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A general proof of consistency of heuristic classification for cognitive diagnosis models

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The Asymptotic Classification Theory of Cognitive Diagnosis (Chiu *et al*., 2009, *Psychometrika*, *74*, 633–665) determined the conditions that cognitive diagnosis models must satisfy so that the correct assignment of examinees to proficiency classes is guaranteed when non-parametric classification methods are used. These conditions have only been proven for the Deterministic Input Noisy Output AND gate model. For other cognitive diagnosis models, no theoretical legitimization exists for using non-parametric classification techniques for assigning examinees to proficiency classes. The specific statistical properties of different cognitive diagnosis models require tailored proofs of the conditions of the Asymptotic Classification Theory of Cognitive Diagnosis for each individual model – a tedious undertaking in light of the numerous models presented in the literature. In this paper a different way is presented to address this task. The unified mathematical framework of general cognitive diagnosis models is used as a theoretical basis for a general proof that under mild regularity conditions any cognitive diagnosis model is covered by the Asymptotic Classification Theory of Cognitive Diagnosis.

1. Introduction

Cognitive diagnosis models (CDMs) for educational assessment (DiBello, Roussos, & Stout, 2007; Haberman & von Davier 2007; Leighton & Gierl, 2007; Rupp, Templin, & Henson, 2010) decompose an examinee's ability in a domain into binary cognitive skills called *attributes*, each of which an examinee may or may not have mastered. Distinct profiles of attributes define classes of intellectual proficiency. An examinee's class membership is estimated from his or her observed item scores using frequentist or Bayesian methods such as expectation maximization (EM; de la Torre, 2009) and Markov chain Monte Carlo (MCMC; DiBello *et al.*, 2007; Henson, Templin, & Willse, 2009; von Davier, 2005 2008). These parametric methods often encounter difficulties in practice – for example,the possible misspecification of the CDM that is supposed to underlie the data; CPU time and convergence issues; and the requirement of large samples that are often not available in small- or medium-sized testing programs. In response to these difficulties, non-parametric classification techniques that do not incorporate fully specified parametric models have been proposed as approximate or heuristic methods for assigning examinees to proficiency classes (Ayers, Nugent, & Dean, 2008; Chiu & Douglas, 2013; Chiu Douglas, & Li, 2009; Park & Lee, 2011; Willse, Henson, & Templin,

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2007). The Asymptotic Classification Theory of Cognitive Diagnosis (ACTCD) by Chiu *et al.* (2009) provided a foundation for this approach. Chiu *et al.* (2009) determined the conditions that CDMs must satisfy so that the correct assignment of examinees to proficiency classes is guaranteed when non-parametric classification methods are used. These conditions have only been proven for the Deterministic Input Noisy Output 'AND' gate (DINA) model (Junker & Sijtsma, 2001; Macready & Dayton, 1977). For other CDMs, no theoretical legitimization exists for using non-parametric classification as a heuristic for assigning examinees to proficiency classes. The specific statistical properties of different CDMs require tailored proofs of the conditions of the ACTCD for each individual model – a tedious undertaking in light of the numerous models presented in the literature (a survey by Fu & Li, 2007, identified more than 60 different models).

In this paper, a different way is presented to address this task. The unified mathematical framework of general cognitive diagnosis models (de la Torre, 2011; Henson *et al.*, 2009; Rupp *et al.*, 2010; von Davier, 2005 2008, 2011, 2014) is used as a theoretical basis for a general proof that under mild regularity conditions any CDM is covered by the ACTCD.

The next section reviews definitions and technical key concepts of (general) CDMs and the ACTCD. The theoretical propositions and proofs are presented in the subsequent sections. The discussion provides a synthesis of these results and summarizes their practical implications.

2. Technical background

2.1. Cognitive diagnosis models

Models for cognitive diagnosis are constrained latent class models that are equivalent to a special form of finite mixture models (Fraley & Raftery, 2002; Grim, 2006; McLachlan & Basford, 1988; McLachlan & Peel, 2000). Let Y_{ij} denote the response to binary test item j, $j = 1,\ldots,J$, obtained for examinee i, $i = 1,\ldots, N$; the J-dimensional item-score profile of examinee i is written as $\mathbf{Y}_i = (Y_{i1},\ldots,Y_{ij},\ldots,Y_{iJ})^{\mathrm{T}}$. [Throughout the text, the superscript T denotes the transpose of vectors or matrices; the 'prime' notation is reserved for distinguishing between vectors or their scalar entries. For brevity, the examinee index, i, is often omitted if the context permits; for example, $\mathbf{Y}_i$ is simply written as $\mathbf{Y} = (Y_1,\ldots,Y_j,\ldots,Y_J)^{\mathrm{T}}$.] Consider N examinees who belong to M distinct latent classes of intellectual proficiency. The general latent class model (Bartholomew, 1987; Bartholomew & Knott, 1999; Langeheine & Rost, 1988; Lazarsfeld & Henry, 1968; Heinen, 1996; Vermunt, 1997) defines the (conditional) probability of examinee i in proficiency class $\mathcal{C}_m$, $m = 1,\ldots, M$, answering correctly binary item j by the response function, $P(Y_{ij} = 1|i \in \mathcal{C}_m) = \pi_{jm}$, where π_{jm} is constant for item j across all examinees i in proficiency class $\mathcal{C}_m$. For J items, the response function is characterized by $J \times M$ parameters, π_{jm}. The proficiency-class memberships of examinees are estimated from the observed responses, Y_{ij}, using maximum likelihood methods, with the likelihood function defined as

$$L(\boldsymbol{\pi},\boldsymbol{\zeta};\mathbf{Y}) = \prod_{i=1}^{N}\left(\sum_{m=1}^{M}\zeta_m\prod_{j=1}^{J}\pi_{jm}^{y_{ij}}(1-\pi_{jm})^{(1-y_{ij})}\right),$$

where ζ_m is the unknown population proportion of proficiency class $\mathcal{C}_m$ such that $\sum_{m=1}^{M}\zeta_m = 1$. By convention, after fitting all model parameters, an examinee is assigned to the proficiency class with the maximal (posterior) probability of membership, $P(i \in \mathcal{C}_m | \mathbf{Y}_i)$. (For additional details, consult the above references.) In summary, the general latent class model assumes only that the Y_{ij} are independent, conditional on proficiency-class membership (i.e., local independence); no further restrictions are imposed on the relation between the latent variable – proficiency-class membership – and the observed response.

In contrast, CDMs constrain the relation between the latent variable and the observed response so that the mastery of cognitive attributes characteristic of distinct latent proficiency classes determines the observed response (correct or incorrect) to a test item. Suppose that K latent binary attributes constitute a certain ability domain; there are then 2^K distinct attribute profiles composed of these K attributes representing $M = 2^K$ distinct latent proficiency classes. (Note that an attribute profile for a proficiency class can consist of all zeros, because it is possible for an examinee not to have mastered any attributes at all.) Let the K-dimensional vector, $\boldsymbol{\alpha}_m = (\alpha_1, \ldots, \alpha_K)^{\mathrm{T}}$, represent the binary attribute profile of proficiency class $\mathcal{C}_m$, where the kth entry indicates whether the respective attribute has been mastered. [For brevity, the attribute profile of examinee $i \in \mathcal{C}_m$, $\boldsymbol{\alpha}_{i \in \mathcal{C}_m}$, is usually written $\boldsymbol{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{iK})^{\mathrm{T}}$.] Consider a test of J items for assessing ability in the domain. Each individual item j is associated with a binary attribute profile that specifies or constrains the particular skills required for answering it correctly. Item-attribute profiles that consist entirely of zeros, however, are inadmissible, because they correspond to items whose answers require no skills at all; hence, given K attributes, there are at most $2^K - 1$ distinct item-attribute profiles. The entire set of constraints specifying the associations between J items and K attributes constitutes the Q-matrix, $\mathbf{Q} = \{q_{jk}\}_{(J \times K)}$, $k = 1, \ldots, K$, where $q_{jk} = 1$ if a correct answer to the jth item requires mastery of the kth attribute, and 0 otherwise; thus, the row vectors of $\mathbf{Q}$, $\mathbf{q}_j$, represent the item-attribute profiles (Tatsuoka, 1985). The Q-matrix is an integral component of all CDMs, although these models differ according to the way in which mastery and non-mastery of the attributes under consideration are believed to influence performance on a test item (i.e., compensatory models versus non-compensatory models; conjunctive models versus disjunctive models; for a detailed discussion, see Henson *et al.*, 2009). Examinees' proficiency class memberships are determined based on the posterior likelihoods computed from the maximum likelihood estimates of the model parameters. de la Torre (2009) provides a didactic on implementing the EM algorithm for maximum likelihood estimation (MLE) of the parameters of the DINA model (Junker & Sijtsma, 2001; Macready & Dayton, 1977).

2.2. *General cognitive diagnosis models*

General CDMs are meta-models of the functional relation between attribute mastery and the probability of a correct item response. They allow diverse CDMs to be expressed in unified mathematical form and parametrization, thereby establishing a general standard for model comparison and evaluation that applies to 'recognizable' CDMs (de la Torre, 2011, p. 181), as discussed previously in the literature ('core models'; Rupp *et al.*, 2010) and CDMs 'that have not yet been defined' (Henson *et al.*, 2009, p. 199).

The "kernel" of item j (Rupp *et al.*, 2010, p. 135) is defined as a linear combination of the K attribute main effects, α_k, and all their interactions

$$g(\mathbf{q}_j,\boldsymbol{\alpha}) = \beta_{j0} + \sum_{k=1}^{K}\beta_{jk}q_{jk}\alpha_k + \sum_{k'=k+1}^{K}\sum_{k=1}^{K-1}\beta_{j(kk')}q_{jk}q_{jk'}\alpha_k\alpha_{k'} + \cdots + \beta_{j(12\ldots K)}\prod_{k=1}^{K}q_{jk}\alpha_k. \tag{1}$$

The q_{jk} terms are included in the main effects and interactions because they determine whether mastery of attribute α_k is required for item j. [The order of an interaction is indicated by the number of parenthetical attribute subscripts of the associated coefficient, $\beta_{j(\ldots)}$]. Attribute main effects and interactions can be removed from the kernel by constraining the corresponding entries in the parameter vector, $\boldsymbol{\beta}_j = (\beta_{j0}, \beta_{jk}, \beta_{j(kk')}, \ldots, \beta_{j(12\ldots K)})^{\mathrm{T}}$, to zero.

When expressing the item response function, $P(Y_j = 1|\boldsymbol{\alpha})$, in terms of $g(\mathbf{q}_j,\boldsymbol{\alpha})$, it must be guaranteed that $0 \le P(Y_j = 1|\boldsymbol{\alpha}) \le 1$, for example, by using the logit link, $logit\big(P(Y_j = 1|\boldsymbol{\alpha})\big) = g(\mathbf{q}_j,\boldsymbol{\alpha})$, which results in the general logistic item response function,

$$P(Y_j = 1 \mid \boldsymbol{\alpha}) = \frac{\exp\{g(\mathbf{q}_j,\boldsymbol{\alpha})\}}{1 + \exp\{g(\mathbf{q}_j,\boldsymbol{\alpha})\}}. \tag{2}$$

Von Davier (2005, 2008, 2011, 2014) defined the item response function of his General Diagnostic Model (GDM) by constraining all interaction terms of $g(\mathbf{q}_j,\boldsymbol{\alpha})$ in equation (2) to be zero (see von Davier, 2005, equations (1) and (2), pp. 3–4). Von Davier's GDM is the archetypal general CDM. Henson *et al.* (2009) retained all attribute main and interaction effects in equation (2) and proposed it as the item response function of a general CDM termed the Loglinear Cognitive Diagnosis Model (LCDM; see Henson *et al.*, 2009, equation (11), p. 197). By constraining the coefficients in $\boldsymbol{\beta}_j$, Henson *et al.* then derived from equation (2) the item response functions of several recognizable CDMs such as the DINA and Deterministic Input Noisy Output 'OR' gate (DINO) models (Templin & Henson, 2006), and the Reduced Reparametrized Unified Model (Reduced RUM; Hartz, 2002; Hartz & Roussos, 2008). The logit link was also used by de la Torre (2011) for the item response function of a general CDM called the generalized DINA (G-DINA) model (see de la Torre, 2011, equation (2), p. 181). In addition, de la Torre (2011) proposed the identity link and the log link for two alternative expressions of the item response function of the G-DINA model:

$$\text{(identity link)} \quad P(Y_j = 1 \mid \boldsymbol{\alpha}) = g(\mathbf{q}_j,\boldsymbol{\alpha}), \tag{3}$$

$$\text{(log link)} \quad P(Y_j = 1 \mid \boldsymbol{\alpha}) = \exp\{g(\mathbf{q}_j,\boldsymbol{\alpha})\} \tag{4}$$

(see de la Torre, 2011, equations (1) and (3), pp. 181–182). The identity link and the log link require additional constraints on the parameter vector, $\boldsymbol{\beta}_j$, to guarantee that $P(Y_j = 1|\boldsymbol{\alpha})$ is bounded by 0 and 1. de la Torre derived the item response functions of various recognizable CDMs by constraining $\boldsymbol{\beta}_j$ in equations (2), (3), and (4) above.

The flexibility offered by general CDMs can be summarized by two observations. First, the three link functions – identity, logit, and log link – define three prototypes of general CDMs that allow CDMs to be expressed in several ways. Examples are the DINO and DINA models that can be derived using the logit link (Henson *et al.*, 2009) and the identity link (de la Torre, 2011), provided adequate constraints are imposed on the parameter vector $\boldsymbol{\beta}_j$; and the reduced RUM that can be expressed based on the logit link (Henson *et al.*, 2009) and the log link (de la Torre, 2011).

Second, if $\boldsymbol{\alpha}$ is known, then the item response functions of general CDMs using different link functions can be converted by appropriate reparametrization. For example, consider item j that requires mastery of two attributes (i.e., $\mathbf{q}_j = (11)$). The item response function based on the logit link is

$$P(Y_j = 1|\boldsymbol{\alpha}) = \frac{\exp\{\beta_{j0} + \beta_{j1}\alpha_1 + \beta_{j2}\alpha_2 + \beta_{j(12)}(\alpha_1\alpha_2)\}}{1 + \exp\{\beta_{j0} + \beta_{j1}\alpha_1 + \beta_{j2}\alpha_2 + \beta_{j(12)}(\alpha_1\alpha_2)\}}$$

which, for given $\boldsymbol{\alpha}$, can be re-expressed based on the identity link as

$$P(Y_j = 1|\boldsymbol{\alpha}) = \beta'_{j0} + \beta'_{j1}\alpha_1 + \beta'_{j2}\alpha_2 + \beta'_{j(12)}\alpha_1\alpha_2$$

by reparametrizing

$$\begin{aligned}
\beta'_{j0} &= \frac{\exp\{\beta_{j0}\}}{1 + \exp\{\beta_{j0}\}},\\
\beta'_{j1} &= \frac{\exp\{\beta_{j0} + \beta_{j1}\}}{1 + \exp\{\beta_{j0} + \beta_{j1}\}} - \beta'_{j0},\\
\beta'_{j2} &= \frac{\exp\{\beta_{j0} + \beta_{j2}\}}{1 + \exp\{\beta_{j0} + \beta_{j2}\}} - \beta'_{j0},\\
\beta'_{j(12)} &= \frac{\exp\{\beta_{j0} + \beta_{j1} + \beta_{j2} + \beta_{j(12)}\}}{1 + \exp\{\beta_{j0} + \beta_{j1} + \beta_{j2} + \beta_{j(12)}\}} - \beta'_{j0} - \beta'_{j1} - \beta'_{j2}.
\end{aligned}$$

2.3. *The Asymptotic Classification Theory of Cognitive Diagnosis*

The Asymptotic Classification Theory of Cognitive Diagnosis (Chiu *et al.*, 2009) provided a foundation for using non-parametric classification techniques that do not incorporate fully specified parametric models as approximate methods for assigning examinees to proficiency classes. The ACTCD consists of three lemmas, each of which determines a condition for a consistency theorem of classification to hold. Lemma 1 specifies the general condition, under which the proficiency classes of a CDM are identifiable (Chiu *et al.*, 2009, p. 643). Lemma 2 states the general condition, under which well-separated centres of the proficiency classes of a CDM are guaranteed (Chiu *et al.*, 2009, pp. 643–644). Lemmas 1 and 2 have only been proven for the DINA model. Lemma 3 specifies the condition under which a non-parametric classification method assigns examinees to their true proficiency classes if a finite mixture model with M latent classes underlies the data. Lemma 3 has only been proven for complete linkage hierarchical agglomerative clustering (Johnson, 1967; Chiu *et al.*, 2009, pp. 644–645). Building on these three lemmas, the consistency theorem of classification states that the probability of assigning examinees to their true proficiency classes approaches 1 as the length of a test (i.e., the number of items, J) increases (Chiu *et al.*, 2009, pp. 645–647).

Lemmas 1 and 2 determine regularity conditions that must be satisfied by any CDM for the consistency theorem of classification to hold. Their general proofs based on the framework of general CDMs are provided in the following sections. (Lemma 3 is of no concern here, because it specifies the conditions a non-parametric classification method must satisfy.)

3. A general proof of Lemma 1 (identifiability of the proficiency classes)

Let $S_j(\boldsymbol{\alpha}) = E(Y_j|\boldsymbol{\alpha}) = P(Y_j = 1|\boldsymbol{\alpha})$ denote the (conditional) expected item response and $\mathbf{S}(\boldsymbol{\alpha}) = E(\mathbf{Y}|\boldsymbol{\alpha})$ the expectation of an examinee's item-score profile, $\mathbf{Y} = (Y_1, \ldots, Y_j, \ldots, Y_J)^{\mathrm{T}}$, given attribute profile $\boldsymbol{\alpha}$. Define the Q-matrix to be complete if it allows identification of all possible attribute profiles: $\mathbf{S}(\boldsymbol{\alpha}) = \mathbf{S}(\boldsymbol{\alpha}') \Rightarrow \boldsymbol{\alpha} = \boldsymbol{\alpha}'$. (Recall that each proficiency class $\mathcal{C}_m$ is uniquely characterized by its attribute profile $\boldsymbol{\alpha}_m$; hence, the identification of all possible attribute profiles is equivalent to identifying all proficiency classes.) Completeness of the Q-matrix is a general requirement for any CDM; an incomplete Q-matrix prevents identification of all possible attribute profiles among examinees, regardless of whether parametric or non-parametric methods are used to group examinees into proficiency classes.

Lemma 1 Let $\mathbf{e}_k$ denote a unit vector with the kth element equal to 1, and the remaining entries all 0. (Items having $\mathbf{q} = \mathbf{e}_k$ as their attribute profiles are called single-attribute items.) A J $\times$ K matrix $\mathbf{Q}$ is complete if it includes the K vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$ among its J rows.

Proof:

First, consider the item response function of the general CDM using the identity link

$$P(Y_j = 1|\boldsymbol{\alpha}) = \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} q_{jk} \alpha_k \\ + \sum_{k'=k+1}^{K} \sum_{k=1}^{K-1} \beta_{j(kk')} q_{jk} q_{jk'} \alpha_k \alpha_{k'} + \cdots + \beta_{j(12\ldots K)} \prod_{k=1}^{K} q_{jk} \alpha_k, \tag{5}$$

subject to

$$\begin{aligned} &0 \leq \beta_{j0} \leq 1, \\ &0 < \beta_{jk}, \beta_{j(kk')}, \ldots, \beta_{j(12\ldots K)} \leq 1, \\ &0 \leq \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} + \sum_{k'=k+1}^{K} \sum_{k=1}^{K-1} \beta_{j(kk')} + \cdots + \beta_{j(12\ldots K)} \leq 1. \end{aligned}$$

(The three constraints are required to guarantee that $0 \leq P(Y_j = 1|\boldsymbol{\alpha}) \leq 1$.) Assume that K of the J rows of $\mathbf{Q}$ consist of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$. Reorder the rows of $\mathbf{Q}$ by moving these K rows to the first K row positions. From equation (5), the (conditional) expected response to item k is derived as

$$S_k(\boldsymbol{\alpha}) = \begin{cases} \beta_{k0} + \beta_{kk}, & \text{if } \alpha_k = 1, \\ \beta_{k0}, & \text{if } \alpha_k = 0. \end{cases} \tag{6}$$

Define the partition of the vector $\mathbf{S}(\boldsymbol{\alpha}) = \left(\mathbf{S}_{1:K}(\boldsymbol{\alpha})|\mathbf{S}_{(K+1):J}(\boldsymbol{\alpha})\right)^{\mathrm{T}}$. Consider two attribute profiles, $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$. Thus, $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ must differ in at least one element k, say, $\alpha_k = 1$ and $\alpha'_k = 0$. Equation (6) then implies that $S_k(\boldsymbol{\alpha}) > S_k(\boldsymbol{\alpha}')$ because $\beta_{kk} > 0$. Therefore, $\mathbf{S}_{1:K}(\boldsymbol{\alpha}) \neq \mathbf{S}_{1:K}(\boldsymbol{\alpha}')$, and so $\mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$, independent of whether $\mathbf{S}_{(K+1):J}(\boldsymbol{\alpha})$ is identical to $\mathbf{S}_{(K+1):J}(\boldsymbol{\alpha}')$ – that is, regardless of which additional multi-attribute items are included in the test. Thus, $\mathbf{Q}$ is complete. □

For general CDMs using the logit link or the log link, the proof of Lemma 1 follows the same logic. The item response function based on the logit link is

$$\begin{aligned} logit\left(P(Y_j = 1|\boldsymbol{\alpha})\right) = \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} q_{jk} \alpha_k \\ + \sum_{k'=k+1}^{K} \sum_{k=1}^{K-1} \beta_{j(kk')} q_{jk} q_{jk'} \alpha_k \alpha_{k'} + \cdots + \beta_{j(12\ldots K)} \prod_{k=1}^{K} q_{jk} \alpha_k, \end{aligned} \tag{7}$$

subject to $0 < \beta_{jk}$. The expected item response (corresponding to equation (6)) is

$$S_k(\boldsymbol{\alpha}) = \begin{cases} \dfrac{\exp\{\beta_{k0} + \beta_{kk}\}}{1 + \exp\{\beta_{k0} + \beta_{kk}\}}, & \text{if } \alpha_k = 1, \\ \dfrac{\exp\{\beta_{k0}\}}{1 + \exp\{\beta_{k0}\}}, & \text{if } \alpha_k = 0. \end{cases} \tag{8}$$

Because the logistic function is monotonically increasing, $\beta_{kk} > 0$ guarantees that

$$\frac{\exp\{\beta_{k0} + \beta_{kk}\}}{1 + \exp\{\beta_{k0} + \beta_{kk}\}} > \frac{\exp\{\beta_{k0}\}}{1 + \exp\{\beta_{k0}\}}.$$

When the log link is used, the item response function is

$$\begin{aligned} \log(P(Y_j = 1|\boldsymbol{\alpha})) = \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} q_{jk} \alpha_k \\ + \sum_{k'=k+1}^{K} \sum_{k=1}^{K-1} \beta_{j(kk')} q_{jk} q_{jk'} \alpha_k \alpha_{k'} + \cdots + \beta_{j(12\ldots K)} \prod_{k=1}^{K} q_{jk} \alpha_k, \end{aligned} \tag{9}$$

subject to

$$\begin{aligned} &\beta_{j0} < 0, \\ &0 < \beta_{jk} < |\beta_{j0}|, \\ &0 < \sum_{k=1}^{K} \beta_{jk} + \sum_{k'=k+1}^{K} \sum_{k=1}^{K-1} \beta_{j(kk')} + \cdots + \beta_{j(12\ldots K)} < |\beta_{j0}|. \end{aligned}$$

The expected item response (corresponding to equations (6) and (8)) is

$$S_k(\boldsymbol{\alpha}) = \begin{cases} \exp\{\beta_{k0} + \beta_{kk}\}, & \text{if } \alpha_k = 1, \\ \exp\{\beta_{k0}\}, & \text{if } \alpha_k = 0. \end{cases} \tag{10}$$

Because the exponential function is monotonically increasing, $\beta_{kk} > 0$ guarantees that $\exp\{\beta_{k0} + \beta_{kk}\} > \exp\{\beta_{k0}\}$. If $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ such that they differ in at least one element k, $\alpha_k = 1$ and $\alpha'_k = 0$, then equations (8) and (10) – like equation (6) – guarantee that $S_k(\boldsymbol{\alpha}) > S_k(\boldsymbol{\alpha}')$. Thus, if $\mathbf{Q}$ contains the profiles of all K single-attribute items, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$, then $\mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$, regardless whether a test contains additional multi-attribute items. This completes the proof.

In summary, the general proof of Lemma 1 demonstrates that completeness of $\mathbf{Q}$ – the identifiability of all proficiency classes – is guaranteed for the three prototypes of general CDMs as long as the Q-matrix contains all K single-attribute items. Two aspects of Lemma 1 remain to be addressed in greater detail in the next sections.

4. Completeness of the Q-matrix: Further considerations

The general proof of Lemma 1 assumed that all single-attribute items, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$, are included in the Q-matrix – a condition that may seem rather restrictive for many practical applications. In fact, alternative compositions of the Q-matrix that do not include all single-attribute items can also guarantee the completeness of $\mathbf{Q}$. More to the point, the inclusion of all single-attribute items in $\mathbf{Q}$ is a sufficient but not a necessary condition of Q-completeness for the three prototypes of general CDMs. As an example, consider the Q-matrix $\mathbf{Q}^*$, with $K = 3$ attributes and $J = 3$ items,

$$\mathbf{Q}^* = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

that does not contain the single-attribute items, but that is complete, as the computation of the expected item-response profiles, $\mathbf{S}(\boldsymbol{\alpha}) = (S_1(\boldsymbol{\alpha}), S_2(\boldsymbol{\alpha}), S_3(\boldsymbol{\alpha}))^{\mathrm{T}}$, for the general CDM with identity link demonstrates:

$$S_j(\boldsymbol{\alpha}) = \beta_{j0} + \sum_{k=1}^{3} \beta_{jk} q_{jk} \alpha_k + \sum_{k'=k+1}^{3} \sum_{k=1}^{2} \beta_{j(kk')} q_{jk} q_{jk'} \alpha_k \alpha_{k'} + \beta_{j(123)} \prod_{k=1}^{3} q_{jk} \alpha_k. \tag{11}$$

$\boldsymbol{\alpha}$	Q-vectors		
	$\mathbf{q}_1^* = (011)$	$\mathbf{q}_2^* = (101)$	$\mathbf{q}_3^* = (110)$
	$S_1(\boldsymbol{\alpha})$	$S_2(\boldsymbol{\alpha})$	$S_3(\boldsymbol{\alpha})$
(000)	β_{10}	β_{20}	β_{30}
(100)	β_{10}	$\beta_{20} + \beta_{21}$	$\beta_{30} + \beta_{31}$
(010)	$\beta_{10} + \beta_{12}$	β_{20}	$\beta_{30} + \beta_{32}$
(001)	$\beta_{10} + \beta_{13}$	$\beta_{20} + \beta_{23}$	β_{30}
(110)	$\beta_{10} + \beta_{12}$	$\beta_{20} + \beta_{21}$	$\beta_{30} + \beta_{31} + \beta_{32} + \beta_{3(12)}$
(101)	$\beta_{10} + \beta_{13}$	$\beta_{20} + \beta_{21} + \beta_{23} + \beta_{2(13)}$	$\beta_{30} + \beta_{31}$
(011)	$\beta_{10} + \beta_{12} + \beta_{13} + \beta_{1(23)}$	$\beta_{20} + \beta_{23}$	$\beta_{30} + \beta_{32}$
(111)	$\beta_{10} + \beta_{12} + \beta_{13} + \beta_{1(23)}$	$\beta_{20} + \beta_{21} + \beta_{23} + \beta_{2(13)}$	$\beta_{30} + \beta_{31} + \beta_{32} + \beta_{3(12)}$

Indeed, $\mathbf{Q}^*$ is complete because $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$. (Using the logit link or the log link changes the specific mathematical expressions of $\mathbf{S}(\boldsymbol{\alpha})$, but not the general result.)

Inspection of $\mathbf{S}(\boldsymbol{\alpha})$ suggests that the presence of the attribute main effects in equation (11) is important for the identifiability of the proficiency classes. If all main effects are removed from equation (11),

$$S_j(\boldsymbol{\alpha}) = \beta_{j0} + \sum_{k'=k+1}^{3} \sum_{k=1}^{2} \beta_{j(kk')} q_{jk} q_{jk'} \alpha_k \alpha_{k'} + \beta_{j(123)} \prod_{k=1}^{3} q_{jk} \alpha_k, \tag{12}$$

then four of the proficiency classes are no longer identifiable and $\mathbf{Q}^*$ is incomplete – that is, $\mathbf{S}(\boldsymbol{\alpha}) = \mathbf{S}(\boldsymbol{\alpha}')$ despite $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$:

$\boldsymbol{\alpha}$	Q-vectors		
	$q_1^* = (011)$	$q_2^* = (101)$	$\mathbf{q}_3^* = (110)$
	$S_1(\boldsymbol{\alpha})$	$S_2(\boldsymbol{\alpha})$	$S_3(\boldsymbol{\alpha})$
(000)	β_{10}	β_{20}	β_{30}
(100)	β_{10}	β_{20}	β_{30}
(010)	β_{10}	β_{20}	β_{30}
(001)	β_{10}	β_{20}	β_{30}
(110)	β_{10}	β_{20}	$\beta_{30} + \beta_{3(12)}$
(101)	β_{10}	$\beta_{20} + \beta_{2(13)}$	β_{30}
(011)	$\beta_{10} + \beta_{1(23)}$	β_{20}	β_{30}
(111)	$\beta_{10} + \beta_{1(23)}$	$\beta_{20} + \beta_{2(13)}$	$\beta_{30} + \beta_{3(12)}$

(The same result is obtained for the logit link and the log link; of course, the specific expressions of the expected item-response profiles, $\mathbf{S}(\boldsymbol{\alpha})$, are different.) The inclusion of the attribute main effects in equation (11) apparently compensates for the lack of single-attribute items in $\mathbf{Q}^*$. To illustrate, consider the attribute profile $\boldsymbol{\alpha} = (100)^{\mathrm{T}}$ of an examinee who only masters the first attribute. If the model in equation (11) is used, then the main effect of the first attribute, α_1, 'picks up' a β_{j1} for the second and third items (with attribute profiles $\mathbf{q}_2^* = (101)$ and $\mathbf{q}_3^* = (110)$, respectively), which results in $\mathbf{S}(100) = (\beta_{10}, \beta_{20} + \beta_{21}, \beta_{30} + \beta_{31})^{\mathrm{T}}$. In contrast, the model in equation (12) does not include any main effects. Consequently, none of the β_{j1} can be retrieved for the second and third items (this would only occur if an examinee mastered both attributes, α_1 and α_2), hence $\mathbf{S}(100) = (\beta_{10}, \beta_{20}, \beta_{30})^{\mathrm{T}}$. The same argument applies to the attribute profiles $\boldsymbol{\alpha} = (010)^{\mathrm{T}}$ and $\boldsymbol{\alpha} = (001)^{\mathrm{T}}$, so that these proficiency classes are no longer distinguishable among each other and from $\boldsymbol{\alpha} = (000)^{\mathrm{T}}$ when the attribute main effects are missing from the model.

From the prototypes of general CDMs, several important recognizable CDMs can be derived that include the attribute main effects (e.g., the GDM and the reduced RUM). This property, however, is not necessarily obvious from the traditional parametrization of these CDMs, as the example of the reduced RUM shows. The traditional item response function of the reduced RUM is

$$P(Y_i = 1|\boldsymbol{\alpha}) = \pi_j^* \prod_{k=1}^{K} r_{jk}^{*q_{jk}(1-\alpha_k)},$$

where $0 < \pi_j^* < 1$ is the probability of a correct answer for an examinee who has mastered all the attributes required by item j, and $0 < r_{jk}^* < 1$ is a penalty parameter for not mastering the kth attribute. Using the log link, the item response function of the reduced RUM in the form of a general CDM is

$$\begin{aligned} P(Y_j = 1|\boldsymbol{\alpha}) &= \exp\left\{ \log(\pi_j^*) + \sum_{k=1}^{K} \log\left(r_{jk}^{*q_{jk}(1-\alpha_k)} \right) \right\} \\ &= \exp\left\{ \log(\pi_j^*) + \sum_{k=1}^{K} \log(r_{jk}^*) q_{jk} + \sum_{k=1}^{K} -\log(r_{jk}^*) q_{jk}\alpha_k \right\}. \end{aligned}$$

Setting

$$\begin{aligned} \beta_{j0} &= \log(\pi_j^*) + \sum_{k=1}^{K} \log(r_{jk}^*) q_{jk}, \\ \beta_{jk} &= -\log(r_{jk}^*) \end{aligned}$$

results in

$$P(Y_j = 1|\boldsymbol{\alpha}) = \exp\left\{ \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} q_{jk}\alpha_k \right\}$$

subject to

$$\begin{aligned} &(1)\ \beta_{j0} < 0, \\ &(2)\ 0 < \sum_{k=1}^{K} \beta_{jk} q_{jk}\alpha_k < |\beta_{j0}|. \end{aligned}$$

The two constraints are implied by $0 < \pi_j^*, r_{jk}^* < 1$. (Alternatively, the item response function of the reduced RUM as a general CDM can be based on the logit link, but it must then include, in addition to the attribute main effects, all interactions up to order K.)

In summary, the examples suggest that the inclusion of the attribute main effects in a CDM is important for preserving the completeness of the Q-matrix when the single-attribute items are absent. Whether a CDM includes the attribute main effects can only be determined after its conversion to a general CDM. For many recognizable CDMs like the reduced RUM, the GDM, the Noisy Input Deterministic Output 'AND' gate (NIDA) model (Maris, 1999), the generalized NIDA model (Maris, 1999), and the additive cognitive diagnosis model (A-CDM), it is known that they contain the attribute main effects. For these models, the inclusion of the K single-attribute items in the Q-matrix is a sufficient but not a necessary condition of Q-completeness.

5. Completeness of the Q-matrix: Two special cases, the DINA model and the DINO model

Two popular recognizable CDMs come to mind as instances where the main effects are missing from the item response function: the DINA and DINO models.

The traditional form of the item response function of the DINA model is $P(Y_j = 1|\boldsymbol{\alpha}) = (1 - s_j)^{\eta_j} g_j^{(1-\eta_j)}$, with the conjunction parameter η_j defined as $\eta_j = \prod_{k=1}^{K} \alpha_k^{q_{jk}}$. The item-related parameters $s_j = P(Y_j = 0|\eta_j = 1)$ and $g_j = P(Y_j = 1|\eta_j = 0)$ formalize the probabilities of *slipping* and *guessing*, respectively. The DINA model is converted to a general CDM using the identity link function. Because the DINA model is a non-compensatory CDM, all attribute main effects and interactions in equation (5) must be restricted to zero except for the highest-order interaction involving all attributes α_k, $k \in \mathcal{L}_j$, where the set $\mathcal{L}_j = \{k|q_{jk} = 1\}$ contains the non-zero elements in $\mathbf{q}_j$. The item response function of the DINA model in the form of a general CDM is then

$$P(Y_j = 1|\boldsymbol{\alpha}) = \beta_{j0} + \beta_{j(\forall k \in \mathcal{L}_j)} \prod_{k \in \mathcal{L}_j} \alpha_k,$$

subject to

$$(1)\ 0 \leq \beta_{j0} < 0.5,$$
$$(2)\ 0 < \beta_{j(\forall k \in \mathcal{L}_j)} \leq 1,$$
$$(3)\ 0.5 < \beta_{j0} + \beta_{j(\forall k \in \mathcal{L}_j)} \leq 1.$$

(If $k \in \mathcal{L}_j = \{k|q_{jk} = 1\}$, then $q_{jk} = 1$ is always true; hence, q_{jk} has been dropped from the item response function.) The three constraints are implied by the traditional parametrization of the DINA model. From $P(Y_j|\boldsymbol{\alpha}) = g_j$ if $\eta_j = 0$ it follows that $\beta_{j0} = g_j$; and $P(Y_j|\boldsymbol{\alpha}) = 1 - s_j$ if $\eta_j = 1$ suggests that $\beta_{j0} + \beta_{j(\forall k \in \mathcal{L}_j)} = 1 - s_j$; the constraints then follow from the restriction $0 \leq g_j, s_j < 0.5$.

For the DINA model, the previously used Q-matrix $\mathbf{Q}^*$ is incomplete. But Q-completeness can be restored by adding the $K = 3$ single-attribute items to $\mathbf{Q}^*$. The expected item response for the DINA model is

$$S_j(\boldsymbol{\alpha}) = \beta_{j0} + \beta_{j(\forall k \in \mathcal{L}_j)} \prod_{k \in \mathcal{L}_j} \alpha_k.$$

Hence, we have:

	Q-vectors					
$\boldsymbol{\alpha}$	$\mathbf{q}_1^* = (011)$	$\mathbf{q}_2^* = (101)$	$\mathbf{q}_3^* = (110)$	$\mathbf{e}_1 = (100)$	$\mathbf{e}_2 = (010)$	$\mathbf{e}_3 = (001)$
	$S_1(\boldsymbol{\alpha})$	$S_2(\boldsymbol{\alpha})$	$S_3(\boldsymbol{\alpha})$	$S_4(\boldsymbol{\alpha})$	$S_5(\boldsymbol{\alpha})$	$S_6(\boldsymbol{\alpha})$
(000)	β_{10}	β_{20}	β_{30}	β_{40}	β_{50}	β_{60}
(100)	β_{10}	β_{20}	β_{30}	$\beta_{40} + \beta_{41}$	β_{50}	β_{60}
(010)	β_{10}	β_{20}	β_{30}	β_{40}	$\beta_{50} + \beta_{52}$	β_{60}
(001)	β_{10}	β_{20}	β_{30}	β_{40}	β_{50}	$\beta_{60} + \beta_{63}$
(110)	β_{10}	β_{20}	$\beta_{30} + \beta_{3(12)}$	$\beta_{40} + \beta_{41}$	$\beta_{50} + \beta_{52}$	β_{60}
(101)	β_{10}	$\beta_{20} + \beta_{2(13)}$	β_{30}	$\beta_{40} + \beta_{41}$	β_{50}	$\beta_{60} + \beta_{63}$
(011)	$\beta_{10} + \beta_{1(23)}$	β_{20}	β_{30}	β_{40}	$\beta_{50} + \beta_{52}$	$\beta_{60} + \beta_{63}$
(111)	$\beta_{10} + \beta_{1(23)}$	$\beta_{20} + \beta_{2(13)}$	$\beta_{30} + \beta_{3(12)}$	$\beta_{40} + \beta_{41}$	$\beta_{50} + \beta_{52}$	$\beta_{60} + \beta_{63}$

Obviously, $\mathbf{S}_{1:3}(\boldsymbol{\alpha})$ does not allow for distinguishing between all $\boldsymbol{\alpha}$. However, by adding the attribute profiles of the single-attribute items, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, to $\mathbf{Q}^*$, the identifiability of the proficiency classes is restored because then the Q-matrix is complete so that $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$. This result is due to the fact that the interaction, $\beta_{j(\forall k \in \mathcal{L}_j)} \prod_{k \in \mathcal{L}_j} \alpha_k$, in the case of a single-attribute item with $\mathbf{q}_j = \mathbf{e}_k$ is reduced to the attribute main effect $\beta_{jk}\alpha_k$, for $k \in \mathcal{L}_j$. As an illustration, consider again the attribute profile $\boldsymbol{\alpha} = (100)^{\mathrm{T}}$. For item 3, $\mathcal{L}_3 = \{1, 2\}$ because $\mathbf{q}_3^* = (110)$. Hence, item 3 requires mastery of α_1 and α_2, which, in the case of the DINA model translates into the two-way interaction $q_{31}q_{32}\alpha_1\alpha_2$. But an examinee with $\boldsymbol{\alpha} = (100)^{\mathrm{T}}$ simply cannot 'pick up' this required two-way interaction because he or she lacks α_2. Thus, $S_3(100) = \beta_{30}$. For item 4, $\mathcal{L}_4 = \{1\}$ because $\mathbf{q}_4 = \mathbf{e}_1 = (100)$. Thus, for item 4, the highest-order interaction can only be the attribute main effect, $q_{41}\alpha_1$, which the examinee with $\boldsymbol{\alpha} = (100)^{\mathrm{T}}$ can 'pick up' because he or she masters α_1. Thus, $S_4(100) = \beta_{40} + \beta_{41}$. The reduction of the interaction, $\beta_{j(\forall k \in \mathcal{L}_j)} \prod_{k \in \mathcal{L}_j} \alpha_k$, to the main effect $\beta_{jk}\alpha_k$ for single-attribute items is the reason why their addition to the Q-matrix induces completeness for the DINA model. In other words, for the DINA model, the inclusion of all single-attribute items in the Q-matrix is a necessary condition for completeness. This is also true for the DINO model, as is proved next.

The item response function of the DINO model as a general CDM based on the identity link is

$$P(Y_j = 1|\boldsymbol{\alpha}) = \beta_{j0} + \beta_{jk} - \beta_{jk} \prod_{l \in \mathcal{L}_j} (1 - \alpha_l), \quad \text{for some } k \in \mathcal{L}_j,$$

subject to

$$\begin{aligned} &(1)\ 0 \leq \beta_{j0} < 0.5, \\ &(2)\ 0 < \beta_{jk} \leq 1, \\ &(3)\ 0.5 < \beta_{j0} + \beta_{jk} \leq 1. \end{aligned}$$

The three constraints are implied by the traditional parametrization of the DINO model, $P(Y_j = 1|\boldsymbol{\alpha}) = (1 - s_j)^{\omega_j} g_j^{(1-\omega_j)}$, where the disjunction parameter, ω_j, is defined as $\omega_j = 1 - \prod_{k=1}^{K} (1 - \alpha_k)^{q_{jk}}$. Because $P(Y_j = 1|\boldsymbol{\alpha}) = 1 - s_j$ if $\omega_j = 1$, and $P(Y_j = 1|\boldsymbol{\alpha}) = g_j$ if $\omega_j = 0$, $\beta_{j0} = g_j$, and $\beta_{j0} + \beta_{jk} = 1 - s_j$, for some $k \in \mathcal{L}_j$, which implies $\beta_{jk} = 1 - s_j - g_j$. As the slipping and guessing parameters are restricted to $0 \leq g_j, s_j < 0.5$, the three constraints follow directly.

Claim For the DINO and DINA models, a $J \times K$ matrix $\mathbf{Q}$ is complete if and only if it contains the K vectors, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$, of all single-attribute items among its J rows.
Proof:
($\Rightarrow$) The expected response of the DINO model is

$$S_j(\boldsymbol{\alpha}) = \beta_{j0} + \beta_{jk} - \beta_{jk} \prod_{l \in \mathcal{L}_j} (1 - \alpha_l),$$

for some $k \in \mathcal{L}_j$. Let attribute profiles $\boldsymbol{\alpha} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $\boldsymbol{\alpha}' = \mathbf{e}_k^c$, where $\mathbf{e}_k^c$ is the complement vector of $\mathbf{e}_k$, with the kth element equal to 0 and all other elements equal to 1. For some $k' \neq k$,

$$S_j(\boldsymbol{\alpha}) = \beta_{j0} + \beta_{jk'}$$

and

$$S_j(\boldsymbol{\alpha}') = \begin{cases} \beta_{j0}, & \text{if } \mathbf{q}_j = \mathbf{e}_k, \\ \beta_{j0} + \beta_{jk'}, & \text{otherwise.} \end{cases}$$

Assume that $\mathbf{e}_k$ is missing from $\mathbf{Q}$. Then, $S_j(\boldsymbol{\alpha}) = S_j(\boldsymbol{\alpha}') = \beta_{j0} + \beta_{jk'}$ for all j, which implies that $\mathbf{S}(\boldsymbol{\alpha}) = \mathbf{S}(\boldsymbol{\alpha}')$. Therefore, $\mathbf{Q}$ is not complete.

For the DINA model, consider the attribute profiles $\boldsymbol{\alpha} = (0, 0, \ldots, 0)^{\mathrm{T}}$ and $\boldsymbol{\alpha}' = \mathbf{e}_k$. Thus, for all j,

$$S_j(\boldsymbol{\alpha}) = \beta_{j0}$$
$$S_j(\boldsymbol{\alpha}') = \begin{cases} \beta_{j0} + \beta_{jk}, & \text{if } \mathbf{q}_j = \mathbf{e}_k, \\ \beta_{j0}, & \text{otherwise.} \end{cases}$$

Assume that $\mathbf{e}_k$ is missing from $\mathbf{Q}$; thus, $S_j(\boldsymbol{\alpha}) = S_j(\boldsymbol{\alpha}') = \beta_{j0}$ for all j, which implies that $\mathbf{S}(\boldsymbol{\alpha}) = \mathbf{S}(\boldsymbol{\alpha}')$. Hence, $\mathbf{Q}$ is not complete. □

($\Leftarrow$) For the DINO and DINA models, the proof of sufficiency is identical to the general proof of Lemma 1 presented earlier and is therefore repeated only in abbreviated form.

Assume that the first K of the J rows of $\mathbf{Q}$ consist of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$. Thus,

$$S_k(\boldsymbol{\alpha}) = \begin{cases} \beta_{k0} + \beta_{kk}, & \text{if } \alpha_k = 1, \\ \beta_{k0}, & \text{if } \alpha_k = 0. \end{cases} \tag{13}$$

Consider $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ such that $\alpha_k = 1$ and $\alpha'_k = 0$. Then, according to equation (13), $S_k(\boldsymbol{\alpha}) > S_k(\boldsymbol{\alpha}')$ because $\beta_{kk} > 0$, which implies that $\mathbf{S}_{1:K}(\boldsymbol{\alpha}) \neq \mathbf{S}_{1:K}(\boldsymbol{\alpha}')$. Hence, $\mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$, regardless of whether $\mathbf{S}_{(K+1):J}(\boldsymbol{\alpha})$ is identical to $\mathbf{S}_{(K+1):J}(\boldsymbol{\alpha}')$. Therefore, $\mathbf{Q}$ is complete. □

In summary, for the DINO and DINA models the inclusion of all single-attribute items is required as a necessary (i.e., indispensable) condition to guarantee completeness of the Q-matrix. Thus, if a researcher wants to analyse data from a test containing no or too few single-attribute items with one of these models, then he or she must be prepared for an incomplete Q-matrix, resulting in lack of identifiability of all proficiency classes. (Recall that completeness of the Q-matrix is a required condition for identifiability, regardless of whether parametric or nonparametric methods are used to assign examinees to proficiency classes.)

6. A general proof of Lemma 2 (separation of proficiency-class centres)

Lemma 2 of the ACTCD states that, if the Q-matrix is complete, then for two attribute profiles, $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, the centres of the corresponding proficiency classes are guaranteed to be well separated.

For educational data conforming to the DINA model, Chiu *et al.* (2009) used the K-dimensional vector of attribute-related sum scores, $\mathbf{W} = (W_1, \ldots, W_K)^{\mathrm{T}} = \mathbf{YQ}$, as a statistic for $\boldsymbol{\alpha}$. The conditional expectation of the attribute sum-score profile $\mathbf{W}$, $\mathbf{T}(\boldsymbol{\alpha}) =$

$E(\mathbf{W}|\boldsymbol{\alpha})$, corresponds to the centre of the proficiency class characterized by $\boldsymbol{\alpha}$. Chiu *et al.* (2009; pp. 643–644) proved that, if $\mathbf{Q}$ is complete, then $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{T}(\boldsymbol{\alpha}) \neq \mathbf{T}(\boldsymbol{\alpha}')$.

However, for other CDMs like the Reduced RUM, the GDM, or the A-CDM (de la Torre, 2011), $\mathbf{W}$ is not a legitimate statistic for $\boldsymbol{\alpha}$ because distinct proficiency classes can have identical conditional expectations of $\mathbf{W}$, which invalidates Lemma 2 – that is, $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{T}(\boldsymbol{\alpha}) \neq \mathbf{T}(\boldsymbol{\alpha}')$. (A proof of this claim for the A-CDM is provided in the Appendix) The inability of the attribute sum-score profile, $\mathbf{W}$, to guarantee well-separated proficiency-class centres can be remedied by using instead an *augmented attribute sum-score profile*, $\mathbf{W}_{\mathrm{aug}}$, as a statistic for $\boldsymbol{\alpha}$ defined as $\mathbf{W}_{\mathrm{aug}} = \mathbf{Y}\mathbf{Q}_{\mathrm{aug}}$. The $J \times 2K$ partitioned matrix $\mathbf{Q}_{\mathrm{aug}} = (\mathbf{Q}|\mathbf{Q}_{\mathbf{e}})$ is obtained by augmenting the Q-matrix with the $J \times K$ matrix $\mathbf{Q}_{\mathbf{e}}$. The latter is constructed by retaining the row vectors $\mathbf{q}_j = \mathbf{e}_k$ in $\mathbf{Q}$ and replacing the remaining $\mathbf{q}_j$ by (0,0,. . .,0). So, $\mathbf{Q}_{\mathrm{aug}}$ preserves the information in the original Q-matrix, but increases the impact of the single-attribute items in a test. Using $\mathbf{W}_{\mathrm{aug}}$ as a statistic for $\boldsymbol{\alpha}$ enhances their effect, resulting in well-separated proficiency-class centres.

Lemma 2 Define the augmented attribute sum-score statistic for $\boldsymbol{\alpha}$ as $\mathbf{W}_{\mathrm{aug}} = \mathbf{Y}\mathbf{Q}_{\mathrm{aug}} = \mathbf{Y}(\mathbf{Q}|\mathbf{Q}_{\mathbf{e}}) = (\mathbf{W}|\mathbf{W}_{\mathbf{e}})$ and its expectation as $\mathbf{T}_{\mathrm{aug}}(\boldsymbol{\alpha}) = E(\mathbf{W}_{\mathrm{aug}}|\boldsymbol{\alpha}) = E((\mathbf{W}|\mathbf{W}_{\mathbf{e}})|\boldsymbol{\alpha}) = (\mathbf{T}(\boldsymbol{\alpha})|\mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha}))$. Assume that $\mathbf{Q}$ is complete. If $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, then $\mathbf{T}_{\mathrm{aug}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\mathrm{aug}}(\boldsymbol{\alpha}')$.
Proof:
First, consider the general CDM using the identity link (see equation (5)). For a single-attribute item j (i.e., $\mathbf{q}_j = \mathbf{e}_k$), equation (6) showed that the expected item response is

$$S_j(\boldsymbol{\alpha}) = \begin{cases} \beta_{j0} + \beta_{jk}, & \text{if } \alpha_k = 1, \\ \beta_{j0}, & \text{if } \alpha_k = 0. \end{cases}$$

Define on $\mathbf{Q}$ the set $\mathcal{B}_{\mathbf{e}_k} = \{j|\mathbf{q}_j = \mathbf{e}_k\}$, which allows the kth element of $\mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha})$ to be written as

$$T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}) = \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} E(Y_j|\boldsymbol{\alpha}) q_{jk} = \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} S_j(\boldsymbol{\alpha}) q_{jk} = \begin{cases} \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} \beta_{j0} + \beta_{jk}, & \text{if } \alpha_k = 1, \\ \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} \beta_{j0}, & \text{if } \alpha_k = 0. \end{cases} \tag{14}$$

(Recall that $\mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha}) = E(\mathbf{W}_{\mathbf{e}}|\boldsymbol{\alpha}) = E(\mathbf{Y}|\boldsymbol{\alpha})\mathbf{Q}_{\mathbf{e}} = \mathbf{S}(\boldsymbol{\alpha})\mathbf{Q}_{\mathbf{e}}$.) If $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ must differ in at least one element k, say $\alpha_k = 1$ and $\alpha'_k = 0$. As $\mathbf{Q}$ is complete, $\mathcal{B}_{\mathbf{e}_k} \neq \emptyset$ so that at least one β_{jk} is retained in equation (14) if $\alpha_k = 1$. Thus, $T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}) > T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}')$ because $0 < \beta_{jk} \leq 1$ for all j. Therefore, $\mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha}')$, and consequently, $\mathbf{T}_{\mathrm{aug}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\mathrm{aug}}(\boldsymbol{\alpha}')$ independent of whether $\mathbf{T}(\boldsymbol{\alpha}) = \mathbf{T}(\boldsymbol{\alpha}')$. □

The same argument applies when the logit link and the log link are used (the corresponding item response functions were defined by equations (7) and (9)). The expected responses to a single-attribute item j, with $\mathbf{q}_j = \mathbf{e}_k$, are

$$S_j(\boldsymbol{\alpha}) = \begin{cases} \frac{\exp\{\beta_{j0}+\beta_{jk}\}}{1+\exp\{\beta_{j0}+\beta_{jk}\}}, & \text{if } \alpha_k = 1, \\ \frac{\exp\{\beta_{j0}\}}{1+\exp\{\beta_{j0}\}}, & \text{if } \alpha_k = 0, \end{cases}$$

for the logit link (see equation (8)), and

$$S_j(\boldsymbol{\alpha}) = \begin{cases} \exp\{\beta_{j0} + \beta_{jk}\}, & \text{if } \alpha_k = 1, \\ \exp\{\beta_{j0}\}, & \text{if } \alpha_k = 0, \end{cases}$$

for the log link (see equation (10)). The set $\mathcal{B}_{\mathbf{e}_k}$ defined on $\mathbf{Q}$ allows the kth element of $\mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha})$ to be written for the logit link as

$$T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}) = \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} S_j(\boldsymbol{\alpha}) q_{jk} = \begin{cases} \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} \frac{\exp\{\beta_{j0}+\beta_{jk}\}}{1+\exp\{\beta_{j0}+\beta_{jk}\}}, & \text{if } \alpha_k = 1, \\ \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} \frac{\exp\{\beta_{j0}\}}{1+\exp\{\beta_{j0}\}}, & \text{if } \alpha_k = 0, \end{cases} \tag{15}$$

and for the log link as

$$T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}) = \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} S_j(\boldsymbol{\alpha}) q_{jk} = \begin{cases} \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} \exp\{\beta_{j0} + \beta_{jk}\}, & \text{if } \alpha_k = 1, \\ \sum_{j \in \mathcal{B}_{\mathbf{e}_k}} \exp\{\beta_{j0}\}, & \text{if } \alpha_k = 0, \end{cases} \tag{16}$$

Consider $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ differing in at least one element k, $\alpha_k = 1$ and $\alpha'_k = 0$. Completeness of $\mathbf{Q}$ implies that $\mathcal{B}_{\mathbf{e}_k} \neq \emptyset$; hence, at least one β_{jk} is kept in equations (15) and (16). Therefore, $T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}) > T_{\mathbf{e}_{(k)}}(\boldsymbol{\alpha}')$ because the logistic and the exponential functions are monotonically increasing. Consequently, $\mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\mathbf{e}}(\boldsymbol{\alpha}')$ and $\mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}')$, regardless of whether $\mathbf{T}(\boldsymbol{\alpha}) = \mathbf{T}(\boldsymbol{\alpha}')$, which completes the proof.

In summary, the general proof of Lemma 2 demonstrates that, if $\mathbf{Q}$ is complete, then using $\mathbf{W}_{\text{aug}}$ as a statistic for $\boldsymbol{\alpha}$ guarantees well-separated proficiency classes for any CDM derivable from one of the three prototypes of general CDMs: $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}')$.

7. Discussion and conclusion

Three prototypes of general CDMs can be distinguished based on the link function (i.e., identity, logit, and log link) used with the kernel, $g(\mathbf{q}_j, \boldsymbol{\alpha})$, to construct the item response function. Specific CDMs, whether recognizable models or models that have not yet been identified, can be derived from these prototypes by constraining the parameter vector, $\boldsymbol{\beta}_j$. Lemmas 1 and 2 of the ACTCD were proved for the three prototypes of general CDMs. The general proof of Lemma 1 demonstrated that, if the Q-matrix contains the attribute profiles, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_K$, of all single-attribute items, then it is guaranteed to be complete for any CDM that can be derived from one of the prototypes of general CDMs: $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$ – that is, the identifiability of the $M = 2^K$ proficiency classes is guaranteed. The general proof of Lemma 2 demonstrated that, if the Q-matrix is complete, then $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}')$ immediately follows – that is, the centres of the $M = 2^K$ proficiency classes are guaranteed to be well separated for any CDM that can be derived from one of the prototypes of general CDMs, provided the augmented attribute sum-score profile, $\mathbf{W}_{\text{aug}}$, is used as a statistic for $\boldsymbol{\alpha}$. Thus, Lemmas 1 and 2 can be used with Lemma 3 for a general proof of the consistency theorem of classification: the probability that examinees are correctly assigned to their true proficiency classes based on $\mathbf{W}_{\text{aug}}$ approaches 1 for any CDM derivable from one of the prototypes of general CDMs as the length of a test (i.e., the number of items, J) increases. (This proof is omitted here because it is nearly identical to that given by Chiu *et al.*, 2009, pp. 645–647, requiring only the trivial change of doubling the number of columns in the Q-matrix from K to $2K$ to accommodate $\mathbf{W}_{\text{aug}}$.)

This summarizes the general aspects of the proofs presented; two points remain to be addressed in greater detail. The general proof of Lemma 1 established the sufficiency condition for Q-completeness: if the attribute profiles of the single-attribute items, $\mathbf{e}_1, \ldots, \mathbf{e}_K$, are contained in the Q-matrix, then it is guaranteed to be complete for any CDM. But this condition might be difficult to meet in practice because, for certain domains, it may simply be impossible to construct genuine single-attribute items that represent all K attributes. This can become an issue especially when existing data that were originally collected with a test not based on the cognitive diagnosis framework are analysed *post hoc* with a CDM (*retro-fitting*; Leighton & Gierl, 2007). Then, it is not unlikely that the original test contains no or too few single-attribute items. The classic example is the famous fraction-subtraction data set (Tatsuoka, 1984) that has been retro-fitted by several researchers with the DINA model (see de la Torre, 2008 2009; de la Torre & Douglas, 2008; DeCarlo, 2011; Mislevy, 1996; Tatsuoka, 2002). But for the DINA model, the Q-matrix of the fraction-subtraction data is incomplete because the underlying test has only a few single-attribute items. (DeCarlo, 2011, proposed several interesting strategies for analysing educational data when the Q-matrix is incomplete.)

The good news, however, is that not all recognizable CDMs require the inclusion of the single-attribute items for the Q-matrix to be complete. For example, for the GDM, the NIDA model, the generalized NIDA model, the Reduced RUM, and the A-CDM, the inclusion of the single-attribute items in the Q-matrix is a sufficient but not a necessary condition for completeness. For these models, other compositions of the Q-matrix that do not contain the single-attribute items can guarantee completeness and thus, identifiability of the proficiency classes, as was shown by an example. This example also suggested that Q-completeness is preserved as long as the attribute main effects are included in the model – a property that is shared by all these models.

In summary, when using non-parametric classification methods as heuristics for assigning examinees to proficiency classes, three guidelines can be given. First, any CDM derivable from one of the prototypes of general CDMs is covered by the ACTCD if the Q-matrix contains all single-attribute items. Second, if the single-attribute items are not included in the Q-matrix, then completeness of the Q-matrix typically still holds if a CDM contains the attribute main effects (as is the case with many recognizable CDMs). Then coverage by the ACTCD follows because of the logical connection between Lemma 1, $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$, and Lemma 2, $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}) \neq \mathbf{T}_{\text{aug}}(\boldsymbol{\alpha}')$. Third, this connection also suggests that if it is difficult to determine whether a CDM is covered by the ACTCD, then, after the conversion to a general CDM, it suffices to check whether $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{S}(\boldsymbol{\alpha}) \neq \mathbf{S}(\boldsymbol{\alpha}')$ is satisfied, because then coverage by the ACTCD is guaranteed (as the examples of the DINA and DINO models showed).

What further recommendations can be given to practitioners who wish to use general CDMs in their testing programs and empirical research? The ulterior goal of general CDMs is to establish a standard for comparison and evaluation of different CDMs based on a unified parametrization. Thus, general CDMs have been proposed as a theoretical framework for identifying the best-fitting model among candidate CDMs when information on the true CDM underlying the data is not available (e.g., Rupp *et al.*, 2010; Templin & Bradshaw, 2014), which is often the case in practice. (A misspecified model causes examinees to be assigned to the wrong proficiency classes.) What are the computational options for educational researchers who wish to use general CDMs for model identification, but do not feel comfortable writing their own code? Two choices exist for (exact) MLE methods: MCMC and EM. If a researcher wants to use the former, then he or she can refer to the MCMC routine implemented, for example, in OpenBUGS (Lunn, Spiegelhalter, Thomas, & Best, 2009). Alternatively, general CDMs can be fitted by the EM algorithm using the implementation in a commercial package for fitting (constrained) latent class models, for example, Latent GOLD (Vermunt & Magidson, 2000) and Mplus (Muthén & Muthén, 1998–2012). Latent class analysis routines typically require the general CDM to be re-expressed as a logit model, with constraints imposed on the parameters of the logistic function (for details, consult Rupp *et al.*, 2010). The stand-alone package mdltm by von Davier (2005, 2006) provides an implementation of the EM algorithm for fitting the GDM.

Fit statistics for model comparisons were reviewed in Rupp *et al.* (2010). The available procedures either required unrealistic CPU times (e.g., chi-squared test of model fit), advanced skills in statistical computing (e.g., posterior predictive model checking), or their applicability in cognitively diagnostic modelling had not (yet) been well researched (e.g., uni- and multivariate limited-information fit statistics). As Rupp *et al.* (2010) concluded, practical methods for model-fit evaluation in cognitively diagnostic modelling "remain elusive, as many research questions about their optimum performance are just now being explored in depth" (p. 272) – an assessment that also seems applicable to recently discussed methods, such as relative and absolute fit indices for comparing saturated general CDMs and reduced models (Chen, de la Torre, & Zhang, 2013), and the Wald test for model comparison at the level of individual items (de la Torre & Lee, 2013). In summary, researchers should be cautious when considering the general CDM framework as a means for identifying the best-fitting CDM as long as the statistical test procedures necessary for comparing multiple candidate CDMs have not yet been fully developed and evaluated and the statistical theory for comparing the fit of various CDMs is still at the development stage. At the same time, one should recall that MLE methods are particularly vulnerable to the misspecification of the CDM believed to underlie the data: using the wrong CDM causes examinees to be assigned to proficiency classes to which they do not belong.

As an alternative to the model identification procedures based on general CDMs, non-parametric classification using hierarchical agglomerative cluster analysis (HACA) is a viable approximate method for assigning examinees to proficiency classes when the true CDM underlying the data is unknown or cannot properly be identified, especially as the logical requirement for using HACA as a heuristic – the coverage of general CDMs by the ACTCD – is now fulfilled. The R package ACTCD (Chiu & Ma, 2013) provides the functions used to perform HACA for cognitive diagnosis. Simulation studies showed (e.g., for the DINA model; Chiu *et al.*, 2009) that, depending on the number of attributes K and the complexity of the CDM underlying the data, a test should contain at least about 30–40 items to ensure that HACA assigns examinees to their true proficiency classes at a reasonable rate. (A test consisting of only J=20 items is definitely too short.)

By definition, non-parametric classification using HACA cannot provide (item) parameter estimates and their standard errors. Thus, these methods are suitable for the classification of examinees, but should not be considered when the evaluation of individual items and specific models is the primary goal of the analysis. In addition, educational practitioners interested in determining the proficiency classes of examinees by using HACA should be aware that the resulting examinee clusters serve as proxies for the proficiency classes. But HACA methods cannot estimate the attribute profiles underlying the clusters. Hence, the clusters must be interpreted or *labelled* – that is, their underlying attribute profiles must be reconstructed from the chosen input data, which can be tedious if the number of examinees is large. Chiu *et al.* (2009; pp. 661–663) proposed an automatic cluster labelling algorithm when $\mathbf{W}$ is used as input to clustering. This algorithm has been further developed so that now also $\mathbf{W}_{\mathrm{aug}}$ is accommodated as input to clustering such that an optimal match is sought between candidate cluster attribute profiles and examinees' within-cluster augmented sum-score profiles.

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Appendix: The deficiency of W as a statistic for $\boldsymbol{\alpha}$ for the additive cognitive diagnosis model

The response function of the A-CDM using the identity link is

$$P(Y_j = 1 \mid \boldsymbol{\alpha}) = \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} q_{jk} \alpha_k.$$

In certain cases, the sum-score statistic, **W**, fails to guarantee well-separated proficiency-class centres for the A-CDM, which invalidates Lemma 2: $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Rightarrow \mathbf{T}(\boldsymbol{\alpha}) \neq \mathbf{T}(\boldsymbol{\alpha}')$. As an example, consider a test with 17 items underlain by three attributes. The Q-matrix and the parameter settings are as follows:

		Parameter			
Frequency of item type	Item-attribute profile	β_{j0}	β_{j1}	β_{j2}	β_{j3}
2	1 0 0	0.269	0.313	0	0
2	0 1 0	0.269	0	0.354	0
2	0 0 1	0.450	0	0	0.172
2	1 1 0	0.450	0.045	0.025	0
2	1 0 1	0.147	0.765	0	0.172
2	0 1 1	0.162	0	0.633	0.080
5	1 1 1	0.555	0.025	0.025	0.440

For attribute profiles $\boldsymbol{\alpha} = (0,0,1)^{\mathrm{T}}$ and $\boldsymbol{\alpha}' = (1,1,0)^{\mathrm{T}}$, identical proficiency-class centres are obtained: $\mathbf{T}((0,0,1)) = \mathbf{T}((1,1,0)) = (7.05, 6.90, 7.34)^{\mathrm{T}}$. Detailed calculations are provided only for $\mathbf{T}((0,0,1))$:

$$\begin{aligned} T_1((0,0,1)) &= 2\beta_{10} + 2\beta_{40} + 2(\beta_{50} + \beta_{53}) + 5(\beta_{70} + \beta_{73}) \\ &= 2 \times 0.269 + 2 \times 0.450 + 2(0.147 + 0.172) + 5(0.555 + 0.440) \\ &= 7.051, \end{aligned}$$

$$\begin{aligned} T_2((0,0,1)) &= 2\beta_{20} + 2\beta_{40} + 2(\beta_{60} + \beta_{63}) + 5(\beta_{70} + \beta_{73}) \\ &= 2 \times 0.269 + 2 \times 0.450 + 2(0.162 + 0.080) + 5(0.555 + 0.440) \\ &= 6.897, \end{aligned}$$

$$\begin{aligned} T_3((0,0,1)) &= 2(\beta_{30} + \beta_{33}) + 2(\beta_{50} + \beta_{53}) + 2(\beta_{60} + \beta_{63}) + 5(\beta_{70} + \beta_{73}) \\ &= 2(0.450 + 0.172) + 2(0.147 + 0.172) + 2(0.162 + 0.080) + \\ &\quad 5(0.555 + 0.440) \\ &= 7.341. \end{aligned}$$

The various conditions under which **W** fails to guarantee well-separated proficiency-class centres for distinct attribute profiles are studied below. The concept of *nested* attribute profiles is crucial for the subsequent proofs. When $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, attribute profile $\boldsymbol{\alpha}$ is said to be *nested* within attribute profile $\boldsymbol{\alpha}'$ if there exist some k and k' ($k \neq k'$; $k, k' \in \{1, 2, \ldots, K\}$) such that $\alpha_k = 0$, $\alpha'_k = 1$, and $\alpha_{k'} \leq \alpha'_{k'}$. The Q-matrix is assumed to contain at least one of each of the possible $2^K - 1$ item-attribute profiles.

Scenario I: $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}$ is nested within $\boldsymbol{\alpha}'$

The expected item response for the A-CDM is

$$S_j(\boldsymbol{\alpha}) = E(Y_j|\boldsymbol{\alpha}) = \beta_{j0} + \sum_{k=1}^{K} \beta_{jk} q_{jk} \alpha_k.$$

Hence,

$$T_k(\boldsymbol{\alpha}) = \sum_{j=1}^{J} \left(\beta_{j0} + \sum_{l=1, l\neq k}^{K} \beta_{jl} q_{jl} \alpha_l \right) I(q_{jk} = 1) \tag{A17}$$

and

$$T_k(\boldsymbol{\alpha}') = \sum_{j=1}^{J} \left(\beta_{j0} + \sum_{l=1}^{K} \beta_{jl} q_{jl} \alpha'_l \right) I(q_{jk} = 1), \tag{A18}$$

where $I(\cdot)$ denotes the indicator function. By assuming that (a) the slope, $\beta_{jk} > 0$, $\alpha'_k = 1$ and (b) $\alpha_{k'} \leq \alpha'_{k'}$ for all $k' \neq k$, the term $\sum_{l=1}^{K} \beta_{jl} q_{jl} \alpha'_l$ in equation (A18) can be decomposed into

$$\sum_{l=1}^{K} \beta_{jl} q_{jl} \alpha'_l = \sum_{l=1, l\neq k}^{K} \beta_{jl} q_{jl} \alpha'_l + \beta_{jk} \geq \sum_{l=1, l\neq k}^{K} \beta_{jl} q_{jl} \alpha_l + \beta_{jk} > \sum_{l=1, l\neq k}^{K} \beta_{jl} q_{jl} \alpha_l. \tag{A19}$$

According to equations (A17–19), $T_k(\boldsymbol{\alpha}) < T_k(\boldsymbol{\alpha}')$, which implies that $\mathbf{T}(\boldsymbol{\alpha}) \neq \mathbf{T}(\boldsymbol{\alpha}')$. Thus, as long as the attribute profiles of two proficiency classes are nested, their centres are guaranteed to be separated.

Scenario II: $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}$ is not nested within $\boldsymbol{\alpha}'$

If $\boldsymbol{\alpha}$ is not nested within $\boldsymbol{\alpha}'$, then there must be some k and k' ($k \neq k'$; $k, k' \in \{1, 2, \ldots, K\}$) such that $\alpha_k = 1$ and $\alpha_{k'} = 0$; $\alpha'_k = 0$ and $\alpha'_{k'} = 1$. Hence, the kth components of $\mathbf{T}(\boldsymbol{\alpha})$ and $\mathbf{T}(\boldsymbol{\alpha}')$ equal

$$\begin{aligned}
T_k(\boldsymbol{\alpha}) &= \sum_{j=1}^{J} \left(\beta_{j0} + \beta_{jk} + \sum_{l=1, l\neq k}^{K} \beta_{jl} q_{jl} \alpha_l \right) I(q_{jk} = 1) \\
&= \sum_{j=1}^{J} \left(\beta_{j0} + \beta_{jk} + \sum_{l=1, l\neq k,k'}^{K} \beta_{jl} q_{jl} \alpha_l \right) I(q_{jk} = 1, q_{jk'} = 0) \\
&\quad + \sum_{j=1}^{J} \left(\beta_{j0} + \beta_{jk} + \sum_{l=1, l\neq k,k'}^{K} \beta_{jl} q_{jl} \alpha_l \right) I(q_{jk} = 1, q_{jk'} = 1), \\
T_k(\boldsymbol{\alpha}') &= \sum_{j=1}^{J} \left(\beta_{j0} + \sum_{l=1, l\neq k}^{K} \beta_{jl} q_{jl} \alpha'_l \right) I(q_{jk} = 1) \\
&= \sum_{j=1}^{J} \left(\beta_{j0} + \sum_{l=1, l\neq k,k'}^{K} \beta_{jl} q_{jl} \alpha'_l \right) I(q_{jk} = 1, q_{jk'} = 0) \\
&\quad + \sum_{j=1}^{J} \left(\beta_{j0} + \beta_{jk'} + \sum_{l=1, l\neq k,k'}^{K} \beta_{jl} q_{jl} \alpha'_l \right) I(q_{jk} = 1, q_{jk'} = 1).
\end{aligned} \tag{A20}$$

The components k' of $\mathbf{T}(\boldsymbol{\alpha})$ and $\mathbf{T}(\boldsymbol{\alpha}')$ are

$$\begin{aligned}
T_{k'}(\boldsymbol{\alpha}) &= \sum_{j=1}^{J}\left(\beta_{j0} + \sum_{l=1,l\neq k'}^{K}\beta_{jl}q_{jl}\alpha_l\right)I(q_{jk'}=1)\\
&= \sum_{j=1}^{J}\left(\beta_{j0} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l\right)I(q_{jk'}=1, q_{jk}=0)\\
&+ \sum_{j=1}^{J}\left(\beta_{j0} + \beta_{jk} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l\right)I(q_{jk'}=1, q_{jk}=1),\\
T_{k'}(\boldsymbol{\alpha}') &= \sum_{j=1}^{J}\left(\beta_{j0} + \beta_{jk'} + \sum_{l=1,l\neq k'}^{K}\beta_{jl}q_{jl}\alpha'_l\right)I(q_{jk'}=1)\\
&= \sum_{j=1}^{J}\left(\beta_{j0} + \beta_{jk'} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l\right)I(q_{jk'}=1, q_{jk}=0)\\
&+ \sum_{j=1}^{J}\left(\beta_{j0} + \beta_{jk'} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l\right)I(q_{jk'}=1, q_{jk}=1).
\end{aligned} \tag{A21}$$

Note that the right-hand sides of the pairs of equations in (A20) and (A21) contain the sums of triple products involving the remaining entries, α_l and α'_l, with $l \neq k, k'$. However, there is no way to predict which specific elements, α_l, α'_l, equal 0 or 1. Therefore, it is impossible (as opposed to the case of the nested attribute profiles) to determine whether any equality relation exists between the right-hand sides of the pair of equations in either (A20) or (A21). Consequently, no conclusions about the equality or inequality of the components $T_k(\boldsymbol{\alpha})$ and $T_k(\boldsymbol{\alpha}')$ or the components $T_{k'}(\boldsymbol{\alpha})$ and $T_{k'}(\boldsymbol{\alpha}')$ can be drawn, and no conclusive statement as to whether $\mathbf{T}(\boldsymbol{\alpha})$ equals $\mathbf{T}(\boldsymbol{\alpha}')$ can be made when attribute profiles are not nested, with the exception of the special case described next.

Special case: $||\boldsymbol{\alpha}||^2 = ||\boldsymbol{\alpha}'||^2 = K'$; $K' = 1, 2, \ldots, K-1$, when $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ is not nested within $\boldsymbol{\alpha}'$. If two attribute profiles, $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$, have the same length, $||\boldsymbol{\alpha}||^2 = ||\boldsymbol{\alpha}'||^2 = K'$, with $K' = 1, 2, \ldots, K-1$, but differ in only two elements, k and k' (i.e., all other elements in $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ must be identical), then the following equality holds:

$$\sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l = \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l. \tag{A22}$$

From equation (A22) it follows that (a) for the the kth components of $\mathbf{T}(\boldsymbol{\alpha})$ and $\mathbf{T}(\boldsymbol{\alpha}')$ in equation (A20),

$$\beta_{j0} + \beta_{jk} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l > \beta_{j0} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l; \tag{A23}$$

and (b) for the the components k' of $\mathbf{T}(\boldsymbol{\alpha})$ and $\mathbf{T}(\boldsymbol{\alpha}')$ in equation (21),

$$\beta_{j0} + \beta_{jk'} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l > \beta_{j0} + \sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l. \tag{A24}$$

Recall that whether $T_k(\boldsymbol{\alpha})$ equals $T_k(\boldsymbol{\alpha}')$ or whether $T_{k'}(\boldsymbol{\alpha})$ equals $T_{k'}(\boldsymbol{\alpha}')$ (and thus, whether $\mathbf{T}(\boldsymbol{\alpha})$ equals $\mathbf{T}(\boldsymbol{\alpha}')$) cannot be determined. However, two possibilities can be distinguished.

First, if it is assumed that $T_k(\boldsymbol{\alpha}) \neq T_k(\boldsymbol{\alpha}')$, no further examination of $T_{k'}(\boldsymbol{\alpha})$ and $T_{k'}(\boldsymbol{\alpha}')$ is necessary because then $\mathbf{T}(\boldsymbol{\alpha}) \neq \mathbf{T}(\boldsymbol{\alpha}')$ must hold. If, on the other hand, it is assumed that $T_k(\boldsymbol{\alpha}) = T_k(\boldsymbol{\alpha}')$, then due to (A23) it must be true that

$$\begin{aligned}&\sum_{j=1}^{J}\left(\beta_{j0}+\beta_{jk}+\sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l\right)I(q_{jk}=1,q_{jk'}=1)\\ &<\sum_{j=1}^{J}\left(\beta_{j0}+\beta_{jk'}+\sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l\right)I(q_{jk}=1,q_{jk'}=1),\end{aligned} \tag{A25}$$

otherwise the equality assumption is violated. By inspecting $T_{k'}(\boldsymbol{\alpha})$ and $T_{k'}(\boldsymbol{\alpha}')$, it can be deduced from equations (A24) and (A25) that $T_{k'}(\boldsymbol{\alpha}) < T_{k'}(\boldsymbol{\alpha}')$, which implies that $\mathbf{T}(\boldsymbol{\alpha}) \neq \mathbf{T}(\boldsymbol{\alpha}')$.

In summary, with the exception of the case just discussed, **W** cannot guarantee well-separated proficiency-class centres for data conforming to the A-CDM unless the attribute profiles are nested.

Closer examination of the pairs of equations in (A20) and (A21) suggests the reason for this lack of predictability. Note that the summation terms driving all four equations link the distinct attribute profiles directly to the item-attribute requirements specified in the Q-matrix. These summation terms appear in, or disappear from, the equations depending on whether the attribute profile of a given item j meets the condition defined by the associated indicator function. For example, if j represents a single-attribute item (i.e., the condition $[q_{jk}=1, q_{jk'}=1]$ is violated), then the second summation terms on the right-hand side in all four equations in (A20) and (A21),

$$\sum_{j=1}^{J}(\beta_{j0}+\beta_{jk}+\sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha_l)I(q_{jk}=1,q_{jk'}=1)$$

or

$$\sum_{j=1}^{J}(\beta_{j0}+\beta_{jk'}+\sum_{l=1,l\neq k,k'}^{K}\beta_{jl}q_{jl}\alpha'_l)I(q_{jk}=1,q_{jk'}=1),$$

vanish. In addition, any single-attribute item j always satisfies the condition $(q_{jk}=1,$ $q_{jk'}=0)$ or $(q_{jk}=0, q_{jk'}=1)$ of the indicator functions associated with the first summation terms on the right-hand side of the equations, while all remaining elements in $\mathbf{q}_j$ (i.e., with index $l \neq k, l \neq k'$) equal 0 by definition, so the entire summation term reduces to 1. Therefore, for single-attribute items, $T_k(\boldsymbol{\alpha}) \neq T_k(\boldsymbol{\alpha}')$ and $T_{k'}(\boldsymbol{\alpha}) \neq T_{k'}(\boldsymbol{\alpha}')$ are always satisfied; hence, $\mathbf{T}(\boldsymbol{\alpha})$ can never equal $\mathbf{T}(\boldsymbol{\alpha}')$.

These results suggest solving the proficiency-class separation problem of the A-CDM by increasing the impact of the single-attribute items in the Q-matrix. Specifically, the Q-matrix can be augmented by a matrix of the same dimensionality, $\mathbf{Q_e}$, resulting in the $J \times 2K$ partitioned matrix $\mathbf{Q}_{\text{aug}} = (\mathbf{Q}|\mathbf{Q_e})$. The matrix $\mathbf{Q_e}$ is constructed by retaining the row vectors $\mathbf{q}_j = \mathbf{e}_k$ in $\mathbf{Q}$ and replacing the remaining $\mathbf{q}_j$ by (0,0,. . .,0). So, $\mathbf{Q}_{\text{aug}}$ preserves the information in the original Q-matrix, but increases the impact of the single-attribute items in a test.