Fuzzy Mathematics and Its in Technology

By

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- A Bit of History
- Fuzzy Thinking
- Fuzzy Sets and Some Examples
- Fuzzy Boundaries , Crisp sets v/s Fuzzy sets
- Fuzzy Sets Operations
- Fuzzy Set Representation
- Fuzzy Relations
- Linguistic Variables and Hedges
- Fuzzy Applications
- Working of Washing machine using fuzzy logic

SET THEORY

GEORGE CANTOR

George Cantor, in 1870's, gave the concept of set theory which is of great importance in mathematics.

- GEORGE CANTOR:His Mathematics and philosophy of infinite,Boston.

Fuzzy Thinking

- The concept of a set and set theory are powerful concepts in mathematics. However, the principal notion underlying set theory, that an element can (exclusively) either belong to set or not belong to a set, makes it well nigh impossible to represent much of human discourse. How is one to represent notions like:
  - large profit
  - high pressure
  - tall man
  - moderate temperature
- Ordinary set-theoretic representations will require the maintenance of a crisp differentiation in a very artificial manner:
  - high
  - not quite high
  - very high … etc.

FUZZY SET THEORY

LOTFI ZADEH

- Fuzzy Set Theory was formalised by Professor Lotfi Zadeh at the University of California in 1965 to generalise classical set theory. Zadeh was almost single handedly responsible for the early development in this field.

REFERENCES:

- Zadeh L.A.(1965)Fuzzy sets. Information and Control,8(1965),338-353.
- Zadeh L.A.(1978)Fuzzy Sets as the Basis for a Theory of Possibility. Fuzzy Sets and Systems

A Fuzzy Set has Boundaries

- Let X be the universe of discourse and its elements be denoted as x. In the classical set theory, **crisp set A of X is defined as function χ_A called the characteristic function of A**

$\chi_A : X \rightarrow \{0, 1\}$, where $$\chi_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

This set maps universe X to a set of two elements. For any element x of universe X, characteristic function $\chi_A(x)$ is equal to 1 if x is an element of set A, and is equal to 0 if x is not an element of A.

- In the fuzzy theory, **fuzzy set A of universe X is defined by function μ_A called the membership function of set A**

μ_A : X → [0, 1], where $\mu_A(x) = 1$ if x is totally in A;

$\mu_A(x) = 0$ if x is not in A;

$0 < \mu_A(x) < 1$ if x is partly in A.

- This definition of set allows a continuum of possible choices. For any element x of universe X, membership function $\mu_A(x)$ equals the degree to which x is an element of set A. This degree, a value between 0 and 1, represents the **degree of membership**, also called **membership value**, of element x in set A.

Fuzzy Sets

- **Formal definition:**

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set

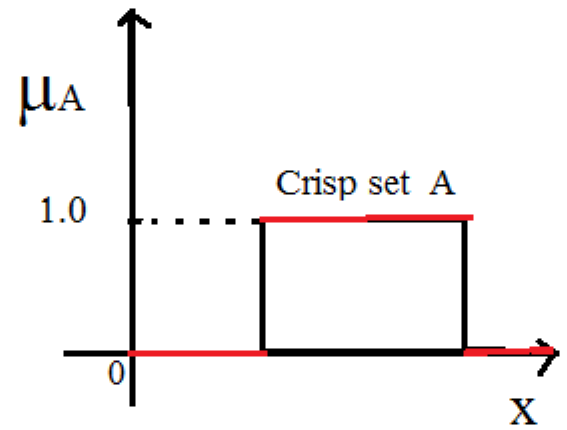
Membership function (MF)

Universe or universe of discourse

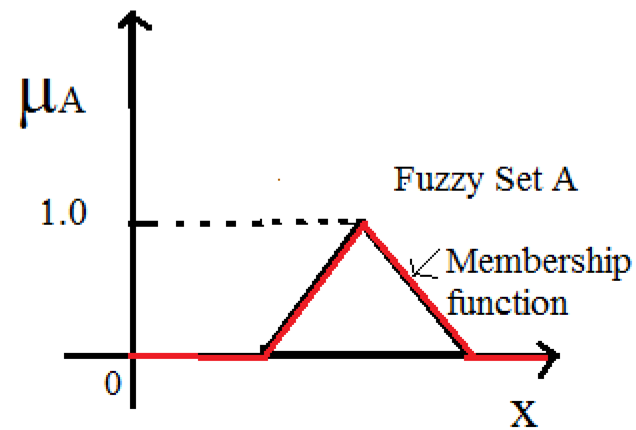
A fuzzy set is totally characterized by a membership function (MF).

Definition of Crisp Set and Fuzzy Sets

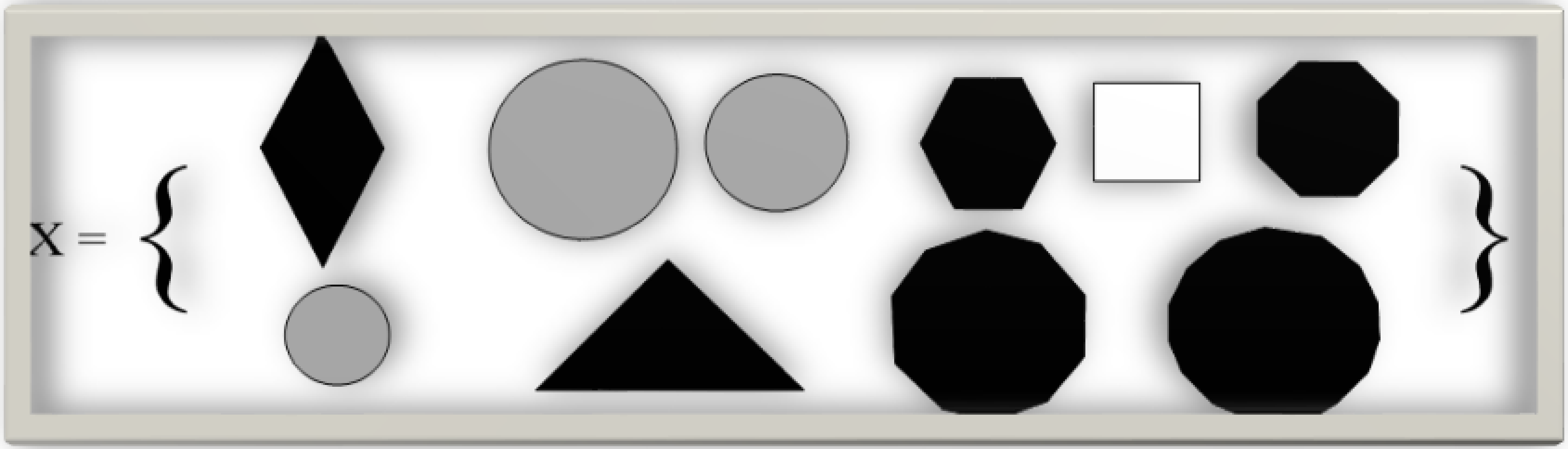
- A 'crisp' set, A, can be defined as a set which consists of elements with either full or no membership at all in the set.
- Each item in its universe is either in the set, or not.



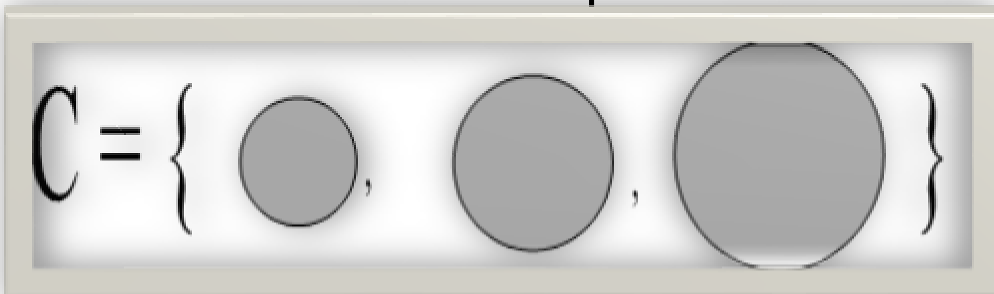
- A "fuzzy set" is defined as a class of objects with a continuum of grades of membership. It is characterized by a "membership function" or "characteristic function" that assigns to each member of the fuzzy set a degree of membership in the unit interval [0,1].



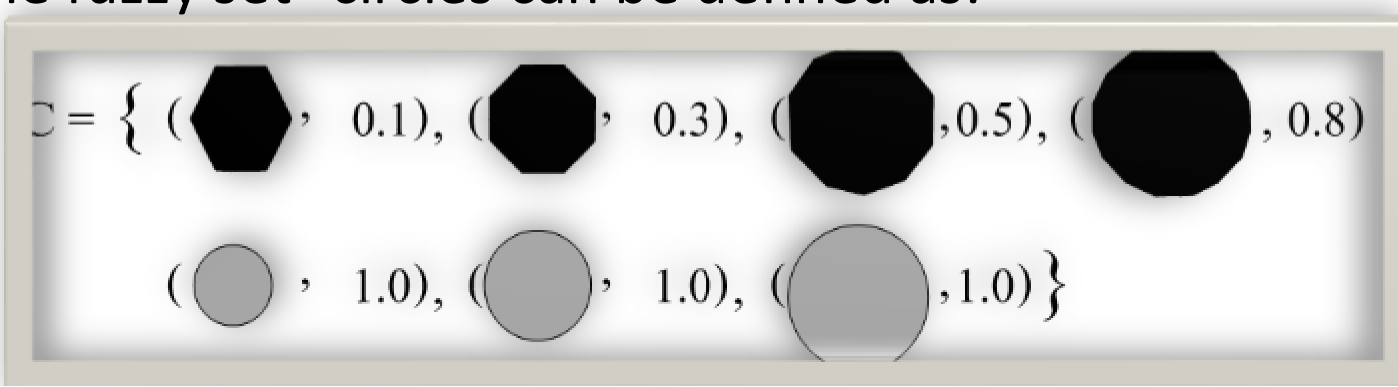
Crisp and Fuzzy example



One can define the crisp set “circles” as:



The fuzzy set “circles can be defined as:



Fuzzy Membership Functions

- One of the key issues in all fuzzy sets is how to determine fuzzy membership functions
- The membership function fully defines the fuzzy set
- A membership function provides a measure of the degree of similarity of an element to a fuzzy set
- Membership functions can take any form, but there are some common examples that appear in real applications

- **Membership functions can**

  - either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)

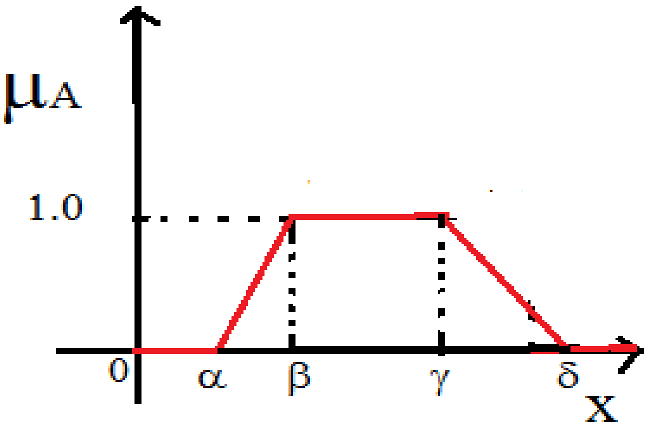
  - Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)

- There are different shapes of membership functions; triangular, trapezoidal, piecewise-linear, Gaussian, bell-shaped, etc.

Membership Functions

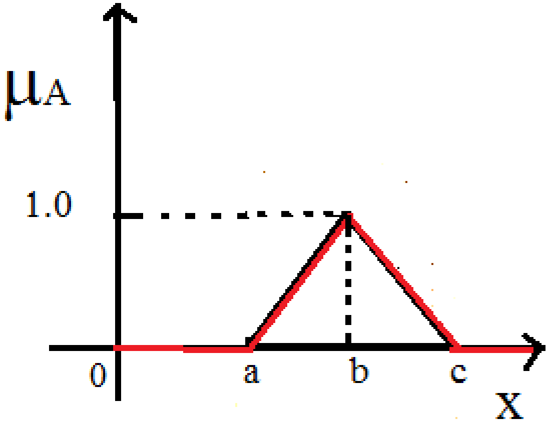
❖ Trapezoidal Membership Function

$$\prod(X,\alpha,\beta,\gamma,\delta)=\begin{cases}0 & \text{for } x<\alpha\\ (x-\alpha)/(\beta-\alpha) & \text{for } \alpha\le x\le\beta\\ 1 & \text{for } \beta\le x\le\gamma\\ (\delta-x)/(\delta-\gamma) & \text{for } \gamma\le x\le\delta\\ 0 & \text{for } x>\delta\end{cases}$$



❖ Triangular Membership Function

$$T(X,\mathrm{a},\mathrm{b},\mathrm{c})=\begin{cases}0 & \text{for } x<\mathrm{a}\\ (x-a)/(b-a) & \text{for } \mathrm{a}\le x\le b\\ (c-x)/(c-b) & \text{for } \mathrm{b}\le x\le c\\ 0 & \text{for } x>c\end{cases}$$

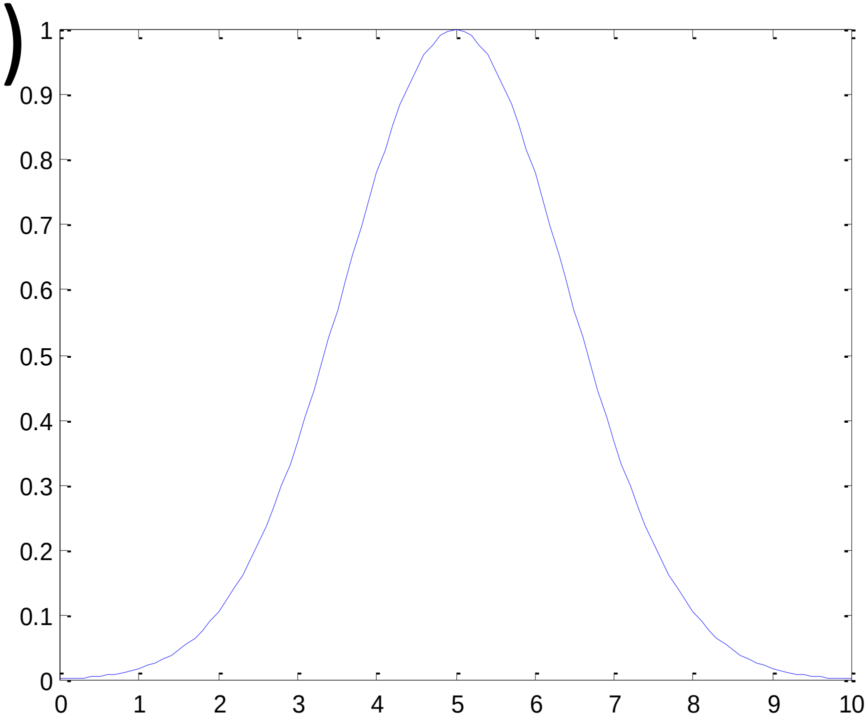


• Gaussian membership function

$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

Where c – centre , s - width and m - fuzzification factor

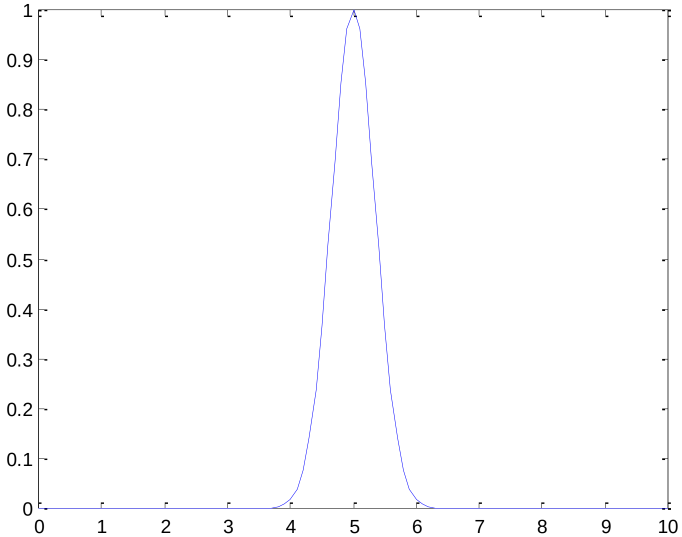
$\mu_A(x)$



$c=5$

$s=2$

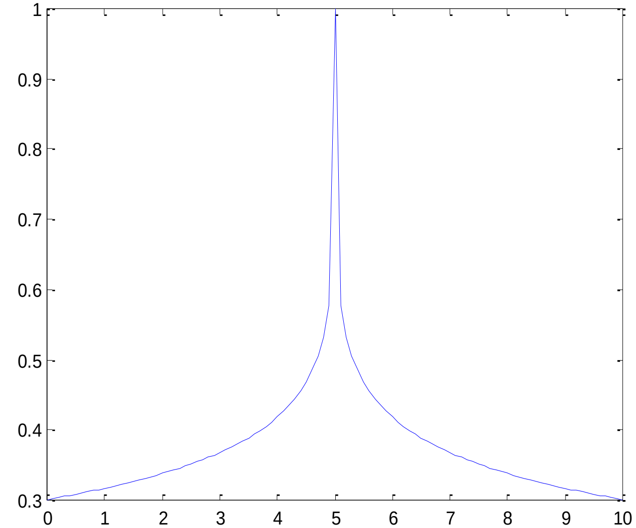
$m=2$



$c=5$

$s=0.5$

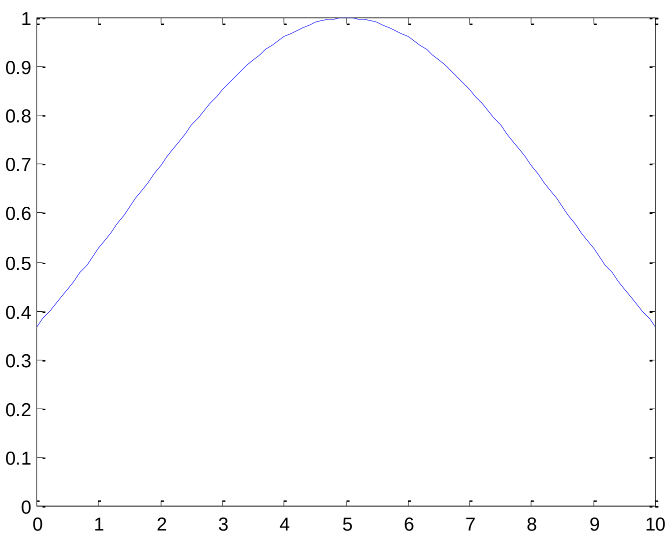
$m=2$



$c=5$

$s=2$

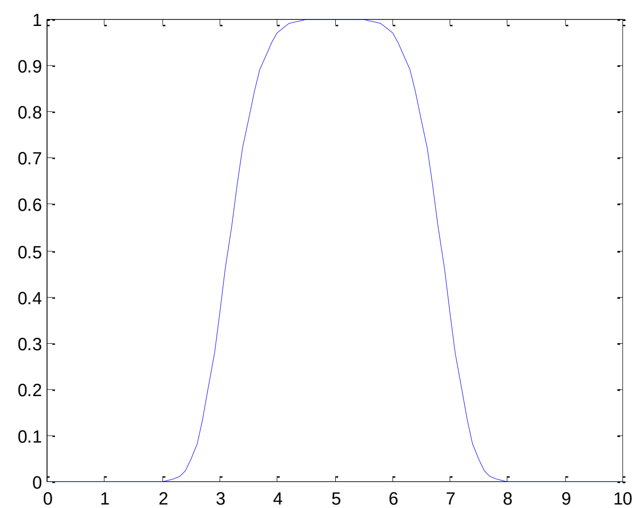
$m=0.2$



$c=5$

$s=5$

$m=2$



$c=5$

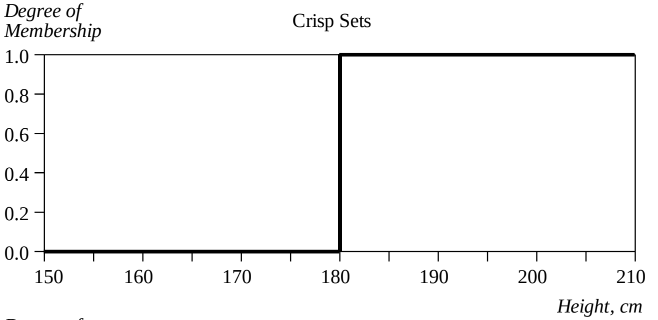
$s=5$

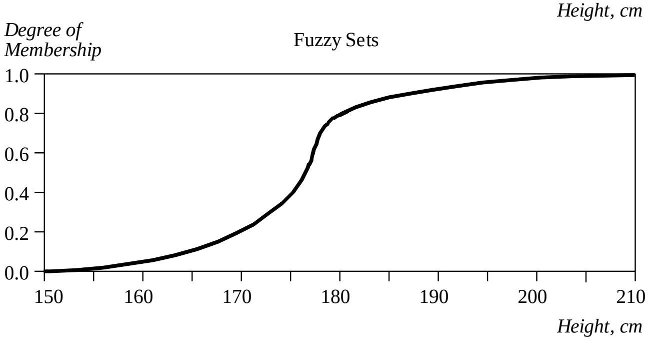
$m=5$

Crisp Sets vs. Fuzzy Sets

The classical example in fuzzy sets is "tall men". The elements of the fuzzy set "tall men" are all men, but their degrees of membership depend on their height.

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Ram	208	1	1.00
Sham	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00



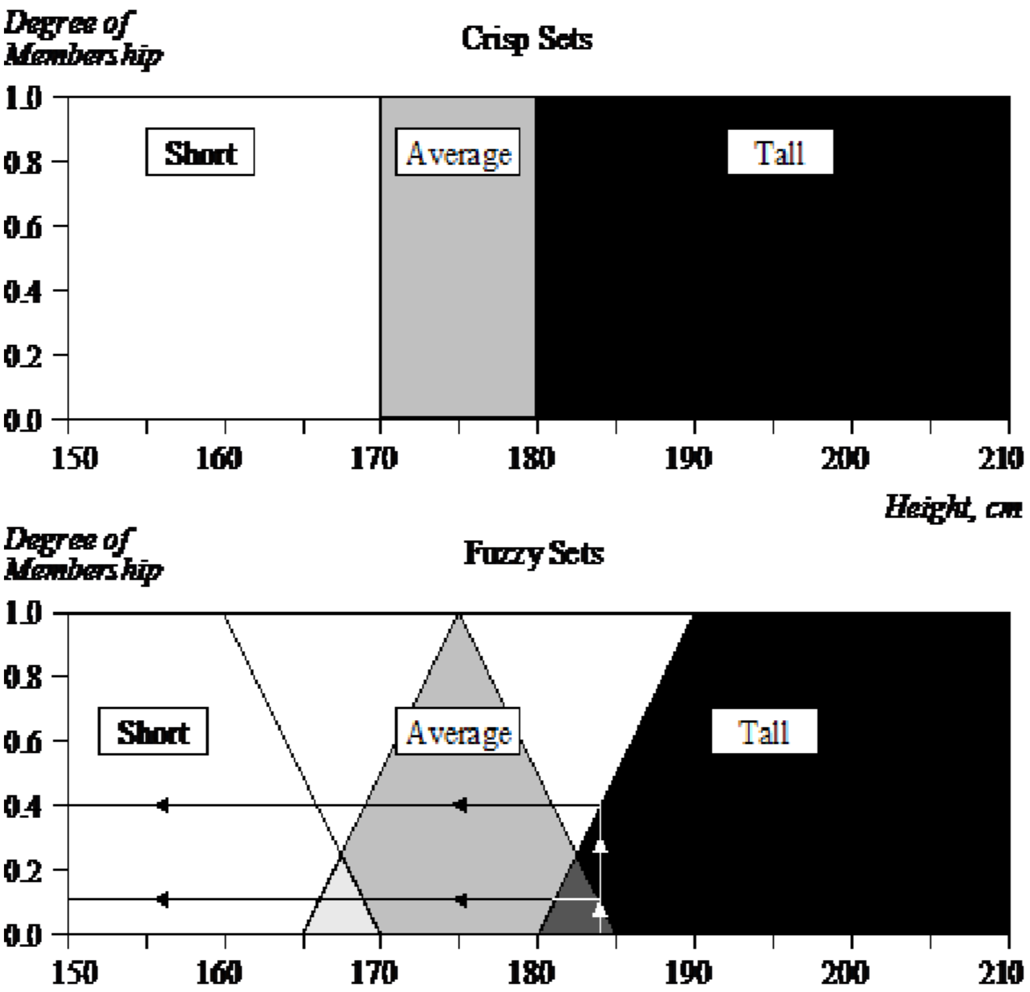


The x-axis represents the **universe of discourse** – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.

The y-axis represents the **membership value of the fuzzy set**. In our case, the fuzzy set of "tall men" maps height values into corresponding membership values.

Fuzzy Set Representation

- First, we determine the membership functions. In our "tall men" example, we can define fuzzy sets of *tall*, *short* and *average* men.
- The universe of discourse for three defined fuzzy sets consist of all possible values of the men's heights.
- For example, a man who is 184 cm tall is a member of the *average* men set with a degree of membership of 0.1, and at the same time, he is also a member of the *tall* men set with a degree of 0.4.



CRISP SET V/S FUZZY SET(Cont.)

The most obvious limiting feature of bivalent sets that can be seen clearly from the diagram is that they are mutually exclusive - it is not possible to have membership of more than one set

Fuzzy sets however define degree of membership.

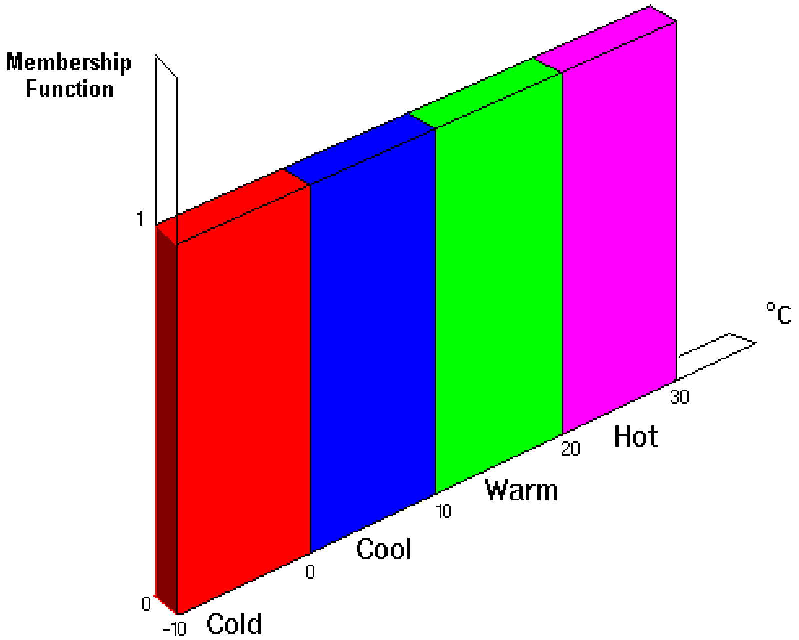


Fig. 1 : Bivalent Sets to Characterize the Temp. of a room.

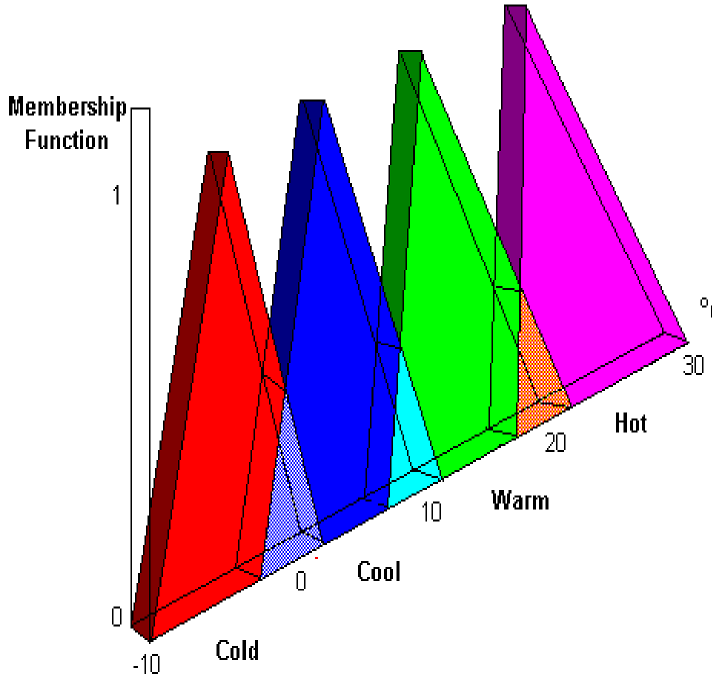


Fig. 2 - Fuzzy Sets to characterize the Temp. of a room.

CRIPS SET OPERATIONS

The Crips set operations of union, intersection and complementation are defined in terms of characteristic functions as follows:

- Union:

  $\chi_{A \cup B}(x) = \max(\chi_A(x), \chi_B(x))$

- Intersection:

  $\chi_{A \cap B}(x) = \min(\chi_A(x), \chi_B(x))$

- Complement:

  $\chi_{\text{not } A}(x) = 1 - \chi_A(x)$

The other set theory constructs that are essential are:

- Crips Set Inclusion:

  $A \subset B$ if and only if $\forall x$ (for all x) $\chi_A(x) = 1$ implies $\chi_B(x) = 1$

- Crips Set Equality:

  $A = B$ if and only if $\forall x$ (for all x) $\chi_A(x) = \chi_B(x)$.

FUZZY SET OPERATIONS

The fuzzy set operations of union, intersection and complementation are defined in terms of membership functions as follows:

- Union:

  $\mu_{A \cup B}(x) = \max(\mu_A(x),\ \mu_B(x))$

- Intersection:

  $\mu_{A \cap B}(x) = \min(\mu_A(x),\ \mu_B(x))$

- Complement:

  $\mu_{\text{not } A}(x) = 1 - \mu_A(x)$

The other fuzzy set theory constructs that are essential are:

- Fuzzy Set Inclusion:

  $A \subset B$ if and only if $\forall x$ (for all x) $\mu_A(x) \leq \chi_B(x)$

- Fuzzy Set Equality:

  $A = B$ if and only if $\forall x$ (for all x) $\mu_A(x) = \mu_B(x)$.

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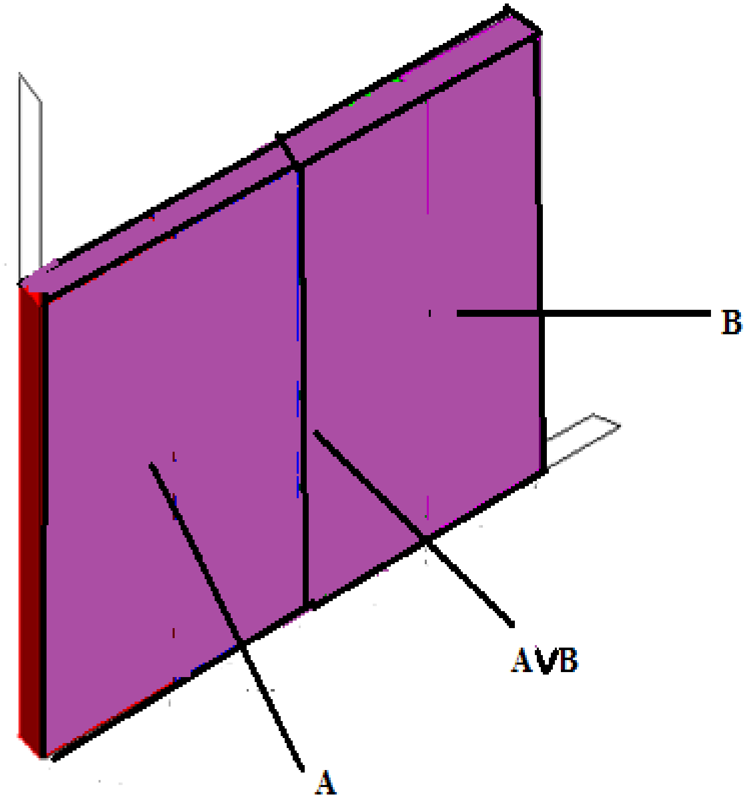
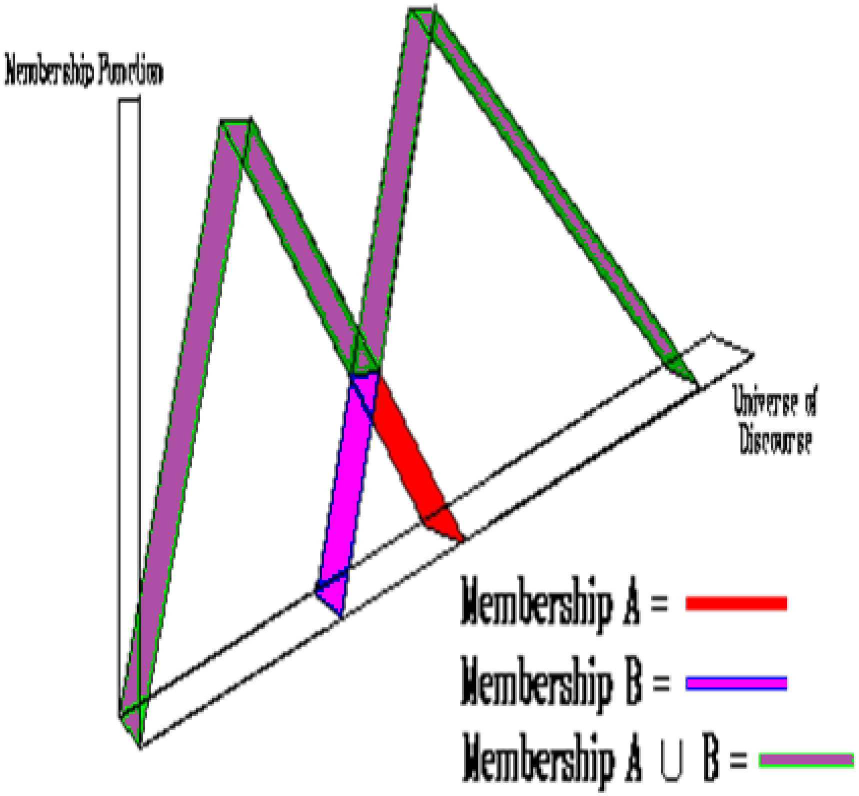
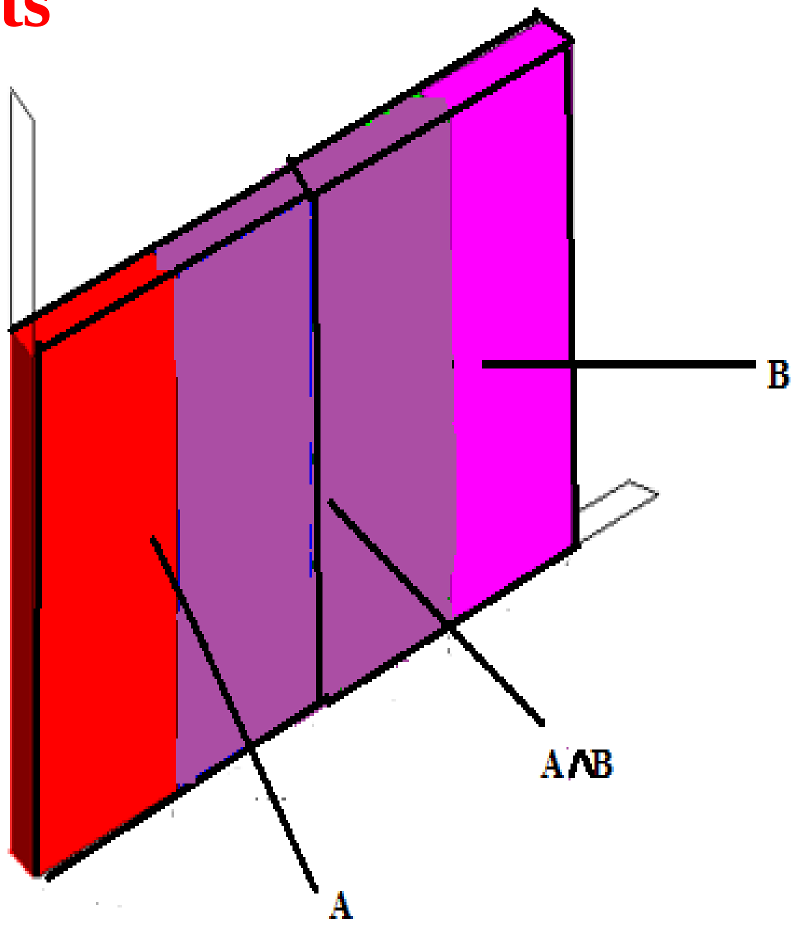
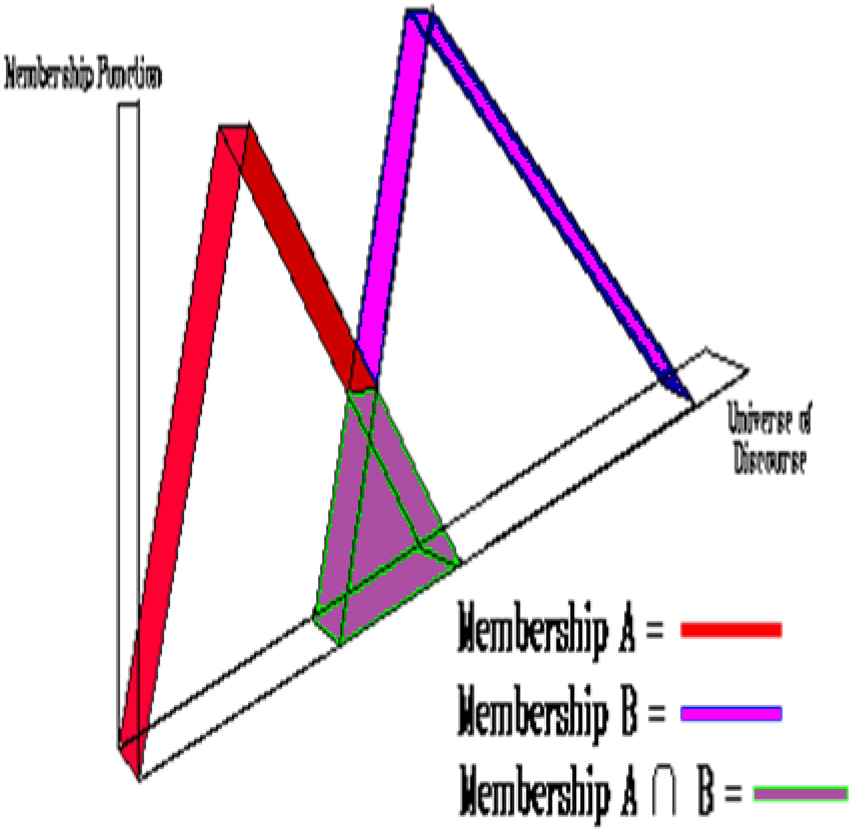
Representation of Union of two crisp sets and fuzzy sets



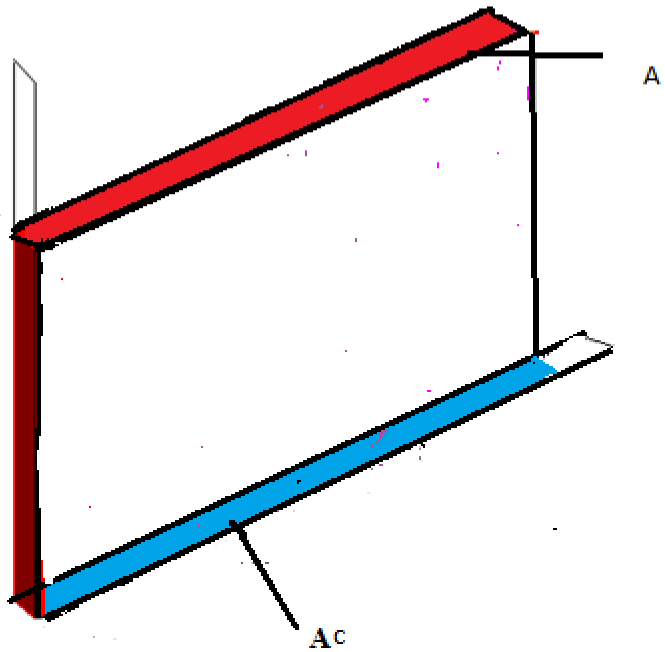


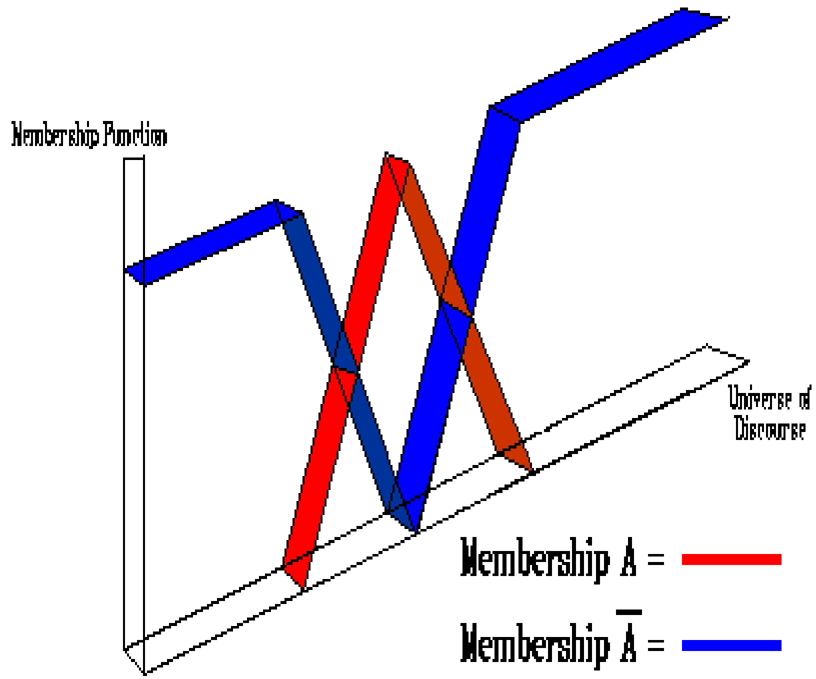
Representation of Intersection of two crisp sets and fuzzy sets





Representation of Complement of a crisp set and a Fuzzy set

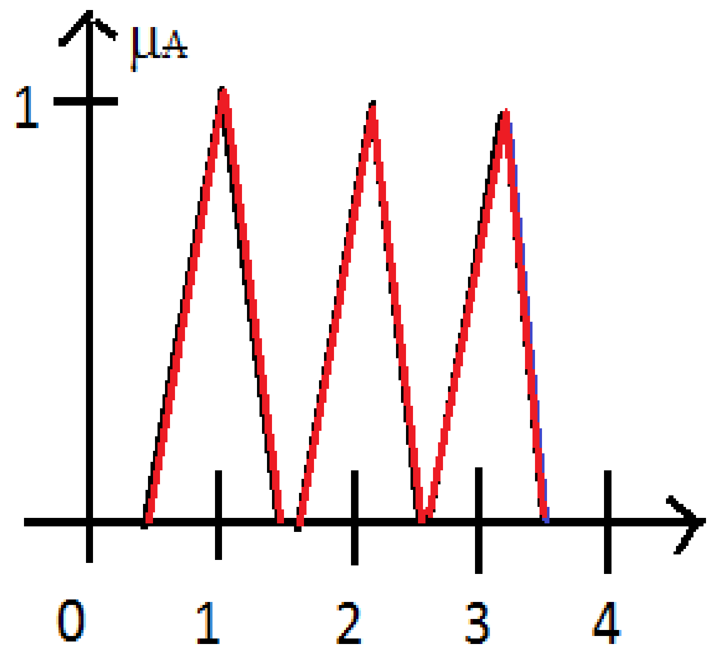


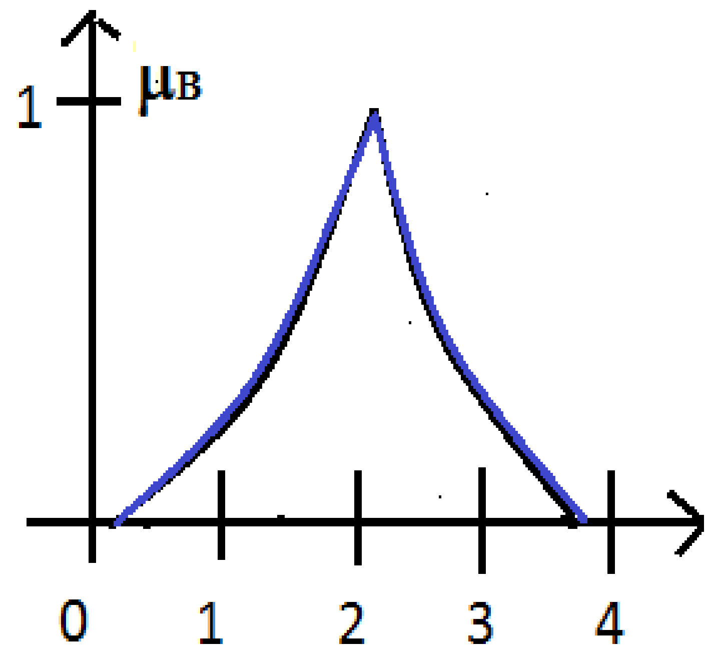


Examples of Fuzzy Sets

A = { x : x is near an integer }

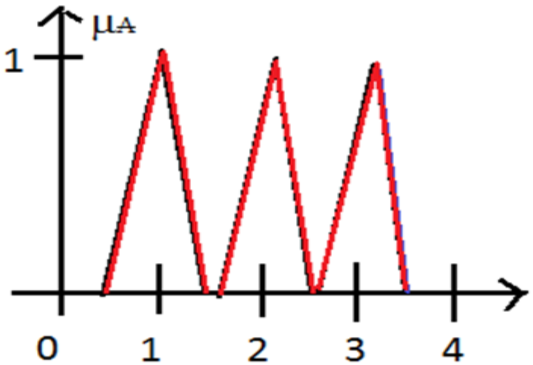
B = { x : x is close to 2 }

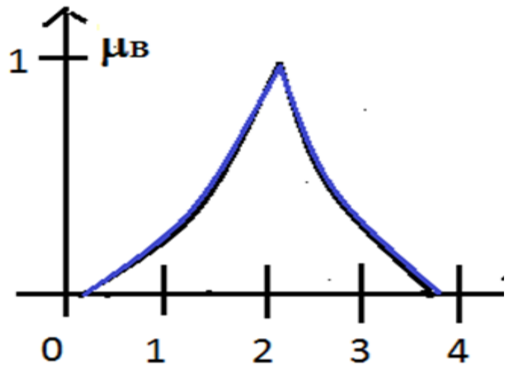


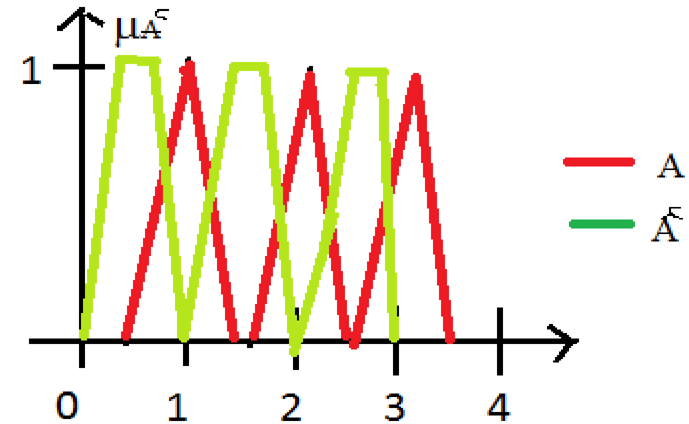


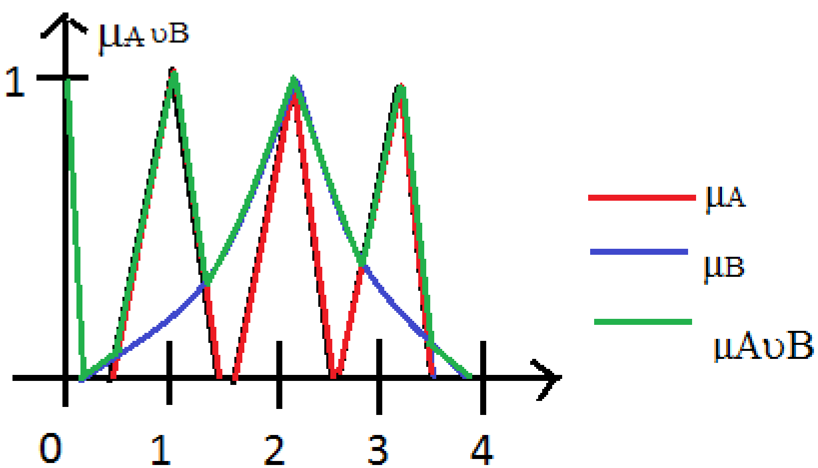
A = { x : x is near an integer }

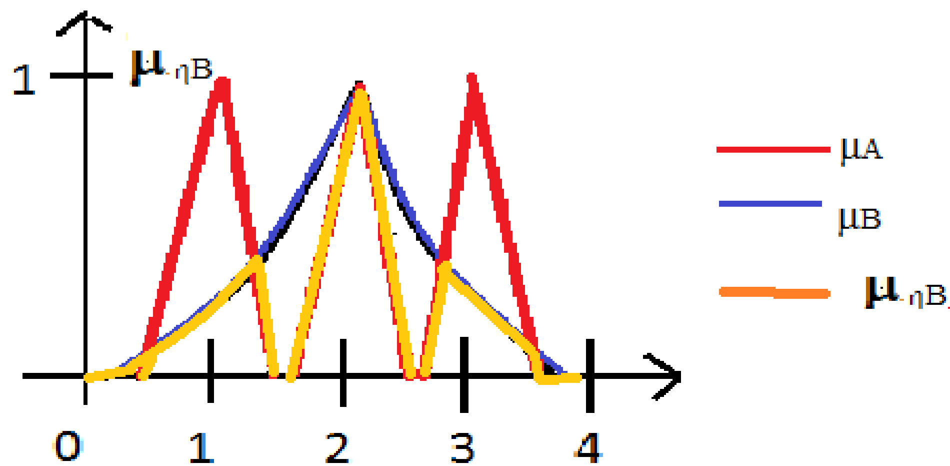
B = { x : x is close to 2 }











Fuzzy and Excluded Middle

Min-Max fuzzy logic fails: ***The Law of Excluded Middle.***

$$A \cap \bar{A} \neq \varnothing$$

Since $\min\{\, \mu_A(x)\, ,\, 1-\mu_A(x)\} \neq 0$

Thus, (the set of numbers *close* to 2) AND (the set of numbers <u>not</u> *close* to 2) ≠ null set

Fuzzy and the Law of Contradiction

Min-Max fuzzy logic fails: ***The Law of Contradiction*.**

$$A \cup \bar{A} \neq X$$

Since max $\{\mu_A(x)\ ,\ 1-\mu_A(x)\} \neq 1$

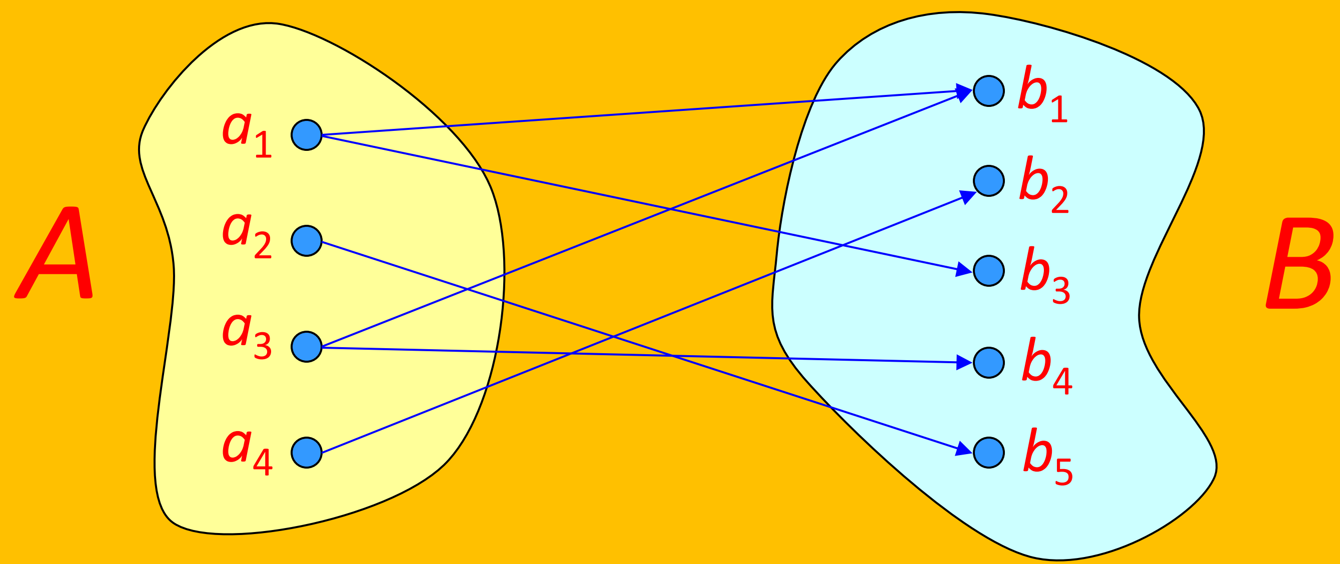
Thus, (the set of numbers *close* to 2) OR (the set of numbers not *close* to 2) $\neq$ universal set

Fuzzy Relations

Representations of fuzzy relation

- List of ordered pairs with their membership grades
- Matrix representation
- Composition of Fuzzy Relations

Binary Relation (R)



$$R \subseteq A \times B$$

Matrix Representation

Fuzzy relation R on X×Y

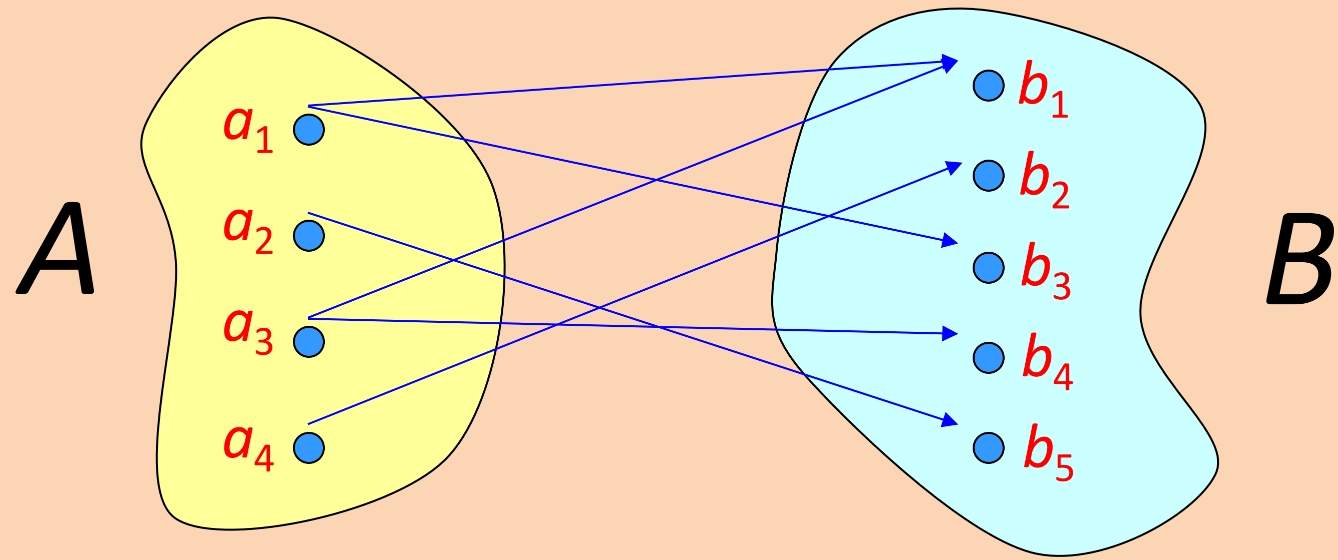
$X=\{x_1,\ x_2, \ldots, x_n\}$, $Y=\{y_1,\ y_2, \ldots, y_m\}$

$r_{ij}=R(x_i, y_j)$ is the membership degree of pair (x_i, y_j)

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix}$$

$$R \subseteq A \times B$$

Binary Relation (*R*)



$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

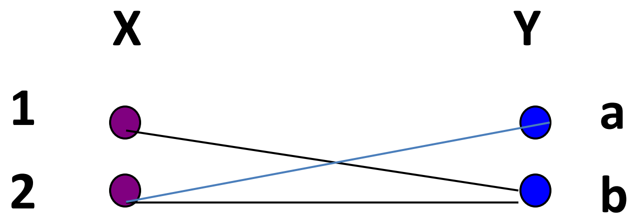
$$a_1 R b_1 \quad a_1 R b_3 \quad a_2 R b_5$$

$$R = \begin{Bmatrix} (a_1, b_1), (a_1, b_3), (a_2, b_5) \\ (a_3, b_1), (a_3, b_4), (a_4, b_2) \end{Bmatrix}$$

$$a_3 R b_1 \quad a_3 R b_4 \quad a_4 R b_2$$

Crips Relations

$X = \{1\ ,\ 2\}$ $Y = \{a,\ b\}$



$$R = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

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Fuzzy Relations

A fuzzy relation *R* is a 2D MF:

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \mid (x, y) \in X \times Y \right\}$$

Example of Fuzzy Relations

For example, let: $X = \{x_1, x_2, x_3\}, \; Y = \{y_1, y_2\}$

$$A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 1)\} \text{ and } B = \{(y_1, 0.3), (y_2, 0.9)\}$$

$$\mu_R(x,y) = \mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y))$$

$$A \times B = R = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Operations on Crips Relations

$$O = Null\ relation = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = Complete\ relation = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$Union \quad R \cup S \rightarrow \chi_{R \cup S}(x, y): \quad \chi_{R \cup S}(x, y) = \max[\chi_R(x, y), \chi_S(x, y)]$$

$$Intersection \quad R \cap S \rightarrow \chi_{R \cap S}(x, y): \quad \chi_{R \cap S}(x, y) = \min[\chi_R(x, y), \chi_S(x, y)]$$

$$Complement \quad \bar{R} \rightarrow \chi_{\bar{R}}(x, y): \quad \chi_{\bar{R}}(x, y) = 1 - \chi_R(x, y)$$

$$Containment \quad R \subset S \rightarrow \chi_R(x, y): \quad \chi_R(x, y) \leq \chi_S(x, y)$$

$$Identity \quad (\phi \rightarrow 0\ and\ X \rightarrow E)$$

Properties of Crips Relations

The properties of commutativity, assosiativity, idempotency and distributivity all hold for crisp relations.

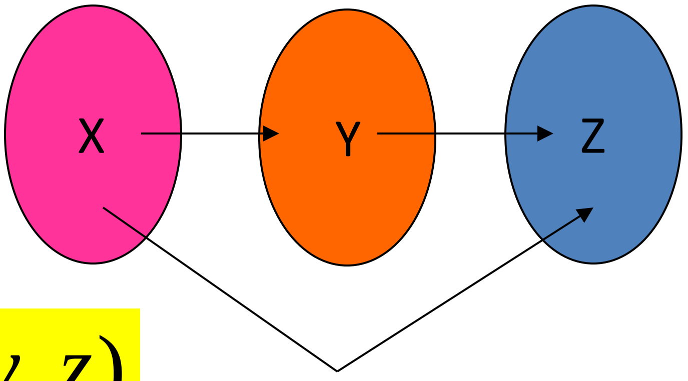
De Morgan’s laws and the excluded middle laws also hold for crisp

Composition of Crips Relations

Let R be a relation that relates elements from universe X To universe Y, and let S be a relation that relates elements From Universe Y to universe Z.

A useful question we seek to answer is whether we can find a relation, T, that relates the same elements in universe X that R contains to the same elements in universe Z that S contains.

$$T = R \circ S$$



$$\chi_T(x,z) = \bigvee_{y \in Y} (\chi_R(x,y) \wedge \chi_S(y,z))$$

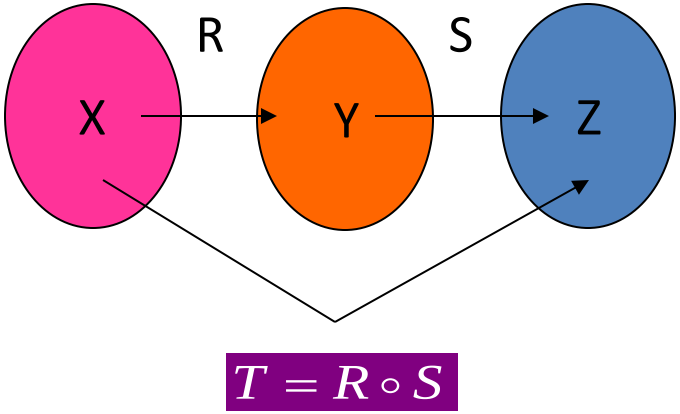
$T = R \circ S$

Composition of Fuzzy Relations

Fuzzy composition can be defined just as it is for crisp relations. Suppose R be a fuzzy relation that relates elements from universe X to universe Y, and let S be a fuzzy relation that relates elements from universe Y to universe Z and T is a fuzzy relation that relates the same elements in universe X that R contains to the same elements in universe Z that S contains. Then fuzzy max-min composition is defined as:

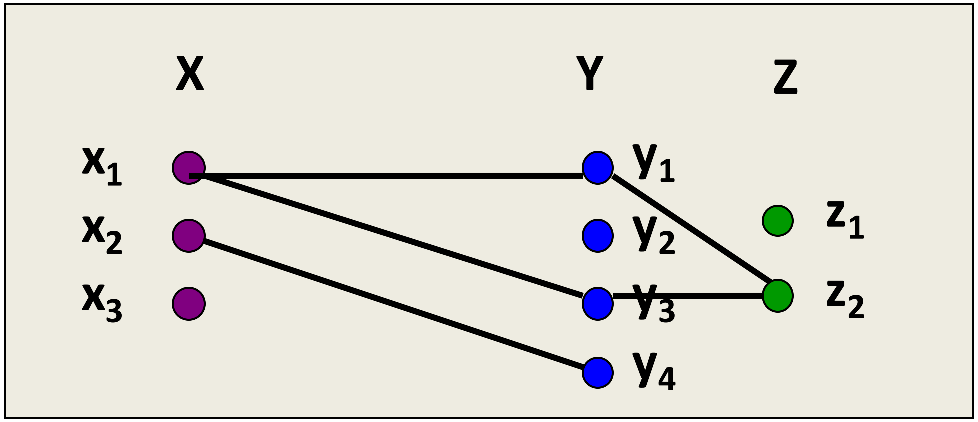
$$\mu_T(x,z) = \vee_{y \in Y} (\mu_R(x,y) \wedge \mu_S(y,z))$$

$$T = R \circ S$$



Composition of Crips Relations

$$T = R \circ S$$



$$\chi_T(x,z) = \vee_{y \in Y}(\chi_R(x,y) \wedge \chi_S(y,z))$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \circ S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} = T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\chi_T(x_1, z_1) = \max[\min(1,0), \min(0,0), \min(1,0), \min(0,0)] = 0$$

$$\chi_T(x_1, z_2) = \max[\min(1,1), \min(0,0), \min(1,1), \min(0,0)] = 1$$

And for rest follows similarly :

Operations on Fuzzy Relations

$$O = Null\ relation = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = Complete\ relation = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$Union \quad R \cup S \rightarrow \mu_{R \cup S}(x, y): \qquad \mu_{R \cup S}(x, y) = \max[\mu_R(x, y), \mu_S(x, y)]$$

$$Intersection \quad R \cap S \rightarrow \mu_{R \cap S}(x, y): \qquad \mu_{R \cap S}(x, y) = \min[\mu_R(x, y), \mu_S(x, y)]$$

$$Complement \quad \bar{R} \rightarrow \mu_{\bar{R}}(x, y): \qquad \mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$$

$$Containment \quad R \subset S \rightarrow \mu_R(x, y): \qquad \mu_R(x, y) \leq \mu_S(x, y)$$

Properties of Fuzzy Relations

Just as for crisp relation, the properties of commutativity, assosiativity, idempotency and distributivity all hold for fuzzy relations.

De Morgan's laws hold for fuzzy but the excluded middle laws for relation do not result in the null relation, O, or the complete relation, E.

$$R \cup \overline{R} \neq E$$
$$R \cap \overline{R} \neq 0$$

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \mid (x, y) \in X \times Y \right\}$$

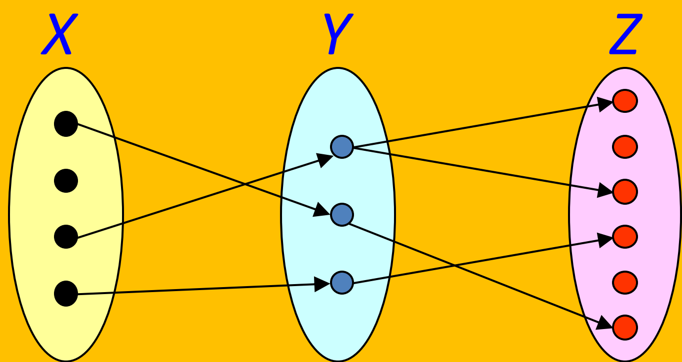
Example (Approximate Equal)

$$X = Y = U = \{1, 2, 3, 4, 5\}$$

$$\mu_R(x, y) = \begin{cases} 1 & |x - y| = 0 \\ 0.8 & |x - y| = 1 \\ 0.3 & |x - y| = 2 \\ 0 & otherwise \end{cases}$$

$$M_R = \begin{bmatrix} 1 & 0.8 & 0.3 & 0 & 0 \\ 0.8 & 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.8 & 1 & 0.8 & 0.3 \\ 0 & 0.3 & 0.8 & 1 & 0.8 \\ 0 & 0 & 0.3 & 0.8 & 1 \end{bmatrix}$$

Max-Min Composition



R: fuzzy relation defined on *X* and *Y*.

S: fuzzy relation defined on *Y* and *Z*.

$\underbrace{\textit{RoS}}_{\text{A fuzzy relation defined on } X \text{ and } Z.}$: the composition of *R* and *S*.

A fuzzy relation defined on *X* and *Z*.

$$\mu_{RoS}(x,z) = \max_y \min\left(\mu_R(x,y), \mu_S(y,z)\right)$$

$$= \vee_y \left(\mu_R(x,y) \wedge \mu_S(y,z)\right)$$

$$\mu_{S \circ R}(x,y) = \max_v \min\left(\mu_R(x,v), \mu_S(v,y)\right)$$

Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

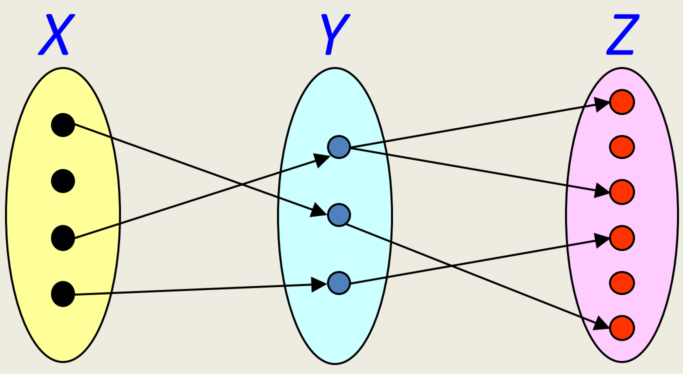
S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

	0.1	0.2	0.0	1.0
min	0.9	0.2	0.8	0.4
max	0.1	0.2	0.0	0.4

$R \circ S$	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

Max-Product Composition

Max-min composition is not mathematically tractable, therefore other compositions such as max-product composition have been suggested.



R: fuzzy relation defined on X and Y.

S: fuzzy relation defined on Y and Z.

$\underbrace{R \circ S}$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{RoS}(x, y) = \max_v \left(\mu_R(x, v) \mu_S(v, y) \right)$$

Example of Fuzzy Composition

For example, let:

$$X = \{x_1, x_2\}$$

$$Y = \{y_1, y_2\}$$

$$Z = \{z_1, z_2, z_3\}$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$\mu_T(x,z) = \vee_{y \in Y} (\mu_R(x,y) \wedge \mu_S(y,z))$$

$$T = R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

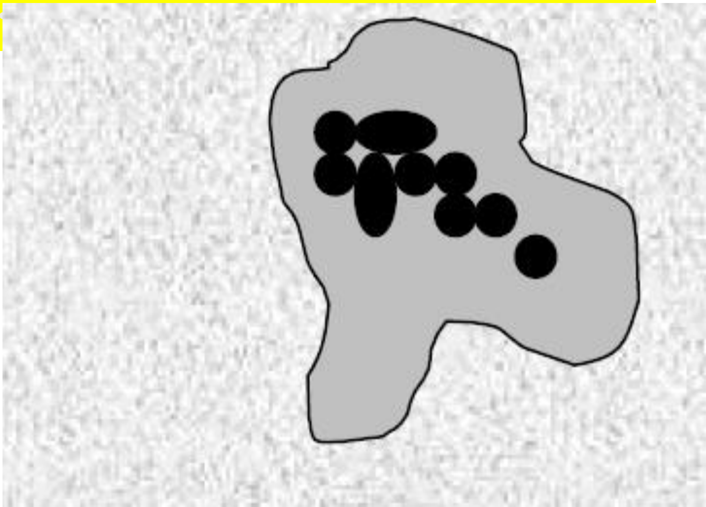
$$\mu_T(x,z) = \vee_{y \in Y} (\mu_R(x,y) \bullet \mu_S(y,z))$$

$$T = R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}$$

Example : A certain type of virus attacks cells of the human body. The infected cells can be visualized using a special microscope. The microscope generates digital images that medical doctors can analyze and identify the infected cells. The virus causes the infected cells to have a black spot, within a darker grey region.

A digital image process can be applied to the image. This processing generates two variables: the first variable, P, is related to black spot quantity (black pixels), and the second variable, S, is related to the shape of the black spot, i.e., if they are circular or elliptic. In these images it is often difficult to actually count the number of black pixels, or to identify a perfect circular cluster of pixels; hence, both these variables must be estimated in a linguistic way.

An infected cell shows black spots with different shapes in a micrograph.



Suppose that we have two fuzzy sets, P which represents the number of black pixels (e.g., none with black pixels, C1, a few with black pixels, C2, and a lot of black pixels, C3), and S which represents the shape of the black pixel clusters, e.g., S1 is an ellipse and S2 is a circle. So we have

$$P = \{(C_1, 0.1), (C_2, 0.5), (C_3, 1.0)\} \text{ and } S = \{(S_1, 0.3), (S_2, 0.8)\}$$

and we want to find the relationship between quantity of black pixels in the virus and the shape of the black pixel clusters. Using a Cartesian product between P and S gives:

$$R = P \times S = \begin{array}{c c} & \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} & \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \\ 0.3 & 0.8 \end{bmatrix} \end{array}$$

Now, suppose another microscope image is taken and the number of black pixels is slightly different; let the new black pixel quantity be represented by a fuzzy set P':

$$P' = \{(C_1, 0.4)\ ,(C_2, 0.7)\ ,(C_3, 1.0)\ \}$$

Using max–min composition with the relation R will yield a new value for the fuzzy set of pixel cluster shapes that are associated with the new black pixel quantity:

$$S' = P' \circ R = \begin{bmatrix} 0.4 & 0.7 & 1.0 \end{bmatrix} \circ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.8 \end{bmatrix}$$

Linguistic Variables

- At the root of fuzzy set theory lies the idea of linguistic variables.
- **A linguistic variable is a fuzzy variable**. For example, the statement "Ram is tall" implies that the linguistic variable Ram takes the linguistic value tall.
- In fuzzy expert systems, linguistic variables are used in fuzzy rules. For example:

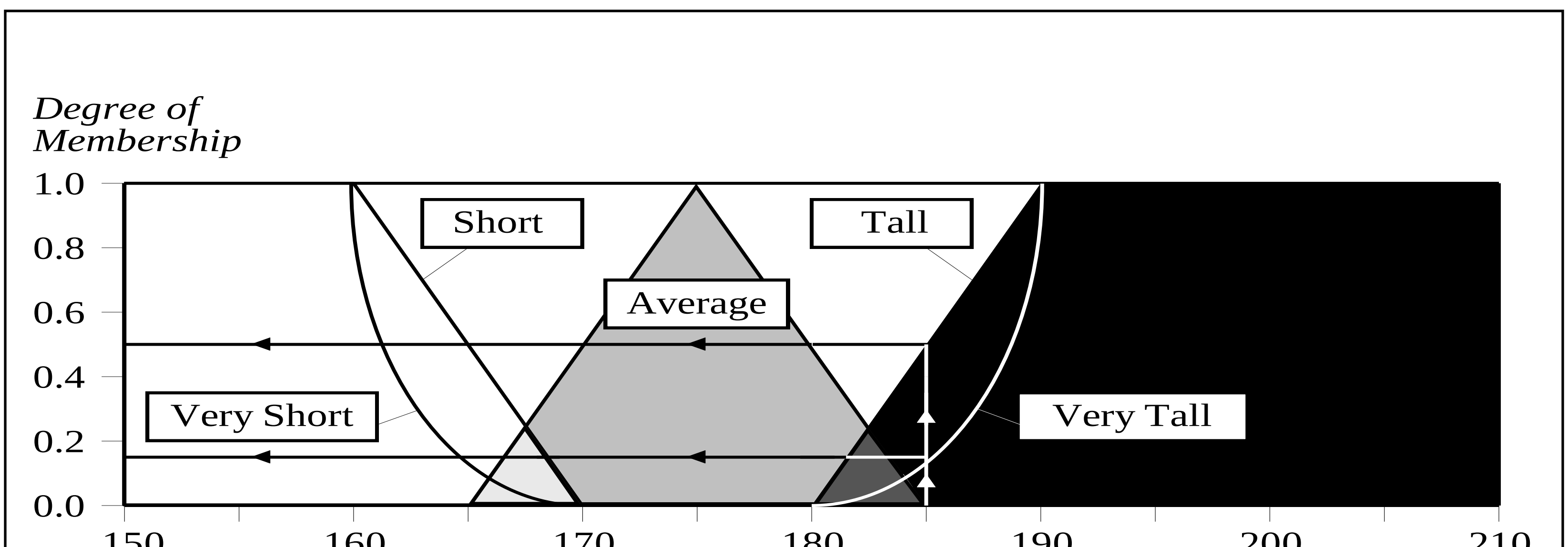
IF wind is strong THEN sailing is good

IF project duration is long THEN completion risk is high

IF speed is slow THEN stopping distance is short

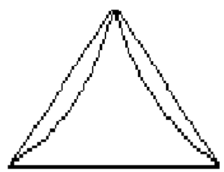
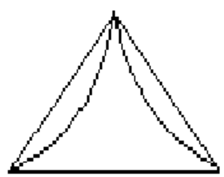
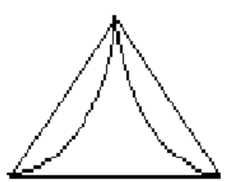
Linguistic Variables and Hedges

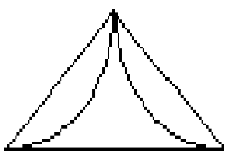
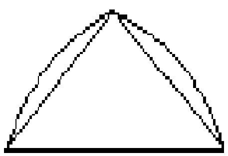
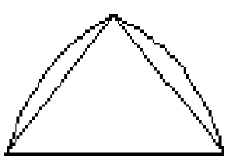
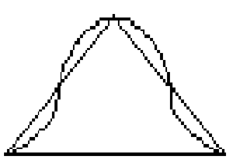
- The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.
- A linguistic variable carries with it the concept of fuzzy set qualifiers, called **hedges**.
- Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as *very*, *somewhat*, *quite*, *more or less* and *slightly*.



Linguistic Variables and Hedges

Typical hedges:

Hedge	Mathematical Expression	Graphical Representation
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^2$	
Extremely	$[\mu_A(x)]^3$	

Hedge	Mathematical Expression	Graphical Representation
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2[\mu_A(x)]^2$ if $0 \le \mu_A \le 0.5$ $1-2[1-\mu_A(x)]^2$ if $0.5 < \mu_A \le 1$	

The Term "Fuzzy Logic"

- **Why "fuzzy" ?**

  As Zadeh said, the term is concrete, immediate and descriptive; we all know what it means. However, many people were repelled by the word fuzzy, because it is usually used in a negative sense.

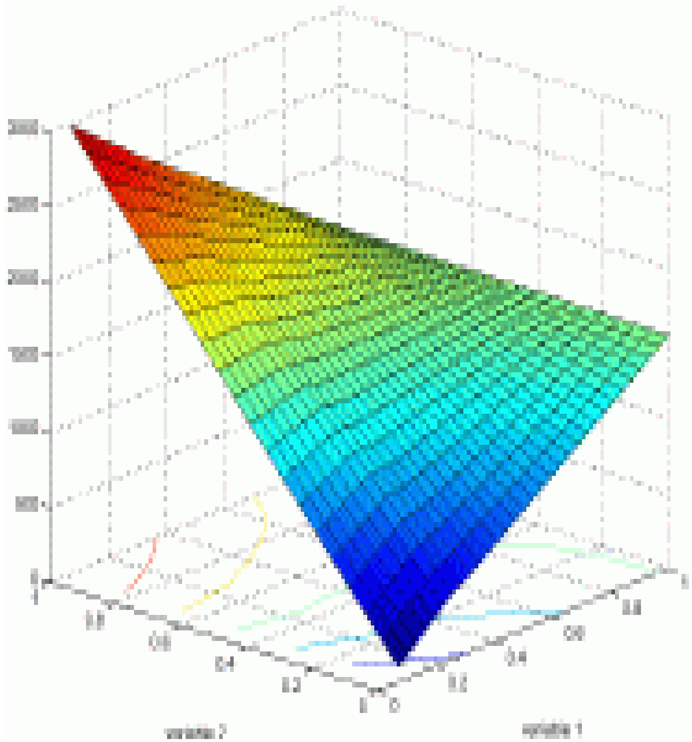
- **Why "logic"?**

  Fuzziness rests on fuzzy set theory, and fuzzy logic is just a small part of that theory.

- The term fuzzy logic is **used in two senses**:
  - **Narrow sense**: Fuzzy logic is a branch of fuzzy set theory, which deals (as logical systems do) with the representation and inference from knowledge. Fuzzy logic, unlike other logical systems, deals with imprecise or uncertain knowledge. In this narrow, and perhaps correct sense, fuzzy logic is just one of the branches of fuzzy set theory.
  - **Broad Sense**: fuzzy logic synonymously with fuzzy set theory.

How is Fuzzy Logic Used in Maths?

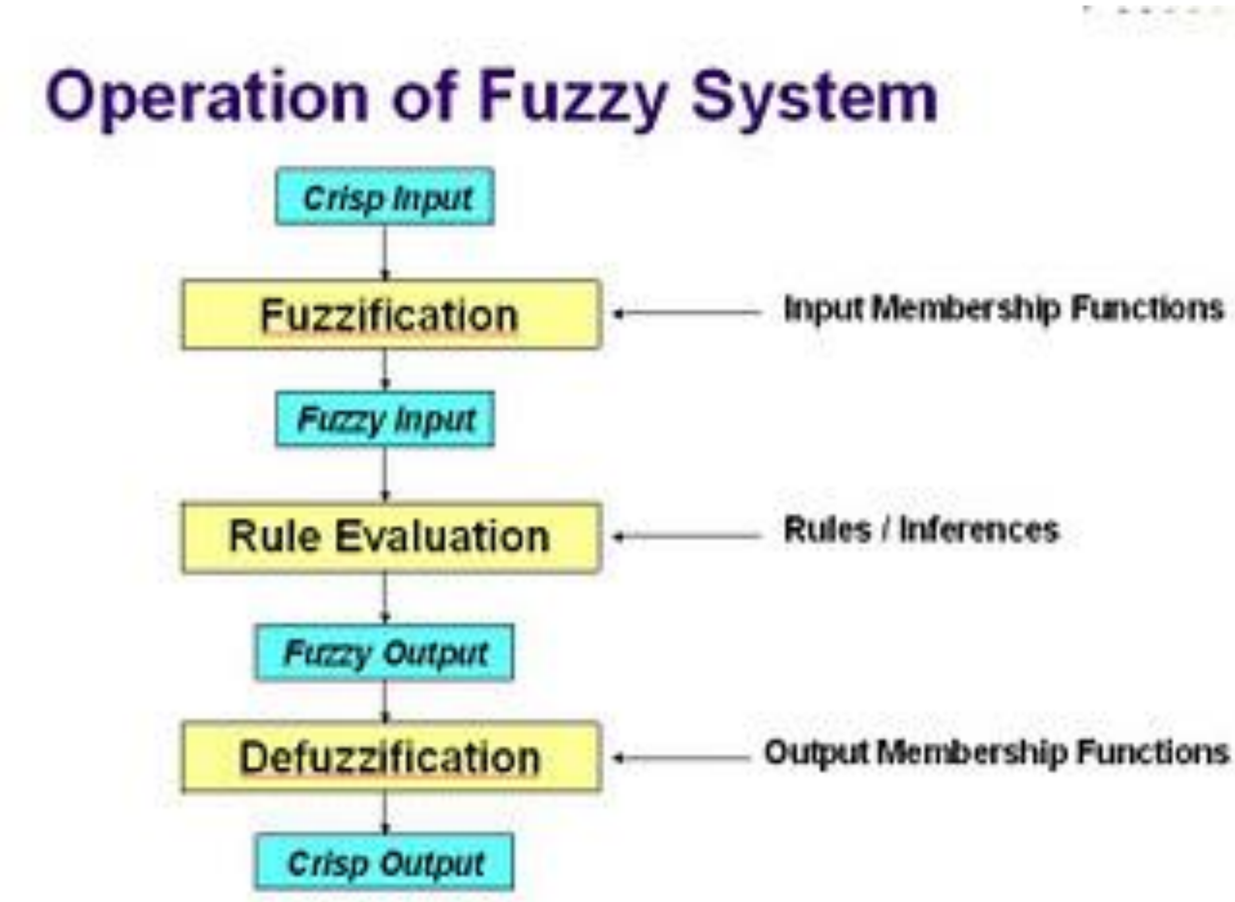
Fuzzy Mathematics



- Fuzzy Numbers – almost 5, or more than 50
- Fuzzy Geometry – Almost Straight Lines
- Fuzzy Calculus– Not quite a parabola
- Fuzzy Algebra – based on fuzzy relations
- Fuzzy Graphs – based on fuzzy points

Fuzzification and Defuzzification

- Fuzzification is to transform crisp inputs into fuzzy subsets.
- Defuzzification is to map fuzzy subsets of real numbers into real numbers.



Simple example of Fuzzy Logic

Controlling a fan:

- Conventional model –

  if temperature > X, run fan , else stop fan

- Fuzzy System -

  if temperature = hot, run fan at full speed

  if temperature = warm, run fan at moderate speed

  if temperature = comfortable, maintain fan speed

  if temperature = cool, slow fan

  if temperature = cold, stop fan

Fuzzy Applications

- Theory of fuzzy sets and fuzzy logic has been applied to problems in a variety of fields:
  - pattern recognition, decision support, data mining & information retrieval, medicine, law, taxonomy, topology, linguistics, automata theory, game theory, etc.

- And more recently fuzzy machines have been developed including:
  - automatic train control, tunnel digging machinery, home appliances: washing machines, air conditioners, etc.

Fuzzy Classification

The classifier is described by fuzzy IF-THEN rules.

An example of fuzzy classification rules for a 2-dimensional feature space is:

R_1 *IF* x_1 is **small** *AND* x_2 is **very large** *THEN* $x = (x_1, x_2)$ belongs to class C_1

R_2 *IF* x_1 is **large** *AND* x_2 is **small** *THEN* $x = (x_1, x_2)$ belongs to class C_2

R_3 *IF* x_1 is **small** *AND* x_2 is **large** *THEN* $x = (x_1, x_2)$ belongs to class C_3

R_4: *IF* x_1 is **very small** *AND* x_2 is **very large** *THEN* $x = (x_1, x_2)$ belongs to class C_4

where R_i is the i-th classification rule, C_k indicates an output class, x_1 and x_2 are the features **very small**, **small**, **large** and **very large** are linguistic terms characterized by appropriate membership functions

Inference of Fuzzy rule based Classifiers

For simplicity, let the rule base contain 9 fuzzy IF-THEN rules and have two inputs x and y as follows:

Rule 1:	IF input1 is A_1	and input2 is B_1	THEN output is C_1
Rule 2:	IF input1 is A_1	and input2 is B_2	THEN output is C_2
Rule 3:	IF input1 is A_1	and input2 is B_3	THEN output is C_3
Rule 4:	IF input1 is A_2	and input2 is B_1	THEN output is C_4
Rule 5:	IF input1 is A_2	and input2 is B_2	THEN output is C_1
Rule 6:	IF input1 is A_2	and input2 is B_3	THEN output is C_6
Rule 7:	IF input1 is A_3	and input2 is B_1	THEN output is C_7
Rule 8:	IF input1 is A_3	and input2 is B_2	THEN output is C_8
Rule 9:	IF input1 is A_3	and input2 is B_3	THEN output is C_6

fact: input1 is x_0 and input2 is y_0

---

consequence: output is C

The firing levels of the rules, denoted by α_i, $i = 1, 2, \ldots, 9$, are computed by:

$$\alpha_1 = \min\{A_1(x_0), B_1(y_0)\}$$
$$\alpha_2 = \min\{A_1(x_0), B_2(y_0)\}$$
$$\alpha_3 = \min\{A_1(x_0), B_3(y_0)\}$$
$$\alpha_4 = \min\{A_2(x_0), B_1(y_0)\}$$
$$\alpha_5 = \min\{A_2(x_0), B_2(y_0)\}$$
$$\alpha_6 = \min\{A_2(x_0), B_3(y_0)\}$$
$$\alpha_7 = \min\{A_3(x_0), B_1(y_0)\}$$
$$\alpha_8 = \min\{A_3(x_0), B_2(y_0)\}$$
$$\alpha_9 = \min\{A_3(x_0), B_3(y_0)\}$$

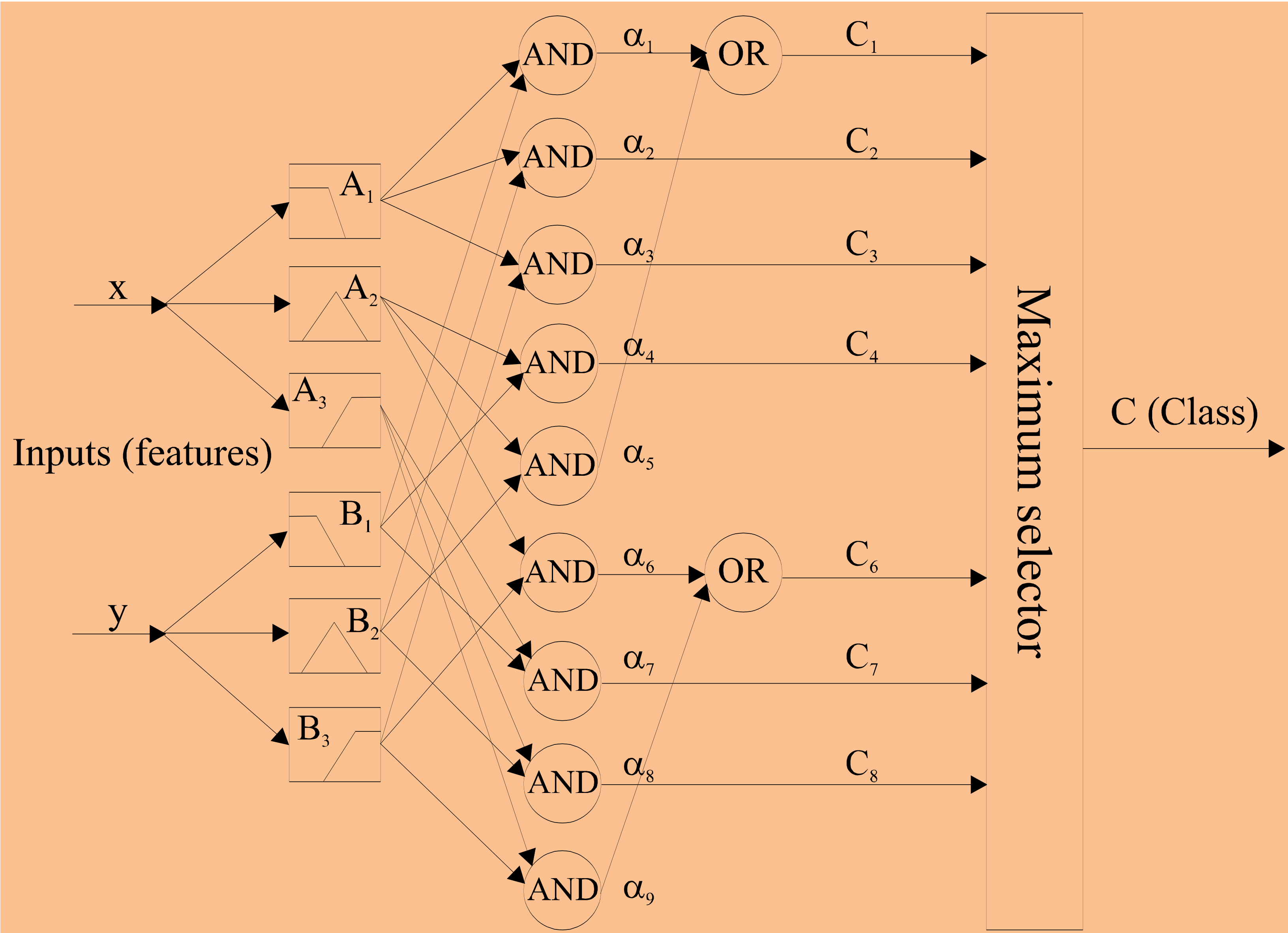
If several fuzzy rules have the same consequence class, their firing strengths have to be combined. Usually, the OR operation is used.
The individual rule outputs are computed by:

$C_1 = \alpha_1$ OR $\alpha_5 = \max \{\alpha_1 , \alpha_5 \}$
$C_2 = \alpha_2$
$C_3 = \alpha_3$
$C_4 = \alpha_4$
$C_6 = \alpha_6$ OR $\alpha_9 = \max \{\alpha_6 , \alpha_9 \}$
$C_7 = \alpha_7$
$C_8 = \alpha_8$

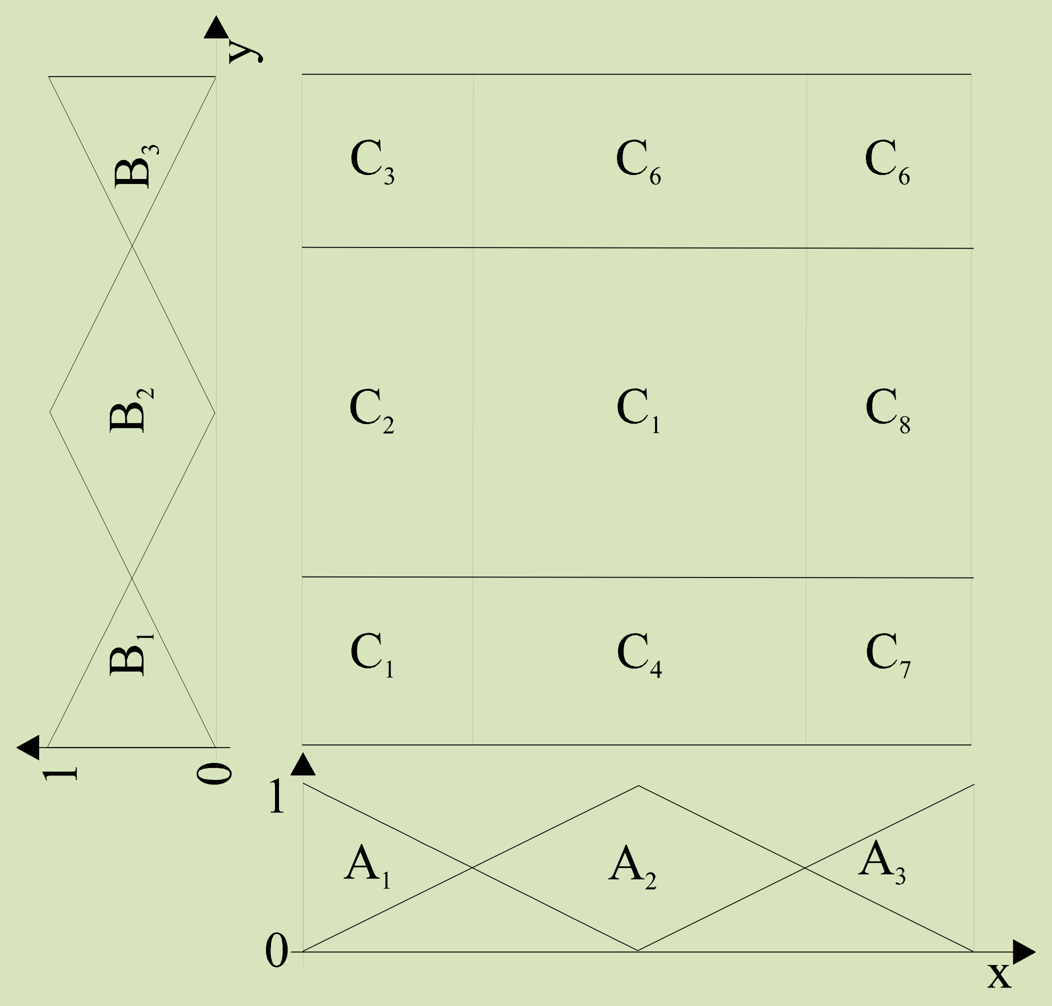
the overall classifier output is selected by

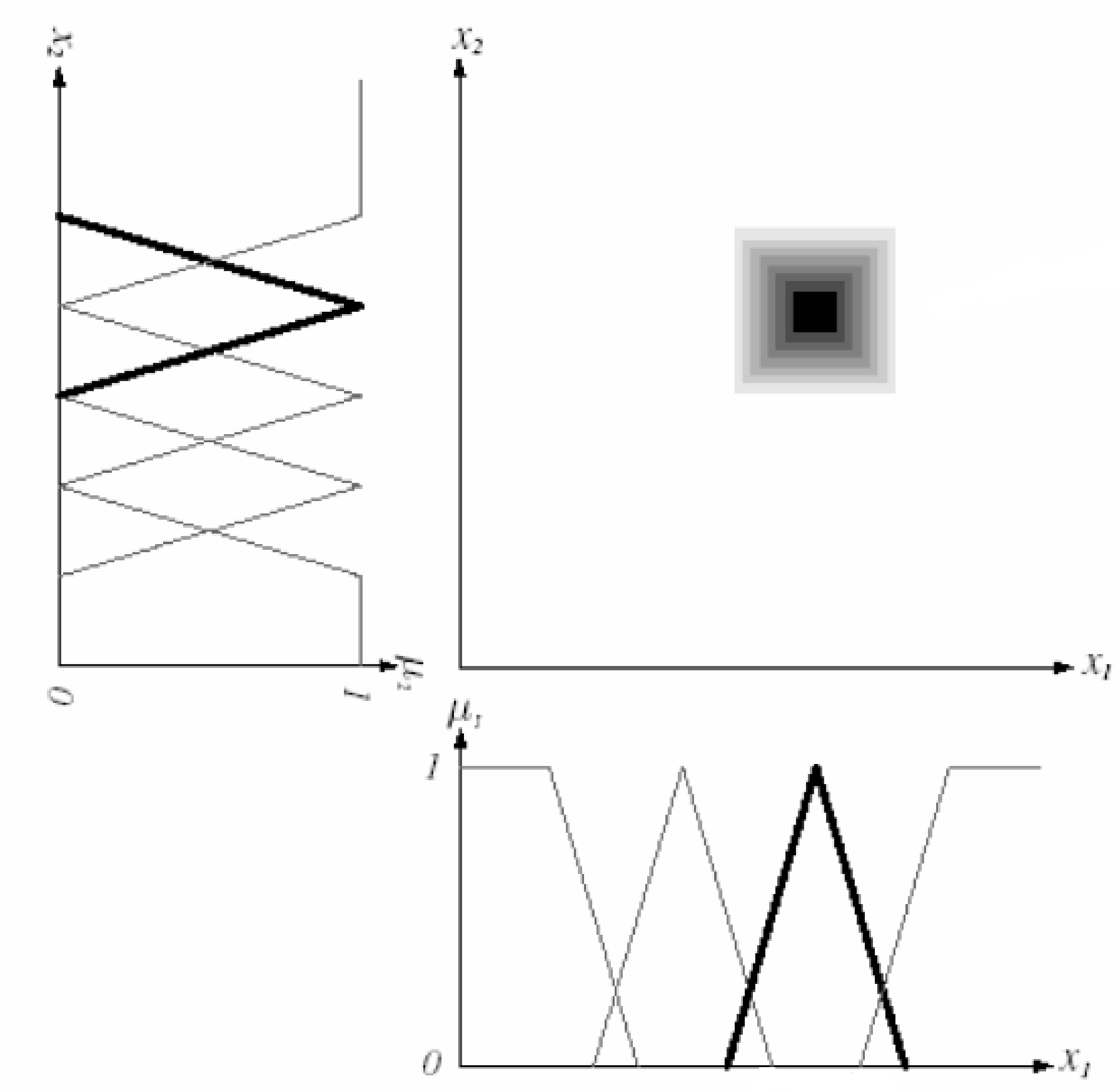
$C = \max\{ C1, C2, C3, C4, C6, C7, C8 \}$

The 2-input fuzzy rule based classifier with 9 rules.



Since each input feature is associated with three memberships, the input space (feature space) is partitioned into 9 fuzzy subspace, each of which is governed by a fuzzy IF-THEN rule.





x_2
μ_2
0
1
x_2
x_1
μ_1
1
0
x_1

Here I am explaining how the washing machine work when there is only two inputs these are:

1. Degree of dirt
2. Type of dirt

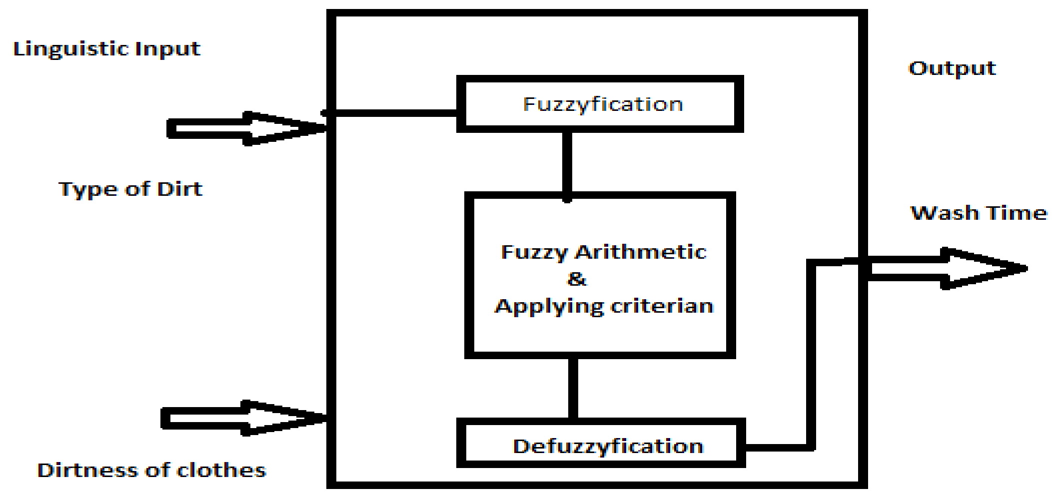
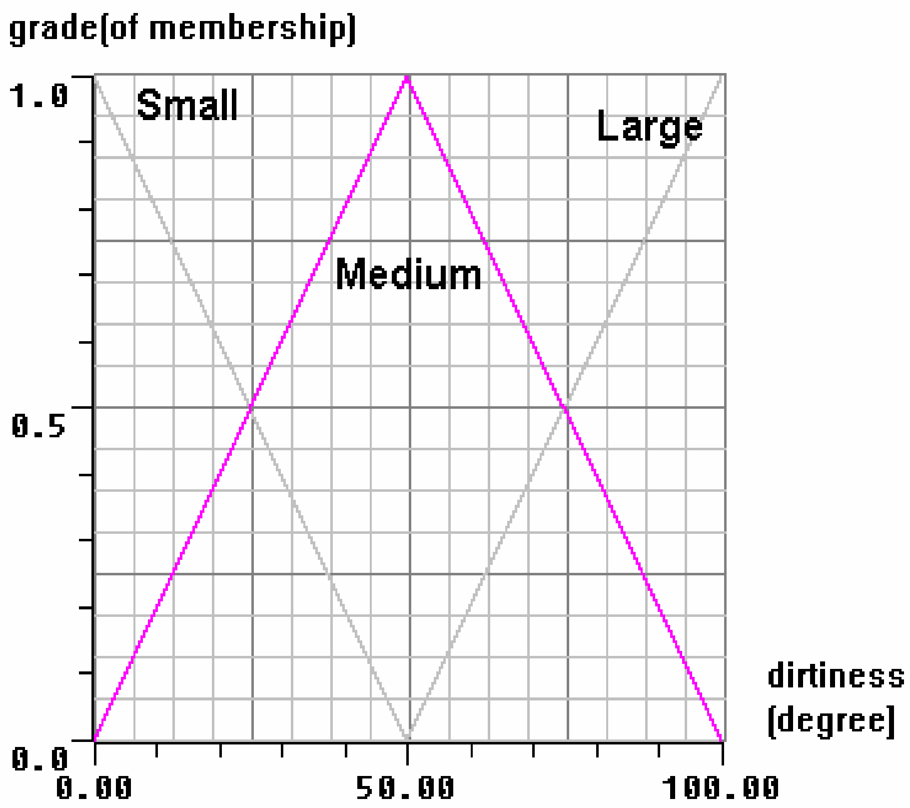


Fig.1. Basic block diagram of the process

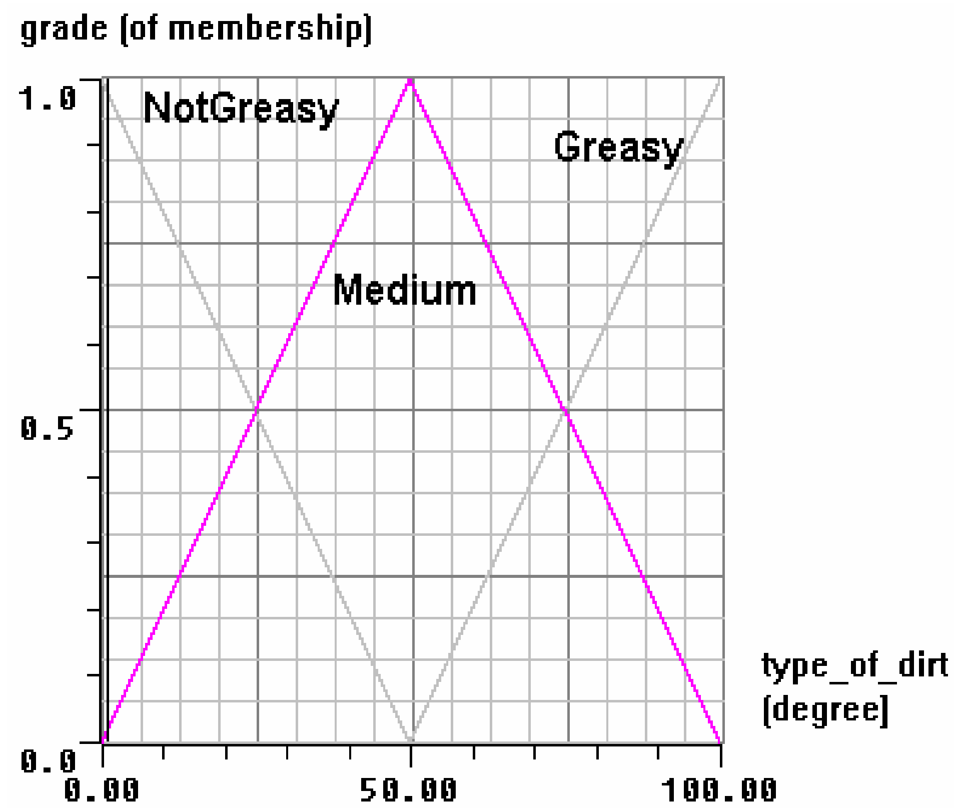
Above Fig. shows the basic approach to the problem. The fuzzy controller takes two inputs (as stated for simplification), processes the information and outputs a wash time. How to get these two inputs can be left to the sensors (optical, electrical or any type). Anyway the two stated points need a bit of introduction which follows. The degree of dirt is determined by the transparency of the wash water. The dirtier the clothes, less transparent the water being analyzed by the sensors is. On the other hand, type of dirt is determined by the time of saturation, the time it takes to reach saturation. Saturation is a point, at which there is no more appreciable change in the colour of the water. Degree of dirt determines how much dirty a cloth is. Where as Type of dirt determines the quality of dirt. Greasy cloths, for example, take longer for water transparency to reach transparency because grease is less soluble in water than other forms of dirt. Thus a fairly straight forward sensor system can provide us the necessary input for our fuzzy controller.

Details about the set applied

- Before the details of the fuzzy controller are dealt with, the range of possible values for the input and output variables are determined. These (in language of Fuzzy Set theory) are the membership functions used to map the real world measurement values to the fuzzy values, so that the operations can be applied on them.
- Figure 2 shows the labels of input and output variables and their associated membership functions. Values of the input variables degree of dirt and type of dirt are normalized range 1 to 100) over the domain of optical sensor.
- The decision which the fuzzy controller makes is derived from the rules which are stored in the database. These are stored in a set of rules. Basically the rules are if then statements that are intuitive and easy to understand, since they are nothing but common English statements. Rules used here are derived from common sense, data taken from typical home use, and experimentation in a controlled environment.
- The sets of rules used here to derive the output are:



Membership functions for dirtness of clothes
Fig.2(i)

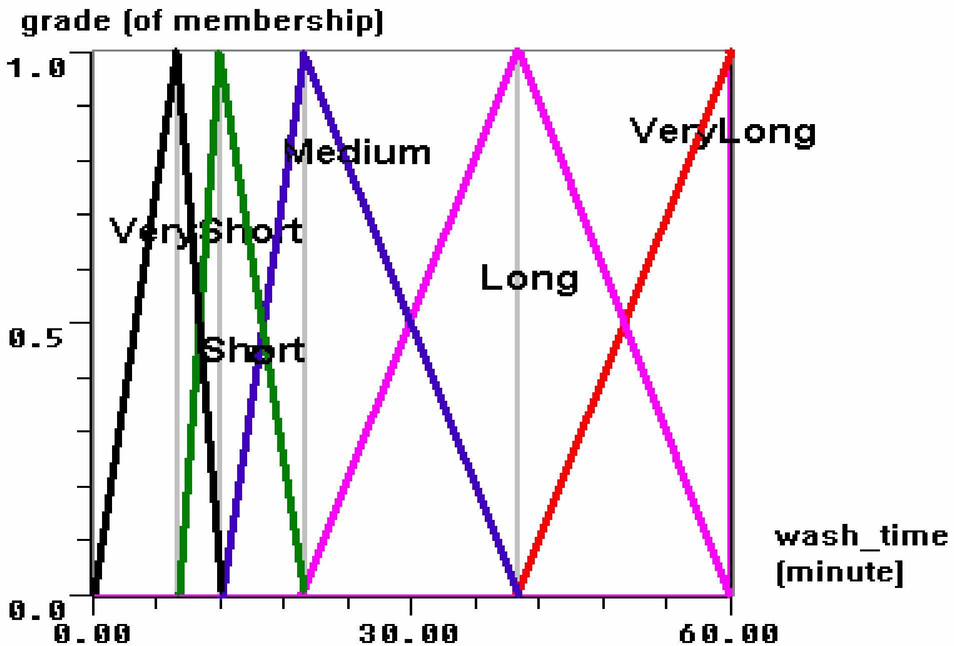


Membership function of types of dirt
Fig.2(ii)

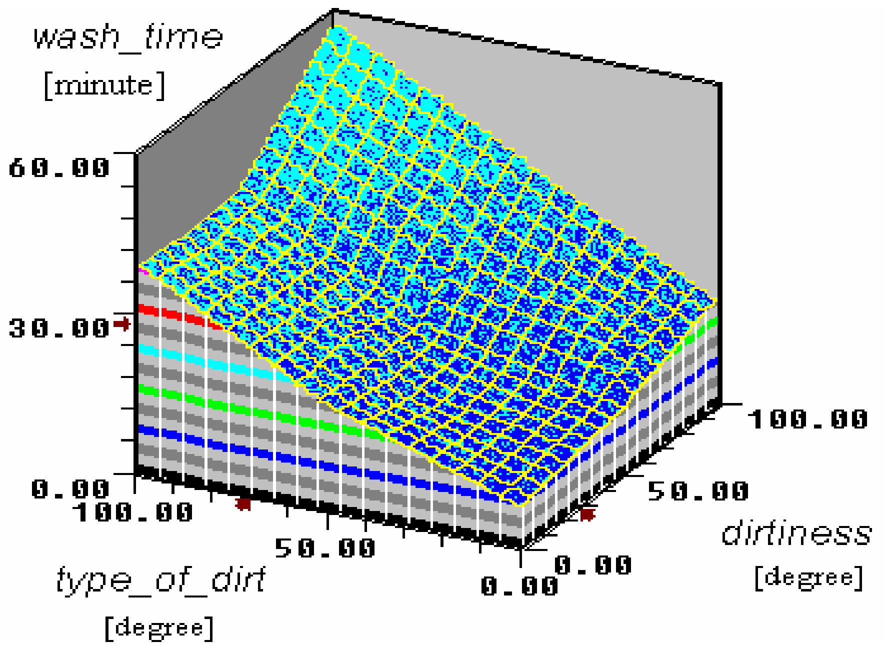
The sets of rules used here to derive the output are:

1. If dirtness of clothes is Large and type of dirt is Greasy then wash time is Very Long;
2. If dirtness of clothes is Medium and type of dirt is Greasy then wash time is Long;
3. If dirtness of clothes is Small and type of dirt is Greasy then wash time is Long;
4. If dirtness of clothes is Large and type of dirt is Medium then wash time is Long;
5. If dirtness of clothes is Medium and type of dirt is Medium then wash time is Medium;
6. If dirtness of clothes is Small and type of dirt is Medium then wash time is Medium;
7. If dirtness of clothes is Large and type of dirt is Not Greasy then wash time is Medium;
8. If dirtness of clothes is Medium and type of dirt is Not Greasy then wash time is Short;
9. If dirtness of clothes is Small and type of dirt is Not Greasy then wash time is Very Short

These rules have been shown as membership functions in figure below:



Labels and membership functions for output variable wash time. (Fig. 2)



Input/Output response surfaces

Summary:-

• By the use of fuzzy logic control we have been able to obtain a wash time for different type of dirt and different degree of dirt. The conventional method required the human interruption to decide upon what should be the wash time for different cloths.

In other words this situation analysis ability has been incorporated in the machine which makes the machine much more automatic and represents the decision taking power of the new arrangement. Though the analysis in this manner is very crude, but this clearly depicts the advantage of adding the fuzzy logic controller in the conventional washing machine.

END OF THE SLIDE

THANK YOU VERY MUCH

Differences between fuzzy logic and probability?

- Degrees of truth are often confused with probabilities.
- Fuzzy truth represents membership in vaguely defined sets, not likelihood of some event or condition.
- Fuzzy sets are based on vague definitions of sets, not randomness.
- Fuzzy logic is a new way of expressing probability
- Fuzzy logic and probability refer to different kinds of uncertainty
- Fuzzy logic is specifically designed to deal with imprecision of facts (fuzzy logic statements)
- while probability deals with chances of that happening (but still considering the result to be precise).
- Bart Kosko argues that probability is a sub-theory of fuzzy logic, as probability only handles one kind of uncertainty. He also claims to have proven a theorem demonstrating that Bayes' theorem can be derived from the concept of fuzzy sub-set-hood.
- Lotfi Zadeh, the creator of fuzzy logic, argues that fuzzy logic is different in character from probability, and is not a replacement for it. He has created a fuzzy alternative to probability, which he calls possibility theory.