicular to x-direction

$\overline{ds}_y = dx\,dz\,\bar{a}_y$ differential surface area perpendicular to y-axis

$\overline{ds}_z = dx\,dy\,\bar{a}_z$ differential surface area perpendicular to z-axis.

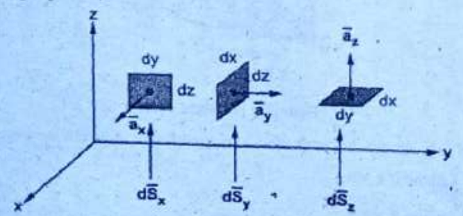


Figure (b): Differential Surface Elements in Cartesian System

Q19. Explain cylindrical co-ordinate system and differential elements in cylindrical co-ordinate system.

Ans:

The cylindrical co-ordinate system is defined by the following surfaces,

(i) A plane parallel to xy plane known as constant z plane

(ii) The radius 'r' of the cylinder, the axis cylinder lies on z-axis.

(iii) A plane moving an azimuthal angle 'ϕ' with xz plane and normal to xy plane as shown in figure (a).

The point 'P' is a function of these three polar co-ordinates 'r', 'ϕ' and 'z'. These co-ordinates are defined over the following range,

$0 \le r \le \infty$

$0 \le \phi \le 2\pi$

$-\infty \le z \le \infty$

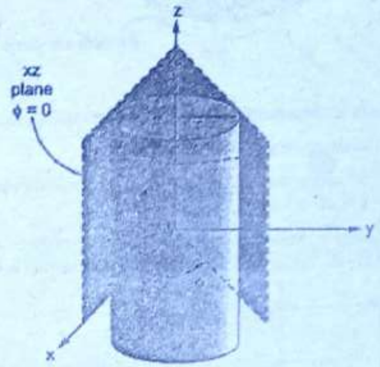


Figure (a) Cylindrical Co-ordinate System

The point where all the three surfaces intersect is known as point 'P'. The behaviour of the surfaces is as shown in figure (b).

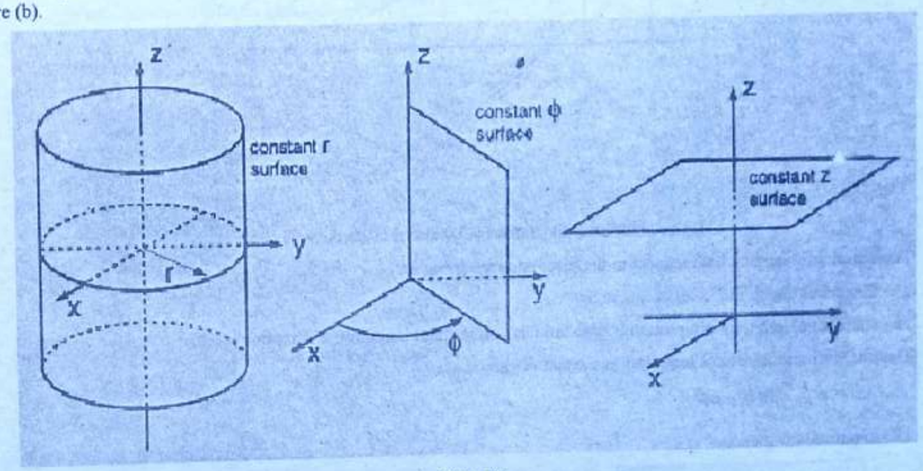


Figure (b)

(i) If 'r' is constant, then the resultant will be a circular cylinder about z-axis

(ii) If 'ϕ' is constant, then the resultant will be a vertical plane perpendicular to xy-plane

(iii) If 'z' is constant, the resultant is a parallel plane to xy plane.

If two constant planes i.e., 'r' and 'z' intersect each other, then the resultant will be a circle. If constant planes, 'ϕ' and 'r' intersect each other, then the resultant will be a line. The point of intersection of all the constant planes is at 'P' as shown in figure (c).

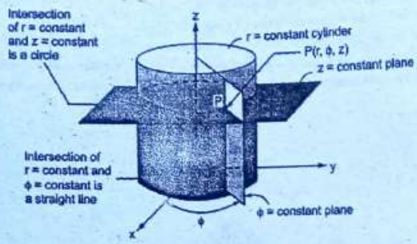


Figure (c): Representing Point P in Cylindrical System

Differential Elements in Cylindrical Co-ordinate System

In cylindrical co-ordinate the point 'P' is represented using (r, ϕ, z). If these co-ordinates are incremented by small length. Then the new co-ordinates will be $(r + dr, \phi + d\phi, z + dz)$.

Therefore, two cylinders each of radius 'r' and '$r + dr$' respectively will result. Similarly two radial planes with angle ϕ and $d\phi$ and two horizontal planes at z and $z + dz$. Enclosing these changes is as shown in figure (a).

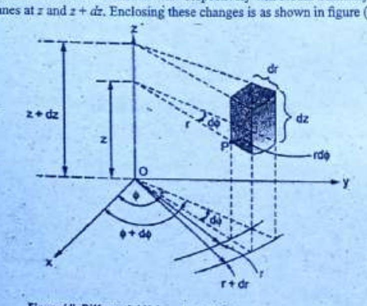


Figure (d): Differential Volume in Cylindrical Co-ordinate System

The differential lengths with respect to the directions are given as,

The differential length in 'r' direction is "dr".

The differential length in ϕ direction is '$rd\phi$' and the differential length in 'z' direction is 'dz'.

The total differential vector length in this system is given as,

$$\overline{dl} = dr\bar{a}_r + rd\phi\bar{a}_\phi + dz\bar{a}_z$$

The magnitude is given as,

$$\boxed{|\overline{dl}| = \sqrt{(dr)^2 + (rd\phi)^2 + (dz)^2}}$$

The differential volume is,

$$\boxed{dr = rdrd\phi dz}$$

The differential surface areas in cylindrical co-ordinate system is given as,

(i) Differential vector surface area perpendicular to r-direction is,

$$\boxed{d\bar{s}_r = rd\phi dz\bar{a}_r}$$

(ii) Differential vector surface area perpendicular to ϕ direction is,

$$\boxed{d\bar{s}_\phi = drdz\bar{a}_\phi}$$

(iii) Differential vector surface area perpendicular to z direction is,

$$\boxed{d\bar{s}_z = rdrd\phi\bar{a}_z}$$

These surface areas are represented with respect to the directions is as shown in figure (5).

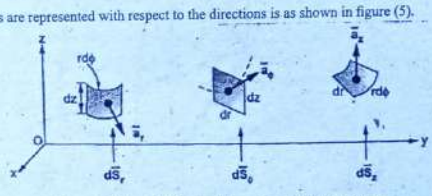


Figure (e): Differential Surface Elements in Cylindrical System

Q20. Explain spherical co-ordinate system and differential elements in spherical co-ordinate system.

Ans:

The spherical co-ordinate system is defined by three different surfaces,

(i) Radius of the sphere (r), where the centre of the sphere is origin.

(ii) Right circular cone along z-axis apex is located at the origin. Its angle is 2θ

(iii) Half plane which is normal to the xy plane and angle between these planes is 'ϕ' (Azimuthal angle).

Thus, surfaces are depicted in figure (a),

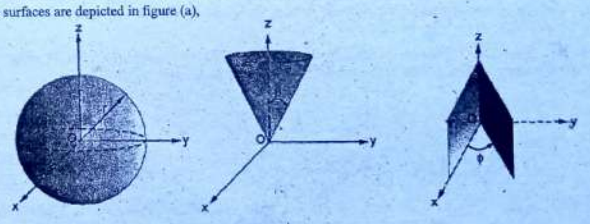


Figure (a)

These surfaces are defined over the ranges given below,

(i) $0 \leq \gamma < \infty$

(ii) $0 \leq \phi \leq 2\pi$

(iii) $0 \leq \theta \leq \pi$ as half angle.

The point of intersection of all these surfaces is known as point $P(\gamma, \theta, \phi)$ is as shown in figure (b).

If 'r' remains constant, this results in a sphere with origin as its centre.

If 'θ' remains constant, this results in a right circular cone with apex located at the origin.

If 'ϕ' remains constant, this results in a plane normal to the xy-plane.

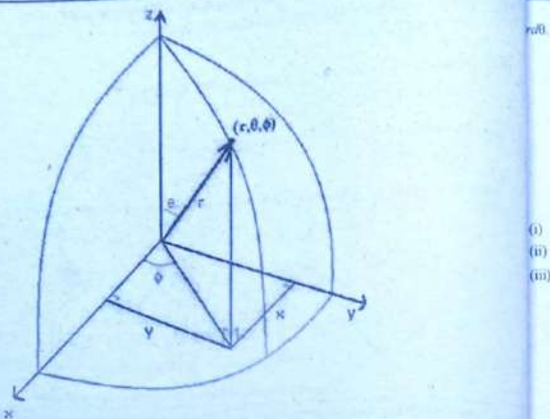


Figure (b): Representing Point P in Spherical Co-ordinate System

If the sphere (r) and the right circular cone (θ) both get intersected, then a horizontal circle will result with radius '$r\sin\theta$'.

If the plane perpendicular to xy-plane (ϕ = constant) and the sphere (r = constant) and the right circular cone intersects at a point, that point is represented as P.

Differential Elements in Spherical Co-ordinate System

The point 'P' is represented by the sperical co-ordinates (r, θ, ϕ). If these co-ordinates are incremented with small amount then the new co-ordinates will be $r + dr, d + d\theta, \phi + d\phi$"

Due to this increment, the two spheres of radius 'r' and $r + dr$, two cones with half angles θ and $\theta + d\theta$and two plane at angles ϕ and $\phi + d\phi$ are formed. A small volume enclosing all these surfaces is as shown in figure (c).

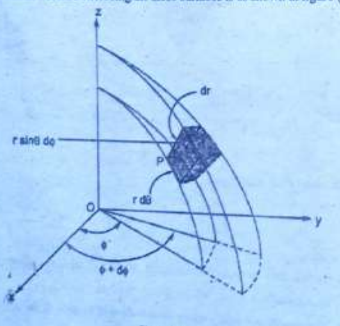


Figure (c)

The incremental change i.e., the differential length in r-direction is 'dr', in ϕ-direction is '$r\sin\theta d\phi$' and in θ-directionis $rd\theta$. Therefore, the differential vector length is given as,

$$\boxed{\overline{dl} = d_r a_r + rd\theta\overline{a}_\theta + \sin\theta d\phi\overline{a}_\phi}$$

The magnitude is given as,

$$|\overline{dl}| = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2}$$

The differential volume for the differential element is,

$$\boxed{dv = r^2 \sin\theta dr d\theta d\phi}$$

The differential surface area in all the directions are represented as,

(i) $\overline{ds}_r = r^2 \sin\theta d\theta d\phi$ differential vector surface area perpendicular to r-direction

(ii) $\overline{ds}_\theta = r\sin\theta dr d\phi$ differential vector surface area perpendicular to θ-direction

(iii) $\overline{ds}_\phi = rdrd\theta$ differential vector surface area perpendicular to ϕ-direction.

This is as shown in figure (d).

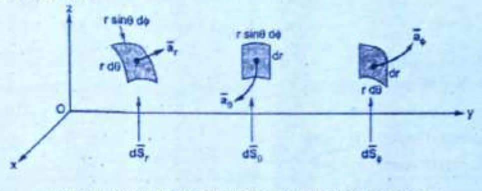


Figure (d): Differential Surface Elements in Spherical Co-ordinate System

Q21. What are different types of vector transformations. Explain any two of them.

Ans: **Model Paper-I, Q2(a)**

Different transformations of vector from one co-ordinate system to the another co-ordinate system are as follows,

1. Cartesian to cylindrical
2. Cylindrical to cartesian
3. Cartesian to spherical
4. Spherical to cartesian
5. Spherical to cylindrical
6. Cylindrical to spherical

1. Cartesian to Cylindrical

Let the vector $\overline{A}$ in cartesian co-ordinate system, which can be represented as, $\overline{A} = A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z$ ... (1)

The same vector in cylindrical co-ordinate system can be represented as,

$$\overline{A} = A_r\overline{a}_r + A_\phi\overline{a}_\phi + A_z\overline{a}_z \quad \text{... (2)}$$

The component of $\overline{A}$ in $\overline{a}_r$ direction is given by,

$$A_r = \overline{A}.\overline{a}_r$$

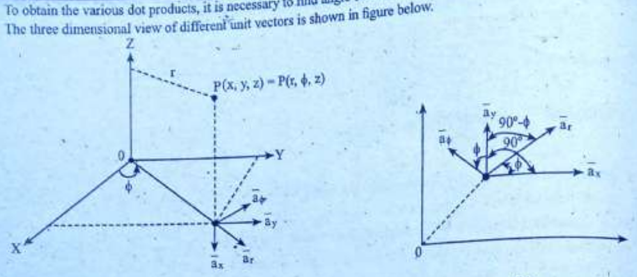
$$A_r = [A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z].\overline{a}_r = A_x(\overline{a}_x.\overline{a}_r) + A_y(\overline{a}_y.\overline{a}_r) + A_z(\overline{a}_z.\overline{a}_r) \quad \text{... (3)}$$

Similarly,

$$A_\phi = [A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z].\overline{a}_\phi = A_x(\overline{a}_x.\overline{a}_\phi) + A_y(\overline{a}_y.\overline{a}_\phi) + A_z(\overline{a}_z.\overline{a}_\phi) \quad \text{... (4)}$$

$$A_z = [A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z].\overline{a}_z = A_x(\overline{a}_x.\overline{a}_z) + A_y(\overline{a}_y.\overline{a}_z) + A_z(\overline{a}_z.\overline{a}_z) \quad \text{... (5)}$$

To obtain the various dot products, it is necessary to find angle between the unit vectors. The three dimensional view of different unit vectors is shown in figure below.



(a) (b)

Figure

$$\bar{a}_x.\bar{a}_r = |\bar{a}_x||\bar{a}_r|\cos\phi$$

Here the magnitude of all vector is unity.

$$\bar{a}_x.\bar{a}_r = (1)(1)\cos\phi = \cos\phi$$

$$\bar{a}_x.\bar{a}_\phi = (1)(1)\cos(90+\phi) = -\sin\phi$$

$$\bar{a}_y.\bar{a}_r = (1)(1)\cos(90-\phi) = \sin\phi$$

$$\bar{a}_y.\bar{a}_\phi = (1)(1)\cos\phi = \cos\phi$$

$$\bar{a}_z.\bar{a}_z = 1$$

$\bar{a}_z$ is perpendicular to $\bar{a}_r$ and $\bar{a}_\phi$ so;

$$\bar{a}_z.\bar{a}_r = \bar{a}_z.\bar{a}_\phi = 0$$

Now substituting these values in equations (3), (4) and (5) we get,

$$A_r = A_x\cos\phi + A_y\sin\phi + A_z(0)$$

$$A_r = A_x\cos\phi + A_y\sin\phi \quad \text{...(}$$

Similarly,

$$A_\phi = -A_x\sin\phi + A_y\cos\phi + A_z(0)$$

$$= -A_x\sin\phi + Ay\cos\phi \quad \text{...(}$$

$$A_z = A_x(0) + A_y(0) + A_z(1)$$

$$A_z = A_z$$

The above result can be represented in matrix form as,

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

(2) **Cylindrical to Cartesian**

Now the component $\bar{A}$ is in the direction of $\bar{a}_x$

$$\bar{A}_x = [\bar{A}.\bar{a}_x]$$

$$= [A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z].\bar{a}_x$$

$$= A_r(\bar{a}_r.\bar{a}_x) + A_\phi(\bar{a}_\phi.\bar{a}_x) + A_z(\bar{a}_z.\bar{a}_x) \quad \text{...(}$$

Similarly,

$$\bar{A}_y = A_r(\bar{a}_r.\bar{a}_y) + A_\phi(\bar{a}_\phi.\bar{a}_y) + A_z(\bar{a}_z.\bar{a}_y) \quad \text{...(9)}$$

$$\bar{A}_z = A_r(\bar{a}_r.\bar{a}_z) + A_\phi(\bar{a}_\phi.\bar{a}_z) + A_z(\bar{a}_z.\bar{a}_z) \quad \text{...(10)}$$

$\because$ In dot product $\bar{a}_r.\bar{a}_x = \bar{a}_x.\bar{a}_r = \cos\phi$ and so on.

The result of dot product can be summerized in table format as,

Dot product (•)	$\bar{a}_r$	$\bar{a}_\phi$	$\bar{a}_z$
$\bar{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\bar{a}_y$	$\sin\phi$	$\cos\phi$	0
$\bar{a}_z$	0	0	1

Hence by referring above the equations (8) (9) and (10) can be written as,

$$A_x = A_r\cos\phi - A_\phi\sin\phi + A_z(0) = A_r\cos\phi - A_\phi\sin\phi$$

$$A_y = A_r\sin\phi + A_\phi\cos\phi + A_z(0) = A_r\sin\phi + A_\phi\cos\phi$$

$$A_z = A_r(0) + A_\phi(0) + A_z(1) = A_z$$

So, the above result can be expressed in matrix form as,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Q22. Write about the transformation of vectors from spherical to cartesian.

Ans:

Spherical to Cartesian Transformation of Vectors

Consider the vector $\bar{A}$ in spherical co-ordinate system as,

$$\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$$

The same vector $\bar{A}$ in cartesian co-ordinate system is given as,

$$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

The component of $\bar{A}$ in $\bar{a}_x$ direction is given by,

$$A_x = \bar{A}.\bar{a}_x = [A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi].\bar{a}_x$$

$$= A_r(\bar{a}_r.\bar{a}_x) + A_\theta(\bar{a}_\theta.\bar{a}_x) + A_\phi(\bar{a}_\phi.\bar{a}_x) \quad \text{...(1)}$$

$$A_y = \bar{A}.\bar{a}_y = [A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi].\bar{a}_y$$

$$= A_r(\bar{a}_r.\bar{a}_y) + A_\theta(\bar{a}_\theta.\bar{a}_y) + A_\phi(\bar{a}_\phi.\bar{a}_y) \quad \text{...(2)}$$

$$A_z = \bar{A}.\bar{a}_z = [A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi].\bar{a}_z$$

$$= A_r(\bar{a}_r.\bar{a}_z) + A_\theta(\bar{a}_\theta.\bar{a}_z) + A\phi(\bar{a}_\phi.\bar{a}_z) \quad \text{...(3)}$$

The dot products for spherical to cartesian transformation are as follows,

Dot product (•)	$\bar{a}_x$	$\bar{a}_y$	$\bar{a}_z$
$\bar{a}_r$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\bar{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\bar{a}_\phi$	$-\sin\phi$	$\cos\phi$	0

Table

By referring the above table the equations (1), (2) and (3) can be written as,

$$A_x = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi \quad \text{... (4)}$$

$$A_y = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi \quad \text{... (5)}$$

$$A_z = A_r\cos\theta - A_\theta\sin\theta + A_\phi(0)$$

$$= A_r\cos\theta - A_\theta\sin\theta \quad \text{... (6)}$$

Therefore, the result of transformation can be written in matrix form as,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Q23. Write about the transformation of vectors from cartesian to spherical.

Ans:

Cartesian to Spherical Transformation of Vectors

Consjder the vector $\overline{A}$ in the cartesian co-ordinate system as,

$$\overline{A} = A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z \quad \text{... (1)}$$

The same vector in spherical co-ordinate system is represented as,

$$\overline{A} = A_r\overline{a}_r + A_\theta\overline{a}_\theta + A_\phi\overline{a}_\phi \quad \text{... (2)}$$

The component of $\overline{A}$ in $\overline{a}_r$ direction is given by,

$$A_r = \overline{A}.\overline{a}_r = \left[A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z\right].\overline{a}_r$$

$$= A_x(\overline{a}_x.\overline{a}_r) + A_y(\overline{a}_y.\overline{a}_r) + A_z(\overline{a}_z.\overline{a}_r) \quad \text{... (3)}$$

Here, representation of 'r' in cylindrical and spherical system is same, but the direction of $\overline{a}_r$ in both systems are different. Therefore the value of $\overline{a}_r.\overline{a}_r$ is different for spherical and cylindrical system.

Similarly, $A_\theta = \overline{A}.\overline{a}_\theta = \left[A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z\right].\overline{a}_\theta$

$$= A_x(\overline{a}_x.\overline{a}_\theta) + A_y(\overline{a}_y.\overline{a}_\theta) + A_z(\overline{a}_z.\overline{a}_\theta) \quad \text{... (4)}$$

$$A_\phi = \overline{A}.\overline{a}_\phi = \left[A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z\right].\overline{a}_\phi$$

$$= A_x(\overline{a}_x.\overline{a}_\phi) + A_y(\overline{a}_y.\overline{a}_\phi) + A_z(\overline{a}_z.\overline{a}_\phi) \quad \text{... (5)}$$

The dot products can be determined by projecting spherical unit vector on xy plane and then taking it on desired axis.

For $\overline{a}_x.\overline{a}_r$, projection of $\overline{a}_r$ on xy plane and on x axis is given by,

$$\overline{a}_x.\overline{a}_r = \overline{a}_x.\overline{a}_r = \sin\theta\cos\phi$$

Similarly, the other dot products can be determined.

$$\overline{a}_\theta.\overline{a}_x = \overline{a}_x.\overline{a}_\theta = \cos\theta\cos\phi$$

$$\overline{a}_\phi.\overline{a}_x = \overline{a}_x.\overline{a}_\phi = -\sin\phi$$

$$\overline{a}_r.\overline{a}_y = \overline{a}_y.\overline{a}_r = \sin\theta\sin\phi$$

$$\overline{a}_y.\overline{a}_\theta = \overline{a}_\theta.\overline{a}_y = \cos\theta\sin\phi$$

$$\overline{a}_y.\overline{a}_\phi = \overline{a}_\phi.\overline{a}_y = \cos\phi$$

$$\overline{a}_z.\overline{a}_r = \overline{a}_r.\overline{a}_z = \cos\theta$$

$$\overline{a}_z.\overline{a}_\theta = \overline{a}_\theta.\overline{a}_z = -\sin\theta$$

$$\overline{a}_z.\overline{a}_\phi = \overline{a}_\phi.\overline{a}_z = 0$$

Summarizing the result of dot products in the tubular form as,

Dot product (•)	$\overline{a}_r$	$\overline{a}_\theta$	$\overline{a}_\phi$
$\overline{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\overline{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\overline{a}_z$	$\cos\theta$	$-\sin\theta$	θ

Table

By referring the above table, the equations (3), (4) and (5) can be modifed as,

$$A_r = A_x(\sin\theta\cos\phi) + A_y(\sin\theta\sin\phi) + A_z\cos\theta \quad \text{... (6)}$$

$$A_\theta = A_x(\cos\theta\cos\phi) + A_y(\cos\theta\sin\phi) - A_z\sin\theta \quad \text{... (7)}$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi - A_z(0)$$

$$= -A_x\sin\phi + A_y\cos\phi \quad \text{... (8)}$$

The result of transformation can be summarize in matrix form as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Q24. Write about the transformation of vectors from cylindrical to spherical co-ordinate system.

Ans:

Vector Transformation from Cylindrical to Spherical Co-ordinate System

Consider the vector $\overline{A}$ in cylindrical co-ordinate system as,

$$\overline{A} = A_\rho\overline{a}_\rho + A_\phi\overline{a}_\phi + A_z\overline{a}_z$$

The same vector in spherical co-ordinate system is given by,

$$\overline{A} = A_r\overline{a}_r + A_\theta\overline{a}_\theta + A_\phi\overline{a}_\phi$$

The component of $\overline{A}$ in $\overline{a}_r$ direction is given by,

$$A_r = \overline{A}.\overline{a}_r$$

$$= [A_\rho\overline{a}_\rho + A_\phi\overline{a}_\phi + A_z\overline{a}_z].\overline{a}_r$$

$$= A_\rho(\overline{a}_\rho.\overline{a}_r) + A_\phi(\overline{a}_\phi.\overline{a}_r) + A_z(\overline{a}_z.\overline{a}_r) \quad \text{... (1)}$$

$$A_\theta = \overline{A}.\overline{a}_\theta$$

$$= [A_\rho\overline{a}_\rho + A_\phi\overline{a}_\phi + A_z\overline{a}_z].\overline{a}_\theta$$

$$= A_\rho(\overline{a}_\rho.\overline{a}_\theta) + A_\phi(\overline{a}_\phi.\overline{a}_\theta) + A_z(\overline{a}_z.\overline{a}_\theta) \quad \text{... (2)}$$

$$A_\phi = \overline{A}.\overline{a}_\phi$$

$$= [A_\rho\overline{a}_\rho + A_\phi\overline{a}_\phi + A_z\overline{a}_z].\overline{a}_\phi$$

$$= A_\rho(\overline{a}_\rho.\overline{a}_\phi) + A_\phi(\overline{a}_\phi.\overline{a}_\phi) + A_z(\overline{a}_z.\overline{a}_\phi) \quad \text{... (3)}$$

The dot products of cylindrical to spherical transformation are as follows,

Dot product (·)	$\bar{a}_r$	$\bar{a}_\theta$	$\bar{a}_\phi$
$\bar{a}_\rho$	$\sin\theta$	$\cos\theta$	0
$\bar{a}_\phi$	0	0	1
$\bar{a}_z$	$\cos\theta$	$-\sin\theta$	0

Table

By referring the above table the equations (1), (2) and (3) can be written as,

$$A_r = A_\rho \sin\theta + A_\phi(0) + A_z(\cos\theta)$$

$$= A_\rho \sin\theta + A_z \cos\theta \quad \text{... (4)}$$

$$A_\theta = A_\rho(\cos\theta) + A_\phi(0) + A_z(-\sin\theta)$$

$$= A_\rho \cos\theta - A_z \sin\theta \quad \text{... (5)}$$

$$A_\phi = A_\rho(0) + A_\phi(1) + A_z(0)$$

$$= A\phi \quad \text{... (6)}$$

Therefore, the result of transformation can be written in matrix form as,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Q25. Write about the transformation of vectors from spherical to cylindrical co-ordinate system.

Ans:

Vectors Transformation from Spherical to Cylindrical Co-ordinate System

Consider the vector $\bar{A}$ in spherical co-ordinate system as,

$$\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$$

The same vector in cylindrical co-ordinate system is represented as,

$$\bar{A} = A_\rho\bar{a}_\rho + A_\phi\bar{a}_\phi + A_z\bar{a}_z$$

The component of $\bar{A}$ in $\bar{a}_\rho$ direction is given by,

$$A_\rho = \bar{A}.\bar{a}_\rho = [A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi].\bar{a}_\rho$$

$$= A_r(\bar{a}_r.\bar{a}_\rho) + A_\theta(\bar{a}_\theta.\bar{a}_\rho) + A_\phi(\bar{a}_\phi.\bar{a}_\rho) \quad \text{... (1)}$$

Similarly,

$$A_\phi = \bar{A}.\bar{a}_\phi = [A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi].\bar{a}_\phi$$

$$= A_r(\bar{a}_r.\bar{a}_\phi) + A_\theta(\bar{a}_\theta.\bar{a}_\phi) + A_\phi(\bar{a}_\phi.\bar{a}_\phi)$$

$$= A_r(\bar{a}_r.\bar{a}_\phi) + A_\theta(\bar{a}_\theta.\bar{a}_\phi) + A_\phi \quad [\because \bar{a}_\phi.\bar{a}_\phi = 1] \quad \text{... (2)}$$

$$A_z = \bar{A}.\bar{a}_z = [A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi].\bar{a}_z$$

$$= A_r(\bar{a}_r.\bar{a}_z) + A_\theta(\bar{a}_\theta.\bar{a}_z) + A_\phi(\bar{a}_\phi.\bar{a}_z) \quad \text{... (3)}$$

The dot products of spherical to cylindrical transformation are as follows,

Dot product (·)	$\bar{a}_\rho$	$\bar{a}_\phi$	$\bar{a}_z$
$\bar{a}_r$	$\sin\theta$	0	$\cos\theta$
$\bar{a}_\theta$	$\cos\theta$	0	$-\sin\theta$
$\bar{a}_\phi$	0	1	0

Table

By referring the above table the equations (1), (2) and (3) can be written as,

$$A_\rho = A_r\sin\theta + A_\theta\cos\theta + A_\phi(0)$$

$$= A_r\sin\theta + A_\theta\cos\theta \quad \text{... (4)}$$

$$A_\phi = A_r(0) + A_\theta(0) + A_\phi$$

$$= A\phi \quad \text{... (5)}$$

$$A_z = A_r\cos\theta + A_\theta(-\sin\theta) + A_\phi(0)$$

$$= A_r\cos\theta - A_\theta\sin\theta \quad \text{... (6)}$$

Therefore the result of transformation can be written in matrix form as,

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

1.2 COULOMB'S LAW, ELECTRIC FIELD INTENSITY, ELECTRIC FIELD DUE TO POINT CHARGES

Q26. Define and explain the following terms,

(i) Electrostatic field

(ii) Electric field intensity

(iii) Electric potential

Ans:

(i) **Electrostatic Field** — Model Paper-III, Q2(a)

The electric field produced by a static charge distribution is said to be electrostatic field. Basically, this field is conservative in nature i.e., it obeys conservative property. It means that, the work done in carrying a unit charge around any closed path within the field is zero. These fields are governed by two basic laws namely, Coulomb's law and Gauss's law.

(ii) **Electric Field Intensity (E)**

For answer refer Unit-I, Q10.

(iii) **Electric Potential (V)**

Electric potential is defined as the work done per unit charge in bringing a point charge from infinity or a zero reference point into the vicinity of an electric field.

$$\text{i.e., } W = -Q\int_{\infty}^{r} \bar{E}.d\bar{l} \quad \text{... (1)}$$

We know that, $V = \frac{W}{Q}$ ... (2)

Substituting equation (1) in equation (2), we get,

$$V = \frac{-Q\int_{\infty}^{r} \bar{E}.d\bar{l}}{Q}$$

$$\Rightarrow \quad V = -\int_{\infty}^{r} \bar{E}.d\bar{l}$$

$$\Rightarrow \quad V = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R \bar{a}_R dr \qquad \left[\because E = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R \text{ and } d\bar{l} = \bar{a}_R dR\right]$$

$$\Rightarrow \quad V = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 R^2}.1.dR \qquad [\because \bar{a}_R \bar{a}_R = 1]$$

$$\Rightarrow \quad V = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 R^2}.1.dR$$

$$\Rightarrow \quad V = \frac{-Q}{4\pi\epsilon_0}\left[\frac{R^{-2+1}}{(-2+1)}\right]_{\infty}^{r}$$

$$\Rightarrow \quad V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R}\right]_{\infty}^{r}$$

$$\therefore \quad V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{\infty}\right] = \frac{Q}{4\pi\epsilon_0 r}$$

Hence, it is a scalar quantity with units of J/C or V.

Q27. State and express Coulomb's law in vector form.

Dec.-11, Set-4, Q2(a)

OR

State and explain Coulomb's law for the vector force between two point charges in free space.

Ans:

Coulomb's law states that, "the force between two charged bodies is directly proportional to the product of their charges and inversely proportional to the square of the distance between them". Provided that, the sizes of the charged bodies are quite negligible when compared with the distance between them, due to which those charged bodies are usually termed as point charges.

Consider two point charges Q_1 and Q_2 separated by a distance R. Then from Coulomb's law, the force between Q_1 and Q_2 can be determined as,

$$F \propto Q_1 Q_2 \text{ and}$$

$$F \propto \frac{1}{R^2}$$

$$\Rightarrow \quad F = K\frac{Q_1 Q_2}{R^2}$$

Where,

K = Constant of proportionality

$$= \frac{1}{4\pi\epsilon_0} \qquad \text{[For free space]}$$

$$\therefore F = \frac{1}{4\pi\epsilon_0} \times \frac{Q_1 Q_2}{R^2} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \text{ N}$$

The above expression gives the magnitude of force exerted on Q_1 by Q_2 or on Q_2 by Q_1. If the magnitude of force is a negative value then, it indicates that the force acting between the charges is an attracting force and if it is positive value then, it indicates that the force acting between the charges is a repulsive force. This is because of the fact that like charges repel and unlike one attracts each other. Either kind of force will act along the straight line joining the charges.

Hence, Coulomb's law can be defined in its vector form as,

$$\bar{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\bar{a}_r$$

Where, $\bar{a}_r$ is the unit vector which specifies the direction of force between the charges.

So, the force exerted on Q_1 by Q_2 can be expressed as,

$$\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\bar{a}_{21}$$

Where,

$$\bar{a}_{21} = \frac{\bar{R}_{21}}{|\bar{R}_{21}|} = \frac{\bar{R}_{21}}{R} \text{ and}$$

$$\bar{R}_{21} = \bar{R}_1 - \bar{R}_2$$

The vectors $\bar{R}_1$ and $\bar{R}_2$ indicates the location of the charges Q_1 and Q_2 respectively.

Coulomb's law obeys linearity and also the superposition principle due to which, the force exerted on a point charge by 'n' number of different point charges can be given by the vector summation of all 'n' individual forces acting on that point charge i.e.,

$$\bar{F}_{total} = \sum_{i=1}^{n} \bar{F}_i$$

Q28. Explain the concept of electric field intensity.

Ans:

Nov.-18, Set-1, Q1(a)

The space or region around an electric charge in which the effect of charge is felt is termed as electrostatic field. A static electric field may be due to a positive charge or a negative charge. The presence of electric field or the absence of electric field in a region is confirmed by bringing the test charge into that region. If a test charge experiences a force, then it means that the field is present. If a small positive charge is brought into the field of a positive charge, it experiences a repulsive force and if the field is of negative charge and the test charge is positive, it experiences the force of attraction.

Electric Field Due to a Static Charge Configuration

"Electric field is defined as the force per unit charge provided the charge being as small as possible".

If 'F' is the force acting on the charge 'q' then electric field intensity is given by,

$$\boxed{E = \underset{q \to 0}{Lt}\left(\frac{F}{q}\right) \text{N/C}}$$

The test charge 'q' must be so small that it should not disturb the properties of the field, i.e., it should exert negligible force on the other charges. So, 'q' is too vanishingly small.

Consider a point charge which is positive i.e., $+Q$ as shown in figure. The test charge or the static charge $(+q)$ is located at 'S' at a distance 'r' from point charge $+Q$.

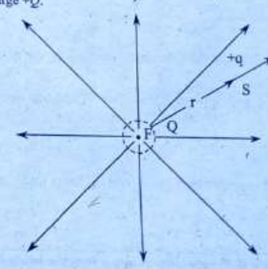


Figure: Electric Field at a Point Charge +Q

According to Coulomb's law, the test charge experiences a force and an electric field is produced around a charge $+Q$, where the force acts and if any charge is brought in this region, it also experiences a force. The force is repulsive as field and test charge both are positive and this force is directed radially outwards.

The force experienced by the test charge is given as,

$$\vec{F} = \left[\frac{Qq}{4\pi\epsilon_o R^2}\right]\bar{a}_r$$

Where,

$\bar{a}_r$ = A unit vector directed away from point charge.

As electric field,

$$\bar{E} = \frac{\bar{F}}{q}$$

$$\boxed{\bar{E} = \frac{Q}{4\pi\epsilon_o R^2}\bar{a}_r \text{ N/C}}$$

Q29. Explain the superposition principle governing the forces between charges at rest.

Model Paper-II, Q2(a)

Ans:

Consider two point charges Q_1 and Q_2 be located at points P_1 and P_2 respectively.

According to Coulomb's law, the force exerted on charge Q_1 due to charge Q_2 is,

$$\bar{F}_{21} = \frac{Q_1Q_2}{4\pi\epsilon_0|\bar{R}_1 - \bar{R}_2|^2}\bar{a}_{21}$$

Where, the vectors $\bar{R}_1$ and $\bar{R}_2$ indicates the location of the charges Q_1 and Q_2 respectively, $\bar{a}_{21}$ is the unit vector which indicates the direction of force (i.e., from Q_2 to Q_1).

Let us consider another charge Q_3 at point P_3.

Now, the force exerted on charge Q_1 due to charge Q_3 is,

$$\bar{F}_{31} = \frac{Q_1Q_3}{4\pi\epsilon_0|\bar{R}_1 - \bar{R}_3|^2}\bar{a}_{31}$$

According to superposition principle i.e., the forces between charges at rest.

We have, the total force exerted on charge Q_1 due to the charges Q_2 and Q_3 is the vectorial sum of the force exerted on Q_1 due to individual point charge in the absence of other charges i.e.,

$$\bar{F}_1 = \bar{F}_{21} + \bar{F}_{31}$$

$$= \frac{Q_1Q_2}{4\pi\epsilon_0|\bar{R}_1 - \bar{R}_2|^2}\bar{a}_{21} + \frac{Q_1Q_3}{4\pi\epsilon_0|\bar{R}_1 - \bar{R}_3|^2}\bar{a}_{31}$$

$$= \frac{Q_1}{4\pi\epsilon_0}\left[\frac{Q_2}{|\bar{R}_1 - \bar{R}_2|^2}\bar{a}_{21} + \frac{Q_3}{|\bar{R}_1 - \bar{R}_3|^2}\bar{a}_{31}\right]$$

In general according to superposition principle, the force exerted on charge Q_1 due to '$n-1$' point charges $Q_2, Q_3, ..., Q_n$ is,

$$\bar{F}_1 = \frac{Q_1}{4\pi\epsilon_0}\left[\frac{Q_2}{|\bar{R}_1 - \bar{R}_2|^2}\bar{a}_{21} + \frac{Q_3}{|\bar{R}_1 - \bar{R}_3|^2}\bar{a}_{31} + ... + \frac{Q_{n-1}}{|\bar{R}_1 - \bar{R}_{n-1}|^2}\bar{a}_{(n-1)1} + \frac{Q_n}{|\bar{R}_1 - \bar{R}_n|^2}\bar{a}_{n1}\right]$$

$$= \frac{Q_1}{4\pi\epsilon_0}\sum_{k=2}^{n}\frac{Q_k}{|\bar{R}_1 - \bar{R}_k|^2}\bar{a}_{k1}$$

Above formula can be generalized to calculate the force exerted on charge Q_i due to all the other charges as,

$$\bar{F}_i = \frac{Q_i}{4\pi\epsilon_0}\sum_{\substack{k=1 \\ k\neq i}}^{n}\frac{Q_k}{|\bar{R}_i - \bar{R}_k|^2}\bar{a}_{ki}$$

Q30. Derive an expression for electric field intensity at any point due to number of point charges.

Nov.-10, Set-1, Q1(b)

Ans:

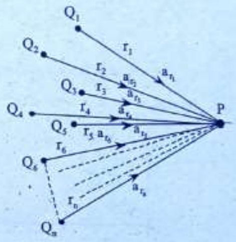
Electric Field Intensity Due to Number of Point Charges

For answer refer Unit-I, Q28, Topic: Electric Field Due to a Static Charge Configuration.

$\therefore$ The electric field intensity for point is given by,

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\vec{a}_r \text{ N/C}$$

Assume that, $E_1 E_2 E_3 ... E_n$ be the electric field intensities due to point charges $Q_1 Q_2 Q_3 ... Q_n$, which are placed at a distance $r_1 r_2 r_3 ... r_n$ with respect to point P as shown in figure.



Figure

We know that,

Electric field intensity for a point charge is given by,

$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon_0 r_1^2}\bar{a}_{r_1}$$

Similarly,

$$\bar{E}_2 = \frac{Q_2}{4\pi\epsilon_0 r_2^2}\bar{a}_{r_2}$$

$$\bar{E}_3 = \frac{Q_3}{4\pi\epsilon_0 r_3^2}\bar{a}_{r_3}$$

$$\vdots$$

$$\bar{E}_n = \frac{Q_n}{4\pi\epsilon_0 r_n^2}\bar{a}_{r_n}$$

Now, the resultant electric field intensity $\bar{E}_T$ at the point P due to n number of charges is given by the vector sum of individual electric field intensities produced by various charges at point P. i.e.,

$$\bar{E}_T = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + ... + \bar{E}_n$$

$$= \frac{Q_1}{4\pi\epsilon_0 r_1^2}\bar{a}_{r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2^2}\bar{a}_{r_2} + \frac{Q_3}{4\pi\epsilon_0 r_3^2}\bar{a}_{r_3} + ... + \frac{Q_n}{4\pi\epsilon_0 r_n^2}\bar{a}_{r_n}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{r_1^2}\bar{a}_{r_1} + \frac{Q_2}{r_2^2}\bar{a}_{r_2} + \frac{Q_3}{r_3^2}\bar{a}_{r_3} + ... + \frac{Q_n}{r_n^2}\bar{a}_{r_n}\right]$$

$$\boxed{\therefore \bar{E}_T = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{Q_i}{r_i^2}\bar{a}_{r_i}}$$

Q31. Two point charges $q_1 = 250\ \mu C$ and $q_2 = 300\ \mu C$ are located at (5, 0, 0) m and (0, 0, –5) m respectively. Find the force on q_1.

Dec.-11, Set-2, Q6(b)

Ans:

Given that,

Point charge, $q_1 = 250\ \mu C$

At point, $P_1 = (5, 0, 0)$ m

Point charge, $q_2 = 300\ \mu C$

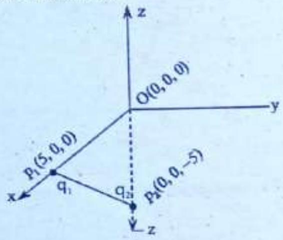
At point, $P_2 = (0, 0, -5)$ m

To determine,

Force on point charge, $q_1 = ?$

i.e., $F = ?$

The two point charges q_1 and q_2 are shown in figure



Figure

From figure, we have,

$$P_2P_1 = OP_1 - OP_2$$
$$= 5a_x - (-5a_z)$$
$$= 5a_x + 5a_z$$
$$= 5(a_x + a_z)$$

The distance between P_1 and P_2 is given by,

$$R_{P_1P_2} = |P_2P_1|$$
$$= \sqrt{5^2 + 5^2}$$
$$= 5\sqrt{2}$$

Now, the unit vector along P_2P_1 is,

$$a_{P_{21}} = a_{P_2P_1} = \frac{5(a_x + a_z)}{5\sqrt{2}}$$

The force on point charge q_1 due to point charge q_2 is given by,

$$F_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 R_{P_{12}}^2} a_{P_{21}}$$
$$= \frac{(250\times10^{-6})(300\times10^{-6})}{(4\pi)(8.854\times10^{-12})(5\sqrt{2})^2} \times \left(\frac{a_x + a_z}{\sqrt{2}}\right)$$
$$= 13.48\left(\frac{a_x + a_z}{\sqrt{2}}\right)$$

$\therefore \quad F_{12} = 9.53(a_x + a_z)$ N

Q32. Two identical charges of 500 μC each are spaced equally around a circle of diameter 2 m. Find the force on a charge of –10 μC located on the axis 2 m from the plane of the circle.

Nov.-10, Set-4, Q4

Ans:

Given that,

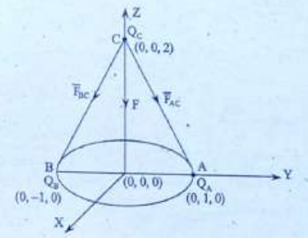
Charge, $Q_A = 500\ \mu C$

Charge, $Q_B = 500\ \mu C$

Charge, $Q_C = -10\ \mu C$

Diameter of circle, $d = 2$ m

Let us consider the circle consisting of charges placed in XY-plane and charge of $-10\ \mu C$ is on Z-axis, 2 m from the plane of the circle as shown in figure,



Figure

Due to charge Q_A the force on the charge Q_C is given by,

$$\bar{F}_{AC} = \frac{Q_A Q_C}{4\pi\epsilon_0 R_{AC}^2} a_{AC}$$

From figure,

$$\overline{AC} = (1-0)a_y + (0-2)a_z$$
$$= 1a_y - 2a_z$$
$$R_{AC} = |\overline{AC}| = \sqrt{(1)^2 + (-2)^2}$$
$$= \sqrt{5}$$
$$\therefore a_{AC} = \frac{a_y - 2a_z}{\sqrt{5}}$$
$$\therefore \bar{F}_{AC} = \frac{500\times10^{-6} \times -10\times10^{-6}}{4\pi\epsilon_0 (\sqrt{5})^2}\left[\frac{a_y - 2a_z}{\sqrt{5}}\right]$$
$$= -8.9\left(\frac{a_y - 2a_z}{\sqrt{5}}\right) \text{N}$$

Similarly, the force on charge Q_C due to charge Q_B

$$\bar{F}_{BC} = \frac{Q_B Q_C}{4\pi\epsilon_0 R_{CB}^2} a_{BC}$$
$$\overline{BC} = (-1-0)a_y + (0-2)a_z$$
$$= -(a_y + 2a_z)$$

$$|\overline{BC}| = \sqrt{(-1)^2+(-2)^2}$$

$$= \sqrt{5}$$

$$\therefore \; a_{BC} = \frac{-(a_y+2a_z)}{\sqrt{5}}$$

$$\overline{F}_{BC} = \left(\frac{500\times10^{-6}\times-10\times10^{-6}}{4\pi\epsilon\left(\sqrt{5}\right)^2}\right)\left[-\left(\frac{a_y+2a_z}{\sqrt{5}}\right)\right]$$

$$= 8.9\left(\frac{a_y+2a_z}{\sqrt{5}}\right)\text{N}$$

Total force at C is,

$$\overline{F}_T = \overline{F}_{BC} + \overline{F}_{AC}$$

$$= 8.9\left(\frac{a_y+2a_z}{\sqrt{5}}\right) - 8.9\left(\frac{a_y-2a_z}{\sqrt{5}}\right)$$

$$= \frac{8.9}{\sqrt{5}}\,[a_y+2a_z-a_y+2a_z]$$

$$= \frac{8.9}{\sqrt{5}}(4a_z)$$

$$= 15.92\, a_z\,\text{N}$$

Q33. If V = 2 x² y +20z–(4/(x² + y²)) Volts, find E and D at P(6, –2.5, 3).

Ans: Nov./Dec.-17, (R16), Q2(a)

Note: In question instead of $V = 2x^2y+20z - \frac{4}{x^2+y^2}$ V, it is printed as $V = 2x\,2y+20z-\frac{4}{x^2+y^2}$

Given data,

Potential field, $V = 2x^2y + 20z - \frac{4}{x^2+y^2}$ volts

Electric flux intensity at point, $p(6, -2.5, 3)$, $E = ?$ ... (1)

Electric flux density at point, $p(6, -2.5, 3)$, $\overline{D} = ?$

We know that,

Potential gradient in cartesian form is given as,

$$\nabla V = \frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial V}{\partial z}\overline{a}_z \quad \text{... (2)}$$

Substituting equation (1) in equation (2), we get,

$$\nabla V = \frac{\partial}{\partial x}\left[2x^2y+20z-\frac{4}{x^2+y^2}\right]\overline{a}_x + \frac{\partial}{\partial y}\left[2x^2y+20z-\frac{4}{x^2+y^2}\right]\overline{a}_y + \frac{\partial}{\partial z}\left[2x^2y+20z-\frac{4}{x^2+y^2}\right]\overline{a}_z$$

$$= \left\{2y(2x)+0-4\left[\frac{-(2x)}{(x^2+y^2)^2}\right]\right\}\overline{a}_x + \left\{2x^2+0-4\left[\frac{-2y}{(x^2+y^2)^2}\right]\right\}\overline{a}_y + \{0+20-0\}\,\overline{a}_z$$

$$\nabla V = \left\{4xy+\frac{8x}{(x^2+y^2)^2}\right\}\overline{a}_x + \left\{2x^2+\frac{8y}{(x^2+y^2)^2}\right\}\overline{a}_y + 20a_z$$

We know that,

Electric field intensity, $\overline{E} = -\nabla V$

'$\overline{E}$' at point P(6, – 2.5, 3) [$\because$ here $x = 6$, $y = -2.5$, $z = 3$]

$$\overline{E} = -\left\{\left[4(6)(-2.5)+\frac{8(6)}{((6^2+(-2.5)^2)^2}\right]\overline{a}_x + \left[2(6)^2+\frac{8(-2.5)}{(6^2+(-2.5)^2)^2}\right]\overline{a}_y + [20]\overline{a}_z\right\}$$

$$= -\{[-60+0.0268]\overline{a}_x[72-0.0112]\overline{a}_y + 20\,\overline{a}_z\}$$

$$= 59.9732\,\overline{a}_x - 71.9888\,\overline{a}_y - 20\,\overline{a}_z \text{ V/m}$$

We know that,

Electric flux density, $\overline{D}$ at $P = \overline{E}$ at $P \times \epsilon_0$

$$\overline{D} \text{ at } P = [59.9732\,\overline{a}_x - 71.9888\,\overline{a}_y - 20\,\overline{a}_z]\times 8.8541\times10^{-2} \qquad [\because \epsilon_0 = 8.8541\times10^{-12}]$$

$$= 0.531\,\overline{a}_x - 0.6373\,\overline{a}_y - 0.177\,\overline{a}_z \text{ nc/m}^2$$

Q34. Three equal positive charges of 4 × 10⁻⁹ coulomb each are located at three corners of a square, side 20 cm. Determine the electric field intensity at the vacant corner point of the square.

Ans: Nov./Dec.-18, R16, Q2(a)

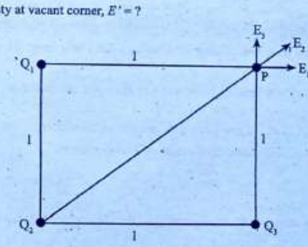
Given that,

Three equal positive charges, $Q_1 = Q_2 = Q_3 = 4\times10^{-9}$ C

Each side of the squarel, $l = 20$ cm $= 20\times10^{-2}$ m

To determine,

Electric field intensity at vacant corner, $E' = ?$



Let the test charge at point 'P'

Field intensity at point 'P'. due to charge Q_1 is given by,

$$E_1 = \frac{Q_1}{4\pi\epsilon_0 l^2} \text{ along positive } x\text{-axis}$$

Field intensity at point 'P' due to charge Q_3 is given by,

$$E_3 = \frac{Q_3}{4\pi\epsilon_0 l^2} \text{ along positive } y\text{-axis}$$

Resultant of above two fields is given by,

$$E' = \sqrt{E_1^2 + E_3^2}$$

$$= \sqrt{\left(\frac{Q_1}{4\pi \in_0 l^2}\right)^2 + \left(\frac{Q_3}{4\pi \in_0 l^2}\right)^2}$$

$$= \sqrt{\left(\frac{Q_1}{4\pi \in_0 l^2}\right)^2 + \left(\frac{Q_1}{4\pi \in_0 l^2}\right)^2} \qquad [\because Q_1 = Q_3]$$

$$= \frac{\sqrt{2}Q_1}{4\pi \in_0 l^2}$$

Now, the field intensity at point 'P' due to charge Q_2 with an angle 45° to the horizontal axis is given by,

$$E_2 = \frac{Q_2}{4\pi \in_0 (\sqrt{2}l)^2} = \frac{Q_2}{8\pi \in_0 l^2}$$

∴ The total electric field at vacant corner (i.e., corner 4) is given by,

$$E = E_2 + E' = \frac{Q_2}{8\pi \in_0 l^2} + \frac{\sqrt{2}Q_1}{4\pi \in_0 l^2}$$

$$= \frac{Q_1}{8\pi \in_0 l^2} + \frac{\sqrt{2}Q_1}{4\pi \in_0 l^2} \qquad [\because Q_1 = Q_2]$$

$$= \frac{Q_1}{4\pi \in_0 l^2}\left(\frac{1}{2} + \sqrt{2}\right)$$

$$= \frac{4 \times 10^{-9} \times (1.914)}{4\pi \times 8.85 \times 10^{-12} \times (20 \times 10^{-2})^2} \qquad [\because \in_0 = 8.85 \times 10^{-12}]$$

∴ Electric field intensity, $E = 1721.03$ V/m

1.3 LINE SURFACE AND VOLUME CHARGE DISTRIBUTIONS

Q35. Find the E at any point due to a line charge of density λ C/m and length L meter.

Model Paper-III, Q3(a)

Ans:

Consider a uniformly charged wire of length L meter. Let λ be the linear charge density in Coulombs/meter. Let A be any point at a distance 'd' from the wire.

Consider a differential element dy at a distance x m from one end of the wire as shown in figure (1).

The electric field intensity due to differential charge element $\lambda\ dx$ is given by,

$$dE = \frac{\lambda\, dx}{4\pi\epsilon_0 y^2} \qquad \text{...(1)}$$

The elemental field dE can be resolved into two components, i.e., the horizontal component along X-axis and the vertical component along Y-axis is given as,

$$\therefore \quad dE_x = dE \cos\phi \qquad \text{...(2)}$$

$$dE_y = dE \sin\phi \qquad \text{...(3)}$$

Substituting equation (1) in equation (2), we get,

$$dE_x = \frac{\lambda\, dx}{4\pi\epsilon_0 y^2}\cos\phi$$

$$= \frac{\lambda \cos\phi}{4\pi\epsilon_0 y^2}dx \qquad \text{...(4)}$$

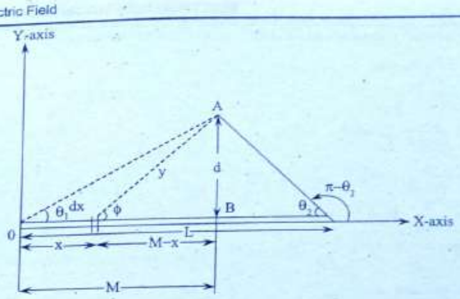


Figure (1)

Consider Δ^{le} ABC as shown in figure (2),

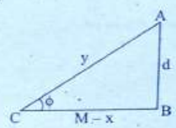


Figure (2)

From figure (2),

$$\cot\phi = \frac{M - x}{d}$$

$$M - x = d\cot\phi \qquad \text{...(5)}$$

Differentiating equation (5), we get,

$$0 - dx = -d\,\text{cosec}^2\phi\, d\phi$$

$$dx = d\,\text{cosec}^2\phi\, d\phi \qquad \text{...(6)}$$

From figure (2),

$$\text{cosec}\phi = \frac{y}{d}$$

$$y = d\,\text{cosec}\phi \qquad \text{...(7)}$$

Substituting equations (6) and (7) in equation (4), we get,

$$dE_x = \frac{\lambda\cos\phi\, d\,\text{cosec}^2\phi\, d\phi}{4\pi\epsilon_0 d^2\,\text{cosec}^2\phi}$$

$$dE_x = \frac{\lambda\cos\phi\, d\phi}{4\pi\epsilon_0 d} \qquad \text{...(8)}$$

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In order to get the electric field over the whole length of the wire, integrating equation (8),

$$E_h = \int_{\phi=\theta_1}^{\pi-\theta_2} \frac{\lambda \cos\phi \, d\phi}{4\pi\epsilon_0 d}$$

$$E_h = \frac{\lambda}{4\pi\epsilon_0 d} \int_{\theta_1}^{\pi-\theta_2} \cos\phi \, d\phi$$

$$E_h = \frac{\lambda}{4\pi\epsilon_0 d} [\sin\phi]_{\theta_1}^{\pi-\theta_2}$$

$$E_h = \frac{\lambda}{4\pi\epsilon_0 d} [\sin\theta_2 - \sin\theta_1] \text{ V/m}$$

Considering equations (3), i.e.,

$$dE_v = dE \sin\phi$$

$$dE_v = \frac{\lambda \, dx \sin\phi}{4\pi\epsilon_0 y^2}$$

$$dE_v = \frac{\lambda \sin\phi \, dx}{4\pi\epsilon_0 y^2} \quad \text{... (9)}$$

Substituting equations (6) and (7) in equation (9), we get,

$$dE_v = \frac{\lambda \sin\phi \, d \operatorname{cosec}^2\phi \, d\phi}{4\pi\epsilon_0 d^2 \operatorname{cosec}^2\phi}$$

$$dE_v = \frac{\lambda \sin\phi \, d\phi}{4\pi\epsilon_0 d}$$

In order to get the vertical component of E along the length of the wire, integrating equation (10),

$$dE_v = \frac{\lambda}{4\pi\epsilon_0 d} \int_{\theta_1}^{\pi-\theta_2} \sin\phi \, d\phi$$

$$= \frac{\lambda}{4\pi\epsilon_0 d} [-\cos\phi]_{\theta_1}^{\pi-\theta_2}$$

$$= \frac{\lambda}{4\pi\epsilon_0 d} [\cos\theta_1 + \cos\theta_2] \text{ V/m}$$

Q36. Explain,

(i) Electric field intensity due to a line charge.

(ii) Electric field due to a charged disc.

Ans:

(i) Electric Field Intensity due to a Line Charge

For answer refer Unit-I, Q35.

(ii) Electric Field due to a Charged Disc

Consider a charged disc made up of several annular rings as shown in figure.

Let,

a = Radius of charged disc

x = Radius of annular disc or ring

ρ_s = Charge density of annular ring

dS = Area of an annular ring

dQ = Charge in Coulomb-meter

Now, the electric field intensity due to annular ring in horizontal direction is given by,

$$dE = \frac{dQ}{4\pi\epsilon_0 r^2} \cos\theta \quad \text{... (1)}$$

But, the charge on annular ring is given by,

$$dQ = \rho_s dS$$

Therefore, equation (1) becomes,

$$dE = \frac{\rho_s dS}{4\pi\epsilon_0 r^2} \cos\theta \quad \text{... (2)}$$

The Area of an annular disc is given by,

$$dS = \text{Area of outer surface} - \text{Area of inner surface}$$

$$= \pi[(x+dx)^2 - x^2] = \pi[(x^2 + 2x.dx + (dx)^2 - x^2]$$

$$dS = \pi[2x.dx + (dx)^2]$$

Let, $dx >> (dx)^2$ then,

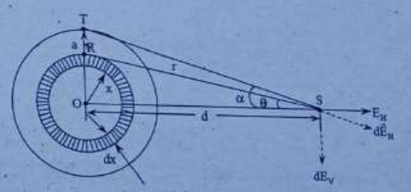
$$dS = \pi[2x.dx]$$

$$dS = 2\pi x.dx \quad \text{... (3)}$$

Substituting equation (3) in equation (2), we get,

$$dE = \frac{\rho_s . 2\pi x.dx}{4\pi\epsilon_0 r^2} \cos\theta$$

$$\Rightarrow \quad dE = \frac{\rho_s . x . dx}{2\epsilon_0 r^2} \cos\theta \quad \text{... (4)}$$



Figure

From figure,

Triangle OSR, we have,

$$\cos\theta = \frac{d}{r} \text{ and } \tan\theta = \frac{x}{d}$$

$$\Rightarrow \quad r = \frac{d}{\cos\theta} \Rightarrow x = d\tan\theta$$

$$\Rightarrow \quad r = d\sec\theta \Rightarrow dx = d\sec^2\theta . d\theta$$

Substituting the values of r, dx in equation (4), we have,

$$dE = \frac{\rho_s d \tan\theta . d\sec^2\theta . d\theta}{2\epsilon_0 (d\sec\theta)^2} . \cos\theta$$

$$dE = \frac{\rho_s d^2 . \tan\theta . \sec^2\theta}{2\epsilon_0 d^2 \sec^2\theta} . \cos\theta . d\theta$$

$$dE = \frac{\rho_s \tan\theta}{2\epsilon_0}\cos\theta.d\theta$$

$$dE = \frac{\rho_s \cdot \frac{\sin\theta}{\cos\theta}}{2\epsilon_0} \times \cos\theta.d\theta$$

$$dE = \frac{\rho_s.\sin\theta}{2\epsilon_0\cos\theta} \times \cos\theta.d\theta$$

$$dE = \frac{\rho_s \sin\theta}{2\epsilon_0}\,d\theta$$

Now the total electric field intensity on the charged disc is given by,

$$E = \int_0^\alpha dE$$

$$\therefore \quad E = \int_0^\alpha \frac{\rho_s \sin\theta}{2\epsilon_0}.d\theta$$

$$E = \frac{\rho_s}{2\epsilon_0}\int_0^\alpha \sin\theta\,.d\theta$$

$$E = \frac{\rho_s}{2\epsilon_0}[-\cos\theta]_0^\alpha$$

$$E = \frac{-\rho_s}{2\epsilon_0}[\cos\alpha - \cos 0]$$

$$E = \frac{-\rho_s}{2\epsilon_0}[\cos\alpha - 1]$$

$$\Rightarrow \quad E = \frac{\rho_s}{2\epsilon_0}(1-\cos\alpha) \qquad \dots(5)$$

From figure,

Triangle OST,

$$\cos\alpha = \frac{d}{\sqrt{a^2+d^2}}$$

Substituting $\cos\alpha$ value in equation (5), we have,

$$E = \frac{\rho_s}{2\epsilon_0}\left(1-\frac{d}{\sqrt{a^2+d^2}}\right)$$

Case 1

For an infinite radius, the EFI is given by,

$$E = \frac{\rho_s}{2\epsilon_0}\left(1-\frac{d}{\sqrt{(\infty)^2+d^2}}\right) \quad [\because \text{radius } a = \infty]$$

$$E = \frac{\rho_s}{2\epsilon_0}\left(1-\frac{d}{\infty}\right)$$

$$E = \frac{\rho_s}{2\epsilon_0}(1-0) \quad \left[\because \frac{1}{\infty}=0\right]$$

$$E = \frac{\rho_s}{2\epsilon_0}(1)$$

$$E = \frac{\rho_s}{2\epsilon_0}$$

Hence

$$\overline{E} = \frac{\rho_s}{2\epsilon_0}\overline{a}_r$$

Where, $\overline{a}_r$ is an unit vector normal to the plane of the disc.

Case 2

When two infinite parallel charge sheets are considered,

Let the charge density due to positive charge be $+\rho_s$ and due to negative charge be $-\rho_s$. Then the electric field intensity is given by,

$$\overline{E} = \frac{+\rho_s}{2\epsilon_0}(\overline{a}_r) - \frac{\rho_s}{2\epsilon_0}(\overline{a}_r)$$

$$\overline{E} = \frac{\rho_s}{2\epsilon_0}(\overline{a}_r) + \frac{\rho_s}{2\epsilon_0}(\overline{a}_r)$$

$$\overline{E} = \frac{2\rho_s}{2\epsilon_0}\overline{a}_r$$

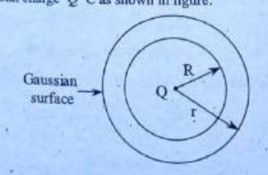
$$\boxed{\overline{E} = \frac{\rho_s}{\epsilon_0}\overline{a}_r}$$

Q37. $\overline{E}$ is the electric field due to a point charge Q C at the origin in free space. Find $\int_S \overline{E}.\overline{da}$ where S is a spherical surface of radius R m and center at origin.

Ans:

The Gauss's law is used to find the electric field due to a point charge 'Q' at the origin in free space.

Consider a uniformly charged sphere of radius "R" m and of total charge 'Q' C as shown in figure.



Figure

A spherical surface of radius 'r' is drawn such that ($r > R$) and this surface is known as 'Gaussian surface'. "According to Gauss's law, the flux through any surface enclosing the charge is $\frac{Q}{\epsilon_0}$".

Then, for closed surface 'S'

$$\oint_S E.da = \frac{1}{\epsilon_0}Q_{enc} \qquad \dots(1)$$

Where 'da' is the infinitesimal area and it is perpendicular to field 'E'.

'E' is present inside the surface integral and its direction is radially outwards, some as 'da' which also points radially outwards, so we can cancel the dot product of 'E' and 'da'.

$$\int_S E.da = \int_S |E|\,da$$

As the magnitude of 'E' is constant over the Gaussian surface, so it comes outside the integral.

$$\int_S |E|\,da = |E|\int_S da$$

As the area of circle is πr^2 so the area of the spherical surface S is $4\pi r^2$.

i.e., $\int_S da = 4\pi r^2$

$$\therefore \quad |E|\int_S da = |E|4\pi r^2 \qquad \dots(2)$$

Substituting equation (2) in equation (1),

Thus,

$$|E|4\pi r^2 = \frac{1}{\epsilon_0}Q$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}$$

Where, $\hat{r}$ is a vector which points radially outward.

As the medium is free space so, $\epsilon_r = 1$ and $\epsilon = \epsilon_0$.

Q38. A charge of Q is uniformly distributed in a half-circular ring of radius 'a'. Determine 'E' at the centre.

Ans:

Given that,

Uniformly distributed charge = Q

Radius of the half-circular ring = a

Magnitude of electric field intensity at the centre, E = ?

As, the charge is distributed uniformly, the line charge density along the length of half-circular ring is,

$$\rho_l = \frac{Q}{l}$$

Where,

l = Length of half-circular ring

$$= \frac{1}{2} \times \text{Perimeter of circle} = \frac{1}{2} \times 2\pi r = \pi r$$

$$\therefore \quad \rho_l = \frac{Q}{\pi r} = \frac{Q}{\pi a}$$

Consider the EFI at centre due to a differential charge element of length dl lying on the half-circular ring as shown in the figure.

By definition of EFI, we have,

$$d\overline{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\,\overline{a}_r$$

Where,

$$a_r = \frac{\overline{R}}{|\overline{R}|} = \frac{-a\overline{a}_r}{a} = -\overline{a}_r$$

$$\therefore \quad d\overline{E} = \frac{(\rho_l\, d\overline{l})}{4\pi\epsilon_0 a^2}(-\overline{a}_r) \quad [\because R = a]$$

$$= \frac{-\rho_l\, a d\phi}{4\pi\epsilon_0 a^2}\,\overline{a}_r \quad [\because d\overline{l} = a d\phi \text{ (from figure)}]$$

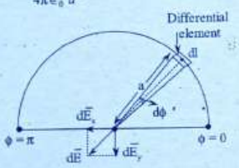


Figure: Half Circular Ring

From the figure, it is clear that by integrating $d\overline{E}$ in the interval $\phi = 0$ to π radius, we get $\overline{E}$.

$$\Rightarrow \quad \overline{E} = \int_{\phi=0}^{\pi} d\overline{E}$$

$$= \int_{\phi=0}^{\pi} \frac{-\rho_l}{4\pi\epsilon_0 a}\, d\phi\, \overline{a}_r$$

$$= \frac{-\rho_l \overline{a}_r}{4\pi\epsilon_0 a}\int_{\phi=0}^{\pi} d\phi = \frac{-\rho_l \overline{a}_r}{4\pi\epsilon_0 a} \times [\phi]_{\phi=0}^{\pi}$$

$$= \frac{-\rho_l \overline{a}_r}{4\pi\epsilon_0 a} \times [\pi - 0] = \frac{-\rho_l \overline{a}_r}{4\epsilon_0 a}$$

$$= \frac{-Q}{\pi a} \times \frac{1}{4\epsilon_0 a}\,\overline{a}_r \left(\because \rho_l = \frac{Q}{\pi a}\right)$$

$$= \frac{-Q}{4\pi\epsilon_0 a^2}\,\overline{a}_r$$

$\therefore$ The magnitude of electric field intensity at the centre is,

$$E = |\overline{E}| = \frac{Q}{4\pi\epsilon_0 a^2}$$

Q39. Determine electric field E at the origin due to a point charge of 54.9 nC located at (– 4, 5, 3) m in Cartesian coordinates.

Ans:

Given that,

Charge, $Q = 54.9$ nC

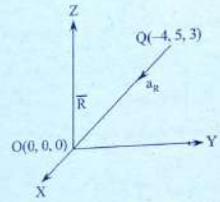
Point at $(-4, 5, 3)$

To find

Electric field, $E = ?$

The electric field intensity due to a point charge Q is given as,

$$\overline{E} = \frac{Q}{4\pi\epsilon_o R^2}\,\overline{a}_r \qquad \text{... (1)}$$



$$\therefore \quad \overline{R} = [0-(-4)]\overline{a}_x + [0-5]\overline{a}_y + [0-3]\overline{a}_z$$

$$= 4\overline{a}_x - 5\overline{a}_y - 3\overline{a}_z$$

$$|R| = \sqrt{[4^2 + (-5)^2 + (-3)^2]}$$

$$= \sqrt{50}$$

We know that,

$$\overline{a}_R = \frac{\overline{R}}{|\overline{R}|}$$

$$= \frac{4\overline{a}_x - 5\overline{a}_y - 3\overline{a}_z}{\sqrt{(4)^2 + (-5)^2 + (-3)^2}}$$

$$= \frac{4\overline{a}_x - 5\overline{a}_y - 3\overline{a}_z}{\sqrt{50}}$$

Substituting the values in equation (1), we get,

$$E = \frac{54.9\times10^{-9}}{4\pi\times8.854\times10^{-12}\times50}\times\left[\frac{4\overline{a}_x - 5\overline{a}_y - 3\overline{a}_z}{\sqrt{50}}\right]$$

$$= 9.868[0.565\overline{a}_x - 0.707\overline{a}_y - 0.424\overline{a}_z]$$

$$\overline{E} = (5.575\overline{a}_x - 6.976\overline{a}_y - 4.184\overline{a}_z)\text{ V/m}$$

Q40. Find the total electric field at the origin due to 10^{-8} C charge at P (0, 4, 4)m and -0.5×10^{-8} C charge at P(4, 0, 2)m.

Ans: Dec.-14, (R13), Q2(a)

Given that,

$Q_1 = 10^{-8}$ C at $P(0, 4, 4)$m

$Q_2 = -0.5 \times 10^{-8}$ C at $P(4, 0, 2)$m

Total electric field at origin, $\overline{E} = ?$

Electric field intensity at origin due to charge Q_1 is given by,

$$\overline{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 |\overline{R}_1|^2}\,\overline{a}_1$$

$$\overline{R}_1 = (0-0)\overline{a}_x + (0-4)\overline{a}_y + (0-4)\overline{a}_z$$

$$= -4\overline{a}_y - 4\overline{a}_z$$

$$|\overline{R}_1| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32}$$

$$\overline{a}_1 = \frac{\overline{R}_1}{|R_1|} = \frac{(-4\overline{a}_y - 4\overline{a}_z)}{\sqrt{32}}$$

$$\therefore \overline{E}_1 = \frac{10^{-8}}{4\times\pi\times8.854\times10^{-12}\times(\sqrt{32})^2}\cdot\frac{(-4\overline{a}_y - 4\overline{a}_z)}{\sqrt{32}}$$

$$= 0.4965(-4\overline{a}_y - 4\overline{a}_z)$$

$$= (-1.986\overline{a}_y - 1.986\overline{a}_z)\text{V/m}$$

Electric field intensity at origin due to charge Q_2 is given by,

$$\overline{E}_2 = \frac{Q_2}{4\pi\varepsilon_0|\overline{R}_2|^2}\,\overline{a}_2$$

$$\overline{R}_2 = (0-4)\overline{a}_x + (0-0)\overline{a}_y + (0-2)\overline{a}_z$$

$$= -4\overline{a}_y - 2\overline{a}_z$$

$$|\overline{R}_2| = \sqrt{(-4)^2 + (-2)^2} = \sqrt{20}$$

$$\overline{a}_2 = \frac{\overline{R}_2}{|R_2|} = \frac{-4\overline{a}_y - 2\overline{a}_z}{\sqrt{20}}$$

$$\therefore \quad \overline{E}_2 = \frac{-0.5\times10^{-8}}{4\times\pi\times8.854\times10^{-12}\times(\sqrt{20})^2}\cdot\frac{(-4\overline{a}_y - 2\overline{a}_z)}{\sqrt{20}}$$

$$= 0.5(-4\overline{a}_y - 2\overline{a}_z) = (2\overline{a}_y + \overline{a}_z)\text{V/m}$$

$\therefore$ Total electric field intensity at origin is,

$$\overline{E} = \overline{E}_1 + \overline{E}_2$$

$$= -1.986\overline{a}_y - 1.986\overline{a}_z + (2\overline{a}_x + \overline{a}_z)$$

$$= (2\overline{a}_x - 1.986\overline{a}_y - 0.986\overline{a}_z)\text{V/m}$$

Q41. Determine $\overline{E}$ at the origin due to a uniform line charge distribution with linear charge density $\rho_L = 3.3 \times 10^{-9}$ C/m located at (3, 4, 0) m.

Ans: Dec.-11, Set-1, Q7

Given that,

Linear charge density, $\rho_L = 3.3 \times 10^{-9}$ C/m

Charge density located at, $P = (3, 4, 0)$ m

To Determine

Electric field intensity, $E = ?$

At origin (0, 0, 0)

Electric field intensity due to a uniform line charge is given by,

$$E = \frac{\rho_L}{2\pi\epsilon_0 d}\,\overline{a}_{12} \qquad \text{... (1)}$$

Where,

$$d = |\overline{R}_{12}|$$

$$R_{12} = (0-3)a_x + (0-4)\overline{a}_y$$

$$= -3\overline{a}_x - 4\overline{a}_y$$

$$d = \sqrt{3^2 + 4^2}$$

$$= 5$$

$$\text{Unit vector, } a_{12} = \frac{-(3a_x + 4a_y)}{5}$$

Substituting the values in equation (1), we get,

$$E = \frac{3.3\times10^{-9}}{(2\pi)(8.854\times10^{-12})(5)}\times\left[\frac{-(3a_x + 4a_y)}{5}\right]$$

$$= \frac{-11.86}{5}(3a_x + 4a_y)$$

$$= -2.372(3a_x + 4a_y)$$

$$= (-7.116\,a_x - 9.488\,a_y)\text{ V/m}$$

1.4 GAUSS LAW AND ITS APPLICATIONS

Q42. State and prove Gauss's law.

Ans: May/June-13, (R09), Q1(a)

Statement

It states that the net flux coming out of any closed surface is equal to the total charge enclosed by that surface.

Mathematically,

$$\boxed{\psi = Q}$$

Where,

- ψ = Net flux coming out of the surface.
- Q = Charge enclosed by the surface.

Explanation

Consider a charge 'Q' enclosed at the centre of a sphere of radius 'r' as shown in the figure. The charge 'Q' causes the 'Q' coulombs of flux and hence electric flux density exists on the surface of the sphere. Here, instead of sphere, any closed surface can be assumed. Consider a differential surface area $\overline{ds}$ on the sphere as shown in the figure.

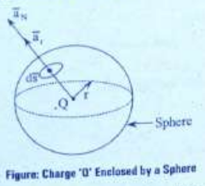


Figure: Charge 'Q' Enclosed by a Sphere

Let $\overline{a}_N$ be the unit vector passing through the differential surface area $\overline{ds}$ and whose direction is normal to the surface of the sphere.

$$\therefore \quad \overline{a}_r . \overline{a}_N = 1$$

We know that, electric field density ($\overline{D}$) is given by,

$$\overline{D} = \epsilon_0 \overline{E} \quad \text{... (1)}$$

By the definition for a sphere, we have,

$$\overline{E} = \frac{Q}{4\pi\epsilon_0 r^2}\overline{a}_r \quad \text{... (2)}$$

Substituting equation (2) in equation (1), we get,

$$\overline{D} = \epsilon_0 \times \frac{Q}{4\pi\epsilon_0 r^2}\overline{a}_r$$

$$\overline{D} = \frac{Q}{4\pi r^2}\overline{a}_r \quad \text{... (3)}$$

Applying dot product with $\overline{ds}$ on both sides of the equation (3), we get,

$$\overline{D}.\overline{ds} = \frac{Q}{4\pi r^2}\overline{a}_r . ds\,\overline{a}_N$$

$$\overline{D}.\overline{ds} = \frac{Q}{4\pi r^2}\overline{ds} \quad \text{... (4)}$$

Applying surface integral on both sides, we get,

$$\oint_s \overline{D}.\overline{ds} = \oint_s \frac{Q}{4\pi r^2}\overline{ds}$$

$$\oint_s \overline{D}.\overline{ds} = \frac{Q}{4\pi r^2}\oint_s \overline{ds}$$

But, $\oint_s \overline{ds}$ = Surface area of the sphere = $4\pi r^2$

$$\oint_s \overline{D}.\overline{ds} = \frac{Q}{4\pi r^2} \times 4\pi r^2$$

$$\oint_s \overline{D}.\overline{ds} = Q \quad \text{... (5)}$$

Where,

$\oint_s \overline{D}.\overline{ds}$ = Total flux passing through the surface = ψ

$$\boxed{\therefore \ \psi = Q}$$

Hence, it can be inferred that the total flux (ψ) passing through the surface is equal to the charge (Q) enclosed by that surface.

Q43. Using Gauss law, find E at any point due to long infinite cherge wire.

OR

Derive an expression for the electric field intensity due to an infinite length line charge along the z-axis at an arbitrary point Q(x, y, z).

Model Paper-I, Q3(a)

Ans:

Electric Field Due to Long Infinite Charge Wire

Consider a conducting wire, carrying a uniformly distributed charge 'Q' of linear charge density 'ρ_l' is placed along the Z-axis as shown in the figure. Consider a cylindrical Gaussian surface around the infinite line charge so that it is symmetrical about the infinite line charge as depicted in the figure. Let 'R' be the radius of the cylinder and 'l' be its length.

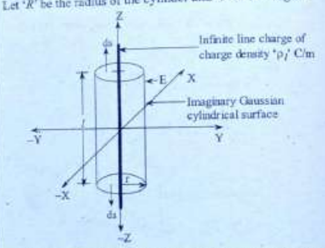
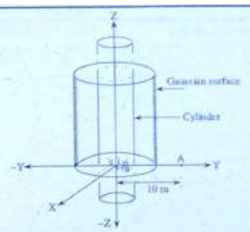


Figure: Imaginary Cylindrical Gaussian Surface Around the Line Charge

Derivation

Assume that the zero potential occurs at point B and the potential at point A is to be calculated such that A is 10 m away from B, as shown in the figure.



Figure

Assume a Gaussian surface around the cylinder as shown in the figure. Since, the charge uniformly distributed throughout the cylinder, the cylinder can be assumed as an infinite line charge.

According to the Gauss's law, total flux emanating (coming out) from a closed surface area is equal to the charge enclosed by that surface area.

$$\text{i.e.,} \quad Q = \oint_s \overline{D}.\overline{ds} \quad \text{... (1)}$$

Here, the imaginary cylinder has three areas namely top, bottom and curved surface area. Consequently, the total flux emanating is equal to the sum of the fluxes emanating (coming out) from these three areas. Hence, equation (1) can be written as,

$$Q = \oint_{\substack{\text{curved}\\\text{surface}}} \overline{D}.\overline{ds} + \oint_{\text{top}} \overline{D}.\overline{ds} + \oint_{\text{bottom}} \overline{D}.\overline{ds}$$

$$\Rightarrow \quad Q = \oint_{\substack{\text{curved}\\\text{surface}}} \epsilon_0 \overline{E}.\overline{ds} + \oint_{\text{top}} \epsilon_0 \overline{E}.\overline{ds} + \oint_{\text{bottom}} \epsilon_0 \overline{E}.\overline{ds} \qquad [\because \overline{D} = \epsilon \overline{E} \text{ for free space}]$$

$$\Rightarrow \quad Q = \epsilon_0 \overline{E}\left[\oint_{\substack{\text{curved}\\\text{surface}}} \overline{ds} + \oint_{\text{top}} \overline{ds} + \oint_{\text{bottom}} \overline{ds}\right]$$

$$\Rightarrow \quad Q = \epsilon_0 \overline{E}\,[2\pi R\,\overline{a}_r + 2\pi R\,\overline{a}_z - 2\pi R\,\overline{a}_z]$$

$$\therefore \quad Q = 2\pi Rl\epsilon_0 \overline{E}.\overline{a}_r \quad \text{... (2)}$$

In cylindrical coordinates, $\overline{E}$ can be expressed as,

$$\overline{E} = E_r\overline{a_r} + E_\phi\overline{a_\phi} + E_z\overline{a_z}$$

As electric field intensity($\overline{E}$) acts radial to the Gaussian surface, only 'r' components exists.

$$\therefore \quad \overline{E} = E\overline{a}_r \quad \text{... (3)}$$

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Substituting equation (3) in equation (2), we get,

$$Q = 2\pi R l \in_0 E \bar{a}_r \cdot \bar{a}_r$$

$$Q = 2\pi R l \in_0 E_r .1 \quad [\because \bar{a}_r \cdot \bar{a}_r = 1]$$

$$Q = 2\pi R l \in_0 E_r \quad \ldots (4)$$

But, the total charge is given by,

$$Q = \rho_l \times l$$

$$\therefore \quad \rho_l \times l = 2\pi R l \in_0 E_r$$

$$E_r = \frac{\rho_l}{2\pi \in_0 R} \quad \ldots (5)$$

Substituting equation (5) in equation (3), we get,

$$\bar{E} = \frac{\rho_l}{2\pi \in_0 R} \bar{a}_r \text{ V/m}$$

$\therefore$ Above equation gives the field intensity at any point due to infinite length line charge.

Q44. What are the criteria for selection of Gaussian surface?

Ans: (Dec.-11, Set-3, Q2(a) | Model Paper-II, Q2(b))

Gauss law states that the net flux coming out of any closed surface is equal to the total charge enclosed by that surface i.e.,

$$\psi = Q$$

Therefore in order to determine the electric field vector simple Gaussian surfaces are chosen, which in turn makes the calculation simple and easy.

There are two parts of the flux integral which varies. They, are,

(i) Variation of the magnitude of the field and

(ii) The dot product.

In order to maintain the constant dot product and constant electric field, the Gaussian surface is selected to the following criteria.

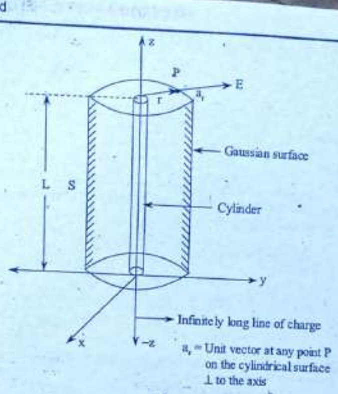
- The Gaussian surface should be a closed surface like sphere-cylinder or box etc.
- Choose the Gaussian surface right angles to the electric field such that E and dt are parallel to each other.
- The electric field point must and should lie on the Gaussian surface where the electric field vector is to be determined.
- In order to maintain a constant magnitude of E the point, which is selected on Gaussian surface must have a constant distance from the charge.
- In case if the above points are not satisfying then choose the Gaussian surface parallel to 'E' such that the dot product is zero and E and dt are right angles to each other.

Q45. Using Gauss's law, show that the electric field due to an infinite straight line of uniform charge density λ C/m along the z-axis in free space is ($\lambda/2\pi\in_0 r$) a_r N/C.

Ans:

Electric Field Due to Infinite Line

Consider a conductor of infinite length. Let 'λ' C/m be the line charge density. Assuming that the conductor is lying along the z-axis as shown in figure.



Figure

The conductor carries a uniformly distributed charge, the charge density being 'λ' C/m. Assume a Gaussian surface around an infinite line charge, which is imagined coaxial cylinder. Let 'r' be the radius of the cylinder and 'L' be the length of the cylinder.

The directions of electric flux density 'D' and electric field intensity 'E' are radial everywhere.

$\therefore$ Total charge, Q = Length × Charge density

$$= L \times \lambda$$

According to Gauss's law, total flux coming out from a closed surface is equal to the charge enclosed by the surface area.

i.e., $Q = \oint_S \bar{D} . d\bar{s}$

Here, the imaginary cylinder comprises of three areas, i.e., top, bottom and curved surface area. So, the total flux is the sum of the fluxes coming out from these three areas.

$$Q = \oint_{\text{Curved}} \bar{D} . d\bar{s} + \oint_{\text{Top}} \bar{D} . d\bar{s} + \oint_{\text{Bottom}} \bar{D} . d\bar{s}$$

$$= D \int_{\text{Curved}} ds + \int_{\text{Top}} ds + \int_{\text{Bottom}} ds$$

$\because$ Components for the top and bottom surfaces are zero.

$$\therefore \quad Q = \int_{\text{Curved}} ds = D \int_0^L 2\pi r dL$$

$$Q = D(2\pi r L) \Rightarrow D = \frac{Q}{2\pi r L}$$

As, $Q = L \times \lambda$

$$D = \frac{L \times \lambda}{2\pi r L} \frac{\lambda}{2\pi r} \quad \ldots (1)$$

We know that,

$$D = \epsilon_0 E$$

$$\Rightarrow \quad E = \frac{D}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 r}$$

At a distance 'r' field intensity is,

$$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 r} a_r \text{ N/C}}$$

Q46. Using the Gauss's law in differential form, obtain the 'E' and 'D' at different points due to the following charge distribution in spherical coordinates.

$$\rho(r, \theta, \phi) = \rho_0 (r/a) \qquad 0 < r < a$$
$$= 0 \qquad a < r < \infty$$

May/June-13, (R09), Q1(b)

Ans:

Given that,

The charge distribution,

$$\rho_v = \rho(r, \theta, \phi) = \rho_0 (r/a) \qquad 0 < r < a$$
$$= 0 \qquad a < r < \infty$$

Electric flux density, $D = ?$

Electric field intensity, $E = ?$

The Gauss's law in differential form is given by, ... (1)

$$\nabla . \overline{D} = \rho_v$$

Also, the Guass's law in spherical co-ordinate system, ... (2)

$$\nabla . \overline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \Delta_\theta) + \frac{1}{r\sin\theta}\frac{\partial D\phi}{\partial\phi}$$

The $\overline{D}$ varies only in radial direction.

Thus, all the other terms in equation (2) becomes zero. ... (3)

$$\nabla . \overline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)$$

Comparing equations (1) and (3), we get,

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)$$

$$\rho_0\left(\frac{r}{a}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)$$

$$\rho_0\left(\frac{r^3}{a}\right) = \frac{\partial}{\partial r}(r^2 D_r)$$

On integrating the above equation, we get,

$$\int \rho_0\left(\frac{r^3}{a}\right) dr = \int \frac{\partial}{\partial r}(r^2 D_r)$$

$$\frac{\rho_0}{a}\int r^3 dr = r^2 D_r$$

$$\frac{\rho_0}{a}\frac{r^4}{4} + C = r^2 D_r$$

$$r^2 D_r = \frac{\rho_0 r^4}{4a} + C$$

At $r = 0, \rho_v = 0$

$$\therefore \quad C = 0$$

$$r^2 D_r = \frac{\rho_0 r^4}{4a}$$

$$D_r = \frac{\rho_0 r^2}{4a}$$

$$\boxed{\therefore \overline{D} = \frac{\rho_0 r^2}{4a}\overline{a}_r} \qquad \text{For } 0 < r < a$$

The electric field intensity E is given by,

$$E = \frac{\overline{D}}{\epsilon_0} = \frac{\rho_0 r^2}{4a}\overline{a}_r \times \frac{1}{\epsilon_0}$$

$$\boxed{\overline{E} = \frac{\rho_0 r^2}{4a\epsilon_0}\overline{a}_r} \qquad \text{For } 0 < r < a$$

For $a < r < \infty$, the charge enclosed is upto $r = a$ as $\rho_v = 0$ For $a < r < \infty$

$$\therefore \rho_v = \frac{1}{r^2}\frac{\partial}{\partial_r}(r^2 D_r)$$

$$\rho_0\left(\frac{r}{a}\right) = \frac{1}{r^2}\frac{\partial}{\partial_r}(r^2 D_r)$$

$$\rho_0\left(\frac{r^3}{a}\right) = \frac{\partial}{\partial_r}(r^2 D_r)$$

On integrating the above equation and taking the limits from 0 to a, we get,

$$\int_{r=0}^{a} \rho_0\left(\frac{r^3}{a}\right) dr = \int \frac{\partial}{\partial r}(r^2 D_r)$$

$$r^2 D_r = \frac{\rho_0}{a}\int_0^a r^3 dr$$

$$= \frac{\rho_0}{a}\left[\frac{r^4}{4}\right]_0^a$$

$$r^2 D_r = \frac{\rho_0}{4a}(a^4 - 0)$$

$$r^2 D_r = \frac{\rho_0 a^3}{4}$$

$$D_r = \frac{\rho_0 a^3}{4r^2}$$

$$\boxed{\therefore \overline{D} = \frac{\rho_0 a^3}{4r^2}\overline{a}_r} \qquad \text{For } a < r < \infty$$

$$\therefore \overline{E} = \frac{\overline{D}}{\epsilon_0}$$

$$\boxed{\overline{E} = \frac{\rho_0 a^3}{4r^2\epsilon_0}\overline{a}_r}$$

Result

For $0 < r < a$, $\overline{D} = \frac{\rho_0 r^2}{4a}\overline{a}_r$ and $\overline{E} = \frac{\rho_0 r^2}{4a \epsilon_0}\overline{a}_r$

For $a < r < \infty$, $\overline{D} = \frac{\rho_0 a^3}{4r^2}\overline{a}_r$ and $\overline{E} = \frac{\rho_0 a^3}{4r^2 \epsilon_0}\overline{a}_r$

Q47. The flux density $\overline{D} = \frac{r}{3}\overline{a}_r$ nC/m² is in the free space.

(a) Find $\overline{E}$ at r = 0.2 m.

(b) Find the total electric flux leaving the sphere of r = 0.2 m.

(c) Find the total charge within the sphere of r = 0.3 m.

Nov.-10, Set-2, C

Ans:

Give that,

Flux density, $\overline{D} = \frac{r}{3}\overline{a}_r$ nC/m² ... (1)

(a) Electric field intensity, $\overline{E} = ?$ at $r = 0.2$ m

(b) Total electric flux, $\Psi = ?$ at $r = 0.2$ m

(c) Total charge within the sphere, $Q = ?$ at $r = 0.3$ m

(a) Electric Field Intensity

The electric field intensity in terms of electric flux density is given as,

$$\overline{E} = \frac{\overline{D}}{\epsilon_0} \quad \text{... (2)}$$

Substituting equation (1) in equation (2), we have,

$$\overline{E} = \frac{r}{3\epsilon_0} \times 10^{-9}\,\overline{a}_r$$

Now, substituting $r = 0.2$ m in equation (3), we get,

$$\overline{E} = \frac{0.2\times10^{-9}}{3\times8.854\times10^{-12}}\overline{a}_r$$

$$= 7.529\,\overline{a}_r$$

$$\therefore \quad \overline{E} = 7.53\,\overline{a}_r \text{ V/m}$$

(b) Total Electric Flux

From Gauss's law, we have,

$$\psi = Q$$

$$= \oint_s \overline{D}.d\overline{s} \quad \text{... (4)}$$

Assuming the differential vector surface normal to r direction as,

$$d\overline{s} = r^2 \sin\theta \, d\theta . d\overline{a}_r \quad \text{... (5)}$$

Substituting equations (1) and (5) in equation (4), we get,

$$\psi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{r}{3}\right)(r^2\sin\theta)(1)d\theta\, d\phi \qquad (\because \overline{a}_r\,\overline{a}_r = 1)$$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{r^3}{3}\sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\frac{r^3}{3}[-\cos\theta]_0^{\pi}d\phi$$

$$= \frac{r^3}{3}\int_0^{2\pi}[1+1]d\phi$$

$$= \frac{2r^3}{3}\int_0^{2\pi}d\phi$$

$$= \frac{2}{3}r^3[\phi]_0^{2\pi}$$

$$= \frac{2r^3}{3}[2\pi - 0]$$

$$= \frac{4}{3}\pi r^3$$

$$\therefore \quad \psi = \frac{4}{3}\pi r^3 \text{ nC}$$

Now substituting $r = 0.2$, we get,

$$\psi = \frac{4}{3}\pi(0.2)^3 \times 10^{-9}$$

$$= 3.351 \times 10^{-11}$$

$$= 33.51 \text{ pC}$$

$\therefore$ The total electric flux leaving the sphere at $r = 2$ m is,

$$\psi = 33.51 \text{ pC}$$

(c) Total Charge

From equation (6), we have,

$$\psi = Q = \frac{4}{3}\pi r^3 \text{ nC}$$

When, $r = 0.3$

$$Q = \frac{4}{3}\pi(0.3)^3 \times 10^{-9}$$

$$= 1.1309 \times 10^{-10}$$

$$= 0.113 \text{ nC}$$

Q48. Calculate the 'D' at a point A(6, 4, –5) caused by,

(i) A point charge of 20 mC at the origin.

(ii) A uniform line charge density ρ_l = 20 μ C/m on the z-axis.

(iii) A uniform charge density ρ_s = 60 μC/m² at a plane x = 4.

Ans: May/June-13, (R09), Q3(b)

Given that,

Point $A = (6, 4, -5)$

To determine electric flux density, D for the following conditions,

(i) Point charge, $Q = 20$ mC at a point $P_1 = (0, 0, 0)$

(ii) Uniform line charge density, $\rho_l = 20$ μc/m on z-axis

(iii) Uniform charge density, $\rho_s = 60$ μC/m² at $x = 4$

(i) The electric field intensity at a point P_1 (0, 0, 0) due to the point charge $Q = 20$ mC is given by,

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_r \text{ V/m}$$

$$\bar{R} = (6-0)\,a_x + (4-0)\,a_y + (-5-0)\,a_z$$

$$= 6a_x + 4a_y - 5a_z$$

$$|\bar{R}| = \sqrt{6^2 + 4^2 + (-5)^2}$$

$$|\bar{R}| = \sqrt{77}$$

$$\bar{a}_r = \frac{\bar{R}}{|\bar{R}|} = \frac{6a_x + 4a_y - 5a_z}{\sqrt{77}}$$

$$\therefore \quad \bar{E} = \frac{20\times10^{-3}}{4\pi\times\epsilon_0\times77} \times \frac{6a_x + 4a_y - 5a_z}{\sqrt{77}}$$

The electric flux density, D is given as,

$$\bar{D} = \epsilon_0 \bar{E}$$

$$\bar{D} = \frac{20\times10^{-3}(6a_x + 4a_y - 5a_z)}{4\pi\times77\times\sqrt{77}}$$

$$= \frac{0.12\bar{a}_x + 0.08a_y - 0.1}{8490.75}$$

$$= 14.13\times10^{-6}a_x + 9.42\times10^{-6}a_y - 11.78\times10^{-6}a_z$$

$$\bar{D} = 14.13\,a_x + 9.42\,a_y - 11.78\,a_z \text{ μC/m}^2$$

(ii) The electric field intensity in z-direction due to an infinite line charge ρ_l is given by,

$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_r \text{ V/m}$$

As the direction is along z-axis, the point A is considered as (6, 4).

$$\bar{R} = (6-0)\,a_x + (4-0)\,a_y$$

$$\bar{R} = 6a_x + 4a_y$$

$$|\bar{R}| = \sqrt{6^2 + 4^2} = \sqrt{52}$$

$$\bar{a}_r = \frac{\bar{R}}{|\bar{R}|} = \frac{6a_x + 4a_y}{\sqrt{52}}$$

$$\therefore \quad \bar{E} = \frac{20\times10^{-6}}{2\pi\times\epsilon_0\times\sqrt{52}} \times \frac{6a_x + 4a_y}{\sqrt{52}}$$

The electric flux density, D is given as,

$$\bar{D} = \epsilon_0 \bar{E}$$

$$\bar{D} = \frac{20\times10^{-6}(6a_x + 4a_y)}{2\pi\times\sqrt{52}\times\sqrt{52}}$$

$$= \frac{1.2\times10^{-4}a_x + 8\times10^{-5}a_y}{326.725}$$

$$= 0.367\times10^{-6}a_x + 0.244\times10^{-6}a_y$$

$$\bar{D} = 0.367\,a_x + 0.244\,a_y \text{ μC/m}^2$$

(iii) The electric field intensity due to infinite plane of uniform charge density is given by,

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_x \text{ V/m} \qquad (\because x\text{-plane})$$

The plane is at $x = 4$ and the point A is at $x = 6$. Thus, the point A is above the plane,

$$\therefore E = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$$

$$\bar{E} = \frac{60\times10^{-6}}{2\epsilon_0}\bar{a}_x$$

The electric flux density, D is given as,

$$\bar{D} = \epsilon_0 \bar{E}$$

$$\bar{D} = \frac{60\times10^{-6}}{2}\bar{a}_x$$

$$\bar{D} = 30\,\bar{a}_x \text{μ C/m}^2$$

Q49. Given $\bar{D} = 2xy\bar{\partial}x + z\bar{\partial}y + yz^2\bar{\partial}z$, find $\nabla.\bar{D}$ at P(2, –1, 3).

Ans: Dec.-11, Set-3, Q2(b)

Given that,

$$\bar{D} = 2xy\bar{\partial}x + z\bar{\partial}y + yz^2\bar{\partial}z \quad \text{... (1)}$$

To determine,

$$\nabla.\bar{D} = ?$$

At $P = (2, -1, 3)$

The expression of divergence, in Cartesian form is given as,

$$\nabla.\bar{D} = div\bar{D} = \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z) \quad \text{... (2)}$$

Comparing equation (1) and (2), we get,

$$D_x = 2xy; \quad D_y = z; \quad D_z = yz^2$$

Substituting the above values in equation (2), we get,

$$\nabla.\bar{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz^2)$$

$$= 2y + 0 + 2yz$$

At (2, –1, 3)

$$(\nabla.\bar{D})_{(2,-1,3)} = 2(-1) + 2(-1)(3)$$

$$= -2 - 6$$

$$= -8$$

$$\therefore \quad \nabla.\bar{D} = -8$$

1.5 ABSOLUTE ELECTRIC POTENTIAL, POTENTIAL DIFFERENCE, CALCULATION OF POTENTIAL DIFFERENCES FOR DIFFERENT CONFIGURATIONS

Q50. What is a potential function? List out the properties of a potential function.

Model Paper-I, Q2(b)

Ans:

Potential Function (V)

Potential function 'V' is not uniquely defined. Any quantity which is independent of the coordinates can be added to it without effecting the electric field in any way.

"The electrostatic field 'E' can be described completely by means of a potential function 'V'(x, y, z), which is known as electric potential". Thus,

$$E = -\nabla V$$

Where, 'V' is a scalar point function.

Electric field intensity is the derivative of potential function i.e., $\frac{dV}{dx} = -Ea_x$

It means that if the two points have non-zero potential difference, then only the electric lines of forces should be present. These lines of forces travel from higher potential conductor to lower potential conductor.

Properties of Potential Function

For answer refer Unit-I, Q9.

Q51. Explain,

(i) Electric potential due to point charge

(ii) Absolute potential

(iii) Electric potential due to n point charges.

Ans:

(i) Electric Potential Due to Point Charge

Consider a point charge located in a system, producing electric field radially in all directions figure (1) represents the potential due to a point charge.

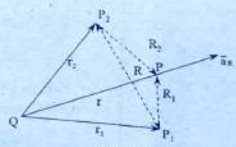


Figure (1)

Let,

Q be a point charge in the fied

P_2, P_2 be the arbitrary path in the field

R_1,R_2 are the radial distances of p_1,p_2 from charge Q

P be a point on the path

R be a distance between point P and charge

$\bar{E}$ be the electric field intensity

$\bar{a}_R$ be the vector directed from the point and charge to p.

V be the potential between point and charge.

In the free space, the field intensity $\overline{E}$ due to point charge Q at a point with raidus R is given by,

$$\overline{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\,\overline{a}_R \qquad \dots (1)$$

The potential difference between the points P_1 and p_2 is given by,

$$V_{12} = -\int_{initial}^{final} \overline{E}.d\overline{L}$$

$$V_{12} = -\int_{R_1}^{R_2} \overline{E}\,d\overline{L} \qquad \dots (2)$$

$$= -\int_{R_1}^{R_2}\left(\frac{Q}{4\pi\varepsilon_0 R^2}\,\overline{a}_R\right)d\overline{L} \qquad (\because \text{ from equation (1)})$$

$$= \int_{-R_1}^{R_2} \frac{Q}{4\pi\varepsilon_0 R^2}\,dR \qquad (\because dR = \overline{a}_R\ d\overline{L})$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_{R_1}^{R_2}\frac{1}{R^2}\,dR$$

$$= \frac{-Q}{4\pi\varepsilon_0}\int_{R_1}^{R_2} R^{-2}\,dR$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{R^{-1}}{-1}\right)_{R_1}^{R_2}$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{R}\right)_{R_1}^{R_2}$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{R_2} - \left(\frac{-1}{R_1}\right)\right)$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\therefore\ V_1 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \qquad \dots (3)$$

When $R_1 > R_2$, $\frac{1}{R_1} < \frac{1}{R_2}$, Thus V_{12} is positive.

Thus equation (3) indicates the potential due to any two point charges.

(ii) Absolute Potential

At any point in an Electric filed the absolute potential is defined by the work done in moving a unit charge from infinity to the point, against the direction of the field.

From equation (3), Let us assume point P_1 is located at infinity, thus V_1,

We have,

$$V_{12} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

$$\Rightarrow\ V_{12} = V_2 - V_1 \qquad \dots (4)$$

Here,

$$V_1 = \frac{Q}{4\pi\varepsilon_0 R_1}, \text{ and}$$

$$V_2 = \frac{Q}{4\pi\varepsilon_0 R_2}$$

To obtain above terms, let us assume point p_1 is located at infinity, then V_1 becomes zero and V_2 gives absolute potential at R_2.

$$\text{i.e}\quad R_1 = \infty \Rightarrow \frac{1}{R_1} = \frac{1}{\infty} = 0$$

$$\therefore\quad V_{12} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{\infty}\right)$$

$$\Rightarrow\quad V_{12} = \frac{Q}{4\pi\varepsilon_0 R_2}\text{ volts} \qquad \dots (5)$$

Thus equation (5) is called as absolute potential at point P_2.

Similarly, absolute potential at point P_1 is given as,

$$V_1 = \frac{-Q}{4\pi\varepsilon_0 R_1}\text{ volts} \qquad \dots (6)$$

Thus, the potential at point which is located at a distance R from point charge is given by,

$$V = \int_{\infty}^{R}\frac{Q}{4\pi\varepsilon_0 R^2}\,dR$$

$$\therefore\quad V = \frac{Q}{4\pi\varepsilon_0 R}\text{ volts} \qquad \dots (7)$$

Hence, equation (7), indicates the V abosolute potential due to point charge

(iii) Electric Potential Due to n Point Charges

Consider a linear, homegenous and istropic dielectric region consists of n number of point charges. The arrangement is shown in figure (2).

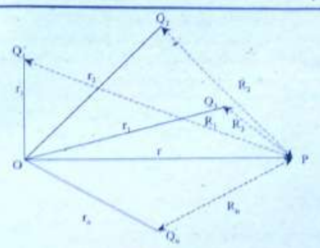


Figure (2)

Let,

O be the origin of the region

P be the point at distance r from origin

V_p be the potential at P due to n point charges

$Q_1, Q_2, \dots, Q_n$ be the various point charges

$r_1, r_2, \dots, r_n$ be the radial distances from origin 'o' to respective point charges

$R_1, R_2, \dots, R_n$ are the distance between point P and the point charges $Q_1, Q_2, \dots, Q_n$

Bu using super position principle, the potential due to n point charges at point P is can be determined.

Now, the potential V_{p1} due to Q_1 is given by,

$$V_{p1} = \frac{Q_1}{4\pi\varepsilon_0|r - r_1|}$$

$$\therefore\ V_{p1} = \frac{Q_1}{4\pi\varepsilon_0 R_1}\text{ volts}\quad [\because |r - r_1| = R_1] \qquad \dots (8)$$

Now, the potential V_{p2} due to piont charge Q_2 is given by,

$$V_{p2} = \frac{Q_2}{4\pi\varepsilon_0|r - r_2|}$$

$$\therefore\quad V_{p2} = \frac{Q_2}{4\pi\varepsilon_0 R_2}\text{ volts}\quad [\because |r - r_2| = R_2] \qquad \dots (9)$$

Similarly, the potential V_{pn} due to point charge Q_n is given by,

$$V_{pn} = \frac{Q_n}{4\pi\varepsilon_0|r - r_n|}$$

$$\therefore\ V_{pn} = \frac{Q_n}{4\pi\varepsilon_0 R_n}\text{ volts}\quad [\because |r - r_n| = R_n] \qquad \dots (10)$$

Now, the potential due to n charges are added to get the total potential at point p.

$$V_p = V(r) = V_{p1} + V_{p2} + \dots + V_{pn}$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \dots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{R_1} + \frac{Q_2}{R_2} + \dots + \frac{Q_n}{R_n}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k}{R_k}$$

$$\therefore\quad = \sum_{k=1}^{n}\frac{Q_k}{4\pi\varepsilon_0 R_k}\text{ volts} \qquad \dots (11)$$

Hence, equation (11) gives the potential due to n point charges.

Q52. Derive the expression for electric potential due to volume charge.

Ans:

Electric Potential due to volume charge

Let,

ρ_v be the volume charge density (C/m³)

P be the point

dv be the differential volume

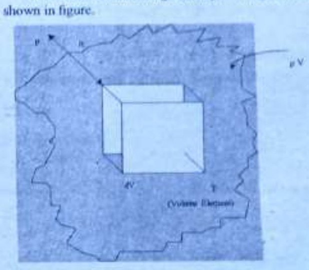
R be the distance between point p and differential charge

dQ be the differential charge

$\rho_v(r)$ be the charge density

T be the cubic volume surface

Consider a volume charge density over a given volume as shown in figure.



Figure

Consider the differential volume dV at point T with charge density of $\rho_v(r)$.

The charge dQ on the differential volume dV is given by, ... (1)

$$dQ = \rho_v(r)dV$$

Now, the differential potential at point P due to dQ is given by,

$$dV_p = \frac{dQ}{4\pi\varepsilon_0 R}$$

Where,

ε_0 = Absolute permittivity

By substituting equation (1) in dV_p, we get,

$$dV_p = \frac{\rho_v(r)}{4\pi\varepsilon_0 R} dv \qquad \text{... (2)}$$

On integrating equation (2) over the given volume, the total potential at point p is obtained as,

$$V_p = \int_v \frac{\rho_v(r)}{4\pi\varepsilon_0 R} dV$$

$$V_p = \frac{1}{4\pi\varepsilon_0 R} \int_v \frac{\rho_v(r)}{R} dV \text{ volts} \qquad \text{... (3)}$$

In equation (3), only one integral sign is indicated, but the differentiation (dV) signifies through out the volume. Therefore this integration will be a tripple integration.

Now, equation (3) becomes as,

$$V_{vol} = V_p = \frac{1}{4\pi\varepsilon_0} \iiint_{vol} \frac{\rho_v(r)}{R} dV \text{ volts} \qquad \text{... (4)}$$

Note: For a uniform volume charge density $\rho_v(r) = \rho_v$

Thus absolute potential due to volume charge is,

$$V_{vol} = \int_v \frac{\rho_v dV}{4\pi\varepsilon_0 R} \text{ volts} \qquad \text{... (5)}$$

$\therefore$ The required eletric potential due to volume charge at point p is,

$$V_{vol} = \frac{1}{4\pi\varepsilon_0} \iiint_{vol} \frac{\rho_v(r)}{R} dV \text{ volts}$$

Q53. Define potential difference? Mention the characteristics of potential difference.

Ans:

Potential Difference

It is defined as the amount of work done to move a charge from point A to point B in an electric field as shown in figure.

Point A Potential difference Point B

V_1 ⟷ V_2

Figure: Potential Difference between Two Points

It is the difference of potential between two terminals (i.e., point A and point B). It is also known as the voltage between the two points and is a measure of energy used by one Coulomb of charge in moving from one point to another. Unit of potential difference is volt.

Potential difference, $V_{AB} = \frac{W}{\theta} = -\oint_A^B E.dl$

Where, W – Workdone,

θ – Point charge

E – Electric fied intensity

dl – Displacement of charge

The characteristics of potential difference are,

(i) A is considered the initial point and B is considered the final point while evaluating V_{AB}.

(ii) The negative V_{AB} indicates there is a loss in potential energy in moving charge q from A to B that means, the work is done by field.

(iii) The positive V_{AB} indicates that there is gain in potential energy in moving charge q from A to B.

(iv) V_{AB} is independent of the path travelled.

(v) The unit of V_{AB} is Joules/Coloumb and standard unit is volts (V).

Q54. Explain the concept of potential with necessary equations.

Ans: Model Paper-III, Q2(b)

Consider a point charge 'Q' moving from point A to point B in an electric field (E) as shown in figure below.

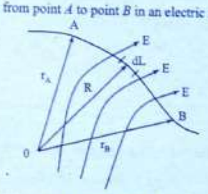


Figure: Displacement of Point Charge Q in an Electric Field (E)

According to Coulomb's law,

Force, $F = QE$ ... (1)

Work done in displacing the charge by $\overline{dl}$ is,

$$dw = -\overline{F}\cdot\overline{dl} \qquad \text{... (2)}$$

Substituting equation (1), in equation (2),

$$dw = -Q\vec{E}.\overrightarrow{dl}$$

Where,

– ve sign represents the work done by external agent.

Therefore, the total work done in moving a point charge from A to B is,

$$W = -Q\int_A^B \vec{E}.\overrightarrow{dl}$$

$$\therefore\ V_{AB} = \frac{W}{Q} = -\int_A^B E.\overrightarrow{dl} \qquad \text{... (3)}$$

Where,

V_{AB} – Potential difference between points A and B.

From figure, the electric field due to a point charge Q at origin is,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R \qquad \text{... (4)}$$

$$\vec{V}_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R\, dR\, \hat{a}_R$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\therefore\ V_{AB} = V_B - V_A$$

Where,

V_A – Absolute potential at A

V_B – Absolute potential at B.

If $V_A = 0$ as $r_A \to \infty$, then the potential at any point due to a point charge Q is,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

Q55. Derive the relation between electric field (E) and potential (V).

Ans:

The potential differential between points A and B is,

$$V_{AB} = -V_{BA}$$

$$\Rightarrow V_{AB} + V_{BA} = 0 \quad \text{... (1)}$$

$$\Rightarrow \oint_L E.dl = 0$$

From equation (1), it is observed that the line integral along the closed path is zero.

Applying Stoke's theorem,

$$\oint_L E.dl = \int_S (\nabla \times E).dS = 0 \quad \text{... (2)}$$

$$\Rightarrow \nabla \times E = 0$$

The above equation represents the Maxwell's equation for electrostatic fields.

The expression for electric potential is,

$$V = -\int E.\,dl$$

$$\Rightarrow dv = -E.\,dl \quad \text{... (3)}$$

$$\Rightarrow dv = -(E_x d_x + E_y d_y + E_z d_z)$$

Since,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \quad \text{... (4)}$$

Comparing equations (3) and (4),

$$E_x = \frac{-\partial v}{\partial x}$$

$$E_y = \frac{-\partial v}{\partial y}$$

$$E_z = \frac{-\partial v}{\partial z}$$

And $E = E_x i + E_y j + E_z k$

$$= -\left(\frac{\partial v}{\partial x}i + \frac{\partial v}{\partial y}j + \frac{\partial v}{\partial z}k\right)$$

$$= -\nabla V$$

$$\therefore \quad E = -\nabla V$$

1.6 ELECTRIC DIPOLE, ELECTROSTATIC ENERGY AND ENERGY DENSITY

Q56. In electrostatics, what is meant by a physical dipole?

Ans:

Physical Dipole

A physical dipole is a pair of equal and opposite charges ($\pm Q$) separated by a distance d.

The product of magnitude of either point charges and the distance of separation is called as dipole moment.

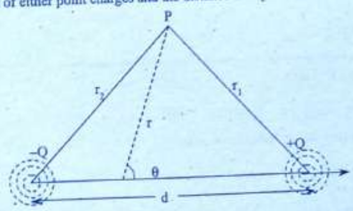


Figure (1)

Consider, two equal and opposite charges $\pm Q$ separated by a small distance 'd', then electric dipole moment

$$\boxed{P = Qd}$$

Example

When an atom or molecule is placed in an electric field, the positive and negative charges experience opposite forces and they are displaced slightly forming a dipole.

Let r_1 be the distance from $+Q$ and r_2 be the distance from $-Q$ then the potential is given by,

$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r_1} - \frac{Q}{r_2}\right) \quad \text{... (1)}$$

From the law of cosines,

$$r_1^2 = r^2 + (d/2)^2 - 2(d/2)\, r \cos\theta.$$

Where $+d/2$ and $-d/2$ are the distances from the origin on the z-axis where the charges are placed. If 'θ' is the angle between the z-axis and position of vector of 'P'. Then,

$$r_1^2 = r^2\left(1 + \frac{d^2}{4r^2} - \frac{d\cos\theta}{r}\right)$$

$$\Rightarrow \frac{1}{r_1} = \frac{1}{r}\left[1 + \frac{d^2}{4r^2} - \frac{d\cos\theta}{r}\right]^{-1/2}$$

$$\Rightarrow \frac{1}{r_1} = \frac{1}{r}\left[1 + \frac{d\cos\theta}{2r} - \ldots\right]$$

$\because$ $r >> d$, so the 3rd term is negligible

$$\frac{1}{r_1} \equiv \frac{1}{r} + \frac{d\cos\theta}{2r^2} \quad \text{... (2)}$$

$$\text{Similarly, } \frac{1}{r_2} \equiv \frac{1}{r} - \frac{d\cos\theta}{2r^2} \quad \text{... (3)}$$

Substitute equation (2) and (3) in equation (1).

$$V(r) \cong \frac{1}{4\pi\epsilon_0}\left[Q\left(\frac{1}{r}+\frac{d\cos\theta}{2r^2}\right)-Q\left(\frac{1}{r}-\frac{d\cos\theta}{2r^2}\right)\right]$$

$$V(r) \cong \frac{1}{4\pi\epsilon_0} Q\left(\frac{d\cos\theta + d\cos\theta}{2r^2}\right) = \frac{1}{4\pi\epsilon_0} Q\frac{2d\cos\theta}{2r^2}$$

$$\boxed{V(r) = \frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}} \text{ Volts}$$

Q57. What is electric dipole? Explain.

OR

Find the electric potential due to electric dipole.

Ans:

Electric Dipole

The combination of a pair of equal (in magnitude) and opposite (in sign) point charges which are separated by a very small distance is termed as electric dipole.

Electric Dipole Moment

The product of magnitude of either of the point charge and the distance of separation is called as dipole moment or electric dipole moment.

i.e., if $+Q$ and $-Q$ are two point charges separated by distance 'd' then the electric dipole moment,

$$P = Qd \quad \text{... (1)}$$

Derivation

Let an electric dipole be lying on X-axis, such that, charge $+Q$ is at $d/2$ and charge $-Q$ at $-d/2$. Consider a point 'O' in X-Y plane at a distance of R_1, R and R_2 from the positive charge ($+Q$), origin and negative charge ($-Q$) respectively as shown in the figure (1).

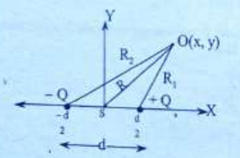


Figure (1)

The electric potential at point 'O' due to charge '$+Q$' can be given by,

$$V_1 = \frac{Q}{4\pi\epsilon_0 R_1}$$

Similarly, the electric potential due to charge '$-Q$' will be,

$$V_2 = \frac{-Q}{4\pi\epsilon_0 R_2}$$

$\therefore$ The electric potential due to electric dipole is,

$$V = V_1 + V_2$$

$$= \frac{Q}{4\pi\epsilon_0 R_1} - \frac{Q}{4\pi\epsilon_0 R_2}$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{R_2 - R_1}{R_1 R_2}\right] \quad \text{... (2)}$$

From the definition of electric dipole itself, it is clear that the distance between the charges is too small. Hence, the lines joining the point 'O' with the origin and charges can be treated approximately as parallel lines as shown in figure (2).

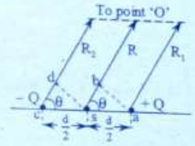


Figure (2)

From triangle sba of figure (2), we have,

$$\cos\theta = \frac{\overline{sb}}{\overline{sa}} = \frac{\overline{sb}}{d/2}$$

$$\Rightarrow \quad \overline{sb} = \frac{d}{2}\cos\theta$$

Similarly, from triangle cds, we have,

$$\cos\theta = \frac{\overline{sb}}{\overline{cs}} = \frac{\overline{cd}}{-d/2}$$

$$\Rightarrow \quad \overline{cd} = \frac{-d}{2}\cos\theta$$

From figure (2), we also have that,

$R_1 = R - \overline{sb}$ and

$R_2 = R - \overline{cd}$

$\therefore R_1 = R - \frac{d}{2}\cos\theta$ and

$R_2 = R + \frac{d}{2}\cos\theta$

Substituting R_1 and R_2 in equation (2), we have,

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{\left(R+\frac{d}{2}\cos\theta\right)-\left(R-\frac{d}{2}\cos\theta\right)}{\left(R+\frac{d}{2}\cos\theta\right)\left(R-\frac{d}{2}\cos\theta\right)}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{R+\frac{d}{2}\cos\theta-R+\frac{d}{2}\cos\theta}{R^2-\left(\frac{d}{2}\cos\theta\right)^2}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{\frac{2d}{2}\cos\theta}{R^2-\left(\frac{d}{2}\right)^2\cos^2\theta}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2} \qquad \left\{\because d \text{ is too small}, \left(\frac{d}{2}\right)^2\cos^2\theta \ll R^2\right\}$$

$$= \frac{Qd\cos\theta}{4\pi\epsilon_0 R^2}$$

$$= \frac{P\cos\theta}{4\pi\epsilon_0 R^2} \qquad \text{[From equation (1)]}$$

Hence, the electric potential due to an electric dipole,

$$V = \frac{P\cos\theta}{4\pi\epsilon_0 R^2} \text{ Volts}$$

Q58. Show that the potential due to an electric dipole satisfies Laplace's equation.

Ans:

An electric dipole is the name given to two point charges of equal and opposite sign, separated by a distance.

Electric potential due to an electric dipole is given by the relation,

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} \qquad \text{... (1)}$$

Equation (1) is in spherical coordinates.

Laplace's equation in spherical coordinates is given as,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad \text{... (2)}$$

Hence, it is to be proved that equation (1) satisfies equation (2).

Proof

Equation (1) varies with r and θ, but not with ϕ. Therefore, 3rd term in Laplace's equation can be neglected.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}\right)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{Qd\cos\theta}{4\pi\varepsilon_0}\left(\frac{-2}{r^3}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{Qd}{4\pi\varepsilon_0 r^2}(-\sin\theta)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-Qd\cos\theta}{2\pi\varepsilon_0 r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{Qd}{4\pi\varepsilon_0 r^2}(-\sin^2\theta)\right]$$

$$= \frac{1}{r^2}\left[\frac{-Qd\cos\theta}{2\pi\varepsilon_0}\left(\frac{-1}{r^2}\right)\right]\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{-Qd}{4\pi\varepsilon_0 r^2}\frac{(1-\cos 2\theta)}{2}\right] \qquad (\because \cos 2\theta = 1-2\sin^2\theta)$$

$$= \frac{1}{r^2}\left[\frac{Qd\cos\theta}{2\pi\varepsilon_0 r^2}\right] + \frac{1}{r^2\sin\theta}\left[\frac{-Qd}{4\pi\varepsilon_0 r^2}\frac{\partial}{\partial\theta}\left(\frac{1-\cos 2\theta}{2}\right)\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^4} - \frac{1}{r^2\sin\theta}\left[\frac{-Qd}{4\pi\varepsilon_0 r^2}\left(\frac{-2\sin 2\theta}{2}\right)\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^4} - \frac{Qd(2\sin\theta\cos\theta)}{4\pi\varepsilon_0 r^4\sin\theta} \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^4} - \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^4}$$

$$\nabla^2 V = 0$$

Hence, it is proved that electric potential due to an electric dipole satisfies Laplace's equation.

Q59. Find the electric field at any point due to electric dipole.

Ans: Model Paper-III, Q[illegible](b)

For derivation of "Electric potential at any point due to electric dipole" refer Unit-I, Q57, Topic: Derivation.

We know that, the electric field intensity is the negative gradient of potential.

i.e. $\bar{E} = -\nabla V$ ... (1)

∇V can be expanded in spherical co-ordinate system as,

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\bar{a}_\phi$$

As 'V' does not varies with 'ϕ', $\frac{\partial V}{\partial\phi} = 0$ ... (2)

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(\frac{P\cos\theta}{4\pi\epsilon_0 R^2}\right)$$

$$= \frac{P\cos\theta}{4\pi\epsilon_0}\times\frac{\partial}{\partial r}(r^{-2}) \qquad (\because R \text{ and } r \text{ are same})$$

$$= \frac{P\cos\theta}{4\pi\epsilon_0}\times(-2)\times r^{-2-1}$$

$$= \frac{-P\cos\theta}{2\pi\epsilon_0 r^3} \qquad \text{... (3)}$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{P\cos\theta}{4\pi\epsilon_0 R^2}\right)$$

$$= \frac{P}{4\pi\epsilon_0 R^2}\frac{\partial}{\partial\theta}(\cos\theta)$$

$$= \frac{P}{4\pi\epsilon_0 r^2}\times(-\sin\theta) \qquad (\because R = r)$$

$$= \frac{-P\sin\theta}{4\pi\epsilon_0 r^2} \qquad \dots (4)$$

Substituting equations (2), (3) and (4) in equation (1).

$$\overline{E} = -\left[\frac{-P\cos\theta}{2\pi\epsilon_0 r^3}\overline{a}_r + \frac{1}{r}\left(\frac{-P\sin\theta}{4\pi\epsilon_0 r^2}\right)\overline{a}_\theta + \frac{1}{r\sin\theta}\times 0\times\overline{a}_\phi\right]$$

$$= \frac{P\cos\theta}{2\pi\epsilon_0 r^3}\overline{a}_r + \frac{P\sin\theta}{4\pi\epsilon_0 r^3}\overline{a}_\theta$$

$$= \frac{P}{4\pi\epsilon_0 r^3}(2\cos\theta\,\overline{a}_r + \sin\theta\,\overline{a}_\theta)\text{ V/m}$$

The above expression gives the electric field at any point due to an electric dipole.

Q60. A dipole having moment $\overline{P} = 3\overline{a}_x - 5\overline{a}_y + 10\overline{a}_z$ ncm is located at Q(1, 2, –4) in free space. Find potential v at p(2, 3, 4).

Ans: Dec.-11, Set-2, Q1(b)

Given that,

Dipole moment,

$$\overline{P} = 3\overline{a}_x - 5\overline{a}_y + 10\overline{a}_z \text{ ncm}$$

Point in free space,

$$Q = (1, 2, -4)$$

To determine,

Potential, v = ? at

Point, $p = (2, 3, 4)$

The potential due to dipole moment is given as,

$$v = \frac{\overline{P}\,\overline{a}_r}{4\pi\epsilon_0 r^2} \qquad \dots (1)$$

Where,

$$\overline{a}_r = \frac{\overline{r}}{|\overline{r}|}$$

We have,

$$\overline{r} = (2-1)\overline{a}_x + (3-2)\overline{a}_y + (4-(-4))\overline{a}_z$$

$$= \overline{a}_x + \overline{a}_y + 8\overline{a}_z$$

And

$$|\overline{r}| = \sqrt{1^2+1^2+8^2}$$

$$= \sqrt{1+1+64}$$

$$= \sqrt{66}$$

$$\therefore a_r = \frac{1}{|\overline{r}|}(\overline{a}_x + \overline{a}_y + 8\overline{a}_z)$$

Substituting the values in equation (1), we get,

$$v = \frac{(3\overline{a}_x - 5\overline{a}_y + 10\overline{a}_z)\frac{1}{\sqrt{66}}(\overline{a}_x + \overline{a}_y + 8\overline{a}_z)}{(4\pi)(8.854\times10^{-12})(\sqrt{66})^2}\times 10^{-9}$$

$$= \frac{(3-5+80)\times10^{-9}}{\sqrt{66}(4\pi\times8.854\times10^{-12})66}$$

$$\therefore v = 1.3074 \text{ volts}$$

Q61. Point charges of +3 μC and –3 μC are located at (0, 0, 1) mm and (0, 0, –1) mm respectively in free space.

(i) Find dipole moment $\overline{P}$.

(ii) Find $\overline{E}$ in spherical components at P(r = 2, θ = 40°, φ = 50°).

Ans: Nov.-10, Set-4, Q5(b)

Given that,

Charge, $Q_1 = +3\ \mu C$

Charge, $Q_2 = -3\ \mu C$

Point, $P_1 = (0, 0, 1 \text{ mm})$

Point, $P_2 = (0, 0, -1 \text{ mm})$

(i) **Dipole Moment ($\overline{P}$)**

We know that,

$$\overline{P} = Q\overline{d}$$

Where,

$\overline{d}$ is a vector

$$\overline{d} = (0.00\overline{a}_z) - (-0.00\overline{a}_z)$$

$$= 0.002\overline{a}_z$$

$$\therefore \text{Dipole moment, } \overline{P} = 3\times10^{-6}(0.00\overline{a}_z)$$

$$= 6\times10^{-}\overline{a}_z$$

$$= 6\overline{a}_z \text{ nC-m}$$

(ii) **Electric Field Intensity ($\overline{E}$)**

The dipole center is located at origin, thus we know that,

$$\overline{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\overline{a}_r + \sin\theta\,\overline{a}_\theta)$$

We have,

$$\theta = 40^\circ$$

$$r = 2$$

$$\phi = 50^\circ$$

Substituting the values in the above expression we get,

$$E = \frac{3\times10^{-6}\times0.002}{4\pi\times8.854\times10^{-12}\times(2)^3}\times(2\cos 4\,\overline{a}_r + \sin 4\,\overline{a}_\theta)$$

$$= 6.74(1.53\overline{a}_r + 0.64\overline{a}_\theta)$$

$$= (10.32\overline{a}_r + 4.32\overline{a}_\theta)\text{ V/m}$$

$\therefore$ Electric field intensity, $E = (10.32\overline{a}_r + 4.32\overline{a}_\theta)$ V/m

Q62. Derive the energy stored in electrostatic field in terms of field quantities.

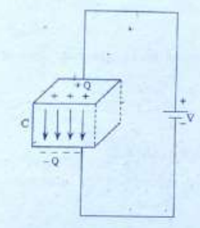
OR

Prove that the energy required for charging a capacitor C by a voltage V is W = 0.5 CV * V

Ans:

Capacitor is a device which acquires charge, when it is connected to a source of potential. Therefore, a capacitor can store energy and this energy gets stored in the electrostatic field of the capacitor.

Consider a capacitor 'C' charged to a potential difference 'V' as shown in figure.



Figure

The charge 'Q' acquired by capacitor is given by,

$$Q = CV \qquad \dots (1)$$

It involves work in order to charge a capacitor. Potential is the work done per unit charge. It is given as,

$$V = \frac{dW}{dQ} \qquad \dots (2)$$

Where,

dW = Infinitesimal work done

dQ = Infinitesimal charge.

From equations (1) and (2), we get,

$$dW = \frac{Q}{C}dQ \qquad \dots (3)$$

The charge 'Q' is at zero value at the starting and the charging proceeds until a final value of the charge 'Q' is attained.

$\therefore$ Total work done in charging the capacitor is given as,

$$W = \int dW$$

$$= \int_0^Q \frac{Q}{C}dQ$$

$$= \frac{1}{C}\int_0^Q Q\,dQ$$

$$= \frac{1}{C}\left[\frac{Q^2}{2}\right]_0^Q$$

$$= \frac{1}{2C}[Q^2 - 0]$$

$$= \frac{Q^2}{2C}$$

$$= \frac{1}{2}QV = \frac{1}{2}CV^2 \qquad [\because Q = CV]$$

Energy stored in a capacitor or energy stored in static electric field is given by,

$$W = \frac{1}{2}CV^2 = 0.5\,CV{*}V \text{ Joules}$$

Q63. Derive an expression for energy stored in the electrostatic field of a section of a coaxial cable.

Ans:

For answer refer Unit-II, Q41.

The general expression for energy stored in the electrostatic field is given as,

$$W = \frac{1}{2}CV^2$$

Substituting the value of capacitance of coaxial cable in the above equation, we get,

$$W = \frac{1}{2}\left(\frac{2\pi\varepsilon_r L}{\ln\left(\frac{b}{a}\right)}\right)V^2 = \frac{\pi\varepsilon_r L V^2}{\ln\left(\frac{b}{a}\right)}\text{ Joules}$$

Q64. Derive the expression for electrostatic energy density.

Ans:

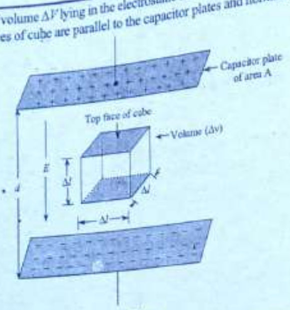
Energy density in an electrostatic field is the amount of energy stored per unit volume for a particular space. It is a scalar quantity. The unit of energy density is Joule/m³.

Consider a parallel plate capacitor of capacitance C, plate area A and distance between the plates d. When it is charged to a potential of V volts, energy gets stored in it. The stored energy is in the form of electrostatic field, which is given by,

$$W = \frac{1}{2}CV^2 \text{J}$$

Therefore 'W' amount of work is done to charge the upper plate and lower plate of the capacitor to a value $+Q$ and $-Q$ respectively. The electrostatic field of capacitor is directed from positively charged plate to negatively charged plate.

Now, consider a small cube of volume ΔV lying in the electrostatic field of capacitor as shown in the figure. Let, Δl be the length of the cube and the top and bottom faces of cube are parallel to the capacitor plates and normal to the electrostatic field.



Figure

If a thin metal foil is placed on the top and bottom face of the cube, then it acts like a parallel plate capacitor of plate area (Δl^2). Then the capacitance of this cube is given by,

$$\Delta C = \frac{\epsilon_0 A}{d} \text{ F}$$

$$\Delta C = \frac{\epsilon_0 (\Delta l)^2}{(\Delta l)} \text{ F} \qquad [\because \text{Area, } A = (\Delta l)(\Delta l) \text{ and distance, } d = (\Delta l)]$$

$$\Delta C = \epsilon_0 (\Delta l) \text{ F} \qquad \dots (1)$$

Potential difference in an electrostatic field is given by,

$$V = Ed \text{ V}$$

Where, E = Electric field intensity.

Potential difference 'ΔV' of the cube of length 'Δl' is given by,

$$\Delta V = E\Delta l \text{ V} \qquad \dots (2)$$

$\therefore$ Energy stored in the cube of volume 'ΔV' is given by,

$$\Delta W = (\Delta C)(\Delta V)^2 \text{ J} \qquad \left[\because W = \frac{1}{2}CV^2\right]$$

Substituting equations (1) and (2) in the above equation, we get,

$$\Delta W = \frac{1}{2}(\epsilon_0 \Delta l)(E\Delta l)^2$$

$$\Rightarrow \quad \Delta W = \frac{1}{2}\epsilon_0 E^2 (\Delta l)^3$$

$$\Rightarrow \quad \Delta W = \frac{1}{2}\epsilon_0 E^2 (\Delta V) \qquad [\because \text{Volume, } \Delta V = (\Delta l)(\Delta l)(\Delta l) \Rightarrow \Delta V = (\Delta l)^3]$$

$$\Rightarrow \quad \frac{\Delta W}{\Delta V} = \frac{1}{2}\epsilon_0 E^2$$

$\therefore$ Energy per unit volume, which is energy density is given by,

$$W = \frac{1}{2}\epsilon_0 E^2 \qquad [\because \Delta V = 1]$$

$$W = \frac{1}{2}(E\epsilon_0)E$$

$$W = \frac{1}{2}DE \text{ J/m}^3 \quad [\because D = \epsilon_0 E] \qquad \dots (3)$$

Since, the direction of both D and E is the same, equation (3) can be written in vector form as,

$$W = \frac{1}{2}\bar{D}.\bar{E} \qquad \text{J/m}^3$$

Where,

$\bar{D}$ = Electric flux density, C/m^2

$\bar{E}$ = Electric field intensity, V/m

Q65. Determine the capacitance of a capacitor consisting of two parallel metal plates 30 cm × 30 cm surface area separated by 5 mm in air. What is the total energy stored by the capacitor if the capacitor charged to a potential difference of 500 V? What is the energy density?

Ans:

Given that,

Dimensions of two plates = 30 cm × 30 cm

Distance between two plates, $d = 5 \text{ mm} = 5\times10^{-3}$ m

Potential difference between plates, $V = 500$ V

Required to determine,

Capacitance, C = ?

Energy stored, W_e = ?

Energy density, W_d = ?

Capacitance

The area of plates,

$$A = 30\times10^{-2}\times30\times10^{-2}$$
$$= 9\times10^{-2} \text{ m}^2$$

The capacitance of the parallel plate capacitor is given by,

$$C = \frac{A\epsilon_0\epsilon_r}{d}$$

$$= \frac{9\times10^{-2}\times8.854\times10^{-12}\times1}{5\times10^{-3}} \qquad [\because \epsilon_r = 1 \text{ for air dielectric}]$$

$$= 1.59372\times10^{-10}$$
$$= 159.372\times10^{-12} \text{ F}$$
$$= 159.372 \text{ pF}$$

Energy Stored

The energy stored by a capacitor is given by,

$$W_e = \frac{1}{2}CV^2$$

$$= \frac{1}{2}\times159.372\times10^{-12}\times(500)^2$$

$$= 1.9922\times10^{-5} = 19.922\times10^{-6} \text{ J} = 19.922 \text{ µJ}$$

Energy Density

The energy density is given by,

$$W_d = \frac{1}{2}|\bar{D}||\bar{E}|$$

$$= \frac{1}{2}|\epsilon\bar{E}||\bar{E}|$$

$$= \frac{1}{2}\epsilon_0\epsilon_r E^2 \qquad [\text{Where } |\bar{E}| = E]$$

We know that,

$$E = \frac{V}{d}$$

$$\Rightarrow E = \frac{500}{5 \times 10^{-3}}$$

$$\Rightarrow E = 100 \times 10^{3}\ \text{V/m}$$

$$\therefore \quad W_d = \frac{1}{2} \times 8.854 \times 10^{-12} \times 1 \times (100 \times 10^{3})^2$$

[$\because \in_r = 1$ for air dielectric]

$$= \frac{1}{2} \times 8.854 \times 10^{-12} \times (100)^2 \times 10^6$$

$$= \frac{1}{2} \times 8.854 \times 10000 \times 10^{-6}$$

$$= \frac{1}{2} \times 8.854 \times 10^4 \times 10^{-6}$$

$$= \frac{1}{2} \times 8.854 \times 10^{-2}$$

$$= 4.427 \times 10^{-2} = 0.04427\ \text{J/m}^3$$

Q66. What is an electric dipole? Obtain expression for torque experienced by an electric dipole in a uniform electric field.

Ans: (Nov./Dec.-17, (R16), Q3(b) | Model Paper-I, Q3(b))

Electric Dipole

For answer refer Unit-I, Q14.

Expression for Torque

Consider an electric dipole in a uniform electric field $\overline{E}$, which makes an angle θ with the dipole axis as shown in the figure.

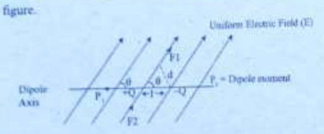


Figure: Torque on a Dipole

The two charges $+Q$ and $-Q$ of an electric dipole experiences forces F_1 and F_2 which are equal and opposite and are of magnitude "QE". Therefore, the forces F_1 and F_2 neutralizes each other. Hence, resulting in the zero net translational force. But, these forces form a couple, whose torque is given by,

Torque,

T = Magnitude of force × Arm of a couple

$$T = (QE) \times d \quad \ldots (1)$$

From the figure, we have,

$$\sin\theta = \frac{d}{l}$$

$$d = l\sin\theta \quad \ldots (2)$$

Substituting equation (2) in equation (1), we get,

Torque,

$$T = (QE) \times l\sin\theta$$

$$T = (Ql)\,E\sin\theta$$

$T = P_1 E\sin\theta$ in magnitude.

Where,

$P_1 = Ql$ = Dipole moment

In the vector form, torque can be written as,

$$\overline{T} = \overline{P_1} \times \overline{E}$$

UNIT 2

Conductors, Dielectrics And Capacitance

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Define current.

Ans:

Current

The continuous flow of free electrons through the medium under the influence of an external energy or force is known as an electric current. It is denoted by I or i.

(or)

The rate of flow of charges is defined as current.

It is denoted by I.

$$\boxed{I = \frac{Q}{t}}$$

Where,

Q = Charge in Coulombs and

t = Time in seconds.

Unit of current is Amperes (A)

Also, $i = \dfrac{dq}{dt}$

Where,

dq = Small change in charge and

dt = Small change in time.

Q2. Define current density. Mention the different types of current densities.

Ans:

Current Density

At a particular point, the current passing through a unit normal area is referred as the current density. It is denoted by 'J' and unit is Ampere/(meter)^2.

Types of Current Densities

The different types of current densities are,

1. Convection current density
2. Conduction current density
3. Displacement current density.

Q3. Define,

(i) Conduction current density

(ii) Convection current density.

Ans:

Conduction Current Density

It is defined as the conduction current at a given point, passing through a unit surface area, when the surface is normal to the direction of current.

Convection Current Density

It is defined as the convection current at a given point passing through a unit surface area normal to the direction of convection current.

Q4. Define current density. Write the relation between current and current density.

Ans: (Nov./Dec.-17, (R16), Q1(d) | Model Paper-I, Q1(c))

Current Density

For answer refer Unit-II, Q21, Topic: Current Density($\bar{J}$).

Relation between Current and Current Density

From above statement,

$$J = \frac{I}{A} \text{ Amp/m}^2$$

Where J = Current density in Amp/m^2

I – Current in amps

A = Area of cross section of conductor in m^2.

Q5. Derive Ohm's law in point form.

Nov./Dec.-18, (R16), Q1(d)

OR

State Ohm's law in point form.

Ans: Model Paper-III, Q1(c)

Ohm's Law

It states that the current (I) flowing through a conductor is directly proportional to the potential difference (V) between the ends of that conductor.

i.e., $\bar{I} \propto \bar{V}$

$$\Rightarrow \quad \bar{I} = \frac{1}{R}\bar{V} \quad \text{...(1)}$$

$$= G\bar{V}$$

Where,

G = Conductance, and is a proportionality constant its units is mho (℧)

R = Resistance measured in ohms (Ω)

$\therefore \quad \bar{V} = \bar{I}R$

Q6. State continuity equation.

Ans:

Continuity equation states that the charge, flowing outwards per second is equal to the rate of decrease of charge per unit volume.

i.e., $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$

Where,

$\nabla \cdot \bar{J}$ – Outward current flow

$\frac{\partial \rho_v}{\partial t}$ – Rate of decrease of charge per unit volume.

Q7. What are conductors and insulators? Give examples.

Nov./Dec.-18, (R16), Q1(c)

Ans:

Conductor

Materials having very high value of conductivity 'σ' are known as conductors. Conductors support a generous flow of charge, when a voltage source is applied across its terminals. Usually, conductors contain approximately 10^{21} free electrons per cubic centimeter of the material

Example of conductor is 'copper'.

Insulator

Materials having very low value of conductivity 'σ' are known as insulators. Insulators completely opposes the flow of charge, when a voltage source is applied across its terminals. Usually, insulators contain less than 10^{6} free electrons per cubic centimeter.

Examples of insulators are 'Mica', Glass etc.

Q8. List the properties of conductors.

Ans:

Some of the following properties of conductors are as follows,

- In a good conductor both electric and magnetic fields get reflected completely.
- At any point on the conductor the potential will be same.
- A good conductor will have high conductivity and low resistivity.
- There will be no electric field and magnetic field within a conductor.

Q9. Write atleast three properties of a dielectric material.

Ans: Model Paper-I, Q1(d)

Some of the following properties of a dielectric material are as follows:

- The dielectric materials will have infinite resistivity.
- No free electrons exists in dielectric materials.
- Both volume charge density (ρ_v) and conductivity are zero.
- Due to the presence of both electric and magnetic fields the dielectric materials penetrate freely.

Q10. Define the boundary conditions for the conductor - free space boundary in electrostatic and interface between two dielectrics.

Ans: Nov.-15, (R13), Q1(c)

Boundary conditions between conductor and free space in electrostatics

1. The tangential component of electric field intensity and electric flux density is zero at boundary.

$$\boxed{E_{tan} = D_{tan} = \epsilon_0 \epsilon_{tan} = 0}$$

2. The normal component of flux density is equal to the surface charge density and the flux leaves the surface normally.

$$\boxed{\begin{array}{l} D_N = P_s \\ \epsilon_N = \dfrac{P_s}{\epsilon_0} \end{array}}$$

Boundary conditions between two dielectrics

1. The normal components of electric field, $\bar{E}$ is continuous across the interface and inversely propostional to relative permittivities of two media.

$$\boxed{\frac{E_{n_1}}{E_{n_2}} = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_{r_2}}{\epsilon_{r_1}}}$$

2. The tangential components of are discontinuous across the boundary i.e., the tangential components of electric field intensity are equal.

$$\boxed{\frac{D_{t_1}}{D_{t_2}} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r_1}}{\epsilon_{r_2}}}$$

Q11. What are the boundary conditions of dielectrics?

Ans: Dec.-14, (R13), Q1(c)

Basically there are three types of boundary conditions. They are,

(i) The boundary conditions between two dielectrics:

$E_{t1} = E_{t2}$

$D_{n1} = D_{n2}$

(ii) The boundary conditions between a dielectrics and a conductor:

$E_{t1} = O$

$D_{n1} = \rho_s$

(iii) The boundary conditions between conductor and free space:

$E_t = O$

$D_n = \rho_s$

Q12. State Poisson's and Laplace's equation.

Ans: Model Paper-III, Q1(d)

Poisson's Equation

The Poisson's equation for homogeneous medium is defined as,

$$\nabla^2 V = \frac{-\rho_v}{\epsilon}$$

Where,

V – Potential function

ρ_v – Volume charge density

ϵ – Permittivity of medium.

Laplace's Equation

The Laplace's equation for a charge free region ($\rho_v = 0$) is defined as,

$$\nabla^2 V = 0$$

Q13. Write the Laplace equations in all three coordinates.

Ans:

The Laplace equations in all three coordinates are as follows,

In Cartesian coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Q14. What is electromotive force?

Ans:

The electrical pressure that tends to cause the flow of current in a circuit is known as electromotive force (E. M. F).

(or)

It can also be defined as the energy supplied by a source of electric power in driving a unit charge around the circuit.

Q15. Find the electric flux density in free space if the electric field, $E = 4a_x - a_y + 6a_z$ V/m.

Ans:

Given electric field is,

$E = (4a_x - a_y + 6a_z)$ V/m

To find,

Electric flux density, $D = ?$

The expression for electric flux density is given as,

$$D = \epsilon_0 E$$
$$= (8.854 \times 10^{-12})[4a_x - a_y + 6a_z]$$
$$= (35.41a_x - 8.854a_y + 53.12a_z) \times 10^{-12}\ \text{C/m}^2$$
$$= (35.41\,a_x - 8.854\,a_y + 53.12\,a_z)\ \text{pC/m}^2$$

Q16. Find the polarization, if a dielectric material of $\epsilon_r = 4.0$ is kept in an electric field $E = 2a_x + 4a_y + 3a_z$ V/m.

Ans:

Given electric field is,

$E = (2a_x + 4a_y + 3a_z)$ V/m

Relative permittivity,

$\epsilon_r = 4.0$

To determine,

Polarization, $P = ?$

Polarization in dielectric material is given as,

$$P = x_e \epsilon_0 E$$
$$= (4-1)(8.854 \times 10^{-12})(2a_x + 4a_y + 3a_z) \qquad [\because x_e = \epsilon_r - 1]$$
$$= (26.562 \times 10^{-12})(2a_x + 4a_y + 3a_z)$$
$$= (53.12\,a_x + 106.24\,a_y + 79.68\,a_z) \times 10^{-12}\ \text{C/m}^2$$
$$= (53.12\,a_x + 106.24\,a_y + 79.68\,a_z)\ \text{PC/m}^2$$

Q17. If the polarization $P = 3a_x$ nC/m² in a homogeneous and isotropic dielectric material whose $x_e = 3.2$, find E in the material.

Ans:

Given that,

Polarization,

$P = 3a_x$ nC/m²

Dielectric susceptibility,

$x_e = 3.2$

To find,

Electric field, $E = ?$

We know that,

Polarization, $P = x_e \epsilon_0 E$

$\therefore$ Electric field,

$$E = \frac{P}{x_e \epsilon_0}$$
$$= \frac{3\times10^{-9}a_x}{3.2\times8.854\times10^{-12}}$$
$$= 105.88\,a_x\ \text{V/m}$$

Q18. If the plates are separated by 1 mm in air and have potential difference of 1000 V. What is the energy stored per unit area?

Ans:

Given data,

Distance, $d = 1$ mm

$= 1\times10^{-3}$ m

Potential difference, $V = 1000$ V

To find,

Energy stored/unit area, $W_E = ?$

Capacitance, $C = \dfrac{S\epsilon_0}{d}$

(As the medium is air, $\epsilon = \epsilon_0$)

Capacitance/unit area, $\dfrac{C}{S} = \dfrac{\epsilon_0}{d}$

Where, $S \rightarrow$ Surface area of the plate

$$\therefore\ C = \frac{8.854\times10^{-12}}{1\times10^{-3}} \qquad [\because S = 1]$$
$$= 8.854\times10^{-9}\ \text{F}$$

Energy stored, $W_E = \dfrac{1}{2}CV^2$

$$= \frac{1}{2}\times 8.854\times10^{-9}\times(1000)^2$$
$$= 4.427\times10^{-3}\ \text{J}$$
$$\boxed{\therefore W_E = 4.427\times10^{-3}\ \text{J}}$$

Q19. Find the capacitance of an isolated sphere of diameter 4 cm.

Ans:

Given that,

Diameter of sphere,

$d = 4$ cm

$\therefore$ Radius,

$r = 2\ \text{cm} = 2\times10^{-2}$ m

To find,

Capacitance of the sphere,

$C = ?$

The capacitance of an isolated sphere is given as,

$$C = 4\pi\epsilon_0 r$$
$$= 4\pi\times8.854\times10^{-12}\times(2\times10^{-2})$$
$$= 2.225\times10^{-12}\ \text{F} = 2.225\ \text{pF}$$

PART-B
ESSAY QUESTIONS WITH SOLUTIONS
2.1 CURRENT AND CURRENT DENSITY

Q20. Derive an expression for current density.

Ans:

Conduction Current Density

It is defined as the conduction current at a given point, passing through a unit surface area, when the surface is normal to the direction of current.

It is denoted by $\bar{J}$ and measured in ampere per square meter (A/m²).

Explanation

Conduction current results in the conductor in the presence of an electric field ($\bar{E}$). Since, the conductor consists of a large amount of free electrons, these free electrons constitute a current, when an electric field is applied. Consequently, the positively charged particles experience a force, which is given by,

$$\bar{F} = Q\bar{E} \text{ N}$$

In a conductor, the charges are free to move thereby the charges are accelerated within the conductor which is given by,

$$\bar{a} = \frac{\bar{F}}{m} \text{ m/s}^2$$

Where,

m = Mass of charged particle in kg.

Due to the lattice structure of the conductor, the charged particles suffers collisions and lose part of their energy. Therefore, the particles will have an average velocity, which is nothing but drift velocity (V_d). The drift velocity of charged particle is similar to that of the electric field and is given by,

$$V_d = \mu \bar{E}$$

Where,

μ = Mobility of charged particles.

If the conductor is having charged particles of volume charged density ρ_v and uniform cross-section area 'A' then the conduction current passing through a given point is given by,

$$\bar{I} = \bar{V}_d \, \rho_v A \text{ C/s}$$

Where,

ρ_v = Charge density, C/m³
V_d = Drift velocity, m/s
A = Area of cross-section, m².

Now, the current density ($\bar{J}$) is defined as the conduction current passing through a unit surface area normal to the direction of conduction current i.e.,

$$\bar{J} = \frac{\bar{I}}{A}$$

$$\Rightarrow \quad \bar{J} = \frac{V_d \rho_v A}{A} = V_d \rho_v$$

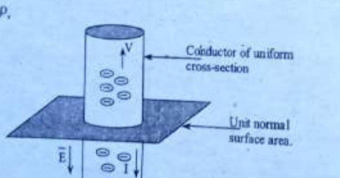


Figure (1): Representation of Conduction of Current Density

If current density is not uniform, then it is given by,

$$|\bar{J}| = \underset{\Delta s \to 0}{Lt} \frac{\Delta I}{\Delta s}$$

Where,

Δs = Incremental normal surface area.

Thus, the total current density is given by

$$I = \int_s \bar{J} . d\bar{s}$$

Convection Current Density

It is defined as the convection current at a given point, passing through a unit surface area normal to the direction of convection current.

It is measured in ampere per square meter (A/m²).

Explanation

Convection current is different from the conduction current, as the former does not involve conductors like the latter. Convection current occurs when the current flows through the insulating medium. For example, a beam of electrons in a Cathode Ray Tube (CRT).

The charge in motion contributes flow of current. This can be shown by the following.

Consider a cube of charge ΔQ placed on the XYZ coordinates as shown in the figure 2(a). Let 'Δl' represents the length of each side of a cube and 'Δa' represents the surface area. Let 'ρ_v' denotes the volume charge density of the cube.

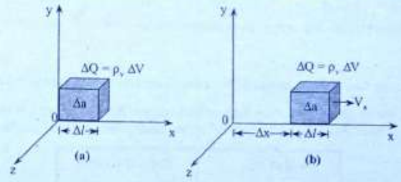


Figure (2): Illustration of Convection Current Density

The charge 'ΔQ' on the cube is given by,

$$\Delta Q = \rho_v \Delta V$$

$$\Delta Q = \rho_v \Delta a \, \Delta l \qquad \text{... (1)}$$

In incremental time Δt, let the cube is moved in X-direction to a distance of Δx as depicted in figure 2(b) the resultant current is given by,

$$\Delta I = \frac{\Delta Q}{\Delta t} = \int_V \frac{\Delta a \, \Delta x}{\Delta t}$$

$$\Delta I = \rho_v \Delta a \frac{\Delta x}{\Delta t}$$

$$\Delta I = \rho_v \, \Delta a \, \Delta x$$

Where, V_x = Component of velocity in X-direction.

By the definition, current density is given by,

$$|\bar{J}| = \frac{\Delta I}{\Delta a}$$

Considering the X-components, we can write,

$$|\bar{J}_x| = \frac{\rho_v \, \Delta a V_x}{\Delta a} \quad \text{... (2)}$$

$$|\bar{J}_x| = \rho_v V_x$$

In general, we can write equation (2) as,

$$\bar{J} = \rho_v \bar{V} \text{ A/m}^2$$

Where,

$\bar{J}$ = Convection current density or displacement current density.

Q21. Explain duality between $\overline{D}$ and $\overline{J}$. Model Paper-I, Q4(a)

Ans:

Electric Flux Density ($\overline{D}$)

It is defined as electric flux per unit surface area. It is denoted by $\overline{D}$. The unit of $\overline{D}$ is Coulomb/m².

Current Density ($\overline{J}$)

It is defined as electric current per unit area of cross-section. It is denoted by $\bar{J}$. The unit of current density is Amp/m².

Duality

Two entities are said to be dual if the equations of two systems are represented in same mathematical form.

In the similar fashion, the entities $\overline{D}$ and $\bar{J}$ are duals of each other as their equations and boundary condition can be represented in same mathematical form. From the table given below, the duality between them can be clearly seen.

Field of Flux Density ($\overline{D}$)	Field of Current Density ($\overline{J}$)
1. $D = \epsilon E$	1. $J = \sigma E$
2. $\overline{\nabla} D = 0$	2. $\overline{\nabla} J = 0$
3. $\overline{\nabla} \times D = 0$	3. $\overline{\nabla} \times J = 0$
4. $\frac{D_{t_1}}{\epsilon_1} = \frac{D_{t_2}}{\epsilon_2}$	4. $\frac{J_{t_1}}{\sigma_1} = \frac{J_{t_2}}{\sigma_2}$
5. $D_{n_1} = D_{n_2}$	5. $J_{n_1} = J_{n_2}$

From the equations given above, it can be seen that one set of equations can be obtained from the other set by just replacing $\overline{D}$ with $\bar{J}$, ϵ with σ and vice versa.

Application of Duality

Let us now understand the application of duality by comparing the relations of capacitance C and conductance G.

The total conductance for any particular conductor configuration in a medium of conductivity σ can be obtained by replacing ϵ with σ in the expression of capacitance for this conductor configuration.

We know that,

$$C = \frac{Q}{V}$$

$$\therefore \quad C = \frac{\oint_s Q_s . ds}{\int_1^2 E.dl}$$

But at the surface of conductor,

$$Q_s = \epsilon E_n$$

$$\therefore \quad C = \frac{\oint_s \epsilon E_n . dr}{\int_1^2 E.dl} \quad \text{... (1)}$$

For the case of current flow,

Conductance,

$$G = \frac{1}{R} = \frac{I}{V}$$

$$G = \frac{I}{V} = \frac{\oint_s J_n . ds}{\int_1^2 E.dl}$$

$$\therefore \quad G = \frac{\sigma \oint_s E_n . ds}{\int_1^2 E.dl} \quad \text{... (2)}$$

Comparing equations (1) and (2), we get,

$$\boxed{G = \frac{\sigma}{\epsilon} C} \quad \text{... (3)}$$

Hence, from the above equation it can be concluded that if we just know the value of capacitance C, it is possible to find the value of conductance (G) and resistance (R).

The capacitance of parallel plates is given by,

$$C = \frac{\epsilon A}{d}$$

We know that,

$$G = \frac{\sigma}{\epsilon} C$$

$$= \frac{\sigma}{\epsilon}\left(\frac{\epsilon A}{d}\right)$$

$$\boxed{\therefore G = \frac{\sigma A}{d}}$$

The capacitance of a concentric sphere for a single dielectric is,

$$C = 4\pi\epsilon\left(\frac{ab}{b-a}\right)$$

$$G = \frac{\sigma}{\epsilon} C$$

$$\therefore \quad G = \frac{\sigma}{\epsilon} \times 4\pi\epsilon\left(\frac{ab}{b-a}\right)$$

$$\therefore \quad G = 4\pi\sigma\left(\frac{ab}{b-a}\right)$$

and $R = \frac{1}{G}$

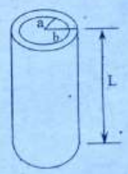
$$= \frac{1}{4\pi\sigma}\left(\frac{b-a}{ab}\right)$$

Q22. Determine the resista' ce of insulation in length 'L' of coaxial cable as inner and outer radii are 'a' and 'b' respectively.

Ans:

Derivation

Consider a coaxial cable of length 'L' meters with inner radius 'a' and outer radius 'b' as shown in figure below.



Figure

Let,

I = Current flowing from inner conductor to outer conductor

$\bar{J}_c$ = Conduction current density

σ = Conductivity

$\bar{E}$ = Electric field intensity

R = Resistance of insulation

V_{ab} = Potential difference between inner conductor and outer conductor

r = Radial distance.

Conduction current density is given by the relation,

$$\bar{J}_c = \sigma \bar{E}$$

$$\therefore \bar{E} = \frac{\bar{J}_c}{\sigma} \quad \text{... (1)}$$

But, conduction current density is defined as current per unit area,

i.e., $\bar{J}_c = \dfrac{I}{A}$

$$\bar{J}_c = \frac{I}{2\pi r L} \quad (\because \text{ Area of a cylinder, } A = 2\pi r L) \quad \text{... (2)}$$

Substitute equation (2) in equation (1), we get,

$$\bar{E} = \frac{\frac{I}{2\pi r L}}{\sigma}$$

$$\bar{E} = \frac{I}{2\pi r L\sigma} \quad \text{... (3)}$$

Now, resistance of insulation is given by,

$$R = \frac{V_{ab}}{I} \quad \text{... (4)}$$

V_{ab} can be obtained by integrating equation (3),

$$\therefore V_{ab} = -\int_b^a E.dr$$

$$V_{ab} = -\int_b^a \frac{I}{2\pi r L\sigma} dr$$

$$V_{ab} = \frac{-I}{2\pi L\sigma} \int_b^a \frac{1}{r} dr$$

$$V_{ab} = \frac{-I}{2\pi L\sigma} (\ln r)_b^a$$

$$V_{ab} = \frac{-I}{2\pi L\sigma} (\ln(a) - \ln(b))$$

$$V_{ab} = \frac{I}{2\pi L\sigma} (\ln(b) - \ln(a))$$

$$V_{ab} = \frac{I}{2\pi L\sigma} \ln\left(\frac{b}{a}\right) \quad \text{... (5)}$$

Substitute equation (5) in equation (4), we get,

$$R = \frac{\frac{I}{2\pi L\sigma} \ln\left(\frac{b}{a}\right)}{I}$$

$$\boxed{\therefore R = \frac{1}{2\pi L\sigma} \ln\left(\frac{b}{a}\right)}$$

Q23. Find the total current in a circular conductor of radius 4 mm if the current density varies according to $J = \dfrac{10^4}{r} \bar{a}_r$ A/m².

Ans: Nov.-15, (R13), Q4(b)

Given that,

Current density,

$$J = \frac{10^4}{r} \bar{a}_r \text{ A/m}^2$$

Radius of conductor,

$$r = 4 \text{ mm}$$
$$= 4 \times 10^{-3} \text{ m}$$
$$= 0.004 \text{ m}$$

Total current in conductor, I = ?

We know from continuity equation,

$$I = \int_s \bar{J} ds$$

$$ds = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$I = \int_s \left(\frac{10^4}{r} \bar{a}_r\right)(r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r)$$

$$= \int_s 10^4 r \sin\theta . d\theta . d\phi \qquad [\because \bar{a}_r . \bar{a}_r = 1]$$

$$= 10^4 r \int_{\theta=0}^{\pi/2} \sin\theta . d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 10^4 r [-\cos\theta]_0^{\pi/2} [\phi]_0^{2\pi}$$

$$= 10^4 r \left[-\cos\frac{\pi}{2} + \cos 0\right][2\pi - 0]$$

$$I = 10^4 r (2\pi)(1)$$

Substitute,

$$r = 4 \text{ mm} = 0.004 \text{ m}$$

$$\therefore I = 10^4 (0.004)(2\pi)$$

$$= 0.025 \times 10^4 \text{ Amps}$$

$$\boxed{I = 251.32 \text{ Amps}}$$

Q24. A circular conductor has an internal magnetic field $\bar{H} = \dfrac{1}{\rho}\left[\dfrac{1}{a^2}\sin(a\rho) - \dfrac{\rho}{a}\cos(a\rho)\right]\bar{a}_\phi$ A/m. Find the current density in the conductor.

Ans: Nov.-10, Set-1, Q8

Given that,

$$H = \frac{1}{\rho}\left(\frac{1}{a^2}\sin a\rho - \frac{\rho}{a}\cos a\rho\right) a_\phi \text{ A/m}$$

Here,

$$H = \bar{H}_\phi \text{ A/m}$$

$$H_\phi = \frac{1}{\rho}\left(\frac{1}{a^2}\sin a\rho - \frac{\rho}{a}\cos a\rho\right) a_\phi \quad \text{... (1)}$$

We know that,

$$\nabla \times H = J$$

$$J = \begin{vmatrix} a_\rho & a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & H_\phi & H_z \end{vmatrix}$$

$$J = \begin{vmatrix} a_\rho & a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & H_\phi & 0 \end{vmatrix}$$

$$= \left(-\frac{\partial}{\partial z} H_\phi\right) a_\rho - a_\phi (0 - 0) + a_z \left(\frac{\partial}{\partial \rho} H_\phi\right)$$

$$= \left(-\frac{\partial}{\partial z} H_\phi\right) a_\rho + \left(\frac{\partial}{\partial \rho} H_\phi\right) a_z$$

Since H_ϕ is independent of 'z',

$$\therefore \frac{\partial}{\partial z} H_\phi = 0$$

$$\therefore J = \left(\frac{\partial}{\partial \rho} H_\phi\right) a_z \quad \text{... (2)}$$

Substituting equation (1) in equation (2), we have,

$$\frac{\partial}{\partial z} H_\phi = \frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\left(\frac{1}{a^2}\sin a\rho - \frac{\rho}{a}\cos a\rho\right)\right]$$

$$= \frac{\partial}{\partial \rho}\left[\frac{1}{a^2 \rho}\sin a\rho - \frac{1}{a}\cos a\rho\right]$$

$$= \frac{1}{a^2}\left[\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\sin a\rho\right)\right] - \frac{1}{a}\left[\frac{\partial}{\partial \rho}\cos a\rho\right]$$

$$= \frac{1}{a^2}\left(\frac{a\cos a\rho}{\rho} - \frac{\sin a\rho}{\rho^2}\right) - \frac{1}{a}(-a\sin a\rho)$$

$$= \frac{\cos a\rho}{a\rho} - \frac{\sin a\rho}{a^2\rho^2} + \sin a\rho \quad \text{... (3)}$$

Substituting equation (3) in equation (2), we get,

$$\therefore J = \left\{\frac{\cos a\rho}{a\rho} - \frac{\sin a\rho}{a^2\rho^2} + \sin a\rho\right\} a_z$$

2.2 OHM'S LAW IN POINT FORM, CONTINUITY EQUATION

Q25. Derive an expression for Ohm's law in point form.

OR

State and explain Ohm's law in point form.

Ans: (Dec.-11, Set-2, Q3(a) | Model Paper-III, Q4(a))

Ohm's Law in Point Form

For answer refer Unit-II, Q5.

Derivation

Consider a cylindrical conductor shown in the figure. Let 'l' denotes the length of the conductor, 'S' be the cross-sectional area. Further, assume that the conductor has uniform current density ($\bar{J}$).

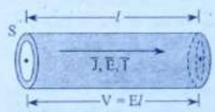


Figure: Current Carrying Cylindrical Conductor

When an electric field ($\bar{E}$) is applied, the conductor carries a current I proportional to the potential difference (V), which is in accordance with the Ohm's law.

Mathematically,

$$\bar{V} = \bar{I} R$$

$$\bar{I} = \frac{\bar{V}}{R} \quad \text{... (1)}$$

Where,

V = Potential difference between the ends of the cylindrical conductor.

$\overline{V} = \overline{E}\, l$

R = Resistance of the conductor.

The current flowing in the conducting block is given by, ... (2)

$\overline{I} = |\overline{J}|S$

Where,

$\overline{J}$ = Current density, A/m²

S = Cross-sectional area of conductor.

The resistance of the conductor is given by, ... (3)

$$R = \frac{l}{\sigma S}\,\Omega$$

Where, σ = Conductivity of the conductor.

Substituting equations (1) and (3) in equation (2), we get,

$$\overline{I} = \overline{J}.S$$

$$\Rightarrow \quad \frac{\overline{V}}{R} = \overline{J}\,S$$

$$\Rightarrow \quad \frac{\overline{E}l}{\left(\frac{l}{\sigma S}\right)} = \overline{J}\,S$$

$$\Rightarrow \quad \overline{E}\,\sigma = \overline{J}$$

$$\boxed{\overline{J} = \sigma\overline{E}}$$

The above equation is called as the Ohm's law in point form. Thereby, we can define Ohm's law in point form as "the current density at any point in a conducting medium is proportional to the electric field intensity ($\overline{E}$)".

$$\therefore \quad \overline{J} \propto \overline{E}$$

$$\Rightarrow \quad \overline{J} = \sigma\overline{E}$$

Where,

σ = Conductivity, a proportionality constant.

Q26. Obtain an expression for Ohm's law in point form and integral form.

Ans: Dec.-14, (R13), Q4

Expression for Ohm's Law in Point Form

For answer refer Unit-II, Q25. Topic : Derivation.

Expression for Ohm's Law in Integral Form

The expression for ohm's law in integral form can be obtained from point form i.e.,

$$\overline{J} = \sigma\overline{E} \qquad \text{... (4)}$$

Form equation (2), we have current density

$$\overline{J} = \frac{\overline{I}}{S}$$

Also,

$$\overline{E} = \frac{\overline{V}}{l} \qquad (\because \overline{V} = \overline{E}l)$$

Substituting these values in equation (4), we get.

$$\frac{\overline{I}}{S} = \sigma\left(\frac{\overline{V}}{l}\right)$$

$$\overline{I}\left(\frac{l}{\sigma S}\right) = \overline{V}$$

$$\overline{V} = \overline{I}R \qquad (\because \text{From equation (3)})$$

$\therefore$ The obtain equation represents the integral form of Ohm's law.

Q27. Derive equation of continuity. What is its significance?

Nov./Dec.-18, (R16), Q5(b)

OR

Derive the integral form of continuity equation and also write its meaning.

Ans: Model Paper-II, Q4(a)

Integral Form of Continuity Equation

Continuity equation refers to the "law of conservation of charge, which states that charge can neither be created nor be destroyed".

Consider a small region R bounded by closed surface as shown in the figure. Let Q denotes the charge enclosed by the surface. $\overline{J}$ denotes the current density and I is the current flowing through the surface.

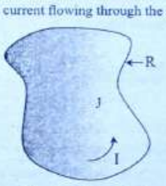


Figure: Current Flowing Through the Closed Surface

Current is given by the rate of change of charge. Current is flowing outwards from the closed surface causing the decrease in the current in the closed surface. The decrease in current is given by,

$$I = -\frac{dQ}{dt}\text{ Amps} \qquad \text{... (1)}$$

But, total current is given by

$$I = \oint_S \overline{J}.d\overline{S} \qquad \text{... (2)}$$

Substituting equation (2) in equation (1), we get,

$$\oint_S \overline{J}.d\overline{S} = -\frac{dQ}{dt} \qquad \text{... (3)}$$

According to the Divergence theorem, we have,

$$\oint_S \overline{J}.d\overline{S} = \int_v \nabla.\overline{J}.dv \qquad \text{... (4)}$$

Substituting equation (4) in equation (3), we get,

$$\int_v \nabla.\overline{J}.dv = \frac{-dQ}{dt} \qquad \text{... (5)}$$

But, the charge 'Q' is expressed as,

$$Q = \int_v \rho_v . dv$$

Where,

ρ_v = Volume charge density

Substituting the above value in equation (5), we get,

$$\int_v \nabla . \bar{J}\, dv = \frac{-d}{dt}\left[\int_v \rho_v . dv\right] \quad ...(6)$$

$$\boxed{\therefore \int_v \nabla . \bar{J}\, dv = -\int_v \frac{d\rho_v}{dt} . dv}$$

Here, the partial derivative is taken because the surface is constant and there is only one variable.

In equation (6), the two integrals are equal if, and only if the integrands are equal.

Therefore,

$$\nabla . \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

Equation (6) gives the integral form of continuity equation.

Meaning

In continuity equation, the term '$\nabla . \bar{J}$' signifies the outward current flow and the term $\frac{\partial \rho_v}{\partial t}$ denotes the rate of decrease of charge per unit volume.

On the whole, the continuity equation states that the "charge, flowing outwards per second is equal to the rate of decrease of charge per unit volume", which is in accordance with the law of conversation of charge.

Q28. z < 0 is a region of a linear dielectric of relative permittivity 6.5; and z > 0 is a free space. Electric field in the free space region is $(-3a_x + 4a_y - 2a_z)$ V/m. Find,

(i) $\bar{D}$ for z > 0

(ii) Tangential components of $\bar{D}$ & $\bar{E}$ for z < 0, on the boundary.

Ans:

Given data,

For $z < 0$: Relative primittivity, $\epsilon_{r1} = 6.5$

For $z > 0$: Medium is free space so relative permittivity, $\epsilon_{r2} = 1$

Electric field $\bar{E}_2 = (-3a_x + 4a_y - 2a_z)$ V/m

To determine,

(i) Electric flux density $\bar{D}$ for $z > 0$

(ii) The tangential components of Electric flux density $\bar{D}$ and electric field intensity $\bar{E}$ i.e., D_{t1} and E_{t1} for $z < 0$

$$z < 0 : \epsilon_{r1} = 6.5 \qquad D_{t1}, E_{t1}$$

$$\xrightarrow{\hspace{6em}} z$$

$$z > 0 : \epsilon_{r2} = 1 \qquad D_{t2}, E_{t2}$$

$$\overrightarrow{E_2} = -3a_x + 4a_y = 2a_z$$

(i) $\bar{D}$ for z > 0

$$\bar{D} = \epsilon \bar{E}$$

$$\bar{D} = \epsilon_o \epsilon_{r2} \bar{E}_2$$

For $z > 0$

$$\epsilon_{r2} = 1$$

$$\bar{E}_2 = -3a_x + 4a_y - 2a_z \text{ V/m}$$

$$\epsilon_o = 8.854 \times 10^{-12}$$

$$\therefore \quad \bar{D} = 8.854 \times 10^{-12}(-3a_x + 4a_y - 2a_z) = [-3\bar{a}_x + 4\bar{a}_y - 2\bar{a}_z] \times 8.854 \times 10^{-12}$$

$$\bar{D} = [-26.56\bar{a}_x + 35.41\bar{a}_y - 17.7\bar{a}_z] \times 10^{-12}$$

$$\boxed{\therefore \bar{D} = [-26.56\bar{a}_x + 35.41\bar{a}_y - 17.7\bar{a}_z] \times 10^{-12} \text{ C/m}^2}$$

(ii) D_{t1}, E_{t1} for z < 0

$$\bar{E}_{t1} = \bar{E}_1 - \bar{E}_{N1}$$

$\bar{E}_1$ for z < 0 is not known.

$$\therefore \quad \bar{E}_{t2} = \bar{E}_2 - \bar{E}_{N2}$$

$$\bar{E}_{N2} = \bar{E}_2 . \bar{a}_N \qquad \text{(Where, } E_{N2} = \text{Normal components of } \bar{E}_2\text{)}$$

$$\because \quad \bar{E}_2 = -3\bar{a}_x + 4\bar{a}_y - 2\bar{a}_z$$

For z < 0;

$$\bar{E}_{N2} = -2\bar{a}_z \text{ V/m}$$

$$\bar{E}_{t2} = \bar{E}_2 - \bar{E}_{N2}$$

$$= (-3\bar{a}_x + 4\bar{a}_y - 2\bar{a}_z) - (-2\bar{a}_z)$$

$$= -3\bar{a}_x + 4\bar{a}_y$$

$$|\bar{E}_{t2}| = [(-3)^2 + (4)^2]^{1/2}$$

$$|\bar{E}_{t2}| = [9 + 16]^{1/2}$$

$$|\bar{E}_{t2}| = [25]^{1/2}$$

$$|\bar{E}_{t2}| = 5 \text{ V/m}$$

For two linear dielectrics $E_{t1} = E_{t2}$ $\boxed{\therefore E_{t1} = 5 \text{ V/m}}$

$$\bar{D}_{t1} = D_1 - D_{N1}$$

We know that, D_2 for $z > 0$

$$E_{t1} = \frac{D_{t1}}{\epsilon_o \epsilon_{R1}}$$

$$\therefore \quad D_{t1} = E_{t1} \times \epsilon_o \times \epsilon_{R1}$$

$$E_{t1} = 5 \text{ V}$$

$$E_{R1} = 6.5$$

$$\epsilon_o = 8.854 \times 10^{-12}$$

$$D_{t1} = 5 \times 6.5 \times 8.854 \times 10^{-12}$$

$$D_{t1} = 2.877 \times 10^{-10}$$

$$D_{t1} = 0.2877 \times 10^{-9}$$

$$\boxed{D_{t1} = 0.287 \text{nC/m}^2}$$

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Q29. Given $\bar{J} = 10^3 \sin\theta\, \bar{a}_r$ A/m² in spherical co-ordinates, find the current crossing the spherical shell of $r = 0.02$ m, where r = Radius of shell.

Nov./Dec.-12, (R09), Q3(b)

Ans:

Given that,

Current density, $\bar{J} = 10^3 \sin\theta\, \bar{a}_r$ A/m²

Radius of spherical shells, $r = 0.02$ m

Current crossing the spherical shell, $I = ?$

From the integral form of continuity equation, we have,

Current, $I = \int_s \bar{J}.d\bar{s}$ Amps

Where,

$d\bar{s} = r^2 \sin\phi\, d\theta\, d\phi\, \bar{a}_r$

$$\therefore I = \int_s (10^3 \sin\theta \bar{a}_r)(r^2 \sin\theta.d\theta.d\phi\bar{a}_r)$$

$$= \int_s (10^3 \sin\theta)(r^2 \sin\theta)d\theta.d\phi$$

$$= \int_s 10^3 r^2 \sin^2\theta d\theta.d\phi$$

$$= 10^3 r^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin^2\theta d\theta$$

$$= 10^3 r^2[2\pi - 0] \int_{\theta=0}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta \qquad (\because \cos 2\theta = 1 - 2\sin^2\theta)$$

$$= 2\times 10^3 \pi r^2 \left[\frac{1}{2}\int_{\theta=0}^{\pi} d\theta - \frac{1}{2}\int_{\theta=0}^{\pi} \cos 2\theta.d\theta\right]$$

$$= 2\times 10^3 \pi r^2 . \frac{1}{2}\left[[\theta]_0^{\pi} - \left[\frac{\sin 2\theta}{2}\right]_0^{\pi}\right]$$

$$= 10^3 \pi r^2 \left[(\pi - 0) - \frac{1}{2}(\sin 2\pi - \sin 0)\right]$$

$$= 10^3 \pi r^2 [(\pi - 0)]$$

$$= 10^3 \pi^2 r^2$$

We have,

$r = 0.02$ m

$\therefore I = 10^3 \pi^2 (0.02)^2$

$= 3.948$ A

Q30. A boundary exists at z = 0 between two dielectrics $\in_{r_1} = 2.5$ region $z < 0$ and the $\in_{r_2} = 4$ region $z > 0$. The electric field intensity in region $z < 0$ is $E = -30I_x + 50I_y + 70I_z$ V/m. Find,

(i) E_{n_1}

(ii) E_{t_1}

(iii) D_{n_2}

(iv) D_{t_2}

(v) D_2

(vi) The angle 'α_1' between E_1 and normal to the surface

(vii) Polarization in $z > 0$ region

(viii) The angle 'α_2' between D_2 and normal to the surface.

Ans:

May/June-13, (R09), Q5(a)

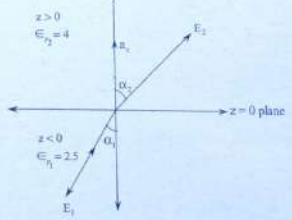
Given that,

Two dielectrics, $\in_{r_1} = 2.5 \qquad Z < 0$

$\in_{r_2} = 4 \qquad Z > 0$

Electric field intensity, $\bar{E} = -30\bar{a}_x + 50\bar{a}_y + 70\bar{a}_z$ V/m

The boundary between the two dielectrics is shown in figure,



Figure

(i) $\mathbf{E_{n_1}}$

The normal direction to the plane are $\pm\bar{a}_z$

The normal component of $\bar{E}_1 = \bar{E}_{n_1}$

$\therefore \bar{E}_{n_1} = 70\,\bar{a}_z$ V/m

$|\bar{E}_{n_1}| = E_{n_1} = 70$ V/m

(ii) $\mathbf{E_{t_1}}$

The tangential component of $\bar{E}_1 = \bar{E}_{t_1}$

$\therefore \bar{E}_{t_1}$ = x and y components of E_1

$\bar{E}_{t_1} = -30\bar{a}_x + 50\bar{a}_y$ V/m

$|\bar{E}_{t_1}| = E_{t_1} = \sqrt{(-30)^2 + (50)^2}$

$E_{t_1} = 58.31$ V/m

(iii) $\mathbf{D_{n_2}}$

The normal component of $\bar{D}_2 = \bar{D}_{n_2} = \bar{D}_{n_1}$

But, $\bar{D}_{n_2} = \epsilon_0 \epsilon_{r_2} \bar{E}_{n_2}$

According to the boundary conditions, we have,

$$\frac{\bar{E}_{n_1}}{\bar{E}_{n_2}} = \frac{\epsilon_{r_2}}{\epsilon_{r_1}}$$

$$\bar{E}_{n_2} = \frac{\bar{E}_{n_1} \times \epsilon_{r_1}}{\epsilon_{r_2}} = \frac{70 \times 2.5}{4} a_z$$

$\bar{E}_{n_2} = 43.75\, a_z$

$\bar{D}_{n_2} = 8.854 \times 10^{-12} \times 4 \times 43.75\, a_z = 1.55 \times 10^{-9} a_z$

$D_{n_2} = 1.55$ nC/m²

(iv) $\mathbf{D_{t_2}}$

The tangential component of $\bar{D}_2 = \bar{D}_{t_2}$

$\therefore \bar{D}_{t_2} = \epsilon_0 \epsilon_{r_2} \bar{E}_{t_2}$

But,

$\bar{E}_{t_2} = \bar{E}_{t_1} = -30\bar{a}_x + 50\bar{a}_y$

$\bar{D}_{t_2} = 8.854 \times 10^{-12} \times 4(-30\bar{a}_x + 50\bar{a}_y)$

$\bar{D}_{t_2} = -1.062 \times 10^{-9}\bar{a}_x + 1.77 \times 10^{-9}\bar{a}_y$

$|\bar{D}_{t_2}| = D_{t_2} = \sqrt{(-1.062\times 10^{-9})^2 + (1.77\times 10^{-9})^2}$

$= 2.064$ nC/m²

$D_{t_2} = 2.064$ nC/m²

(v) $\mathbf{D_2}$

We know that,

$\bar{D}_2 = \bar{D}_{t_2} + \bar{D}_{n_2}$

$\bar{D}_2 = -1.062 \times 10^{-9}\bar{a}_x + 1.77 \times 10^{-9}\bar{a}_y + 1.55 \times 10^{-9}\bar{a}_z$

$|\bar{D}_2| = D_2 = \sqrt{(-1.062\times 10^{-9})^2 + (1.77\times 10^{-9})^2 + (1.55\times 10^{-9})^2}$

$D_2 = 2.581$ nC/m²

(vi) The angle α_1 between E_1 and Normal to the Surface

The angle α_1 between E_1 and normal to the surface is given as,

$$\tan\alpha_1 = \frac{|E_{t_1}|}{|E_{n_1}|}$$

$$= \frac{58.31}{70}$$

$$\tan\alpha_1 = 0.833$$

$$\alpha_1 = \tan^{-1}(0.833)$$

$$\alpha_1 = 39.8°$$

(vii) Polarization in $z > 0$ Region

In $z > 0$ region $\epsilon_r = 4$

The polarization is given as,

$$\overline{P} = x_e \epsilon_0 \overline{E}$$

But, $x_e = \epsilon_r - 1$

$$= 4 - 1$$

$$x_e = 3$$

$$\therefore \quad \overline{P} = 3 \times 8.854 \times 10^{-12}(-30\overline{a}_x + 50\overline{a}_y + 70\overline{a}_z)$$

$$= -7.96 \times 10^{-10}\,\overline{a}_x + 1.328 \times 10^{-9}\,\overline{a}_y + 1.86 \times 10^{-9}\,\overline{a}_z$$

$$|\overline{P}| = P = \sqrt{(-7.96\times10^{-10})^2 + (1.328\times10^{-9})^2 + (1.86\times10^{-9})^2}$$

$$P = 2.42 \text{ nC/m}^2$$

(viii) The angle α_2 between D_2 and Normal to the Surface

The angle α_2,

$$\tan\alpha_2 = \frac{|E_{t_2}|}{|E_{n_2}|}$$

Since, $\dfrac{E_{t_2}}{E_{n_2}} = \dfrac{D_{t_2}}{D_{n_2}}$

$$\therefore \tan\alpha_2 = \frac{|D_{t_2}|}{|D_{n_2}|} = \frac{2.064\times10^{-9}}{1.55\times10^{-9}}$$

$$\tan\alpha_2 = 1.331$$

$$\alpha_2 = \tan^{-1}(1.331)$$

$$\alpha_2 = 53.09°$$

2.3 BOUNDARY CONDITIONS OF CONDUCTORS AND DIELECTRIC MATERIALS

Q31. Derive the boundary conditions of the normal and tangential components of electric field at the interface of two media with different dielectrics.

Nov.-15, (R13), Q5(a)

OR

Get the conditions at a boundary between two dielectrics.

Nov./Dec.-13, Q3(a)

OR

State and prove the conditions at the boundary between two dielectrics.

Dec.-11, Set-1, Q2(b)

OR

Obtain the dielectric boundary conditions at the boundary between two composite dielectrics.

Ans: Model Paper-I, Q5(a)

Dielectric Boundary Conditions between Two Composite Dielectrics

Consider the boundary between two composite dielectrics as shown in the figure (1). Let the permittivity of two mediums be ϵ_1 and ϵ_2 such that,

$$\epsilon_1 = \epsilon_0\epsilon_{r_1} \text{ and}$$

$$\epsilon_2 = \epsilon_0\epsilon_{r_2}$$

The electric flux densities in the two mediums be $\overline{D}_1$ and $\overline{D}_2$ and $\overline{E}_1$ and $\overline{E}_2$ represents electric field intensity in the two mediums respectively.

Consider a small pill box, so placed that it encloses both the mediums. Let Δs represents the surface area of top and bottom as shown in the figure (1). Assuming that there is no charge sheet present at the boundary between two dielectrics i.e., there is no free charge present at the interface.

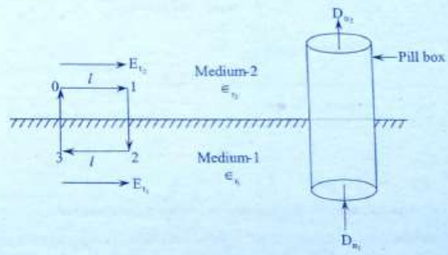


Figure (1): Boundary Between Two Dielectrics

According to Gauss's law, the total flux coming out of a charged body is equal to the charge enclosed by that body i.e.,

$$D_{n_1}\Delta s - D_{n_2}\Delta s = Q \qquad \text{... (1)}$$

Where, $D_{n_1}\Delta s$ = Flux entering the pill box and

$D_{n_2}\Delta s$ = Flux leaving the pill box.

But,

$$Q = \rho_s\Delta s \qquad (\because \rho_s = \text{Surface charge density})$$

Substituting 'Q' in equation (1), we get,

$$D_{n_1}\Delta s - D_{n_2}\Delta s = \rho_s\Delta s$$

$$\Rightarrow \quad D_{n_1} - D_{n_2} = \rho_s$$

The surface charge density $\rho_s = 0$ as there is no free charge between two perfect dielectrics.

$\therefore D_{n_1} - D_{n_2} = 0$

$$\boxed{D_{n_1} = D_{n_2}}$$

Therefore, the normal components of flux density $\overline{D}$ are continuous at the boundary between two perfect dielectrics.

As $\quad D = \in E$

$\therefore \quad D_{n_1} = \in_1 E_{n_1}$ and $D_{n_2} = \in_2 E_{n_2}$

$$\frac{D_{n_1}}{D_{n_2}} = \frac{\in_1 E_{n_1}}{\in_2 E_{n_2}} = 1$$

$$\boxed{\frac{E_{n_1}}{E_{n_2}} = \frac{\in_2}{\in_1} = \frac{\in_{r_2}}{\in_{r_1}}}$$

Therefore, the normal components of electric field $\overline{E}$ are inversely proportional to relative permittivities of two media.

The reflection of electric field intensity and electric flux density lines are shown in figure (2).

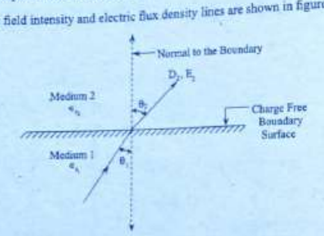


Figure (2): Reflection Phenomenon at the Interface of Two Dielectrics

Now, consider the small rectangular closed path '01230', as shown in the figure (1), so placed that it encloses both mediums. Let E_1 and E_2 be the electric field intensities in medium-1 and medium-2 respectively. Consequently, E_{t_1} and E_{t_2} be the tangential components of E_1 and E_2 respectively.

Since, the static electric field is a conservative field, the work done in moving a unit charge around the closed path is zero. Thus, we have,

$$\oint \overline{E}.d\overline{l} = 0$$

Applying the above equation to the closed path '01230', we get,

$$\int_0^1 E.dl + \int_1^2 E.dl + \int_2^3 E.dl + \int_3^0 E.dl = 0$$

Here, the work done in moving the charge from the path 1 to 2 and 3 to 0 is zero.

$$\therefore \quad \int_0^1 E.dl + \int_2^3 E.dl = 0$$

$$E_{t_1} \Delta l - E_{t_2} \Delta l = 0$$

$$E_{t_1} \Delta l = E_{t_2} \Delta l$$

$$\boxed{E_{t_1} = E_{t_2}}$$

Now, consider the flux density $\overline{D}$

As we know, $\overline{D} = \in \overline{E}$

$\therefore \quad D_{t_1} = \in_1 E_{t_1}$ and $D_{t_2} = \in_2 E_{t_2}$

Where, D_{t_1} and D_{t_2} are tangential components of flux density respectively.

$$\text{As, } E_{t_1} = E_{t_2} \Rightarrow \frac{D_{t_1}}{\in_1} = \frac{D_{t_2}}{\in_2}$$

$$\boxed{\frac{D_{t_1}}{D_{t_2}} = \frac{\in_1}{\in_2} = \frac{\in_{r_1}}{\in_{r_2}}}$$

Therefore, the tangential components of $\overline{D}$ are discontinuous across the boundary. i.e., the tangential components of electric field intensity are equal.

Q32. What are boundary conditions? Explain.

Ans:

Boundary Conditions

Electric field inside a particular medium is almost constant and there is a minute change from one point to the other. But when it moves from one medium to the other medium, there is a drastic change in the electric field at the boundary between these two mediums. Hence, boundary conditions can be defined as the conditions prevailing at the boundary of two mediums when the field moves from one medium to another.

There are basically three types of boundary conditions. They are,

(i) Boundary conditions between two dielectrics

(ii) Boundary condition between a dielectric and a conductor

(iii) Boundary condition between conductor and free space.

For remaining answer refer Unit-II, Q31.

Q33. One medium is a dielectric with permittivity $\in$, and the other is a conductor. Find the angle θ_1 between the normal and a field line in medium 1 incident on the conductor (medium 2).

Ans:

Refraction Phenomenon

The bending of lines of force at the boundary between two different mediums, say medium-1 and medium-2, is known as refraction phenomenon or law of refraction.

The magnitude and direction of lines of force may change sharply at the boundary.

Consider the two mediums namely, medium-1 and medium-2, with permittivity $\in_1$ and $\in_2$ respectively. Charge free plane is assumed at the boundary between the two mediums.

According to the law of refraction, the field lines gets refracted at the boundary as shown in figure (1) further, it is assumed that the nature of both the mediums is isotropic.

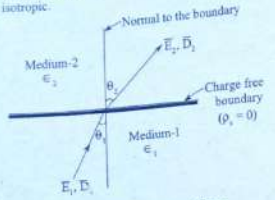


Figure (1): Bending of Lines of Force

Let θ_1, be the angle made by the field line with the normal in medium-1 and θ_2 be the angle made by the field line with the normal in medium-2.

We know that,

$$\overline{D_1} = \in_1 \overline{E_1} \text{ and}$$

$$\overline{D_2} = \in_2 \overline{E_2}$$

Where,

$\overline{D_1}$ = Flux density in medium-1

$\overline{E_1}$ = Field intensity in medium-1

$\overline{D_2}$ = Flux density in medium-2

$\overline{E_2}$ = Field intensity in medium-2.

Consider a Gaussian surface at the boundary between two mediums, such that it encloses both the mediums as shown in the figure (2). Let 'h' denotes the height of the Gaussian surface i.e., cylinder.

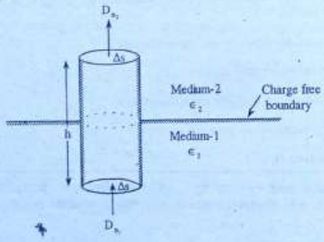


Figure (2): Gaussian Cylindrical Surface at the Boundary between Two Mediums

Applying Gauss's law to the cylinder, i.e.,

$$\int_s \overline{D}.ds = Q = \text{Charge enclosed.}$$

But, charge enclosed is given by,

$$Q = \int_{vol} \rho_v \, dv$$

$$\Rightarrow \int_s \overline{D}.ds = \int_{vol} \rho_v \, dv \qquad ...(1)$$

As we approach towards boundary, the height 'h' of the cylinder, gradually becomes zero and hence, the flux coming out of the curved surface of the cylinder is zero. Therefore, the net flux coming out of the cylinder is due to top and bottom surface of the cylinder. Therefore, on expanding the surface integral of equation (1), we get,

$$D_{n_1} \Delta s - D_{n_2} \Delta s = \rho_s \Delta s = 0$$

[$\because$ Charge free boundary is assumed]

$$\Rightarrow D_{n_1} \Delta s - D_{n_2} \Delta s = 0$$

$$\Rightarrow D_{n_1} = D_{n_2} \qquad ...(2)$$

If the medium-2 is a conductor, then flux density inside a conductor is zero. Therefore, we have,

$$D_{n_2} = 0$$

$$D_{n_1} = 0 \qquad ...(3)$$

Now, consider a rectangular coil $ABCDA$ placed at the boundary between the two mediums such that it encloses both the mediums as shown in the figure (3). Let 'Δl', 'Δw' denotes the length and breadth of the coil respectively.

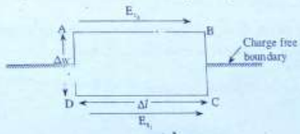


Figure (3): Rectangular Coil ABCDA at the Boundary between Two Mediums

Since, the static electric field is conservative in nature, the work done in moving unit charge around the closed path is zero. i.e.,

$$\oint \overline{E}.d\overline{l} = 0$$

Applying the above value to the coil $ABCDA$, we get,

$$\int_A^B \overline{E}.d\overline{l} + \int_B^C \overline{E}.d\overline{l} + \int_C^D \overline{E}.d\overline{l} + \int_D^A \overline{E}.d\overline{l} = 0$$

As we approach towards the boundary, the width 'Δw' becomes zero i.e., the second and third integral of above equation

[illegible]

From figure (3), it is to be noted that the lengths AB and CD of the coil are tangential to the boundary. Hence, on expanding the integral, we get,

$$E_{t_1} \Delta l - E_{t_2} \Delta l = 0$$

$$\Rightarrow E_{t_1} - E_{t_2} = 0$$

$$\therefore E_{t_1} = E_{t_2}$$

If medium-2 is a conductor, then the tangential component of electrical field intensity i.e., E_{t_2} is zero.

$$\therefore E_{t_1} = 0 \qquad ...(4)$$

From figure (1), we have,

$$\cos\theta_1 = \frac{D_{n_1}}{D_1} \Rightarrow D_{n_1} = D_1 \cos\theta_1$$

$$\sin\theta_1 = \frac{E_{t_1}}{E_1} \Rightarrow E_{t_1} = E_1 \sin\theta_1 \text{ and}$$

$$\cos\theta_2 = \frac{D_{n_2}}{D_2} \Rightarrow D_{n_2} = D_2 \cos\theta_2$$

$$\sin\theta_2 = \frac{E_{t_2}}{E_2} \Rightarrow E_{t_2} = E_2 \sin\theta_2$$

We have,

$$D_{n_1} = D_{n_2} \text{ and } E_{t_1} = E_{t_2}$$

$$\Rightarrow D_1 \cos\theta_1 = D_2 \cos\theta_2 \qquad ...(5)$$

and

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \qquad ...(6)$$

Dividing equation (6) by equation (5), we get,

$$\frac{E_1 \sin\theta_1}{D_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{D_2 \cos\theta_2}$$

$$\Rightarrow \left(\frac{E_1}{D_1}\right) \tan\theta_1 = \left(\frac{E_2}{D_2}\right) \tan\theta_2$$

$$\Rightarrow \quad \frac{\tan\theta_1}{\epsilon_1} = \frac{\tan\theta_2}{\epsilon_2} \qquad [\because D = \epsilon E]$$

$$\Rightarrow \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$\Rightarrow \quad \tan\theta_1 = \frac{\epsilon_1}{\epsilon_2}\tan\theta_2$$

$$\Rightarrow \quad \tan\theta_1 = \frac{D_1E_2}{D_2E_1}\tan\theta_2$$

$$\Rightarrow \quad \tan\theta_1 = 0\times\tan\theta_2 \quad [\because D_2 = 0; E_2 = 0]$$

$$\Rightarrow \quad \tan\theta_1 = 0$$

$$\Rightarrow \quad \theta_1 = 0$$

Therefore the angle θ_1 made by the field line with the normal in medium-1 is zero.

Q34. Explain and derive the boundary conditions for a conductor free space interface.

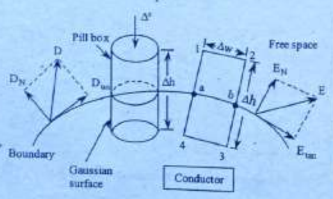
Nov.-15, (R13), Q4(a)

OR

Derive the boundary conditions between conductor and free space.

Ans: (Nov./Dec.-12, (R09), Q2(a) | Model Paper-III, Q5(a))

Consider the boundary between conductor and free space is as shown in figure,



Figure

To derive the boundary conditions for electric field intensity $(\overline{E})$ and for electric flux density, closed path and Gaussian surface are used respectively.

Electric Field Intensity $(\overline{E})$ at the Boundary

Let,

Δh be the elementary height

Δw be the elementary width

$\overline{E}$ be the electric field intensity

E_N be the electric field intensity to the normal component

E_{tan} be the electric field intensity to the tangential component.

Consider a rectangular closed path 1 - 2 - 3 - 4 - 1 as shown in figure. Here the sides 1-2 and 3-4 are in parallel to tangential direction and 1-4 and 2-3 are perpendicular to the surface at the boundary. The electric field intensity can be resolved in two components,

1. Tangential component of electric field intensity (E_{tan})
2. Normal component of electric flux intensity (E_N).

We know that,

$$\oint \overline{E}.\overline{dL} = 0 \qquad \text{... (1)}$$

Equation (1) can be divided into four parts i.e.,

$$\oint \overline{E}.\overline{dL} = \int_1^2 \overline{E}.\overline{dL} + \int_2^3 \overline{E}.\overline{dL} + \int_3^4 \overline{E}.\overline{dL} + \int_4^1 \overline{E}.\overline{dL} = 0 \qquad \text{... (2)}$$

From figure it is observed that the elementary height Δh is equally divided in to conductor and in free space i.e.,

$\frac{\Delta h}{2}$ is in conductor

$\frac{\Delta h}{2}$ is in free space.

Electric field intensity inside the conductor is zero and the path 3-4 is in conductor.

$$\therefore \int_3^4 \overline{E}.\overline{dL} = 0$$

Thus, equation (1) becomes,

$$\int_1^2 \overline{E}.\overline{dL} + \int_2^3 \overline{E}.\overline{dL} + \int_4^1 \overline{E}.\overline{dL} = 0 \qquad \text{... (3)}$$

The portion 1-2 is in free space and electric field intensity $\overline{E}$ over it can be assumed constant.

$$\int_1^2 \overline{E}.\overline{dL} = \overline{E}(\Delta w)$$

$$\int_1^2 \overline{E}.\overline{dL} = E_{tan}(\Delta w) \quad (\because \overline{E} = \overline{E}_{tan}) \qquad \text{... (4)}$$

Now, the portion 2-3 is parallel to normal component and $\overline{E}$ can be constant.

$$\therefore \int_2^3 \overline{E}.\overline{dL} = \int_2^3 E_N.\overline{dL} \qquad (\because \overline{E} = E_N)$$

$$= E_N \int_2^3 \overline{dL}$$

From figure, it can be observed that the portion 2-b is in free space and the portion b-3 is in conductor, where electric field intensity is zero.

$$\therefore \quad E_N \int_2^3 \overline{dL} = E_N\left[\int_2^b \overline{dL} + \int_b^3 \overline{dL}\right]$$

$$\int_2^3 \overline{dL} = \frac{\Delta h}{2} + 0$$

$$= \frac{\Delta h}{2} \qquad \text{... (5)}$$

$$\therefore \quad \int_2^3 \overline{E}.\overline{dL} = E_N\left(\frac{\Delta h}{2}\right)$$

As the path 1-4 and 2-3 are in parallel, the condition is same only, the direction is opposite.

$$\therefore \quad \int_4^1 \overline{E}.\overline{dL} = -E_N\left(\frac{\Delta h}{2}\right) \qquad \text{... (6)}$$

Substituting equations (4), (5) and (6) in equation (3), we get,

$$E_{tan}\Delta w + E_N\left(\frac{\Delta h}{2}\right) + \left(-E_N\left(\frac{\Delta h}{2}\right)\right) = 0$$

$$E_{tan}\Delta w + E_N\left(\frac{\Delta h}{2}\right) - E_N\left(\frac{\Delta h}{2}\right) = 0$$

$$E_{tan}\Delta w = 0$$

We know that,

$$\Delta w \neq 0$$

$$\boxed{\therefore E_{tan} = 0}$$

Therefore, the tangential component of electric field intensity is zero.

Flux Density $\overline{D}$ at the Boundary

Let,

Δh be the elementary height

Δs be the surface area of top and bottom of small pill box.

ρ_s be the surface charge density

Q be the charge

r be the radius of cylinder.

Consider a small pill box, so placed that it encloses both the medium as shown in figure. It can be observed that the elementary height is equally divided in the conductor and in free space.

According to Gauss's law,

$$\oint_s \overline{D}.\overline{ds} = Q \quad \text{... (7)}$$

The surface integral is divided into three surfaces.

$$\therefore \quad \oint \overline{D}.\overline{ds} = \int_{top} \overline{D}.\overline{ds} + \int_{bottom} \overline{D}.\overline{ds} + \int_{lateral} \overline{D}.\overline{ds} = Q \quad \text{... (8)}$$

The flux density inside the conductor is zero.

$$\therefore \quad \int_{Bottom} \overline{D}.\overline{ds} = 0$$

Equation (8) becomes,

$$\int_{top} \overline{D}.\overline{ds} + \int_{lateral} \overline{D}.\overline{ds} = Q \quad \text{... (9)}$$

The lateral surface area is,

$$= 2\pi r \Delta h$$

$$= 0 \quad (\because \Delta h = 0)$$

Therefore, the lateral integral becomes zero,

Equation (9) reduces to,

$$\int_{top} \overline{D}.\overline{ds} = Q$$

The top surface area is in free space and the electric flux density $\overline{D}$ over it can be assumed constant.

$$\therefore \int_{top} \overline{D}.\overline{ds} = \int_{top} D_N \overline{ds} \quad (\because \overline{D} = D_N)$$

$$= D_N \int_{top} \overline{ds}$$

$$= D_N \Delta s$$

According to Gauss's law,

$$D_N \Delta s = Q \quad \text{... (10)}$$

No charge can exists within a conductor. The charge appearing on the surface is in the form of surface charge density (ρ_s)

$$\therefore \quad Q = \rho_s \Delta s \quad \text{... (11)}$$

Equating equations (10) and (11), we get,

$$D_N \Delta s = \rho_s \Delta s$$

$$\boxed{D_N = \rho_s}$$

Therefore, the normal component of flux density is equal to the surface charge density.

Q35. Show that the electric field intensities on the two sides of a boundary and normal to the boundary will be uncharged, if the relative permittivity of the two sides of boundary is charged to unity and a surface density of charge,

$$\mathbf{r_s} = \left[\frac{\epsilon_{r_2} - \epsilon_{r_1}}{\epsilon_{r_1}}\right] \epsilon_0 \mathbf{E_2} = \left[\frac{\epsilon_{r_2} - \epsilon_{r_1}}{\epsilon_{r_2}}\right] \epsilon_0 \mathbf{E_1}$$

Where, ϵ_{r_1} and ϵ_{r_2} are the relative primitivities of the two mediums in which the normal electric field is E_1 and E_2 respectively.

Ans:

To show that the electric field intensities on the two sides of a boundary will be uncharged if their permittivities are charged to unity.

Consider the case in which E_1 and E_2 represents the electric field intensities in the two mediums respectively and ϵ_{r_1} and ϵ_{r_2} be their corresponding relative permittivities. Assume that the boundary interface between the two media is charge free as shown in figure (1).

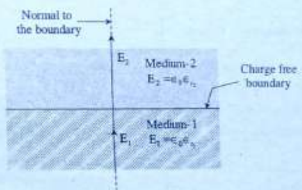


Figure (1): Electric Field Intensities

For a charge free boundary, the normal components of electric flux densities, in both the mediums are equal.

i.e., $D_{n_1} - D_{n_2} = 0$

$$\Rightarrow D_{n_1} = D_{n_2} \quad \text{... (1)}$$

here,

D_{n_1} = Normal component of flux density in medium-1

D_{n_2} = Normal component of flux density in medium-2

But, flux density 'D', is given by,

$$D = \epsilon_0 \epsilon_r E \quad \text{... (2)}$$

Substituting equation (2) in equation (1), we get,

$$\epsilon_0 \epsilon_{r_1} E_1 = \epsilon_0 \epsilon_{r_2} E_2 \quad \text{... (3)}$$

Now, consider the case in which the relative permittivities of both the mediums are charged to unity, i.e.,

$$\epsilon_{r_1} = \epsilon_{r_2} = 1$$

Substituting the above condition in equation (3), we get,

$$\epsilon_0 \epsilon_{r_1} E_1 = \epsilon_0 \epsilon_{r_2} E_2$$

$$\Rightarrow \quad \epsilon_0 (1) E_1 = \epsilon_0 (1) E_2$$

$$\therefore \quad E_1 = E_2$$

Figure (2), illustrates the above condition.

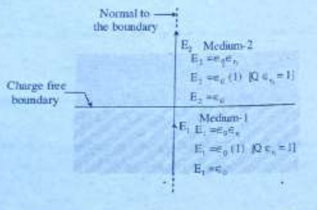


Figure (2): Electric Field Intensities after Charging ϵ_r to Unity

Let a charge sheet of surface charge density ρ_s C/m^2 is placed at the boundary between the two mediums, so that the field intensity in medium-2 remains same as that of in the previous case. Due to this surface charge density, the boundary condition changes i.e.,

$$D_{n_1} - D_{n_2} = \rho_s$$

$$\Rightarrow \quad D_1 - D_2 = \rho_s \quad [\because D_{n_1} = D_1 \text{ and } D_{n_2} = D_2]$$

$$\Rightarrow \quad \epsilon_0 \epsilon_{r_1} E_1 - \epsilon_0 \epsilon_{r_2} E_2 = \rho_s \quad \text{[From equation (2)]}$$

$$\Rightarrow \quad \epsilon_0 E_1 - \epsilon_0 E_2 = \rho_s \quad [\because \epsilon_{r_1} = \epsilon_{r_2} = 1] \quad \text{... (4)}$$

But, from equation (3), we have,

$$\epsilon_0 E_1 = \frac{\epsilon_0 \epsilon_{r_2} E_2}{\epsilon_{r_1}}$$

Substituting the above value in equation (4), we get,

$$\Rightarrow \quad \frac{\epsilon_0 \epsilon_{r_2} E_2}{\epsilon_{r_1}} - \epsilon_0 E_2 = \rho_s$$

$$\epsilon_0 E_2 \left[\frac{\epsilon_{r_2}}{\epsilon_{r_1}} - 1\right] = \rho_s$$

$$\epsilon_0 E_2 \left[\frac{\epsilon_{r_2} - \epsilon_{r_1}}{\epsilon_{r_1}}\right] = \rho_s \quad \text{... (5)}$$

Similarly, from equation (3), we have,

$$\epsilon_0 E_2 = \frac{\epsilon_0 \epsilon_{r_1} E_1}{\epsilon_{r_2}}$$

Substituting the above value in equation (4), we get,

$$\epsilon_0 E_1 - \frac{\epsilon_0 \epsilon_{r_1} E_1}{\epsilon_{r_2}} = \rho_s$$

$$\Rightarrow \quad \epsilon_0 E_1 \left[1 - \frac{\epsilon_{r_1}}{\epsilon_{r_2}}\right] = \rho_s$$

$$\epsilon_0 E_1 \left[\frac{\epsilon_{r_2} - \epsilon_{r_1}}{\epsilon_{r_2}}\right] = \rho_s \quad \text{... (6)}$$

$\therefore$ From equations (5) and (6), we can conclude that,

$$\rho_s = \epsilon_0 E_1 \left[\frac{\epsilon_{r_2} - \epsilon_{r_1}}{\epsilon_{r_2}}\right]$$

$$= \epsilon_0 E_2 \left[\frac{\epsilon_{r_2} - \epsilon_{r_1}}{\epsilon_{r_1}}\right]$$

Hence proved.

2.4 CAPACITANCE, CAPACITANCE OF A TWO WIRE LINE

Q36. Derive the expression for the energy stored in the charged condenser.

Nov./Dec.-18, (R16), Q4(a)

OR

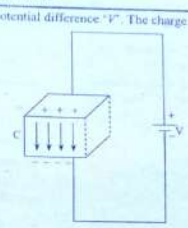
Derive the expression for the energy stored in a capacitor.

Ans:

Energy Stored in a Capacitor

Capacitor is a device which acquires charge, when it is connected to a source of potential. Therefore, a capacitor can store energy and this energy gets stored in the electrostatic field of the capacitor.

Consider a capacitor 'C' charged to a potential difference 'V'. The charge 'Q' is given by,



Figure

$$Q = CV \quad \text{... (1)}$$

Potential is the work done per unit charge. It is given as,

$$V = \frac{dW}{dQ} \quad \text{... (2)}$$

Where,

dW = Infinitesimal work done

dQ = Infinitesimal charge

From equations (1) and (2), we get,

$$dW = V\,dQ$$

$$= \frac{Q}{C} dQ \quad \left(\because V = \frac{Q}{C}\right) \quad \text{... (3)}$$

The charge 'Q' is at a zero value at the starting and the charging proceeds until a final value of the charge 'Q' is delivered.

$\therefore$ Total work done in charging the capacitor is,

$$W = \int dW$$

$$= \int_0^Q \frac{Q}{C} dQ = \frac{1}{C} \int_0^Q Q\,dQ$$

$$= \frac{1}{C}\left[\frac{Q^2}{2}\right]_0^Q$$

$$= \frac{1}{2C}[Q^2 - 0]$$

$$= \frac{Q^2}{2C} \text{ Joules}$$

$$\frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2 \quad [\because Q = CV]$$

Energy stored in a capacitor is given by,

$$\boxed{W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV} \text{ Joules}$$

Q37. (i) Define capacitance. Express its units in 2 different ways.

(ii) As per the usual definition, show that a capacitance is always positive.

(iii) Sometimes, capacitance of a single conductor is referred to, what does this mean?

Ans:

(i) **Capacitance**

Capacitance is defined as the ratio of the amount of charge stored by the device to the potential across device

i.e., $C = \frac{Q}{V}$ C/V ...(1)

So, the unit of capacitance is coulomb per volt.

From Gauss's law,

$$Q = \iint_s \overline{D}.\overline{ds} = \in \iint_s \overline{E}.\overline{ds} \quad ...(2)$$

Where,

$\in$ = Permittivity of the medium between the plates

From the definition of potential,

$$V = -\int_A^B \overline{E}.\overline{dL} \quad ...(3)$$

Substituting equations (2) and (3) in equation (1), we get,

$$\therefore \quad C = \frac{\in \iint \overline{E}.\overline{ds}}{-\int_A^B \overline{E}.\overline{dL}} \text{ F/m} \quad ...(4)$$

Although, equation (1) relates capacitance with charge and voltage, but the capacitance is independent of both charge and voltage and depends only on the physical dimensions of device and the properties of mediums present (i.e., dielectric) as shown in from equation (4).

From equation (4) capacitance can be expressed in Farad/meter.

$\therefore$ The two units of capacitance are,

1. Coulomb/Volt
2. Farad/Meter.

(ii) The capacitance is usually defined as the magnitude of the charge on the body (since, the other is equal in magnitude) divided by the magnitude of the potential difference between the two bodies.

i.e., $C = \frac{|Q|}{|V_B - V_A|}$

Hence, capacitance is always positive.

iii) Whenever, the capacitance of a single conductor is referred it means that the given conductor is assumed to have a positive charge or positive charge density with zero reference at infinity i.e., the conductor possess the positive potential with zero potential reference at infinity.

Q38. Derive the expression for capacitance of a two-wire line.

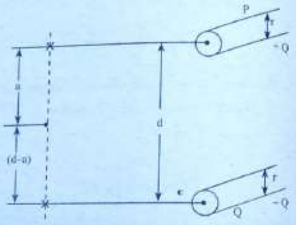
OR

Find the capacitance between two parallel conductors. The radius of conductor is 'r' separated by a distance 'd' m. Both wires are carrying the current in opposite direction.

Model Paper-II, Q5(b)

Ans:

Consider two parallel wires 'P' and 'Q' with each of radius 'r' m separated by a distance 'd' m such that 'd' is greater than 'r' and carry opposite polarity charge of '$+Q$' and '$-Q$' C/m as shown in figure.



Figure

Consider a straight line path of integration from 'Q' to 'P' in order to calculate the potential difference between P and Q.

Let 'E_a' be the field intensity at any point 'S' on this line at a distance of 'a' from 'P' and at a distance of $(d - a)$ from 'Q'.

Then, the potential difference between the points P and Q is given by,

$$V_{PQ} = -\int_{d-r}^{r} E_a \, da \quad ...(1)$$

Now,

The electric field due to a positive charge 'Q' at a distance of 'a' directed from 'P' to 'Q' is given by,

$$E_1 = \frac{\rho_L}{2\pi\epsilon_0 a}$$

Similarly, the electric field due to a negative charge '$-Q$' at a distance of '$(d-a)$' directed from 'P' to 'Q' is given by,

$$E_2 = \frac{\rho_L}{2\pi\epsilon_0 (d-a)}$$

The total field at point 'S' is given by,

$$E_a = E_1 + E_2$$

$$= \frac{\rho_L}{2\pi\epsilon_0 a} + \frac{\rho_L}{2\pi\epsilon_0 (d-a)} \quad ...(2)$$

Substituting equation (2) in equation (1), we get,

$$V_{PQ} = -\int_{(d-r)}^{r} \frac{\rho_L}{2\pi\epsilon_0 a} + \frac{\rho_L}{2\pi\epsilon_0 (d-a)} da$$

$$= \frac{-\rho_L}{2\pi\epsilon_0} \int_{d-r}^{r} \left(\frac{1}{a} + \frac{1}{d-a}\right) da$$

$$= \frac{-\rho_L}{2\pi\epsilon_0} [\ln a + \ln(d-a)(-1)]_{d-r}^{r}$$

$$= \frac{-\rho_L}{2\pi\epsilon_0} [\ln a - \ln(d-a)]_{d-r}^{r}$$

$$= \frac{-\rho_L}{2\pi\epsilon_0} [(\ln(r) - \ln(d-r)) - (\ln(d-r) - \ln(d-d+r))] \ln(d-d+r))]$$

$$= \frac{-\rho_L}{2\pi\epsilon_0} \left[\ln\frac{r}{d-r} - \ln\frac{d-r}{r}\right]$$

$$= \frac{-\rho_L}{2\pi\epsilon_0} \left[\ln\left(\frac{r}{d-r}\right) + \ln\left(\frac{r}{d-r}\right)\right]$$

$$= \frac{\rho_L}{2\pi\epsilon_0} \left[-\ln\left(\frac{r}{d-r}\right) - \ln\left(\frac{r}{d-r}\right)\right]$$

$$= \frac{\rho_L}{2\pi\epsilon_0} \left[-\ln\left(\frac{d-r}{r}\right) + \ln\left(\frac{d-r}{r}\right)\right]$$

$$= \frac{\rho_L}{2\pi\epsilon_0} \left[2\ln\left(\frac{d-r}{r}\right)\right]$$

$$V_{PQ} = \frac{\rho_L}{\pi\epsilon_0} \ln\frac{d-r}{r}$$

As $d >> r$, we have,

$$\frac{d-r}{r} \cong \frac{d}{r}$$

$$V = \frac{\rho_L}{\pi\epsilon_0} \ln\frac{d}{r}$$

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Now,

The capacitance of two parallel wires 'P' & 'Q' is given by,

$$C_{PQ} = \frac{\rho_L}{V}$$

$$= \frac{\rho_L}{\frac{\rho_L}{\pi\epsilon_0}\ln\frac{d}{r}}$$

$$\therefore \quad C_{PQ} = \frac{\pi\epsilon_0}{\ln\frac{d}{r}} \text{ F/m}$$

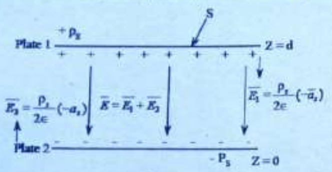
The capacitance per km length is given by,

$$C = \frac{0.01205}{\log_{10}\frac{d}{r}} \mu F$$

Q39. Derive the capacitance of a parallel plate capacitor.

Ans:

Consider a parallel plate capacitor as shown in figure.



Figure

Let the area of the plate be 'S' m^2 and 'd' be the distance between plate 1 and plate 2. Both the plates carry charges $+Q$ and $-Q$ respectively. The space between the plates is filled with dielectric of permittivity 'ϵ'.

Assuming the plates to be ideal and field to be uniform, the electric field intensity from Gauss's law is given as,

$$\bar{E}_1 = \frac{\rho_s}{2\epsilon}(-\bar{a}_z) \rightarrow \text{(below the surface charge } (+\rho_s))$$

$$\bar{E}_2 = \frac{\rho_s}{2\epsilon}(-\bar{a}_z) \rightarrow \text{(above the surface charge } (-\rho_s))$$

Where, ρ_s = Surface charge density.

The total electric field intensity is given by,

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

$$\bar{E} = \frac{\rho_s}{2\epsilon}(-\bar{a}_z) + \frac{\rho_s}{2\epsilon}(-\bar{a}_z)$$

$$\bar{E} = \frac{\rho_s}{\epsilon}(-\bar{a}_z)$$

The surface charge density,

$$\rho_s = \frac{Q}{S} \text{ C/m}^2$$

$$\therefore \quad \bar{E} = \frac{Q}{S\epsilon}(-\bar{a}_z) \text{ V/m}$$

$$d\bar{l} = dz\bar{a}_z$$

We know that,

Potential,

$$V = -\int_2^1 \bar{E}.d\bar{l}$$

$$V = -\int_{z=0}^{z=d} \frac{Q}{S\epsilon}(-\bar{a}_z).dz\bar{a}_z \quad [\because d\bar{l} \text{ is along } d\bar{z}]$$

$$V = \frac{Q}{S\epsilon}\int_{z=0}^{z=d} (\bar{a}_z.\bar{a}_z)dz$$

$$V = \frac{Q}{S\epsilon}(z)_{z=0}^{z=d} \qquad [\because \bar{a}_z.\bar{a}_z = 1]$$

$$\therefore \quad V = \frac{Qd}{S\epsilon}$$

The capacitance,

$$C = \frac{Q}{V}$$

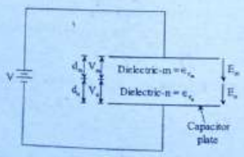
$$\therefore \quad C = \frac{Q}{\frac{Qd}{S\epsilon}}$$

$$\boxed{\therefore C = \frac{S\epsilon}{d} \text{ Farads}}$$

Q40. Deduce an expression for the capacitance of a parallel plate capacitor having two dielectric media.

Ans:

Consider a parallel plate capacitor with two different and perfect dielectrics, m and n in between the plates with relative permittivities of ϵ_{r_m} and ϵ_{r_n} respectively as shown in the figure.



Figure

Let,

E_m and E_n be the electric field intensities

V_m and V_n be the potential difference between the plates

a be the area covered by the plates

d_m and d_n be the thickness of dielectrics.

From the figure, we have,

$$V = V_m + V_n$$

$$= E_m d_m + E_n d_n \; [\because V = Ed]$$

$$= \frac{D_{N_m} d_m}{\epsilon_m} + \frac{D_{N_n} d_n}{\epsilon_n} \quad \left[\because E = \frac{D_N}{\epsilon}\right]$$

Where, D_{N_m} and D_{N_n} are the normal components of flux densities at the dielectrics m and n respectively.

But, from the boundary conditions, we know that at the boundary between two perfect dielectrics the normal component of electric flux density (D_N) is continuous.

Thus,

$$D_{N_m} = D_{N_n} = D \text{ (say)}$$

$$\therefore \quad V = \frac{Dd_m}{\epsilon_m} + \frac{Dd_n}{\epsilon_n} = D\left[\frac{d_m}{\epsilon_m} + \frac{d_n}{\epsilon_n}\right]$$

$$= \frac{Q}{A}\left[\frac{d_m}{\epsilon_m} + \frac{d_n}{\epsilon_n}\right] \qquad \left[\because D = \frac{Q}{A}\right]$$

By the definition of capacitance, we have,

$$C = \frac{Q}{V}$$

$$= \frac{Q}{\frac{Q}{A}\left[\frac{d_m}{\epsilon_m} + \frac{d_n}{\epsilon_n}\right]}$$

$$= \frac{1}{\frac{d_m}{A\epsilon_m} + \frac{d_n}{A\epsilon_n}} \quad \dots (1)$$

$$= \frac{1}{\frac{1}{C_m} + \frac{1}{C_n}} \qquad \left[\because C = \frac{\epsilon A}{d}\right]$$

$$\Rightarrow \quad \boxed{\frac{1}{C} = \frac{1}{C_m} + \frac{1}{C_n}} \quad \dots (2)$$

Where,

C_m = Capacitance contributed by dielectric m

$$= \frac{\epsilon_m A}{d_m} = \frac{\epsilon_0\epsilon_{r_m} A}{d_m}$$

C_n = Capacitance contributed by dielectric n

$$= \frac{\epsilon_n A}{d_n} = \frac{\epsilon_0\epsilon_{r_n} A}{d_n}$$

Equation (2) is the generalized formula and is same as the total capacitance equation of the series connected capacitors.

Now, equation (1) can be written as,

$$C = \frac{1}{\frac{d_m}{A\epsilon_0\epsilon_{r_m}} + \frac{d_n}{A\epsilon_0\epsilon_{r_n}}}$$

$$= \frac{1}{\frac{1}{A\epsilon_0}\left[\frac{d_m}{\epsilon_{r_m}} + \frac{d_n}{\epsilon_{r_n}}\right]}$$

$$\therefore \quad \boxed{C = \frac{\epsilon_0 A}{\frac{d_m}{\epsilon_{r_m}} + \frac{d_n}{\epsilon_{r_n}}}}$$

Q41. Find the capacitance for a coaxial capacitor with inner radius a and outer radius b with length L.

OR

A cylindrical capacitor consists of an inner conductor of radius 'a' and an outer conductor whose inner radius is 'b'. The space between conductors is filled with dielectric permittivity ϵ_r and length of capacitor is L. Find the value of the capacitance.

OR

Derive an expression for capacitance of cylindrical capacitor.

Ans:

Consider a coaxial capacitor with outer and inner conductors. Let 'L' be the length, 'a' be the inner radius and 'b' be the outer radius (where $b > a$) as shown in figure below.

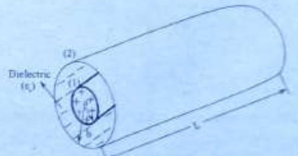


Figure: Coaxial Capacitor with Inner and Outer Conductor Radii as a and b Respectively

Let us assume that outer and inner conductors carry a uniform charge of $+Q$ and $-Q$ respectively. Also assume that a homogeneous dielectric with permittivity is filled between two conductors.

By applying Gaussian law to an arbitrary Gaussian cylindrical surface of radius ρ, where ρ lies between a and b i.e., $a < \rho < b$, we get,

$$Q = \epsilon_r \oint E\, ds$$

$$= \epsilon_r E 2\pi\rho L \quad \ldots (1)$$

Where, L is the length and

ρ is the radius of cylindrical surface

From equation (1), we get,

$$E = \frac{Q}{2\pi\epsilon_r \rho L} a_\rho$$

Where, a_ρ is a unit vector.

By neglecting flux at the cylindrical ends, we get,

$$V = -\int_2^1 E \cdot dl$$

$$= -\int_b^a \left(\frac{Q}{2\pi\epsilon_r \rho L} a_\rho\right) \cdot d\rho \cdot a_\rho \quad [\text{Here, } dl = d\rho \cdot a_\rho]$$

The negative sign here indicates limits from outer to inner radius i.e., from b to a.

$$V = -\int_b^a \left(\frac{Q}{2\pi\epsilon_r \rho L} a_\rho\right) \cdot d\rho \cdot a_\rho$$

$$= \frac{Q}{2\pi\epsilon_r L} \int_a^b \frac{1}{\rho} d\rho \quad [\text{As } a_\rho \text{ is a unit vector}]$$

$$= \frac{Q}{2\pi\epsilon_r L} [\ln(\rho)]_a^b \quad \left[\because \int \frac{1}{x} dx = \ln(x)\right]$$

$$= \frac{Q}{2\pi\epsilon_r L} [\ln(b) - \ln(a)]$$

$$V = \frac{Q}{2\pi\epsilon_r L}\left[\ln\left(\frac{b}{a}\right)\right] \quad \ldots (1)$$

The expression for capacitance of a cylindrical conductors is given as,

$$\text{Capacitance, } C = \frac{Q}{V} \quad \ldots (2)$$

On substituting equation (1) in equation (2), we get,

$$C = \frac{2\pi\epsilon_r L}{\ln\left(\frac{b}{a}\right)}$$

Therefore, the capacitance of a coaxial cylinder is given by,

$$C = \frac{2\pi\epsilon_r L}{\ln\left(\frac{b}{a}\right)}$$

Q42. Derive the capacitance of a spherical capacitor.

Ans:

Consider a spherical capacitor consisting of two concentric conducting shells with radii of inner shell a_1 and the radii of outer shell a as shown in the figure.

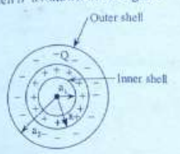


Figure: Concentric Spherical Shell

Let a charge 'Q' is placed on the outer surface of the inner shell of radius a_1. This charge will then induce an equal and opposite sign charge (i.e., $-Q$) on the inner surface of the outer shell of radius a_2.

Let V be the potential difference between the two spheres. Here, the electric field $(\overline{E})$ due to the charge 'Q' is directed radially outwards at all points throughout the surface of the inner sphere. Similarly, the electric flux density $(\overline{D})$ also acts in the same direction i.e., radially outwards.

Consider the surface of the sphere at a radius 'R' such that $a_1 < R < a_2$ as shown in the figure.

According to the Gauss's law,

Total flux diverging from the surface = Total charge enclosed within the sphere.

i.e., Total electric flux, $\phi = \int_s D\, ds$

Where,

$\int_s ds$ – Total surface area of spheres.

We know that, total surface area of sphere is $4\pi R^2$.

$$\therefore \int_s ds = 4\pi R^2$$

Where,

R = Radius of sphere considered.

$\therefore$ Total electric flux, $\phi = 4\pi R^2 D$.

Since, the total flux is created by the total charge enclosed by the sphere,

$$Q = 4\pi R^2 D \quad [\because \phi = Q]$$

$$\Rightarrow D = \frac{Q}{4\pi R^2} \quad \ldots (1)$$

By the definition of electric flux density, we have,

$$\overline{D} = \epsilon \overline{E}$$

Where,

$\overline{E}$ – Electric field intensity.

$$\therefore \overline{E} = \frac{\overline{D}}{\epsilon} \quad \ldots (2)$$

Substituting equation (1) in equation (2), we get,

$$\overline{E} = \frac{Q}{4\pi\epsilon R^2}$$

$$\text{Since, } E = \frac{-dv}{dR}$$

$$\Rightarrow dv = -E dR \quad \ldots (3)$$

Substituting the value of E in equation (3), we get,

$$dv = \frac{-Q}{4\pi\epsilon R^2} dR$$

The total voltage obtained in moving a charge 'Q' from a_2 to a_1 ranges from '0' at a_1 to 'V' at a_2

$$\therefore \int_0^V dv = \int_{a_2}^{a_1} \frac{-Q}{4\pi\epsilon R^2} dR$$

$$\Rightarrow [v]_0^V = \frac{-Q}{4\pi\epsilon} \int_{a_2}^{a_1} \frac{1}{R^2} dR$$

$$\Rightarrow V = \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{R}\right]_{a_2}^{a_1}$$

$$= \frac{Q}{4\pi\epsilon}\left[\frac{1}{a_1} - \frac{1}{a_2}\right]$$

$$= \frac{Q(a_2 - a_1)}{4\pi\epsilon a_1 a_2}$$

By definition, capacitance is the ratio of charge present on each surface to the potential difference between them.

$$\text{i.e., } C = \frac{Q}{V}$$

$$= \frac{Q}{\left[\frac{Q(a_2 - a_1)}{4\pi\epsilon a_1 a_2}\right]}$$

$$\boxed{\therefore C = \frac{4\pi\epsilon a_1 a_2}{(a_2 - a_1)} \text{ Farads}}$$

Thus, the capacitance is directly proportional to the product of radius of inner and outer shells and inversely proportional to the difference between their radii.

Q43. Show the expression of the capacitance for a spherical capacitor consists of 2 concentric spheres of radius 'a' & 'b' also obtain the capacitance for an isolated sphere.

Ans: (Nov./Dec.-17, (R16), Q4(a) | Model Paper-I, Q4(b))

Expression for Capacitance with Concentric Spheres

For answer refer Unit-II, Q42.

Capacitance of an Isolated Sphere

Consider a spherical capacitor containing of two concentric sphere of radius, R and r (i.e., inner radius and outer radius) as shown in figure (i).

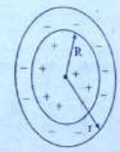


Figure (i): Spherical Capacitor

The capacitance of a charge sphere having radius, R (inner radius) can be determine by making the outer radius (say 'r') infinitely large. A charged sphere of such configuration can be referred as an isolated sphere, as shown in figure (ii).

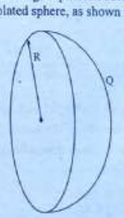


Figure (ii): Isolated Sphere

Therefore, the capacitance of the spherical capacitor is given as,

$$C = \frac{Q}{V}$$

$$= \frac{4\pi \in}{\frac{1}{R} - \frac{1}{r}}$$

$$= 4\pi \in \left(\frac{Rr}{r-R}\right) \quad \text{... (1)}$$

Where, R = Inner radius

r = Outer radius.

Applying limits to the outer radius i.e., $r \to \infty$, we get,

$$\underset{r\to\infty}{Lt}\ C = \underset{r\to\infty}{Lt}\ 4\pi \in \left(\frac{Rr}{r-R}\right)$$

$$= \underset{r\to\infty}{Lt}\ 4\pi \in \left(\frac{R}{1-\frac{R}{r}}\right) = 4\pi \in R$$

Therefore, the capacitance of charged sphere with radius, R is given as,

$$\boxed{C = 4\pi \in R}$$

From the above expression, it can be concluded that the charged sphere completely relies on its geometry i.e., the radius, R.

Q44. Derive an expression for capacitance of co-axial cable.

Ans: Nov./Dec.-17, (R16), Q5

Consider two co-axial conductors with radius of inner and outer radius as 'a' and 'b' respectively as shown in figure.

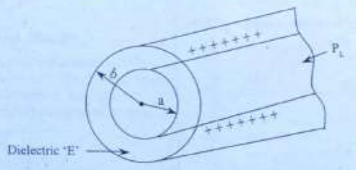


Figure: Co-axial cable

The inner cable is charged uniformly and P_L is the charge density. The outer cable is taken as reference. The potential difference between the two conductors be 'V'. Let the positive charge at inner conductor be '$+Q$' and the negative charge at outer conductor be '$-Q$' along their infinite length. Let 'E' be the permittivity of the medium between the two conductors. The potential difference V_{ab} is determined as,

$$V_{ab} = -\int_b^a E\, d\rho$$

It is obtained as a line integral from b to a.

But, $E = \frac{\rho l}{2\pi \in \rho}$

$$\therefore \quad V_{ab} = -\int_b^a \frac{\rho_l}{2\pi \in \rho}\, d\rho$$

$$= -\frac{\rho_l}{2\pi \in}\int_b^a \frac{d\rho}{\rho}$$

$$= -\frac{\rho_l}{2\pi \in}\left[ln\, \rho\right]_b^a$$

$$= -\frac{\rho_l}{2\pi \in}(ln\, a - ln\, b)$$

$$V_{ab} = -\frac{\rho_l}{2\pi \in} ln\left(\frac{a}{b}\right)$$

$$\therefore \quad V = -\frac{\rho_l}{2\pi \in} ln\left(\frac{b}{a}\right) \quad \text{... (1)}$$

Capacitance per unit length is given as,

$$C = \frac{Q}{V} \text{ and}$$

$$Q = \rho_l$$

$$\therefore \quad C = \frac{\rho_l}{V} \quad \text{... (2)}$$

Substitute equation (1) in equation (2), we get,

$$C = \frac{\rho_l}{\frac{\rho_l}{2\pi \in} ln\left(\frac{b}{a}\right)}$$

$$C = \frac{2\pi \in}{ln\left(\frac{b}{a}\right)}$$

$\therefore$ The expression for capacitance of co-axial conductors of infinite length is given by,

$$C = \frac{2\pi \in}{ln\left(\frac{b}{a}\right)}$$

Q45. The capacitance of a parallel plate condenser is 0.2 μF. Potential difference between the plates is 2V. Calculate the energy stored by the charged condenser.

Ans: Nov./Dec.-18, (R16), Q4(b)

Given that,

Capacitance, $C = 0.2\ \mu F = 0.2 \times 10^{-6}$ F

Potential difference, $V = 2$V

Energy stored in charged condenser, $W = ?$

We know that,

Energy stored in charged condenser is given by,

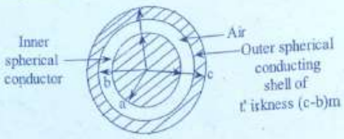
$$W = \frac{1}{2}CV^2$$

$$= \frac{1}{2} \times 0.2 \times 10^{-6} \times (2)^2$$

$$= 4 \times 10^{-7}\ \text{J}$$

$$= 0.4\ \mu\text{J}$$

Q46. A concentric spherical conductor arrangement is shown in figure. If the capacitance of the arrangement is 0.1 nF, and a is 10 cm, find b.



Figure

Ans:

Given that,

Capacitance,

$$C = 0.1\ \text{nF}$$

$$C = 0.1 \times 10^{-9}\ \text{F}$$

$$\text{cm} = 10 \times 10^{-2}\ \text{m}$$

$$= 0.1\ \text{m}$$

To calculate, $b = ?$

Figure shows two concentric spherical conductors of radii 'a' and 'c' placed concentrically and space between them is filled with dielectric material of relative permittivity $\in_r$. A Gaussian surface in the form of sphere of radius 'b' is concentric with these spheres.

The capacitance of spherical conductors is given by,

$$C = 4\pi\in\left[\frac{ac}{c-a}\right]\text{F}$$

$$\in = \in_0 \in_r \quad (\text{As } \in_r = 1 \text{ for medium air and}$$

$$\in_0 = 8.854 \times 10^{-12}\ \text{F/m})$$

$$\therefore \in = 8.854 \times 10^{-12} \times 1$$

$$C = 4\pi \times 8.854 \times 10^{-12}\left[\frac{0.1c}{c-0.1}\right]$$

$$0.1 \times 10^{-9} = 4\pi \times 8.854 \times 10^{-12}\left[\frac{0.1c}{c-0.1}\right]$$

$$0.1 \times 10^{-9} = 1.1120 \times 10^{-10}\left[\frac{0.1c}{c-0.1}\right]$$

$$c - 0.1 = \frac{1.1120 \times 10^{-10} \times 0.1c}{0.1 \times 10^{-9}}$$

$$c - 0.1 = \frac{1.1120 \times 10^{-11}c}{0.1 \times 10^{-9}}$$

$$c - 0.1 = 0.11120\,c$$

$$c - 0.1 = 0.11120\,c$$

$$c - 0.11120c = 0.1$$

$$c(1 - 0.11120) = 0.1$$

$$c(0.8888) = 0.1$$

$$c = \frac{0.1}{0.8888}$$

$$c = 0.1125\ \text{m}$$

$$\boxed{c = 11.2\text{cm}}$$

Given that, the thickness between outer spherical conductor and Gaussian spherical conductor is $(c-b)$ m. As the conductors are concentrically placed, so the thickness between sphere b and c and the thickness between a and b are same.

$$\therefore \quad b = a + (c - b)$$

$$2b = a + c$$

$$b = \frac{10 + 11.2}{2}$$

$$b = \frac{21.2}{2}$$

$$b = 10.6\ \text{cm}$$

$$\boxed{b = 10.6\ \text{cm}}$$

Q47. Find the capacitance of a conducting sphere of 2 cm in diameter, covered with a layer of polyethelene with $\in_r$ = 2.26 and 3 cm thick.

Ans: (Nov./Dec.-17, (R16), Q4(b) | Nov.-15, (R13), Q5(b) | Model Paper-III, Q4(b))

Given that,

Diameter of conducting sphere, $d = 2\text{cm} = 2 \times 10^{-2}\text{m}$

$\therefore$ Radius of conducting sphere, $r_1 = \frac{d}{2} = 1 \times 10^{-2}\text{m}$

Permittivity of polythelene, $\in_r = 2.26$

Thickness, $t_1 = 3\text{cm} = 3 \times 10^{-2}\text{m}$

$$r_2 = r_1 + t_1$$

$$= 1 \times 10^{-2} + 3 \times 10^{-2}$$

$$= 4 \times 10^{-2}\ \text{m}$$

To calculate,

The capacitance of conducting spheres.

Figure (1)

The capacitance of concentric sphere is given by,

$$C_1 = \frac{4\pi \in r_1 r_2}{(r_2 - r_1)}$$

$$= \frac{4\pi \times 2.26 \times 8.854 \times 10^{-12} \times 1 \times 10^{-2} \times 4 \times 10^{-2}}{(4 \times 10^{-2} - 1 \times 10^{-2})} \quad [\because \in = \in_o \in_r \text{ Where } \in_o = 8.854 \times 10^{-12}]$$

$$C_1 = 3.3527 \times 10^{-12} \text{ F}$$

Similarly, capacitance of air gap is given by,

$$C_2 = 4\pi \in r_1$$

$$= 4\pi \times 1 \times 8.854 \times 10^{-12} \times 4 \times 10^{-2} \quad [\because \in_r = 1, \text{ for air}]$$

$$C_2 = 4.4505 \times 10^{-12} \text{ F}$$

Capacitance of conducting sphere, C is given by,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \quad (C_1 \text{ and } C_2 \text{ are in series})$$

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

$$= \frac{3.3527 \times 10^{-12} \times 4.4505 \times 10^{-12}}{[3.3527 \times 10^{-12} + 4.4504 \times 10^{-12}]}$$

$$= \frac{3.3527 \times 10^{-12} \times 4.4505 \times 10^{-12}}{10^{-12}[3.3527 + 4.4504]}$$

$$C = 1.9121 \times 10^{-12} \text{ F}$$

$$C = 1.9121 \text{ pF}$$

$\therefore$ Capacitance of conducting sphere, $C = 1.9121$ pF.

Q48. What is the capacitance of a capacitor consisting of two parallel plates of 30 cm by 30 cm, separated by 5 mm in air? What is the energy stored by the capacitor if it is charged to a potential difference of 500 volts?

Ans: Dec.-11, Set-1, Q2(a)

Given that,

Dimensions of two plates = 30 cm × 30 cm

Distance between two plates, $d = 5$ mm

Capacitance, $C = ?$

Potential difference between plates, $V = 500$ V

Energy stored, $W_e = ?$

The area of plates,

$$A = 30 \times 10^{-2} \times 30 \times 10^{-2}$$

$$= 90 \times 10^{-3} \text{ m}^2$$

The capacitance of the parallel plates capacitor is given by,

$$C = \frac{A \in_0 \in_r}{d}$$

$$= \frac{90 \times 10^{-3} \times 8.854 \times 10^{-12} \times 1}{5 \times 10^{-3}} \quad [\because \in_r = 1 \text{ for air dielectric}]$$

$$= 159.375 \times 10^{-12} \text{ F}$$

$$= 159.375 \text{ pF}$$

We know that,

The energy stored by a capacitor is,

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2} \times 159.375 \times 10^{-12} \times (500)^2$$

$$= 19.922 \times 10^{-6} \text{ J} = 19.922 \text{ μJ}$$

Q49. A spherical condenser has a capacity of 54 pF. It consists of two concentric spheres differing in radii by 4 cm and having air as dielectric. Find their radii.

Ans: (Nov.-10, Set-2, Q4(b) | Model Paper-II, Q5(b))

Given that,

Capacitance, $C = 54$ pF

Difference in radii of two spheres,

$$b - a = 4 \text{ cm}$$

$$b - a = 4 \times 10^{-2} \text{ m} \quad \ldots (1)$$

Assume, the relative permittivity, $\in_r = 1$

The expression for capacitance of a spherical capacitor is given by,

$$C = \frac{4\pi \in}{\left(\frac{1}{a} - \frac{1}{b}\right)} \quad \ldots (2)$$

Here, $b > a$

Substituting the values in the equation (2), we get,

$$\Rightarrow \quad 54 \times 10^{-12} = \frac{4\pi \times 8.854 \times 10^{-12}}{\left(\frac{b-a}{ab}\right) \times 10^{-2}} \quad (\because \text{The difference in radii is in cm})$$

$$\Rightarrow \quad \frac{(b-a) \times 10^{-2}}{ab} = 2.06$$

$$\Rightarrow \quad ab = \frac{4 \times 10^{-2}}{2.06} \quad (\because b - a = 4 \text{ cm})$$

$$\Rightarrow \quad ab = 1.941 \times 10^{-2} \quad \ldots (3)$$

From equation (1), we have,

$$b - a = 4 \times 10^{-2}$$

$$b = (4 \times 10^{-2}) + a \quad \ldots (4)$$

Substituting equation (4) in equation (3), we get,

$$\Rightarrow \quad a[(4 \times 10^{-2}) + a] = 1.941 \times 10^{-2}$$

$$\Rightarrow \quad 0.04a + a^2 = 0.01941$$

$$\Rightarrow \quad a^2 + 0.04a - 0.01941 = 0$$

On solving the above equation, we get,

$$a = 0.12 \text{ m and}$$
$$= -0.16 \text{ m}$$

Neglecting the negative value and substituting $a = 0.12$ m in equation (4), we get,

$$b = (4 \times 10^{-2}) + a$$
$$= (4 \times 10^{-2}) + 0.12$$
$$= 0.16 \text{ m}$$

$$\therefore \quad a = 0.12 \text{ m and}$$
$$b = 0.16 \text{ m}$$

Q50. A co-axial cable is required to transmit electric power. The potential difference between the inner and outer conductors is to be filled mainly with nitrogen gas under pressure whose dielectric strength is 25 × 10⁻⁶ V/m. The radius of the outer conductors is double that of inner conductors.

(a) Determine the capacitance of the cable.

(b) Determine energy stored in the electric field of this cable when potential difference is 2 × 10⁵.

Ans: Nov.-10, Set-3, Q8

Given that,

Dielectric strength, $\epsilon_r = 25 \times 10^{-6}$ V/m

Radius of outer conductor, $r_2 = 2r_1$

Potential difference, $V = 2 \times 10^5$ V

(a) Capacitance of Cable

For co-axial cable the capacitance per unit length is given by,

$$C/l = \frac{2\pi\epsilon}{\ln\left(\frac{r_2}{r_1}\right)}$$

$$= \frac{2\pi\epsilon_0\epsilon_r}{\ln\left(\frac{2r_1}{r_1}\right)}$$

$$= \frac{(2\pi)\times(8.854\times10^{-12})\times(25\times10^{-6})}{\ln(2)}$$

$$= 2 \times 10^{-15} \text{ F/m}$$

$$\therefore \quad \frac{C}{l} = 0.002 \text{ pF/m}$$

(b) Energy Stored

We know that,

$$E = \frac{1}{2}CV^2 \quad \ldots (1)$$

Substituting the values of "C" and 'V' in equation (1), we get,

$$E = \frac{1}{2} \times (0.002 \times 10^{-12}) \times (2 \times 10^5)^2$$
$$= 4 \times 10^{-5} \text{ J}$$
$$= 40 \text{ µJ}$$

$\therefore$ The energy stored in electric field of this cable is 40 µJ.

2.5 POISSON'S EQUATION, LAPLACE EQUATION, SOLUTION OF LAPLACE AND POISSON'S EQUATION

Q51. Write down the general procedure for solving Poisson's and Laplace's equation.

Ans: (Nov./Dec.-17, (R13), Q12(b)(ii) | Model Paper-II, Q4(b))

The general procedure for solving Poisson's and Laplace's equation is explained as follows,

Step-1

The Laplace and Poisson's equations are to be solved for different variables using different methods viz.,

(a) When V is the function of one variable then direct integration method is used.

(b) When V is the function of more than one variable then an advanced method i.e., separation method of variables is used.

The solution obtained in this step is not unique as the integration constants are unknown.

Step-2

The unique solution is now determined by applying boundary conditions to the unknown integration constants.

Step-3

In this step, electric field '$\overline{E}$' is determined from potential field 'V' using the gradient relation i.e., $\overline{E} = -\Delta V$, and for homogeneous medium, flux density $\overline{D}$ can be determined from V i.e., $\overline{D} = \epsilon\overline{E}$.

Step-4

Then, calculate the charge induced on the surface of the conductor from the relation,

$$Q = \int \rho_s \, ds$$

But,

$$\rho_s = D_n$$

D_n is the normal component which can be obtained when once D is known i.e., $D = \epsilon E_n$

Here,

ρ_s = Induced surface charge density

E_n = Normal component of the field.

Step-5

Determine the capacitance between the two conductors by knowing the values of potential field and charge induced Q by using the relation,

$$C = \frac{Q}{V}$$

Q52. Derive Poisson's equation for homogeneous medium with constant ∈.

Ans:

For a volume charge distribution, the relation between the volume charge density 'ρ_v' and the electric flux density '$\overline{D}$' is given by Maxwell's first law as,

$$\text{div}(\overline{D}) = \rho_v \quad \ldots (1)$$

The electric flux density is,

$$\overline{D} = \epsilon\,\overline{E}$$
$$= \epsilon(-\Delta V) \quad [\because \overline{E} = -\Delta V]$$
$$= -\epsilon\,\Delta V \quad \ldots (2)$$

Substituting equation (2) in equation (1), we get,

$$\Rightarrow \quad \text{div}(-\epsilon\nabla V) = \rho_v$$

$$\Rightarrow \quad \nabla.(-\epsilon\nabla V) = \rho_v$$

$$\Rightarrow \quad -\epsilon(\nabla.\nabla V) = \rho_v$$

[$\because \epsilon$ is constant for homogeneous medium]

$$\Rightarrow \quad \nabla.\nabla V = \frac{-\rho_v}{\epsilon}$$

$$\Rightarrow \quad \nabla^2 V = \frac{-\rho_v}{\epsilon} \quad \ldots (3)$$

The above equation is called as the Poisson's equation for homogeneous medium.

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Q53. Derive Laplace's equation.

Ans:

The Poisson's equation for homogeneous media is,

$$\nabla^2 V = \frac{-\rho_v}{\in} \quad \text{...(1)}$$

Consider a charge distribution in which the volume charge density is absent, while the charge distribution may contain the combination of point, line and surface charge densities.

$$\therefore \quad \rho_v = 0 \quad \text{...(2)}$$

Substituting equation (2) in equation (1), we get,

$$\Rightarrow \quad \nabla^2 V = \frac{-0}{\in} = 0$$

$$\nabla^2 V = 0 \quad \text{...(3)}$$

The above equation is called as the Laplace's equation which is obtained from the Poisson's equation for a charge distribution free of volume charges.

Q54. Give the significant physical differences between Poisson's and Laplace's equation.

Model Paper-I, Q5(b)

Ans:

Laplace's and Poisson's equations are very much analogous to each other and hence there is no much difference between them. But some of them are tabulated below.

S.No	Poisson's Equation	S.No	Laplace's Equation
1.	This equation is used for the study of electrostatic fields.	1.	This equation is also used for the study of electrostatic fields.
2.	Mathematical expression is given by, $\nabla^2 V = \frac{-\rho_v}{\in}$	2.	Mathematical expression is given by, $\nabla^2 V = 0$
3.	This equation is applicable for more than one field of technological science.	3.	It is also used for more than one field of technological science.
4.	This equation must be satisfied with suitable boundary conditions.	4.	In this also boundary conditions must be satisfied.
5.	This equation is used to evaluate the charge densities such as vacuum tubes ion propulsion, transistor model etc.		This equation is used for evaluating line charges, point charges or surface charge densities at a single location.

Q55. What are the applications of Poisson's and Laplace's equations?

Ans:

The following are the applications of Poisson's and Laplace's equations.

The solutions of Poisson and Laplace equations are very much useful in describing the behaviour of various physical phenomena. A few physical phenomena that apply Laplace and Possion equations to describe their behaviour are, .

(a) Steady heat conduction.

(b) Seepage through porous media.

(c) Irrotational flow of an ideal fluid.

(d) Distribution of electrical and magnetic potential.

(e) Torsion of prismatic shaft.

(f) Bending of prismatic leaves.

(g) Distribution of gravitational potential.

Q56. Illustrate with an example, to apply Poisson's and Laplace equation.

Ans:

Consider an example to determine E field both inside and inside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \leq R \leq b$ and $\rho = 0$ for $R > b$ by solving Poisson's and Laplace equations for V.

In the assumed example as there is no change in the directions of θ and ϕ, considering the spherical co-ordinates to solve the expressions.

(i) Inside a Spherical Cloud

Inside a spherical cloud, the general boundary conditions are given as,

$$0 \leq R \leq b, \rho = -\rho_0$$

In this region, Poisson's equation holds. The spherical co-ordinates of Poisson's equation are given as,

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \frac{\rho_0}{\in_0}$$

$$\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \frac{\rho_0 R^2}{\in_0} \quad \text{...(1)}$$

On Integrating equation (1) on both sides, we get,

$$\int \frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \int \frac{\rho_0 R^2}{\in_0}$$

$$\Rightarrow R^2 \frac{dV_i}{dR} = \frac{\rho_0}{\in_0}\left(\frac{R^3}{3}\right) + C_1$$

$$\Rightarrow \quad \frac{dV_i}{dR} = \frac{\rho_0 R}{\in_0 3} + \frac{C_1}{R^2}$$

$$\Rightarrow \quad \frac{dV_i}{dR} = \frac{\rho_0}{3\in_0} R + \frac{C_1}{R^2} \quad \text{...(2)}$$

We know that,

$$E_{inside} = -\Delta V_i$$

$$= -a_R\left(\frac{dV_i}{dR}\right) \quad \text{...(3)}$$

Substituting equation (2) in equation (3), we get,

$$E_{inside} = -a_R\left(\frac{\rho_0}{3\in_0} R + \frac{C_1}{R^2}\right)$$

The value of constant $C_1 = 0$, since E_i cannot be infinite.

$$\therefore \quad E_{inside} = -a_R \frac{\rho_0}{3\in_0} R, \quad 0 \leq R \leq b \quad \text{...(4)}$$

(ii) Outside a Spherical Cloud

Outside a spherical cloud, the general boundary conditions are given as,

$$R \geq b, \rho = 0$$

In this region, Laplace equation is. The spherical co-ordinates of Laplace equation are given as,

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{dV_o}{dR}\right)=0$$

$$\Rightarrow \frac{d}{dR}\left(R^2\frac{dV_o}{dR}\right)=0 \qquad ...(4)$$

On integrating equation (4), we get,

$$\int\frac{d}{dR}\left(R^2\frac{dV_o}{dR}\right)=0$$

$$\Rightarrow R^2\frac{dV_o}{dR}=C_2$$

$$\Rightarrow \frac{dV_o}{dR}=\frac{C_2}{R^2} \qquad ...(5)$$

We know that,

$$E_{outside}=-\Delta V_o$$

$$=-a_R\left(\frac{dV_o}{dR}\right)$$

$$=-a_R\left(\frac{C_2}{R^2}\right) \qquad [\because \text{From equation (5)}] \qquad ...(6)$$

Equating equation (4) and equation (6) at $R=b$, we get,

$$-a_R\frac{\rho_0}{3\epsilon_0}b=-a_R\left(\frac{C_2}{R^2}\right)$$

$$\frac{\rho_0}{3\epsilon_0}b=\frac{-C_2}{b^2}$$

$$C_2=\frac{\rho_0 b^3}{3\epsilon_0} \qquad ...(7)$$

Substituting equation (7) in equation (6), we get,

$$E_{outside}=-a_R\frac{\rho_0 b^3}{3\epsilon_0 R^2},\ R\geq b \qquad ...(8)$$

We know that,

The total charge in a spherical cloud is given as,

$$Q=-\rho_0\frac{4\pi}{3}b^3 \qquad ...(9)$$

Substituting equation (9) in equation (8), we get,

$$E_{outside}=a_R\frac{Q}{4\pi\epsilon_0 R^2} \qquad ...(10)$$

Q57. Write Laplace's equation in Cartesian coordinates. And obtain the solution when V is function of x only for the boundary condition $V=V_1$ at $x=x_1$ and $V=V_2$ at $x=x_2$.

Ans: Model Paper-III, Q5(b)

The Laplace's equation is given by,

$$\nabla^2 V=0$$

Where, V represents the potential function.

In Cartesian coordinates, $\nabla^2 V$ can be expressed as,

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}$$

Therefore, Laplace's equation in Cartesian coordinates is,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2}{\partial z^2}=0 \qquad ...(1)$$

Since, 'V' is a function of 'x', we have,

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial z^2}=0$$

Now, equation (1) becomes,

$$\frac{\partial^2 V}{\partial x^2}=0$$

Integrating with respect to 'x', we get,

$$\int\frac{\partial^2 V}{\partial x^2}=\int 0dx$$

$$\Rightarrow \frac{\partial V}{\partial x}=A$$

Where, A is an integration constant.

Integrating again with respect to 'x', we get,

$$\int\frac{\partial V}{\partial x}=\int Adx\ V=Ax+B \qquad ...(2)$$

Where, B is an integration constant

Substituting the given boundary conditions i.e., $V=V_1$ at $x=x_1$ and $V=V_2$ at $x=x_2$ simultaneously in equation (2), we get,

$$V_1=Ax_1+B$$

$$\Rightarrow B=V_1-Ax_1 \qquad ...(3)$$

And, $$V_2=Ax_2+B \qquad ...(4)$$

Substituting equation (3) in equation (4), we get,

$$V_2=Ax_2+V_1-Ax_1$$

$$\Rightarrow V_2=A(x_2-x_1)+V_1$$

$$\Rightarrow A=\frac{V_2-V_1}{x_2-x_1} \qquad ...(5)$$

Substituting equation (5) in equation (3), we get,

$$B=V_1-\left(\frac{V_2-V_1}{x_2-x_1}\right)x_1$$

$$\Rightarrow B=\frac{V_1(x_2-x_1)-(V_2-V_1)x_1}{x_2-x_1}$$

$$\Rightarrow B=\frac{V_1x_2-V_1x_1-V_2x_1+V_1x_1}{x_2-x_1}$$

$$\Rightarrow B=\frac{V_1x_2-V_2x_1}{x_2-x_1} \qquad ...(6)$$

Now, Substituting the values of A and B in equation (2), we get,

$$V = \left(\frac{V_2 - V_1}{x_2 - x_1}\right)x + \frac{V_1 x_2 - V_2 x_1}{x_2 - x_1} \quad \dots (7)$$

Therefore, equation (7) gives the required solution.

UNIT 3

Static Magnetic Fields and Magnetic Forces

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. State Biot-Savart's law.

Ans:

Statement

It states that the magnetic field intensity $\overline{dH}$ at a point 'P' due to a current element Idl is,

(i) Directly proportional to the current element Idl i.e., $dH \propto Idl$.

(ii) Directly proportional to the sine of the angle between Idl and the line joining the point 'P' to the element i.e., $dH = \sin\theta$.

(iii) Inversely proportional to the square of the distance between the point and the element i.e., $dH = \frac{1}{r^2}$.

From the above three points, we have,

$$|dH| \propto \frac{|Idl|\sin\theta}{|r|^2}$$

$$|dH| = \frac{|Idl|\sin\theta}{4\pi r^2} \text{ A/m}$$

Q2. Give the applications of Biot Savart's Law.

Ans: Model Paper-I, Q1(e)

Biot Savart's law is widely used for finding magnetic field strength and magnetic flux density. In the field of electromagnetism, this law is used for the applications such as,

1. Magnetic flux density, B at the center of the circular current carrying conductor.
2. Magnetic flux density, B at any point on the axis of a circular current carrying conductor.
3. Magnetic flux density, B due to straight current carrying conductor.

Q3. State Ampere's circuital law.

Ans:

Statement

It states that the line integral of magnetic field intensity $(\overline{H})$ taken around a single closed loop is equal to the current enclosed by that path.

Mathematically,

$$\oint \overline{H}.\overline{dl} = I$$

Where,

H = Magnetic field intensity, A/m

I = Current enclosed, Amps.

Q4. Define magnetic flux.

Ans:

Magnetic Flux

The total number of lines of induction cutting through a surface is called magnetic flux through that surface.

The unit of magnetic flux is weber.

Q5. Define magnetic flux density.

Ans: Model Paper-II, Q1(e)

Magnetic flux density is defined as the magnetic flux lines per unit cross-sectional area. It is denoted by B and its unit is Weber/m² (or) Tesla.

Mathematically, it is expressed as,

$$B = \frac{\phi}{A} \text{ Weber/m}^2$$

Q6. Give the applications of magnetostatic field.

Ans:

Some of the magnetostatic field applications of are as follows,

- It is employed in transformers.
- Microphones.
- Compasses.
- High speed velocity devices.
- Telephone ringers, etc.

Q7. Mention the limitations of scalar magnetic potential.

Refer Only Topic : Limitation of Scalar Magnetic Potential

Nov./Dec.-17, (R16), Q1(g)

OR

Define scalar magnetic potential and state its limitations.

Ans: Model Paper-III, Q1(e)

Scalar Magnetic Potential

It is defined as the negative of magnetic potential difference between any two points m and n along the selected differential path (dl), provided the current density in that particular region is zero i.e.,

$$V_m = -\int_m^n H.dl \text{ ; provided, } \bar{J} = 0$$

Limitations of Scalar Magnetic Potential

1. It is defined only in the regions where no electric current flows i.e., where current density does not exists.
2. Unlike electrostatic potential, it is not a conservative field. It means that, for each selected path between two points in the field, different values of V_m is obtained.

Q8. Define:
(i) Scalar magnetic potential
(ii) Vector magnetic potential.

Model Paper-I, Q1(f)

Ans:

(i) Scalar Magnetic Potential

For answer refer Unit-IV, Q1.

Vector Magnetic Potential

It is defined in such a way the curl of it gives the magnetic flux density ($\bar{B}$)

Mathematiacally it is expressed as,

$$\bar{B} = \nabla \times \bar{A}$$

Where,

$\bar{B}$ = Magnetic flux density

$\bar{A}$ = Vector magnetic potential.

Q9. Define:
(i) Self inductance
(ii) Mutual inductance.

Ans:

(i) Self Inductance

It is defined as the flux linkage per unit current due to the current flowing through the same coil. Mathematically it is given as,

$$L = \frac{N\phi}{I} H$$

Where,

N = Number of turns

ϕ = Flux

I = Current

(ii) Mutual Inductance

It is defined the flux linkage per unit current due to the current in some other coil. It is denoted by M and mathematically it is given as,

$$M_{12} = \frac{N_1\phi_{21}}{I_2}$$

(or)

$$M_{21} = \frac{N_2\phi_{12}}{I_1}$$

Q10. Derive the relation between L_{SO}, L_{SA} and M.

Ans:

Relation between L_{SO}, L_{SA} and M

Let L_{SA}, L_{SO} be the equivalent inductance of series aiding and series opposing of two self inductances (L_1 and L_2) respectively. M be the mutual inductance between the two self inductances (L_1 and L_2).

The expression for series aiding of coils L_1 and L_2 is given as,

$$L_{SA} = L_1 + L_2 + 2M \quad \text{...(1)}$$

The expression for series opposing of coils L_1 and L_2 is given as,

$$L_{SO} = L_1 + L_2 - 2M \quad \text{...(2)}$$

Solving equations (1) and (2), we get,

$$\begin{array}{r} L_{SA} = L_1 + L_2 + 2M \\ \underline{L_{SO} = L_1 + L_2 - 2M} \\ L_{SA} - L_{SO} = 4M \end{array}$$

$$\therefore \quad M = \left(\frac{L_{SA} - L_{SO}}{4}\right)$$

Q11. Write the expression for the magnetic force between an electromagnet and an armature to be attracted.

OR

Mention the force between two current elements.

Ans: (May/June-16, (R13), Q7 | Model Paper-II, Q1...)

The expression for the magnetic force between an electromagnet and an armature to be attracted is given by,

$$F = \frac{B^2 A}{2\mu_0} \text{ N}$$

Where,

B = Flux density in Tesla

A = Area of the air gap between the magnetic poles

μ_0 = magnetic permeability in free space.

Q12. Write the Lorentz force equation for a moving charge.

Ans: Nov./Dec.-17, (R13), Q8

The Lorentz force equation for a moving charge is given as,

$$\bar{F} = Q(\bar{E} + \bar{V} \times \bar{B}) = m\frac{d\bar{u}}{dt}$$

Where,

Q = Magnitude of charged particle

$\bar{E}$ = Electric field intensity

$\bar{V}$ = Velocity of charged particle

$\bar{B}$ = Magnetic flux density.

Q13. Derive Maxwell's equation derived from Ampere's law.

Ans: (Nov./Dec.-17, (R16), Q1(j) ; Model Paper-III, Q1(f))

According to Ampere's circuit law, we have,

$$\oint \bar{H}.d\bar{L} = I_{enclosed} \quad \text{...(1)}$$

Where,

$\bar{H}$ is magnetic field intensity

$I_{enclosed}$ is current enclosed by the path.

And also,

The current in an enclosed path is given as,

$$I_{enclosed} = \int_S \bar{J}.d\bar{S} \quad \text{...(2)}$$

Now, substituting equation (2) in equation (1), we get,

$$\oint \bar{H}.d\bar{L} = \int_S \bar{J}.d\bar{S} \quad \text{...(3)}$$

Equation (3) is more simplified by adding displacement current density to conduction current density as follows,

$$\oint \bar{H}.d\bar{L} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right].d\bar{S} \quad \text{...(4)}$$

Equation (4) is Maxwell's equation derived from Ampere's circuit law in integral form.

By applying stoke's theorem to L.H.S of equation (4) we get,

$$\int_S (\bar{\nabla} \times \bar{H}).d\bar{S} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right].d\bar{S}$$

Assuming, surface to be same,

We get,

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{...(5)}$$

Equation (5) is Maxwell's equation derived from Ampere's circuit law in differential form or point form.

Q14. Two coupled coils with L_1 = 20 mH, L_2 = 10 mH and k = 0.5 are connected in series aiding. Find their equivalent inductance.

Ans:

Given that,

Self inductance of 1st coil, L_1 = 20 mH

Self inductance of 2nd coil, L_2 = 10 mH

Coefficient of coupling, k = 0.5

To determine,

Equivalent inductance when connected in series aiding i.e.,

$$L_{SA} = ?$$

Mutual inductance,

$$\begin{aligned} M &= K\sqrt{L_1 L_2} \\ &= (0.5) \times \sqrt{(20 \times 10^{-3})(10 \times 10^{-3})} \\ &= 7.07 \times 10^{-3} \text{ H} \end{aligned}$$

The expression for series aiding is given as,

$$\begin{aligned} L_{SA} &= L_1 + L_2 + 2M \\ &= 20 \times 10^{-3} + 10 \times 10^{-3} + 2 \times 7.07 \times 10^{-3} \\ &= (20 + 10 + 14.14) \times 10^{-3} \\ &= 44.14 \times 10^{-3} \text{ H} \\ &= 44.14 \text{ mH.} \end{aligned}$$

Q15. A solenoid has an inductance of 20 mH. If the length of the solenoid is increased by two times and the radius is decreased to half of its original value, find the new inductance.

Ans: (Nov./Dec.-17, (R16), Q1(h) | Nov.-15, (R13), Q1(h))

Given that,

Inductance of solenoid, $L = 20$ mH

We know that inductance of a solenoid is given by,

$$L = \frac{\mu N^2 A}{l} = 20 \text{ mH} = 20 \times 10^{-3} \text{ H}$$

Where N = number of turns

$A = \pi r^2$ = Area of solenoid

l = length of solenoid

From given data,

Length is increased by two times i.e.,

$l = 2l$

Radius is decreased to half i.e.,

$r = \frac{r}{2}$

New inductance, L_{new} = ?

The inductance of new solenoid, L_{new} is given by,

$$L_{new} = \frac{\mu N^2 A}{l}$$

$$= \frac{\mu \times N^2 (\pi r^2)}{l}$$

$$= \frac{\mu \times N^2 \times \left(\pi\left(\frac{r}{2}\right)^2\right)}{2l}$$

$$= \frac{\mu \times N^2 \times \frac{\pi r^2}{4}}{2l}$$

$$= \frac{\mu \times N^2 \times \pi r^2}{8l}$$

$$= \frac{1}{8}\left[\frac{\mu N^2 \pi r^2}{l}\right]$$

$$= \frac{1}{8}[L] = \frac{1}{8}[20 \times 10^{-3}] \quad \left[\because L = \frac{\mu N^2 A}{l}\right]$$

$$= 2.5 \times 10^{-3} \text{ H}$$

$$\boxed{\therefore \text{ New inductance, } L_{new} = 2.5 \times 10^{-3} \text{ H}}$$

Q16. A long straight wire carries a current I = 1 amp. At what distance is the magnetic field H = 1 A/m.

Ans: (Nov./Dec.-17, (R16), Q1(f) | Nov.-15, (R13), Q1(f))

Given that,

Current, $I = 1$ amp

Magnetic field, $H = 1$ A/m

Distance between magnetic field intensity and long straight wire carrying current, a = ?

Magnetic field of a long straight wire carrying current is given by,

$$H = \frac{I}{2\pi a}$$

$$1 \text{ Amp/m} = \frac{1 \text{amp}}{2\pi \times a}$$

$$a = \frac{1}{2\pi \times 1} = 0.159 \text{ m}$$

$\therefore$ The magnetic field, H is placed at a distance of 0.159 m from long straight wire.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

3.1 BIOT-SAVART LAW

Q17. State and explain Biot-Savart's law.

Ans: Model Paper-I, Q6(a)

Biot-Savart's Law

For answer refer Unit-III, Q1.

Explanation

Consider a differential element $Id\bar{l}$ within a straight conductor placed on Z-axis carrying current 'I' in positive Z-direction as shown in figure below.

Let a point 'P' where the differential magnetic field $(d\bar{H})$ is to be found, is at a distant '$\bar{r}$' from the current element $Id\bar{l}$.

Let 'θ' be the angle between $Id\bar{l}$ and unit vector $\bar{a}_r$ in the direction of $\bar{r}$.

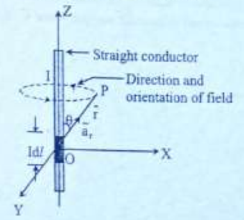


Figure: Geometry of the Figure

According to the Biot-Savart's law, we have,

$$|d\bar{H}| \propto \frac{|Id\bar{l}|.\sin\theta}{|\bar{r}|^2}$$

$$\Rightarrow \quad |d\bar{H}| \propto \frac{|Id\bar{l}|1.\sin\theta}{|\bar{r}|^2}$$

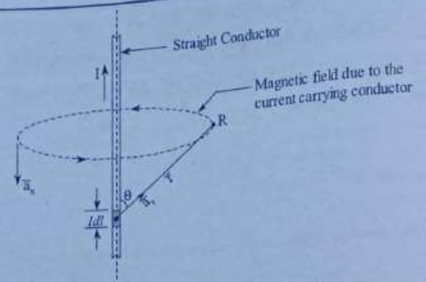
$$\Rightarrow \quad |d\bar{H}| \propto \frac{|Id\bar{l}|\overline{ar}}{|\bar{r}|^2} \quad [\because \bar{a} \times \bar{b} = |\bar{a}|.|\bar{b}|\sin\theta]$$

$$\therefore \quad d\bar{H} = \frac{1}{4\pi}\frac{Id\bar{l} \times \overline{ar}}{|\bar{r}|^2} \text{ A/m}$$

Q18. State Biot-Savart's law for the magnetic field $\bar{B}$ due to a steady line current in free space.

Ans:

Consider a current element '$Id\bar{l}$' within a straight conductor carrying current 'I' in the upward direction. Let a point 'R' where the magnetic field intensity $(d\bar{H})$ is to be found, is at a distance of 'r' from the current element '$Id\bar{l}$', as shown in the figure. Let 'θ' be the angle between current element, $I\,d\bar{l}$ and unit vector $\bar{a}_r$.



Figure

Biot-Savart's law states that the magnitude of magnetic field due to current element $I\bar{dl}$ is proportional to magnitude of current element $|I\bar{dl}|$ and is inversely proportional to square of the distance between the current element and the point 'R' and proportional to sine of the angle between the current element $I\bar{dl}$ and unit vector $\bar{a}_r$ in the direction of $\bar{r}$.

Mathematically,

$$|d\bar{H}| \propto |I\bar{dl}|$$

$$|d\bar{H}| \propto \frac{1}{|\bar{r}|^2} \text{ and}$$

$$|d\bar{H}| \propto \sin\theta$$

Combinedly, we can write it as,

$$|d\bar{H}| \propto \frac{|I\bar{dl}|\sin\theta}{|\bar{r}|^2}$$

$$|d\bar{H}| = \frac{1}{4\pi}\frac{|I\bar{dl}|\sin\theta}{|\bar{r}|^2}$$

Where, $\frac{1}{4\pi}$ = Proportionality constant.

In vector form, it is expressed as,

$$d\bar{H} = \frac{|I\bar{dl}|\sin\theta}{4\pi|\bar{r}|^2}\bar{a}_n$$

Where,

$\bar{a}_n$ = Unit vector in the direction of field.

$$\Rightarrow \quad d\bar{H} = \frac{|I\bar{dl}|\times|\bar{a}_r|\times\sin\theta}{4\pi|\bar{r}|^2}\bar{a}_n$$

From vector calculus, we have,

$$I\bar{dl}\times\bar{a}_r = |I\bar{dl}|.|\bar{a}_r|\sin\theta\,\bar{a}_n$$

$$\therefore \quad d\bar{H} = \frac{|I\bar{dl}|\times\bar{a}_r}{4\pi|\bar{r}|^2}$$

Multiplying and dividing by $|\bar{r}|$, we get,

$$\Rightarrow d\bar{H} = \frac{|I\bar{dl}|\times\bar{a}_r}{4\pi|\bar{r}|^2}\times\frac{|\bar{r}|}{|\bar{r}|}$$

$$\Rightarrow d\bar{H} = \frac{I\bar{dl}\times\bar{r}}{4\pi|\bar{r}|^3} \qquad [\because \bar{r} = |\bar{r}|.\bar{a}_r]$$

$\therefore$ Magnetic field intensity $|\bar{H}|$ due to the entire length of the conductor is given by,

$$\bar{H} = \int d\bar{H} = \int\frac{I\bar{dl}\times\bar{r}}{4\pi|\bar{r}|^3}$$

$$\Rightarrow \quad \bar{H} = \frac{1}{4\pi}\int\frac{I\bar{dl}\times\bar{r}}{|\bar{r}|^3} \text{ A/m}$$

Q19. Derive Biot-Savart's law and obtain Maxwell's second equation from it.

Ans:

For answer refer Unit-III, Q18.

Maxwell's Second Equation From Biot-Savart's Law

From Biot-Savart's law, we have,

$$\bar{H} = \frac{1}{4\pi}\int\frac{I\bar{dl}\times\bar{r}}{|\bar{r}|^3}$$

We know that,

$$\bar{B} = \mu\bar{H}$$

$$\Rightarrow \quad \bar{B} = \frac{\mu}{4\pi}\int\frac{I\bar{dl}\times\bar{r}}{|\bar{r}|^3}$$

$$\Rightarrow \quad \bar{B} = \frac{\mu}{4\pi|\bar{r}|^3}\int(I\bar{dl}\times\bar{r})$$

Taking divergence on both sides,

$$div(\bar{B}) = div\left(\frac{\mu}{4\pi|\bar{r}|^3}\int(I\bar{dl}\times\bar{r})\right)$$

$$\Rightarrow \quad div(\bar{B}) = \frac{\mu}{4\pi|\bar{r}|^3}\int div(I\bar{dl}\times\bar{r})$$

$$\Rightarrow \quad div(\bar{B}) = \frac{\mu}{4\pi|\bar{r}|^3}\int\left[\bar{r}.(\nabla\times I\bar{dl}) - I\bar{dl}.(\nabla\times\bar{r})\right] \quad [\because div(\bar{u}\times\bar{v}) = \bar{v}.\nabla\times\bar{u} - \bar{u}.\nabla\times\bar{v}]$$

But, the curl is applied for rotational quantities.

Here, $I\bar{dl}$ and $\bar{r}$ are not rotating vectors.

$\therefore \quad \nabla\times I\bar{dl} = 0$ and

$\nabla\times\bar{r} = 0$

$$\Rightarrow \quad div(\bar{B}) = \frac{\mu}{4\pi|\bar{r}|^3}\int(0-0)$$

$$\Rightarrow \quad div(\bar{B}) = 0$$

$$\Rightarrow \quad \nabla.\bar{B} = 0$$

$\therefore$ The above equation represents the Maxwell's second equation in point form.

Q20. State Biot-Savart's law and derive the expression for magnetic field intensity of straight current carrying conductor.

April/May-18, (R13), Q13(a)

OR

Using Biot-Savart's law, determine the magnetic field intensity due to a straight current filamentary conductor of finite length AB.

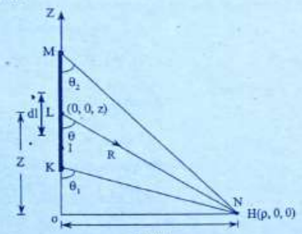
Ans: (Nov./Dec.-17, (R13), Q13(b) | Model Paper-II, Q7(a))

Biot-Savart's Law

For answer refer Unit-III, Q1

Magnetic Field Intensity of Straight Current Carrying Conductor

Consider a current carrying finite length conductor placed along the z-axis as shown in figure. The point N at which MFI is to be found is on x-axis, whose coordinates are given as $(\rho, 0, 0)$.



Figure

Now consider a current carrying element Idl at a distance z meters from the origin and assume that the distance vector between current vector and point N to be $\bar{R}$.

Hence, the magnetic field intensity due to small current element Idl at point N is given by,

$$\overline{dH} = \frac{Id\bar{l}\times\bar{R}}{4\pi R^3} \quad \text{... (1)}$$

But,

$$dl = dz\,\bar{a}_z \text{ and} \quad \text{... (2)}$$

$$\bar{R} = (\rho-0)\bar{a}_\rho + (0-0)\bar{a}_\phi + (0-z)\bar{a}_z$$

$$= \rho\bar{a}_\rho + 0\bar{a}_\phi + (-z)\bar{a}_z$$

$$= \rho\bar{a}_\rho - z\bar{a}_z \quad \text{... (3)}$$

$$|\bar{R}| = \sqrt{\rho^2+z^2} = (\rho^2+z^2)^{1/2} \quad \text{... (4)}$$

Substituting the values of equations (2), (3) and (4) in equation (1), we get,

$$\overline{dH} = \frac{Idz\,\bar{a}_z\times(\rho\bar{a}_\rho - z\bar{a}_z)}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\overline{dH} = \frac{Idz[\rho(\bar{a}_z\times\bar{a}_\rho) - z(\bar{a}_z\times\bar{a}_z)]}{4\pi(\rho^2+z^2)^{3/2}}$$

But, $\bar{a}_z\times\bar{a}_\rho = \bar{a}_\phi$ and

$$\bar{a}_z\times\bar{a}_z = 0$$

Therefore,

$$\overline{dH} = \frac{Idz[\rho\bar{a}_\phi - z(0)]}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\overline{dH} = \frac{Idz[\rho\bar{a}_\phi - 0]}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\overline{dH} = \frac{Idz\,\rho\bar{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}}$$

Hence, the total magnetic field intensity H is given by,

$$\bar{H} = \int d\bar{H}$$

$$\bar{H} = \int \frac{Idz\,\rho}{4\pi(\rho^2+z^2)^{3/2}}\bar{a}_\phi \quad \text{... (5)}$$

From the figure, we have,

$$\tan\theta = \frac{ON}{OL}$$

$$\Rightarrow \tan\theta = \frac{\rho}{z}$$

$$\Rightarrow z = \frac{\rho}{\tan\theta}$$

$$\Rightarrow z = \rho\cot\theta$$

$$\Rightarrow z^2 = \rho^2\cot^2\theta \quad \text{... (6)}$$

$$\Rightarrow dz = -\rho\,\text{cosec}^2\theta\, d\theta \quad \text{... (7)}$$

Substituting equations (6) and (7) in equation (5), we get,

$$\bar{H} = \int_{\theta_1}^{\theta_2} \frac{I\rho(-\rho\,\text{cosec}^2\theta)}{4\pi(\rho^2+\rho^2\cot^2\theta)^{3/2}}\bar{a}_\phi\,d\theta$$

$$= \int_{\theta_1}^{\theta_2} \frac{-I\rho^2\,\text{cosec}^2\theta}{4\pi[\rho^2(1+\cot^2\theta)]^{3/2}}\bar{a}_\phi\,d\theta$$

$$= \int_{\theta_1}^{\theta_2} \frac{-I\rho^2\,\text{cosec}^2\theta}{4\pi(\rho^2)^{3/2}(\text{cosec}^2\theta)^{3/2}}\bar{a}_\phi\,d\theta$$

$$[\because \text{cosec}^2\theta = 1+\cot^2\theta]$$

$$= \frac{-I\rho^2}{4\pi\rho^3}\int_{\theta_1}^{\theta_2}\frac{\text{cosec}^2\theta}{\text{cosec}^3\theta}\,d\theta\,.\,\bar{a}_\phi$$

$$= \frac{-I}{4\pi\rho}\int_{\theta_1}^{\theta_2}\frac{1}{\text{cosec}\,\theta}\,d\theta\,.\,\bar{a}_\phi$$

$$= \frac{-I}{4\pi\rho}\int_{\theta_1}^{\theta_2}\sin\theta\,.d\theta.\bar{a}_\phi \quad \left[\because \sin\theta = \frac{1}{\text{cosec}\,\theta}\right]$$

$$= \frac{-I}{4\pi\rho}\bar{a}_\phi[-\cos\theta]_{\theta_1}^{\theta_2}$$

$$= \frac{I}{4\pi\rho}\bar{a}_\phi[\cos\theta_2 - \cos\theta_1]$$

$$\therefore \quad \bar{H} = \frac{I}{4\pi\rho}[\cos\theta_2 - \cos\theta_1]\,\bar{a}_\phi \text{ A/m} \quad \text{... (8)}$$

Hence, the above equation represents magnetic field intensity $\bar{H}$ due to straight conductor of finite length.

3.2 AMPERE LAW

Q21. State and explain Ampere's force law.

Ans: (Nov./Dec.-15, (R13), 14(a)(ii) | Model Paper-III, Q6(a))

Statement

Ampere's force law states that the line integral of magnetic field intensity ($\bar{H}$) taken around a single closed loop is equal to the current enclosed by that path.

Mathematically it is given as,

$$\oint \bar{H}.d\bar{l} = I$$

Where,

$\bar{H}$ = Magnetic field intensity in A/m

I = Current enclosed in Amps.

Proof

Consider an infinitely long straight current carrying conductor placed on z-axis as shown in figure. The direction of current I in the conductor is upward. The flow of current causes the magnetic field to be produced around the conductor and the magnetic field lines are closed circles. Consider a magnetic field line as shown in figure.

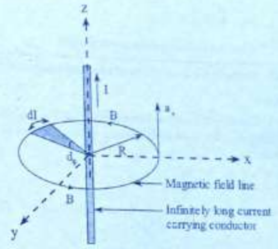


Figure: Geometry of Figure

The magnetic flux density ($\bar{B}$) is given by the relation,

$$\bar{B} = \frac{\mu I}{2\pi R}\bar{a}_\phi \quad \text{... (1)}$$

Where,

$\bar{a}_\phi$ = Unit vector in the direction of $\bar{B}$

R = Radius of the magnetic field line

I = Current carrying by the conductor.

Magnetic flux density around the circular path of radius 'R' is obtained by integrating equation (1),

i.e.,

$$\oint \bar{B}.d\bar{l} = \oint\left[\frac{\mu I}{2\pi R}\bar{a}_\phi\right]d\bar{l}$$

But, we have,

$$d\bar{l} = (Rd\phi)\bar{a}_\phi$$

$$\therefore \quad \oint \bar{B}.d\bar{l} = \oint \frac{\mu I}{2\pi R}\bar{a}_\phi(Rd\phi)\bar{a}_\phi$$

$$\Rightarrow \quad \oint \bar{B}.d\bar{l} = \oint \frac{\mu I}{2\pi R}Rd\phi \quad [\because \bar{a}_\phi.\bar{a}_\phi = 1]$$

$$\Rightarrow \quad \oint \bar{B}.d\bar{l} = \frac{\mu I}{2\pi}\int_0^{2\pi}d\phi$$

$$\Rightarrow \oint \bar{B}.\bar{dl} = \frac{\mu I}{2\pi}[\phi]_0^{2\pi} = \frac{\mu I}{2\pi}[2\pi - 0]$$

$$\Rightarrow \oint \bar{B}.\bar{dl} = \mu I$$

$$\Rightarrow \oint \mu\bar{H}.\bar{dl} = \mu I \qquad [\because \bar{B} = \mu\bar{H}]$$

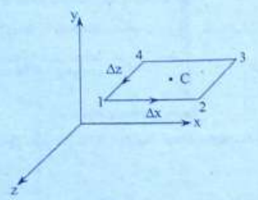
$$\Rightarrow \oint \bar{H}.\bar{dl} = I$$

Thus, Ampere's circuital law is proved.

Q22. Deduce the point form of Ampere's circuital law.

Ans:

Consider a closed rectangular path 1-2-3-4-1 as shown in figure. Let, the sides of differential surface element be Δx and Δz.



Figure

Let I be the current flowing through the closed path. The direction of this current will be along y-axis, as the surface is lying in xz plane, so that H is normal to it. The magnetic field intensity at the centre of this closed path is given as,

$$H_c = H_{xc}a_x + H_{yc}a_y + H_{zc}a_z$$

The line integral of H for this closed rectangular path can be calculated by calculating the individual line integrals for the paths 1-2, 2-3, 3-4, 4-1 and then finally summing them.

Line integration of path 1-2 is,

$$(H.\Delta L)_{1-2} = H_{x_{1-2}} . \Delta x$$

$$H_{x_{1-2}} = H_{xc} + \frac{\partial H_x}{\partial z}\left(\frac{\Delta z}{2}\right) \qquad \ldots (1)$$

Where,

H_{xc} is initial value at point C

$\frac{\partial H_x}{\partial z}$ is rate of change of H_x along z-axis

$\frac{\Delta z}{2}$ is distance between point C and midpoint of path 1-2.

$$\therefore (H.\Delta L)_{1-2} = \left(H_{xc} + \frac{1}{2}\frac{\partial H_x}{\partial z}\Delta z\right)\Delta x \qquad \ldots (2)$$

Line integration of path 2-3 is,

$(H.\Delta L)_{2-3} = H_{z_{2-3}}(-\Delta z)$ (As the path 2-3 is opposite to direction of z)

$$H_{z_{2-3}} = H_{zc} - \frac{1}{2}\frac{\partial H_z}{\partial x}\Delta x$$

$$(H.\Delta L)_{2-3} = \left(H_{zc} - \frac{1}{2}\frac{\partial H_z}{\partial x}\Delta x\right)(-\Delta z)$$

$$= -\left(H_{zc} + \frac{1}{2}\frac{\partial H_z}{\partial x}\Delta x\right)(\Delta z) \qquad \ldots (3)$$

Line integration for path 3-4 is, ,

$$(H.\Delta L)_{3-4} = H_{x_{3-4}}(-\Delta x)$$

$$H_{x_{3-4}} = H_{xc} - \frac{1}{2}\frac{\partial H_x}{\partial z}\Delta z$$

$$(H.\Delta L)_{3-4} = \left(H_{xc} - \frac{1}{2}\frac{\partial H_x}{\partial z}\Delta z\right)(-\Delta x)$$

$$= -\left(H_{xc} - \frac{1}{2}\frac{\partial H_x}{\partial z}\Delta z\right)(\Delta x) \qquad \ldots (4)$$

Line integration for path 4-1

$$(H.\Delta L)_{4-1} = H_{z_{4-1}}(\Delta z)$$

$$H_{z_{4-1}} = H_{zc} - \frac{1}{2}\frac{\partial H_z}{\partial x}\Delta x$$

$$(H.\Delta L)_{4-1} = \left(H_{zc} - \frac{1}{2}\frac{\partial H_z}{\partial x}\Delta x\right)(\Delta z) \qquad \ldots (5)$$

Adding equations (2), (3), (4) and (5), we get,

$$\oint H.dl = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\Delta z.\Delta x$$

From Ampere's circuital law, we have,

$$\oint H.dl = I \qquad \ldots (6)$$

We know that,

$$J = \frac{I}{A}$$

$$I = JA$$

$$I = J.\Delta x\,\Delta z \qquad \ldots (7)$$

Substituting equation (6) in equation (7), we get,

$$\therefore \oint H.dl = J_y\Delta x\,\Delta z$$

$$\oint H.dl = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\Delta z.\Delta x = J_y\Delta x\,\Delta z$$

$$\oint \frac{H.dl}{\Delta z.\Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

$$\underset{\Delta z\Delta x \to 0}{Lt}\oint \frac{H.dl}{\Delta z.\Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \qquad \ldots (8)$$

The above equation is obtained by considering a closed rectangular path in xz plane.

In the same way, considering closed paths in xy and yz planes, we get,

$$\underset{\Delta x\Delta y \to 0}{Lt}\oint \frac{Hdl}{\Delta x.\Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \qquad \ldots (9)$$

$$\underset{\Delta y\Delta z \to 0}{Lt}\oint \frac{Hdl}{\Delta y.\Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \qquad \ldots (10)$$

Current density,

$$J = J_x a_x + J_y a_y + J_z a_z$$

$$\therefore \quad J = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)a_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)a_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)a_z$$

R.H.S can be represented in a determinant form as,

$$J = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Which is termed as *curl H*,

$$\boxed{\therefore J = Curl\ H = \nabla \times H}$$

Q23. Derive the expression for magnetic field intensity due to infinitely long straight conductor carrying a current of I amps along z-axis.

Ans:

Consider an infinite line charge carrying current I in the upward direction and is placed along the z-axis as shown in figure.

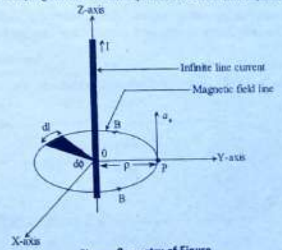


Figure: Geometry of Figure

Let P be the point, where the magnetic field intensity is to be found is placed on the y-axis and consider a magnetic field line passing through the point 'P'. Let the point P is at a distance of 'ρ' from the origin as depicted in figure.

According to Ampere's law, we have,

$$I = \oint \bar{H}.d\bar{l} \quad \text{...(1)}$$

Where,

I = Current enclosed in Amp

$\bar{H}$ = Magnetic field intensity in A/m

$= H_\phi . \bar{a}_\phi$

$d\bar{l}$ = Differential elementary length

$= \rho . d\phi . \bar{a}_\phi$

Substituting the above equations in equation (1), we get,

$$I = \oint H_\phi \bar{a}_\phi (\rho\, d\phi \bar{a}_\phi)$$

$$\Rightarrow \quad I = \int_{\phi=0}^{2\pi} H_\phi (\rho\, d\phi)(\bar{a}_\phi . \bar{a}_\phi)$$

$$\Rightarrow \quad I = H_\phi \int_{\phi=0}^{2\pi} \rho d\phi\, (1) \qquad [\because \bar{a}_\phi . \bar{a}_\phi = 1]$$

$$\Rightarrow \quad I = H_\phi \rho [\phi]_0^{2\pi}$$

$$\Rightarrow \quad I = H_\phi \rho (2\pi)$$

$$\Rightarrow \quad H_\phi = \frac{I}{2\pi\rho}$$

$$\therefore \quad \bar{H} = \frac{I}{2\pi\rho} . \bar{a}_\phi \quad \text{...(2)}$$

$\therefore$ Equation (2) gives the magnetic field intensity ($\bar{H}$) at a point P due to an infinite line charge.

Q24. An infinitely long, straight conductor with a circular cross section of radius 'b' carries a steady current 'I'. Determine magnetic flux density both inside and outside the conductor.

Ans: (Nov./Dec.-15, (R13), Q13(a) | Model Paper-II, Q6(a))

Consider a solid cylindrical conductor of radius 'b' and current I is uniformly distributed over the cross section as shown in figure (1).

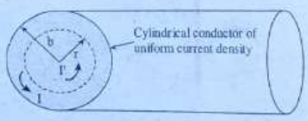


Figure (1)

Case (i): Within the Conductor (r < b)

Since, the current density is uniform throughout the conductor, it is given by,

$$J = \frac{I}{A}$$

Where,

A = Area of the cylinder

$$J = \frac{I}{\pi b^2}$$

Now, consider an area within the cylinder of radius 'r' such that $r < a$, then the current enclosed I' by this area is given by,

$$I' = \left(\frac{\pi r^2}{\pi b^2}\right) I$$

$$\Rightarrow \quad I' = \left(\frac{r^2}{b^2}\right) I$$

We know that magnetic field intensity ($\bar{H}$) of radius r is given by,

$$\oint H' \cdot dl = I'$$

We have, $dl = r d\phi$

$$\oint H' r d\phi = I\left[\frac{r^2}{b^2}\right]$$

$$\Rightarrow \quad \int_{\phi=0}^{2\pi} H' r d\phi = I\left[\frac{r^2}{b^2}\right]$$

$$\Rightarrow \quad H' r \int_{\phi=0}^{2\pi} d\phi = I\left[\frac{r^2}{b^2}\right]$$

$$\Rightarrow \quad H' r \times [\phi]_0^{2\pi} = I\left[\frac{r^2}{b^2}\right]$$

$$\Rightarrow \quad H' r (2\pi) = I \frac{r^2}{b^2}$$

$$\Rightarrow \quad H' = \frac{I}{2\pi r} \times \frac{r^2}{b^2}$$

$$\therefore \quad H' = \frac{r}{2\pi b^2} \text{ A/m for } r < a$$

At the surface of the conductor i.e., $r = b$, H is given by,

$$H = \frac{I b}{2\pi b^2}$$

$$\Rightarrow \quad H = \frac{I}{2\pi b} \text{ A/m for } r = b$$

Case (ii): For Outside the Conductor (r > b)

For outside the conductor i.e., $r > b$, the corresponding current will be $I' = I$.

Applying Ampere's circuital law, we get,

$$\int \overline{H}.d\overline{l} = I$$

$$H(2\pi r) = I$$

$$H = \frac{I}{2\pi r} \text{ A/m for } r > b$$

Variation of H against r

The magnetic field intensity H is obtained for inside and outside of the conductor as $H = \frac{I}{2\pi b}$ and $H = \frac{I}{2\pi r}$ respectively. Therefore, the variation of field intensity with respect to distance 'r' from the axis of conductor is represented in figure (2).

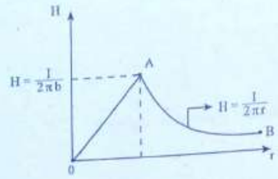


Figure (2)

Magnetic flux density of infinitely long, straight conductor with circular cross-section is given as,

$$B = \mu \overline{H}$$

$$\therefore\ B = \frac{r}{2\pi b^2} \text{ A/m for } r < b$$

$$= \frac{\mu I}{2\pi b} \text{ A/m for } r = b$$

$$= \frac{\mu I}{2\pi r} \text{ A/m for } r > b$$

3.3 MAGNETIC-FLUX AND MAGNETIC FLUX DENSITY

Q25. Explain the relationship between magnetic flux and magnetic flux density.

Ans:

Relation Between Magnetic Flux (ϕ), Magnetic Flux Density ($\overline{B}$)

Magnetic flux (ϕ) constitutes the magnetic lines of force produced by a permanent magnet or a current carrying conductor. In case of current carrying conductor, the magnetic lines of force surround the conductor to form concentric circles and they do not intersect each other as shown in the figure (1).

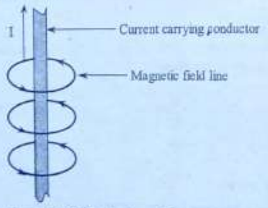


Figure (1): Field Due to Current Carrying Conductor

The direction and orientation of this magnetic field can be determined with the help of Fleming's right hand rule or right hand screw rule.

Magnetic flux is expressed in Webers (Wb).

Magnetic flux density ($\overline{B}$) is the measure of magnetic field lines. It is defined as the magnetic flux per unit area.

Mathematically,

$$|\overline{B}| = \frac{\phi}{A} \text{ Wb/m}^2 \text{ or } T \quad \ldots (1)$$

Where,

ϕ = Magnetic flux, Wb

A = Surface area, m^2.

Magnetic flux density ($\overline{B}$) can be explained with the help of figure (2).

Consider a thin sheet of surface area 'A' is placed in a magnetic field. Let 'θ' be the angle made by the field lines with the normal to surface as shown in figure (2).

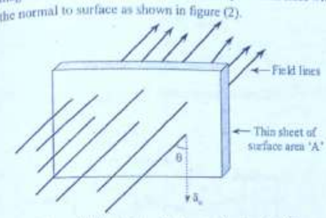


Figure (2): Illustration of Magnetic Flux Density ($\overline{B}$)

The magnetic flux crossing the surface of area 'A' is given by,

$$\phi = |\overline{B}|\, A \cos\theta \text{ Wb}$$

If the field is not uniform, then the flux crossing the surface is given by,

$$\phi = \int_s \overline{B}\, d\overline{s} \text{ Wb} \quad \ldots (2)$$

Q26. Discuss the relation between magnetic flux (ϕ), magnetic flux density ($\overline{B}$), magnetic field intensity ($\overline{H}$).

Ans:

Relation between Magnetic Flux (ϕ), Magnetic Flux Density ($\overline{B}$) and Magnetic Flux Intensity ($\overline{H}$)

For answer refer Unit-III, Q25.

Consider a conductor 'R' carrying current of 1 amp, which produces a magnetic field, in the form of concentric circles, around the conductor. Consider a flux path at a distance of 1 m from the conductor 'R'. The magneto motive force producing the flux is given by,

$$|\overline{F}_R| = NI = 1 \times 1 = 1 \text{ N} \quad [\because \text{Number of turns} = 1]$$

$$\therefore\ |\overline{F}_R| = 1 \text{ N}$$

Magnetic field intensity ($\overline{H}$) is defined as the force experienced by a unit north pole, when it is placed in a magnetic field.

The field intensity at all points on the flux path is given by,

$$\overline{H} = \frac{F}{l} = \frac{1}{2\pi(1)} = \frac{1}{2\pi} \text{ A/m} \quad \ldots (1)$$

Now, consider another conductor, S, placed at a distance of 1 m from the conductor R. The force on the conductor-S due to the current in conductor R is given by,

$$\overline{F}_1 = \overline{B}Il \text{ N}$$

$$\Rightarrow |\overline{F}_1| = |\overline{B}| \times 1 \times 1 \text{ N}$$

$$\Rightarrow |\overline{F}_1| = |\overline{B}| \times \text{N}$$

From the definition of ampere, the force per meter on charge 'Q' is 2×10^{-7} N.

$$|\overline{F}_1| = 2 \times 10^{-7} \text{ N}$$

$$\Rightarrow |\overline{B}| = 2 \times 10^{-7} \text{ Wb/m}^2 \quad \ldots (2)$$

Dividing equation (2) by equation (1), we get,

$$\Rightarrow \frac{|\overline{B}|}{|\overline{H}|} = \frac{2 \times 10^{-7}}{\left(\frac{1}{2\pi}\right)}$$

$$\Rightarrow \frac{|\overline{B}|}{|\overline{H}|} = 4\pi \times 10^{-7}$$

$$\Rightarrow |\overline{B}| = 4\pi \times 10^{-7} (\overline{H})$$

$$\Rightarrow |\overline{B}| = \mu_0 (\overline{H})$$

Where,

μ_0 = Permeability of free space, H/m.

In vector form,

$$\overline{B} = \mu_0 \overline{H}$$

In any other medium,

$$\overline{B} = \mu_0 \mu_r \overline{H}$$

Where,

μ_r = Relative permeability of the medium, H/m.

Q27. Calculate the magnetic flux density due to a coil of 100 amperes and area 50 cm² on the axis of the coil at a distance 10 m from the centre.

Ans: (Dec.-11, Set-1, Q3(b) | Model Paper-I, Q7(a))

Given that,

Current, I = 100 Amperes

Area of cross-section, A = 50 cm^2

$= 50 \times 10^{-4}$ m^2

Distance from centre, x = 10 m

To determine

Magnetic flux density, B = ?

We know that,

$$\text{Radius, } R = \frac{A}{2}$$

$$= \frac{50\times10^{-4}}{2}$$

$$= 25\times10^{-4}\ m^2$$

$\therefore$ Magnetic flux density at a distance of 10 m from the centre on the axis of the coil,

$$B = \frac{\mu_0 I R^2}{2(R^2+x^2)^{3/2}}$$

$$= \frac{(4\pi\times10^{-7})(100)(25\times10^{-4})^2}{2[(25\times10^{-4})^2+(10)^2]^{3/2}}$$

$\therefore$ $B = 3.926\times10^{-13}$ Wb/m² (or) Tesla.

Q28. Calculate the magnetic flux density due to a coil carrying 60 amperes and area 20 cm² on the axis of the coil at a distance 6 m from the centre.

Ans: Nov.-10, Set-3, Q4(a)

Given that,

Current, $I = 60$ Amperes

Area of cross-section, $A = 20\ cm^2$

$= 20\times10^{-4}\ m^2$

Distance from the centre, $x = 6$ m

Magnetic flux density, $B = ?$

$$\text{Radius, } R = \frac{A}{2}$$

$$= \frac{20\times10^{-4}}{2}$$

$$= 10\times10^{-4} \text{ meters.}$$

$\therefore$ Magnetic flux density at a distance of 6 m from the coil and on the axis of the coil,

$$B = \frac{\mu_0 I R^2}{2(R^2+x^2)^{3/2}}$$

$$= \frac{(4\pi\times10^{-7})\times60\times(10\times10^{-4})^2}{2[(10\times10^{-4})^2+6^2]^{3/2}}$$

$$(\because \mu_0 = 4\pi\times10^{-7})$$

$$= \frac{7.54\times10^{-11}}{432}$$

$$= 1.74\times10^{-13}\ Wb/m^2$$

Q29. A solenoid of radius 8 cm and length 16 cm is wound uniformly with 200 turns of wire and carries a current of 6 A. Calculate the flux density at the point on the axis at a distance 8 cm from the middle of the coil.

Ans: Nov.-10, Set-3, Q4(b)

Given that,

Radius of solenoid, $r = 8$ cm

$= 0.08$ m

Length of the solenoid, $l = 16$ cm

$= 0.16$ m

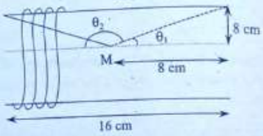
Number of turns, $N = 200$

Current, $I = 6$ A

Flux density, $B = ?$

$$\text{Number of turns/meter, } N = \frac{200}{0.16}$$

$$= 1250$$



Figure

From figure,

$$\tan\theta_1 = \frac{8}{8}$$

$$= 1$$

$$\theta_1 = 45°$$

Similarly,

$$\tan(\pi-\theta_2) = \frac{8}{8}$$

$$= 1$$

$$\pi - \theta_2 = 45$$

$$\theta_2 = 135$$

The magnetic field intensity at M is given by,

$$H = \frac{NI}{2}(\cos\theta_1 - \cos\theta_2)$$

$$= \frac{1250\times6}{2}(\cos45° - \cos135°)$$

$$= 5303.3$$

We know that,

Magnetic flux density,

$$B = \mu_o H$$

$$= 4\pi\times10^{-7}\times5303.3$$

$$= 6.66\times10^{-3}\ Wb/m^2$$

$\therefore$ Magnetic flux density,

$$B = 6.66\times10^{-3}\ Wb/m^2$$

Q30. Find the magnetic field intensity at the centre of square loop of side 5 m carrying 10 A of current.

Ans: Nov./Dec.-17, (R16), Q6(b)

Given that,

Current, $I = 10$ A

Each side of square = 5 m

Consider square loop in XY plane with length 5 m.

Let 'O' be the centre of square.

Now, consider the side AD or square, lying along the y-axis.

If a perpendicular is drawn from O to AD, then it bisects AD at midpoint P.

We know that,

$$AD = 5\ m$$

$$AP = \frac{5}{2}\ m$$

i.e., $r = 2.5$ m

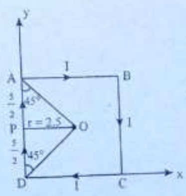


Figure (1)

$$\alpha_2 = \tan^{-1}\left(\frac{2.5}{2.5}\right)$$

$$= \tan^{-1}(1)$$

$$= 45°$$

$$\alpha_1 = 180° - 45°$$

$$= 135°$$

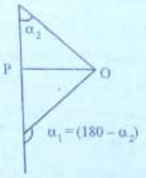


Figure (2)

Magnetic field intensity at O due to the current carrying element AD is given as,

$$H_1 = \frac{I}{4\pi r}[\cos(\alpha_2) - \cos(\alpha_1)]$$

$$= \frac{10}{4\pi\times2.5}[\cos45° - \cos135°]$$

$$= 0.45\ Wb/m^2$$

From figure (1) the field intensities from sides H_2, H_3, H_4 and H_1 due to AB, BC, CD and DA respectively are equal. Hence, they all add up.

$\therefore$ Total field intensity is,

$$H = H_1 + H_2 + H_3 + H_4$$

$$= 4H_1$$

$$= 4(0.45)$$

$$= 1.8\ b/m^2$$

3.4 SCALAR AND VECTOR MAGNETIC POTENTIALS

Q31. Derive the expression for vector magnetic potential in terms of current density.

Ans:

Definition

Vector magnetic potential is defined in such a way that the curl of it gives the magnetic flux density. Mathematically, it is expressed as,

$$\overline{B} = \nabla\times\overline{A}$$

Where,

$\overline{B}$ = Magnetic flux density

$\overline{A}$ = Vector magnetic potential.

Derivation

From Maxwell's equation, we have,

$$\nabla.\overline{B} = 0$$

From the definition of vector magnetic potential, we have,

$$\overline{B} = \nabla \times \overline{A} \quad \text{...(1)}$$

We know that,

$$\overline{B} = \mu_0 \overline{H}$$

$$\Rightarrow \quad \overline{H} = \frac{\overline{B}}{\mu_0}$$

Substituting equation (1), in the above expression we get,

$$\overline{H} = \left(\frac{1}{\mu_0}\right) \nabla \times \overline{A}$$

Taking curl on both sides, we get,

$$\Rightarrow \nabla \times \overline{H} = \nabla \times \left(\frac{1}{\mu_0} \nabla \times \overline{A}\right)$$

$$\Rightarrow \nabla \times \overline{H} = \frac{1}{\mu_0} (\nabla \times \nabla \times \overline{A}) \quad \text{...(2)}$$

But, we know that,

$$\nabla \times \nabla \times \overline{A} = \nabla(\nabla.\overline{A}) - \nabla^2 \overline{A} \quad \text{...(3)}$$

Substituting equation (3) in equation (2), we get,

$$\nabla \times \overline{H} = \frac{1}{\mu_0} [\nabla(\nabla.\overline{A}) - \nabla^2 \overline{A}] \quad \text{...(4)}$$

We know that,

$$\nabla \times \overline{H} = \overline{J} \text{ and } \nabla(\nabla.\overline{A}) = 0$$

Substituting above values in equation (4), we get,

$$\Rightarrow \quad \overline{J} = \frac{1}{\mu_0} [-\nabla^2.\overline{A}]$$

$$\therefore \quad \nabla^2 \overline{A} = -\mu_0 \overline{J}$$

Where,

$\overline{A}$ = Vector magnetic potential (Wb/m)

$\overline{J}$ = Current density (A/m²).

Q32. Compare and contrast scalar and vector potentials.

Ans:

Model Paper-III, Q7(a)

	Magnetic Scalar Potential		Magnetic Vector Potential
1.	Scalar magnetic potential is one whose negative gradient equals the magnetic field intensity ($\overline{H}$) i.e., $\overline{H} = -\nabla V_m$	1.	Vector magnetic potential is one whose curl gives the magnetic flux density ($\overline{B}$) i.e., $\overline{B} = \nabla \times \overline{A}$
2.	It is a scalar quantity.	2.	It is a vector quantity.
3.	The unit of scalar magnetic potential is Amperes.	3.	The unit of vector magnetic potential is Wb/meter.
4.	Scalar magnetic potential finds its application only for regions where current density, $\overline{J} = 0$.	4.	Vector magnetic potential finds its application for the regions where current density $\overline{J} = 0$ as well as current densify, $\overline{J} \neq 0$.
5.	Scalar magnetic potential concept cannot be applied to the field produced by steady current.	5.	Vector magnetic potential concept can be applied to the field produced by steady magnetic fields.
6.	It is used in permanent magnet applications and magnetic circuits where the area covered by the current carrying elements is negligibly smaller in comparison to the magnetic field.	6.	It is used in calculating radiation characteristics and near and far fields of antennas.

Q33. Give the applications and limitations of scalar magnetic potential.

Ans:

Applications of Scalar Magnetic Potential

Scalar magnetic potential is used to find the magnetic field intensity and hence the flux density. It is also used to plot the flux lines in the following areas of application,

(i) Permanent magnet applications (Where the current carrying elements are not at all required).

(ii) Magnetic circuits where the area covered by the current carrying elements is negligibly smaller when compared with that of the magnetic field.

(iii) Magnetic circuits in which only a constant amount of direct current flows through the conductors. Because, the current density within the field is zero except in the conductor region.

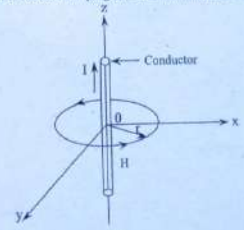
Limitations of Scalar Magnetic Potential

1. Scalar magnetic potential is defined only in the regions where no electric current flows i.e., where current density does not exists.
2. Unlike electrostatic potential, it is not a conservative field. It means that, for each selected path between two points in the field, different values of V_m is obtained.

Q34. Derive the expression for the magnetic vector potential in the case of an infinitely long straight conductor in free space.

Ans:

Consider an infinitely long straight conductor carrying current I and is placed along the z-axis as shown in figure (1).



Figure

From the definition of magnetic vector potential, we have,

$$\bar{B} = \nabla \times \bar{A} \quad \text{...(1)}$$

The magnetic flux density of the conductor placed in free space is given by,

$$\bar{B} = \frac{\mu_0 I}{2\pi r}\bar{a}_\phi \quad \text{...(2)}$$

Substituting equation (2) in equation (1) and considering the cylindrical coordinate system for $\bar{A}$, we get,

$$\frac{\mu_0 I}{2\pi r}\bar{a}_\phi = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A}{\partial z}\right]\bar{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi + \frac{1}{r}\left[\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\bar{a}_z$$

Comparing the coefficients of $\bar{a}_\phi$, we get,

$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

As the magnetic flux density $(\bar{B})$ is a function of r only, the vector magnetic potential $(\bar{A})$ will change with r only and will be constant with respect to z.

i.e., $\frac{\partial A_r}{\partial z} = 0$

$$\therefore \quad \frac{\partial A_z}{\partial r} = -\frac{\mu_0 I}{2\pi r}$$

Now, integrating on both sides, we get,

$$\int \frac{\partial A_z}{\partial r} = \int -\frac{\mu_0 I}{2\pi r} d_r$$

$$A_z = \frac{-\mu_0 I}{2\pi}\int \frac{1}{r}\, dr$$

$$\therefore \quad A_z = -\frac{\mu_0 I}{2\pi}\log r + C_1 \quad \text{...(3)}$$

To calculate 'C_1', consider the boundary condition A_z as zero at $r = r_0$

Now equation (3) becomes,

$$0 = -\frac{\mu_0 I}{2\pi}\log r_0 + C_1$$

$$\therefore \quad C_1 = \frac{\mu_0 I}{2\pi}\log r_0$$

Substituting the value of C_1 in equation (3), we get,

$$A_z = -\frac{\mu_0 I}{2\pi}\log r + \frac{\mu_0 I}{2\pi}\log r_0$$

$$= \frac{\mu_0 I}{2\pi}[\log r_0 - \log r]$$

$$A_z = \frac{\mu_0 I}{2\pi}\left[\log\left(\frac{r_0}{r}\right)\right]$$

The vector magnetic potential is given by,

$$\bar{A} = A_z\bar{a}_z$$

$$\boxed{\therefore \bar{A} = \frac{\mu_0 I}{2\pi}\left[\log\left(\frac{r_0}{r}\right)\right]\bar{a}_z \text{ Wb/m}}$$

Q35. Derive both Biot-Savart's law and Ampere's law using the concept of magnetic vector potential.

Ans:

Biot-Savart's Law

By the definition of vector magnetic potential, we have,

$$\bar{B} = \nabla \times \bar{A} \quad \text{...(1)}$$

The vector magnetic potential for the distribution of line current is given by,

$$A = \int_L \frac{\mu I dl}{4\pi r} \quad \text{...(2)}$$

Substituting equation (2) in equation (1) we get,

$$\bar{B} = \nabla \times \left[\frac{\mu}{4\pi}\int \frac{I d\bar{l}}{r}\right]$$

$$= \frac{\mu I}{4\pi}\int \left(\nabla \times \frac{d\bar{l}}{r}\right) \quad \text{...(3)}$$

From vector identity, we know that,

$$\nabla \times (f\,\bar{F}) = f\,\nabla \times \bar{F} + (\nabla f) \times \bar{F}$$

$$\therefore \quad \nabla \times \frac{d\bar{l}}{r} = \frac{1}{r}(\nabla \times d\bar{l}) + \nabla\left(\frac{1}{r}\right) \times d\bar{l}$$

But, the differential length $d\bar{l}$ is an independent variable.

$$\therefore \quad \nabla \times d\bar{l} = 0$$

$$\Rightarrow \quad \nabla \times \frac{d\bar{l}}{r} = \nabla\left(\frac{1}{r}\right) \times d\bar{l}$$

$$= \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\bar{a}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\right)\bar{a}_\phi + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\bar{a}_z\right] \times d\bar{l}$$

$$= \left[\frac{-1}{r^2}\bar{a} + 0 + 0\right] \times d\bar{l}$$

$$= \frac{-1}{r^2}\bar{a}_r \times d\bar{l}$$

$$= \frac{-1}{r^2}(-d\bar{l} \times \bar{a}_r)$$

$$= \frac{d\bar{l} \times \bar{a}_r}{r^2} \quad \text{...(4)}$$

Substituting equation (4) in equation (3), we get,

$$\bar{B} = \frac{\mu I}{4\pi}\int \frac{d\bar{l} \times \bar{a}_r}{r^2}$$

Appl, ing differentiation on both sides, we get,

$$d\bar{B} = \frac{\mu I d\bar{l} \times \bar{a}_r}{4\pi r^2} \quad \text{...(5)}$$

We know that,

$$\bar{B} = \mu\bar{H}$$

$$\Rightarrow \quad d\bar{B} = \mu d\bar{H} \quad \text{...(6)}$$

Substituting equation (6) in (5), we get,

$$\Rightarrow \quad \mu\, d\overline{H} = \frac{\mu I d\overline{l} \times \overline{a_r}}{4\pi r^2}$$

$$\Rightarrow \quad d\overline{H} = \frac{I d\overline{l} \times \overline{a_r}}{4\pi r^2}$$

The above equation represents Biot-Savart's law.

Ampere's Law

Applying curl on both sides of equation (1), we get, ... (7)

$$\nabla \times \overline{B} = \nabla \times \nabla \times \overline{A}$$

From vector identity, we know that, ... (8)

$$\nabla \times \nabla \times \overline{A} = \nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A} \quad \text{... (9)}$$

For steady magnetic fields, $\nabla \cdot \overline{A} = 0$

Substituting equations (8) and (9) in equation (7), we get,

$$\nabla \times \overline{B} = -\nabla^2 \overline{A}$$

$$\Rightarrow \quad \nabla^2 A = \mu_0 \nabla \times \overline{H} \qquad [\because \overline{B} = \mu_0 \overline{H}]$$

$$\Rightarrow \quad \nabla^2 \overline{A} = -\mu_0 \overline{J} \qquad [\because \nabla \times \overline{H} = \overline{J}] \quad \text{... (10)}$$

The above equation represents vector Poisson's equation.

In Cartesian coordinates, equation (10) is dissolved into scalar equations known as scalar Poisson's equation.

i.e., $\nabla^2 A_x = -\mu_0 J_x$

$\nabla^2 A_y = -\mu_0 J_y$

$\nabla^2 A_z = -\mu_0 J_z$

According to Stoke's theorem, we have,

$$\oint_L \overline{H} \cdot d\overline{l} = \int_S \nabla \times H \cdot ds \quad \text{... (11)}$$

We know that,

$$\overline{B} = \mu_0 \overline{H}$$

$$\nabla \times \overline{A} = \mu_0 \overline{H}$$

$$\therefore \quad \overline{H} = \frac{1}{\mu_0} \nabla \times \overline{A} \quad \text{... (12)}$$

Substituting equation (12) in equation (11), we get,

$$\oint_L \overline{H} \cdot d\overline{l} = \frac{1}{\mu_0} \int_s \nabla \times \nabla \times \overline{A} \cdot d\overline{s}$$

$$= \frac{1}{\mu_0} \int_s \mu_0 \overline{J} \cdot d\overline{s} \qquad [\because \text{From equations (8), (9) and (10)}]$$

$$= \int_s \overline{J} \cdot d\overline{s}$$

$$\therefore \quad \int_L \overline{H} \cdot d\overline{l} = I \qquad \left[\because \int_s \overline{J} \cdot d\overline{s} = I\right]$$

The above equation represents Ampere's circuit law.

Q36. At point P (x, y, z) the components of vector magnetic potential $\hat{A}$ are given $A_x = 4x + 3y + 2z$, $A_y = 5x + 6y + 3z$, $A_z = 2x + 3y + 5z$. Find B at point P.

Ans:

April/May-15, (R13), Q13(b)(ii)

Given that,

The components of vector magnetic potential are,

$A_x = 4x + 3y + 2z$

$A_y = 5x + 6y + 3z$

$A_z = 2x + 3y + 5z$

$\bar{B}$ at point $P = ?$

The expression for $\bar{B}$ from vector magnetic potential is given as,

$$B = \nabla \times \bar{A}$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$J = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x+3y+2z & 5x+6y+3z & 2x+3y+5z \end{vmatrix}$$

$$= \hat{a}_x\left[\frac{\partial}{\partial y}(2x+3y+5z) - \frac{\partial}{\partial z}(5x+6y+3z)\right] - \hat{a}_y\left[\frac{\partial}{\partial x}(2x+3y+5z) - \frac{\partial}{\partial z}(4x+3y+2z)\right]$$

$$+ \hat{a}_z\left[\frac{\partial}{\partial x}(5x+6y+3z) - \frac{\partial}{\partial y}(4x+3y+2z)\right]$$

$$= \hat{a}_x[3-3] - \hat{a}_y[2-2] + \hat{a}_z[5-3]$$

$$= 2\hat{a}_z \text{ Wb/m}^2$$

$\therefore$ $\bar{B}$ at point $P = 2\hat{a}_z$ Wb/m²

Q37. Magnetic vector potential $\bar{A} = -\frac{\rho^2}{4}\hat{a}_z$ Wb/m, Calculate the total magnetic flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \le \rho \le 2$m, $0 \le z \le 5$m?

Ans:

April/May-15, (R13), Q13(a)(i)

Given that,

Magnetic vector potential, $\bar{A} = \hat{a}_z = -\frac{\rho^2}{4}\hat{a}_z$ Wb/m $= -0.25\rho^2\hat{a}_z$ Wb/m

$\phi = \frac{\pi}{2}$, $1 \le \rho \le m$ $0 \le z \le 5$m

Magnetic flux, $\psi = ?$

The expression of magnetic flux through a surface is given by,

$$\psi = \int B\,ds$$

Where,

$$B = \nabla \times A,\ ds = dx\,dy\,\hat{a}_z$$

$$\therefore \quad \psi = \iint (\nabla \times A)\, d\rho\, dz\, a_\phi \quad \text{... (1)}$$

We know that,

$$\nabla \times A = -\frac{\partial}{\partial \rho}(\bar{A})\hat{a}_\phi$$

$$= \frac{\partial}{\partial \rho}(-0.25\rho^2)\hat{a}_\phi$$

$$= 0.25 \times 2\rho\,\hat{a}_\phi$$

$$\therefore \quad \boxed{\nabla \times A = 0.5\rho\phi}$$

By substituting the above value in equation (1),

$$\psi = \int_0^5\int_1^2 (0.5\rho)\, d\rho\, dz\, \hat{a}_\phi$$

$$= \int_0^5\int_1^2 (\rho)\, d\rho\, dz$$

$$= (0.5)\int_0^5 \left[\frac{\rho^2}{2}\right]_1^2 dz$$

$$= \frac{1}{2}\int_0^5 \left(\frac{3}{2}\right) dz = \frac{3}{4}[z]_0^5$$

$$= \frac{3}{4}[5-0]$$

$$= \frac{15}{4}$$

$\boxed{\therefore \text{ Total flux crossing the surface } \psi = \frac{15}{4}\text{ Wb}}$

3.5 STEADY MAGNETIC FIELD PRODUCED BY CURRENT CARRYING CONDUCTORS

Q38. Explain the concept of static magnetic fields and magnetic field intensity.

Ans:

Static Magnetic Fields

Static magnetic fields are produced by either permanent magnets or by a current carrying element.

Consider a wire carrying current 'I' in the upward direction as shown in figure (1).

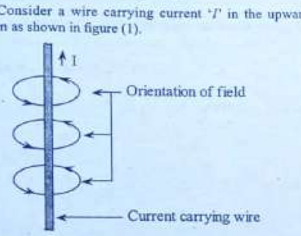


Figure (1): Magnetic Field due to Current Carrying Wire

Whenever a current is passed through a conductor, magnetic field is created round about the conductor. Now, if a compass is placed in the vicinity of the conductor, the needle gets deflected and confirms the presence of magnetic field. The direction of it is given by right hand thumb rule. If the wire is held in the right hand such that the fingers encircles the wire, then the thumb shows the direction of current i.e., upwards in this care, and the fingers point in the direction of magnetic field. The directions of current and magnetic field are shown in the figure (1).

This magnetic field constitutes magnetic lines of force and are magnetic flux, denoted by ϕ and measured in webers (Wb). In case of current carrying wire, there lines forms concentric circles round the conductor as shown in the figure (2).

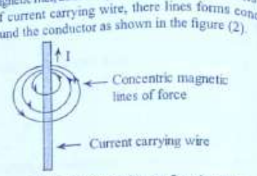


Figure (2): Field around a Current Carrying Wire

Magnetic Field Intensity

Magnetic field intensity at any point is defined as the force experienced by a unit north pole at that point when it is placed in a uniform magnetic field.

It is denoted by $\overline{H}$ and measured in ampere per meter (A/m). It is vector quantity and in analogous to electric field intensity.

Q39. Find the magnetic field intensity due to a current carrying conductor with finite length.

Nov./Dec.-18, (R16), Q1(f)

OR

Find the magnetic field intensity due to a straight current carrying filament.

Nov./Dec.-18, (R16), Q6(a)

OR

Derive the expression for H due to finite length wire carrying a steady current I.

Dec.-14, (R13), Q6

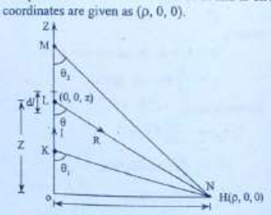
OR

Explain magnetic field intensity H due to finite conductor.

Ans: Model Paper-I, Q6(b)

Consider a current carrying finite conductor along the z-axis as shown in figure.

The point N at which MFI is to be found is on x-axis, whose coordinates are given as $(\rho, 0, 0)$.



Figure

Now consider a current carrying element Idl at a distance z meters from the origin and assume that the distance vector between current vector and point N to be $\overline{R}$.

Hence, the magnetic field intensity due to small current element Idl at point N is given by,

$$dH = \frac{Idl\times\overline{R}}{4\pi R^3} \quad \text{...(1)}$$

But,

$$dl = dz\,\overline{a}_z \text{ and} \quad \text{...(2)}$$

$$\overline{R} = (\rho-0)\overline{a}_\rho + (0-0)\overline{a}_\phi + (0-z)\overline{a}_z$$

$$\overline{R} = \rho\overline{a}_\rho + 0\overline{a}_\phi + (-z)\overline{a}_z$$

$$\overline{R} = \rho\overline{a}_\rho - z\overline{a}_z \quad \text{...(3)}$$

$$|\overline{R}| = \sqrt{\rho^2+z^2} = (\rho^2+z^2)^{1/2} \quad \text{...(4)}$$

Substitute the values of equations (2), (3) and (4) in equation (1), we get,

$$dH = \frac{Idz\;\overline{a}_z\times(\rho\overline{a}_\rho - z\overline{a}_z)}{4\pi(\rho^2+z^2)^{3/2}}$$

$$dH = \frac{Idz\,[\rho(\overline{a}_z\times\overline{a}_\rho) - z(\overline{a}_z\times\overline{a}_z)]}{4\pi(\rho^2+z^2)^{3/2}}$$

But, $\overline{a}_z\times\overline{a}_\rho = \overline{a}_\phi$ and

$\overline{a}_z\times\overline{a}_z = 0$

Therefore,

$$\overline{dH} = \frac{Idz[\rho\overline{a}_\phi - z(0)]}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\overline{dH} = \frac{Idz[\rho\overline{a}_\phi - 0]}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\overline{dH} = \frac{Idz\;\rho\overline{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}}$$

Hence, the total magnetic field intensity H is given by,

$$\overline{H} = \int\overline{dH}$$

$$H = \int\frac{Idz\;\rho}{4\pi(\rho^2+z^2)^{3/2}}\overline{a}_\phi \quad \text{...(5)}$$

From the figure, we have,

$$\tan\theta = \frac{ON}{OL}$$

$$\Rightarrow \quad \tan\theta = \frac{\rho}{z}$$

$$\Rightarrow \quad z = \frac{\rho}{\tan\theta}$$

$$\Rightarrow \quad z = \rho\cot\theta \quad \text{...(6)}$$

$$\Rightarrow \quad dz = -\rho\,\text{cosec}^2\theta\, d\theta \quad \text{...(7)}$$

On substituting equations (6) and (7) in equation (5), we get.

$$H = \int_{\theta_1}^{\theta_2}\frac{I\rho(-\rho\,\text{cosec}^2\theta)}{4\pi(\rho^2+\rho^2\cot^2\theta)^{3/2}}\overline{a}_\phi\, d\theta$$

$$H = \int_{\theta_1}^{\theta_2}\frac{-I\rho^2\,\text{cosec}^2\theta}{4\pi[\rho^2(1+\cot^2\theta)]^{3/2}}\overline{a}_\phi\, d\theta$$

$$H = \int_{\theta_1}^{\theta_2}\frac{-I\rho^2\,\text{cosec}^2\theta}{4\pi(\rho^2)^{3/2}(\text{cosec}^2\theta)^{3/2}}\overline{a}_\phi\, d\theta$$

$$[\because \text{cosec}^2\theta = 1+\cot^2\theta]$$

$$H = \frac{-I\rho^2}{4\pi\rho^3}\int_{\theta_1}^{\theta_2}\frac{\text{cosec}^2\theta}{\text{cosec}^3\theta}\,d\theta\,.\,\overline{a}_\phi$$

$$H = \frac{-I}{4\pi\rho}\int_{\theta_1}^{\theta_2}\frac{1}{\text{cosec}\,\theta}\,d\theta\,.\,\overline{a}_\phi$$

$$= \frac{-I}{4\pi\rho}\int_{\theta_1}^{\theta_2}\sin\theta\,.\,d\theta\,.\,\overline{a}_\phi \quad \left[\because \sin\theta = \frac{1}{\text{cosec}\,\theta}\right]$$

$$H = \frac{-I}{4\pi\rho}\overline{a}_\phi[-\cos\theta]_{\theta_1}^{\theta_2}$$

$$H = \frac{I}{4\pi\rho}\overline{a}_\phi[\cos\theta_2 - \cos\theta_1]$$

$$H = \frac{I}{4\pi\rho}[\cos\theta_2 - \cos\theta_1]\,\overline{a}_\phi \quad \text{...(8)}$$

Hence, the above equation represents magnetic field intensity $\overline{H}$ due to straight wire carrying conductor of finite length.

Q40. Using Biot-Savart's law, find the magnetic field intensity on the axis of a circular loop with radius R and carrying a steady current I.

Nov./Dec.-17, (R16), Q5(a)

OR

Obtain the expression for H at any point on the axis of circular loop carrying current I and deduce H in the center of the circular loop.

Dec.-14, (R13), Q7

OR

Derive expression for magnetic field intensity on the axis of a circular loop carrying a current of I-amps.

Nov./Dec.-13, (R09) Q4(a)

OR

Derive an expression for magnetic field intensity due to a circular wire.

OR

Find an expression for field intensity at the centre of a circular wire carrying a current I in the anti-clockwise direction. The radius of circle is 'r' and the wire in the x-y plane.

Ans:

Magnetic Field Intensity ($\overline{H}$) Due to Circular Current Carrying Wire

Consider a wire of length 'L' bent in the form of a circle of radius 'r' and carrying current of 'I' amperes as shown in the figure. Let the circular wire is placed in the xy plane.

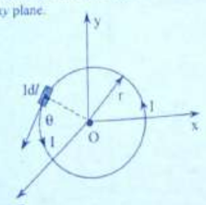


Figure: Circular Current Carrying Wire

Consider a differential current element $Id\bar{l}$ at a point Q on the circumference of the circular wire, which causes field intensity $d\overline{H}$ at the centre of the circular wire.

Apply Biot-Savart's law to the circular wire, we get,

$$d\overline{H} = \frac{Id\bar{l}\times\bar{a}_r}{4\pi r^2}\ \text{A/m}$$

Where, $\bar{a}_r$ = Unit vector, whose direction is radially inwards and which makes an angle 90° with $Id\bar{l}$.

$$d\overline{H} = \frac{Id\bar{l}\times 1\times\sin 90}{4\pi r^2}\bar{a}_z$$

$$d\overline{H} = \frac{Id\bar{l}}{4\pi r^2}\bar{a}_z\ \text{A/m} \qquad \text{... (1)}$$

Total magnetic field intensity at the centre of the circular wire is obtained by integrating equation (1).

i.e., $\overline{H} = \oint H = \oint\frac{Id\bar{l}}{4\pi r^2}\,\bar{a}_z$

$$\overline{H} = \frac{I}{4\pi r^2}\times\oint d\bar{l}\,\bar{a}_z$$

$$\overline{H} = \frac{I}{4\pi r^2}(L)\,\bar{a}_z$$

$$\overline{H} = \frac{I}{4\pi r^2}\times 2\pi r\,\bar{a}_z \qquad [\because \text{Length '}L\text{' = Circumference of circle}]$$

$$\overline{H} = \frac{I}{2r}\,\bar{a}_z\ \text{A/m} \qquad [\because \text{The field intensity is perpendicular to plane of the circular wire}]$$

$$\overline{H} = \frac{\pi I}{L}\bar{a}_z\ \text{A/m} \qquad \left[\because 2\pi r = L,\ r = \frac{L}{2\pi}\right]$$

Q41. Differentiate static electric and magnetic fields.

Nov./Dec.-18, (R16), Q5(a)

OR

What is fundamental difference between static electric and magnetic field lines?

Nov./Dec.-17, (R16), Q1(e)

OR

Give the differences between static electric field and static magnetic field.

Ans:

The differences between static electric field and static magnetic field are as follows.

	Static Electric Field		Static Magnetic Field
1.	Static electric field is produced by a stationary charge.	1.	Static magnetic field is produced by a permanent magnet.
2.	Electric field exerts force on stationary as well as on moving charges.	2.	Magnetic field exerts force only on moving charges.
3.	The force acting on a charge, placed in an electric field, is given by, $\overline{F}_e = Q\overline{E}$ Where, $\overline{E}$ = Electric field intensity, V/m. The force, $\overline{F}_e$ is independent of the velocity of charge.	3.	The force acting on a charge, placed in a magnetic field is given by, $\overline{F}_m = Q\overline{V}\times\overline{B}$ Where, $\overline{V}$ = Velocity of charge, m/s; $\overline{B}$ = Magnetic flux density, Wb/m². The force, $\overline{F}_m$ depends on the velocity of charge ($\overline{V}$).
4.	The force acting on a unit positive charge, placed at a point in an electric field is called electric field intensity at that point. It is denoted by $\overline{E}$ and expressed in Volts per meter (V/m).	4.	The force experienced by a unit north pole at any point in a magnetic field is called as the magnetic field intensity at that point. It is denoted by $\overline{H}$ and expressed in Ampere per meter (A/m).
5.	Electric field lines, always leaves a positive charge and enters a negative charge. They doesn't form a closed path.	5.	Magnetic lines of force always form a closed path.
6.	Electric field lines do not intersect each other. Because, at the point of intersection the field has two directions.	6.	Magnetic field lines do not intersect each other for the same reason because, at the point of intersection the field has two directions.

3.6 FORCE ON A MOVING CHARGE, FORCE ON A DIFFERENTIAL CURRENT ELEMENT, FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS

Q42. Explain the concept of forces acting on a moving charge placed in a uniform magnetic field.

Ans:

Model Paper-III, Q7(b)

Forces Acting on a Moving Charge Placed in a Uniform Magnetic Field

A magnetic field exerts force only on moving charges. Therefore, a charged particle present in a magnetic field $\overline{B}$, experiences a force which is proportional to the product of the magnitude of charge (Q), velocity of the charged particle ($\bar{v}$), flux density ($\overline{B}$) and sine of the angle between $\bar{v}$ and $\overline{B}$.

Mathematically,

Force, $\overline{F}_{mag} = Q\bar{v}\times\overline{B}$ N.

In a magnetic field, force applied in a direction perpendicular to the direction of motion of charged particle, will never change the magnitude of velocity of charged particle i.e., the acceleration vector is always at right angles to the velocity vector. Hence, the kinetic energy of the charged particle remains unchanged.

Thus, we can conclude that the steady magnetic field do not transfer energy to the moving charged particle.

Q43. A solenoid has an inductance of 20 mH. If the length of the solenoid is increased by two times and the radius is decreased to half of its original value, find the new inductance.

Nov.-15, (R13), Q1(h)

Ans:

Given that,

Inductance of solenoid, $L = 20$ mH

We know that inductance of a solenoid is given by,

$$L = \frac{\mu N^2 A}{l} = 20 \text{ mH} = 20 \times 10^{-3} \text{ H}$$

Where N = number of turns

$A = \pi r^2$ = Area of solenoid

l = length of solenoid

From given data,

Length is increased by two times i.e.,

$l = 2l$

Radius is decreased to half i.e.,

$r = \frac{r}{2}$

New inductance, $L_{new} = ?$

The inductance of new solenoid, L_{new} is given by,

$$L_{new} = \frac{\mu N^2 A}{l}$$

$$= \frac{\mu \times N^2 (\pi r^2)}{l}$$

$$= \frac{\mu \times N^2 \times \left(\pi\left(\frac{r}{2}\right)^2\right)}{2l}$$

$$= \frac{\mu \times N^2 \times \frac{\pi r^2}{4}}{2l}$$

$$= \frac{\mu \times N^2 \times \pi r^2}{8l}$$

$$= \frac{1}{8}\left[\frac{\mu N^2 \pi r^2}{l}\right]$$

$$= \frac{1}{8}[L] = \frac{1}{8}[20 \times 10^{-3}] \quad \left[\because L = \frac{\mu N^2 A}{l}\right]$$

$$= 2.5 \times 10^{-3} \text{ H}$$

$$\boxed{\therefore \text{New inductance}, L_{new} = 2.5 \times 10^{-3} \text{H}}$$

Q44. Derive an expression for force on a current element.

OR

Derive an expression for force on a current element in a magnetic field.

Ans:

Force Experienced by a Differential Current Element Placed in Steady Magnetic Field

In an electric field ($\overline{E}$) a charged particle is accelerated in the direction of field ($\overline{E}$). If a charged particle is placed in a magnetic field, no acceleration takes place. However, if the charge particle is moving in a magnetic field, it experiences acceleration perpendicular to the direction of its motion. Thus, the particle experiences a force at right angle to its velocity.

Consider a differential charge 'dQ' in a magnetic field ($\overline{B}$) moving with a velocity $\overline{V}$ as shown in the figure.

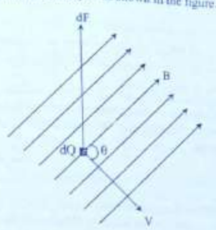


Figure: Force on a Differential Current Element

The differential force $d\overline{F}$ acting on a differential charge dQ, moving with a velocity $\overline{V}$ in a steady magnetic field is given by,

$$d\overline{F} = dQ(\overline{V} \times \overline{B}) \text{ N} \quad \text{... (1)}$$

We know that, current 'I' is the ratio of rate of change of charge with respect to time i.e.,

$$I = \frac{dQ}{dt}$$

$$\Rightarrow dQ = Idt$$

Substituting this in equation (1), we get,

$$d\overline{F} = Idt(\overline{V} \times \overline{B})$$

$$\Rightarrow dF = I(dt\overline{V} \times \overline{B})$$

Consider the term $dt\overline{V}$ in the above equation, it has the dimensions of length and represents the distance travelled by the charge particle in time dt.

$$\therefore dt\overline{V} \equiv d\overline{l}$$

$$\Rightarrow d\overline{F} = I(d\overline{l} \times \overline{B}) \text{ N} \quad \text{... (2)}$$

$\therefore$ Equation (2) gives the force acting on a differential charge 'dQ'.

If a straight conductor of length 'l' is considered, then the force acting on the entire length of the conductor is given by,

$$F = \int_0^l d\overline{F}$$

$$\Rightarrow \overline{F} = \int_0^l I(d\overline{l} \times \overline{B})$$

$$\Rightarrow \overline{F} = \int_0^l |\overline{B}|.|Idl| \sin\theta . \overline{a}_n$$

Where,

θ = Angle between $\overline{V}$ and $\overline{B}$

$$\Rightarrow |\overline{F}| = |\overline{B}| I \sin\theta \int_0^l d\overline{l}$$

$$\Rightarrow |\overline{F}| = |\overline{B}| Il \sin\theta \text{ N}$$

The direction of force can be determined by Fleming's left hand rule.

Q45. Prove that the force on a closed filamentary circuit in a uniform magnetic field is zero.

Ans:

The force exerted on a current filament in a magnetic field is given as,

$$\overline{F} = -\overline{I}\oint \overline{B} \times \overline{dL} \text{ Newton} \quad \text{... (1)}$$

If we assume that magnetic flux density is uniform throughout the field, then, equation (1) can be written as,

$$\overline{F} = -I\overline{B} \times \oint \overline{dL} \quad \text{... (2)}$$

Since the circuit is forming a closed loop, the term $\oint dL$ will be zero.

Substituting $\oint dL = 0$ in equation (2), we get,

$$F = -IB \times 0 = 0$$

Hence the force acting on a circuit forming a closed loop is zero provided B is uniform.

However, if the magnetic flux density $\overline{B}$ is not uniform then the force is not equal to zero.

Q46. Explain the phenomena why a current carrying conductor kept in a magnetic field experience force.

Ans:

Force on Current Carrying Conductor in a Magnetic Field

Whenever a current carrying conductor is placed in a magnetic field, it experiences a force which acts in a direction perpendicular to both the direction of current 'I' and the uniform horizontal field of flux density 'B' Wb/m^2.

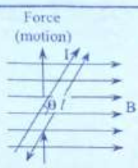


Figure (1)

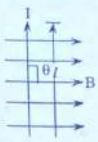


Figure (2)

A current carrying conductor placed in a magnetic field experiences a force because, we know that whenever two equal forces are perpendicular to each other, a couple is formed. The couple so formed produces a twisting moment i.e., torque due to which the conductor experiences a force and hence rotates.

This mechanical force depends upon the magnitude of current, the length of the conductor 'l' and the flux density of the magnetic field 'B'.

The magnitude of the force experienced is given by,

$$\boxed{F = BIl\text{Newton}}$$

Using vector notation, $\bar{F} = I\bar{l} \times \bar{B}$ and

$$\boxed{F = BIl\sin\theta}$$

Where 'θ' is the angle between $\bar{l}$ and $\bar{B}$ which is 90° in the above case i.e., figure (2).

So, $F = BIl \sin 90°$

$F = BIl$ Newtons $\quad (\because \sin 90° = 1)$

No force is exerted on a conductor when it lies parallel to the magnetic field. If the conductor lies at an angle 'θ' with the direction of the field, then B can be resolved into two components, $B\cos\theta$ parallel to and $B\sin\theta$ perpendicular to the conductor. There is no effect produced due to $B\cos\theta$, but the component $B\sin\theta$ is responsible for the motion. In this case,

$$\boxed{F = BIl\sin\theta\text{Newton}}$$

The force experienced is directly proportional to,

(i) Current flowing in the conductor, I

(ii) Length of the conductor, l

(iii) Flux density of the uniform magnetic field, B and

(iv) Sine of the angle between the conductor and the uniform magnetic field, 'θ'.

Electrical motors producing mechanical power, work basically on this principle.

Direction of the Force

It can be determined by Fleming's left hand rule.

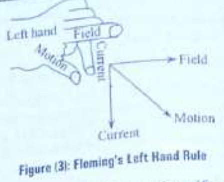


Figure (3): Fleming's Left Hand Rule

Hold your left hand with forefinger, second finger and thumb at right angles to one another as shown in figure (3).

"If the forefinger points in the direction of the magnetic field, second finger in the direction of current through the conductor, then the thumb will indicate the direction of force, i.e., the motion of the conductor".

Q47. Derive an expression for force on a straight long current carrying conductor placed in a magnetic field.

Dec.-11, Set-1, Q1(a)

OR

Obtain the expression for the force experie-nced by a conductor kept in magnetic field.

Ans:

The force $\overline{F}$ on a charge 'Q' moving with a velocity $\overline{u}$ in a magnetic field of magnetic flux density $\overline{B}$ is given as,

$$\overline{F} = Q\,\overline{u} \times \overline{B}$$

Force due to differential element of charge 'dQ' is,

$$d\overline{F} = dQ\,\overline{u} \times \overline{B} \quad \text{... (1)}$$

Let, $\overline{J}$ = Conduction current density of a conductor whose volume charge density is ρ_v.

If differential element of charge of the conductor is 'dQ', then we have,

$$dQ = \rho_v\,dv \quad \text{... (2)}$$

Where, dv is differential volume enclosing the differential charge 'dQ'.

We also have,

$$\overline{J} = \rho_v \overline{u} \quad \text{... (3)}$$

Substituting equation (2) in equation (1), we get,

$$d\overline{F} = \rho_v\,dv\,\overline{u} \times \overline{B} \quad \text{... (4)}$$

Substituting equation (3) in equation (4), we get,

$$d\overline{F} = \overline{J}\,dv \times \overline{B} \quad \text{... (5)}$$

Considering $\overline{J}\,dv$ as differential current element i.e.,

$$\overline{J}\,dv = I\,d\overline{L} \quad \text{... (6)}$$

and substituting equation (6) in equation (5), we get,

$$d\overline{F} = I\,d\overline{L} \times \overline{B} \quad \text{... (7)}$$

Integrating equation (7) over a closed path, we get,

$$\overline{F} = \oint I\,d\overline{L} \times \overline{B}$$

More appropriately,

$$\boxed{\overline{F} = -I\oint \overline{B} \times d\overline{L}}$$

This is the expression for the force experienced by a conductor kept in magnetic field.

Q48. Show that the force between two parallel conductors carrying current in the same direction is attractive

Nov./Dec.-13, (R09), Q6(

OR

Derive an expression for the force between parallel wires carrying currents in the same direction.

Ans:

Model Paper-II, Q6(

Expression for Force between Two Straight Long Parallel Conductors Carrying Current in Same Direction

Consider two straight long parallel conductors (conductor (a) and conductor (b)) carrying current (I_a and I_b respectivel in same direction.

Let the distance between the conductors be 'd' m. Also, let the conductors lie in YZ plane with currents along positive direction as shown in figure (a).

To find the magnetic field intensity at any point due to a long straight current carrying conductor, consider a point 'P' a distance of 'd' m from the conductor as shown in figure (b).

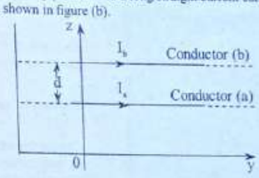


Figure (a)

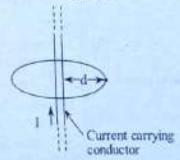


Figure (b)

Applying ampere's circuital law by constructing an amperian circular loop with radius 'd'

i.e., $\oint H.dL = I$

$\Rightarrow H\oint dL = I$

$\Rightarrow H \times (2\pi d) = I$

$\Rightarrow H = \dfrac{I}{2\pi d}$ and $\overline{H} = \dfrac{I}{2\pi d}\,\overline{a}_\phi$

$\therefore$ The magnitude of magnetic field intensity is $\dfrac{I}{2\pi d}$ and it will be perpendicular to the plane in which the conductor and point 'P' are lying.

$\therefore$ In present case,

$$\overline{H} = \frac{I}{2\pi d}\,\overline{a}_x \quad [\because H \text{ is perpendicular to } YZ \text{ plane}]$$

Magnetic flux density is given by,

$$\overline{B} = \mu\overline{H}$$
$$= \mu_0\mu_r\overline{H}$$
$$= \mu_0\overline{H} \quad [\because \mu_r = 1 \text{ in free space}]$$
$$= \frac{\mu_0 I}{2\pi d}\overline{a}_x \quad \dots (1)$$

$\therefore$ The magnetic flux density due to current (I_a) through conductor (a) at conductor (b),

$$\overline{B}_a = \frac{\mu_0 I_a}{2\pi d}\,\overline{a}_x$$

$\therefore$ Force exerted on conductor (b) by the field of conductor (a),

$$\overline{F}_b = \overline{I}_b l \times \overline{B}_a$$
$$= l\,I_b\overline{a}_y \times \frac{\mu_0 I_a}{2\pi d}\overline{a}_x$$
$$= \frac{\mu_0 l\,I_a\,I_b}{2\pi d}\,\overline{a}_y \times \overline{a}_x$$
$$= \frac{\mu_0 l\,I_a\,I_b}{2\pi d}(-\overline{a}_z)$$
$$= \frac{-\mu_0 l\,I_a\,I_b}{2\pi d}\overline{a}_z \quad \dots (2)$$

Similarly, the magnetic flux density due to current (I_b) through conductor (b) at conductor (a),

$$\overline{B}_b = \frac{\mu_0 I_b}{2\pi d}\,\overline{a}_x$$

$\therefore$ Force exerted on conductor (a) by the field of conductor (b),

$$\overline{F}_a = \overline{I}_a l \times \overline{B}_b$$
$$= l\,I_a\overline{a}_y \times \frac{\mu_0 I_b}{2\pi d}\overline{a}_x$$
$$= \frac{\mu_0 l\,I_a\,I_b}{2\pi d}(-\overline{a}_z)$$
$$= \frac{-\mu_0 l\,I_a\,I_b}{2\pi d}\overline{a}_z \quad \dots (3)$$

From equations (1) and (2),

$$\overline{F}_a = \overline{F}_b = \overline{F} = \frac{-\mu_0 l\,I_a\,I_b}{2\pi d}\overline{a}_z$$

$$\Rightarrow \frac{|F|}{l} = \frac{\mu_0 I_a I_b}{2\pi d}\ \text{N/m}$$

$\therefore$ The magnitude of force between the conductors is $\dfrac{\mu_0 I_a I_b}{2\pi d}$ N/m

As both the forces $\overline{F}_a$ and $\overline{F}_b$ are acting towards same negative direction of z-axis, it is an attractive force directed normally to the plane in which the conductors are present.

Q49. Derive an expression for force between two straight parallel current carrying conductors.

Dec.-14, (R13), Q9(a)

OR

Derive an expression for force between two straight long parallel current carrying conductors. What will be the nature of force if the currents are in the same and opposite directions?

Nov./Dec.-12, (R09), Q6(a)

OR

Explain force between two parallel current carrying conductors.

OR

Explain force between two current elements.

OR

Obtain the expression for the force experienced by two current carrying conductors. What is the direction of force when they are carrying current in similar direction and opposite direction?

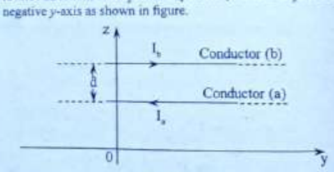
Ans:

Expression for Force between Two Straight Long Parallel Conductors Carrying Current in Same Direction

For answer refer Unit-IV, Q48.

Expression for Force Between Two Straight Long Parallel Conductors Carrying Current in Opposite Direction

Now, let the currents through conductors be flowing in reverse direction, with I_a towards positive y-axis and I_b towards negative y-axis as shown in figure.



Figure

From equation (1), the magnetic flux density at conductor (b) due to current (I_a) through conductor (a) is,

$$\overline{B}_a = \frac{\mu_0 I_a}{2\pi d}\,\overline{a}_x$$

$\therefore$ Force applied on conductor (b) by the field of conductor (a),

$$\overline{F}_b = \overline{I}_b l \times \overline{B}_a$$
$$= l\,I_b(-\overline{a}_y) \times \frac{\mu_0 I_a}{2\pi d}\,\overline{a}_x$$
$$= \frac{\mu_0\, l\,I_a\,I_b}{2\pi d}\,\overline{a}_z \quad \dots (1)$$

The magnetic flux density at conductor (b) due to current (I_b) through conductor (a),

$$\overline{B}_b = \frac{\mu_0 I_b}{2\pi d}\,\overline{a}_x$$

$\therefore$ Force applied on conductor (a) by the field of conductor (b),

$$\overline{F}_a = \overline{I}_a l \times \overline{B}_b$$
$$= l\,I_a(\overline{a}_y) \times \frac{\mu_0 I_b}{2\pi d}\,\overline{a}_x$$
$$= \frac{-\mu_0\, l\,I_a\,I_b}{2\pi d}\,\overline{a}_z \quad \dots (2)$$

From equations (1) and (2),

$$\overline{F}_a = -\overline{F}_b$$

$\therefore$ The magnitude of force between the conductors is $\dfrac{\mu_0 I_a I_b}{2\pi d}$ N/m

As both the forces $\overline{F}_a$ and $\overline{F}_b$ are acting in opposite directions i.e., $\overline{F}_b$ towards positive z axis and $\overline{F}_a$ towards negative z-axis, the force between the conductors is a repulsive force.

Q50. Calculate the force on a straight conductor of length 30 cm carrying a current of 5 Amp in the magnetic field is $B = 3.5 \times 10^{-3}(\overline{a}_x - \overline{a}_y)$ Tesla, where $\overline{a}_x$ and $\overline{a}_y$ are unit vectors.

Ans:

Given that,

Current, $I = 5$ Amp

Length of conductor, $l = 30$ cm $= 30 \times 10^{-2}$ m

Magnetic field, $B = 3.5 \times 10^{-3}(\overline{a}_x - \overline{a}_y)$

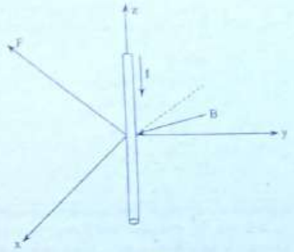
To determine,

Force on conductor, $F = ?$

Length,

$l = 30$ cm $= 0.3$ m

According to the data $I = 5$ A along z-axis as shown in figure



Figure

If current is considered as vector, then current I in vector form can be written as,

$$I = 0\overline{a}_x + 0\overline{a}_y + 5\overline{a}_z$$
$$B = 3.5 \times 10^{-3}\overline{a}_x - 3.5 \times 10^{-3}\overline{a}_y$$

The force on conductor is given by,

$$F = (B \times I)\,l \text{ Newtons}$$

$$B \times I = \begin{vmatrix} a_x & a_y & a_z \\ 3.5\times10^{-3} & -3.5\times10^{-3} & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= a_x(-3.5 \times 10^{-3} \times 5) - a_y(3.5 \times 10^{-3} \times 5) + a_z(0)$$
$$= -0.0175\overline{a}_x - 0.0175\overline{a}_y$$

$$\therefore\ F = (-0.0175\overline{a}_x - 0.0175\overline{a}_y) \times 0.3$$
$$= -5.25 \times 10^{-3}\,\overline{a}_x - 5.25 \times 10^{-3}\,\overline{a}_y$$

Magnitude of,

$$F = \sqrt{(-5.25\times10^{-3})^2 + (-5.25\times10^{-3})^2}$$
$$= 7.42 \times 10^{-3} \text{ Newton}$$
$$= 7.42 \text{ mN}$$

Q51. A conductor 6 m long lies along z-direction with a current of 2 A in $\overline{a}_z$ direction. Find the force experienced by the conductor if $\overline{B} = 0.08\ \overline{a}_x$ Tesla.

Dec.-11, Set-1, Q1(b)

Ans:

Given that,

Length of conductor, $dl = 6$ m

Current in conductor, $I = 2$ A

Magnetic flux density, $\overline{B} = 0.08\,\overline{a}_x$ Tesla

The current is in $\bar{a}_y$ direction i.e.,

$$I = 2\bar{a}_y$$

The force exerted on a current filament in magnetic field is,

$$\bar{F} = Id\bar{l} \times \bar{B}$$
$$= (2\bar{a}_y)(6) \times (0.08\bar{a}_x)$$
$$= 12\bar{a}_y \times 0.08\bar{a}_x$$

We have,

$$\bar{a}_y \times \bar{a}_x = -\bar{a}_z$$
$$\therefore \bar{F} = (12 \times 0.08)\bar{a}_z$$
$$= 0.96\bar{a}_z \text{ Newton}$$

Q52. A conductor of length 4 m, with current held at 10 A in the $\bar{a}_y$ direction laid along the y-axis between y = ± 2. If the field is $\bar{B} = 0.05\bar{a}_x$ T, find the work done in moving the conductor parallel to itself at constant speed to x = y = 2 m. Derive the formula used.

Nov./Dec.-12, (R09), Q6(b)

Ans: Dec.-11, Set-3, Q8

Given that,

Length of the conductor, = 4 m

Current, $I = 10$ A

Magnetic flux density, $B = 0.05\ \bar{a}_x$ Tesla

To determine

Workdone in moving the conductor parallel to itself,

$W = ?$

The conductor lies along the y-axis between y = ± 2 m as shown in figure (1).

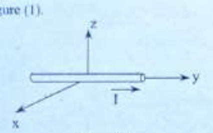


Figure (1)

Given that, the conductor is moving parallel to itself at a constant speed to $x = y = 2$ m which is incorrect. The conductor will move parallel to itself if and only of $x = z = 2$ m as shown in figure (2).

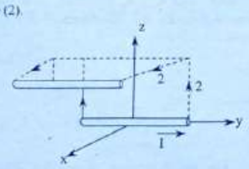


Figure (2)

From the entire motion we have,

$$\text{Force} = IL \times B$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ 0 & 4\times10 & 0 \\ 0.05 & 0 & 0 \end{vmatrix}$$

$$= -(4 \times 10)(0.05)\, a_z$$
$$= -2\bar{a}_z$$

The applied force is equal and opposite

$$\therefore F_a = 2\bar{a}_z$$

$\therefore$ The workdone in moving a conductor parallel to itself is along z direction only.

$$\therefore \quad W = \int_0^2 F_a dz\bar{a}_z$$

$$= \int_0^2 (2a_z) dz\bar{a}_z$$

$$= 2[z]_0^2 = 2(2-0)$$
$$= 4 \text{ Joules}$$

Derivation of Workdone

Consider a current carrying conductor in a_z direction as shown in figure (3).

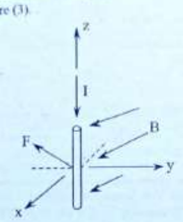


Figure (3)

Equal and opposite forces must be applied in order to counter magnetic forces and to establish equilibrium. If a motion occurs, then the work done on the system is given in integral form.

$$\boxed{\therefore W = \int_{\text{initial}L}^{\text{final}L} F_a . dL}$$

Q53. A current element 2 m length lies along the y-axis centred at the origin. The current is 6 amps in the $\bar{a}_y$ direction. If it experiences a force $2.5\left(\frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right)$ N due to a uniform field $\bar{B}$, determine B.

Ans: Dec.-11, Set-4, Q6(a)

Given that,

Length of the current element, $L = 2$ m

Current element lies on y-axis i.e., $\bar{L} = 2\bar{a}_y$

Current in the element-L, $I = 6$ A

Force, $\bar{F} = 2.5\left(\frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right)$N

To determine,

Uniform magnetic field, $\bar{B} = ?$

We know that,

$$\text{Force, } \bar{F} = I\bar{L} \times \bar{B}$$
$$= 6(2\bar{a}_y)\times(B_x\bar{a}_x + B_y\bar{a}_y + B_z\bar{a}_z)$$
$$= (12\bar{a}_y)\times(B_x\bar{a}_x + B_y\bar{a}_y + B_z\bar{a}_z) \quad \ldots (1)$$

We have,

$$\bar{F} = \frac{2.5}{\sqrt{2}}(\bar{a}_x + \bar{a}_z) \quad \ldots (2)$$

Equating equations (1) and (2), we get,

$$\frac{2.5}{\sqrt{2}}(\bar{a}_x + \bar{a}_z) = (12\bar{a}_y)\times(B_x\bar{a}_x + B_y\bar{a}_y + B_z\bar{a}_z)$$

$$1.7677(\bar{a}_x + \bar{a}_z) = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 12 & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

$$1.7677(\bar{a}_x + \bar{a}_z) = \bar{a}_x[12B_z - 0] - \bar{a}_y[0-0] + \bar{a}_z[0 - 12B_x]$$

$$1.7677\ \bar{a}_x + 1.7677\ \bar{a}_z = 12B_z\bar{a}_x - 12B_x\bar{a}_z$$

Comparing the coefficients on both sides, we have,

$$-12B_x = 1.7677$$
$$B_x = -0.1473 \text{ Tesla}$$

Similarly,

$$12B_z = 1.7677$$
$$B_z = 0.1473 \text{ Tesla}$$

$\therefore$ The net uniform magnetic field is,

$$\bar{B} = B_x + B_z$$
$$= -0.1473\ \bar{a}_x + 0.1473\ \bar{a}_z$$
$$= 0.1473\ (\bar{a}_z - \bar{a}_x)$$

Q54. A magnetic field, B = 3.5 × 10 – 2a_z Tesla, exerts a force on a 0.3 m conductor along the x-axis. If the conductor current is 5 A in the –a_x direction, what force must be applied to hold the conductor in position.

Nov./Dec.-17, (R16), Q8(b)

Ans:

Given that,

Length of the conductor, $L = 0.30$ m

Magnetic flux density, $\bar{B} = 3.5 \times 10^{-2}\ \bar{a}_z$ T

Current in the conductor, $I = 5$ A

To find,

Force must be applied to hold the conductor, $\overline{F}_a$ = ?

Since, the current is flowing in the negative x-direction, the current vector $\bar{I}$ can be written as,

$$\bar{I} = -5\bar{a}_x + 0\bar{a}_y + 0\,\bar{a}_z \text{ A}$$

$$\Rightarrow \quad \bar{I} = -5\,\bar{a}_x \text{ A}$$

$$\Rightarrow \quad \bar{I}L = -5\,\bar{a}_x \times 0.3$$

$$\Rightarrow \quad \bar{I}L = -1.5\bar{a}_x$$

The force $\overline{F}$ on the conductor is given by,

$$\overline{F} = \bar{I}L \times \overline{B} \text{ N}$$

$$\Rightarrow \quad \overline{F} = -1.5\bar{a}_x \times 3.5 \times 10^{-2}\,\bar{a}_z$$

$$\Rightarrow \quad \overline{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -1.5 & 0 & 0 \\ 0 & 0 & 3.5\times10^{-2} \end{vmatrix}$$

$$\Rightarrow \quad \overline{F} = \bar{a}_x\,(0) - \bar{a}_y\,(-1.5 \times 3.5 \times 10^{-2}) + \bar{a}_z\,(0)$$

$$\Rightarrow \quad \overline{F} = 5.25 \times 10^{-2}\,\bar{a}_y \text{ N}$$

∴ Equal and opposite force must be applied in order to hold the conductor in position.

∴ Force to be applied, $\overline{F}_a = -5.25 \times 10^{-2}\,\bar{a}_y$ N

3.7 MAGNETIC BOUNDARY CONDITIONS

Q55. Derive the magnetic boundary conditions.

Ans: Model Paper-I, Q7(b)

Statement

There are two magnetic boundary conditions. They are,

(i) The tangential components of magnetic field intensity are continuous across the boundary.

i.e., $H_{t_a} = H_{t_b}$

(ii) The normal components of magnetic flux density are continuous across the boundary.

i.e., $B_{a_N} = B_{b_N}$

Proof

(a) For Magnetic Field Intensity H

Consider the boundary surface between the two different media i.e., medium a and medium b as shown in figure (a).

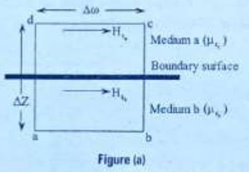


Figure (a)

Let,

H_{t_a} = Tangential component of magnetic field intensity of medium 'a'

H_{t_b} = Tangential component of magnetic field intensity of medium 'b'

μ_{r_a}, μ_{r_b} = Relative permeabilities of medium 'a' and medium 'b' respectively

$\Delta\omega, \Delta Z$ = Length and breadth of the rectangular area.

We know that, Maxwell's equation obtained from Ampere's circuital law is given by,

$$\oint H.dL = \int_s \left(J + \frac{\partial D}{\partial t}\right) \partial s \quad \text{... (1)}$$

Where,

J = Current density

$\frac{\partial D}{\partial t}$ = Displacement current density.

If the breadth ΔZ of the rectangular area approaches zero, then the term $\frac{\partial D}{\partial t}$ also approaches zero. And also, the current density 'J' is zero at a dielectric-dielectric interface.

So, substituting $J = 0$ and $\frac{\partial D}{\partial t} = 0$ in equation (1), we get,

$$\oint H.dL = 0$$

$$\Rightarrow \quad H_{t_a}\,\Delta\omega - H_{t_b}\,\Delta\omega = 0$$

$$\boxed{H_{t_a} = H_{t_b}}$$

The necessary boundary condition is that the tangential components of magnetic field intensity must be equal or continuous.

(b) For Magnetic Flux Density, B

Consider a cylindrical surface enclosing the boundary surface between the two media i.e., medium a and medium b as shown in figure (b).

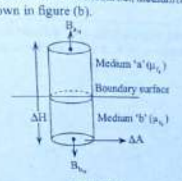


Figure (b)

Let,

B_{a_N}, B_{b_N} = Normal components of magnetic flux density of medium 'a' and medium 'b' respectively.

μ_{r_a}, μ_{r_b} = Relative permeabilities of medium 'a' and medium 'b' respectively.

ΔH = Height of the cylindrical conductor.

We know that, the Maxwell's equation obtained from Gauss law for magnetic fields is given by,

$$\oint_s B.ds = 0 \quad \text{... (2)}$$

So, as height of the cylindrical surface approaches zero, the total cylindrical surface area becomes zero.

∴ Equation (2) becomes as,

$$\Rightarrow \quad B_{a_N}\,\Delta S - B_{b_N}\,\Delta S = 0$$

$$\Rightarrow \quad B_{a_N} = B_{b_N}$$

Hence, the necessary boundary condition for 'B' is that the normal components of B of both the media must be equal i.e., the normal component of B must be continuous.

∴ The boundary conditions for time-varying fields are,

$$H_{t_a} = H_{t_b}$$

$$B_{a_N} = B_{b_N}$$

Q56. Derive the boundary conditions for magnetostatic fields at the interface of two different medium with permeability μ_1 and μ_2.

Nov./Dec.-17, (R13), Q14(a)

OR

Derive the boundary conditions of static magnetic field at the interface of two different magnetic medium.

Ans: Nov./Dec.-15, (R13), Q14(b)

Consider a magnetic flux line passing through a boundary between the two magnetic materials as shown in the figure. Let μ_1 and μ_2 be the permeabilities of medium-1 and 2 respectively.

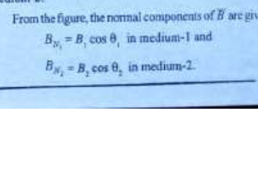


Figure: Refraction of a Flux Line

Since the media considered are isotropic in nature, the direction of both $\overline{H}$ and $\overline{B}$ will be same in both mediums.

Let θ_1 be the angle of incidence i.e., angle made by the flux line with the normal in medium-1 and θ_2 be the angle of emergence i.e., angle made by the flux line with the normal in medium-2.

From the figure, the normal components of $\overline{B}$ are given by,

$B_{N_1} = B_1 \cos\theta_1$ in medium-1 and

$B_{N_2} = B_2 \cos\theta_2$ in medium-2.

Similarly, the tangential components of $\bar{H}$ are given by,

$$\sin\theta_1 = \frac{H_{t_1}}{H_1}$$

$$\Rightarrow \quad H_{t_1} = H_1 \sin\theta_1 \text{ in medium-1}$$

$$= \frac{B_1 \sin\theta_1}{\mu_1} \qquad \left[\because H = \frac{B}{\mu}\right]$$

$$\sin\theta_2 = \frac{H_{t_2}}{H_2}$$

$$\Rightarrow \quad H_{t_2} = H_2 \sin\theta_2 \text{ in medium-2}$$

$$= \frac{B_2 \sin\theta_2}{\mu_2}$$

But, from the boundary condition, we know that,

$$B_{N_1} = B_{N_2} \text{ and} \qquad \text{...(1)}$$

$$H_{t_1} = H_{t_2} \qquad \text{...(2)}$$

Dividing equation (1) by equation (2), we get,

$$\Rightarrow \quad \frac{B_{N_1}}{H_{t_1}} = \frac{B_{N_2}}{H_{t_2}}$$

Substituting the corresponding values, we get,

$$\Rightarrow \quad \frac{B_1 \cos\theta_1}{\frac{B_1}{\mu_1}\sin\theta_1} = \frac{B_2 \cos\theta_2}{\frac{B_2}{\mu_2}\sin\theta_2}$$

$$\Rightarrow \quad \left(\frac{1}{\mu_1}\right) \times \tan\theta_1 = \left(\frac{1}{\mu_2}\right) \times \tan\theta_2$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

Hence, proved.

Q57. The xy-plane serves as the interface between two different media. Medium 1(z < 0) is filled with a material whose μ_r = 6 and medium 2(z >0) is filled with a material whose μ_r = 4. If the interface carries linear current density of $(1/\mu_0)\hat{a}_y$ mA/m, and $\bar{B}_2 = 5\hat{a}_x + 8\hat{a}_z$ mWb/m², find $\bar{H}_1$ and B_1.

Ans:

Given that,

For medium $1(z < 0) : \mu_r = 6$

For medium $2(z > 0) : \mu_r = 4$

Linear current density, $K = \frac{1}{\mu_0}\hat{a}_y$ mA/m.

Magnetic flux density, $\bar{B}_2 = 5\hat{a}_x + 8\hat{a}_z$ mWb/m²

Magnetic field intensity, $\bar{H}_1 = ?$

Let,

$$B_1 = (B_x, B_y, B_z) \text{ mWb/m}^2$$

$$\Rightarrow \quad B_1 = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z \text{ mWb/m}^2 \qquad \text{...(1)}$$

The normal component of B is continuous. Therefore,

$$B_{1N} = B_{2N}$$

$$\Rightarrow \quad B_{1N} = B_{2N} = 8\hat{a}_z$$

$$\boxed{\therefore B_z = 8}$$

The relation between 'B' and 'H' is given as,

$$B = \mu H$$

$$\Rightarrow \quad B_2 = \mu_2 H_2 \qquad \text{...(2)}$$

$$\Rightarrow \quad H_2 = \frac{B_2}{\mu_2} = \frac{B_2}{\mu_0\mu_r} = \frac{5\hat{a}_x + 8\hat{a}_z}{4\mu_0} \qquad [\because \mu_r = 4 \text{ for medium 2}]$$

$$\therefore \quad H_2 = \frac{1}{4\mu_0}(5\hat{a}_x + 8\hat{a}_z) \text{ mA/m} \qquad \text{...(3)}$$

From equation (2), we have,

$$B_1 = \mu_1 H_1$$

$$\Rightarrow \quad H_1 = \frac{B_1}{\mu_1} = \frac{B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z}{\mu_0\mu_r} \qquad [\because \text{From equation (1)}]$$

$$= \frac{1}{6\mu_0}(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z) \text{ mA/m} \qquad [\because \mu_r = 6 \text{ for medium 1}] \qquad \text{...(4)}$$

The tangential component of 'H' can be obtained as,

$$(H_1 - H_2) \times a_{n12} = K$$

$$\Rightarrow \quad H_1 \times a_{n12} - H_2 \times a_{n12} = K \qquad \text{...(5)}$$

Substituting the corresponding values in equation (5), we get,

$$\frac{1}{6\mu_0}(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z) \times \hat{a}_z - \frac{1}{4\mu_0}(5_x\hat{a}_x + 8\hat{a}_z) \times \hat{a}_z = K$$

$$\Rightarrow \quad \frac{1}{6\mu_0}[B_x(\hat{a}_x \times \hat{a}_z) + B_y(\hat{a}_y \times \hat{a}_z) + B_z(\hat{a}_z \times \hat{a}_z)] = \frac{1}{4\mu_0}(5(\hat{a}_x \times \hat{a}_z) + 8(\hat{a}_z \times \hat{a}_z)) + \frac{1}{\mu_0}\hat{a}_y$$

$$\Rightarrow \quad \frac{1}{6\mu_0}[-B_x\hat{a}_y + B_y\hat{a}_x] = \frac{1}{4\mu_0}[-5\hat{a}_y] + \frac{1}{\mu_0}\hat{a}_y \qquad \left[\because K = \frac{1}{\mu_0}\hat{a}_y\right]$$

Comparing both sides of above equation, we get,

$$B_y = 0$$

$$\frac{-1}{6\mu_0}B_x = \frac{-5}{4\mu_0} + \frac{1}{\mu_0}$$

$$\Rightarrow \quad \frac{-B_x}{6} = \frac{-5}{4} + 1$$

$$\Rightarrow \quad \frac{-B_x}{6} = \frac{-1}{4}$$

$$\boxed{\therefore B_x = 1.5}$$

Substituting the values of B_x, B_y and B_z in equation (1), we get,

$$B_1 = 1.5\ \hat{a}_y + 8\ \hat{a}_z \text{ mWb/m}^2 \quad ...(6)$$

Since, $B_1 = \mu_1 H_1$

$$\Rightarrow \quad H_1 = \frac{B_1}{\mu_1}$$

$$= \frac{(1.5\hat{a}_x + 8\hat{a}_z)\times 10^3}{6\mu_0} \quad [\because \text{ From equation (6)}]$$

$$\therefore \quad H_1 = \frac{1}{\mu_0}(0.25\,\hat{a}_x + 1.33\,\hat{a}_z) \text{ mA/m}$$

3.8 MAGNETIC CIRCUIT, SELF INDUCTANCES AND MUTUAL INDUCTANCES

Q58. Explain in detail about magnetic circuit.

Ans:

A magnetic circuit constitutes the closed path traced by the magnetic lines of force (f) that leave the north pole and flows through the entire circuit and terminate at the point from where they have started. In order to maintain the flow of magnetic flux, the magnetic circuit contains high permeability materials such as iron, soft steel etc. The reluctance these materials is low.

Consider a magnetic circuit of length 'L' with 'N' turns and cross-sectional area 'A' as shown in figure (1).

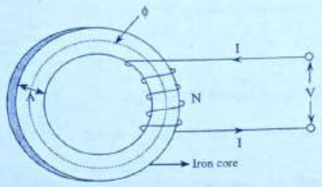


Figure (1): Magnetic Circuit

The magnetic field (H) in the circuit is given as,

$$H = \frac{NI}{L} \text{ A/m} \quad ...(1)$$

The flux density (B) in the circuit is given as,

$$B = \mu H \quad ...(2)$$

Where,

μ = Permeability

Substituting equation (1) in equation (2), we get,

$$B = \mu\frac{NI}{L} \quad ...(3)$$

Now,

The flux in the circuit is given by,

$$\phi = \int_s B.d\bar{s}$$

$$\Rightarrow \quad \phi = BA \quad ...(4)$$

Substituting equation (3) in equation (4), we get,

$$\phi = \mu\frac{NI}{L}A \quad ...(5)$$

We know that,

$$\text{Reluctance, } R = \frac{F}{\phi}$$

$$\Rightarrow \quad \text{m.m.f, } F = \phi R \quad ...(6)$$

According to the Ampere's law, the current enclosed by the integration path is given by,

$$\int H.dl = NI$$

$$\therefore \quad F = \left[\frac{NI}{L}\right]L = NI$$

Substituting 'F' and f in equation (6), we get,

$$NI = \left[\mu\frac{NI}{L}A\right]R$$

$$\Rightarrow \quad 1 = \frac{\mu A}{L}R$$

$$R = \frac{L}{\mu A} \quad ...(7)$$

If a series magnetic circuit has two parts, then,

Total reluctance, $R = R_1 + R_2$

Where,

R_1 = Reluctance of part 1

R_2 = Reluctance of part 2

Now, the total m.m.f needed to force flux 'ϕ' through the circuit is given by,

$$F = \phi(R_1) + \phi(R_2)$$

For example, consider a toroid having two series magnetic paths i.e., a mean iron path of length L_i and a saw cut with an air gap of length L_G as shown in figure (2).

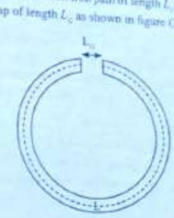


Figure (2): Toroid

The resultant m.m.f required is given by,

$$F = F_I + F_G$$

$$= \phi(R_1 + R_2) \quad ...(8)$$

Where,

R_1 = Reluctance in magnetic path 1

$$= \frac{L_I}{\mu_0\mu_r A}$$

R_2 = Reluctance in magnetic path 2

$$= \frac{L_G}{\mu_0 A}$$

Thus,

$$F = \phi\left(\left[\frac{L_I}{\mu_0\mu_r A}\right] + \left[\frac{L_G}{\mu_0 A}\right]\right)$$

$$F = \frac{\phi}{\mu_0 A}\left[\frac{L_I}{\mu_r} + L_G\right] \quad ...(9)$$

If the two magnetic paths are in parallel, then the equivalent reluctance is given as,

$$R = \frac{R_1 R_2}{(R_1 + R_2)}$$

And the m.m.f is given by,

$$F = \phi R$$

$$F = \phi\frac{R_1 R_2}{(R_1 + R_2)}$$

Q59. Differentiate magnetic and electric circuits.

Model Paper-II, Q7(b)

Ans:

	Magnetic Circuit		Electric Circuit
1.	It is the closed path formed by magnetic flux.	1.	It is the closed path formed by electric current.
2.	**Reluctance:** The opposition to magnetic flux in a magnetic circuit is called reluctance. It is represented by a letter S and it is given by, $\boxed{S = \frac{l}{\mu_0 \mu_r A} \text{ AT/Wb}}$ Where, l = Length of magnetic path in metres μ_r = Permeability of the magnetic material A = Area of cross-section of magnetic path.	2.	**Resistance:** The opposition to electric field in an electric circuit is called resistance. It is represented by a letter R and is given by, $\boxed{R = \frac{\rho l}{A} \text{ Ohm}}$ Where, ρ = Specific resistance or resistivity of the conductor in metres l = Length of the conductor in metres A = Area of cross-section of conductor in m^2.
3.	**Magnetic Flux:** It is the amount of magnetic field produced by a magnetic source. It is represented by a Greek letter (ϕ) and is given by, $\boxed{\phi = \frac{\text{MMF}}{\text{Reluctance}} \text{ Wb}}$ Where, M.M.F = Magnetomotive Force in ampere turn.	3.	**Electric Current:** It is the amount of electric current produced by an electric source (voltage or current source). It is represented by letter I and is given by, $\boxed{I = \frac{V}{R} \text{ Amp}}$ Where, V = Voltage in volts R = Resistance in ohm.
4.	**Magnetic Flux Density:** It is defined as flux per unit cross-sectional area (A). It is represented by letter B and is given by, $\boxed{B = \frac{\phi}{A} \text{ Wb/m}^2}$ Where, ϕ = Flux in Wb A = Cross-sectional area in m^2.	4.	**Electric Current Density:** It is defined as current per unit cross-sectional area (A). It is represented by letter J and is given by, $\boxed{J = \frac{I}{A} \text{ A/m}^2}$ Where, I = Current in amperes A = Cross-sectional area in m^2.
5.	**Magnetomotive Force (M.M.F):** It is the force or pressure which tends to produce the magnetic flux in a magnetic circuit. It is represented by M.M.F and is given by, $\boxed{\text{M.M.F} = NI \text{ (Ampere-Turn) or (AT)}}$ Where, N = Number of turns of the coil I = Current in ampere.	5.	**Electromotive Force (E.M.F):** It is the energy which tends to produce the electric current in an electric circuit. It is represented by E.M.F and is given by, $\boxed{\text{E.M.F} = IR \text{ Volts}}$ Where, I = Current in amperes R = Resistance in ohm.
6.	**Magnetic Intensity:** It is defined as magnetomotive force per unit length of magnetic circuit. It is represented by letter H and is given by, $\boxed{H = \frac{NI}{l} \text{ AT/m}}$ Where, N = Number of turns of coil I = Current in ampere l = Length of magnetic path in metres.	6.	**Electric Intensity:** It is defined as electromotive force per unit length of electric circuit. It is represented by letter 'E' and is given by, $\boxed{E = \frac{V}{d} \text{ Volt/m}}$ Where, V = Voltage in volts d = Distance in metres.

Q60. Explain self and mutual inductance.

OR

Explain the self and mutual inductance. Obtain the expression for same.

Nov./Dec.-17, Q7(a)

Ans:

It is the property of the circuit by virtue of which the change in current in the circuit is opposed by the e.m.f. induced by change in current.

It is denoted by 'L' and expressed in Henry.

Consider a circuit carrying current of 'I' ampere's as shown in figure (1). This current produces a magnetic field $\bar{B}$, which in turn causes a flux 'ϕ' to pass through the coil.

Let 'N' be the number of turns of coil.

Flux 'ϕ' is given by,

$$\phi = \int_s \bar{B}\, d\bar{s} \text{ Wb}$$

And, flux linkage (λ) of the coil is given by,

$$\lambda = N\phi \text{ Wb-turns} \quad \ldots (1)$$

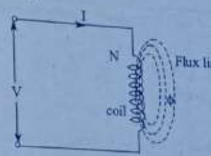


Figure (1): Self-Inductance of a Coil

Self-inductance of a coil is defined as the ratio of magnetic flux linkage (λ) to the current through the coil (I) i.e.,

$$L = \frac{\lambda}{I}$$

$$\Rightarrow \quad L = \frac{N\phi}{I} \text{ H} \quad \text{[From equation (1)]}$$

Mutual Inductance

Mutual inductance exists due to the magnetic interaction between the two coils placed side by side in such a manner that the flux produced by the current in one coil, say coil-1, links the second coil, say coil-2.

Consider the two circuits, namely circuit-1 and circuit-2, carrying current I_1 and I_2 respectively. Let N_1, N_2 denotes number of turns of coil-1 and coil-2 respectively as shown in figure (2).

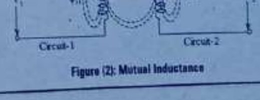


Figure (2): Mutual Inductance

Here, some part of the flux (ϕ_{12}) produced by the current I_2 in a coil-2, links the coil-1 which leads to the magnetic interaction between the two circuits. Similarly, part of the flux (ϕ_{21}) produced by the current I_1 in coil-1 links with the coil-1

If $\overline{B}_2$ is the field due to current I_2 and s_1 is the area circuit-1, then,

$$\phi_{12} = \int_{s_1} \overline{B}_2 . ds \text{ Wb}$$

Flux linkage (λ_{12}) is given by,

$$\lambda_{12} = N_1 \phi_{12}$$

Finally, the mutual inductance (M_{12}) of the coil is given by,

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\phi_{12}}{I_2} \text{ H}$$

Similarly, the mutual inductance (M_{21}) due to flux linkages (λ_{21}) of circuit-2 due to the current I_1 in circuit-1 is given by,

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\phi_{21}}{I_1} \text{ H.}$$

If the medium surrounding the two circuits is linear, then we can assume that $M_{12} = M_{21} = M$

Q61. Derive the expression for self inductance of a coaxial cable of inner radius 'a' and outer radius 'b'.

Nov./Dec.-17, (R16), Q9(a)

OR

Show that inductance of the cable $L = \frac{\mu l}{2p} \ln b/a$ H.

Nov.-15, (R13), Q9(b)

Ans:

Consider a coaxial cable of length 'l' carrying current 'I' amperes. Let the radius of the inner conductor be 'a' and the radius of the outer conductor be 'b' as shown in the figure.

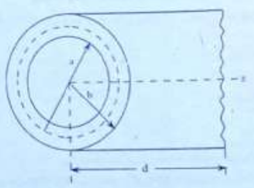


Figure: Co-axial Cable

Assume that the current of I amp is along z-axis.

The magnetic field intensity (H) between the inner and outer conductors at any point is given by,

$$H = \frac{1}{2\pi r} \quad a < r < b$$

The magnetic field intensity (b) is given by,

$$B = \mu H$$

$$= \frac{\mu I}{2\pi r}$$

If the cable is along z-axis, then the flux density extends from $r = a$ to b and $z = 0$ to l

$$\therefore \quad \overline{B} = \frac{\mu I}{2\pi r}\overline{a}_\phi$$

But, the total magnetic flux produced (ϕ) is given by,

$$\phi = \int_s \overline{B}.d\overline{s}$$

Where,

ds is the differential current element and is given as,

$$ds = dr\, dz\, \overline{a}_\phi$$

$$\therefore \quad \phi = \int_{z=0}^{l}\int_{r=a}^{b} B\, dr\, dz\, \overline{a}_\phi = \int_{z=0}^{l}\int_{r=a}^{b} \frac{\mu I}{2\pi r}\overline{a}_\phi . dr\, dz\, \overline{a}_\phi$$

$$= \frac{\mu I}{2\pi}\int_{z=0}^{l}\int_{r=0}^{b}\frac{1}{r} dr\, dz \quad [\because \overline{a}_\phi . \overline{a}_\phi = 1]$$

$$= \frac{\mu I}{2\pi}\int_{z=0}^{l}[\ln r]_{r=a}^{b} dz = \frac{\mu I}{2\pi}\int_{z=0}^{l}\ln(b-a)dz$$

$$= \frac{\mu I}{2\pi}\int_{z=0}^{l}\ln\left(\frac{b}{a}\right)dz = \frac{\mu I}{2\pi}\ln\left(\frac{b}{a}\right)\int_{z=0}^{l} 1 dz$$

$$= \frac{\mu I}{2\pi}\ln\left(\frac{b}{a}\right)(Z)_{z=0}^{l} = \frac{\mu I}{2\pi}\ln\left(\frac{b}{a}\right)(l-0)$$

$$\phi = \frac{\mu I}{2\pi}\ln\left(\frac{b}{a}\right)$$

The inductance of a co-axial cable is given by,

$$L = \frac{\text{Total flux }(\phi)}{\text{Total current }(I)}$$

$$= \frac{\frac{\mu Il}{2\pi}\ln\left(\frac{b}{a}\right)}{I}$$

$$\boxed{\therefore L = \frac{\mu l}{2\pi}\ln\left(\frac{b}{a}\right)\text{H}}$$

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Q62. Derive the expression for coefficient of coupling between two circuits.

OR

Prove that in the case of two mutually coupled coils $M = K\sqrt{L_1 L_2}$ with usual notations.

Ans:

When two coils placed side by side such that the flux produced by one coil links the other, then the coils are said to be "magnetically coupled" or "mutually coupled."

Consider two mutually coupled coils as shown in the figure.

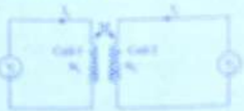

Figure: Mutually Coupled Coils

The various parameters in coil-1 and coil-2 are,

N_1 = Number of turns in coil-1

I_1 = Current flowing through coil-1

Φ_{11} = Flux produced in coil-1

M_{12} = Mutual inductance of coil-1 due to the flux in coil-2

N_2 = Number of turns in coil-2

I_2 = Current flowing through coil-2

Φ_{22} = Flux produced in coil-2. Due to induced current

M_{21} = Mutual inductance of coil-2 due to the flux in coil-1.

The self inductance of coil-1 is given by,

$$L_1 = \frac{N_1 \Phi_{11}}{I_1} \quad \text{... (1)}$$

The self inductance of coil-2 is given by,

$$L_2 = \frac{N_2 \Phi_{22}}{I_2} \quad \text{... (2)}$$

Now, the mutual inductance between two coils is defined as the total flux linkage of one coil per unit change in the other coil.

∴ Mutual inductance of coil-1 due to the flux produced by coil-2 is given by,

$$M_{12} = \frac{N_1 \Phi_{12}}{I_2} \quad \text{... (3)}$$

Where, Φ_{12} = Flux linking the coil-1 and produced by coil-2.

Similarly, mutual inductance of coil-2 due to the flux produced by coil-1 is given by,

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1} \quad \text{... (4)}$$

Where, Φ_{21} = Flux linking the coil and produced by coil

We know that, degree of coupling is the ratio of mutually induced flux to the total flux produced in a coil.

K_1 = Degree of coupling with respect to coil-1

$$= \frac{\Phi_{21}}{\Phi_{11}}$$

K_2 = Degree of coupling with respect to coil-2

$$= \frac{\Phi_{12}}{\Phi_{22}}$$

Since, $K_1 = K_2 = K$, the coefficient of coupling (K) is given by,

$$K = \sqrt{K_1 \times K_2}$$

$$K = \sqrt{\frac{\Phi_{21}}{\Phi_{11}} \times \frac{\Phi_{12}}{\Phi_{22}}} \quad \text{... (5)}$$

From equation (1), we have,

$$\Phi_{11} = \frac{L_1 I_1}{N_1} \quad \text{... (6)}$$

From equation (2), we have,

$$\Phi_{22} = \frac{L_2 I_2}{N_2} \quad \text{... (7)}$$

From equation (3), we have,

$$\Phi_{12} = \frac{M_{12} I_2}{N_1} \quad \text{... (8)}$$

From equation (4), we have,

$$\Phi_{21} = \frac{M_{21} I_1}{N_2} \quad \text{... (9)}$$

Substituting equations (6), (7), (8) and (9) in equation (5), we get,

$$\Rightarrow \quad K = \sqrt{\frac{M_{12} \times M_{21}}{L_1 L_2}}$$

But, $M_{12} = M_{21} = M$, we get,

$$\Rightarrow \quad K = \sqrt{\frac{M \times M}{L_1 L_2}} = \frac{M}{\sqrt{L_1 L_2}}$$

$$\boxed{M = K\sqrt{L_1 L_2}}$$

Where,

L_1 = Self-inductance of Coil-1

L_2 = Self-inductance of Coil-2

M = Mutual Inductance

K_1 = Coefficient of coupling.

Q63. Distinguish between self inductance and mutual inductance.

Ans: (Dec.-11, Set-3, Q3(a) | Dec.-11, Set-2, Q2(a) | Model Paper-III, Q6(b))

	Self Inductance		Mutual Inductance
1.	Self inductance of a coil is defined as the flux linkages per unit current due to the current flowing through the same coil.	1.	Mutual inductance of a coil is defined as the flux linkages per unit current due to the current in some other coil.
2.	It is associated with only one coil.	2.	It is associated with more than one coil.
3.	It is denoted by 'l'.	3.	It is denoted by 'M'.
4.	It is the actual value of an inductor element.	4.	It is not the actual value of an inductor, but it is only property associated with it.
5.	It is given by, $L = \frac{N\phi}{I}$ H. Where, N = Number of turns, ϕ = Flux, I = Current.	5.	It is given by, $M_{12} = \frac{\text{(Flux linkage with coil 1)}}{\text{(Current in coil 2)}} = \frac{N_1 \phi_{21}}{I_2}$ (or) $M_{21} = \frac{N_2 \phi_{12}}{I_1}$
6.	The self inductance of a coil can be increased by increasing the number of turns (N), area of cross section (A) and by selecting the material of high permeability ($\in$).	6.	The mutual inductance of a coil can be increased by increasing the relative permeability ($\in_r$), maintaining low spacing between two coils.
7.	It depends, on the current I passing through its own coil.	7.	Mutual inductance depends on current in its neighbouring coil.

Q64. Two mutually coupled coils are connected in series. L_1 = 0.5 H, L_2 = 0.6 H, m = 0.1 H, a D.C current of 2 amps is parallel through this system in such a way that the current increases at a uniform rate of 1 amp per sec. What is the voltage developed across the end points if,

(i) The coils are connected in a magnetically adding conditions.

(ii) The coils are connected in a magnetically opposing condition.

Nov.-10, Set-3, Q6(b)

OR

Two mutually coupled coils are connected in series.

L_1 = 0.5 H, L_2 = 0.6 H, M = 0.1 H

A D.C. current of 2 Amps is passed through this system in such a way that the current increases at a uniform rate of 1 Amper per sec. What is the voltage developed across the end points if,

(a) The coils are connected in a magnetically aiding condition.

(b) The coils are connected in a magnetically opposing condition.

Derive formula used.

Ans:

Given that,

Two mutually coupled coils connected in series

Self-inductance of 1st coil, L_1 = 0.5 H

Self-inductance of 2nd coil, L_2 = 0.6 H

Mutual inductance of two coils, M = 0.1 H

Current passing through the systems, i = 2 Amps

Rate of change of current, $\frac{di}{dt}$ = 1 Amp

Two series coils are as shown in the figure.

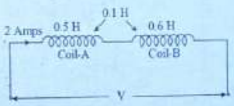

Figure: Geometry of the Figure

We know that, the voltage developed across the system is given by,

$$V = L_{eff} \frac{di}{dt} \quad [\because \text{ Neglecting resistance of the two coils}]$$

Where,

L_{eff} = Effective inductance of systems.

(a) If the coils are connected in magnetically aiding condition then effective inductance is given by,

$$L_{SA} = L_{eff} = L_A + L_B + 2M$$

$$= 0.5 + 0.6 + 2(0.1)$$

$$L_{eff} = 1.3 \text{ H}$$

$\therefore$ Voltage developed is given by,

$$V = L_{eff} \times \frac{di}{dt}$$

$$V = 1.3 \times 1$$

$$V = 1.3 \text{ Volts}$$

(b) If the two coils are connected in magnetically opposing condition then the effective inductance is given by,

$$L_{SO} = L_{eff} = L_A + L_B - 2M$$

$$\Rightarrow L_{eff} = 0.5 + 0.6 - 2(0.1)$$

$$\Rightarrow L_{eff} = 1.1 - 0.1$$

$$\Rightarrow L_{eff} = 0.9 \text{ H}$$

$\therefore$ Voltage developed is given by,

$$V = L_{eff} \times \frac{di}{dt}$$

$$\Rightarrow V = 0.9 \times 1$$

$$\Rightarrow V = 0.9 \text{ Volts}$$

Derivation

Effective Inductance of Two Series Connected Coils

Consider the two coils, coil-A and coil-B, connected in series as shown in figure. In practice, a coil is represented by the combination of inductance and resistance because, practically the coil possess certain value of resistance in addition to inductance. Hence, 'R_A' and 'L_A' forms coil-A, where as 'R_B' and 'L_B' forms coil-B as shown in figure.

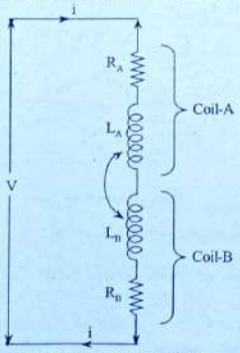

Figure: Two Series Connected Coils

Let a potential difference of 'V' volts is applied across the combination of coils, which results in the flow of current 'i' amperes through the coils. This current sets up flux and an e.m.f to be induced in the coils. The induced e.m.f depends on self-inductance and mutual inductance of the coil.

Self-inductance is the property of a coil which decides the induced e.m.f for a given rate of change of current. It is denoted by 'L'.

Similarly, mutual inductance decides the induced e.m.f in a coil by changing the current in the adjacent coil. It is denoted by 'M'.

Let 'e_A' be the e.m.f induced in coil A and it is given by,

$$e_A = -\left[L_A \frac{di}{dt} + M \frac{di}{dt}\right] \quad \text{... (1)}$$

Similarly, let e_B be the e.m.f induced in coil-B and it is given by,

$$e_B = -\left[L_B \frac{di}{dt} + M \frac{di}{dt}\right] \quad \text{... (2)}$$

The total voltage drop in the circuit is equal to the sum of applied potential 'V' and induced e.m.f's 'e_A' and 'e_B'. Mathematically, it is expressed as,

$$V + e_A + e_B = R_Ai + R_Bi$$

From equations (1) and (2), we have,

$$\Rightarrow V - \left[L_A \frac{di}{dt} + M \frac{di}{dt}\right] - \left[L_B \frac{di}{dt} + M \frac{di}{dt}\right] = R_Ai + R_Bi$$

$$\Rightarrow \quad V = R_Ai + R_Bi + L_A \frac{di}{dt} + M \frac{di}{dt} + L_B \frac{di}{dt} + M \frac{di}{dt}$$

On rearranging the terms, we get,

$$\Rightarrow \quad V = i(R_A + R_B) + (L_A + M + L_B + M) \frac{di}{dt}$$

$$\Rightarrow \quad V = i(R_A + R_B) + (L_A + L_B + 2M) \frac{di}{dt} \quad \text{... (3)}$$

Above equation is similar to,

$$V = Ri_{eff} + L_{eff} \frac{di}{dt}$$

Where,

$$R_{eff} = R_A + R_B \text{ and}$$

$$L_{eff} = L_A + L_B + 2M$$

The magnitude of effective inductance L_{eff} depends on the mutual inductance 'M'. The value of mutual inductance 'M' can be taken as positive or negative depending up on the direction of currents in two windings and the type of connection. If the coil-B in the figure is reversed, then the value of mutual inductance is taken as '$-M$'.

Hence, the effective inductance is more precisely given as,

$$L_{eff} = L_A + L_B + 2|M|$$

For positive coupling, mutual inductance 'M' is taken as positive, which results in the higher value of 'L_{eff}' i.e.,

$$L_{eff} = L_A + L_B + 2M$$

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For negative coupling, mutual inductance 'M' is taken as negative, which results in the lower value of 'L_{eff}' i.e.,

$$L_{eff} = L_A + L_B - 2M$$

Further, the mutual inductance is expressed in terms of self-inductances using the coupling coefficient as given below.

Mutual inductance, $M = k\sqrt{L_A L_B}$

Where, k = Coefficient of coupling

$= -1 \le k \le 1$

By varying the value of k, effective inductance of the system can be varied. This concept is used in the tuning of radio receiver.

Q65. If a coil of 800 μH is magnetically coupled to another coil of 200 μH. The coefficient of coupling between the coils is 0.05. Calculate inductance if two coils are connected in,

(a) Series aiding

(b) Series opposing

(c) Parallel aiding

(d) Parallel opposing.

Ans: Dec.-11, Set-4, Q5

Given that,

Self inductance of coil, $L_1 = 800\ \mu H$

Self inductance of coil, $L_2 = 200\ \mu H$

Coefficient of coupling, $K = 0.05$

To determine,

The effective inductance, L = ?, when the coils are in,

(a) Series aiding

(b) Series opposing

(c) Parallel aiding

(d) Parallel opposing.

If L_1 and L_2 are the self inductances then the mutual inductance between L_1 and L_2 is,

$$M = K\sqrt{L_1 L_2}$$

$$= (0.05)\left(\sqrt{(800\times10^{-6})(200\times10^{-6})}\right)$$

$$= (0.05)\left(\sqrt{1.6\times10^{-7}}\right)$$

$$= 2 \times 10^{-5}\ H$$

$$= 20\ \mu H$$

(a) Effective Inductance for Series Aiding

The effective inductance for series aiding of L_1 and L_2 is given by,

$$L = L_1 + L_2 + 2M$$

$$= 800 \times 10^{-6} + 200 \times 10^{-6} + 2\,(20 \times 10^{-6})$$

$$= 1 \times 10^{-3} + 4 \times 10^{-5}$$

$$= 1.04 \times 10^{-3}\ H$$

$$= 1040\ \mu H \quad \text{...(1)}$$

(b) Effective Inductance for Series Opposing

The effective inductance for series opposing of L_1 and L_2 is given by,

$$L = L_1 + L_2 - 2M$$

$$= 800 \times 10^{-6} + 200 \times 10^{-6} - 2\,(20 \times 10^{-6})$$

$$= 1 \times 10^{-3} - 4 \times 10^{-5}$$

$$= 9.6 \times 10^{-4}\ H$$

$$= 960\ \mu H \quad \text{...(2)}$$

(c) Effective Inductance for Parallel Aiding

The effective inductance for parallel aiding is given by,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$= \frac{(800\times10^{-6})(200\times10^{-6})-(20\times10^{-6})^2}{960\times10^{-6}} \quad [\because \text{From equation (2)}]$$

$$= \frac{1.6\times10^{-7}-4\times10^{-10}}{960\times10^{-6}}$$

$$= 1.6625 \times 10^{-4}$$

$$= 166.25 \times 10^{-6}$$

(d) Effective Inductance for Parallel Opposing

The effective inductance for parallel opposing is given by,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

$$= \frac{(800\times10^{-6})(200\times10^{-6})-(20\times10^{-6})^2}{1040\times10^{-6}} \quad [\because \text{From equation (1)}]$$

$$= \frac{1.6\times10^{-7}-4\times10^{-10}}{1040\times10^{-6}}$$

$$= 1.5346 \times 10^{-4}\ H$$

$$= 153.46\ \mu H$$

Q66. Determine the inductance of a solenoid of 2500 turns wound uniformly over a length of 0.25 m on a cylindrical paper tube, 4 cm in diameter and the medium is air.

Ans: Nov./Dec.-17, (R16), Q9(b)

Given that,

Number of turns, $N = 2500$

Length, $l = 0.25$ m

Diameter of cylindrical paper tube, $d = 4$ cm

Medium is air, therefore, $\mu_0 = 4\pi \times 10^{-7}$

We need to find inductance, L of the solenoid.

The inductance of a solenoid is given as,

$$L = \frac{\mu_0 N^2 S}{l} \quad \text{...(1)}$$

Where 'S' is the area of cross-section of the tube.

Since the tube is cylindrical, its area of cross-section represents a circle whose area,

$$= \pi r^2$$

$$= \pi \times (d/2)^2$$

$$= \frac{\pi d^2}{4}$$

$$= \frac{\pi}{4} \times (4 \times 10^{-2})^2$$

$$\therefore S = 1.256 \times 10^{-3}\ m^2$$

Substituting the values of μ_0, N, S and l in equation (1), we get

$$L = \frac{4\pi \times 10^{-7} \times (2500)^2 \times 1.256 \times 10^{-3}}{0.25}$$

$$\therefore L = 39.49 \text{ mH}$$

Therefore, the inductance of solenoid is 39.49 mH.

UNIT 4

Time Varying Fields and Maxwell's Equations

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Explain time varying fields.

Ans: Model Paper-I, Q1(g)

Time varying fields are the electromagnetic fields, which are produced either by time varying current (charges) or by moving magnets.

When a stationary conductor is placed in the vicinity of a time varying field then, according to Faraday's law, an e.m.f is induced in the conductor which is proportional to the rate of change of field with respect to time $\left(\frac{d\phi}{dt}\right)$ i.e., the frequency with which the field is varying. This is the basic principle of operation of generators, transformers etc.

Unlike static fields, the time varying fields are not conservative fields. It means that, the work done in moving a charge around a closed path within the field is not zero.

Q2. Explain statically induced e.m.f.

Ans: Model Paper-II, Q1(g)

When a stationary conductor is placed in the vicinity of time varying field, produced by the time varying current (or charges), an e.m.f is induced in the conductor, according to Faraday's law. This induced e.m.f is called as statically induced e.m.f. It is so called because, there is no physical movement between the conductor and source producing the field (i.e., electromagnet). The e.m.f induced in a transformer is the best example of statically induced e.m.f.

The statically induced e.m.f can be expressed by the integral form of Faraday's law as

$$e = \oint \bar{E}\, d\bar{L} = -\iint \frac{\partial \bar{B}}{\partial t} d\bar{s}$$

Q3. Differentiate between statically and dynamically induced e.m.f.

Ans: Model Paper-III, Q1(g)

Differences between Statically and Dynamically Induced E.M.F

	Statically Induced E.M.F		Dynamically Induced E.M.F
1.	The E.M.F induced in a conductor by the time varying current when the stationary conductor is placed in vicinity of time varying field is called static E.M.F.	1.	The E.M.F induced in a conductor moving in a stationary magnetic field due to the physical motion of conductors is called dynamically induced E.M.F.
2.	It is also called as transformer E.M.F.	2.	It is also called as motional e.m.f or flux cutting E.M.F.
3.	Here, the conductor is stationary whereas field is moving.	3.	Here, the field is stationary whereas the Conductor is moving.

Q4. State Faraday's law of electromagnetic induction.

Ans: (Nov./Dec.-18, (R16), Q1(i) | Model Paper-III, Q1(h))

Faraday's First Law of Electromagnetic Induction

It states that "Whenever the number of magnetic lines of force (called as magnetic flux) passing through a conductor changes, then an e.m.f is induced in it.

Faraday's Second Law of Electromagnetic Induction

It states that "The magnitude of the induced e.m.f is proportional to the rate of change of magnetic lines of forces".

Q5. What is the significance of displacement current? Nov./Dec.-17, (R16), Q1(j)

OR

What is displacement current.

Ans: Model Paper-I, Q1(h)

Displacement current is the current which is the resultant of time varying electric field. In case of a capacitor displacement current is the current which is passing through the capacitor (externally) when it is connected across a voltage source. Therefore, it indicates the rate of flow of charge between the capacitor plates in the external part of the capacitor circuit.

Q6. What is the significance of the ratio of magnitude of the conduction current density to the displacement current density?

Ans: Nov.-15, (R13), Q1(i)

Loss Tangent

In a lossy medium, the ratio of magnitudes of conduction current density to the displacement current density is known as Loss tangent. It can be mathematically expressed as,

$$\frac{|\bar{J}_c|}{|\bar{J}_d|} = \frac{\sigma}{\omega\varepsilon} \quad \text{...(1)}$$

Loss tangent is used to classify materials as conductors and dielectrics based on the value of $\tan\theta$. It acts as a demarcation line between the conductor and dielectrics as shown in figure below.

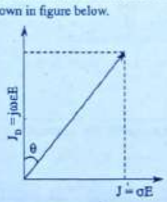


Figure Loss Tangent

If $\tan\theta$ is very small i.e., $\sigma << \omega\varepsilon$, then the medium is said to be a good or perfect dielectric. On the other hand if $\tan\theta$ is very large i.e., $\sigma >> \omega\varepsilon$ then the medium is a good conductor.

Q7. The total flux at the end of a long bar magnet is 360 μWb. The end of the magnet is withdrawn through a 1000 turn coil in 1/80 sec. What is the e.m.f generated in the coil?

Ans: Model Paper-II, Q1(h)

Given that,

Number of turns, $N = 1000$

Change in flux, $d\phi = 360\ \mu Wb = 360 \times 10^{-6}$ Wb

Change in time, $dt = \frac{1}{80}$ sec

E.m.f generated in the coil,

$$E = N\frac{d\phi}{dt}$$

$$= 1000 \times \frac{360\times10^{-6}}{\frac{1}{80}} = 28.8 \text{ V}$$

$\therefore$ E.m.f generated in the coil is 28.8 V.

Q8. Determine the e.m.f induced about the path r = 0.5, z = 0, t = 0. If B = 0.01 sin377t.

Ans: (Nov./Dec.-18, (R16), Q1(j) | Nov.-15, (R13), Q1(j))

Given that,

Radius, $r = 0.5$ m

Plane is $Z = 0$

Time, $t = 0$

Magnetic flux density, $B = 0.01 \sin 377t$

Induced e.m.f, $e = ?$

According to Faraday's law, induced e.m.f is given by,

$$e = \frac{d\phi}{dt}$$

$$= \frac{d(B.A)}{dt} \qquad [\because = B.A]$$

$$= -A\frac{dB}{dt}$$

Where,

$$A = \pi r^2$$

$$= \pi (0.5)^2$$

$$= 0.7853 \text{ m}^2$$

$\therefore$ E.M.F induced is given by,

$$e = -0.7853 \frac{d}{dt}(0.01 \sin 377t)$$

$$= -(0.7853)(0.01)\frac{d}{dt}\sin 377t$$

$$= -(7.853 \times 10^{-3})\, 377 \cos 377t$$

$$= -2.96 \cos 377t$$

Now,

$$e_{t=0} = -2.96 \cos 377(0)$$

$$\boxed{\therefore e_{t=0} = -2.96 \text{ V}}$$

Q9. Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2 \times 10^{-4}$ mho/m and $\epsilon_r = 81$.

Ans:

Given that,

Conductivity, $\sigma = 2 \times 10^{-4}$ mho/m

Relative permittivity, $\epsilon_r = 81$

Frequency at $J_c = J_d$

Let,

$E = E_m \sin \omega t$ (Where $\omega = 2\pi f$)

$$|\bar{J}_c| = |\bar{J}_d| \quad \dots (1)$$

We know that,

$$|\bar{J}_c| = \sigma E_m \quad \dots (2)$$

And
$$|\bar{J}_d| = \left|\frac{\partial \bar{D}}{\partial t}\right| = \left|\frac{\partial \epsilon \bar{E}}{\partial t}\right|$$

$$= \epsilon \left|\frac{\partial \bar{E}}{\partial t}\right| = \epsilon \left|\frac{\partial}{\partial t}(E_m \sin \omega t)\right|$$

$$= \epsilon | E_m \omega |$$

$$= \epsilon E_m \omega \quad \dots (3)$$

Substituting equations (2) and (3) in equation (1), we get,

$$\sigma E_m = \epsilon E_m \omega$$

$$\Rightarrow \quad \omega = \frac{\sigma}{\epsilon}$$

$$\Rightarrow \quad 2\pi f = \frac{\sigma}{\epsilon}$$

$\therefore$ Frequency,

$$f = \frac{\sigma}{2\pi \epsilon_0 \epsilon_r} = \frac{2 \times 10^{-4}}{2\pi \times 8.854 \times 10^{-12} \times 81}$$

$$= 44.38 \times 10^3 \text{ Hz}$$

$$\therefore \quad \boxed{f = 44.38 \text{ kHz}}$$

Q10. Find the conduction and displacement current densities in a material having conductivity of 10^{-3} S/m and $\epsilon_r = 3$, if the electric field in material is, $E = 2 \times 10^{-6} \sin(5.0 \times 10^9 t)$ V/m.

Ans:

Given that,

Conductivity, $\sigma = 10^{-3}$ S/m

Relative permittivity, $\epsilon_r = 3$

Electric field intensity, $E = 2 \times 10^{-6} \sin(5.0 \times 10^9 t)$ V/m

(i) Conduction current density, $\bar{J}_c = ?$

(ii) Displacement current density, $\bar{J}_d = ?$

(i) **Conduction Current Density, $(\bar{J}_c)$**

The conduction current density is given by,

$$\bar{J}_c = \sigma \bar{E}$$

$$= 10^{-3} \times 2 \times 10^{-6} \sin(5.0 \times 10^9 t)$$

$$= 2 \times 10^{-9} \sin(5.0 \times 10^9 t)$$

$$= 2 \sin(5.0 \times 10^9 t) \text{ nA/m}^2$$

(ii) **Displacement Current Density, $(\bar{J}_d)$**

The displacement current density is given by,

$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon \bar{E}) \qquad (\because \bar{D} = \epsilon \bar{E})$$

$$= \epsilon \frac{\partial \bar{E}}{\partial t}$$

$$= \epsilon_0 \epsilon_r \frac{\partial}{\partial t}(2 \times 10^{-6} \sin(5.0 \times 10^9 t))$$

$$= 8.854 \times 10^{-12} \times 3 \times (2 \times 10^{-6} \cos(5 \times 10^9 t) \times 5 \times 10^9)$$

$$= 2.655 \times 10^{-7} \times \cos(5 \times 10^9 t)$$

$$= 0.2655 \cos(5 \times 10^9 t) \ \mu\text{A/m}^2$$

$\therefore$ Conduction current density,

$$\bar{J}_c = 2 \sin(5 \times 10^9 t) \text{ A/m}^2$$

Displacement current density,

$$\bar{J}_d = 0.2655 \cos(5 \times 10^9 t) \ \mu\text{A/m}^2.$$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

4.1 FARADAY'S LAWS OF ELECTROMAGNETIC INDUCTION – MOTIONAL ELECTROMOTIVE FORCES

Q11. State and explain Faraday's laws of electromagnetic induction.

Dec.-14, (R13), Q10

OR

Explain Faraday's law of electromagnetic induction and derive the expression for induced e.m.f.

Model Paper-I, Q8

Ans:

Faraday's First Law of Electromagnetic Induction

Faraday's first law of electromagnetic induction states that "Whenever the number of magnetic lines of force (called as magnetic flux) passing through a conductor changes, an e.m.f is induced in it".

Faraday's Second Law of Electromagnetic Induction

Faraday's second law of electromagnetic induction states that "The magnitude of the induced e.m.f. is proportional to the rate of change of magnetic lines of force".

Suppose the flux linking with a coil having N turns, changes from initial value of ϕ_1 Wb to a final value of ϕ_2 Wb in 't' seconds. Then,

Initial flux linkages = (Number of turns) × (Initial flux linking with coil) = $N \times \phi_1 = N\phi_1$

Final flux linkages = (Number of turns) × (Final flux linking with coil) = $N \times \phi_2 = N\phi_2$

∴ From Faraday's laws of EMI, e.m.f induced in the coil,

$$e = \frac{\text{Change in flux linkages}}{\text{Time in seconds}} = \frac{N\phi_2 - N\phi_1}{t} = \frac{N(\phi_2 - \phi_1)}{t}$$

The above equation can be written in differential form as,

$$e = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt}$$

Since, the induced e.m.f (e) sets up a current in the direction opposing the very cause producing magnetic field, so a negative sign is given to it (by Lenz law).

$$\therefore \quad e = -N\frac{d\phi}{dt} \text{ Volts.}$$

Q12. State and explain Faraday's laws of electromagnetic induction.

Nov./Dec.-17, (R16), Q1(a)

OR

State Faraday's law of electromagnetic induction in its point form and derive the same in its integral form.

Ans:

Faraday's Law of Electromagnetic Induction

Point Form

Faraday's law of electromagnetic induction states that "the e.m.f induced in a conductor, due to change in magnetic flux passing through it, is directly proportional to the rate of change of flux linking the conductor".

$$\text{i.e.,} \quad e = -\frac{d\phi}{dt} \text{ V} \qquad \dots (1)$$

Integral Form

Faraday's law of electromagnetic induction states that "the closed line integral of electric field intensity around the loop of conductor (which may or may not have a closed path) is equal to the negative of surface integral of the rate of change of flux density with respect to time, over the loop area.

$$\text{i.e.,} \quad e = \oint \bar{E} \, d\bar{L} = -\frac{d}{dt}\int_s \bar{B} \cdot d\bar{s} \qquad \dots (2)$$

Derivation

According to Faraday's law the e.m.f induced is given as,

$$e = \oint \bar{E} . d\bar{L} \qquad \dots (3)$$

The magnetic flux ϕ passing through a specified area is given by,

$$\phi = \int_s \bar{B} \cdot d\bar{s}$$

Substituting the value of ϕ in equation (1),

$$e = -\frac{d}{dt}\left(\int_s \bar{B} d\bar{s}\right) \qquad \dots (4)$$

From equations (3) and (4), we get,

$$e = \oint \bar{E} . d\bar{L} = -\frac{d}{dt}\int_s \bar{B} \cdot d\bar{s}$$

Hence proved.

Q13. Explain the Faraday's disc generator and derive an expression for finding the unknown magnetic field.

Model Paper-III, Q8(a)

Ans:

Faraday's Disc Generator

The setup of Faraday's disc generator is shown in figure (1). It essentially consists of a circular disc of radius 'r', mounted on a shaft, which is supported by bearings on both the sides. The disc is rotated at a constant angular velocity of ω rad/sec with the help of a prime mover.

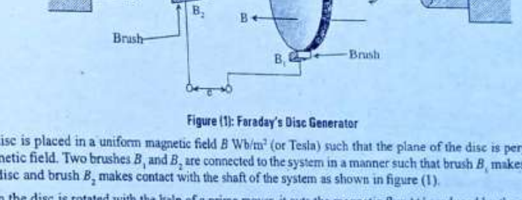


Figure (1): Faraday's Disc Generator

The disc is placed in a uniform magnetic field B Wb/m² (or Tesla) such that the plane of the disc is perpendicular to the applied magnetic field. Two brushes B_1 and B_2 are connected to the system in a manner such that brush B_1 makes contact with the edge of the disc and brush B_2 makes contact with the shaft of the system as shown in figure (1).

When the disc is rotated with the help of a prime mover, it cuts the magnetic flux 'ϕ' produced by the uniform magnetic field (B) there by inducing an e.m.f in the disc, which is collected by the brushes.

Derivation

According to Faraday's law of electromagnetic induction, the e.m.f induced is equal to rate of change of flux.

Mathematically,

$$\text{e.m.f, } e = \frac{d\phi}{dt} \qquad \dots (1)$$

Where, ϕ = Magnetic flux, Wb

Now, consider a segment 'OP' at time, say t_1 on the disc. After a fraction of time 'dt', this segment occupies the position 'OQ' as shown in figure 2(a).

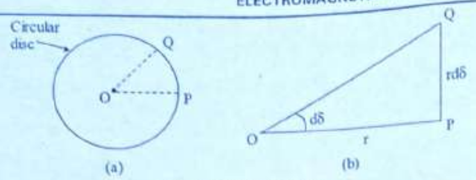


Figure (2): Segment on the Disc

The area OPQ on the disc is assumed to be a right angled triangle as shown in the figure 2(b). From the triangle OPQ, we have

$$\text{Area} = \frac{1}{2} \times \text{Base} \times \text{Height}$$

$$\Rightarrow dS = \frac{1}{2} \times r \times r\,d\delta$$

$$\Rightarrow dS = \frac{1}{2} \times r^2 d\delta \qquad \text{... (2)}$$

From equation (1), we have,

$$\text{e.m.f, } e = \frac{d\phi}{dt}$$

$$e = \frac{d}{dt}(B \times S) \qquad [\because \text{Flux, } \phi = B \times S]$$

$$e = B \times \frac{dS}{dt} \qquad \text{... (3)}$$

Substituting equation (2) in equation (3), we get,

$$e = B \times \frac{1}{2} r^2 \times \frac{d\delta}{dt}$$

$$e = B \times \frac{1}{2} r^2 \times (\omega) \qquad \left[\because \omega = \frac{d\delta}{dt}\right]$$

$$e = \frac{B r^2 \omega}{2} \text{ volts}$$

The unknown magnetic field 'B' can be found as,

$$\boxed{\therefore B = \frac{2e}{r^2 \omega} \text{ Wb/m}^2 \text{ (or) Tesla}}$$

Q14. State Lenz's law and prove it.

Ans: Nov.-10, Set-3, Q1(a)

Lenz's Law

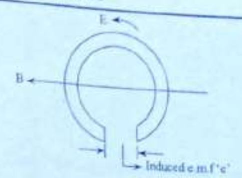
Lenz's law states that the current formed by the induced e.m.f will flow in opposite direction i.e., in the direction that opposes the very cause (for example the cause is supply voltage) which is responsible to produce the e.m.f.

Therefore, from Lenz's law the induced e.m.f expression changes as follows,

$$e = -N \frac{d\phi}{dt}$$

Here, the negative sign indicates that e.m.f induced opposes the very cause which is responsible to produce it.

Consider a coil which has 'N' turns. Let the coil represents an open circuited loop. Now, if a permanent magnet is brought closer to this loop, the flux density 'B' through the coil increases. If the magnet is moved away from the coil, the flux density through the coil decreases. This process leads to a rate of change of flux ($d\phi/dt$) through the coil resulting in the induction of voltage or e.m.f 'e' inside the loop.



Figure

This induced e.m.f can be measured by a voltmeter across the two open ends of the loop. On connecting voltmeter, the loop becomes a closed circuit which leads to the flow of induced current.

According to Faraday's law of magnetism, the magnitude of the induced voltage is given by,

$$e = N \frac{d\phi}{dt} \qquad \text{... (1)}$$

Hence,

According to Lenz's law, the induced e.m.f 'e', which is actually a scalar quantity is given by,

$$e = -N \frac{d\phi}{dt} \qquad \text{... (2)}$$

If the induced current inside the loop produces an electric field 'E', then the total 'E' is obtained by integrating it throughout the loop i.e.,

$$e = \oint E.dl \qquad \text{... (3)}$$

Also,

The total flux is given by,

$$\phi = \int_s B.ds \qquad \text{... (4)}$$

Where,

B = Normal component of flux density

ds = Differential elemental surface.

Substituting the value of ϕ from equation (4) into equation (2), we get,

$$e = -N \frac{d}{dt} \int_s B.ds \qquad \text{... (5)}$$

Thus, from equations (3) and (5), we get,

$$e = \oint E.dl = -N \frac{d}{dt} \int_s B.ds.$$

Q15. Define dynamically induced e.m.f

(Refer Only Topic: Dynamically Induced E.M.F) Dec.-14, (R13), Q1(i)

OR

Explain the following terms in detail,

(i) Time varying fields

(ii) Statically induced e.m.f

(iii) Dynamically induced e.m.f.

Ans:

(i) **Time Varying Fields**

For answer refer Unit-IV, Q1.

(ii) **Statically Induced e.m.f**

For answer refer Unit-IV, Q2.

(iii) **Dynamically Induced e.m.f**

The e.m.f induced in a conductor due to the physical motion of either conductor or source of field or both is said to be dynamically induced e.m.f.

(a) E.M.F induced in a moving conductor placed in a stationary field or in a stationary conductor, placed in a moving field.

From Lorentz force equation, we have,

$$\bar{F} = Q(\bar{E} + \bar{V} \times \bar{B})$$

$$= Q(\bar{V} \times \bar{B}) \qquad [\because \bar{E} = 0]$$

$\therefore$ Electric field intensity,

$$\bar{E}_m = \frac{\bar{F}}{Q} = \bar{V} \times \bar{B}$$

$\therefore$ The e.m.f induced in the conductor is,

$$e_m = \int \bar{E}_m . d\bar{L} = \int (\bar{V} \times \bar{B}).d\bar{L}$$

This induced e.m.f is also called as motional e.m.f. Since, the relation gives the e.m.f induced in a moving conductor placed in a stationary field or in a stationary conductor placed in a moving field. The former kind of induced e.m.f can be found in D.C generators and later one can be found in alternators (A.C generators).

(b) E.M.F induced in a moving loop placed in a time varying field.

This type of induced e.m.f consists of both statically and motionally induced e.m.f's and so it is given by sum of expressions.

$$e = \oint \bar{E}\, d\bar{L} = -\int_s \frac{\partial \bar{B}}{\partial t} . d\bar{s} + \oint (\bar{V} \times \bar{B}) . d\bar{L}$$

Q16. Write short notes on statically and dynamically induced e.m.f's.

Nov./Dec.-13, (R09), Q8(a)

OR

Explain the terms,

(i) Motional e.m.f

(ii) Static e.m.f.

Ans:

Model Paper-II, Q9(a)

(i) **Motional e.m.f or Dynamically Induced e.m.f**

"The e.m.f induced in a moving conductor placed in a stationary magnetic field due to the physical motion of conductor is called motional e.m.f".

- It is caused by the linking of stationary field with a moving conductor.
- Since the e.m.f is induced in a moving conductor placed in a stationary field so it is found in D.C. generators, motors and alternators.
- It is derived from the Lorentz force equation and Faraday's law.

From Lorentz force equation, we have,

$$\bar{F} = Q(\bar{E} + \bar{V} \times \bar{B})$$

$$\bar{F} = Q(\bar{V} \times \bar{B}) \qquad [\because \bar{E} = 0]$$

$\therefore$ Electric field intensity,

$$\bar{E}_m = \frac{\bar{F}}{Q} = \bar{V} \times \bar{B}$$

$\therefore$ The e.m.f induced in the conductor is,

$$\boxed{e_m = \oint \bar{E}_m \, \overline{dL} = \oint (\bar{V} \times \bar{B}) . \overline{dL}}$$

It is also called as flux-cutting e.m.f due to its motional action.

To understand it more clearly, consider the following example

Example

Suppose a conductor of length 'L' is placed in a uniform static magnetic field and the conductor is normal to the field as shown. 'B' be the flux density.

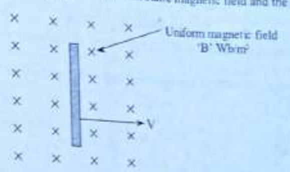


Figure (1)

The conductor moves in the field at right angles to both field and its axis with a velocity v.

Assume that the conductor covers an elementary distance 'dx' in time dt

Then,

Motional e.m.f. $e = \frac{-d\phi}{dt}$

Flux cut in time, $dt = B \times$ area swept in time dt

$$= B \times L dx$$

And $\frac{dx}{dt}$ = Velocity v

$$\boxed{\therefore \text{Motional e.m.f } e = \frac{-d\phi}{dt} = -BLv \text{ volts}}$$

In magnitude, $e = BLv$ volts.

(ii) **Static e.m.f or Statically Induced E.M.F**

"The e.m.f. induced in the conductor by the time varying current when the stationary conductor is placed in the vicinity of time varying field is called Static e.m.f".

- It is also called as transformer e.m.f
- It is caused by the linking of time varying field with a stationary conductor.
- Since, there is no physical movement between the conductor and source producing field, so it is called statically induced e.m.f. and is found in transformers.
- It is derived from Faraday's law.

From Faraday's integral form.

$$\boxed{e = \oint \bar{E}\, \overline{dL} = -\iint \frac{\partial \bar{B}}{\partial t} d\bar{s}}$$

Let us consider a simple example.

Example

Suppose a single open circuited loop is placed in the vicinity of a permanent magnet and assume that magnet either moves inwards or outwards the loop. The flux density 'B' increases when magnet moves inwards and decreases when it moves outwards the loop. Thus in this way the flux, changes and gives rise to induced e.m.f 'e' in the loop as shown in figure (2).

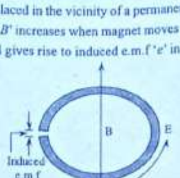


Figure (2)

The e.m.f which is induced appears across the ends of the open loop. The induced current flows as the loop forms a closed circuit.

The electric field 'E' associated with induced current, when integrated throughout the loop gives the induced e.m.f.

$$\therefore \quad e = \oint E.dL \qquad \text{... (1)}$$

We know total flux $\phi = \int_S B.ds$

B = Flux density

ds = Differential elemental surface.

Putting $\phi = \int_S B.ds$ in the expression $e = \frac{-d\phi}{dt}$, we get,

$$e = \frac{-d}{dt}\int_S B.ds$$

If the loop is stationary then,

$$e = -\int_S \frac{\partial B}{\partial t}.ds \text{ volts} \qquad \text{... (2)}$$

From equations (1) and (2), we get,

$$\boxed{e = \oint E.dL = -\int_S \frac{\partial B}{\partial t}.ds \text{ volts}}$$

Q17. An area of 0.65 m² in the z = 0 plane is enclosed by a filamentary conductor. Find the induced voltage, given that

$$\mathbf{B = 0.05\cos 10^3 t\left(\frac{\bar{a}_y + \bar{a}_z}{V_z}\right) Tesla.}$$

Ans: Dec.-11, Set-1, Q4(b)

Given that,

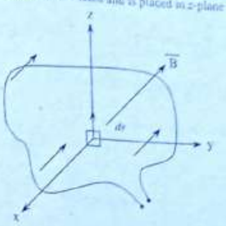
Area = 0.65 m²

Plane is $z = 0$

Magnetic field intensity,

$$B = 0.05\cos 10^3 t\left(\frac{a_y + a_z}{\sqrt{2}}\right) \text{Tesla}$$

According to the data, the filamentary conductor is fixed and is placed in z-plane i.e., $z = 0$ as shown in figure



Figure

From figure,

The filamentary conductor encloses an area of 0.65 m² (given)

$$\therefore \quad d\bar{s} = ds\bar{a}_z$$

According to Faraday's law, induced e.m.f is given by,

$$e = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$$

$$\Rightarrow \quad e = -\int_s \frac{\partial}{\partial t}\left[0.05\cos 10^3 t\left[\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right]\right]\cdot(ds\bar{a}_z)$$

$$\Rightarrow \quad e = -\int_s \frac{0.05(10^3)(-\sin 10^3 t)}{\sqrt{2}}ds \qquad [\because \bar{a}_y \cdot \bar{a}_z = 0, \bar{a}_z \cdot \bar{a}_z = 1]$$

$$\Rightarrow \quad e = 35.355\sin 10^3 t\left[\int_s ds\right]$$

We have,

$$\int_s ds = 0.65 \text{ m}^2$$

$$\Rightarrow \quad e = 35.355\sin 10^3 t\,[0.65]$$

$$\Rightarrow \quad e = 22.98\sin 10^3 t$$

$$\Rightarrow \quad e \simeq 23\sin 10^3 t \text{ Volts}$$

Q18. In free space, $\bar{E} = 50\cos(\omega t - \beta x)\bar{a}_z$ V/m. Find the average power crossing a circular area of radius 2.5 m in the plane z = 0. Assume $E_m = H_m \cdot \eta_o$ and $\eta_o = 120\pi\Omega$.

Ans: Nov.-15, (R13), Q11(a)

Given that,

Electric field intensity, $\bar{E} = 50\cos(\omega t - \beta x)\bar{a}_z$ ... (1)

Radius of circle, $r = 2.5$ m

$E_m = H_m \eta_o$, $\eta_o = 120\,\pi\Omega$

Average power crossing circular area, $P_{avg} = ?$

Magnetic field intensity,

$$\overline{H} = \frac{50}{\eta_0}\cos(\omega t - \beta x)\bar{a}_z \quad \left[\because H_m = \frac{E_m}{\eta_0}\right] \quad ...(2)$$

$$\overline{H} = \frac{50}{120\pi}\cos(\omega t - \beta x)\bar{a}_z \quad [\because \eta_0 = 120\pi] \quad ...(3)$$

Converting equation (1) and (3) into complex form

$$\overline{E} = 50\, e^{j(\omega t - \beta x)}\bar{a}_x$$

$$\overline{H} = \frac{50}{120\pi} e^{j(\omega t - \beta x)}\bar{a}_z$$

$$\overline{P}_{avg} = \frac{1}{2} R_e[\overline{E} \times \overline{H}^*]$$

Where $\overline{H}^*$ is the conjugate of $\overline{H}$

$$\overline{H}^* = \frac{50}{120\pi} e^{j(-\omega t + \beta x)}$$

$$\overline{P}_{avg} = \frac{1}{2}(50)\left(\frac{50}{120\pi}\right)\left[e^{j(\omega t - \beta x)}\bar{a}_x \times e^{j(-\omega t + \beta x)}\right]$$

$$= \frac{2500}{2}\left(\frac{1}{120\pi}\right)\left[e^0 \times \bar{a}_x \times \bar{a}_z\right]$$

$$\overline{P}_{avg} = 3.3157\ \bar{a}_y\ \text{W/m}^2$$

Average power crossing circular area is given by

$$\overline{P}_{avg} = \left(|\overline{P}_{avg}|\right)(A)$$

Where, A = Area of circle

$$= \pi r^2 = \pi(2.5)^2 = 19.6349\ \text{m}^2$$

$$\overline{P}_{avg} = 3.3157 \times 19.6349 = 65.103\ \text{W}$$

$\therefore$ The average power crossing a circular area in the plane, $z = 0$ is 65.103W. As the area is in $z = 0$ plane, the direction normal to plane is $\bar{a}_y$.

Q19. A variable loop consisting of two stationary parallel conductors connected at one end to a voltmeter and a moving bar at the other end moving with a uniform velocity 'V' m/s. The loop is situated in a uniform flux density 'B' $\bar{a}_z$ Tesla. Derive an expression for the E.M.F induced in the loop for L = 10 cm, B = 0.2 $\bar{a}_z$ Tesla and V = 20 $\sqrt{y}$ $\bar{a}_y$ m/s and if y = 2 cm at t = 0, find for t = 0.006s.

(i) The 'V' of the moving bar

(ii) 'y' of the moving bar

(iii) The E.M.F induced in the loop and its polarity.

Ans: **May/June-13, (R09), Q7(a)**

Given that,

Length of the loop, L = 10 cm

Flux density, $B = 0.2\ \bar{a}_z$ Tesla

Velocity, $V = 20\sqrt{y}\ \bar{a}_y$ m/s

At t = 0.006 sec

Determine,

(i) Velocity, V of the moving bar

(ii) Position, y of the moving bar

(iii) EMF in the loop and polarity.

A variable loop consisting of two stationary parallel conductors connected at one end to a voltmeter is shown in figure.

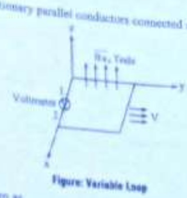


Figure: Variable Loop

The e.m.f induced in the loop is given as,

$$V_{emf} = \int \overline{E}.dl$$

$$= -N\frac{\partial\phi}{\partial t}$$

If it is a single turn circuit, $N = 1$

$$V_{emf} = -\frac{\partial\phi}{\partial t}$$

The flux ϕ at any time is given as,

$$\phi = ByL$$

$$\therefore \quad V_{emf} = \frac{-\partial}{\partial t} ByL$$

$$= -BL\frac{\partial y}{\partial t}$$

The differential of the position is the velocity i.e., $\frac{\partial y}{\partial t} = V$

$$V_{emf} = -BLV \quad ...(1)$$

According to Lenz's law, the point 2 will have higher potential than the point 1.

Thus, the reading of the voltmeter,

$$V_{emf(1)} = -BLV \quad ...(2)$$

From equation (1), we have,

$$V = \frac{\partial y}{\partial t}$$

$$20\sqrt{y} = \frac{\partial y}{\partial t} \quad [\because \text{Given } V = 20\sqrt{y}\,]$$

$$\frac{\partial y}{\sqrt{y}} = 20\,\partial t$$

On integrating the above equation, we get,

$$\int \frac{\partial y}{\sqrt{y}} = 20\int \partial t$$

$$2\sqrt{y} = 20t + C$$

At $t = 0, y = 2$ cm

$$2\sqrt{2\times10^{-2}} = 20(0) + C$$

$$C = 0.28$$

$$\therefore\ 2\sqrt{y} = 20t + 0.28$$

$$\sqrt{y} = \frac{20t + 0.28}{2}$$

$$\sqrt{y} = 10t + 0.14$$

$$y = (10t + 0.14)^2$$

Substituting $t = 0.006$ sec in above equation, we get,

$$y = [10(0.006) + 0.14]^2$$

$$\boxed{y = 0.04\,\text{m}}$$

(a) Velocity of the Moving Bar

$$V = 10\sqrt{y}$$

$$= 10\sqrt{0.04}$$

$$V = 2\bar{a}_y \text{ m/s}$$

(b) Position of the Moving Bar

$$y = 0.04 \text{ m}$$

(c) Induced EMF

$$V_{emf_{12}} = -BLV \qquad \text{[From equation (2)]}$$

$$= -(0.2 \times 10 \times 10^{-2} \times 2)$$

$$V'_{emf_{12}} = -0.04 \text{ volts}$$

The e.m.f induced in the loop is negative. This shows the point 2 will be at a higher voltage than point 1.

4.2 DISPLACEMENT CURRENT

Q20. Explain what is meant by the term displacement current. Deduce equation of continuity of current div (J + ∂D/∂t) = 0.

Ans: Model Paper-I, Q9(a)

Displacement Current

Displacement current is the current which is the resultant of time varying electric field. In case of a capacitor, displacement current is the current which is passing through the capacitor (externally) when it is connected across an alternating voltage source. Therefore, it indicates the rate of flow of charge between the capacitor plates in the external part of capacitor circuit.

Equation of Continuity

From Ampere's circuital law,

$$\nabla \times H = J \qquad \ldots (1)$$

Taking divergence on both sides

$$\Rightarrow\ \nabla \cdot \nabla \times H = \nabla \cdot J$$

We know that the divergence of curl of any vector is zero.

$$\therefore\ \nabla \cdot \nabla \times H = 0$$

But in case of time varying conditions,

$$\nabla \cdot J = \frac{-\partial \rho_v}{\partial t} \neq 0$$

$\therefore$ Equation (1) is modified as

$$\nabla \times H = J + J_d \qquad \ldots (2)$$

Taking divergence on both sides,

$$\nabla(\nabla \times H) = \nabla \cdot (J + J_d)$$

$$0 = \nabla \cdot J + \nabla \cdot J_d$$

$$\nabla \cdot J_d = -\nabla \cdot J \qquad \ldots (3)$$

Now, $$\nabla \cdot J = \frac{-\partial \rho_v}{\partial t} = \frac{-\partial}{\partial t}(\nabla \cdot D)$$

$$\nabla \cdot J = -\nabla \cdot \frac{\partial D}{\partial t} \qquad \ldots (4)$$

From equations (3) and (4), we have,

$$\nabla \cdot J_d = -\left(-\nabla \frac{\partial D}{\partial t}\right) = \nabla \cdot \frac{\partial D}{\partial t}$$

$$\Rightarrow\ \nabla \cdot J_d = \nabla \cdot \frac{\partial D}{\partial t}$$

$$\Rightarrow\ J_d = \frac{\partial D}{\partial t} \qquad \ldots (5)$$

Where,

J_d = Displacement current density

Substituting equation (5) in equation (2)

$$\boxed{\nabla \times H = J + \frac{\partial D}{\partial t}} \qquad \ldots (6)$$

Above equation is called as Maxwells equation for time varying fields.

On taking divergence on both sides we get,

$$\nabla \cdot \nabla \times H = \nabla \cdot \left(J + \frac{\partial D}{\partial t}\right)$$

$$\Rightarrow\ 0 = \nabla \cdot \left(J + \frac{\partial D}{\partial t}\right)$$

i.e., $$\text{div}\left(J + \frac{\partial D}{\partial t}\right) = 0$$

Hence proved.

Q21. Derive an expression for displacement current density.

Dec.-14, (R13), Q11

OR

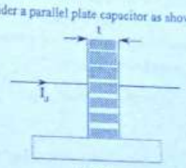
Obtain the expression for displacement current density.

Ans:

The current through a capacitor is called displacement current. The displacement current per unit area is called displacement current density, given as,

$$J_d = \frac{I_d}{A} \text{ A/m}^2$$

Consider a parallel plate capacitor as shown in figure



Figure

Let,

I_d = Current through the capacitor

t = Separation between the plates

$\in$ = Permittivity of the dielectric

A = Area of cross-section

Q = Amount of charge stored

$$I_d = \frac{dQ}{dt} = C\frac{dV}{dt} \quad [\because Q = CV] \qquad \ldots (1)$$

But, C is also given as

$$C = \frac{\in A}{t} \qquad \ldots (2)$$

Substituting equation (2) in equation (1), we get,

$$I_d = \frac{\in A}{t}\frac{dV}{dt}$$

But, $dV = t\,dE$

Where,

E is electric field intensity.

$$\therefore\ I_d = \frac{\in A}{t} t \frac{dE}{dt}$$

$$I_d = \frac{\in A\,dE}{dt}$$

$$\therefore\ J_d = \frac{I_d}{A} = \frac{\in A\,dE}{A\,dt}$$

$$J_d = \frac{\in dE}{dt} \qquad [\because \bar{D} = \bar{E}\in]$$

$$\boxed{J_d = \frac{d\bar{D}}{dt}}$$

Where,

$\bar{D}$ = Electrical displacement vector

Q22. Show that the ratio of the amplitude of the conduction current and displacement current density is $\frac{\sigma}{\omega\epsilon}$ for the applied field $E = E_{max} \cos \omega t$ V/m.

Ans:

Conduction Current

Conduction current is the current caused by the free electrons in a conductor, when the conductor is conducting.

Conduction current density is defined as the current at a given point, passing through the unit surface area, which is normal to the direction of conduction current. It is denoted by '$\bar{J}_C$' and measured in ampere per square meter.

From Ohm's law, we have

$$\boxed{\bar{J}_c = \sigma\bar{E}}$$

Where,

$\bar{J}_c$ = Conduction current density, A/m²

σ = Conductivity, ℧

$\bar{E}$ = Electric field intensity, V/m

$\bar{E} = E_{max} \cos\omega t$, V/m

$$\bar{J}_c = \sigma\,(E_{max} \cos\omega t)$$

$$\Rightarrow \bar{J}_c = (\sigma E_{max}) \cos\omega t$$

$$\Rightarrow |\bar{J}_c| = \sigma E_{max} \quad \text{... (1)}$$

Displacement Current

Displacement current is the current through a capacitor (externally) when an alternating voltage source is connected to its plates.

Consequently, the displacement current density is defined as the rate of displacement of electric flux density($\bar{D}$) with time. It is denoted by '$\bar{J}_d$' and measured in ampere per square meter.

$$\therefore \bar{J}_d = \frac{\partial \bar{D}}{\partial t} \text{ A/m}^2$$

$$\Rightarrow \bar{J}_d = \frac{\partial}{\partial t}(\epsilon \bar{E}) \qquad [\because \bar{D} = \epsilon \bar{E}]$$

$$\Rightarrow \bar{J}_d = \epsilon \frac{\partial}{\partial t}(\bar{E})$$

$$\Rightarrow \bar{J}_d = \epsilon \frac{\partial}{\partial t}(E_{max} \cos \omega t)$$

$$\Rightarrow \bar{J}_d = \epsilon E_{max} \frac{\partial}{\partial t}(\cos \omega t)$$

$$\Rightarrow \bar{J}_d = \epsilon E_{max}(-\omega \sin \omega t)$$

$$\Rightarrow \bar{J}_d = (-\omega \epsilon E_{max}) \sin \omega t$$

$$\Rightarrow |\bar{J}_d| = |-\omega\epsilon E_{max}| \quad \text{... (2)}$$

The ratio of amplitude of conduction current density ($|\bar{J}_c|$) to displacement current density ($|\bar{J}_d|$) is given by,

$$\frac{|\bar{J}_c|}{|\bar{J}_d|} = \frac{\sigma E_{max}}{|-\omega\epsilon E_{max}|} = \frac{\sigma E_{max}}{\omega\epsilon E_{max}} = \frac{\sigma}{\omega\epsilon}$$

$$\boxed{\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\epsilon}}$$ Hence showed

Q23. Explain why conduction current is absent through the capacitor.

OR

Explain how the current flowing through capacitor differs from the normal conduction current.

Ans: Model Paper-III, Q9(a)

We have,

$$\bar{J}_c = \frac{I_c}{A}$$

$$= \sigma\bar{E}$$

I_c = Conduction current

$$\therefore I_c = \sigma\bar{E}A$$

From Ampere's circuit law,

$$\nabla \times H = J + J_d \quad \text{... (1)}$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times H) = \nabla (J + J_d)$$

$$\Rightarrow \nabla (J + J_d) = 0$$

$$\therefore \nabla \cdot J_d = -\nabla \cdot J \quad \text{... (2)}$$

Now, $\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$

$$\nabla \cdot J = \frac{-\partial}{\partial t}(\nabla \cdot D) \qquad [\because \nabla \cdot D = \rho_v]$$

$$\nabla \cdot J = -\nabla \cdot \frac{\partial D}{\partial t} \quad \text{... (3)}$$

Substituting equation (3) in equation (2), we get,

$$\nabla \cdot J_d = -\left[-\nabla \cdot \frac{\partial D}{\partial t}\right] = \nabla \cdot \frac{\partial D}{\partial t}$$

$\therefore$ Displacement current density, $J_d = \frac{\partial D}{\partial t}$

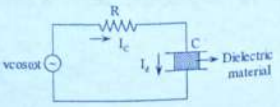
Then, displacement current, $I_d = J_d \cdot A$

$$\therefore I_d = \frac{\partial D}{\partial t} A$$

Where, A = Plate area cross section, m^2

In the circuit shown, the current through the resistor (also wires) is called conduction current and the current flowing through the capacitor is called as displacement current. Generally, the electrical or physical connection is must for the flow of normal conduction current. But, the capacitor is a passive element with two parallel plates separated by a dielectric material of constant 'k'. As, the dielectric being an insulator, which does not allow the flow of conduction current by providing infinite impedance in order of mega ohms. Thus, the conduction current is absent through the capacitor. Therefore, only displacement current flows through it.

Even though, the resistor oppose the flow of current, because of electrical contact conduction current flows through it, as shown in figure.



Figure

In this way, the current flowing through capacitor differs from the normal conduction current.

Q24. Explain how the displacement current density differs from conduction current density.

Model Paper-III, Q9(b)

Ans:

	Displacement Current Density		Conduction Current Density
1.	It is defined as the displacement current at given point, passing through a unit surface area when the surface is normal to the direction of displacement current. It is denoted by J_D.	1.	It is defined as the convection current at a given point, passing through a unit surface area normal to the direction of current. It is denoted by J_c.
2.	Displacement current results when the current flows through the insulating medium.	2.	Conduction current results in the conductors in the presence of electric field.
3.	Displacement current density is given by, $\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$	3.	Conduction current density is given by, $\bar{J}_c = \frac{I_c}{A} = \sigma\bar{E}$.
4.	Displacement current is greater when compared to conduction current in a dielectric medium.	4.	Conduction current is greater when compared to displacement current in a conductor.
5.	Displacement current density is proportional to the derivative of electric displacement. to the electric field intensity.	5.	Conduction current density is proportional to the conduction current and inversely proportional to the surface area (or) it is proportional to the product of conductivity (σ) and electric field intensity ($\bar{E}$).

Q25. ...conduction, convection and displacement currents.

Model Paper-II, Q5(a)

Ans:

	Conduction Current		Convention Current		Displacement Current
1.	Conduction current is the flow of electrons through any conducting medium.	1.	Convection current is the flow of electrons through a non-conducting (insulating) medium.	1.	Displacement current is the flow of charge which results due to time varying electric field.
2.	conduction current is the current passing through the resistor and wires.	2.	It is the leakage current passing through the dielectric (insulating) medium of the capacitor.	2.	It is the rate of flow of charge between the capacitor plates in the external part of capacitor circuit.
3.	It is given by $I_c = \sigma\bar{E}.\bar{S}$ Where, σ = Conductivity $\bar{E}$ = Electric field intensity $\bar{S}$ = Surface area charge	3.	It is given by $I = \iint \bar{J}.d\bar{S} = \iint \rho_v \bar{V}.d\bar{S}$ Where, $\bar{J} = \rho_v\bar{V}$ = Convection current density, ρ_v = Volume density and $\bar{V}$ = Velocity.	3.	It is given by $I_d = \epsilon S \frac{\partial E}{\partial t}$ Where, ϵ = Permittivity of dielectric material.
4.	It is independent of frequency.	4.	It's value increases with frequency.	4.	It is directly proportional to frequency.
5.	If obeys ohm's law and hence has linear charge characteristics	5.	It does not obey's ohm's law and so have non-linear characteristics.	5.	It also does not obey ohm's law and so have non-linear characteristic.
6.	Exists in both, time varient and invarient	6.	It also exists in both time varient cases and invarient cases	6.	It exists only in time varient case.
7.	Typical examples are current through conductors, resistors etc.	7.	Typical examples are electron beam through vacuum, CRT, liquids etc.	7.	Typical examples are the current flowing through capacitor (externally from its plate to plate) and all imperfect conductors are a time varying conduction current.
8.	It flows only through the closed paths in the circuits.	8.	It flows only through the open paths in the circuits.	8.	It provides a closed path in the circuits having capacitor elements or open circuits where, the conduction current cannot flow further.

Q26. Find the displacement current density next to your radio, in air, where the local FM station provides a carrier having H = 0.2 cos [210 (3 × 10⁸t – x)] a_z A/m.

Ans:

Given that,

Magnetic field intensity

$$H = 0.2\cos[210\,(3\times10^8\,t - x)]\,a_z = 0.2\cos[6.3\times10^{10}\,t - 210x]a_z$$

To determine,

Displacement current density 'J_d' = ?

From Ampere's circuital law, we have

$$J_d = \frac{\partial \overline{D}}{\partial t} = \frac{\partial}{\partial t}(\in \overline{E}) \text{ (As } \in = \in_0 \in_r \text{, In air } \in_r = 1 \text{ so } \in = \in_0)$$

$$J_d = \frac{\partial}{\partial t}(\in_0 \overline{E}) \qquad (\because \overline{D} = \in_0 \overline{E})$$

$$J_d = \in_0 \frac{\partial \overline{E}}{\partial t} \qquad \text{...(1)}$$

$$\overline{\nabla} \times \overline{E} = \frac{-\partial \overline{B}}{\partial t}$$

$$= \frac{-\partial}{\partial t}(\mu_0 \overline{H})$$

$$= -\mu_0 \frac{\partial \overline{H}}{\partial t}$$

$$= -\mu_0 \frac{\partial}{\partial t}[0.2 \cos(6.3 \times 10^{10} t - 210 x)] a_y$$

$$= -\mu_0 \times 0.2(-6.3 \times 10^{10}) \sin(6.3 \times 10^{10} t - 210 x)\overline{a}_z$$

$$\overline{\nabla} \times \overline{E} = 4\pi \times 10^{-7} \times 1.26 \times 10^{10} \sin(6.3 \times 10^{10} t - 210 x)\overline{a}_z$$

$$\overline{\nabla} \times \overline{E} = 15833.6 \sin(6.3 \times 10^{10} t - 210 x)\ \overline{a}_z$$

We should consider 'z' component of $\overline{\nabla} \times \overline{E}$ as $\overline{a}_z$ is present on RHS

$$\therefore \left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right] \overline{a}_z = 15833.6 \sin(6.3 \times 10^{10} t - 210 x)\ \overline{a}_z$$

$$\frac{\partial E_y}{\partial x} = 0 \text{ as } y \text{ is present on RHS}$$

$$\therefore \frac{\partial E_x}{\partial y} = 15833.6 \sin(6.3 \times 10^{10} t - 210 x)$$

$$\Rightarrow E_x = 15833.6 \left[\frac{-\cos(6.3 \times 10^{10} t - 210x)}{210}\right]$$

$$\overline{E} = -75.39 \cos(6.3 \times 10^{10} t - 210 x)\ \overline{a}_x$$

Substituting $\overline{E}$ value in equation (1),

$$\overline{J}_d = \in_0 \frac{\partial \overline{E}}{\partial t}$$

$$= \in_0(-75.39)(-6.3 \times 10^{10}) \sin(6.3 \times 10^{10} t - 210 x)\ \overline{a}_x$$

$$= 8.854 \times 10^{-12} \times 4.74768 \times 10^{12} \sin(6.3 \times 10^{10} t - 210 x)\overline{a}_x$$

$$\boxed{\overline{J}_d = 42.05 \sin(6.3 \times 10^{10} t - 210 x)\overline{a}_x \text{ A/m}^2}$$

Q27. In a material for which $\sigma = 5.0$ S/m and $\in_r = 1$, the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction and displacement current densities, and the frequency at which both have equal magnitudes.

Ans: (Nov./Dec.-18, (R16), Q11(b) | Nov./Dec.-17, (R16), Q5(b) | Nov.-15, (R13), Q10(b))

Given that,

Conductivity, $\sigma = 5.0$ S/m

Relative Permittivity, $\in_r = 1$

Electric field intensity, $\in = 250 \sin 10^{10} t$ V/m

To determine,

(i) Conduction current density, $\overline{J}_c$

(ii) Displacement current density, $\overline{J}_d$

(iii) Frequency at $J_c = J_d$

(i) Conduction current density is given by,

$$\overline{J}_c = \sigma \overline{\in}$$

$$= 5.0 \times (250 \times \sin 10^{10} t) = 1250 \sin 10^{10} t \text{ A/m}^2$$

$$\boxed{\therefore \overline{J}_c = 1250 \sin 10^{10} t \text{ A/m}^2}$$

(ii) Displacement current density is given by,

$$\overline{J}_d = \frac{\partial \overline{D}}{\partial t} = \frac{\partial}{\partial t}(\overline{E}) \qquad [\because \overline{D} = \in \overline{E}]$$

$$= \in \frac{\partial \overline{E}}{\partial t} = \in_0 \in_r \frac{\partial \overline{E}}{\partial t} \qquad [\because \in = \in_0 \in_r]$$

$$= 8.854 \times 10^{-12} \times 1 \times \frac{\partial}{\partial t}[250 \times \sin 10^{10} t]$$

$$= 8.854 \times 10^{-12} \times 1 \times 250 \frac{\partial}{\partial t}[\sin 10^{10} t]$$

$$= 8.854 \times 10^{-12} \times 250 \times \cos 10^{10} t \times 10^{10}$$

$$= 22.135 \cos 10^{10} t \text{ A/m}^2$$

$$\boxed{\therefore \overline{J}_d = 22.135 \cos 10^{10} t \text{ A/m}^2}$$

(iii) Frequency at $J_c = J_d$ is given by,

$$f = \frac{\sigma}{2\pi \in_0 \in_r} = \frac{5.0}{2 \times \pi \times 8.854 \times 10^{-12} \times 1}$$

$$= 8.9877 \times 10^{10} = 89.877 \times 10^9 \text{ Hz}$$

$$\boxed{\therefore f = 89.877 \text{ GHz}}$$

Q28. A parallel plate capacitor with plate area of 5 cm² and plate separation of 3 mm has a voltage 50 sin 10³t volts applied to its plates. Calculate the displacement current assuming $\in = 2\in_r$.

Ans:

Given that,

Plate area, $A = 5 \text{ cm}^2 = 5 \times 10^{-4} \text{ m}^2$

Plate separation, $d = 3 \text{ mm} = 3 \times 10^{-3}$ m

Capacitor voltage, $V = 50 \sin 10^3 t$

Permittivity, $\in = 2\in_0$

Displacement current, $I_d = ?$

We know,

Electric flux density, $D = \in E$

$$\Rightarrow D = \frac{\in V}{d} \qquad \left(\because E = \frac{V}{d}\right)$$

Displacement current density,

$$J_d = \frac{\partial D}{\partial t}$$

$$\therefore J_d = \frac{\partial}{\partial t}\left(\frac{\in V}{d}\right)$$

$$J_d = \frac{\in}{d}\frac{\partial}{\partial t}(V)$$

Displacement current,

$$I_d = J_d A$$

$$I_d = \frac{\in}{d}\frac{\partial V}{\partial t} A$$

$$I_d = \frac{\in A}{d}\frac{\partial V}{\partial t}$$

$$\therefore I_d = C \frac{\partial V}{\partial t}$$

Where capacitance,

$$C = \frac{\in A}{d}$$

$$= \frac{2\in_0 \times 5 \times 10^{-4}}{3 \times 10^{-3}}$$

$$= \frac{2 \times 8.85 \times 10^{-12} \times 5 \times 10^{-4}}{3 \times 10^{-3}}$$

$$= 2.95 \times 10^{-12}$$

$$= 2.95 \text{ pF}$$

∴ Displacement current,

$$I_d = C\frac{\partial V}{\partial t}$$
$$= 2.95 \times 10^{-12} \times \frac{\partial}{\partial t}(50 \sin 10^3 t)$$
$$= 2.95 \times 10^{-12} \times 50 \cos 10^3 t \times 10^3$$
$$= 1.475 \times 10^{-7} \cos 10^3 t$$
$$= 0.147 \times 10^{-6} \cos 10^3 t \text{ Amps}$$
$$\boxed{\therefore I_d = 0.147 \cos 10^3 t \ \mu A}$$

Q29. Find the displacement current density within a parallel plate capacitor where $\in = 100 \in_0$, a = 0.01 m², d = 0.05 mm and the capacitor voltage is 100 sin 200 πt volts.

Ans:

Given that,

Permittivity, $\in = 100 \in_0$

Plate area, $a = 0.01$ m²

Plate separation, $d = 0.05$ mm

Capacitor voltage, $V = 100 \sin 200 \pi t$ V

Displacement current density, $J_d = ?$

We know that

Capacitance,

$$C = \frac{\in a}{d} = \frac{100\in_0 \times 0.01}{0.05\times10^{-3}}$$
$$= \frac{100\times8.854\times10^{-12}\times0.01}{0.05\times10^{-3}}$$
$$= 0.1771 \times 10^{-6} \text{ F}$$
$$= 0.1771 \ \mu\text{F}$$

Displacement current,

$$I_d = C\frac{\partial V}{\partial t}$$
$$= 0.1771 \times 10^{-6} \times \frac{d}{dt}(100 \sin 200 \pi t)$$
$$= 0.1771 \times 10^{-6} \times 100 \times \cos 200 \pi t \times 200 \pi$$
$$= 11.127 \times 10^{-3} \cos 200 \pi t \ A$$

∴ Displacement current density,

$$J_d = \frac{I_d}{a} = \frac{11.127\times10^{-3}\cos200\pi t}{0.01}$$
$$= 1.1127 \cos 200 \pi t \text{ A/m}^2$$

Alternate Method

Displacement current density,

$$J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\in E)$$
$$= \in\frac{\partial E}{\partial t} = \in\frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$
$$= \frac{\in}{d}\frac{\partial V}{\partial t}$$
$$= \frac{100\in_0}{0.05\times10^{-3}}\times\frac{\partial}{\partial t}(100\sin 200\pi t)$$
$$= \frac{100\times8.854\times10^{-12}}{0.05\times10^{-3}}\times100 \cos 200\pi t\times\frac{200}{\pi}$$
$$= 1.1127 \cos 200 \pi t \text{ A/m}^2$$

Q30. Verify that the displacement current in the capacitor is same as that of the conduction current in wires.

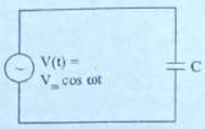
May/June-13, (R09), Q7(b)

OR

Show that in a capacitor, the conduction current and displacement current are equal.

Ans: Model Paper-II, Q8(b)

The two a parallel plate capaciter with capacitance C are considered and if they are connected across a voltage source as shown in figure.



Figure

Then, voltage $v(t)$ is mathematically represented as,

$$v(t) = V_m \cos \omega t$$

The current flowing through the capacitor called as conduction current is given by,

$$i_c = C\frac{dv(t)}{dt}$$
$$= C\frac{d}{dt}[V_m \cos \omega t]$$
$$= -CV_m\omega \sin \omega t$$

∴ Conduction current, $i_c = -V_m C\omega \sin \omega t$ ... (1)

Displacement current density,

$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$$
$$= \frac{\partial}{\partial t}(\in \bar{E})$$

But we know that, $\bar{E} = \frac{v(t)}{d}$ ... (2)

Substituting equation (3) in equation (2), we get, ... (3)

$$\bar{J}_d = \frac{\partial}{\partial t}\left(\in\frac{v(t)}{d}\right)$$
$$= \frac{\in}{d}\frac{\partial}{\partial t}[V_m \cos \omega t]$$
$$= -\frac{\in}{d}V_m \sin \omega t$$

Displacement current,

$$i_d = \bar{J}_d \times A$$
$$= -\frac{\in V_m\omega}{d}\sin \omega t \times A$$
$$= -\frac{\in A}{d}V_m\omega \sin \omega t$$
... (4)

Capacitance of parallel plate capacitor, $C = \frac{\in A}{d}$ ... (5)

Substituting equation (5) in equation (4), we get,

$$i_d = -CV_m\omega \sin \omega t$$
... (6)

Comparing equations (1) and (6), we get,

$$i_c = i_d$$

∴ In a u7capacitor, conduction current is equal to displacement current.

Q31. A parallel plate capacitor with plates of radius R and separation 'd' has a voltage applied at the centre as given by V = V_s sinωt, as a function of radius r (for r < R), find,

(a) The displacement current density J_d(r).

(b) Magnetic field H(r). Take d << R.

Ans:

Given that,

Radius of plates = R

Distance between plates = 'd'

Supply voltage, $V = V_s \sin\omega t$

(a) $J_d(r) = ?$

(b) $H(r) = ?$

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(a) Displacement Current Density

Displacement current density,

$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\in \bar{E}) \quad (\because \bar{D} = \in \bar{E})$$

$$= \in \frac{\partial}{\partial t}\left(\frac{V}{d}\right) \left(\because \bar{E} = \frac{V}{d}\right)$$

$$= \frac{\in}{d}\frac{\partial}{\partial t}[V_0 \sin \omega t]$$

$$= \frac{\in}{d}(V_0 \cos\omega t)\omega$$

$$= \frac{\in \omega V_0}{d}\cos\omega t \quad \text{... (1)}$$

$\therefore$ Displacement current density, $\bar{J}_d = \frac{\epsilon_0 \omega V_0}{d}\cos\omega t$

(b) Magnetic Field

According to Ampere's circuital law,

$$\nabla \times \bar{H} = \bar{J} \quad \text{... (2)}$$

Representing $\nabla \times \bar{H}$ in cylindrical coordinates as,

$$\nabla \times \bar{H} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial_r} & \frac{\partial}{\partial_\phi} & \frac{\partial}{\partial_z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

Assuming that magnetic field intensity $\bar{H}$ is in a_ϕ direction.

$$\therefore \quad \nabla \times \bar{H} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial_r} & \frac{\partial}{\partial_\phi} & \frac{\partial}{\partial_z} \\ 0 & rH_\phi & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[\bar{a}_r\left(0 - \frac{\partial}{\partial_z}(rH_\phi)\right) - r\bar{a}_\phi(0-0) + \bar{a}_z\left(\frac{\partial}{\partial_r}(rH_\phi) - 0\right)\right]$$

$\bar{H}$ is a function of $\bar{r}$ only,

$$\therefore \quad \nabla \times \bar{H} = \frac{1}{r}\left[\bar{a}_r(0) - r\bar{a}_\phi(0) + \bar{a}_z\left(\frac{\partial}{\partial_r}(rH_\phi)\right)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial_r}(rH_\phi)\bar{a}_z \quad \text{... (3)}$$

Substituting equations (1) and (3) in equation (2), we get,

$$\frac{1}{r}\frac{\partial}{\partial_r}(r\bar{H}_\phi) = \frac{\in \omega V_0}{d}\cos\omega t$$

$$\frac{\partial}{\partial_r}(rH_\phi) = \frac{\in \omega V_0 r}{d}\cos\omega t$$

Integrating on both sides with respect to 'r' we get,

$$\int \frac{\partial}{\partial r}(rH_\phi)d_r = \int \frac{\epsilon_0 \omega V_0}{d} r\cos\omega t dr$$

$$rH_\phi = \frac{\epsilon_0 \omega V_0}{d}\left(\frac{r^2}{2}\right)\cos\omega t$$

$$\therefore \quad H_\phi = \frac{\epsilon_0 \omega V_0 r}{2d}\cos\omega t$$

$$\therefore \quad H(r) = \frac{\epsilon_0 \omega V_0 r}{2d}\cos\omega t$$

Q32. A coaxial capacitor with inner radius 5 mm, outer radius 6 mm and a length of 500 mm has a dielectric for which $\in_r$ = 6.7 and an applied voltage 250 sin 377 t volts. Determine the displacement current i_D and compare with conduction current I_c.

Ans: Nov.-10, Set-1, Q3(a)

Given that,

Applied voltage, $V = 250 \sin 377\,t$ (volts)

Length, $l = 500$ mm

$= 0.5$ m

Inner radius, $a = 5$ mm

$= 5 \times 10^{-3}$ m

Outer radius, $b = 6$ mm

$= 6 \times 10^{-3}$ m

Let the axis of the cable lie along z-axis as shown in figure.

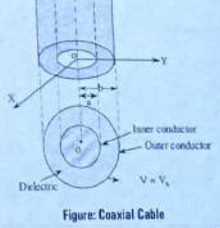

Figure: Coaxial Cable

From Laplace's equation, we have,

$$\nabla^2 V = 0$$

In cylindrical coordinate system, $\nabla^2 V$ can be expanded as,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

It is evident that the potential neither varies with length of the cable (i.e., along z) nor with angle (ϕ) and,instead it varies with radial distance 'r'.

$$\therefore \quad \frac{\partial V}{\partial z} = \frac{\partial V}{\partial \phi} = 0$$

$$\Rightarrow \quad \nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)$$

Substituting the above equation in Laplace's equation,

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

Integrating on both sides with respect to 'r',

$$\int \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)dr = \int 0\, dr$$

$$\Rightarrow \quad r\frac{\partial V}{\partial r} = C$$

Where, C = Integration constant

$$\Rightarrow \quad \frac{\partial V}{\partial r} = \frac{C}{r} \Rightarrow \partial V = \frac{C}{r}\partial r$$

Again integrating on both sides with respect to 'r',

$$\int \partial V = \int \frac{C}{r}\partial r$$

$$V = C\, ln\,(r) + D \quad \text{... (1)}$$

Where, D – Integration constant

Assuming the inner conductor at zero potential at $r = a$

i.e.,

at $r =$ a, $V = 0$ and

at $r =$ b, $V = 250 \sin 377\, t$

Substituting, $r = a$, and $V = 0$ in equation (1), we get,

$$\Rightarrow \quad 0 = C\, ln\, a + D$$

$$\Rightarrow \quad D = -C\, ln\, a \quad \text{... (2)}$$

Similarly at $r = b$ and $V = 250 \sin (377t)$ equation (1) becomes,

$$\Rightarrow \quad 250 \sin (377t) = C\, ln\, b + D$$

$$\Rightarrow \quad 250 \sin (377t) = C\, ln\, b - C\, ln\, a$$

$$\Rightarrow \quad 250 \sin (377t) = C\,(ln\, b - ln\, a) \qquad [\because \text{from equation (2)}]$$

$$\Rightarrow \quad 250 \sin (377t) = C\, ln\,(b/a)$$

$$\Rightarrow \quad C = \frac{250\sin(377t)}{ln\left(\frac{b}{a}\right)}$$

$$\therefore \quad C = \frac{250\sin(377t)}{ln\left(\frac{6\times10^{-3}}{5\times10^{-3}}\right)} = 1371.2 \sin (377t)$$

Substituting the value of C in equation (2) we get,

$$D = -1371.2 \sin (377t)\, ln\,(5 \times 10^{-3})$$

Substituting D value in equation (1), we have,

$$V = 1371.2 \sin (377t)\, ln\,(r) - 1371.2 \sin (377t)\, ln\,(5 \times 10^{-3})$$

We know that, electric field intensity,

$$\bar{E} = -\nabla V$$

$$= -\frac{\partial V}{\partial r}\bar{a}_r$$

$$= -\frac{\partial}{\partial r}[1371.2 \sin (377t)\, ln\,(r)]\,\bar{a}_r$$

$$= -1371.2 \sin (377t)\frac{1}{r}\,\bar{a}_r \text{ V/M}$$

We have, electric field density,

$$\bar{D} = \in \bar{E}$$

$$= \in\left[-1371.2\sin(377t)\frac{1}{r}\right]\bar{a}_r$$

$$= (-1371.2 \sin (377t))\left(\frac{\in_0 \in_r}{r}\right)\bar{a}_r$$

$$= \frac{-1371.2\sin(377t)(8.854\times10^{-12})(6.7)}{r}\,\bar{a}_r \qquad [\because \in_r = 1]$$

$$= (-81.342 \times 10^{-9})\left(\frac{\sin(377t)}{r}\right)\bar{a}_r$$

We know that,

$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[(-81.342\times10^{-9})\left(\frac{\sin(377t)}{r}\right)\bar{a}_r\right]$$

$$= -\frac{81.342\times10^{-9}}{r}(\cos(377t)(377)\,\bar{a}_r$$

$$= -\frac{30.66\times10^{-6}}{r}\cos(377t)\,\bar{a}_r \text{ A/m}^2$$

Displacement current,

$$i_d = J_d (2\pi r L)$$

$$= \left(-\frac{30.66\times10^{-6}}{r}\cos(377t)\right)(2\pi r \times 0.5)$$

$$= -96.32\times10^{-6}\cos(377t)\,A$$

$$= -96.32\cos(377t)\,\mu A$$

The circuit analysis method for conduction current requires capacitance.

$$\therefore \quad C = \frac{2\pi\epsilon L}{ln\left(\frac{b}{a}\right)}$$

$$= \frac{2\pi\times8.854\times10^{-12}\times6.7\times0.5}{ln\left(\frac{6\times10^{-3}}{5\times10^{-3}}\right)}$$

$$= 1.022\times10^{-9}\,F$$

$$= 1.022n\,F$$

We know that conduction current,

$$i_c = C\frac{dV}{dt}$$

We have,

$$V = 250\sin(377t)$$

$$\therefore \quad i_c = 1.022\times10^{-9}\frac{d}{dt}(250\sin(377t))$$

$$= (1.022\times10^{-9})(250)(377)\cos(377t)$$

$$= 96.32\times10^{-6}\cos(377t)$$

$$= 96.32\cos(377t)\,\mu A$$

Therefore, the displacement current is equal to the conduction current, i.e.,

$$\boxed{I_d = I_c}$$

Q33. Find the frequency at which conduction current density and displacement current density are equal in,

(i) Distilled water, for which $\epsilon_r = 18$ and $\sigma = 2.0\times10^{-4}$ mho/m

(ii) Sea water for which $\epsilon_r = 1$ and $\sigma = 4.0$ mho/m.

Ans: Nov.-10, Set-2, Q1(b)

Note

In the given problem, the value of relative permittivity ϵ_r for sea water is misprinted. Thus assuming $\epsilon_r = 1$.

(i) Given that,

Conductivity, $\sigma = 2\times10^{-4}$ mho/m

Relative permittivity, $\epsilon_r = 18$

Peak value of condition current density is,

$$|\bar{J}_c| = \sigma E_m \qquad \ldots(1)$$

Peak value of displacement current density is,

$$|\bar{J}_d| = \epsilon E_m \omega \qquad \ldots(2)$$

Given condition is,

Conduction current density = Displacement current density

$\therefore$ Equating equations (1) and (2), we get,

$$\sigma E_m = \epsilon E_m \omega$$

$$\omega = \frac{\sigma}{\epsilon}$$

$$2\pi f = \frac{\sigma}{\epsilon}$$

$$\therefore \quad f = \frac{\sigma}{2\pi\epsilon_0\epsilon_r} \quad (\because \epsilon = \epsilon_0\epsilon_r) \qquad \ldots(3)$$

Substituting the given values in equation (3), we get,

$$f = \frac{2\times10^{-4}}{2\times\pi\times8.854\times10^{-12}\times18}$$

$$= 199727.609$$

$$= 199.727\text{ kHz}$$

$\therefore$ Frequency at which $|\bar{J}_c| = |\bar{J}_d|$ is,

$$f = 199.727\text{ kHz}$$

(ii) Given that,

Conductivity, $\sigma = 4$ mho/m

Relative permittivity, $\epsilon_r = 1$

Substituting the given values in equation (3), we get,

$$f = \frac{4}{2\pi\times8.854\times10^{-12}\times1}$$

$$= 71.9\text{ GHz}$$

$\therefore$ Frequency at which $|\bar{J}_c| = |\bar{J}_d|$ is,

$$f = 71.9\text{ GHz}$$

Q34. In a given lossy dielectric medium, conduction current density $J_c = 0.02\sin10^9 t$ (A/m²). Find the displacement current density if $\sigma = 10^3$ V/m and $\epsilon_r = 6.5$.

Ans: Nov.-10, Set-3, Q1(b)

Given that,

Conduction current density, $J_c = 0.02\sin10^9 t$ (A/m²)

Conductivity, $\sigma = 10^3$ V/m

Relative permittivity, $\epsilon_r = 6.5$

For lossy dielectric medium, the ratio of magnitudes of the conduction current density to the displacement current density is given by,

$$\frac{|\bar{J}_c|}{|\bar{J}_d|} = \frac{\sigma}{\omega\epsilon}$$

$$\therefore \quad J_d = \frac{\omega\epsilon}{\sigma}J_c$$

$$= \frac{10^9\times6.5\times8.854\times10^{-12}}{10^3}\times0.02\sin(10^9 t) \qquad [\because \omega = 10^9]$$

$$= 1.151\times10^{-6}\sin(10^9 t)\text{ A/m}^2$$

$$= 1.151\sin(10^9 t)\,\mu\text{A/m}^2$$

We know that $\bar{J}_c$ and $\bar{J}_d$ are always right angles to each other. Thus, the displacement current can be written as,

$$\bar{J}_d = 1.151\cos10^9 t\,\mu\,(\text{A/m}^2).$$

4.3 MAXWELL'S EQUATION IN POINT AND INTEGRAL FORM

Q35. State Maxwell's equations in their general point form and derive their form for harmonically varying fields.

Ans: Model Paper-III, Q8(b)

The Maxwell's equations in their general point (differential) form are,

(i) $\nabla\times\bar{H} = \bar{J} + \dfrac{\partial\bar{D}}{\partial t}$ ...(1)

(ii) $\nabla\times\bar{E} = \dfrac{-\partial\bar{B}}{\partial t}$ ...(2)

(iii) $\nabla.\bar{D} = \rho$ ...(3)

(iv) $\nabla.\bar{B} = 0$ ...(4)

For harmonically varying electric and magnetic fields, with respect to time a time factor $e^{j\omega t}$ is to be multiplied to $\bar{D}$ and $\bar{B}$ i.e., to electric flux density and magnetic flux density.

$$\bar{D} = D_m e^{j\omega t} \text{ and } \bar{B} = B_m e^{j\omega t}$$

$$\therefore \quad \frac{\partial\bar{D}}{\partial t} = \frac{\partial}{\partial t}(D_m e^{j\omega t}) = D_m e^{j\omega t}\times(j\omega)$$

$$= j\omega D_m e^{j\omega t}$$

$$= j\omega\bar{D} \qquad \ldots(5)$$

$$\frac{\partial\bar{B}}{\partial t} = \frac{\partial}{\partial t}(B_m e^{j\omega t}) = j\omega B_m e^{j\omega t}$$

$$= j\omega\bar{B} \qquad \ldots(6)$$

Substituting equation (5) in equation (1), we get,

$$\Rightarrow \quad \nabla\times\bar{H} = \bar{J} + j\omega\bar{D}$$

We know that,

$$\bar{J} = \sigma\bar{E} \text{ and } \bar{D} = \epsilon\bar{E}$$

$$\nabla\times\bar{H} = \sigma\bar{E} + j\omega\epsilon\bar{E}$$

$$= (\sigma + j\omega\epsilon)\bar{E} \qquad \ldots(7)$$

Substituting equation (6) in equation (2), we get,

$$\nabla\times\bar{E} = -j\omega\bar{B}$$

$$\therefore \nabla\times\bar{E} = -j\omega\mu\bar{H} \quad [\because B = \mu H] \qquad \ldots(8)$$

Equations (3) and (4) remains same for harmonically varying fields also.

$\therefore$ Equations (7), (8), (3) and (4) are the Maxwell's equations in point (or) differential form, for harmonically varying fields.

Q36. Write Maxwell's equation for static fields. Explain how they are modified for time varying electric and magnetic fields.

Nov./Dec.-18, (R16), Q10

OR

Write Maxwell's equations in integral form for time varying fields.

Nov./Dec.-17, (R16), Q10(a)

OR

Write down Maxwell's equations in their general integral form. Derive the corresponding equations for fields varying harmonically with time.

Ans:

Maxwell's Equations in Integral Form

1. Maxwell's first equation is derived from Gauss's law and integral form of which is,

$$\int_s \bar{D}.d\bar{s} = \int_v \rho_v\,dv$$

Where,

D = Electric flux density c/m²

ρ_v = Volume charge density.

Maxwell's first equation gives the relation between electric flux density and volume charge density and is applied for electrostatic fields.

2. Maxwell's second equation is also derived from Gauss's law and it is given by,

$$\int_s \bar{B}\,d\bar{s} = 0$$

Where,

B = Magnetic flux density, Wb/m²

Maxwell's second equation is applied for magnetostatic fields.

3. Maxwell's third equation is derived from Ampere's law and in integral form it is expressed as,

$$\oint \bar{H}.d\bar{l} = \int_s \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) d\bar{s}$$

Where,

$\bar{H}$ = Magnetic field intensity, A/m

$\bar{J}$ = Conduction current density, A/m^2

$\frac{\partial \bar{D}}{\partial t}$ = Displacement current density, A/m^2.

Maxwell's third equation is also applied for magnetostatic fields.

4. Maxwell's fourth equation is derived from Faraday's law of electromagnetic induction and the integral form of which is,

$$\oint \bar{E}.d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t} d\bar{s}$$

Where,

E = Electric field intensity, V/m

B = Electric flux density, Wb/m^2

Hence, maxwell's fourth equation is applicable for electromagnetic fields.

For harmonically varying fields, electric flux density ($\bar{D}$) and magnetic flux density ($\bar{B}$) varies with time. They are given by,

$$\bar{D} = \bar{D}_m e^{j\omega t}$$

Differentiating w.r.t time, we get

$$\frac{\partial \bar{D}}{\partial t} = \bar{D}_m\, j\omega\, e^{j\omega t}$$

$$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_m e^{j\omega t}$$

$$\therefore \frac{\partial \bar{D}}{\partial t} = j\omega \bar{D} \qquad \text{... (1)}$$

Similarly,

$$\bar{B} = \bar{B}_m e^{j\omega t}$$

Differentiating w.r.t time, we get

$$\frac{\partial \bar{B}}{\partial t} = \bar{B}_m\, j\omega . e^{j\omega t}$$

$$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_m e^{j\omega t}$$

$$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B} \qquad \text{... (2)}$$

Substituting equations (1) and (2) in the Maxwell's equations, we get,

It is to be noted that the Maxwell's first and second equations remains same for harmonically varying fields. Since, there are no '$\frac{\partial \bar{B}}{\partial t}$' or '$\frac{\partial \bar{D}}{\partial t}$' terms present. Maxwell's first and second equations for harmonically varying fields are,

$$\oint \bar{D}.d\bar{s} = \int_v \rho_v\, dv \text{ and}$$

$$\oint \bar{B}.d\bar{s} = 0$$

From Maxwell's third equation, we have

$$\oint \bar{H}.d\bar{l} = \int_s \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) d\bar{s}$$

But, from point form of Ohm's law, we have

$$\bar{J} = \sigma \bar{E} \text{ and from equation (1),}$$

$$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}$$

$$\therefore \oint \bar{H}.d\bar{l} = \int_s (\sigma\bar{E} + j\omega\bar{D})\, d\bar{s} \quad \text{[From equation (1)]}$$

$$\Rightarrow \oint \bar{H}.d\bar{l} = \int_s (\sigma\bar{E} + j\omega\epsilon\, \bar{E})\, d\bar{s}$$

$$\oint \bar{H}.d\bar{l} = \int_s (\sigma + j\omega\epsilon)\bar{E}\, d\bar{s}$$

$$\oint \bar{H}.d\bar{l} = (\sigma + j\omega\epsilon) \int_s \bar{E}\, d\bar{s}$$

From Maxwell's fourth equation, we have

$$\oint \bar{E}.d\bar{l} = -\int \frac{\partial \bar{B}}{\partial t} d\bar{s}$$

$$\Rightarrow \oint \bar{E}.d\bar{l} = -\int j\omega\bar{B}\, d\bar{s} \quad \text{[From equation (2)]}$$

$$\oint \bar{E}.d\bar{l} = -\int j\omega(\mu\bar{H})\, d\bar{s} \quad [\because \bar{B} = \mu\bar{H}]$$

$$\oint \bar{E}.d\bar{l} = -j\omega\mu \int \bar{H}\, d\bar{s}$$

Hence, Maxwell's equations for time varying fields are summarized as below.

$$\oint \bar{D}.d\bar{s} = \int_v \rho_v\, dv$$

$$\oint \bar{B}.d\bar{s} = 0$$

$$\oint \bar{H}.d\bar{l} = (\sigma + j\omega\epsilon) \int_s \bar{E}\, d\bar{s}$$

$$\oint \bar{E}.d\bar{l} = -j\omega\mu \int_s \bar{H}\, d\bar{s}$$

Q37. Derive the boundary conditions for time varying fields.

Ans: Model Paper-I, Q9(b)

The behaviour of the time varying fields across the surfaces of discontinuity is same as that of static fields. The reason for their similar behaviour is that the tangential components magnetic flux density (B) and the electric flux density (D) remain finite at such surfaces.

For time-varying fields, the characterization of a perfect conductor is analogous to that of a conductor in electrostatics and a super conductor in magnetostatics. The electromagnetic effect in time varying fields arises due to the charges and currents existing solely on its surface.

Boundary Conditions for Time Varying Fields

As the time varying fields are electromagnetic fields, boundary conditions in those fields exists due to the following reasons. They are,

(i) Electric field

(ii) Magnetic field

(I) Boundary Conditions for Electric Field

(a) For Electric Field Intensity E

Consider the boundary surface between two different media i.e., medium 'a' and medium 'b' as shown in figure (a).

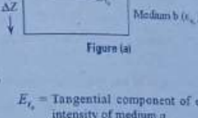

Figure (a)

Let,

E_{t_a} = Tangential component of electric field intensity of medium a

E_{t_b} = Tangential component of electric field intensity of medium b

ϵ_{r_a} = Relative permittivity of medium a

ϵ_{r_b} = Relative permittivity of medium b

$\Delta\omega$ = Length of the rectangular area considered along the surface of the boundary

ΔZ = Breadth of the rectangular area.

According to Maxwell's equations obtained from Faraday's law, the total induced e.m.f is given as,

$$\oint E.dL = -\int_s \frac{\partial B}{\partial t} ds \qquad \text{... (1)}$$

If the breadth ΔZ of the rectangular area approaches zero, then the total area in the closed path also approaches zero. Hence, the term on the right hand side in the equation (1) approaches zero if area approaches zero.

i.e., $\int_s \frac{dB}{dt} ds = 0$, even if magnetic flux density 'B' has a finite value.

Substituting $-\int_s \frac{dB}{dt} ds = 0$ in equation (1), we get,

$$\oint E.dL = 0$$

$$\Rightarrow E_{t_a}\Delta\omega - E_{t_b}\Delta\omega = 0$$

$$\boxed{\therefore E_{t_a} = E_{t_b}}$$

The necessary boundary condition is that the tangential components of electric field intensity in both the media are equal.

(b) For Electric Flux Density D

Consider a cylindrical conductor enclosing the boundary surface between two different media i.e., medium 'a' and medium 'b' as shown in figure (b).

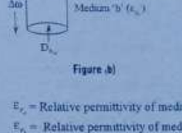

Figure (b)

Let,

ϵ_{r_a} = Relative permittivity of medium a

ϵ_{r_b} = Relative permittivity of medium b

D_{n_a} = Normal component of electric flux density of medium a

D_{n_b} = Normal component of electric flux density of medium b

$\Delta\omega$ = Height of cylindrical surface.

By using Maxwell's equation derived from Gauss law, we have,

$$\int_s D.ds = \int_{Volume} \rho_v\, dV \qquad \text{... (2)}$$

Where,

ρ_v = Volume charge density.

As $\Delta\omega$ approaches zero in figure (b), the volume of the cylinder is reduced and reaches zero finally at $\Delta\omega = 0$.

From equation (2), we have,

$$D_{a_N}\Delta S - D_{b_N}\Delta S = \rho_s \Delta S \quad \text{... (3)}$$

Where,

ρ_s = Surface charge density.

In general, if both media are made up of dielectrics, then there will be no charge on the surface of the boundary. Hence, $\rho_s = 0$. Substituting $\rho_s = 0$ in equation (3), we get,

$$D_{a_N}\Delta S - D_{b_N}\Delta S = 0$$

$$\boxed{D_{a_N} = D_{b_N}}$$

Hence, the necessary boundary condition for electric flux density 'D' is that, the normal component of D must be equal for both the media i.e., the normal component of 'D' must be continuous.

(ii) Boundary Conditions for Magnetic Field

(a) For Magnetic Field Intensity H

Consider the boundary surface between the two different media i.e., medium a and medium b as shown in figure (c).

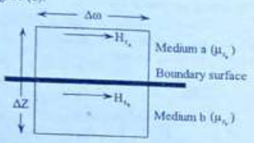

Figure (c)

Let,

H_{t_a} = Tangential component of magnetic field intensity of medium 'a'

H_{t_b} = Tangential component of magnetic field intensity of medium 'b'

μ_{r_a}, μ_{r_b} = Relative permittivities of medium 'a' and medium 'b' respectively

$\Delta\omega, \Delta Z$ = Length and breadth of the rectangular area.

We know that, Maxwell's equation obtained from Ampere's circuital law is given by,

$$\oint H.dL = \int_S \left(J + \frac{\partial D}{\partial t}\right)\partial s \quad \text{... (4)}$$

Where,

J = Current density

$\frac{\partial D}{\partial t}$ = Displacement current density.

If the breadth ΔZ of the rectangular area approaches zero, then the term $\frac{\partial D}{\partial t}$ also approaches zero. And also, the current density 'J' is zero at a dielectric-dielectric interface.

So, substituting $J = 0$ and $\frac{\partial D}{\partial t} = 0$ in equation (4), we get,

$$\oint H.dL = 0$$

$$\Rightarrow \quad H_{t_a}\Delta\omega - H_{t_b}\Delta\omega = 0$$

$$\boxed{H_{t_a} = H_{t_b}}$$

The necessary boundary condition is that the tangential components of magnetic field intensity must be equal.

(b) For Magnetic Flux Density B

Consider a cylindrical surface enclosing the boundary surface between the two media i.e., medium a and medium b as shown in figure (d).

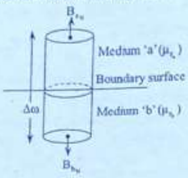

Figure (d)

Let,

B_{a_N}, B_{b_N} = Normal components of magnetic flux density of medium 'a' and medium 'b' respectively.

μ_{r_a}, μ_{r_b} = Relative permittivities of medium 'a' and medium 'b' respectively.

$\Delta\omega$ = Height of the cylindrical conductor.

We know that, the Maxwell's equation obtained from Gauss law for magnetic fields is given by,

$$\int_s B.ds = 0 \quad \text{... (5)}$$

So, as height of the cylindrical surface approaches zero, the total cylindrical surface area becomes zero.

$\therefore$ Equation (5) becomes as,

$$\Rightarrow \quad B_{a_N}\Delta S - B_{b_N}\Delta S = 0$$

$$\Rightarrow \quad B_{a_N} = B_{b_N}$$

Hence, the necessary boundary condition for 'B' is that the normal components of B of both the media must be equal i.e., the normal component of B must be continuous.

The boundary conditions for time-varying fields are,

$$\boxed{\begin{aligned} E_{t_a} &= E_{t_b} \\ H_{t_a} &= H_{t_b} \\ D_{a_N} &= D_{b_N} \\ B_{a_N} &= B_{b_N} \end{aligned}}$$

Q38. Discuss the physical interpretation of Maxwell's equations.

Ans:

Maxwell's equations are a set of four differential equations that describes the properties of electric and magnetic fields. These equations are derived from Ampere's circuital law, Faraday's law and two equations are derived from Gauss's law. These equations are of utmost importance and together with Lorentz force equation, boundary and continuity equations form the complete basic laws for analyzing electromagnetic problems.

The Maxwell's equations in both the point and integral form are listed in the tabular column below.

Name of the Law	Point Form	Integral Form
Gauss's law	$\nabla.D = \rho$	$\int_s \bar{D}.ds = \int_v \rho.dv$
Gauss law for magnetic fields	$\nabla.B = 0$	$\int_s B.ds = 0$
Faraday's law	$\nabla \times E = -\frac{\partial B}{\partial t}$	$\oint_l E.dl = -\int_s \frac{\partial B}{\partial t}.ds$
Ampere's circuital law	$\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint_l H.dl = \int_s \left(J + \frac{\partial D}{\partial t}\right)ds$

Physical Interpretation of Maxwell's Equations

Consider the equation derived from Faraday's law i.e.,

$$\nabla \times E = \frac{-\partial B}{\partial t} \text{ and } \oint_l E.dl = -\int_s \frac{\partial B}{\partial t}.ds$$

But, from Stoke's theorem, we know,

$$\oint E.dl = -\int_s (\nabla \times E).ds$$

Where,

$$\bar{ds} = \hat{n}|\bar{ds}|$$

$\hat{n}$ is a unit vector perpendicular to the direction of ds.

$$\therefore \quad (\nabla \times E).\hat{n} = \frac{\oint_s E.dl}{|ds|}$$

If the direction of $\hat{n}$ is such that $\oint E.dl$ is maximum then magnitude is given by $\frac{\oint E.dl}{ds}$ and the direction is given by $\hat{n}$.

The above discussion relates well for a loop of wire with single turn. If this loop of wire is connected to a voltmeter then, the voltage measured by the voltmeter is,

$$V = \oint E.dl$$

The maximum reading in the voltmeter is achieved by changing the direction of loop. At this instant magnitude of,

$$\nabla \times E = \frac{\text{Voltmeter reading}}{\text{Loop area}}$$

$$= \frac{\oint E.dl}{\text{Loop area}}$$

And the direction is same as the direction of axis of loop.

But from Faraday's law,

$$\nabla \times E = \frac{-\partial B}{\partial t}$$

For non-time varying magnetic flux i.e. $\phi = 0$

$$B = 0 \qquad \left(\because B = \frac{\phi}{A}\right)$$

$$\therefore \quad \frac{\partial B}{\partial t} = 0$$

Hence, $\nabla \times E = \frac{-\partial B}{\partial t} = 0$

Such electric field is said to be irrotational.

From Maxwell's 2nd equation we know,

$$\nabla.B = 0$$

This says that the number of lines of flux entering any closed path is equal to the number of lines of force leaving the closed path or the total magnetic flux through any closed surface is equal to zero.

The reason behind this is basically the magnetic charges exists in pairs (i.e., dipoles) and these charges creates divergence opposite to each other and gets cancelled.

Q39. Write Maxwell's equation in integral and differential form.

Nov./Dec.-12, (R09), Q8(a)

OR

Write Maxwell's equations in differential and integral form for time varying fields and mention the law it represents.

Ans:

Dec.-11, Set-3, Q6

For remaining answer refer Unit-IV, Q35 and Q36.

Q40. Write Maxwell's equations in good conductors for time varying fields and static fields both in differential and integral forms.

Ans:

Model Paper-II, Q9(b)

The conductivity of a good conductors is high, so they carry large conduction current and negligible displacement current. Hence, for a good conductor,

(i) $J >> \frac{\partial D}{\partial t}$

(ii) $\rho_v = 0$

By modifying the original Maxwell's equations using these two conditions, Maxwell's equations for a good conductor is obtained.

Maxwell's equations for a good conductor for time varying fields in both point form and integral form are as given below.

Point Form

(i) $\nabla \times \overline{H} = \overline{J}$

(ii) $\nabla \times \overline{E} = \frac{-\partial \overline{B}}{\partial t}$

(iii) $\nabla.\overline{D} = 0$

(iv) $\nabla.\overline{B} = 0$

Integral Form

(i) $\oint \overline{H}.dl = \oint_s \overline{J}.ds$

(ii) $\oint \overline{E}.dl = -\int_s \frac{\partial \overline{B}}{\partial t} ds$

(iii) $\oint_S D.ds = 0$

(iv) $\oint_S B.ds = 0$

Maxwell's equation for static fields in both point form and integral form are as given below.

Point Form

(i) $\nabla \times \overline{H} = 0$

(ii) $\nabla \times \overline{E} = 0$

(iii) $\nabla.\overline{D} = 0$

(iv) $\nabla.\overline{B} = 0$

Integral Form

(i) $\oint \overline{H}.dl = 0$

(ii) $\oint \overline{E}.dl = 0$

(iii) $\oint_S \overline{D}.ds = 0$

(iv) $\oint_S \overline{B}.ds = 0$

Q41. Starting with Faraday's law, derive Maxwell's equation in integral form based on this law. Also, obtain the corresponding differential relation by applying strokes theorem.

OR

Write and explain Maxwell's fourth equation.

OR

Derive the expression for one of the Maxwell's equation $\nabla \times E = \frac{-\partial B}{\partial t}$.

Ans:

Maxwell's Fourth Equation

Maxwell's fourth equation states that "the curl of electric field intensity is negative of time rate of change of electric flux density".

It is derived from the integral form of Faraday's law, according to which we have

$$e = \oint \bar{E}\, d\bar{L} = -\iint \frac{\partial \bar{B}}{\partial t}\, d\bar{s} \quad ...(1)$$

For derivation of Faraday's law in its integral form refer Unit-IV, Q12, Topic: Derivation.

Stoke's theorem states that the line integral of any vector function over a closed path is equal to the surface integral of the curl of that vector function within that closed path.

$$\text{i.e., } \oint \bar{A}\, d\bar{L} = \iint (\nabla \times \bar{A})\, d\bar{s} \quad ...(2)$$

Substituting equation (2) in equation (1),

$$\Rightarrow \quad \iint (\nabla \times \bar{E})\, d\bar{s} = -\iint \frac{\partial \bar{B}}{\partial t}\, d\bar{s} = \iint \frac{-\partial \bar{B}}{\partial t}\, d\bar{s}$$

As the surface integrals on both sides are taken over identical surfaces, they can be removed and can be written as,

$$\nabla \times \bar{E}\, d\bar{s} = \frac{-\partial \bar{B}}{\partial t}\, d\bar{s}$$

$$\Rightarrow \quad \nabla \times \bar{E} = \frac{-\partial \bar{B}}{\partial t}$$

Hence proved.

Q42. Derive the wave equation starting from the Maxwell's equation for free space.

Nov.-15, (R13), Q10(a)

Ans:

For free space the charge density '$\rho = 0$' and the current density '$J = 0$'. Therefore, the Maxwell's equation for free space are,

$$\nabla \times \bar{E} = \frac{-\partial \bar{B}}{\partial t} \quad ...(1)$$

$$\nabla \times \bar{B} = \mu_0 \varepsilon_0 \frac{\partial \bar{B}}{\partial t} \quad ...(2)$$

$$\left[\because \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c\right] = 0 \quad ...(3)$$

$$\nabla \bar{B} = 0 \quad ...(4)$$

$$\nabla \times (\nabla \times \bar{E}) = -\nabla \times \frac{\partial \bar{B}}{\partial t}$$

$$\Rightarrow \quad \nabla \times (\nabla \times \bar{E}) = \frac{-\partial}{\partial t}(\nabla \times \bar{B}) \quad ...(5)$$

Substituting equation (2) in equation (5), we get,

$$\nabla \times (\nabla \times \bar{E}) = \frac{-\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}\right) \quad ...(6)$$

Since,

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla \bar{E}) - \nabla^2 \bar{E},$$

Equation (6) can be written as,

$$\nabla(\nabla \bar{E})\,] - \nabla^2 \bar{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} \quad ...(7)$$

Substituting $\nabla \bar{E} = 0$ from equation (3) in equation (7), we get,

$$-\nabla^2 \bar{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\therefore \nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} \quad ...(8)$$

A similar derivation gives the same equation for $\bar{B}$

$$\nabla^2 \bar{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2} \quad ...(9)$$

Equations (8) and (9) represent electromagnetic wave equations for free space. This equations can also be written as,

$$\nabla^2 \bar{E} = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} \quad \left[\because \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c\right] \quad ...(10)$$

$$\nabla^2 \bar{B} = \frac{1}{c^2} \frac{\partial^2 \bar{B}}{\partial t^2} \quad ...(11)$$

This equations indicate that electromagnetic waves travel with speed of light in free space.

Q43. If $E = 200\, e^{[illegible]}\, \bar{a}_x$ V/m in free space, use Maxwell's equation to find H, knowing that all fields varying harmonically for which $\nabla \times E = -j\omega\mu H$.

Ans:

Given that,

In free space,

$$E = 200 e^{[illegible]} \bar{a}_x \text{ V/m} \quad ...(1)$$

$$\nabla \times E = -j\omega\mu H \quad ...(2)$$

Magnetic field intensity, $H = ?$

From equation (2) we have,

$$\nabla \times E = -j\omega\mu H$$

$$\Rightarrow \quad H = \frac{-1}{j\omega\mu}(\nabla \times E)$$

Now,

$$\nabla \times E = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right) \times (200e^{(4y-kt)}\bar{a}_x)$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 200e^{(4y-kt)} & 0 & 0 \end{vmatrix}$$

$$= \bar{a}_x(0) + \bar{a}_y\left[\frac{\partial}{\partial z}(200e^{(4y-kt)})\right] + \bar{a}_z\left[-\frac{\partial}{\partial y}(200e^{(4y-kt)})\right]$$

$$= -800e^{(4y-kt)}\bar{a}_z$$

Substituting $(\nabla \times E)$ in equation (3) we get,

$$H = -\frac{1}{j\omega\mu}[-800e^{(4y-kt)}]\bar{a}_z$$

$$\therefore \quad H = \frac{800}{j\omega\mu}e^{(4y-kt)}\bar{a}_z \text{ A/m} \qquad \text{... (4)}$$

From equation (4), it is evident that,

$$\nabla . H = \frac{\partial H_z}{\partial z} = 0$$

Thus, Gauss's law for magnetic fields is also satisfied. Finally, from the Ampere's law.

$$\nabla \times H = j\omega \in E$$

$$\Rightarrow \quad E = \frac{1}{j\omega\in}(\nabla \times H) \qquad \text{... (5)}$$

But,

$$\nabla \times H = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right) \times \left(\frac{800}{j\omega\mu}e^{(4y-kt)}\bar{a}_z\right)$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{800}{j\omega\mu}e^{(4y-kt)} \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial y}\left(\frac{800}{j\omega\mu}e^{(4y-kt)}\right)\right] - \bar{a}_y\left[\frac{\partial}{\partial x}\left(\frac{800}{j\omega\mu}e^{(4y-kt)}\right)\right] + \bar{a}_z(0)$$

$$= 4\left[\frac{800}{j\omega\mu}e^{(4y-kt)}\right]\bar{a}_x - 0$$

$$= \frac{3200}{j\omega\mu}e^{(4y-kt)}\bar{a}_x$$

Substituting $(\nabla \times H)$ value in equation (5), we get,

$$E = \frac{1}{j\omega\in}(\nabla \times H)$$

$$E = \frac{1}{j\omega\in}\left[\frac{3200}{j\omega\mu}e^{(4y-kt)}\right]\bar{a}_x \quad [\because j \times j = -1]$$

$$\therefore \quad E = -\frac{3200}{\omega^2\mu\in}e^{(4y-kt)}\bar{a}_x \text{ A/m}$$

$$E = -\frac{3200}{\omega^2\mu_0\in_0}e^{(4y-kt)}\bar{a}_x \text{ A/m} \qquad [\because \mu = \mu_0, \in = \in_0] \qquad \text{... (6)}$$

Comparing the equations (1) and (6), we have,

$$\frac{3200}{\omega^2\mu_0\in_0} = 200$$

$$\Rightarrow \quad \omega^2\mu_0\in_0 = 16$$

$$\omega^2 = \frac{16}{\mu_0\in_0}$$

$$\therefore \quad \omega = \pm\frac{4}{\sqrt{\mu_0\in_0}}$$

Then, equation (4) becomes as,

$$H = \pm\frac{800}{j \times \frac{4}{\sqrt{\mu_0\in_0}}\mu_0}e^{(4y-kt)}\bar{a}_z$$

$$= \pm\frac{200}{j\sqrt{\frac{\mu_0}{\in_0}}}e^{(4y-kt)}\bar{a}_z$$

Intrinsic impedance for free space,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\,\pi$$

$$= \pm \frac{200}{j120\pi} e^{(4y-kt)} \overline{a}_z$$

$$= \pm \frac{5}{3\pi j} e^{(4y-kt)} \overline{a}_z \text{ A/m}$$

$$\boxed{\therefore H = \pm \frac{5}{3\pi j} e^{(4y-kt)} \overline{a}_z \text{ A/m}}$$

UNIT 5

Electromagnetic Waves

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Define Electromagnetic waves (or) EM waves.

Ans: Model Paper-III, Q1(i)

Electromagnetic (EM) waves can be defined as the waves produced by the variations in electric and magnetic fields aligned perpendicular to each.

Q2. Write the wave equations for conducting and perfect dielectric media.

Ans: Model Paper-I, Q1(i)

Wave Equations for Conducting Media

The wave equations for conducting medium are,

1. $\nabla^2 \overline{E} = \mu\varepsilon \dfrac{\partial^2 \overline{E}}{\partial t^2} + \mu\sigma \dfrac{\partial \overline{E}}{\partial t}$

2. $\nabla^2 \overline{H} = \mu\varepsilon \dfrac{\partial^2 \overline{H}}{\partial t^2} + \mu\sigma \dfrac{\partial \overline{H}}{\partial t}$

Wave Equations for Perfect Dielectric Media

The wave equations for perfect dielectric medium are,

1. $\nabla^2 \overline{E} = \mu\varepsilon \dfrac{\partial^2 \overline{E}}{\partial t^2}$

2. $\nabla^2 \overline{H} = \mu\varepsilon \dfrac{\partial^2 \overline{H}}{\partial t^2}$

Q3. Write the wave equations in phasor form.

Ans: Model Paper-II, Q1(i)

The phasor notation for wave equations in free space is given as,

$$\nabla^2 H = -\omega^2 \mu_0 \epsilon_0 H$$

$$\nabla^2 E = -\omega^2 \mu_0 \epsilon_0 E$$

The phasor notation for wave equations in conducting medium is given as,

$$\nabla^2 E = (-\omega^2 \mu\epsilon + j\,\omega\mu\sigma)\, E$$

$$\nabla^2 E = (-\omega^2 \mu\varepsilon + j\omega\mu\sigma)\, H.$$

Q4. Define the term "uniform plane waves".

Ans:

The electromagnetic (EM) waves for which the electric and magnetic field vectors lie in the same plane such that their values are same (uniform) throughout the plane are termed as uniform plane waves.

Q5. Define intrinsic impedance.

Model Paper-II, Q1(i)

Ans:

Intrinsic impedance or characteristic impedance is defined as the ratio of amplitudes of electric and magnetic fields (E and H) in a uniform plane wave. This intem equal to the square root of ratio of permeability and dielectric constant of the medium.

i.e., $\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$

This, equation gives the relationship between E and H.

Q6. Give an expression for intrinsic impedance in phasor form. What are its magnitude and phase components?

Ans:

Basically, the intrinsic impedance is a complex entity and it can be expressed in phasor form as,

$$\eta = |\eta| e^{j\theta_\eta}$$

Where,

Magnitude component, $\eta = \frac{\sqrt{\alpha/\epsilon}}{[1+(\sigma/\omega\epsilon)^2]^{\frac{1}{4}}}$

Phase component, $\theta_\eta = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right)$ $(0 \le \theta \le 45°)$.

Q7. What are good conductors and good dielectrics?

Model Paper-I, Q1(j)

Ans:

Good Conductors

The materials which have conductivity greater than 1 i.e., $\sigma >> \omega\varepsilon$ are known as good conductors.

Good Dielectrics

The materials which have conductivity lesser than 1 i.e., $\sigma << \omega\varepsilon$ are known as good dielectrics.

Q8. Define poynting vector and poynting theorem.

Ans:

Poynting Vector

The cross product of electric and magnetic field vectors '$\overline{E}$' and '$\overline{H}$' gives the poynting vector. It is denoted by $\overline{P}$ and is given as,

$$\overline{P} = \overline{E} \times \overline{H}$$

Poynting Theorem

According to poynting theorem, the total power 'W' flowing out of a closed surface is equal to the closed surface integral of poynting vector '$\overline{P}$'.

i.e., $W = \oint \overline{P}.d\overline{S}$

Q9. Give the applications of poynting vector and poynting theorem.

Model Paper-III, Q1(j)

Ans:

Few applications of poynting vector and poynting theorem are,

1. Poynting theorem is useful in determining the flow of energy in a uniform plane wave travelling in free space
2. Using poynting theorem, power flow in standard conductors like coaxial cable can be obtained
3. Poynting vector is very helpful in wave propagation, because it gives both the magnitude and direction of power flow
4. The total power dissipated in a region can be determined using poynting vector.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

5.1 DERIVATION OF WAVE EQUATION

Q10. Derive the wave equations for conducting medium, perfect dielectric medium and free space.

Model Paper-II, Q10

Ans:

Wave Equations for Conducting Medium

In conductors, the net flow of charge is zero. Thus, for a uniform conducting medium charge density is zero.

i.e., $\nabla.\overline{D} = \rho_v = 0$

Where, ρ_v - Charge density

$\Rightarrow \nabla.\varepsilon\overline{E} = 0$

$\Rightarrow \nabla.\overline{E} = 0$ ... (1)

From Maxwell's third equation for time varying fields,

$$\nabla \times \overline{E} = \frac{-\partial\overline{B}}{\partial t} \quad \text{... (2)}$$

Applying curl on both sides of equation (2), we get,

$$\nabla \times (\nabla \times E) = \frac{-\partial(\nabla \times \overline{B})}{\partial t}$$

$$\nabla \times (\nabla \times E) = -\mu\frac{\partial(\nabla \times \overline{H})}{\partial t} \quad (\because \overline{B} = \mu\overline{H}) \quad \text{... (3)}$$

From Maxwell's fourth equation for time varying fields,

$$\nabla \times \overline{H} = \frac{\partial\overline{D}}{\partial t} + \overline{J} \quad \text{... (4)}$$

On substituting equation (4) in equation (3), we get,

$$\nabla \times (\nabla \times E) = -\mu\frac{\partial}{\partial t}\left(\frac{\partial\overline{D}}{\partial t} + \overline{J}\right)$$

$$\Rightarrow \nabla^2 E - \nabla(\nabla.E) = -\mu\frac{\partial}{\partial t}\left(\frac{\partial(\varepsilon\overline{E})}{\partial t} + \sigma\overline{E}\right) \quad \left[\begin{array}{l}\because \nabla\times(\nabla\times\overline{A}) = \nabla(\nabla.\overline{A}) - \nabla^2\overline{A},\\ D = \varepsilon E \text{ and } J \doteq \varepsilon E\end{array}\right]$$

$$\Rightarrow \nabla^2\overline{E} - \nabla(\nabla.\overline{E}) = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} + \mu\sigma\frac{\partial\overline{E}}{\partial t}$$

$$\Rightarrow \nabla^2\overline{E} - \nabla(0) = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} + \mu\sigma\frac{\partial\overline{E}}{\partial t} \quad [\because \text{From equation (1)}]$$

$$\boxed{\therefore \nabla^2\overline{E} = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} + \mu\sigma\frac{\partial\overline{E}}{\partial t}} \quad \text{... (5)}$$

Applying curl on both sides of equation (4),

$$\nabla \times \nabla \times H = \frac{\partial(\nabla \times \overline{D})}{\partial t} + \nabla \times \overline{J}$$

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$$\Rightarrow \quad \nabla\times\nabla\times H = \frac{\partial(\nabla\times\varepsilon\,\overline{E})}{\partial t} + (\nabla\times\sigma\,\overline{E})$$

$$= \frac{\partial(\nabla\times\overline{E})}{\partial t} + \sigma(\nabla\times\overline{E}) \qquad [\because D=\varepsilon\overline{E}, J=\sigma\overline{E}]$$

$$= -\varepsilon\frac{\partial^2\overline{B}}{\partial t^2} - \sigma\frac{\partial\overline{B}}{\partial t} \qquad [\because \text{From equation (2)}]$$

$$\Rightarrow \nabla\times(\nabla\times H) = -\mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2} - \mu\varepsilon\frac{\partial\overline{H}}{\partial t}$$

$$\Rightarrow \nabla^2 H - \nabla(\nabla.H) = \mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2} - \mu\sigma\frac{\partial\overline{H}}{\partial t} \qquad [\because \nabla\times(\nabla\times\overline{A}) = \nabla(\nabla.\overline{A}) - \nabla^2\overline{A}] \qquad \text{... (6)}$$

From Maxwell's second equation,

$$\nabla.\overline{B} = 0$$

$$\Rightarrow \quad \nabla.\mu\overline{H} = 0$$

$$\Rightarrow \quad \nabla.\overline{H} = 0 \qquad \text{... (7)}$$

On substituting equation (7) in equation (6), we get,

$$\nabla^2 H - \nabla(0) = \mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2} - \mu\sigma\frac{\partial\overline{H}}{\partial t}$$

$$\boxed{\therefore \nabla^2 H = \mu\varepsilon\frac{\partial^2 H}{\partial t^2} + \mu\sigma\frac{\partial\overline{H}}{\partial t}} \qquad \text{... (8)}$$

Wave Equations for Perfect Dielectric Medium

In perfect dielectrics, the conductivity of material is zero, i.e., $\sigma = 0$. From the wave equations of conducting medium,

$$\nabla^2\overline{E} = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} + \mu\sigma\frac{\partial\overline{H}}{\partial t}$$

$$\nabla^2\overline{H} = \mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2} + \mu\sigma\frac{\partial\overline{H}}{\partial t}$$

On substituting $\sigma = 0$ in above equations, we get,

$$\nabla^2\overline{E} = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} \qquad \text{... (9)}$$

$$\nabla^2\overline{H} = \mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2} \qquad \text{... (10)}$$

Equations (9) and (10) represent the wave equations of a perfect dielectric medium.

Wave Equations for Free Space

In free space, sources such as charges or currents does not exist. Thus, it acts as a perfect dielectric with zero conductivity and charge density. The wave equations for free space can be obtained from the wave equations of perfect dielectric media as,

$$\nabla^2\overline{H} = \mu\varepsilon\frac{\partial^2\overline{H}}{\partial t^2}$$

$$\nabla^2\overline{E} = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}$$

Replacing μ with μ_0 and ε and ε_0 in above equations

$$\Rightarrow \quad \nabla^2\overline{H} = \mu_0\varepsilon_0\frac{\partial^2\overline{H}}{\partial t^2} \qquad \text{... (11)}$$

$$\nabla^2\overline{E} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}}{\partial t^2} \qquad \text{... (12)}$$

Where,

μ_0 - Permeability of free space

ε_0 - Permittivity of free space.

Equations (11) and (12) represent the wave equations for free space.

5.2 MAXWELLS EQUATION IN PHASOR FORM, WAVE EQUATIONS IN PHASOR FORM

Q11. How to convert 4-Maxwell's equations into phasor form? Explain.

Ans: Model Paper-I, Q10

The Maxwell's equations (in differential form) are,

1. Maxwell's first law states that the total electric flux passing through any closed surface is equal to the total volume charge enclosed by that surface.

In phasor form,

$$\nabla.D = \rho_v$$

$$\nabla.\tilde{D} = \rho_v \text{ and } D = Re(De^{j\omega t})$$

$$\Rightarrow \quad \nabla.\text{Re}(De^{j\omega t}) = \rho_v$$

$$\Rightarrow \quad \boxed{\nabla.D = \rho_v}$$

2. Maxwell's second equation states that the total magnetic flux coming out of any closed surface is zero.

$$\nabla.B = 0$$

$$\nabla.\tilde{B} = 0$$

$$\Rightarrow \quad \nabla.\text{Re}(Be^{j\omega t}) = 0$$

$$\boxed{\therefore \nabla.B = 0}$$

3. Maxwell's third equation states that "the curl of electric field intensity is negative of time rate of change of magnetic flux density".

$$\nabla\times E = \frac{-\partial B}{\partial t}$$

In phasor approch, $\nabla\times\tilde{E} = \frac{-\delta\tilde{B}}{\delta t}$

In phasor form, $\tilde{E} = Re[Ee^{j\omega t}]$ and $\tilde{B} = Re[Be^{j\omega t}]$

$\therefore \nabla\times E = \frac{-\partial B}{\partial t}$ is written as,

$$\nabla\times\text{Re}\{Ee^{j\omega t}\} = \frac{-\partial}{\partial t}\text{Re}(Be^{j\omega t})$$

$$\text{Re}[(\nabla\times E + j^{\omega B})e^{j\omega t}] = 0$$

$$\boxed{\Rightarrow \nabla\times E = -j\omega B}$$

4. Maxwell's fourth equation states that, the closed integral of the Magneto Motive Force (M.M.F) is equal to the surface integral of current density taken over the surface enclosed by the closed path.

$$\nabla\times H = \frac{\partial D}{\partial t} + J$$

In phasor form,

$$\tilde{H} = \text{Re}(H.e^{j\omega t}),\ \tilde{D} = \text{Re}(De^{j\omega t}),\ \tilde{J} = \text{Re}(Je^{j\omega t})$$

$$\therefore \quad \nabla\times H = \frac{\partial D}{\partial t} + J$$

$$\Rightarrow \quad \nabla\times\tilde{H} = \frac{\partial\tilde{D}}{\partial t} + \tilde{J}$$

$$\Rightarrow \quad \nabla\times\text{Re}\{He^{j\omega t}\} = \frac{\partial}{\partial t}\text{Re}(De^{j\omega t}) + \text{Re}(Je^{j\omega t})$$

$$\Rightarrow \quad \text{Re}[(\nabla\times H - j\omega D - J)e^{j\omega t}] = 0$$

$$\boxed{\therefore \nabla\times H = j\omega D + J}$$

$\therefore$ Maxwell equations in phasor form are,

$$\nabla\times H = j\omega D + J$$

$$\nabla\times E = -j\omega B$$

$$\nabla.D = \rho_v \text{ and } \nabla.B = 0$$

5.3 UNIFORM PLANE WAVES - PLANE WAVE IN FREE SPACE AND IN A HOMOGENEOUS MATERIAL, WAVE EQUATION FOR A CONDUCTING MEDIUM, PLANE WAVES IN LOSSY DIELECTRICS, PROPAGATION IN GOOD CONDUCTORS

Q12. Determine the general solution of uniform plane wave equation.

Ans:

Consider, the wave equation in free space as,

$$\nabla^2\overline{E} = \mu_0\varepsilon_0 = \frac{\partial^2\overline{E}}{\partial t^2} \qquad \text{... (1)}$$

Since, for a uniform plane, wave travels only in z direction i.e., for $E_z = 0$.

$$\Rightarrow \quad \nabla^2 E = \frac{\partial^2\overline{E}}{\partial z^2} = \mu_0\varepsilon_0\overline{E}\frac{\partial^2\overline{E}}{\partial t^2} \qquad \text{... (2)}$$

For $E_z = 0$, equation (2) with their respective components can be written as,

$$\frac{\partial^2\overline{E}_x}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}_x}{\partial t^2}$$

And, $$\frac{\partial^2\overline{E}_y}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\overline{E}_y}{\partial t^2}$$

The general solution for equation (2) can be written as,

$$E = f_1(z - vt) + f_2(z + vt) \quad \text{... (3)}$$

Where,

f_1 - Function of $z - vt$

f_2 - Function of $z + vt$

For free space condition equation (3) becomes,

$$E = f(z - v_0 t)$$

Where,

$v_0 = c =$ Velocity of light.

$$\boxed{\therefore E = f(z - ct)} \quad \text{... (4)}$$

Equation (4) gives the general solution of uniform plane wave.

Q13. Derive the equation for uniform plane in free space condition.

Ans:

Uniform Plane Waves in Free Space

Consider, the wave equations for $\overline{E}$ and $\overline{H}$ fields as,

$$\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} + \mu\sigma \frac{\partial \overline{E}}{\partial t} \quad \text{... (1)}$$

$$\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{H}}{\partial t^2} + \mu\sigma \frac{\partial \overline{H}}{\partial t} \quad \text{... (2)}$$

For free space condition $\sigma = 0$, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$. Thus, equations (1) and (2) reduces to the form,

$$\nabla^2 \overline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2} \quad \text{... (3)}$$

$$\nabla^2 \overline{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{H}}{\partial t^2} \quad \text{... (4)}$$

Equation (3) can be written as,

$$\nabla^2 \overline{E} = \frac{\partial^2 \overline{E}}{\partial x^2} + \frac{\partial^2 \overline{E}}{\partial y^2} + \frac{\partial^2 \overline{E}}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2} \quad \text{... (5)}$$

$$\left[\because \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right]$$

As $\overline{E}$ is independent of x and y and the wave travels in z-direction, equation (5) gets reduced to the form,

$$\frac{\partial^2 \overline{E}}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 \overline{E}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 \overline{E}}{\partial z^2} \quad \text{... (6)}$$

$$\Rightarrow \frac{\partial^2 \overline{E}}{\partial t^2} = v^2 \frac{\partial^2 \overline{E}}{\partial z^2} \quad \left[\because v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right]$$

Where,

v - Velocity.

Similarly equation (4) gets reduced to the form,

$$\frac{\partial^2 \overline{H}}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 \overline{H}}{\partial z^2} \quad \text{... (7)}$$

$$\frac{\partial^2 \overline{H}}{\partial t^2} = v^2 \frac{\partial^2 \overline{H}}{\partial z^2}$$

Equations (6) and (7) represent the wave equations for uniform plane waves in free space as a function of z and t.

Q14. What is meant by wave propagation? Derive the expressions for α, β and γ in a lossy dielectric medium.

Ans: Model Paper-III, Q10

Wave Propagation

Wave propagation basically refers to the ways in which a wave can travel. The propagation of wave can be generally illustrated for a lossy dielectric medium wherein the progress of wave in a direction leads to loss of power. To understand wave propagation in any kind of medium, few parameters are needed which can be determined using Maxwell's equations in combination with the relations $J = \sigma \overline{E}$, $D = \varepsilon \overline{E}$ as,

$$\nabla \times \overline{H} = \sigma \overline{E} + j\omega\varepsilon \overline{E} \quad \text{... (1)}$$

And, $\nabla \times \overline{E} = -j\mu\omega \overline{H}$ ... (2)

On differentiating equation (1) with respect to time, we get,

$$\nabla \times \left(\frac{\partial \overline{H}}{\partial t}\right) = j\omega(\sigma + j\omega\varepsilon)\overline{E} \quad \text{... (3)}$$

From Maxwell's equations,

$$\nabla \times E = -\mu\left(\frac{\partial \overline{H}}{\partial t}\right)$$

On taking curl of the above equation, we get,

$$\nabla \times \nabla \times E = -\mu \nabla \times \left(\frac{\partial \overline{H}}{\partial t}\right)$$

$$= -j\mu\omega(\sigma + j\omega\varepsilon)\overline{E}$$

[$\because$ From equation (3)]

$$\Rightarrow -\nabla^2 \overline{E} + \nabla(\nabla.\overline{E}) = -j\mu\omega(\sigma + j\omega\varepsilon)\overline{E}$$

$$[\because \nabla \times (\nabla \times \overline{A}) = -\nabla^2 \overline{A} + \nabla(\nabla.\overline{A})]$$

$$\Rightarrow -\nabla^2 E = -j\mu\omega(\sigma + j\omega\varepsilon)\overline{E} \quad [\because \nabla.\overline{E} = 0]$$

$$\Rightarrow \nabla^2 E = j\mu\omega(\sigma + j\omega\varepsilon)\overline{E}$$

$$\Rightarrow \nabla^2 E = \gamma^2 \overline{E} \quad \text{... (4)}$$

Where,

γ - Propagation constant.

From equation (4),

$$\gamma^2 = j\mu\omega(\sigma + j\omega) \quad \text{... (5)}$$

Similarly,

$$\nabla^2 H = \gamma^2 \overline{H} \quad \text{... (6)}$$

The propagation constant is also written as,

$$\gamma = \alpha + j\beta \quad \text{... (7)}$$

Where,

α - Attenuation constant

β - Phase constant.

Since, from equation (5),

$$\gamma^2 = j\mu\omega(\sigma + j\omega\varepsilon)$$

$$\Rightarrow \gamma^2 = j\mu\omega\sigma - \omega^2\mu\varepsilon \quad \text{... (8)}$$

$$\Rightarrow (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + j^2\alpha\beta \quad \text{... (9)}$$

On equating the real and imaginary parts of equations (8) and (9), we get,

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad \text{... (10)}$$

And, $2\alpha\beta = \omega\mu\sigma$ ... (11)

The relation between $(\alpha^2 - \beta^2)$ and $2\alpha\beta$ can be given as,

$$\alpha^2 + \beta^2 = \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}$$

$$[\because a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2}]$$

$$= \sqrt{(\omega^2\mu\varepsilon)^2 + (\omega\mu\sigma)^2} \quad \text{... (12)}$$

$$\alpha^2 + \beta^2 = \omega^2\mu\varepsilon\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2}$$

On adding equations (10) and (12), we get,

$$2\alpha^2 = \omega^2\mu\varepsilon\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)$$

$$\boxed{\therefore \alpha = \omega\left(\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}\right) - 1}\right)} \quad \text{... (13)}$$

On subtracting equation (10) from equation (13), we get,

$$2\beta^2 = \omega^2\mu\varepsilon\left(\sqrt{1 + \left(1 + \frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)$$

$$\boxed{\therefore \beta = \omega\left(\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right) - 1}\right)} \quad \text{... (14)}$$

If a uniform plane wave travels in z-direction, then equation (4) can be written as,

$$\frac{\partial^2 \overline{E}}{\partial z^2} = \gamma^2 \overline{E} \quad \text{... (15)}$$

$$\Rightarrow \overline{E}(z) = \overline{E}_0 e^{-\gamma z}$$

In case of time-varying fields,

$$\overline{E}(z, t) = Re\,(\overline{E}_0 e^{-\gamma z + j\omega t})$$

$$= Re\,(\overline{E}_0 e^{-(\alpha z) - j\beta z + j\omega t})$$

$$\therefore \overline{E}(z, t) = e^{-\alpha z}\, Re\,(\overline{E}_0 e^{j(\omega t - \beta z)})$$

Q15. Describe wave propagation in lossless media.

Ans:

In lossless medium, EM wave propagates with no loss of energy. The wave equation in lossless medium is,

$$\nabla^2 \overline{E} = -\omega^2\mu\varepsilon\overline{E}$$

The attenuation constant and phase constant for a general medium are,

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \quad \text{... (1)}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} \quad \text{... (2)}$$

In case of lossless medium conductivity, $\sigma = 0$.

On substituting $\sigma = 0$ in equations (1) and (2) the wave parameters of lossless media are obtained as,

$$\boxed{\begin{array}{l}\alpha = 0 \\ \text{and } \beta = \omega\sqrt{\mu\varepsilon}\end{array}}$$

Q16. Describe the propagation of wave in conducting medium.

Ans:

The wave equation for electric field vector in conducting medium is given as,

$$\nabla^2 \overline{E} = \mu\sigma\frac{\partial \overline{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \overline{E}}{\partial t^2} \quad \text{... (1)}$$

Since, $\frac{\partial \overline{E}}{\partial t} = j\omega\overline{E}$ and $\frac{\partial^2 \overline{E}}{\partial t^2} = -\omega^2\overline{E}$

$$\Rightarrow \nabla^2 \overline{E} = j\omega\mu\sigma\overline{E} - \omega^2\mu\varepsilon\overline{E}$$

$$= j\omega\mu(\sigma + j\omega\varepsilon)\overline{E}$$

$$\nabla^2 \overline{E} = \gamma^2 \overline{E}$$

Where,

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

For remaining answer refer Unit-V, Q14, from equation (8).

Q17. Describe wave propagation in good conductors and derive α, β, γ, δ and η for the same.

Ans:

In good conductors, the conductivity of material is very large i.e., $\sigma >> \omega\varepsilon$.

$$\Rightarrow \quad \frac{\sigma}{\omega\varepsilon} >> 1 \text{ or } \frac{\omega\varepsilon}{\sigma} << 1$$

Since,

$$\gamma^2 = j\mu\omega(\sigma + j\omega\varepsilon)$$

$$\Rightarrow \quad \gamma = \sqrt{j\mu\omega\sigma\left(1+\frac{j\omega\varepsilon}{\sigma}\right)}$$

$$= \sqrt{j\mu\omega\sigma} \quad \left[\because \frac{\omega\varepsilon}{\sigma} << 1\right]$$

$$\boxed{\therefore \gamma = \sqrt{\mu\omega\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)}$$

Since,

$$\gamma = \alpha + j\beta$$

$$\Rightarrow \quad \boxed{\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}}}$$

As the value of α is large, the attenuation of EM wave in conductors is very high. As a result the wave travels for a very short distance and vanishes. This condition is referred to as skin depth or depth of penetration (δ).

Let E_0 be the initial value of field and $E_0 e^{-\alpha z}$ be the field value at a distance z inside the surface. Then for a distance δ, the field becomes $E_0 e^{-1}$.

On equating the two terms, we get,

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

$$\Rightarrow \quad \alpha\delta = 1$$

$$\Rightarrow \quad \delta = \frac{1}{\alpha}$$

$$\boxed{\therefore \delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

The velocity of wave is given by the expression

$$v = \frac{\omega}{\beta}$$

$$\therefore v = \sqrt{\frac{2\omega}{\mu\sigma}}$$

The characteristic impedance of the wave is given by the expression.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$\Rightarrow \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma\left(1+\frac{j\omega\varepsilon}{\sigma}\right)}}$$

$$\Rightarrow \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma}} \quad \left[\because \frac{\omega\varepsilon}{\sigma} << 1\right]$$

$$\boxed{\therefore \eta = \sqrt{\frac{j\omega\mu}{\sigma}}}$$

Q18. The electric field in free space is given by E = 50 cos (10^8 t + βx) a_y V/m. Find the direction of wave propagation. Calculate β and the time it takes to travel a distance of $\lambda/2$.

Ans:

Given that,

For an EM wave in free space,

Electric field, $E = 50\cos(10^8 t + \beta x)a_y$ V/m ... (1)

(i) Direction of wave propagation = ?

(ii) Phase constant, β = ?

(iii) Time taken to travel a distance of $\lambda/2$, t_1 = ?

(i) Direction of Wave Propagation

Equation (1) is of the form,

$$E = A\cos(\omega t + \beta x) \quad \text{... (2)}$$

On comparing equations (1) and (2), it can be observed that, the term $(\omega t + \beta x)$ i.e., $(10^8 t + \beta x)$ has a positive sign. This indicates that the wave propagates along the negative direction of a_x i.e, $-a_x$.

(ii) Phase Constant, β

The expression for phase constant, β is,

$$\beta = \frac{\omega}{u}$$

Where, u - Wave velocity.

$$\Rightarrow \quad \beta = \frac{\omega}{u} \quad [\because u = c \text{ in free space}]$$

$$= \frac{10^8}{3\times10^8} = \frac{1}{3}$$

$$\boxed{\therefore \beta = 0.333 \text{ rad/m}}$$

(iii) Time Taken to Travel $\lambda/2$ Distance

Let, the time taken by a wave to travel a distance λ be T where T is the time period of wave then, the time taken to travel a distance of $\lambda/2$ can be obtained as t_1.

i.e., t_1 2 2(ω)

$$= \frac{\pi}{\omega} = \frac{\pi}{10^8} \quad [\because \text{From equation (1)}]$$

$$= 3.144 \times 10^{-8}$$

$$= 31.44 \times 10^{-9} \text{s}$$

$$\boxed{\therefore t_1 = 31.44 \text{ ns}}$$

Q19. A medium is characterised by $\sigma = 0$ and $\mu = 2\mu_0$ and $\varepsilon = 5\varepsilon_0$. If H = 2cos $\hat{z}$ (ωt – 3y) A/m, calculate ω and E.

Model Paper-I, Q11

Ans:

Given that,

For an EM wave,

Magnetic field intensity, $H = 2\cos\hat{z}(\omega t - 3y)$ A/m ... (1)

Permeability, $\mu = 2\mu_0$

Permitivity, $\varepsilon = 5\varepsilon_0$

Conductivity, $\sigma = 0$

Angular frequency, ω = ?

Electric field, E = ?

Angular frequency ω is given as,

$$\omega = \frac{\beta}{\sqrt{\mu\varepsilon}} \quad \left[\begin{array}{l}\because \text{In free space,} \\ \text{Permittivity, } \varepsilon_0 = 8.85\times10^{-12} \text{ F/m} \\ \text{Permeability, } \mu_0 = 4\pi\times10^{-7} \text{ H/m}\end{array}\right]$$

$$\Rightarrow \quad \omega = \frac{\beta}{\sqrt{(2\mu_0)(5\varepsilon_0)}}$$

$$\Rightarrow \quad \omega = \frac{\beta}{\sqrt{10\mu_0\varepsilon_0}} \quad \text{... (2)}$$

From, the expression of magnetic field of EM wave,

$$H = H_0\cos(\omega t - \beta z)a_z \quad \text{... (3)}$$

By comparing equations (1) and (3), we get,

$$\beta = 3$$

On substituting the values of β, μ_0 and ε_0 in equation (2), we get,

$$\omega = \frac{3}{\sqrt{(2\times4\pi\times10^{-7})(5\times8.85\times10^{-12})}}$$

$$= 284475380.7$$

$$\boxed{\therefore \omega = 2.845\times10^8 \text{ rad/sec}}$$

The expression for electric field is,

$$E = \frac{D}{\epsilon} = \frac{D}{5\epsilon_0} \quad \text{... (4)}$$

Where,

$$D = \int \nabla\times H \text{ for free space}$$

Consider another vector identity given as,

$$\frac{\partial(\overline{A}.\overline{B})}{\partial t} = \overline{A}.\frac{\partial \overline{B}}{\partial t} + \overline{B}.\frac{\partial \overline{A}}{\partial t}$$

If $\overline{B} = \overline{A}$,

$$\Rightarrow \quad \frac{\partial \overline{A}^2}{\partial t^2} = 2\,\overline{A}.\frac{\partial \overline{A}}{\partial t}$$

$$\therefore \quad \overline{A}.\frac{\partial \overline{A}}{\partial t} = \frac{1}{2}\frac{\partial \overline{A}^2}{\partial t^2} \quad \text{--- (6)}$$

Applying the above identity for $\overline{E}$ and $\overline{H}$ fields, we get,

$$\overline{E}.\frac{\partial \overline{E}}{\partial t} = \frac{1}{2}\frac{\partial \overline{E}^2}{\partial t} \quad \text{--- (7)}$$

And, $\overline{H}.\frac{\partial \overline{H}}{\partial t} = \frac{1}{2}\frac{\partial \overline{H}^2}{\partial t}$ --- (8)

On substituting equations (7) and (8) in equation (5), we get,

$$\nabla(\overline{E}\times\overline{H}) = \frac{-\mu\partial \overline{H}^2}{2\partial t} - \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t} - \overline{E}.\overline{J} \quad \text{--- (9)}$$

Equation (9) gives the relationship between E and H for a single point. In order to obtain such relationship for all points in space, consider its volume integral as,

$$\int_v \nabla(\overline{E}\times\overline{H})dv = \int_v \left(\frac{-\mu}{2}\frac{\partial \overline{H}^2}{\partial t} - \frac{\varepsilon}{2}\frac{\partial \overline{E}^2}{\partial t} - \overline{E}.\overline{J}\right)dv \quad \text{--- (10)}$$

From Divergence theorem,

$$\oint_s (\overline{E}\times\overline{H}).d\overline{S} = \int_v \nabla.(\overline{E}\times\overline{H})\,dv \quad \text{--- (11)}$$

On substituting equation (11) in (10) and interchanging integral and $\frac{\partial}{\partial t}$, we get,

$$\oint_s (\overline{E}\times\overline{H}).d\overline{S} = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon \overline{E}^2 + \frac{1}{2}\mu \overline{H}^2\right]dv - \int_v \overline{E}.Jdv \quad \text{--- (12)}$$

Here,

$\oint_s (\overline{E}\times\overline{H}).d\overline{S}$ - Total power leaving the volume

$\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon \overline{E}^2 + \frac{1}{2}\mu \overline{H}^2\right]dv$ - Rate of decrease in stored energy in electric and magnetic fields

$\int_v \overline{E}.Jdv$ - Ohmic power dissipated.

From law of conservation of energy, the rate of decrease in electromagnetic energy is equal to the total output power coming out of the volume. Thus, the total outward power is given as,

$$W = \oint_s (\overline{E}\times\overline{H}).d\overline{S}$$

$$\boxed{\therefore W = \oint_s \overline{P}.d\overline{S}} \quad \text{... (13)}$$

Where,

$\overline{P} = \overline{E}\times\overline{H}$ (Poynting vector).

Hence proved.