Unit-II

25/Aug/17

→ **Solⁿ of Differential Eqⁿs-**

I) First order & second order homogeneous eqⁿ-

$\frac{dy(x)}{dt} + y(t) = 0$	$\frac{d^2y(x)}{dt} + 3\frac{dy(x)}{dt} + 2y(t) = 0$

replace $D + 1 = 0$

$\boxed{P_1 = -1}$

$C.F. = ke^{P_1 t}$

$C.F. = ke^{-t}$

$D^2 + 3D + 2 = 0$

$(D+1)(D+2) = 0$

$P_1 = -1, \quad P_2 = -2$

$C.F. = k_1 e^{P_1 t} + k_2 e^{P_2 t}$

$= k_1 e^{-t} + k_2 e^{-2t}$

II) First/second order Non-homogeneous eqⁿ containing constant or exponential fn on right hand side-

Case I: ~~Constant~~ Exponential fn on right side

~~Make~~ Put RHS=0

Find PI and C.F.

Solⁿ = PI + C.F.

eg. $\frac{dy(t)}{dt} + y = 5e^{2t} / e^{at}$

A.E. $D + 1 = 0$

$D = -1$

1) $C.F. = ke^{-t}$

2) $P.I. = \frac{5e^{2t}}{D+1}$

Replace D by 2 (i.e. replace D by a for e^{at})

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$\Rightarrow$ $P.I. = \dfrac{5e^{2t}}{2+1} = \dfrac{5e^{2t}}{3}$

Now, $y = CF + PI = ke^{-t} + \dfrac{5e^{2t}}{3}$

eg. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = e^{2t}$

A.E $\Rightarrow$ $D^2 + 3D + 2 = 0$

$(D+1)(D+2) = 0$

$D = -2, -1$

Now, $P.I = \dfrac{e^{2t}}{D^2+3D+2}$

Put $D =$

II] Constant on RHS —

eg. $\dfrac{dy}{dt} + y = 5$

A.E] $D + 1 = 0$

$D = -1$

$C.F = ke^{-t}$

$P.I. = \dfrac{5\,e^{0t}}{D+1}$

eg.) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 5$

A.E. $D^2 + 3D + 2 = 0$

$D = -1, -2$

$P.I. = \dfrac{5\,e^{0t}}{D^2+3D+2}$

Put $D = 0$

$\Rightarrow P.I. =$

Type 3: First & Second order Non-homogeneous eq^n with RHS as a sine or cosine fn.

eg. $\frac{dy}{dt} + yt = \sin 5t$

A.E $\Rightarrow$ $D+1=0$

$D=-1$

$C.F = Ke^{-t}$

$P.I. = \frac{\sin 5t}{D+1}$

Replace D^2 by $-a^2$,

To do so, we need to make $D \to D^2$,

$\therefore$ Rationalizing the fraction,

$= \frac{\sin 5t}{(D+1)} \frac{(D-1)}{(D-1)}$

$= \frac{(D-1)\sin 5t}{D^2-1}$

Put $D^2 = -a^2 = -(5)^2$

$= \frac{(D-1)\sin 5t}{-25-1}$

$= \frac{(D-1)\sin 5t}{-26}$

Equating $D = \frac{d}{dt}$,

$P.I. = \frac{d}{dt}\left(\frac{\sin 5t}{-26}\right) - \frac{\sin 5t}{(-26)}$

$= \frac{-1}{26} 5\cos 5t + \frac{\sin 5t}{26}$

$P.I. = \frac{\sin 5t}{26} - \frac{5\cos 5t}{26}$

and, $\{ y = C.F. + P.I. \}$

IInd Order-

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin 5t$$

$$D^2 + 3D + 2 = 0$$

$$P.I. = \frac{\sin 5t}{D^2 + 3D + 2}$$

Put $D^2 = -(5)^2$

$$= \frac{\sin 5t}{-25 + 3D + 2}$$

$$= \frac{\sin 5t}{3D - 23}$$

$$= \frac{\sin 5t}{(3D - 23)} \cdot \frac{(3D + 23)}{(3D + 23)}$$

$$= \frac{(3D + 23)\sin 5t}{9D^2 - 529}$$

$$= \frac{(3D + 23)\sin 5t}{-225 - 529}$$

Again equate $D = \frac{d}{dt}$ & solve.

→ Circuit Analysis by Classical Method -

Transient Response - Whenever a circuit is switched from one condition to another, then there is a transition period during which the current & voltage change from former values to new one. This period is known as "Transition time" & the value of current & voltage are known as "Transient Response".

Steady State Response - When the transient period is over OR the transition has died out, the circuit is said to be in steady state.
~~Zero input response~~

Zero Input Response - The value of current & voltage which results from initial condition when applied input is zero.

Zero State Response - The values of current & voltage for an input which is applied when all the initial conditions are zero. Such networks/circuits are said to be at rest.

→ Order of Differential Eq^n -
If a circuit contains only one storage element (L or C) ⇒ 1st order
" two " ⇒ 2nd order

- The circuit changes are assumed ~~a~~to occur at time $t=0$ & represented by a switch.

- Switching ON/OFF at $t=0$ does not disturb the storage elements so that $I(0^-) = I(0^+)$.
In case of capacitor $V_c(0^-) = V_c(0^+)$

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→ Voltage-current relationship for storage elements -

1) $V = IR$; $I = V/R$

2) $V_L(t) = L\frac{dI(t)}{dt}$; $I(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(t)dt$

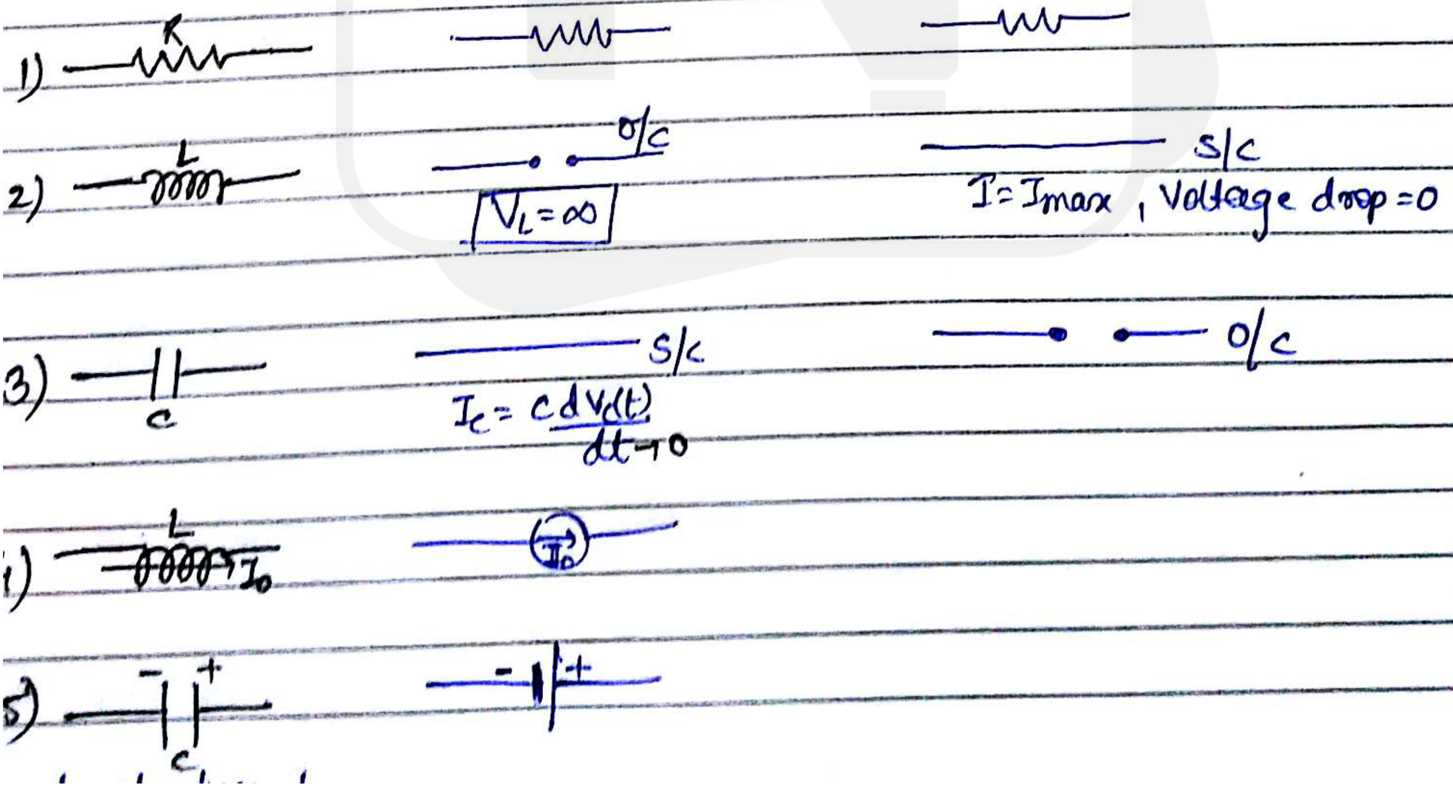
or $I(t) = \frac{1}{L}\int_{0}^{t} V_L(t)dt + I_L(0^-)$

3) $V_C(t) = \frac{1}{C}\int_{-\infty}^{t} I_C(t)dt$; $I_C(t) = C\cdot\frac{dV_C(t)}{dt}$

$= \frac{1}{C}\int_{0}^{t} I_C(t)dt + V_C(t)$

→ Initial / Boundary conditions in a circuit -

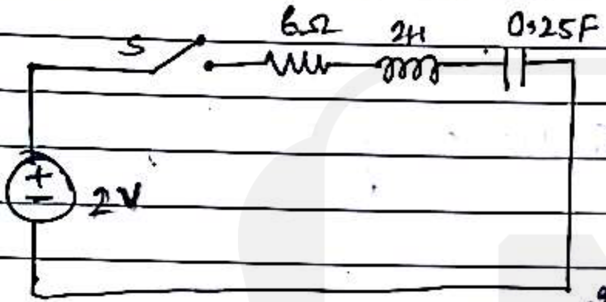
Circuit Element Eq. ckt at $t=0^+$ Eq. ckt. at $t=\infty$



→ Circuit Analysis

Ques) Calculate $i(0^+)$, $\frac{di}{dt}(0^+)$, $\frac{d^2i}{dt^2}(0^+)$, $i(t)$ — transient response

of a given circuit where source is of 2V and resistance is of 6Ω, Inductor = 2H, capacitor = 0.25F when the switch S is closed at time t=0.



Applying KVL,

$$V_s = 6i(t) + 2\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt + V_c(0^-)$$

$$2 = 6i(t) + 2\frac{di}{dt} + \frac{4}{8}\int_0^t i(t)dt + \underbrace{V_c(0^-)}_{\to 0} \quad — (1)$$

(no initial charge)

$$2 = v(t)16 + 16\int_0^t v(t)dt + 4\frac{dv(t)}{dt}$$

$$2 = 4\frac{dv(0^+)}{dt} \Rightarrow \boxed{\frac{dv(0^+)}{dt} = 1/2}$$

Double diff.,

$$0 = 16\frac{dv(t)}{dt} + \frac{1}{2} v(t)dt + 4\frac{d^2v(t)}{dt^2}$$

$$0 = \overset{8}{\cancel{16}}\left(\frac{1}{\cancel{2}}\right) + 0 + 4\frac{d^2v(t)}{dt^2}$$

$$\boxed{\frac{d^2v(t)}{dt^2} = -2}$$

Applying KVL,

$$V = Ri(t) + L\frac{di(t)}{dt}$$

$$V = Ri + LDi$$

$$LD + R = 0$$

$$D = -R/L$$

$$\Rightarrow i(t) = Ke^{-\frac{R}{L}t} + V/R$$

$$C.F. = Ke^{D_1 t}$$

$$= Ke^{-\frac{R}{L}t}$$

$$P.I. = \frac{V}{LD+R}$$

Put $D = 0$

$$= \frac{V}{0+R} = \frac{V}{R}$$

—— x ——

Applying $i(0^+) = 0$

Because a $t = 0^+$, L. behaves as open ckt. (initial condⁿ)

Putting $t = 0^+$,

i.e. at $i(0^+)$

eqⁿ (1) becomes, $2 = 6\underbrace{i(0^+)}_{\to 0} + 4\int_0^t \underbrace{i(0^+)}_{\to 0}dt + 2\frac{di(0^+)}{dt}$

$$\cancel{2} = \cancel{2}\frac{di(0^+)}{dt}$$

or $\boxed{\frac{di(0^+)}{dt} = 1}$ —— (II)

Double diff. of eqⁿ (1),

$$0 = 6\frac{di(t)}{dt} + 2\frac{d^2 i(t)}{dt^2} + 4i(t) \quad \text{—— (III)}$$

Put $\frac{d}{dt} = D$,

or $6Di + 2D^2(i) + 4(i) = 0$

or $3D + D^2 + 2 = 0$

$$D^2 + 3D + 2 = 0$$
$$(D+1)(D+2) = 0$$
$$D = -1, \quad D = -2$$

Now, $C.F = K_1 e^{-D_1 t} + K_2 e^{-D_2 t}$

$$i(t) = K_1 e^{-t} + K_2 e^{-2t} \quad \text{--- (IV)}$$

Apply initial condition in eq^n (IV), [To determine values of K_1 & K_2]

i.e. $t=0$

$$\underbrace{i(0^+)}_{0} = K_1 + K_2 \quad \text{--- (V)}$$

Diff. eq^n (IV) w.r.t t,

$$\frac{di(t)}{dt} = -K_1 e^{-t} - 2K_2 e^{-2t}$$

Put $t=0$,

$$\underbrace{\frac{di(0^+)}{dt}}_{1 \text{ (from (III))}} = -K_1 - 2K_2$$

$$1 = -K_1 - 2K_2$$

and, $0 = K_1 + K_2$

$$1 = -K_2$$

$$\Rightarrow K_1 = 1, \quad K_2 = -1$$

Putting these values in eq^n (IV),

$$\boxed{i(t) = e^{-t} - e^{-2t}}$$

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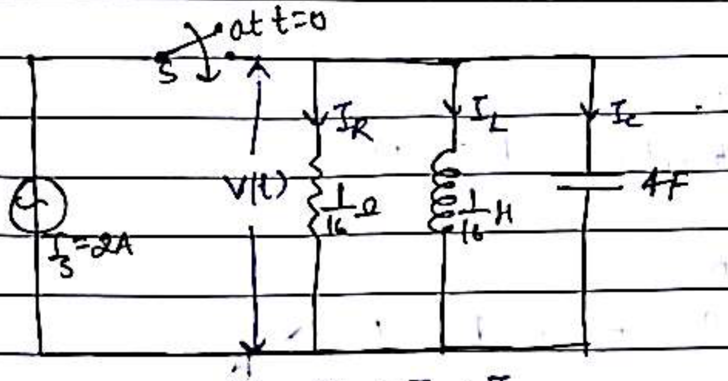
Ques)



$i(0^+) = ?$

$v(t) = ?$

$\frac{dv(0^+)}{dt} = ?$

$\frac{d^2v(0^+)}{dt^2} = ?$

$I_S = I_R + I_L + I_C$

Applying KCL,

$$\underset{I_S}{\underset{\parallel}{2}} = \underset{I_R}{\underset{\parallel}{\frac{v(t)}{R}}} + \underset{I_L}{\underset{\parallel}{\left\{\frac{1}{L}\int_0^t v(t) + \underbrace{I_L(0^-)}_{\to 0}\right\}}} + \underset{I_C}{\underset{\parallel}{C \cdot \frac{dv(t)}{dt}}}$$

Since all three elements are connected in parallel, voltage across each is same.

$$\Rightarrow \quad 2 = 16v(t) + 16\int_0^t v(t)dt + 4\frac{dv(t)}{dt} \quad \text{—} (i)$$

At initial condⁿ, C acts as short circuited element.

$\Rightarrow R_C = 0$

$\Rightarrow$ Entire current flows through this branch

$\Rightarrow$ Voltage across s/c = 0.

$\Rightarrow \quad v(0^+) = 0$

Putting $v(0^+) = 0$ in eqⁿ (i),

$$2 = 0 + 0 + 4\frac{dv(0^+)}{dt}$$

or $\boxed{\frac{dv(0^+)}{dt} = \frac{1}{2}}$ — (ii)

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Double diff. of eq^n 1,

$$0 = 16\frac{dv(t)}{dt} + 16v(t) + 4\frac{d^2v(t)}{dt^2}$$

Put $v(0^+) = 0$

$$0 = 16 \cdot \underbrace{\frac{dv(0^+)}{dt}}_{=\frac{1}{2}} + 0 + 4\frac{d^2v(0^+)}{dt^2}$$

$$0 = 8 + 4\frac{d^2v(0^+)}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2v(0^+)}{dt^2} = -2}$$

~~0 = 16~~ Diff. eq^n 1,

$$0 = 16\frac{dv(t)}{dt} + 16v(t) + 4\frac{d^2v(t)}{dt^2}$$

$$0 = 16D + 4D^2 + 16$$

or $D^2 + 4D + 4 = 0$

$(D+2)^2 = 0$

or $D = -2$

$$CF = K_1e^{-Dt} + K_2e^{-Dt}$$

$$v(t) = k_1e^{-2t} + tk_2e^{-2t} \quad \text{— (iii)}$$

Applying initial cond^n, ~~to~~ $v(0^+) = 0$

$t = 0^+$

$$\underbrace{v(0^+)}_{0} = K_1 + 0K_2$$

$$\boxed{K_1 = 0}$$

Diff. eq^n (iii) w.r.t. t,

$$\frac{dv(t)}{dt} = -2k_1e^{-2t} + \left(k_2e^{-2t} - 2k_2te^{-2t}\right)$$

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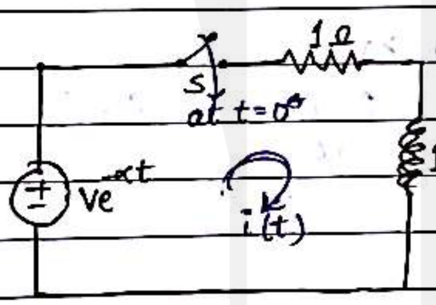
Again put $t = 0^+$,

$$\frac{dv(0^+)}{dt} = -\alpha \underbrace{k_1}_{0} + k_2$$

or $\boxed{\frac{1}{\alpha} = k_2}$ {from (II)}

$$\Rightarrow \boxed{v(t) = \frac{1}{2} t e^{-\alpha t}}$$

Ques) For a given ckt., voltage source is an exponential fn $\{v(t) = Ve^{-\alpha t}$ where α is constant, the switch S is closed at $t=0$, $R = 1\Omega$, $L = 1H$. The ckt. is shown below. Calculate $i(t)$.



Applying KVL,

$$Ve^{-\alpha t} = i(t)R + L\frac{di(t)}{dt}$$

$$\Rightarrow Ve^{-\alpha t} = i(t) + \frac{di(t)}{dt} \quad \text{—(1)}$$

or $0 = D + 1$

$D = -1$

$\therefore$ $CF = ke^{-t}$

$$P.I. = \frac{Ve^{-\alpha t}}{-\alpha + 1}$$

or $$i(t) = ke^{-t} + \frac{Ve^{-\alpha t}}{1-\alpha} \quad \text{—(II)}$$

Put $t = 0$,

$$0 = K + \frac{V}{1-\alpha} \quad \text{or} \quad K = \frac{V}{\alpha - 1}$$

$$\Rightarrow i(t) = \frac{Ve^{-t}}{\alpha - 1} + \frac{Ve^{-\alpha t}}{1-\alpha}$$

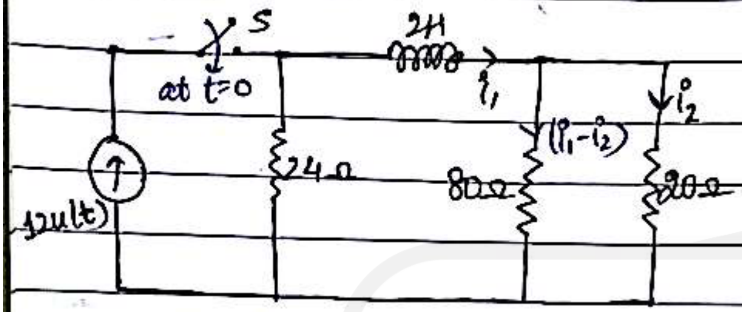
Ques) Calculate the current i_1, i_2 and voltage V_1 of the given network

i) at $t=0^+$

ii) at $t=0^-$

iii) at $t=\infty$

iv) at $t = 50ms$



Since input ~~volt~~ signal is an unit signal (which exists only for $t \geq 0$);

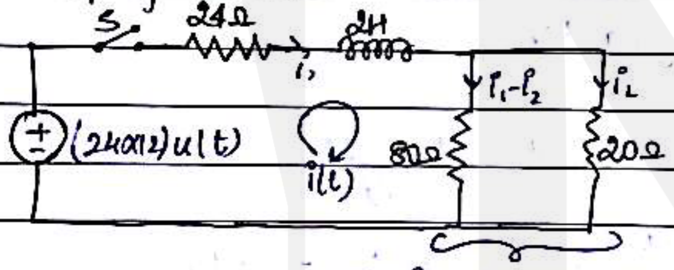
$\Rightarrow i_1(0^-)=0$, $i_2(0^-)=0$

$V(0^-)=0$

~~Applying KVL,~~ Also at $t=0^+$, L behaves as open ckt.

$i_1(0^+)=0$; $i_2(0^+)=0$

Simplified ckt:



Applying KVL,

$$288 = 24i_1 + 2\frac{di_1}{dt} + 80(i_1 - i_2) \quad \text{—— (1)}$$

$$i_2 = i_1 \times \frac{80}{80+20} = 0.8i_1$$

$\Rightarrow$ eq^n (1) becomes, $288 = 24i_1 + 2\frac{di_1}{dt} + 80(i_1 - 0.8i_1)$

$$288 = 24i_1 + 2\frac{di_1}{dt} + 16i_1$$

$$144 = 20i_1 + \frac{di_1}{dt}$$

Now, $D + 20 = 0$

$D = -20$

Now, CF $= ke^{-20t}$

$$P.I. = \frac{144}{20} = 7.7$$

~~$i(t) =$~~

Now, $i_1(t) = ke^{-20t} + 7.7$

at $t=0^+$, $i(0^+)=0$

$0 = k + 7.7$

$\Rightarrow k = -7.7$

$$\boxed{i_1(t) = -7.7e^{-20t} + 7.7}$$

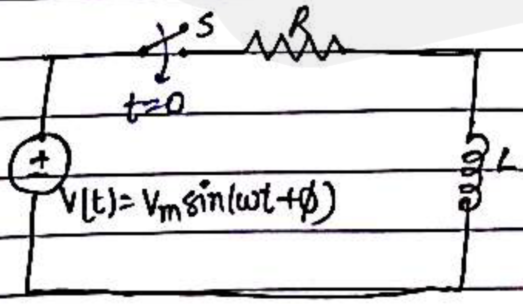
Now, at $t=\infty$, $i_1(\infty) = -7.7\underbrace{e^{-\infty}}_{\to 0} + 7.7$

$i_1(\infty) = 7.7mA.$

and, $i_2(t) = \frac{4}{5} i_1$

$\therefore i_2(\infty) = \frac{4}{5} \times \frac{144}{20\cancel{5}} = \frac{144}{25}$

Ques) Consider a series RL ckt. excited by sinusoidal voltage $v(t) = V_m \sin(\omega t + \phi)$ as shown in the ckt.



Applying KVL,

$v(t) = iR + L\frac{di}{dt}$

$\frac{v(t)}{L} = \frac{R}{L}i + \frac{di}{dt}$

or $D + \frac{R}{L} = 0$

$\rightarrow D = -R/L$

$C.F = Ke^{-\frac{R}{L}t}$

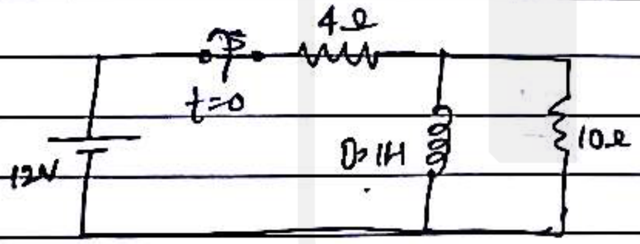
$P.I. = \frac{V(t)}{(D+R/L)} \times \frac{(D-R/L)}{(D-R/L)}$

$= \frac{(D-R/L)\,V(t)}{D^2-(R/L)^2}$

Replace D^2 with $-\omega^2$,

$i(t) = Ke^{-\frac{R}{L}t} - \frac{V_m}{R^2+\omega^2L^2}\left[\omega L\cos(\omega t+\phi) - R\sin(\omega t+\phi)\right]$

Ques) A 12V battery is suddenly disconnected at $t=0$. Find the inductor current ~~at t=0~~ and $V_L(t)$.



at $t=\infty$, L behaves as short circuit

$\Rightarrow$ no current flows through 10Ω

$\Rightarrow i = \frac{12}{4} = 3A$

Applying KVL at $t=0^-$,

Ques)

→ Solutions of Circuits Using Laplace Transform:-

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 5u(t) \quad \text{—— (I)}$$

Taking Laplace transform on both sides,

$$L\left(\frac{d^2y}{dt^2}\right) = s^2Y(s) - sy(0^+) - y'(0^+)$$

$$L\left(\frac{dy}{dt}\right) = sY(s) - y(0^+)$$

$$Ly(t) = Y(s)$$

$$L\{5u(t)\} = \frac{5}{s}$$

∴ Eq^n (I) becomes,

$$s^2Y(s) - sy(0^+) - y'(0^+) + 3sY(s) - 3y(0^+) + 2Y(s) = \frac{5}{s} \quad \text{—— (II)}$$

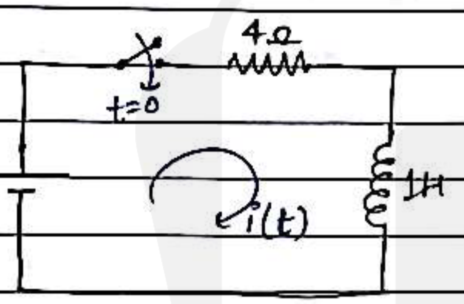
Given that: Value of $y(0^+) = -1$
$y'(0^+) = 2$

∴ Eq^n (II) becomes,

$$s^2Y(s) - s(-1) - (2) + 3sY(s) + 2Y(s) = 5/s$$

or $Y(s)\left[s^2 + 3s + 2\right]$

Ques)



Assume initial current through inductor $= 3A$

Applying KVL,

$$48 = 4i(t) + \frac{di(t)}{dt} \quad \text{—— (1)}$$

Taking Laplace on both sides,

$$\frac{48}{s} = 4\,I(s) + sI(s) - i(0^+)$$

$$\frac{48}{s} = (s+4)\,I(s) - 3$$

$$I(s) = \frac{3s+48}{s(s+4)}$$

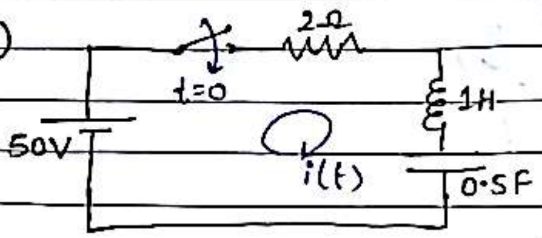
$$= \frac{A}{s} + \frac{B}{s-4}$$

$$= \frac{12}{s} - \frac{9}{(s+4)}$$

→ Applying Inverse LT,

$$I(t) = 12 - 9e^{-4t}$$

Ques)



$$50 = 2i(t) + \frac{di(t)}{dt} + 2\int_0^t i\,dt$$

Applying Laplace,

$$2I(s) + [sI(s) - \underbrace{i(0^+)}_{0}] + \frac{2I(s)}{s} = \frac{50}{s}$$

$$I(s)\left[2 + s + \frac{2}{s}\right] = \frac{50}{s}$$

$$I(s)\left[s^2 + 2s + 2\right] = 50$$

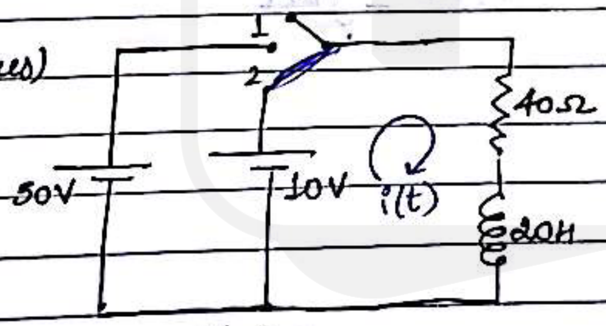
or $$I(s) = \frac{50}{s^2+2s+2}$$

$$= \frac{50}{\{(s+1)^2+1\}}$$

Applying Inverse LT,

$$= 50(\sin t)\,e^{-t}$$

Ques)



Switch is held in pos. 1 for a long time, then it is moved to pos. 2.

$$i(0^+) = i(0^-) = \frac{50}{40} = 5/4$$

Applying KVL,

$$10 = 40i(t) + 20\frac{di(t)}{dt}$$

Applying LT,

$$\frac{10}{s} = 4I(s) + 20[sI(s) - i(0^+)]$$

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$$\frac{10}{s} = 40\,I(s) + 20s\,I(s) - 20\left(\frac{5}{4}\right)$$

$$\frac{10}{s} + 25 = I(s)\left[40 + 20s\right]$$

$$\frac{25s + 10}{20s^2 + 40s} = I(s)$$

$$I(s) = \frac{5s+2}{4s(s+2)}$$

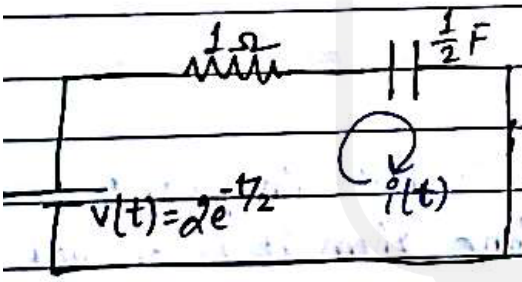
Applying partial fraction,

$$I(s) = \frac{1}{4s} + \frac{1}{s+2}$$

Applying ILT,

$$\boxed{i(t) = \frac{1}{4} + e^{-2}}$$

Ques) Find the current $i(t)$ if the voltage source is $v(t) = 2e^{-t/2}$, $v_c(0^-) = 0$, $R = 1$, $C = \frac{1}{2}$ F.



$$v(t) = 1\,i(t) + 2\int_0^t i\,dt$$