Eigen Values And Eigen Vectors :-

Find the Eigen Values and Eigen Vector of matrix A

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\therefore$ $\lambda = 2, 3, 5$

Thus the given values of A are 2, 3, 5

the corresponding to $\lambda = 2$, the given vector is given by

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + 4x_3 = 0$$

$$6x_3 = 0 \quad \Rightarrow x_3 = 0$$

$$\Rightarrow \quad x_1 + x_2 + 0 = 0$$

$x_2 = K$

Hence $x_1 = -K$, $x_2 = K$, $x_3 = 0$

Hence Eigen vector $= \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(ii) corresponding to $\lambda = 3$ the given vector is given by

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + 4x_3 = 0 \; , \; -x_2 + 6x_3 = 0 \; , \; 2x_3 = 0$

Hence Eigen Vector $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Corresponding to $\lambda = 5$, the Eigen vector is given by

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 + 4x_3 = 0$

$-3x_2 + 6x_3 = 0$

$\frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6} \Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$

Hence Eigen vector $= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Find the Eigen Values and Eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol^n The characteristic Equation of the given matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)[-\lambda(1-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + 1(1-\lambda)] = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By trial $\lambda = -3$ satisfies it

$\therefore$ $(\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

$\Rightarrow$ $(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\Rightarrow$ $\lambda = -3; -3, 5$

Thus the Eigen values of A are $-3, -3, 5$

Corresponding to $\lambda = -3$ the Eigen Vector is given by

$$\begin{bmatrix} -2-(-3) & 2 & -3 \\ 2 & 1-(-3) & -6 \\ -1 & -2 & -(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives one Independent equation

$x_1 + 2x_2 - 3x_3 = 0$

$\therefore$ Choosing $x_2 = 0$, We have $x_1 - 3x_3 = 0$

$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{1}$ giving the Eigen vector.

$$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Choosing $x_3 = 0$ We have $x_1 + 2x_2 = 0$

$\frac{x_2}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$

giving the Eigen vector

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Any other Eigen vector Corresponding to $\lambda = -3$ will be linear Combination of these two

ii) Corresponding to $\lambda = 5$ the Eigen Vector is given by

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_3 = 0$

$$7x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 - 5x_3 = 0$$

Since only two of them are independent, we can omit one of them from first two equations we have

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

giving the Eigen vector (1, 2, -1)

Determine the characteristic roots and the corresponding characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Power Method :-

Determine the largest Eigen value and the Corresponding Eigen vector of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Let the initial approximation to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then $$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = d^{(1)} x^{(1)}$$

So the first approximation to the eigen value is $d^{(1)} = 5$ and the corresponding eigen vector is $x^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$

Now $$Ax^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8 \begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = d^{(2)} x^{(2)}$$

thus the 2nd approximation to the eigen value is $d^{(2)} = 5.8$ and the corresponding eigen vector is $x^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$

Repeating the above process, we get

$$Ax^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966 \begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = d^{(3)} x^{(3)}$$

$$A x^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994 \begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$A x^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999 \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$A x^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{6} x^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $x^{(5)} \simeq x^{(6)}$ upto 3 decimal places.

Hence the largest Eigen Value is 6 and the Corresponding Eigen Vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$

Ex: Find the largest Eigen Value and the corresponding Eigen vector of the matrix

$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using power method. Take $[1\ 0\ 0]^T$ as initial Eigen Vector

Sol^n Let the initial approximation to the required eigen vector be $X = [1, 0, 0]$

then $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

So the first approximation to the ~~required~~ Eigen ~~vector~~ ~~be~~ value is 2 and the corresponding eigen vector

$$x^{(1)} = [1, -0.5, 0]'$$

Hence

$$A x^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$= \lambda^{(2)} x^{(2)}$$

Repeating the above process, we get

$$A x^{(2)} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$A x^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$A (x^{(4)}) = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$A x^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

$$A x^{(6)} = 3.41 \begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)} x^{(7)}$$

clearly $\lambda^{(6)} = \lambda^{(7)}$ and $x^{(6)} = x^{(7)}$ approximately

ue the largest Eigen value 3.41 corresponding to Eigen ctor $[0.74, -1, 0.67]'$

Ex Find the Power Method, the largest Eigen value of the following matrices:

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ [Ans 5.38, $\begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$]

(ii) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ [Ans. 4.618, $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$]

Ex. Find the largest Eigen value and the corresponding Eigen vector of the matrices:

(i) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ [11.66; [0.025, 0.422, 1]]

(ii) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ Ans [7, [$\frac{2.099}{7}$, $\frac{0.467}{7}$, 1]]