UNIT - III

The Laplace Transforms

A transformation is a matematical device which converts one function into another.

Laplace transform majorly used in ~~OD~~ solving ODE & PDE.

Definition :- Let $f(t)$ be a function of t defined for all $t \geq 0$. Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ / $\bar{F}(s)$, is defined by

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

provided that the integral exists. 's' is a parameter which may be real or complex.

Linearity Property :- If c_1, c_2 are constants and f, g are functions of t, then

$$\boxed{L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}}$$

Laplace transforms of some Elementary functions :-

① $L\{1\} = \frac{1}{s}$, $s > 0$

$f(t) = 1$

$$L\{f(t)\} = L\{1\} = \int_0^\infty e^{-st} \times 1\, dt = \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{(-s)}\right]_0^\infty = \frac{-1}{s}(0-1)$$

$$= \frac{1}{s}$$

$$\Rightarrow \boxed{L\{1\} = \frac{1}{s}}$$

② $L\{t^n\} = \frac{n!}{s^{n+1}}$, $n \in I^+$

$$\boxed{\int_0^\infty x^n e^{-x}\, dx = \overline{(n+1)}}$$

$$L\{t^n\} = \int_0^\infty e^{-st} \times t^n\, dt$$

let $t = \frac{x}{s}$

$\Rightarrow dt = \frac{dx}{s}$

$$= \int_0^\infty e^{-x} \times \left(\frac{x}{s}\right)^n \times \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$= \frac{1}{s^{n+1}} \times \Gamma(n+1) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\Rightarrow \boxed{L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}} \quad , s > 0 \ \& \ n+1 > 0$$

Note :- ① $\boxed{\Gamma(n+1) = n!}$

② $\boxed{\int_0^\infty e^{-x} x^n \, dx = \Gamma(n+1)}$

③ $L\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$

$$L\{e^{at}\} = \int_0^\infty e^{-st} \times e^{at} \, dt = \int_0^\infty e^{-t(s-a)} \, dt$$

$$= \frac{-1}{(s-a)} \left(e^{-t(s-a)}\right)_0^\infty$$

$$= \frac{-1}{(s-a)} (0-1) = \frac{1}{(s-a)}$$

$$\Rightarrow \boxed{L\{e^{at}\} = \frac{1}{(s-a)}} \quad , s > a$$

④ $\cancel{L\{\sin at\} = \dfrac{a}{s^2+a^2}}$, $s > 0$

$$\cancel{I = L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt = \left\{\sin at \times \frac{e^{-st}}{(-s)} - \int a\cos at \times \frac{e^{-st}}{(-s)} dt\right\}_0^\infty}$$

$$\cancel{= \left\{\frac{-1}{s} \sin at \times e^{-st} + \frac{a}{s} \int e^{-st} \cos at \, dt\right\}_0^\infty}$$

$$=$$

④ $L\{\sin at\} = \dfrac{a}{s^2+a^2}$, $s > 0$, $L\{\cos at\} = \dfrac{s}{s^2+a^2}$, $s > 0$

$$e^{iat} = \cos at + i \sin at$$

$L\{e^{iat}\} = L\{\cos at\} + i \, L\{\sin at\}$ (By Linearity prop)

$$L\{e^{iat}\} = \frac{1}{s-ia\cancel{t}} = \frac{(s+ia\cancel{t})}{(s^2+a^2\cancel{t^2})} = \frac{s}{s^2+a^2\cancel{t^2}} + i\frac{a\cancel{t}}{s^2+a^2\cancel{t^2}}$$

$$L\{\cos at\} = Re\{L\{e^{iat}\}\}$$

$$\Rightarrow \boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

& $L\{\sin at\} = Im\{L\{e^{iat}\}\}$

$$\boxed{L\{\sin at\} = \frac{a\cancel{t}}{s^2+a^2}}$$

⑤ $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s>|a|$, $L\{\cosh at\} = \frac{s}{s^2+a^2}$, $s>0$

$$\sinh at = \frac{e^{at}-e^{-at}}{2}$$

$$L\{\sinh at\} = \frac{1}{2}\{L\{e^{at}\} - L\{e^{-at}\}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right\}$$

$$\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$$

$$\cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$L\{\cosh at\} = \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$$

$$\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$$

First Shifting theorem :-

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

Change of scale property :-

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Ex. Find the Laplace transforms of

(i) $\sin^2 t$ (ii) $\cos^3 t$ (iii) $e^{-2t}(3\cos 4t - 2\sin 5t)$

(iv) $L(\sinh at \cos at)$ (v) $f(t) = \begin{cases} 2+t^2, & 0<t<2 \\ 6, & 2<t<3 \\ 2t-5, & 3<t<\infty \end{cases}$

(vi) $f(t) = \begin{cases} 0, & 0<t<3 \\ (t-3)^2, & t>3 \end{cases}$

Solⁿ (i) $f(t) = \sin^2 t$

$\because \cos 2t = 1 - 2\sin^2 t$

$\Rightarrow 2\sin^2 t = 1 - \cos 2t$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\Rightarrow L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2}\left\{L\{1\} - L\{\cos 2t\}\right\}$

$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$

$= \frac{1}{2}\left\{\frac{\cancel{s^2}+4-\cancel{s^2}}{s^2+4}\right\} = \boxed{\frac{2}{s^2+4}}$

(ii) $\cos^3 t$

$\because \cos 3t = 4\cos^3 t - 3\cos t$

$\Rightarrow \cos^3 t = \frac{\cos 3t + 3\cos t}{4}$

$\Rightarrow L\{\cos^3 t\} = L\left\{\frac{\cos 3t + 3\cos t}{4}\right\}$

$$\Rightarrow L\{\cos^3 t\} = \frac{1}{4}\left[L\{\cos 3t\} + 3L\{\cos t\}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+9} + 3 \times \frac{s}{s^2+1}\right]$$

$$= \frac{1}{4}\left[\frac{s(s^2+1)+3s(s^2+9)}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{4}\left[\frac{s^3+s+3s^3+27s}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{4}\left[\frac{4s^3+28s}{(s^2+9)(s^2+1)}\right]$$

$$\boxed{L\{\cos^3 t\} = \frac{s^3+7s}{(s^2+9)(s^2+1)}}$$

(iii) $f(t) = e^{-2t}(3\cos 4t - 2\sin 5t)$

$$= 3e^{-2t}\cos 4t - 2e^{-2t}\sin 5t$$

$$L\{f(t)\} = 3L\{e^{-2t}\cos 4t\} - 2L\{e^{-2t}\sin 5t\}$$

By first shifting thm.

if $L\{\cos 4t\} = \frac{s}{s^2+16}$

& $L\{\sin 5t\} = \frac{5}{s^2+25}$

$$\Rightarrow L\{e^{-2t}\cos 4t\} = \frac{(s+2)}{(s+2)^2+16}$$

& $L\{e^{-2t}\sin 5t\} = \frac{5}{(s+2)^2+25}$

$$\Rightarrow \boxed{L\{f(t)\} = \frac{3(s+2)}{(s+2)^2+16} - \frac{10}{(s+2)^2+25}}$$

(iv) $f(t) = \sinh at \cos at$

$$= \left(\frac{e^{at}-e^{-at}}{2}\right) \times \cos at$$

$$= \frac{1}{2}(e^{at}\cos at - e^{-at}\cos at)$$

$L\{f(t)\} = \frac{1}{2}L\{e^{at}\cos at - e^{-at}\cos at\}$ $\quad \because L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{f(t)\} = \frac{1}{2}[L\{e^{at}\cos at\} - L\{e^{-at}\cos at\}]$

~~& L{~~ By first shifting theorem

$= \frac{1}{2} \cancel{[\frac{s}{s^2+}}$

$$\boxed{= \frac{1}{2}\left[\frac{(s-a)}{(s-a)^2+a^2} - \frac{(s+a)}{(s+a)^2+a^2}\right]}$$

(V) $f(t) = \begin{cases} 2+t^2, & 0<t<2 \\ 6, & 2<t<3 \\ 2t-5, & 3<t<\infty \end{cases}$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}(2+t^2)\,dt + \int_2^3 e^{-st} \times 6\,dt + \int_3^\infty e^{-st}(2t-5)\,dt$$

$$= 2\int_0^2 e^{-st}\,dt + \int_0^2 t^2 e^{-st}\,dt + 6\int_2^3 e^{-st}\,dt + 2\int_3^\infty e^{-st} t\,dt - 5\int_3^\infty e^{-st}\,dt$$

$$= -\frac{2}{s}(e^{-st})_0^2 + \left[t^2 \times \frac{e^{-st}}{(-s)} - \int 2t \times \frac{e^{-st}}{(-s)}\,dt\right]_0^2 + \frac{6}{(-s)}(e^{-st})_2^3 + 2\left[t \times \frac{e^{-st}}{(-s)} + \frac{1}{s}\int e^{-st}\,dt\right]_3^\infty + \frac{5}{s}(e^{-st})_3^\infty$$

$$= -\frac{2}{s}(e^{-2s}-1) + \left[-\frac{t^2 e^{-st}}{s} + \frac{2}{s}\left\{\frac{t\,e^{-st}}{(-s)} + \frac{1}{s}\int e^{-st}\,dt\right\}\right]_0^2 - \frac{6}{s}(e^{-3s} - e^{-2s}) + 2\left[\frac{-t e^{-st}}{s} - \frac{1}{s^2}e^{-st}\right]_3^\infty + \frac{5}{s}(0 - e^{-3s})$$

$$= \frac{2}{s} - \frac{2}{s}e^{-2s} + \left[\frac{-4e^{-2s}}{s} - \frac{2}{s^2}\times 2e^{-2s} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}\right] - \frac{6}{s}e^{-3s} + \frac{6}{s}e^{-2s} + 2\left(0 - 0 + \frac{3e^{-3s}}{s} + \frac{1}{s^2}e^{-3s}\right) - \frac{5}{s}e^{-3s}$$

$$= \frac{2}{s} - \frac{2}{s}e^{-2s} - \frac{4}{s}e^{-2s} - \frac{4}{s^2}e^{-2s} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3} - \frac{6}{s}e^{-3s}$$
$$+ \frac{6}{s}e^{-2s} + \frac{6e^{-3s}}{s} + \frac{2}{s^2}e^{-3s} - \frac{5}{s}e^{-3s}$$

$$= \frac{2}{s} + \frac{2}{s^3} \mp e^{-2s}\left(\cancel{+\frac{2}{s}} + \cancel{\frac{4}{s}} + \frac{4}{s^2} + \frac{2}{s^3} - \cancel{\frac{6}{s}}\right) + e^{-3s}\left(\cancel{\frac{6}{s}} - \cancel{\frac{6}{s}} + \frac{2}{s^2} - \frac{5}{s}\right)$$

$$\boxed{= \frac{2}{s} + \frac{2}{s^3} - e^{-2s}\left(\frac{4}{s^2} + \frac{2}{s^3}\right) + e^{-3s}\left(\frac{2}{s^2} - \frac{5}{s}\right)}$$

Properties of Laplace Transform :-

(i) Change of scale :- If $L\{f(t)\} = \bar{f}(s)$

then $\boxed{L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)}$

(ii) If $L\{f(t)\} = \bar{f}(s)$

then $L\{t\,f(t)\} = \frac{-d}{ds}\bar{f}(s)$

In general,

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)\ , \quad n = 1, 2, 3 \ldots}$$

(iii) $L\{f(t)\} = \bar{f}(s)$

then $\boxed{L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds}$

$$\boxed{L\left\{\frac{1}{t^2}f(t)\right\} = \int_s^\infty \left\{\int_s^\infty \bar{f}(s)\,ds\right\} ds}$$

(iv) Laplace transform of derivatives :–

If $L\{f(t)\} = \bar{f}(s)$ &

$f'(t)$ is continuous then

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)}$$

In general,

$$\boxed{L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$$

(v) Laplace transform of Integrals :–

If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)}$$

Q. Find the Laplace transform of

(i) $\sin^3 2t$ (H.W.)

(ii) $t^2 e^{-2t}$ (H.W.)

(iii) $\dfrac{e^{-t}\sin t}{t}$

(iv) $te^{-t}\cosh t$

(v) Show that $L\left\{\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right\} = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$

vi) Show that $L\left\{\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}(s-1)$

vii) Evaluate $\int_0^\infty t\, e^{-2t}\cos t\,dt$

viii) $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt$.

(iv) $f(t) = t e^{-t} \cosh t$

$\because L\{\cosh t\} = \frac{s}{s^2-1}$

By first shifting thm

$L\{e^{-t}\cosh t\} = \frac{(s+1)}{(s+1)^2-1}$

By (ii) property, we get

$L\{t e^{-t} \cosh t\} = -\frac{d}{ds}\left\{\frac{(s+1)}{(s+1)^2-1}\right\}$

$= -\left\{\frac{[(s+1)^2-1]\times 1 - (s+1)\times 2(s+1)}{[(s+1)^2-1]^2}\right\}$

$= \frac{2(s+1)^2-(s+1)^2+1}{[(s+1)^2-1]^2}$

$= \frac{(s+1)^2+1}{[(s+1)^2-1]^2}$

(v) $L\left\{\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right\} = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$

$\because L\{\cos at\} = \frac{s}{s^2+a^2}$ & $L\{\cos bt\} = \frac{s}{s^2+b^2}$

By (iii) prop.

$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{(s^2+a^2)} - \frac{s}{(s^2+b^2)}\right) ds$

$s^2+a^2 = y$

$2s\,ds = dy$

$= \frac{1}{2}\int_{s^2+a^2}^\infty \frac{dy}{y} - \frac{1}{2}\int_{s^2+b^2}^\infty \frac{dx}{x}$

$= \frac{1}{2}[\cancel{\log\infty} - \log(s^2+a^2) - \cancel{\log\infty} + \log(s^2+b^2)]$

$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Hence by (v) prop.

$L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}F(s)$

(iii) $f(t) = \dfrac{e^{-t}\sin t}{t}$

$\because L\{\sin t\} = \dfrac{1}{s^2+1}$, By first shifting theorem

$L\{e^{-t}\sin t\} = \dfrac{1}{(s+1)^2+1}$

Now from (iii) property

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$$

let $s+1 = y$

$ds = dy$

$$= \int_{s+1}^{\infty} \frac{dy}{y^2+1}$$

$$= \left[\tan^{-1} y\right]_{s+1}^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(s+1)$$

$$= \pi/2 - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1).$$

(iv) $f(t) = t e^{-t} \cosh t$

$\because L\{\cosh t\} = \dfrac{s}{s^2-1}$

By first shifting thm

$L\{e^{-t}\cosh t\} = \dfrac{(s+1)}{(s+1)^2-1}$

By (ii) property, we get

$$L\{t e^{-t}\cosh t\} = -\frac{d}{ds}\left\{\frac{(s+1)}{(s+1)^2-1}\right\}$$

$$= -\left\{\frac{[(s+1)^2-1]\times 1 - (s+1)\times 2(s+1)}{[(s+1)^2-1]^2}\right\}$$

$$= \frac{2(s+1)^2-(s+1)^2+1}{[(s+1)^2-1]^2}$$

$$= \frac{(s+1)^2+1}{[(s+1)^2-1]^2}$$

(v) $$L\left\{\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right\} = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$$

$\because L\{\cos at\} = \dfrac{s}{s^2+a^2}$ & $L\{\cos bt\} = \dfrac{s}{s^2+b^2}$

By (iii) prop.

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{(s^2+a^2)} - \frac{s}{(s^2+b^2)}\right) ds$$

$s^2+a^2 = y$

$2s\,ds = dy$

$$= \frac{1}{2}\int_{s^2+a^2}^{\infty}\frac{dy}{y} - \frac{1}{2}\int_{s^2+b^2}^{\infty}\frac{dx}{x}$$

$$= \frac{1}{2}\left[\cancel{\log\infty} - \log(s^2+a^2) - \cancel{\log\infty} + \log(s^2+b^2)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Hence by (v) prop.

$$L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$$

$$\Rightarrow L\left\{\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right\} = \frac{1}{s} \times \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$
$$= \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

(vii) $$I = \int_0^\infty t e^{-2t}\cos t\,dt$$

$\because L\{\cos t\} = \frac{s}{s^2+1}$

By (ii) property

$$L\{t\cos t\} = -\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\} = -\left\{\frac{(s^2+1)\times 1 - s\times 2s}{(s^2+1)^2}\right\}$$
$$= \frac{2s^2 - s^2 - 1}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

$\because$ we know

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2} = \int_0^\infty e^{-st}\, t\cos t\,dt$$

if $s = 2$

then

$$\int_0^\infty t e^{-2t}\cos t\,dt = \frac{4-1}{(4+1)^2} = \frac{3}{25}$$

$$\boxed{\int_0^\infty t e^{-2t}\cos t\,dt = \frac{3}{25}}$$

Q. $t^2 e^t \sin 4t$

Q. $\int_0^t e^t \sin t\,dt$

Q. $\frac{e^{at} - \cos bt}{t}$

Q. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}$

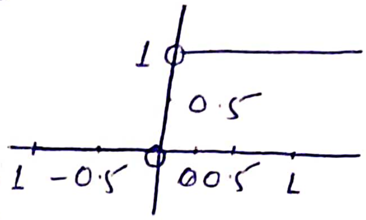
Q. If $L(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}$. Evaluate $L[\omega t\cos\omega t + \sin\omega t]$

Q. Show that $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$ and hence show that

$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \left(\frac{\pi}{p}\right)^{1/2} e^{-1/4p}$$

Unit step function :- The unit step function at time t, $u(t)$, is defined as

$$u(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t \geq 0 \end{cases}$$



That is, u or H is a function of time t

a) When time is -ve, u is zero.

b) When time is +ve, u is 1.

Laplace transformation of unit step fun :-

$$L\{u(t)\} = \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty = \frac{1}{s}$$

Displayed unit step fun :-

$$u(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases}$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} \times 0\, dt + \int_a^\infty e^{-st} \times 1\, dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^\infty = \frac{e^{-as}}{s}$$

$\boxed{L\{f(t)u(t-a)\} = e^{-as} L\{f(t+a)\}}$ $\boxed{L\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s)}$

Q. Find $L\{\sin t\, u(t-\pi/2) - u(t-3\pi/2)\}$

Sol^n By linearity property.

$$L\{\sin t\, u(t-\pi/2) - u(t-3\pi/2)\}$$

$$= L\{\sin t\, u(t-\pi/2)\} - L\{u(t-3\pi/2)\}$$

from

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$\Rightarrow L\{\sin t\, u(t-\pi/2)\} = e^{-\pi/2 s} L\{\sin(t+\pi/2)\}$$

$$\cancel{= e^{-\pi/2 \cdot s} \times e^{-\pi/2 s} L\{\sin t\}}$$

$$\cancel{= e^{-\pi s} \times}$$

$$= e^{-\pi/2 \cdot s} \times L\{\cos t\}$$

$$= e^{-\pi/2 \cdot s} \times \frac{s}{s^2+1}$$

$$= \frac{e^{-\pi s/2} \times s}{s^2+1}$$

& $$L\{u(t-3\pi/2)\} = \frac{e^{-3\pi/2 \cdot s}}{s}$$

so $L\{\sin t\, u(t-\pi/2) - u(t-3\pi/2)\}$

$$\boxed{= \frac{s e^{-\pi s/2}}{s^2+1} - \frac{s e^{-3\pi/2}}{s}}$$

Q. $L(f(t)) = ?$ where $f(t) = \begin{cases} \cos 2t, & 0 < t < \pi \\ \cos 4t, & \pi < t < 2\pi \\ \cos 6t, & t > 2\pi \end{cases}$

$$f(t) = \cos 2t\{u(t-0) - u(t-\pi)\} + \cos 4t\{u(t-\pi) - u(t-2\pi)\} + \cos 6t\{u(t-2\pi)\}$$

$$= u(t)\cos 2t - u(t-\pi)\cos 2t + u(t-\pi)\cos 4t - u(t-2\pi)\cos 4t + u(t-2\pi)\cos 6t$$

$$L\{f(t)\} = L\{u(t)\cos 2t\} - L\{u(t-\pi)\cos 2t\} + L\{u(t-\pi)\cos 4t\} - L\{u(t-2\pi)\cos 4t\} + L\{(t-2\pi)\cos 6t\}$$

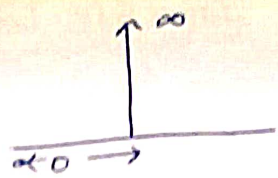
$$= L\{\cos 2(t)\} - e^{-\pi s}L\{\cos 2(t+\pi)\} + e^{-\pi s}L\{\cos 4(t+\pi)\} - e^{-2\pi s}L\{\cos 4(t+2\pi)\} + e^{-2\pi s}L\{\cos 6(t+2\pi)\}$$

$$= \frac{s}{s^2+4} - e^{-\pi s}L(\cos 2t) + e^{-\pi s}L(\cos 4t) - e^{-2\pi s}L(\cos 4t) + e^{-2\pi s}L(\cos 6t)$$

$$\boxed{= \frac{s}{s^2+4} + e^{-\pi s}\left(\frac{s}{s^2+16} - \frac{s}{s^2+4}\right) + e^{-2\pi s}\left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right)}$$

Dirac Delta function :-

$$\delta(t-a) = \begin{cases} \infty & , \text{ if } t = a \\ 0 & , \text{ otherwise} \end{cases}$$



We define $\int_0^\infty \delta(t-a)\, dt = 1.$

& $\int_0^\infty f(t)\,\delta(t-a)\, dt = f(a)$

~~Laplace of Dirac Delta function :-~~

$$\cancel{L\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\, dt}$$

$$\cancel{= \left\{ e^{-st} \times 1 + s \int e^{-st} \times 1 \, dt \right\}_0^\infty}$$

$$\cancel{= \left\{ e^{-st} - \frac{e^{-st}}{s} \times s \right\}_0^\infty}$$

$$\cancel{= t}$$

Laplace of Dirac Delta fun :-

$$L\{\delta(t-a)\} = e^{-as}$$

Hence $L^{-1}\{e^{-as}\} = \delta(t-a)$

& $L\{f(t)\,\delta(t-a)\} = f(a)\,e^{-as}$

Q. $L\{6 \cdot \delta(t-a)\} = 6e^{-as}$

Q. $f(t) = \begin{cases} t & , 0 \le t < 3 \\ 5 & , t \ge 3 \end{cases}$

$$f(t) = t\{\delta(t-0) - \delta(t-3)\} + 5\{\delta(t-3)\}$$

$$L\{f(t)\} = 0 - 3e^{-3s} + 5 \times 1 e^{-3s}$$

$$= e^{-3s}(5-3) = 2e^{-2s}$$

Inverse Laplace Transform :-

If $L[f(t)] = \bar{f}(s)$

then $f(t) = L^{-1}[\bar{f}(s)]$

$$\Rightarrow \boxed{L^{-1}[\bar{f}(s)] = f(t)}$$

Formulae of I.L.T. :-

1. $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n} = \frac{t^{n-1}}{(n-1)!}$

3. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

4. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

5. $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$ / $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

6. $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

7. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

8. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

Q. find $L^{-1}\left[\frac{3}{s^2} + \frac{2}{s^2+9}\right] = 3t + \frac{2}{3}\sin 3t$

Q. find $L^{-1}\left[\frac{3s+4}{s^2+16} + \frac{2}{s+3}\right] = 3\cos 4t + \sin 4t + 2e^{-3t}$

Inverse of first shifting theorem :-

Laplace $\longrightarrow$ If $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{at}f(t)\} = \bar{f}(s-a)$

I.L.T. $\longrightarrow$ If $L^{-1}\{\bar{f}(s)\} = f(t)$

then $\boxed{L^{-1}\{\bar{f}(s-a)\} = e^{at}L^{-1}\{\bar{f}(s)\}}$

Q. find $L^{-1}\left\{\frac{s}{(s-2)^6}\right\}$

Sol. Here $\bar{f}(s) = \frac{s}{(s-2)^6} = \frac{s-2+2}{(s-2)^6} = \frac{(s-2)}{(s-2)^6} + \frac{2}{(s-2)^6}$

$\Rightarrow \bar{f}(s) = \frac{1}{(s-2)^5} + \frac{2}{(s-2)^6}$

Taking I.L.T. on both sides

$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{(s-2)^5}\right\} + 2L^{-1}\left\{\frac{1}{(s-2)^6}\right\}$

By first shifting thm of I.L.T.

$f(t) = e^{2t}L^{-1}\left\{\frac{1}{s^5}\right\} + 2e^{2t}L^{-1}\left\{\frac{1}{s^6}\right\}$

$= e^{2t} \times \frac{t^{5-1}}{\Gamma 5} + 2e^{2t} \times \frac{t^{6-1}}{\Gamma 6}$

$= e^{2t}\left\{\frac{t^4}{4!} + 2\frac{t^5}{5!}\right\} = e^{2t}t^4\left(\frac{1}{24} + \frac{t}{120}\right)$

Q. $\bar{f}(s) = \frac{2s^2-1}{(s^2+1)(s^2+4)}$

By partial fractions

$\frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+4)} = \frac{2s^2-1}{(s^2+1)(s^2+4)}$

$(As+B)(s^2+4) + (Cs+D)(s^2+1) = 2s^2-1$

$As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D = 2s^2 - 1$

$(A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D) = 2s^2 - 1$

$A+C=0$ $\Rightarrow A=-C$

$B+\cancel{D} = 2$
$4B+\cancel{D} = -1$
$-3B = 3$

$4A+C=0$ $\Rightarrow C=-4A$

$4B+D=$

$\boxed{B=-1}$ $\boxed{D=3}$ $\boxed{A=0}$ $\boxed{C=0}$

$\Rightarrow \bar{f}(s) = \frac{-1}{s^2+1} + \frac{3}{s^2+4}$

$L^{-1}\{\bar{f}(s)\} = 3L^{-1}\left\{\frac{1}{s^2+4}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{3}{2}\sin 2t - \sin t$

Q. $F(S) = \dfrac{S}{S^4+4a^4}$

$$= \frac{S}{(S^2)^2+(2a^2)^2+4a^2S^2-4a^2S^2} = \frac{S}{(S^2+2a^2)^2-(2aS)^2}$$

$$= \frac{S}{(S^2+2a^2+2aS)(S^2+2a^2-2aS)}$$

$$= \frac{1}{4a}\left\{\frac{1}{S^2+2a^2-2aS} - \frac{1}{S^2+2a^2+2aS}\right\}$$

$$= \frac{1}{4a}\left\{\frac{1}{(S-a)^2+a^2} - \frac{1}{(S+a)^2+a^2}\right\}$$

Taking inverse Laplace on both sides

$$L^{-1}\{F(S)\} = \frac{1}{4a}\left\{L^{-1}\left(\frac{1}{(S-a)^2+a^2}\right) - L^{-1}\left(\frac{1}{(S+a)^2+a^2}\right)\right\}$$

$$= \frac{1}{4a}\left\{e^{at}L^{-1}\left\{\frac{1}{S^2+a^2}\right\} - e^{-at}L^{-1}\left\{\frac{1}{S^2+a^2}\right\}\right\}$$

$$= \frac{1}{4a}\left\{\frac{e^{at}}{a}\sin at - \frac{e^{-at}}{a}\sin at\right\}$$

$$= \frac{\sin at}{4a^2}(e^{at}-e^{-at})$$

$$\boxed{= \frac{\sin at \sinh at}{2a^2}}$$

~~Multiplication prop. of I.L.T.~~

Q. $L^{-1}\left\{\dfrac{1}{S^3(S^2+1)}\right\} \rightarrow \dfrac{t^2}{2}+\cos t-1$

Q. $L^{-1}\left\{\dfrac{S}{S^4+S^2+1}\right\} \longrightarrow \dfrac{2}{\sqrt{3}}\sin\left(\dfrac{\sqrt{3}t}{2}\right)\sinh\dfrac{t}{2}$

Properties of I.L.T. :-

① Multiplication

$$\left\{\begin{array}{l} \text{If } L\{f(t)\} = F(S) \\ \text{then} \\ L\{t f(t)\} = -\dfrac{d}{ds}F(S) \end{array}\right\}$$

If $L^{-1}\{F(S)\} = f(t)$

then

$$\boxed{L^{-1}\left\{\frac{d}{ds}F(S)\right\} = -t f(t)}$$

② **Division :-**

If $L\{f(t)\} = \bar{f}(s)$
then
$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

If $L^{-1}\{\bar{f}(s)\} = f(t)$
then
$$L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t}$$

③ **Integration :-**

If $L\{f(t)\} = \bar{f}(s)$
then
$$L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$$

If $L^{-1}\{\bar{f}(s)\} = f(t)$
then
$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

Q. $\bar{f}(s) = \dfrac{1}{s^3(s^2+1)} = \dfrac{1}{s}\left\{\dfrac{1}{s^2(s^2+1)}\right\} = \dfrac{1}{s}\left\{\dfrac{1}{s^2} - \dfrac{1}{s^2+1}\right\}$

By $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\} = \displaystyle\int_0^t L^{-1}\{\bar{f}(s)\}\,dt$

$$f(t) = L^{-1}\{\bar{f}(s)\} = \int_0^t L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} dt$$

$$= \int_0^t (t - \sin t)\,dt = \boxed{\frac{1}{2}t^2 + \cos t - 1}$$

Q. $L^{-1}\{\tan^{-1}(s-1)\} = ?$

Since $-\displaystyle\int_s^\infty \frac{1}{(s-1)^2+1}\,ds = \tan^{-1}(s-1)$

then by $L^{-1}\left\{\displaystyle\int_s^\infty \bar{f}(s)\,ds\right\} = \dfrac{f(t)}{t}$

$$-L^{-1}\left\{\int_s^\infty \frac{1}{(s-1)^2+1}\right\} = \frac{L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}}{t}$$

$$= \frac{1}{t} \times e^t L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{= \frac{-e^t \sin t}{t}}$$

Q. $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$

Since $\int_s^\infty \frac{2}{s(s^2+1)}\,ds = 2\int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds$

$$= 2\left[\frac{1}{2}\log|s^2+1| - \log s\right]$$

$$= \log\frac{|s^2+1|}{s^2} = \log\left(1+\frac{1}{s^2}\right)$$

Hence by

$$L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t}$$

$$\therefore 2L^{-1}\left\{\int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds\right\} = 2\left\{\frac{L^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}}{t}\right\}$$

$$= 2\left(\frac{1-\cos t}{t}\right)$$

$$\boxed{= \frac{-(2\cos t-2)}{t}}$$

Q. find $L^{-1}\left\{s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s\right\}$

Ans. $\frac{1-\cos t}{t^2}$

Q. find $L^{-1}\{\tan^{-1}(s-1)\}$

Ans. $\frac{-1}{t}e^t \sin t$.

Q. find $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

Ans. $\frac{t\sinh at}{2a}$

Convolution & Convolution Theorem :-

(To convolute → To multiply)

The function $\int_0^t f_1(u) f_2(t-u)\,du$ is called the convolution of the function f_1 and f_2 and is denoted by $f_1 * f_2$. It is easy to verify that

$$f_1 * f_2 = f_2 * f_1$$

Theorem :- Let $f_1(t)$ & $f_2(t)$ be two functions of t and

$$L\{f_1(t)\} = \bar{f}_1(s) \quad \&$$
$$L\{f_2(t)\} = \bar{f}_2(s)$$

then the theorem states that

$$\boxed{\begin{aligned} L^{-1}\{\bar{f}_1(s)\bar{f}_2(s)\} &= \int_0^t f_1(u) f_2(t-u)\,du \\ &= \int_0^t f_2(u) f_1(t-u)\,du \end{aligned}}$$

Steps :-

1. Identify $\phi_1(s)$ & $\phi_2(s)$
2. $L^{-1}\{\phi_1(s)\} = f_1(t) \Rightarrow$ put $t = u$
3. $L^{-1}\{\phi_2(s)\} = f_2(t) \Rightarrow$ put $t = t-u$
4. $L^{-1}\{\phi_1(s)\phi_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,du$

Q.1 find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$$\bar{f}(s) = \frac{1}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)(s^2+a^2)}$$

take $f_1(s) = \frac{1}{s^2+a^2}$ & $f_2(s) = \frac{1}{s^2+a^2}$

$$L^{-1}\{f_1(s)\} = \frac{1}{a}\sin at \quad \& \quad L^{-1}\{f_2(s)\} = \frac{1}{a}\sin at$$

By Convolution theorem

put $t = u$ in $f_1(s)$

& $t = t-u$ in $f_2(S)$

so

$$L^{-1}\{\bar{f}_1(s)\,\bar{f}_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{a}\sin a(t-u)\, du$$

$$= \frac{1}{a^2}\int_0^t \sin au \cdot \sin a(t-u)\, du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au \cdot \sin a(t-u)\, du$$

$$\boxed{\begin{aligned} &2\sin A \sin B \\ &= \cos(A-B) - \cos(A+B) \end{aligned}}$$

$$= \frac{1}{2a^2}\int_0^t \cos(au-at+au) - \cos(\cancel{au}+at-\cancel{au})\, du$$

$$= \frac{1}{2a^2}\left\{ \int_0^t \cos(2au-at)\, du - \int_0^t \cos at\, du \right\}$$

$$= \frac{1}{2a^2}\left\{ \frac{1}{2a}\left(\sin(2au-at)\right)_0^t - t\cos at \right\}$$

$$= \frac{1}{2a^2}\left\{ \frac{1}{2a}(\sin at + \sin ta) - t\cos at \right\}$$

$$= \frac{1}{2a^2}\left\{ \frac{1}{a}\sin at - t\cos at \right\}$$

$$\boxed{= \frac{1}{2a^3}\{\sin at - at\cos at\}}$$

Q.2 find $L^{-1}\left\{\frac{1}{(s^2+4)(s+2)}\right\}$

let $\bar{f}(s) = \frac{1}{(s^2+4)(s+2)} = \frac{1}{(s^2+4)} \times \frac{1}{(s+2)}$

take $\phi_1(s) = \frac{1}{s^2+4}$ & $\phi_2(s) = \frac{1}{s+2}$

$L^{-1}\{\phi_1(s)\} = \frac{1}{2}\sin 2t$ & $L^{-1}\{\phi_2(s)\} = e^{-2t}$

By convolution theorem

take $t = u$ in $\phi_1(s)$

& $t = t-u$ in $\phi_2(s)$

$$\Rightarrow L^{-1}\{\phi_1(s)\,\phi_2(s)\} = \int_0^t f_1(t)\, f_2(t-u)\,du$$

$$= \int_0^t \frac{1}{2}\sin 2u \times e^{-2(t-u)}\,du$$

$$= \frac{1}{2}\int_0^t \sin 2u \,.\, e^{-2t}.\,e^{2u}\,du$$

$$= \frac{e^{-2t}}{2}\underbrace{\int_0^t \sin 2u \,.\, e^{2u}\,du}_{I}$$

$$I = \int_0^t \sin 2u . e^{2u}\,du$$

$$= \left(\sin 2u \times \frac{1}{2}e^{2u} - \cancel{2}\int \cos 2u \times \frac{1}{\cancel{2}} e^{2u}\,du\right)_0^t$$

$$= \left(\frac{1}{2}e^{2u}\sin 2u - \cos 2u \times \frac{1}{2}e^{2u} + \int (-2\sin 2u)\times\frac{1}{2}e^{2u}\,du\right)_0^t$$

$$= \left(\frac{1}{2}e^{2u}\sin 2u - \frac{1}{2}e^{2u}\cos 2u . - I\right)_0^t$$

$$2I = \cancel{\frac{1}{2}}\left(\frac{e^{2t}\sin 2t}{2} - \frac{1}{2}e^{2t}\cos 2t - 0 + \frac{1}{2}\right)$$

$$\boxed{= \frac{1}{4}\left(e^{2t}\sin 2t - e^{2t}\cos 2t + 1\right)}$$

$$\Rightarrow \boxed{L^{-1}\{\phi_1(s)\phi_2(s)\} = \frac{1}{8}\left(\sin 2t - \cos 2t + e^{-2t}\right)}$$

Q.3 Find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

let $F(s) = \frac{1}{s^2(s+1)^2}$

take $\phi_1(s) = \frac{1}{s^2}$ & $\phi_2(s) = \frac{1}{(s+1)^2}$

$L^{-1}\{\phi_1(s)\} = 1 \times t$ & $L^{-1}\{\phi_2(s)\} = e^{-t} L^{-1}\left\{\frac{1}{s^2}\right\}$

$L^{-1}\{\phi_1(s)\} = t$ & $L^{-1}\{\phi_2(s)\} = te^{-t}$

By convolution thm

take $t = u$ in $\phi_1(s)$

& $t = (t-u)$ in $\phi_2(s)$

$\Rightarrow L^{-1}\{\phi_1(s)\phi_2(s)\} = \int_0^t f_1(t)\, f_2(t-u)\, du$

$= \int_0^t \cancel{t}u \times (t-u)e^{-(t-u)}\, du$

$= \int_0^t (ute^{-t+u} - u^2 e^{-t+u})\, du$

$= te^{-t}\int_0^t ue^u\, du - e^{-t}\int_0^t e^u . u^2\, du$

$= te^{-t}\left\{u.e^u - e^u\right\}_0^t$

$-e^{-t}\left\{u^2.e^u - \int 2ue^u du\right\}_0^t$

$= te^{-t}\{te^t - e^t - 0 + 1\}$

$-e^{-t}\{u^2.e^u - 2ue^u + 2e^u\}_0^t$

$= t^2 - t + te^{-t} - e^{-t}\{t^2e^t - 2te^t + 2e^t - 0 + 0 - 2\}$

$= \cancel{t^2} - t + te^{-t} - \cancel{t^2} + 2t - 2 + 2e^{-t}$

$\boxed{= te^{-t} + 2e^{-t} + t - 2}$

Q.4 Applying convolution theorem show that

$$\int_0^t \sin u \cos(t-u)\, du = \frac{1}{2} t \sin t$$

Q.5 $L^{-1}\left\{\frac{s^2}{(s^2+\omega^2)^2}\right\} \longrightarrow \frac{1}{2}\left[t\cos\omega t + \frac{1}{\omega}\sin\omega t\right]$

Q.6 $L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} \longrightarrow \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

Q.7 $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} \longrightarrow \frac{a\sin at - b\sin bt}{a^2-b^2}$

Application to solve differential equations :-

$$L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Q Solve $\frac{d^4x}{dt^4} - a^4 x = 0$, where a is constant, using Laplace transform, given that $x=1$, $x'=x''=x'''=0$ at $t=0$.

A taking laplace on both sides we get

$$L\left\{\frac{d^4x}{dt^4}\right\} - a^4 L\{x\} = 0$$

$$s^4\bar{x} - s^3 x(0) - s^2 x'(0) - s x''(0) - x'''(0) - a^4\bar{x} = 0$$

$$\bar{x}(s^4 - a^4) = s^3 x(0) + s^2 x'(0) + s x''(0) + x'''(0)$$

using initial conditions.

$$\bar{x} = \frac{s^3}{s^4 - a^4} = \frac{s^3}{(s^2+a^2)(s^2-a^2)} = \frac{s^3}{(s^2+a^2)(s+a)(s-a)}$$

$$= \frac{1}{4(s-a)} + \frac{1}{4(s+a)} + \frac{1}{2}\frac{s}{(s^2+a^2)}$$

taking inverse laplace transform on both sides

$$L^{-1}\{\bar{x}\} = \frac{1}{4}L^{-1}\left\{\frac{1}{(s-a)}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s+a}\right\} + \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$$

$$x(t) = \frac{1}{4}\times e^{at} + \frac{1}{4}e^{-at} + \frac{1}{2}\times \cos at$$

or

$$x(t) = \frac{1}{2}[\cosh at + \cos at]$$

<u>Q</u>. Using L.T., solve

ⓐ $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ under

$y(0)=1, y'(0)=2, y''(0)=2.$

ⓑ $(D-1)(D-2)(D-3)x = 5$; $x=0$, $\frac{dx}{dt}=1$, $\frac{d^2x}{dt^2}=0$ at $t=0$

ⓐ Taking L.T. on both sides

$$L\{y'''(t)\} + 2L\{y''(t)\} - L\{y'(t)\} - 2L\{y(t)\} = 0$$

$$s^3\bar{y} - s^2y(0) - sy'(0) - y''(0) + 2[s^2\bar{y} - sy(0) - y'(0)]$$
$$- [s\bar{y} - y(0)] - 2\bar{y} = 0$$

$y(0)=1, y'(0)=2, y''(0)=2$

$$s^3\bar{y} - s^2 - 2s - 2 + 2[s^2\bar{y} - s - 2] - [s\bar{y} - 1] - 2\bar{y} = 0$$

$$\bar{y}(s^3+2s^2-s-2) = s^2+2s+2+2s+4+1$$

$$\bar{y} = \frac{s^2+4s+7}{s^3+2s^2-s-2}$$

$$= \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

$$\bar{y} = \frac{5}{3(s-1)} - \frac{1}{(s+1)} + \frac{1}{3(s+2)}$$

Taking Laplace inverse on both sides

$$\boxed{y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}}$$

Q. Find the solⁿ of the Initial value problem

$$y'' + 4y' + 4y = 12t^2 e^{-2t},$$

$y(0) = 2$ & $y'(0) = 1$.

Q. Solve the foll. differential eqⁿ, using L.T.

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t, \quad y(0) = 1$$

Q. Solve the simultaneous equations

$$\frac{dx}{dt} + \frac{dy}{dt} + x = -e^{-t}$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0$$ given that $x(0) = -1$, $y(0) = 1$.

Solⁿ Taking L.T. on both sides

$$L\{x'(t)\} + L\{y'(t)\} + L\{x(t)\} = -L\{e^{-t}\}$$

& $L\{x'(t)\} + 2L\{y'(t)\} + 2L\{x(t)\} + 2L\{y(t)\} = 0$

$$\Rightarrow \quad s\bar{x} - x(0) + s\bar{y} - y(0) + \bar{x} = \frac{-1}{s+1}$$

& $s\bar{x} - x(0) + 2[s\bar{y} - y(0)] + 2\bar{x} + 2\bar{y} = 0$

using $x(0) = -1$ & $y(0) = 1$

$$s\bar{x} + \cancel{1} + s\bar{y} - \cancel{1} + \bar{x} = \frac{-1}{s+1} \Rightarrow \bar{x}(s+1) + s\bar{y} = \frac{-1}{s+1} \quad (1)$$

$$s\bar{x} + 1 + 2[s\bar{y} - 1] + 2\bar{x} + 2\bar{y} = 0$$

$$\Rightarrow \bar{x}(s+2) + \bar{y}(2s+2) = 1 \quad (2)$$

(1) × (2s+2) − (2) × s

$$\bar{x}(s+1)(2s+2) + s(2s+2)\bar{y} = -\frac{(2s+2)}{(s+1)}$$

$$\underline{-}\bar{x}\, s(s+2) \quad \underline{+}\, \cancel{s(2s+2)\bar{y}} = \underline{-}\, s$$

$$\bar{x}\left[(s+1)(2s+2) - s(s+2)\right] = -\frac{(2s+2)}{(s+1)} - s$$

$$\bar{x}(2s^2+2+4s-s^2-2s) = \frac{-2s-2-s^2-s}{(s+1}$$

$$\bar{x}(s^2+2s+2) = \frac{-s^2-3s-2}{(s+1)}$$

$$\bar{x} = \frac{-(s^2+3s+2)}{(s^2+2s+2)(s+1)}$$

$$\boxed{\bar{x} = -\frac{(s+2)}{(s^2+2s+2)}} = -\left[\frac{(s+1)}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right] \quad \text{—— (3)}$$

from (1)

$$-\frac{(s+2)(s+1)}{(s^2+2s+2)} + s\bar{y} = \frac{-1}{(s+1}$$

$$s\bar{y} = \frac{-1}{(s+1)} + \frac{(s+2)(s+1)}{(s^2+2s+2)}$$

$$s\bar{y} = \frac{(s+2)(s+1)^2 - (s^2+2s+2)}{(s+1)(s^2+2s+2)}$$

$$\bar{y} = \frac{s^2+2s+2+s+1}{(s^2+2s+2)(s+1)} = \frac{1}{(s+1)} + \frac{1}{(s+1)^2+1}$$

$$\boxed{\bar{y} = \frac{1}{(s+1)} + \frac{1}{(s+1)^2+1}} \quad \text{—— (4)}$$

taking Laplace inverse on both sides of (3) & (4) we get

$$\boxed{x(t) = -e^{-t}(\cos t + \sin t)} \quad \& \quad \boxed{y(t) = e^{-t}(1+\sin t)}$$

Q: Solve the foll. simultaneous eqⁿ using Laplace Transform.

$$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$$

$$\frac{dx}{dt} + 2y = e^{-t}.$$

—X—X—X—X—

Q: Solve the foll. simultaneous eqⁿ using Laplace Transform

$$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$$

$$\frac{dx}{dt} + 2y = e^{-t}$$

—x—x—x—x—

FOURIER SERIES

Periodic function :- A fuⁿ $y = f(x)$ is said to be periodic fuⁿ if there exists a +ve real number P s.t.

$$\boxed{f(x+P) = f(x)} \quad \forall x \in \mathbb{R}$$

The least value of the +ve real no. P is called the fundamental period of the function.

or period of a fuⁿ at which fuⁿ repeats itself.

e.g.

$\sin(x+2\pi) = \sin x$

$\cos(x+2\pi) = \cos x$

~~$\tan(x+\pi) = \tan x$~~

Note :-

$\sin(n\pi) = 0 \quad \forall n \in \mathbb{Z}$

$\cos(n\pi) = 1$ (even)

$\cos(n\pi) = -1$ (odd)

Fourier Series :- fourier series is an infinite series representation of periodic fuⁿ in terms of trignometric fuⁿ (Sine & cosine).

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]}$$

$$\boxed{-l < x < l}$$

$$\boxed{a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx}$$

$$\boxed{b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

Even function :- If we put $-x$ in place of x then

$$f(-x) = f(x)$$

Hence the funⁿ is even

e.g. $\cos x,\ x^2$, etc.

Odd function :- $f(-x) = -f(x)$

Hence the funⁿ is odd

e.g. ~~cos x, x~~ $\sin x,\ x,\ x^3$, etc.

Note :-

→ for even function

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

→ & for odd function

$$\int_{-l}^{l} f(x)\,dx = 0.$$

$\boxed{\text{If } f(x) \text{ is even}}$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$\boxed{a_0 = \frac{1}{l}\int_{0}^{l} f(x)\,dx}$$

$$\boxed{a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx}$$

$\boxed{b_n = 0}$ (∵ sine funⁿ is odd)

$$\Rightarrow \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)}$$

$\boxed{\text{If } f(x) \text{ is odd}}$

$\boxed{a_0 = 0}$, $\boxed{a_n = 0}$

$$\boxed{b_n = \frac{2}{\pi}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

$$\Rightarrow$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)}$$

Types :-

1. 0 to 2π
2. $-\pi$ to π
3. 0 to $2l$
4. $-l$ to l

General formula :-

for $c < x < c+2l$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

if $0 < x < 2\pi$

then $c = 0$

$c + 2l = 2\pi$

$2l = 2\pi \Rightarrow \boxed{l = \pi}$

if $-\pi < x < \pi$

the $c = -\pi$

$c + 2l = \pi$

$\Rightarrow -\pi + 2l = \pi$

$2l = 2\pi \Rightarrow \boxed{l = \pi}$

~~if if $-l < x < l$~~

~~$c = -l$~~

~~& $c + 2l = l$~~

~~$2l = 2l$~~ $\Rightarrow$

Q. find a fourier series to represent $x-x^2$ from $x=-\pi$ to π and hence show that $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\frac{\pi^2}{12}$.

Sol^n

$$f(x)=x-x^2 \quad \text{when} \quad -\pi\le x\le \pi$$

Step-I:- ~~for~~ first check fu^n is even or odd

$$f(-x)=-x-x^2$$

this show $f(x)$ is neither even nor odd

Step-II :- find a_0, a_n & b_n accordingly

$$a_0=\frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx$$

$$=\frac{1}{2\pi}\left\{\frac{1}{2}(x^2)-\frac{1}{3}(x^3)\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\left\{\frac{1}{2}(\pi^2-\pi^2)-\frac{1}{3}(\pi^3+\pi^3)\right\}$$

$$=-\frac{1}{\cancel{6}_3\pi}\times\cancel{2}\pi^3=\boxed{\frac{-\pi^2}{3}} \quad -(1)$$

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{1}{\cancel{l}\pi}\int_{-\pi}^{\pi}(x-x^2)\cos\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)dx$$

$$=\frac{1}{\pi}\left\{(x-x^2)\times\frac{\sin(nx)}{n}-\int(1-2x)\times\frac{\sin(nx)}{n}\,dx\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\frac{(x-x^2)\sin(nx)}{n}-\left(\frac{(1-2x)}{n}\times\frac{(-\cos nx)}{n}+\frac{(-2)}{n}\int\frac{(-\cos(nx))}{n}\,dx\right)\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\frac{(x-x^2)\sin(nx)}{n}-\left(-\frac{\cos nx(1-2x)}{n^2}+\frac{2\sin nx}{n^3}\right)\right\}_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{(\pi-\pi^2)\sin(n\pi)}{n} + \frac{\cos(n\pi)(1-2\pi)}{n^2} - \frac{2\sin n\pi}{n^3} - \frac{(-\pi-\pi^2)\sin(-n\pi)}{n} - \frac{\cos(-n\pi)(1+2\pi)}{n^2} + \frac{2\sin(-n\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left(0 + \frac{(-1)^n}{n^2}(1-2\pi) - 0 - 0 - \frac{(-1)^n(1+2\pi)}{n^2} + 0\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{(-1)^n}{n^2}} - \frac{2\pi(-1)^n}{n^2} - \cancel{\frac{(-1)^n}{n^2}} - \frac{2\pi(-1)^n}{n^2}\right]$$

$$\boxed{= \frac{-4}{n^2}(-1)^n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin(nx)\,dx.$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} x\sin(nx)\,dx - \underbrace{\int_{-\pi}^{\pi} x^2\sin(nx)\,dx\right\}}_{\text{since odd} \Rightarrow 0}$$

$$= \frac{1}{\pi}\left\{x\frac{(-\cos nx)}{n} + \frac{1}{n}\int\cos(nx)\,dx\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{x}{n}\cos nx + \frac{\sin(nx)}{n^2}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{n}\cos(n\pi) + \frac{\sin(n\pi)}{n^2} + \frac{(-\pi)}{n}\cos(n\pi) - \frac{\sin(-n\pi)}{n^2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{n}(-1)^n - \frac{\pi}{n}(-1)^n\right\}$$

$$= \frac{-2}{n}(-1)^n$$

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow f(x) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{-4}{n^2}(-1)^n \cos nx - \frac{2}{n}(-1)^n \sin nx\right]$$

$$= \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$

putting $x = 0$ on both sides, we get

$$f(0) = 0 = \frac{-\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]$$

or

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}}.$$

H.P.

Q. Expand $x \sin x$ in fourier series in $(-\pi, \pi)$.

S. $f(x) = x \sin x$

$f(-x) = (-x)\sin(-x) = x \sin x$

$\Rightarrow f(-x) = f(x)$

this shows $f(x)$ is an even funⁿ

Hence $b_n = 0$

$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$ & $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\left\{-x\cos x + \int \cos x\,dx\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-x\cos x + \sin x\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\pi\cos\pi + \sin\pi + 0 - 0\right\} = \frac{1}{\pi}\{\pi\} = 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos(nx)\, dx$$

$$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\pi} x\,[2\sin x \cos(nx)]\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \times [\sin(x+nx) - \sin(nx-x)]\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(n+1)x\, dx - \int_0^{\pi} x \sin(n-1)x\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{(n+1)} + \int \frac{\cos(n+1)x}{(n+1)}\, dx + x\frac{\cos(n-1)x}{(n-1)} - \int \frac{\cos(n-1)x}{(n-1)}\, dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x}{(n+1)}\cos(n+1)x + \frac{1}{(n+1)^2}\sin(n+1)x + \frac{x}{(n-1)}\cos(n-1)x - \frac{1}{(n-1)^2}\sin(n-1)x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{(n+1)}\cos(n+1)\pi + \frac{1}{(n+1)^2}\sin(n+1)\pi + \frac{\pi}{(n-1)}\cos(n-1)\pi - \frac{1}{(n-1)^2}\sin(n-1)\pi + 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{(n+1)}(-1)^{n+1} + \frac{\pi}{(n-1)}(-1)^{n-1}\right]$$

$$= \frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)} = (-1)^n\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right] = \frac{(-1)^{n+1}.2}{(n^2-1)}$$

for $n = 2, 3, 4, \dots$

for $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\, dx = \frac{\cancel{2}}{\cancel{\pi}}\,\frac{1}{\pi}\int_0^{\pi} x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left\{\frac{-x\cos 2x}{2} + \int \frac{\cos 2x}{2}\, dx\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-x}{2}\cos 2x + \frac{1}{4}\sin 2x\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{2}\cos 2\pi + \frac{1}{4}\sin 2\pi + 0 - 0\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{2} + 0\right\} = \frac{-1}{2}.$$

$$\boxed{\Rightarrow f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{(n^2-1)}\cos nx}$$

Q The function $f(x)$ is given by $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ find its fourier series and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}, \qquad f(0) = \frac{f(0-) + f(0+)}{2} = \frac{-\pi + 0}{2} = \frac{-\pi}{2}$$

Sol Let the fourier series of the given fu^n be

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right].$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{-\pi(0+\pi) + \frac{1}{2}(\pi^2 - 0)\right\}$$

$$= \frac{1}{2\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = \frac{-\pi^{\cancel{2}}}{2} \times \frac{1}{2\cancel{\pi}} = \boxed{\frac{-\pi}{4}}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-\pi)\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{0} + \left[\frac{x\sin(nx)}{n} - \int\frac{\sin(nx)}{n}dx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi}{n}[0+0] + \frac{1}{n}\left[x\sin(nx) + \frac{\cos(nx)}{n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}\left[\pi \sin(n\pi) + \frac{\cos(n\pi)}{n} - 0 - \frac{1}{n}\right]\right\}$$

$$= \frac{1}{n\pi}\left(0 + \frac{(-1)^n}{n} - \frac{1}{n}\right)$$

$$\boxed{= \frac{1}{n^2\pi}\left((-1)^n - 1\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-\pi)\sin(nx)\,dx + \int_0^{\pi} x\sin(nx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-(-\pi)}{n}(\cos(nx))_{-\pi}^{0} + \left\{\frac{-x\cos(nx)}{n} + \int\frac{\cos(nx)}{n}dx\right\}_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}(1-(-1)^n) + \frac{1}{n}\left(-x\cos(nx) + \frac{1}{n}\sin(nx)\right)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}(1-(-1)^n) + \frac{1}{n}\left(-\pi(-1)^n + 0 + 0 - 0\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n} - \frac{(-1)^n\pi}{n} - \frac{\pi}{n}(-1)^n\right\}$$

$$= \frac{\cancel{\pi}}{\cancel{\pi}n}\left(1 - (-1)^n - (-1)^n\right) \boxed{= \frac{1}{n}(1-2(-1)^n)}$$

$$\Rightarrow \boxed{f(x) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]\cos(nx) + \sum_{n=1}^{\infty}\frac{1}{n}[1-2(-1)^n]\sin(nx)}$$

If we put $x=0$

$$f(0) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1] + \sum_{n=1}^{\infty}\frac{1}{n}\times 0$$

$$\frac{-\pi}{2} = \frac{-\pi}{4} + \frac{1}{\pi}\left[(-1-1) + \frac{1}{4}(1-1) + \frac{1}{9}(-1-1) + 0 + \frac{1}{25}(-1-1) + \cdots\right]$$

$$\frac{-\pi}{2} = \frac{-\pi}{4} + \frac{1}{\pi}\left[-2 - \frac{2}{9} - \frac{2}{25} + \cdots\right]$$

$$\frac{-\pi}{2}\left[\frac{\pi}{4} - \frac{\pi}{2}\right] = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

Q. Find the Fourier series for the funⁿ

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

Q. Obtain the fourier series for the funⁿ $f(x)$ given by

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

[only even case will be considered because of interval]

Q. Expand $f(x) = |\cos x|$ as a fourier series in $(-\pi, \pi)$

$|\cos x| \rightarrow$ even funⁿ.

Note :-

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx - b \sin bx\right]$$

Types of Limits

- $(0 \text{ to } 2\pi)$
- $(-\pi \text{ to } \pi)$ $\rightarrow$ Even/odd
- $(0 \text{ to } 2l)$
- $(-l \text{ to } l)$ Half range Sine/cosine series

Half Range Series :-

① Half Range fourier Sine series :- (Odd)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad , 0<x<l$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

② Half Range fourier cosine series :- (even)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad , 0<x<l$$

where

$$a_0 = \frac{1}{l}\int_0^l f(x)\, dx$$

& $a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

Q. Obtain the half range sine series for $f(x) = 2 - x$ for $0 < x < 2$ and hence deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Solⁿ.

for $0 < x < l$, $l = 2$

& for half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

$$\& \quad b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{2}\int_0^2 (2-x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= 2\int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= -2\left[\cos\left(\frac{n\pi x}{2}\right) \Big/ \frac{n\pi}{2}\right]_0^2 - \left[-x\cos\left(\frac{n\pi x}{2}\right) \times \frac{2}{n\pi}\right.$$

Q. Express $f(x) = x$ as a half range cosine series for $0 < x < 2$.

Sol

Here for $0 < x < l$

$l = 2$

half range cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx \quad \&$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow \quad a_0 = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{4}(x^2)_0^2 = 1$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left\{ x\left[\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right] + \int \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}dx \right\}_0^2$$

$$= \left\{ \frac{-2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right) \right\}_0^2$$

$$= \left\{ \frac{-4}{n\pi}\sin(n\pi) + \left(\frac{2}{n\pi}\right)^2\cos(n\pi) + 0 - \left(\frac{2}{n\pi}\right)^2\cos 0 \right.$$

$$= \left\{ 0 + \frac{4}{n^2\pi^2}(-1)^n - \left(\frac{4}{n^2\pi^2}\right) \right\}$$

$$= \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$$

Q.

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{2}\right)}$$

$$+\int \cos\left(\frac{n\pi x}{2}\right) \times \frac{2}{n\pi}\, dx \Big]_0^2$$

$$= -2\left[\frac{2}{n\pi}\left(\cos(n\pi)-1\right)\right] - \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -2\left[\frac{2}{n\pi}\{(-1)^n-1\}\right] - \left[\frac{-4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\sin(n\pi) + 0 - 0\right]$$

$$= \frac{-4}{n\pi}[(-1)^n-1] + \frac{4}{n\pi}(-1)^n - \frac{4}{n^2\pi^2}\times 0.$$

$$= \frac{4}{n\pi}(-1)^n[1-1] + \frac{4}{n\pi} \quad \boxed{=\frac{4}{n\pi}}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

$$= \frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) + \frac{4}{2\pi}\sin\left(\frac{2\pi x}{2}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{2}\right) + \cdots$$

if $x = 1$

$\cancel{f(1) = 1 = \frac{4}{\pi}\cos(\pi/2) + \frac{2}{\pi}\sin}$

$$f(1) = 1 = \frac{4}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{2}{\pi}\sin(\pi) + \frac{4}{3\pi}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{\pi}\sin(2\pi) + \cdots$$

$$1 = \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} - \cdots$$

$$\Rightarrow 1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots}$$

Q. Express $f(x) = x^2$ as half range cosine series for $0 < x < 2$.

Fourier series of functions with Arbitrary interval :-

for $c < x < c+2l$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Q. Find the Fourier Series of the function

$$f(x) = \begin{cases} x & , 0<x<1 \\ 1-x & , 1<x<2. \end{cases}$$

Solⁿ for $0<x<2$.

Hence $c=0$ & $c+2l = 2$.

$\Rightarrow l = 1$.

Hence

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\left\{\int_0^1 x\,dx + \int_1^2 (1-x)\,dx\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} + (2-1) - \frac{1}{2}(4-1)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} + 1 - \frac{3}{2}\right\} = 0.$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (1-x)\cos(n\pi x)\,dx$$

$$= \left[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1 + \left[\frac{(1-x)\sin(n\pi}{n\pi}\right.$$

$$+ \int \frac{\sin(n\pi x)}{n\pi} dx \Bigg]_1^2$$

$$= \left[\frac{\sin(n\pi)}{n\pi} + \frac{\cos(n\pi)}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2} \right] + \left[\frac{(1-x)\sin(n\pi x)}{n\pi} - \frac{\cos(n\pi x)}{n^2\pi^2} \right]_1^2$$

$$= \left(0 + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} \right) + \left(-\frac{\sin(2n\pi)}{n\pi} - \frac{\cos(2n\pi)}{n^2\pi^2} - 0 + \frac{\cos(n\pi)}{n^2\pi^2} \right)$$

$$= \frac{1}{n^2\pi^2}\left((-1)^n - 1 \right) + \left(0 - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2} \right)$$

$$= \frac{1}{n^2\pi^2}\left((-1)^n - 1 \right) + \frac{1}{n^2\pi^2}\left((-1)^n - 1 \right)$$

$$= \frac{2}{n^2\pi^2}\left((-1)^n - 1 \right)$$

$$\Rightarrow \cancel{f(x) = \sum_{n=1}^{\infty}}$$

$$b_n = \int_0^2 f(x) \sin(n\pi x) \, dx$$

$$= \int_0^1 x \sin(n\pi x) \, dx + \int_1^2 (1-x) \sin(n\pi x) \, dx$$

$$= \left[-\frac{x\cos(n\pi x)}{n\pi} + \int \frac{\cos(n\pi x)}{n\pi} dx \right]_0^1 + \left[-\frac{(1-x)\cos(n\pi x)}{n\pi} + \int -\frac{\cos(n\pi x)}{n\pi} dx \right]_1^2$$

$$= \left[\frac{-x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x) \right]_0^1 + \left[\frac{(x-1)}{n\pi}\cos(n\pi x) - \frac{\sin(n\pi x)}{n^2\pi^2} \right]_1^2$$

$$= \left[\frac{-1}{n\pi}\cos(n\pi) + 0 + 0 - 0 \right] + \left[\frac{1}{n\pi}\cos(2n\pi) - 0 - 0 + 0 \right]$$

$$= \frac{-1}{n\pi}(-1)^n + \frac{1}{n\pi} = \frac{1}{n\pi}[1-(-1)^n].$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(n\pi x) + \frac{1}{n\pi}[1-(-1)^n] \sin(n\pi x)$$

Q. Express $f(x) = x^2$ in Fourier series for $0 < x < 2$.

— x ——— x ——— x ——— x ———

Fourier Integral and Fourier Integral theorem

Fourier Integral Theorem :-

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda}$$

The integral on the R.H.S. is called Fourier Integral of $f(x)$.

Fourier cosine & sine integrals :-

cosine integral (even funⁿ)

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda x\cos\lambda t\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\cos\lambda x\left\{\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right\}d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty}\cos\lambda x\int_0^{\infty} f(t)\cos\lambda t\,dt\,d\lambda \quad \text{(even)}$$

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda}$$

or

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\lambda x\left\{\int_0^{\infty} f(t)\cos\lambda t\,dt\right\}d\lambda}$$

Sine Integral (Odd funⁿ)

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\sin\lambda t \sin\lambda x \, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\sin\lambda x\left\{\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt\right\} d\lambda$$

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t\, dt\, d\lambda}$$

or

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\sin\lambda t\sin\lambda x\, dt\, d\lambda}$$

FOURIER TRANSFORMS :-

The function $F(s)$, defined by

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$ } fourier transform.

is called Fourier Transform of $f(x)$.

Also the funⁿ $f(x)$, defined by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$ } Inversion Fourier Transform

is called Inverse Fourier Transform of $F(s)$

Q. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & , |x|<1 \\ 0 & , |x|>1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx$

A.

$$F\{f(x)\} = F(s) = \int_{\infty}^{\infty} f(x) e^{-isx}\, dx$$

$$= \int_{-\infty}^{-1} f(x) e^{-isx}\, dx + \int_{-1}^{1} f(x) e^{-isx}\, dx + \int_{1}^{\infty} f(x) e^{-isx}\, dx$$

$$= 0 + \int_{-1}^{1} 1 \cdot e^{-isx}\, dx + 0$$

$$= \left[\frac{e^{-isx}}{(-is)} \right]_{-1}^{1} = \frac{-1}{is}\left(e^{-is} - e^{is}\right)$$

$$= \frac{i}{s}\left(e^{-is} - e^{is}\right)$$

$$= \frac{2\sin s}{s} \qquad \left[\because \sin x = \frac{e^{ix} - e^{-ix}}{2} \right.$$

$$\Rightarrow F(s) = \frac{2\sin s}{s}$$

from inversion of Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{\infty}^{\infty} F(s) e^{-isx}\, ds$$

if $x = 0$, then

$$1 = f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin s}{s} \times e^{0}\, ds$$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds$$

$\sin s \to$ odd
$s \to$ odd
$\Rightarrow \sin s/s \to$ eve

$$\Rightarrow 2\int_0^\infty \frac{\sin s}{s}\,ds = \pi$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\sin s}{s}\,ds = \frac{\pi}{2}}$$

Q. Obtain the Fourier transform of the function $f(x)$ given by

$$f(x) = \begin{cases} 1-x^2, & \text{for } |x| \le 1 \\ 0, & \text{o/w} \end{cases}$$

and hence evaluate $\displaystyle\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx.$

S

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{-isx}\,dx + \int_{-1}^{1} f(x)\, e^{isx}\,dx + \int_1^\infty f(x)\, e^{isx}\,dx$$

$$= 0 + \int_{-1}^{1} (1-x^2)\, e^{isx}\,dx + 0$$

$$= \int_{-1}^{1} e^{isx}\,dx - \int_{-1}^{1} x^2 e^{isx}\,dx$$

$$= \left(\frac{e^{isx}}{is}\right)_{-1}^{1} - \left[x^2 \frac{e^{isx}}{is} - \int 2x \frac{e^{isx}}{is}\,dx\right]_{-1}^{1}$$

$$= \frac{1}{is}\left\{e^{is} - e^{-is}\right\} - \frac{1}{is}\left[x^2 e^{isx} - 2\left\{x\frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2}\right\}\right]_{-1}^{1}$$

$$= \frac{2\sin s}{s} - \frac{1}{is}\left\{x^2 e^{isx} - \frac{2x e^{isx}}{is} - \frac{2e^{isx}}{s^2}\right\}_{-1}^{1}$$

$$= \frac{2\sin s}{s} - \frac{1}{is}\left\{e^{is} - e^{-is} - \frac{2e^{is}}{is} - \frac{2e^{-is}}{is} - \frac{2e^{is}}{s^2} + \frac{2e^{-is}}{s^2}\right\}$$

$$\frac{2\sin s}{s} - \frac{2}{is}\left\{\sin s + \frac{2\cos s}{s} - \frac{2\sin s}{s^2}\right\}$$

$$\cancel{\frac{2\sin s}{s}} - \cancel{\frac{2\sin s}{s}} - \frac{4\cos s}{s^2} + \frac{4\sin s}{s^3}$$

$$= \frac{-4}{s^3}(s\cos s - \sin s)$$

from inversion of Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)(\cos sx - i\sin sx)\,ds$$

$$= \frac{-4}{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)\cos sx}{s^3}\,ds}_{\text{even}}$$

$$+ \frac{4i}{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\sin sx\,ds}_{\text{odd}}$$

$$= \frac{-2}{\pi}\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos sx\,dx + 0$$

$$= \frac{-4}{\pi}\int_{0}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos sx\,dx$$

let $x = \frac{1}{2}$

$$\frac{3}{4} = f(1/2) = \frac{-4}{\pi}\int_{0}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}\,ds$$

$$\boxed{\frac{-3\pi}{16} = \int_{0}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\left(\frac{s}{2}\right)ds}$$

Q. Find the Fourier transform of the function e^{-ax^2}. $a>0$.

Q. Find the Fourier transform of the fun xe^{-ax^2}, $a>0$.

Fourier Sine Transform:-

The function $F_s(s)$, defined by

$$F_s(s) = \int_0^\infty f(x)\sin sx\, dx$$

is called fourier sine transform of $f(x)$ in $0<x<\infty$

Also the fun $f(x)$, defined by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin sx\, ds$$

is called inverse fourier sine transform of $F_s(s)$.

Q Find the Fourier sine transform of $\frac{1}{x}$.

Sol

$$F_s(s) = \int_0^\infty f(x)\sin sx\, dx$$

$$= \int_0^\infty \frac{\sin sx}{x}\, dx$$

let $sx = t$
$s\,dx = dt$

$$\Rightarrow = \int_0^\infty \frac{\sin t}{t/\cancel{s}} \times \frac{dt}{\cancel{s}}$$

$$\Rightarrow F_s(s) = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\sqrt{\pi}}{2}$$

$$\boxed{F_s(s) = \frac{\sqrt{\pi}}{2}}$$

Q. Find Fourier Sine transform of $e^{-|x|}$. Hence show that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx = \frac{\pi e^{-m}}{2}, \quad m>0$$

Sol

$$f(x) = \begin{cases} e^{x}, & x>0 \\ e^{-x}, & x<0 \end{cases}$$

$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\infty e^{-x} \sin sx \, dx$$

use — $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

$$= \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left\{ \frac{e^{-x}}{1+s^2} (\sin sx - s \cos sx) \right\}_0^\infty$$

$$= \left(0 - \frac{1}{1+s^2}(0-s)\right)$$

$$\boxed{= \frac{s}{1+s^2}}$$

By inversion of fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, dx$$

$$e^{-s} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} (\sin sx) \, dx$$

$$\frac{\pi e^{-s}}{2} = \int_0^\infty \frac{s \cdot \sin(sx)}{1+s^2} \, dx$$

or

$$\boxed{\int_0^\infty \frac{x \sin(mx)}{1+x^2} \, dx = \frac{\pi e^{-m}}{2}}$$

Fourier cosine transform :-

The funⁿ $F_c(s)$, defined by

$$F_c(s) = \int_0^\infty f(x) \cos sx \, dx$$

is called Fourier cosine transform of $f(x)$ in $0 < x < \infty$

Also the fu^n $f(x)$. defined by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\,dx$$

~~Q. Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$.~~

~~Sol~~

$$\cancel{F_c(s) = \int_0^\infty f(x)\cos sx\,dx}$$

$$\cancel{= \int_0^\infty \frac{\cos sx}{1+x^2}\,dx}$$

Q. Find the Fourier cosine transforms of

$$f(x) = \begin{cases} 1 & , 0 < x < a \\ 0 & , x > a. \end{cases}$$

Sol

$$F_c(s) = \int_0^\infty f(x)\cos sx\,dx$$

$$= \int_0^a f(x)\cos sx\,dx + \int_a^\infty f(x)\cos sx\,dx$$

$$= \int_0^a \cos sx\,dx + 0$$

$$= \left[\frac{\sin(sx)}{s}\right]_0^a = \boxed{\frac{\sin(as)}{s}}$$

Q. Do Fourier sine & cosine transform of e^x exist? Expla-

Sol e^x is not a bounded fu^n. & $\int_{-\infty}^\infty |e^x|\,dx$ doesn't exist.

$\Rightarrow$ Fourier sine & cosine transform of e^x doesn't exist.

Q. Find the cosine & sine transform of $f(x) = 2e^{-5x} + 5e^{-2x}$.

Properties of Fourier Transforms :-

① Linearity property :-

$F_1(\lambda)$ & $F_2(\lambda)$ are Fourier Transforms of functions $f_1(x)$ & $f_2(x)$ resp. then

$$\boxed{F[c_1 f_1(x) + c_2 f_2(x)] = c_1 F_1(\lambda) + c_2 F_2(\lambda)}$$

fourier transform

② Change of Scale :-

$$\boxed{F\{f(ax)\} = \frac{1}{a} F\left(\frac{\lambda}{a}\right)}$$

③ Shifting prop. :-

$$\boxed{F[f(x-a)] = e^{-i\lambda a} F(\lambda)}$$

④ Modulation prop :-

$$\boxed{F[f(x)e^{iax}] = F(\lambda - a)}$$

⑤ Convolution thm. :-

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

If $F[f(x)] = F(\lambda)$ &

$F[g(x)] = G(\lambda)$ then

$$\boxed{\begin{aligned} F[f(x) * g(x)] &= F(f(x))\, F(g(x)) \\ &= F(\lambda)\, G(\lambda) \end{aligned}}$$

Fourier Transforms of the derivatives of a fu^n

$u(x,t) \rightarrow$ fu^n of x & t

$\bar{u}(s,t) \longrightarrow$ Fourier Transform of u w.r.t. to x

$$\bar{u}(s,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-isx}\, dx$$

① Fourier transform of $\frac{\partial u}{\partial x}$ w.r.t. to x if $u \rightarrow 0$ as $x \longrightarrow \pm\infty$

$$F\left(\frac{\partial u}{\partial x}\right) = \int_{-\infty}^{\infty} \underset{II}{\frac{\partial u}{\partial x}}\, \underset{I}{e^{-isx}}\, dx$$

$$= \cancel{\partial u} \left[e^{-isx} \times u(x,t) + is \int \frac{e^{-isx}}{\cancel{(-is)}} \times U\, dx \right]_{-\infty}^{\infty}$$

$$= (0-0) + \frac{is}{\cancel{is}} \int_{-\infty}^{\infty} e^{-isx}\, u(x,t)\, dx$$

$$\boxed{F\left(\frac{\partial u}{\partial x}\right) = is\, \bar{u}(s,t)}$$

② Fourier Transform of $\frac{\partial^2 u}{\partial x^2}$ w.r.t. to x. If $u \to 0$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\, e^{-ixs}\, dx$$

$$= \left[e^{-isx} \times \frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} + is \int_{-\infty}^{\infty} e^{-isx}\, \frac{\partial u}{\partial x}\, dx$$

$$= 0 + is \times is\, \bar{u}(s,t)$$

$$\boxed{F\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 \bar{u}(s,t)}$$

In general,

$$\boxed{F\left(\frac{\partial^n u}{\partial x^n}\right) = \cancel{(-i\tfrac{d}{ds})^n}\; (-is)^n\, \bar{u}(s,t)}$$

③ Fourier transform of $\frac{\partial u}{\partial t}$ w.r.t. to x

$$F\left(\frac{\partial u}{\partial t}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\, e^{-isx}\, dx$$

$$= \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\, e^{-isx}\, dx$$

$$\boxed{F\left(\frac{\partial u}{\partial t}\right) = \frac{d}{dt}\, \bar{u}(s,t)}$$

④ Fourier sine and cosine transforms of $\frac{\partial^2 u}{\partial x^2}$ w.r.t. to x

Fourier Sine transform of u

$$\boxed{\bar{u}_s(s,t) = \int_0^{\infty} u(x,t) \sin sx\, dx}$$

$$\Rightarrow \boxed{F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2 F_c[u(x,t)]}$$

or

$$\boxed{F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\frac{\partial u(0,t)}{\partial x} - s^2 \bar{u}(s,t)}$$

Applications of Fourier transforms to Boundary Value Problems :-

① for the interval $-\infty < x < \infty$ we use infinite Fourier Transform.

② For the interval $0 < x < \infty$ we use Fourier sine or cosine transform depending upon the boundary conditions as listed below

(i) $u \to 0$ & $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$.

or

If $u(x,t)$ is given at $x=0$, then use infinite Fourier sine transform

(ii) If $\left[\frac{\partial u}{\partial x}(x,t)\right]$ is given at $x=0$, then use infinite fourier cosine ~~sin~~ transform.

Q. Solve : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t \geq 0$ with conditions $u(x,0) = f(x)$.

Solⁿ Here $u(x,t)$ & $-\infty < x < \infty$ then we apply Fourier transform on both sides

$$F\left(\frac{\partial u}{\partial t}\right) = \alpha^2 F\left(\frac{\partial^2 u}{\partial x^2}\right)$$

Fourier Sine transform of $\frac{\partial^2 u}{\partial x^2}$

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$= \left[\sin sx \times \frac{\partial u}{\partial x}\right]_0^\infty - s\int_0^\infty \cos sx \times \frac{\partial u}{\partial x} dx$$

$$= 0 - s\int_0^\infty \cos sx \times \frac{\partial u}{\partial x} dx$$

$$= -s\left[\left(\cos sx \times u(x,t)\right)_0^\infty + s\int_0^\infty \sin sx \times u \, dx\right]$$

$$= -s\left[0 + s \times \bar{u}_s(s,t) - u(0,t)\right]$$

$$= -s^2 \bar{u}_s(s,t) + s\,u(0,t)$$

$$\Rightarrow \boxed{F\left(\frac{\partial^2 u}{\partial x^2}\right) = s\,u(0,t) - s^2 \bar{u}_s(s,t)}$$

Fourier cosine transform of u

$$\bar{u}_c(s,t) = \int_0^\infty u(x,t) \cos sx \, dx$$

Fourier cosine transform of $\frac{\partial^2 u}{\partial x^2}$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos sx \, dx$$

$$= \left[\cos sx \times \frac{\partial u}{\partial x}\right]_0^\infty + s\int_0^\infty \sin sx \times \frac{\partial u}{\partial x} dx$$

$$= \left[0 - \frac{\partial u(0,t)}{\partial x}\right] + s\left[\left\{\sin sx \times u(x,t)\right\}_0^\infty - s\int_0^\infty \cos sx \times u(x,t) dx\right]$$

$$= -\frac{\partial u(0,t)}{\partial x} + s\left[0 - s\bar{u}_c(x,t)\right]$$

$$\frac{d}{dt}\bar{u}(s,t) = \alpha^2[-s^2\bar{u}(s,t)]$$

$$\frac{d\bar{u}}{dt} = -\alpha^2 s^2 \bar{u}$$

$$\frac{d\bar{u}}{\bar{u}} = -\alpha^2 s^2 dt$$

Integrating both sides

$$\int \frac{d\bar{u}}{\bar{u}} = -\alpha^2 s^2 \int dt$$

$$\log \bar{u} = -\alpha^2 s^2 t + \log c$$

$$\Rightarrow \log \frac{\bar{u}}{c} = -\alpha^2 s^2 t$$

$$\frac{\bar{u}}{c} = e^{-\alpha^2 s^2 t}$$

$$\Rightarrow \bar{u} = c.e^{-\alpha^2 s^2 t}$$

given $\bar{u}(x,0) = f(x)$

$$\Rightarrow \bar{u}(s,0) = c \quad \text{--- (1)}$$

$\cancel{\Rightarrow \bar{u} = f(x)\, e^{-\alpha^2 s^2 t}}$ $\qquad F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\bar{u}(s,t) = F\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t) e^{isx} dx$$

$$\bar{u}(s,0) = \int_{-\infty}^{\infty} u(x,0) e^{isx} dx = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\Rightarrow \bar{u}(s,0) = F(s) \quad \text{--- (2)}$$

from (1) & (2)

$$c = F(s)$$

$$\Rightarrow \bar{u} = F(s)\, e^{-\alpha^2 s^2 t}$$

from inversion of F.T.

$$\boxed{u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-\alpha^2 s^2 t} e^{-isx} ds}$$

Q. Solve the eqn $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $(x>0, t>0)$ subjected to conditions

(i) $u=0$, when $x=0, t>0$

(ii) $u = \begin{cases} 1 & , 0<x<1 \\ 0 & , x \geq 1, \text{ when } t=0 \end{cases}$

(iii) $u(x,t)$ is bdd

Sol

$u(0,t)=0$, given. So we apply F. Sine transform on both sides

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{d}{dt}\bar{u}_s(s,t) = s\,u(0,t) - s^2\bar{u}_s(s,t)$$

$$\frac{d\bar{u}_s}{dt} = 0 - s^2\bar{u}_s(s,t)$$

$$\frac{d\bar{u}_s}{\bar{U}_s} = -s^2 dt$$

On Integrating

$$\log \bar{u}_s = -s^2t + \log c$$

$$\log \frac{\bar{u}_s}{c} = -s^2t$$

$$\Rightarrow \frac{\bar{u}_s}{c} = e^{-s^2t}$$

$$\Rightarrow \bar{u}_s = ce^{-s^2t} \Rightarrow \bar{u}_s(s,t) = ce^{-s^2t}$$

put $t=0$

$$\bar{u}_s(s,0) = c \quad \text{—(1)}$$

from fourier sine transform

$$F(s) = \int_0^\infty f(x)\sin sx\,dx$$

$$\bar{u}_s(s,t) = \int_0^\infty u(x,t)\sin sx\,dx$$

put $t=0$

$$\bar{U}_s(s,0) = \int_0^\infty u(x,0)\sin sx\,dx$$

$$= \int_0^1 1\times \sin sx\,dx + \int_1^\infty 0\times \sin sx\,dx$$

$$= \left[-\frac{\cos sx}{s}\right]_0^1$$

$$= \frac{-1}{s}[\cos s - 1]$$

$$\Rightarrow \bar{U}_s(s,0) = \frac{1-\cos s}{s} \quad \text{---} \; (2)$$

Hence from (1) & (2)

$$C = \frac{1-\cos s}{s}$$

$$\Rightarrow \bar{U}_s(s,t) = \left(\frac{1-\cos s}{s}\right)e^{-s^2t}$$

from inversion of f.s.t.

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s}\right) e^{-s^2t}\sin sx\,ds}$$

Q. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0<x<\infty, t>0$, where

(i) $u(0,t)=0$, $t>0$

(ii) $u(x,0) = e^{-x}$, $x>0$

(iii) u & $\frac{\partial u}{\partial x}$ both tends to 0 as $x\to\infty$

Q. Employ f.t. to solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<\infty, t>0$, where $u(x,t)$ satisfies the condn

(i) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$, $t>0$

(ii) $u(x,0) = \begin{cases} x, & 0<x<1 \\ 0, & x>1 \end{cases}$

(iii) $|u(x,t)| < M$, i.e. bdd.