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Assignment - 5

Sol) → MA(3) is $y_t = \mu + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3}$

$\{ u_t \overset{iid}{\sim} N(0, \sigma^2) \}$

we have mean, $E(y_t) = \mu$

$\because E(u_t) = 0$

and variance $= V(y_t)$

$= V(\mu + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3})$

$= \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 + \theta_3^2 \sigma^2$

$= \sigma^2 (1 + \theta_1^2 + \theta_2^2 + \theta_3^2) = \gamma(0)$

· ACVF at $k=1$ (lag $=1$)

$\gamma(1) = E[(y_t - \mu)(y_{t-1} - \mu)]$

$= E[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3})$

$(u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3} + \theta_3 u_{t-4})]$

$= \theta_1 \sigma^2 + \theta_1 \theta_2 \sigma^2 + \theta_2 \theta_3 \sigma^2$

$\Rightarrow \gamma(1) = \sigma^2 (\theta_1 + \theta_1 \theta_2 + \theta_2 \theta_3)$

$[\because E(u_i^2) = V(u_i) - [E(u_i)]^2 = V(u_i) - \sigma^2$ and $E(u_i u_j)_{i \neq j} = 0]$

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At lag = 2

$$\gamma(2) = E[(y_t - u)(y_{t-2} - u)]$$

$$= E[(u_t + \theta_1 u_t + \theta_2 u_{t-2} + \theta_3 u_{t-3})$$

$$(u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4} + \theta_4 u_{t-5})]$$

$$\Rightarrow \gamma(2) = \theta_2 \sigma^2 + \theta_1 \theta_3 \sigma^2 = \sigma^2 (\theta_2 + \theta_1 \theta_3)$$

At lag = 3

$$\gamma(3) = E[(y_t - u)(y_{t-3} - u)]$$

$$= E[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3})$$

$$(u_{t-3} + \theta_1 u_{t-4} + \theta_2 u_{t-5} + \theta_3 u_{t-6})]$$

$$= \theta_3 \sigma^2$$

Clearly $\gamma(q) = 0$ if $|q| > 3$

Then
$$\gamma(K) = \begin{cases} (1 + \theta_1^2 + \theta_2^2 + \theta_3^3)\sigma^2, & K = 0 \\ (\theta_1 + \theta_1\theta_2 + \theta_2\theta_3)\sigma^2, & K \pm 1 \\ (\theta_2 + \theta_1\theta_3)\sigma^2, & K \pm 2 \\ (\theta_3 \sigma^2) & K = \pm 3 \\ 0 & |K| > 3 \end{cases}$$

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• ACF, $P(k) = \begin{cases} 1 & k=0 \\ \dfrac{\theta_1+\theta_1\theta_2+\theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & k=\pm 1 \\ \dfrac{\theta_2+\theta_1\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & k=\pm 2 \\ \dfrac{\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & k=\pm 3 \\ 0 & |k|>3 \end{cases}$

∴ Auto correlation coefficient becomes zero when $|k|>3$ for MA(3)

→ MA(5) is $y_t = \mu + u_t + \theta u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3} + \theta_4 u_{t-4} + \theta_5 u_{t-5}$

Then Mean $(y_t) = E(y_t) = \mu$

∵ $E(u) = \mu$ as u iid $N(0, \sigma^2)$

and Variance $(y_t) = 0 + \sigma^2(1+\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_5^2) = V(0)$

∵ $Var(u) = \sigma^2$

and $Var(au) = a^2\sigma^2$

(a is constant)

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- ACVF: From the ACVF of MA(3) model we can conclude the ACVF of MA(5) at lag = K as

$$\gamma(K) = \begin{cases} \sigma^2(1+\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_5^2), & K=0 \\ \sigma^2(\theta_1+\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_4+\theta_4\theta_5), & K=\pm 1 \\ \sigma^2(\theta_2+\theta_1\theta_3+\theta_2\theta_4+\theta_3\theta_5), & K=\pm 2 \\ \sigma^2(\theta_3+\theta_1\theta_4+\theta_2\theta_5), & K=\pm 3 \\ \sigma^2(\theta_4+\theta_1\theta_5), & K=\pm 4 \\ \sigma^2\theta_5, & K=\pm 5 \\ 0, & K>5 \end{cases}$$

Then ACF, $\rho(K) = \dfrac{\gamma(K)}{\gamma(0)}$

$$\rho(K) = \begin{cases} 1, & K=0 \\ (\theta_1+\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_4+\theta_4\theta_5)/K, & K=\pm 1 \\ (\theta_2+\theta_1\theta_3+\theta_2\theta_4+\theta_3\theta_5)/K, & K=\pm 2 \\ (\theta_3+\theta_1\theta_4+\theta_2\theta_5)/K, & K=\pm 3 \\ (\theta_4+\theta_1\theta_5)/K, & K=\pm 4 \\ \theta_5/K, & K=\pm 5 \\ 0 & K>5 \end{cases}$$

where $K = 1+\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_5^2$

For a generalized MA model of order q, say MA(q), we can conclude that

ACVF = $\gamma(K)$ at lag K can be written as

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$$\gamma(K) = \begin{cases} \sigma^2\left(1 + \sum_{i=1}^{q} \theta_i^2\right), & K = 0 \\ \sigma^2(\theta_K + \theta_1\theta_{1+K} + \theta_2\theta_{2+K} + \dots + \theta_{q-K}\theta_q), & K < q \\ \sigma^2\theta_q, & K = \pm q \\ 0, & |K| > q \end{cases}$$

and ACF $= P(K) = \dfrac{\gamma(K)}{\gamma(0)}$

$$\therefore P(K) = \begin{cases} 1, & K = 0 \\ \dfrac{\theta_K + \theta_1\theta_{1+K} + \dots + \theta_{q-K}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2}, & K < q \\ \dfrac{\sigma^2\theta_q}{1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2}, & K = \pm q \\ 0 & K > q \end{cases}$$