Example 1.2. **Determine the decimal numbers represented by the following binary numbers:**

(*a*) 110101 (*b*) 101101 (c) 11111111.

Solution: (*a*)

$$(1101101)_2 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 32 + 16 + 0 + 4 + 0 + 1 = (53)_{10}$$

(*b*)

$$(101101)_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 32 + 0 + 8 + 4 + 0 + 1 = (45)_{10}$$

(*c*)

$$(11111111)_2 = 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$
$$= 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = (255)_{10}$$

EXERCISE 1.1

1. What are binary and decimal numbers ?
2. Convert the decimal integer – 349 to binary. [Ans. – $(101011101)_2$]
3. Convert the number with decimal fraction 0.3125 to binary fraction. [Ans. 0.0101]
4. Convert the binary number $(101011101)_2$ to decimal integer. [Ans. 349]
5. Convert the number given as a binary fraction $(0.11001010)_2$ to decimal. [Ans. 0.7890625]
6. Convert the given number in binary as $(101011101.1001010)_2$ [Ans. 349.7890625]

1.3 ACCURACY OF NUMBERS

1.3.1 Approximate Numbers

It will be appropriate for our discussion if we make a distinction between numbers which are exact (in the absolute sense) and those which express approximate values. The numbers like 2, 5, 9/2, 4.28, etc., are exact numbers because there is no approximation or uncertainty associated with them. However, we have numbers such as $\sqrt{2}$, π, e, etc., which are exact but cannot be expressed exactly by finite number of digits. When written in digital form, π must be written as 3.1416, e as 2.7183, etc. Such numbers are, therefore, approximations to the true values, and in such cases, they are called approximate numbers. **Approximate numbers** may, therefore be defined as a number which is used as an approximation to an exact number and differs only slightly from the exact number of which it stands.

1.3.2 Significant Figures

The digits that are used to express a number are called **significant digits or figures**. The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant figures and 0 is a **significant figure** when it is used to fix the decimal point or to fill the places of unknown digits, *i.e.*, 0 may or may not be a significant figure. It depends upon the position in which zero has been used.

Rules for Deciding the Number of Significant Figures in a Measured Quantity

1. All nonzero digits are significant: For example, 1.234 g has 4 significant figures, and 1.2 g has 2 significant figures.
2. Zeroes between nonzero digits are significant: For example, 1002 kg has 4 significant figures, and 3.07 ml has 3 significant figures.

3. Leading zeroes to the left of the first nonzero digits are not significant; such zeroes merely indicate the position of the decimal point:
   For example, 0.001 °C has only 1 significant figure, and 0.012 g has 2 significant figures.
4. Trailing zeroes that are also to the right of a decimal point in a number are significant. For example, 0.0230 ml has 3 significant figures, and 0.20 g has 2 significant figures.
5. When a number ends in zeroes that are not to the right of a decimal point, the zeroes are not necessarily significant.
   For example, 190 miles may be 2 or 3 significant figures, 50,600 calories may be 3, 4 or 5 significant figures.

The potential ambiguity in the last rule can be avoided by the use of standard exponential or "scientific," notation. For example, depending on whether the number of significant figures is 3, 4 or 5, we would write 50,600 calories as:

5.06×10^4 calories (3 significant figures)

5.060×10^4 calories (4 significant figures), or

5.0600×10^4 calories (5 significant figures).

By writing a number in scientific notation, the number of significant figures is clearly indicated by the number of *numerical figures* in the 'digit' term as shown by these examples. This approach is a reasonable convention to follow.

1.3.3 Rounding off

If we try to divide 22 by 7 we get $\frac{22}{7} = 3.142857143 \ldots$ a quotient that never terminates. If we have to use numbers in a practical computation, we must cut it down to manageable form, such as 3.14 or 3.143 or 3.1432, etc. This process of cutting off superfluous digits and retaining as many as we desire, is called rounding off.

The round off of a number is to retain a certain number of digits, counted from the left and dropped the pothers. Thus to round off *e* to three, four, five and six figures respectively, we have 2.72, 2.718, 2.7183, 2.71828.

Numbers are rounded off so as to cause the *least possible error*.

Generally, we use the following rule to round-off a number to n significant figures.

(*a*) Discard all digits to the right of the nth digit.

(*b*) If this discarded digit is less than half a unit in the nth place, leave the nth digit unaltered and if greater than half a unit in the nth place, increase the nth digit by unity.

(*c*) If this discarded digit is exactly half a unit in the nth place, increase nth digit by unit if it is odd, otherwise leave it unchanged.

SOLVED EXAMPLES

Example 1.3. Round off the following numbers correct to four significant figures: 3.26425, 687.543, 4985561, 0.70035, 0.00032217, 18.265101.

Solution: Here we have to retain first four significant figures, therefore

3.26425 becomes 3.264

687.543 becomes 687.5

4985561 becomes 4986000

0.70035 becomes 0.7004
0.00032217 becomes 0.0003222
18.265101 becomes 18.26

Example 1.4. **Find the sum of the following approximate numbers, each being correct to its last figures.**

396.56, 657.2, 758.9286, 3.052

Solution: Since the number 657.2 is correct to one decimal place, it is not worthwhile to retain digits beyond two decimal places. Hence we rounded-off the given number upto two decimal places, and then found the sum. Therefore the required sum

$$= 396.56 + 675.20 + 758.98 + 3.05$$
$$= 1815.79$$
$$= 1815.8$$

EXERCISE 1.2

1. Round-off the following numbers correct to four significant figures.
68.3643, 878.367, 8.7265, 56.395 **[Ans.** 68.36, 878.4, 8.726, 56.40]
2. Find the sum of the following approximate numbers, each being correct to its last figures.
281.45, 523.3, 6.051, 438.8836 **[Ans.** 1258.7]

1.4 ERRORS

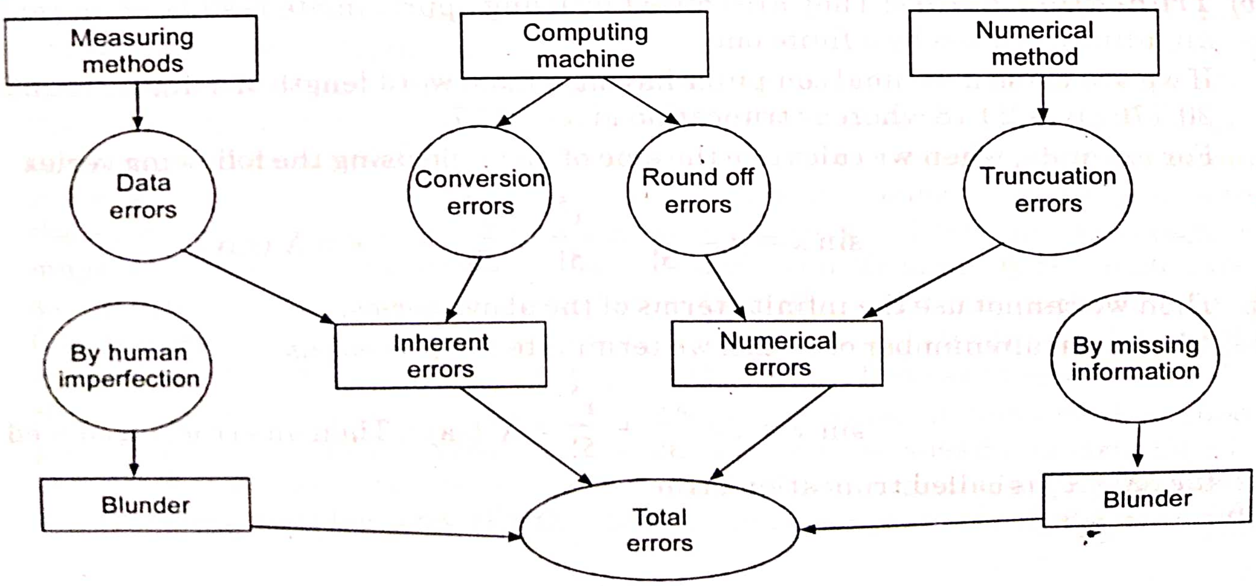
One of the most important aspects of numerical analysis is the error analysis. Error may occur at any stage of the process of solving a problem.

By error we mean the difference between true value and approximate value.

$\therefore$ $\boxed{\text{Error} = \text{True value} - \text{Approximate value}}$

1.5 TYPES OF ERRORS

In any numerical computation results, we come across the following types of errors:



(i) ***Inherent Errors:*** Inherent errors are the errors that pre exist in the problem statement itself before its solution is obtained. Inherent errors exist because the data is approximate or due to the limitations of the calculations using digital computers. Inherent errors cannot be completely eliminated but can be minimized if we select better data or by employing high precision computing aids.

(ii) ***Rounding Errors:*** It occurs from the process of rounding off the numbers during the computations, *i.e.*, it occurs when a fixed number of digits is used to represent exact numbers. Such errors are unavoidable in most of the calculations due to the limitations of the computing aids.

These errors can be reduced however by

(*a*) Changing the calculation procedures so as to avoid subtraction of nearly equal numbers or division by a small number.

(*b*) Retaining atleast one more significant digit at each step and rounding off at last step.

(iii) ***Chopping:*** In chopping, the extra digits are dropped. This is called truncating the number. If we are using a computer with a fixed word length of four digits, then a number like 42.7893 will be written as 42.78 and the digit 93 will be dropped. This is called chopping. We can express the number 42.7893 in floating point form as

$$x = 0.427893 \times 10^2 = (0.4278 + 0.000093) \times 10^2$$
$$= [(0.4278 + (0.93 \times 10^{-4})] \times 10^2$$

This can be expressed in general form as

True $$x = (f_x + g_x \times 10^{-d})\, 10^E = f_x \times 10^E + g_x \times 10^{E-d}$$
$$= \text{approximate } x + \text{error}$$

Where f_x is the mantissa, d is the length of the mantissa permitted and E is the exponent. In chopping, g_x is ignored entirely and therefore,

$$\text{Error} = g_x \times 10^{E-d},\ 0 \le g_x < 1$$

The absolute error introduced depends on the following:

1. The size of the digits dropped.
2. Number of digits in mantissa.
3. The size of the number.

Since the maximum value of g_x is less than 1.0. $\therefore$ **Absolute error $\le 10^{E-d}$**

(iv) ***Truncation Errors:*** They are caused by using approximate results or on replacing an infinite process by a finite one.

If we are using a decimal computer having a fixed word length of 4 digits; rounding of 20.776 gives 20.78 whereas truncation gives 20.77.

For example, when we calculate the sine of an angle using the following series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \infty = X'(\text{say})$$

Then we cannot use the infinite terms of the above series.

After a certain number of terms, we terminate the process as

$$\sin x = x = \frac{x^3}{3!} + \frac{x^5}{5!} = X' \text{ (say)}.$$ Then an error introduced here (*i.e.*, $X - X'$) is called truncation error.

(v) ***Accumulated Errors:*** Consider the following:

$$X_{i+1} = 1000\,X_i\ ;\ i = 0, 1, 2, 3, \ldots$$

$$X_1 = 1000\,X_0;\ X_2 = 1000\,X_1;\ X_3 = 1000\,X_2$$

Let the exact value of $X_0 = 4.99$. Suppose we start with $X_0 = 5$. So there is an Inherent Error of 0.01.

$$X_1 = 1000\,X_0 = 1000 \times 5 = 5000$$

$$X_2 = 1000\,X_1 = 1000 \times 5000 = 5000000$$

$$X_3 = 1000\,X_2 = 1000 \times 5000000 = 5000000000 \text{ and so on.}$$

The following table shows the errors between exact values and computed values.

Variable	Exact Value	Computed Value	Accumulated Error
X_0	4.99	5	0.01
X_1	4990	5000	10
X_2	4990000	5000000	10000
X_3	4990000000	5000000000	10000000

Note that table shows how the error gets accumulated. A small error of 0.01 at X_0 leads to an error of 10000000 in X_3. This is called **accumulation of errors.**

In a sequence of computations, the error in one value may affect the computation of the next value and the error gets accumulated. This is called the ***accumulated error.***

The ***Relative Accumulated Error*** is the ratio of the accumulated error to the exact value of that iteration. For the above example, the relative accumulated error is shown in the table below.

Variable	Exact Value	Computed Value	Accumulated Error	Relative Accumulated Error
X_0	4.99	5	0.01	0.01/4.99 = 0.002004
X_1	4990	5000	10	10/4990 = 0.002004
X_2	4990000	5000000	10000	10000/4990000 = 0.002004
X_3	4990000000	5000000000	10000000	10000000/4990000000 = 0.002004

(vi) ***Modelling Errors:*** Mathematical models are the basis for numerical solutions. They are formulated to represent physical processes using certain parameters involved in the situations. In many situations, it is impossible to include all the real problem and therefore, certain simplifying assumptions are made. For example, while developing a model for calculating the four acting on a falling body, we may not be able to estimate the air resistance coefficient (drag coefficient), properly or determine the direction and magnitude of wind force acting on the body and so on. To simplify the model, we may assume that the force due to air resistance is linearly proportional to the velocity of the falling body or we may assume that there is no wind force acting on the body. All such simplifications certainly result in errors in the output from such models.

Since a model is a basic input to the numerical process, no numerical method will provide adequate results if the model is erroneously conceived and formulated. It is obvious that we can reduce these types of errors by refining or enlarging the models by incorporating more features. But the enhancement may make the model more difficult to solve or may take more time to implement the solution process. It is also not always true that an enhanced model will provide better results. Note that modelling, data quality and computation go hand in hand. An overly refined model with inaccurate data or an inadequate computer may not be meaningful. On the other hand, an over-simplified model may produce a result that is unacceptable. It is, therefore, necessary to strike a balance between the level of accuracy and the complexity of the model. A model must incorporate only those features that are essential to reduce the error to an acceptable level.

(vii) ***Blunders:*** Blunders are errors that are caused due to human imperfection. As the name indicates, such errors may cause a very serious disaster in the result. Since these errors are due to human mistakes, it should be possible to avoid them to a large extent by acquiring a sound knowledge of all the aspects of problems as well as the numerical process.

Human errors can occur at any stage of numerical processing cycle. Some common types of errors are:

1. Lack of understanding of the problem.
2. Wrong assumptions.
3. Overlooking of some basic assumptions required for formulating the model.
4. Errors are deriving the mathematical equation or using a model that does not describe adequately the physical system under study.
5. Selecting a wrong numerical method for solving the mathematical problems.
6. Selecting a wrong algorithm for implementing the numerical method.
7. Making mistakes in the computer programs, such as testing a real number for zero and using < symbol in place of > symbol.
8. Mistakes in data input, such as misprints, giving value columnwise instead of rowwise to a matrix, forgetting a negative sign, etc.
9. Wrong guessing of initial values.

All these mistakes can be avoided through a reasonable understanding after problems and the numerical solution methods, and use of good programming techniques and tools.

1.6 MEASUREMENT OF ACCURACY

In the field of Science and Engineering, basic requirements are numerical results which are used in many applications. We use different kinds of numbers and formulas for obtaining the best possible results. But in most cases, the data being used to solve such type of problems is not always exact. They are approximated correct to certain number of significant digits. But these approximations introduced errors in the calculated results. Hence our main objective is to obtain a approximate result as close to the exact result as possible.

In a numerical computation, the round off errors are difficult to estimate, so its effect on the final results has to be reduced by some specific rules. However, the truncation errors can be easily estimated and can be reduced effectively. Thus in any case, we need some measures of accuracy of the results which are discussed below.

1.6.1 Absolute, Relative and Percentage Errors

If X_E is the exact or true value of a quantity and X_A is its approximate value, then $|X_E - X_A|$ is called the absolute error E_a.

Therefore, $E_a = |X_E - X_A|$

and relative error is defined as

$$E_r = \left|\frac{X_E - X_A}{X_E}\right|$$

provided $X_E \neq 0$ or X_E is not close to 0. The percentage relative error is

$$E_p = 100\, E_r = 100 \left|\frac{X_E - X_A}{X_E}\right|$$

Remarks:

1. The relative and percentage errors are independent of units used while absolute error is expressed in terms of these units.
2. If X_A is the approximate value of X_E correctly rounded to m decimal places then
$$|X_E - X_A| \leq \frac{1}{2} \times 10^{-m}$$
3. If X_A is the approximate value of X_E after truncation to k digits, then
$$\left|\frac{X_E - X_A}{X_E}\right| < 10^{-k+1}$$
4. If X_A is the approximate value of X_E after rounding-off to k digit, then
$$\left|\frac{X_E - X_A}{X_E}\right| < \frac{1}{2} \times 10^{-k+1}$$
5. If X_A is the approximate value of X_E correct to m significant digits, then
$$\left|\frac{X_E - X_A}{X_E}\right| < 10^{-m}$$
6. If a number is correct to n significant figures, and the first significant digit of the number is α_m, then the relative error
$$E_r < \frac{1}{\alpha_m 10^{n-1}}.$$

SOLVED EXAMPLES

Example 1.5. If 0.333 is the approximate value of $\frac{1}{3}$, find absolute, relative and percentage errors.

Solution: True or exact value $(X_E) = \frac{1}{3}$

Approximate value $(X_A) = 0.333$

$\therefore$ Absolute error, $E_a = |X_E - X_A| = \left|\frac{1}{3} - 0.333\right|$

$= |0.333333 - 0.333| = 0.000333$

Relative error, $E_r = \left|\frac{X_E - X_A}{X_E}\right| = \left|\frac{0.000333}{0.333333}\right| = 0.000999$

Percentage error, $E_p = 100 \times E_r$

$= 100 \times 0.000999 = 0.099\%.$

Example 1.6. **If $\pi = \frac{22}{7}$ is approximated as 3.14, find the absolute error, relative error and percentage error.**

Solution: True or exact value $(X_E) = 3.14$

$$\text{Approximate value } (X_A) = \frac{22}{7}$$

$$\therefore \quad \text{Absolute error } (E_a) = |X_E - X_A|$$

$$= \left|\frac{22}{7} - 3.14\right| = \left|\frac{22 - 21.98}{7}\right|$$

$$= \left|\frac{0.02}{7}\right| = 0.002857$$

$$\text{Relative error } (E_r) = \left|\frac{X_E - X_A}{X_E}\right|$$

$$= \left|\frac{0.002857}{\frac{22}{7}}\right| = 0.0009$$

Percentage error, $E_p = 100 \times E_r$

$= 100 \times 0.0009 = 0.09\%.$

Example 1.7. **Round off the numbers 865250 and 37.46235 to four significant figures and compute E_a, E_r and E_p.**

Solution: The rounded off numbers are 865200 and 37.46 correct to four significant figures.

Now

For 865250

$$\text{Absolute error } (E_a) = |X_E - X_A| = |865250 - 865200| = 50$$

$$\text{Relative error } (E_r) = \left|\frac{X_E - X_A}{X_E}\right| = \left|\frac{50}{865250}\right| = 6.71 \times 10^{-5}$$

and Percentage error $(E_p) = 100 \times E_r = 100 \times 6.71 \times 10^{-5} = 6.71 \times 10^{-3}$

For 37.46235

$$\text{Absolute error } (E_a) = |X_E - X_A| = |37.46235 - 37.46| = 0.00235$$

$$\text{Relative error } (E_r) = \left|\frac{X_E - X_A}{X_E}\right| = \left|\frac{0.00235}{37.46235}\right| = 6.27 \times 10^{-5}$$

and Percentage error $(E_p) = 100 \times E_r$

$= 100 \times 6.27 \times 10^{-5} = 6.27 \times 10^{-3}$

Example 1.8. **Find the absolute error and the relative error for the sum,**

$$S = \sqrt{3} + \sqrt{5} + \sqrt{7} \text{ to four significant digits.}$$

Solution: We know that

$$\sqrt{3} = 1.732, \ \sqrt{5} = 2.236 \text{ and } \sqrt{7} = 2.646$$

$\therefore$ $$S = 1.732 + 2.236 + 2.646 = 6.614$$

and $$E_a = 0.0005 + 0.0005 + 0.0005 = 0.0015$$

The total absolute error shows that the sum is correct to 3 significant figures only.

$\therefore$ We take, $$S = 6.61$$

and therefore, $$E_r = \frac{0.0015}{6.61} = 0.0002.$$

Example 1.9. **Two given numbers 2.5 and 48.287 are correct to three significant figures given. Evaluate this product.**

Solution: Between the given numbers 2.5 has the greater absolute error. Therefore, round-off the other number to three significant digits (one digit more than the first number) to get 48.3.

The required product is given by

$$48.3 \times 2.5 = 120.75$$

Since one of the numbers, 2.5 has only two digits, with product retained only to two significant digits, the product to be taken is 120.

That is, $$= 1.2 \times 10^2.$$

Example 1.10. **If $X_E = \frac{8}{9}$ and the exact decimal representation of X is 0.888 ... Verify that relative error is less than $\frac{1}{2} \times 10^{-3}$.**

Solution: We have $$X = \frac{8}{9}, \ k = 3$$

The decimal representation of X rounded-off to three decimal digits is $X_A = 0.889$.

Then
$$E_a = \left|\frac{8}{9} - 0.889\right|$$
$$= \left|\frac{8}{9} - \frac{889}{1000}\right|$$
$$= \left|\frac{8000 - 8001}{9 \times 10^3}\right|$$
$$= \left|\frac{-1}{9 \times 10^3}\right|$$
$$= \frac{1}{9} \times 10^{-3} < \frac{1}{2} \times 10^{-3}$$

$\therefore$ $$E_a < \frac{1}{2} \times 10^{-3}. \quad \text{Hence verified.}$$

Chapter 2

Solution of Algebraic and Transcendental Equations

INSIDE THIS CHAPTER

2.1 INTRODUCTION

In applied mathematics, the most frequently occurring problem is to find the roots of equations of the form

$$f(x) = 0$$

The equation $f(x) = 0$ is called *Algebraic* or *Transcendental* according as $f(x)$ is purely a polynomial in x or contains some other functions such as logarithmic, exponential and trigonometric functions, etc. For example, $1 + \cos x - 5x$, $x \tan x - \cosh x$, $e^{-x} - \sin x$ are transcendental functions. A polynomial in x of degree n is an expression of the form $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$, where a's are constants and n is a positive integer. The zeroes or roots of the polynomial $f(x)$ are those values of x for which $f(x)$ is zero. Geometrically, if the graph of $f(x)$ crosses the x-axis at the point $x = a$, then $x = a$ is a root of the equation $f(x) = 0$. We conclude that 'a' is a root of the equation $f(x) = 0$ if and only if $f(a) = 0$.

By finding the solution to an equation $f(x) = 0$, we mean to find zeroes of $f(x)$. In this chapter, we shall discuss graphical and some numerical methods for the solution of equations of the form $f(x) = 0$. Here $f(x)$ may be algebraic or transcendental or a combination of both.

Theorem 1. If a function $f(x)$ assumes values of opposite sign at the end points of internal (a, b), *i.e.*, $f(a) f(b) < 0$ then the internal will contain at least one root of the equation $f(x) = 0$, in other words there will be at least one number $c \in (a, b)$ such that $f(c) = 0$. Throughout our discussion in this chapter we assume that:

(*i*) $f(x)$ is continuous and continuously differentiable upto sufficient number of times.

(*ii*) $f(x) = 0$ has no multiple root that is if c is a real root $f(x) = 0$ then $f(c) = 0$ and $f'(x) < 0$, $f'(x) > 0$ in (a, b), see Fig. 2.1

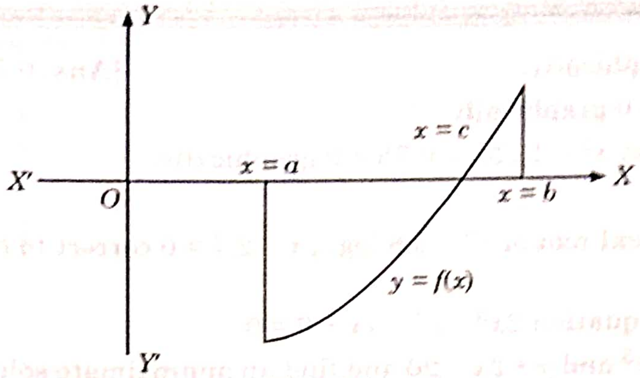


Fig. 2.1

2.2 GRAPHICAL SOLUTION TO EQUATIONS

The real root of the equation $f(x) = 0$ can be determined approximately as the abscissa of the points of intersection of the graph of the function. $y = f(x)$ with the X-axis of $f(x)$ is simple we shall draw the graph of $y = f(x)$ with respect to a rectangular axis X'OX and Y'OY. The points at which the graph meets the X-axis are the location of the roots of (1) if $f(x)$ is not simple then replace equation (1) by an equivalent equation say $\phi(x)$ and $y = \psi(x)$ where $\phi(x)$ and $\psi(x)$ are simpler than $f(x)$. Then we construct the graph of $y = \phi(x)$ and $y = \psi(x)$ then the X-coordinate of the point of intersection of the graphs gives the crude approximation of the real roots of the equation $f(x) = 0$.

SOLVED EXAMPLES

Example 2.1. **Solve the equation $x \log_{10} x = 1$ graphically.**

Solution: The given equation $x \log_{10} x = 1$

Can be written as $\log_{10} x = \dfrac{1}{x}$

where $\log_{10} x$ and $\dfrac{1}{x}$ simpler than $x \log_{10} x$.

then $y = \log_{10} x$ and $y = \dfrac{1}{x}$, we get x-coordinate of the point of intersection

$\therefore$ The approximate values of the root of $x \log_{10} x = 0.1$ is $c = 2.5$.

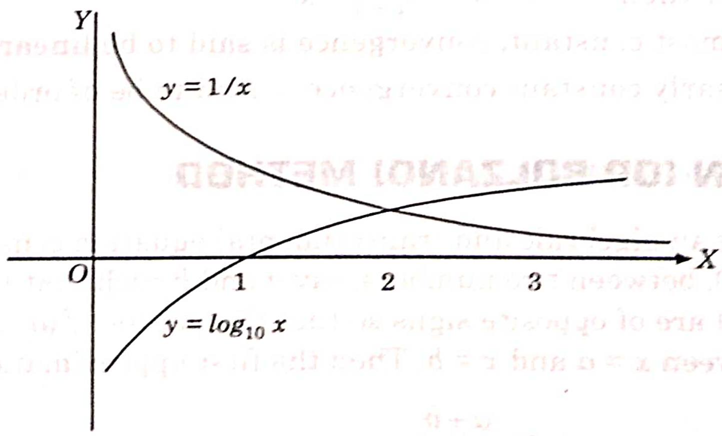


Fig. 2.2.

EXERCISE 2.1

1. Solve $x^2 + x - 1 = 0$ graphically. [Ans. 0.6 and – 1.6 (approximate
2. Solve $-e^{2x} + 2x + 0.1 = 0$ graphically. [Ans. 0.3 (approximate
3. Solve the cubic equation $x^3 - 1.75x + 0.75 = 0$ graphically. [Ans. – 1.5, 0.5 and
4. Solve $x^3 + 2x + 7.8 = 0$. [Ans. – 1.
5. Solve graphically the real root of $x^3 - 3.6 \log_{10} x - 2.7 = 0$ correct to two decimal places. [Ans. 1.
6. Solve graphically the equation $2x^3 - x^2 - 7x + 6 = 0$. [Ans. 1, 1.5, –
7. Draw the graph of $y = x^3$ and $y + 2x = 20$ and find an approximate solution to the equation $x^3 +$ $- 20 = 0$. [Ans. 2.4
8. Solve $x^3 + 10x - 15 = 0$ graphically. [Ans. 1.2
9. Solve graphically the following equations in the range $(0, \pi/2)$; (*i*) $x = \cos x$ (*ii*) $e^x =$ (*iii*) $x = \tan x$. [Ans. (*i*) 0.74 (*ii*) 0.36, 2.15 (*iii*) 4.49

2.3 RATE OF CONVERGENCE

Let $x_0, x_1, x_2, \ldots$ be the values of a root (α) of an equation at the 0th, 1st, 2nd,... iterations whil its actual value is 3.5567, the values of this root calculated by three different methods are a given below:

Root	Ist method	2nd method	3rd method
x_0	5	5	5
x_1	5.6	3.8527	3.8327
x_2	6.4	3.5693	3.56834
x_3	8.3	3.55798	3.55784
x_4	9.7	3.55687	3.55672
x_5	10.6	3.55676	
x_6	11.9	3.55671	

The values in the 1st method do not converge towards the root 3.5567. In the 2nd and 3rd methods, the values converge to the root after 6th and 4th iterations respectively. Clearly 3rd method converges faster than the 2nd method. This fastness of convergence in any method is represented by its rate of convergence.

- if e be the error then $e_i = \alpha - x_i = x_{i+1} - x_i$
- if e_{i+1}/e_i is almost constant, convergence is said to be linear, *i.e.*, slow.
- if e_{i+1}/e_i^p is nearly constant convergence is said to be of order p, *i.e.*, faster.

2.4 BISECTION (OR BOLZANO) METHOD

This method of solving an algebraic and transcendental equation consists in locating the root of the equation $f(x) = 0$, between two numbers, say a and b such that $f(x)$ is continuous for $a \le$ $x \le b$ and $f(a)$ and $f(b)$ are of opposite signs so that the product $f(a) \cdot f(b) < 0$, *i.e.*, the curve crosses the x-axis between $x = a$ and $x = b$. Then the first approximation to the root is

$$x_1 = \frac{a+b}{2}.$$

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$, otherwise root lies between a and x_1 or x_1 and b according as $f(x_1)$ is +ve or –ve.

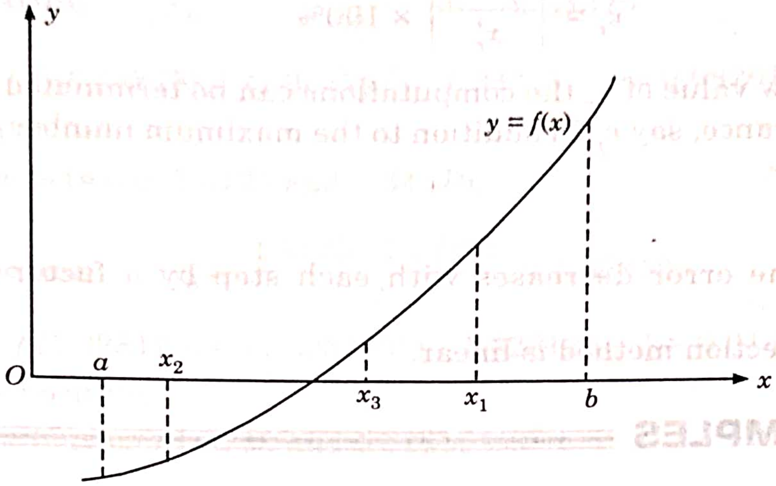


Fig. 2.3.

Then as before we bisect the interval and continue the process until the root is found to desired accuracy. In the figure above, $f(x_1)$ is +ve, therefore root lies between a and x_1. Then, the second approximation to the root is

$$x_2 = \frac{a + x_1}{2}$$

If $f(x_2)$ is –ve, the root lies between x_1 and x_2 and the third approximation to the root is

$$x_3 = \frac{x_1 + x_2}{2}$$

We repeat the process until the root is known to the desired accuracy.

Remark 1: Since the new interval containing the root, is exactly half the length of the previous one, the interval width is reduced by a factor of $\frac{1}{2}$ at each step. At the end of the nth step, the new interval will therefore, be of length $\left(\frac{b-a}{2^n}\right)$ if on repeating this process n times the latest interval is as small as given ε, then $\left(\frac{b-a}{2^n}\right) \leq \varepsilon$.

or $$n \geq [\log(b - a) - \log \varepsilon]/\log 2$$

This gives the number of iterations required for achieving an accuracy ε. If $|b - a| = 1$ and $\varepsilon = 0.001$ then it can be $n \geq 0$.

In particular, the minimum number of iterations required for converging to a root in the interval (0, 1) for a given ε are as under

ε:	10^{-2}	10^{-3}	10^{-4}
n:	7	10	14

A convenient criterion is to compute the percentage error ϵ_r defined by

$$\varepsilon_r = \left|\frac{x'_r - x_r}{x'_r}\right| \times 100\%$$

where x'_r is the new value of x_r the computations can be terminated when ε_r becomes less than a prescribed tolerance, say ε_p in addition to the maximum number of iterations may also be specified in advance.

Remark 2. As the error decreases with each step by a factor at $\frac{1}{2}$ $\left(i.e., \frac{e_{n+1}}{e_n} = \frac{1}{2}\right)$ convergence in the bisection method is linear.

SOLVED EXAMPLES

Example 2.2. Find a real root of the equation $x^3 - x = 1$ lying between 1 and 2 by bisection method.

Solution: Let $f(x) = x^3 - x - 1 = 0$

Since $f(1) = 1^3 - 1 - 1 = -1$, which is negative.

and $f(2) = 2^3 - 2 - 1 = 5$, which is positive.

Hence root lies between 1 and 2.

First Approximation: $x_1 = \frac{1+2}{2} = 1.5$

Then $f(1.5) = (1.5)^3 - 1.5 - 1 = 3.375 - 1.5 - 1 = 0.875$

$\therefore$ $f(1.5)$ is positive.

Hence root lies between 1 and 1.5.

Second Approximation: $x_2 = \frac{1+1.5}{2} = 1.25$

Then $f(1.25) = (1.25)^3 - 1.25 - 1$

$= 1.953 - 2.25 = -0.297 < 0$

$\therefore$ $f(1.25)$ is negative.

Hence the root lies between 1.25 and 1.5.

Third Approximation: $x_3 = \frac{1.25+1.5}{2} = 1.375$

Now $f(1.375) = (1.375)^3 - 1.375 - 1 = 0.2246$

$\therefore$ $f(1.375)$ is positive.

Hence the root lies between 1.25 and 1.375.

Fourth Approximation: $x_4 = \frac{1.25+1.375}{2} = 1.3125$

Now, $f(1.3125) = (1.3125)^3 - 1.3125 - 1 = -0.0515$

$\therefore$ $f(1.3125)$ is negative.

Hence the root lies between 1.3125 and 1.375.

Fifth Approximation: $x_5 = \dfrac{1.3125 + 1.375}{2} = 1.34375$

Now, $f(1.34375) = (1.34375)^3 - 1.34375 - 1 = 0.08261$

$\therefore$ $f(1.34375)$ is positive.

Hence the root lies between 1.3125 and 1.34375.

Sixth Approximation: $x_6 = \dfrac{1.3125 + 1.34375}{2} = 1.328125$

Then $f(1.328125) = (1.328125)^3 - 1.328125 - 1 = 0.01457$

$\therefore$ $f(1.328125)$ is positive.

Hence the root lies between 1.3125 and 1.328125.

Seventh Approximation: $x_7 = \dfrac{1.3125 + 1.328125}{2} = 1.3203125$

Hence the real root of the given equation is 1.32 correct to two decimal places after computing seven iterations.

Example 2.3. **Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages.**

Solution: Let $f(x) = x^3 - 4x - 9$

Since, $f(2.706) = -0.009488$ which is negative.

and $f(2.707) = 0.008487$ which is positive.

Hence root lies between 2.706 and 2.707.

First Approximation: $x_1 = \dfrac{2.706 + 2.707}{2} = 2.7065$

Now, $f(x_1) = -0.0005025$

$\therefore$ $f(2.7065)$ is negative.

Hence root lies between 2.7065 and 2.707.

Second Approximation: $x_2 = \dfrac{2.7065 + 2.707}{2} = 2.70675$

Now, $f(x_2) = 0.003992$

$\therefore$ $f(2.70675)$ is positive.

Hence root lies between 2.7065 and 2.70675.

Third Approximation: $x_3 = \dfrac{2.7065 + 2.70675}{2} = 2.706625$

Now, $f(x_3) = 0.001744$

$\therefore$ $f(2.706625)$ is positive.

Hence root lies between 2.7065 and 2.706625.

Fourth Approximation: $x_4 = \dfrac{2.7065 + 2.706625}{2} = 2.7065625$

Hence the root is 2.706 correct to three decimal places.

Example 2.4. **Find the real root of the equation $x \log_{10} x = 1.2$ by bisection method correct to four decimal places.**

Solution: Let $f(x) = x \log_{10} x - 1.2$

Since, $f(2.74) = -0.000563$ which is negative.

and $f(2.75) = 0.0081649$ which is positive.

Hence a root lies between 2.74 and 2.75.

First Approximation: $x_1 = \dfrac{2.74 + 2.75}{2} = 2.745$

Now, $f(x_1) = f(2.745) = 0.003798$

$\therefore$ $f(2.745)$ is positive.

Hence root lies between 2.74 and 2.745.

Second Approximation: $x_2 = \dfrac{2.74 + 2.745}{2} = 2.7425$

Now, $f(x_2) = f(2.7425) = 0.001617$

$\therefore$ $f(2.7425)$ is positive.

Hence root lies between 2.74 and 2.7425.

Third Approximation: $x_3 = \dfrac{2.74 + 2.7425}{2} = 2.74125$

Now, $f(x_3) = f(2.74125) = 0.0005267$

$\therefore$ $f(2.74125)$ is positive.

Hence root lies between 2.74 and 2.74125.

Fourth Approximation: $x_4 = \dfrac{2.74 + 2.74125}{2} = 2.740625$

Now, $f(x_4) = f(2.740625) = -0.00001839$

$\therefore$ $f(2.740625)$ is negative.

Hence root lies between 2.740625 and 2.74125.

Fifth Approximation: $x_5 = \dfrac{2.740625 + 2.74125}{2} = 2.7409375$

Now, $f(x_5) = f(2.7409375) = 0.000254$

$\therefore$ $f(2.7409375)$ is positive.

Hence root lies between 2.740625 and 2.7409375.

Sixth Approximation: $x_6 = \dfrac{2.740625 + 2.7409375}{2} = 2.74078125$

Now, $f(x_6) = f(2.74078125) = 0.0001178$

$\therefore$ $f(2.74078125)$ is positive.

Hence root lies between 2.740625 and 2.74078125.

Seventh Approximation: $x_7 = \dfrac{2.740625 + 2.74078125}{2} = 2.740703125$

Now, $f(x_7) = f(2.740703125) = 0.00004973$

$\therefore$ $f(2.740703125)$ is positive.

Hence root lies between 2.740625 and 2.740703125.

Eighth Approximation: $x_8 = \dfrac{2.740625 + 2.740703125}{2} = 2.74064063$

Now, $f(x_8) = f(2.740664063) = 0.00001567$

$\therefore$ $f(2.740664063)$ is positive.

Hence root lies between 2.740625 and 2.740664063.

Ninth Approximation: $x_9 = \dfrac{2.740625 + 2.740664063}{2} = 2.740644532$

Since x_8 and x_9 are same upto four decimal places hence the approximate real root is 2.7406.

Example 2.5. **Use bisection method to find out the positive square root of 30 correct to 4 decimal places.**

Solution: Let $f(x) = x^2 - 30$

Since $f(5.477) = -0.00247$ which is negative.

and $f(5.478) = 0.00848$ which is positive.

Hence root lies between 5.477 and 5.478

First Approximation: $x_1 = \dfrac{5.477 + 5.478}{2} = 5.4775$

Now, $f(x_1) = 0.003$

$\therefore$ $f(5.4775)$ is positive

Hence root lies between 5.477 and 5.4775.

Second Approximation: $x_2 = \dfrac{5.477 + 5.4775}{2} = 5.47725$

Now, $f(x_2) = 0.00026$

$\therefore$ $f(5.47725)$ is positive.

Hence root lies between 5.477 and 5.47725.

Third Approximation: $x_3 = \dfrac{5.477 + 5.47725}{2} = 5.477125$

Now, $f(x_3) = -0.0011$

$\therefore$ $f(5.477125)$ is negative.

Hence root lies between 5.477125 and 5.47725.

Fourth Approximation: $x_4 = \dfrac{5.477125 + 5.47725}{2} = 5.4771875$

Since x_3 and x_4 are same upto four decimal places hence the positive square root of 30 correct to 4 decimal places is 5.4771.

Example 2.6. **Find the positive root of $x - \cos x = 0$ by bisection method correct to three decimal places.**

Solution: Let

$$f(x) = x - \cos x$$

$$f(0.05) = 0.5 - \cos(0.5) = -0.37758 \text{ which is negative.}$$

$$f(1) = 1 - \cos(1) = 0.45970 \text{ which is positive.}$$

Hence the root lies between 0.5 and 1.

First Approximation: $x_1 = \dfrac{0.5+1}{2} = 0.75$

Now, $f(0.75) = 0.75 - \cos(0.75)$
$= 0.018311$

$\therefore$ $f(0.75)$ is positive.

Hence the root lies between 0.5 and 0.75.

Second Approximation: $x_2 = \dfrac{0.5+0.75}{2} = 0.625$

Now, $f(0.625) = 0.625 - \cos(0.625)$
$= -0.18596$

$\therefore$ $f(0.625)$ is negative.

Hence the root lies between 0.625 and 0.750.

Third Approximation: $x_3 = \dfrac{0.625+0.750}{2} = 0.6875$

Now, $f(0.6875) = 0.6875 - \cos(0.6875)$
$= -0.08533$

$\therefore$ $f(0.6875)$ is negative.

Hence the root lies between 0.6875 and 0.75.

Fourth Approximation: $x_4 = \dfrac{0.6875+0.75}{2} = 0.71875$

Now, $f(0.71875) = 0.71825 - \cos(0.71875)$
$= -0.033879$

$\therefore$ $f(0.71875)$ is negative.

Hence the root lies between 0.71875 and 0.75.

Fifth Approximation: $x_5 = \dfrac{0.71875+0.75}{2} = 0.73438$

Now, $f(0.73438) = -0.0078664$

$\therefore$ $f(0.73438)$ is negative.

Hence the root lies between 0.73438 and 0.75.

Sixth Approximation: $x_6 = \dfrac{0.73438+0.75}{2} = 0.742190$

Now, $f(0.742190) = 0.0051999$

$\therefore$ $f(0.742190)$ is positive.

Hence the root lies between 0.73438 and 0.742190.

Seventh Approximation: $x_7 = \dfrac{0.73438 + 0.742190}{2} = 0.73829$

Now, $f(0.73829) = -0.0013305$

$\therefore$ $f(0.73829)$ is negative.

Hence the root lies between 0.73829 and 0.74219.

Eighth Approximation: $x_8 = \dfrac{0.73829 + 0.74219}{2} = 0.7402$

Now, $f(0.7402) = 0.7402 - \cos(0.7402) = 0.0018663$

$\therefore$ $f(0.7402)$ is positive.

Hence the root lies between 0.73829 and 0.7402.

Ninth Approximation: $x_9 = \dfrac{0.73829 + 0.7402}{2} = 0.73925$

Now, $f(0.73925) = 0.00027593$

$\therefore$ $f(0.73925)$ is positive.

Hence the root lies between 0.73829 and 0.73925.

Tenth Approximation: $x_{10} = \dfrac{0.73829 + 0.73925}{2} = 0.7388$

Hence the root is 0.739 correct to three decimal places.

EXERCISE 2.2

1. Find a root of the equation $x^3 - 2x - 5$ correct to three places of decimal by bisection method. [Ans. 2.094]
2. Using bisection method determine a real root of the equation
$$f(x) = 8x^3 - 2x - 1.$$ [Ans. 0.66]
3. Compute one positive root of $2x - 3\sin x - 5 = 0$ by bisection method correct to three significant figures. [Ans. 2.88]
4. Find the approximate value of the root of the equation $3x - \sqrt{1 + \sin x} = 0$ by bisection method. [Ans. 0.39188]
5. Compute one root of $e^x - 3x = 0$ correct to two decimal places, using bisection method. [Ans. 1.51]
6. Find the root of $\tan x + x = 0$ upto two decimal places which lies between 2 and 2.1. [Ans. 2.03]
7. Find square root of 12 to four decimal places by bisection method upto five iterations. [Ans. 3.4687]
8. Compute one root of $x + \log x - 2 = 0$ correct to two decimal places which lies between 1 and 2. [Ans. 1.56]
9. Find the root of the equation $x^4 - x - 10 = 0$, using bisection method. [Ans. 1.8125]
10. Solve the equation $x e^x = 1$, by bisection method. [Ans. 0.567]
11. Find real root of $\cos x = x e^x$ correct to four decimal places by bisection method. [Ans. 0.5156]

3. Solve the equation $f(x) = 2\cos x - x$, using secant method. [Ans. 1.0297]
4. Find the approximate root of the equation $f(x) = x e^x - 3$ using secant method. [Ans. 1.0499]
5. Find the root of $e^{-x} = 3 \log x$ using Secant method. [Ans. 1.24682]
6. Find the real root of $\cos x - x e^x = 0$ using secant method. [Ans. 0.51776]

2.11 NEWTON-RAPHSON METHOD

Given an approximate value of a root of an equation, a better and closer approximation to the root can be found by using an iterative process called Newton's method or Newton-Raphson method.

Let x_0 be an approximate value of a root of the equation $f(x) = 0$

Let x_1 be the exact root nearer to x_0

Then $x_1 = x_0 + h$ where h is very small, positive or negative.

$$\therefore \quad f(x_1) = f(x_0 + h) = 0 \text{ since } x_1 \text{ is the exact root of } f(x) = 0$$

By Taylor's expansion

$$f(x_1) = f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

i.e., if h is small neglecting $h^2, h^3, \dots$, etc, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\therefore \quad h = -\frac{f(x_0)}{f'(x_0)} \text{ if } f'(x_0) \neq 0$$

$$\therefore \quad x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ approximately}$$

x_1 is a better approximate root than x.

Starting with this x_1, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ which is still better.}$$

Continuing like this, we iterate this process until $|x_{r+1} - x_r|$ is less than the quantity desired.

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \dots$$

This is the iterative formula of Newton-Raphson method.

2.12 GEOMETRICAL INTERPRETATION OF NEWTON-RAPHSON METHOD

Let $y = f(x)$ be the equation to the curve. Let $A_0(x_0, y_0)$ be any point on it. Now slope of the tangent to the curve $y = f(x)$ at the point $A_0(x_0, y_0)$ is

$$\left(\frac{dy}{dx}\right)_{A_0} = f'(x_0)$$

Tangent to the curve at the point A_0 (x_0, y_0) is given by

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0,\, y_0)} (x - x_0)$$

or
$$y - f(x_0) = f'(x_0)\,(x - x_0)$$
$$[\because\ y = f(x),\ y_0 = f(x_0)]$$

If $f'(x_0) \neq 0$, then the above tangent will intersect the x-axis at some point $B(x_1, 0)$ where $y = 0$. Thus

$$0 - f(x_0) = f'(x_0)\,(x_1 - x_0)$$

or
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

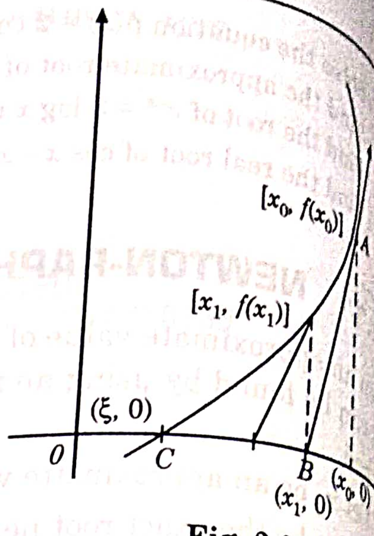


Fig. 2.6.

which is the Newton-Raphson formula for 1st approximation.

This shows that the point C which gives the exact value $x = \xi$ (say), is approximated to x_1 corresponding to the point where the tangent meets the x-axis.

Again, taking x_1 as starting point, the tangent to the curve at (x_1, y) meets the x-axis at the point, say, $(x_2, 0)$, giving.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

which is the second approximate value of the root.

Proceeding in the same way, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The points of intersection of tangents with the x-axis at points on the curve correspond to successive approximations.

$$x_0, x_1, \ldots x_n, x_{n+1}, \ldots$$

gradually come closer and closer to the exact root of the equation.

2.13 CRITERIA FOR CONVERGENCE IN NEWTON-RAPHSON METHOD

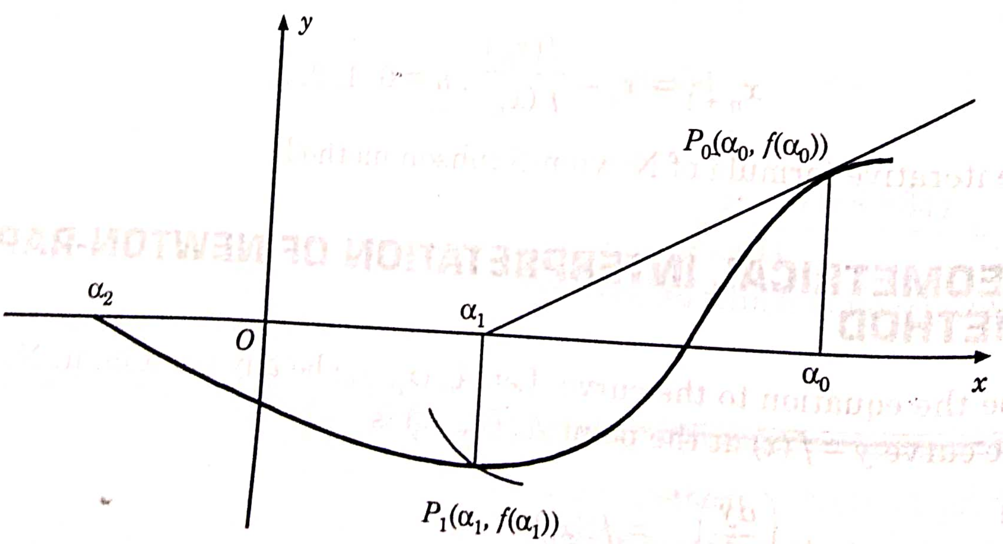


Fig. 2.7.

Here, in Newton's method,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This is really an iteration method where

$$x_{i+1} = \phi(x_i) \text{ and } \phi(x_i) = x_i - \frac{f(x_i)}{f'(x_i)}$$

Hence the equation is $\quad x = \phi(x)$ where $\phi(x) = x - \dfrac{f(x)}{f'(x)}$

The sequence $x_1, x_2, x_3, \ldots$ converges to the exact value if $|\phi'(x)| < 1$.

i.e., if $\quad \left|1 - \dfrac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}\right| < 1$

i.e., if $\quad \left|\dfrac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$

i.e., if $\quad |f(x)\ f''(x)| < 1\ [f'(x)^2]$ ... (*i*)

This is the criterion for the convergence. The interval containing a should be selected in which (*i*) is satisfied.

2.14 ORDER OF CONVERGENCE OF NEWTON-RAPHSON METHOD

The Newton-Raphson method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let β be the actual root of the equation $f(x) = 0$ then $f(\beta) = 0$. Let e_n be the error in the nth iteration, then $e_n = x_{n-\beta}$, where x_n be the approximate root.

This gives $\quad x_n = \beta + e_i$

Similarly for $(n + 1)$th iteration,

$x_{n+1} = \beta + e_{n+1}$ where e_{n+1} is the error in the $(n + 1)$th iteration.

Putting x_{n+1} and x_n in Newton's-Raphson formula

$$\beta + e_{n+1} = \beta + e_n - \frac{f(\beta + e_n)}{f'(\beta + e_n)}$$

$$e_{n+1} = e_n - \frac{f(\beta + e_n)}{f'(\beta + e_n)}$$

Expanding by Taylor's series, we get

$$e_{n+1} = e_n - \frac{\left[f(\beta) + e_n f'(\beta) + \frac{e_n^2}{2!} f''(\beta) + \ldots\right]}{\left[f'(\beta) + e_n f''(\beta) + \frac{e_n^2}{2!} f'''(\beta) + \ldots\right]}$$

$$= e_n - \frac{\left[0 + e_n f'(\beta) + \frac{e_n^2}{2!} f''(\beta) + \ldots\right]}{[f'(\beta) + e_n f''(\beta) + \ldots]}$$

{Being β as actual root, $f(\beta)$

$$e_{n+1} = e_n - \frac{\left[e_n f'(\beta) + \frac{e_n^2}{2!} f''(\beta) + \ldots\right]}{\left[1 + e_n \frac{f''(\beta)}{f'(\beta)} + \frac{e_n^2}{2!} \frac{f'''(\beta)}{f'(\beta)} + \ldots\right]}$$

If e_n is very small then we can ignore the higher order terms of e_n.

Therefore,
$$e_{n+1} = e_n - \frac{e_n + \frac{e_n^2}{2} \frac{f''(\beta)}{f'(\beta)}}{1 + e_n \frac{f''(\beta)}{f'(\beta)}} = e_n - e_n \left[1 + \frac{e_n}{2} \frac{f''(\beta)}{f'(\beta)}\right] \left[1 + \frac{e_n f''(\beta)}{f'(\beta)}\right]$$

$$= e_n - e_n \left[1 + \frac{e_n f''(\beta)}{2f'(\beta)}\right] \left[1 - \frac{e_n f''(\beta)}{f'(\beta)}\right]$$

$$= e_n - e_n \left[1 - e_n \frac{f''(\beta)}{f'(\beta)} + \frac{e_n}{2} \frac{f''(\beta)}{f'(\beta)} - \frac{e_n^2}{2} \left\{\frac{f''(\beta)}{f'(\beta)}\right\}^2\right]$$

$$= e_n - e_n \left[1 - \frac{e_n}{2} \frac{f''(\beta)}{f'(\beta)}\right] - \frac{e_n^2}{2} \left\{\frac{f''(\beta)}{f'(\beta)}\right\}^2$$

$$e_{n+1} = \frac{e_n^2}{2} \frac{f''(\beta)}{f'(\beta)} \quad \text{(ignoring the higher order terms)}$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\beta)}{2f'(\beta)}$$

$$\frac{e_{n+1}}{e_n^2} = K$$

Comparing with $\lim_{n \to \infty} \left[\frac{e_{n+1}}{e_n^n}\right] \le K$

We get $n = 2$.

Thus the order of convergence of Newton-Raphson method is 2, *i.e.*, Newton-Raphson method is quadratic convergent.

If we compare equation (*i*) with the relation $x_{n+1} = \phi(x_n)$ of the iterative method.

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general $$\phi(x) = x - \frac{f(x)}{f'(x)}$$

which gives $$\phi'(x) = \frac{f(x)\,f'(x)}{[f'(x)]^2}$$

To examine the convergence we assume that $f(x)$ and $f'(x)$ are continuous and bounded on any interval containing the root $x = \xi$ of the equation $f(x) = 0$. If ξ is a simple root, then $f(\xi) \neq 0$. Further, since $f(x)$ is continuous, $|f(x)| \geq \varepsilon$ for some $\varepsilon > 0$ in a suitable neighbourhood of ξ. Within this neighbourhood we can select an interval such that $|f(x)\,f'(x)| < \varepsilon^2$ and this is possible since $f(\xi) = 0$ and since $f(x)$ is continuously twice differentiable. Hence, in this interval we have

$$|\phi'(x)| < 1$$

The Newton-Raphson formula (i) converges, provided that the initial approximation x_0 is chosen sufficiently close to ξ. When ξ is a multiple root, the Newton-Raphson method still converges but slowly.

SOLVED EXAMPLES

Example 2.18. Using Newton-Raphson method, find the root of $x^4 - x - 10 = 0$, which is near 2, correct to three decimal places.

Solution: Here $f(x) = x^4 - x - 10$ so that $f'(x) = 4x^3 - 1$.

Now, by Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} = \frac{3x_n^4 + 10}{4x_n^3 - 1} \qquad \ldots (i)$$

Since the root is near to 2 (given), taking $x_0 = 2$ as the initial approximation and $n = 0$ in (i), we have

First Approximation $$x_1 = \frac{3x_0^4 + 10}{4x_0^3 - 1} = \frac{3(2^4) + 10}{4(2^3) - 1} = 1.871$$

Second Approximation $$x_2 = \frac{3x_1^4 + 10}{4x_1^3 - 1} = \frac{3(1.871)^4 + 10}{4(1.871)^3 - 1} = 1.856$$

Third Approximation $$x_3 = \frac{3x_2^4 + 10}{4x_2^3 - 1} = \frac{3(1.856)^4 + 10}{4(1.856)^3 - 1} = 1.856$$

So $x_3 = x_2$, consequently we stop at this stage and hence the required root is 1.856.

Example 2.19. Solve the equation $\log x = \cos x$ to five decimals by Newton-Raphson method.

Solution: $$f(x) = \log x - \cos x \text{ so that } f'(x) = \frac{1}{x} + \sin x.$$

Since $f(x)$ is –ve for $x = 1.3$ and $f(x)$ is +ve for $x = 1.4$, hence the root lies between 1.3 and 1.4. Putting $n = 0$ and taking $x_0 = 1.3$ as the initial value, the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\log x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n}$$

Putting $n = 0$, in (*i*), we have

First Approximation:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{\log x_0 - \cos x_0}{\frac{1}{x_0} + \sin x_n} = 1.3 - \frac{\log_e 1.3 - \cos 1.3}{\frac{1}{1.3} + \sin 1.3}$$

$$= 1.3 - \frac{(2.3026)(0.11394) - 0.26750}{0.76923 + 0.96356} \quad [\log_e x = 2.3026 \times \log_{10} x]$$

or $\quad x_1 = 1.3029$

Putting $n = 1$ in (*i*), we have

Second Approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3029 - \frac{\log_e 1.3029 - \cos(1.3029)}{\frac{1}{1.3029} + \sin(1.3029)} = 1.30295$$

Putting $n = 2$, in (*i*), we have

Third Approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.30295 - \frac{\log_e 1.30295 - \cos(1.30295)}{\frac{1}{1.30295} + \sin(1.30295)}$$

$$= 1.30295$$

So $x_2 = x_3$, consequently we stop at this stage and hence the desired root is 1.30295.

Example 2.20. Find the real root of the equation $3x - \cos x - 1 = 0$ by Newton-Raphson method.

Solution:

$$f(x) = 3x - \cos x - 1$$

$$f(0) = -2 = -\text{ve}$$

and

$$f(1) = 3 - 0.5403 - 1 = 1/4597 = +\text{ve}$$

So a root of $f(x) = 0$ lies between 0 and 1

Let us take

$$x_0 = 0.6$$

Also

$$f'(x) = 3 + \sin x$$

$\therefore$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$\Rightarrow$

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad \ldots(i)$$

Putting $n = 0$ in (*i*), we have

First Approximation:

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= \frac{0.6 \times 5729 + 0.82533 + 1}{3 + 5729} = 0.6071$$

Putting $n = 1$ in (*i*), we have.

Second Approximation:

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$= \frac{0.6071 \times 57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071$$

Clearly $x_1 = x_2$.

Hence the desired root is 0.6071 correct to four decimal places.

Example 2.21. Find the approximate value for the real root of $x \log_{10} x - 1.2 = 0$, correct to five decimal places by Newton-Raphson method.

Solution: $f(x) = x\log_{10} x - 1.2$

Therefore, $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \cdot \log_{10} e = \log_{10} x + \log_{10} e$

$= \log_{10} x + 0.43429$ $\qquad [\because e = 2.71825]$

Now, $f(2) = 2 \log_{10} 2 - 1.2 = -0.59794$ (–ve)

and $f(3) = 3 \log_{10} 3 - 1.2 = 0.23136$ (+ve)

Hence the root of $f(x) = 0$ lies between 2 and 3. We notice that out of $|f(2)|$ and $|f(3)|$, $|f(3)|$ is minimum.

We take $x_0 = 3$ so as to minimise the number of iterations.

Now, putting $n = 0$ in the Newtons-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.43429} = \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429} \quad \ldots(i)$$

Putting $n = 0, 1, 2, 3, \ldots$ in (*i*), we get

First Approximation: $x_1 = \dfrac{0.43429 x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \dfrac{0.43429 \times 3 + 1.2}{\log_{10} 3 + 0.43429}$

or $x_1 = \dfrac{1.30287 + 1.2}{0.47712 + 0.43429} = \dfrac{2.50287}{0.91141} = 2.74615$

Second Approximation: $x_2 = \dfrac{0.43429 x_1 + 1.2}{\log_{10} x_1 + 0.43429} = \dfrac{1.19262 + 1.2}{0.43872 + 0.43429}$

$$= \frac{2.39263}{0.87301} = 2.74066$$

Third Approximation: $x_3 = \dfrac{0.43429 x_2 + 1.2}{\log_{10} x_2 + 0.43429} = \dfrac{1.19023 + 1.2}{0.43785 + 0.43429}$

$$= \frac{2.39025}{0.87215} = 2.74065 \text{ (approx.)}$$

Fourth Approximation: $x_4 = \dfrac{0.43429x_3 + 1.2}{\log_{10} x_3 + 0.43429} = \dfrac{1.19023 + 1.2}{0.43785 + 0.43429} = 2.74065$

So $x_4 = x_3$, we stop at this stage. Hence the required root correct to five decimal places, 2.74065.

Remarks. If we take $x_0 = 2$, five number of iterations gives the value of the root correct, the desirable accuracy.

Example 2.22. Use Newton-Raphson method to obtain a root of sin $x = 1 - x$ to three decimals.

Solution: Here $\quad f(x) = 1 - x - \sin x$ giving $f'(x) = -1 - \cos x$.

We observe that

$$f(0.5) = 1 - 0.5 - \sin(0.5)$$
$$= 1 - 0.5 - 0.4794255 = 0.0205744 \text{ (+ve)}$$

and

$$f(0.6) = 1 - 0.6 - \sin(0.6)$$
$$= 1 - 0.6 - 0.5646424 = -0.1646424 \text{ (–ve)}.$$

Hence the root of $f(x) = 0$ lies between 0.5 and 0.6. Let us take $x_0 = 0.5$.

The Newton-Raphson formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1 - x_n - \sin x_n}{-1 - \cos x_n}$$

$$= \frac{1 + x_n \cos x_n - \sin x_n}{1 + \cos x_n}$$

Putting $n = 0, 1, 2, 3, \ldots$ in (*i*), we have

First Approximation: $\quad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = \dfrac{1 + x_0 \cos x_0 - \sin x_0}{1 + \cos x_0}$

$$= \frac{1 + (0.5)\cos(0.5) - \sin(0.5)}{1 + \cos(0.5)}$$

$$= \frac{1 + (0.5)(0.8775825) - 0.4794255}{1 + 0.8775825}$$

$$= \frac{1 + 0.4387912 - 0.4794255}{1.8775825} = 0.5109579$$

Second Approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_2)} = \frac{1 + x_1 \cos x_1 - \sin x_1}{1 + \cos x_1}$$

$$= \frac{1 + (0.5109579)\cos(0.5109579) - \sin(0.5109579)}{1 + \cos(0.5109579)}$$

$$= 0.5109734$$

Third Approximation: $\quad x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = \dfrac{1 + x_2 \cos x_2 - \sin x_2}{1 + \cos x_2} = 0.15109734$

SOLVED EXAMPLES

Example 2.23. **Evaluate $\sqrt{12}$ to four decimal places by Newton-Raphson method.**

Solution: Since $9 < 12 < 16 \Rightarrow \sqrt{9} < \sqrt{12} < \sqrt{16}$

$$\Rightarrow 3 < \sqrt{12} < 4.$$

$\therefore$ $\sqrt{12}$ lies between 3 and 4.

Here we may take $x_0 = 3.5$ as the initial approximation to $\sqrt{12}$. The iterative formula for finding the successive approximations of the square root of a number 'N' is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right),$$

gives

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) \text{ where } N = 12$$

or

$$x_1 = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.4643 \text{ approx.}$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{12}{x_1}\right) = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

and

$$x_3 = \frac{1}{2}\left(x_2 + \frac{12}{x_2}\right) = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641}\right) = 3.4641$$

Thus $x_2 = x_3$. Hence $\sqrt{12} = 3.4641$.

Aliter: Let $x = \sqrt{12}$ $\therefore$ $x^2 = 12$ or $x^2 - 12 = 0$.

Here $f(x) = x^2 - 12$ so that $f'(x) = 2x$.

Taking $x_0 = 3.5$ as the initial approximation and $n = 0$, the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 12}{2x_0}$$

or

$$x_1 = 3.5 - \frac{(3.5)^2 - 12}{2(3.5)} = 3.4643 \text{ approx.}$$

Again, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.4643 - \frac{(3.4643)^2 - 12}{2(3.4643)} = 3.4641 \text{ (approx.)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4641 - \frac{(3.4641)^2 - 12}{2(3.4641)} = 3.4641 \text{ (approx.)}$$

Hence $$\sqrt{12} = 3.4641.$$

Example 2.24. **Use Newton's formula to prove that square root of N can be obtained by the recursion formula**

$$x_{n+1} = x_n \left(1 - \frac{x_n^2 - N}{2N}\right).$$

Hence find the square root of 26.

Solution: Let us take $f(x) = 1 - \dfrac{N}{x^2}$, then $f(x) = 0$ gives $x = \sqrt{N}$, and $f'(x) = \dfrac{2N}{x^3}$.

Now, the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

gives
$$x_{n+1} = x_n - \frac{\left(\dfrac{x_n^2 - N}{x_n^2}\right)}{\left(\dfrac{2N}{x_n^2}\right)} = x_n - \frac{x_n^3}{2N}\left(\frac{x_n^2 - N}{x_n^2}\right)$$

or
$$x_{n+1} = x_n - \left(1 - \frac{x_n^2 - N}{2N}\right)$$

To find the square root of 26, we put $n = 0$; $x_0 = 5.0$, $N = 26$, to get the first approximation x_1 as given by

$$x_1 = 5\left(1 - \frac{25 - 26}{52}\right) = 5.096.$$

The second approximation x_2 is given by

$$x_2 = 5.096\ (1 + 0.000592) = 5.099$$

Hence $$\sqrt{26} \cong 5.099.$$

Example 2.25. **Show that the following two sequences both have convergence of second order with the same limit $\sqrt{a}$.**

$$x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right) \text{ and } x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right).$$