Unit - IV

Numerical Solution of Ordinary Differential Equations

Taylor's Series Method:-

$$\boxed{y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} + \cdots}$$

Solve $\frac{dy}{dx} = x+y$, $y(1) = 0$, numerically

upto $x = 0.2$ with $h = 0.1$

We have $x_0 = 1$, $y_0 = 0$

$\frac{dy}{dx} = y' = x+y \Rightarrow y_0' = 1+0 = 1$

$\frac{d^2y}{dx^2} = y'' = 1+y' \Rightarrow y_0'' = 1+1 = 2$

$\frac{d^3y}{dx^3} = y''' = y'' \Rightarrow y_0''' = 2$

$\frac{d^4y}{dx^4} = y^{iv} = y''' \Rightarrow y^{iv} = 2$

Substituting the above values in

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \frac{h^5}{5!} y_0^{v} \quad (1)$$

$$y_1 = 0 + (0.1) + \frac{(0.1)^2}{2} \cdot 2 + \frac{(0.1)^3}{6} \cdot 2 + \frac{(0.1)^4}{24} \cdot 2$$

$$y_1 = 0.11033047$$

$y_1 = y(0.1) \approx 0.110$

Now $x_1 = x_0 + h = 1 + 0.1 = 1.1$

$y_1' = x_1 + y_1 = 1.1 + 0.110 = 1.21$

$y_1'' = 1 + y = 1 + 1.21 = 2.21$

$y_1''' = y_1'' = 2.21$

$y_1^{iv} = 2.21$

Substituting the above values in (i) we get

$$y_2 = 0.110 + (0.1)(1.21) + \frac{(0.1)^2}{2}(2.21) + \frac{(0.1)^3}{6}(2.21) + \frac{(0.1)^4}{24}(2.21) + \ldots$$

$\therefore y_2 = 0.24205$

$\therefore y(0.2) = 0.242$

Given $\frac{dy}{dx} = 1 + xy$ with initial condition that $y = 1$, $x = 0$ compute $y(0.1)$ correct to four places of decimal by using Taylor's series Method.

$\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$

$\therefore y'(0) = 1 + 0 \times 1 = 1$

Differentiating the given eq^n w.r.t. 'x' we get

$$\frac{d^2y}{dx^2} = y + x\frac{dy}{dx}$$

$$y_0'' = 1 + 0 \times 1 = 1 + 0 = 1$$

IIly $$\frac{d^3y}{dx^3} = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$$

$$\Rightarrow \quad y_0''' = 2$$

and $$\frac{d^4y}{dx^4} = x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad y_0^{iv} = 3$$

from Taylor's Series Method, we have

$$y_1 = 1 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{24} y_0^{iv} + \ldots$$

$$y(0.1) = 1 + (0.1) \times 1 + \frac{(0.1)^2}{2} \times 1 + \frac{(0.1)^3}{6} \times 2 + \frac{(0.1)^4}{24} \times 3 + \ldots$$

$$= 1.1053425$$

$y(0.1) = 1.1053$ correct to four decimal places.

Apply the Taylor's series method to find the value of $y(1.1)$ and $y(1.2)$ correct to three decimal places given that $\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$ taking the first three terms of the Taylor's series expansion.

Given

$$\frac{dy}{dx} = xy^{1/3}, \quad y_0 = 1, \quad x_0 = 1$$

$$h = 0.1$$

$$y_0' = x_0 y_0^{1/3} = (1)^{1/3} = 1$$

Differentiating the given equation w.r.t. x we get

$$\frac{d^2y}{dx^2} = \frac{1}{3} x y^{-2/3} \frac{dy}{dx} + y^{1/3}$$

$$= \frac{1}{3} x^2 y^{-1/3} + y^{1/3}$$

$$y_0'' = \frac{1}{3}(1 \times 1) + 1 = \frac{4}{3}$$

Taking the first three terms of the Taylor's formula we get

$$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' \quad \text{(i)}$$

$$y_1 = y(1.1) = 1 + (0.1) + \frac{(0.1)^2}{2} \times \frac{4}{3} = 1.1066$$

$\therefore\ y(1.1) = 1.1066$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

$y_1' = (x_1 \cdot y_1^{1/3}) = (1.1) * (1.1066)^{1/3} = 1.138$

$y_1'' = \frac{1}{3} x_1^2 y_1^{-1/3} + y_1^{1/3}$

$= \frac{1}{3}(1.1)^2 (1.1066)^{-1/3} + (1.1066)^{1/3} = 1.4249$

Substituting in

$y_2 = y_1 + h y_1' + \frac{h^2}{2} y_1''$

$y_2 = y(1.2) = 1.1066 + 0.1 \times 1.138$
$+ \frac{(0.1)^2}{2} \times 1.4249$

$= 1.2275245 \approx 1.228$

$y_2 = y(1.2) = 1.228$

Using Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation

$\frac{dy}{dx} = 2y + 3e^x$, $y(0) = 0$, Compare the numerical solution obtained with exact solution.

Sol^n Taylor's Series is

$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0)$
$+ \ldots$

Here $y(x) = 1$ when $x_0 = 0$

Given $y'(x) = 2y + 3e^{x}$ $\quad y'(x_0) = 2(0) + 3e^{0} = 0+3=$

$y''(x) = 2y' + 3e^{x}$ $\quad y''(x_0) = 2(3) + 3e^{0} = 6+3=9$

$y'''(x) = 2y'' + 3e^{x}$. $\quad y'''(x_0) = 2(9) + 3e^{0} = 18+3 = 2$

$y''''(x) = 2y''' + 3e^{x}$ $\quad y''''(x_0) = 2(21) + 3e^{0} = 42+3=4$

Substituting these values in the Taylor's Series, we get

$$y = 0 + x(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45) + \cdots$$

$$y = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \cdots$$

At $x = 0.2$, $y = 0.811$ (approximate)

Exact Solution:-

Given equation $\frac{dy}{dx} - 2y - 3e^{x}$ is a Leibnitz linear in x. Its integrating factor is e^{-2x} and the solution is

$$ye^{-2x} = \int 3e^{x}e^{-2x}\,dx.$$

$$ye^{-2x} = 3\int e^{-x}\,dx.$$

$$ye^{-2x} = -3e^{-x} + c.$$

when $x = 0$ then $y = 0$

$0 = -3 + c \Rightarrow c = 3$

Therefore, the particular solution is

$$ye^{-2x} = -3e^{-x} + 3$$

$$y = -3e^{x} + 3e^{2x}$$

At $x = 0.2$

$$y = -3e^{0.2} + 3e^{0.4}$$

$$= -3(1.2214) + 3(1.4918)$$

$$y = 0.8112 \text{ (Exact}$$

Find the solution $y(0.1)$ to the initial value problem $\frac{dy}{dx} = -2xy^2$ given $y(0) = 1$ with $h = 0.1$. using Taylor's series method of order four.

Using Taylor's series method, obtain the solution $u(0.1)$ to the initial value problem

$u' = x(1-2u^2)$; $u(0) = 1$ with first three non zero terms.

Therefore, the Particular Solution is

$$y e^{-2x} = -3e^{-x} + 3$$

$$y = -3e^{x} + 3e^{2x}$$

At $x = 0.2$

$$y = -3e^{0.2} + 3e^{0.4}$$

$$= -3(1.2214) + 3(1.4918)$$

$$y = 0.8112 \text{ (Exact}$$

Ex Find the Solution $y(0.1)$ to the initial Value problem $\frac{dy}{dx} = -2xy^2$ given $y(0) = 1$ with $h = 0.1$, Using Taylor's Series Method of order four.

Ex. Using Taylor's Series method, obtain the Solution $u(0.1)$ to the initial Value problem $u' = x(1-2u^2)$; $u(0) = 1$ with first three non zero terms.