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Assignment - 6

i) Derive conditions of stationarity of AR(2) model.

ii) AR(2) process : $Y_t = 0.8\, Y_{t-1} + 0.09\, Y_{t-2} + \varepsilon_t$. Obtain mean, variance, ACVF and ACF. Is the process stationary?

iii) Generate a time series of size 500 from AR(1) process with Co-efficient $\alpha = 0.8$ and $\alpha = -0.75$. Obtain mean, variance, ACVF, ACF. Is the process stationary? {R Program}

Solution)

(i)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \mu \rightarrow (1)$$

$$\{\varepsilon_t \sim N(0, \sigma^2)$$

$$\Rightarrow (1 - \phi_1 B - \phi_2 B^2) y_t = \varepsilon_t + \mu$$

where B is the Backward shift operator such that $B^n y_t = y_{t-n}$.

Let $\Phi(B) = (1 - \phi_1 B - \phi_2 B^2) \rightarrow (2)$, the characteristic polynomial of process (1).

For eq^n (1) to be stationary, the root of $\Phi(B) = 0$ should lie outside the unit circle.

Let $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}$ be the reciprocal of $\Phi(B) = 0$.

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Then $\Phi(B) = \left(1 - \frac{1}{\alpha_1}B\right)\left(1 - \frac{1}{\alpha_2}B\right)$

$$= 1 - \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)B + \frac{1}{\alpha_1\alpha_2}B^2 \longrightarrow (3)$$

Comparing (2) and (3)

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \phi_1 \quad , \quad \frac{1}{\alpha_1\alpha_2} = -\phi_2 \longrightarrow (4)$$

Now, for stationarity, $|\alpha_1| > 1$, $|\alpha_2| > 1 \longrightarrow (5)$

$|\alpha_1\alpha_2| > 1$ {from eq^n (5)

$\Rightarrow |\phi_2| < 1$ {from eq^n (4)

$\Rightarrow -1 < \phi_2 < 1$

Again from (5), we can write

$$\left|\frac{1}{\alpha_1}\right| < 1 \quad , \quad \left|\frac{1}{\alpha_2}\right| < 1$$

$$\Rightarrow \left|\frac{1}{\alpha_1}\right| + \left|\frac{1}{\alpha_2}\right| < 2 \Rightarrow \left|\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right| < 2$$

{$\because |A+B| < |A| + |B|$}

$\Rightarrow |\phi_1| < 2 \Rightarrow -2 < \phi_1 < 2$

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$\therefore$ Condition of stationarity for model (1) is

$-2 < \phi_1 < 2$, $-1 < \phi_2 < 1$

(ii) Given AR(2) process: $y_t = 0.8y_{t-1} + 0.09y_{t-2} + \varepsilon_t \rightarrow (1)$

$\varepsilon_t \sim WN(0, \sigma^2)$

From the above concluded condition of stationarity, we see here
$|\phi_1| = |0.08| < 2$ and $|\phi_2| = |0.09| < 1$

Hence, model (1) represents a stationary model.
Then $E(y_t) = E(y_{t-1}) = \dots = \mu$ (let)
and
$Var(y_t) = Var(y_{t-1}) = \dots = \sigma_y^2$ (say)

· Mean,
From (1), $E(y_t) = E(0.8y_{t-1} + 0.09y_{t-2} + \varepsilon_t)$

$\Rightarrow \mu = 0.8\mu + 0.09\mu + 0$
$\Rightarrow \mu = 0$

· Variance,
From (1), $Var(y_t) = Var(0.8y_{t-1} + 0.09y_{t-2} + \varepsilon_t)$

$\Rightarrow \sigma_y^2 \{1 - (0.8)^2 - (0.09)^2\} = \sigma^2$
$\Rightarrow \sigma_y^2 = 2.8417\sigma^2$

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· ACVF,

$$\gamma(\tau) = E[(y_t - \mu)(y_{t-\tau} - \mu)] = E[y_t \cdot y_{t-\tau}]$$

$$= E[(0.8y_{t-1} + 0.09y_{t-2} + \varepsilon_t) y_{t-\tau}]$$

$$= 0.8 E(y_{t-1} \cdot y_{t-\tau}) + 0.09 E(y_{t-2} \cdot y_{t-\tau}) + E(\varepsilon_t \cdot y_{t-\tau})$$

$$\Rightarrow \gamma(\tau) = \begin{cases} 0.8\gamma(\tau-1) + 0.09\gamma(\tau-2), & \tau \neq 0 \\ 0.8\gamma(\tau-1) + 0.09\gamma(\tau-2) + \sigma^2, & \tau = 0 \end{cases}$$

∴ The initial condition is obtained by solving

$$\gamma(0) = 0.8\gamma(1) + 0.09\gamma(2) + \sigma^2 \rightarrow (2)$$

$$\gamma(1) = 0.8\gamma(0) + 0.09\gamma(1) \rightarrow (3)$$

$$\gamma(2) = 0.8\gamma(1) + 0.09\gamma(0) \rightarrow (4)$$

$$\Rightarrow 0.91\gamma(1) = 0.8\gamma(0) \Rightarrow \gamma(1) = \frac{80}{91}\gamma(0) \quad \{\text{Using } (3)\} \rightarrow (5)$$

Again $\gamma(2) = 0.8 \times \frac{80}{91}\gamma(0) + 0.09\gamma(0)$ {Using (4) and (5)}

$$= \frac{7219}{9100}\gamma(0) \rightarrow (6)$$

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$\therefore$ Using ⑤ and ⑥ in eqⁿ ②

$$\gamma(0) = \left\{1 - \frac{64}{91} - \frac{9}{100} \times \frac{64}{91} - \frac{81}{100^2}\right\} = \sigma^2$$

$$\Rightarrow \gamma(0) = 4.4443\,\sigma^2$$
$$\gamma(1) = 3.9071\,\sigma^2$$
$$\gamma(2) = 3.5257\,\sigma^2$$

$\therefore$ ACVF of ①: $\gamma(\tau) = 0.8\,\gamma(\tau-1) + 0.09\,\gamma(\tau-2), \quad \tau > 2$

and $\gamma(0) = 4.4443\,\sigma^2$
$\gamma(1) = 3.9071\,\sigma^2$
$\gamma(2) = 3.5257\,\sigma^2$ $\longrightarrow$ ⑦

- ACF,

From eqⁿ ⑤ and ⑥

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{80}{91} = 0.8791$$

and $$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{7219}{9100} = 0.7933$$

respectively.

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Then dividing $\gamma(0)$ on both sides of eqⁿ (7),
we obtain the ACF

$$\rho(\tau) = 0.8\rho(\tau-1) + 0.09\rho(\tau-2) \quad , \tau = 3, 4, 5 \ldots$$

having $\rho(1) = 0.8791$, $\rho(2) = 0.7933$

$\longrightarrow (8)$