# Consistency of system of Linear Equations.

Consider a system of $m$ simultaneous linear equations in $n$ unknowns $x_1, x_2 \dots x_n$ given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 - - - - - + a_{mn}x_n = b_m$$

The system of equations can be written in matrix form as.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ \vdots & \vdots & & & \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$AX = B$$

Augmented matrix

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ \vdots & & & \vdots & : & b_2 \\ \vdots & & & \vdots & : & \vdots \\ a_{m1} & a_{m2} & - & a_{mn} & : & b_m \end{bmatrix}$$

→ If $b_1, b_2 \ldots b_m = 0$ then $B = 0$

⇒ $AX = 0$

Such a system of equation is called a system of homogeneous linear Equation.

→ Homogeneous linear Equation always Consistent.

$AX = B$ Such a system of equation is called a system of non-homogeneous linear equation.

Working Rule for finding the solution to the Equation $AX = B$

Case-I Rank of A ≠ Rank of [A : B]

⇒ In this case the Equation $AX=B$ is inconsistent i.e It has no Solution.

Case-II Rank of A = Rank of [A : B] = no of unknown

⇒ System has Unique solution.

Case III Rank of A = Rank of [A : B] < no of unknown

$\Rightarrow$ System of Equations has an Infinite no of Solutions.

Q.1 Test for Consistency and Solve the system

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

Solution We have

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

and hence $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

Then the augmented matrix is given by

$$[A : B] = \left[\begin{array}{ccc:c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

Operate $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

Operate $R_2 \to R_2 - 2R_1$ , $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{5}{7}R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8/7 & : & -8/7 \end{bmatrix}$$

$\Rightarrow$ Rank of $[A:B] = 3$ and Rank of $A = 3$

$\Rightarrow$ Rank of A = Rank of $[A:B]$, = number of unknown

Hence the given system is consistent

In matrix form the above system reduces to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 8/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -8/7 \end{bmatrix}$$

$\Rightarrow$ $x + 2y + z = 4$, $-7y - 3z = -11$, $\frac{8}{7}z = -\frac{8}{7}$

$\Rightarrow$ $\boxed{x = 1}$ $\boxed{y = 2}$ and $\boxed{z = -1}$

Example-2 Show that equations

$$2x + 6y = -11$$

$$6x + 20y - 6z = -3$$

$$6y - 18z = -3$$

are not consistant.

Sol$^n$ We have

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

and hence

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

then the augmented matrix is given by

$$[A : B] = \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

Operate $R_1 \leftrightarrow R_2 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

operate $R_3 \rightarrow R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{array}\right]$$

$\Rightarrow$ Rank of $[A:B] = 3$ Rank of $A = 2$

$\Rightarrow$ Rank of $A \neq$ Rank of $[A:B]$ hence the given system is Consistant and Possesses no solution.

Q. Test for Consistency and Solve the system.

$$5x + 3y + 7z = 4$$
$$3x + 26y - 2z = 9$$
$$7x + 2y + 10z = 5$$

Solution We have

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Then the augmented matrix is given by

$$[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

operate $R_1 \to \frac{1}{5} R_1$

$$\sim \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

operate $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & -11/5 & 1/5 & : & -3/5 \end{bmatrix}$$

Operate $R_3 \to R_3 + \frac{1}{11}R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 3/5 & 7/5 & 4/5 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ Rank of $[A:B] = 2$ , Rank of $A = 2$

$\Rightarrow$ Rank of $A$ = Rank of $[A:B]$ < number of unknowns.

Hence the given system is Consistent and possesses Infinite no of Solutions.

$$\begin{bmatrix} 1 & 3/5 & 7/5 \\ 0 & 121/5 & -11/5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4/5 \\ 33/5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + \frac{3y}{5} + \frac{7z}{5} = \frac{4}{5}$$

$$\frac{121}{5}y - \frac{11}{5}z = \frac{33}{5}$$

Let $z = k$ then We have,

$$11y - k = 3$$

$$y = \frac{3}{11} + \frac{k}{11}$$

Also $\quad x = \frac{-16}{11}k + \frac{7}{11}$

Here $k$ can take infinite number of values so $x, y, z$ also take infinite Values

Thus there exists infinite ~~values~~ number of Solutions.

Ex for what value of $\lambda$ and $\mu$ do the system Equations:

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) no Solution (ii) Unique Solution (iii) more than One Solution ?

Sol$^n$ The given system of Eq$^n$ in matrix notation Can be written as $AX = B$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

operate $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-6 \end{bmatrix}$$

operate $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

Case - I . There is no solution if Rank of A $\neq$ Rank of [A:B]

i.e $\lambda - 3 = 0$ or $\lambda = 3$

and $\mu - 10 \neq 0$ or $\mu \neq 10$

Case - II there is a unique solution if Rank of A = Rank of [A:B] = number of unknowns

i.e $\lambda - 3 \neq 0$ or $\lambda \neq 3$. $\mu$ may have any value.

Case - III there are infinite solutions if Rank of A = Rank of [A:B] < number of unknowns.

i.e $\lambda - 3 = 0$ or $\lambda = 3$

$\mu - 10 = 0$ or $\mu = 10$

Ex for what value of $\eta$ the equations

$x + y + z = 1$

$x + 2y + 4z = \eta$

$x + 4y + 10z = \eta^2$

have a solution and solve them completely in each case?

Solⁿ the matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \eta \\ 1 & 4 & 10 & \eta^2 \end{array}\right]$$

Operate $R_2 \to R_2 - R_1$, $R_3 - R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 3 & 9 & \eta^2-1 \end{array}\right]$$

Operate $R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 0 & 0 & \eta^2-3\eta+2 \end{array}\right]$$

Now the given Equation will be consistent iff

$$\eta^2 - 3\eta + 2 = 0$$

if $$(\eta-2)(\eta-1) = 0$$

if $$\eta = 2 \text{ or } \eta = 1$$

Case I if $\eta = 2$

$$y + 3z = \eta - 1 \Rightarrow y + 3z = 1$$

and $x + y + z = 1$

$$\therefore y = -3z, \; x = 2z$$

Thus $x = 2k$, $y = 1-3k$, $z = k$ Consititute the general solution, where k is any arbitrary constant.

Case - II if $\eta = 1$ the Eq$^n$

$$y + 3z = 0, \quad x + y, z = 1$$

$$\therefore \quad y = -3z, \quad x = 1 + 2z$$

thus $x = 1 + 2c$, $y = -3c$, $z = c$ constitute the general solution where c is any arbitrary constant

Ex- For what values of k the equations

$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2$$

have a solution and solve them completely in each case?

Ex. Solve the Eq$^n$

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - y + z = -1$$

Ex. Test the consistency and solve.

$$2x + 5y + 3z = 1$$
$$-x + 2y + z = 2$$
$$x + y + z = 0$$