Gauss - Jordan Method

Ex. Apply Gauss-Jordan method to solve:

$$x+y+z=9$$
$$2x-3y+4z=13$$
$$3x+4y+5z=40$$

Solution- We have

$$x+y+z=9 \quad \text{— (I)}$$
$$2x-3y+4z=13 \quad \text{— (II)}$$
$$3x+4y+5z=40 \quad \text{— (III)}$$

Step-(i) to eliminate x from (II) and (III) operate

[(ii) − 2(i)], [(iii) − 3(i)]

$$x+y+z=9 \quad \text{— (iv)}$$
$$-5y+2z=-5 \quad \text{— (v)}$$
$$y+2z=13 \quad \text{— (vi)}$$

Step-(ii) To eliminate y from (iv) and (v)

operate [(iv) − (vi)], [(v) + 2(vi)]

$$x-z=-4 \quad \text{— (vii)}$$
$$-5y+2z=-5 \quad \text{— (viii)}$$
$$12z=60 \quad \text{— (ix)}$$
$$z=5$$

Step (iii) To eliminate z from (vii) and (viii)

operate [(viii) − 5(ix)] [(viii) + (ix)]

$$x=1$$
$$-5y=-15 \quad \therefore y=3$$

∴ The Solution is $x=1$, $y=3$, $z=5$

Ex Gauss - Jordan method to solve the Equations

$$2x - 2y + z = 13 \quad \text{—— (i)}$$

$$2x + 3y + 4z = 20 \quad \text{—— (ii)}$$

$$3x - y + 3z = 10 \quad \text{—— (iii)}$$

Step-I Dividing (i) by 2, we get

$$x - y + \frac{5}{2}z = \frac{13}{2} \quad \text{—— (iv)}$$

To Eliminate x from (ii) and (iii) operate.

[(ii) - 2(iv)] and [(iii) - 3(iv)]

$$x - y + \frac{5}{2}z = \frac{13}{7} \quad \text{—— (v)}$$

$$5y - z = 7 \quad \text{—— (vi)}$$

$$2y - \frac{9}{2}z = -\frac{19}{2} \quad \text{—— (vii)}$$

Step-2 Dividing (vi) by 5, we get

$$y - \frac{1}{5}z = \frac{7}{5} \quad \text{—— (viii)}$$

To Eliminate y from (v) and (vii), operate.

[(v) - (-1)(viii)] and [(vii) - 2(viii)]

$$x + \frac{23}{10}z = \frac{79}{10}$$

$$y - \frac{1}{5}z = \frac{7}{5}$$

$$-\frac{41}{10}z = -\frac{123}{10}$$

Step-3 Dividing (xi) by $-\frac{41}{10}$, we get

$$z = 3$$

To Eliminate z from (ix) and (x)

operate $\left[(ix) - \frac{23}{10}(xiii)\right]$ and $\left[(x) - \left(-\frac{1}{5}\right)(xii)\right]$

we get

$$x_1 = 1$$

$$y = 2$$

$$z = 3$$

Ex. Solve the system of equations by Gauss - Jordan method :-

1. $x + 2y + z = 3$

$2x + 3y + 3z = 10$

$3x - y + 2z = 13$

2. $10x + y + z = 13$

$2x + 10y + z = 10$

$x + y + 5z = 7$

3. $2x + 3y - z = 5$

$4x + 4y - 3z = 3$

$2x - 3y + 2z = 2.$

Crout's Method (LU Decomposition)

This method is also called triangularisation or Factorization Method.

Here we factorise the given matrix as $A = LU$

where L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix. Then

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}. \quad \text{—— (i)}$$

Working Rule :—

Consider the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3.$$

The above system can be written as

$$Ax = b \quad \text{—— (ii)}$$

Let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

—— (iii)

Here L is a lower triangular matrix and U is an upper triangular matrix with diagonal elements Equal to Unity.

$$A = LU$$

$$\Rightarrow A^{-1} = U^{-1}L^{-1}$$

Now

$$A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & u_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

Equating the corresponding elements we obtain

$$l_{11} = a_{11} \qquad l_{21} = a_{21} \qquad l_{31} = a_{31} \quad \text{—(v)}$$

$$l_{11}u_{12} = a_{12} \qquad l_{11}u_{13} = a_{13} \quad \text{—(vi)}$$

$$l_{21}u_{12} + l_{22} = a_{22} \qquad l_{31}u_{12} + l_{32} = a_{32} \quad \text{—(vii)}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \quad \text{—(viii)}$$

and $l_{31}u_{13} + l_{32}u_{32} + l_{33} = a_{33}$ —(ix)

from (vi) we find

$$u_{12} = a_{12}/l_{11} = a_{12}/a_{11}$$

from (vii) we obtain

$$l_{22} = a_{22} - l_{21} u_{12} \quad \text{——(x)}$$

$$l_{32} = a_{32} - l_{31} u_{12} \quad \text{——(xii)}$$

(viii) gives

$$u_{23} = (a_{33} - l_{31} u_{13} - l_{32} u_{23}) \quad \text{——(xiii)}$$

from the relation (ix) we get

$$l_{33} = (a_{33} - l_{31} u_{13} - l_{32} u_{23})$$

Thus we have determined all the elements of L and U

from (ii) and (iii) we have.

$$LUx = b$$

Let $Ux = V$

where $V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

from (xiv) we have $LV = b$ which on forward substitution yields V.

from $Ux = V$ we find x (by backward substitution) ex Solve the following set of equations by Crout's method.

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

Sol^n We have

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$AX = B$$
$$A = LU$$

Let

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{32} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$l_{11}u_{12} = 1 \quad \Rightarrow \quad \boxed{u_{12}} = \frac{1}{2}$$

$$l_{11}u_{13} = 4 \quad \Rightarrow \quad \boxed{u_{13}} = \frac{4}{2} = 2$$

8.

$l_{21}u_{12}+l_{22}=-3 \Rightarrow \boxed{l_{22}}=-3-8\left(\frac{1}{2}\right)=-7$

$l_{21}u_{12}+l_{32}=-3 \Rightarrow 11-4\left(\frac{1}{2}\right)=9$

$l_{21}u_{13}+l_{22}u_{23}=2 \Rightarrow \boxed{u_{23}}=\frac{2-8\times 2}{-7}=2$

$l_{31}u_{13}+l_{32}u_{23}+l_{33}-1 \Rightarrow \boxed{u_{33}}=-1-4\times 2-9\times 2=-27$

$$L=\begin{bmatrix}2&0&0\\8&-7&0\\4&9&-27\end{bmatrix} \quad \text{and} \quad U=\begin{bmatrix}1&\frac{1}{2}&2\\0&1&2\\0&0&1\end{bmatrix}$$

$LV=B$

$$\begin{bmatrix}2&0&0\\8&-7&0\\4&9&-27\end{bmatrix}\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\begin{bmatrix}12\\20\\33\end{bmatrix}$$

$2V_1=12 \Rightarrow \boxed{V_1=6}$

$8V_1-7V_2=20 \Rightarrow V_2=\frac{-20+8\times 6}{7}$

$\Rightarrow \boxed{V_2=4}$

$4V_1+9V_2-27V_3=33$

$\Rightarrow V_3=\frac{-33+4\times 6+9\times 4}{27}=1$

$$V=\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\begin{bmatrix}6\\4\\1\end{bmatrix}; \quad Ux=V$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

$$x + \frac{1}{2}y + 2z = 6$$

$$y + 2z = 4$$

$$\boxed{z = 1}$$

$$y = 4 - 2 \times 1$$

$$\boxed{y = 2}$$

$$x = 6 - \frac{1}{2} \times 2 - 2 \times 1$$

$$\Rightarrow \boxed{x = 3}$$

Ex Solve the following set of equations by using the Crout's method

$$2x_1 + x_2 + x_3 = 7$$

$$x_1 + 2x_2 + x_3 = 8$$

$$x_1 + x_2 + 2x_3 = 9$$

Sol^n

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Let $A = LU$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+u_{32} & u_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$l_{11} = \frac{1}{2}$ $\qquad l_{21} = 1$, $\qquad l_{31} = 1$

$u_{12} = \frac{1}{2}$ $\qquad u_{31} = \frac{1}{2}$

$$l_{22} = 2 - l_{21}u_{12} = 2 - 1 \times \frac{1}{2} = \frac{3}{2}$$

$$l_{32} = 1 - l_{31}u_{12} = 1 - 1 \times \frac{1}{2} = \frac{1}{2}$$

$$u_{23} = \frac{1 - l_{21}u_{13}}{l_{22}} = \frac{1}{3}$$

$$l_{33} = 2 - l_{31}u_{13} = l_{32}u_{23} = 2 - \frac{1}{2} - \frac{1}{2} \times \frac{1}{3} = \frac{4}{3}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3/2 & 0 \\ 1 & 1/2 & 4/3 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$AX = B$, $LUX = B$, $UX = V$

$$LV = B \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3/2 & 0 \\ 1 & 1/2 & 4/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$2v_1 = 7 \Rightarrow v_1 = 3.5$

$v_1 + \frac{3}{2}v_2 = 8 \Rightarrow v_2 = 3$

$v_1 + \frac{1}{2}v_2 + \frac{4}{3}v_3 = 9 \Rightarrow v_3 = 3$

$$\Rightarrow V = \begin{bmatrix} 3.5 \\ 3 \\ 3 \end{bmatrix}$$

$$Ux = V \Rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3 \\ 3 \end{bmatrix}$$

$x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 3.5$ —(i)

$x_2 = \frac{1}{3}x_3 = 3$ —(ii)

$\boxed{x_3 = 3}$ —(iii)

from Eqn (ii) and (iii) we have

$\boxed{x_2 = 2}$

from Eqn (i) we get

$\boxed{x_1 = 1}$

Ex find the inverse of matrix Using factorization method (Crout's method):

1. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

Ex Apply Crout's method to obtain the inverse of

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

Solution :- Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

Then $A = LU$ and $A^{-1} = U^{-1}L^{-1}$, where L is a lower triangular matrix and U and an upper triangular matrix.

Now, let $L = \begin{bmatrix} l_{12} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$A = L \Rightarrow \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

where by Equality principle of matrices, we easily obtain

$l_{11} = 2$ $\quad l_{21} = 2$ $\quad l_{22} = 5$ $\quad l_{31} = -1$ $\quad l_{32} = 0$ $\quad l_{33} = 1$

and $u_{12} = -1$ $\quad u_{13} = 2$ $\quad u_{23} = -\left(\frac{2}{5}\right)$

$$\Rightarrow L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -(2/5) \\ 0 & 0 & 1 \end{bmatrix}$$

To Compute L^{-1} and U^{-1}

Let the inverse of L (the Lower triangular matrix) be X (a lower triangular matrix) then

$$LX = I$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_{11} & 0 & 0 \\ 2x_{11}+5x_{21} & 5x_{22} & 0 \\ -x_{11}+x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad 2x_{11} = 1 \qquad 2x_{11} + 5x_{21} = 0$

$5x_{22} = 1$

$x_{31} - x_{11} = 0 \qquad x_{32} = 0 \qquad x_{33} = 1$

$\boxed{x_{11} = \frac{1}{2}} \qquad \boxed{x_{21} = -\frac{1}{5}} \qquad , \boxed{x_{22} = \frac{1}{5}}$

$\boxed{x_{31} = \frac{1}{2}} \qquad \boxed{x_{32} = 0} \qquad \boxed{x_{33} = 1}$

$$\therefore \quad X = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/5 & 1/5 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

Similarly, let the inverse of U (the upper triangular matrix) by Y (an upper triangular matrix) then we have

$$UY = I \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & y_{12}-1 & y_{13}-y_{23}+2 \\ 0 & 1 & y_{33}-2/5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow y_{12}-1=0, \quad y_{13}-y_{23}+2=0, \quad y_{23}=+\frac{2}{5}$$

$$\Rightarrow y_{12}=1 \qquad y_{13}=\frac{2}{5}-2 \qquad y_{23}=\frac{2}{5}$$

$$\therefore\ y = U^{-1} = \begin{bmatrix} 1 & 1 & -8/5 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1 \end{bmatrix}$$

whence $A^{-1} = U^{-1}L^{-1}$

$$= \begin{bmatrix} 1 & 1 & -8/5 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1/2 & 0 & 0 \\ -1/5 & 1/5 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/5 & 1/5 & -8/5 \\ 0 & 1/5 & 2/5 \\ 1/2 & 0 & 1 \end{bmatrix}$$