# Numerical solution of Partial Differential Equations.

A general Second-order linear partial Differential $Eq^n$ is of type –

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G}$$

where A, B, C, D, E, F, G are all functions of $x$ and $y$. This Equation can be classified with respect to the sign of discriminant

$$\boxed{\Delta S = B^2 - 4AC}$$

if $\Delta S < 0$ the equation is said to be of Elliptic type.

if $\Delta S > 0$ the $Eq^n$ is said to be Hyperbolic - type.

if $\Delta S = 0$ the $eq^n$ is said to be Parabolic - type.

Ex

Classify the following partial Differential equations.

(a) $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 2$

(b) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$

(c) $5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial t^2} = 0$

Sol$^n$

General 2nd order Partial Differential Eq$^n$ is

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G} \quad (1)$$

(a)

$$2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 2$$

Comparing this Equation with (1)

$A = 2 \quad B = 4 \quad C = 3$

$$B^2 - 4AC = (4)^2 - 4 \times 2 \times 3$$

$$= -8 < 0$$

$\Rightarrow$ showing that the given equation is elliptic at all points.

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$$

Comparing this equation with (1)

$$A = 1 \quad , \quad B = 4 \quad , \quad C = 4$$

$$B^2 - 4AC = 4^2 - 4 \times 4 = 0$$

Showing that the given equation is Parabolic at all points.

$$5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial t^2} = 0$$

Comparing this equation with (1)

$$A = 5 \quad , \quad B = -9 \quad , \quad C = 4$$

$$B^2 - 4AC = (-9)^2 - 4 \times 5 \times 4$$

$$= 1 > 0$$

Showing that the given eqⁿ is hyperbolic of all points.

Ex Find whether the following operators are hyperbolic, parabolic and elliptic.

(a) $x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u$ (b) $t \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$.

(c) $x \frac{\partial^2 u}{\partial x^2} + t \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

Solⁿ (a) $x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u \Rightarrow x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -u$

Here $A = x^2$, $B = 0$, $C = -1$

Now $B^2 - 4AC = 0^2 - 4x^2 \cdot (-1) = 4x^2$

$\therefore$ the operator is hyperbolic if

$4x^2 > 0$ i.e $x > 0$ and $x < 0$

the operator is parabolic if

$4x^2 = 0$ i.e $x = 0$

Since $4x^2$ can not be negative (being a square)

hence operator can not be elliptic.

(b) $t \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$

Here $A = 1$, $B = 2$, $C = x$.

Now $B^2 - 4AC = 4 - 4tx$

$\therefore$ the operator is hyperbolic if

$4-4tx > 0$ i.e $4tx < 4$

$tx < 1$

the operator is elliptic if $4-4tx < 0$

$4tx > 4$

$tx > 1$

the operator is parabolic if $4-4tx = 0$

i.e $tx = 1$

(c) $x\frac{\partial^2 u}{\partial x^2} + t\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

Here, $A = x$, $B = t$, $C = 1$

Now $B^2 - 4AC = t^2 - 4 \times x \times 1 = t^2 - 4x$.

$\therefore$ The operator is hyperbolic if $t^2 - 4x > 0$

i.e $t^2 > 4x$

The operator is elliptic if $t^2 - 4x < 0$

i.e $t^2 < 4x$

the operator is parabolic if $t^2 - 4x = 0$

i.e $t^2 = 4x$.