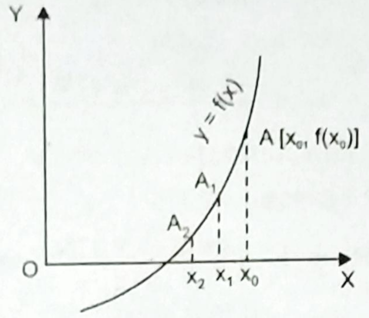
NEW TOPIC ADDED FROM ACADEMIC SESSION [2022-23]
THIRD SEMESTER [B.TECH]
COMPUTATIONAL METHODS [ES-201]
UNIT - I

Q.1 Give geometrical interpretation of Newton Raphson Method. (2015)

Ans. Let x_0 be a point near the root α of equation $f(x) = 0$, then tangent at $A(x_0, f(x_0))$ is



$$y - f(x_0) = f'(x_0)(x - x_0)$$

It cuts x–axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, which is I approximation to root α. If A_1 corresponds to x_1 on the curve, then tangent at A_1 will cut x-axis at x_2, nearer to α and is therefore II approximate to root α. Repeating this process, we approach the root a quite rapidly. Hence the method consists in replacing the part of the curve between A and x-axis by the means of the tangent to the curve at A_0.

Q.2 Using Bisection method find the value of negative root of the equation $x^3 - x + 11 = 0$ upto 2 decimal places. (2015)

Ans.
$$f(x) = x^3 - x + 11 = 0$$
$$f(-2) = -8 + 2 + 11 = 5$$
$$f(-3) = -27 + 3 + 11 = -13$$

$\therefore$ Root lies between – 2 and – 3.

$$x_1 = \frac{-2+(-3)}{2} = -2.5$$
$$f(x_1) = -15.625 + 2.5 + 11 = -2.125 \text{ (–ve)}$$

Root lies between – 2 and – 2.5

$$x_2 = \frac{-2+(-2.5)}{2} = -2.25$$
$$f(x_2) = 1.85 \text{ (+ve)}$$

$\therefore$ Root lies between – 2.25 and – 2.5

$$x_3 = \frac{-2.25+(-2.5)}{2} = -2.375$$
$$f(x_3) = -0.02 \text{ (–ve)}$$

$\therefore$ root lies between – 2.25 and – 2.375

$$x_4 = \frac{-2.25+(-2.375)}{2} = -2.312$$

$$f(x_4) = 0.953\ (+ve)$$

$\therefore$ root lies between – 2.312 and – 2.375

$$x_5 = \frac{-2.312+(-2.375)}{2} = -2.343$$

$$f(x_5) = 0.4807\ (+ve)$$

$\therefore$ root lies between – 2.343 and – 2.375

$$x_6 = \frac{-2.343+(-2.375)}{2} = -2.359$$

$$f(x_6) = 0.2314\ (+ve)$$

$\therefore$ root lies between – 2.359 and – 2.375

$$x_7 = \frac{-2.359+(-2.375)}{2} = -2.367$$

$$f(x_7) = 0.1054\ (+ve)$$

$\therefore$ root lies between – 2.367 and –2.375

$$x_8 = \frac{-2.367+(-2.375)}{2} = -2.371$$

$\therefore$ Required root is – 2.37.

Q.3 Use Newton Rephson method to solve the equation 3x–cos x–1 = 0 correct to four decimal places. (2015)

Ans.

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

$$f(0) = -2\ (-ve),\ f(1) = 1.4597\ (+ve)$$

$\therefore$ root lies between 0 and 1

Again $f(0.5) = -0.377\ (-ve)$

$$f(0.7) = 0.33\,515\ (+ve)$$

thus root lies between 0.6 and 0.7

Let $x_o = 0.6$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3+\sin x_n}$$

$$\Rightarrow \quad x_{n+1} = \frac{3x_n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3+\sin x_n} \qquad \ldots(1)$$

Let first approximation be $x_o = 0.6$

$$x_1 = \frac{x_o \sin x_o + \cos x_o + 1}{3+\sin x_o}$$

$$= \frac{0.6\sin 0.6 + \cos 0.6 + 1}{3+\sin 0.6} = 0.6071$$

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3+\sin x_1}$$

$$\Rightarrow \quad x_2 = 0.6071.$$

$\therefore$ root is 0.6071.

Q.4 Evaluate $\sqrt{12}$ to three places of decimals by Newton Raphson method. (2017)

Ans. Let $x = \sqrt{12} \Rightarrow x^2 - 12 = 0$

$$\therefore \quad f(x) = x^2 - 12 = 0 \qquad \ldots(i)$$

$$f(3) = -3,\ f(4) = 4$$

Thus root lies between 3 and 4.

Let $x_0 = 3.5$

Iterative formula $x_{n+1} = x_n - \dfrac{x_n^2 - 12}{2x_n}$

$$\Rightarrow \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right) \qquad \ldots(ii)$$

Putting $x_o = 3.5$ in equation (ii), we get

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right) = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.4643$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{12}{x_1}\right) = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

$$x_3 = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641}\right) = 3.4641$$

Thus root is 3.4641.

Q.5 Using Newton Raphson method, find a root of the equation x sin x + cos x = 0 which is near x = π correct to 3 decimal places. (2018)

Ans. Let $f(x) = x \sin x + \cos x$

$$f'(x) = \sin x + x\cos x - \sin x = x\cos x$$

Let $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}(+ve)$

$$f(\pi) = -1\ (-ve)$$

$\therefore$ root lies between $\frac{\pi}{2}$ and π.

Let $x_0 = \pi$

By iterative formula $x_{n+1} = x_n - \dfrac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$

$$x_{n+1} = \frac{x_n^2 \cos x_n - x_n \sin x_n - \cos x_n}{x_n \cos x_n}$$

Let $x_0 = \pi$

$$x_1 = \frac{x_0^2 \cos x_0 - x_0 \sin x_0 - \cos x_0}{x_0 \cos x_0}$$

$$= \frac{-\pi^2+1}{-\pi} = 2.8233$$

$$x_2 = \frac{(2.8238)^2 \cos(2.8233) - 2.8233 \sin(2.8233) - \cos(2.8233)}{2.8233 \cos(2.8233)}$$

$$x_2 = 2.7986$$

$$x_3 = \frac{(2.7986)^2 \cos(2.7986) - 2.7986 \sin(2.7986) - \cos 2.7986}{2.7986 \cos(2.7986)}$$

$$= 2.7983$$

Required root is 2.798.

Q.6. Use Bisection method to find a root of the equation $x^3 - 3x - 5 = 0$. **(2017)**

Ans.

$$f(x) = x^3 - 3x - 5$$
$$f(2) = 8 - 6 - 5 = -3 \ (-\text{ve})$$
$$f(3) = 27 - 9 - 5 = 13 \ (+\text{ve})$$

Root lies between 2 and 3

Let $x_0 = 2, x_1 = 3$

$$x_2 = \frac{2+3}{2} = 2.5$$
$$f(x_2) = f(2.5) = (2.5)^2 - 3 \times 2.5 - 5$$
$$= 3.125 \ (+\text{ve})$$

Root lies between 2 and 2.5

$$x_3 = \frac{2+2.5}{2} = 2.25$$
$$f(x_3) = (2.25)^3 - 3 \times 2.5 - 5$$
$$= 0.359 \ (-\text{ve})$$

Root lies between 2.25 and 2.5

$$x_4 = \frac{2.25+2.5}{2} = 2.375$$
$$f(x_4) = (2.375)^3 - 3 \times 2.375 - 5$$
$$= 1.2715 \ (+\text{ve})$$

Root lies between 2.25 and 2.375

$$x_5 = \frac{2.25+2.375}{2} = 2.3125$$

$$f(x_5) = 2.3125^3 - 3 \times 2.3125 - 5$$
$$= 0.429 \ (+\text{ve})$$

Root lies between 2.25 and 2.3125

$$x_6 = \frac{2.25+2.3125}{2} = 2.2813$$
$$f(x_6) = (2.2813)^3 - 3 \times 2.2813 - 5$$
$$= 0.0287 \ (+\text{ve})$$

Root lies between 2.25 and 2.2813

$$x_7 = \frac{2.25+2.813}{2} = 2.2657$$
$$f(x_7) = (2.2657)^3 - 3 \times 2.2657 - 5$$
$$= -0.1664 \ (-\text{ve})$$

Root lies between 2.2657 and 2.2813

$$x_8 = \frac{2.2657+2.813}{2} = 2.2735$$
$$f(x_8) = 2.2735^3 - 3 \times 2.2735 - 5$$
$$= -0.0692 \ (-\text{ve})$$

Root lies between 2.2735 and 2.2815

$$x_9 = \frac{2.2735+2.2813}{2} = 2.2774$$
$$f(x_9) = 2.2774^3 - 3 \times 2.2774 - 5$$
$$= -0.0203 \ (-\text{ve})$$

Root lies between 2.2774 and 2.2813

$$x_{10} = \frac{2.2774+2.2813}{2} = 2.2794$$
$$f(x_{10}) = 2.2794^3 - 3 \times 2.2794 - 5$$
$$= 0.0048 \ (+\text{ve})$$

Root lies between 2.2774 and 2.2794

$$x_{11} = \frac{2.2774+2.2794}{2} = 2.2784$$
$$f(x_{11}) = 2.2784^3 - 3 \times 2.2784 - 5$$
$$= -0.0078 \ (-\text{ve})$$

Root lies between 2.2784 and 2.2794

$$x_{12} = \frac{2.2784+2.2794}{2} = 2.2789$$
$$f(x_{12}) = 2.2789^3 - 3 \times 2.2789 - 5 = -\text{ve}$$

Root lies between 2.2784 and 2.2789

$$x_{13} = \frac{2.2784 + 2.2789}{2} = 2.2787$$

$$f(x_{13}) = -ve$$

Root lies between 2.784 and 2.2787

$$x_{14} = \frac{2.2784 + 2.2787}{2} = 2.2786$$

$$f(x_{14}) = -ve$$

$\therefore$ Root is 2.278.

Q.7. Find the Newton Raphson iterative method to find the pth root of a positive number N and hence find the cube root of 17. **(2017)**

Ans. Let $x = N^{1/q}$

$\Rightarrow$ $x^q - N = 0$

$\Rightarrow$ $f(x) = x^q - N = 0$

$\Rightarrow$ $f'(x) = qx^{q-1}$

Iterative formula

$$x_{n+1} = x_n - \frac{x_n^q - N}{qx_n^{q-1}}$$

$$\Rightarrow \quad x_{n+1} = \frac{qx_n^q - x_n^q + N}{qx_n^{q-1}}$$

$$\Rightarrow \quad x_{n+1} = \frac{(q-1)x_n^q + N}{qx_n^{q-1}} \quad \ldots(1)$$

Four $N = 17$, $q = 3$

$$x_{n+1} = \frac{2x_n^3 + 17}{3x_n^2} \quad \ldots(2)$$

$$f(2) = x^3 - 17 = 0 = 2^3 - 17 = (-ve)$$

$$f(3) = 3^3 - 17\ (+ve)$$

Let $x_0 = 2.5$ by (2), we get

$$x_1 = \frac{2 \times 2.5^3 + 17}{3 \times 2.5^2} = 2.5733$$

$$x_2 = \frac{2 \times (2.5733)^3 + 17}{3 \times (2.5733)^2} = 2.5713$$

$$x_3 = \frac{2 \times (2.5713)^3 + 17}{3 \times (2.5713)^2} = 2.57128$$

$$x_4 = \frac{2 \times (2.57128)^3 + 17}{3 \times (2.57128)^2} = 2.57128$$

$\therefore$ Root is 2.5712.

Q.8. Define absolute, relative and percentage errors.

Ans. Absolute Error: If X is the true value of a quantity and X′ is its approximate value, then $|X - X'|$ is called the absolute error E_a.

Relative Error: The relative error is defined by

$$E_r = \left|\frac{X - X'}{X}\right|$$

Percentage Error: The percentage error is

$$E_p = 100E_r = 100\left|\frac{x - x'}{x}\right|$$

Q.9. Find the absolute error and relative error in $\sqrt{6} + \sqrt{7} + \sqrt{8}$ correct to 4 significant digits. **(10)**

Ans. Let $X = \sqrt{6} + \sqrt{7} + \sqrt{8}$

$\Rightarrow$ $X = 7.923668$

Let $X' = 7.923$ up to four significant figures

$$E_a = |X - X'| = |7.923668 - 7.923|$$

$$= 0.000668$$

$$E_r = \left|\frac{X - X'}{X}\right| = \frac{0.000668}{7.923668} = 8.432 \times 10^{-5}$$

Q.10. Explain Newton-Raphson method with convergence. Find the positive root of $x^4 - x = 10$ correct to three decimal places. Using Newton-Raphson method. **(20)**

Ans. Newton Raphson method: Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ Expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{21_0}f''(x_0) + \ldots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow \quad h = -\frac{f(x_0)}{f'(x_0)}.$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_0)}{f'(x_1)}$$

In general, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, 3, \ldots$$

Which is known as Newton-Raphson formula.

Convergence of Newton-Raphson Method

Suppose x_n differs from the root a by a small quantity en so that

$$x_0 = \alpha + \varepsilon_n \text{ and } x_{n+1} = \alpha + \varepsilon_{n+1}$$

Then the general equation becomes

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha)+\varepsilon_n f'(\alpha)+\frac{1}{2!}\varepsilon_n^2 f''(\alpha)+\ldots}{f'(\alpha)+\varepsilon_n f''(\alpha)+\ldots} \quad \text{[By Talyor's expansion]}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha)+\frac{1}{2}\varepsilon_n^2 f''(\alpha)+\ldots}{f'(\alpha)+\varepsilon_n f''(\alpha)+\ldots} \quad \text{[as f(x) = 0]}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha)+\varepsilon_n f''(\alpha)]} \quad \text{[neglecting third and higher powers of } x_n\text{]}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}$$

This shows that the subsequent errors at each step is proportional to the square of the previous error and as such the convergence s quadratic. Thus Newton-Raphson method has second order convergence.

Now, we solve $x^4 - x - 10 = 0$ ...(1)

Let $f(x) = x^4 - x - 10$

$\Rightarrow$ $f'(x) = 4x^3 - 1$

Put $x = 2$, we get $f(2) = 2^4 - 2 - 10 = 4$

Put $x = 1$, we get $f(1) = 1^4 - 1 - 19 = -10$

So, root of equation (1) lies between 1 and 2, but root is nearer to 2.

Let $x_0 = 1.8$

By Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3,\ldots$$

Put $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.8 - \frac{(-1.3024)}{22.328} = 1.8583$$

$$f(x_1) = f(1.8583) = 0.0668$$

Put $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.8583 - \frac{(0.0668)}{24.668} = 1.8555$$

$$f(x_2) = f(1.8555) = -0.00207$$

Put $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.8555 - \frac{(-0.00207)}{24.553} = 1.8555$$

So, the difference between the values of x_2 and x_3 are negligible upto three decimal places.

Hence the root of the equation (1) is $x = 1.855$

Q.11. Define different types of errors.

Ans. 1. Inherent errors: Errors which are already present in the statement of a problem before its solution are called inherent errors. These errors occur either due to the given data being approximate or due to limitations of calculators, mathematical tables or digital computer.

2. Rounding errors: Errors which arise from the process of rounding off the numbers during the computations are called rounding errors. Also called as procedual error or numerical error. They can be reduced either by changing the calculation procedure or retaining atleast one more significant digit at each step and rounding off at last step.

3. Truncation errors: Errors caused by using approximate results or an replacing an infinite process by a finite one are called truncation errors i.e., if $s = \sum_{i=1}^{\infty} a_i x_i$ is replaced or truncated to $s = \sum_{i=1}^{\infty} a_i x_i$ then error develop is called truncation error.

Tuncation error is a type of algorithm error.

For e.g. If $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \infty$ X (say) is truncated to

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + X' \text{ (say) then truncation error} = X - X'.$$

4. Absolute error: It is the numerical difference between the true values of a quantity and its approximate value let if X is the true value of a quantity and X′ is its approximate value then $| X - X' |$ is called the absolute error, denoted by e_a.

i.e., $e_a = | X - X' | = \text{error}$

5. Relative error: It is defined as

$$e_r = \frac{|\text{error}|}{\text{True value}} = \left|\frac{X - X'}{X}\right|$$

where X is true value. It is independent of the units used.

6. Percentage error: It is defined as

$$e_p = 100e_r = 100\left|\frac{X - X'}{X}\right|$$

It is independent of the units used.

Q.12. Round off the numbers 8650250 and 37.46235 to four significant figures and compute E_a, E_r, E_p in each case.

Ans. 1. Number rounded off to four significant figures = 865200

$$\therefore \quad E_a = |X - X_1| = 86250 - 865200 = 50$$

$$E_r = \left|\frac{X - X_1}{X}\right| = \frac{50}{865250} = 6.71 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.71 \times 10^{-3}$$

2. Number rounded off to four significant figures = 37.46

$$E_a = |X - X_1| = 37.46235 - 37.46000 = 0.00235$$

$$E_r = \left|\frac{X - X_1}{X}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.27 \times 10^{-3}$$

Q.13. Explain secant method with order of convergence. Find a root of the equation $x^3 - 2x - 5 = 0$. Using Secant method correct to three decimal places.

Ans. Secant Method: This method is an improvement over the method of false position as it does not require the condition $f(x_0)\, f(x_1) < 0$. Also it is not necessary that the interval must contain the root.

Taking x_0, x_1 as the initial limits of the interval, we write equation of the chord joining these as

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Then the abscissa of the point where it crosses the x-axis (y = 0) is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximation is, therefore given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n),\ n \geq 1$$

If at any iteration $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over the method of false position which always converge. But if the secant method once converge, its rate of convergence is 1.6 which is faster than that of the method of false position.

Now, let

$$f(x) = x^3 - 2x - 5$$

$$f(2) = 2^3 - 2 \times 2 - 5 = 8 - 4 - 5 = -1 = -\text{ve}$$

$$f(3) = 3^3 - 2 \times 3 - 5 = 27 - 6 - 5 = 16 = +\text{ve}$$

So, the roots lie between 2 and 3.

Let $x_0 = 2$ and $x_1 = 3$

Then by secant method, we have

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 3 - \frac{3-2}{16-(-1)}(16) = 3 - \frac{1}{17} \times 16 = 2.0588$$

$$f(x_2) = -0.3910$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \times f(x_2)$$

$$= 2.0588 - \frac{2.0588}{-0.3910 - 16} \times (-0.3910) = 2.0812$$

$$f(x_3) = -0.1479$$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} \times f(x_3)$$

$$= 2.0812 - \frac{(2.0812 - 2.0588)}{-0.1479 - (-0.3910)} \times (-0.1479) = 2.0948$$

$$f(x_4) = 0.0027$$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} \times f(x_4)$$

$$= 2.0948 - \frac{(2.0948 - 2.0812)}{0.0027 - (-0.1479)} \times 0.0027 = 2.0945$$

$$f(x_5) = -0.00057$$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f(x_5) - f(x_4)} \times f(x_5)$$

$$= 2.0945 - \frac{(2.0945 - 2.0948)}{-0.0057 - 0.0027} \times (-0.00057)$$

$$= 2.0945 \text{ and so on.}$$

Hence, the root is 2.094 correct to three decimal places.

Q.14. If $u = 4x^2y^3/z^4$ and errors in x, y, z be 0.001, compute the relative maximum error in u when x = y = z = 1.

Ans. Since

$$\frac{\partial u}{\partial x} = \frac{8xy^3}{z^4},\ \frac{\partial u}{\partial y} = \frac{12x^2y^2}{z^4},\ \frac{\partial u}{\partial z} = \frac{-16x^2y^3}{z^5}$$

$$\therefore \quad \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

$$= \frac{8xy^3}{z^4}\delta x + \frac{12x^2y^2}{z^4}\delta y - \frac{16x^2y^3}{z^5}\delta z$$

Since the errors $\delta x, \delta y, \delta z$ may be positive or negative, we take the absolute values of the turns on the right side, giving.

$$(\delta u)_{max} = \left|\frac{8xy^3}{z^4}\delta x\right| + \left|\frac{12x^2y^2}{z^4}\delta y\right| - \left|\frac{16x^2y^3}{z^5}\delta z\right|$$

$$= 8(0.001) + 12(0.001) + 16(0.001)$$

$$= 0.036$$

Hence the maximum relative error = (δu) max/u

$$= \frac{0.036}{4} = 0.009$$

Q.15. Given a = 9.00 ± 0.05, b = 0.0356 ± 0.002, c = 15300 ± 100, d = 62000 ± 500. Find the maximum value of absolute error in (i) a + b + c (ii) a + b + d.

Ans. (i) The maximum absolute error in a + b + c + d is

$$E_a = |\ 77909.0858 - 77309.0358\ |$$

$$= 600.0502$$

(ii) The maximum absolute Error is a + b + d is

$$E_a = |\ 62509.0858 - 62009.0356\ |$$

$$= 500.0502$$

Q.16. By applying Newton's method upto two iteration, find the real root near to 2 for the equation $x^4 - 12x + 7 = 0$. (2019)

Ans. Let

$$f(x) = x^4 - 12x + 7$$

$$f(x) = 4x^3 - 12$$

Let the initial approximation be $x_0 = 2$

Using Newton-Raphson method first iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(2)^4 - 12(2) + 7}{4(2)^3 - 12} = 2.05$$

Second iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.05 - \frac{(2.05)^4 - 12(2.05) + 7}{4(2.05)^3 - 12} = 2.0473$$

$\therefore$ Real root is 2.0473.

Q.17. (a) Find roots of the equation $xe^x = \cos x$ using secant method upto 4 decimal places.

Ans. Let

$$f(x) = \cos x - xe^x = 0$$

$$f(0) = 1 = f(x_0)$$

$$f(1) = -2.178 = f(x_1)$$

Let $x_0 = 0, x_1 = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow \quad x_2 = \frac{0-1}{-2.178-1} = 0.31466$$

$$f(x_2) = \cos 0.31466 - 0.31466e^{0.31466}$$

$$= 0.51988$$

Consider x_1 and x_2

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$\Rightarrow \quad x_3 = \frac{0.51988 - 0.31466(-2.178)}{0.51988 + 2.178}$$

$$\Rightarrow \quad x_3 = 0.44672$$

$$f(x_3) = \cos(0.44672) - 0.44672e^{0.44672} = 0.2035$$

Consider x_2 and x_3

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$\Rightarrow \quad x_4 = \frac{0.31466(0.2035) - 0.44672(0.51988)}{0.2035 - 0.51988}$$

$$x_4 = 0.53168$$

$$f(x_4) = \cos(0.53168) - 0.53168e^{-0.53168} = -0.0428$$

Consider x_3 and x_4

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$x_5 = \frac{0.44672(-0.0428) - 0.53168(0.2035)}{-0.0428 - 0.2035}$$

$$\Rightarrow \quad x_5 = 0.51691$$

$$f(x_5) = \cos(0.51691) - 0.51691e^{0.51691}$$

$$= 0.0025$$

Consider x_4 and x_5

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$$

$$= \frac{0.53168(0.0025) - 0.51691(-0.0428)}{0.0025 + 0.0428}$$

$$x_6 = 0.51772$$

$$f(x_6) = 0.00011$$

Consider x_5 and x_6

$$x_7 = \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)}$$

$$= \frac{0.51691(0.00011) - 0.51772 \times 0.0025}{0.00011 - 0.0025}$$

$= 0.51775$ Root is 0.5177.

Q.18. (a) How many digits are to be taken in computing $\sqrt{20}$ so that error does not exceed 0.1%?

Ans. The first digit of $\sqrt{20}$ is 4

$$a_m = 4$$

As $E_r < \dfrac{1}{a_m 10^{n-1}}$

as error must not exceed 0.1% i.e. 0.001

$$\frac{1}{a_m 10^{n-1}} = \frac{1}{4 \cdot 10^{n-1}} \leq 0.001$$

$$\Rightarrow \quad 10^{n-1} \geq \frac{1}{0.004}$$

$$\Rightarrow \quad 10^{n-1} \geq 250$$

$$\Rightarrow \quad n \geq 4$$

Q.19. Use golden section search to find the value of x that minimizes $f(x) = x^4 - 14x^3 + 60x^2 - 70x$ in the range [0, 2]. Locate this value of x to within a range of 0.3.

Ans. After N stage the range [0, 2] is reduced by $(0.61803)^N$.

Choose N so that

$$(0.61803)^N \leq \frac{0.3}{2}$$

$$\Rightarrow \quad N = 4 \text{ will do}$$

Iteration 1: Evaluate f at two intermediate points a_1 and b_1.

$$\Rightarrow \quad a_1 = a_0 + \rho(b_0 - a_0) = 0.7639 \qquad \left[\rho = \frac{3-\sqrt{5}}{2}\right]$$

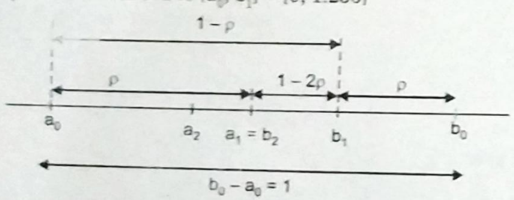
$$b_1 = a_0 + (1-\rho)(b_0 - a_0) = 1.236$$

$$f(a_1) = -24.36$$

$$f(b_1) = -18.96$$

$$f(a_1) < f(b_1)$$

So, uncertainty level is reduced to $[a_0, b_1] = [0, 1.236]$



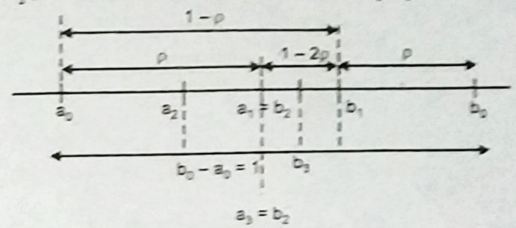
Iteration 2: We choose b_2 to coincide with a_1 and f need only be evaluated at one new point.

$$a_2 = a_0 + \rho(b_1 - a_0) = 0.4721$$

$$f(a_2) = -21.1$$

$$f(b_2) = f(a_1) = -24.36$$

Now, $f(b_2) < f(a_2)$, so uncertainity level is reduced to $[a_2, b_1] = [0.4721, 1.236]$



Iteration 3: We set $a_3 = a_2$ and compute b_3

$$b_3 = a_2 + (1-\rho)(b_1 - a_2) = 0.9443$$

We have

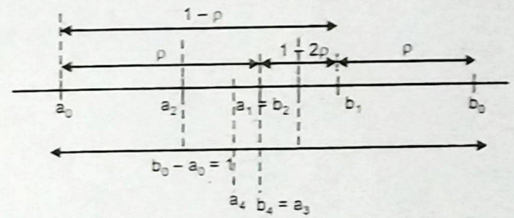
$$f(a_3) = f(b_2) = -24.36$$

$$f(b_3) = -23.59$$

So,

$$f(b_3) > f(a_3)$$

Hence, uncertainity interval is reduced to $[a_2, b_3] = [0.4721, 0.9443]$



Iteration 4: We set $b_4 = b_3$

and

$$a_4 = a_2 + \rho(b_3 - a_2) = 0.6525$$

We have

$$f(a_4) = -23.84$$

$$f(b_4) = f(a_3) = -24.36$$

$$f(a_4) > f(b_4)$$

Thus value of x that minimizes f is located in interval $[a_4, b_3] = [0.6525, 0.9443]$

Now,

$$b_3 - b_4 = 0.292 < 0.3$$

UNIT – II

Q.20 Compute $\log_{10} 323.5$ from the table (2014)

x :	**321.0**	**322.8**	**324.2**	**325.0**
$\log_{10}$:	**2.5065**	**2.5089**	**2.5108**	**2.5119**

Ans.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$
321.0	2.5065	$\frac{0.0024}{1.8} = 0.0012$	
322.8	2.5089	0.00125	0
324.2	2.5108	0.00127	0
325.0	2.5119		

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0)$$
$$= 2.5065 + (x - 321) \times 0.0012$$
$$\text{for } x = 323.5$$
$$f(323.5) = 2.5065 + (323.5 - 321) \times 0.0012$$
$$= 2.50975$$

Q.21 Using simpson's one third rule evaluate $\int_0^{\pi/2} \sin x\, dx$ upto 4 decimal place in 10 intervals. (2014)

Ans.

$$h = \pi/20$$
$$x = 0, \frac{\pi}{20}, \frac{\pi}{10}, \frac{3\pi}{20}, \frac{\pi}{5}, \frac{\pi}{4}, \frac{3\pi}{10}, \frac{7\pi}{20}, \frac{2\pi}{5}, \frac{9\pi}{20}, \frac{\pi}{2}$$

x	y = sin x	
0	0	$= y_0$
$\pi/20$	0.1564	$= y_1$
$\pi/10$	0.3090	$= y_2$
$3\pi/20$	0.4539	$= y_3$
$\pi/5$	0.5878	$= y_4$
$\pi/4$	0.7071	$= y_5$
$3\pi/10$	0.8090	$= y_6$
$7\pi/20$	0.8910	$= y_7$
$2\pi/5$	0.9510	$= y_8$
$9\pi/20$	0.9876	$= y_9$
$\pi/2$	1	$= y_{10}$

simpson's $\frac{1}{3}$rd rule

$$\int_0^{\pi/2} \sin x\, dx = \frac{h}{3}[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9)$$
$$+ 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{\pi}{60}[(0 + 1) + 4(0.1564 + 0.4539 + 0.7071$$
$$+ 0.8910 + 0.9876) + 2(0.3090 + 0.5878 + 0.8090 + 0.9510)]$$
$$= \frac{\pi}{60}[1 + 12.784 + 5.3136]$$
$$= \frac{\pi}{60} \times 19.0976 = 0.9999$$

Q.22 Evaluate $\int_0^4 e^x\, dx$ by Simpson's one third rule given that $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.6$ and compare it with the actual value. (2015)

Ans.

x	$y = e^x$		
0	1	=	y_0
1	2.72	=	y_1
2	7.39	=	y_2
3	20.09	=	y_3
4	54.6	=	y_4

$$h = 1$$

By Simpson's $\frac{1}{3}$rd rule

$$\int_0^4 e^x\, dx = \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$
$$= \frac{1}{3}[(1 + 54.6) + 2 \times 7.39 + 4(2.72 + 20.09)]$$
$$= \frac{1}{3}[55.6 + 14.78 + 22.81] = \frac{93.19}{3} = 31.06$$

Actual Value $\quad \int_0^4 e^x\, dx = |e^x|_0^4 = e^4 - e^0 = 54.598 - 1 = 53.598$

Q.23 Prove the following relations (2015)

(i) $\Delta\nabla = \delta^2$, (ii) $\mu\delta = \frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - E^{-1})$

Ans. (i) $\quad \Delta\nabla f(x) = \Delta\{\nabla f(x)\}$

$$= (\Delta\Delta E^{-1}) f(x) = \Delta^2 E^{-1} f(x)$$

Now $\quad \delta^2\{f(x)\} = (\Delta E^{-1/2})^2 f(x)$

$$= \Delta^2 E^{-1} f(x)$$
$$\therefore \quad \Delta\nabla = \delta^2$$

(ii) $\quad \mu\delta\{f(x)\} = \frac{1}{2}[E^{1/2} + E^{-1/2}][E^{1/2} - E^{-1/2}] f(x)$

$$= \frac{1}{2}[E - E^{-1}]f(x)$$

$$= \frac{1}{2}\left[E - \frac{1}{E}\right]f(x) = \frac{1}{2}\left[\frac{E^2-1}{E}\right]f(x)$$

$$= \frac{1}{2}\left[\frac{(E-1)(E+1)}{E}\right]f(x)$$

$$= \frac{1}{2}(E-1)\left[\frac{E+1}{E}\right]fx$$

$$= \frac{1}{2}\Delta[1+E^{-1}]f(x) = \frac{1}{2}[\Delta + \Delta E^{-1}]f(x)$$

$$= \frac{1}{2}[\Delta + \nabla]f(x)$$

$$\Rightarrow \quad \mu\delta = \frac{1}{2}(\Delta + \nabla)$$

Consider $\frac{1}{2}(\Delta + \nabla)f(x) = \frac{1}{2}[(1+\Delta)-(1-\nabla)]f(x)$

$$= \frac{1}{2}[(E - E^{-1})]f(x)$$

$$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - E^{-1})$$

$$\mu\delta = \frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - E^{-1})$$

Q.24 Find the number of men getting wages between Rs 100 and 150 from the following data **(2015)**

Wages in	Rs. 0–100	100–200	200–300	300–400
Frequency	9	30	35	42

Ans. Table is rearranged as

Wages	No of men
less than 100	9
less than 200	39
less than 300	74
less than 400	116

Difference table is

wages (x) less than	No. of men (y)	Δy	$\Delta^2 y$	$\Delta^3 y$
100	9			
		30		
200	39		5	2
		35		
300	74		7	
		42		
400	16			

Between 100 – 150

To find f(100) and f(150)

f(100), $\quad a = 100, h = 100$

$$100 + 100\mu = 100$$

$$\mu = 0$$

$$f(100) = 9$$

f(150), $\quad a = 100, h = 100$

$$100 + 100\mu = 150$$

$$100\mu = 50 \Rightarrow \mu = 0.5$$

By forward interpolation

$$f(150) = 9 + 0.5 \times 30 + \frac{0.5(0.5-1)}{2} \times 5 + \frac{0.5(0.5-1)(0.52)}{6} \times 2$$

$$f(150) = 9 + 15 - 0.625 + 0.125 = 23.5 \approx 23$$

No. of men between 100 and 150

$$= 23 - 9 = 14.$$

Q.25. Find the interpolating polynomial for (0, 2), (1, 3) (2, 12) and (5, 147) using Lagrange's interpolation formula. **(2015)**

Ans.

x	0	1	2	5
f(x)	2	3	12	147

According to formula

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3 + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$\Rightarrow \quad f(x) = \frac{-2}{10}(x^2 - 3x + 2)(x-5) + \frac{3}{4}(x^2 - 2x)(x-5) - \frac{12}{6}(x^2 - x)(x-5) + \frac{147}{60}(x^2 - x)(x-2)$$

$$\Rightarrow \quad f(x) = -\frac{1}{5}(x^3-8x^2+17x-10)+\frac{3}{4}(x^3-7x^2+10x)$$
$$-2(x^3-6x^2+5x)+\frac{49}{20}(x^3-3x^2+2x)$$

$$\Rightarrow \quad f(x) = \frac{-4(x^3-8x^2+17x-10)+15(x^3-7x^2+10x) -20(x^3-6x^2+5x)+49(x^3-3x^2+2x)}{20}$$

$$\Rightarrow \quad f(x) = \frac{40x^3-100x^2+80x+40}{20}$$

$$f(x) = 2x^3-5x^2+4x+2$$

Q.26 Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ by using simpson's $\frac{1}{3}$ rule taking $h = \frac{1}{4}$ and by using Simpson's $\frac{3}{8}$ rule taking $h = \frac{1}{6}$ where h is interval of differencing. **(2015)**

Ans. (i) $f(x) = \frac{1}{1+x^2}, h = \frac{1}{4}$

x	0	1/4	1/2	3/4	1
f(x)	1	16/17	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rd rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2y_2]$$
$$= \frac{1}{12}\left[(1+0.5)+4\left(\frac{16}{17}+0.64\right)+2\times 0.8\right]$$
$$= 0.785392$$

(ii) $f(x) = \frac{1}{1+x^2}, h = 1/6$

x	0	1/6	2/6	3/6	4/6	5/6	1
f(x)	1	36/37	9/10	4/5	9/13	36/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{3}$ rule

$$\int_0^1 f(x)dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$
$$= \frac{1}{16}\left[\left(1+\frac{1}{2}\right)+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)+2\times\frac{4}{5}\right]$$

$= 0.785395.$

Q.27 Express $y = 2x^3 - 3x^2 + 3x - 10$ in a factorial notation hence show that $^3y = 12$. **(2016)**

Ans. $y = 2x^3 - 3x^2 + 3x - 10$

$$y = 2x(x-1)(x-2) + Ax(x-1) + Bx + C$$

$\Rightarrow$ $2x^3 - 3x^2 + 3x - 10 = 2x(x-1)(x-2) + Ax(x-1) + Bx + C$

For $x = 0$ $\Rightarrow$ $-10 = C$

For $x = 1$ $\Rightarrow$ $B + C = -8$

$\Rightarrow$ $B = 2$

For $x = 2$ $\Rightarrow$ $2A + 2B + C = 0$

$\Rightarrow$ $A = 3$

$\therefore$ $y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

Now, $\Delta y = 6[x]^2 + 6[x] + 2$

$\Delta^2 y = 12[x] + 6$

$\Delta^3 y = 12.$

Q.28 From the following table, estimate the number of students who obtained marks between 40 and 45. **(2016)**

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Ans. Table is rearranged as

Marks obtained	No. of students (C.F.)
less than 40	31
less than 50	73
less than 60	125
less than 70	159
less than 80	190

Difference table is

Marks (x) (Less than)	Students (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$\boxed{31}$				
		$\boxed{42}$			
50	73		$\boxed{9}$		
		51		$\boxed{-25}$	
60	124		−16		$\boxed{37}$
		35		12	
70	159		−4		
		31			
80	190				

We calculate f(40) and f(45)

$$f(40) = 31$$

$$f(45) = y(40) + \mu\Delta y(40) + \frac{u(u-1)}{2!}\Delta^2 y(40) + \frac{u(u-1)(u-2)}{3!}\Delta^3 y(40) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y(40) + \ldots$$

$$a = 40,\ h = 10$$

$$\mu = \frac{45-40}{10} = 0.5$$

$$f(45) = 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2!}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!}\times(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\times 37 + \ldots$$

$$= 52 - 1.125 - 1.5625 - 1.4453125$$

$$= 47.867 \approx 48$$

Number of students between 40 and 45

$$= 48 - 31 = 17.$$

Q.29 Using Lagrange's formula, find the form of the function f(x) given that. (2016)

x	0	2	3	6
f(x)	659	705	729	804

Ans. $$f(x) = \frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)}\times 659 + \frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)}\times 705 + \frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)}\times 729 + \frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)}\times 804$$

$$f(x) = \frac{(x^2-5x+6)(x-6)}{-36}\times 659 + \frac{(x^2-3x)(x-6)}{8}\times 705 + \frac{(x^2-2x)(x-6)}{-1}\times 81 + \frac{(x^2-2x)(x-3)}{6}\times 67$$

$$= \frac{-(x^3-11x^2+36x-36)}{36}\times 659 + \frac{(x^3-9x^2+18x)}{8}\times 705 - (x^3-8x^2+12x)81 + \frac{(x^3-5x^2+6x)}{6}\times 67$$

$$= \frac{\begin{array}{c}-1318x^3+14498x^2-47448x+47448+6345x^3 \\ -57105x^2+114210x-5832x^3+46656x^2 \\ -69984x+804x^3-4020x^2+4824x\end{array}}{72}$$

$$f(x) = \frac{-x^3+29x^2+1602x+47448}{72}$$

Q.30 Use Lagrange's interpolation formula, express the function $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions. (2017)

Ans. Let $$f(x) = \frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$$

Consider $$\phi(x) = 3x^2+x+1$$

Its tabulate values for x = 1, 2, 3.

x	:	1	2	3
$3x^2+x+1$	:	5	15	31

Using lagranges interpolation formula, we get

$$f(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}\times 5 + \frac{(x-1)(x-3)}{(2-1)(2-3)}\times 15 + \frac{(x-1)(x-2)}{(3-1)(3-2)}\times 31$$

$$\Rightarrow \quad f(x) = \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

$$\Rightarrow \quad f(x) = \frac{5}{2(x-1)} - \frac{15}{(x-2)} + \frac{31}{2(x-3)}.$$

Q.31 Using simpson's one third rule, evaluate $\int_0^1 \frac{dx}{1+x}$ Hence find value value of $\log_e 2$. (2017)

Ans. We divide interval in 8 equal parts.

Here $$y = \frac{1}{1+x} = f(x)$$

x :	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1
y :	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{8}{11}$	$\frac{2}{3}$	$\frac{8}{13}$	$\frac{4}{7}$	$\frac{8}{15}$	$\frac{1}{2}$
	y_o	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's $\frac{1}{3}$rd rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}[(y_o+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$= \frac{1}{24}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{8}{9}+\frac{8}{11}+\frac{8}{13}+\frac{8}{15}\right)+2\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)\right] = 0.693154$$

Since $$\int_0^1 \frac{dx}{1+x} = [\log_e(1+x)]_0^1 = \log_e 2$$

$$\therefore \quad \log_e 2 = 0.693154.$$

Q.32 The following table gives the values of x and y.

x	1.2	2.1	2.8	4.1	4.9	6.2
y	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to y = 12, using Lagrange's interpolation formula. **(2018)**

Ans. According to formula

$$f(x) = \frac{(x-2.1)(x-2.8)(x-4.1)(x-4.9)(x-6.2)\times 4.2}{(1.2-2.1)(1.2-2.8)(1.2-4.1)(1.2-4.9)(1.2-6.2)}$$

$$+\frac{(x-1.2)(x-2.8)(x-4.1)(x-4.9)(x-6.2)\times 6.8}{(2.1-1.2)(2.1-2.8)(2.1-4.1)(2.1-4.9)(2.1-6.2)}$$

$$+\frac{(x-1.2)(x-2.1)(x-4.1)(x-4.9)(x-6.2)\times 9.8}{(2.8-1.2)(2.8-2.1)(2.8-4.1)(2.8-4.9)(2.8-6.2)}$$

$$+\frac{(x-1.2)(x-2.1)(x-2.8)(x-4.9)(x-6.2)\times 13.4}{(4.1-1.2)(4.1-2.1)(4.1-2.8)(4.1-4.9)(4.1-6.2)}$$

$$+\frac{(x-1.2)(x-2.1)(x-2.8)(x-4.1)(x-6.2)\times 15.5}{(4.9-1.2)(4.9-2.1)(4.9-2.8)(4.9-4.1)(4.9-6.2)}$$

$$+\frac{(x-1.2)(x-2.1)(x-2.8)(x-4.1)(x-4.9)\times 19.6}{(6.9-1.2)(6.9-2.1)(6.9-2.8)(6.9-4.1)(6.9-4.9)}$$

$$f(12) = \frac{(12-2.1)(12-2.8)(12-4.1)(12-4.9)(12-6.2)\times 4.2}{-77.2560}$$

$$+\frac{(12-1.2)(12-2.8)(12-4.1)(12-4.9)(12-6.2)\times 6.8}{14.4648}$$

$$+\frac{(12-1.2)(12-2.1)(12-4.1)(12-4.9)(12-6.2)\times 9.8}{570.5237}$$

$$+\frac{(12-1.2)(12-2.1)(12-2.8)(12-4.9)(12-6.2)\times 13.4}{12.6672}$$

$$+\frac{(12-1.2)(12-2.1)(12-2.8)(12-4.1)(12-6.2)\times 15.5}{-22.6262}$$

$$+\frac{(12-1.2)(12-2.1)(12-2.8)(12-4.1)(12-4.9)\times 19.6}{628.1856}$$

$$f(12) = -1610.84 + 15195.727 + 597.482 + 42850.637 - 30876.064 + 1721.473$$

$$= 27878.4$$

Q.33. Find the value of $\int_0^1 \frac{(\cos x - x)\,dx}{(1+x)}$ by using trapezoidal rule with number of steps equal to 6. **(2017)**

Ans. Dividing the interval (0, 1) nto 6 equal steps

$$h = \frac{1-0}{5} = 0.2,\ f(x) = \frac{\cos x - x}{1+x}$$

x	0	0.2	0.4	0.6	0.8	1
f(x)	1	0.6501	0.3722	0.1408	−0.0574	−0.2298
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule

$$= \int_0^1 \frac{(\cos x - x)}{1+x}\,dx = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}\,[(1 - 0.2298) + 2(0.6501 + 0.3722 + 0.1408 - 0.0574)]$$

$$= 0.1\,[0.7702 + 2.2114] = 0.2982$$

Q.34. Using Newton's interpolation formula find the value of y when x = 218 given that **(2018)**

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Ans.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	Δ^5	Δ^6
100	$\boxed{10.63}$	$\boxed{2.4}$					
			$\boxed{-0.39}$				
150	13.03	2.01		$\boxed{0.15}$			
			−0.24		$\boxed{-0.07}$		
200	15.04	1.77		0.08		$\boxed{0.02}$	
			−0.16		−0.05		$\boxed{0.02}$
250	16.81	1.61		0.03		0.04	
			−0.13		−0.01		
300	18.42	1.48		0.02			
			−0.11				
350	19.9	1.37					
400	21.27						

Here

$$a = 100,\ h = 50$$

$$a + \mu h = 218$$

$$\Rightarrow \quad 100 + 50\,u = 218$$

$$\Rightarrow \quad u = 2.36$$

By Newton's forward formula

$$f(a + \mu h) = y_0 + \mu\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y_{218} = 10.68 + 2.36 \times .2.4 + \frac{(2.36)(2.36-1)}{2!}(-0.39)$$

$$+\frac{(2.36)(2.36-1)(2.36-2)}{3!}(0.15)$$

$$+\frac{(2.36)(2.36-1)(2.36-2)(2.36-3)}{4!}(-0.07)$$

$$+\frac{(2.36)(2.36-1)(2.36-2)(2.36-3)(2.36-4)}{5!}\times(0.02)$$

$$= 10.63 + 5.664 - 0.6259 + 0.0289 + 0.0009 + 0.0001$$

$$= 15.698$$

Q.35. Given the value of

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Evaluate f(9), using lagrange's formula. (2018)

Ans. Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$f(x) = \frac{(x-7)(x-11)(x-13)(x-17)}{(5-7)(5-11)(5-13)(5-17)}\times 150$$

$$+\frac{(x-5)(x-11)(x-13)(x-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392 + \frac{(x-5)(x-7)(x-13)(x-17)}{(11-5)(11-7)(11-13)(11-17)}\times 1452$$

$$+\frac{(x-5)(x-7)(x-11)(x-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366 + \frac{(x-5)(x-7)(x-11)(x-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202$$

$$f(x) = \frac{(x-7)(x-11)(x-13)(x-17)}{0.1302} - \frac{(x-5)(x-11)(x-13)(x-17)}{0.8167}$$

$$+\frac{(x-5)(x-7)(x-13)(x-17)}{5.0417} - \frac{(x-5)(x-7)(x-11)(x-17)}{6.1615}$$

$$+\frac{(x-5)(x-7)(x-11)(x-13)}{1.8063}$$

At x = 9

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{0.1302} - \frac{(9-5)(9-11)(9-13)(9-17)}{0.8167}$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{5.0417} - \frac{(9-5)(9-7)(9-11)(9-17)}{6.1615}$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{1.8063}$$

$$= -983.1029 + 313.4566 + 50.7765 + 20.7742 + 35.4315$$

$$f(9) = 562.66.$$

Q.36. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Trapeziodal rule and compare the result with its actual value. (2018)

Ans. Divide the interval (0, 6) into 6 parts each of width h = 1.

Let $f(x) = \frac{1}{1+x^2}$

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	1/17	1/26	1/37
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezodial rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{37}\right)+2\left(0.5+0.2+0.1+\frac{1}{17}+\frac{1}{26}\right)\right]$$

$$= 1.41079$$

Actual value

$$\int_0^6 \frac{dx}{1+x^2} = \tan^{-1}x\,\Big]_0^6 = \tan^{-1}6 - \tan^{-1}0$$

$$= 1.4056 - 0 = 1.4056.$$

$$\text{Error} = 1.41079 - 1.4056 = 0.0052$$

Q.37. Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ by using

1. Trapezoidal **2. Simpson's $\frac{1}{3}$ rule**

3. Simpson $\frac{3}{8}$ rule **4. Weddle rule**

and compare the results with its actual value.

Ans. Let $y(x) = \frac{x^2}{1+x^3}$

and $I = \int_0^1 y(x)\,dx$

Divide the interval [0, 1] into five subinterval by taking h = 0.2

x_0	x_1	x_2	x_3	x_4	x_5

0	0.2	0.4	0.6	0.8	1
y_0	y_1	y_2	y_3	y_4	y_5
0	0.039	0.15	0.290	0.423	0.5

3. Trapezoidal Rule

$$I = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[0.5 + 2(0.9040)] = 0.2316$$

4. Simpson's $\frac{1}{3}$ Rule

$$I = \frac{h}{3}[(y_0 + y_5) + 4(y_1 + y_3) + 2(y_2 + y_4)]$$

$$I = \frac{0.2}{3}[0.5 + 4(0.330) + 2(0.5731)] = 0.199$$

5. Simpson's $\frac{3}{8}$ Rule

$$I = \frac{3h}{8}[(y_0 + y_5) + 3(y_1 + y_2 + y_3) + 2(y_4)]$$

$$= \frac{3 \times 0.2}{8}[0.5 + 3(0.433) + 2(0.2961)] = 0.2196$$

6. Weddle's Rule

$$I = \frac{3h}{10}[y_1 - 5y_2 + y_3 + 6y_4 + y_5 + 5y_6]$$

$$= \frac{3 \times 0.2}{10}[0 + 0.195 + 0.33 + 1.776 + 0.423 + 2.5] = 0.3$$

$$I = \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}[\log_e(1+x^3)]_0^1$$

$$= \frac{1}{3}\log_e 2 = 0.231$$

Q.[illegible] [illegible] f(x) as a polynomial in x for the following data.

x	–4	–1	0	2	5
y	1245	33	5	9	1335

by using Divided Difference Table. Hence [illegible] f(1).

Ans. The divided difference table is:

x	f(x)	1st divided differences	2nd divided differences	3rd divided differences	4th divided difference
–4	1245				
		–404			

–1	33		94		
		–28		–14	
0	5		10		2
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula:

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \ldots$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$

$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(2)$$

$$f(x) = 2x^4 - 5x^3 - 6x^2 - 14x + 5$$

Put x = 1

$$f(1) = 2 - 5 + 6 - 14 + 5 = -2 + 6 - 9 = -5$$

Q.29. If f(1.15) = 1.0723, f(1.20) = 1.0954, f(1.25) = 1.1180 and f(1.30) = 1.1401, then find the value of f(1.28). **(2019)**

Ans. The difference table is:

x	y = f(x)	$\nabla y(x)$	$\nabla^2 y(x)$	$\nabla^3 y(x)$
1.15	1.0723			
		0.0231		
1.20	1.0954		–0.0005	
		0.0226		0
1.25	1.1180		–0.0005	
		0.0221		
1.30	1.1401			

To find f(1.28), we choose x_n = 1.30, x = 1.28, h = 0.05

so that
$$p = \frac{x - x_n}{h} = \frac{1.28 - 1.30}{0.05} = -0.4$$

Using Newton's backward formula

$$y = f(x)$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{6}\nabla^3 y_n + \ldots$$

$$y_{p=} = f(1.28) = 1.1401 + (-0.4)(0.0221)$$

$$+\frac{(-0.4)(-0.4+1)}{2}(-0.0005)+\frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(0)$$

$$= 1.13132$$

Q.40. Evaluate the integral $\int_0^{\pi} x \sin x dx$ using Simpson's $\frac{1}{3}$ rule by dividing the interval [0, p] in sin subintervals. **(2019)**

Ans. We have $f(x) = x \sin x$, $a = 0$, $b = \pi$, $n = 6$

$$h = \frac{b-a}{h} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
y = f(x)	0	0.2619	0.9073	1.5714	1.8145	1.3095	0

Here $y_0 = 0$, $y_1 = 0.2619$, $y_2 = 0.9073$, $y_3 = 1.5714$, $y_4 = 1.8145$, $y_5 = 1.3095$, $y_6 = 0$

Using Simpson's $\frac{1}{3}$ rule

$$\int_0^{\pi} x \sin x\, dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{18}[(0 + 0) + 4(0.2619 + 1.5714 + 1.3095)$$

$$+ 2(0.9073 + 1.8145)]$$

$$= 3.1454$$

Q.41. Find the data:

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Evaluate f(9) by using Newton's divided difference method. **(2019)**

Ans. The dividend table is:

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	Δ^4y(x)
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

Using Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)\, \Delta f(x_0) + (x - x_0)(x - x_1)\, \Delta^2 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\, \Delta^3 f(x_0) + \ldots$$

$$f(x) = 150 + (9.5)(121) + (9.5)(9 - 7)(24) + (9 - 5)(9 - 7)(9 - 11)(1)$$

$$= 810$$

Q.42. A second degree polynomial passes through the points (1, –1), (2, –1), (3, 1) and (4, 5). Find the polynomial. **(2019)**

Ans. The difference table is

x	y = f(x)	Δy	Δ^2y	Δ^3y
1	–1			
		0		
2	–1		2	
		2		0
3	1		2	
		4		
4	5			

We take $x_0 = 1$

so $$P = \frac{x - x_0}{h} = \frac{x-1}{1} = x - 1$$

Using Newton's forward interpolation formula we get

$$f(x) = y_0 + p\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots$$

$$= -1 + (x-1)(0) + \frac{(x-1)(x-2)}{2!}(2) + \frac{(x-1)(x-2)(x-3)}{3!}(0)$$

$$= -1 + 0 + (x - 1)(x - 2) + 0$$

$$= x^2 - 3x + 1$$

Q.43. Find the value of $\int_0^{0.6} e^x dx$ using Simpson's 1/3 and 3/8 rule by dividing the interval [0, 0.6] is suitable number of subintervals. **(2019)**

Ans. We have $f(x) = e^x$, $x_0 = 0$, $x_n = 0.6$, $n = 6$

$$h = \frac{x_n - x_0}{h} = \frac{0.6 - 0}{6} = 0.1$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
y = f(x)	1.0000	1.10517	1.22140	1.34986	1.49182	1.64872	1.82212

Here $y_0 = 1.0000$, $y_1 = 1.10517$, $y_2 = 1.22140$, $y_3 = 1.34986$, $y_4 = 1.49182$, $y_5 = 1.64872$, $y_6 = 1.82212$

Using Simpson's one third rule

$$\int_0^{0.6} x\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}[(1.0000 + 1.82212) + 4(1.10517 + 1.34986 - 1.64872) + 2(1.22140 + 1.49182)]$$

$= 0.82212$

By Simpson's three eight rule

$$\int_0^{0.6} e^x\,dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{3(0.1)}{8}[(1.0000 + 1.82212) + 3(1.10517 + 1.22140) + (1.49182 + 1.64872) + 2(1.34986)]$$

$= 0.82211$

Q.44. (a) Compute $\int_0^{\pi/2} \sin x\,dx$ significant to five digits by Romberg's method.

Ans. Take $h = \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}$ successively and compute integral using Trapezoidal rule.

(i) When $h = \frac{\pi}{4}$, values of $y = \sin x$

x	0	$\pi/4$	$\pi/2$
y	0	0.7071	1.0
	y_0	y_1	y_2

Thus

$$I(h) = \frac{h}{2}[(y_0 + y_2) + 2y_1]$$

$$= \frac{\pi}{8}[1 + 2 \times 0.7071] = 0.94805$$

(ii) When $h = \frac{\pi}{8}$, values of $y = \sin x$

x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
y	0	0.3827	0.7071	0.9239	1

y_0 y_1 y_2 y_3 y_4

$$I(h/2) = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{\pi}{16}[1 + 2(0.3827 + 0.7071 + 0.9239)]$$

$= 0.98712$

(iii) When $h = \frac{\pi}{16}$

x	y	
0	0	y_0
$\pi/16$	0.1951	y_1
$\pi/8$	0.3827	y_2
$3\pi/16$	0.5556	y_3
$\pi/4$	0.7071	y_4
$5\pi/16$	0.8315	y_5
$3\pi/8$	0.9239	y_6
$7\pi/16$	0.9808	y_7
$\pi/2$	1	y_8

$$I(h/4) = \frac{h}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{\pi}{32}[1 + 2(0.1951 + 0.3827 + 0.5556 + 0.7071 + 0.8315 + 0.9239 + 0.9808)]$$

$= 0.99680$

First order extrapolated values are

$$I(h, h/2) = \frac{4I(h/2) - I(h)}{4-1}$$

$$= \frac{4 \times 0.9871 - 0.9480}{4-1} = 1.00015$$

$$I(h/2, h/4) = \frac{4I(h/4) - I(h/2)}{4-1}$$

$$= \frac{4 \times 0.9968 - 0.98712}{4-1} = 1.000034$$

$$I(h, h/2, h/4) = \frac{4I(h/2, h/4) - I(h, ih/2)}{4-1}$$

$$= \frac{4 \times 1.000034 - 1.000152}{4-1} = 0.99995$$

Value of integral = 1

Q.45. Evaluate $\int_2^4 (x^2 + 2x)\,dx$ by Gauss Quadrature formula.

Ans. Here $a = 2, b = 4$

$$f(x) = x^2 + 2x$$

Let us transform the interval [2, 4] to [– 1, 1] by putting

$$x = \frac{1}{2}[(a + b) + (b - a)y]$$

$$x = \frac{1}{2}[(2 + 4) + (4 - 2)y]$$

$$x = 3 + y \quad ...(1)$$

$\therefore$ $dx = dy$

and $I = \int_2^4 f(x)\,dx$

$$= \int_{-1}^{1} \phi(y)\,dy \quad ...(2) \quad [\phi(y) = f(3+y)]$$

By taking $n = 2$ in Guass formula

$$I = \frac{b-a}{2}\sum_{i=1}^{n} R_i\phi(y_i)$$

$$I = [R_1\phi(y_1) + R_2\phi(y_2)] \quad ...(3)$$

where $R_1 = 1, R_2 = 1$

$$y_1 = -0.57735, y_2 = -y_1$$

$$\phi(y) = f(3+y)$$

$$= (3+y)^2 + (3+y).2 = y^2 + 8y + 15$$

Now $\phi(y_1) = y_1^2 + 8y_1 + 15$

$\Rightarrow$ $\phi(y_1) = (-0.57735)^2 + 8(-0.57735) + 15$

$$= 0.333333 - 4.6188024 + 15 = 10.714531$$

$$\phi(y_2) = (-y_1)^2 + 8(-y_1) + 15 = y_1^2 - 8y_1 + 15 = 19.952135$$

By (3)

$$I = 1 \times 10.714531 + 1 \times 19.952135$$

$$= 30.66666$$

UNIT - III

Q.46. Solve the following equations by Gauss Jordan method. **(2017)**

$$2x + y - 4z = 1$$
$$5x + 9y - 3z = 17$$
$$-8x - 2y + z = -9$$

Ans. In matrix form, we have

$$\begin{bmatrix} 2 & 1 & -4 \\ 5 & 9 & 3 \\ -8 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \\ -9 \end{bmatrix}$$

$$AX = B$$

Consider

$$[A/B] = \begin{bmatrix} 2 & 1 & -4 \\ 5 & 9 & 3 \\ -8 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 17 \\ -9 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & 1/2 & -2 \\ 5 & 9 & 3 \\ -8 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 17 \\ -9 \end{bmatrix}$$

$$R_2 \to R_2 - 5R_1, R_3 \to R_3 + 8R_1$$

$$\sim \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 13/2 & 13 \\ 0 & 2 & -15 \end{bmatrix}\begin{bmatrix} 1/2 \\ 29/2 \\ -5 \end{bmatrix}$$

$$R_3 \to R_2 - \frac{4}{13}R_2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 13/2 & 13 \\ 0 & 0 & -19 \end{bmatrix}\begin{bmatrix} 1/2 \\ 29/2 \\ 103/13 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{2}{26}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 13/2 & 13 \\ 0 & 0 & -19 \end{bmatrix}\begin{bmatrix} 6/13 \\ 29/2 \\ 103/13 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{3}{19}R_3, R_2 \to R_2 + \frac{13}{19}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 03 \\ 0 & 13/2 & 0 \\ 0 & 0 & -19 \end{bmatrix}\begin{bmatrix} -195/247 \\ 29/2 \\ 757/38 \end{bmatrix}$$

$$R_2 \to \frac{2}{13}R_2, R_3 \to \frac{-1}{19}R_3$$

$$- \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-195/247\\29/13\\-757/722\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-195/247\\29/13\\-757/722\end{bmatrix}$$

$$\Rightarrow \qquad x = \frac{-195}{247}, y = \frac{29}{13}, z = \frac{-757}{722}$$

Q.47. Explain matrix inversion method.

Ans. Let A be a non-singular matrix. Let B be another matrix of the same order such that

$$AB = BA = I$$

Where I being a unit matrix of the same order.

There are two methodsL

1. Gauss elimination method: The method involves a unit matrix of the same order as the given matrix A and write it as AI.

Now making simultaneous row operations on AI, we try to convert A into an upper triangular matrix and then to a unit matrix, Ultimately when A is transformed into unit matrix the adjacent matrix gives the inverse of A.

2. Gauss-Jordan method: This is similar to the Gauss elimination method except that instead of first converting A into upper triangular form, it is directly converted into the unit matrix.

Q.48. Explain UV factorization method and hence apply factorization method to solve the equations.

$2x + 3y + z = 9, x + 2y + 3z = 6, 3x + y + 2z = 8$.

Ans. UV Factorization method: This method is based on the fact that every square matrix. A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of matrix A are non-singular.

If $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{vmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{vmatrix} \neq 0 \begin{vmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{vmatrix} \neq 0 \text{ and so on.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \qquad ...(1)$$

Which can be written as AX = B.

Where, $$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}, X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \text{ and } B = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \qquad ...(2)$$

Let A = LU

$$\text{Where } L = \begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 0\end{bmatrix} \text{ and } U = \begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix}$$

Then (1) becomes

$$LUX = B \qquad ...(3)$$

Let $$UX = Y \qquad ...(4)$$

Then (3) becomes

$$LY = B$$

Which is equivalent to

$$y_1 = b_1$$
$$l_{21}y_1 + y_2 = b_2$$
$$l_{31}y_1 + l_{32}y_2 + y_3 = b_3$$

Solving these for y_1, y_2, y_3, then we get Y.

$\therefore$ (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

From which x_3, x_2, and x_1 can be found by back substitution.

Now, we solve

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

It can be written in matrix form as

$$AX = B \qquad ...(1)$$

$$\text{where } A = \begin{bmatrix}2 & 3 & 1\\1 & 2 & 3\\3 & 1 & 2\end{bmatrix}, X = \begin{bmatrix}x\\y\\z\end{bmatrix}, B = \begin{bmatrix}9\\6\\8\end{bmatrix}$$

Let $$A = LU \qquad ...(2)$$

$$\text{where } L = \begin{bmatrix}1 & 0 & 0\\21 & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix} \text{ and u } \begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{31}\end{bmatrix}$$

By equation (2), we get

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{31} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

so that,

(i) $u_{11} = 2, \quad u_{12} = 3 \quad u_{13} = 1$

(ii) $l_{21}u_{11} = 1 \quad l_{31}u_{11} = 3$

$\Rightarrow \quad l_{21} = \frac{1}{2}, \qquad \Rightarrow \quad l_{31} = \frac{3}{2}$

(iii) $l_{21}u_{12} + u_{22} = 2 \;\Rightarrow\; u_{22} = 2 - \frac{3}{2} = \frac{1}{2}$

$l_{21}u_{13} + u_{23} = 3 \;\Rightarrow\; u_{23} = 3 - \frac{1}{2} = \frac{5}{2}$

(iv) $\qquad l_{31}u_{12} + l_{32}u_{22} = 1$

$$\frac{9}{2} + \frac{1}{2}l_{32} = 1$$

$\Rightarrow \qquad l_{32} = 2\left[1 - \frac{9}{2}\right] = -\frac{14}{2} = -7$

(iv) $\qquad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$

$$\left(\frac{3}{2}\right)(1) + (-7)\left(\frac{5}{2}\right) + u_{33} = 2$$

$$\frac{3}{2} - \frac{35}{2} + u_{33} = 2$$

$$u_{33} = 2 - \frac{3}{2} + \frac{35}{2} = 4\frac{-3+35}{2}$$

$$u_{33} = 18$$

Thus, $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$

Let $\qquad UX = Y \qquad \ldots(1)$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then equation (1) becomes

$$LUX = B$$

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$\Rightarrow \qquad y_1 = 9$

$$\frac{1}{2}y_1 + y_2 = 6 \;\Rightarrow\; y_2 = 6 - \frac{9}{2} = \frac{3}{2}$$

$$-y_1 - 7y_2 + y_3 = 8$$

$\Rightarrow \qquad \frac{27}{2} - \frac{21}{2}y_3 = 8 \;\Rightarrow\; y_2 = 8 - 3 = 5$

$\therefore \qquad Y = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$

From (2), we get

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$\Rightarrow \qquad 2x + 3y + z = 9$

$\Rightarrow \qquad x = \frac{35}{18}$

$\Rightarrow \qquad \frac{1}{2}y + \frac{5}{2}z = \frac{3}{2}$

$\Rightarrow \qquad y = \frac{29}{18}$

$$18z = 5$$

$\Rightarrow \qquad z = \frac{5}{18}$

Hence, the solution is $\quad x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$

Q.49. Explain power method. Determine the largest Eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ using power method.

Ans. Power method: If $X_1, X_2, \ldots, X_n$ be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then an arbitrary column vector can be written as

$$X = K_1X_1 + K_2X_2 + \ldots + K_nX_n$$

Then, $\qquad AX = K_1AX_1 + K_2AX_2 + \ldots + K_nAX_n$

$$= K_1\lambda_1 X_1 + K_2\lambda_1 X_2 + \dots + K_n\lambda_n X_n$$

$$A^2X = K_1\lambda_1^2 X_1 + K_2\lambda_2^2 X_2 + \dots + K_n\lambda_n^2 X_n$$

Similarly,

$$A^rX = K_1\lambda_1^r X_1 + K_2^r\lambda_1 X_2 + \dots + K_n\lambda_n^r X_n$$

and

If $|\lambda_1|\,|\lambda_2| > \dots > |\lambda_n|$, then λ_1 is the largest root and the contribution of the term $K_1\lambda_1^r X_1$ to the sum on the right increases with r and therefore every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor K_1.

Thus we start with a column vector X which is as near the solution to possible and evaluate AX which is written as $\lambda_1 X_1$ after normalisation. This gives the first approximation λ_1 to the eigen value and X_1 to the eigen vector. Similarly we evaluate $AX_1 = \lambda_2 X_2$ which gives the second approximation. We repeat this process till $[X_r - X_{r-1}]$ becomes negligible. Then λ_r will be the largest eigen value and X_r, the corresponding eigen vector.

This process for finding the dominant eigen value of a matrix is known as power method.

Now, we determine the largest eigen vector of $\begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}$

Let $$A = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}$$

Let the initial eigen vector be $\begin{bmatrix}1\\0\end{bmatrix}$

then $$AX = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} = 5\begin{bmatrix}1\\0.2\end{bmatrix}$$

So, the first approximation to the eigen value is 5 and the corresponding vector

$$X_1 = \begin{bmatrix}1\\0.2\end{bmatrix}$$

$$AX_1 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.2\end{bmatrix} = \begin{bmatrix}5.8\\1.4\end{bmatrix} = 5.8\begin{bmatrix}1\\0.24\end{bmatrix}$$

$$\lambda_2 = 5.8 \text{ and } X_2 = \begin{bmatrix}1\\0.24\end{bmatrix}$$

$$AX_2 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.24\end{bmatrix} = \begin{bmatrix}5.96\\1.48\end{bmatrix} = 5.96\begin{bmatrix}1\\0.248\end{bmatrix}$$

$$\lambda_3 = 5.96 \text{ and } X_3 = \begin{bmatrix}1\\0.248\end{bmatrix}$$

$$AX_3 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.248\end{bmatrix} = \begin{bmatrix}5.99\\1.49\end{bmatrix} = 5.99\begin{bmatrix}1\\0.248\end{bmatrix}$$

$$\lambda_4 = 5.99 \text{ and } X_4 = \begin{bmatrix}1\\0.248\end{bmatrix}$$

$$AX_4 = \begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}1\\0.248\end{bmatrix} = \begin{bmatrix}5.99\\1.49\end{bmatrix} = 5.99\begin{bmatrix}1\\0.248\end{bmatrix}$$

Hence the largest eigenvalue is 5.99 and the corresponding eigen vector is $\begin{bmatrix}1\\0.248\end{bmatrix}$.

Q.50. Explain Guass Elimination method.

Ans. Guass elimination method is a direct method to find solution of tedious linear algebraic equation. This method produce the exact solution after a finite number of steps.

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system for which the unknowns are found by back substitution.

For this method, produces is

1. Write down the equation in matrix form AX = B.
2. Write argument matrix.
3. Convert the matrix A into an upper triangular form by using only row operations.

Find unknowns by back substitution.

If at any stage of elimination, any one of the pivot elements $a_{11}, a_{22}, \dots, a_{nn}$ vanishes or become very small compared to other elements in that row, then we interchange the i^{th} row (called the pivot now) with any other row below it, preferably with the row having numerically greatest element in the i^{th} column.

Elimination is performed in (n – 1) steps in case of n unknowns. It is possible to count the total number of addition, subtraction, multiplications and divisions called as operation count of the method. For Guass elimination method operation count for n system is $n(n^2 + 3n - 1/3)$.

For large n it is approx. $n^3/3$.

Q.51. Find the natural cubic spline to fit the data.

x	1	2	3	4
y	0	1	0	0

Ans. Since the points are equispaced with h = 1 and n = 3 we obtain.

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}) i = 1, 2$$

$$M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1}), i = 1, 2$$

$$\therefore \quad M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + 2y_2) = 6(-2) = -12$$

and $$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3) = 6$$

Since $M_0 = 0$ and $M_3 = 0$

$$\Rightarrow \quad 4M_1 + M_2 = -12 \text{ and } M_1 + 4M_2 = 6$$

On solving, we get

$$M_1 = \frac{-18}{5}, M_2 = \frac{36}{15}$$

$$M_1 = -3.6M_2 = \frac{12}{5} = 2.4$$

i.e.,

Thus the cubic splines are

$$f(x) = \begin{cases} \frac{6}{10}(-x^3+3x^2-2x), & 1 \le x \le 2 \\ x^3-7.8x^2+19x-3, & 2 \le x \le 3 \\ \frac{4}{10}(-x^3+12x^2-47x+60), & 3 \le x \le 4 \end{cases}$$

Q.52. Solve the system

$$\begin{aligned} 2x+4y+z &= 3 \\ 3x+2y-2z &= -2 \\ x-y+z &= 6 \end{aligned}$$

by using Gauss Elimination method.

Ans.

$$\begin{aligned} 2x+4y+z &= 3 \\ 3x+2y-2z &= -2 \\ x-y+z &= 6 \end{aligned}$$

In Matrix Form:

$$[A/B] = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{3}{2}R_1$$

$$R_3 \to R_3 - \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 2 & 4 & 1 \\ 0 & 4 & 7/2 \\ 0 & 3 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 13/2 \\ -9/2 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{3}{4}R_2$$

$$\sim \begin{bmatrix} 2 & 4 & 1 \\ 0 & 4 & 7/2 \\ 0 & 0 & -25/8 \end{bmatrix}\begin{bmatrix} 3 \\ 13/2 \\ -75/8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x+4y+z &= 3 \\ 4y+\frac{7}{2}z &= \frac{13}{2} \\ \frac{25}{2}z &= \frac{75}{8} \end{aligned}$$

$$\Rightarrow \quad x = 2, y = -1, z = 3$$

Q.53. Find the largest eigen value of the matrix, using Power method.

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Ans. Let $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an approximation vector.

Consider

$$AX_0 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 0 \\ 1/5 \end{bmatrix} = 5X_1$$

$$AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5+0.2 \\ 0 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 0 \\ 0.38 \end{bmatrix}$$

$$= 5.2X_2$$

$$AX_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.38 \end{bmatrix} = \begin{bmatrix} 5.38 \\ 0 \\ 2.9 \end{bmatrix} = 5.38\begin{bmatrix} 1 \\ 0 \\ 0.54 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 5.54 \\ 0 \\ 3.7 \end{bmatrix} = 5.54\begin{bmatrix} 1 \\ 0 \\ 0.66 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.66 \end{bmatrix} = \begin{bmatrix} 5.66 \\ 0 \\ 4.3 \end{bmatrix} = 5.66\begin{bmatrix} 1 \\ 0 \\ 0.75 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 5.75 \\ 0 \\ 4.75 \end{bmatrix} = 5.75\begin{bmatrix} 1 \\ 0 \\ 0.82 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.82 \end{bmatrix} = \begin{bmatrix} 5.82 \\ 0 \\ 5.1 \end{bmatrix} = 5.82\begin{bmatrix} 1 \\ 0 \\ 0.87 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.87 \end{bmatrix} = \begin{bmatrix} 5.87 \\ 0 \\ 5.35 \end{bmatrix} = 5.87\begin{bmatrix} 1 \\ 0 \\ 0.91 \end{bmatrix}$$

$$AX_8 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.91 \end{bmatrix} = \begin{bmatrix} 5.91 \\ 0 \\ 5.55 \end{bmatrix} = 5.91\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix}$$

$$AX_9 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 5.93 \\ 0 \\ 5.65 \end{bmatrix} = 5.93\begin{bmatrix} 1 \\ 0 \\ 0.95 \end{bmatrix}$$

$$AX_{10} = \begin{bmatrix}5 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 5\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0.95\end{bmatrix} = \begin{bmatrix}5.95\\ 0\\ 5.75\end{bmatrix} = 5.95\begin{bmatrix}1\\ 0\\ 0.96\end{bmatrix}$$

$$AX_{11} = \begin{bmatrix}5 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 5\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0.96\end{bmatrix} = \begin{bmatrix}5.95\\ 0\\ 5.8\end{bmatrix} = 5.96\begin{bmatrix}1\\ 0\\ 0.97\end{bmatrix}$$

Since the difference between 5.95 and 5.96 are negligible.

Hence the largest eigen value of matrix A is 5.96

Q.54. Solve the equation using Gauss Jordan method $2x - 2y + 5z = 13$, $2x + 3y + 4z = 20$, $3x - y + 3z = 10$.

Ans. Given equations are

$$2x - 2y + 5z = 13 \quad \text{...(1)}$$

$$2x + 3y + 4z = 20 \quad \text{...(2)}$$

$$3x - y + 3z = 10 \quad \text{...(3)}$$

Dividing (1) by 2, we get

$$x - y + \frac{5}{2}z = \frac{13}{2} \quad \text{...(4)}$$

To eliminate x from (2) and (3) operate [(2) – 2(4)] and [(3) – 3(4)]

$$\Rightarrow \quad x - y + \frac{5}{2}z = \frac{13}{7} \quad \text{...(5)}$$

$$5y - z = 7 \quad \text{...(6)}$$

$$2y - \frac{9}{2}z = \frac{-19}{2} \quad \text{...(7)}$$

Dividing (6) by (5), we get

$$y - \frac{1}{5}z = \frac{7}{5} \quad \text{...(8)}$$

To eliminate y from (5) and (7), operate [(5) – (–1)(8)], [(7) – 2(8)]

$$\Rightarrow \quad x + \frac{23}{10}z = \frac{79}{10} \quad \text{...(9)}$$

$$y - \frac{1}{5}z = \frac{7}{5} \quad \text{...(10)}$$

$$-\frac{41}{10}z = \frac{-123}{10} \quad \text{...(11)}$$

Dividing (11) by $-\frac{41}{10}$

$$\Rightarrow \quad z = 3 \quad \text{...(12)}$$

To eliminate z from (9) and (10) operate $\left[(9) - \frac{23}{10}(12)\right]$ and $\left[(10) + \frac{1}{5}(12)\right]$

$\Rightarrow$ $x = 1, y = 2, z = 3$

Q.55. Solve the following set of equations using Crout's method

$2x_1 + x_2 + x_3 = 7$, $x_1 + 2x_2 + x_3 = 8$, $x_1 + x_2 + 2x_3 = 9$

Ans. $A = \begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}, X = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, B = \begin{bmatrix}7\\ 8\\ 9\end{bmatrix}$

Let A = LU

$$L = \begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$$

$$U = \begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix} = \begin{bmatrix}l_{11} & l_{11}u_{12} & l_{11}u_{13}\\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23}\\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{33} + l_{33}\end{bmatrix}$$

$$\Rightarrow \quad l_{11} = 2, l_{21} = 1, l_{31} = 1$$

$$u_{12} = \frac{1}{2}, u_{31} = \frac{1}{2}$$

$$l_{22} = 2 - l_{21} \cdot u_{12} = \frac{3}{2}$$

$$l_{32} = 1 - l_{31} \cdot u_{12} = 1 - 1 \times \frac{1}{2} = \frac{1}{2}$$

$$u_{23} = \frac{1 - l_{21}u_{13}}{l_{22}} = \frac{1}{3}$$

$$l_{33} = 2 - l_{31}u_{13} = l_{32}u_{33}$$

$$= 2 - \frac{1}{2} - \frac{1}{2} \times \frac{1}{3} = \frac{4}{3}$$

$$L = \begin{bmatrix}2 & 0 & 0\\ 1 & \frac{3}{2} & 0\\ 1 & \frac{1}{2} & \frac{4}{3}\end{bmatrix}$$

$$U = \begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \frac{1}{3}\\ 0 & 0 & 1\end{bmatrix}$$

$AX = B, LU \cdot X = B, UX = V$

$LV = B$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{3}{2} & 0 \\ 1 & \frac{1}{2} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$2V_1 = 7 \Rightarrow V_1 = 3.5$$

$$V_1 + \frac{3}{2}V_2 = 8 \Rightarrow V_2 = 3$$

$$V_1 + \frac{1}{2}V_2 + \frac{4}{3}V_3 = 9 \Rightarrow V_3 = 3$$

$$\Rightarrow V = \begin{bmatrix} 3.5 \\ 3 \\ 5 \end{bmatrix}$$

$$UX = V$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 3.5 \quad \text{...(1)}$$

$$x_2 + \frac{1}{3}x_3 = 3 \quad \text{...(2)}$$

$$\Rightarrow x_3 = 3 \quad \text{...(3)}$$

From eqn. (2) and (3), we have $x_2 = 2 \Rightarrow x_1 = 1$

$\therefore \; x_1 = 1, x_2 = 2, x_3 = 3$

Q.56. Solve 6x + 15y + 55z = 76, 15x + 55y + 225z = 295, 55x + 225y + 979z = 1259 using Cholesky's decomposition method.

Ans. We have

$$6x + 15y + 55z = 76 \quad \text{...(1)}$$

$$15x + 55y + 225z = 295 \quad \text{...(2)}$$

$$55x + 225y + 979z = 1259 \quad \text{...(3)}$$

Let AX = B

$$A = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 255 \\ 55 & 225 & 979 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 76 \\ 295 \\ 1259 \end{bmatrix}$$

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} \cdot l_{kj}}{l_{ii}}$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

Now, $l_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.4495$

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{1.5}{2.4495} = 6.1237$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{55 - (6.1237)^2} = 4.1833$$

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{55}{2.4495} = 24.4537$$

$$l_{32} = \frac{a_{32} - l_{31} \times l_{21}}{l_{22}}$$

$$= \frac{225 - (24.4537 \times 6.1237)}{4.1833} = 20.9165$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$

$$= \sqrt{979 - (22.4537)^2 - (20.9165)^2}\; l_{33} = 6.1101$$

Now,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 6.1101 \end{bmatrix}$$

$$L \times L^T = \begin{bmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 6.1101 \end{bmatrix} \times \begin{bmatrix} 2.4495 & 6.1237 & 22.4537 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.1101 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$

Now, AX = B and A = LL^T

$$\Rightarrow LL^TX = B$$

Let $L^TX = Y$ then $LY = B$

$$\begin{bmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 6.1101 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 76 \\ 295 \\ 1259 \end{bmatrix} \quad ...(4)$$

$$\Rightarrow \quad 2.4495y_1 = 76 \quad ...(5)$$

$$6.1237y_1 + 4.1833y_2 = 295 \quad ...(6)$$

$$22.4537y_1 + 20.9165y_2 + 6.1101y_3 = 1259$$

By back substitution

From (4)

$$2.4495y_1 = 76$$

$$\Rightarrow \quad y_1 = 31.0269$$

Eq. (5) gives

$$6.1237(31.0269) + 4.1833y_2 = 295$$

$$\Rightarrow \quad y_2 = 25.0998$$

Eq. (6) gives

$$22.4537(31.0269) + 20.9165(25.0998) + 6.1101y_3 = 1259$$

$$\Rightarrow \quad y_3 = 6.1101$$

Now, $L^TX = Y$

$$\begin{bmatrix} 2.4495 & 6.1237 & 22.4537 \\ 0 & 4.1883 & 20.9165 \\ 0 & 0 & 6.1101 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 31.0269 \\ 25.0998 \\ 6.1101 \end{bmatrix}$$

$$\Rightarrow 2.4495x + 6.1237y + 22.4537z = 31.0269 \quad ...(7)$$

$$4.1833y + 20.9165z = 25.0998 \quad ...(8)$$

$$6.1101z = 6.1101 \quad ...(9)$$

Again by back substitution Eq. (7) gives

$$z = 1$$

Eq. (8) gives

$$4.1833y + 20.9165 = 25.0998$$

$$\Rightarrow \quad y = 1$$

By (7), we have

$$2.4495x + 6.1237 + 22.4537 = 31.0269$$

$$\Rightarrow \quad x = 1$$

Thus, $x = 1, y = 1, z = 1$

Q.57. The following values of x and y are given

x	1	2	3	4
y	1	2	5	11

Find the cubic splines and evaluate y(1.5) and y′(3).

Ans. Here $h = 1, n = 3$

$$M_{j-1} + 4M_j + M_{j+1} = \frac{6}{h_2}(y_{j+1} - 2y_j + y_{j-1}) \text{ where } j = 1, 2$$

for $j = 1$,

$$M_0 + 4M_1 + M_2 = 6(y_2 - 2y_1 + y_0)$$

$$\Rightarrow \quad M_0 + 4M_1 + M_2 = 6(5 - 2.2 + 1) = 12$$

for $j = 2$, $\quad M_1 + 4M_2 + M_3 = 6(y_3 - 2y_2 + y_1)$

$$= 6(11 - 2 \times 5 + 2) = 18$$

Since $M_0 = 0 = M_3$

$$4M_1 + M_2 = 12$$

$$M_1 + 4M_2 = 18$$

$$\Rightarrow \quad M_1 = 2, M_2 = 4$$

Cubic spline can be obtained by

$$f(x) = \frac{1}{6h}[(x_i - x)^3 M_{j-1} + (x_i - x_{i-1})^3 M_j]$$

$$+ \frac{1}{h}(x_j - x)\left[f(x_{j-1}) - \frac{h^2}{6}M_{j-1}\right] + \frac{1}{h}(x - x_{j-1})\left[f(x_j) - \frac{h^2}{6}M_j\right], j = 1, 2, 3$$

Now, for $j = 1$ in [1, 2]

$$f(x) = \frac{1}{3}(x^3 - 3x^2 + 5x)$$

For $j = 2$ in [2, 3)

$$f(x) = \frac{1}{3}(x^3 - 3x^2 + 5x)$$

For $j = 3$ in [3, 4]

$$f(x) = \frac{1}{3}(-2x^3 + 24x^2 - 76x + 81)$$

$$\therefore \quad y(1.5) = f(1.5) = \frac{11}{8}$$

Also, $y'(3) = \frac{14}{8}$ from both the splines of interval [2, 3] and [3, 4].

UNIT - IV

Q.58 Apply Taylor's method to find y(0.2) from y′ – 4y = 0 given that y(0) = 1. **(2014)**

Ans. Let

$$y' = 4y = f(x, y)$$

$$x_0 = 0, y_0 = 1$$

Here

$$y' = 4y \; y'_0 = 4 y_0 = 4$$

$$y'' = 4 \; y''_0 = 4$$

By Taylor's series

$$y(x) = y_0 + (x - x_0) y_0^1 + \frac{(x-x_0)^2}{26} y_0^{11} + \ldots$$

$$y(x) = 1 + x.4 + \frac{x^2}{2}.4$$

$$y(x) = 1 + 4x + 2x^2$$

At

$$x = 0.2$$

$$y(0.2) = 1 + 4 \times 0.2 + 2 \times (0.2)^2$$

$$= 1.88$$

Q.59 Apply modified Euler's method to find y(0.3) give that $\frac{dy}{dx}$ = x + y, y (0) = 1. **(2014)**

Ans. Let

$$x_0 = 0, y_0 = 1, n = 0.3$$

for

$$x_1 = 0.3$$

$$f(x, y) = x + y$$

$$f(x_0, y_0) = 1$$

Now

$$y^{(1)}_1 = y_0 + h + f(x_0, y_0) = 1 + 0.3 = 1.3$$

$$f[x_1, y_1^{(1)}] = 0.3 + 1.3 = 1.6$$

$\therefore$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.3}{2}[1 + 1.6]$$

$$= 1.39$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.3}{2}[1 + (0.3 + 1.39)] = 1.4035$$

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.3}{2}[1 + (0.3 + 1.4035)] = 1.4055$$

$$y_1^{(5)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(4)})\right]$$

$$= 1 + \frac{0.3}{2}[1 + (0.3 + 1.4055)] = 1.4058$$

$$y_1^{(6)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(5)})]$$

$$= 1 + \frac{0.3}{2}[1 + (0.3 + 1.4058)]$$

$$= 1.4058$$

At

$$x_1 = 0.3, y_1 = 1.4058$$

Q.60 Use Runge-Kutta method of 4th order to solve $\frac{dy}{dx}$ = 1 + y², y(0) = 0 to find y(0.6) in 3 steps.

Ans.

$$\frac{dy}{dx} = 1 + y^2$$

$$x_0 = 0, y_0 = 0, h = 0.2$$

$\therefore$

$$f(x_0, y_0) = 1 + y_0^2 = 1$$

$$k_1 = h_f(x_0, y_0) = 0.2 \times 1 = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, \frac{0 + 0.2}{2}\right)$$

$$= 0.2 f(0.1, 0.1)$$

$$= 0.2(1 + 0.1^2) = 0.202$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 0 + \frac{0.202}{2}\right)$$

$$= 0.2 f(0.1, 0.101)$$

$$= 0.2(1 + 0.101^2) = 0.202$$

$$= 0.2 + f(0.2, 0.202)$$
$$= 0.2(1 + 0.202^2) = 0.208$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.2026$$

At $x_1 = 0.2, y_1 = y_0 + k$
$$= 0.2026$$

For $x_1 = 0.2, y_1 = 0.2026$

$$f(x_1, y_1) = 1 + y_1^2 = 1.0410$$
$$k_1 = h f(x_1, y_1) = 0.2 \times 1.0410 = 0.2082$$
$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.2 + \left(0.2 + \frac{0.2}{2}, 0.2026 + \frac{0.2082}{2}\right)$$
$$= 0.2 f(0.3, 0.3067)$$
$$= 0.2(1 + 0.3067^2) = 0.2188$$
$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.2 f\left(0.2 + \frac{0.2}{2}, 0.2026 + \frac{0.2188}{2}\right)$$
$$= 0.2 f(0.3, 0.312)$$
$$= 0.2(1 + 0.312^2) = 0.2194$$
$$k_4 = h f(x_1 + h, y_1 + k_3)$$
$$= 0.2 f(0.4, 0.422)$$
$$= 0.2(1 + 0.422^2) = 0.2356$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.2200$$

At $x_2 = 0.4, y_2 = y_1 + k = 0.4226$

$$f(x_2, y_2) = 1 + y_2^2 = 1.1785$$
$$k_1 = h f(x_2, y_2)$$
$$= 0.2 \times 1.1785 = 0.2357$$

$$= 0.2 \times 1.1785 = 0.2357$$
$$k_2 = h f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$
$$= 0.2 f\left(0.4 + \frac{0.2}{2}, 0.4226 + \frac{0.2357}{2}\right)$$
$$= 0.2 f(0.5, 0.54045)$$
$$= 0.2584$$
$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$
$$= 0.2 f(0.5, 0.5518)$$
$$= 0.2608$$
$$k_4 = h f(x_2 + h, y_2 + k_3)$$
$$= 0.2 f(0.6, 0.6834) = 0.2934$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.26125$$

At $x_3 = 0.6$

$$y_3 = y_2 + k$$
$$= 0.4226 + 0.26125$$
$$= 0.6838$$

Q.61 Using modified Euler's method, find y(0.2) and y(0.4) correct to 3 decimal places, given that **(2015)**

$$\frac{dy}{dx} = y - x^2, y(0) = 1.$$

Ans. Given $\frac{dy}{dx} = y - x^2, x_0 = 0, y_0 = 1, h = 0.2$

for $x_1 = 0.2$

$$f(x_0, y_0) = y_0 - x_0^2 = 1$$

Now

$$y'_1 = y_0 + h + (x_0, y_0) = 1 + 0.2 = 1.2$$
$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})$$
$$= 1 + \frac{0.2}{2}[1 + (1.2 - 0.2^2)] = 1.216$$
$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})$$

$$= 1+\frac{0.2}{2}[1+(1.216-0.2^2)]$$

$$= 1.2176$$

$$y_1^{(4)} = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(3)})]$$

$$= 1+\frac{0.2}{2}[1+(1.2176-0.2^2)]$$

$$= 1.2177$$

Thus, at $x_1 = 0.2, y_1 = 1.217$

At $x_2 = 0.4$

$$y_2^{(1)} = y_1 + h\, f(x_1, y_1)$$

$$= 1.217 + 0.2\,(1.217 - 0.2^2) = 1.4524$$

$$y_2^{(2)} = y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(1)})]$$

$$= 1.217+\frac{0.2}{2}[1.177+(1.4524-0.4^2)]$$

$$= 1.4639$$

$$y_2^{(3)} = y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(2)})]$$

$$= 1.217+\frac{0.2}{2}[1.77+(1.4634-0.4^2)]$$

$$= 1.46509$$

$$y_2^{(4)} = y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(3)})$$

$$= 1.217+\frac{0.2}{2}[1.177+(1.46509-0.4^2)]$$

$$= 1.4652$$

Thus at $x_2 = 0.4, y_2 = 1.465.$

Q.62 Solve the differential equation $\frac{dy}{dx}=\frac{y-x}{y+x}$ by Euler's method with the initial condition y(o) = 1 for x = 0.06 taking interval of differencing h = 0.02 **(2015)**

Ans.

$$\frac{dy}{dx} = f(x,y)=\frac{y-x}{y+x}$$

$$x_o = 0, y_o = 1, h = 0.02$$

$$y_1 = y_o+hf(x_o,y_o) = 1+0.02\left(\frac{1-0}{1+0}\right) = 1.02$$

$\therefore$

$$y(0.02) = 1.02$$

Again

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$= 1.02+0.02\left(\frac{1.02-0.02}{1.02+0.02}\right)$$

$$= 1.02+\frac{0.02}{1.04} = 1.0392$$

$\therefore$

$$y(0.04) = 1.00\,392$$

Again

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.0392+0.02\left(\frac{1.0392-0.04}{1.0392+0.04}\right)$$

$$= 1.0392+0.02\left(\frac{0.9992}{1.0792}\right)$$

$$= 1.0577$$

$\therefore$

$$y(0.06) = 1.0577$$

Q.63 Apply the fourth order R.K method to solve $\frac{dy}{dx} = x^2 + y^2$, y (0) = 1 Taking h = 0.1 and determine approximation to y(0.1) and y(0.2) correct to 4 decimal places. **(2015)**

Ans. Here

$$f(x,y) = x^2 + y^2, x_0 = 0, y_0 = 1, h = 0.01$$

Now

$$k_1 = hf(x_0, y_0) = 0.1\,(0^2 + 1^2) = 0.1$$

$$k_2 = hf\left(x_o+\frac{h}{2},y_o+\frac{k_1}{2}\right) = 0.1f(0.05,1.05)$$

$$= 0.1[(0.05)^2+(1.05)^2] = 0.1105$$

$$k_3 = hf\left(x_o+\frac{h}{2},y_o+\frac{k_2}{2}\right) = 0.1f(0.05,1.0553)$$

$$= 0.1[(0.05)^2+(1.0553)^2] = 0.1116$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

$$= 0.1\,f(0.1,\ 1.1116) = 0.1\,[0.1^2 + (1.1116)^2]$$

$$= 0.1246$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1 + 2\times 0.1105 + 2\times 0.1116 + 0.1246]$$

$$= 0.1115$$

At $x = x_1 = 0.1, y_1 = y_0 + k$

$$= 1 + 0.1115 = 1.115$$

To find y_2 at $x_2 = 0.2, x_1 = 0.1, y_1 = 1.1115, h = 0.1$

$$k_1 = h f(x_1, y_1) = 0.1\ (0.1^2 + 1.1115^2) = 0.1245$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f[0.15, 1.1738]$$

$$= 0.1\,[0.15^2 + 1.1738^2] = 0.1400$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f[0.15, 1.1815]$$

$$= 0.1\,[0.15^2 + 1.1815^2] = 0.1418$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1\,f(0.2, 1.2533)$$

$$= 0.1\,[0.2^2 + 1.2533^2] = 0.1611$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1245 + 2\times 0.14 + 2\times 0.1418 + 0.1611]$$

$$= 0.1415$$

At $x_2 = 0.2, y_2 = y_1 + k = 1.1115 + 0.1415$

$$y_2 = 1.2530$$

Q.64 Use Picard method to approximate y when $x = 0.2$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = x - y$. **(2016)**

Ans. $\frac{dy}{dx} = x - y = f(x, y)$

Given $x_0 = 0, y_0 = 1$

By Picard's method

$$y = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y = 1 + \int_0^x (x-1)\,dx = 1 + \frac{x^2}{2} - x$$

Ist iteration: $y_1^{(1)}(0.2) = 1 + \frac{(0.2)^2}{2} - 0.2$

$$= 0.82$$

IInd iteration $y_1^{(2)} = 1 + \int_0^x \left(1 + \frac{x^2}{2} - x\right)dx$

$$= 1 + x + \frac{x^3}{6} - \frac{x^2}{2}$$

$$y_1^{(2)}(0.2) = 1 + 0.2 + \frac{(0.2)^3}{6} - \frac{(0.2)^2}{2}$$

$$= 1.1813$$

IIIrd iteration: $y_1^{(3)} = 1 + \int_0^x \left(1 + x - \frac{x^2}{2} + \frac{x^3}{6}\right)dx$

$$y_1^{(3)} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

$$y_1^{(3)}(0.2) = 1.2187$$

IVth iteration: $y_1^{(4)} = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right)dx$

$$y_1^{(4)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^5}{120}$$

$$y_1^{(4)}(0.2) = 1.221$$

$$y = 1.221 \text{ at } x = 0.2.$$

Q.65 Use Taylor's series method to solve the equation $\frac{dy}{dx} = -xy, y(0) = 1$ at $x = 0.1$. **(2016)**

Ans. Let $y' = -xy = f(x, y)$

Here

$x_0 = 0,$	$y_0 = 1$
$y' = -xy,$	$y_0' = 0$
$y'' = -xy' - y,$	$y_0'' = -x_0y_0' - y_0 = -1$
$y''' = -xy'' - y' - 1,$	$y_0''' = -x_0y_0'' - y_0' - 1 = -1$
$y^{iv} = -xy''' - y'' - y'',$ $= -xy''' - 2y'',$	$y_0^{iv} = -x_0y_0''' - 2y_0'' = 2$

By Taylor's series

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots$$

$$y(x) = 1 + x \times 0 + \frac{x^2}{2}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(2) + \ldots$$

$$y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \ldots$$

At $x = 0.1$

$$y(0.1) = 1 - \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} + \frac{(0.1)^4}{12} + \ldots$$

$= 0.9948$ (Approx).

Q.66 Use the Runge – Kutta fourth order method to find $y_{(0.2)}$ with h = 0.1 for the initial value problem , $\frac{dy}{dx} = \sqrt{x+y}$ y (0) = 1. **(2016)**

Ans. $\frac{dy}{dx} = \sqrt{x+y} = f(x, y)$

$$x_0 = 0,\ y_0 = 1,\ h = 0.1$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1\sqrt{x_0 + y_0} = 0.1\sqrt{0+1} = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1\,f(0.05, 1.05) = 0.1 \times 1.0488 = 0.1049$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\,f(0.05, 1.0525) = 0.1 \times 1.05$$

$$= 0.105$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1\,f(0.1, 1.105) = 0.1098$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.1049 \times 2 + 2 \times 0.105 + 0.1098)$$

$$= 0.1049$$

At $x_1 = 0.1,\ y_1 = 1 + 0.1049 = 1.1049$

At $x_2 = 0.2$

$$k_1 = hf(x_1, y_1) = 0.1\sqrt{0.1 + 1.1049}$$

$$= 0.1098$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.15, 1.1598) = 0.1144$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1\,f(0.15, 1.1621) = 0.1145$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1\,f(0.2, 1.2194) = 0.1191$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1098 + 2 \times 0.1144 + 2 \times 0.1145 + 0.1191)$$

$$= 0.1144.$$

At $x_2 = 0.2,\quad y_2 = 1.1049 + 0.1144$

$$= 1.2193.$$

Q.67 Using fourth order R – K method to solve $\frac{dy}{dx} = xy + y^2, y(0) = 1$, Take interval h = 0.1. Compute y(0.1) and y(0.2). **(2017)**

Ans. Here $f(x, y) = xy + y^2$

$h = 0.1, x_0 = 0, y_0 = 1$

Now, $K_1 = h\,f(x_0, y_0) = 0.1\,(0 + 1) = 0.1$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1f(0.05, 1.05)$$

$$= 0.1\,(0.05 \times 1.05 + 1.052) = 0.1155$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1155}{2}\right)$$

$$= 0.1f(0.05, 1.0578) = 0.1172$$

$$K_4 = h\,f(x_0 + h, y_0 + K_3) = 0.1\,f(0.1, 1.1172)$$

$$= 0.1360$$

$\therefore$ $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$$= \frac{1}{6}(0.1 + 2 \times 0.1155 + 2 \times 0.1172 + 0.1360)$$

$= 0.1169$

Thus at $x_1 = 0.1, y_1 = y_0 + K = 1 + 0.1169 = 1.1169$

For x = 0.2

$$K_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1169) = 0.1359$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1169 + \frac{0.1359}{2}\right) = 0.1\,f(0.15, 1.1849) = 0.1582$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.1f\left(0.1 + \frac{0.1}{2}, 1.1169 + \frac{0.1582}{2}\right) = 0.1\,f(0.15, 1.1960) = 0.1610$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.1\,f(0.2, 1.2779) = 0.1889$$

Now

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = \frac{1}{6}(0.1359 + 2 \times 0.1582 + 2 \times 0.1610 + 0.1889)$$

$$K = 0.1605$$

At

$$x_2 = 0.2$$

$$y_2 = y_1 + K = 1.1169 + 0.1605$$

$$y_2 = 1.2774.$$

Q.68 Solve the following by Euler's modified method $\frac{dy}{dx} = \log(x + y)$, y (0) = 2 at x = 0.2 and 0.4 with h = 0.2. (2018)

Ans. Let $x_0 = 0, y_0 = 2, h = 0.2$

for

$$x_1 = 0.2$$

$$f(x, y) = \log(x + y)$$

$$f(x_0, y_0) = \log(0 + 2) = 0.3010$$

Now

$$y_1(1) = y_0 + hf(x_0, y_0) = 2 + 0.2 \times 0.3010 = 2.0602$$

$$f[x_1, y_1(1)] = \log(0.2 + 2.0602) = \log(2.2602) = 0.3541$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 2 + \frac{0.2}{2}[0.3010 + 0.3541] = 2.0655$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] = 2 + \frac{0.2}{2}[0.3010 + \log(0.2 + 2.0655)] = 2.0656$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] = 2 + \frac{0.2}{2}[0.3010 + \log(0.2 + 2.0656)] = 2.0656$$

$\therefore$ at $x_1 = 0.2, y_1 = 2.0656$

To find y_2 at $x_2 = 0.4$, h = 0.2

$$y_2^{(1)} = y_1 + hf(x_1, y_1) = 2.0656 + 0.2 \log(0.2 + 2.0656) = 2.1366$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right] = 2.0656 + \frac{0.2}{2}[0.3552 + \log(0.4 + 2.1366)] = 2.1415$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2) y_2^{(2)}\right] = 2.0656 + \frac{0.2}{2}[0.3552 + \log(0.4 + 2.1415)] = 2.1416$$

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2) y_2^{(3)}\right] = 2.0656 + \frac{0.2}{2}[0.3552 + \log(0.4 + 2.1416)] = 2.1416$$

At $x_2 = 0.4$, $y2 = 2.1416$

Thus

x	0.2	0.4
y	2.0656	2.1416

Q.69 Apply Range-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with y (0) = 1 at x = 0.2, 0.4. (2018)

Ans.

$$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}, x_0 = 0, y_0 = 1, h = 0.2$$

$$\therefore f(x_0, y_0) = \frac{1 - 0}{1 + 0} = 1$$

$$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0 + 0.1, 1 + 0.1)$$

$$= 0.2f(0.1, 1.1) = 0.2\frac{(1.1^2 - 0.1^2)}{(1.1^2 + 0.1^2)}$$

$$= 0.1967$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f\left(0 + 0.1, 1 + \frac{0.1967}{2}\right)$$

$$= 0.2f(0.1, 1.0984) = 0.2\left(\frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2}\right)$$

$$= 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0 + 0.2, 1 + 0.1967)$$

$$= 0.2f(0.2, 1.1967) = 0.2\left(\frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2}\right)$$

$$= 0.1891$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.2 + 2 \times 0.1967 + 2 \times 0.1967 + 0.1891]$$

$$= 0.1960$$

At $x_1 = 0.2, y_1 = y_0 + k = 1 + 0.1960 = 1.196$

Now $x_2 = 0.4, y_1 = 1.196, x_1 = 0.2, h = 0.2$

$$f(x_1, y_1) = \frac{1.196^2 - 0.2^2}{1.196^2 + 0.2^2} = 0.9456$$

$$k_1 = hf(x_1, y_1) = 0.2 \times 0.9456 = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f\left(0.2 + 0.1, 1.196 + \frac{0.1891}{2}\right)$$

$$= 0.2f(0.3, 1.2906) = 0.2\left(\frac{1.2906^2 - 0.3^2}{1.2906^2 + 0.3^2}\right)$$

$$= 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.2f\left(0.3, 1.196 + \frac{0.1795}{2}\right) = 0.2f(0.3, 1.2858)$$

$$= 0.2\left(\frac{1.2858^2 - 0.3^2}{1.2858^2 + 0.3^2}\right) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$k_4 = 0.2f(0.2 + 0.2, 1.196 + 0.1793)$$

$$= 0.2f(0.4, 1.3753) = 0.2\left(\frac{1.3753^2 - 0.4^2}{1.3753^2 + 0.4^2}\right)$$

$$= 0.1688$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1891 + 2 \times 0.1795 + 2 \times 0.1793 + 0.1688]$$

$$= 0.1793$$

At $x_2 = 0.4, y_2 = 1.196 + 0.1793 = 1.3753$

Q.70. Using Range-Kutta Method of Order 4, find y for x = 0.1, 0.2, 0.3. Given that:

$$\frac{dy}{dx} = xy + y^2, y(0) = 1$$

Continue the solution at x = 0.4 using Milne's method.

Ans. $f(x, y) = xy + y^2$

To find $y(0, 1)$

Here $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = (0.1)f(0.1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= (0.1)f(1.05, 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= (0.1)f(0.05, 1.0577) = 0.1172$$

$$k_4 = h(f(x_0 + h, y_0 + k_3) = (0.1)f(0.01, 1.1172) = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$

To find y(0.2)

Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$

$$k_1 = h(f(x_1, y_1)) = (0.1)\,f(0.1, 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= (0.1)f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$$= (0.1)\,f(0.15, 1.1959) = 0.1609$$

$$k_4 = h(f(x_1 + h, y_1 + k_3)) = (0.1)\,f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

Thus $\quad y(0.2) = y_2 = y_1 + k = 1.2733$

To find y(0.3)

Here $x_2 = 0.2, y_2 = 1.2773, h = 0.1$

$$k_1 = hf(x_2, y_2) = (0.1)\,f(0.2, 1.2773) = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= (0.1)\,f(0.25, 1.3716) = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= (0.1)\,f(0.25, 1.3885) = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3, 1.5048) = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2267$$

Thus $\quad y(0.3) = y_3 = y_2 + k = 1.504$

Now the starting values for the Milne's method are:

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the predictor

$$y_4 = y_0 = \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$x_4 = 0.4, y_4 = 1.8344, f_4 = 4.0988$$

and the corrector

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4 = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8386$$

$$f_4 = 4.1159$$

Again using the corrector

$$y_4 = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159] = 1.8391$$

$$f_4 = 4.1182$$

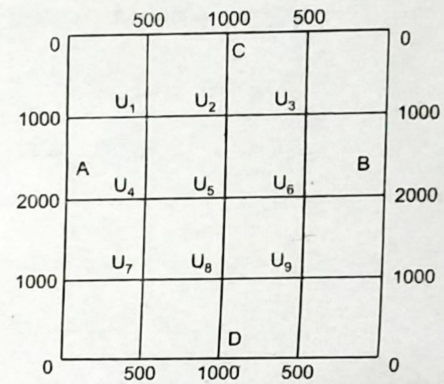
Again using the corrector

$$y_4 = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182] = 1.8392$$

Hence $\quad y(0.4) = 1.8392$

Q.71. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square meshwith boundary, values or shown:

Ans.



Let $u_1, u_2, u_3, \ldots, u_9$ be the value of u at the interior mesh-points.

Since the boundary values of u are symmetrical about AB

$$u_1 = u_7, u_2 = u_8, u_3 = u_9$$

Also, the value of u beign symmetrical abov. CD.

$$u_1 = u_3, u_4 = u_6, u_7 = u_9$$

Thus it is sufficient to find the values u_1, u_2, u_4, u_5

$$u_5 = \frac{1}{4}[1000 + 2000 + 1000 + 2000] = \frac{6000}{4} = 1500$$

$$u_1 = \frac{1}{4}[0 + 1500 + 1000 + 2000] = \frac{1}{4}[4500] = 1125$$

$$u_2 = \frac{1}{4}[1000 + 1125 + 1500 + 1125] = 1188$$

$$u_4 = \frac{1}{4}[1125 + 2000 + 1125 + 1500] = 1438$$

Now, we have got the rough values at all interior mesh pts. We will improve the values by using Gauss-seidal method (or SFPF).

First Iteration:

$$u_1^{(1)} = \frac{1}{4}[500 + 1000 + u_4^{(0)} + u_2^{(0)}]$$
$$= \frac{1}{4}[500 + 1000 + 1438 + 1188] = 1032$$
$$u_2^{(1)} = \frac{1}{4}[1000 + u_1^{(0)} + u_5^{(0)} + u_3^{(0)}]$$
$$= \frac{1}{4}[1000 + 1032 + 1500 + 1125] = 1164$$
$$u_4^{(1)} = \frac{1}{4}[u_1^{(1)} + 2000 + u_7^{(0)} + u_5^{(0)}]$$
$$= \frac{1}{4}[1032 + 2000 + 1125 + 1500] = 1414$$
$$u_5^{(1)} = \frac{1}{4}[u_2^{(1)} + u_4^{(1)} + u_8^{(0)} + u_6^{(0)}]$$
$$= \frac{1}{4}[1164 + 1414 + 1188 + 1438] = 1301$$

Second Iteration:

$$u_1^{(2)} = \frac{1}{4}[1000 + 1164 + 500 + 1414] = 1020$$
$$u_2^{(2)} = \frac{1}{4}[1020 + 1032 + 1000 + 1301] = 1088$$
$$u_4^{(2)} = \frac{1}{4}[2000 + 1301 + 1020 + 1032] = 1388$$
$$u_5^{(2)} = \frac{1}{4}[1338 + 1414 + 1088 + 1164] = 1251$$

Third Iteration:

$$u_1^{(3)} = \frac{1}{4}[1000 + 1088 + 500 + 1338] = 982$$
$$u_2^{(3)} = \frac{1}{4}[982 + 1020 + 1000 + 1251] = 1063$$
$$u_4^{(3)} = \frac{1}{4}[2000 + 1251 + 982 + 1020] = 1313$$
$$u_5^{(3)} = \frac{1}{4}[1313 + 1338 + 1063 + 1088] = 1201$$

Fourth Iteration:

$$u_1^{(4)} = \frac{1}{4}[1000 + 1063 + 500 + 1313] = 969$$
$$u_2^{(4)} = \frac{1}{4}[969 + 982 + 1000 + 1201] = 1038$$
$$u_4^{(4)} = \frac{1}{4}[2000 + 1201 + 969 + 982] = 1288$$
$$u_5^{(4)} = \frac{1}{4}[1288 + 1313 + 1038 + 1063] = 1176$$

Fifth Iteration:

$$u_1^{(5)} = \frac{1}{4}[1000 + 1038 + 500 + 1288] = 957$$
$$u_2^{(5)} = \frac{1}{4}[957 + 969 + 1000 + 1176] = 1026$$
$$u_4^{(5)} = \frac{1}{4}[2000 + 1176 + 957 + 969] = 1276$$
$$u_5^{(5)} = \frac{1}{4}[1276 + 1288 + 1026 + 1038] = 1157$$

Similarly,

$$u_1^{(6)} = 951,\ u_2^{(6)} = 1016,\ u_4^{(6)} = 1266,\ u_5^{(6)} = 1146$$
$$u_1^{(7)} = 946,\ u_2^{(7)} = 1011,\ u_4^{(7)} = 1260,\ u_5^{(7)} = 1138$$
$$u_1^{(8)} = 943,\ u_2^{(8)} = 1007,\ u_4^{(8)} = 1257,\ u_5^{(8)} = 1134$$
$$u_1^{(9)} = 941,\ u_2^{(9)} = 1005,\ u_4^{(9)} = 1255,\ u_5^{(9)} = 1131$$
$$u_1^{(10)} = 940,\ u_2^{(10)} = 1003,\ u_4^{(10)} = 1253,\ u_5^{(10)} = 1129$$
$$u_1^{(11)} = 939,\ u_2^{(11)} = 1002,\ u_4^{(11)} = 1252,\ u_5^{(11)} = 1128$$
$$u_1^{(12)} = 939,\ u_2^{(12)} = 1001,\ u_4^{(12)} = 1251,\ u_5^{(12)} = 1126$$

Thus there is negligible difference between the values obtained in the 11th and 12th iterations.

Hence $u_1 = 939$, $u_2 = 1001$, $u_4 = 1251$, $u_5 = 1126$

Q.72. Apply Adams-Bashforth Method, to find a solution to the differential equation $\frac{dy}{dx} = x^2(1 + y)$ at $x = 1.4$ given $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979$.

Ans. Here $h = 0.1$ and $f(x, y) = x^2(1 + y)$

$x_0 = 1, y_0 = 1$

$$f(x_0, y_0) = x_0^2(1 + y_0) = 2$$

$x_1 = 1.1, y_1 = 1.233$

$$f(x_1, y_1) = x_1^2(1 + y_1) = 2.702$$

$x_2 = 1.2, y_2 = 1.548$

$$f(x_2, y_2) = x_2^2(1 + y_2) = 3.669$$

$x_3 = 1.3, y_3 = 1.979$

$$f(x_3, y_3) = x_3^2(1 + y_3) = 5.035$$

Use predictor formula to find y(1.4)

$$y_4^p = y_3 + \frac{h}{24}[-9f(x_0, y_0) + 37f(x_1, y_1) - 59f(x_2, y_2) + 55f(x_3, y_3)]$$

$$y_4^p = 1.979 + \frac{0.1}{24}[-9 \times 2 + 37 \times 2.702 - 59 \times 3.669 + 55 \times 5.035] = 2.573$$

Use corrector formula

$$y_4^c = y_3 + \frac{h}{24}[f(x_1, y_1) - 5f(x_2, y_2) + 19f(x_3, y_3) + 9f(x_4, y_4^P)]$$

$$f(x_4, y_4^P) = x_4^2(1 + y_4^P) = 7.004$$

$$y_4^c = 1.979 + \frac{0.1}{24}[2.702 - 5 \times 3.669 + 19 \times 5.035 + 9 \times 7.004]$$

$$\Rightarrow \quad = 2.575$$

$\therefore$ Final value at x = 1.4 is y = 2.575

Q.73. Evaluate the pivotal values of the following equations taking h = 1 and upto one half of the period of vibrations $16\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

given $u(0, t) = u(5, t) = 0$

$u(x, 0) = x^2(5 - x)$

$u_t(x, 0) = 0$

Ans. $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ is the equation of vibrating string of length l, having period of vibration $= \frac{2l}{a}$

Here $l = 5$, $a^2 = 16$

$$\text{Period} = 2 \times \frac{5}{4} = \frac{5}{2} \text{ sec.}$$

Evaluate u upto to $t = \frac{5}{4}$ sec.

$$16u_{xx} = u_{tt}$$

$$\Rightarrow \quad \frac{16(u_{i-1} - 2u_{i,j} + u_{i-1,j})}{h^2} = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

Taking h = 1, we get

$$16k^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j-1}) = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

$$\Rightarrow \quad u_{i,j+1} = 2(1 - 16k^2)u_{i,j} + 16k^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

Choose k, so that coefficient of $u_{i,j} = 0$

$$16k^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j-1}) = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

$$\Rightarrow \quad u_{i,j+1} = 2(1 - 16k^2)u_{i,j} + 16k^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

Choose k, so that coefficient of $u_{i,j} = 0$

$$1 - 16k^2 = 0 \Rightarrow k^2 = \frac{1}{16} \Rightarrow k = \frac{1}{4}$$

$\Rightarrow$

By above equation

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \quad ...(2)$$

As $k = \frac{1}{4} < h = 1$

$\therefore$ Solution (2) is convergent

Also initial conditions

$$u(0, t) = 0 = u(5, t)$$

$$u_{0,j} = 0 \text{ and } u_{n,j} = 0 \quad ...(3)$$

Thus first and last column enteries giving values of $u_{i,j}$ are all zero.

The condition $u_t(x, 0) = 0$

$$\Rightarrow \quad \frac{u_{i,j+1} - u_{i,j}}{k} = 0 \text{ when } t = 0$$

i.e., when j = 0

$$\Rightarrow \quad u_{i,1} - u_{i,0} = 0 \text{ or } u_{i,1} = u_{i,0} \quad ...(4)$$

Also initial condition

$$u(x, 0) = x^2(5 - x)$$

$$\Rightarrow \quad u_{i,0} = i^2(5 - i) \quad ...(5)$$

Putting i = 1, 2, 3, 4 in (5), we get

$$u_{1,0} = 1(5 - 1) = 4$$

$$u_{2,0} = 2^2(5 - 2) = 12$$

$$u_{3,0} = 3^2(5 - 3) = 18$$

$$u_{4,2} = 4^2(5 - 4) = 16$$

First row is complete.

But $u_{i,1} = u_{i,0}$

Second row is same as first row.

Again putting j = 1 in (2)

$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0} \quad ...(6)$$

Put i = 1, 2, 3, 4 in (6), we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,2}$$
$$= u_{2,0} + 0 - 4 = 12 - 4 = 8$$

$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0}$$
$$= u_{3,0} + u_{1,0} - 12 = 18 + 4 - 12 = 10$$

$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0}$$
$$= u_{4,0} + u_{2,0} - 18 = 16 + 12 - 18 = 10$$

$$u_{4,2} = u_{5,1} + u_{3,1} - u_{4,0}$$
$$= u_{5,0} + u_{3,0} - u_{4,0} = 0 + 18 - 16 = 2$$

The table is as follows:

j \ i	0	1	2	3	4	5
0	0	4	12	8	16	0
1	0	4	12	18	16	0
2	0	0 + 12 – 4 = 8	10	10	2	0
3	0	6	6	–6	–6	0
4	0	–2	–10	–10	–8	0
5	0	–16	–18	–12	–4	0