CM - formula Sheet :-

UNIT-01

Taylor's Series

$y' = \frac{dy}{dx} = f(x,y)$

$$y = y_0 + x y_0' + \frac{x^2 y_0''}{2!} + \frac{x^3 y_0'''}{3!} + \cdots$$

Rolle's Theorem

- (i) $f(x)$ is Continous in $[a,b]$
- (ii) $f(x)$ is differentiable in (a,b)
- (iii) $f(a) = f(b)$

then $\exists$ atleast one point $c \in (a,b)$ so that $f'(c) = 0$

Mean Value theorem

- (i)
- (ii) same Rolle's theorem
- (iii) $f(a) \neq f(b)$

then $\exists\, c \in (a,b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Bisection Method

Let $f(x) = 0$

Suppose, $f(0) = -ve$

$\left.\begin{array}{l} f(1) = -ve \\ f(2) = +ve \end{array}\right\}$ root lie b/w them

1st Approx.

$x_1 = \frac{1+2}{2} = 1.5$; $f(1.5) = -ve$

2nd Approx.

$x_2 = \frac{1.5+2}{2} = 1.75$; $f(1.75) = +ve$

3rd Approx.

$x_3 =$ - - - - - - - - -

so on till required accuracy.

Secant Method

Let $f(x) = 0$

Suppose, $f(0) = -ve$

$f(1) = -ve \rightarrow x_0$

$f(2) = +ve \rightarrow x_1$

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}}$$

$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$; $f(x_2) =$

$x_3 =$ similarly -

Newton Raphson Method

Let $f(x) = 0$

Suppose, $f(0) = -ve$

$\left.\begin{array}{l} f(1) = -ve \\ f(2) = +ve \end{array}\right\}$ root lies b/w them

Let $x_0 = 1$ or 2 or $\frac{1+2}{2} = 1.5$

also find $f'(x)$

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}}$$

$n=1$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Absolute Error : If x is true value & x' its Approximate value.

$E_a = |X - X'|$

Relative Error : $E_R = \left|\frac{X - X'}{X}\right|$

% Error : $E_P = \left|\frac{X - X'}{X}\right| \times 100$

Inherent errors : Errors which are already present in the statement of a problem before its solution.

Rounding errors : Errors which arise from the process of rounding off the numbers.

Truncation errors : Errors caused by using Approximate results.

! Do it Yourself :-)

- fibonacci
- Golden Search
- Newton's Method
- Steepest
- Nelder-Mead Algo.

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UNIT-02

Interpolation

→ Newton forward :

$$f(\underset{\text{initial / Base}}{x}+\underset{\text{Interval}}{h}u) = f(x) + \frac{u}{1!}\Delta f(x) + \frac{u(u-1)}{2!}\Delta^2 f(x) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x) + \cdots$$

→ Newton Backward :

$$f(x+hu) = f(x) + u\nabla f(x) + \frac{u(u+1)}{2!}\nabla^2 f(x) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x) + \cdots$$

$$u = \frac{x - x_0}{h}$$

Base · initial

Lagrange's interpolation for unequal interval :

Ex:-

x	5	6	9	11
y	12	13	14	16

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

Newton Divided Difference :

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	5	12	$\frac{13-12}{6-5}=1$	$\frac{\frac{1}{3}-1}{9-5} = -\frac{1}{6}$	$\frac{\frac{2}{15}-(-\frac{1}{6})}{11-5} = \frac{1}{20}$
x_1	6	13	$\frac{14-13}{9-6}=\frac{1}{3}$	$\frac{1-\frac{1}{3}}{11-6} = \frac{2}{15}$	
x_2	9	14	$\frac{16-14}{11-9}=1$		
x_3	11	16			

Given : $x=10$

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \cdots$$

Numerical integration :

$$I = \int_a^b f(x)\,dx \quad ; \quad h = \frac{b-a}{n}$$

① Trapezoidal Rule :-

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots y_{n-1})\right]$$

∴ for any no. of interval.

② Simpson $\frac{1}{3}$ Rule : (Even interval)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots) + 2(y_2+y_4+y_6+\cdots)\right]$$

③ Simpson $\frac{3}{8}$ Rule : (for Multiple of 3)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\cdots) + 2(y_3+y_6+y_9+\cdots)\right]$$

Gauss quadature formula

Gauss → $I = \int_a^b f(x)\,dx$

Step ①: change interval $[a,b]$ into $[-1,1]$ using $\boxed{x = \frac{b-a}{2}t + \frac{b+a}{2}}$

Step ②: Substitute x in $f(x)$ integrand.

Step ③: formula -

one-point : $\int_{-1}^{1} f(x)\,dx = 2f(0)$
$(n=0)$

two-point : $\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

three-point : $\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{3/5}\right)\right]$

UNIT - 03

Gauss Elimination Method.

Consider the system of eq^ns.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Convert in Matrix form $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Make Augmented matrix, $C = [A : B]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$$

∴ use Row transformations,

$$C = \begin{bmatrix} - & - & - & : & - \\ 0 & - & - & : & - \\ 0 & 0 & - & : & - \end{bmatrix}$$

Convert back to system of eqs.

$$c_{11}x + c_{12}y + c_{13}z = d_1$$
$$c_{22}y + c_{23}z = d_2$$
$$c_{33}z = d_3$$

Now, solve these to get ur ~~Answers~~ :)

Gauss Jordan

Similar to Gauss Elimination

$$\begin{bmatrix} 1 & 0 & 0 & : & - \\ 0 & 1 & 0 & : & - \\ 0 & 0 & 1 & : & - \end{bmatrix} \qquad x = d_1, \quad y = d_2, \quad z = d_3$$

Eigen Value Problem & Power Method.

Ex.

$$AX_0 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \underset{\text{Eigen Value}}{2} \underset{\text{Eigen Vector}}{\begin{bmatrix} -1 \\ +1 \end{bmatrix}} \Rightarrow 2X_1$$

$$AX_1 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} +8 \\ -6 \end{bmatrix} = \underset{\text{bigger one}}{8} \begin{bmatrix} 1 \\ -0.75 \end{bmatrix} = 8X_2$$

$$AX_2 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -0.75 \end{bmatrix} = \_ X_3$$

So on till we get same value of successive iteration

LU Decomposition Methods.

DoLittle :

① $AX = B$

② $A = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

③ $LY = B$ ∵ $L\underbrace{UX}_{Y} = B$

where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ Solve to find y_1, y_2, y_3

④ $UX = Y$

from this we can find X (x, y, z) req. sol.

Crout's :

Same Procedure as DoLittle

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad U = \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

Cholesky's :

$A \to$ Matrix.

Conditions
- → A is Symmetric i.e $A^t = A$
- → A is positive definite.

① $AX = B$

② $A = LL^T$

$$A = \underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}} \underset{L^T}{\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}}$$

$$= \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ec & d^2+e^2+f^2 \end{bmatrix}$$

③ $LY = B$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

④ $L^T X = Y$

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UNit-04

Picard's Method

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$$

1st Approx. —

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

2nd Approx. —

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

Doing this till the desired degree of accuracy obtained.

Taylor's Series $\left(f(x,y) = y' = \frac{dy}{dx}\right)$

$$\boxed{y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots}$$

$h = (x - x_0)$

$x = x_0 - h$

Similarly

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

and so on ---

Euler's Method

Consider, $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

To find : $y(x_n) = y_n$ (Initial)

Acc. to Euler's Method,

$$\boxed{y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1}), \; n = 1, 2, 3 \ldots} \leftarrow \text{eq}^n \text{ (1)}$$

h ← width of differencing

$h = \frac{x_n - x_0}{n} = x_n - x_{n-1} = x_1 - x_0$

n ← no. of interval. (Let any $n = 5, 10$, etc)

Put $n=1$, $y_1 = y_0 + h f(x_0, y_0)$

Put $n=2$, $y_2 = y_1 + h f(x_1, y_1)$

and so on.

from Euler's Method, Starting from y_0 and using eqn (1), we can get $y_1, y_2, y_3, \ldots y_n$.

Modified Euler's Method. $(x = x_0 + h)$

Step (1):- $y_1^{(0)} = y_0 + h f(x_0, y_0)$ (Euler Method)

$$\boxed{y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(n)})\right]} \text{ (Modified)}$$

Put $n = 0, 1, 2, 3, \ldots$

Step (2): 1st Approx. $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

Similarly.

Step (3): 2nd Approx. $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$

and so on. till required accuracy.

- if two consecutive values of $y_1^{(k)}$ & $y_1^{(k+1)}$ are Same then $y_1 = y_1^{(k)}$

Preceding in the Same manner, $y_2, y_3, \ldots$ are Calculated.

Runge-Kutta (Second Order) $(x = x_0 + h)$

$\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

To find $y(x_1)$

$$\boxed{\begin{aligned} k_1 &= h f(x_n, y_n) \\ k_2 &= h f(x_n + h, y_n + k_1) \\ y_{n+1} &= y_n + \tfrac{1}{2}(k_1 + k_2) \end{aligned}}$$

Runge-Kutta (4th Order)

$k_1 = h f(x_n, y_n)$

$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = h f(x_n + h, y_n + k_3)$

then Compute ;

Weighted mean of k_1, k_2, k_3 & k_4 → $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_{n+1} = y_n + k$

Adams-Bashforth Predictor Corrector Method :

Consider, $\frac{dy}{dx} = f(x,y)$ with initial $y(x_0) = y_0$

To find : $y(x_n)$ * must be 4 starting Values of y.

find $y(x_1), y(x_2), y(x_3)$ using Any Methods.

Then Calculate — $f_0 = f(x_0, y_0), f_1, f_2, f_3$

Predictor → $y_4 = y_3 + \frac{h}{24}(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0)$

find $f_4 = f(x_4, y_4)$

Corrector → $y_4 = y_3 + \frac{h}{24}(9 f_4 + 19 f_3 - 5 f_2 + f_1)$

again find f_4 & again corrector.

Milne Predictor-Corrector Method:

$\frac{dy}{dx} = f(x,y)$ with initial Condⁿ $y(x_0) = y_0$

To find $y(x_4)$

find $y(x_1), y(x_2), y(x_3)$ using Any Method!

Then Calculate:

$f_0 = f(x_0, y_0)$ $\quad f_1 = f(x_1, y_1)$

$f_2 = f(x_2, y_2)$ $\quad f_3 = f(x_3, y_3)$

<u>Milne's Corrector formullar.</u>

$$\boxed{y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)}$$

find f_4 and again Apply Milne Method to find better value of y_4,

we repeat this step untill y_4 remains unchanged.

Numerical Solution of Partial Diff. eqⁿ.

(Parabolic, Hyperbolic, elliptical)

$$\boxed{a\frac{\partial^2\phi}{\partial x} + b\frac{\partial^2\phi}{\partial x\,\partial y} + c\frac{\partial^2\phi}{\partial y^2}}$$

$$D = b^2 - 4ac$$

$D > 0$	Hyperbola,	2 characⁿ Solⁿ
$D = 0$	Parabola,	"
$D < 0$	Ellipse.,	No Solⁿ.

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