operators

(i) Negation:

p: Today is monday

¬p: Today is not monday ✓

¬p: Today is wednesday ✗

¬p: Today is anyday of week except Monday ✓

Negation operator is a UNARY_OPERATOR...

symbol: [$\neg$, $\sim$, $\bar{p}$, $\bar{p}$]

Conjunction: $\wedge$

$p \wedge q$ → is true when both p and q are true.

Eg) : 10 is divisible by 3 = f

4 is an odd no. = f

$P \wedge Q$ = f

In logic: [AND = BUT = HOWEVER = EVEN THOUGH = YET]
= THOUGH = NEVERTHLESS
= Nonetheless
= comma.

All are same
But in Eng. they
are different.

Delhi is capital of India, However Kerla is Beautiful
OR
BUT
OR
EVEN THOUGH

Mary is poet, singer ✓

Conditional Statement

If Sunday then Holiday
P Q

logic expression

$P \to Q$

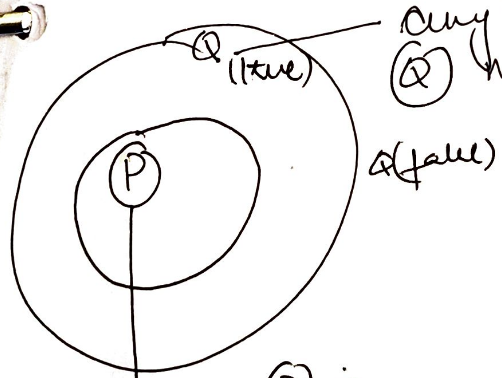
Antecedant OR Premise

Conclusion OR Consequence.

Note:
If ≡ when ≡ whenever ≡ provided that.

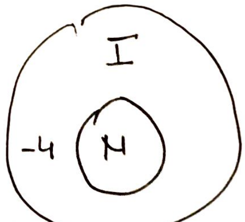
Diagram Representation of Implication:

anytime P is false Q may [True / false].



anytime P is True Q also true.

Eg)



If x is Integer then x is natural no's..

$I \to N$ (True) ? } no ✗

$P \to Q$ is True when

$$\boxed{\begin{array}{l} P \text{ is false} \\ \quad \text{OR} \\ Q \text{ is True} \end{array}}$$

$P \to q$ is false only when

$$\boxed{\begin{array}{l} P = \text{True} \\ \quad \text{AND} \\ Q = \text{FALSE} \end{array}}$$

<u>Ques)</u> $\underset{\text{condition}}{P} \to \underset{\text{conclusion}}{Q} \not\equiv \overset{\text{condition}}{Q} \to \underset{\text{conclusion}}{P}$??

Eg) P: You are PM of India
Q: You are Indian

$P \to Q$ ✓
$Q \to P$ ✗

Eg) A: x is a Natural No.
B: x is an integer.

$A \to B$ ✓
$B \to A$ ✗

$A \to B \not\equiv B \to A$

conditionals are not arguments

If A then B

"not claiming that antecedent is true"

A
∴ B

"Premise is true & implies conclusion"

"Sufficent & Necessary CONDITION"

Sufficent

A is a sufficent condition for B whenever the occurance of A is all that is needed for the occurance of B...

- what is enough for a conclusion to be TRUE...

If A then B

Necessary

B is necessary condition for A whenever A cannot occur w/o the occurance of B.

+ something that must be present for conclusion to be True...

- They are not enough by themselves to make conclusion TRUE

- Being a dog is a ______ condition for Being an animal.

Sufficent

Necessary condition for being a dog.

- Winning the lottery is ______ condition for Becoming RICH.

{ $P \to Q$: different ways }

(i) If P then Q ✓

(ii) P implies Q ✓

(iii) If P, Q ✓

(iv) When P, Q ✓

(v) Whenever P, Q ✓

(vi) Provided that P, Q ✓

~~Q if P~~ ~~vs~~ ~~Q→P~~

(vii) P is sufficient condition for Q.

(viii)

Eg) If x is 2 then x is even.

Sol^n: 2 is sufficient for even No. can be - 2, 4

P if Q i.e $Q \to P$

(ix) P only if Q i.e $P \to Q$

(x) Q is necessary condition for P.

Eg) P only if Q : $P \to Q$

You will crack GATE only if STUDY..

necessary condition.

Bi-conditional

P only $\boxed{\text{if}}$ Q

- P will happen $\boxed{\text{only if}}$ Q happens.
- Q is necessary for P.

Note

$$\boxed{\begin{array}{l} \# \ (P \leftrightarrow q) \equiv (\overline{P \oplus Q}) \\ \quad \text{OR} \\ (P \oplus Q) = (\overline{P \leftrightarrow Q}) \end{array}}$$

Eg) A number is even no. if it is integer multiple of 2.
($\hookrightarrow m_2$)

$m_2 \rightarrow$ even

even $\rightarrow$ m_2 ?? can say it is also correct.

So the correct statⁿ :-
even if and only if m_2.

Definition: are always "If and only if" statements ...
(Even if they are written as "If").

Implication tells us "PROPERTY"
Bi-implication ———— "DEFINATION".

$\boxed{\text{PM of India} \rightarrow \text{Indian}}$

So PM has property that He/she is an Indian. But Being Indian is not the defination of PM of India..

Implication (1-way stat^n) tells you about "PROPERTY"...
But
Bi-Implication (defination)..

$\boxed{P \rightarrow Q}$

× P is property of Q
✓ Q is property of P

Defination of a even No.

1) Integer multiple of 2.
2) Integer divisible by 2.
3) n mod 2 = 0

$\boxed{\text{Prime} > 2 \rightarrow \text{odd}}$

"Being odd" defination of prime > 2 ??
L NO.
But
Being odd is a property.

Boy → Human

Being Human is property of being Boy.

Define a boy ??

- Boy is a human.

Properties of Natural No's

1) Integer
2) > 0
3) Rational
4) Real
5) odd or even.

Integer is property of Natural No...

$N \to I$

$N \to$ Real

$N \to$ Rational

Eg) Prime No. greater than 2 $\to$ $\boxed{\text{odd No.}}$

$\underbrace{\text{Prime No. greater than 2}}_{P}$ $\to$ $\underbrace{\boxed{\text{odd No.}}}_{Q}$

Q is property of P.

Eg) $x = 2 \longleftrightarrow x =$ even prime

So $P \longleftrightarrow Q$

P is definition of Q

Q is definition of P

P, Q are definition of each other.

Defination of even prime

1) Being 2
2) prime < 3
3) is prime & divisible by 2

- even prime $\longleftrightarrow$ Being 2
- even prime $\longleftrightarrow$ prime < 3
- Even prime $\longleftrightarrow$ is prime & divisible By 2.

RECAP

propositional variable is a variable that is either True | false.

logical connectives are

- Negation $\neg p$
- conjunction $p \wedge q$
- Disjunction $(p \vee q)$
- Implication $(p \rightarrow q)$
- Biconditional $(p \leftrightarrow q)$

Truth tables for statements

p	q	$p \wedge (p \to q)$	$p \to q$
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

final result.

Note:
ORDER of combination matters in truth table ??

↓

No

p	q	α
T	T	T
T	F	T
F	F	T
F	T	F

=

p	q	α
F	T	T
T	F	F
T	T	T
F	F	T

Eg)

P	Q	R	$(p \vee q) \to R$	$(p \vee q)$

helper column can be omitted ...

Eg) $\boxed{p \vee q \wedge r}$

$(p \vee q) \wedge r$ ✗ $\quad$ $p \vee (q \wedge r)$ ✓

Ques) suppose the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false. find truth values P, Q, R, S.

$$\underbrace{((P \wedge Q) \vee R) \Rightarrow (R \vee S)}_{\text{false.}}$$

- T $\quad$ f.
- R = f, S = f.
- P = T, Q = T } R = f, S = f }

Ques) P = Q = T, R = S = f.

more priority.

(A) $P \to (Q \wedge \neg R) \vee \neg S$

$T \to T \wedge T \vee T$.

$T \to T \quad \equiv$ True.

(B) $\neg P \to \neg Q \vee \neg R \wedge \neg S$

$F \to \underbrace{\quad}_{T}$

True.

TAUTOLOGY, CONVERSE, EQUIVALENCE, TRUTH TABLE, ENG-LOGIC TRANSLATION.

$P = T \quad Q = R = F$.

Q) Let p represent a true statⁿ, while q, r represent false. Find truth value of compound statⁿ...

(i) $\sim[(\sim p \wedge \sim q) \vee \sim q]$

$\sim[(\sim T \wedge \sim F) \vee \sim F]$

$\sim[F \vee \sim F]$

F.

(2) $\sim(p \wedge q) \wedge (r \vee \sim q)$

$\sim(T \wedge F) \wedge (F \vee T)$

$\sim(F) \wedge T$

$T \wedge T = T$.

(3) $\sim(\sim p \wedge \sim q) \vee (\sim r \vee \sim p)$

$\sim(\sim T \wedge \sim F) \vee (\sim F \vee \sim T)$

$\sim(F \wedge T) \vee (T \vee F)$

$T \vee T = T$ Truth value = True.

Note PRIORITY ORDER OF CONNECTIVES:

$\boxed{\neg > \wedge > \vee > \rightarrow > \leftrightarrow}$

↓ highest priority

$\neg p \vee q$

$\neg(p \vee q)$ ✗ $\quad (\neg p) \vee q$ ✓

negation has more priority..

$(p \vee \neg(p \wedge q))$

TAUTOLOGY

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Ques) Build a truth table to verify that the proposition $(p \leftrightarrow q) \wedge (\neg p \wedge q)$ is a CONTRADICTION ...

Ques) TAUTOLOGIES-

- $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$
- $p \vee \neg p$
- $(p \wedge q) \rightarrow p$
- $q \rightarrow (p \vee q)$
- $(p \vee q) \leftrightarrow (q \vee p)$

Q) CONTRADICTIONS

- $p \wedge \neg p$
- $(p \vee q) \wedge (\neg p) \wedge (\neg q)$

P1

PROPOSITIONAL EXPRESSION

Always TRUE. → Tautology.

Always false. → Contradiction

Atleast one time true & atleast one time false. → Contingency

- these 3 categories are completely disjoint.

Tautology is a proposition that is always TRUE..

Ques) $p \vee q \rightarrow \bar{r}$

L Contingency

P = T

r = T.

~~Q = T~~.

$T \rightarrow f = f$.

P = f

q = f.

K = T

True.

Ques) Which of the following propositional logic expressions are tautology?

- P – contingency
- $\sim P$ – contingency
- $T \to P$ – contingency
- $P \to P$ – Tautology.
- $P \to \sim P$ – contingency
- $\sim P \to P$ – contingency.
- T – Tautology
- F – contradiction.
- $P \to F$ – contingency.

LOGICAL Equivalence

2-expressions $\alpha \equiv \beta$ iff they have same truth table.

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a TAUTOLOGY...

$p \equiv q$

$p \vee (p \to q)$

P	Q	$p \vee (p \to q)$ = True
F	F	T
F	T	T
T	F	T
T	T	T

Eg) $p \wedge (p \to q)$

P	Q	$P \wedge (P \to q)$
F	F	F
F	T	F
T	F	F
T	T	T

$p \wedge (p \to q) \equiv p \wedge q$

Eq) Two propositional expressions α, β are equivalent iff

- Same truth tables $\Rightarrow$ in every situation
- $\alpha \leftrightarrow \beta$ is tautology.

Eg) $\alpha: p \to q$

$\beta: \bar{p} \vee q$ $\qquad \alpha \equiv \beta$??

p	q	$p \to q$	$\bar{p} \vee q$	$\alpha \leftrightarrow \beta$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	T	T	T

$\alpha \equiv \beta$ ...

$\begin{bmatrix} \alpha = \text{True} & \alpha = f & \beta = f & \beta = T \\ \beta = \text{True} & \beta = f & \alpha = F & \alpha = T \end{bmatrix}$

Ques) $\left.\begin{matrix} p \to q \\ \neg q \to \neg p \end{matrix}\right\}$ are they equivalent?

Method 1: Truth table. Stupid & inefficient.

Method-2

Case 1: $P = T$

Case 2

1)

Ques) $\alpha: p \to q$

$\beta: \neg q \to \neg p$

Truth table ✓

Method 2:

Case 1

$p = T$

- $T \to q \equiv q$

  $F = F$

  $T = T$

  final ans $= q$

- $q \to F = \bar{q}$

q	$\bar{q}$
F	T
T	F

case-2

$p = F$

$\boxed{\alpha = T \quad \beta = T}$ ✓

$\alpha: p \to q$ $\qquad q \vee \neg p$

$\boxed{T \to q = q}$

$\beta:$ $\boxed{\neg q \to \neg p \equiv q}$ ✓

$T \to F. = q$

Note:

You can determine whether compound proposition (r) and (s) are logically equivalent by building a single truth table for both propositions & checking to see that they have exactly the same truth value

Q) $\underbrace{p \vee (q \wedge r)}_{\alpha} \equiv \underbrace{(p \vee q) \wedge (q \vee R)}_{\beta}$

Case 1)

$P = T$

$\alpha : T$ } same.
$\beta : T$

Case 2

$P = f$

$\alpha : q \wedge r$ } same.
$\beta : q \wedge r$

LOGICAL EQUIVALENCE

Propositions r and s are logically equivalent if the statement $r \leftrightarrow s$ is a tautology.

$\alpha : P \rightarrow Q$
$\beta : \bar{P} \vee Q$

- $\alpha \leftrightarrow \beta$ is Tautology
- $\alpha \equiv \beta$

P	Q	α	β	$\alpha \leftrightarrow \beta$
		f	f	T
		T	T	T

they should same only then they are equivalent.

logically equivalent

- $\alpha \equiv \beta$ iff $\alpha \leftrightarrow \beta$ is Tautology
- $\alpha \equiv \beta$ iff α, β have same truth table.
- $\alpha \equiv \beta$ iff { whenever (α) True, β (True
  ——— (β) True, α (True.

Some std logical equivalences (LOGICAL IDENTITIES)

Identity LAWS...

$$\boxed{\begin{array}{l} P \wedge T \equiv P \\ P \vee F = P \end{array}}$$

T = T
F = F.

Proof 1

Proof 2

P = T | P = F.

$\underbrace{P \wedge T}_{\alpha} \equiv \underbrace{P}_{\beta}$

$\alpha = T$
$\beta = T$

$\alpha = F$
$\beta = F$

Domination LAWS...

$$\boxed{\begin{array}{l} P \vee T = T \\ P \wedge F = F \end{array}}$$

Idempotent law.

$$\boxed{P \# P = P}$$

In prop. logic which connective is Idempotent?

$P \wedge P = P$ ✓
$P \vee P = P$ ✓
$P \rightarrow P \neq P$ ✗
$P \rightarrow P = T$ ✗
$\boxed{P \leftrightarrow P = T}$ ✗

$\underbrace{P \leftrightarrow P}_{\text{LHS}} = \underbrace{T}_{\text{RHS}}$

Proof: LHS

case 1: P = T
LHS = T
RHS = T

case 2: P = F
LHS = T
RHS = T

$\cancel{P \wedge T = P}$
$\cancel{P \vee F = P}$

①

* COMMUTATIVE LAW

$a \# b = b \# a$

No. Theory: $+$, $\times$
But subtraction not commutative. .

In prop. logic which connective is COMMUTATIVE?

- $\wedge$
- $\vee$
- $\rightarrow$ \ $p \rightarrow q \not\equiv q \rightarrow p$
- $\leftrightarrow$
- $\oplus$
- $\uparrow$
- $\downarrow$

Associative

$(a \# b) \# c = a \# (b \# c)$

In Prop. logic which connective is Associative?

- $\wedge$
- $\vee$
- × $\rightarrow$
- $\leftrightarrow$
- $\oplus$
- × $\uparrow$
- × $\downarrow$

$$(P \to Q) \to R \not\equiv P \to (Q \to R)$$

Proof: Truth Table Method.

$$(P \to Q) \to R \not\equiv P \to (Q \to R)$$

These have diff. truth table.

$$\underbrace{(P \to Q) \to R}_{\alpha} \not\equiv \underbrace{P \to (Q \to R)}_{\beta}$$

Proof by case Method:

$T \to y = y$.

Case 1) $P =$ True

$\alpha = Q \to R$
$\beta = Q \to R$
same.

Case 2: $P =$ false.

$F \to \bigcirc \equiv T$.

$\alpha = T \to R \equiv R$.
$\beta = F \to \bigcirc = T$
not same.

Note

$\oplus$ is Associative

$$\underbrace{(P \oplus Q)}_{\alpha} \oplus R \equiv P \oplus \underbrace{(Q \oplus R)}_{\beta}$$

Proof by case method ...

Case 1

$P = T$ $\qquad T \oplus Y = \bar{Y}$

$\alpha = \bar{Q} \oplus R$

$\beta = \overline{q \oplus R}$

equivalent.

$$\underbrace{(\overline{q \oplus R})}_{\alpha} = \underbrace{\bar{q} \oplus R}_{\beta}$$

$q = T$

$\alpha = R$

$\beta = R$

same

$q = F$

$\alpha = \bar{R}$

$b = \bar{R}$

same

Case 2

$P = F$

~~Et~~ $\alpha = q \oplus r$

$\beta = q \oplus r$.

DISTRIBUTIVE PROPERTY.

distribute × over +

$a \times (b+c) = (a \times b) + (a \times c)$ ✓

$a + (b \times c) = (a+b) \times (a+c)$ ✗

$a + bc = (a+b)(a+c)$ ✗

$$\boxed{a \# (b * c) = (a \# b) * (a \# c)}$$

Note →

$\wedge$ is Distribute over $\vee$

$\vee$ ——— $\wedge$

$$\boxed{\begin{aligned} p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \end{aligned}}$$

Prove that: $\vee$ is Dist. over $\wedge$

$$\underbrace{P \vee (Q \wedge R)}_{LHS} \equiv \underbrace{(P \vee Q) \wedge (P \vee R)}_{RHS}$$

Proof: P = True

LHS = T

RHS = T

P = false

LHS = $Q \wedge R$

RHS = $Q \wedge R$

Equivalent

HW:

1. $\rightarrow$ is Dist. over $\wedge$
2. $\wedge$ is $\longleftrightarrow$ $\rightarrow$
3. $\rightarrow$ ________ $\vee$
4. $\vee$ ________ $\rightarrow$
5. $\oplus$ ________ $\rightarrow$ $\oplus$
6. $\rightarrow$

Demorgan's Law

* $\overline{(P \wedge Q)} = \overline{P} \vee \overline{Q}$
* $\overline{(P \vee Q)} = \overline{P} \wedge \overline{Q}$

formulae of: $\wedge, \vee, \neg$

$\overline{G} = \left\{ \begin{array}{l} \vee \rightarrow \wedge \\ \wedge \rightarrow \vee \\ P \rightarrow \overline{P} \\ \overline{P} \rightarrow P \end{array} \right\}$.

$\underbrace{\overline{(P \wedge Q)}}_{\alpha} \equiv \underbrace{\overline{P} \vee \overline{Q}}_{\beta}$

$P = T$

$\alpha = \overline{Q}$

$\beta = \overline{Q}$

$P = F$

$\alpha = T$

$\beta = T$

equivalent.

Ques) $G: (\bar{P} \wedge Q) \vee R$.

$\bar{G} = ?$

$\overline{(\bar{P} \wedge Q) \vee R} = \overline{(\bar{P} \wedge Q)} \wedge \bar{R}$ ✓

$= (P \vee \bar{Q}) \wedge \bar{R}$

Ques) $G: P \vee Q \wedge R$

$\bar{G} = ?$ $\quad \overline{P \vee (Q \wedge R)} \equiv \bar{P} \wedge (\bar{Q} \vee \bar{R})$

conjunction more priority.

✗ → $(\bar{P} \wedge \bar{Q}) \vee \bar{R}$

✓ → $\bar{P} \wedge (\bar{Q} \vee \bar{R})$

Absorption LAW.

$$\boxed{\begin{aligned} P + PQ &\equiv P \\ P \wedge (P + Q) &= P \end{aligned}}$$

Proof: $P \vee PQ$

$\equiv P(1 + Q)$

$= P.$ ✓

Q) $P + PQ + PQ\bar{R}S \equiv P$ ✓

* TAUTOLOGY :- $p \vee \neg p \Leftrightarrow T$.

* CONTRADICTION :- $p \wedge \neg p \Rightarrow F$

* EQUIVALENCE : $(p \to q) \wedge (q \to p) \Leftrightarrow (p \leftrightarrow q)$

* CONTRA POSITIVE -LAW = $(p \to q) \Leftrightarrow (\neg q \to \neg p)$

Domination LAW :

$p \vee T = T$
$p \wedge F = F$

(Any thing) $\wedge F = F$ . . .

"Imp. property."

$$\boxed{\alpha + \alpha' \beta \equiv \alpha + \beta.}$$

LAW of common identities

$A + \bar{A} B = A + B$

$A(1 + B) + \bar{A} B$

$A + AB + \bar{A} B = A + B(A + \bar{A})$

$= A + B$

Ques) [illegible]

SUMMARY

$P \vee T = T$ | $P \vee \bar{P} = T$ | $P \wedge Q = Q \wedge P$

$P \wedge F = F$ | $P \wedge \bar{P} = F$ | $PQ = QP$

$P \wedge T = P$ | $P \wedge P = P$ | $P + Q = Q + P$.

$P \vee F = P$ | $P \vee P = P$

~~αβ~~: V.Imp

$$\boxed{\begin{aligned} \alpha \to \beta &= \bar{\alpha} + \beta \\ P + PQ &= P \\ P + \bar{P}Q &= P + Q \end{aligned}}$$

Application of LAW :

Note $\underbrace{(P \leftrightarrow Q)}_{\text{True.}} \equiv \underbrace{(P \to Q)}_{\text{True}} \wedge \underbrace{(Q \to P)}_{\text{True.}}$

$$\boxed{\alpha \equiv \beta \text{ iff } \alpha \leftrightarrow \beta \text{ is a TAUTOLOGY.}}$$

Q) Check Tautology or Not ??

$\alpha : \underbrace{(P \wedge Q)} \rightarrow \underbrace{(P \vee Q)}$

Method 1: Truth Table.
Method 2: Case - Method...

Case 1 $P = T$	$P = F$
$\alpha : Q \rightarrow T$.	$\alpha : F \rightarrow$ anything.
$\alpha : \underbrace{\quad}$ True.	α : True.

α = always True.
So it's a TAUTOLOGY...

Simplification

↳ we are the property that we studied.

α:

Q) $\alpha \rightarrow \beta$ is Tautology or Not ?

Try to Make { If $\alpha = T$, $\beta = F$ } simultaneously false Hence then it will become Not a Tautology..

$\alpha \to \beta$

Try to make $\alpha = T,\ \beta = F$

Approach-1

1st make $\alpha = T$ then Try to make $\beta = F$

Approach-2

1st make $\beta = F$ then try to make $\alpha = T$

Ques) $y: (P \wedge Q) \to (P \vee Q)$

Approach-1 ✓

Make LHS = T.

$P = T$
$Q = T$.

$T \to F$ } not possible

So tautology ...

Q) $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ check TAUTOLOGY.

Simplification Method-2.

$\overline{(\bar{P}+Q)(\bar{Q}+R)} + \bar{P} + R$

$(P\bar{Q}) + (Q\bar{R}) + \bar{P} + R$

$\bar{P} + P\bar{Q} + R + \bar{R}Q$

$\underbrace{\bar{P} + \bar{Q} + R + Q}_{T} = \bar{P} + R + T$

$= T$ ✓

Analysis of Implication

CONVERSE, SATISFIABILITY & VALIDITY

SATISFIABILITY

Y: $P \vee (P \to Q)$

Can we satisfy Y?

↓

In prop. logic making TRUE ...

$P = T$ will satisfy Y ..

$P \wedge \neg P$

Not satisfiable ≡ unsatisfiable.

$P = T$; $P \wedge \neg P = F$

$P = F$; $P \wedge \neg P = F$.

Note

A compound proposition is satisfiable If there is an assignment of truth values to its variables that makes it true otherwise unsatisfiable (when compound proposition is false for all assignments of truth values to its variable) ...

SATISFIABLE (Can you make it TRUE ...)

① $P \rightarrow$ Ⓢ ✓ $P = T$ | contingency

② $P \wedge Q \rightarrow$ Ⓢ ✓ $P = T, Q = T$ | contingency.

③ $P \wedge (Q \wedge \bar{P}) \rightarrow$ ✗ unsatisfiable $(P \wedge \bar{P}) \wedge Q$
$F \wedge Q = f$.

④ $P \vee \bar{P} \rightarrow$ satisfiable ✓ $(P = T)$

⑤ $P \rightarrow P$ satisfiable.

$\boxed{\text{SATISFIABLE} \equiv \text{TAUTOLOGY}}$ ✓ ✗ $\neq$? OR. $\boxed{\text{CONTIGENCY}}$ ✓ ✗

Coz atleast one time true.

$\boxed{\underbrace{\text{UNSATISFIABLE}} = \text{CONTRADICTION}}$ = Always false.

Cannot satisfy
Cannot be True.

Q) Assume $\bar{G}$ is CONTRADICTION...
Then what is $\bar{G}$?

- TAUTOLOGY ✓
- CONTINGENCY
- SATISFIABLE ✓

- Every TAUTOLOGY is satisfiable...
- But every satisfiable not a tautology...

$$\boxed{\text{TAUTOLOGY} \underset{\times}{\overset{\checkmark}{\rightleftarrows}} \text{SATISFIABLE}}$$

Note

Compound proposition is unsatisfiable iff its negation is TRUE for all assignments of truth values to the variables that is iff its negation is TAUTOLOGY.

VALID

In prop. logic

VALID ≡ TAUTOLOGY

NOT VALID ≡ NOT TAUTOLOGY ..

≡ CONTINGENCY OR CONTRADICTION.

- TAUTOLOGY — VALID — Always TRUE. (equivalent to TRUE)
- INVALID — NOT TAUTOLOGY — CONTRADICTION OR CONTINGENCY
- CONTRADICTION — UNSATISFIABLE — Always FALSE (equivalent to false)
- SATISFIABLE — Atleast one time TRUE... — TAUTOLOGY OR CONTINGENCY.

FALSIFIABLE

Atleast one time FALSE ≡ CONTRADICTION OR CONTINGENCY. (neither TAUTOLOGY NOR CONTRADICTION)

CONTINGENCY → atleast one time TRUE & atleast one time false..

Ques) Assume G is falsifiable then $\bar{G}$?

falsifiable → G can be made false in atleast one case.

- TAUTOLOGY
- CONTINGENCY
- NOT VALID)
- CONTRADICTION
- SATISFIABLE.

G	$\bar{G}$
F	T

Ques) G_1: $P \wedge \bar{P}$ — falsifiable, contradiction

G_2 : $P \rightarrow Q$ — falsifiable, contingency.

G_3: $P \vee \bar{P}$ — Not falsifiable

falsifiable ≡ NOT VALID ≡ NOT TAUTOLOGY

Contradiction
OR
contingency

- $\overline{\text{TAUTOLOGY}}$ = CONTRADICTION
- $\overline{\text{CONTRADICTION}}$ = TAUTOLOGY.
- $\overline{\text{CONTINGENCY}}$ = CONTINGENCY.

(a) Checking satisfiability ?

Make it TRUE ...

1) $(p \lor \neg p)$ ✓

2) $(p \to p)$ ✓

3) $(p \to (q \to p)) = T$ ✓

Can make True.

4) $\neg(p \land \neg p)$ ✓ $\neg(p \land \bar{p}) = (\bar{F}) = T$.

5) $\neg(p \lor \neg p)$ ✗

6) $(p \land \neg p)$ ✗

7) p ✓

8) $((p \to q) \to p)$ ✓ $p = T$.

$$\boxed{P \to Q}$$

Antecedent — Consequent

Checking TAUTOLOGY for conditional stc

$X \to Y$
$\boxed{T \quad F}$ try to make $X \to Y$ (false)

If you can then $X \to Y$ is NOT Tautology
Cannot — $X \to Y$ is TAUTOLOGY.

Approach ①

$X \to Y$.

- Make X = True.
- Try to make F = false.

Yes — Not a TAUTOLOGY

No — TAUTOLOGY.

Approach ②

$X \to Y$

- Make Y = false.
- Try to make X = True.

Yes — Not a Tautology

No — TAUTOLOGY.

↓

Approach ② is very imp Method when we study Arguments

Check TAUTOLOGY ?

Ques) $(P \oplus Q) \to (P \vee Q)$

Approach-2

T

- Make RHS = false $P = f$
- Try to make LHS = True $Q = f$.

It cannot happen.

∴ Not a Tautology.

Propositional variable vs Propositional definition

Proposition
Proposition variable
logical connectives

} are studied in propositional logic.

! prop. logic is collection / set of all prop. formulae {" p, q, r ... T, F"} — ¬p∨r∧q —— p∨q ?

What is Propositional logic ? ↳ every var. is T/F; every var. call prop var; every var. call Boolean var; only take T/F that either a fact

↓

Is a sentence T/F.

- is a sentence that declares a fact T/F but not both. so it is declarative sentence.

Eg) $1+2=4$ prop; yes coz (False)
$2\times3=6$ ⟶ yes coz (True).

Eg) Grass is green (T) prop

Eg) 2 is odd prime AND 3 is an even no (F) prop

Eg) GRASS ✗

Eg) Hello! ✗

ATOMIC Proposition ⟶ [No part is a proposition]

GRASS is green
SNOW is white

- By itself T/F.

- ~~Not~~ a T/F values does not ... on anyone ...

COMPOUND Proposition

"GRASS is green & SNOW is white" ..

(mix that is proposit...

Propositional formula / Expression / form

⇓

"Well-formed formula"

Prop. variable : p, q, r

Prop. Expression : p ∨ q

⇓

Truth value of exp. depends on the truth value of atomic proposition.

PROPOSITIONAL FORMULA :

1) Every prop. variable (atomic proposition) is a prop. formula.

(Eg) p, q, r.

2) T, F are prop. formula.

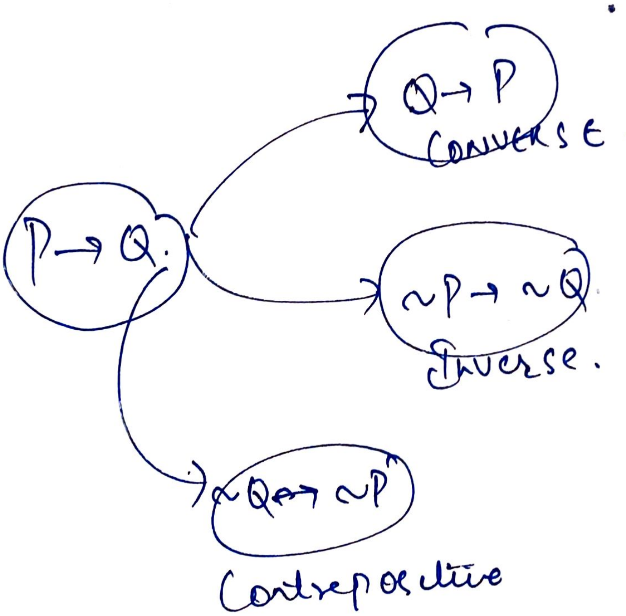
3) If G, H are prop formula then
$\neg G$,
$\neg H$,
$G \wedge H$ } prop. formula.

Eng-logic translation
- Converse, unless word
- Contrapositive
- Inverse.

Consider the conditional stat^n : $P \to Q$.

Ex: If the weather (P) is nice, then I will wash the car (Q).

- If I'll wash the car, then the weather is nice. ($Q \to P$) **CONVERSE**
- If the weather is not nice, then I'll not wash the car. **INVERSE** $\sim P \to \sim Q$.
- If I'll not wash the car then weather is not nice. $\sim Q \to \sim P$ — **CONTRAPOSITIVE**



Eg) The home team wins whether it is Raining

P / Q

Y: P if Q

If Q, R. $\equiv$ $Q \rightarrow P$

$\sim Q \rightarrow \sim P$ — Converse

$P \rightarrow Q$ — Inverse

$\sim P \rightarrow \sim Q$ — Contra +

$\Downarrow$

If the home team wins, then it is Raining.

Eg) If This Book is Interesting then I am staying at home.

$P \rightarrow Q$

CONVERSE : $Q \rightarrow P$ [If I am staying at home then This book is interesting]

Eg CONTRAPOSITIVE : $\sim Q \rightarrow \sim P$: If I am not staying at home then This Book is not Interesting ...

INVERSE : $\sim P \rightarrow \sim Q$: If This Book is not Interesting then I am not staying at home.

cannot make ...

$\boxed{P \to Q}$

- Premise
- hypothesis
- antecedents

- conclusion
- consequent.

Note →

CONVERSE & INVERSE $\not\equiv$ Implication

CONTRAPOSITIVE $\equiv$ IMPLICATION

equivalent

same truth value...

CONVERSE $\equiv$ INVERSE

Proof

α: $P \to Q$ false.

$\equiv$ equivalent.

β: $\neg Q \to \neg P$ false.

$P = T$

$P=T$, $Q=f$

$\alpha : q$

$\beta : T$

~~P=f~~

$\neg Q = T = F$

$\neg P = f = T$.

Ques) Assume P, Q are propo var then which is correct?

~~(i) $P \to Q$~~ // never equivalent to $Q \to P$

~~(ii)~~ ______ always ______

P	Q	$P \to Q$	$Q \to P$
F	F	T	T
F	T	T	F
T	F	F	T
T	T	F	T

$P \to Q \not\equiv Q \to P$.

Negation

P: This book is Interesting

- This book is not interesting
- It is not the case that This book is Interesting
- It is not True that This book is INTERESTING.
- It is false that This book is INTERESTING
- This book is Interesting is false.

Eg) The sun is out : P

The sun is out, TRUE. : $\neg P$

Eg) P is false but Q is True.

$\neg P \wedge Q$

Eg) not P, but Q

$\neg P \wedge Q$

How to write formula / Expression

$(P \to Q)$ is TRUE $\equiv$ P is false OR Q is True.

$P \to Q \equiv \neg P \vee Q = \bar{P} + Q$.

$P \oplus Q$ is TRUE $\equiv$ $P = T, Q = F$.

OR

$P = F, Q = T$.

$$\boxed{P \oplus Q \equiv P\bar{Q} + \bar{P}Q}$$

Being 18+ is necessary for joining army.
(Q = Being 18+, P = joining army)

$P \to Q$

✓ If you are in army then you are 18+.

✗ If you are 18+ then you are in army.

means

- Without being 18+, you cannot join army.
- If you are not 18+, you cannot join army.

same $\neg Q \to \neg P$.

(Contrapositive)

PROVIDED THAT

- Provided that P, Q $\equiv$ If P, Q
- $\equiv P \to Q$

cannot make ...

- P if Q $\equiv$ Q → P

- P only if Q = P → Q.

  Q means Q is Necessary

UNLESS in Eng

- Imagine a stubborn child "JEN"...

- JEN won't go to the party (P) UNLESS Mary goes to the party. (Q)

  this will happen

  If this not happen

If not Q, P.

\# JEN won't go to the party unless Mary goes to the party.

↓

- If Mary goes to the Party then JEN won't go to the party.

\# - If not-(Mary goes to the party) then JEN won't go to the party.

logical Inference / Arguments

Check Tautologies ?

* $\underset{T}{p} \to (\underbrace{\neg q \lor r}_{F.} \lor \neg r)$ $\qquad \neg q = F \;,\; q = T$, $\; r = T$

$\downarrow$

can't $r \lor \neg r$.

$\therefore$ Tautology .

* $[\overbrace{\overset{T}{\neg p} \land (\overset{F}{p} \lor \overset{T}{q})}^{}] \to q$

$\underbrace{\neg p \land (p \lor q)}_{T.} \qquad q \downarrow F.$

$\therefore$ not tautology .

* $\underset{T}{\neg p} \to (p \to q)$

$F \to T$

True .

$\therefore$ tautology .

* $\underset{T}{p} \to (\underbrace{\overset{T}{p} \to \overset{F}{q}}_{F})$

$\therefore$ not tautology . . .

Approach 1

$\underset{T}{P} \to Q$

Approach 2

$P \to \underset{F.}{Q}$

Q) infer ≡ Deduce ≡ Imply ≡ Deduction.

Knowledge.
If it is raining, He'll take umbrella. It is raining ✓
"He'll take umbrella" ? (infer).

Q). If I work all night on his homework (H), then I can ans. all the exercises (E).

. If I ans. all the exercises (E), I will understand the material (M).

. I work all night on this h/w (H).

"I will understand the material (M)"?

Knowledge Base $\{ H \to E,\ E \to M,\ H \}$

Conclusion $\{M\}$

$H \to E,\ E \to M,\ H \models M.$

$\boxed{\text{INFERENCE} \equiv \text{DEDUCTION} \equiv \text{Derivation}}$

Knowledge Base:

Set of premises | hypothesis on assumptions

- KB $\models$ Q || Q is logically inferred by KB.
  This is TRUE.
- If KB = True then Q should be True.
- KB $\models$ y iff KB $\rightarrow$ y is Tautology...

→ never possible $\boxed{\text{KB = True But y = false}}$

Q) Which is correct?

* $P \rightarrow Q$, $Q \models P$ × P need not be true.
* $P \oplus Q$, $Q \models \neg P$ ✓
* $P \leftrightarrow Q$, $\neg Q \models \neg P$ ✓
* $P \leftrightarrow Q \models P$ × can be false.
* $P \wedge Q \models P$ ✓
* $P \rightarrow \bar{Q}$, $\bar{P} \models \bar{Q}$ ×

logical consequence ≡ logical inference ≡ logical implication ≡ logical entailment.

$$P \wedge Q \models Q$$

Q is logical consequence of $P \wedge Q$

$P \wedge Q$ logically implies Q.

ARGUMENTS.

If it is raining, He'll take umbrella }
It is not raining } premise

Hence;
He'll not take umbrella - conclusion

Argument.

Argument: seq of statements in which you have some premise followed by 1-conclusion

P_1
P_2
P_3
P_4 } Premise

C } conclusion | consequence.

Argument.

Argument

An argument in prop. logic is a seq of propositions.

All but final proposition in the argument are called premises & final proposition is called "CONCLUSION".

An argument is valid if truth of all its premises implies that conclusion is TRUE.

An argument form in prop. logic is a seq. of compound propositions involving propositional variables.

An argument form is valid no matter which particular propositions are substituted for the propositional var. in its premises the conclusion is TRUE if all premises are all TRUE...

Valid = ?

P_1
P_2
P_3
———
C

If all premises True
then
conclusion is TRUE.

i.e. is valid iff $(P_1 \wedge P_2 \wedge P_3) \longrightarrow C$
is Tautology.

* VALID
By an argument, we mean a sequence of statements that end with a conclusion

* By valid, we mean that the conclusion must follow from truth of preceding stetn / premises.

* That is, an argument is valid iff it is impossible for all the premises to be TRUE & conclusion to be false.

To check validity of Argument ...

P_1
P_2
P_3
———
C

make conclusion false ...
Then try to make all premises TRUE.

If possible

Arg 1) P (F) ~~false~~ false to premise false

$\overset{F}{P} \to \overset{F}{Q}$ } Make its true.

Q (false

∴ valid argument.

- $P, P \to Q \models Q$.
- $(P \wedge (P \to Q)) \to Q$ is Tautology.

Arg 2

$\overset{F}{P} \oplus \overset{T}{Q}$ → True

P

$\neg Q$ (false).

} TRUE. false (not possible.

∴ valid argument.

Arg 3

$\overset{T}{P} \vee \overset{F}{Q}$

$\overset{T}{R} \vee \overset{F}{S}$

$\overset{F}{\neg P \vee \neg R^F}$ → false

$Q \vee S$ (False)

$\overset{F}{Q}$, $\overset{F}{S}$

} True. false

Cannot be true.

∴ valid argument.

Arg4)

$\overset{T}{P} \to \overset{T}{Q}$

$^{T}Q \to R_{F}$ } True

false

$P \to R$ false

T F

∴ valid

Note

Implication is TRANSITIVE . . .

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right] & B \arrow[r] & C \end{tikzcd}$$

$(A \to B,\ B \to C)$ then $A \to C$ ..

Args 5)

$\overset{T}{P} \to \overset{F}{Q} \longrightarrow F$

$^{F}R \to S^{F} \longrightarrow T$

$\underset{T}{P} \vee \underset{F}{R} \longrightarrow T$ ∴ valid

$S \vee Q$ (false)

F F.

Ques

$\overset{T}{P} \to \overset{T}{Q} \to T.$

$^{T}R \to S^{T} \to T.$

$\neg \underset{F}{Q} \vee \neg \underset{F}{S} = f.$ ∴ valid

$\neg P^{F} \vee \neg R^{F} \to F$

$P = T,\ R = T.$

Rule of Inference

* Some popular valid Argument forms. frequently used.

P (study well.

$P \to Q$

$\underline{}$

Q (Crack gate exam)

} Modus Ponens.

Some popular (frequently used) arguments forms have special names :

$$\begin{array}{l} P \to Q \\ \neg Q \\ \hline \neg P \end{array}$$

Modus Tollens.

#

$$\begin{array}{l} P \to Q \\ Q \to R \\ \hline P \to R \end{array}$$

Transitivity of Implication
OR
Hypothetical syllogism.

Std. rules of Inference

1) Modus Ponens
$$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore p \\ \hline \end{array}$$

2) Modus Tollens
$$\begin{array}{c} \neg q \\ p \rightarrow q \\ \hline \neg p \\ \hline \end{array}$$

3) Conjunctive Simplification
$$\begin{array}{c} p \\ q \\ \hline p \\ \hline \end{array}$$

4) Disjunctive Syllogism
$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \\ \hline \end{array}$$

5) Hypothetical Syllogism
$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \\ \hline \end{array}$$

6) Addition
$$\begin{array}{c} p \\ \hline p \vee q \\ \hline \end{array}$$

7) Conjunctive simplification.
$$\begin{array}{c} p \wedge q \\ \hline p \end{array}$$

8) Resolution
$$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline q \vee r \\ \hline \end{array}$$

DOMAIN, PREDICATES, QUANTIFIERS

Toh Problem kya hai ??

↓

Problem yeh hai ki every var. True/false.

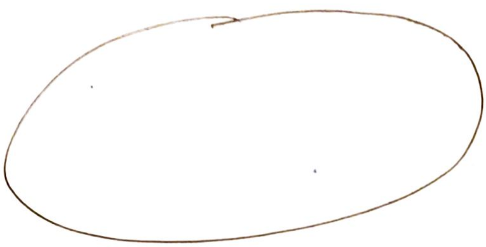
UNIVERSE of Propositional LOGIC:

$\boxed{p \wedge q \to \neg r \vee \neg s}$

$\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$. { $\underbrace{p, q, r, s}$ take value T/F.

A Better logic for real world.. → Talk about more T/F...

PRICE ≡ variable.



take real No's...

Set of Non-negative real no's... (price can be 0)...

* We want to talk about OBJECT & their properties
  L Priyanka, age, profession...

* Relationship B/W objects.

Ram Sita

Object.

marriage (x, y) ≡ x married y.

Our variable should be able to take some specific set of values of our choice.

↓

Domain / universe of that var.

SUM (x, y, z) : x + y = Z ....

Domain 2, -1, 0, 1, 2, ... object / constant

Integers

var x, y, z can take any value from the DOMAIN / VARIABLE ...

SUM (2, 3, 5) : TRUE

* TRANSFORMATION of objects
or
function

$f(x) = x^2$
$f(2) = 4.$

$2 \xrightarrow{func?} 4$

* Quantification

↓

we want to be able to talk about multiple objects...

(all, SOME, NONE).

Frege & Pierce noted the need for more than Prop. LOGIC and they started developing 1st order LOGIC (Predicate calculus).

So what does FOL have that Prop. Logic does not?

Object = element / values in a DOMAIN.

First order logic

* A world of objects, their properties, their relationships, their transformation.

* (1) FOL speaks about objects, which are the DOMAIN of Discourse / OR UNIVERSE.

↓

each var. refers to some object in a set.

$x \rightarrow \{1, 2, 3, \ldots, \cancel{\alpha}\}$ } Domain | universe of var. x.

Not a No.

x (2) $x = 2$ ✓
$x = 3$ ✗

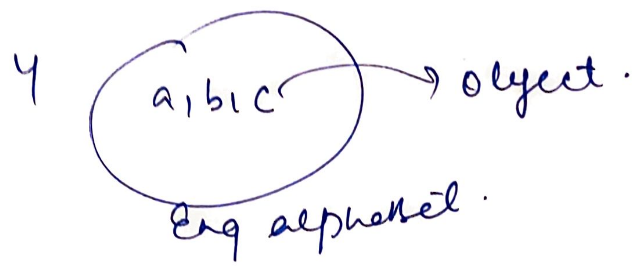
* (2) Every var. has a DOMAIN . . .
Domain | universe | Domain of Discourse | universe of Discourse . . .

Z (T, F) → this domain also possible..

(3) Every var. can refer to objects (3)
from its own DOMAIN ...

set of values that a var. can take.

* FIRST ORDER LOGIC is concerned about (Properties of these objects)



define property of these objects.

vowel(y) : y is vowel
vowel(a) : TRUE.
vowel(b) : FALSE....

DOMAIN : values var. can take.

IN FOL, we also have relations over
between / among objects called PREDICATES

// unary relations / predicates are called "properties"

N

x, y — $\{1, 2, 3 \ldots\}$

$D(x,y)$: $\boxed{x < y}$

$D(1,3)$: True.

$D(3,1)$: FALSE.

$D(1,1)$: FALSE.

$D'(0,2)$: invalid.

↓

$x \neq 0$.

<u>Note</u>:

Two elements < same / Different.

$\boxed{\text{Relation | PREDICATE}}$

x — $\{1, 2, 3\}$

y — $\{2, 3, 5\}$

z — $\{2, 3, 4, 5, 6, 7, 8\}$

SUM(x, y, z) : $x + y = z$

SUM$(2, 3, 6)$: false.

SUM$(4, 2, 6)$: Invalid

$x \neq 4$.

Quantifiers

PREDICATE

↳ is a sentence in which you have variable once you replace all the variables you will get proposition

↳ It is also called "propositional variable"

Eg) $\boxed{\begin{array}{l} D = N \text{ (set of Natural No's)} \\ \text{Natural\_No. x is even.} \end{array}}$

↳ T/F depends on value of x. so not a proposition, it is a predicate coz it involves a variable...

· Substituting any particular value for the variable x will make it a proposition.

$\left.\begin{array}{ll} \text{Natural No. (2) is even.} & E(2) : \text{True} \\ \text{Natural No (3) is even} & E(3) : \text{false} \end{array}\right\}$

SO
PREDICATE is a statement that contains propositional variables (predicate variables) & that may be T/F depending on the values of these variables

$\boxed{\text{Object} = \text{element in Domain} = \text{constant}}$

DOMAIN

Variable: X → a, e, i, o, u → Object / constant / element

this var. can take any of this value.

Eg

DOMAIN

PRICE → Non-neg. real no's. 5, 0.5, 1000 → CONSTANT / object / element

Variable can have any of these values from the DOMAIN.

* PREDICATES take objects as "ARGUMENTS" & evaluate T/F.

* Hence predicates are also called "Proposition-function" becoz predicates become proposition when we replace every var. by objects from Domain of their variable.

...ant

/ constant / element

...t / object / element

"ARGUMENTS"

called

...edicates become

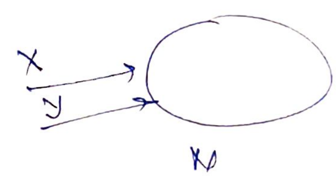
... every var.

of their variable

Note: ① Domain in FOL is always Non-empty, unless explicitly stated. ②

② By default (if nothing is mentioned about Domain or domain of Diff var.) Domain of every var. in a FOL expression is same & it is called "DOMAIN / Universe of the expression.

Eg) DOMAIN is N.
$P(x,y) : x > y$ } // Both set of Natural No.



$P(1,1) : 1 > 1$ false.
$P(2,1)$: True.
$P(3,2)$: True.

Propositional functions

- DOMAIN : N
- Let $Q(x,y,z)$ denote "$x^2+y^2=z^2$" — ternary predicate.

  What is the truth value of $Q(3,4,5)$?

  $Q(3,4,5)$ is TRUE ...

Tab Newnormal - LC

1x on x10 day.

(2) $\boxed{E(x) : x \text{ is even}}$ – not a proposition coz Truth values depend on value of x.

Quantification

* "for every x, $E(x)$ is True" → false.

* "for some value of x, $E(x)$ is TRUE" coz this is a proposition

So quantification is any other way to create "proposition"...

Quantification words in English:

[few, all, Many, some, any.

Std. Quantification words in logic

{ All ≡ Every // universal quantifier

{ SOME ≡ atleast one // Existential Quantifier.

If you will understand these 2- quantifiers you can handle any quantifiers (How one in exam they mention definition".

$P(x, y)$: ~.

Quantifiers

Universal Quantifiers

talks about universal Quantification over all elements in the DOMAIN.

Existential Quantifiers

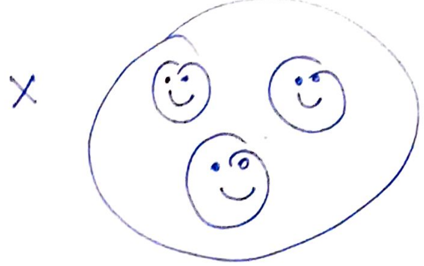
talks about "Existence" Does there exist some element.

UNIVERSAL QUANTIFIERS

Eng.: for every x (in Domain of x) property $P(x)$ is TRUE.

Logic:

$\forall_x P(x)$: for every x, $P(x)$ is T



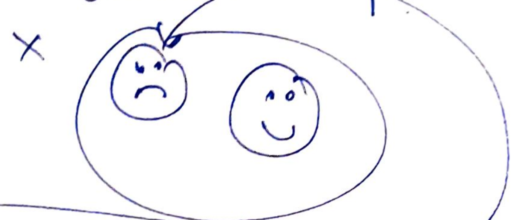
$\left(\forall_x \boxed{\text{Smiling}(x)}\right) \equiv$ True.

for every x, x is smiling ...

Since SMILING(x) is TRUE for every choice of x, this stat^n TRUE...

// in exam they can give diff. domain of every var. but if not mention then by default DOMAIN is same for every var.

counter example.



$\boxed{\forall_x \text{ Smiling}(x)} \equiv$ false.

....

Definition

The UNIVERSAL QUANTIFICATION of P(x) is the proposition ...

$\boxed{\text{P(x) is TRUE for all values x in the universe of DISCOURSE}}$

"for all x P(x)" or "for every x P(x)".

$\boxed{\forall_x P(x)}$

* The universal quantification $P(x)$

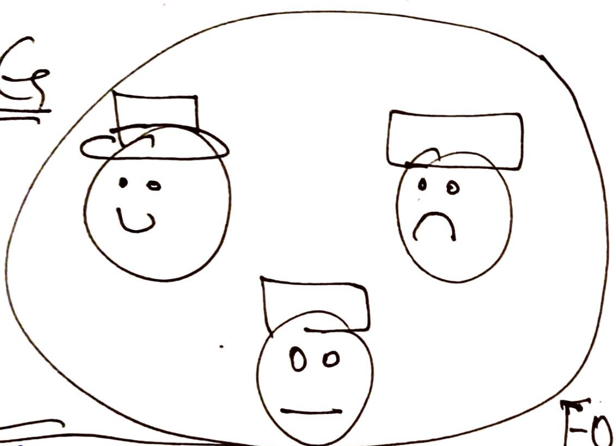
"$P(x)$ for all values of x in the DOMAIN."

$\forall_x P(x)$ $\xrightarrow{\text{read}}$ "for all x $P(x)$
OR
"for every x $P(x)$.

$\forall$ ↓ universal quantifier

An element for which $P(x)$ is false is called "COUNTER-EXAMPLE" of $\forall_x P(x)$...

Eg



FOL...

$$\underbrace{\left(\forall x.\ \text{Smiling}(x)\right)}_{A} \longrightarrow \underbrace{\left(\forall y\ \text{wearing HAT}(y)\right)}_{B}$$

A: False...
B: TRUE
} TRUE.

Eg) $\forall x \underbrace{\boxed{\left(\text{SMILING}(x) \rightarrow \text{wearingHAT}(x) \right)}}_{P(x)}$

"For every x, if x is SMILING then x is wearing a HAT"...

$\equiv$ every SMILING element or person is wearing a HAT.

~~false~~ ...

TRUE (SMILE → HAT) ✓

Note: // universal quantifier basically it is "conjuction"

If the universe of Discourse is finite $\{x_1, x_2, \ldots, x_k\}$ then universe quantifier is simply conjunction of all elements.

$$\forall x\, P(x) \Longleftrightarrow P(x_1) \wedge P(x_2) \wedge \ldots P(x_k).$$

---

a, b, c, d

DOMAIN.

$\forall x\, P(x) \equiv P(a) \wedge P(b) \wedge P(c) \wedge P(d)$

$ Only possible if DOMAIN is finite

$ In infinite you can understand but can't write.

Existential Quantifier

Eng: for some x, $P(x)$ is TRUE.

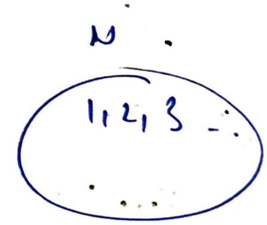
logic: $\exists x (P(x))$. $\equiv$ There exists x, $P(x)$ is true
There exists atleast one.

$\exists x \cdot$ some-formulae
is TRUE if, for some choice of x, the statement some formulae is TRUE when that x is plugged into it.

Eg)

DOMAIN : $\mathbb{N}$

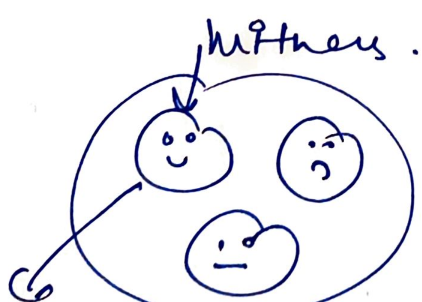
$\exists x \{ \text{Even}(x) \wedge \text{Prime}(x)$



witness : 2

$\text{Even}(2) \wedge \text{Prime}(2) = \text{True}$.
$\downarrow$ TRUE $\qquad \downarrow$ TRUE.

Eg)



$\boxed{\cancel{\exists} x \ \text{Smiling}(x)}$

Definition:

- There exists an element x in the universe of Discourse such that $P(x)$ is TRUE.
- $\exists x\, P(x)$ // There is an x such that $P(x)$ True

OR // There is atleast one x such that $P(x)$ TRUE...

$\boxed{\text{for some } x\ P(x)}$

Ques) Suppose $P(x)$ is the predicate $\underline{x+2=2x}$ and the universe of discourse of x is the set $\{1,2,3\}$... Then...

Counter examples

(A) $\forall x\, P(x) \equiv$ false // $x=1$

(B) $\exists x\, P(x) \equiv$ TRUE // witness $x=2$.

(C) $\forall x\, \neg P(x) =$ false // Counter example $x=2$

(D) $\exists x\, \neg P(x) =$ TRUE // Counter example $x=1$.

(E) $\neg \exists x\, P(x) =$ false

(F) $\neg \forall x\, P(x) =$ TRUE.

DOMAIN = D

	WHEN TRUE	WHEN FALSE
$\forall_x P(x)$	for every element in D, P(x) is TRUE	for atleast one element P(x) is false.
$\exists x P(x)$	for atleast one element, P(x) TRUE.	for all x in D, P(x) is false.
$\forall x \neg P(x)$	for every element in D, P(x) false	for atleast one element P(x) TRUE.
$\exists x \neg P(x)$	for atleast one element P(x) is false	for all x in D, P(x) TRUE ...

Domain X = {2, 4, 9}

Ques) P(x) : x is prime.
Q(x) : x is even.

$\forall x \, (P(x) \rightarrow Q(x))$

$\underset{T}{P(2)} \rightarrow \underset{T}{Q(2)}$ — True.

$\underset{F}{P(9)} \rightarrow Q(9) \longrightarrow$ True.

$\underset{F}{P(4)} \longrightarrow$ True.

Scope, bounded & free variable

Ex)

$D = N$

$E(x)$: x is even.

$\forall x\, E(x)$ - True ?

- false.
- Counter-ex : $x=3$.

$\exists x\, E(x)$?

- True.
- Witness : $x=2$.

Note :

Useful Intuition :

(I) Universally quantified statement are TRUE unless there's a counter example.

unless = If Not.

If there is no counter example then $\forall x\, P(x)$ is TRUE...

- P unless Q (unless = If Not.)
- P if not Q
- If $\neg Q$ then P

- $\neg Q \to P$
- $Q + P$

② Existentially - Quantified statement are false unless there's a witness.

Special - case

* Quantifiers when DOMAIN is EMPTY ??

$\forall_x P(x)$ = True.

⇓

No counter Example.

$\exists x \, P(x)$ = false

⇓

No witness example.

* Can you give a real world appn of Behaviour of universal quantifier when the should be vacuous TRUE when there is an empty DOMAIN?

$\forall_x$ (any thing) = True

D= empty.

|

Sure | suppose you want to check whether you've taken all the pre-requisites for a course.

What should happen if you want to take a class that has no pre-requisites?

↓

You can take the class —

DOMAIN : set of all natural no's

Variable bounded by $\forall$ in $E(x)$ x is bounded.

bounded

- $\neg\forall_x E(x)$: false
- $\neg\forall_m E(m)$: false
- $\neg\forall_y E(y)$: false

SAME

- $\exists_x E(x)$: True
- $\exists_m E(m)$: True
- $\exists_y E(y)$: True

$\forall_x E(x)$: for every natural no x, x is even
$\equiv$ Every natural No. is even.

$\forall_m E(m)$
:- for every natural no. m, m is even
$\equiv$ Every Natural No is even ...

$P(x)$ ✓
Domain : D

- $\forall_x P(x) = \forall_y P(y) = \forall_z P(z)$
- Is var. is AVAILABLE ??
  ↓
  Coz it is a proposition so no variable ...
  so x, m, y has no significance .. } bounded variables

Note
THERE IS NO INDIVIDUAL IMPORTANCE OF BOUNDED VARIABLE, they are just PLACE HOLDER

* $E(x)$: x is even. → free var.
DOMAIN : Z ↳ set of integers

$E(2)$: 2 is even // proposition

* $P(x) : \forall_x E(x)$ — proposition

$P(2) : \cancel{\forall_2} E(2)$

↓

Bounded var.
or
Quantified var.

$E(x)$
↓
- free var.
- Can take any value

$\forall_x (E(x)) \vee E(y) = P(y)$.

$\forall_x$ ↓ Bounded

$E(y)$ ↓ free

$P(y)$ ↓ Predicate

* $P(x,y)$: $\boxed{x + y = \text{even}.}$

x ↓ free, y ↓ free.

- $P(2,4)$: True

$Q(y) : \forall_x P(x,y) \equiv$ for every integer x, $P(x,y)$ is TRUE...

$P(y)$: $\boxed{\forall_y \; P(x, y)}$ = for every integer x $P(x, y)$ is TRUE.

free

Bounded.

$P(2)$: $\boxed{\forall_x \; P(x, 2)}$ $\equiv$ false

counter ex = $x=3$

There is no free var. so it is a proposition

$\forall_x P(x, 2) \equiv \forall_y \; P(y, 2) \equiv \forall_z (P_{z,2})$

Dummy var.

coz it is a proposition So actually there is NO var.

Bounded-variable.

$\boxed{\exists_x \; E(x)}$

* NO individual importance
* can't do anything w/o this Quantifier
* This Bounded var. cannot replace with any value coz LHS quantifier will decide what will happen here.

$\boxed{E(z)}$

- free var.
- individual-identity?

$D = N$ (6)

$\boxed{\exists x \; E(x)}$ → Proposition

BOSS → Slave.

- Bounded var.
- Dummy var.
- place holder.
- No individual identity
- All values are checking or calling By Quantifiers from DOMAIN. (1, 2, 3 — N)

What it means $\exists x \; E(x)$, x is taking value ??

$\boxed{P(x) = \exists x \; E(x)}$

$P(2) = \exists_2 \; E(2)$ X

not make any sense.

(1, 2, 3) $\exists x \, P(x)$

$P(1) \vee P(2) \vee P(3)$.

(4) Can we replace a free var. by some value from the DOMAIN ?? → YES.

$D = Z$

✓ $S(y): \; \exists x \, (x > y)$ — y: free $S(4):$

X $S(x): \; \exists x \, (x > y)$ — x: Bounded

X $S(x): \; \exists x \, (E(x))$ — x: Bounded

Ans) DOMAIN : set of all integers

① $P(m,n)$: $mn>0$
free : m, n
Bounded : NIL.

② $P(0,2)$
No free & no Bounded var.

③ $P(0,n)$ — free = n
Proposition X

④ $\forall n\ P(0,n)$ — free = NIL
Bounded = n

⑤ $\forall m\ P(0,n)$
free = n
Bounded = m
not a proposition...

$\forall m \boxed{P(1,n)}$ DOMAIN = Z.
↳ $n>0$

∘ $\forall m (n>0) \equiv$ for every integer m, $n>0$.

∘ Not a proposition
coz n is not Bounded.

SET

- Subsets
- Set operations
- Venn Diagram
- Element
- Membership
- Cordinality
- EMPTY_SET.

SET (DEFINATION)

$\boxed{\text{Set is an unordered collection of Distinct objects which may be anything}}$

objects → elements.

$S_1 = \{\underbrace{1, 2, 3, 4, 5}_{5 \text{ elements / objects}}\}$ → curly notation seprate by COMMA...

S_1 contains 5 elements...

$\underbrace{\{1,2,3\} = \{3,2,1\}}_{\text{order does not matter}}$

$\underbrace{\{1,2,3\} = \{1,1,2,3\}}_{\text{repetition NOT allow.}}$ ✗

3-elements only.

ISRO

$S = \{a, a, 1, 1, 1, 2, 3\}$

↓

elements = 4.

Ques) Which of the following is set of prime No. less than 5?

Preferable → (1) $\{2, 3\}$

(2) $\{2, 2, 3\}$

(3) $\{2, 2, 2, 3, 3, 3, 3\}$

SAME SET.

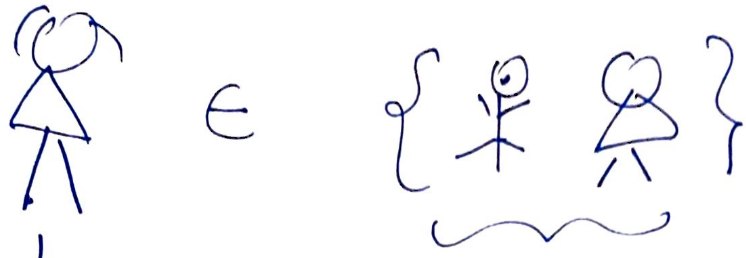
$S = \{a, b, c\}$

- a is an element in S
- a $\boxed{\text{Belongs to}}$ S
- $a \in S$

  $e \notin S$
- $\in$

This symbol is the "Element-of" symbol.

*



Element of set A.

SET(A)

$S = \{a, b, c\}$

Elements of S

- a ✓
- b ✓
- c ✓
- $\{a, b\}$ ✓
- $\{a, b\} \in S$ ✗
- $\{a\} \in S$ ✗ — not an element.
- $a \in S$ ✓

$\underset{\text{only 1}}{1} \neq \underbrace{\{1\}}_{\text{set contain 1}}$

- only 1
- ME

set contain 1

room that contains ME.

*

a → element

$\in$ →
- is element in
- Belongs to

$\{a, b, c\}$ → SET

SET → Finite SET, INFINITE SET

Finite SET

* limited # of elements.

* set in which [# elements = some whole No.]

INFINITE SET

unlimited # elements.

* # elements ≠ some whole No.

∞ → not a no.

* "INFINITE SET":

1. $N = \{1, 2, 3 \text{———}\}$
2. $Z = I = \{ - - 3, -2 \text{———} 0, 1, 2 \text{———}\}$
3. Prime $= \{2, 3, 7 \text{———}\}$
4. Q = Set of Rational No
   $\left\{ \frac{a}{b} \;\middle|\; a, b \text{ are integers }, b \neq 0 \right\}$
5. R = set of real No.
   $= \{0, 1, 1.2, \pi, e, 0.22 \text{......} -0.3, \infty\}$

   ∞ not a Number.

Cardinality

Cardinality of set is the finite Number of elements it contains. $\boxed{\text{\# Cardinality = size of set}}$

$|S|$ ✓

$\{1, 2, 3, 3\} = 3.$

$\underbrace{\{\{a\}, \{b, c\}, \{d\}\}}_{\text{In this set objects are also set.}}$ ——— $|S| = 3$

Note:

for Infinite set we see cardinality in countable

$1 \neq \{1\}$

$\{1\} \neq \{\{1\}\}$

1, 2 — }

$\{\ \}$ = set of good karan Johar movies...

$\boxed{\{\ \} = \phi}$ phi This is a set which has no element

[empty set
[Null set

, b ≠ 0}

..... -0.3, ∞} ∞ not a Number.

$\phi \neq \{\phi\}$ Imp.

$|\phi| = 0$ $|\{\phi\}| = 1$.

of
ty = size of set

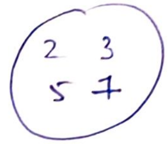
- $\{\ \} = \phi$ ✓
- $\{\phi\} \neq \phi$ ✓
- $\{\phi\} \neq \{\ \}$ ✓
- $|\{\ \}| = 0$ ✓
- $|\phi| = 0$ ✓
- $|\{\phi\}| = 1$ ✓
- $|\{\{\ \}\}| = 1$ ✓

in cardinality

Set Representation.

Set Builder Notation | List | Venn Diagram

$S = \underbrace{\{2,3,5,7\}}_{\text{list representation}}$



(Ques) Consider Eng. descriptions

- All even No's.
- All real No's less than 137.
- All negative integers

We can't list their (infinitely many!) elts..

((# elements very large))

| "SET - BUILDER Notation" | Imp ** |

$S = \{x \mid x \text{ is prime No}, x < 10\}$

$S = \{2,3,5,7\}$.

Ans) $\boxed{S = \{n \mid n \in N \text{ and } n \text{ is even}\}}$

n → set of all n

$\mid$ → where

$n \in N$ → n is Natural No.

and n is even → and n is even.

Subset

$S = \{a, b, c\}$ // collection

↓

sub-collection.

$A = \{a, b\}$
$B = \{b\}$
subset.

A is a subset of S.

$S = \{a, b, c\}$

⇓

Subset of S :-

$\{\}$, $\{b, c\}$, $\{a, c\}$, $\{a, b\}$, $\{c\}$, $\{a\}$, $\{b\}$
$\{a, b, c\}$

Set P, set Q.

$P \subseteq Q$ means every element of P is in Q.

$$\boxed{\forall x\,(x \in P \rightarrow x \in Q)}$$

① $S \subseteq S$

$S = \{a_1, a_2 \text{———} a_n\}$ every set is a subset of itself ?
$S = \{a_1, a_2 \text{———} a_n\}$

② $\phi \subseteq S$

$S = \{a, b, c\}$

$\{\ \}$ ✓ $\equiv \phi$ ✓ | just remove all element of S.

③ $\phi \subseteq \phi$

SUBSET

A set S is called subset of a set T $S \subseteq T$ if all elements of S are also element of T.

1, 2, 3

Q) $S = \{1, 2, 3\}$
$P = \{\{1, 2\}\}$. $\{1, 2\}$ ✗

① $P \subseteq S$ ✗

② $\{\{\ \}\} \subseteq S$ ✗

Note:

$\phi = \{\ \}$

ϕ is a subset of every element.

Ques) $S = \{a, b\}$.

$\{\phi\} \subseteq S$ ✗

$\phi \subseteq S$ ✓

$\boxed{\phi \neq \{\phi\}}$

Ques) $S = \{1, 2, \{2\}, 3\}$.

$1 \in S$ ✓

$2 \in S$ ✓

$\{2\} \in S$ ✓

$\{3\} \in S$ ✗

$\phi \in S$ ✗

$1 \subseteq S$ ✗

$\{1\} \subseteq S$ ✓

$\{1, 2\} \subseteq S$ ✓

$\{2\} \subseteq S$ ✓

$\{\{2\}\} \subseteq S$ ✓

$3 \subseteq S$ ✗

$\phi \subseteq S$ ✓

$\{\phi\} \subseteq S$ ✗

Note:

while checking subset
Always apply definition.

$$\boxed{\begin{gathered} S \subseteq Q \\ \forall x,\ x \in S \rightarrow x \in Q. \end{gathered}}$$

Cardinality

ϕ

$|\{x \mid -2 < x < 5,\ x \in \mathbb{Z}\}| = ?$ 6 $\{-1, 0, 1, 2, 3, 4\}$

$|\phi| = ?$ 0

$|\{x \mid x \in \phi \text{ and } x < 3\}| = ?$ 0 ~~$\{0, 1, 2\}$~~

! Definition of ϕ:

$\forall x,\ x \notin \phi$

$|\{x \mid x \in \{\phi\}\}| = ?$

! $|\{\phi\}| = 1$

$x \in \{\phi\}$ so $x = \phi$.

---

let $A = \{a_1, a_2 \text{ —— } a_n\}$

$\boxed{A \subseteq B}$

every element of A is in B.

then

$B = \{a_1, a_2 \text{ — } a_n, b_1, b_2 \text{ —— } b_m\}$ ..

Subset

Power set

* Set of all subsets of S is called "power set of S".
* $P(S)$: power set of S. $|P(S)| = 2^{|S|}$

eg) $S = \{1, 2\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

PYQ: Which is $P(\phi)$?

Set $\phi = \{\ \}$ $\quad |\phi| = 0$

$P(\phi) = \{\phi\}$

$= |P(\phi)| = 1$

Set $S = \{\phi\}$ $\quad |S| = 1$

$P(S) = \{\phi, \{\phi\}\}$

$P|S| = 2$

of subsets possible:

$S = \{a, b\}$ take it / ignore it

$2^2 = 4$ (2 → choice; 2 in exponent → # elements.)

Subsets of $S = \{\}, \{a\}, \{b\}, \{a, b\}$

subsets = 4

$|S| = n$

$S = \{a_1, a_2 \text{ ——— } a_n\}$

subsets $= 2 \times 2 \text{ ——— } 2 = 2^n$

$$\boxed{|P(S)| = \#\text{ subsets} = 2^n} = 2^{|S|}$$

$$\boxed{P|S| = 2^{|S|}}$$

$P(P(S)) = 2^{P|S|}$ or $2^{2^{|S|}}$

$$2^{2^n} = 2^{(2^n)} \neq (2^2)^n$$

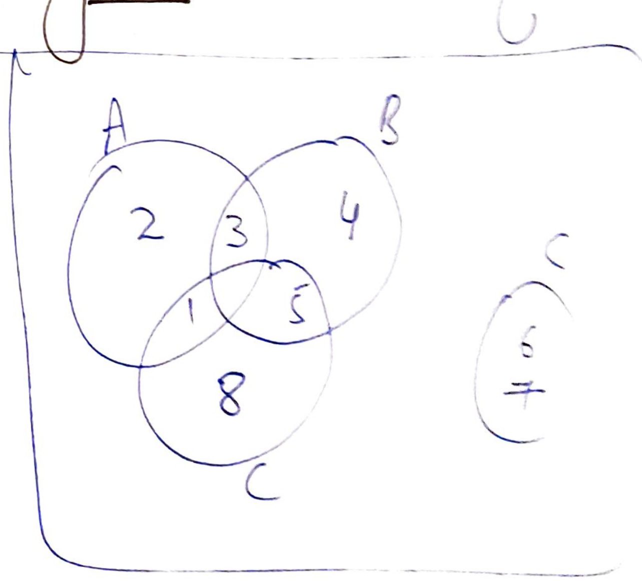
- U is the set containing everything currently under consideration
- Content depends on the context
- Sometimes explicitly stated, sometimes implicit.

(eg) A = {Orange, Apple, banana}
↓
universal set might be the set of names of all fruits.

Venn-Diagram

A = {1, 2, 3}
B = {3, 4, 5}
C = {6, 7}
D = {1, 5, 8}

Some other element



universal sets possible
{1, 2, 3, 4, 5, 6, 7, 8}

N ✓
Z ✓
R ✓
Q ✓
Even no ✗

Set operations

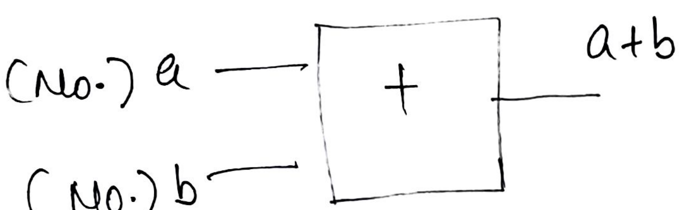
No's operators
- +
- −
- ×
- /

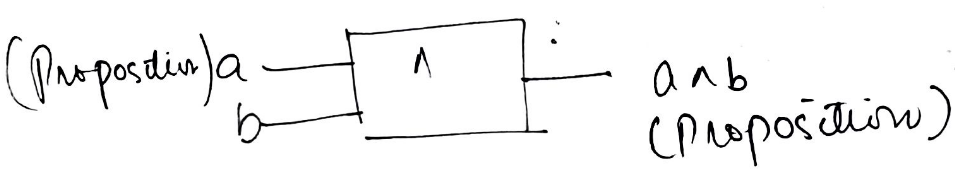
What kind of operations do we have the sets?

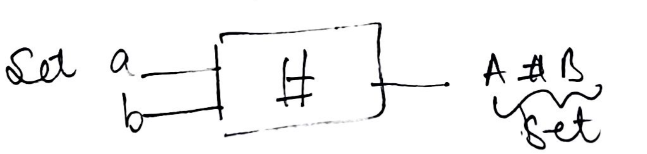
⇓

- UNION
- INTERSECTION
- DIFFERENCE
- COMPLEMENT.

On Numbers :







Ques) Suppose M is the set of students who love Mangoes and K is the set of students who love kiwis.

↓

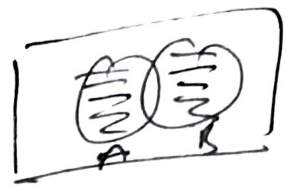
$M \cap K$.

$M \cap K = \{x \mid (x \in M) \wedge (x \in K)\}$.

$M \cup K = \{x \mid (x \in M) \vee (x \in K)\}$.

↓

Set of students who love Mangoes or Kiwis



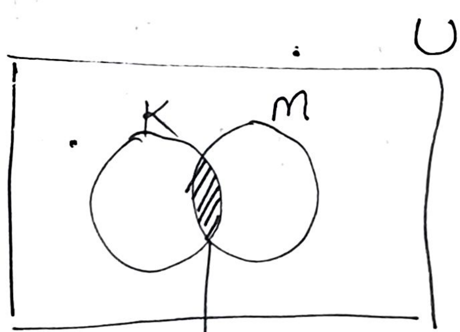
Symmetric-Difference

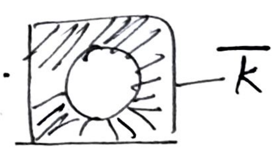
$K \Delta M$: Set of students who love kiwis Or Mangoes but not both.

$\{1,2,3\} \Delta \{3,4,5\} = \{1,2,4,5\}$

$A \Delta B = (A-B) \cup (B-A)$

$\bar{M}$: Set of students who donot love Mangoes...





Eg)

SET DIFFERENCE

M-K :- set of students who love Mangoes but not Kiwis.

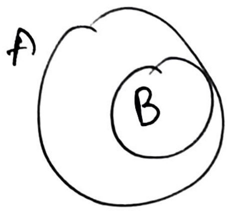
$$A-B = A \setminus B = \{x \mid x \in A, x \notin B\}$$

Eg) $\{1,2,3\}-\{3,4,5\}$
$\{1,2\}$

UNION

$A = \{x \mid x > 0\}$
$B = \{x \mid x > 1\}$.

$A \cup B = \{x \mid x > 0\} = A$.



$B \subseteq A$.

DISJOINT.

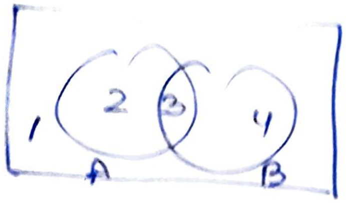
If A and B are sets and $A \cap B = \phi$ then we say that A and B are DISJOINT sets.

$A = \{a\}$
$B = \phi$.

$A \cap B = \phi$ ✓

$\therefore$ DISJOINT.

28)



$A = 2,3$

$B = 3,4$

$A \cap B = 3$

$A \cup B = 2,3,4$

$A - B = 2$

$B - A = 4$

$A \Delta B = 2,4$

$\bar{A} = 1,4$

$\bar{B} = 2,1$

Quick

1. $x \in A \cap B \xrightarrow{\text{means}} x \in A$ AND $x \in B$
2. $x \in A \cup B \rightarrow x \in A$ OR $x \in B$ OR both
3. $x \in \bar{A} \cap B \rightarrow x \in B$ AND $x \notin A$
4. $x \in \bar{A} \cup B \rightarrow x \in B$ OR $x \notin A$ OR both.

- (Equal sets) Set Equality
- SET IDENTITIES

$\{1,2,3\} = \{2,1,3\}$.

$\{1,2,3\} = \{3,3,3,1,2,2\}$

$\{1,2,3\} \neq \{1,2\}$

$\{a\} \neq a$
↳ not a set.

Suppose A and B are sets. Then $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

↓ Useful to prove set identities

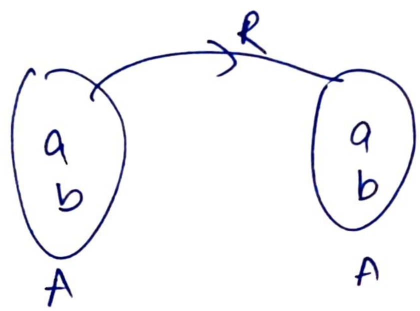
$A = B$
$\left.\begin{array}{l} A \subseteq B \\ B \subseteq A \end{array}\right\}$.

Eg) $A = \{\phi, \{a\}, \{b\}, \{a,b\}\}$
$B = \{x \mid x$ is a subset of set $\{a,b\}\}$

$A = B$ ..

Representation of Relation

① **Diagram representation**



② **SET representation**

$R : A \to A$

$R = \{(a,b)\}$

③ **MATRIX representation**

$R : A \to A$

	a	b
a	✗	✓
b	✗	✗

→ elements of Base set.

④ **Directed graph repⁿ**

$R : A \to A$

$(a) \to (b)$

Note : aRb

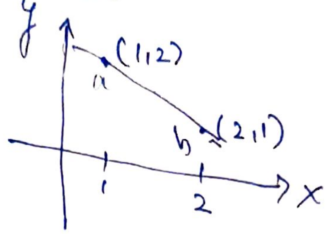
$(a,b) \in R$

$a \to b$

Relation

1

* $\boxed{\text{Cartesian PRODUCT}}$



$a\{1,2\} = \underbrace{b\{2,1\}}$

cannot write like this as a set...

If we write like this $\underline{\underline{a=b}}$

ORDERED-PAIRS

└ Order Matters

(a, b) — a: 1st, b: 2nd.

* (1) = (1,1) ×
* $\{1\} = \{1,1\}$ ✓
* $\{1,2\} = \{2,1\}$ ✓
* $\{1,2,1\} = \{1,2\}$ ✓
* $\underbrace{\{1,2,1\}}_{\text{point on 3D plane.}} = \underbrace{(1,2)}_{\text{point on 2D plane.}}$ × ordered pair.

*

$$\boxed{\begin{array}{l}(a,b) = (c,d) \\ \text{means} \\ a = c \\ b = d.\end{array}}$$

* ## CROSS - PRODUCT

~~Cartesian~~ Cartesian Product

$A = \{1, 2\}$

$B = \{a, b\}$

$A \times B = \{(1,a)\ (2,a)\ (1,b)\ (2,b)\}$

$|A \times B| = 4$

$B = \{c, d\}$

$A \times B = \{(1,c)\ (1,d)\}$

$B \times A = \{(c,1)\ (d,1)\}$

$A \times B \neq B \times A$ // order matters.

$A \times B \times C = \{$

$A \times B \times C =$

* $(1,2,3)$

$((1,2),3)$

$(1,(2,3))$

$((1,2),(3,$

$$\boxed{A \times B = \{\underbrace{(x,y)}_{\text{ordered pair}} \mid x \in A,\ y \in B\}}$$

$|A| = n \quad \{a_1, a_2 \text{ —— } a_n\}$

$|B| = m \quad \{b_1, b_2 \text{ —— } b_m\}$

$A \times B = \{(x,y) \mid x \in A,\ y \in B\}$

$$\boxed{|A \times B| = mn = |A| \underset{\text{Multiplication}}{\cdot} |B|}$$

$|A| = m$

$|B| = n$

$|A \times B| = m \times n$

$|B \times A| = n \times m$

$A = \{1, 2, 3 \text{ —— } m\}$

$B = \{1, 2, 3 \text{ —— } n\}$

* $A = \{a$

$B = \{b$

$C = \{c$

$A \times B \times$

$A \times (B \times$

* $A \times$

$B \times C$

$A \times$

$$\boxed{|A \times B \times C| = |A| \cdot |B| \cdot |C|}$$

$A = \{a\}$

$B = \{1\}$

$C = \{m, n\}$

$=$

$A \times B \times C = \{(a,1,m),\ (a,1,n)\}$

$|A \times B \times C| = 2$

$1 \times 1 \times 2 = 2$

(2)

$A \times B \times C = \{ (x, y, z) \mid x \in A,\ y \in B,\ z \in C \}$

// order matters. $(A \times B) \times C = A \times (B \times C)$ ✗

* IS CROSS-PRODUCT is ASSOCIATIVE ??

$\Downarrow$

$A = \{1\}$
$B = \{a\}$
$C = \{e\}$

$A \times B \times C = \{ \underbrace{(1, a, e)}_{\text{3-elements triplet.}} \}$

$(A \times B) \times C \equiv \{ \underbrace{(1, a), c}_{\text{2-elements}} \}$

$A \times (B \times C) = \{ 1, (a, e) \}$

$\downarrow$

NOT associative.

* $(1, 2, 3)$ — triple

$((1, 2), 3)$ = ordered pair

$(1, (2, 3))$ = ordered pairs

$((1, 2), (3, 1))$ = ordered pairs

* $A = \{a\}$
$B = \{b\}$
$C = \{c\}$

$A \times B \times C = \{ (a, b, c) \}$ ✓

$A \times (B \times C) = \{ (a, (b, c)) \}$ ✓

* $A \times (B \times C)$

$B \times C = \{ (b, c) \}$ $\quad A = \{a\}$

$A \times B \times C = \{ (a, (b, c)) \}$.

Cartesian Product

Ques)

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

- Cartesian product of 2 sets is a set.
- The elements of that set are ordered pairs.
- In each ordered pair, 1st component is an element of A, & the 2nd component is an element of B.

$A = \{\{a\}, b\}$

$B = \{\{1,2\}, \{a\}\}$

$A \times B = \{(\{a\}, \{1,2\}), (\{a\}, \{a\}), \text{------}\}$

$|A \times B| = |A| \cdot |B| = 4.$

Ques) $A = \{1, 2\}$

$S : (A \times A) \times (A \times A)$ // every element of S is an ordered pair.

*① $\times$ $(1,1,1,1) \in S$

② $\times$ $(1,1), (1,1) \in S$ $|S| = 4 \times 4 = 16.$

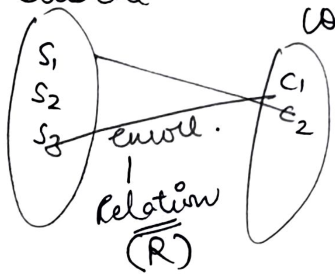
③ ✓ $((1,1), (1,1)) \in S$

Ques) When $A \times B = B \times A$??

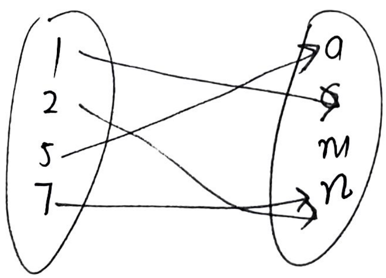
$$\left\{ \begin{array}{l} A = B \\ A = \phi \text{ or } B = \phi \end{array} \right\} \checkmark$$

$A \times B = \{(a, \underset{\text{nothing}}{\_})\}$ // cannot write ordered pair ...

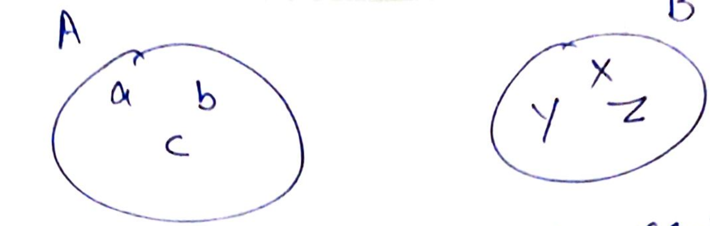
Relation



- S_1 is related to C_2
- $S_1 \, R \, C_2$
- $(S_1, C_2) \in R$.



1 S c
1 $\cancel{S}$ a
2 S n
7 S n
7 $\cancel{S}$ m

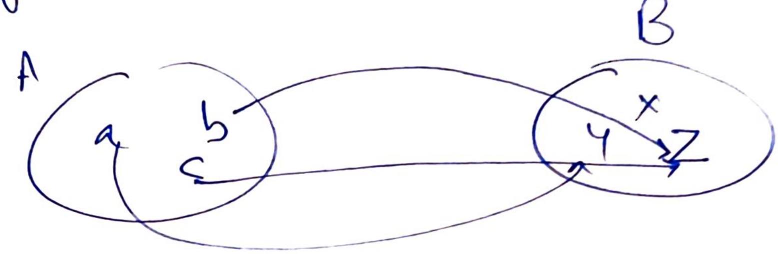


$A \times B = \{ (a,x), (b,x), (c,x), (a,y)\ (b,y)\ (c,y)$
$(a,z), (b,z)\ (c,z) \}$.

↓

Cross Product.

" every elt of (A) is related to every elt of (B) } LARGEST relationships b/w A & B



$R = \{ (a,y), (b,z), (c,z) \}$

↓

Subset.

$\boxed{R \subseteq A \times B}$ Relation

↓

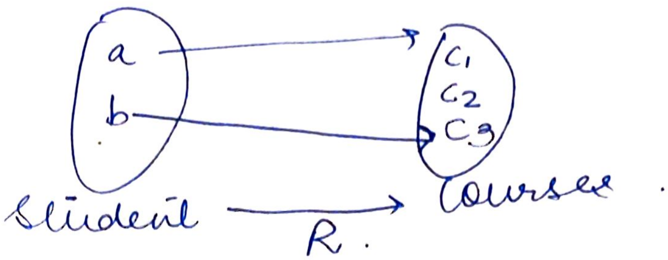
Subset

$\boxed{R : A \to B}$ || relation create.

BINARY RELATION ⇒ is a subset of the cartesian product $A \times B$.

① Relation from A to B are one-way.

y)



B

- $x \, R \, y$ — x ↓ Stud, y ↓ course.
- xRy iff x takes y.
- aRc_1 $\quad c_1 \not R a$ // one-way.

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$R: A \to B$

$\underset{A}{x} \, R \, \underset{B}{y}$ iff $x \le y$

$2 R 3$ ✓

$3 R 4$ ✓

$3 R 2$ ✗

$4 R 5$ ✗ // not in A.

$|A| = m$

$|B| = n$

$|A \times B| = mn$

$|B \times A| = nm$

$|A \times A| = m^2$

$|B \times B| = n^2$

$\#$ relations $_{A \to B}$ $= 2^{m \times n}$

$\#$ relations $_{B \to A}$ $= 2^{n \times m}$

$R : A \to B$

means $R \subseteq A \times B$

$S : B \to A$ means $S \subseteq B \times A$

$T : A \times B \to B \times A$ means $T \subseteq (A \times B) \times (B \times A)$.

Relation on $N \times N$

$R : \boxed{N \times N \to N \times N}$

Brave set.

$R \subseteq N \times N$ ✗

$R \subseteq (N \times N) \times (N \times N)$ ✓

Ques) SET A, SET B

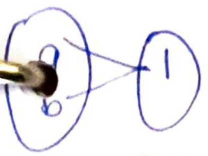
relations from A to B ?

$R : A \to B = ?$ $\underline{R \subseteq A \times B}$

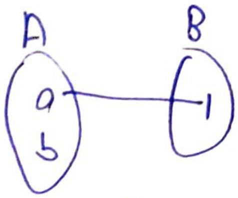
$A = \{a, b\}$

$B = \{1\}$

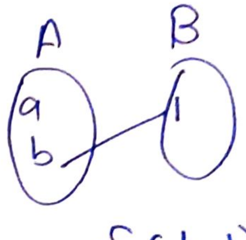
$A \to B$



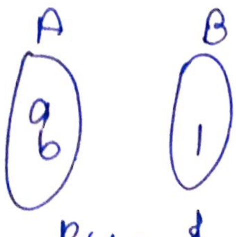
$R_1 = \{(a,1), (b,1)\}$



$R_2 = \{(a,1)\}$



$R_3 = \{(b,1)\}$



$R_4 = \phi$.

$R \subseteq A \times B$

- Every relation from A to B is subset of $A \times B$
- Every subset of $A \times B$ is a relation from A to B.

$|A \times B| = |A| \cdot |B|$

subsets of $A \times B$ = # relations from A to B $= 2^{|A| \cdot |B|}$

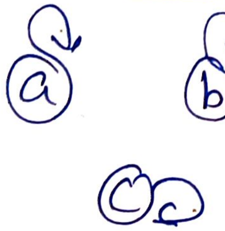
Reflexive relation

— Every element of Base set should be related to itself.

R on set A is called Reflexive.

iff $\boxed{\forall_{x \in A} ((x,x) \in R)}$

graph repn



Every Node has self loop

	a	b	c
a	a		
b		b	
c			c

don't care.

Ques) S = {1, 2, 3, 4} Reflexive.

R_1: R✗ {(1,1) (1,2) (2,1) (2,2) (3,4) (4,1) (4,4)} S✗

R_2: R✗ {(1,1) (1,2), (2,1)} ✓S

R_3: ✓ {(1,1) (1,2) (1,4) (2,1) (2,2) (3,3) (4,1) (4,4)} ✓S

R_4 = R✗ {(3,4)} ✗S

R_5 = ✓ {(1,1) (1,2) (1,3) (1,4) (2,2) (2,3) (2,4) (3,3) (3,4) (4,4)} S✗

SYMMETRIC : $R : A \to A$

If $a \to b$ then $b \to a$

$$\forall_{a,b \in A} \ (aRb \to bRa)$$

When SYMMETRY violates?

If $a \to b$
$b \not\to a$

Graph repⁿ

$a \leftrightarrow b$ (bi-directional arrows).

Matrix repⁿ

$M = M^T$.

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & & 1 & \\ b & 1 & & \\ c & & & \end{array}$$

Note

Reflexive relation – a relation in which ALL reflexive pairs are present.

Note

I. Presence of reflexive pairs SYMMETRY is never violated. So in SYMMETRIC relation, reflexive pairs may or may not be present.

Antisymmetric

There should be no symmetry b/w different elements.

If $a \neq b$ and $a \to b$ then $b \not\to a$.

Graph

$a \leftrightarrow b$ (Bidirectional arrow not allowed.

Reflexive pairs allow $a \to b$ ✓

$$\boxed{\forall_{a,b \in A} \left[(aRb \text{ and } bRa) \to a = b \right]}$$

Asymmetric.

Antisymmetric + no reflexive pair.

Asymmetric violates ?

Antisymm. violates OR There is atleast 1 reflexive pair.

Q) consider these relations on the set of integers Base set Z

$R_1 = \{(a,b) \mid a \le b\}$

aR_1b iff $a \le b$

$1R1$

$1R2$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a,b) \mid a = b\}$

$R_5 = \{(a,b) \mid a = b+1\}$

$R_6 = \{(a,b) \mid a+b \le 3\}$.

aR_6b iff $a+b \le 3$

$-1R_64$ ✓ $1.5 R_6 6$ ✗ not in Base set.

which of these relations contains each of the pair =

$(1,1)$ — R_1, R_3, R_4, R_6

$(1,2)$ — R_1, R_6

$(2,1)$ — R_2, R_5, R_6

$(1,-1)$ — R_2, R_3, R_6

$(2,2)$ — R_1, R_3, R_4

Transitive

Bangalore $\rightarrow$ Delhi $\rightarrow$ Jaipur

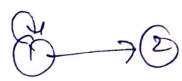
$\forall\, a, b, c \in$ Base-set

If (a,b) and (b,c)

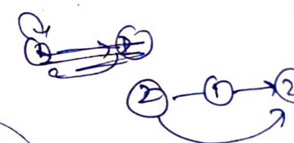
then (a,c)

Ep) $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1)\ (1,2)\}$ ✓



$R_2 = \times\ \{(1,1)\ (1,2)\ (2,1)\}$

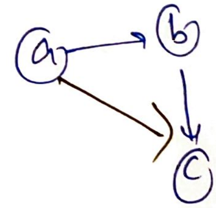


$R_3 = \{(1,1)\ (1,2)\ (2,2)\ (3,3)\}$ ✓✓

Checking for Transitivity

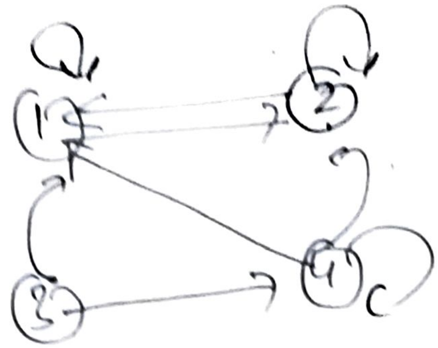
(1) Definition $a \rightarrow b \rightarrow c$

(2) Graph Method:



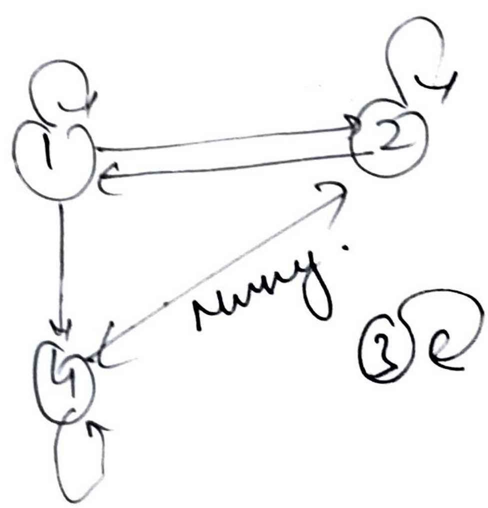
(3) Counter example $(2,2)$ not present.

Eg) $R_1 = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,4)\ (4,1)\ (4,4)\}$



Eg) $\{(1,1)\ (1,2)\ (2,1)\}$

Eg) $\{(1,1)\ (1,2)\ (1,4)\ (2,1)\ (2,2)\ (3,3)\ (4,1)\ (4,4)\}$



LATTICE

When elements are "COMPARABLE"

If aRb then

$$\boxed{\begin{aligned} a \vee b &= b \\ a \wedge b &= a \end{aligned}}$$

b — $a \vee b$
$\uparrow$
a — $a \wedge b$.

When elements are "INCOMPARABLE"

LUB – unique first joining point in upper direction.

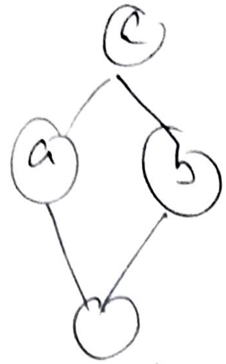
GLB – unique 1st meeting point in down direction.

Case 1) $aRbRc$

c — $a \vee b \vee c \equiv LUB\{a, b, c\}$

a — $a \wedge b \wedge c \equiv GLB\{a, b, c\}$.

Case 2



unique 1st meeting pt.

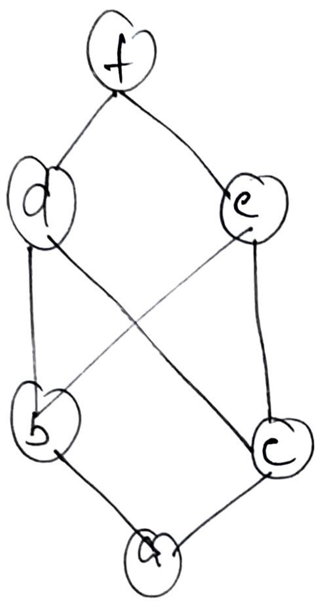
$a \vee b \vee c = LUB\{a,b,c\}$
$= c$.

$GLB\{a,b,c\} =$ ~~DNE~~
$\Downarrow$
$a \wedge b$ {c-transitive connected}.

POSET
$\vee p$
$\wedge p$

HASSE
o Total O
o POSET
o Chai
o Line

Ques)

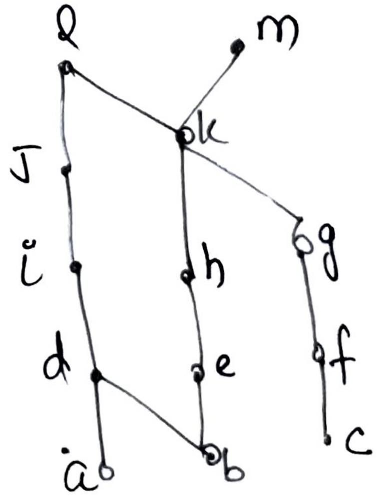


$b \vee c =$ DNE.

$UB\{b,c\} = \{d, e, f\}$.

Ques)

$a \wedge b \wedge c = GLB\{a,b,c\}$
$=$ DNE



$a \vee e \vee l = l$

$a \wedge i \wedge l = a$ $\quad b \wedge h \wedge k = b$.

$b \vee h \vee k = k$.

2

POSET P,

$\vee P$ = LUB of all elements.

$\equiv$ Greatest element.

$\wedge P$ $\equiv$ least element ...

HASSE DIAGRAM of total order relations

- Total ORDER
- TOSET
- Chain
- Linear_ORDER

Ques) $(\{1,2,3,4,5\}, >=)$

1
|
2
|
3
|
4
|
5 chain / straight line.

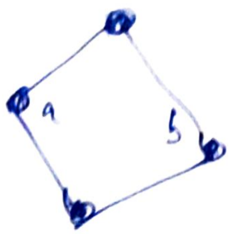
5R4
4R3
$5 \geq 4$
$4 \geq 3$

Total order

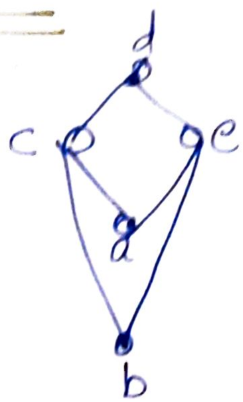
$\forall a,b$, a,b are COMPARABLE...

LATTICE

(1)

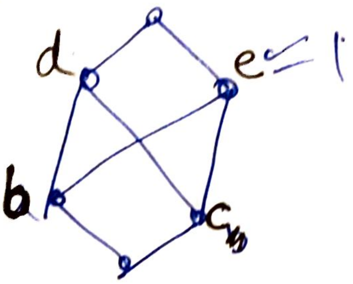


(2)



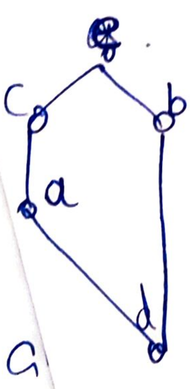
$b \vee a$ = DNE
$b \wedge a$ = DNE.

(3)



$d \wedge e$ = DNE.
$b \vee c$ = DNE
$a \vee b$ = DNE.

Popular example of LATTICE . . .



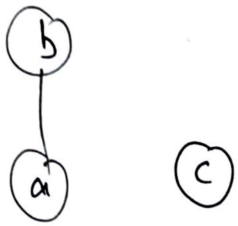
$a \vee b = e$
$a \wedge b = d$
$c \vee b = e$
$c \wedge b = d$

a

$a \wedge$

ds -1

from the HASSE diagram you can find out the CARDINALITY of POSET.



Just a representation

$|POR| = 4$

$POR = \{ (a,a)\ (b,b)\ (c,c)\ (a,b) \}$

HD

$|R| =$

$R =$

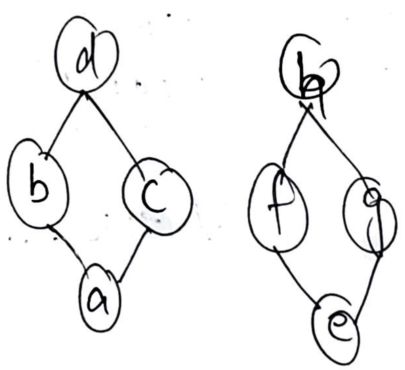
The poset

is a LATTICE

Indeed P.t li

Corresponds

Corresponds



Base set :- $\{a, b, c, d, e, f, g, h\}$

$|POR| = ?$ 18

$\underline{4}\,\cancel{8} + 3 + 1 + 1\ \cancel{8}$

Ref. pairs

a, x b, x) c, x.

$+$ (a, b) (a, c) (a, d)

$4 + 3 + 1 + 1$

$9 + 9 = 18$

(N, |)

POSET

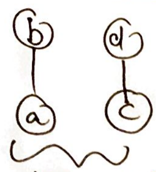
$a \vee b = l$

$a \wedge b = g$

Is this

"HASSE-DIAGRAM Cardinality"

①



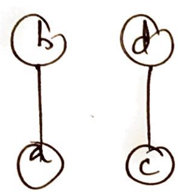
HD of R_1 POR.

$|R_1| = \{(a,a)\ (b,b)\ (a,b)\ (c,c)\ (d,d)\ (c,d)\}$

Base set $= \{a,b,c,d\} = A$

$(A,R_1) = R_1(A)$. // Poset.

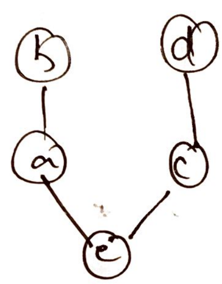
$|R_1| = 6$.



$|R_1| = (a,x),\ (b,x)\ \ (c,x)\ (d,x)$

$2 + 1 + 2 + 1$

$\underline{|R_1| = 6}$



HD of R...

Base set of $R = \{a,b,c,d,e\}$

$|\text{Base-set}| = 5$

edges in HD = 4.

$|R| = ?\ (11)$ ✓

$(e,x)\ \ (a,x)\ \ (b,x)\ \ (c,x)\ \ (d,x)$

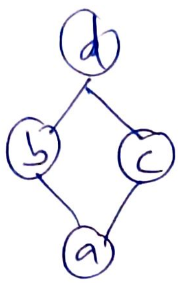
$5 + 2 + 1 + 2 + 1$

(11)..

SUB-LATTICE

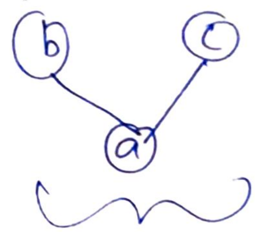
LATTICE-L

Sublattice(S) of L :
1. subset of L
2. LATTICE
3. GLB, LUB for any $a,b \in S$ in L, S must be same

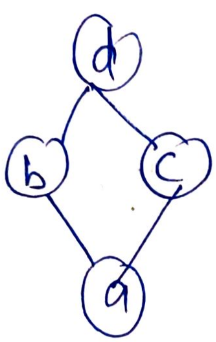


LATTICE-L

$\xrightarrow[\text{delete } d.]{\text{subset } \{a,b,c\}}$



Not a sublattice coz not a LATTICE.



$\xrightarrow[\text{delete } c]{\text{subset } \{a,b,d\}}$

· subset of L ✓
· LATTICE ✓

③

		L	S
GLB:	$a \wedge b$	a	a
	$a \vee b$	b	b
	$a \wedge d$	a	a

∴ sublattice.

f L

for any a,b ∈ S
same in L,S

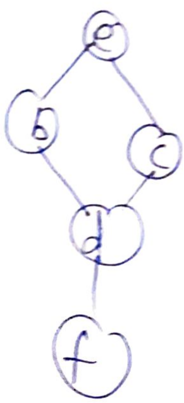
a sublattice
not a LATTICE.

set of L ✓
TICE ✓

sublattice.

(4)

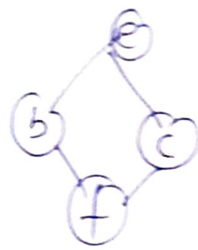
L



Subset {b,c,e,f}

Delete d

S

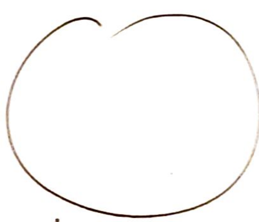


(5)

	L	S
b∨c	d	f

∴ not a sublattice of L.

Intution behind sublattice :-

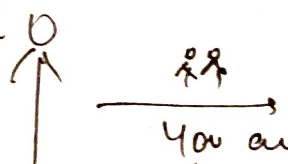


LATTICE

Subset {a,b---}

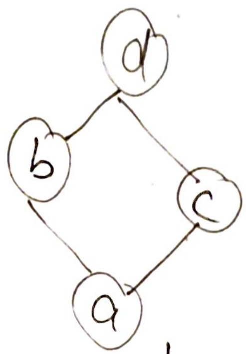
take their GLB, LUB also...

Hindi Serial

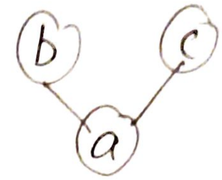


You are going-take your luggage also...

Sub-lattice



Subset {a, b, c} → delete d



Not a sublattice of L ...

Not even la...

Subset {a, b, d} → means delete c

Sublattice ✓

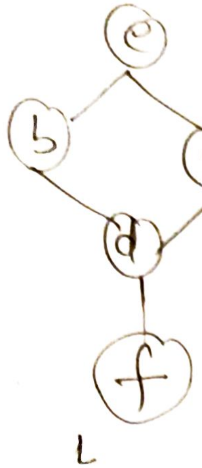
LATTICE ✓

Definition

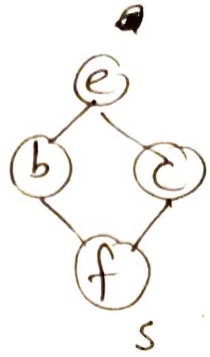
A sublattice of L is a subset of L that is a lattice with the same meet & JOIN operations as L.

	L	S
$a \wedge b$	a	a
$a \vee b$	b	b
$a \wedge d$	a	a

That is, if L is a lattice & M is a subset of L such that for every pair of elements a, b in M both $a \wedge b$ and $a \vee b$ are in M then M is a sublattice of M.



subset {b, c, e, f}
$\xrightarrow[\text{delete } d]{}$



	L	S
b∧c	d	f

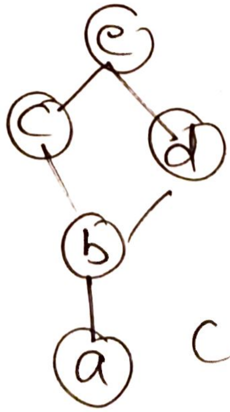
✓ Not a sublattice.

LATTICE L

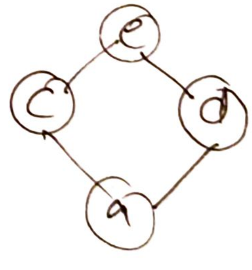
subset {a, b — }

↓

If you are taking these elements then their GLB / ~~ULB~~ LUB ...



subset {a, c, d, e}



	L	S
c∧d =	b	a

× ~~c∧d~~

∴ Not a sublattice.

Bounded lattice

Lattice $(Z, \leq)$ - lattice

greatest - none
least - none

Bounded lattice => lattice which greatest (1) & least (0).

(a) Infinite lattice which is bounded.

$([0,1], \leq)$
greatest = 1
least = 0

Ques) $(P(N), \leq)$
Lattice
bounded

greatest = N : every element subset of N ...
least = ϕ

Finite space

- atleast one maximal
- In the same manner than 1 maximal

No — log in lattice [LUB, GLB]

So more than one
Minimal | maximal
not element.

finite lattice → unique maximal (greatest) | Minimal (least).

Show every finite lattice is bounded?

∴ $\boxed{\text{Every finite lattice is "bounded"}}$

Definition

A lattice L is bounded if it has both

- upper bound (1)
- lower bound (0).

Properties:

$x \vee 1 = 1$ $\quad$ $x \wedge 1 = x$

$x \vee 0 = x$ $\quad$ $x \wedge 0 = 0$

Complement of an Element :

(b) is complement of (a) iff

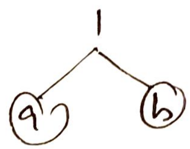
$$\boxed{\begin{aligned} a \vee b &= 1 \\ a \wedge b &= 0 \end{aligned}}$$

1 — greatest, 0 — lattice

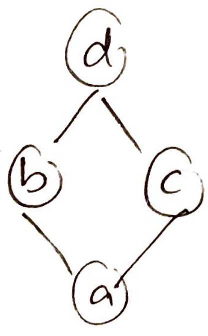
} bounded. lattice

If you want to know complement lattice you should have Bounded lattice ...

In Bounded Lattice



$b = a^{-1}$ } iff $a \vee b = G$, $a \wedge b = L$.



is this a bounded lattice? → Yes...

complement of $b = c$

$b^{-1} = c$, $c^{-1} = b$ } $b \vee c = d = G$, $b \wedge c = a = L$.

$a^{-1} = d$, $d^{-1} = a$ }

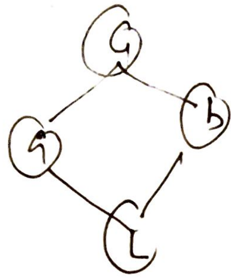
Idea:

In a Bounded lattice !-

Complement of $a = a^{-1}$.

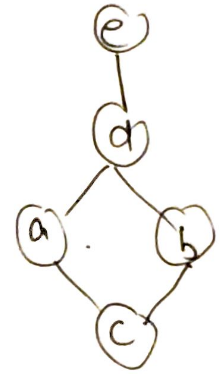
$\downarrow$

b.



$a \vee b = G$.

$a \wedge b = L$.

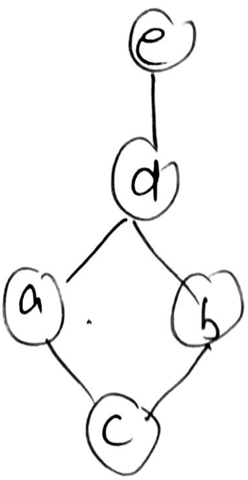


Complements in a Bounded LATTICE.

Complement of an element (x) is an elem... z such that $x \vee z = 1$, $x \wedge z = 0$.

Complemented LATTICE:

A bounded lattice is complemented iff every element has a complement.



$a^{-1} =$

$a \vee b = d$. × (G)

$a \wedge b = c$

$a \vee d = d$ ×

$\therefore$ a^{-1} = complement DNE.

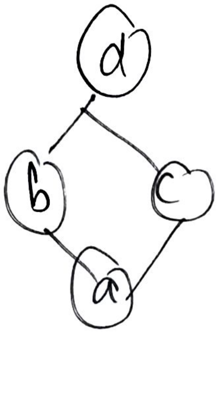
$c^{-1} =$ $c \vee e = e$.

$c \wedge e = e$.

d^{-1} = DNE.

$$\boxed{\begin{aligned}(\text{least})^{-1} &= \text{Greatest}\\ (\text{Greatest})^{-1} &= \text{LEAST.}\end{aligned}}$$

"Some-times some elements complement do not exist..."



$a^{-1} = d$

$d^{-1} = a$

$b^{-1} = c$

$c^{-1} = a$

$\therefore$ Complemented LATTICE.

⇓

iff every elt has complement.

$b' = c, d$

$b \vee c = e$

$b \wedge c = a$

So $b' = c$

$b \vee d = e$

$b \wedge d = c$

So $b' = d$

Complemented lattice

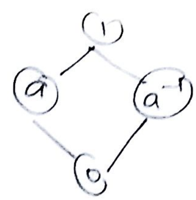
$c' = b, d$

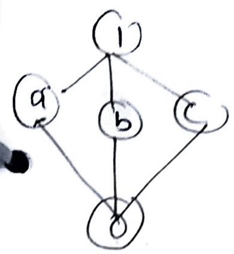
$d' = c, b$

$a' = e$

$e' = a$

Every elt should have atleast one complement





$1^{-1} = 0$

$0^{-1} = 1$

$a^{-1} = b, c$

$b^{-1} = a, c$

$c^{-1} = a, b$

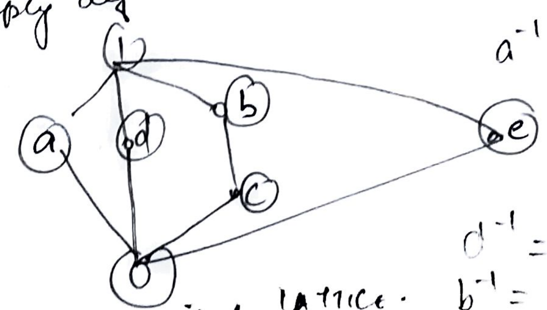
Note:

A complemented lattice where some complements are not unique ....

Ques) Complement of an element (a) in the lattice.

– Then apply def^n.

$0^{-1} = 1$, $1^{-1} = 0$ $\Big\}$ $0 \wedge 1 = 0$, $0 \vee 1 = 1$



$a^{-1} = d, b, c$.

$a \vee b = 1$

$a \wedge b = 0$

$d^{-1} = a, b, c, e$ | $e^{-1} = a, d, b, c$

$b^{-1} = a, d, e$. | $c^{-1} = a, d, e$.

∴ Complemented lattice.

...plemented

...ted lattice

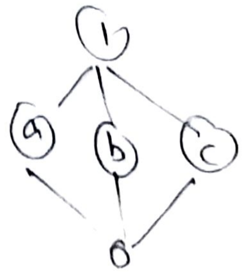
...ld have

...complement ...

DISTRIBUTIVE LATTICE ... Valid

$a \vee (b \wedge c) = (a \vee b) \wedge (a \wedge c)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$



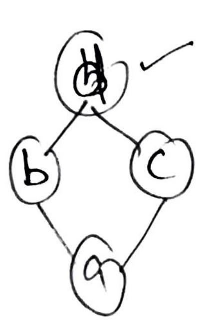
Distributive LATTICE

→ 1) If you were paying attention to the sort of properties that were being proved earlier, you may have been wondering what happened to the DISTRIBUTIVE property.

2) Why was not that included?

3) Well there's a good reason. ~~Also~~

4) Not all lattice are DISTRIBUTIVE. Only distributive ones are.



$b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$

$b \wedge d = a \vee b$

$b = b$

∴ Distributive

∴ Not ...

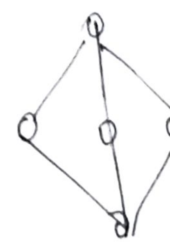
$a \wedge ($ ...

Distributive LATTICE:

iff $\forall_{x,y,z}$

$x \vee (y$ ...

$x \wedge (y$ ...



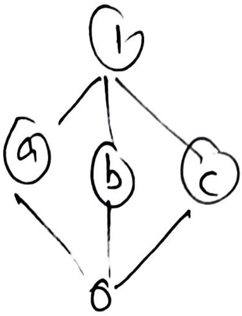
• KITE ... or Diamond

Non-D... LATTIC...

) $b \vee (a$ ...

$b \vee ($ ...

5



$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \vee 0 = 1 \wedge 1$$

$$a \neq 1$$

$\therefore$ Not a DISTRIBUTIVE LATTICE.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \neq 0$$

Distributive LATTICE:

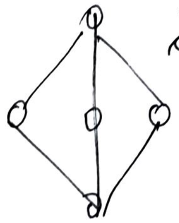
if $\forall_{x,y,z}$.

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ ✓

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ ✓

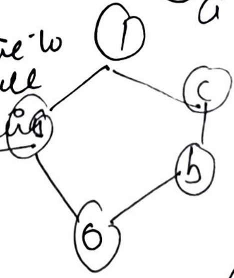
IMP Non-Distributive LATTICE

4 special



- KITE LATTICE. or Diamond
- Non-Distributive LATTICE...

We have to check all the pairs



Pentagon (N_5) LATTICE..

(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \vee c = 1 \wedge 1$

$1 = 1$

(1) $c \vee (b \wedge a) = (c \vee b) \wedge (c \vee a)$

$c \vee 0 = c \wedge 1$

$c = c$ ✓

(3) $b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c)$

$b \vee 0 = 1 \wedge c$

$b \neq c$

Checking Distributive property now

Theorem

L: Kite / pentagon ⟹ Not a Distributive

Df "L is Distributive iff there is no sublattice of L which is kite / pentagon".

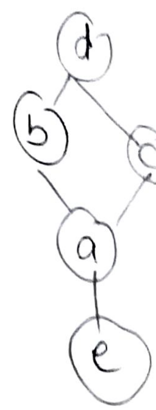
for lattice L,

If ∃ sublattice S = Kite / pentagon structure then L is not Distributive.

(If L is not Distributive then ∃ sublattice which has structure of kite / pentagon.

Q1 If lattice L has ...
then L is ...
coz we can ...

Q) which lat... are Distribu...
Every lattice ...



But If every element has ≤ 1 complement ⟶ find kite, N_5 sublattice.

If find then not a Distributive

★ Complemented distributive lattice have unique complements

- **Note:** In Distributive LATTICE ⟹ every elt has ≤ 1 complement. | Complemented LATTICE . every elt has ≥ 1 complement.

- In DISTRIBUTIVE COMPLEMENTED LATTICE ⟹
  - every elt has exactly one complement.
  - Note: If every elt has exactly 1 complement... then complemented ✓ guaranteed, distributive ✗ not guaranteed

Boolean LATTICE | Boolean Algebra.

Type.
LATTICE
- Bounded &
- complemented &
- Distributive

} → Boolean LATTICE.
|
all 3 properties satisfy...

⇓

If 1- remove still
Bounded then still
PL is Boolean algebra....

↓

Yes...

Complemented lattice → Bounded

- Complements are only defined
- coz complements
Bounded lattice...

Boolean LATTICE : Complemented Distributive LATTICE.

↓ property

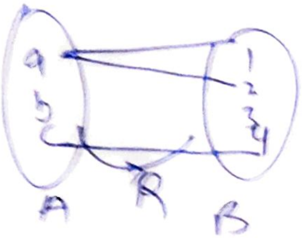
(1) Complemented
(2) Distributive.

FUNCTIONS

Relation from SET A to SET B

$R: A \to B$

$R \subseteq A \times B$



$a \to 1, 2$

$b \not\to$

$c \to 4$

$|A| = 5$

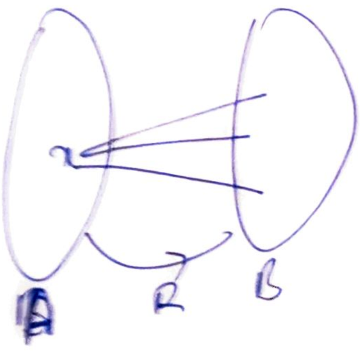
$|B| = 10$

$R: A \to B$

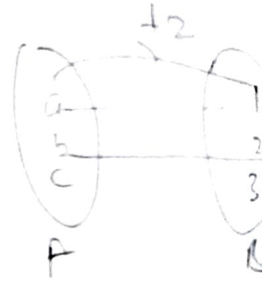
Ques $x \in A$ then $\{ y \mid xRy, y \in B \} \overset{\text{Cardinality}}{=} ?$ 0 to 10

↓

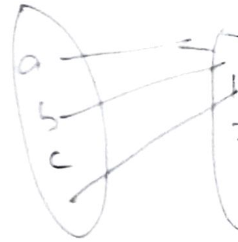
Set of those elements of B to which x is related.



Function : A binary ... is said to Be a function if for every element a in A, there is unique element b in B so that (a,b) is in R.



For a function R from A to B, instead of writing $(a,b) \in R$, we also use the notation $R(a) = b$, where b = image of a.



Set A is called domain of the function & set B is called Range of the function

$f : A \to B$

domain

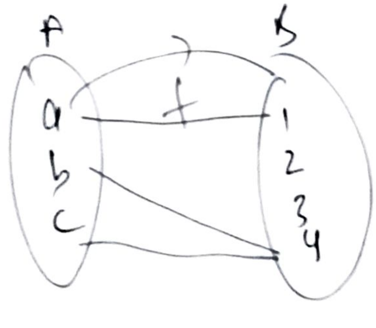
$R : A \to B$

domain

Note

I function from A to B is a relation in which every element of A (individually) related to exactly one element of B.

domain of Relation in case of function



$a \to 1$
$b \to 4$
$c \to 4$ } exactly 1 element of B.

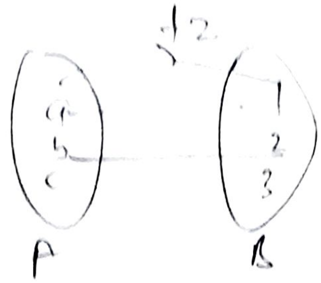
f for
there is a

included
use the
) ≡ image

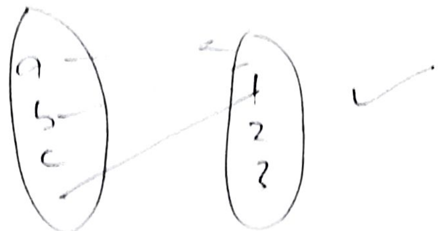
function
function.

relation
of A
ctly one

ctly
element
B.



not a function
3 (c) not
CONNECTED

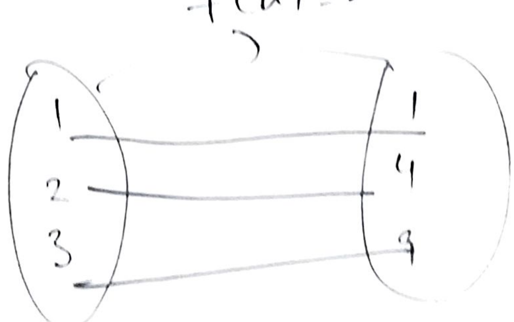


$f : (A) \to (B)$
domain co-domain] function

$R : (A) \to (B)$
domain of Relation codomain of R.] Relation

... called:
function | Mapping | Transformation

$f(x) = x^2$



$f(x) = x^2$

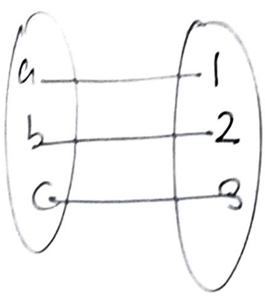
$x \to \boxed{f} \to x^2$

Transformation

Special type of functions

1) One-one func()
2) Onto func()
3) Bijection func().

(1) $\boxed{\text{ONE-ONE FUNC()}}$ / Injection / Injective func



Different people should get different enjoy...

(2) $\boxed{\text{Onto-function}}$

↳ CO-DOMAIN = Range...

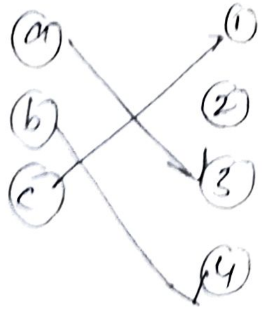
↓

Every element of CO-DOMAIN should have pre-image.

onto function

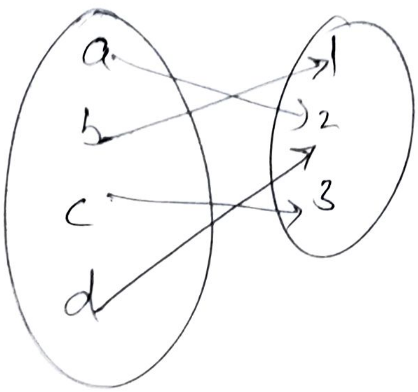
Co domain = Range

Eg)



One-one.
diff-image

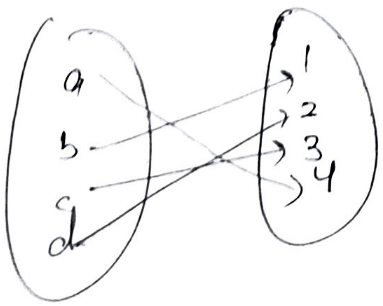
Eg)



a, d - same image
∴ not one to one

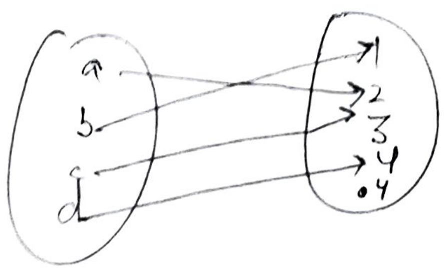
But
Onto coz codomain all
occupied.

Eg)



One-one ✓
Onto ✓

Eg)



neither < one to one / onto.

not a func()

Onto / Surjective

* $f : A \to B$ is called "surjective" if each element of the co-domain of the domain has atleast one element associated with it.

* For any $b \in B$, there exists atleast one $a \in A$, such that $f(a) = b$.

* Surjective functions cover every elt from co domain of B with atleast one element of A.

* function with this property is called "SURJECTION"...

$$\boxed{\forall y \in C \; ; \; \exists x \in D \;\; f(x) = y}$$

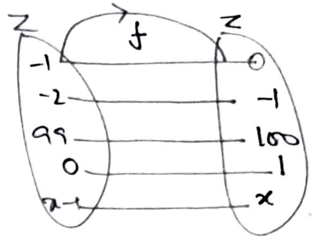
(2)

eg) $f : N \to N$
$f(x) = x^2$

Onto ✗

No Pre-image of 3, 5...

$f : Z \to Z$
$f(x) = x + 1$



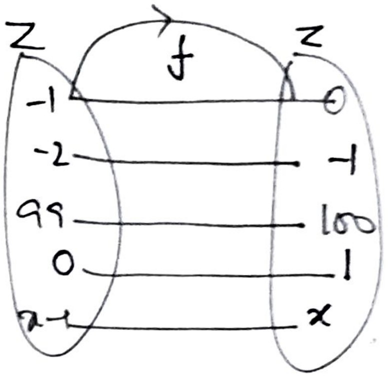
i) Injective
└ every
at most

ii) Surjective
└ every
atle

Surjective
└ every
has

$f : Z \to Z$

$f(x) = x + 1$



$f(99) = 100$

Pre-image $(y) = y - 1$.

FROM co domain Point of view

i) Injective :

Every element of co-DOMAIN has atmost one pre-image.

ii) Surjective

Every element of CO-DOMAIN has atleast one pre-image.

Surjective and injective = Bijection

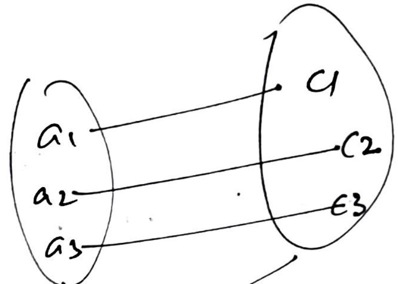
Every element of co-domain has ~~atleast~~ exactly one pre-image ...

BIJECTION

Function that associates each element of the co-domain with a unique element of DOMAIN.

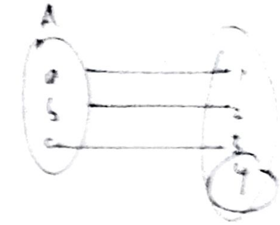
Bijection is a func() that is both INJECTION & SURJECTIVE.

Bijection is a one to one correspondence b/w 2-sets.



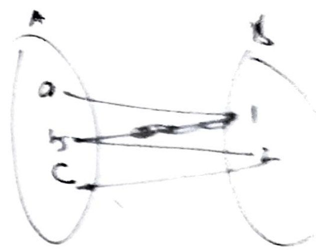
same # elements should be there

$$\boxed{\begin{array}{l} f: A \to B \\ |A| = |B| \end{array}}$$



Onto ×

$|A| < |B|$ — then



|DOMAIN| > |co-domain|

|DOMAIN| < |co-d

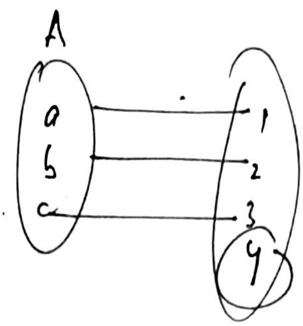
ms) $f: A \to B$

there exists

$\boxed{|A| \geq |B|}$

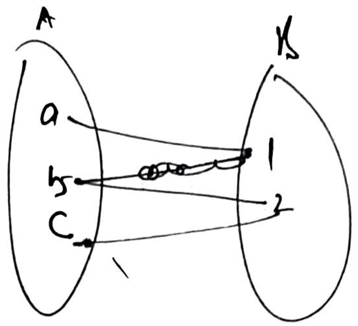
One-one func()

|A



Onto ✗

$|A| < |B|$ — Then onto not possible.



← One one ✗

$|A| > |B|$ then One-one not possible...

$|\text{DOMAIN}| > |\text{Co-DOMAIN}| \Rightarrow$ One-one not possible.

$|\text{DOMAIN}| < |\text{Co-DOMAIN}| \Rightarrow$ Not ~~co-domain~~ Onto.

Ques) $f: A \to B$
There exists atleast one onto fun().

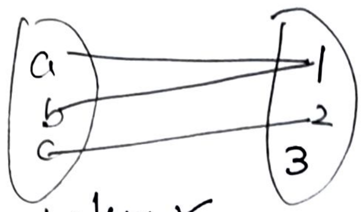
$$\boxed{|A| \geq |B|}$$

One-one fun()

$$\boxed{|A| \leq |B|}$$

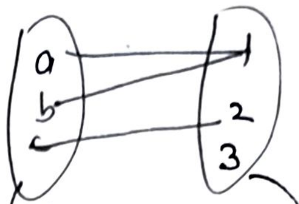
1

NVERSE OF A FUNCTION



func ✓

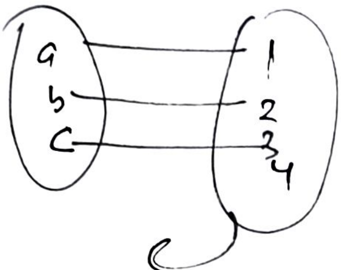
func ×



not a one-one function

not a func().

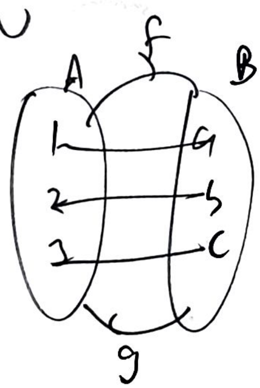
ot onto func)



not a func().

en will function be a func() even in
erse Direction?

ition



$$\left.\begin{array}{l} f: A \to B \\ g: B \to A \end{array}\right\} \text{func()}$$

$$\boxed{g = f^{-1}}$$ ✓