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Assignment - 4

Obtain mean, variance, ACCVF and ACF of following models,

i) AR(1)
ii) AR(3)
iii) AR(p)
iv) Draw ACF and PACF plot

Sol)

i) AR(1) :-

$$y_t = \delta + \phi_1 y_{t-1} + u_t \rightarrow (1) \qquad \{u_t \sim N(0, \sigma^2)\}$$

Recursing back with lag 1

$$y_{t-1} = \delta + \phi_1 y_{t-2} + u_{t-1} \rightarrow (2)$$

putting (2) in (1), we get

$$y_t = \delta + \phi_1 (\delta + \phi_1 y_{t-2} + u_{t-1}) + u_t$$

$$= \delta + \phi_1 \delta + \phi_1^2 y_{t-2} + \phi_1 u_{t-1} + u_t \rightarrow (3)$$

Similarily, $y_{t-2} = \delta + \phi_1 y_{t-3} + u_{t-2} \rightarrow (4)$

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Putting eqⁿ (4) in eqⁿ (3)

$$y_t = \delta + \phi_1 \delta + \phi_1^2 \delta + \phi_1^3 y_{t-3} + \phi_1^2 u_{t-1} + \phi_1 u_{t-1} + u_t$$

Similarity after p^{th} period recursion,

$$y_t = \delta \left[\sum_{j=0}^{p} \phi_1^j \right] + \phi_1^{p+1} y_{t-p-1} + \sum_{j=0}^{p} \phi_1^j u_{t-j} \rightarrow (5)$$

Thus this will be invertible if $|\phi_1| < 1$ as $p \to \infty$

$$y_t = \delta \left[\sum_{j=0}^{\infty} \phi_1^j \right] + 0 + \left[\sum_{j=0}^{\infty} \phi_1^j u_{t-j} \right]$$

$$= \frac{\delta}{1-\phi_1} + \left[\sum_{j=0}^{\infty} \phi_1^j u_{t-j} \right] \rightarrow (6)$$

From eqⁿ (6), we can say that MA (∞) represents stationary AR (1)

Say: MA (∞)

$$y_t = \alpha + \sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k} \rightarrow (7) \quad \left\{ \theta_k = \phi_1^k, \varepsilon_t = u_t \right\}$$

Comparing (6) and (7)

- $E(y_t) = \alpha = \dfrac{\delta}{1-\phi_1}$

- $V(y_t) = \left[\sum_{k=0}^{\infty} \theta_k^2\right] \sigma^2 = \sigma^2 \sum_{j=0}^{\infty} (\phi_1^j)^2$

$$= \sigma^2 (1 + \phi_1^2 + \dots)$$

$$V(y_t) = \frac{\sigma^2}{1-\phi_1^2}$$

- ACVF $\gamma(k) = \sigma^2 \sum_{i=0}^{\infty} \theta_i \theta_{i+k}$

$$= \sigma^2 \sum_{j=0}^{\infty} \phi_1^j \phi_1^{j+k}$$

$$= \sigma^2 \phi_1^k \sum_{i=0}^{\infty} \phi_1^{2j}$$

$$\gamma(0) = \frac{\sigma^2}{1-\phi_1^2} \qquad \gamma(k) = \phi_1^k \gamma(0)$$

- ACF $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$

$$\rho(k) = \phi_1^k$$

(ii) AR(3) :-

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + u_t$$

$$\{u_t \sim N(0, \sigma^2)\}$$

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- Mean $E(y_t) = \delta + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + \phi_3 E(y_{t-3})$

$+ 0$

$E(y_t) = \delta + (\phi_1 + \phi_2 + \phi_3)\mu$

$\mu = \delta + (\phi_1 + \phi_2 + \phi_3)\mu$

$\mu - (\phi_1 + \phi_2 + \phi_3)\mu = \delta$

$\mu(1 - (\phi_1 + \phi_2 + \phi_3)) = \delta$

$$\mu = \frac{\delta}{1 - (\phi_1 + \phi_2 + \phi_3)}$$

- ACVF :

$\gamma(k) = Cov(y_t, y_{t-k})$

$= Cov(\delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + u_t, y_{t-k})$

$= \phi_1 Cov(y_{t-1}, y_{t-k}) + \phi_2 Cov(y_{t-2}, y_{t-k}) +$

$\phi_3 Cov(y_{t-3}, y_{t-k}) + Cov(u_t, y_{t-k})$

$= \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \phi_3 \gamma(k-3) + C$

where $C = \begin{cases} \sigma^2 & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$

$\gamma(0) = \text{Variance} = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \phi_3 \gamma(3) + \sigma^2$

$\gamma(k) = \sum_{i=1}^{3} \phi_i \gamma(k-i) \rightarrow (1)$

Similarily, $\gamma(0) = \sum_{i=1}^{3} \phi_i \gamma(i) + \sigma^2$

$$\sigma^2 = \gamma(0)\left[1 - \sum_{i=1}^{3} \phi_i \frac{\gamma(i)}{\gamma(0)}\right]$$

Variance: $\sigma^2 = \gamma(0)\left[1 - \sum_{i=1}^{3} \phi_i \rho(i)\right]$

- ACF: $\rho(k) = \dfrac{\gamma(k)}{\gamma(0)}$

$$= \frac{\sum_{i=1}^{3} \phi_i \gamma(k-i)}{\sum_{i=1}^{3} \phi_i \gamma(i) + \sigma^2}$$

$$\rho(k) = \sum_{i=1}^{3} \phi_i \rho(k-i) \; ; \; k = 1, 2, \ldots$$

(iii) AR (P) :-

$$y_t = \delta + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + u_t$$

$$E(y_t) = \delta + \phi_1 E(y_{t-1}) + \cdots + \phi_p E(y_{t-p}) + 0$$

$$\mu = \delta + (\phi_1 + \phi_2 + \cdots \phi_p)\mu$$

$$\mu = \frac{\delta}{1 - (\phi_1 + \phi_2 + \cdots \phi_p)}$$

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- ACVF :-

$$\gamma(k) = Cov(y_t, y_{t-k})$$

$$\therefore \gamma(k) = Cov(\delta + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + u_t, y_{t-k})$$

$$= \sum_{i=1}^{p} \phi_i Cov(y_{t-i}, y_{t-k}) + Cov(u_t, y_{t-k})$$

$$= \sum_{i=1}^{p} \phi_i \gamma(k-i) + \begin{cases} \sigma^2 & k=0 \\ 0 & k>0 \end{cases}$$

Now, $\gamma(0) = \sum_{i=1}^{p} \phi_i \gamma(i) + \sigma^2$

$$= \frac{\sigma^2}{1 - \sum_{i=1}^{p} \phi_i \rho(i)}$$

$$\gamma(k) = \sum_{i=1}^{p} \phi_i \gamma(k-i) \qquad k=1,2,\ldots$$

- ACF :

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

$$= \sum_{i=1}^{p} \phi_i \rho(k-i) \quad ; \quad k=1,2,\ldots$$