→ Gauss's Elimination Method

→ Partial pivoting method.

Step-I : The numerically largest coefficient of x is selected from all the equations as pivot by interchanging the first equation with the equation having the largest coefficient of x

Step - II The Numerically largest coefficient of y is selected from all the remaining equation as pivot and the corresponding equation becomes the second equation

(ii) the process is repeated till we arrive at the equation with the single variable.

Complete Pivoting Method:-

In this method we select at each stage the numerically largest coefficient of the complete matrix of coefficient. This procedure leads to an interchange of the equation as well as interchange of the position of variables. it is more complicated and does not appreciably improve the accuracy and is not often used.

Ex Solve the following system by Gauss's Elimination Method.

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

Solⁿ We have

$$2x + y + z = 10 \quad \text{——— (I)}$$
$$3x + 2y + 3z = 18 \quad \text{——— (II)}$$
$$x + 4y + 9z = 16 \quad \text{——— (III)}$$

Dividing (i) by 2 and subtracting multiplied by 3 from (II) then subtracting from (III) we get

$$x + \frac{1}{2}y + \frac{1}{2}z = 5 \quad \text{——— (IV)}$$
$$\frac{1}{2}y + \frac{3}{2}z = 3 \quad \text{——— (V)}$$
$$\frac{7}{2}y + \frac{17}{2}z = 11 \quad \text{——— (VI)}$$

Now dividing (V) by $\frac{1}{2}$ and then subtracting after getting multiplied by $\frac{7}{2}$ from (VI) we have

$$x + \frac{1}{2}y + \frac{1}{2}z = 5 \quad \text{——— (VII)}$$
$$y + 3z = 6 \quad \text{——— (VIII)}$$
$$-2z = 10 \quad \text{——— (ix)}$$

from back substitution from (ix), (viii) and (vii), we get.

$$z = 5$$

$$y + 3z = 6$$

and

$$y + 3(5) = 6$$

$$y = 6 - 15$$

$$y = -9$$

and

$$x + \frac{1}{2}(-9) + \frac{1}{2}(5) = 5$$

$$x = 5 + \frac{9}{2} - \frac{5}{2} = 7$$

Hence the solution is $\boxed{x = 7,\ y = -9,\ z = 5}$

Ex Using Gauss's Elimination method to solve

$$2x_1 + 4x_2 + x_3 = 3$$

$$3x_1 + 2x_2 - 2x_3 = 2$$

$$x_1 - x_2 + x_3 = 6$$

Sol^n Dividing first Equation by 2 we get

$$x_1 + 2x_2 + \frac{1}{2}x_3 = \frac{3}{2} \quad \text{——— (i)}$$

multiplying (i) by (3) and subtracting from 2nd and also subtracting (i) from 3rd Equation, we get

$$4x_2 + \frac{7}{2}x_3 = \frac{5}{2} \quad \text{——— (ii)}$$

$$-3x_2 + \frac{1}{2}x_3 = \frac{9}{2} \quad \text{——— (iii)}$$

Now dividing (II) by 4 and subtracting after multiplying by -3 from (III), we get

$$25x_3 = 51$$

or $$x_3 = \frac{51}{25} = 2.04$$

substituting the value of x_3 into (II) we get.

$$4x_2 + \frac{7}{2}(2.04) = \frac{5}{2}$$

$$4x_2 = \frac{5}{2} - \frac{7(2.04)}{2}$$

$$= \frac{5-14.28}{2}$$

$$x_2 = -\frac{9.28}{8}$$

$$x_2 = -1.16$$

Now substituting the values of x_2 and x_3 into (i) we get

$$x_1 + 2(-1.16) + \frac{1}{2}(2.04) = \frac{3}{2}$$

$$x_1 = \frac{3}{2} + 2(1.16) - \frac{1}{2}(2.04)$$

$$= \frac{3+4.64-2.04}{2}$$

$$= \frac{5.6}{2}$$

$$x_1 = 2.8$$

Hence the solutions are given by

$x_1 = 2.8$, $x_2 = -1.16$, $x_3 = 2.04$

Ex Solve the system of Equations

$$x_1 + x_2 + x_3 = 6$$

$$3x_1 + 3x_2 + 4x_3 = 20$$

$$2x_1 + x_2 + 3x_3 = 13$$

using Gauss Elimination method with partial Pivoting.

Sol^n Note that in the above matrix the second Pivot has the value zero and the elimination procedure cannot be continued further unless, Pivoting is used

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

operate $R_2 = \frac{1}{3}R_1$, $R_3 - \frac{2}{3}R_1$

$$= \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & -1 & 1/3 & -1/3 \end{array}\right]$$

In the second column, 1 is the largest element in magnitude leaving the first element interchanging the 2nd and 3rd row we have,

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{array}\right]$$

you may observe here that the resultant matrix is in triangular form and no further elimination is required using back substitution Method, we obtain the solution

$$x_3 = 2 \quad ; \quad x_2 = 1 \quad , \quad x_1 = 3$$

Q. Solve the following system of equations using Gauss Elimination method. without Pivoting

$$2x_1 + 2x_2 + x_3 - 2x_4 = 10$$
$$x_1 + 2x_2 + x_3 + x_4 = 12$$
$$3x_1 + x_2 - 3x_3 + x_4 = 0$$
$$x_1 - 3x_2 + 3x_3 - 2x_4 = 3$$

We have

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 2 & 1 & -2 & 10 \\ 1 & 2 & 1 & 1 & 12 \\ 3 & 1 & -3 & 1 & 0 \\ 1 & -3 & 3 & -2 & 3 \end{array}\right]$$

Apply row operation $R_2 - (1/2)R_1$

$R_3 - \frac{3}{2}R_1$

$R_4 - \frac{1}{2}R_1$ on $[A|b]$ we get

$$\sim \left[\begin{array}{cccc|c} 2 & 2 & 1 & -2 & 10 \\ 0 & 1 & 1/2 & 2 & 7 \\ 0 & -2 & -9/2 & 4 & -15 \\ 0 & -4 & 5/2 & -1 & -2 \end{array}\right]$$

Similarly $R_3 + 2R_2$, $R_4 + 4R_2$

$$\sim \left[\begin{array}{cccc|c} 2 & 2 & 1 & -2 & 10 \\ 0 & 1 & 1/2 & 2 & 7 \\ 0 & 0 & -7/2 & 8 & -1 \\ 0 & 0 & 9/2 & 7 & 26 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 2 & 2 & 1 & -2 & 10 \\ 0 & 1 & 1/2 & 2 & 7 \\ 0 & 0 & -7/2 & 8 & -1 \\ 0 & 0 & 0 & 121/7 & 173/7 \end{array}\right]$$

Since the matrix A has been reduced to an upper triangular matrix, we do not need more row operations

The linear system corresponding to the reduced augmented matrix is

$$2x_1 + 2x_2 + x_3 - 2x_4 = 10$$

$$x_2 + \frac{1}{2}x_3 + 2x_4 = 7$$

$$-\frac{7}{2}x_3 + 8x_4 = -1$$

$$\frac{121}{7}x_4 = \frac{173}{7}$$

we get $x_4 = \frac{173}{121}$

Using back substitution

$$x_3 = \frac{430}{121}, \quad x_2 = \frac{26}{11}, \quad x_1 = \frac{277}{121}$$

x To highlight the pitfall of Gauss Elimination method and importance of pivoting solve the following linear equations.

$$\frac{2}{3}x_1 + \frac{2}{7}x_2 + \frac{1}{5}x_3 = \frac{43}{15}$$

$$\frac{1}{3}x_1 + \frac{1}{7}x_2 - \frac{1}{2}x_3 = \frac{5}{6}$$

$$\frac{1}{5}x_1 - \frac{3}{7}x_2 + \frac{2}{5}x_3 = -\frac{12}{5}$$

$$[A|b] = \left[\begin{array}{ccc|c} 2/3 & 2/7 & 1/5 & \frac{43}{15} \\ 1/3 & 1/7 & -1/2 & \frac{5}{6} \\ \frac{1}{5} & -3/7 & 2/5 & -\frac{12}{5} \end{array}\right]$$

Apply row operations $R_2 - \frac{1}{2}R_1$, $R_3 - \frac{3}{10}R_1$ we get

$$\sim \left[\begin{array}{ccc|c} \frac{2}{3} & \frac{2}{7} & \frac{1}{5} & \frac{43}{15} \\ 0 & 0 & -\frac{3}{5} & -\frac{3}{5} \\ 0 & -\frac{18}{35} & \frac{17}{50} & -\frac{163}{50} \end{array}\right]$$

$$\frac{2}{3}x_1 + \frac{2}{7}x_2 + \frac{1}{5}x_1 = \frac{43}{15}$$

$$\boxed{-\frac{3}{5}x_3 = -\frac{3}{5}}$$

$$-\frac{18}{35}x_2 + \frac{17}{50}x_3 = -\frac{163}{50}$$

we get $x_3 = 1$, $x_2 = 7$, $x_1 = 1$

Q. Use Gauss Elimination with full pivoting to Solve the following linear system of Equations.

$$2x_1 + x_2 + 3x_3 = 12$$
$$x_1 + 3x_2 + 2x_3 = 6$$
$$3x_1 + 5x_2 - x_3 = 20$$

We have

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 20 \end{bmatrix}$$

The largest magnitude element in the matrix A is 5 which belongs to the third row and second column. interchange the third and first rows so that the system becomes

$$\begin{bmatrix} 3 & 5 & -1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 12 \end{bmatrix}$$

Now interchange the second and first columns so that the element with largest magnitude becomes the first pivot. At the same time, interchange the order of variables x_1 and x_2 to, get

$$\begin{bmatrix} 5 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 12 \end{bmatrix}$$

Apply row operations

$$R_2 - \frac{3}{5}R_1 \,,\quad R_3 - \frac{1}{5}R_1$$

the system transforom to

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & -0.8 & 2.6 \\ 0 & 1.4 & 3.2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -6 \\ 8 \end{bmatrix}$$

Again search the largest magnitude element within the modified matrix A excluding its first row and first column this element is 3.2 and belongs to third row and third column. interchange the second and third rows

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 1.4 & 3.2 \\ 0 & -0.8 & 2.6 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ -6 \end{bmatrix}$$

Now interchange the 2nd and 3rd column, and also Exchange the order of 2nd and 3rd variables of the matrix x i.e x_1 and x_3 the linear system can be written as

$$\begin{bmatrix} 5 & -1 & 3 \\ 0 & 3.2 & 1.4 \\ 0 & 0 & -1.9375 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ -6 \end{bmatrix}$$

Lastly apply row operations

$R_3 - (2.6/3.2) R_2$

$$\begin{bmatrix} 5 & -1 & 3 \\ 0 & 3.2 & 1.4 \\ 0 & 0 & -1.9375 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ -12.5 \end{bmatrix}$$

Coefficient matrix A is an upper triangular form, implying that no more row operations are required

The equivalent linear system can be written as

$$5x_2 - x_3 + 3x_1 = 20$$
$$3.2x_3 + 1.4x_1 = 8$$
$$-1.9375x_1 = -12.5$$

We use back substitution to obtain the solution

$x_1 = 6.45161$ $\quad x_3 = -0.32258$ $\quad x_2 = 0.06451$