Q1. (c) Evaluate $\int_c \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, **where c is the circle $|z| = 1$** ...(3)

Ans. It has singularities at $z = \frac{\pi}{6}$ of order 3. It lies within the circle $|z| = 1$

$$\therefore \quad f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}} dz$$

Here
$$a = \frac{\pi}{6}, n = 2, f(z) = \sin^2 z$$

$$f'(z) = 2 \sin z \cos z$$
$$f'(z) = 2 [\cos^2 z - \sin^2 z] = 2 \cos 2z$$

Now
$$f''\left(\frac{\pi}{6}\right) = \frac{2!}{2\pi i}\int_c \frac{\sin^2 z}{(z-\pi/6)^3} dz$$

$$1 = \frac{1}{\pi i}\int_c \frac{\sin^2 z}{(z-\pi/6)^3} dz$$

$$\Rightarrow \quad \int \frac{\sin^2 z}{(z-\pi/6)^3} dz = \pi i$$

Q.4. (a) Prove that the function $f(z)$ defined by $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \neq 0)$, $f(0) = 0$ is continuous and the C–R equations are satisfied at the origin, yet $f'(o)$ does not exist (5)

Ans.
$$f(z) = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, \quad z \neq 0$$

Let
$$f(z) = u + iv$$

where
$$u = \frac{x^3 - y^3}{x^2 + y^2},$$

$$V = \frac{x^3 + y^3}{x^2 + y^2}$$

At
$$z \neq 0, x \neq 0, \ y \neq 0$$

$\therefore$ u and v are rational functions of x and y with non – zero denominators. Thus u and v and $f(z)$ being polynomials are continous at $z \neq 0$.

for $z = 0$

Let
$$x = r\cos\theta, y = r\sin\theta$$

$\Rightarrow$
$$u = \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^3(\cos^2\theta + \sin^2\theta)}$$

$$u = r(\cos^3\theta - \sin^3\theta)$$

Similarly
$$v = r(\cos^3\theta + \sin^3\theta)$$

when
$$z \to 0$$

$$x, y \to 0 \text{ and } r \to 0$$

$\therefore$
$$\lim_{z\to 0} u = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

similarly
$$\lim_{z\to 0} v = 0$$

$\therefore$
$$\lim_{z\to 0} f(z) = 0 = f(0)$$

$\Rightarrow f(z)$ is continous at $z = 0$

Hence $f(z)$ is continuous for all values of z.

At origin $(0,0)$, we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,o) - u(0,0)}{x} = \lim_{x\to 0}\frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(o,o)}{y} = \lim_{y\to 0}\frac{-y-0}{y} = -1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(o,o)}{y} = \lim_{y\to 0}\frac{y-0}{y} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Hence C–R equations are satisfied at origin

Now
$$f(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z\to 0}\frac{\left(x^3-y^3\right)+i\left(x^3+y^3\right)-0}{\left(x^2+y^2\right)(x+iy)}$$

Let $z \to 0$ along path
$$x = y^3$$

$$f(0) = \lim_{y\to 0}\frac{\left(y^9-y^3\right)+i\left(y^9+y^3\right)}{\left(y^6+y^2\right)\left(y^3+iy\right)}$$

$$= \lim_{y\to 0}\frac{y^3\left[\left(y^6-1\right)+i\left(y^6+1\right)\right]}{y^2\left(y^4+1\right).y\left(y^2+i\right)}$$

$$= \lim_{y\to 0}\frac{\left(y^6-1\right)+i\left(y^6+1\right)}{\left(y^4+1\right)\left(y^2+i\right)}$$

$$= \frac{-1+i}{1\times i} = \frac{i-1}{i} = \frac{i^2-i}{i^2}$$

$$= 1+i$$

Let $z \to 0$ along path $y = x$

$$f(o) = \lim_{x\to 0}\frac{\left(x^3-x^3\right)+i\left(x^3+x^3\right)}{\left(x^2+x^2\right)(x+ix)}$$

$$= \lim_{x\to 0}\frac{2ix^3}{2x^3(1+i)} = \frac{i}{1+i}\frac{(1-i)}{(1-i)}$$

$$= \frac{i-i^2}{1+1} = \frac{1+i}{2}$$

$\therefore$ since limits are different for 2 paths.

$\therefore f(0)$ does not exist.

Q.4. (b) Evaluate $\int_c |z|\,dz$ where c is the left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$ **(5)**

Ans. Circle

$$|z| = 1$$

$$\therefore \quad z = e^{i\theta} \Rightarrow dz = i\,e^{i\theta}\,d\theta.$$

since left half is considered

$\therefore \theta$ varies from $\frac{3\pi}{2}$ to $\frac{\pi}{2}$

$$\therefore \quad \int_c |z|dz = \int_{3\pi/2}^{\pi/2} ie^{i\theta}d\theta$$

$$= \frac{i}{i}\left|e^{i\theta}\right|_{3\pi/2}^{\pi/2}$$

$$= e^{i\pi/2} - e^{i3\pi/2}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= i + i = 2i$$

Q.1. (c) Obtain the residue of $f(z) = \dfrac{(z+1)^3}{(z-1)^3}$ at its pole (3)

Ans. $$f(z) = \frac{(z+1)^3}{(z-1)^3}$$

It has poles at $z = 1$ of order 3.

$$\text{Res}\{f(z), 1\} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{(z-1)^3\frac{(z+1)^3}{(z-1)^3}\right\}$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{(z+1)^3\right\}$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[3(z+1)^2\right]$$

$$= \frac{1}{2}\lim_{z\to 1} 6(z+1) = \frac{1}{2}\times 12 = 6$$

Q.1. (d) Show that when $|z+1|<1$, $\dfrac{1}{z^2} = 1 + \sum\limits_{n=1}^{\infty}(n+1)(z+1)^n$ (3)

Ans. $$f(z) = \frac{1}{z^2} = \frac{1}{[(z+1)-1]^2} = \frac{1}{[1-(z-1)]^2} \text{ as } |z+1| < 1$$

$\therefore$
$$f(z) = [1-(z+1)]^{-2}$$
$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \ldots..$$
$$= 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$$

Q. 6. (a) Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate hormonic and the coorresponding analytic function. (6)

Ans.

$$V = e^x \sin y$$

$$\frac{\partial V}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 V}{\partial x^2} = e^x \sin y$$

$$\frac{\partial V}{\partial y} = e^x \cos y, \quad \frac{\partial^2 V}{\partial y^2} = -e^x \sin y$$

On adding
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = e^x \sin y - e^x \sin y$$
$$= 0$$

$\therefore$ V is harmonic

let
$$f'(z) = u + iV$$

Then
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial V}{\partial x}$$
$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$
$$f'(z) = e^x \cos y + ie^x \sin y$$

Replace x by z and y by 0
$$f'(z) = e^z$$

On integrating, we get
$$f(z) = \int e^z \, dz$$
$$= e^z + c$$
$$f(z) = e^{x+iy} + c$$
$$= e^x . e^{iy} + c.$$
$$f(z) = e^x (\cos y + i \sin y) + c.$$
$\therefore$
$$u = e^x \cos y + c.$$

Q. 6. (b) Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Draw a sketch of the transformed region. (6.5)

Ans. Let

$$w = \frac{1}{z}$$

$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

or

$$u+iv = \frac{x-iy}{x^2+y^2}$$

Equating real and imaginary parts

$$u = \frac{x}{x^2+y^2} \quad ...(1)$$

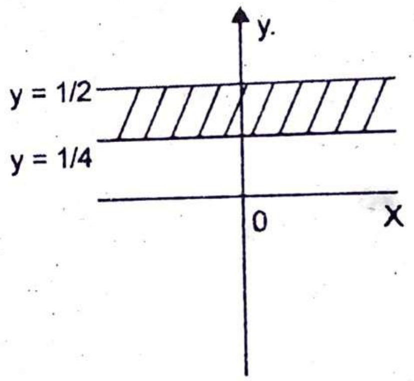
$$v = \frac{-y}{x^2+y^2} \quad ...(2)$$

On solving (1) and (2), we get

$$y = \frac{-v}{u^2+v^2} \quad ...(3)$$

If

$$y < \frac{1}{2}, \text{ then}$$



(3) becomes

$$\frac{-v}{u^2+v^2} < \frac{1}{z}$$

or

$$u^2+v^2+2v > 0$$

or

$$u^2+(v+1)^2 = 1.$$

Which represents outer portion of the circle with centre (0, – 1) and radius 1.

If

$$y > \frac{1}{4}, \text{ then (3) becomes}$$

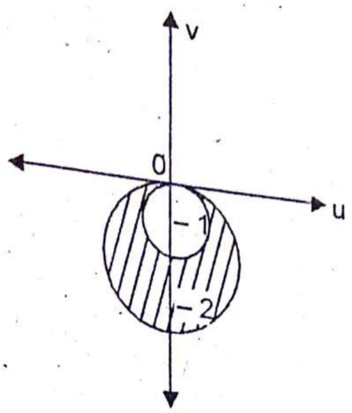
$$\frac{-v}{u^2+v^2} > \frac{1}{4} \text{ or}$$

$$u^2+v^2+4v < 0$$

or

$$u^2+(v+2)^2 < 2^2$$

Which represents the inner portion of circle with centre (0, – 2) and raduse 2. Hence, the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is the shaded portion in the figure.



Q. 7. (a) Evaluate the integral (6)

$$\oint_c \frac{dz}{(z-1)(z-2)(z-3)}, \; c: |z| = 4$$

Ans. It has singularity at z = 1, 2, 3 which lies inside cirle $|z| = 4$.

$$\text{Res of } f(z) \text{ at } z = 1 \text{ is}$$

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to 1}\frac{1}{(z-2)(z-3)} = \frac{1}{2}$$

$$\text{Res}\{f(z), z=2\} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{1}{(z-1)(z-2)(z-3)}$$

$$= \frac{1}{-1} = -1$$

$$\text{Res}\{f(z), z=3\} = \lim_{z\to 3}(z-3)f(z)$$

$$= \lim_{z\to 3}(z-3)\frac{1}{(z-1)(z-2)(z-3)}$$

$$= \frac{1}{2}$$

$\therefore$ By residue theorem

$$\oint_c \frac{dz}{(z-1)(z-2)(z-3)} = 2\pi i\left(\frac{1}{2} - 1 + \frac{1}{2}\right)$$

$$= 0.$$

Q. 7. (b) Apply residue theorem to evaluate the integral $\int_0^{2\pi} \frac{d\theta}{2-\sin\theta}$

Ans. Put

$$z = e^{i\theta}, \text{ so that}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{ and } d\theta = \frac{dz}{iz}$$

Then

$$\int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \oint_c \frac{1}{2-\frac{1}{2i}\left(z-\frac{1}{z}\right)} \frac{dz}{iz}, \; |z| = 1$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \oint_c \frac{1}{2+\frac{iz}{2}-\frac{i}{2z}} \frac{dz}{iz}$$

$$= \oint_c \frac{dz}{\left(\frac{4z+iz^2-i}{2z}\right)} \frac{1}{iz}$$

$$= 2\oint_c \frac{dz}{4iz-z^2+1}$$

Poles are

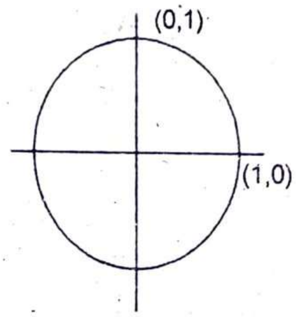
$$z^2 - 4iz - 1 = 0.$$

$$z = \frac{4i \pm \sqrt{16i^2+4}}{2}$$

$$= \frac{4i \pm 2\sqrt{3}i}{2}$$

$$= 2i \pm \sqrt{3}i$$

$i\left(2-\sqrt{3}\right)$ lie inside the circle $|z| = 1$



$$\text{Res}\left\{f(z), z = i\left(2-\sqrt{3}\right)\right\}$$

$$= \lim_{z \to i(2-\sqrt{3})} \left[z - i\left(2-\sqrt{3}\right)\right] \frac{(-2)}{z^2-4iz-1}$$

By (2),

$$\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}s,\ s=0 = \frac{\pi}{2}$$

UNIT-III

Q.6. (a) Find the image of $|z| = 1$ under the transformation $w = \frac{i-z}{i+z}$, onto the w-plane (6.5)

Ans.

$$w = \frac{i-z}{i+z}$$

$$iw + wz = i - z.$$

$$wz + z = i - iw$$

$$z = \frac{i(1-w)}{1+w}$$

As given $|z| = 1$

$$\Rightarrow \quad \left|\frac{i(1-w)}{1+w}\right| = 1$$

$$\Rightarrow \quad |i(1-w)| = |1+w|$$

$$\Rightarrow \quad |i(1-u-iv)| = |1+u+iv|$$

$$\Rightarrow \quad |i|\,|(1-u)-iv| = |(1+u)+iv|$$

$$\Rightarrow \quad (1-u)^2+v^2 = (1+u)^2+v^2$$

$$\Rightarrow \quad 1+u^2-2u = 1+u^2+2u$$

$$\Rightarrow \quad 4u = 0$$

$$\Rightarrow \quad u = 0$$

Thus image of $|z| = 1$ in z plane gives $u = 0$ i.e. imaginary axis in w-plane.

Q.6. (b) Show that the function $z|z|$ is not analytic anywhere. (6)

Ans. Let

$$f(z) = z\,|z|$$

$$f(z) = (x+iy)\sqrt{x^2+y^2}$$

Here

$$u = x\sqrt{x^2+y^2},\ v = y\sqrt{x^2+y^2}$$

Now

$$\frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + \frac{x.2x}{2\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{x.2y}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{y.2x}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y.2y}{2\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}} = \frac{2y^2+x^2}{\sqrt{x^2+y^2}}$$

Now $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq \frac{-\partial v}{\partial x}$

$\therefore$ C – R equations are not satisfied

Thus $f(z) = z\,|z|$ is not analytic anywhere.

Q.7. (a) Expand $f(z) = \frac{1}{(z-1)(z-2)}$, $1 < |z| < 2$ **(6.5)**

Ans. Consider
$$f(z) = \frac{1}{(z-1)(z-2)}$$
$$= \frac{1}{z-2} - \frac{1}{z-1}$$

As $1 < |z| < 2$

$|z| > 1$ and $|z| < 2$

$$\frac{1}{|z|} < 1 \text{ and } \frac{|z|}{2} < 1$$

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{-1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$

This is a Laurent's series.

Q.7. (b) Using contour integration in complex plane evaluate. **(6)**

$$\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}$$

Ans. Let
$$I = \int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} \quad \ldots(1)$$

Let
$$z = e^{i\theta}, \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right), d\theta = \frac{dz}{iz}$$

Consider $$\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \oint_c \frac{1}{3+\frac{2}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}, C : |z| = 1 = \oint_c \frac{1}{3z+z^2+1} \frac{dz}{i}$$

$$= \frac{1}{i} \oint_c \frac{dz}{z^2+3z+1}$$

Now for poles $z^2 + 3z + 1 = 0$,

$$z = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

Let $$\alpha = \frac{-3+\sqrt{5}}{2}, \beta = \frac{-3-\sqrt{5}}{2}$$

for $|z| = 1$ only $z = \alpha$ lies inside the circle.

$$\text{Res}\,\{f(z), \alpha\} = \lim_{z\to\alpha} (z-\alpha) \frac{1}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta}$$

$$= \frac{1}{\alpha-\beta}$$

$$= \frac{2}{-3+\sqrt{5}-(-3-\sqrt{5})}$$

$$= \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$\therefore$ By residue theorem

$$\frac{1}{i} \oint_c \frac{dz}{z^2+3z+1} = 2\pi i \times \frac{1}{i} . \frac{1}{\sqrt{5}} = \frac{2\pi}{\sqrt{5}}$$

By eqn (1)

$$\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2} \times \frac{2\pi}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$$

UNIT-IV

UNIT–III

Q. 6. (a) Determine analytic function $f(z) = u + iv$ in terms of z, if $v = \log(x^2 + y^2) + x - 2y$. (6.5)

Ans. As
$$V = \log(x^2 + y^2) + x - 2y$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$$

As $w = u + iv$ is analytic, then

$$w' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \qquad \text{[by C–R equations]}$$

$$= \left(\frac{2y}{x^2 + y^2} - 2\right) + i\left(\frac{2x}{x^2 + y^2} + 1\right)$$

Replacing x by z and y by 0, we get

$$\frac{dw}{dz} = -2 + i\left(\frac{2z}{z^2} + 1\right)$$

$$= -2 + i\left(\frac{2}{z} + 1\right)$$

$$\Rightarrow \quad \frac{dw}{dz} = (i - 2) + \frac{2i}{z}$$

On integrating w.r.t 'z', we get

$$w = (i - 2)z + 2i \log z + c.$$

Q. 6. (b) Under the transformation $w = \dfrac{1}{z}, z \neq 0$, find the image of $|z - 2i| = 2$ (6)

Ans. Given
$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

To find image of $|z - 2i| = 2$ under transformation. $z = \dfrac{1}{w}$

$$\Rightarrow \quad \left|\frac{1}{w} - 2i\right| = 2$$

$$\Rightarrow \quad \left|\frac{1}{u + iv} - 2i\right| = 2$$

$$\Rightarrow \quad |1 - 2i(u + iv)| = 2|u + iv|$$

$$\Rightarrow \quad |(1 + 2v) - 2iu| = 2|u + iv|$$

On squaring both sides

$$\Rightarrow \quad (1+2v)^2 + 4u^2 = 4(u^2+v^2)$$

$$\Rightarrow \quad 1+4v^2+4v+4u^2 = 4u^2+4v^2$$

$$\Rightarrow \quad 1+4v = 0$$

Thus image of circle $|z-2i| = 2$ is a straight line $1+4v$ in w plane.

Q. 7. (a) If $f(\alpha) = \int_c \frac{3z^2+7z+1}{z-\alpha} dz$, where c is the circle $x^2+y^2=4$, find the value of $f(3)$, $f'(1-i)$ and $f''(1-i)$ **(6.5)**

Ans. Given circle is $x^2+y^2=4$ or $|z| = 2$

The point $z=3$ lies outside the circle $|z| = 2$

while $z = 1-i$ *i.e.* $(1,-1)$ lie inside the circle $|z| = 2$.

Now $f(3) = \oint_c \frac{3z^2+7z+1}{z-3} dz$ and $\frac{3z^2+7z+1}{z-3}$ is analytic everywhere within c.

$\therefore$ By Cauchy integral theorem

$$f(3) = \oint_c \frac{3z^2+7z+1}{z-3} dz = 0$$

$$\Rightarrow \quad f(3) = 0$$

Now, let $\phi(z) = 3z^2+7z+1$ which is analytic any where.

$\therefore$ By Cauchy integral formula

$$\phi(\alpha) = \frac{1}{2\pi i}\oint_c \frac{3z^2+7z+1}{z-\alpha} dz, \text{ a is a point within c.}$$

$$\Rightarrow \quad 2\pi i\,\phi(\alpha) = \oint_c \frac{3z^2+7z+1}{z-\alpha} dz = f(\alpha)$$

$$\Rightarrow \quad f(\alpha) = 2\pi i\,(3\alpha^2+7\alpha+1)$$

$$\Rightarrow \quad f'(\alpha) = 2\pi i\,(6\alpha+7)$$

Now

$$f'(1-i) = 2\pi i\,[6(1-i)+7]$$

$$= 2\pi i\,(6-6i+7) = 2\pi\,(13i+6)$$

$$f''(\alpha) = 2\pi i \times 6$$

$$f''(1-i) = 12\pi i.$$

Q. 7. (b) Prove that if a > 0, then $\int_0^\infty \frac{1}{x^4+a^4} dx = \frac{\pi\sqrt{2}}{2a^3}$. **(6)**

Ans. Given $\int_0^\infty \frac{dx}{x^4+a^4}, a>0$

Let

$$\phi(z) = \frac{1}{z^4+a^4}$$

Poles of $\phi(z)$ are $z^4+a^4=0$

$$z^4 = -a^4 = a^4(-1)$$

$\Rightarrow$
$$z = a(-1)^{1/4} = a(\cos\pi + i\sin\pi)^{1/4}$$

$\Rightarrow$

By De moivre's theorem

$$z = a\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right], n = 0,1,2,3$$

When
$$n = 0, z = a\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)a$$

$$= ae^{i\pi/4}$$

When
$$n = 1,\ z = a\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = ae^{3i\pi/4}$$

When
$$n = 2, z = a\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = ae^{i5\pi/4}$$

When
$$n = 3, z = a\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = ae^{i7\pi/4}$$

Only $z = ae^{i\pi/4}$ and $ae^{i3\pi/4}$ lies in upper half of z – plane

$\therefore$ Residue of $\phi(z)$ at $z = ae^{i\pi/4}$

Res
$$\left\{\phi(z), ae^{i\pi/4}\right\} = \lim_{z\to ae^{i\pi/4}} \left(z - ae^{i\pi/4}\right)\frac{1}{z^4+a^4}\left(\frac{0}{0}\right)$$

$$= \lim_{z\to ae^{i\pi/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3e^{i3\pi/4}}$$

Res
$$\left[\phi(z), ae^{i3\pi/4}\right] = \lim_{z\to a/e^{i3\pi/4}} \left(z - ae^{i3\pi/4}\right)\frac{1}{z^4+a^4}\left(\frac{0}{0}\right)$$

$$= \lim_{z\to ae^{i3\pi/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3e^{i9\pi/4}}$$

By Cauchy Residue theorem.

$$\oint_c \phi(z)dz = 2\pi i\left[\frac{1}{4a^3e^{3i\pi/4}} + \frac{1}{4a^3e^{i9\pi/4}}\right]$$

$$\Rightarrow \quad \int_{-R}^{R} \phi(x)dx + \int_{C_R} \phi(z)dz = \frac{\pi i}{2a^3}\left[e^{-3i\pi/4} + e^{-i9\pi/4}\right] \qquad \ldots(1)$$

Consider $$\left|\int_{C_R} \phi(z)dz\right| \leq \left|\int_{C_R} \frac{|dz|}{|z^4 + a^4|}\right|$$

Let $$z = Re^{i\theta} \quad , |dz| = |Ri\, e^{i\theta}\, d\theta| = R\, d\theta.$$

$$\Rightarrow \quad \left|\int_{C_R.} \frac{dz}{z^4 + a^4}\right| \leq \int_0^{\pi} \frac{1}{R^4 + 1}\, d\theta . R$$

$$\leq \frac{R}{R^4 + 1}\int_0^{\pi} d\theta . \leq \frac{\pi R}{R^4 + 1}$$

$$\rightarrow 0 \text{ as } R \rightarrow \infty.$$

By (1), we have

$$\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi i}{2a^3}\left[e^{-3i\pi/4} + e^{-9i\pi/4}\right]$$

$$= \frac{\pi i}{2a^3}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\right] = \frac{\pi i}{2a^3}\left(\frac{-2i}{\sqrt{2}}\right)$$

$$= \frac{\pi}{\sqrt{2}\, a^3} = \frac{\pi\sqrt{2}}{2a^3}$$

$$\therefore \quad \int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{2a^3}$$

Q.1. (e) Show that the limit of the function $f(z) = \dfrac{\text{Re}(z)}{|z|}, z \neq 0$, and $f(z) = 0, z = 0$ as $z \to 0$ does not exist. **(3)**

Ans. To show $\lim\limits_{z \to 0} \dfrac{\text{Re}(z)}{|z|}$ does not exist

Let $z = x + iy$

$$\Rightarrow \qquad f(z) = \lim_{z \to 0} \frac{x}{\sqrt{x^2 + y^2}}$$

Let $z \to 0$ along path $y = mx$

$$\Rightarrow \qquad \lim_{z \to 0} \frac{\text{Re}(z)}{|z|} = \lim_{x \to 0} \frac{x}{\sqrt{x^2 + m^2 x^2}}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{1 + m^2}} = \frac{1}{\sqrt{1 + m^2}}$$

Since its value depends on m

$\therefore$ For different values of m, we have different limits.

Thus, limit does not exist.

Q.1. (f) Prove that the function $e^x(\cos y + i \sin y)$ is analytic and find its derivative. (3)

Ans. Let

$$f(z) = e^x(\cos y + i \sin y)$$

Here,

$$u = e^x \cos y, v = -e^x \sin y.$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y; \frac{\partial v}{\partial y} = e^x \cos y$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ C – R equations are satisfied. Since e^x, $\cos y$, $\sin y$ are continous functions:

$\therefore$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions satisfying C – R equations.

Hence $f(z)$ is analytic every where

Now

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i \sin y) = e^x \cdot e^{iy} = e^{x+iy}.$$

Q.6. (a) Determine the analytic function $w = u + iv$, if $v = \log(x^2 + y^2) + x - 2y$. **(6.5)**

Ans.

$$v = \log(x^2 + y^2) + x - 2y.$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$$

As $w = u + iv$ is analytic

$$\Rightarrow \quad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad f'(z) = \left(\frac{2y}{x^2 + y^2} - 2\right) + i\left(\frac{2x}{x^2 + y^2} + 1\right)$$

Replacing x by z and y by 0

$$f'(z) = -2 + i\left(\frac{\partial z}{z^2} + 1\right) = -2 + i\left(\frac{2}{z} + 1\right)$$

$$\Rightarrow \quad f'(z) = (i - 2) + \frac{2i}{z}$$

On integrating we get

$$f(z) = (i-2)z + 2i \log z + c.$$

Q.6. (b) Find the bilinear transformation which maps 1, i, –1 to i, 0, –i, respectively. Also find invariant point of this transformation. (6)

Ans. Let $z_1 = 1, z_2 = i, z_3 = -1, z_4 = z$ and $w_1 = i, w_2 = 0, w_3 = -i, w_4 = w$.

By cross ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$$

$$\Rightarrow \quad \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{i(-1-w)}{((i-w)-i)}$$

$$\Rightarrow \quad \frac{(1-i)(1+z)}{(1+i)(1-z)} = \frac{w+i}{i-w}$$

$$\Rightarrow \quad \frac{(1-i)(1+z)}{(1+i)(z-1)} = \frac{w+i}{w-i}$$

By C and D

$$\Rightarrow \quad \frac{(1-i)(1+z)}{(1+i)(z-1)} = \frac{(w+i)+w-i}{w+i-w+i}$$

$$\Rightarrow \quad \frac{2z-2i}{2-2iz} = \frac{2w}{2i}$$

$$\Rightarrow \quad \frac{w}{i} = \frac{z-i}{1-iz}$$

$$\Rightarrow \quad w - i\,wz = iz + 1$$

$$\Rightarrow \quad w = \frac{iz+1}{1-iz}$$

Put

$$w = z$$

$$z = \frac{iz+1}{1-iz}$$

$$\Rightarrow \quad z - iz^2 - iz - 1 = 0$$

$$\Rightarrow \quad iz^2 + iz - z + 1 = 0$$

$$\Rightarrow \quad iz^2 + z(i-1) + 1 = 0$$

Roots are

$$z = \frac{(1-i) \pm \sqrt{(i-1)^2 - 4i}}{2i} = \frac{(1-i) \pm \sqrt{-6i}}{2i} = \frac{(1+i) \pm \sqrt{-6i}}{-2}$$

Q.7. (a) Expand the function. (6.5)

$$f(z) = \frac{1}{z^2 - 4z + 3}, \text{ for } 1 < |z| < 3$$

Ans. Here

$$f(z) = \frac{1}{z^2 - 4z + 3} = \frac{1}{(z-3)(z-1)}$$

$$\frac{1}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$$

$$1 = A(z-1) + B(z-3)$$

$\Rightarrow$

$$Z = 1, 1 = -2B \Rightarrow B = \frac{-1}{2}$$

for

$$Z = 3, 1 = 2A \quad \Rightarrow A = \frac{1}{2}$$

and

$$\frac{1}{(z-3)(z-1)} = \frac{1}{2(z-3)} - \frac{1}{2(z-1)}$$

$\therefore$

$$1 < | z | < 3$$

for

$$\frac{1}{|z|} < 1 \quad \text{and} \quad \frac{|z|}{3} < 1$$

$\Rightarrow$

$\therefore$

$$f(z) = \frac{1}{-6\left(1-\frac{z}{3}\right)} - \frac{1}{2z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\ldots\right) - \frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)$$

$$= \frac{-1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right) - \frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$

(6)

Q.7. (b) Apply calculus of residues to prove that;

$$\int_0^{2x} \frac{1}{1-2a\sin\theta+a^2}d\theta = \frac{2\pi}{1-a^2}, (0 < a < 1)$$

Ans. Put

$$z = e^{i\theta}$$

$$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right) \quad \text{and } d\theta = \frac{dz}{iz}$$

$\Rightarrow$

$$\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2} = \oint_C \frac{1}{1-2a\frac{1}{2i}\left[z-\frac{1}{z}\right]+a^2}\cdot\frac{dz}{iz}$$

$$= \oint_C \frac{dz}{iz\left(1-\frac{az}{i}+\frac{a}{iz}+a^2\right)}$$

$$= \oint \frac{dz}{iz\left(z+aiz^2-ai+a^2z\right)}$$

$$= \oint_C \frac{dz}{iz - az^2 + a + ia^2 z}$$

$$= -\oint_C \frac{dz}{az^2 - a - ia^2 z - iz}$$

$$= -\oint_C \frac{dz}{az(z - ia) - i(z - ia)}$$

$$= -\oint_C \frac{dz}{(z - ia)(az - i)}; \quad C : |z| \, |z|$$

It has simple poles $(a)\ z = \frac{i}{a}$ and $z = ia$

As $\quad 0 < a < 1; \ \& \ |z| = 1;$

$\therefore \quad z = ia$ lies inside the circle C.

$$\text{Res. } \{f(z), ia\} = \underset{z \to ia}{\text{Lt}} (z - ai) - \frac{1}{(z - ia)(az - i)}$$

$$= \frac{-1}{ia^2 - i} = \frac{-1}{i[a^2 - 1]}$$

$$\therefore \quad \oint_C \frac{-dz}{(az - i)(z - ia)} = 2\pi i \frac{-1}{i(a^2 - 1)}$$

$$= \frac{2\pi}{1 - a^2}$$

$$\Rightarrow \quad \oint_C \frac{d\theta}{1 - 2\theta \sin\theta + a^2} = \frac{2\pi}{1 - a^2} \ \textbf{Ans.}$$