UNIT-4 :- Complex Variables :-

Function of complex variable :- If for each value of complex variable $z = x + iy$ there corresponds one or more values of another complex variable $w = u + iv$, then w is said to be a function of z.

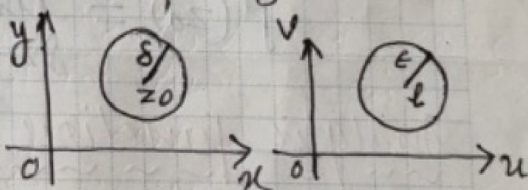
If $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$, then f is univalent otherwise manyvalent.

If $z_1 = z_2 \Rightarrow f(z_1) = f(z_2)$, then f is a single valued function. Eg. $w = f(z) = z^2$.

Eg. If $w = f(z) = z^{1/2}$, then f is multi-valued function

Limit of a function $f(z)$:- A function $w = f(z)$ is said to tend to a limit l as z tends to a point z_0 if for a given small +ve no. ϵ, we can find a positive no. δ such that

$$|f(z) - l| < \epsilon \text{ for } |z - z_0| < \delta$$



Let $f(z) = u(x,y) + iv(x,y)$,
$z = x + iy$, $z_0 = x_0 + iy_0$
and $l = u_0 + iv_0$, then

$$\lim_{z \to z_0} f(z) = l, \text{ iff } \lim_{x \to x_0, y \to y_0} u(x,y) = u_0 \text{ and}$$

$$\lim_{x \to x_0, y \to y_0} v(x,y) = v_0.$$

If $f(z)$ has a finite limit at $z = z_0$, then $f(z)$ is a bounded function in some nbd of z_0.

Limit of function at $z = \infty$

$$\lim_{z \to \infty} f(z).$$

Continuity of function $f(z)$:— A function $f(z)$ is said to be continuous at a point $z = z_0$, if for every $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon, \text{ whenever } |z - z_0| < \delta.$$

Thus a function $f(z)$ is continuous at a point $z = z_0$, if the following three conditions are satisfied

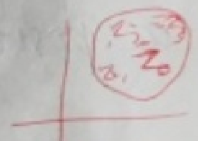
(i) $f(z_0)$ exists

(ii) $\lim_{z \to z_0} f(z)$ exists

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$.

Differentiability of $f(z)$:— The derivative of $w = f(z)$ at z_0 is given by $w' = f'(z) = \lim_{\delta z \to 0} \frac{\delta w}{\partial z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$. Put $z - z_0 = \delta z$ in the above relation. We can write

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$



Analytic function:— A complex function $f(z)$ is said to be analytic at a point z_0, if is differentiable in some nhd. of z_0 and z_0 is called a regular point of $f(z)$.

If $f(z)$ is not analytic at z_0 and every nhd of z_0 contains points at which $f(z)$ is analytic, then z_0 is called a singular point of $f(z)$

A function which is analytic at every point of a region R is called analytic in R or holomorphic in R or regular in R.

Entire function: A complex function f is said to be entire if it is analytic in the whole complex plane:

Eg $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, e^z, polynomial f^n and constant function are entire function.

Cauchy-Riemann Equation: The necessary and sufficient conditions (n.a.s.c) for the function $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in a region 'R' are

$$u_x = v_y \text{ and } u_y = -v_x.$$

n.a.s.c for $f(z)$ to be analytic:- The n.a.s.c for the f^n $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

(i) u_x, u_y, v_x, v_y are continuous of x & y in the region R.

(ii) $u_x = v_y$, $u_y = -v_x$ (called C-R Equation)

$$f(z) = z^3, \quad f'(z) = 3z^2$$
$$f(z) = \sin z, \quad f'(z) = \cos z$$
$$f(z) = \cosh z, \quad f'(z) = \sinh x$$

Q Prove that the function $f(z) = \sinh z$ is analytic find its derivative.

Here, $f(z) = \sinh z = \sinh(x+iy) = \sinh x \cdot \cos y + i\cosh x \cdot \sin y$

Equating the real and imaginary part

$$u = \sinh x * \cos y, \quad v = \cosh x \cdot \sin y$$
$$u_x = \cosh x \cdot \cos y, \quad v_x = \sinh x \cdot \sin y$$
$$u_y = -\sinh x \cdot \sin y, \quad v_y = \cosh x \cdot \cos y$$

$\therefore u_x = v_y$ & $u_y = -v_x$

Thus C-R equation satisfied.

~~$\cos(ix) = \cosh x$~~

$\cosh x = \cos ix$

$\sinh x = -i \sin(ix)$

$\tanh x = \frac{-i}{\tan(ix)}$

Polar form of Cauchy-Riemann equations :-

We know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

$$\therefore f(z) = f(re^{i\theta})$$

$$\Rightarrow \quad u+iv = f(r\cdot e^{i\theta})$$

Differentiating partially w.r. to r, keeping θ constant

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \quad \langle 1 \rangle$$

Differentiating partially w.r. to θ, keeping r constant.

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta}. \quad \langle 11 \rangle$$

Eliminating $\langle 1 \rangle$ & $\langle 2 \rangle$, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \& \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\boxed{\text{or} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.}$$

Harmonic Function: — A function which satisfy the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \; ; \; \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

are known as Harmonic function.

Q. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic/regular at the origin, although C-R equation satisfied at $(0,0)$.

We know that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Now,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

Put $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x + imx} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1 + im}.$$

Here, the limit on the RHS depends on m.

$\therefore f'(0)$ is not unique / analytic

Also,

Let $f(z) = u + iv = \sqrt{|xy|}$

Equating real and imaginary parts, we get

$$u = \sqrt{|xy|}, \quad v = 0$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \left\{ \frac{u(x,0) - u(0,0)}{x} \right\}$$

$$= \lim_{x \to 0} \left\{ \frac{0 - 0}{x} \right\} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \left\{ \frac{u(0,y) - u(0,0)}{y} \right\}.$$

$$= \lim_{y \to 0} \left\{ \frac{0 - 0}{y} \right\} = 0.$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \left\{ \frac{v(x,0) - v(0,0)}{x} \right\}$$

$$= \lim_{x \to 0} \left\{ \frac{0 - 0}{x} \right\} = 0.$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0}\left\{\frac{v(0,y)-v(0,0)}{y}\right\} = 0.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at $(0,0)$.

Q. Prove that the function $e^x(\cos y + i\sin y)$ is analytic and find its derivative

Let $w = u+iv = e^x \cos y + i e^x \sin y$

Equating real and imaginary parts, we get

$u = e^x . \cos y \quad ; \quad v = e^x \sin y$

Now $\frac{\partial u}{\partial x} = e^x \cos y \quad ; \quad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y.$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Hence CR equations are satisfied.

Since e^x, $\sin y$, $\cos y$ are continuous functions hence $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous fns which satisfy C-R equations,

$\therefore f(z)$ is analytic everywhere.

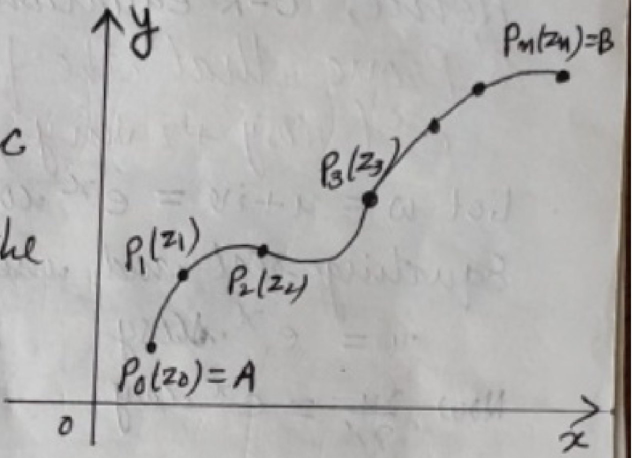
Also, $w = u+iv$

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$$
$$= e^x(\cos y + i\sin y)$$
$$= e^x \times e^{iy} = e^{x+iy} = e^z$$

$$\Rightarrow \frac{\partial w}{\partial z} = e^z \checkmark$$

✓ Complex Integration :- Let $f(z)$ be continuous function of the complex variable $z = x+iy$ defined at all points of a curve 'C' having end points A and B. Divide the curve 'C' into n parts at the points $A = P_0(z_0), P_1(z_1), P_2(z_2), ---, P_n(z_n) = B$.

Let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $P_{i-1} \cdot P_i$, then the limit of the sum



$$\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$$

as $n \to \infty$ and each $\delta z_i \to 0$, if it exists, is called line integral of $f(z)$ along 'C'. It is denoted by

$$\int_C f(z)\,dz$$

In case the point ~~coin~~ P_0 and P_n coincide so that C is closed curve, then the integral is called contour integral. It is denoted by

$$\oint_C f(z)\,dz.$$

Evaluation of a complex integral : The integral $\int_C f(z)\,dz$ is expressed as the sum of two integral of real variables.

Let $z = x + iy \Rightarrow dz = dx + idy$

and $f(z) = u + iv$, then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).$$

Properties of integral of complex variable :-

If $f(z)$ and $g(z)$ are integrable along a curve C, then

ⓐ $\int_C [f(z) \pm g(z)]\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$

ⓑ $\int_C A f(z)\,dz = A\int_C f(z)\,dz$

ⓒ $\int_b^a f(z)\,dz = -\int_b^a f(z)\,dz$

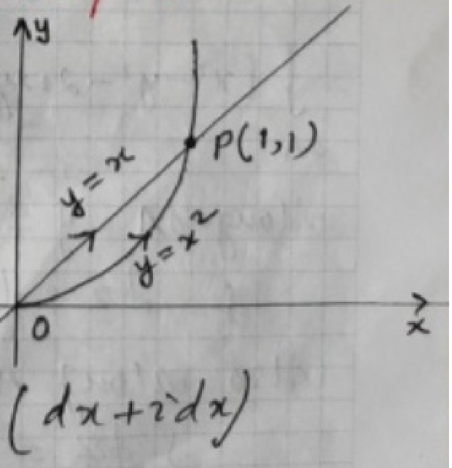
ⓓ If a, b and m are three points on C, then

$$\int_a^b f(z)\,dz = \int_a^m f(z)\,dz + \int_m^b f(z)\,dz$$

Q Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths

① $y = x$ ⓑ $y = x^2$.

ⓐ Along the line $y = x$.



$\because z = x + iy \Rightarrow dz = dx + idy$

and $dy = dx$

$\therefore dz = dx + i\,dx = (1+i)\,dx.$

Now $\int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix)(dx + i\,dx)$

$$= \int_0^1 (x^2 - ix)(1+i)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1+i)\left(\frac{1}{3} - i\frac{1}{2}\right) = \frac{5}{6} - \frac{i}{6}$$

(b) Along the parabola $y = x^2$

$$\Rightarrow dy = 2x\,dx$$

and $z = x + iy$

$$\Rightarrow dz = dx + i\,dy = dx + i\,2x\,dx = (1+2ix)\,dx$$

$$\therefore \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1+2ix)\,dx$$

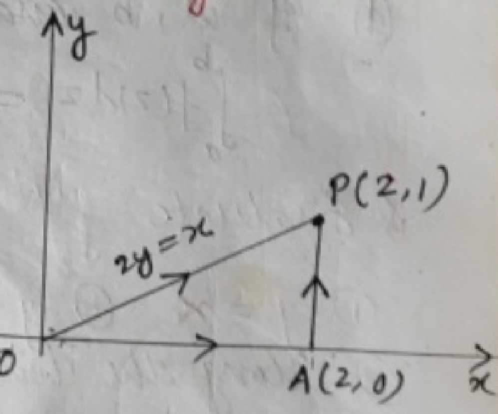
$$= (1-i)\int_0^1 (x^2 + 2ix^3)\,dx$$

$$= (1-i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1 = \frac{5}{6} + \frac{i}{6}$$

Q Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along

(a) the real axis to 2 and then vertically to $2+i$

(b) along the line $2y = x$.



(a) Along the path OAP

$$(\bar{z})^2 = (x-iy)^2 = x^2 - y^2 - 2ixy$$

$$\therefore \underbrace{\int_{OA} (x^2-y^2-2ixy)\,dz}_{I} + \underbrace{\int_{AP} (x^2-y^2-2ixy)\,dz}_{II}$$

Along OA, $y = 0$; and x varies from 0 to 2

$\therefore$ & $dz = dx$; $z = x + iy$

Also along AP, $x = 2$, and y varies from 0 to 1

& $dz = i\,dy$; $z = x + iy \Rightarrow z = 2 + iy$

$dz = i\,dy$.

$$\therefore \int_0^2 (x^2)\,dx + \int_0^1 (4 - y^2 - 4iy)\,i\,dy$$

$$= \left(\frac{x^3}{3}\right)_0^2 + \left(4iy - i\frac{y^3}{3} + 2y^2\right)_0^1$$

$$= \frac{8}{3} + 4i - \frac{i}{3} + 2 = \frac{14}{3} + \frac{11}{3}i$$

Also, along the line OP, $2y = x$
$= 2dy = dx$

and $z = x + iy$
$\Rightarrow dz = dx + idy = 2d\cancel{x}y + idy = (2+i)dy$
and y varies from 0 to 1.

$$\therefore \int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x^2 - y^2 - 2ixy)\, dz$$

$$= \int_0^1 (4y^2 - y^2 - 4iy^2)(2+i)\, dy$$

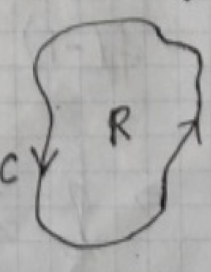
$$= \int_0^1 (3y^2 - 4iy^2)(2+i)\, dy$$

$$= (3-4i)(2+i)\int_0^1 y^2 dy$$

$$= (3-4i)(2+i)\frac{1}{3}\left[y^3\right]_0^1$$

$$= \frac{1}{3}(3-4i)(2+i) \times 1 = \frac{10}{3} - \frac{5}{3}i$$

Cauchy's Integral Theorem:- If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve 'C', then



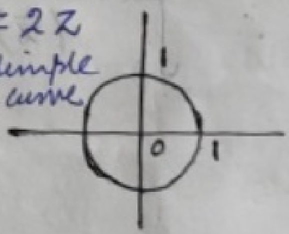
$$\oint_C f(z)\, dz = 0.$$

Cauchy's Integral Formula: If $f(z)$ is analytic within and on a closed curve 'C' and 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Q. Evaluate $\oint_C (x^2 - y^2 + 2ixy)\,dz$, where C is the contour $|z| = 1$.

$f(z) = x^2 - y^2 + 2ixy = (x+iy)^2 = z^2$ is analytic within and on $|z| = 1$ and $f'(z) = 2z$ is continuous at each point within and on a simple close curve.



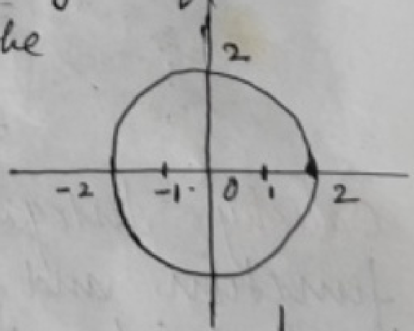
$\therefore$ By Cauchy's integral theorem

$$\oint_C f(z)\,dz = 0.$$

Q. Evaluate $\oint_C \frac{e^{-z}}{z+1}\,dz$, where C is the circle

a $|z| = 2$ b $|z| = \frac{1}{2}$.

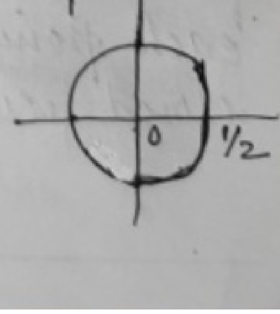
a Here, $f(z) = e^{-z}$ is an analytic function the point $a = -1$ lies inside the circle $|z| = 2$



$\therefore$ By Cauchy's integral formula

$$f(-1) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(-1)}\,dz$$

$$\Rightarrow \oint_C \frac{e^{-z}dz}{z+1} = 2\pi i e. \quad [f(z) = e^{-z}]$$



(b) The point $a = -1$ lies outside the circle $|z| = \frac{1}{2}$

$\therefore$ The function $\frac{e^{-z}}{z+1}$ is analytic within and on C

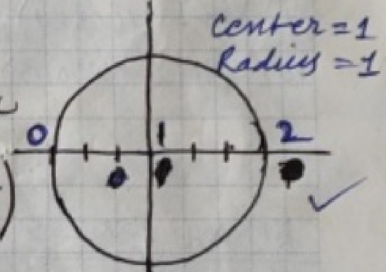
$\therefore$ By Cauchy's integral formula, we have

$$\oint_C \frac{e^{-z}}{z+1}\,dz = 0.$$ (because $a = -1$ lies outside the circle $|z| = \frac{1}{2}$.

Q Evaluate $\oint_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1| = 1$

The integrand has singularities, where $z^2 - 1 = 0$

i.e $z = 1$ & $z = -1$



$f(z) = 3z^2 + z$ is an analytic function

Also, $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$$\therefore \oint_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}\,dz \quad \langle 1\rangle$$

By Cauchy's integral formula

$$f(1) = \frac{1}{2\pi i}\oint_C \frac{3z^2+z}{z-1} \Rightarrow \oint_C \frac{3z^2+z}{z-1}\,dz = 2\pi i f(1)$$
$$= 8\pi i$$

where $f(z) = 3z^2 + z$

By Cauchy's integral formula

$$\oint_C \frac{3z^2+z}{z-1}\,dz = 0$$

From $\langle 1\rangle$, we have

$$\oint_C \frac{3z^2+z}{z-1}\,dz = 4\pi i$$

$|z-1| = 1$
$\Rightarrow |x+iy-1| = 1$
$\Rightarrow |(x-1)+iy| = 1$
$\Rightarrow \sqrt{(x-1)^2+y^2} = 1$
$\Rightarrow (x-1)^2 + y^2 = 1$
Center $= 1$, Radius $= 1$

Singularity:- A point at which a function $f(z)$ fail to be analytic is called a singular point or singularity of $f(z)$.

For example, $z=2$ is a singular point of $f(z)=\frac{1}{z-2}$.

Isolated singular point:- A singular point $z=a$ of a function $f(z)$ is called an isolated singular point if there exists a circle with center 'a' which contain no other singular point of $f(z)$.

For example, $z=-1, 1$ are two isolated singular points of the function $f(z)=\frac{z}{z^2-1}$; $f(z)=\frac{1}{\sin(\pi z)}$, $\pi z = n\pi$, $z=\pm 1, 2, 3,$ Infinite.

Non-isolated Singularity / singular point:- If the function $f(z)=\frac{1}{z-a}$ contains infinite no. of singularities in the nhd. of 'a', it is called non isolated singularity.

Eg $f(z)=\frac{1}{\tan \pi/z}$ | $\sin\frac{\pi}{z}$; $\sin\frac{\pi}{z}=0$, then $\pi/z = n\pi \Rightarrow z=\frac{1}{n}$

For singularity, we take $\tan\pi/z = 0 \Rightarrow \frac{\pi}{z} = n\pi$

$\Rightarrow z = 1/n$

where $n = 1, 2, 3, ---$

$z = 1/n = 1, 1/2, 1/3, 1/4, —— \infty$

All points come in the nhd of 0

Hence, 0 is non-isolated singularity.

Removable singularity:- A singularity which can be remove by defining the function suitably is called removable singularity.

Eg $f(z)=\frac{\sin z}{z}$ has removable singularity at $z=0$

since 0/0 form.

$$\lim_{z\to 0} \frac{\sin z}{z} = 1 \text{ exist.}$$

Eg $f(z) = \frac{e^z - 1}{z}$ $\left[\frac{0}{0} \text{ form}\right]$

$$\therefore \lim_{z\to 0} \frac{e^z-1}{z} = \lim_{z\to 0} \frac{1}{z}\left[\cancel{1} + z + \frac{z^2}{2} + \cdots - \cancel{1}\right]$$

$$= 1 \text{ exist.}$$

Essential Singularity:- A function $f(z) = \frac{1}{z-a}$ is said to be essential ~~removable~~ singularity, if $\lim_{z\to a} f(z)$ does not exist.

Eg $f(z) = \sin\frac{1}{z-a}$

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!}\frac{1}{(z-a)^3} + \frac{1}{5!}\frac{1}{(z-a)^5} - \cdots$$

$$= \lim_{z\to a}\left\{\frac{1}{z-a} - \frac{1}{3!}\frac{1}{(z-a)^3} + \frac{1}{5!}\frac{1}{(z-a)^5} - \cdots\right.$$

$$= \infty.$$

Poles:- When $z=a$ is an isolated singular point of $f(z)$ then we can expand $f(z)$ in Laurent's series about $z=a$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

if $b_n = 0$ for $n > m$, then $f(z)$ has a pole at $z=a$ of order m.

A pole of order one is called simple pole.

Eg. $f(z) = \dfrac{-1}{2-3z+z^2} = \dfrac{-1}{(z-1)(z-2)}$

Here, $f(z)$ has simple poles of order 1 at $z=1$, & $z=2$

Eg: $f(z) = \dfrac{\sin z}{z^2(z^2+1)(z+1)^2} = \dfrac{\sin z}{z} \cdot \dfrac{1}{(z-0)(z^2+1)(z+1)^2}$

Since $\lim\limits_{z\to 0} \dfrac{\sin z}{z} = 1$, hence we can write

$$f(z) = \frac{\sin z}{z} \cdot \frac{1}{z(z^2+1)(z+1)^2} = \frac{\sin z}{z} \cdot \frac{1}{(z-0)(z^2+1)(z+1)^2}$$

For poles, we put

$$z=0 \quad ; \quad (z^2+1=0 \quad ; \quad (z+1)^2=0$$
$$z=0 \quad ; \quad z=\pm i \quad ; \quad z=-1,-1$$

- The function has simple poles of order 1 at $z=0$ and $z=\pm i$.
- The function has double pole of order 2 at $z=-1$.

Zero's of analytic function:- If $f(z)$ is any analytic function, then z_0 is called zero of analytic function, if $f(z_0)=0$.

If $f'(z_0)=0, f''(z_0)=0, \ldots, f^{m-1}(z_0)=0$, but $f^m(z_0)\neq 0$ then $f(z)$ is said to be zero of order m at z_0.

Q Find the zero of $f(z) = \dfrac{(z-3)^3}{(z-1)}$

Here $z-3=0 \Rightarrow z=3$

At $z=3$, $f(z)$ becomes zero, so $z=3$ is a zero of $f(z)$

Now,

$$f(z) = (z-3)^3$$
$$f'(z) = 3(z-3)^2 \quad ; \quad f'(3)=0$$
$$f''(z) = 6(z-3) \quad ; \quad f'(3)=0$$
$$f'''(z) = 6 \neq 0$$

Note: To find singularity and pole, we have to equate denominator 0 and for zero, equate numerator 0.

Hence 3 is the order of the zero at $z=3$.