Time Series Assignments 3

Q3) Obtain MA(1), MA(2) and MA(q) process.

Sol)

(i) MA(1) : 1^{st} Order Moving Average

$$y_t = \mu + u_t - \theta_1 u_{t-1}$$

$$E(y_t) = \mu$$

$$Var(y_t) = (1+\theta_1^2)\sigma^2$$

$$ACVF = \gamma(t) = Cov(y_t, y_{t-n})$$

- for $n=1$

$$\gamma(1) = Cov(y_t, y_{t-1})$$

$$= E\left[(y_t - \mu)(y_{t-1} - \mu)\right]$$

$$= E\left[(\cancel{\mu} + u_t - \theta_1 u_{t-1} - \cancel{\mu})(\cancel{\mu} + u_{t-1} - \theta_1 u_{t-2} - \cancel{\mu})\right]$$

$$= E\left[(u_t - \theta_1 u_{t-1})(u_{t-1} - \theta_1 u_{t-2})\right]$$

$$= E\left[u_t u_{t-1} - \theta_1 (u_{t-1})^2 - \theta_1 u_{t-1} u_{t-2} + \theta_1^2 u_{t-1} u_{t-2}\right]$$

$$= -\theta_1 \sigma^2$$

- for $n=2$,

$$\gamma(2) = Cov(y_t, y_{t-2})$$

$$= E\left[(u_t - \theta_1 u_{t-1})(u_{t-2} - \theta_1 u_{t-3})\right]$$

$$= 0$$

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- Similarly $\gamma(k) = 0$ for $k > 1$

$\because \gamma(k) = \gamma(-k)$

$$\gamma(k) = \begin{cases} -\theta_1 \sigma^2 & \text{if } k = \pm 1 \\ 0 & \text{o.w} \end{cases}$$

$$ACF = \frac{\rho(k)}{\rho(0)}$$

$$P(k) = \frac{\rho(k)}{\rho(0)} = \begin{cases} \dfrac{-\theta_1 \sigma^2}{(1+\theta_1^2)\sigma^2} & \text{if } k=1 \\ 0 & \text{Otherwise} \end{cases}$$

Therefore, Auto correlation becomes 0 when $k > 0$ in MA(1) process.

(ii) MA(2): 2nd Order Moving Average

$$y_t = \mu + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}$$

$$E(y_t) = \mu \quad , \quad V(y_t) = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

$\therefore$ ACVF $= \gamma(t) = Cov(y_t, y_{t-n})$

$\rightarrow$ for $n = 1$,

$$Cov(y_t, y_{t-1}) = E\left[(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}) (u_{t-2} - \theta_1 u_{t-2} - \theta_2 u_{t-3}) \right]$$

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$$= -\theta_1\sigma^2 + \theta_1\theta_2\sigma^2$$
$$= (-\theta_1 + \theta_1\theta_2)\sigma^2$$

→ for $n=2$,
$$Cov(y_t, y_{t-2}) = -\theta_2\sigma^2$$

→ for $n=3$,
$$Cov(y_t, y_{t-3}) = 0$$

So, $ACVF = \begin{cases} (-\theta_1 + \theta_1\theta_2)\sigma^2 & K=\pm 1 \\ -\theta_2\sigma^2 & K=\pm 2 \\ 0 & \text{Otherwise} \end{cases}$

this shows that ~~at~~ autocovariance depends on lag not on time.

$$ACF = \begin{cases} 1 & \text{for } K=0 \\ \dfrac{\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & \text{for } K=\pm 1 \\ \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{for } K=\pm 2 \\ 0 & K>2 \end{cases}$$

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(iii) MA(q):

$$y_t = \mu + u_t - \theta_1 u_{t-2} - \dots - \theta_q u_{t-q}$$

$$E(y_t) = \mu \quad ; \quad Var(y_t) = (1 + \theta_1^2 + \dots + \theta_q^2)\sigma^2$$

$\Rightarrow$ ACVF: $\gamma(k) = E[y_t, y_{t-k}]$

$$= E\left[\left(y_t - E(y_t)\right)\left(y_{t-k} - E(y_{t-k})\right)\right]$$

$$= E\left[(u_t - \theta_1 u_{t-1} - \dots - \theta_q u_{t-q})\right.$$
$$\left.(u_{t-k} - \theta_1 u_{t-k-1} - \dots - \theta_q u_{t-k-q})\right]$$

$$= E\left[u_t(u_{t-k} - \theta_1 u_{t-k-1} - \dots - \theta_q u_{t-k-q})\right.$$
$$- \theta_1 u_{t-2}(u_{t-k} - \dots - \theta_q u_{t-k-q})$$
$$\left. - \dots - \theta_q u_{t-q}(u_{t-k} - \dots - \theta_q u_{t-k-q})\right]$$

$$= E\left[(u_t - \theta_1 u_{t-1} - \dots - \theta_q u_{t-q}) u_{t-k}\right.$$
$$+ (u_t - \theta_1 u_{t-1} - \dots - \theta_q u_{t-q})(-\theta_1 u_{t-k-1})$$
$$+ \dots + (u_t - \theta_1 u_{t-1} - \dots - \theta_q u_{t-q})$$
$$\left.(-\theta_q u_{t-k-q})\right]$$

$$= -\theta_k \sigma^2 + \theta_1 \theta_{k+1} \sigma^2 + \dots + \theta_q \theta_{q+k} \sigma^2 \; ; \quad k < q$$

$$\text{ACVF}: \begin{cases} \left(-\theta_k + \theta_1 \theta_{k+1} + \dots + \theta_{q+k} \theta_q\right) \sigma^2 & \text{for } 0 \le k \le q \\ 0 & \text{for } k > q \end{cases}$$

$$\text{ACF}: \begin{cases} \dfrac{\left(-\theta_k + \sum_{i=1}^{q} \theta_i * \theta_{k+i}\right) \sigma^2}{\left(1 + \sum_{i=1}^{q} \theta_i^2\right) \sigma^2} & ; \; 0 < k \le q \\ 0 & ; \; k > q \end{cases}$$