# 15

# AC Fundamentals

## OBJECTIVES

After studying this chapter, you will be able to

- explain how ac voltages and currents differ from dc,
- draw waveforms for ac voltage and currents and explain what they mean,
- explain the voltage polarity and current direction conventions used for ac,
- describe the basic ac generator and explain how ac voltage is generated,
- define and compute frequency, period, amplitude, and peak-to-peak values,
- compute instantaneous sinusoidal voltage or current at any instant in time,
- define the relationships between $\omega$, $T$, and $f$ for a sine wave,
- define and compute phase differences between waveforms,
- use phasors to represent sinusoidal voltages and currents,
- determine phase relationships between waveforms using phasors,
- define and compute average values for time-varying waveforms,
- define and compute effective values for time-varying waveforms,
- use Electronics Workbench and PSpice to study ac waveforms.

## KEY TERMS

ac
Alternating Voltage
Alternating Current
Amplitude
Angular Velocity
Average Value
Cycle
Effective Value
Frequency
Hertz
Instantaneous Value
Oscilloscope
Peak Value
Period
Phase Shifts
Phasor
RMS
Sine Wave

## OUTLINE

## CHAPTER PREVIEW

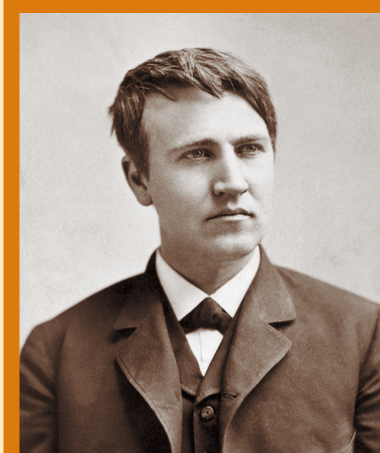
**Alternating currents (ac)** are currents that alternate in direction (usually many times per second), passing first in one direction, then in the other through a circuit. Such currents are produced by voltage sources whose polarities alternate between positive and negative (rather than being fixed as with dc sources). By convention, alternating currents are called *ac currents* and alternating voltages are called *ac voltages.*

The variation of an ac voltage or current versus time is called its waveform. Since waveforms vary with time, they are designated by lowercase letters $v(t)$, $i(t)$, $e(t)$, and so on, rather than by uppercase letters $V$, $I$, and $E$ as for dc. Often we drop the functional notation and simply use $v$, $i$, and $e$.

While many waveforms are important to us, the most fundamental is the sine wave (also called sinusoidal ac). In fact, the sine wave is of such importance that many people associate the term ac with sinusoidal, even though ac refers to any quantity that alternates with time.

In this chapter, we look at basic ac principles, including the generation of ac voltages and ways to represent and manipulate ac quantities. These ideas are then used throughout the remainder of the book to develop methods of analysis for ac circuits.

## PUTTING IT IN PERSPECTIVE



### *Thomas Alva Edison*

NOWADAYS WE TAKE IT FOR GRANTED that our electrical power systems are ac. (This is driven home every time you see a piece of equipment rated "60 hertz ac", for example.) However, this was not always the case. In the late 1800s, a fierce battle—the so-called "war of the currents"—raged in the emerging electrical power industry. The forces favoring the use of dc were led by Thomas Alva Edison, and those favoring the use of ac were led by George Westinghouse (Chapter 24) and Nikola Tesla (Chapter 23).

Edison, a prolific inventor who gave us the electric light, the phonograph, and many other great inventions as well, fought vigorously for dc. He had spent a considerable amount of time and money on the development of dc power and had a lot at stake, in terms of both money and prestige. So unscrupulous was Edison in this battle that he first persuaded the state of New York to adopt ac for its newly devised electric chair, and then pointed at it with horror as an example of how deadly ac was. Ultimately, however, the combination of ac's advantages over dc and the stout opposition of Tesla and Westinghouse won the day for ac.

Edison was born in 1847 in Milan, Ohio. Most of his work was done at two sites in New Jersey—first at a laboratory in Menlo Park, and later at a much larger laboratory in West Orange, where his staff at one time numbered around 5,000. He received patents as inventor or co-inventor on an astonishing 1,093 inventions, making him probably the greatest inventor of all time.

Thomas Edison died at the age of 84 on October 18, 1831.

## 15.1 Introduction

Previously you learned that dc sources have fixed polarities and constant magnitudes and thus produce currents with constant value and unchanging direction, as illustrated in Figure 15–1. In contrast, the voltages of ac sources alternate in polarity and vary in magnitude and thus produce currents that vary in magnitude and alternate in direction.

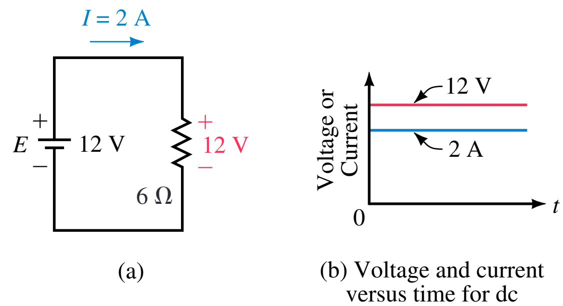



**FIGURE 15–1** In a dc circuit, voltage polarities and current directions do not change.

### Sinusoidal AC Voltage

To illustrate, consider the voltage at the wall outlet in your home. Called a **sine wave** or **sinusoidal ac waveform** (for reasons discussed in Section 15.5), this voltage has the shape shown in Figure 15–2. Starting at zero, the voltage increases to a positive maximum, decreases to zero, changes polarity, increases to a negative maximum, then returns again to zero. One complete variation is referred to as a **cycle.** Since the waveform repeats itself at regular intervals as in (b), it is called a **periodic** waveform.

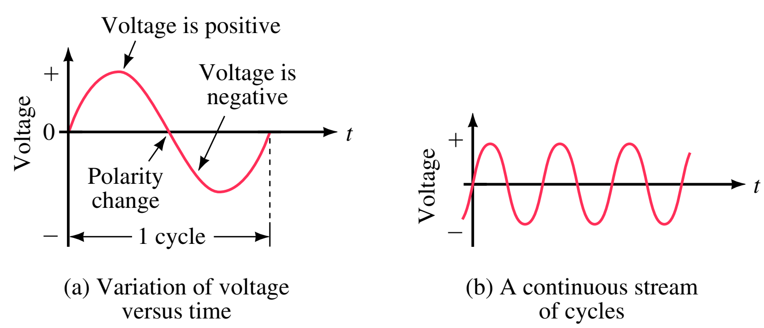



**FIGURE 15–2** Sinusoidal ac waveforms. Values above the axis are positive while values below are negative.

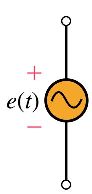



**FIGURE 15–3** Symbol for a sinusoidal voltage source. Lowercase letter *e* is used to indicate that the voltage varies with time.

### Symbol for an AC Voltage Source

The symbol for a sinusoidal voltage source is shown in Figure 15–3. Note that a lowercase *e* is used to represent voltage rather than *E,* since it is a function of time. Polarity marks are also shown although, since the polarity of the source varies, their meaning has yet to be established.

### Sinusoidal AC Current

Figure 15–4 shows a resistor connected to an ac source. During the first half-cycle, the source voltage is positive; therefore, the current is in the clockwise direction. During the second half-cycle, the voltage polarity reverses; therefore, the current is in the counterclockwise direction. Since current is proportional to voltage, its shape is also sinusoidal (Figure 15–5).

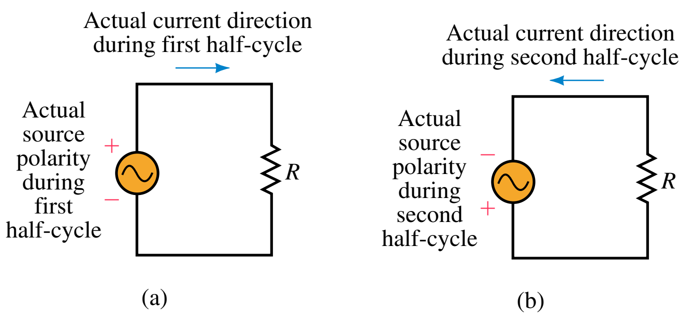



**FIGURE 15–4** Current direction reverses when the source polarity reverses.

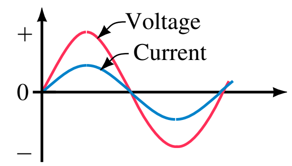
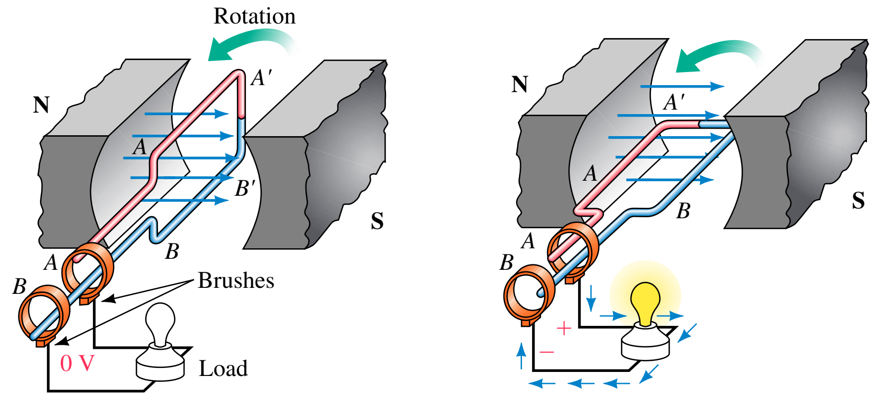



**FIGURE 15–5** Current has the same wave shape as voltage.

## 15.2 Generating AC Voltages

One way to generate an ac voltage is to rotate a coil of wire at constant angular velocity in a fixed magnetic field, Figure 15–6. (Slip rings and brushes connect the coil to the load.) The magnitude of the resulting voltage is proportional to the rate at which flux lines are cut (Faraday's law, Chapter 13), and its polarity is dependent on the direction the coil sides move through the field. Since the rate of cutting flux varies with time, the resulting voltage will also vary with time. For example in (a), since the coil sides are moving parallel to the field, no flux lines being cut and the induced voltage at this instant (and hence the current) is zero. (This is defined as the 0° position of the coil.) As the coil rotates from the 0° position, coil sides $AA'$ and $BB'$ cut across flux lines; hence, voltage builds, reaching a peak when flux is cut at the maximum rate in the 90° position as in (b). Note the polarity of the voltage and the direction of current. As the coil rotates further, voltage decreases, reaching zero at the 180° position when the coil sides again move parallel to the field as in (c). At this point, the coil has gone through a half-revolution.

During the second half-revolution, coil sides cut flux in directions opposite to that which they did in the first half revolution; hence, the polarity of the induced voltage reverses. As indicated in (d), voltage reaches a peak at the 270° point, and, since the polarity of the voltage has changed, so has the direction of current. When the coil reaches the 360° position, voltage is again zero and the cycle starts over. Figure 15–7 shows one cycle of the resulting waveform. Since the coil rotates continuously, the voltage produced will be a repetitive, periodic waveform as you saw in Figure 15–2(b).




(a) 0° Position: Coil sides move parallel to flux lines. Since no flux is being cut, induced voltage is zero.

(b) 90° Position: Coil end *A* is positive with respect to *B*. Current direction is out of slip ring *A*.

(c) 180° Position: Coil again cutting no flux. Induced voltage is zero.

(d) 270° Position: Voltage polarity has reversed, therefore, current direction reverses.

**FIGURE 15–6** Generating an ac voltage. The 0° position of the coil is defined as in (a) where the coil sides move parallel to the flux lines.

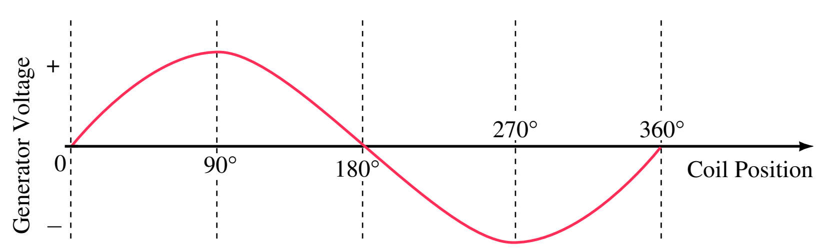



**FIGURE 15–7** Coil voltage versus angular position.

PRACTICAL NOTES...

In practice, the coil of Figure 15–6 consists of many turns wound on an iron core. The coil, core, and slip rings rotate as a unit.

In Figure 15–6, the magnetic field is fixed and the coil rotates. While small generators are built this way, large ac generators usually have the opposite construction, that is, their coils are fixed and the magnetic field is rotated instead. In addition, large ac generators are usually made as three-phase machines with three sets of coils instead of one. This is covered in Chapter 23. However, although its details are oversimplified, the generator of Figure 15–6 gives a true picture of the voltage produced by a real ac generator.

## Time Scales

The horizontal axis of Figure 15–7 is scaled in degrees. Often we need it scaled in time. The length of time required to generate one cycle depends on the velocity of rotation. To illustrate, assume that the coil rotates at 600 rpm (revolutions per minute). Six hundred revolutions in one minute equals 600 rev/60 s = 10 revolutions in one second. At ten revolutions per second, the time for one revolution is one tenth of a second, i.e., 100 ms. Since one cycle is 100 ms, a half-cycle is 50 ms, a quarter-cycle is 25 ms, and so on. Figure 15–8 shows the waveform rescaled in time.

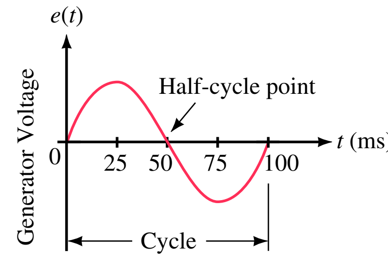
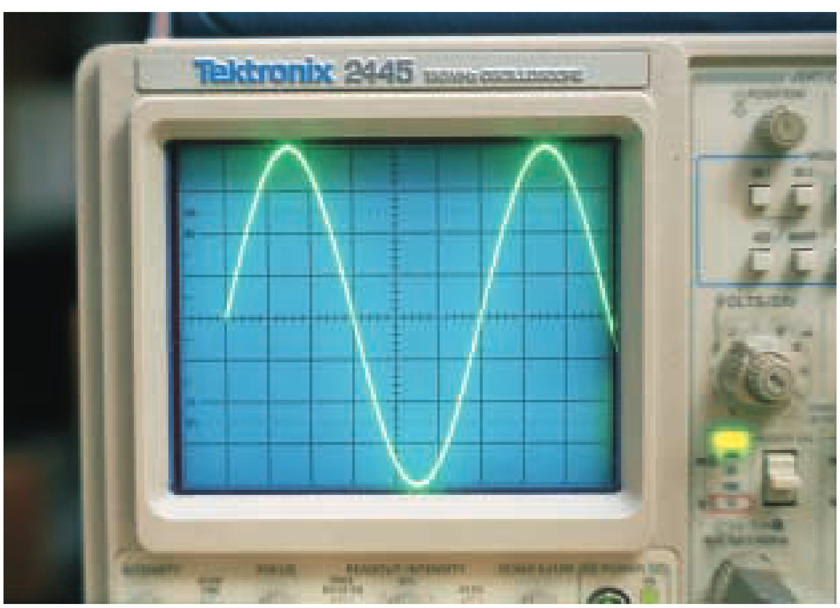


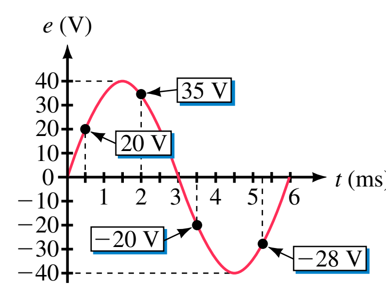

**FIGURE 15–8** Cycle scaled in time. At 600 rpm, the cycle length is 100 ms.

## Instantaneous Value

As Figure 15–8 shows, the coil voltage changes from instant to instant. The value of voltage at any point on the waveform is referred to as its **instantaneous value.** This is illustrated in Figure 15–9. Figure 15–9(a) shows a photograph of an actual waveform, and (b) shows it redrawn, with values scaled from the photo. For this example, the voltage has a peak value of 40 volts and a cycle time of 6 ms. From the graph, we see that at $t$ = 0 ms, the voltage is zero. At $t$ = 0.5 ms, it is 20 V. At $t$ = 2 ms, it is 35 V. At $t$ = 3.5 ms, it is −20 V, and so on.




(a) Sinusoidal voltage



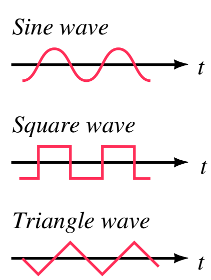

(b) Values scaled from the photograph

**FIGURE 15–9** Instantaneous values.

### Electronic Signal Generators

AC waveforms may also be created electronically using signal generators. In fact, with signal generators, you are not limited to sinusoidal ac. The general-purpose lab signal generator of Figure 15–10, for example, can produce a variety of variable-frequency waveforms, including sinusoidal, square wave, triangular, and so on. Waveforms such as these are commonly used to test electronic gear.

(a) A typical signal generator

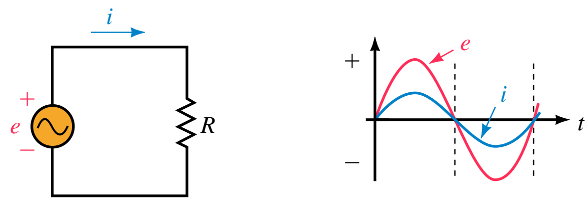



(b) Sample waveforms

**FIGURE 15–10** Electronic signal generators produce waveforms of different shapes.

## 15.3 Voltage and Current Conventions for AC

In Section 15.1, we looked briefly at voltage polarities and current directions. At that time, we used separate diagrams for each half-cycle (Figure 15–4). However, this is unnecessary; one diagram and one set of references is all that is required. This is illustrated in Figure 15–11. First, we assign reference polarities for the source and a reference direction for the current. We then use the convention that, *when e has a positive value, its actual polarity is the same as the reference polarity, and when e has a negative value, its actual polarity is opposite to that of the reference.* For current, we use the convention that *when i has a positive value, its actual direction is the same as the reference arrow, and when i has a negative value, its actual direction is opposite to that of the reference.*

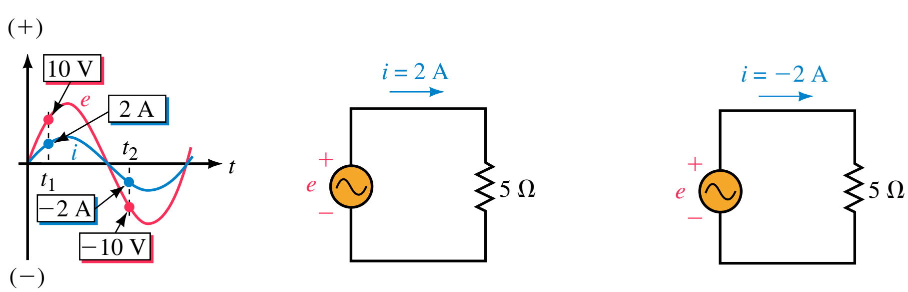



(a) References for voltage and current.

(b) During the first half-cycle, voltage polarity and current direction are as shown in (a). Therefore, *e* and *i* are positive. During the second half-cycle, voltage polarity and current direction are opposite to that shown in (a). Therefore, *e* and *i* are negative.

**FIGURE 15–11** AC voltage and current reference conventions.

To illustrate, consider Figure 15–12. At time $t_1$, $e$ has a value of 10 volts. This means that at this instant, the voltage of the source is 10 V and its top end is positive with respect to its bottom end. This is indicated in (b). With a voltage of 10 V and a resistance of 5 Ω, the instantaneous value of current is $i = e/R = 10\text{ V}/5\ \Omega = 2\text{ A}$. Since $i$ is positive, the current is in the direction of the reference arrow.




(a)

(b) Time $t_1$: $e = 10$ V and $i = 2$ A. Thus voltage and current have the polarity and direction indicated

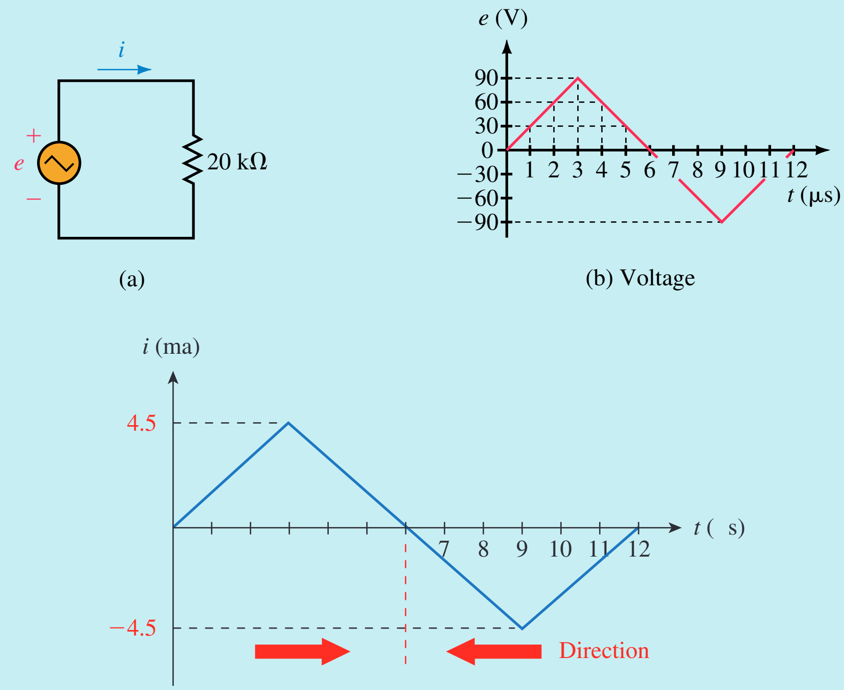
(c) Time $t_2$: $e = -10$ V and $i = -2$ A. Thus, voltage polarity is opposite to that indicated and current direction is opposite to the arrow direction.

**FIGURE 15–12** Illustrating the ac voltage and current convention.

Now consider time $t_2$. Here, $e = -10$ V. This means that source voltage is again 10 V, but now its top end is negative with respect to its bottom end. Again applying Ohm's law, you get $i = e/R = -10\text{ V}/5\ \Omega = -2\text{ A}$. Since $i$ is negative, current is actually opposite in direction to the reference arrow. This is indicated in (c).

The above concept is valid for any ac signal, regardless of waveshape.

**EXAMPLE 15–1** Figure 15–13(b) shows one cycle of a triangular voltage wave. Determine the current and its direction at $t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,$ and $12\ \mu s$ and sketch.




(a)

(b) Voltage

(c) Current

**FIGURE 15–13**

**Solution** Apply Ohm's law at each point in time. At $t = 0\ \mu s$, $e = 0$ V, so $i = e/R = 0\ \text{V}/20\ \text{k}\Omega = 0$ mA. At $t = 1\ \mu s$, $e = 30$ V. Thus, $i = e/R = 30\ \text{V}/20\ \text{k}\Omega = 1.5$ mA. At $t = 2\ \mu s$, $e = 60$ V. Thus, $i = e/R = 60\ \text{V}/20\ \text{k}\Omega = 3$ mA. Continuing in this manner, you get the values shown in Table 15–1. The waveform is plotted as Figure 15–13(c).

**TABLE 15–1** Values for Example 15–1

| $t$ ($\mu$s) | $e$ (V) | $i$ (mA) |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 30 | 1.5 |
| 2 | 60 | 3.0 |
| 3 | 90 | 4.5 |
| 4 | 60 | 3.0 |
| 5 | 30 | 1.5 |
| 6 | 0 | 0 |
| 7 | −30 | −1.5 |
| 8 | −60 | −3.0 |
| 9 | −90 | −4.5 |
| 10 | −60 | −3.0 |
| 11 | −30 | −1.5 |
| 12 | 0 | 0 |

PRACTICE PROBLEMS 1

1. Let the source voltage of Figure 15–11 be the waveform of Figure 15–9. If $R = 2.5\ \text{k}\Omega$, determine the current at $t = 0$, 0.5, 1, 1.5, 3, 4.5, and 5.25 ms.
2. For Figure 15–13, if $R = 180\ \Omega$, determine the current at $t = 1.5$, 3, 7.5, and 9 $\mu$s.

*Answers:*
1. 0, 8, 14, 16, 0, −16, −11.2 (all mA)
2. 0.25, 0.5, −0.25, −0.5 (all A)

## 15.4 Frequency, Period, Amplitude, and Peak Value

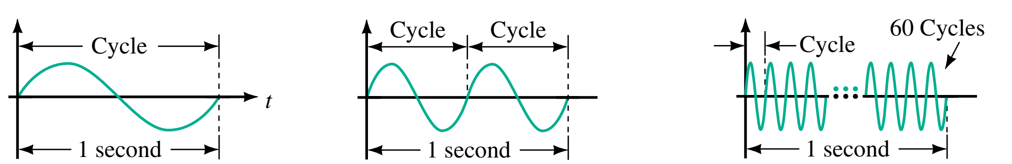
Periodic waveforms (i.e., waveforms that repeat at regular intervals), regardless of their waveshape, may be described by a group of attributes such as frequency, period, amplitude, peak value, and so on.

### Frequency

The number of cycles per second of a waveform is defined as its **frequency.** In Figure 15–14(a), one cycle occurs in one second; thus its frequency is one cycle per second. Similarly, the frequency of (b) is two cycles per second and that of (c) is 60 cycles per second. Frequency is denoted by the lowercase letter *f*. In the SI system, its unit is the **hertz** (Hz, named in honor of pioneer researcher Heinrich Hertz, 1857–1894). By definition,

$$1\ \text{Hz} = 1\ \text{cycle per second} \tag{15–1}$$

Thus, the examples depicted in Figure 15–14 represent 1 Hz, 2 Hz, and 60 Hz respectively.




(a) 1 cycle per second = 1 Hz (b) 2 cycles per second = 2 Hz (c) 60 cycles per second = 60 Hz

**FIGURE 15–14** Frequency is measured in hertz (Hz).

The range of frequencies is immense. Power line frequencies, for example, are 60 Hz in North America and 50 Hz in many other parts of the world. Audible sound frequencies range from about 20 Hz to about 20 kHz. The standard AM radio band occupies from 550 kHz to 1.6 MHz, while the FM band extends from 88 MHz to 108 MHz. TV transmissions occupy several bands in the 54-MHz to 890-MHz range. Above 300 GHz are optical and X-ray frequencies.

### Period

The **period,** *T,* of a waveform, (Figure 15–15) is the duration of one cycle. It is the inverse of frequency. To illustrate, consider again Figure 15–14. In (a), the frequency is 1 cycle per second; thus, the duration of each cycle is $T = 1$ s.

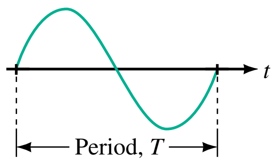



**FIGURE 15–15** Period $T$ is the duration of one cycle, measured in seconds.

In (b), the frequency is two cycles per second; thus, the duration of each cycle is $T = \frac{1}{2}$ s, and so on. In general,

$$T = \frac{1}{f} \quad \text{(s)} \tag{15–2}$$

and

$$f = \frac{1}{T} \quad \text{(Hz)} \tag{15–3}$$

Note that these definitions are independent of wave shape.

**EXAMPLE 15–2**

a. What is the period of a 50-Hz voltage?

b. What is the period of a 1-MHz current?

**Solution**

$$\text{(a)}\quad T = \frac{1}{f} = \frac{1}{50\text{ Hz}} = 20\text{ ms}$$

$$\text{(b)}\quad T = \frac{1}{f} = \frac{1}{1 \times 10^6\text{ Hz}} = 1\ \mu\text{s}$$

**EXAMPLE 15–3** Figure 15–16 shows an oscilloscope trace of a square wave. Each horizontal division represents 50 $\mu$s. Determine the frequency.

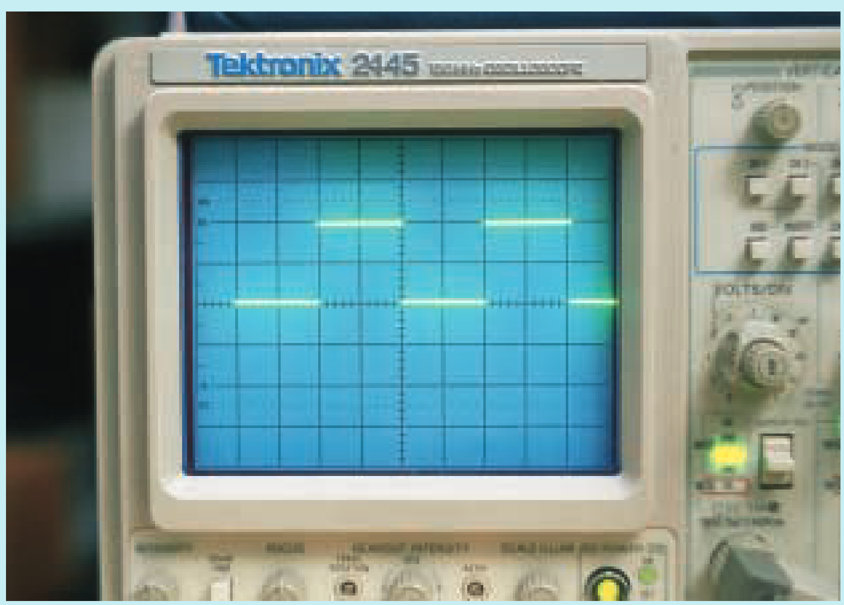



**FIGURE 15–16** The concepts of frequency and period apply to nonsinusoidal waveforms.

**Solution** Since the wave repeats itself every 200 $\mu$s, its period is 200 $\mu$s and

$$f = \frac{1}{200 \times 10^{-6}\ \text{s}} = 5\ \text{kHz}$$

The period of a waveform can be measured between any two corresponding points (Figure 15–17). Often it is measured between zero points because they are easy to establish on an oscilloscope trace.

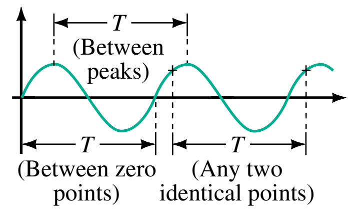
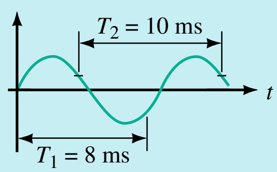



**FIGURE 15–17** Period may be measured between any two corresponding points.

**EXAMPLE 15–4** Determine the period and frequency of the waveform of Figure 15–18.

**FIGURE 15–18**




**Solution** Time interval $T_1$ does not represent a period as it is not measured between corresponding points. Interval $T_2$, however, is. Thus, $T = 10$ ms and

$$f = \frac{1}{T} = \frac{1}{10 \times 10^{-3}\ \text{s}} = 100\ \text{Hz}$$

## Amplitude and Peak-to-Peak Value

The **amplitude** of a sine wave is the distance from its average to its peak. Thus, the amplitude of the voltage in Figures 15–19(a) and (b) is $E_m$.

**Peak-to-peak voltage** is also indicated in Figure 15–19(a). It is measured between minimum and maximum peaks. Peak-to-peak voltages are denoted $E_{p\text{-}p}$ or $V_{p\text{-}p}$ in this book. (Some authors use $V_{pk\text{-}pk}$ or the like.) Similarly, peak-to-peak currents are denoted as $I_{p\text{-}p}$. To illustrate, consider again Figure 15–9. The amplitude of this voltage is $E_m = 40$ V, and its peak-to-peak voltage is $E_{p\text{-}p} = 80$ V.

## Peak Value

The **peak value** of a voltage or current is its maximum value with respect to zero. Consider Figure 15–19(b). Here, a sine wave rides on top of a dc value, yielding a peak that is the sum of the dc voltage and the ac waveform amplitude. For the case indicated, the peak voltage is $E + E_m$.

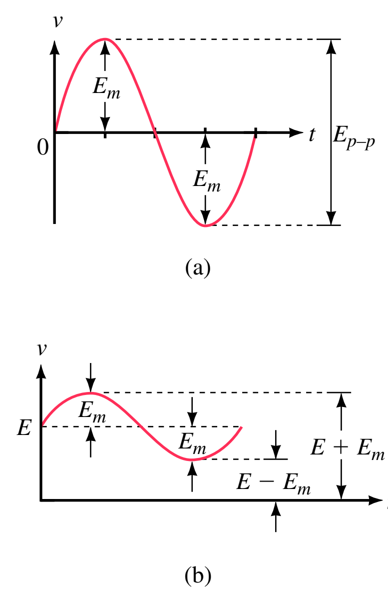
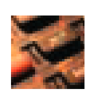



**FIGURE 15–19** Definitions.

1. What is the period of the commercial ac power system voltage in North America?
2. If you double the rotational speed of an ac generator, what happens to the frequency and period of the waveform?



IN-PROCESS LEARNING CHECK 1

3. If the generator of Figure 15–6 rotates at 3000 rpm, what is the period and frequency of the resulting voltage? Sketch four cycles and scale the horizontal axis in units of time.
4. For the waveform of Figure 15–9, list all values of time at which $e = 20$ V and $e = -35$ V. Hint: Sine waves are symmetrical.
5. Which of the waveform pairs of Figure 15–20 are valid combinations? Why?

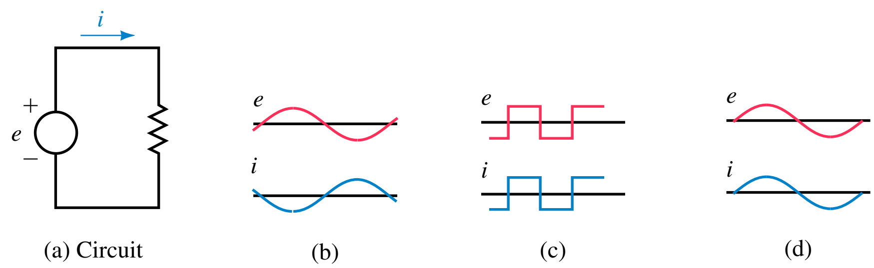



**FIGURE 15–20** Which waveform pairs are valid?

6. For the waveform in Figure 15–21, determine the frequency.

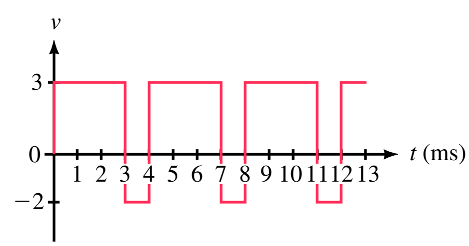
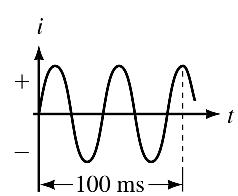



**FIGURE 15–21**

7. Two waveforms have periods of $T_1 = 10$ ms and $T_2 = 30$ ms respectively. Which has the higher frequency? Compute the frequencies of both waveforms.
8. Two sources have frequencies $f_1$ and $f_2$ respectively. If $f_2 = 20f_1$, and $T_2$ is 1 $\mu$s, what is $f_1$? What is $f_2$?
9. Consider Figure 15–22. What is the frequency of the waveform?

**FIGURE 15–22**




10. For Figure 15–11, if $f = 20$ Hz, what is the current direction at $t = 12$ ms, 37 ms, and 60 ms? Hint: Sketch the waveform and scale the horizontal axis in ms. The answers should be apparent.

11. A 10-Hz sinusoidal current has a value of 5 amps at $t = 25$ ms. What is its value at $t = 75$ ms? See hint in Problem 10.

*(Answers are at the end of the chapter.)*

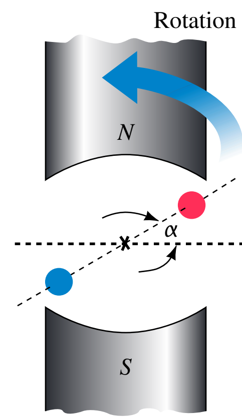
## 15.5 Angular and Graphic Relationships for Sine Waves

### The Basic Sine Wave Equation

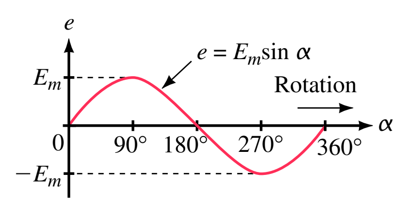
Consider again the generator of Figure 15–6, reoriented and redrawn in end view as Figure 15–23. The voltage produced by this generator is

$$e = E_m \sin \alpha \quad \text{(V)} \tag{15–4}$$

where $E_m$ is the maximum coil voltage and $\alpha$ is the instantaneous angular position of the coil. (For a given generator and rotational velocity, $E_m$ is constant.) Note that $\alpha = 0°$ represents the horizontal position of the coil and that one complete cycle corresponds to 360°. Equation 15–4 states that the voltage at any point on the sine wave may be found by multiplying $E_m$ times the sine of the angle at that point.




(a) End view showing coil position




(b) Voltage waveform

**FIGURE 15–23** Coil voltage versus angular position.

**EXAMPLE 15–5** If the amplitude of the waveform of Figure 15–23(b) is $E_m = 100$ V, determine the coil voltage at 30° and 330°.

**Solution** At $\alpha = 30°$, $e = E_m \sin \alpha = 100 \sin 30° = 50$ V. At 330°, $e = 100 \sin 330° = -50$ V. These are shown on the graph of Figure 15–24.

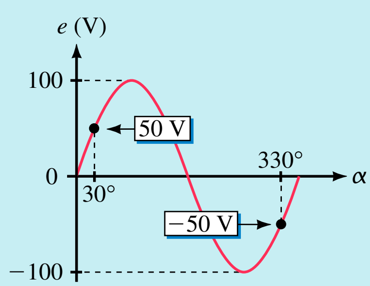



**FIGURE 15–24**

**PRACTICE PROBLEMS 2**

Table 15–2 is a tabulation of voltage versus angle computed from $e = 100 \sin \alpha$. Use your calculator to verify each value, then plot the result on graph paper. The resulting waveshape should look like Figure 15–24.

**TABLE 15–2** Data for Plotting $e = 100 \sin \alpha$

| Angle $\alpha$ | Voltage $e$ |
|---|---|
| 0 | 0 |
| 30 | 50 |
| 60 | 86.6 |
| 90 | 100 |
| 120 | 86.6 |
| 150 | 50 |
| 180 | 0 |
| 210 | −50 |
| 240 | −86.6 |
| 270 | −100 |
| 300 | −86.6 |
| 330 | −50 |
| 360 | 0 |

## Angular Velocity, $\omega$

The rate at which the generator coil rotates is called its **angular velocity.** If the coil rotates through an angle of 30° in one second, for example, its angular velocity is 30° per second. Angular velocity is denoted by the Greek letter $\omega$ (omega). For the case cited, $\omega = 30°/\text{s}$. (Normally angular velocity is expressed in radians per second instead of degrees per second. We will make this change shortly.) When you know the angular velocity of a coil and the length of time that it has rotated, you can compute the angle through which it has turned. For example, a coil rotating at 30°/s rotates through an angle of 30° in one second, 60° in two seconds, 90° in three seconds, and so on. In general,

$$\alpha = \omega t \tag{15–5}$$

Expressions for $t$ and $\omega$ can now be found. They are

$$t = \frac{\alpha}{\omega} \quad \text{(s)} \tag{15–6}$$

$$\omega = \frac{\alpha}{t} \tag{15–7}$$

**EXAMPLE 15–6** If the coil of Figure 15–23 rotates at $\omega = 300°/\text{s}$, how long does it take to complete one revolution?

**Solution** One revolution is 360°. Thus,

$$t = \frac{\alpha}{\omega} = \frac{360 \text{ degrees}}{300 \dfrac{\text{degrees}}{\text{s}}} = 1.2 \text{ s}$$

Since this is one period, we should use the symbol $T$. Thus, $T = 1.2$ s, as in Figure 15–25.

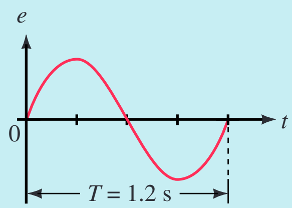



**FIGURE 15–25**

If the coil of Figure 15–23 rotates at 3600 rpm, determine its angular velocity, $\omega$, in degrees per second.

**PRACTICE PROBLEMS 3**

*Answer:* 21 600 deg/s

## Radian Measure

In practice, $\omega$ is usually expressed in radians per second, where radians and degrees are related by the identity

$$2\pi \text{ radians} = 360° \qquad \textbf{(15–8)}$$

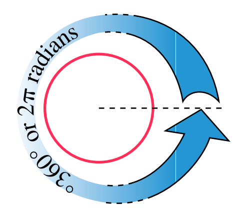
One radian therefore equals $360°/2\pi = 57.296°$. A full circle, as shown in Figure 15–26(a), can be designated as either 360° or $2\pi$ radians. Likewise, the cycle length of a sinusoid, shown in Figure 15–26(b), can be stated as either 360° or $2\pi$ radians; a half-cycle as 180° or $\pi$ radians, and so on.

To convert from degrees to radians, multiply by $\pi/180$, while to convert from radians to degrees, multiply by $180/\pi$.

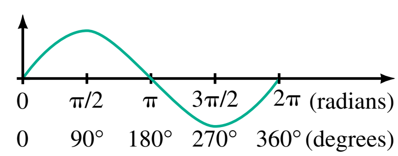
$$\alpha_{\text{radians}} = \frac{\pi}{180°} \times \alpha_{\text{degrees}} \qquad \textbf{(15–9)}$$

$$\alpha_{\text{degrees}} = \frac{180°}{\pi} \times \alpha_{\text{radians}} \qquad \textbf{(15–10)}$$

Table 15–3 shows selected angles in both measures.




(a) 360° = 2$\pi$ radians




(b) Cycle length scaled in degrees and radians

**FIGURE 15–26** Radian measure.

**TABLE 15–3** Selected Angles in Degrees and Radians

| Degrees | Radians |
|---|---|
| 30 | $\pi/6$ |
| 45 | $\pi/4$ |
| 60 | $\pi/3$ |
| 90 | $\pi/2$ |
| 180 | $\pi$ |
| 270 | $3\pi/2$ |
| 360 | $2\pi$ |

**EXAMPLE 15–7**

a. Convert 315° to radians.

b. Convert $5\pi/4$ radians to degrees.

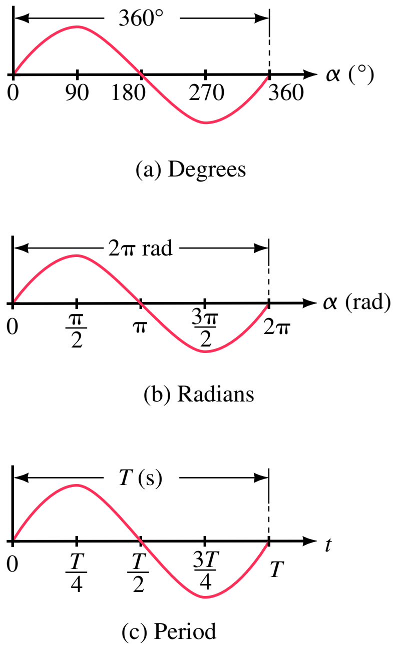



**FIGURE 15–27** Comparison of various horizontal scales. Cycle length may be scaled in degrees, radians or period. Each of these is independent of frequency.

**Solution**

a. $\alpha_{\text{radians}} = (\pi/180°)(315°) = 5.5$ rad

b. $\alpha_{\text{degrees}} = (180°/\pi)(5\pi/4) = 225°$

Scientific calculators can perform these conversions directly. You will find this more convenient than using the above formulas.

## Graphing Sine Waves

A sinusoidal waveform can be graphed with its horizontal axis scaled in degrees, radians, or time. When scaled in degrees or radians, one cycle is always 360° or $2\pi$ radians (Figure 15–27); when scaled in time, it is frequency dependent, since the length of a cycle depends on the coil's velocity of rotation. However, if scaled in terms of period $T$ instead of in seconds, the waveform is also frequency independent, since one cycle is always $T$, as shown in Figure 15–27(c).

When graphing a sine wave, you don't actually need many points to get a good sketch: Values every 45° (one eighth of a cycle) are generally adequate. Table 15–4 shows corresponding values for sin $\alpha$ at this spacing.

**TABLE 15–4** Values for Rapid Sketching

| α (deg) | α (rad) | t (T) | Value of sin α |
|---|---|---|---|
| 0 | 0 | 0 | 0.0 |
| 45 | $\pi/4$ | $T/8$ | 0.707 |
| 90 | $\pi/2$ | $T/4$ | 1.0 |
| 135 | $3\pi/4$ | $3T/8$ | 0.707 |
| 180 | $\pi$ | $T/2$ | 0.0 |
| 225 | $5\pi/4$ | $5T/8$ | −0.707 |
| 270 | $3\pi/2$ | $3T/4$ | −1.0 |
| 315 | $7\pi/4$ | $7T/8$ | −0.707 |
| 360 | $2\pi$ | $T$ | 0.0 |

**EXAMPLE 15–8** Sketch the waveform for a 25-kHz sinusoidal current that has an amplitude of 4 mA. Scale the axis in seconds.

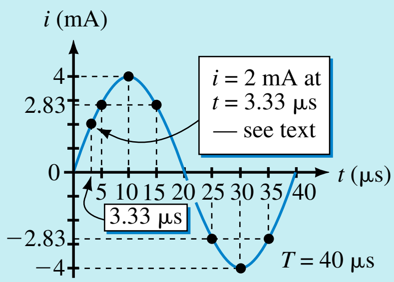
**Solution** The easiest approach is to use $T = 1/f$, then scale the graph accordingly. For this waveform, $T = 1/25$ kHz $= 40\ \mu$s. Thus,

1. Mark the end of the cycle as 40 μs, the half-cycle point as 20 μs, the quarter-cycle point as 10 μs, and so on (Figure 15–28).
2. The peak value (i.e., 4 mA) occurs at the quarter-cycle point, which is 10 μs on the waveform. Likewise, −4 mA occurs at 30 μs. Now sketch.
3. Values at other time points can be determined easily. For example, the value at 5 μs can be calculated by noting that 5 μs is one eighth of a cycle, or 45°. Thus, $i = 4 \sin 45°$ mA $= 2.83$ mA. Alternately, from Table 15–4,

at $T/8$, $i = (4\text{ mA})(0.707) = 2.83\text{ mA}$. As many points as you need can be computed and plotted in this manner.

4. Values at particular angles can also be located easily. For instance, if you want a value at 30°, the required value is $i = 4 \sin 30°\text{ mA} = 2.0\text{ mA}$. To locate this point, note that 30° is one twelfth of a cycle or $T/12 = (40\ \mu\text{s})/12 = 3.33\ \mu\text{s}$. The point is shown on Figure 15–28.

**FIGURE 15–28**




## 15.6 Voltages and Currents as Functions of Time

### Relationship between $\omega$, $T$, and $f$

Earlier you learned that one cycle of sine wave may be represented as either $\alpha = 2\pi$ rads or $t = T$ s, Figure 15–27. Substituting these into $\alpha = \omega t$ (Equation 15–5), you get $2\pi = \omega T$. Transposing yields

$$\omega T = 2\pi \quad \text{(rad)} \qquad \textbf{(15–11)}$$

Thus,

$$\omega = \frac{2\pi}{T} \quad \text{(rad/s)} \qquad \textbf{(15–12)}$$

Recall, $f = 1/T$ Hz. Substituting this into Equation 15–12 you get

$$\omega = 2\pi f \quad \text{(rad/s)} \qquad \textbf{(15–13)}$$

**EXAMPLE 15–9** In some parts of the world, the power system frequency is 60 Hz; in other parts, it is 50 Hz. Determine $\omega$ for each.

**Solution** For 60 Hz, $\omega = 2\pi f = 2\pi(60) = 377$ rad/s. For 50 Hz, $\omega = 2\pi f = 2\pi(50) = 314.2$ rad/s.

**PRACTICE PROBLEMS 4**

1. If $\omega = 240$ rad/s, what are $T$ and $f$? How many cycles occur in 27 s?
2. If 56 000 cycles occur in 3.5 s, what is $\omega$?

*Answers:*

1. 26.18 ms, 38.2 Hz, 1031 cycles
2. $100.5 \times 10^3$ rad/s

## Sinusoidal Voltages and Currents as Functions of Time

Recall from Equation 15–4, $e = E_m \sin \alpha$, and from Equation 15–5, $\alpha = \omega t$. Combining these equations yields

$$e = E_m \sin \omega t \tag{15–14a}$$

Similarly,

$$v = V_m \sin \omega t \tag{15–14b}$$

$$i = I_m \sin \omega t \tag{15–14c}$$

**EXAMPLE 15–10** A 100-Hz sinusoidal voltage source has an amplitude of 150 volts. Write the equation for $e$ as a function of time.

**Solution** $\omega = 2\pi f = 2\pi(100) = 628$ rad/s and $E_m = 150$ V. Thus, $e = E_m \sin \omega t = 150 \sin 628t$ V.

Equations 15–14 may be used to compute voltages or currents at any instant in time. Usually, $\omega$ is in radians per second, and thus $\omega t$ is in radians. You can work directly in radians or you can convert to degrees. For example, suppose you want to know the voltage at $t = 1.25$ ms for $e = 150 \sin 628t$ V.

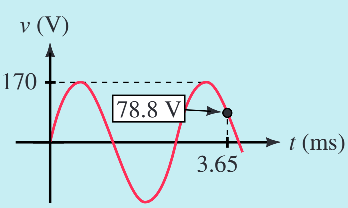
**Working in Rads.** With your calculator in the RAD mode, $e = 150 \sin(628)(1.25 \times 10^{-3}) = 150 \sin 0.785 \text{ rad} = 106$ V.

**Working in Degree.** $0.785 \text{ rad} = 45°$. Thus, $e = 150 \sin 45° = 106$ V as before.

**EXAMPLE 15–11** For $v = 170 \sin 2450t$, determine $v$ at $t = 3.65$ ms and show the point on the $v$ waveform.

**Solution** $\omega = 2450$ rad/s. Therefore $\omega t = (2450)(3.65 \times 10^{-3}) = 8.943 \text{ rad} = 512.4°$. Thus, $v = 170 \sin 512.4° = 78.8$ V. Alternatively, $v = 170 \sin 8.943 \text{ rad} = 78.8$ V. The point is plotted on the waveform in Figure 15–29.

**FIGURE 15–29**




**PRACTICE PROBLEMS 5**

A sinusoidal current has a peak amplitude of 10 amps and a period of 120 ms.

a. Determine its equation as a function of time using Equation 15–14c.

b. Using this equation, compute a table of values at 10-ms intervals and plot one cycle of the waveform scaled in seconds.

c. Sketch one cycle of the waveform using the procedure of Example 15–8. (Note how much less work this is.)

*Answers:*
a. $i = 10 \sin 52.36t$ A

c. Mark the end of the cycle as 120 ms, ½ cycle as 60 ms, ¼ cycle as 30 ms, etc. Draw the sine wave so that it is zero at $t = 0$, 10 A at 30 ms, 0 A at 60 ms, $-10$ A at 90 ms and ends at $t = 120$ ms. (See Figure 15–30.)

## Determining when a Particular Value Occurs

Sometimes you need to know when a particular value of voltage or current occurs. Given $v = V_m \sin \alpha$. Rewrite this as $\sin \alpha = v/V_m$. Then,

$$\alpha = \sin^{-1}\frac{v}{V_m} \tag{15–15}$$

Compute the angle $\alpha$ at which the desired value occurs using the inverse sine function of your calculator, then determine the time from

$$t = \alpha/\omega$$

**EXAMPLE 15–12** A sinusoidal current has an amplitude of 10 A and a period of 0.120 s. Determine the times at which

a. $i = 5.0$ A,
b. $i = -5$ A.

**Solution**

a. Consider Figure 15–30. As you can see, there are two points on the waveform where $i = 5$ A. Let these be denoted $t_1$ and $t_2$ respectively. First, determine $\omega$:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.120 \text{ s}} = 52.36 \text{ rad/s}$$

Let $i = 10 \sin \alpha$ A. Now, find the angle $\alpha_1$ at which $i = 5$ A:

$$\alpha_1 = \sin^{-1}\frac{i}{I_m} = \sin^{-1}\frac{5 \text{ A}}{10 \text{ A}} = \sin^{-1}0.5 = 30° = 0.5236 \text{ rad}$$

Thus, $t_1 = \alpha_1/\omega = (0.5236 \text{ rad})/(52.36 \text{ rad/s}) = 0.01 \text{ s} = 10$ ms. This is indicated in Figure 15–30. Now consider $t_2$. Note that $t_2$ is the same distance back from the half-cycle point as $t_1$ is in from the beginning of the cycle. Thus, $t_2 = 60 \text{ ms} - 10 \text{ ms} = 50$ ms.

b. Similarly, $t_3$ (the first point at which $i = -5$ A occurs) is 10 ms past midpoint, while $t_4$ is 10 ms back from the end of the cycle. Thus, $t_3 = 70$ ms and $t_4 = 110$ ms.

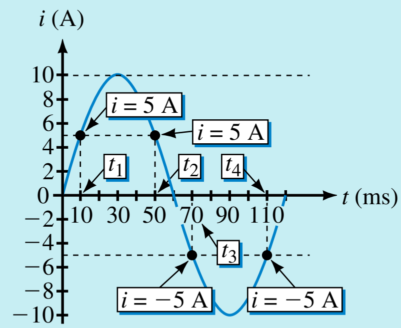



**FIGURE 15–30**

PRACTICE PROBLEMS 6

Given $v = 10 \sin 52.36t$, determine both occurrences of $v = -8.66$ V.

*Answer:* 80 ms 100 ms

## Voltages and Currents with Phase Shifts

If a sine wave does not pass through zero at $t = 0$ s as in Figure 15–30, it has a **phase shift.** Waveforms may be shifted to the left or to the right (see Figure 15–31). For a waveform shifted left as in (a),

$$v = V_m \sin(\omega t + \theta) \tag{15–16a}$$

while, for a waveform shifted right as in (b),

$$v = V_m \sin(\omega t - \theta) \tag{15–16b}$$

NOTES...

When applying equations 15–16(a) and (b), it is customary to express $\omega t$ in radians and $\theta$ in degrees, yielding mixed angular units (as indicated in the following examples). Although this is acceptable when the equations are written in symbolic form, you must convert both angles to the same unit before you make numerical computations.


(a) $v = V_m \sin(\omega t + \theta)$ (b) $v = V_m \sin(\omega t - \theta)$

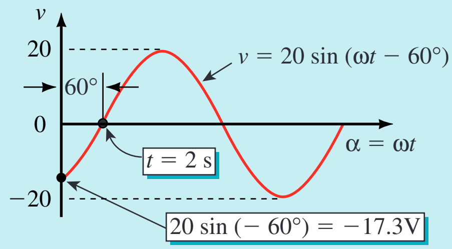
**FIGURE 15–31** Waveforms with phase shifts. Angle $\theta$ is normally measured in degrees, yielding mixed angular units. (See note.)

EXAMPLE 15–13 Demonstrate that $v = 20 \sin(\omega t - 60°)$, where $\omega = \pi/6$ rad/s (i.e, $= 30°$/s), yields the shifted waveform shown in Figure 15–32.

**Solution**

1. Since $\omega t$ and 60° are both angles, $(\omega t - 60°)$ is also an angle. Let us define it as $x$. Then $v = 20 \sin x$, which means that the shifted wave is also sinusoidal.
2. Consider $v = \sin(\omega t - 60°)$. At $t = 0$ s, $v = 20 \sin(0 - 60°) = 20 \sin(-60°) = -17.3$ V as indicated in Figure 15–32.

3. Since $\omega = 30°/\text{s}$, it takes 2 s for $\omega t$ to reach 60°. Thus, at $t = 2$ s, $v = 20 \sin(60° - 60°) = 0$ V, and the waveform passes through zero at $t = 2$ s as indicated.




EWB **FIGURE 15–32**

Summary: Since $v = 20 \sin(\omega t - 60°)$ is a sine wave and since it passes through zero at $t = 2$ s, where $\omega t = 60°$, it represents the shifted wave shown in Figure 15–32.

## EXAMPLE 15–14

a. Determine the equation for the waveform of Figure 15–33(a), given $f = 60$ Hz. Compute current at $t = 4$ ms.

b. Repeat (a) for Figure 15–33(b).

### Solution

a. $I_m = 2$ A and $\omega = 2\pi(60) = 377$ rad/s. This waveform corresponds to Figure 15–31(b). Therefore,

$$i = I_m \sin(\omega t - \theta) = 2 \sin(377t - 120°) \text{ A}$$

At $t = 4$ ms, current is

$$\begin{aligned} i &= 2 \sin(377 \times 4 \text{ ms} - 120°) = 2 \sin(1.508 \text{ rad} - 120°) \\ &= 2 \sin(86.4° - 120°) = 2 \sin(-33.64°) = -1.11 \text{ A}. \end{aligned}$$

b. This waveform matches Figure 15–31(a) if you extend the waveform back 90° from its peak as in (c). Thus,

$$i = 2 \sin(377t + 40°) \text{ A}$$

At $t = 4$ ms, current is

$$\begin{aligned} i &= 2 \sin(377 \times 4 \text{ ms} + 40°) = 2 \sin(126.4°) \\ &= 1.61 \text{ A}. \end{aligned}$$

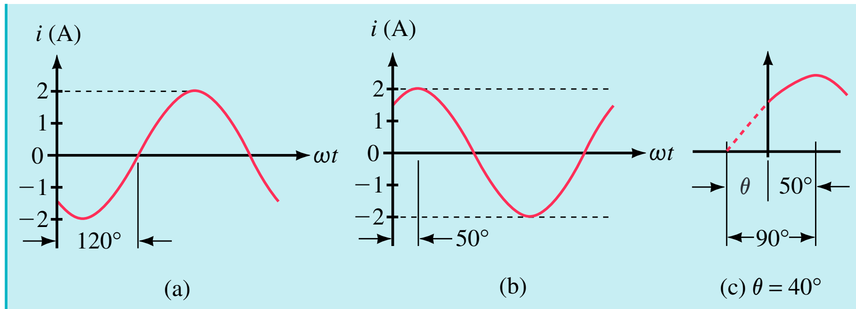



FIGURE 15–33

PRACTICE PROBLEMS 7

1. Given $i = 2 \sin(377t + 60°)$, compute the current at $t = 3$ ms.
2. Sketch each of the following:
   a. $v = 10 \sin(\omega t + 20°)$ V.
   b. $i = 80 \sin(\omega t - 50°)$ A.
   c. $i = 50 \sin(\omega t + 90°)$ A.
   d. $v = 5 \sin(\omega t + 180°)$ V.
3. Given $i = 2 \sin(377t + 60°)$, determine at what time $i = 1.8$ A.

*Answers:*
1. 1.64 A
2. a. Same as Figure 15–31(a) with $V_m = 10$ V, $\theta = 20°$.
   b. Same as Figure 15–31(b) with $I_m = 80$ A, $\theta = 50°$.
   c. Same as Figure 15–39(b) except use $I_m = 50$ A instead of $V_m$.
   d. A negative sine wave with magnitude of 5 V.
3. 0.193 ms

Probably the easiest way to deal with shifted waveforms is to use phasors. We introduce the idea next.

## 15.7 Introduction to Phasors

A **phasor** is a rotating line whose projection on a vertical axis can be used to represent sinusoidally varying quantities. To get at the idea, consider the red line of length $V_m$ shown in Figure 15–34(a). (It is the phasor.) The vertical

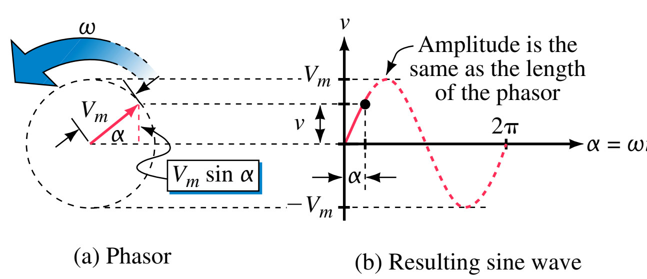



FIGURE 15–34 As the phasor rotates about the origin, its vertical projection creates a sine wave. (Figure 15–35 illustrates the process.)

projection of this line (indicated in dotted red) is $V_m \sin \alpha$. Now, assume that the phasor rotates at angular velocity of $\omega$ rad/s in the counterclockwise direction. Then, $\alpha = \omega t$, and its vertical projection is $V_m \sin \omega t$. If we designate this projection (height) as $v$, we get $v = V_m \sin \omega t$, which is the familiar sinusoidal voltage equation.

If you plot a graph of $v$ versus $\alpha$, you get the sine wave of Figure 15–34(b). Figure 15–35 illustrates the process. It shows snapshots of the phasor and the evolving waveform at various instants of time for a phasor of magnitude $V_m = 100$ V rotating at $\omega = 30°$/s. For example, consider $t = 0$, 1, 2, and 3s:

1. At $t = 0$ s, $\alpha = 0$, the phasor is at its 0° position, and its vertical projection is $v = V_m \sin \omega t = 100 \sin 0° = 0$ V. The point is at the origin.
2. At $t = 1$ s, the phasor has rotated 30° and its vertical projection is $v = 100 \sin 30° = 50$ V. This point is plotted at $\alpha = 30°$ on the horizontal axis.

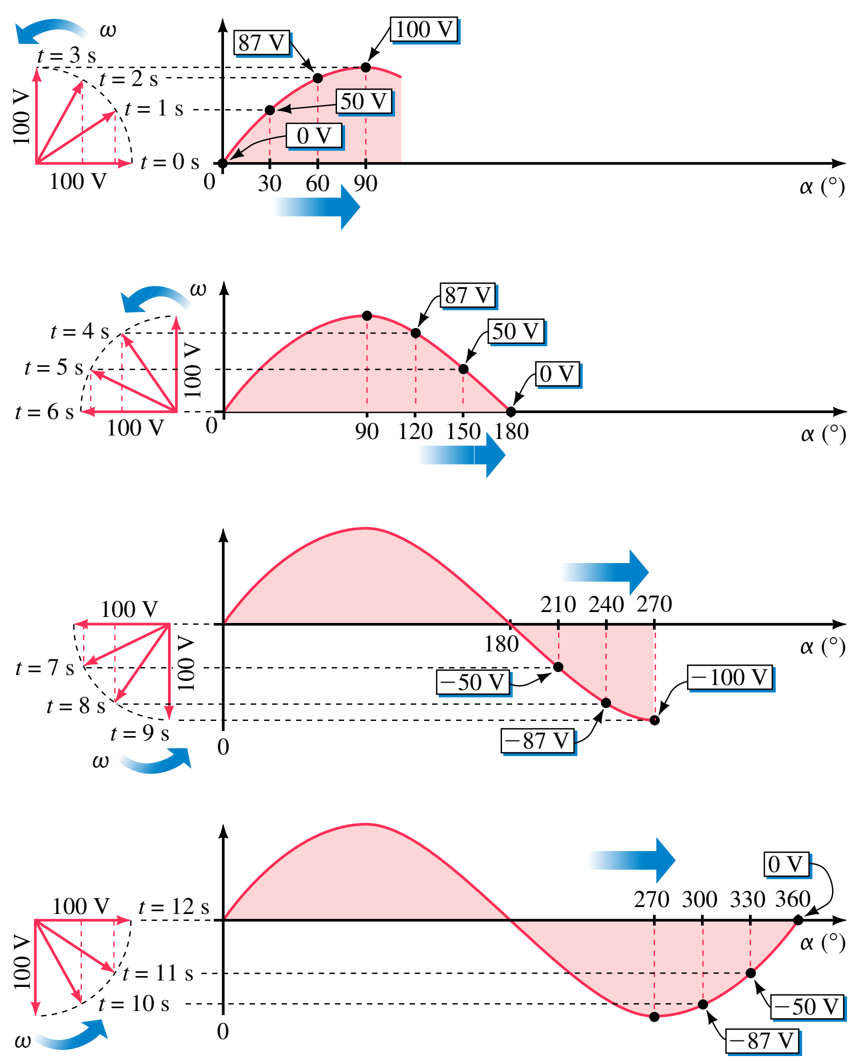



**FIGURE 15–35** Evolution of the sine wave of Figure 15–34.

**NOTES...**

1. Although we have indicated phasor rotation in Figure 15–35 by a series of "snapshots," this is too cumbersome; in practice, we show only the phasor at its $t = 0$ s (reference) position and imply rotation rather than show it explicitly.
2. Although we are using maximum values ($E_m$ and $I_m$) here, phasors are normally drawn in terms of effective values (considered in Section 15.9). For the moment, we will continue to use maximum values. We make the change in Chapter 16.

3. At $t = 2$ s, $\alpha = 60°$ and $v = 100 \sin 60° = 87$ V, which is plotted at $\alpha = 60°$ on the horizontal axis. Similarly, at $t = 3$ s, $\alpha = 90°$, and $v = 100$ V. Continuing in this manner, the complete waveform is evolved.

From the foregoing, we conclude that *a sinusoidal waveform can be created by plotting the vertical projection of a phasor that rotates in the counterclockwise direction at constant angular velocity ω. If the phasor has a length of $V_m$, the waveform represents voltage; if the phasor has a length of $I_m$*, it represents current. Note carefully: **Phasors apply only to sinusoidal waveforms.**

**EXAMPLE 15–15** Draw the phasor and waveform for current $i = 25 \sin \omega t$ mA for $f = 100$ Hz.

**Solution** The phasor has a length of 25 mA and is drawn at its $t = 0$ position, which is zero degrees as indicated in Figure 15–36. Since $f = 100$ Hz, the period is $T = 1/f = 10$ ms.

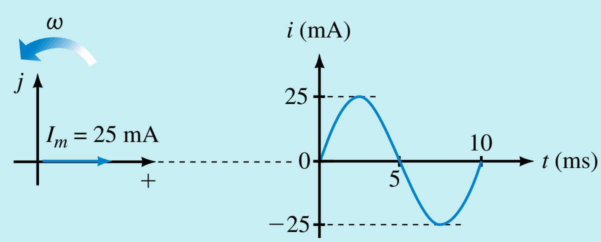



**FIGURE 15–36** The reference position of the phasor is its $t = 0$ position.

## Shifted Sine Waves

Phasors may be used to represent shifted waveforms, $v = V_m \sin(\omega t \pm \theta)$ or $i = I_m \sin(\omega t \pm \theta)$ as indicated in Figure 15–37. Angle $\theta$ is the position of the phasor at $t = 0$ s.

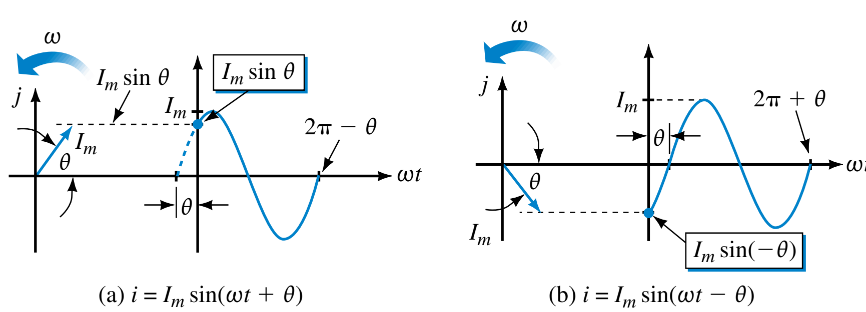
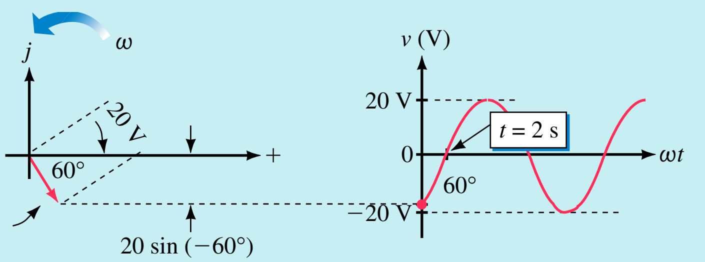



**FIGURE 15–37** Phasors for shifted waveforms. Angle $\theta$ is the position of the phasor at $t = 0$ s.

**EXAMPLE 15–16** Consider $v = 20 \sin(\omega t - 60°)$, where $\omega = \pi/6$ rad/s (i.e., 30°/s). Show that the phasor of Figure 15–38(a) represents this waveform.




(a) Phasor (b) $v = 20 \sin(\omega t - 60°)$, $\omega = 30°/s$

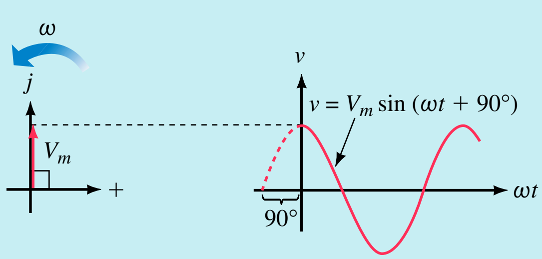
**FIGURE 15–38**

**Solution** The phasor has length 20 V and at time $t = 0$ is at $-60°$ as indicated in (a). Now, as the phasor rotates, it generates a sinusoidal waveform, oscillating between $\pm 20$ V as indicated in (b). Note that the zero crossover point occurs at $t = 2$ s, since it takes 2 seconds for the phasor to rotate from $-60°$ to 0° at 30 degrees per second. Now compare the waveform of (b) to the waveform of Figure 15–32, Example 15–13. They are identical. Thus, the phasor of (a) represents the shifted waveform $v = 20 \sin(\omega t - 60°)$.

**EXAMPLE 15–17** With the aid of a phasor, sketch the waveform for $v = V_m \sin(\omega t + 90°)$.

**Solution** Place the phasor at 90° as in Figure 15–39(a). Note that the resultant waveform (b) is a cosine waveform, i.e., $v = V_m \cos \omega t$. From this, we conclude that

$$\sin(\omega t + 90°) = \cos \omega t$$




(a) Phasor at 90° position (b) Waveform can also be described as a cosine wave

**FIGURE 15–39** Demonstrating that $\sin(\omega t + 90°) = \cos \omega t$.

**PRACTICE PROBLEMS 8**

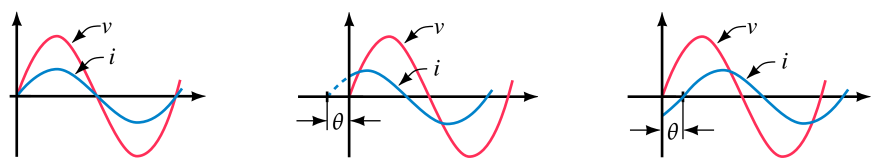
With the aid of phasors, show that

a. $\sin(\omega t - 90°) = -\cos \omega t$,

b. $\sin(\omega t \pm 180°) = -\sin \omega t$,

## Phase Difference

**Phase difference** refers to the angular displacement between different waveforms of the same frequency. Consider Figure 15–40. If the angular displacement is 0° as in (a), the waveforms are said to be **in phase;** otherwise, they are **out of phase.** When describing a phase difference, select one waveform as reference. Other waveforms then lead, lag, or are in phase with this reference. For example, in (b), for reasons to be discussed in the next paragraph, the current waveform is said to lead the voltage waveform, while in (c) the current waveform is said to lag.



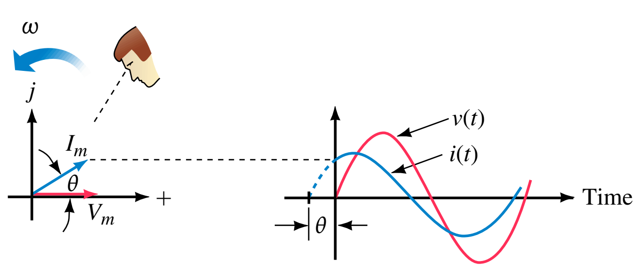

(a) In phase (b) Current leads (c) Current lags

**FIGURE 15–40** Illustrating phase difference. In these examples, voltage is taken as reference.

The terms **lead** and **lag** can be understood in terms of phasors. If you observe phasors rotating as in Figure 15–41(a), the one that you see passing first is leading and the other is lagging. By definition, *the waveform generated by the leading phasor leads the waveform generated by the lagging phasor and vice versa.* In Figure 15–41, phasor $I_m$ leads phasor $V_m$; thus current $i(t)$ leads voltage $v(t)$.

**NOTES...**

If you have trouble determining which waveform leads and which lags when you are solving a problem, make a quick sketch of their phasors, and the answer will be apparent. Note also that the terms *lead* and *lag* are relative. In Figure 15–41, we said that current leads voltage; you can just as correctly say that voltage lags current.




(a) $I_m$ leads $V_m$ (b) Therefore, $i(t)$ leads $v(t)$

**FIGURE 15–41** Defining lead and lag.

**EXAMPLE 15–18** Voltage and current are out of phase by 40°, and voltage lags. Using current as the reference, sketch the phasor diagram and the corresponding waveforms.

**Solution** Since current is the reference, place its phasor in the 0° position and the voltage phasor at −40°. Figure 15–42 shows the phasors and corresponding waveforms.

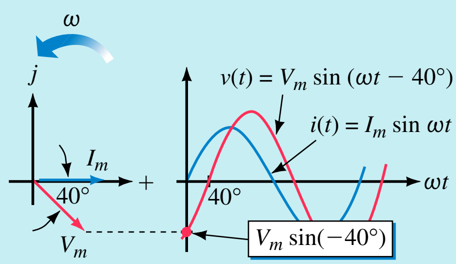



**FIGURE 15–42**

**EXAMPLE 15–19** Given $v = 20 \sin(\omega t + 30°)$ and $i = 18 \sin(\omega t - 40°)$, draw the phasor diagram, determine phase relationships, and sketch the waveforms.

**Solution** The phasors are shown in Figure 15–43(a). From these, you can see that $v$ leads $i$ by 70°. The waveforms are shown in (b).

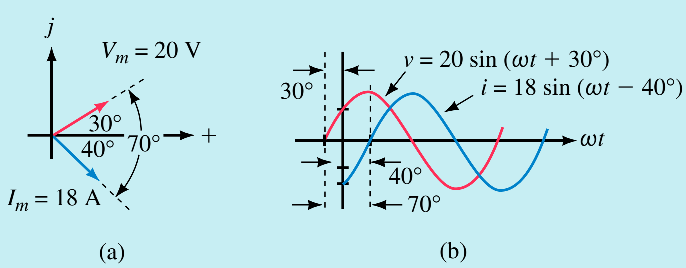
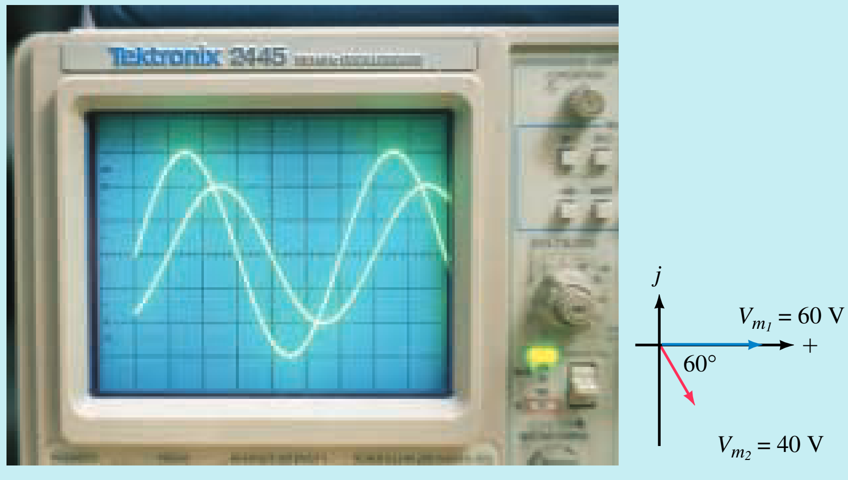



**FIGURE 15–43**

**EXAMPLE 15–20** Figure 15–44 shows a pair of waveforms $v_1$ and $v_2$ on an oscilloscope. Each major vertical division represents 20 V and each major division on the horizontal (time) scale represents 20 μs. Voltage $v_1$ leads. Prepare a phasor diagram using $v_1$ as reference. Determine equations for both voltages.




(a) (b)

**FIGURE 15–44**

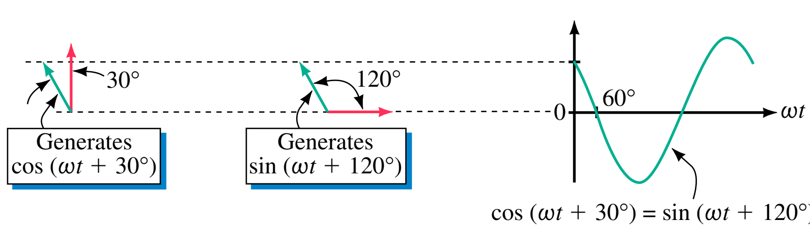
**Solution** From the photograph, the magnitude of $v_1$ is $V_{m_1} = 3$ div $\times$ 20 V/div $= 60$ V. Similarly, $V_{m_2} = 40$ V. Cycle length is $T = 6 \times 20\ \mu s = 120\ \mu s$, and the displacement between waveforms is 20 $\mu$s which is $\frac{1}{6}$ of a cycle (i.e., 60°). Selecting $v_1$ as reference and noting that $v_2$ lags yields the phasors shown in (b). Angular frequency $\omega = 2\pi/T = 2\pi/(120 \times 10^{-6}\text{ s}) = 52.36 \times 10^3$ rad/s. Thus, $v_1 = V_{m_1} \sin \omega t = 60 \sin(52.36 \times 10^3 t)$ V and $v_2 = 40 \sin(52.36 \times 10^3 t - 60°)$ V.

Sometimes voltages and currents are expressed in terms of $\cos \omega t$ rather than $\sin \omega t$. As Example 15–17 shows, a cosine wave is a sine wave shifted by $+90°$, or alternatively, a sine wave is a cosine wave shifted by $-90°$. For sines or cosines with an angle, the following formulas apply.

$$\cos(\omega t + \theta) = \sin(\omega t + \theta + 90°) \quad \textbf{(15–17a)}$$

$$\sin(\omega t + \theta) = \cos(\omega t + \theta - 90°) \quad \textbf{(15–17b)}$$

To illustrate, consider $\cos(\omega t + 30°)$. From Equation 15–17a, $\cos(\omega t + 30°) = \sin(\omega t + 30° + 90°) = \sin(\omega t + 120°)$. Figure 15–45 illustrates this relationship graphically. The red phasor in (a) generates $\cos \omega t$ as was shown




(a) (b) (c)

**FIGURE 15–45** Using phasors to show that $\cos(\omega t + 30°) = \sin(\omega t + 120°)$.

in Example 15–17. Therefore, the green phasor generates a waveform that leads it by 30°, namely cos($\omega t$ + 30°). For (b), the red phasor generates sin $\omega t$, and the green phasor generates a waveform that leads it by 120°, i.e., sin($\omega t$ + 120°). Since the green phasor is the same in both cases, you can see that cos($\omega t$ + 30°) = sin($\omega t$ + 120°). Note that this process is easier than trying to remember equations 15–17(a) and (b).

**EXAMPLE 15–21** Determine the phase angle between $v = 30\cos(\omega t + 20°)$ and $i = 25\sin(\omega t + 70°)$.

**Solution** $i = 25\sin(\omega t + 70°)$ may be represented by a phasor at 70°, and $v = 30\cos(\omega t + 20°)$ by a phasor at (90° + 20°) = 110°, Figure 15–46(a). Thus, $v$ leads $i$ by 40°. Waveforms are shown in (b).

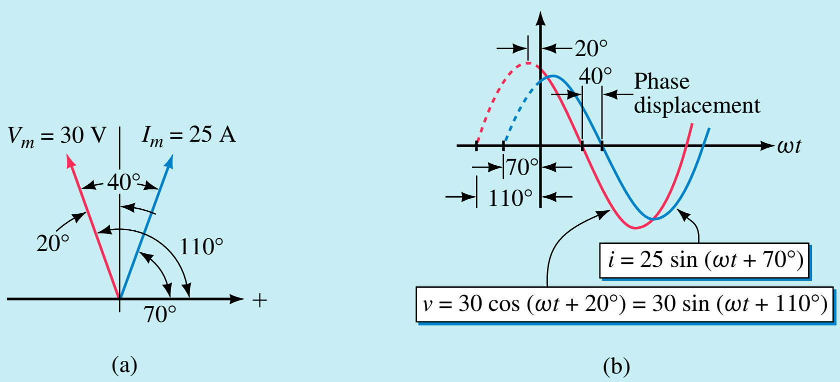
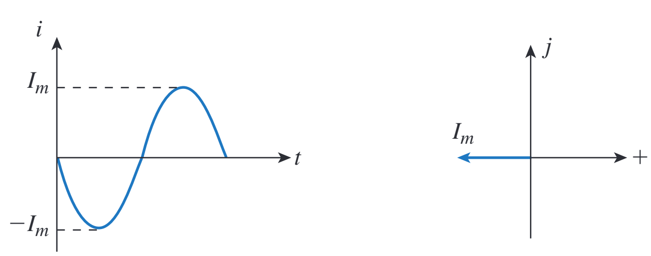



**FIGURE 15–46**

Sometimes you encounter negative waveforms such as $i = -I_m \sin \omega t$. To see how to handle these, refer back to Figure 15–36, which shows the waveform and phasor for $i = I_m \sin \omega t$. If you multiply this waveform by −1, you get the inverted waveform $-I_m \sin \omega t$ of Figure 15–47(a) with corresponding phasor (b). Note that the phasor is the same as the original phasor except that it is rotated by 180°. This is always true—thus, if you multiply a waveform by −1, the phasor for the new waveform is 180° rotated from the original phasor, regardless of the angle of the original phasor.




(a) $i = -I_m \sin \omega t$ (b) Phasor

**FIGURE 15–47** The phasor for a negative sine wave is at 180°.

**EXAMPLE 15–22** Find the phase relationship between $i = -4 \sin(\omega t + 50°)$ and $v = 120 \sin(\omega t - 60°)$.

**Solution** $i = -4 \sin(\omega t + 50°)$ is represented by a phasor at $(50° - 180°) = -130°$ and $v = 120 \sin(\omega t - 60°)$ by a phasor at $-60°$, Figure 15–48. The phase difference is 70° and voltage leads. Note also that $i$ can be written as $i = 4 \sin(\omega t - 130°)$.

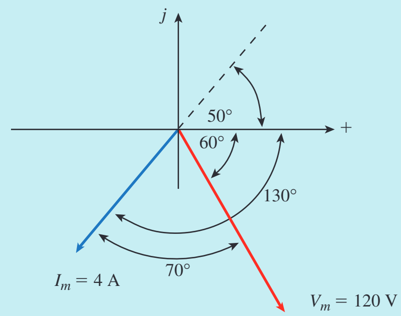



**FIGURE 15–48**

The importance of phasors to ac circuit analysis cannot be overstated—you will find that they are one of your main tools for representing ideas and for solving problems in later chapters. We will leave them for the moment, but pick them up again in Chapter 16.

**IN-PROCESS LEARNING CHECK 2**

1. If $i = 15 \sin \alpha$ mA, compute the current at $\alpha = 0°$, 45°, 90°, 135°, 180°, 225°, 270°, 315°, and 360°.
2. Convert the following angles to radians:

   a. 20° b. 50°

   c. 120° d. 250°
3. If a coil rotates at $\omega = \pi/60$ radians per millisecond, how many degrees does it rotate through in 10 ms? In 40 ms? In 150 ms?
4. A current has an amplitude of 50 mA and $\omega = 0.2\pi$ rad/s. Sketch the waveform with the horizontal axis scaled in

   a. degrees b. radians c. seconds
5. If 2400 cycles of a waveform occur in 10 ms, what is $\omega$ in radians per second?
6. A sinusoidal current has a period of 40 ms and an amplitude of 8 A. Write its equation in the form of $i = I_m \sin \omega t$, with numerical values for $I_m$ and $\omega$.
7. A current $i = I_m \sin \omega t$ has a period of 90 ms. If $i = 3$ A at $t = 7.5$ ms, what is its equation?
8. Write equations for each of the waveforms in Figure 15–49 with the phase angle $\theta$ expressed in degrees.

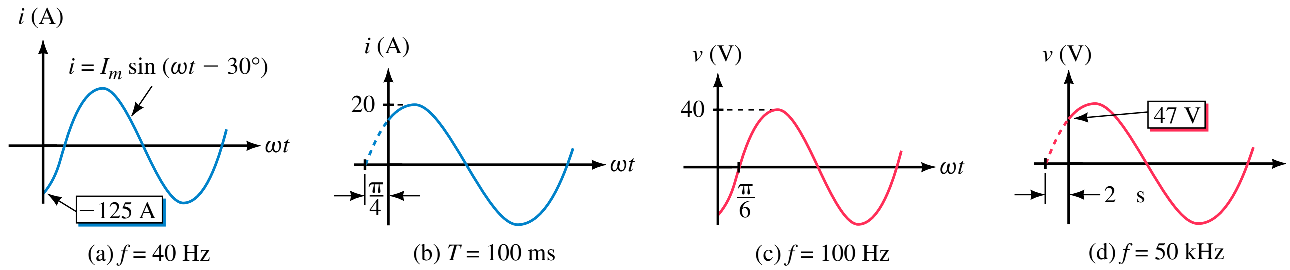



**FIGURE 15–49**

9. Given $i = 10 \sin \omega t$, where $f = 50$ Hz, find all occurrences of
   a. $i = 8$ A between $t = 0$ and $t = 40$ ms
   b. $i = -5$ A between $t = 0$ and $t = 40$ ms
10. Sketch the following waveforms with the horizontal axis scaled in degrees:
    a. $v_1 = 80 \sin(\omega t + 45°)$ V
    b. $v_2 = 40 \sin(\omega t - 80°)$ V
    c. $i_1 = 10 \cos \omega t$ mA
    d. $i_2 = 5 \cos(\omega t - 20°)$ mA
11. Given $\omega = \pi/3$ rad/s, determine when voltage first crosses through 0 for
    a. $v_1 = 80 \sin(\omega t + 45°)$ V
    b. $v_2 = 40 \sin(\omega t - 80°)$ V
12. Consider the voltages of Question 10:
    a. Sketch phasors for $v_1$ and $v_2$
    b. What is the phase difference between $v_1$ and $v_2$?
    c. Determine which voltage leads and which lags.
13. Repeat Question 12 for the currents of Question 10.

*(Answers are at the end of the chapter.)*

## 15.8 AC Waveforms and Average Value

While we can describe ac quantities in terms of frequency, period, instantaneous value, etc., we do not yet have any way to give a meaningful value to an ac current or voltage in the same sense that we can say of a car battery that it has a voltage of 12 volts. This is because ac quantities constantly change and thus there is no one single numerical value that truly represents a waveform over its complete cycle. For this reason, ac quantities are generally described by a group of characteristics, including instantaneous, peak, average, and effective values. The first two of these we have already seen. In this section, we look at average values; in Section 15.9, we consider effective values.

### Average Values

Many quantities are measured by their average, for instance, test and examination scores. To find the average of a set of marks for example, you add them, then divide by the number of items summed. For waveforms, the process is conceptually the same. For example, to find the average of a waveform, you can sum the instantaneous values over a full cycle, then divide by the number of points used. The trouble with this approach is that waveforms do not consist of discrete values.

## Average in Terms of the Area Under a Curve

An approach more suitable for use with waveforms is to find the area under the curve, then divide by the baseline of the curve. To get at the idea, we can use an analogy. Consider again the technique of computing the average for a set of numbers. Assume that you earn marks of 80, 60, 60, 95, and 75 on a group of tests. Your average mark is therefore

$$\text{average} = (80 + 60 + 60 + 95 + 75)/5 = 74$$

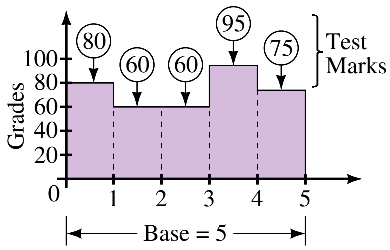



**FIGURE 15–50** Determining average by area.

An alternate way to view these marks is graphically as in Figure 15–50. The area under this curve can be computed as

$$\text{area} = (80 \times 1) + (60 \times 2) + (95 \times 1) + (75 \times 1)$$

Now divide this by the length of the base, namely 5. Thus,

$$\frac{(80 \times 1) + (60 \times 2) + (95 \times 1) + (75 \times 1)}{5} = 74$$

which is exactly the answer obtained above. That is,

$$\text{average} = \frac{\text{area under curve}}{\text{length of base}} \tag{15–18}$$

This result is true in general. Thus, *to find the average value of a waveform, divide the area under the waveform by the length of its base. Areas above the axis are counted as positive, while areas below the axis are counted as negative.* This approach is valid regardless of waveshape.

Average values are also called **dc values,** because dc meters indicate average values rather than instantaneous values. Thus, if you measure a non-dc quantity with a dc meter, the meter will read the average of the waveform, i.e., the value calculated according to Equation 15–18.

### EXAMPLE 15–23

a. Compute the average for the current waveform of Figure 15–51.
b. If the negative portion of Figure 15–51 is −3 A instead of −1.5 A, what is the average?
c. If the current is measured by a dc ammeter, what will the ammeter indicate?

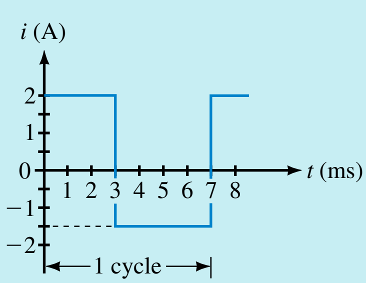



**FIGURE 15–51**

**Solution**

a. The waveform repeats itself after 7 ms. Thus, $T = 7$ ms and the average is

$$I_{avg} = \frac{(2\text{ A} \times 3\text{ ms}) - (1.5\text{ A} \times 4\text{ ms})}{7\text{ ms}} = \frac{6 - 6}{7} = 0\text{ A}$$

b. $$I_{avg} = \frac{(2\text{ A} \times 3\text{ ms}) - (3\text{ A} \times 4\text{ ms})}{7\text{ ms}} = \frac{-6\text{ A}}{7} = -0.857\text{ A}$$

c. A dc ammeter measuring (a) will indicate zero, while for (b) it will indicate $-0.857$ A.

**EXAMPLE 15–24** Compute the average value for the waveforms of Figures 15–52(a) and (c). Sketch the averages for each.

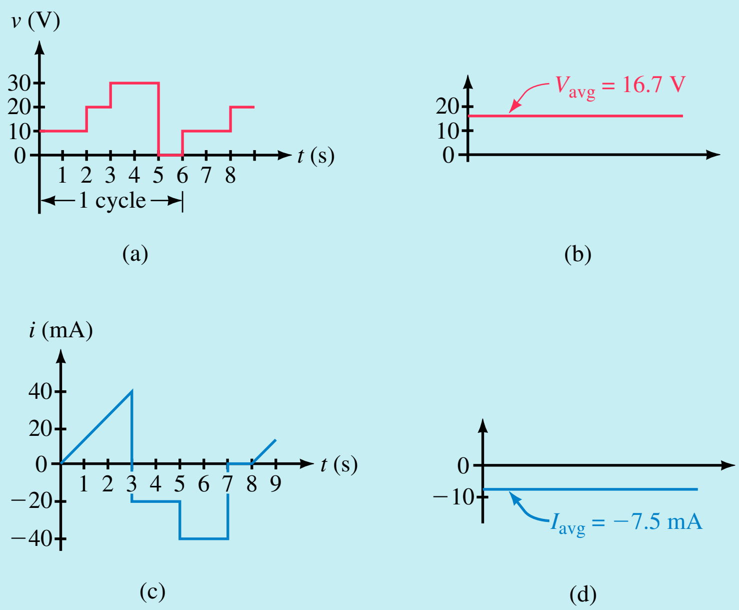



**FIGURE 15–52**

**Solution** For the waveform of (a), $T = 6$ s. Thus,

$$V_{avg} = \frac{(10\text{V} \times 2\text{s}) + (20\text{V} \times 1\text{s}) + (30\text{V} \times 2\text{s}) + (0\text{V} \times 1\text{s})}{6\text{s}} = \frac{100\text{ V-s}}{6\text{s}} = 16.7\text{V}$$

The average is shown as (b). A dc voltmeter would indicate 16.7 V. For the waveform of (c), $T = 8$ s and

$$I_{avg} = \frac{\frac{1}{2}(40\text{mA} \times 3\text{s}) - (20\text{mA} \times 2\text{s}) - (40\text{mA} \times 2\text{s})}{8\text{s}} = \frac{-60}{8}\text{mA} = -7.5\text{mA}$$

In this case, a dc ammeter would indicate $-7.5$ mA.

**PRACTICE PROBLEMS 9**

Determine the averages for Figures 15–53(a) and (b).

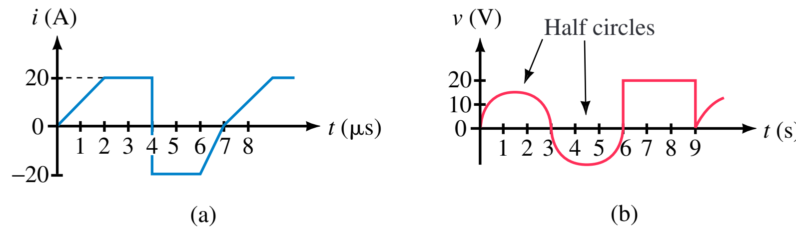



**FIGURE 15–53**

*Answers:* a. 1.43 A b. 6.67 V

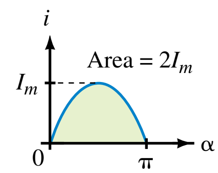



**FIGURE 15–54** Area under a half-cycle.

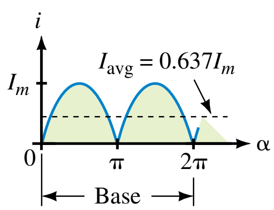



**FIGURE 15–55** Full-wave average.

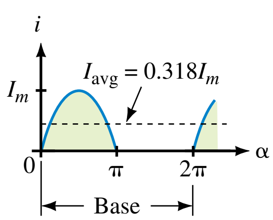



**FIGURE 15–56** Half-wave average.

## Sine Wave Averages

Because a sine wave is symmetrical, its area below the horizontal axis is the same as its area above the axis; thus, over a full cycle its net area is zero, independent of frequency and phase angle. Thus, the average of sin $\omega t$, $\sin(\omega t \pm \theta)$, sin $2\omega t$, cos $\omega t$, $\cos(\omega t \pm \theta)$, cos $2\omega t$, and so on are each zero. The average of half a sine wave, however, is not zero. Consider Figure 15–54. The area under the half-cycle may be found using calculus as

$$\text{area} = \int_0^\pi I_m \sin\alpha \, d\alpha = -I_m \cos\alpha \Big|_0^\pi = 2I_m \qquad \textbf{(15–19)}$$

Similarly, the area under a half-cycle of voltage is $2V_m$. (If you haven't studied calculus, you can approximate this area using numerical methods as described later in this section.)

Two cases are important; full-wave average and half-wave average. The full-wave case is illustrated in Figure 15–55. The area from 0 to $2\pi$ is $2(2I_m)$ and the base is $2\pi$. Thus, the average is

$$I_{avg} = \frac{2(2I_m)}{2\pi} = \frac{2I_m}{\pi} = 0.637I_m$$

For the half-wave case (Figure 15–56),

$$I_{avg} = \frac{2I_m}{2\pi} = \frac{I_m}{\pi} = 0.318I_m$$

The corresponding expressions for voltage are

$$V_{avg} = 0.637V_m \quad \text{(full-wave)}$$

$$V_{avg} = 0.318V_m \quad \text{(half-wave)}$$

## Numerical Methods

If the area under a curve cannot be computed exactly, it can be approximated. One method is to approximate the curve by straight line segments as in Figure 15–57. (If the straight lines closely fit the curve, the accuracy is very good.) Each element of area is a trapezoid (b) whose area is its average height times its base. Thus, $A_1 = ½(y_0 + y_1)\Delta x$, $A_2 = ½(y_1 + y_2)\Delta x$, etc. Summing areas and combining terms yields

$$\text{area} = \left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{k-1} + \frac{y_k}{2}\right)\Delta x \qquad \textbf{(15–20)}$$

This result is known as the **trapezoidal rule.** Example 15–25 illustrates its use.

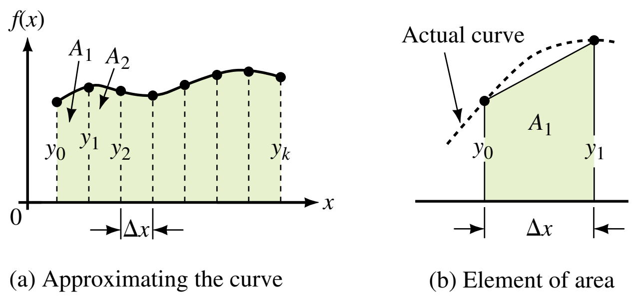



**FIGURE 15–57** Calculating area using the trapezoidal rule.

**EXAMPLE 15–25** Approximate the area under $y = \sin(\omega t - 30°)$, Figure 15–58. Use an increment size of $\pi/6$ rad, i.e., 30°.

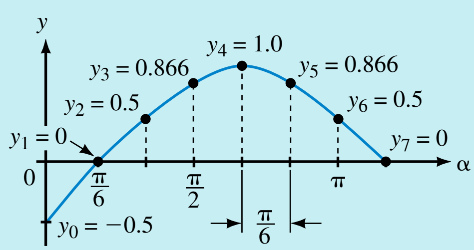



**FIGURE 15–58**

**Solution** Points on the curve $\sin(\omega t - 30°)$ have been computed by calculator and plotted as Figure 15–58. Substituting these values into Equation 15–20 yields

$$\text{area} = \left(\frac{1}{2}(-0.5) + 0 + 0.5 + 0.866 + 1.0 + 0.866 + 0.5 + \frac{1}{2}(0)\right)\left(\frac{\pi}{6}\right) = 1.823$$

The exact area (found using calculus) is 1.866; thus, the above value is in error by 2.3%.

**PRACTICE PROBLEMS 10**

1. Repeat Example 15–25 using an increment size of $\pi/12$ rad. What is the percent error?

2. Approximate the area under $v = 50 \sin(\omega t + 30°)$ from $\omega t = 0°$ to $\omega t = 210°$. Use an increment size of $\pi/12$ rad.

*Answers:*
1. 1.855; 0.59%
2. 67.9 (exact 68.3; error = 0.6%)

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## Superimposed AC and DC

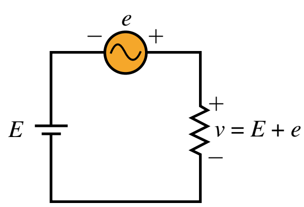
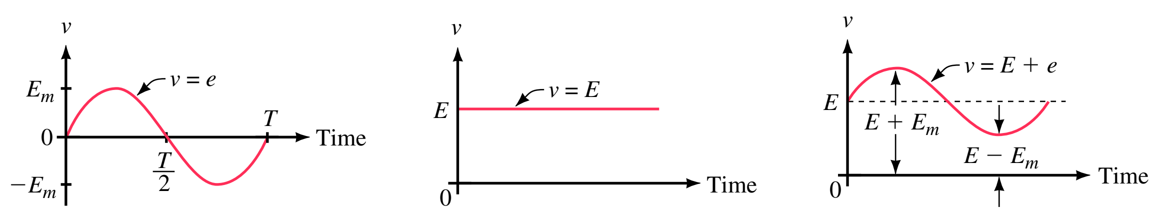



**FIGURE 15–59**

Sometimes ac and dc are used in the same circuit. For example, amplifiers are powered by dc but the signals they amplify are ac. Figure 15–59 shows a simple circuit with combined ac and dc.

Figure 15–60(c) shows superimposed ac and dc. Since we know that the average of a sine wave is zero, the average value of the combined waveform will be its dc component, $E$. However, peak voltages depend on both components as illustrated in (c). Note for the case illustrated that although the waveform varies sinusoidally, it does not alternate in polarity since it never changes polarity to become negative.



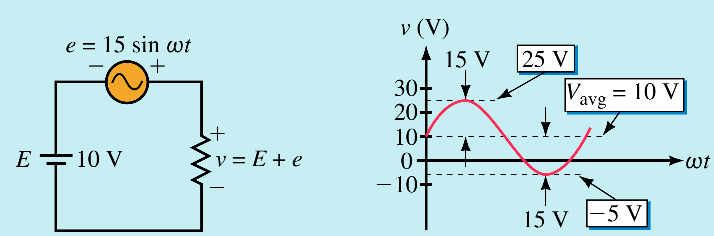

(a) AC alone. $E = 0$ V. $V_{avg} = 0$ V

(b) DC alone. $e = 0$ V. $V_{avg} = E$

(c) Superimposed ac and dc. $V_{avg} = E$

**FIGURE 15–60** Superimposed dc and ac.

**EXAMPLE 15–26** Draw the voltage waveform for the circuit of Figure 15–61(a). Determine average, peak, and minimum voltages.




(a)

(b)

**FIGURE 15–61** $v = 10 + 15 \sin \omega t$.

**Solution** The waveform consists of a 10-V dc value with 15 V ac riding on top of it. The average is the dc value, $V_{avg} = 10$ V. The peak voltage is $10 + 15 = 25$ V, while the minimum voltage is $10 - 15 = -5$ V. This waveform alternates in polarity, although not symmetrically (as is the case when there is no dc component).

PRACTICE PROBLEMS 11

Repeat Example 15–26 if the dc source of Figure 15–61 is $E = -5$ V.

*Answers:* $V_{avg} = -5$ V; positive peak $= 10$ V; negative peak $= -20$ V

## 15.9 Effective Values

While instantaneous, peak, and average values provide useful information about a waveform, none of them truly represents the ability of the waveform to do useful work. In this section, we look at a representation that does. It is called the waveform's **effective value.** The concept of effective value is an important one; in practice, most ac voltages and currents are expressed as effective values. Effective values are also called **rms values** for reasons discussed shortly.

### What Is an Effective Value?

*An effective value is an equivalent dc value: it tells you how many volts or amps of dc that a time-varying waveform is equal to in terms of its ability to produce average power.* Effective values depend on the waveform. A familiar example of such a value is the value of the voltage at the wall outlet in your home. In North America its value is 120 Vac. This means that the sinusoidal voltage at the wall outlets of your home is capable of producing the same average power as 120 volts of steady dc.

### Effective Values for Sine Waves

The effective value of a waveform can be determined using the circuits of Figure 15–62. Consider a sinusoidally varying current, $i(t)$. By definition, the effective value of $i$ is that value of dc current that produces the same average power. Consider (b). Let the dc source be adjusted until its average power is the same as the average power in (a). The resulting dc current is then the effective value of the current of (a). To determine this value, determine the average power for both cases, then equate them.

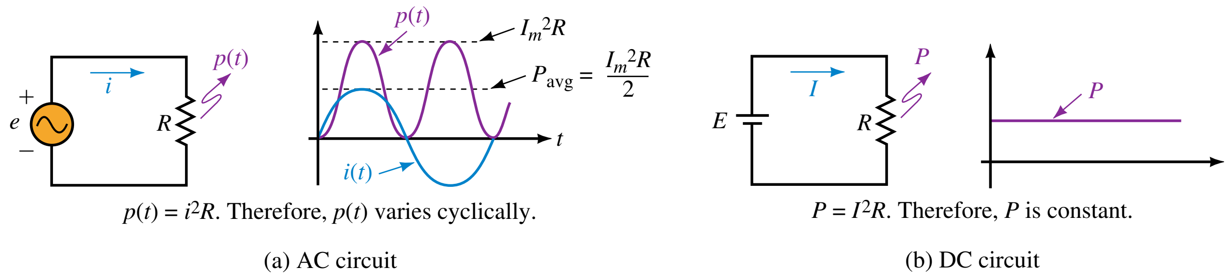



**FIGURE 15–62** Determining the effective value of sinusoidal ac.

First, consider the dc case. Since current is constant, power is constant, and average power is

$$P_{avg} = P = I^2R \tag{15–21}$$

Now consider the ac case. Power to the resistor at any value of time is $p(t) = i^2R$, where $i$ is the instantaneous value of current. A sketch of $p(t)$ is shown in Figure 15–62(a), obtained by squaring values of current at various points along the axis, then multiplying by $R$. Average power is the average of $p(t)$. Since $i = I_m \sin \omega t$,

$$\begin{aligned} p(t) &= i^2R \\ &= (I_m \sin \omega t)^2 R = I_m{}^2 R \sin^2 \omega t \\ &= I_m{}^2 R\left[\frac{1}{2}(1 - \cos 2\omega t)\right] \end{aligned} \quad \textbf{(15–22)}$$

where we have used the trigonometric identity $\sin^2 \omega t = ½(1 - \cos 2\omega t)$, from the mathematics tables inside the front cover to expand $\sin^2 \omega t$. Thus,

$$p(t) = \frac{I_m{}^2 R}{2} - \frac{I_m{}^2 R}{2} \cos 2\omega t \quad \textbf{(15–23)}$$

To get the average of $p(t)$, note that the average of $\cos 2\omega t$ is zero and thus the last term of Equation 15–23 drops off leaving

$$P_{\text{avg}} = \text{average of } p(t) = \frac{I_m{}^2 R}{2} \quad \textbf{(15–24)}$$

Now equate Equations 15–21 and 15–24, then cancel $R$.

$$I^2 = \frac{I_m{}^2}{2}$$

Now take the square root of both sides. Thus,

$$I = \sqrt{\frac{I_m{}^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Current $I$ is the value that we are looking for; it is the effective value of current $i$. To emphasize that it is an effective value, we will initially use subscripted notation $I_{\text{eff}}$. Thus,

$$I_{\text{eff}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m \quad \textbf{(15–25)}$$

Effective values for voltage are found in the same way:

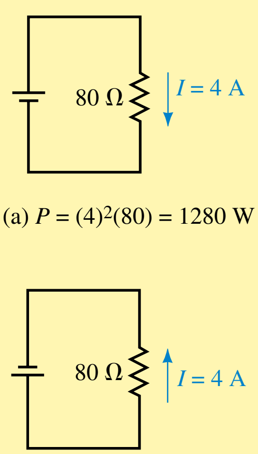
$$E_{\text{eff}} = \frac{E_m}{\sqrt{2}} = 0.707 E_m \quad \textbf{(15–26a)}$$

$$V_{\text{eff}} = \frac{V_m}{\sqrt{2}} = 0.707 V_m \quad \textbf{(15–26b)}$$

As you can see, *effective values for sinusoidal waveforms depend only on magnitude.*

**NOTES...**

Because ac currents alternate in direction, you might expect average power to be zero, with power during the negative half-cycle being equal and opposite to power during the positive half-cycle and hence cancelling. However, as Equation 15–22 shows, current is squared, and hence power is never negative. This is consistent with the idea that insofar as power dissipation is concerned, the direction of current through a resistor does not matter (Figure 15–63).




(a) $P = (4)^2(80) = 1280$ W

(b) $P = (4)^2(80) = 1280$ W

**FIGURE 15–63** Since power depends only on current magnitude, it is the same for both directions.

**EXAMPLE 15–27** Determine the effective values of

a. $i = 10 \sin \omega t$ A,  b. $i = 50 \sin(\omega t + 20°)$ mA,
c. $v = 100 \cos 2\omega t$ V

Solution Since effective values depend only on magnitude,

a. $I_{eff} = (0.707)(10\text{ A}) = 7.07\text{ A}$,

b. $I_{eff} = (0.707)(50\text{ mA}) = 35.35\text{ mA}$,

c. $V_{eff} = (0.707)(100\text{ V}) = 70.7\text{ V}$.

To obtain peak values from effective values, rewrite Equations 15–25 and 15–26. Thus,

$$I_m = \sqrt{2}I_{eff} = 1.414I_{eff} \tag{15–27}$$

$$E_m = \sqrt{2}E_{eff} = 1.414V_{eff} \tag{15–28a}$$

$$V_m = \sqrt{2}V_{eff} = 1.414V_{eff} \tag{15–28b}$$

*It is important to note that these relationships hold only for sinusoidal waveforms.* However, the concept of effective value applies to all waveforms, as we soon see.

Consider again the ac voltage at the wall outlet in your home. Since $E_{eff} = 120\text{ V}$, $E_m = (\sqrt{2})(120\text{ V}) = 170\text{ V}$. This means that a sinusoidal voltage alternating between $\pm 170\text{ V}$ produces the same average power as 120 V of steady dc (Figure 15–64).

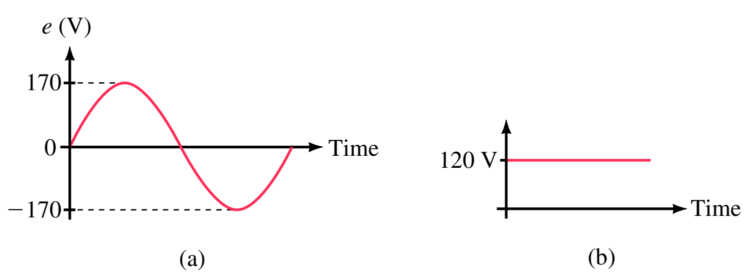



**FIGURE 15–64** 120 V of steady dc is capable of producing the same average power as sinusoidal ac with $E_m = 170\text{ V}$.

## General Equation for Effective Values

The $\sqrt{2}$ relationship holds only for sinusoidal waveforms. For other waveforms, you need a more general formula. Using calculus, it can be shown that for any waveform

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \tag{15–29}$$

with a similar equation for voltage. This equation can be used to compute effective values for any waveform, including sinusoidal. In addition, it leads to a graphic approach to finding effective values. In Equation 15–29, the integral of $i^2$ represents the area under the $i^2$ waveform. Thus,

$$I_{eff} = \sqrt{\frac{\text{area under the } i^2 \text{ curve}}{\text{base}}} \tag{15–30}$$

To compute effective values using this equation, do the following:

**Step 1:** Square the current (or voltage) curve.

**Step 2:** Find the area under the squared curve.

**Step 3:** Divide the area by the length of the curve.

**Step 4:** Find the square root of the value from Step 3.

This process is easily carried out for rectangular-shaped waveforms since the area under their squared curves is easy to compute. For other waveforms, you have to use calculus or approximate the area using numerical methods. For the special case of superimposed ac and dc (Figure 15–60), Equation 15–29 leads to the following formula:

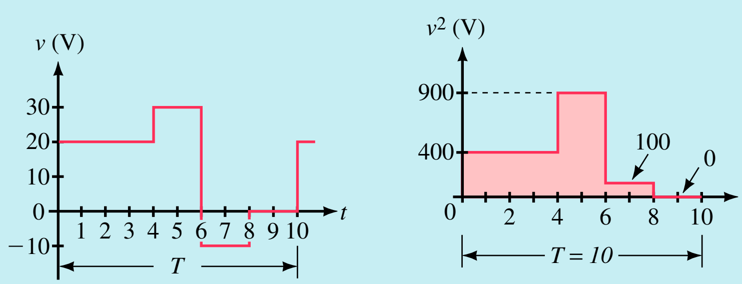
$$I_{\text{eff}} = \sqrt{I_{\text{dc}}^2 + I_{\text{ac}}^2} \tag{15–31}$$

where $I_{\text{dc}}$ is the dc current value, $I_{\text{ac}}$ is the effective value of the ac component, and $I_{\text{eff}}$ is the effective value of the combined ac and dc currents. Equations 15–30 and 15–31 also hold for voltage when $V$ is substituted for $I$.

## RMS Values

Consider again Equation 15–30. To use this equation, we compute the root of the mean square to obtain the effective value. For this reason, effective values are called **root mean square** or **rms** values and **the terms *effective* and *rms* are synonymous.** Since, in practice, ac quantities are almost always expressed as rms values, we shall assume from here on that, unless otherwise noted, *all ac voltages and currents are rms values.*

**EXAMPLE 15–28** One cycle of a voltage waveform is shown in Figure 15–65(a). Determine its effective value.




(a) Voltage waveform

(b) Squared waveform

**FIGURE 15–65**

**Solution** Square the voltage waveform and plot it as in (b). Apply Equation 15–30:

$$V_{\text{eff}} = \sqrt{\frac{(400 \times 4) + (900 \times 2) + (100 \times 2) + (0 \times 2)}{10}}$$

$$= \sqrt{\frac{3600}{10}} = 19.0 \text{ V}$$

The waveform of Figure 15–65(a) has the same effective value as 19.0 V of steady dc.

**EXAMPLE 15–29** Determine the effective value of the waveform of Figure 15–66(a).

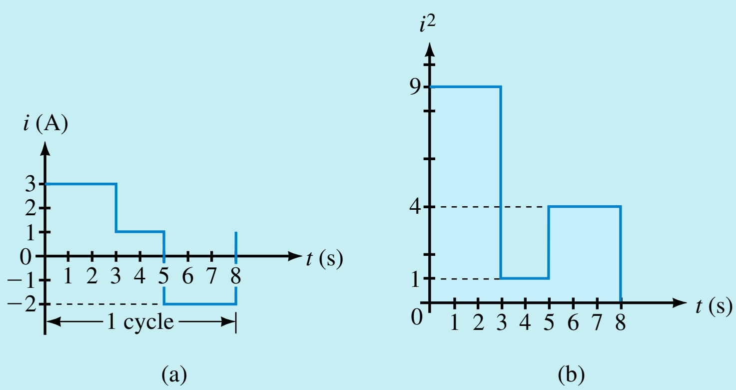



**FIGURE 15–66**

**Solution** Square the curve, then apply Equation 15–30. Thus,

$$I_{\text{eff}} = \sqrt{\frac{(9 \times 3) + (1 \times 2) + (4 \times 3)}{8}}$$

$$= \sqrt{\frac{41}{8}} = 2.26 \text{ A}$$

**EXAMPLE 15–30** Compute the effective value of the waveform of Figure 15–61(b).

**Solution** Use Equation 15–31 (with $I$ replaced by $V$). First, compute the rms value of the ac component. $V_{\text{ac}} = 0.707 \times 15 = 10.61$ V. Now substitute this into Equation 15–31. Thus,

$$V_{\text{rms}} = \sqrt{V_{\text{dc}}^2 + V_{\text{ac}}^2} = \sqrt{(10)^2 + (10.61)^2} = 14.6 \text{ V}$$

**PRACTICE PROBLEMS 12**

1. Determine the effective value of the current of Figure 15–51.
2. Repeat for the voltage graphed in Figure 15–52(a).

*Answers:*
1. 1.73 A
2. 20 V

## One Final Note

The subscripts *eff* and *rms* are not used in practice. Once the concept is familiar, we drop them.

## 15.10 Rate of Change of a Sine Wave (Derivative)

Several important circuit effects depend on the rate of change of sinusoidal quantities. The rate of change of a quantity is the slope (i.e., derivative) of its waveform versus time. Consider the waveform of Figure 15–67. As indicated, the slope is maximum positive at the beginning of the cycle, zero at both its peaks, maximum negative at the half-cycle crossover point, and maximum positive at the end of the cycle. This slope is plotted in Figure 15–68. Note that the original waveform and its slope are 90° out of phase. Thus, if sinusoidal waveform *A* is taken as reference, its slope *B* leads it by 90°, whereas, if the slope *B* is taken as reference, *A* lags it by 90°. Thus, if *A* is a sine wave, *B* is a cosine wave, and so on. (This result is important to us in Chapter 16.)

NOTES...

**∫ The Derivative of a Sine Wave**

The result developed intuitively here can be proven easily using calculus. To illustrate, consider the waveform sin $\omega t$ shown in Figure 15–67. The slope of this function is its derivative. Thus,

$$\text{Slope} = \frac{d}{dt}\sin \omega t = \omega \cos \omega t$$

Therefore, the slope of a sine wave is a cosine wave as depicted in Figure 15–68.

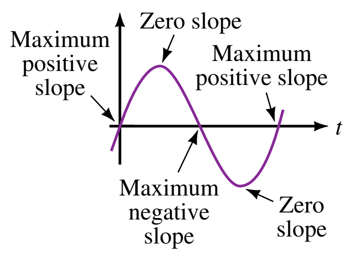



**FIGURE 15–67** Slope at various places for a sine wave.

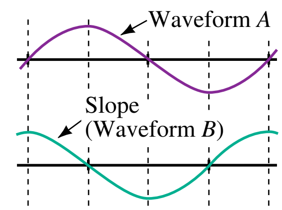



**FIGURE 15–68** Showing the 90° phase shift.

## 15.11 AC Voltage and Current Measurement

Two of the most important instruments for measuring ac quantities are the multimeter and the oscilloscope. Multimeters read voltage, current, and sometimes frequency. Oscilloscopes show waveshape and period and permit determination of frequency, phase difference, and so on.

### Meters for Voltage and Current Measurement

There are two basic classes of ac meters: one measures rms correctly for sinusoidal waveforms only (called "average responding" instruments); the other measures rms correctly regardless of waveform (called "true rms" meters). Most common meters are average responding meters.

#### Average Responding Meters

Average responding meters use a rectifier circuit to convert incoming ac to dc. They then respond to the average value of the rectified input, which, as shown in Figure 15–55, is $0.637V_m$ for a "full-wave" rectified sine wave. However, the rms value of a sine wave is $0.707V_m$. Thus, the scale of such a meter is modified by the factor $0.707V_m/0.637V_m = 1.11$ so that it indicates rms values directly. Other meters use a "half-wave" circuit, which yields the waveform of Figure 15–56 for a sine wave input. In this case, its average is $0.318V_m$, yielding a scale factor of $0.707V_m/0.318V_m = 2.22$. Figure 15–69 shows a typical average responding DMM.


**FIGURE 15–69** A DMM. While all DMMs measure voltage, current, and resistance, this one also measures frequency.

### True RMS Measurement

To measure the rms value of a nonsinusoidal waveform, you need a true rms meter. A true rms meter indicates true rms voltages and currents regardless of waveform. For example, for the waveform of Figure 15–64(a), any ac meter will correctly read 120 V (since it is a sine wave). For the waveform of 15–61(b), a true rms meter will correctly read 14.6 V (the rms value that we calculated earlier, in Example 15–30) but an average responding meter will yield only a meaningless value. True rms instruments are more expensive than standard meters.

## Oscilloscopes

Oscilloscopes (frequently referred to as scopes, Figure 15–70) are used for time domain measurement, i.e., waveshape, frequency, period, phase difference, and so on. Usually, you scale values from the screen, although some higher-priced models can compute and display them for you on a digital readout.

Oscilloscopes measure voltage. To measure current, you need a current-to-voltage converter. One type of converter is a clip-on device, known as a **current gun** that clamps over the current-carrying conductor and monitors its magnetic field. (It works only with ac.) The varying magnetic field induces a voltage which is then displayed on the screen. With such a device, you can monitor current waveshapes and make current-related measurements. Alternately, you can place a small resistor in the current path, mea-

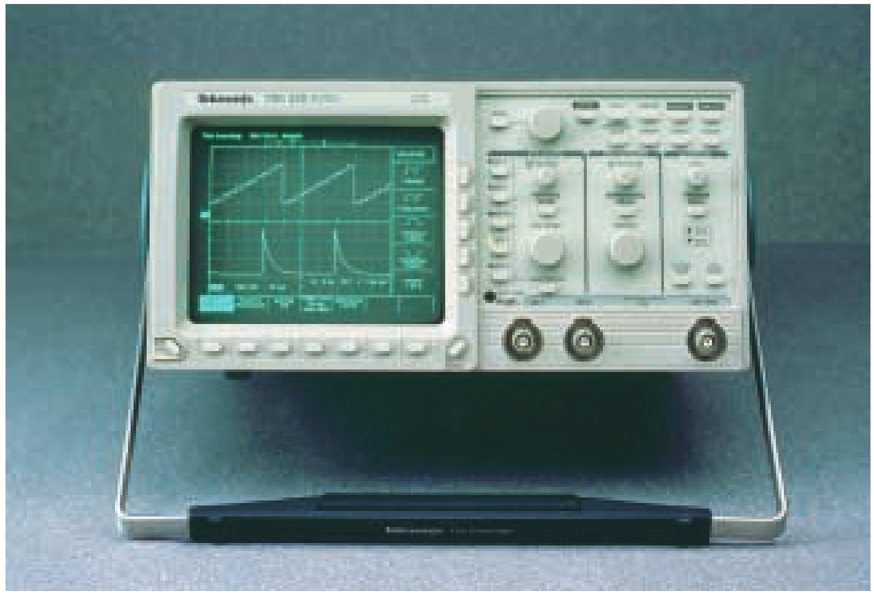


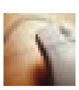
**FIGURE 15–70** An oscilloscope may be used for waveform analysis.

sure voltage across it with the oscilloscope, then use Ohm's law to determine the current.

### A Final Note

AC meters measure voltage and current only over a limited frequency range, typically from 50 Hz to a few kHz, although others are available that work up to the 100-kHz range. Note, however, that accuracy may be affected by frequency. (Check the manual.) Oscilloscopes, on the other hand, can measure very high frequencies; even moderately priced oscilloscopes work at frequencies up to hundreds of MHz.

## 15.12 Circuit Analysis Using Computers



ELECTRONICS WORKBENCH

PSpice

Electronics Workbench and PSpice both provide a convenient way to study the phase relationships of this chapter, as they both incorporate easy-to-use graphing facilities. You simply set up sources with the desired magnitude and phase values and instruct the software to compute and plot the results. To illustrate, let us graph $e_1 = 100 \sin \omega t$ V and $e_2 = 80 \sin(\omega t + 60°)$ V. Use a frequency of 500 Hz.

### Electronics Workbench

For Electronics Workbench, you must specify rms rather than peak values. Thus, to plot $e_1 = 100 \sin \omega t$ V, key in 70.71 V and 0 degrees. Similarly, for $e_2$, use 56.57 V and 60°. Here are the steps: Create the circuit of Figure 15–71 on the screen; Double click Source 1 and enter 70.71 V, 0 deg, and 500 Hz in the dialog box; Similarly, set Source 2 to 56.57 V, 60 deg, and

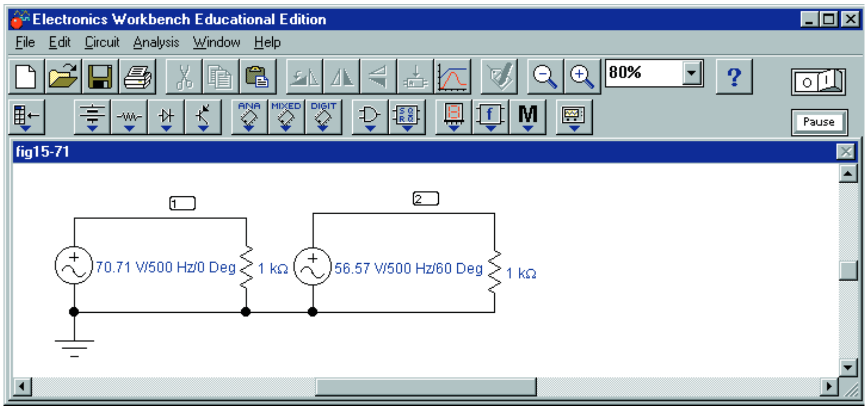



**FIGURE 15–71** Studying phase relationships using Electronics Workbench.

500 Hz; Click **Analysis, Transient,** set **TSTOP** to 0.002 (to run the solution out to 2 ms so that you display a full cycle) and **TMAX** to 2e-06 (to avoid getting a choppy waveform); Highlight Node 1 (to display $e_1$) and click **Add;** Repeat for Node 2 (to display $e_2$); Click the **Simulate** icon; Following simulation, graphs $e_1$ and $e_2$ (Figure 15–72) appear.

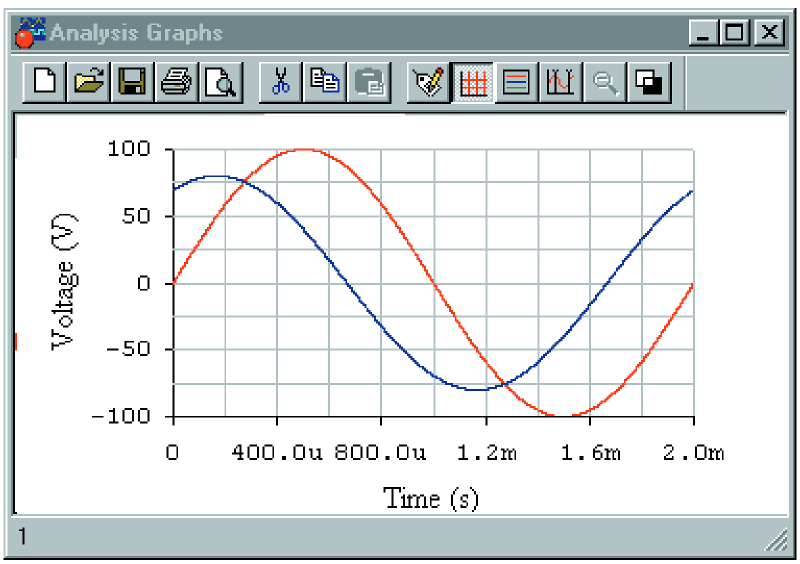



**FIGURE 15–72**

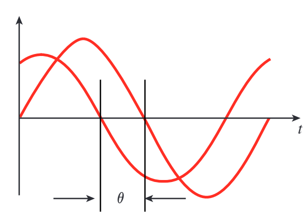



**FIGURE 15–73**

You can verify the angle between the waveforms using cursors. First, note that the period $T = 2$ ms $= 2000\ \mu$s. (This corresponds to 360°.) Expand the graph to full screen, click the **Grid** icon, then the **Cursors** icon. Using the cursors, measure the time between crossover points, as indicated in Figure 15–73. You should get 333 $\mu$s. This yields an angular displacement of

$$\theta = \frac{333\ \mu\text{s}}{2000\ \mu\text{s}} \times 360° = 60°$$

## OrCAD PSpice

For this problem, you need a sinusoidal time-varying ac voltage source. Use **VSIN** (it is found in the **SOURCE** library.) For VSIN, you must specify the magnitude, phase, and frequency of the source, as well as its offset (since we do not want an offset in this problem, we will set it to zero.) Proceed as follows. Build the circuit of Figure 15–74 on the screen. Double click source 1 and in the Properties editor, select the **Parts** tab. Scroll right until you find a list of source properties: then in cell **VAMPL,** enter 100V; in cell **PHASE,** enter 0deg; in cell **FREQ,** enter 500Hz; and in cell **VOFF,** enter 0V (this sets source $e_i = 100 \sin \omega t$ V with $\omega = 2\pi(500 \text{ Hz})$). Click **Apply,** then close the Properties editor window. Similarly, set up source 2 (making sure you use a phase angle of 60°). Click the **New Simulation** icon and enter a name (e.g., **fig15-74**). In the **Simulation Settings** box, select **Time Domain** and **General Settings.** Set **TSTOP** to 2ms (to display a full cycle). Select voltage markers from the toolbar and place as shown. (This causes PSpice to automatically create the photos.) Run the simulation. When the simulation is complete, the waveforms of Figure 15–75 should appear.

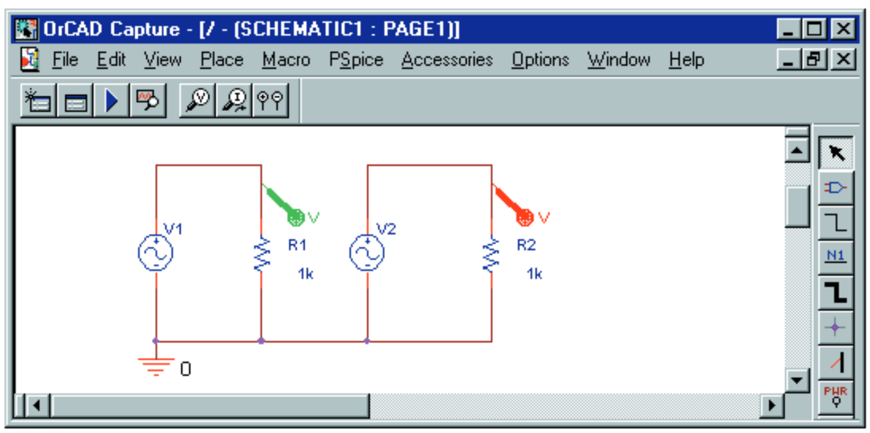



FIGURE 15–74 Studying phase relationships using OrCAD PSpice.

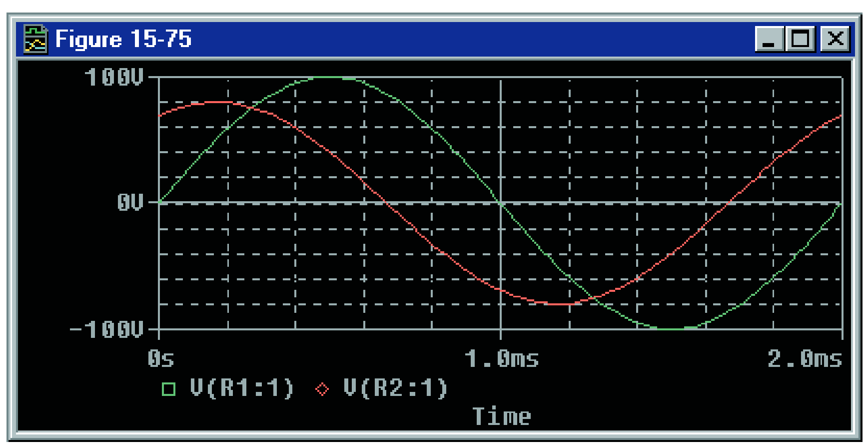



FIGURE 15–75

You can verify the angle between the waveforms using cursors. First, note that the period $T = 2$ ms $= 2000$ $\mu$s. (This corresponds to 360°.) Now using the cursors (see Appendix A if you need help), measure the time between crossover points as indicated in Figure 15–73. You should get 333 $\mu$s. This yields an angular displacement of

$$\theta = \frac{333\ \mu\text{s}}{2000\ \mu\text{s}} \times 360° = 60°$$

which agrees with the given sources.

## PROBLEMS

### 15.1 Introduction

1. What do we mean by "ac voltage"? By "ac current"?

### 15.2 Generating AC Voltages

2. The waveform of Figure 15–8 is created by a 600-rpm generator. If the speed of the generator changes so that its cycle time is 50 ms, what is its new speed?
3. a. What do we mean by instantaneous value?
   b. For Figure 15–76, determine instantaneous voltages at $t = 0, 1, 2, 3, 4, 5, 6, 7$, and 8 ms.

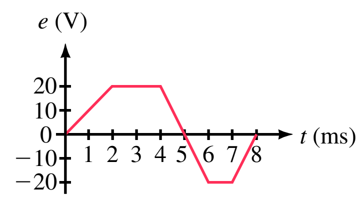



FIGURE 15–76

### 15.3 Voltage and Current Conventions for ac

4. For Figure 15–77, what is $I$ when the switch is in position 1? When in position 2? Include sign.

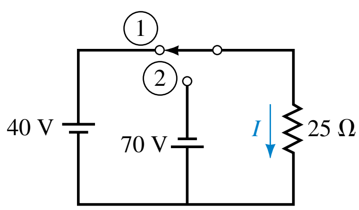



FIGURE 15–77

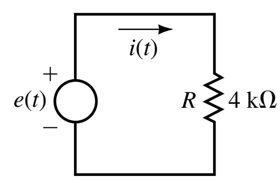



FIGURE 15–78

5. The source of Figure 15–78 has the waveform of Figure 15–76. Determine the current at $t = 0, 1, 2, 3, 4, 5, 6, 7$, and 8 ms. Include sign.

### 15.4 Frequency, Period, Amplitude, and Peak Value

6. For each of the following, determine the period:
   a. $f = 100$ Hz  b. $f = 40$ kHz  c. $f = 200$ MHz
7. For each of the following, determine the frequency:
   a. $T = 0.5$ s  b. $T = 100$ ms  c. $5T = 80$ $\mu$s
8. For a triangular wave, $f = 1.25$ MHz. What is its period? How long does it take to go through $8 \times 10^7$ cycles?

9. Determine the period and frequency for the waveform of Figure 15–79.

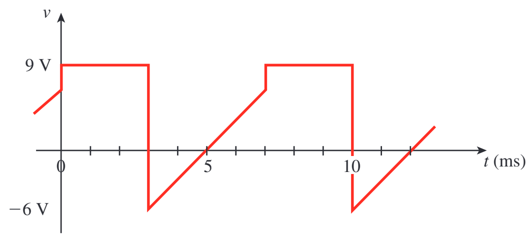



FIGURE 15–79

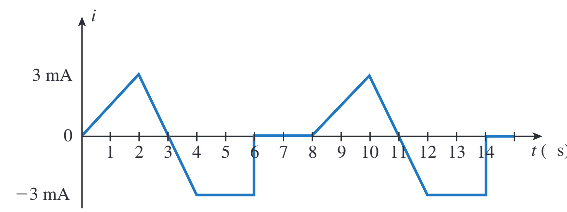



FIGURE 15–80

10. Determine the period and frequency for the waveform of Figure 15–80. How many cycles are shown?
11. What is the peak-to-peak voltage for Figure 15–79? What is the peak-to-peak current of Figure 15–80?
12. For a certain waveform, $625T = 12.5$ ms. What is the waveform's period and frequency?
13. A square wave with a frequency of 847 Hz goes through how many cycles in 2 minutes and 57 seconds?
14. For the waveform of Figure 15–81, determine
    a. period  b. frequency  c. peak-to-peak value

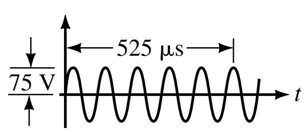



FIGURE 15–81

15. Two waveforms have periods of $T_1$ and $T_2$ respectively. If $T_1 = 0.25\ T_2$ and $f_1 = 10$ kHz, what are $T_1$, $T_2$, and $f_2$?
16. Two waveforms have frequencies $f_1$ and $f_2$ respectively. If $T_1 = 4\ T_2$ and waveform 1 is as shown in Figure 15–79, what is $f_2$?

### 15.5 Angular and Graphic Relationships for Sine Waves

17. Given voltage $v = V_m \sin \alpha$. If $V_m = 240$ V, what is $v$ at $\alpha = 37°$?
18. For the sinusoidal waveform of Figure 15–82,
    a. Determine the equation for $i$.
    b. Determine current at all points marked.

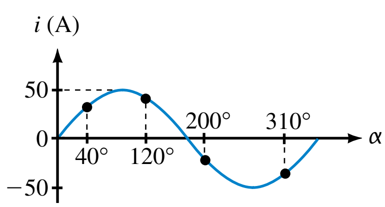



FIGURE 15–82

19. A sinusoidal voltage has a value of 50 V at $\alpha = 150°$. What is $V_m$?
20. Convert the following angles from radians to degrees:
    a. $\pi/12$  b. $\pi/1.5$  c. $3\pi/2$
    d. 1.43  e. 17  f. $32\pi$
21. Convert the following angles from degrees to radians:
    a. 10°  b. 25°  c. 80°
    d. 150°  e. 350°  f. 620°
22. A 50-kHz sine wave has an amplitude of 150 V. Sketch the waveform with its axis scaled in microseconds.
23. If the period of the waveform in Figure 15–82 is 180 ms, compute current at $t = 30$, 75, 140, and 315 ms.
24. A sinusoidal waveform has a period of 60 $\mu$s and $V_m = 80$ V. Sketch the waveform. What is its voltage at 4 $\mu$s?

25. A 20-kHz sine wave has a value of 50 volts at $t = 5\ \mu s$. Determine $V_m$ and sketch the waveform.

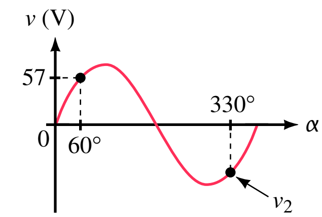



FIGURE 15–83

26. For the waveform of Figure 15–83, determine $v_2$.

## 15.6 Voltages and Currents as Functions of Time

27. Calculate $\omega$ in radians per second for each of the following:
   a. $T = 100$ ns
   b. $f = 30$ Hz
   c. 100 cycles in 4 s
   d. period = 20 ms
   e. 5 periods in 20 ms
28. For each of the following values of $\omega$, compute $f$ and $T$:
   a. 100 rad/s
   b. 40 rad in 20 ms
   c. $34 \times 10^3$ rad/s
29. Determine equations for sine waves with the following:
   a. $V_m = 170$ V, $f = 60$ Hz
   b. $I_m = 40\ \mu A$, $T = 10$ ms
   c. $T = 120\ \mu s$, $v = 10$ V at $t = 12\ \mu s$
30. Determine $f$, $T$, and amplitude for each of the following:
   a. $v = 75 \sin 200\pi t$
   b. $i = 8 \sin 300t$
31. A sine wave has a peak-to-peak voltage of 40 V and $T = 50$ ms. Determine its equation.
32. Sketch the following waveforms with the horizontal axis scaled in degrees, radians, and seconds:
   a. $v = 100 \sin 200\pi t$ V
   b. $i = 90 \sin \omega t$ mA, $T = 80\ \mu s$
33. Given $i = 47 \sin 8260t$ mA, determine current at $t = 0$ s, 80 $\mu$s, 410 $\mu$s, and 1200 $\mu$s.
34. Given $v = 100 \sin \alpha$. Sketch one cycle.
   a. Determine at which two angles $v = 86.6$ V.
   b. If $\omega = 100\pi/60$ rad/s, at which times do these occur?
35. Write equations for the waveforms of Figure 15–84. Express the phase angle in degrees.

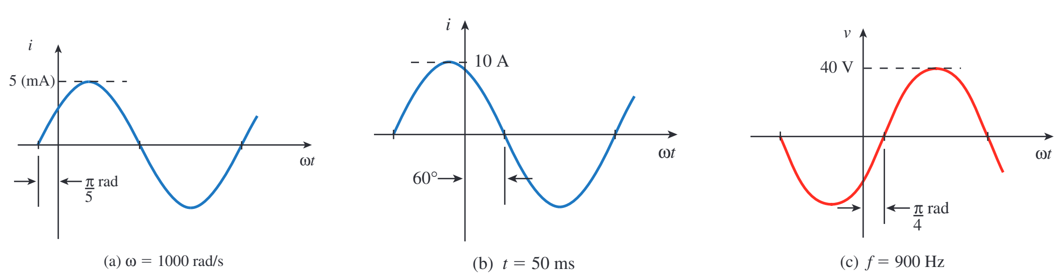
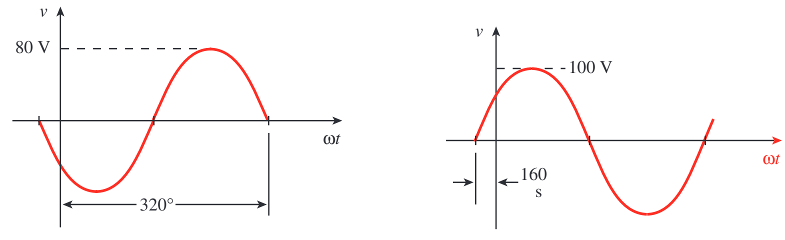



FIGURE 15–84

36. Sketch the following waveforms with the horizontal axis scaled in degrees and seconds:
   a. $v = 100 \sin(232.7t + 40°)$ V
   b. $i = 20 \sin(\omega t - 60°)$ mA, $f = 200$ Hz
37. Given $v = 5 \sin(\omega t + 45°)$. If $\omega = 20\pi$ rad/s, what is $v$ at $t = 20$, 75, and 90 ms?

38. Repeat Problem 35 for the waveforms of Figure 15–85.




(a) Period = 10 s (b) $f$ = 833.3 Hz

**FIGURE 15–85**

39. Determine the equation for the waveform shown in Figure 15–86.

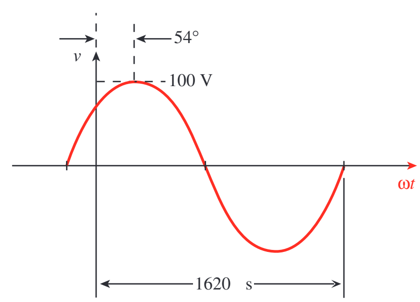



**FIGURE 15–86**

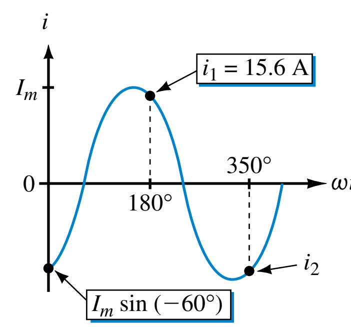


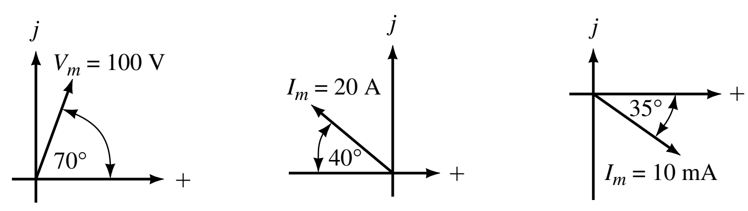

**FIGURE 15–87**

40. For the waveform of Figure 15–87, determine $i_2$.

41. Given $v = 30 \sin(\omega t - 45°)$ where $\omega = 40\pi$ rad/s. Sketch the waveform. At what time does $v$ reach 0 V? At what time does it reach 23 V and −23 V?

## 15.7 Introduction to Phasors

42. For each of the phasors of Figure 15–88, determine the equation for $v(t)$ *or i(t) as applicable, and sketch the waveform.*




(a) (b) (c)

**FIGURE 15–88**

43. With the aid of phasors, sketch the waveforms for each of the following pairs and determine the phase difference and which waveform leads:

a. $v = 100 \sin \omega t$
   $i = 80 \sin(\omega t + 20°)$

b. $v_1 = 200 \sin(\omega t - 30°)$
   $v_2 = 150 \sin(\omega t - 30°)$

c. $i_1 = 40 \sin(\omega t + 30°)$
   $i_2 = 50 \sin(\omega t - 20°)$

d. $v = 100 \sin(\omega t + 140°)$
   $i = 80 \sin(\omega t - 160°)$

44. Repeat Problem 43 for the following.

a. $i = 40 \sin(\omega t + 80°)$
   $v = -30 \sin(\omega t - 70°)$

b. $v = 20 \cos(\omega t + 10°)$
   $i = 15 \sin(\omega t - 10°)$

c. $v = 20 \cos(\omega t + 10°)$
   $i = 15 \sin(\omega t + 120°)$

d. $v = 80 \cos(\omega t + 30°)$
   $i = 10 \cos(\omega t - 15°)$

45. For the waveforms in Figure 15–89, determine the phase differences. Which waveform leads?

46. Draw phasors for the waveforms of Figure 15–89.

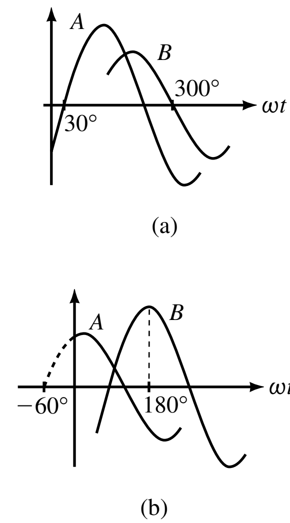


FIGURE 15–89

### 15.8 AC Waveforms and Average Value

47. What is the average value of each of the following over an integral number of cycles?

a. $i = 5 \sin \omega t$

b. $i = 40 \cos \omega t$

c. $v = 400 \sin(\omega t + 30°)$

d. $v = 20 \cos 2\omega t$

48. Using Equation 15–20, compute the area under the half-cycle of Figure 15–54 using increments of $\pi/12$ rad.

49. Compute $I_{avg}$ for the waveforms of Figure 15–90.

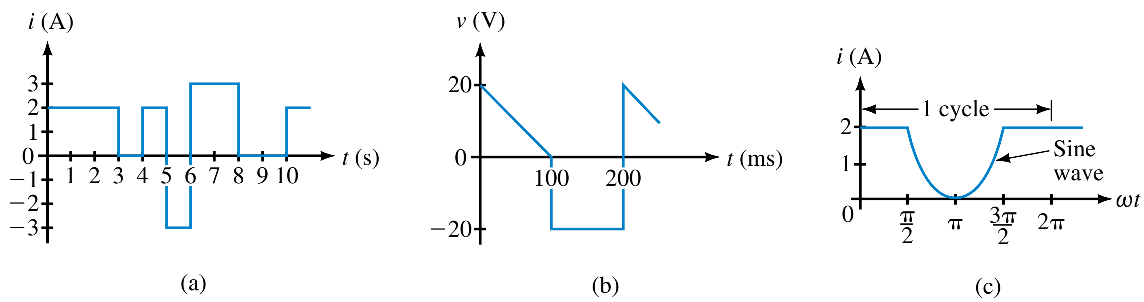


FIGURE 15–90

50. For the waveform of Figure 15–91, compute $I_m$.

51. For the circuit of Figure 15–92, $e = 25 \sin \omega t$ V and period $T = 120$ ms.

a. Sketch voltage $v(t)$ with the axis scaled in milliseconds.

b. Determine the peak and minimum voltages.

c. Compute $v$ at $t = 10, 20, 70$, and $100$ ms.

d. Determine $V_{avg}$.

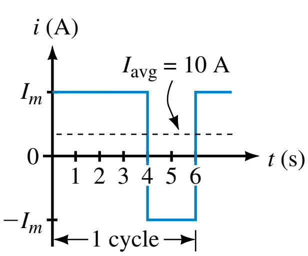


FIGURE 15–91

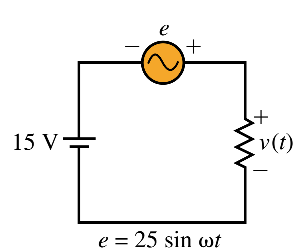


FIGURE 15–92

52. Using numerical methods for the curved part of the waveform (with increment size $\Delta t = 0.25$ s), determine the area and the average value for the waveform of Figure 15–93.

53. Using calculus, find the average value for Figure 15–93.

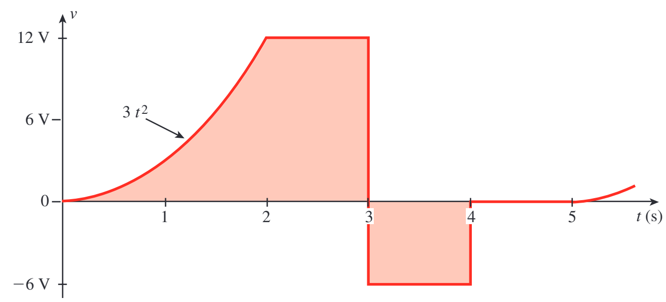



**FIGURE 15–93**

## 15.9 Effective Values

54. Determine the effective values of each of the following:
    a. $v = 100 \sin \omega t$ V  b. $i = 8 \sin 377t$ A
    c. $v = 40 \sin(\omega t + 40°)$ V  d. $i = 120 \cos \omega t$ mA

55. Determine the rms values of each for the following.
    a. A 12 V battery  b. $-24 \sin(\omega t + 73°)$ mA
    c. $10 + 24 \sin \omega t$ V  d. $45 - 27 \cos 2\omega t$ V

56. For a sine wave, $V_{eff} = 9$ V. What is its amplitude?

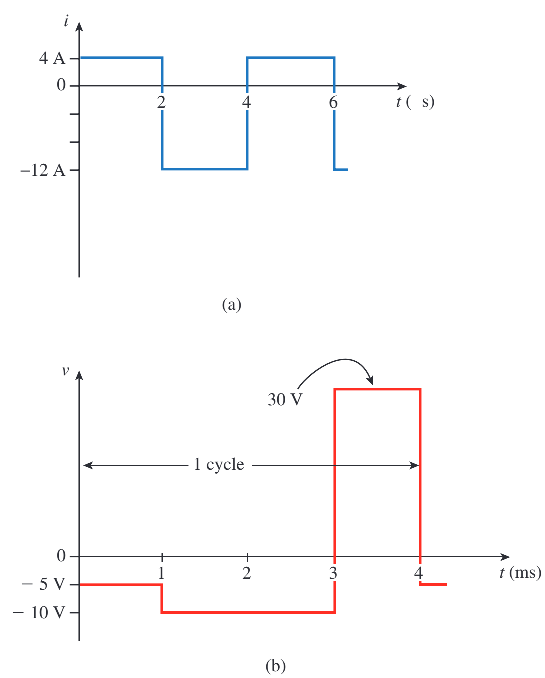



**FIGURE 15–94**

57. Determine the root mean square values for
    a. $i = 3 + \sqrt{2}(4)\sin(\omega t + 44°)$ mA
    b. Voltage $v$ of Figure 15–92 with $e = 25 \sin \omega t$ V
58. Compute the rms values for Figures 15–90(a), and 15–91. For Figure 15–91, $I_m = 30$ A.
59. Compute the rms values for the waveforms of Figure 15–94.
60. Compute the effective value for Figure 15–95.

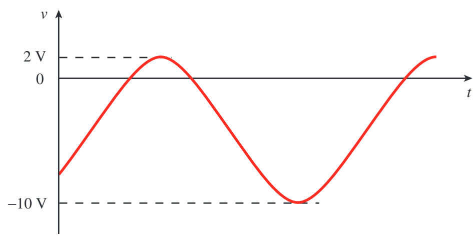



FIGURE 15–95

61. Determine the rms value of the waveform of Figure 15–96. Why is it the same as that of a 24-V battery?

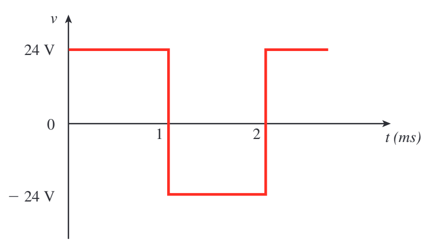



FIGURE 15–96

62. Compute the rms value of the waveform of Figure 15–52(c). To handle the triangular portion, use Equation 15–20. Use a time interval $\Delta t = 1$s.
63. ∫ Repeat Problem 62, using calculus to handle the triangular portion.

### 15.11 AC Voltage and Current Measurement

64. Determine the reading of an average responding AC meter for each of the following cases. (Note: Meaningless is a valid answer if applicable.) Assume the frequency is within the range of the instrument.
    a. $v = 153 \sin \omega t$ V
    b. $v = \sqrt{2}(120)\sin(\omega t + 30°)$ V
    c. The waveform of Figure 15–61
    d. $v = 597 \cos \omega t$ V
65. Repeat Problem 64 using a true rms meter.

### 15.12 Circuit Analysis Using Computers

Use Electronics Workbench or PSpice for the following.

66. **EWB** **PSpice** Plot the waveform of Problem 37 and, using the cursor, determine voltage at the times indicated. Don't forget to convert the frequency to Hz.

67. **EWB** **PSpice** Plot the waveform of Problem 41. Using the cursor, determine the time at which $v$ reaches 0 V. Don't forget to convert the frequency to Hz.

68. **EWB** **PSpice** Assume the equations of Problem 43 all represent voltages. For each case, plot the waveforms, then use the cursor to determine the phase difference between waveforms.

## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

1. 16.7 ms
2. Frequency doubles, period halves
3. 50 Hz; 20 ms
4. 20 V; 0.5 ms and 2.5 ms; −35 V: 4 ms and 5 ms
5. (c) and (d); Since current is directly proportional to the voltage, it will have the same waveshape.
6. 250 Hz
7. $f_1 = 100$ Hz; $f_2 = 33.3$ Hz
8. 50 kHz and 1 MHz
9. 22.5 Hz
10. At 12 ms, direction →; at 37 ms, direction ←; at 60 ms, →
11. At 75 ms, $i = -5$ A

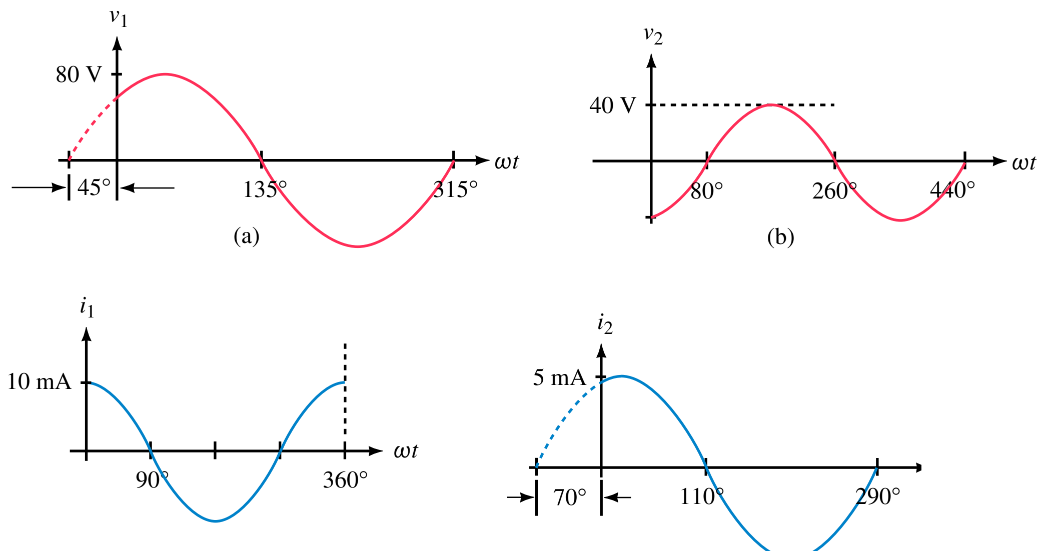
**In-Process Learning Check 2**

1.

| $\alpha$ (deg) | 0 | 45 | 90 | 135 | 180 | 225 | 270 | 315 | 360 |
|---|---|---|---|---|---|---|---|---|---|
| $i$ (mA) | 0 | 10.6 | 15 | 10.6 | 0 | −10.6 | −15 | −10.6 | 0 |

2. a. 0.349 b. 0.873
   c. 2.09 d. 4.36
3. 30°; 120°; 450°
4. Same as Figure 15–27 with $T = 10$ s and amplitude = 50 mA.
5. $1.508 \times 10^6$ rad/s
6. $i = 8 \sin 157t$ A
7. $i = 6 \sin 69.81t$ A
8. a. $i = 250 \sin(251t - 30°)$ A
   b. $i = 20 \sin(62.8t + 45°)$ A
   c. $v = 40 \sin(628t - 30°)$ V
   d. $v = 80 \sin(314 \times 10^3 t + 36°)$ V
9. a. 2.95 ms; 7.05 ms; 22.95 ms; 27.05 ms
   b. 11.67 ms; 18.33 ms; 31.67 ms; 38.33 ms

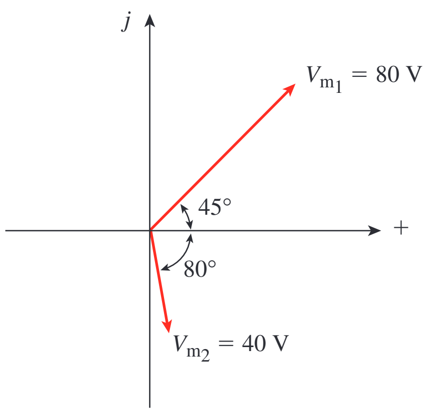
10.



(a) (b) (c) (d)

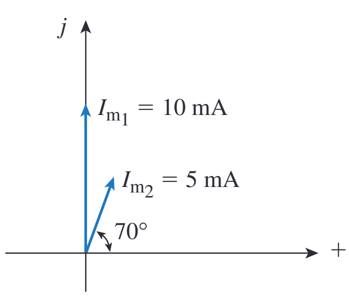
11. a. 2.25 s b. 1.33 s

12. a.




b. 125° c. $v_1$ leads

13. a.




b. 20° c. $i_1$ leads

# 16

# *R, L,* and *C* Elements and the Impedance Concept

## OBJECTIVES

After studying this chapter, you will be able to

- express complex numbers in rectangular and polar forms,
- represent ac voltage and current phasors as complex numbers,
- represent voltage and current sources in transformed form,
- add and subtract currents and voltages using phasors,
- compute inductive and capacitive reactance,
- determine voltages and currents in simple ac circuits,
- explain the impedance concept,
- determine impedance for *R, L,* and *C* circuit elements,
- determine voltages and currents in simple ac circuits using the impedance concept,
- use Electronics Workbench and PSpice to solve simple ac circuit problems.

## KEY TERMS

Capacitive Reactance
Complex Number
Impedance
Inductive Reactance
$j = \sqrt{-1}$
Phasor Domain
Polar Form
Rectangular Form
Time Domain

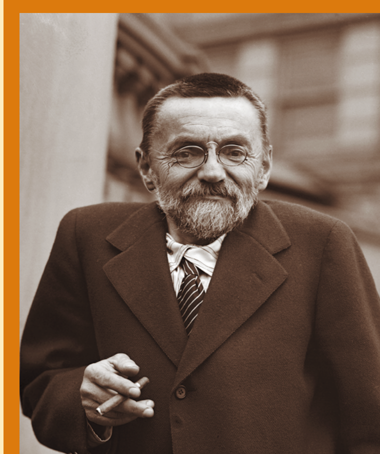
## OUTLINE

## CHAPTER PREVIEW

In Chapter 15, you learned how to analyze a few simple ac circuits in the time domain using voltages and currents expressed as functions of time. However, this is not a very practical approach. A more practical approach is to represent ac voltages and currents as phasors, circuit elements as impedances, and analyze circuits in the phasor domain using complex algebra. With this approach, ac circuit analysis is handled much like dc circuit analysis, and all basic relationships and theorems—Ohm's law, Kirchhoff's laws, mesh and nodal analysis, superposition and so on—apply. The major difference is that ac quantities are complex rather than real as with dc. While this complicates computational details, it does not alter basic circuit principles. This is the approach used in practice. The basic ideas are developed in this chapter.

Since phasor analysis and the impedance concept require a familiarity with complex numbers, we begin with a short review.

## PUTTING IT IN PERSPECTIVE



### *Charles Proteus Steinmetz*

CHARLES STEINMETZ WAS BORN IN Breslau, Germany in 1865 and emigrated to the United States in 1889. In 1892, he began working for the General Electric Company in Schenectady, New York, where he stayed until his death in 1923, and it was there that his work revolutionized ac circuit analysis. Prior to his time, this analysis had to be carried out using calculus, a difficult and time-consuming process. By 1893, however, Steinmetz had reduced the very complex alternating-current theory to, in his words, "a simple problem in algebra." The key concept in this simplification was the phasor—a representation based on complex numbers. By representing voltages and currents as phasors, Steinmetz was able to define a quantity called **impedance** and then use it to determine voltage and current magnitude and phase relationships in one algebraic operation.

Steinmetz wrote the seminal textbook on ac analysis based on his method, but at the time he introduced it he was practically the only person who understood it. Now, however, it is common knowledge and one of the basic tools of the electrical engineer and technologist. In this chapter, we learn the method and illustrate its application to the solution of basic ac circuit problems.

In addition to his work for GE, Charles Steinmetz was a professor of electrical engineering (1902–1913) and electrophysics (1913–1923) at Union University (now Union College) in Schenectady.

## 16.1 Complex Number Review

A **complex number** is a number of the form $\mathbf{C} = a + jb$, where $a$ and $b$ are real numbers and $j = \sqrt{-1}$. The number $a$ is called the **real** part of **C** and $b$ is called its **imaginary** part. (In circuit theory, $j$ is used to denote the imaginary component rather than $i$ to avoid confusion with current $i$.)

### Geometrical Representation

Complex numbers may be represented geometrically, either in rectangular form or in polar form as points on a two-dimensional plane called the **complex plane** (Figure 16–1). The complex number $\mathbf{C} = 6 + j8$, for example, represents a point whose coordinate on the real axis is 6 and whose coordinate on the imaginary axis is 8. This form of representation is called the **rectangular form.**

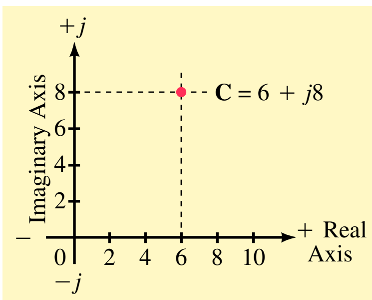



**FIGURE 16–1** A complex number in rectangular form.

Complex numbers may also be represented in **polar form** by magnitude and angle. Thus, $\mathbf{C} = 10\angle 53.13°$ (Figure 16–2) is a complex number with magnitude 10 and angle 53.13°. This magnitude and angle representation is just an alternate way of specifying the location of the point represented by $\mathbf{C} = a + jb$.

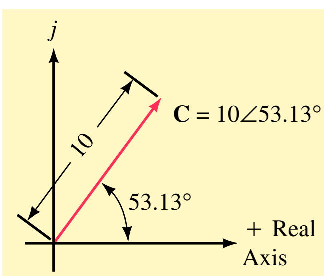



**FIGURE 16–2** A complex number in polar form.

### Conversion between Rectangular and Polar Forms

To convert between forms, note from Figure 16–3 that

$$\mathbf{C} = a + jb \quad \text{(rectangular form)} \tag{16–1}$$

$$\mathbf{C} = C\angle\theta \quad \text{(polar form)} \tag{16–2}$$

where $C$ is the magnitude of **C.** From the geometry of the triangle,

$$a = C\cos\theta \tag{16–3a}$$

$$b = C\sin\theta \tag{16–3b}$$

where

$$C = \sqrt{a^2 + b^2} \tag{16–4a}$$

and

$$\theta = \tan^{-1}\frac{b}{a} \tag{16–4b}$$

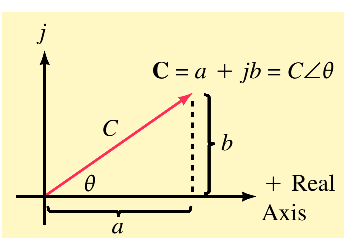
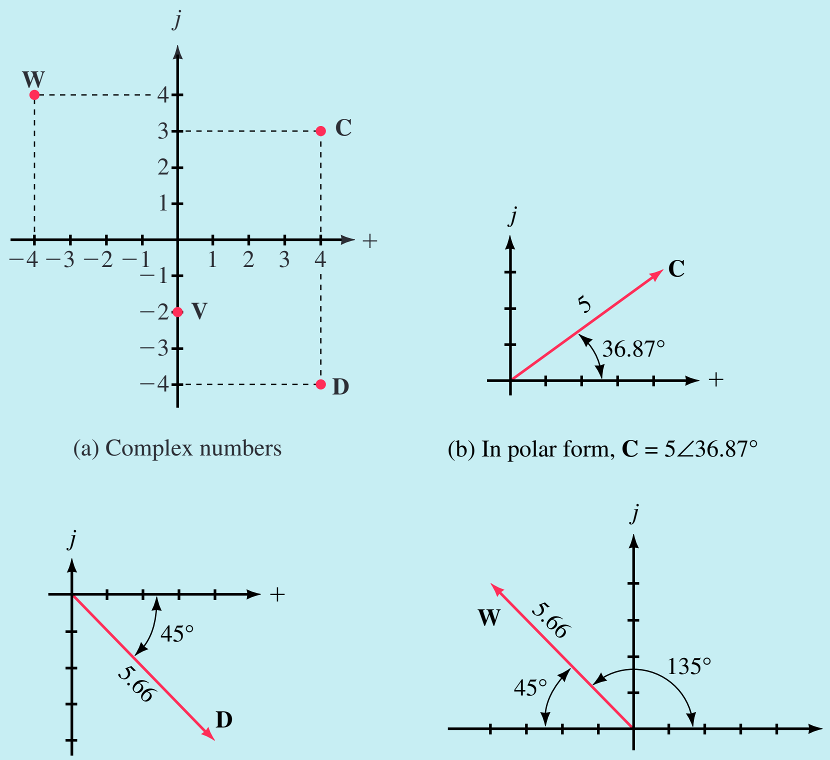



**FIGURE 16–3** Polar and rectangular equivalence.

Equations 16–3 and 16–4 permit conversion between forms. When using Equation 16–4b, however, be careful when the number to be converted is in the second or third quadrant, as the angle obtained is the supplementary angle rather than the actual angle in these two quadrants. This is illustrated in Example 16–1 for the complex number **W.**

**EXAMPLE 16–1** Determine rectangular and polar forms for the complex numbers **C, D, V,** and **W** of Figure 16–4(a)




(a) Complex numbers

(b) In polar form, **C** = 5∠36.87°

(c) In polar form, **D** = 5.66∠−45°

(d) In polar form, **W** = 5.66∠135°

**FIGURE 16–4**

**Solution**

*Point C:* Real part = 4; imaginary part = 3. Thus, **C** = 4 + $j3$. In polar form, $C = \sqrt{4^2 + 3^2} = 5$ and $\theta_C = \tan^{-1}(3/4) = 36.87°$. Thus, **C** = 5∠36.87° as indicated in (b).

*Point D:* In rectangular form, **D** = 4 − $j4$. Thus, $D = \sqrt{4^2 + 4^2} = 5.66$ and $\theta_D = \tan^{-1}(-4/4) = -45°$. Therefore, **D** = 5.66∠−45°, as shown in (c).

*Point V:* In rectangular form, **V** = −$j2$. In polar form, **V** = 2∠−90°.

*Point W:* In rectangular form, **W** = −4 + $j4$. Thus, $W = \sqrt{4^2 + 4^2} = 5.66$ and $\tan^{-1}(-4/4) = -45°$. Inspection of Figure 16–4(d) shows, however, that this 45° angle is the supplementary angle. The actual angle (measured from the positive horizontal axis) is 135°. Thus, **W** = 5.66∠135°.

In practice (because of the large amount of complex number work that you will do), a more efficient conversion process is needed than that described previously. As discussed later in this section, inexpensive calculators are available that perform such conversions directly—you simply enter the complex number components and press the conversion key. With these, the problem of determining angles for numbers such as **W** in Example 16–1 does not occur; you just enter $-4 + j4$ and the calculator returns $5.66\angle 135°$.

## Powers of *j*

Powers of *j* are frequently required in calculations. Here are some useful powers:

$$j^2 = (\sqrt{-1})(\sqrt{-1}) = -1$$

$$j^3 = j^2 j = -j$$

$$j^4 = j^2 j^2 = (-1)(-1) = 1 \tag{16–5}$$

$$(-j)j = 1$$

$$\frac{1}{j} = \frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = -j$$

## Addition and Subtraction of Complex Numbers

Addition and subtraction of complex numbers can be performed analytically or graphically. Analytic addition and subtraction is most easily illustrated in rectangular form, while graphical addition and subtraction is best illustrated in polar form. For analytic addition, add real and imaginary parts separately. Similarly for subtraction. For graphical addition, add vectorially as in Figure 16–5(a); for subtraction, change the sign of the subtrahend, then add, as in Figure 16–5(b).

**EXAMPLE 16–2** Given $\mathbf{A} = 2 + j1$ and $\mathbf{B} = 1 + j3$. Determine their sum and difference analytically and graphically.

**Solution**

$$\mathbf{A} + \mathbf{B} = (2 + j1) + (1 + j3) = (2 + 1) + j(1 + 3) = 3 + j4.$$

$$\mathbf{A} - \mathbf{B} = (2 + j1) - (1 + j3) = (2 - 1) + j(1 - 3) = 1 - j2.$$

Graphical addition and subtraction are shown in Figure 16–5.

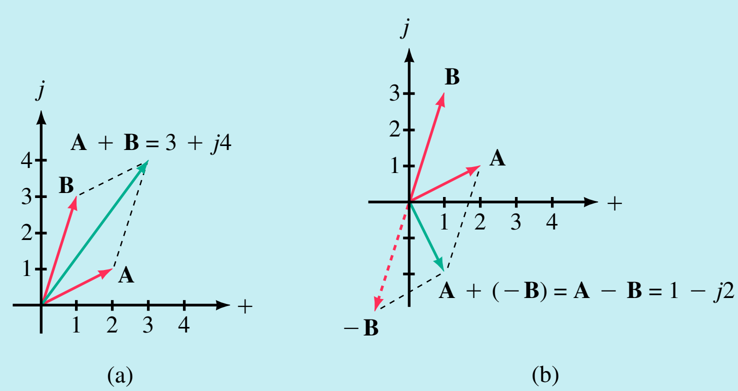



**FIGURE 16–5**

### Multiplication and Division of Complex Numbers

These operations are usually performed in polar form. For multiplication, multiply magnitudes and add angles algebraically. For division, divide the magnitude of the denominator into the magnitude of the numerator, then subtract algebraically the angle of the denominator from that of the numerator. Thus, given $\mathbf{A} = A\angle\theta_A$ and $\mathbf{B} = B\angle\theta_B$,

$$\mathbf{A} \cdot \mathbf{B} = AB\angle\theta_A + \theta_B \tag{16–6}$$

$$\mathbf{A/B} = A/B\angle\theta_A - \theta_B \tag{16–7}$$

**EXAMPLE 16–3** Given $\mathbf{A} = 3\angle 35°$ and $\mathbf{B} = 2\angle -20°$, determine the product $\mathbf{A} \cdot \mathbf{B}$ and the quotient $\mathbf{A/B}$.

**Solution**

$$\mathbf{A} \cdot \mathbf{B} = (3\angle 35°)(2\angle -20°) = (3)(2)\angle 35° - 20° = 6\angle 15°$$

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(3\angle 35°)}{(2\angle -20°)} = \frac{3}{2}\angle 35° - (-20°) = 1.5\angle 55°$$

**EXAMPLE 16–4** For computations involving purely real, purely imaginary, or small integer numbers, it is sometimes easier to multiply directly in rectangular form than it is to convert to polar. Compute the following directly:

a. $(-j3)(2 + j4)$.

b. $(2 + j3)(1 + j5)$.

**Solution**

a. $(-j3)(2 + j4) = (-j3)(2) + (-j3)(j4) = -j6 - j^2 12 = 12 - j6$

b. $(2 + j3)(1 + j5) = (2)(1) + (2)(j5) + (j3)(1) + (j3)(j5)$
$= 2 + j10 + j3 + j^2 15 = 2 + j13 - 15 = -13 + j13$

**PRACTICE PROBLEMS 1**

1. Polar numbers with the same angle can be added or subtracted directly without conversion to rectangular form. For example, the sum of $6\angle 36.87°$ and $4\angle 36.87°$ is $10\angle 36.87°$, while the difference is $6\angle 36.87° - 4\angle 36.87° = 2\angle 36.87°$. By means of sketches, indicate why this procedure is valid.
2. To compare methods of multiplication with small integer values, convert the numbers of Example 16–4 to polar form, multiply them, then convert the answers back to rectangular form.

*Answers:*

1. Since the numbers have the same angle, their sum also has the same angle and thus, their magnitudes simply add (or subtract).

### Reciprocals

The reciprocal of a complex number $\mathbf{C} = C\angle\theta$ is

$$\frac{1}{\mathbf{C}\angle\theta} = \frac{1}{C}\angle-\theta \qquad \textbf{(16–8)}$$

Thus,

$$\frac{1}{20\angle 30^\circ} = 0.05\angle-30^\circ$$

### Complex Conjugates

The **conjugate** of a complex number (denoted by an asterisk *) is a complex number with the same real part but the opposite imaginary part. Thus, the conjugate of $\mathbf{C} = C\angle\theta = a + jb$ is $\mathbf{C}^* = C\angle-\theta = a - jb$. For example, if $\mathbf{C} = 3 + j4 = 5\angle 53.13^\circ$, then $\mathbf{C}^* = 3 - j4 = 5\angle-53.13^\circ$.

### Calculators for AC Analysis


**FIGURE 16–6** This calculator displays complex numbers in standard mathematical notation.

The analysis of ac circuits involves a considerable amount of complex number arithmetic; thus, you will need a calculator that can work easily with complex numbers. There are several inexpensive calculators on the market that are suitable for this purpose in that they can perform all required calculations (addition, subtraction, multiplication, and division) in either rectangular or polar form without the need for conversion. This is important, because it saves you a great deal of time and cuts down on errors. To illustrate, consider Example 16–5. Using a calculator with only basic complex number conversion capabilities requires that you convert between forms as illustrated. On the other hand, a calculator with more sophisticated complex number capabilities (such as that shown in Figure 16–6) allows you to perform the calculation without going through all the intermediate conversion steps. You need to acquire an appropriate calculator and learn to use it proficiently.

**EXAMPLE 16–5** The following illustrates the type of calculations that you will encounter. Use your calculator to reduce the following to rectangular form:

$$(6 + j5) + \frac{(3 - j4)(10\angle 40^\circ)}{6 + 30\angle 53.13^\circ}$$

**Solution** Using a calculator with basic capabilities requires a number of intermediate steps, some of which are shown below.

$$\begin{aligned}\text{answer} &= (6 + j5) + \frac{(5\angle-53.13)(10\angle 40)}{6 + (18 + j24)} \\ &= (6 + j5) + \frac{(5\angle-53.13)(10\angle 40)}{24 + j24}\end{aligned}$$

$$= (6 + j5) + \frac{(5\angle -53.13)(10\angle 40)}{33.94\angle 45}$$
$$= (6 + j5) + 1.473\angle -58.13 = (6 + j5) + (0.778 - j1.251)$$
$$= 6.778 + j3.749$$

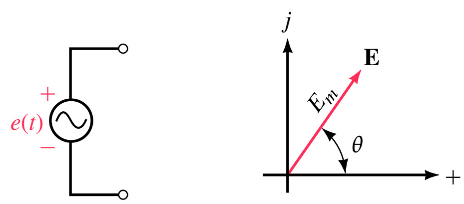
Using a calculator such as that shown in Figure 16–6 saves steps, as it multiplies $(3 - j4)(10\angle 40°)$ and adds $6 + 30\angle 53.13°$, etc., directly, without your having to convert forms.

## 16.2 Complex Numbers in AC Analysis

### Representing AC Voltages and Currents by Complex Numbers

As you learned in Chapter 15, ac voltages and currents can be represented as phasors. Since phasors have magnitude and angle, they can be viewed as complex numbers. To get at the idea, consider the voltage source of Figure 16–7(a). Its phasor equivalent (b) has magnitude $E_m$ and angle $\theta$. It therefore can be viewed as the complex number

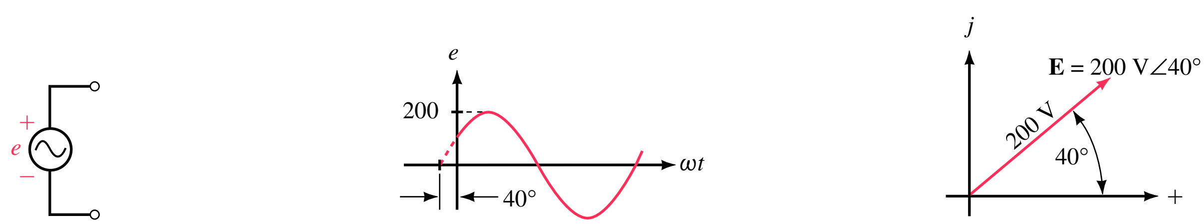
$$\mathbf{E} = E_m\angle\theta \qquad \textbf{(16–9)}$$




(a) $e(t) = E_m \sin(\omega t + \theta)$ (b) $\mathbf{E} = E_m\angle\theta$

**FIGURE 16–7** Representation of a sinusoidal source voltage as a complex number.

From this point of view, the sinusoidal voltage $e(t) = 200 \sin(\omega t + 40°)$ of Figure 16–8(a) and (b) can be represented by its phasor equivalent, $\mathbf{E} = 200\text{ V}\angle 40°$, as in (c).




(a) $e = 200 \sin(\omega t + 40°)$ V (b) Waveform (c) Phasor equivalent

**FIGURE 16–8** Transforming $e = 200 \sin(\omega t + 40°)$ V to $\mathbf{E} = 200\text{ V}\angle 40°$.

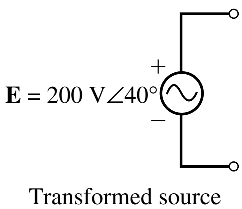



**FIGURE 16–9** Direct transformation of the source.

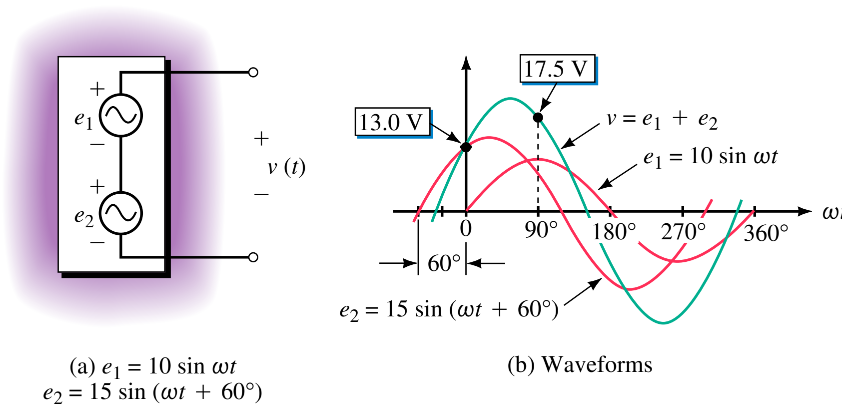
We can take advantage of this equivalence. *Rather than show a source as a time-varying voltage e(t) that we subsequently convert to a phasor, we can represent the source by its phasor equivalent right from the start.* This viewpoint is illustrated in Figure 16–9. Since $\mathbf{E} = 200\text{ V}\angle 40°$, this representation retains all the original information of Figure 16–8 with the exception of the sinusoidal time variation. However, since the sinusoidal waveform and time variation is implicit in the definition of a phasor, you can easily restore this information if it is needed.

The idea illustrated in Figure 16–9 is of fundamental importance to circuit theory. *By replacing the time function e(t) with its phasor equivalent* **E,** *we have transformed the source from the time domain to the phasor domain.* The value of this approach is illustrated next.

Before we move on, we should note that both Kirchhoff's voltage law and Kirchhoff's current law apply in the time domain (i.e., when voltages and currents are expressed as functions of time) and in the phasor domain (i.e., when voltages and currents are represented as phasors). For example, $e = v_1 + v_2$ in the time domain can be transformed to $\mathbf{E} = \mathbf{V}_1 + \mathbf{V}_2$ in the phasor domain and vice versa. Similarly for currents.

## Summing AC Voltages and Currents

Sinusoidal quantites must sometimes be added or subtracted as in Figure 16–10. Here, we want the sum of $e_1$ and $e_2$, where $e_1 = 10 \sin \omega t$ and $e_2 = 15 \sin(\omega t + 60°)$. The sum of $e_1$ and $e_2$ can be found by adding waveforms point by point as in (b). For example, at $\omega t = 0°$, $e_1 = 10 \sin 0° = 0$ and $e_2 = 15 \sin(0° + 60°) = 13$ V, and their sum is 13 V. Similarly, at $\omega t = 90°$, $e_1 = 10 \sin 90° = 10$ V and $e_2 = 15 \sin(90° + 60°) = 15 \sin 150° = 7.5$, and their sum is 17.5 V. Continuing in this manner, the sum of $e_1 + e_2$ (the green waveform) is obtained.



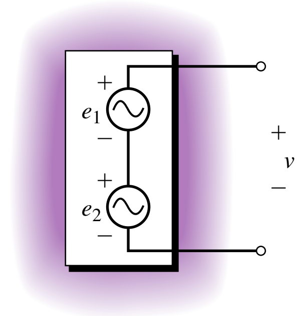

(a) $e_1 = 10 \sin \omega t$
$e_2 = 15 \sin (\omega t + 60°)$

(b) Waveforms

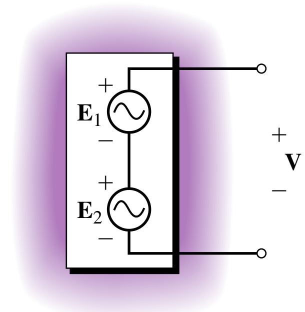
**FIGURE 16–10** Summing waveforms point by point.

As you can see, the process is tedious and provides no analytic expression for the resulting voltage. A better way is to transform the sources and use complex numbers to perform the addition. This is shown in Figure 16–11.

Here, we have replaced voltages $e_1$ and $e_2$ with their phasor equivalents, $\mathbf{E}_1$ and $\mathbf{E}_2$, and $v$ with its phasor equivalent, **V.** Since $v = e_1 + e_2$, replacing $v$, $e_1$, and $e_2$ with their phasor equivalents yields $\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2$. Now **V** can be found by adding $\mathbf{E}_1$ and $\mathbf{E}_2$ as complex numbers. Once **V** is known, its corresponding time equation and companion waveform can be determined.




(a) Original network.
$v(t) = e_1(t) + e_2(t)$




(b) Transformed network.
$\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2$

**FIGURE 16–11** Transformed circuit. This is one of the key ideas of sinusoidal circuit analysis.

**EXAMPLE 16–6** Given $e_1 = 10 \sin \omega t$ V and $e_2 = 15 \sin(\omega t + 60°)$ V as before, determine $v$ and sketch it.

**Solution** $e_1 = 10 \sin \omega t$ V. Thus, $\mathbf{E}_1 = 10\text{ V}\angle 0°$.

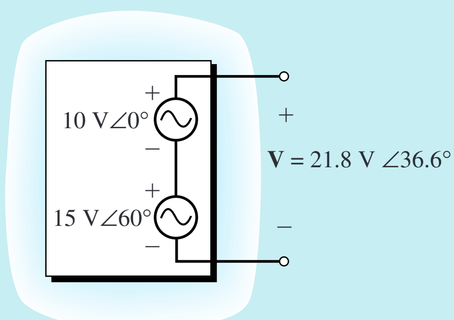
$e_2 = 15 \sin(\omega t + 60°)$ V. Thus, $\mathbf{E}_2 = 15\text{ V}\angle 60°$.

Transformed sources are shown in Figure 16–12(a) and phasors in (b).

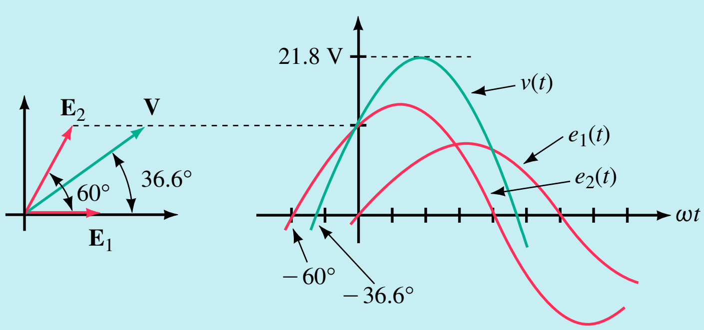
$$\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2 = 10\angle 0° + 15\angle 60° = (10 + j0) + (7.5 + j13)$$
$$= (17.5 + j13) = 21.8\text{ V}\angle 36.6°$$

Thus, $v = 21.8 \sin(\omega t + 36.6°)$ V

Waveforms are shown in (c).




(a) Phasor summation




(b) Phasors (c) Waveforms

**FIGURE 16–12**

PRACTICE PROBLEMS 2

Verify by direct substitution that $v = 21.8 \sin(\omega t + 36.6°)$ V, as in Figure 16–12, is the sum of $e_1$ and $e_2$. To do this, compute $e_1$ and $e_2$ at a point, add them, then compare the sum to $21.8 \sin(\omega t + 36.6°)$ V computed at the same point. Perform this computation at $\omega t = 30°$ intervals over the complete cycle to satisfy yourself that the result is true everywhere. (For example, at $\omega t = 0°$, $v = 21.8 \sin(\omega t + 36.6°) = 21.8 \sin(36.6°) = 13$ V, as we saw earlier in Figure 16–10.)

*Answer:* Here are the points on the graph at 30° intervals:

| $\omega t$ | 0° | 30° | 60° | 90° | 120° | 150° | 180° | 210° | 240° | 270° | 300° | 330° | 360° |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 13 | 20 | 21.7 | 17.5 | 8.66 | −2.5 | −13 | −20 | −21.7 | −17.5 | −8.66 | 2.5 | 13 |

IMPORTANT NOTES...

1. To this point, we have used peak values such as $V_m$ and $I_m$ to represent the magnitudes of phasor voltages and currents, as this has been most convenient for our purposes. In practice, however, rms values are used instead. Accordingly, we will now change to rms. Thus, from here on, the phasor $\mathbf{V} = 120\text{ V} \angle 0°$ will be taken to mean a voltage of 120 volts rms at an angle of 0°. If you need to convert this to a time function, first multiply the rms value by $\sqrt{2}$, then follow the usual procedure. Thus, $v = \sqrt{2}(120) \sin \omega t = 170 \sin \omega t$.
2. To add or subtract sinusoidal voltages or currents, follow the three steps outlined in Example 16–6. That is,
   - convert sine waves to phasors and express them in complex number form,
   - add or subtract the complex numbers,
   - convert back to time functions if desired.
3. Although we use phasors to represent sinusoidal waveforms, it should be noted that sine waves and phasors are not the same thing. Sinusoidal voltages and currents are real—they are the actual quantities that you measure with meters and whose waveforms you see on oscilloscopes. *Phasors, on the other hand, are mathematical abstractions that we use to help visualize relationships and solve problems.*
4. Quantities expressed as time functions are said to be in the **time domain,** while quantities expressed as phasors are said to be in the **phasor (or frequency) domain.** Thus, $e = 170 \sin \omega t$ V is in the time domain, while $\mathbf{V} = 120\text{ V} \angle 0°$ is in the phasor domain.

**EXAMPLE 16–7** Express the voltages and currents of Figure 16–13 in both the time and the phasor domains.

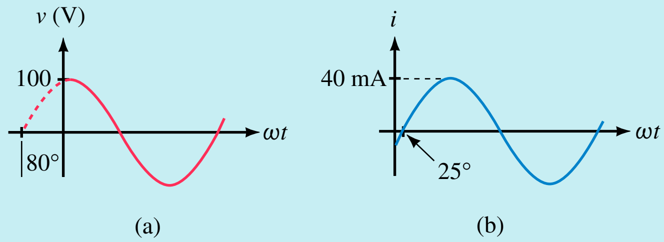



**FIGURE 16–13**

**Solution**

a. Time domain: $v = 100 \sin(\omega t + 80°)$ volts.
   Phasor domain: $\mathbf{V} = (0.707)(100\text{ V}\angle 80°) = 70.7\text{ V}\angle 80°$.

b. Time domain: $i = 40 \sin(\omega t - 25°)$ mA.
   Phasor domain: $\mathbf{I} = (0.707)(40\text{ mA}\angle -25°) = 28.3\text{ mA}\angle -25°$.

**EXAMPLE 16–8** If $i_1 = 14.14 \sin(\omega t - 55°)$ A and $i_2 = 4 \sin(\omega t + 15°)$ A, determine their sum, $i$. Work with rms values.

**Solution**

$$\mathbf{I}_1 = (0.707)(14.14\text{ A})\angle -55° = 10\text{ A}\angle -55°$$
$$\mathbf{I}_2 = (0.707)(4\text{ A})\angle 15° = 2.828\text{ A}\angle 15°$$
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10\text{ A}\angle -55° + 2.828\text{ A}\angle 15°$$
$$= (5.74\text{ A} - j8.19\text{ A}) + (2.73\text{ A} + j0.732\text{ A})$$
$$= 8.47\text{ A} - j7.46\text{ A} = 11.3\text{ A}\angle -41.4$$
$$i(t) = \sqrt{2}(11.3)\sin(\omega t - 41.4°) = 16 \sin(\omega t - 41.4°)\text{ A}$$

While it may seem silly to convert peak values to rms and then convert rms back to peak as we did here, we did it for a reason. The reason is that very soon, we will stop working in the time domain entirely and work only with phasors. At that point, the solution will be complete when we have the answer in the form $\mathbf{I} = 11.3\angle -41.4°$. (To help focus on rms, voltages and currents in the next two examples (and in other examples to come) are expressed as an rms value times $\sqrt{2}$.)

**EXAMPLE 16–9** For Figure 16–14, $v_1 = \sqrt{2}(16)\sin\omega t$ V, $v_2 = \sqrt{2}(24)\sin(\omega t + 90°)$ and $v_3 = \sqrt{2}(15)\sin(\omega t - 90°)$ V. Determine source voltage $e$.

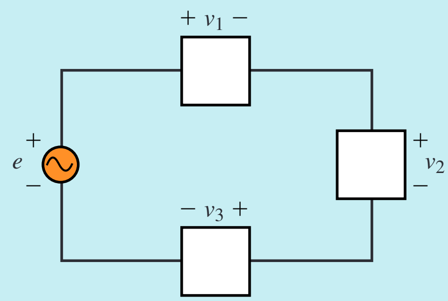



**FIGURE 16–14**

**Solution** The answer can be obtained by KVL. First, convert to phasors. Thus, $\mathbf{V}_1 = 16\text{ V}\angle 0°$, $\mathbf{V}_2 = 24\text{ V}\angle 90°$, and $\mathbf{V}_3 = 15\text{ V}\angle -90°$. KVL yields $\mathbf{E} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 16\text{ V}\angle 0° + 24\text{ V}\angle 90° + 15\text{ V}\angle -90° = 18.4\text{ V}\angle 29.4°$. Converting back to a function of time yields $e = \sqrt{2}(18.4)\sin(\omega t + 29.4°)$ V.

**EXAMPLE 16–10** For Figure 16–15, $i_1 = \sqrt{2}(23)\sin\omega t$ mA, $i_2 = \sqrt{2}(0.29)\sin(\omega t + 63°)$ A and $i_3 = \sqrt{2}(127) \times 10^{-3}\sin(\omega t - 72°)$ A. Determine current $i_T$.

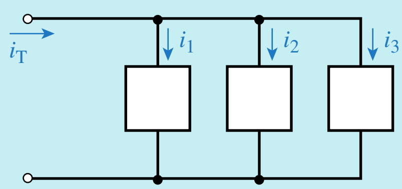


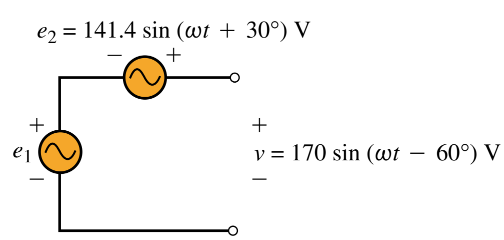

**FIGURE 16–15**

**Solution** Convert to phasors. Thus, $\mathbf{I}_1 = 23\text{ mA}\angle 0°$, $\mathbf{I}_2 = 0.29\text{ A}\angle 63°$, and $\mathbf{I}_3 = 127 \times 10^{-3}\text{ A}\angle -72°$. KCL yields $\mathbf{I}_T = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 23\text{ mA}\angle 0° + 290\text{ mA}\angle 63° + 127\text{ mA}\angle -72° = 238\text{ mA}\angle 35.4°$. Converting back to a function of time yields $i_T = \sqrt{2}(238)\sin(\omega t + 35.4°)$ mA.

## PRACTICE PROBLEMS 3

1. Convert the following to time functions. Values are rms.
   a. $\mathbf{E} = 500\text{ mV}\angle -20°$   b. $\mathbf{I} = 80\text{ A}\angle 40°$
2. For the circuit of Figure 16–16, determine voltage $e_1$.

**FIGURE 16–16**



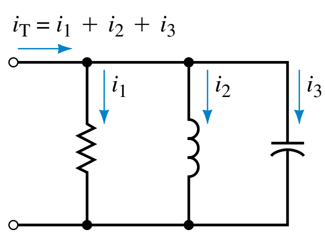

*Answers:*

1. a. $e = 707\sin(\omega t - 20°)$ mV   b. $i = 113\sin(\omega t + 40°)$ A
2. $e_1 = 221\sin(\omega t - 99.8°)$ V

**IN-PROCESS LEARNING CHECK 1**

1. Convert the following to polar form:
   a. $j6$ b. $-j4$ c. $3 + j3$ d. $4 - j6$
   e. $-5 + j8$ f. $1 - j2$ g. $-2 - j3$
2. Convert the following to rectangular form:
   a. $4\angle 90°$ b. $3\angle 0°$ c. $2\angle -90°$ d. $5\angle 40°$
   e. $6\angle 120°$ f. $2.5\angle -20°$ g. $1.75\angle -160°$
3. If $-\mathbf{C} = 12\angle -140°$, what is $\mathbf{C}$?
4. Given: $\mathbf{C}_1 = 36 + j4$ and $\mathbf{C}_2 = 52 - j11$. Determine $\mathbf{C}_1 + \mathbf{C}_2$ and $\mathbf{C}_1 - \mathbf{C}_2$. Express in rectangular form.
5. Given: $\mathbf{C}_1 = 24\angle 25°$ and $\mathbf{C}_2 = 12\angle -125°$. Determine $\mathbf{C}_1 \cdot \mathbf{C}_2$ and $\mathbf{C}_1/\mathbf{C}_2$.
6. Compute the following and express answers in rectangular form:
   a. $\dfrac{6 + j4}{10\angle 20°} + (14 + j2)$ b. $(1 + j6) + \left[2 + \dfrac{(12\angle 0°)(14 + j2)}{6 - (10\angle 20°)(2\angle -10°)}\right]$
7. For Figure 16–17, determine $i_T$ where $i_1 = 10 \sin \omega t$, $i_2 = 20 \sin(\omega t - 90°)$, and $i_3 = 5 \sin(\omega t + 90°)$.

**FIGURE 16–17**




*(Answers are at the end of the chapter.)*

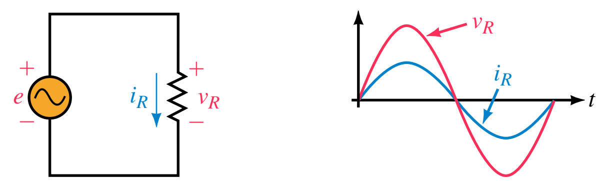
## 16.3 *R, L,* and *C* Circuits with Sinusoidal Excitation

*R, L,* and *C* circuit elements each have quite different electrical properties. Resistance, for example, opposes current, while inductance opposes changes in current, and capacitance opposes changes in voltage. These differences result in quite different voltage-current relationships as you saw earlier. We now investigate these relationships for the case of sinusoidal ac. Sine waves have several important characteristics that you will discover from this investigation:

1. When a circuit consisting of linear circuit elements *R, L,* and *C* is connected to a sinusoidal source, all currents and voltages in the circuit will be sinusoidal.
2. These sine waves have the same frequency as the source and differ from it only in terms of their magnitudes and phase angles.

## 16.4 Resistance and Sinusoidal AC

We begin with a purely resistive circuit. Here, Ohm's law applies and thus, current is directly proportional to voltage. Current variations therefore follow voltage variations, reaching their peak when voltage reaches its peak, changing direction when voltage changes polarity, and so on (Figure 16–18). From this, we conclude that *for a purely resistive circuit, current and voltage are in phase.* Since voltage and current waveforms coincide, their phasors also coincide (Figure 16–19).




(a) Source voltage is a sine wave. Therefore, $v_R$ is a sine wave.

(b) $i_R = v_R/R$. Therefore $i_R$ is a sine wave.

**FIGURE 16–18** Ohm's law applies to resistors. Note that current and voltage are in phase.

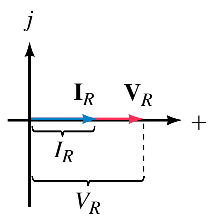



**FIGURE 16–19** For a resistor, voltage and current phasors are in phase.

The relationship illustrated in Figure 16–18 may be stated mathematically as

$$i_R = \frac{v_R}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t \tag{16–10}$$

where

$$I_m = V_m/R \tag{16–11}$$

Transposing,

$$V_m = I_m R \tag{16–12}$$

The in-phase relationship is true regardless of reference. Thus, if $v_R = V_m \sin(\omega t + \theta)$, then $i_R = I_m \sin(\omega t + \theta)$.

**EXAMPLE 16–11** For the circuit of Figure 16–18(a), if $R = 5\ \Omega$ and $i_R = 12 \sin(\omega t - 18)$ A, determine $v_R$.

**Solution** $v_R = Ri_R = 5 \times 12 \sin(\omega t - 18°) = 60 \sin(\omega t - 18°)$ V. The waveforms are shown in Figure 16–20.

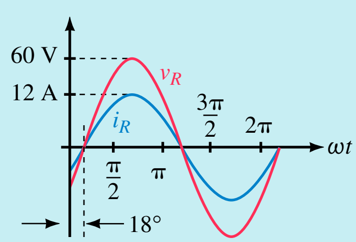



**FIGURE 16–20**

PRACTICE PROBLEMS 4

1. If $v_R = 150 \cos \omega t$ V and $R = 25$ kΩ, determine $i_R$ and sketch both waveforms.
2. If $v_R = 100 \sin(\omega t + 30°)$ V and $R = 0.2$ MΩ, determine $i_R$ and sketch both waveforms.

*Answers:*
1. $i_R = 6 \cos \omega t$ mA. $v_R$ and $i_R$ are in phase.
2. $i_R = 0.5 \sin(\omega t + 30°)$ mA. $v_R$ and $i_R$ are in phase.

## 16.5 Inductance and Sinusoidal AC

### Phase Lag in an Inductive Circuit

As you saw in Chapter 13, for an ideal inductor, voltage $v_L$ is proportional to the rate of change of current. Because of this, voltage and current are not in phase as they are for a resistive circuit. This can be shown with a bit of calculus. From Figure 16–21, $v_L = L di_L/dt$. For a sine wave of current, you get when you differentiate

$$v_L = L\frac{di_L}{dt} = L\frac{d}{dt}(I_m \sin \omega t) = \omega L I_m \cos \omega t = V_m \cos \omega t$$

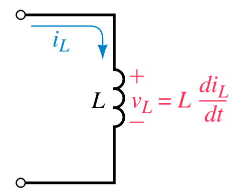



**FIGURE 16–21** Voltage $v_L$ is proportional to the rate of change of current $i_L$.

Utilizing the trigonometric identity $\cos \omega t = \sin(\omega t + 90°)$, you can write this as

$$v_L = V_m \sin(\omega t + 90°) \tag{16–13}$$

where

$$V_m = \omega L I_m \tag{16–14}$$

Voltage and current waveforms are shown in Figure 16–22, and phasors in Figure 16–23. As you can see, *for a purely inductive circuit, current lags voltage by 90°* (i.e., ¼ cycle). Alternatively you can say that voltage leads current by 90°.

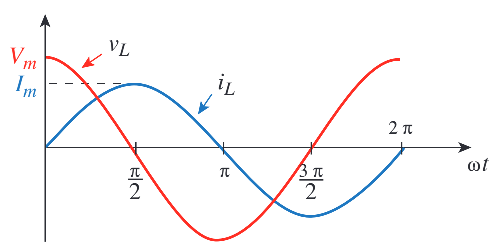



**FIGURE 16–22** For inductance, current lags voltage by 90°. Here $i_L$ is reference.

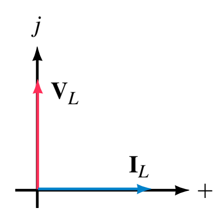
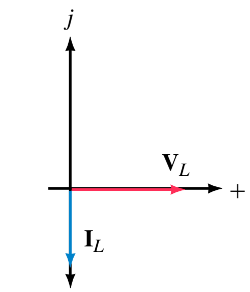



**FIGURE 16–23** Phasors for the waveforms of Fig. 16–22 showing the 90° lag of current.

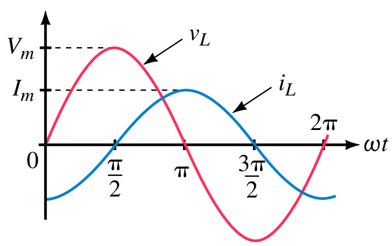
Although we have shown that current lags voltage by 90° for the case of Figure 16–22, this relationship is true in general, that is, current always lags voltage by 90° regardless of the choice of reference. This is illustrated in Figure 16–24. Here, $\mathbf{V}_L$ is at 0° and $\mathbf{I}_L$ at −90°. Thus, voltage $v_L$ will be a sine wave and current $i_L$ a negative cosine wave, i.e., $i_L = -I_m \cos \omega t$. Since

$i_L$ is a negative cosine wave, it can also be expressed as $i_L = I_m \sin(\omega t - 90°)$. The waveforms are shown in (b).

Since current always lags voltage by 90° for a pure inductance, you can, if you know the phase of the voltage, determine the phase of the current, and vice versa. Thus, if $v_L$ is known, $i_L$ must lag it by 90°, while if $i_L$ is known, $v_L$ must lead it by 90°.




(a) Current $\mathbf{I}_L$ always lags voltage $\mathbf{V}_L$ by 90°




(b) Waveforms

**FIGURE 16–24** Phasors and waveforms when $\mathbf{V}_L$ is used as reference.

## Inductive Reactance

From Equation 16–14, we see that the ratio $V_m$ to $I_m$ is

$$\frac{V_m}{I_m} = \omega L \tag{16–15}$$

This ratio is defined as **inductive reactance** and is given the symbol $X_L$. Since the ratio of volts to amps is ohms, reactance has units of ohms. Thus,

$$X_L = \frac{V_m}{I_m} \quad (\Omega) \tag{16–16}$$

Combining Equations 16–15 and 16–16 yields

$$X_L = \omega L \quad (\Omega) \tag{16–17}$$

where $\omega$ is in radians per second and $L$ is in henries. *Reactance $X_L$ represents the opposition that inductance presents to current for the sinusoidal ac case.*

We now have everything that we need to solve simple inductive circuits with sinusoidal excitation, that is, we know that current lags voltage by 90° and that their amplitudes are related by

$$I_m = \frac{V_m}{X_L} \tag{16–18}$$

and

$$V_m = I_m X_L \tag{16–19}$$

**EXAMPLE 16–12** The voltage across a 0.2-H inductance is $v_L = 100 \sin(400t + 70°)$ V. Determine $i_L$ and sketch it.

**Solution** $\omega = 400$ rad/s. Therefore, $X_L = \omega L = (400)(0.2) = 80\ \Omega$.

$$I_m = \frac{V_m}{X_L} = \frac{100\text{ V}}{80\ \Omega} = 1.25\text{ A}$$

The current lags the voltage by 90°. Therefore $i_L = 1.25 \sin(400t - 20°)$ A as indicated in Figure 16–25.

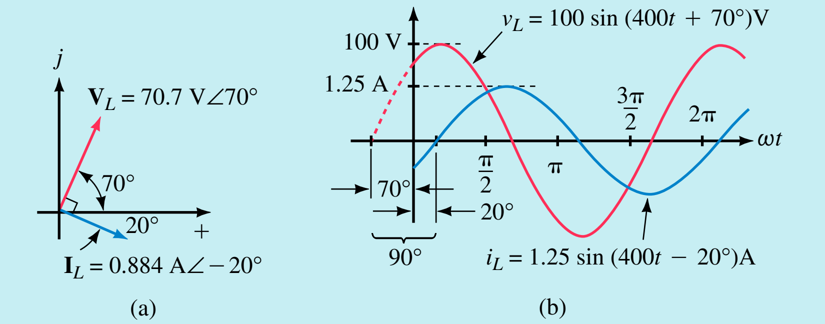



**FIGURE 16–25** With voltage $\mathbf{V}_L$ at 70°, current $\mathbf{I}_L$ will be 90° later at −20°.

**NOTE...**

Remember to show phasors as rms values from now on.

**EXAMPLE 16–13** The current through a 0.01-H inductance is $i_L = 20 \sin(\omega t - 50°)$ A and $f = 60$ Hz. Determine $v_L$.

**Solution**

$$\omega = 2\pi f = 2\pi(60) = 377 \text{ rad/s}$$
$$X_L = \omega L = (377)(0.01) = 3.77\ \Omega$$
$$V_m = I_m X_L = (20\text{ A})(3.77\ \Omega) = 75.4\text{ V}$$

Voltage leads current by 90°. Thus, $v_L = 75.4 \sin(377t + 40°)$ V as shown in Figure 16–26.

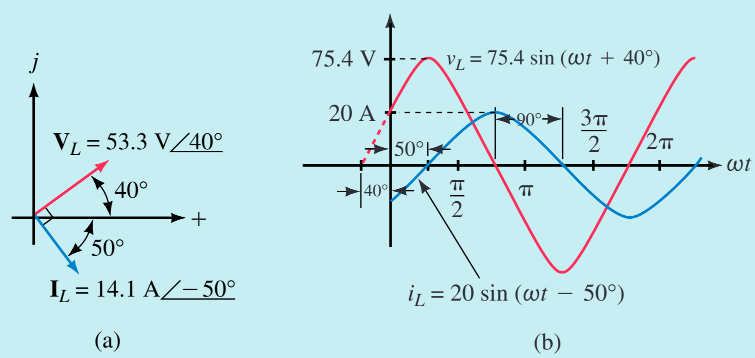
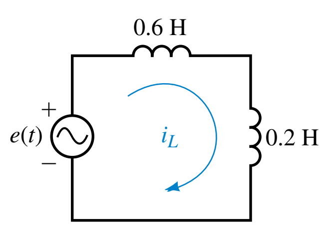



**FIGURE 16–26**

**PRACTICE PROBLEMS 5**

1. Two inductances are connected in series (Figure 16–27). If $e = 100 \sin \omega t$ and $f = 10$ kHz, determine the current. Sketch voltage and current waveforms.

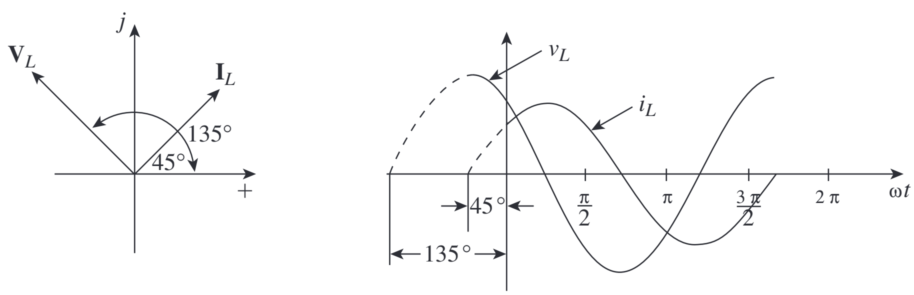
**FIGURE 16–27**



2. The current through a 0.5-H inductance is $i_L = 100 \sin(2400t + 45°)$ mA. Determine $v_L$ and sketch voltage and current phasors and waveforms.

*Answers:*

1. $i_L = 1.99 \sin(\omega t - 90°)$ mA. Waveforms same as Figure 16–24.
2. $v_L = 120 \sin(2400t + 135°$ V). See following art for waveforms.




## Variation of Inductive Reactance with Frequency

Since $X_L = \omega L = 2\pi f L$, inductive reactance is directly proportional to frequency (Figure 16–28). Thus, if frequency is doubled, reactance doubles, while if frequency is halved, reactance halves, and so on. In addition, $X_L$ is directly proportional to inductance. Thus, if inductance is doubled, $X_L$ is doubled, and so on. Note also that at $f = 0$, $X_L = 0\ \Omega$. This means that inductance looks like a short circuit to dc. (We already concluded this earlier in Chapter 13.)

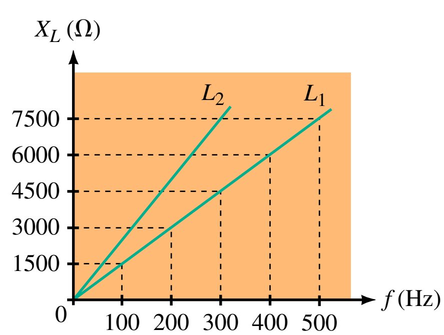



**FIGURE 16–28** Variation of $X_L$ with frequency. Note that $L_2 > L_1$.

PRACTICE PROBLEMS 6

A circuit has 50 ohms inductive reactance. If both the inductance and the frequency are doubled, what is the new $X_L$?

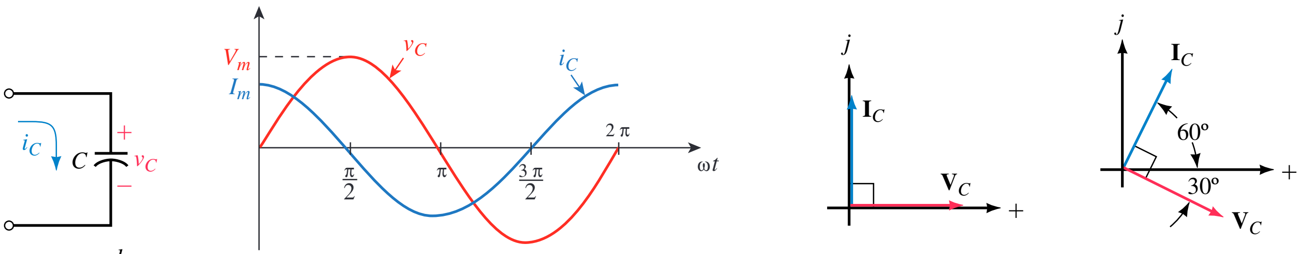
*Answer:* 200 Ω

## 16.6 Capacitance and Sinusoidal AC

### Phase Lead in a Capacitive Circuit

For capacitance, current is proportional to the rate of change of voltage, i.e., $i_C = C\,dv_C/dt$ [Figure 16–29(a)]. Thus if $v_C$ is a sine wave, you get upon substitution

$$i_C = C\frac{dv_C}{dt} = C\frac{d}{dt}(V_m \sin \omega t) = \omega C V_m \cos \omega t = I_m \cos \omega t$$




(a) $i_C = C\dfrac{dv_C}{dt}$ (b) Waveforms with $v_C$ as reference (c) $\mathbf{V}_e$ at 0° (d) $\mathbf{V}_C$ at –30°

**FIGURE 16–29** For capacitance, current always leads voltage by 90°.

Using the appropriate trigonometric identity, this can be written as

$$i_C = I_m \sin(\omega t + 90°) \qquad \textbf{(16–20)}$$

where

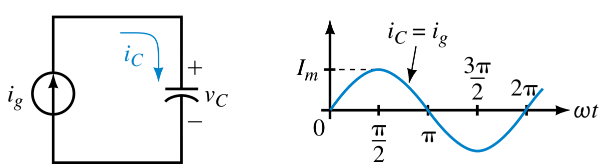
$$I_m = \omega C V_m \qquad \textbf{(16–21)}$$

Waveforms are shown in Figure 16–29(b) and phasors in (c). As indicated, *for a purely capacitive circuit, current leads voltage by 90°,* or alternatively, voltage lags current by 90°. This relationship is true regardless of reference. Thus, if the voltage is known, the current must lead by 90° while if the current is known, the voltage must lag by 90°. For example, if $\mathbf{I}_C$ is at 60° as in (d), $\mathbf{V}_C$ must be at $-30°$.

---

**PRACTICE PROBLEMS 7**

1. The current source of Figure 16–30(a) is a sine wave. Sketch phasors and capacitor voltage $v_C$.




(a) (b)

**FIGURE 16–30**

2. Refer to the circuit of Figure 16–31(a):
   a. Sketch the phasors.
   b. Sketch capacitor current $i_C$.

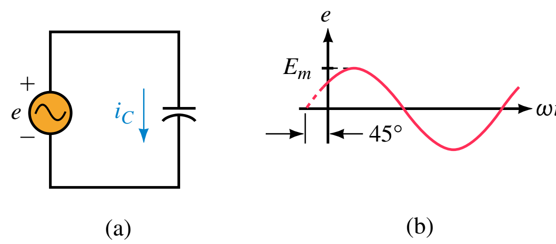



**FIGURE 16–31**

*Answers:*
1. $\mathbf{I}_C$ is at 0°; $\mathbf{V}_C$ is at −90°; $v_C$ is a negative cosine wave.
2. a. $\mathbf{V}_C$ is at 45° and $\mathbf{I}_C$ is at 135°.
   b. Waveforms are the same as for Problem 2, Practice Problem 5, except that voltage and current waveforms are interchanged.

---

## Capacitive Reactance

Now consider the relationship between maximum capacitor voltage and current magnitudes. As we saw in Equation 16–21, they are related by $I_m = \omega C V_m$. Rearranging, we get $V_m/I_m = 1/\omega C$. The ratio of $V_m$ to $I_m$ is defined as **capacitive reactance** and is given the symbol $X_C$. That is,

$$X_C = \frac{V_m}{I_m} \quad (\Omega)$$

Since $V_m/I_m = 1/\omega C$, we also get

$$X_C = \frac{1}{\omega C} \quad (\Omega) \tag{16–22}$$

where ω is in radians per second and $C$ is in farads. *Reactance $X_C$ represents the opposition that capacitance presents to current for the sinusoidal ac case.*

We now have everything that we need to solve simple capacitive circuits with sinusoidal excitation, i.e., we know that current leads voltage by 90° and that

$$I_m = \frac{V_m}{X_C} \tag{16–23}$$

and

$$V_m = I_m X_C \tag{16–24}$$

**EXAMPLE 16–14** The voltage across a 10-$\mu$F capacitance is $v_C = 100 \sin(\omega t - 40°)$ V and $f = 1000$ Hz. Determine $i_C$ and sketch its waveform.

**Solution**

$$\omega = 2\pi f = 2\pi(1000 \text{ Hz}) = 6283 \text{ rad/s}$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(6283)(10 \times 10^{-6})} = 15.92\ \Omega$$

$$I_m = \frac{V_m}{X_C} = \frac{100 \text{ V}}{15.92\ \Omega} = 6.28 \text{ A}$$

Since current leads voltage by 90°, $i_C = 6.28 \sin(6283t + 50°)$ A as indicated in Figure 16–32.

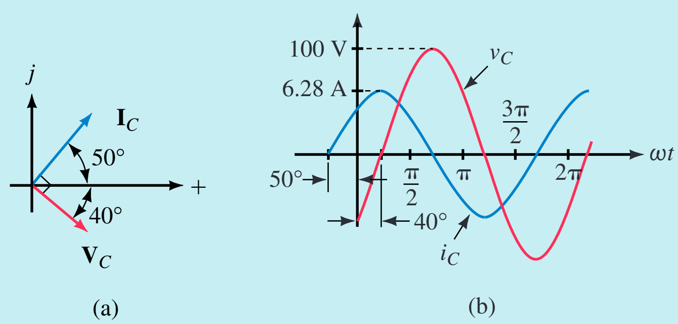



**FIGURE 16–32** Phasors are not to scale with waveform.

**EXAMPLE 16–15** The current through a 0.1-$\mu$F capacitance is $i_C = 5 \sin(1000t + 120°)$ mA. Determine $v_C$.

**Solution**

$$X_C = \frac{1}{\omega C} = \frac{1}{(1000 \text{ rad/s})(0.1 \times 10^{-6}\ F)} = 10\ \text{k}\Omega$$

Thus, $V_m = I_m X_C = (5 \text{ mA})(10\ \text{k}\Omega) = 50$ V. Since voltage lags current by 90°, $v_C = 50 \sin(1000t + 30°)$ V. Waveforms and phasors are shown in Figure 16–33.

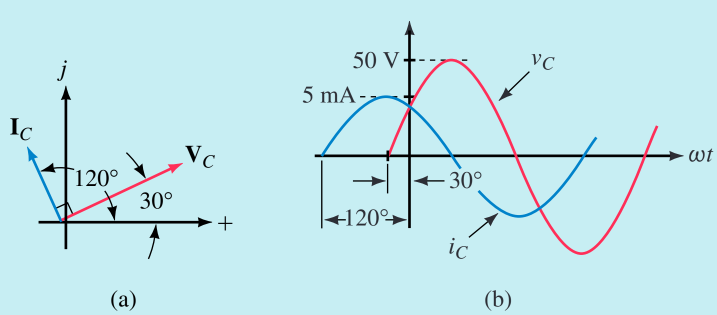
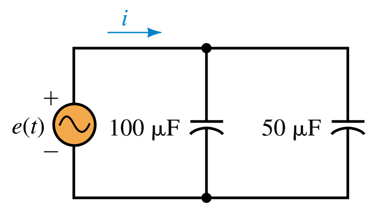



**FIGURE 16–33** Phasors are not to scale with waveform.

**PRACTICE PROBLEMS 8**

Two capacitances are connected in parallel (Figure 16–34). If $e = 100 \sin \omega t$ V and $f = 10$ Hz, determine the source current. Sketch current and voltage phasors and waveforms.

**FIGURE 16–34**




*Answer:* $i = 0.942 \sin(62.8t + 90°) = 0.942 \cos 62.8t$ A
See Figure 16–29(b) and (c).

## Variation of Capacitive Reactance with Frequency

Since $X_C = 1/\omega C = 1/2\pi fC$, the opposition that capacitance presents varies inversely with frequency. This means that the higher the frequency, the lower the reactance, and vice versa (Figure 16–35). At $f = 0$ (i.e., dc), capacitive reactance is infinite. This means that a capacitance looks like an open circuit to dc. (We already concluded this earlier in Chapter 10.) Note that $X_C$ is also inversely proportional to capacitance. Thus, if capacitance is doubled, $X_C$ is halved, and so on.

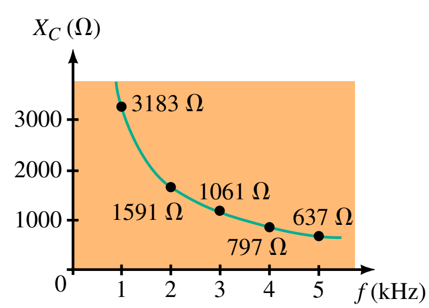



**FIGURE 16–35** $X_C$ varies inversely with frequency. Values shown are for $C = 0.05$ $\mu$F.

**IN-PROCESS LEARNING CHECK 2**

1. For a pure resistance, $v_R = 100 \sin(\omega t + 30°)$ V. If $R = 2\ \Omega$, what is the expression for $i_R$?
2. For a pure inductance, $v_L = 100 \sin(\omega t + 30°)$ V. If $X_L = 2\ \Omega$, what is the expression for $i_L$?
3. For a pure capacitance, $v_C = 100 \sin(\omega t + 30°)$ V. If $X_C = 2\ \Omega$, what is the expression for $i_C$?
4. If $f = 100$ Hz and $X_L = 400\ \Omega$, what is $L$?
5. If $f = 100$ Hz and $X_C = 400\ \Omega$, what is $C$?
6. For each of the phasor sets of Figure 16–36, identify whether the circuit is resistive, inductive, or capacitive. Justify your answers.

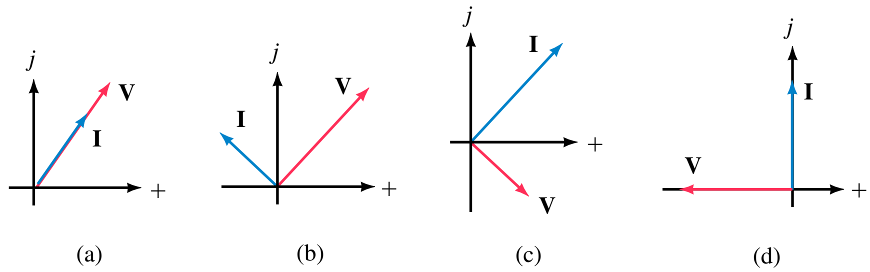



FIGURE 16–36

*(Answers are at the end of the chapter.)*

## 16.7 The Impedance Concept

In Sections 16.5 and 16.6, we handled magnitude and phase analysis separately. However, this is not the way it is done in practice. In practice, we represent circuit elements by their impedance, and determine magnitude and phase relationships in one step. Before we do this, however, we need to learn how to represent circuit elements as impedances.

### Impedance

The opposition that a circuit element presents to current in the phasor domain is defined as its **impedance.** The impedance of the element of Figure 16–37, for example, is the ratio of its voltage phasor to its current phasor. Impedance is denoted by the boldface, uppercase letter **Z.** Thus,

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{(ohms)} \tag{16–25}$$

(This equation is sometimes referred to as Ohm's law for ac circuits.)

Since phasor voltages and currents are complex, **Z** is also complex. That is,

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V}{I}\angle\theta \tag{16–26}$$

where $V$ and $I$ are the rms magnitudes of **V** and **I** respectively, and $\theta$ is the angle between them. From Equation 16–26,

$$\mathbf{Z} = Z\angle\theta \tag{16–27}$$

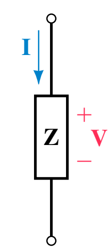
where $Z = V/I$. Since $V = 0.707V_m$ and $I = 0.707I_m$, $Z$ can also be expressed as $V_m/I_m$. Once the impedance of a circuit is known, the current and voltage can be determined using

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} \tag{16–28}$$

and

$$\mathbf{V} = \mathbf{IZ} \tag{16–29}$$

Let us now determine impedance for the basic circuit elements $R$, $L$, and $C$.

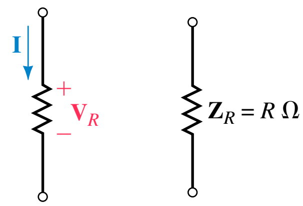



$\mathbf{Z} = \dfrac{\mathbf{V}}{\mathbf{I}}$ ohms

FIGURE 16–37 Impedance concept.

**NOTES...**

Although **Z** is a complex number, it is not a phasor since it does not represent a sinusoidally varying quantity.




(a) Voltage and current (b) Impedance

**FIGURE 16–38** Impedance of a pure resistance.

### Resistance

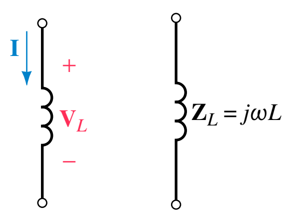
For a pure resistance (Figure 16–38), voltage and current are in phase. Thus, if voltage has an angle $\theta$, current will have the same angle. For example, if $\mathbf{V}_R = V_R\angle\theta$, then $\mathbf{I} = I\angle\theta$. Substituting into Equation 16–25 yields:

$$\mathbf{Z}_R = \frac{\mathbf{V}_R}{\mathbf{I}} = \frac{V_R\angle\theta}{I\angle\theta} = \frac{V_R}{I}\angle 0° = R$$

Thus the impedance of a resistor is just its resistance. That is,

$$\mathbf{Z}_R = R \tag{16–30}$$

This agrees with what we know about resistive circuits, i.e., that the ratio of voltage to current is $R$, and that the angle between them is 0°.




(a) Voltage and current (b) Impedance

**FIGURE 16–39** Impedance of a pure inductance.

### Inductance

For a pure inductance, current lags voltage by 90°. Assuming a 0° angle for voltage (we can assume any reference we want because we are interested only in the angle between $\mathbf{V}_L$ and $\mathbf{I}$), we can write $\mathbf{V}_L = V_L\angle 0°$ and $\mathbf{I} = I\angle -90°$. The impedance of a pure inductance (Figure 16–39) is therefore

$$\mathbf{Z}_L = \frac{\mathbf{V}_L}{\mathbf{I}} = \frac{V_L\angle 0°}{I\angle -90°} = \frac{V_L}{I}\angle 90° = \omega L\angle 90° = j\omega L$$

where we have used the fact that $V_L/I_L = \omega L$. Thus,

$$\mathbf{Z}_L = j\omega L = jX_L \tag{16–31}$$

since $\omega L$ is equal to $X_L$.

**EXAMPLE 16–16** Consider again Example 16–12. Given $v_L = 100 \sin(400t + 70°)$ and $L = 0.2$ H, determine $i_L$ using the impedance concept.

**Solution** See Figure 16–40.

$$\mathbf{V}_L = 70.7\text{ V}\angle 70° \quad \text{and} \quad \omega = 400 \text{ rad/s}$$

$$\mathbf{Z}_L = j\omega L = j(400)(0.2) = j80\ \Omega$$

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{Z}_L} = \frac{70.7\angle 70°}{j80} = \frac{70.7\angle 70°}{80\angle 90°} = 0.884\text{ A}\angle -20°$$

In the time domain, $i_L = \sqrt{2}(0.884)\sin(400t - 20°) = 1.25 \sin(400t - 20°)$ A, which agrees with our previous solution.

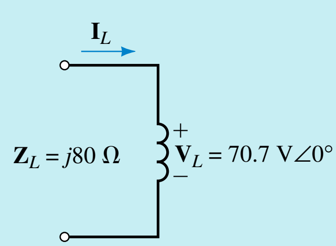



**FIGURE 16–40**

**Capacitance**

For a pure capacitance, current leads voltage by 90°. Its impedance (Figure 16–41) is therefore

$$\mathbf{Z}_C = \frac{\mathbf{V}_C}{\mathbf{I}} = \frac{V_C\angle 0°}{I\angle 90°} = \frac{V_C}{I}\angle -90° = \frac{1}{\omega C}\angle -90° = -j\frac{1}{\omega C} \quad \text{(ohms)}$$

Thus,

$$\mathbf{Z}_C = -j\frac{1}{\omega C} = -jX_C \quad \text{(ohms)} \qquad \textbf{(16–32)}$$

since $1/\omega C$ is equal to $X_C$.

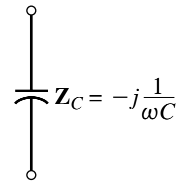



**FIGURE 16–41** Impedance of a pure capacitance.

**EXAMPLE 16–17** Given $v_C = 100 \sin(\omega t - 40°)$, $f = 1000$ Hz, and $C = 10\ \mu$F, determine $i_C$ in Figure 16–42.

**Solution**

$$\omega = 2\pi f = 2\pi(1000 \text{ Hz}) = 6283 \text{ rads/s}$$

$$\mathbf{V}_C = 70.7 \text{ V}\angle -40°$$

$$\mathbf{Z}_C = -j\frac{1}{\omega C} = -j\left(\frac{1}{6283 \times 10 \times 10^{-6}}\right) = -j15.92\ \Omega.$$

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{\mathbf{Z}_C} = \frac{70.7\angle -40°}{-j15.92} = \frac{70.7\angle -40°}{15.92\angle -90°} = 4.442 \text{ A}\angle 50°$$

In the time domain, $i_C = \sqrt{2}(4.442) \sin(6283t + 50°) = 6.28 \sin(6283t + 50°)$ A, which agrees with our previous solution, in Example 16–14.

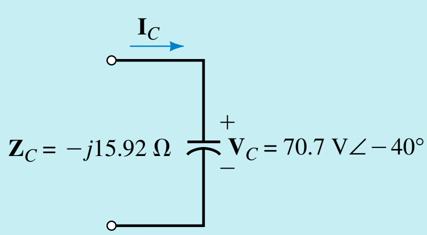



**FIGURE 16–42**

**PRACTICE PROBLEMS 9**

1. If $\mathbf{I}_L = 5 \text{ mA}\angle -60°$, $L = 2$ mH, and $f = 10$ kHz, what is $\mathbf{V}_L$?
2. A capacitor has a reactance of 50 Ω at 1200 Hz. If $v_C = 80 \sin 800t$ V, what is $i_C$?

*Answers:*

1. 628 mV∠30°
2. 0.170 sin(800$t$ + 90°) A

## A Final Note

The real power of the impedance method becomes apparent when you consider complex circuits with elements in series, parallel, and so on. This we do later, beginning in Chapter 18. Before we do this, however, there are some ideas on power that you need to know. These are considered in Chapter 17.

## 16.8 Computer Analysis of AC Circuits

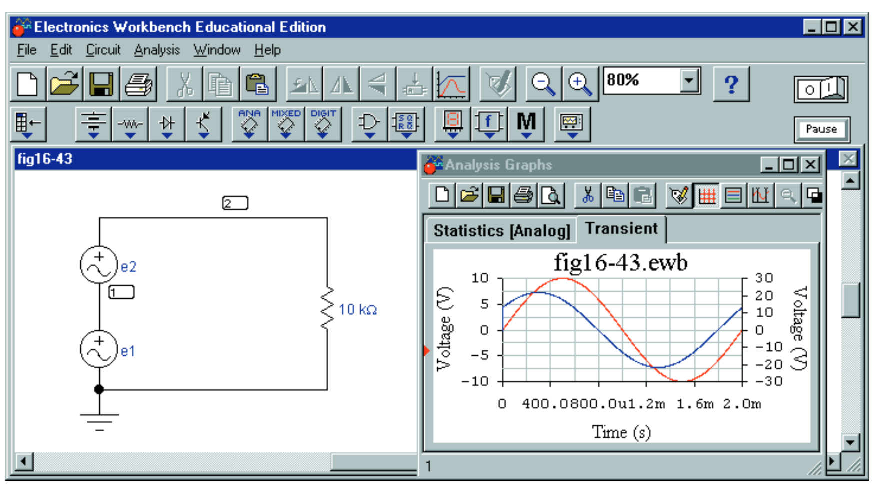

ELECTRONICS WORKBENCH


PSpice

In Chapter 15, you saw how to represent sinusoidal waveforms using PSpice and Electronics Workbench. Let us now apply these tools to the ideas of this chapter. To illustrate, recall that in Example 16–6, we summed voltages $e_1 = 10 \sin \omega t$ V and $e_2 = 15 \sin(\omega t + 60°)$ V using phasor methods to obtain their sum $v = 21.8 \sin(\omega t + 36.6°)$ V. We will now verify this summation numerically. Since the process is independent of frequency, let us choose $f$ = 500 Hz. This yields a period of 2 ms; thus, ½ cycle is 1 ms, ¼ cycle (90°) is 500 $\mu$s, 45° is 250 $\mu$s, etc.

### Electronics Workbench

As noted in Chapter 15, Electronics Workbench requires that you enter rms values for waveforms. Thus, to represent $10 \sin \omega t$ V, enter (0.7071)(10 V) = 7.071 V as its magnitude, and to represent $15 \sin(\omega t + 60°)$ V, enter 10.607 V. You must also enter its angle. Procedure: Create the circuit of Figure 16–43(a) on the screen. Set Source 1 to **7.071 V, 0 deg,** and **500 Hz** using the procedure of Chapter 15. Similarly, set Source 2 to **10.607 V, 60 deg,** and **500 Hz.** Click Analysis, Transient, set TSTOP to 0.002 (to run the solution out to 2 ms so that you display a full cycle) and TMAX to 2e-06 (to avoid getting a choppy waveform.) Highlight Node 1 (to display $e_1$) and click Add. Repeat for Node 2 (to display $v$) Click Simulate. Following simulation, graphs $e_1$ (the red curve, left-hand scale) and $v$ (the blue curve, right-hand scale) appear. Expand the Analysis Graph window to full screen, then click the cursor icon and drag a cursor to 500 $\mu$s or as close as you can get it and read values. (You should get about 10 V for $e_1$ and 17.5 V for $v$.) Scale values from the graph at 200-$\mu$s increments and tabulate. Now replace the sources with a single source of 21.8




(a) The circuit (b) Waveforms $e_1$ and $v$

**FIGURE 16–43** Summing sinusoidal waveforms with Electronics Workbench.

sin($\omega t$ + 36.6°) V. (Don't forget to convert to rms). Run a simulation. The resulting graph should be identical to $v$ (the blue curve) of Figure 16–43(b).

## OrCAD PSpice

With PSpice, you can plot all three waveforms of Fig. 16–10 simultaneously. Procedure: Create the circuit of Fig. 16–44 using source VSIN. Double click Source 1 and select Parts in the Property Editor. Set VOFF to **0V,** VAMPL to **10V,** FREQ to **500Hz,** and PHASE to **0deg.** Click Apply, then close the Property Editor. Set up source 2 similarly. Now place markers as shown (Note 2) so that PSpice will automatically create the plots. Click the New Simulation Profile icon, choose Transient, set TSTOP to **2ms** (to display a full cycle), set Maximum Step Size to **1us** (to yield a smooth plot), then click OK. Run the simulation and the waveforms of Figure 16–45 should

NOTES...

1. Make sure that the polarities of the sources are as indicated. (You will have to rotate V2 three times to get it into the position shown.)
2. To display V2, use differential markers (indicated as + and − on the toolbar).

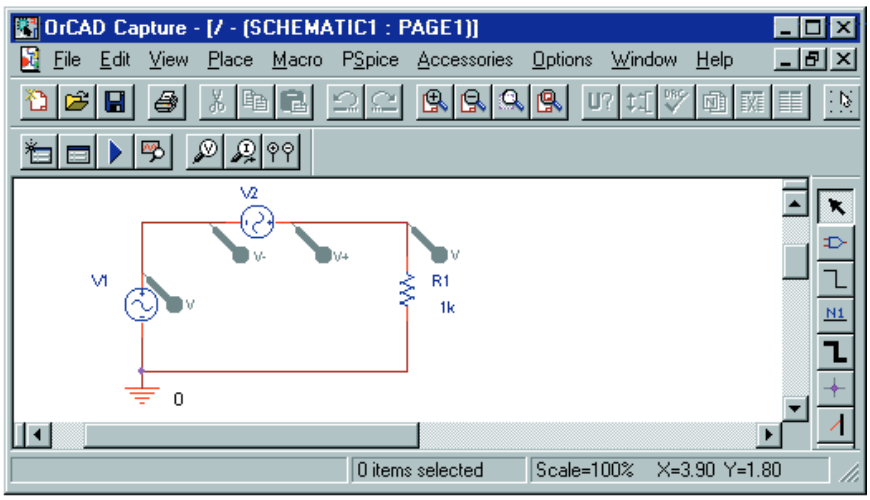



**FIGURE 16–44** Summing sinusoidal waveforms with PSpice. Source 2 is displayed using a differential marker.

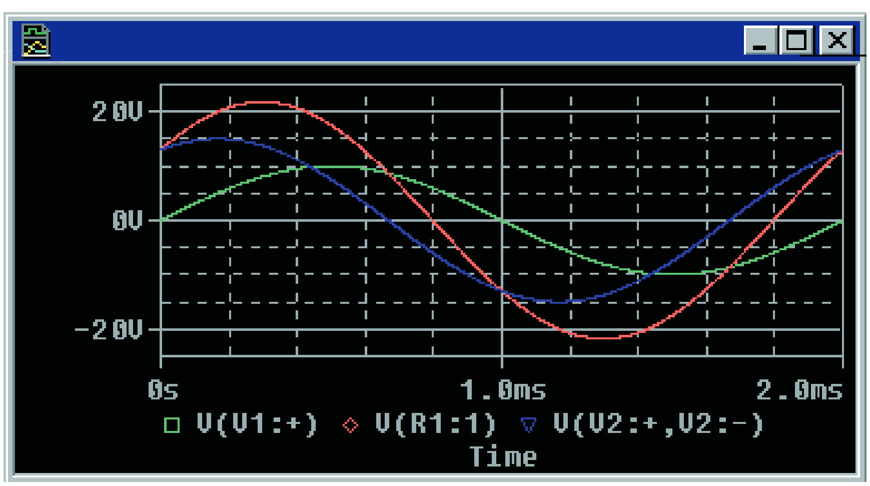



**FIGURE 16–45** PSpice waveforms. Compare to Figure 16–10.

appear. Using the cursor, scale voltages at 500 $\mu$s. You should get 10 V for $e_1$, 7.5 V for $e_2$ and 17.5 V for $v$. Read values at 200-$\mu$s intervals and tabulate. Now replace the sources of Figure 16–44 with a single source of 21.8 sin($\omega t$ + 36.6°) V, run a simulation, scale values, and compare results. They should agree.

**Another Example**

PSpice makes it easy to study the response of circuits over a range of frequencies. This is illustrated in Example 16–18.

**EXAMPLE 16–18** Compute and plot the reactance of a 12-μF capacitor over the range 10 Hz to 1000 Hz.

**Solution** PSpice has no command to compute reactance; however, we can calculate voltage and current over the desired frequency range, then plot their ratio. This gives reactance. Procedure: Create the circuit of Figure 16–46 on the screen. (Use source VAC here as it is the source to use for phasor analyses—see Appendix A). Note its default of 0V. Double click the default value (not the symbol) and in the dialog box, enter **120V,** then click OK. Click the New Simulations Profile icon, enter **fig 16-46** for a name and then in the dialog box that opens, select AC Sweep/Noise. For the Start Frequency, key in **10Hz;** for the End Frequency, key **1kHz;** set AC Sweep type to Logarithmic, select Decade and type **100** into the Pts/Decade (points per decade) box. Run the simultation and a set of empty axes appears. Click Trace, Add Trace and in the dialog box, click **V1(C1),** press the / key on the keyboard, then click **I(C1)** to yield the ratio V1(C1)/I(C1) (which is the capacitor's reactance). Click OK and PSpice will compute and plot the capacitor's reactance versus frequency, Figure 16–47. Compare its shape to Figure 16–35. Use the cursor to scale some values off the screen and verify each point using $X_C = 1/\omega C$.

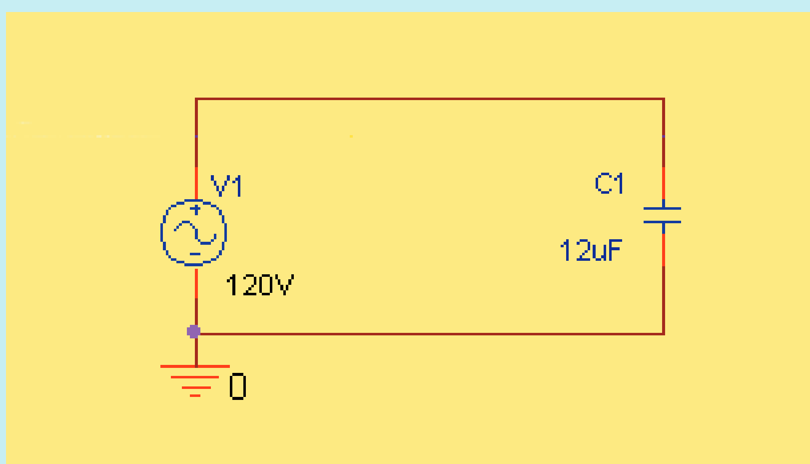



**FIGURE 16–46** Circuit for plotting reactance.

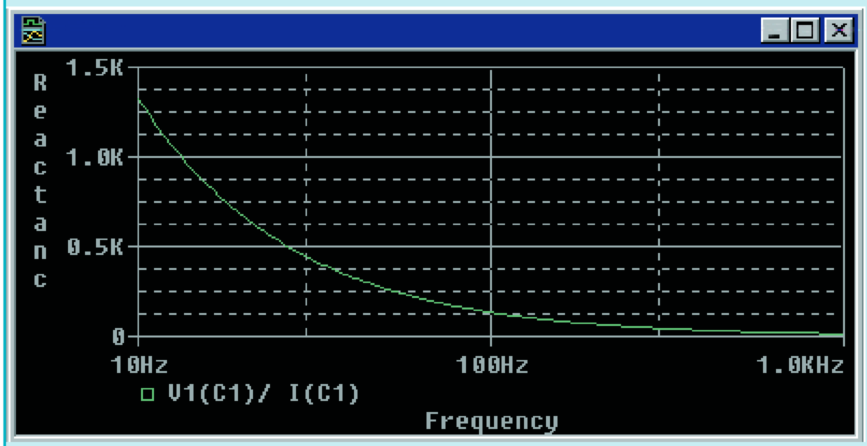



**FIGURE 16–47** Reactance for a 12-$\mu$F capacitor versus frequency.

**Phasor Analysis**

As a last example, we will show how to use PSpice to perform phasor analysis—i.e., to solve problems with voltages and currents expressed in phasor form. To illustrate, consider again Example 16–17. Recall, $\mathbf{V}_C = 70.7\ \text{V}\angle -40°$, $C = 10\ \mu\text{F}$, and $f = 1000$ Hz. Procedure: Create the circuit on the screen (Figure 16–48) using source VAC and component IPRINT (Note 1). Double click the VAC symbol and in the Property Editor, set ACMAG to **70.7V** and ACPHASE to **−40deg.** (See Note 2). Double click IPRINT and in the Property Editor, type **yes** into cells AC, MAG, and PHASE. Click Apply and close the editor. Click the New Simulation Profile icon, select AC Sweep/Noise, Linear, set both Start Frequency and End Frequency to

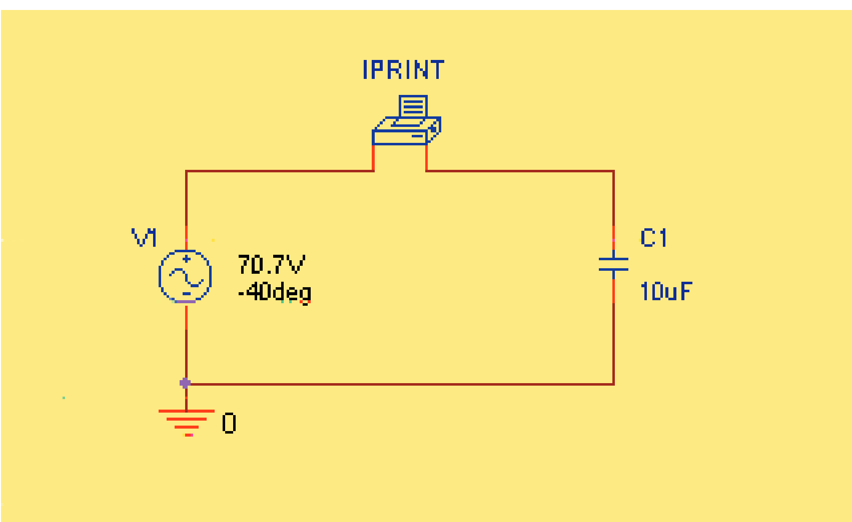



**FIGURE 16–48** Phasor analysis using PSpice. Component IPRINT is a software ammeter.

NOTES...

1. Component IPRINT is a software ammeter, found in the SPECIAL parts library. In this example, we configure it to display ac current in magnitude and phase angle format. Make sure that it is connected as shown in Figure 16–48, since if it is reversed, the phase angle of the measured current will be in error by 180°.
2. If you want to display the phase of the source voltage on the schematic as in Figure 16–48, double click the source symbol and in the Property Editor, click ACPHASE, Display, then select Value Only.
3. The results displayed by IPRINT are expressed in exponential format. Thus, frequency (Figure 16–49) is shown as 1.000E+03, which is $1.000 \times 10^3 = 1000$ Hz, etc.

**1000Hz** and Total Points to **1.** Run the simulation. When the simulation window opens, click View, Output File, then scroll until you find the answers (Figure 16–49 and Note 3). The first number is the frequency (1000 Hz), the second number (IM) is the magnitude of the current (4.442 A), and the third (IP) is its phase (50 degrees). Thus, $\mathbf{I}_C = 4.442\text{ A}\angle 50°$ as we determined earlier in Example 16–17.

| FREQ | IM(V_PRINT1) | IP(V_PRINT1) |
|---|---|---|
| 1.000E+03 | 4.442E+00 | 5.000E+01 |

**FIGURE 16–49** Current for the circuit of Figure 16–48. $\mathbf{I} = 4.442\text{ A}\angle 50°$.

**PRACTICE PROBLEMS 10**

Modify Example 16–18 to plot both capacitor current and reactance on the same graph. You will need to add a second Y-axis for the capacitor current. (See Appendix A if you need help.)

## PROBLEMS

### 16.1 Complex Number Review

1. Convert each of the following to polar form:
   a. $5 + j12$ b. $9 - j6$ c. $-8 + j15$ d. $-10 - j4$
2. Convert each of the following to rectangular form:
   a. $6\angle 30°$ b. $14\angle 90°$ c. $16\angle 0°$ d. $6\angle 150°$
   e. $20\angle -140°$ f. $-12\angle 30°$ g. $-15\angle -150°$
3. Plot each of the following on the complex plane:
   a. $4 + j6$ b. $j4$ c. $6\angle -90°$ d. $10\angle 135°$
4. Simplify the following using powers of *j*:
   a. $j(1 - j1)$ b. $(-j)(2 + j5)$ c. $j[j(1 + j6)]$
   d. $(j4)(-j2 + 4)$ e. $(2 + j3)(3 - j4)$
5. Add or subtract as indicated. Express your answer in rectangular form.
   a. $(4 + j8) + (3 - j2)$ b. $(4 + j8) - (3 - j2)$
   c. $(4.1 - j7.6) + 12\angle 20°$ d. $2.9\angle 25° - 7.3\angle -5°$
   e. $9.2\angle -120° - (2.6 + j4.1)$
6. Multiply or divide as indicated. Express your answer in polar form.
   a. $(37 + j9.8)(3.6 - j12.3)$ b. $(41.9\angle -80°)(16 + j2)$
   c. $\dfrac{42 + j18.6}{19.1 - j4.8}$ d. $\dfrac{42.6 + j187.5}{11.2\angle 38°}$

7. Reduce each of the following to polar form:

   a. $15 - j6 - \left[\dfrac{18\angle 40° + (12 + j8)}{11 + j11}\right]$

   b. $\dfrac{21\angle 20° - j41}{36\angle 0° + (1 + j12) - 11\angle 40°}$

   c. $\dfrac{18\angle 40° - 18\angle -40°}{7 + j12} - \dfrac{16 + j17 + 21\angle -60°}{4}$

## 16.2 Complex Numbers in AC Analysis

8. In the manner of Figure 16–9, represent each of the following as transformed sources.

   a. $e = 100 \sin(\omega t + 30°)$ V
   b. $e = 15 \sin(\omega t - 20°)$ V
   c. $e = 50 \sin(\omega t + 90°)$ V
   d. $e = 50 \cos \omega t$ V
   e. $e = 40 \sin(\omega t + 120°)$ V
   f. $e = 80 \sin(\omega t - 70°)$ V

9. Determine the sinusoidal equivalent for each of the transformed sources of Figure 16–50.

10. Given: $e_1 = 10 \sin(\omega t + 30°)$ V and $e_2 = 15 \sin(\omega t - 20°)$ V. Determine their sum $v = e_1 + e_2$ in the manner of Example 16–6, i.e.,

    a. Convert $e_1$ and $e_2$ to phasor form.
    b. Determine $\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2$.
    c. Convert $\mathbf{V}$ to the time domain.
    d. Sketch $e_1$, $e_2$, and $v$ as per Figure 16–12.

11. Repeat Problem 10 for $v = e_1 - e_2$.

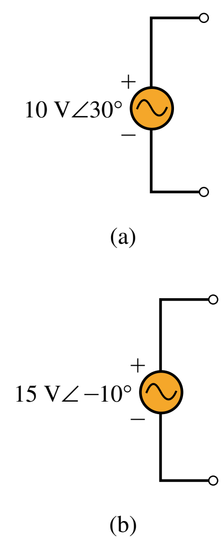



FIGURE 16–50

**Note: For the remaining problems and throughout the remainder of the book, express phasor quantities as rms values rather than as peak values.**

12. Express the voltages and currents of Figure 16–51 as time domain and phasor domain quantities.

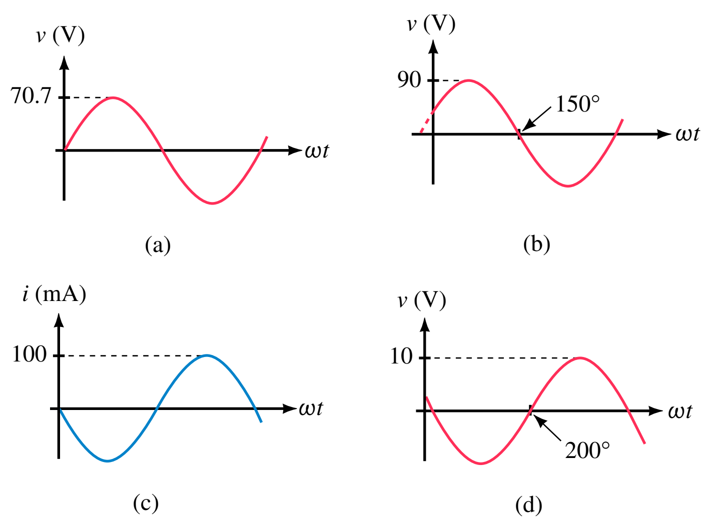



FIGURE 16–51

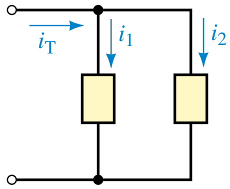



FIGURE 16–52

13. For Figure 16–52, $i_1 = 25 \sin(\omega t + 36°)$ mA and $i_2 = 40 \cos(\omega t - 10°)$ mA.
    a. Determine phasors $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_T$.
    b. Determine the equation for $i_T$ in the time domain.
14. For Figure 16–52, $i_T = 50 \sin(\omega t + 60°)$ A and $i_2 = 20 \sin(\omega t - 30°)$ A.
    a. Determine phasors $\mathbf{I}_T$ and $\mathbf{I}_2$.
    b. Determine $\mathbf{I}_1$.
    c. From (b), determine the equation for $i_1$.
15. For Figure 16–17, $i_1 = 7 \sin \omega t$ mA, $i_2 = 4 \sin(\omega t - 90°)$ mA, and $i_3 = 6 \sin(\omega t + 90°)$ mA.
    a. Determine phasors $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_T$.
    b. Determine the equation for $i_T$ in the time domain.
16. For Figure 16–17, $i_T = 38.08 \sin(\omega t - 21.8°)$ A, $i_1 = 35.36 \sin \omega t$ A, and $i_2 = 28.28 \sin(\omega t - 90°)$ A. Determine the equation for $i_3$.

**16.4 to 16.6**

17. For Figure 16–18(a), $R = 12\ \Omega$. For each of the following, determine the current or voltage and sketch.
    a. $v = 120 \sin \omega t$ V, $i =$ ______
    b. $v = 120 \sin(\omega t + 27°)$ V, $i =$ ______
    c. $i = 17 \sin(\omega t - 56°)$ mA, $v =$ ______
    d. $i = -17 \cos(\omega t - 67°)$ $\mu$A, $v =$ ______
18. Given $v = 120 \sin(\omega t + 52°)$ V and $i = 15 \sin(\omega t + 52°)$ mA, what is $R$?
19. Two resistors $R_1 = 10$ k$\Omega$ and $R_2 = 12.5$ k$\Omega$ are in series. If $i = 14.7 \sin(\omega t + 39°)$ mA,
    a. What are $v_{R_1}$ and $v_{R_2}$?
    b. Compute $v_T = v_{R_1} + v_{R_2}$ and compare to $v_T$ calculated from $v_T = i R_T$.
20. The voltage across a certain component is $v = 120 \sin(\omega t + 55°)$ V and its current is $-18 \cos(\omega t + 145°)$ mA. Show that the component is a resistor and determine its value.
21. For Figure 16–53, $V_m = 10$ V and $I_m = 5$ A. For each of the following, determine the missing quantity:
    a. $v_L = 10 \sin(\omega t + 60°)$ V, $i_L =$ ______
    b. $v_L = 10 \sin(\omega t - 15°)$ V, $i_L =$ ______
    c. $i_L = 5 \cos(\omega t - 60°)$ A, $v_L =$ ______
    d. $i_L = 5 \sin(\omega t + 10°)$ A, $v_L =$ ______

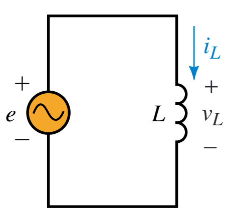



FIGURE 16–53

22. What is the reactance of a 0.5-H inductor at
    a. 60 Hz b. 1000 Hz c. 500 rad/s
23. For Figure 16–53, $e = 100 \sin \omega t$ and $L = 0.5$ H. Determine $i_L$ at
    a. 60 Hz b. 1000 Hz c. 500 rad/s
24. For Figure 16–53, let $L = 200$ mH.
    a. If $v_L = 100 \sin 377t$ V, what is $i_L$?
    b. If $i_L = 10 \sin(2\pi \times 400t - 60°)$ mA, what is $v_L$?

25. For Figure 16–53, if

a. $v_L = 40 \sin(\omega t + 30°)$ V, $i_L = 364 \sin(\omega t - 60°)$ mA, and $L = 2$ mH, what is $f$?

b. $i_L = 250 \sin(\omega t + 40°)$ $\mu$A, $v_L = 40 \sin(\omega t + \theta)$ V, and $f = 500$ kHz, what are $L$ and $\theta$?

26. Repeat Problem 21 if the given voltages and currents are for a capacitor instead of an inductor.

27. What is the reactance of a 5-$\mu$F capacitor at

a. 60 Hz  b. 1000 Hz  c. 500 rad/s

28. For Figure 16–54, $e = 100 \sin \omega t$ and $C = 5$ $\mu$F. Determine $i_C$ at

a. 60 Hz  b. 1000 Hz  c. 500 rad/s

29. For Figure 16–54, let $C = 50$ $\mu$F.

a. If $v_C = 100 \sin 377t$ V, what is $i_C$?

b. If $i_C = 10 \sin(2\pi \times 400t - 60°)$ mA, what is $v_C$?

30. For Figure 16–54, if

a. $v_C = 362 \sin(\omega t - 33°)$ V, $i_C = 94 \sin(\omega t + 57°)$ mA, and $C = 2.2$ $\mu$F, what is $f$?

b. $i_C = 350 \sin(\omega t + 40°)$ mA, $v_C = 3.6 \sin(\omega t + \theta)$ V, and $f = 12$ kHz, what are $C$ and $\theta$?

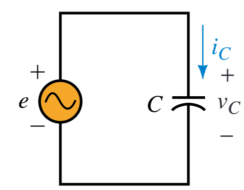


FIGURE 16–54

### 16.7 The Impedance Concept

31. Determine the impedance of each circuit element of Figure 16–55.

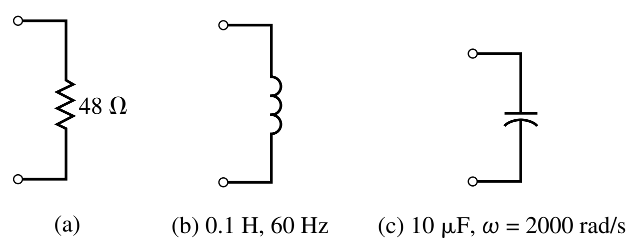
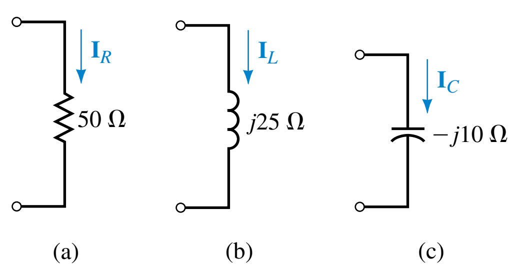


FIGURE 16–55

32. If $\mathbf{E} = 100\text{ V}\angle 0°$ is applied across each of the circuit elements of Figure 16–56:

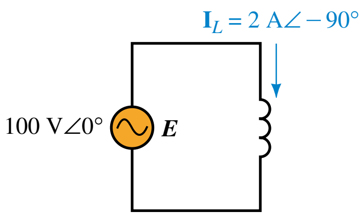
a. Determine each current in phasor form.

b. Express each current in time domain form.

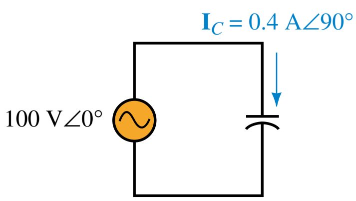


EWB FIGURE 16–56

33. If the current through each circuit element of Figure 16–56 is 0.5 A∠0°:
    a. Determine each voltage in phasor form.
    b. Express each voltage in time domain form.
34. For each of the following, determine the impedance of the circuit element and state whether it is resistive, inductive, or capacitive.
    a. $\mathbf{V} = 240 \text{ V}\angle -30°$, $\mathbf{I} = 4 \text{ A}\angle -30°$.
    b. $\mathbf{V} = 40 \text{ V}\angle 30°$, $\mathbf{I} = 4 \text{ A}\angle -60°$.
    c. $\mathbf{V} = 60 \text{ V}\angle -30°$, $\mathbf{I} = 4 \text{ A}\angle 60°$.
    d. $\mathbf{V} = 140 \text{ V}\angle -30°$, $\mathbf{I} = 14 \text{ mA}\angle -120°$.
35. For each circuit of Figure 16–57, determine the unknown.
36. a. If $\mathbf{V}_L = 120 \text{ V} \angle 67°$, $L = 600\ \mu\text{H}$, and $f = 10$ kHz, what is $\mathbf{I}_L$?
    b. If $\mathbf{I}_L = 48 \text{ mA} \angle -43°$, $L = 550$ mH, and $f = 700$ Hz, what is $\mathbf{V}_L$?
    c. If $\mathbf{V}_C = 50 \text{ V} \angle -36°$, $C = 390$ pF, and $f = 470$ kHz, what is $\mathbf{I}_C$?
    d. If $\mathbf{I}_C = 95 \text{ mA} \angle 87°$, $C = 6.5$ nF, and $f = 1.2$ MHz, what is $\mathbf{V}_C$?




(a) $L = 0.2$ H.
Determine *f*.




(b) $f = 100$ Hz.
Determine *C*.

**FIGURE 16–57**

## 16.8 Computer Analysis of AC Circuits

The version of Electronics Workbench current at the time of writing of this book is unable to measure phase angles. Thus, in the problems that follow, we ask only for magnitudes of voltages and currents.

37. **EWB** Create the circuit of Figure 16–58 on the screen. (Use the ac source from the Sources Parts bin and the ammeter from the Indicators Parts bin.) Double click the ammeter symbol and set Mode to AC. Click the ON/OFF switch at the top right hand corner of the screen to energize the circuit. Compare the measured reading against the theoretical value.

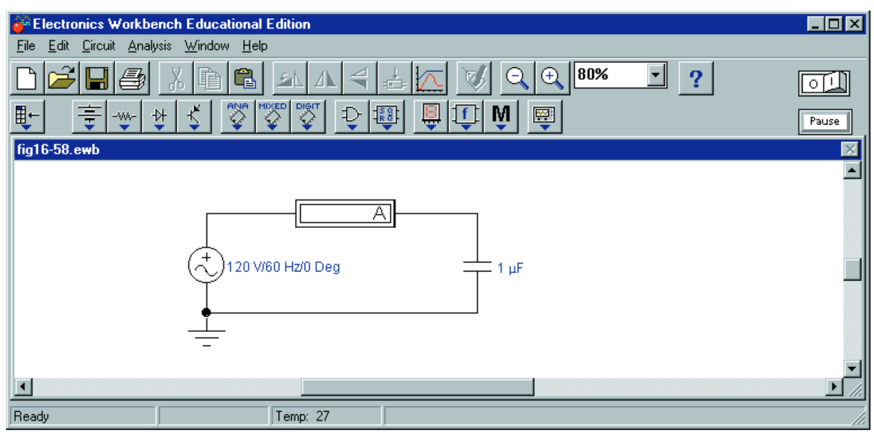



**FIGURE 16–58**

38. **EWB** Replace the capacitor of Figure 16–58 with a 200-mH inductor and repeat Problem 37.
39. **PSpice** Create the circuit of Figure 16–53 on the screen. Use a source of $100 \text{ V}\angle 0°$, $L = 0.2$ H, and $f = 50$ Hz. Solve for current $\mathbf{I}_L$ (magnitude and angle). See note below.

40. **PSpice** Plot the reactance of a 2.39-H inductor versus frequency from 1 Hz to 500 Hz and compare to Figure 16–28. Change the x-axis scale to linear.
41. **PSpice** For the circuit of Problem 39, plot current magnitude versus frequency from $f = 1$ Hz to $f = 20$ Hz. Measure the current at 10 Hz and verify with your calculator.

Note: PSpice does not permit source/inductor loops. To get around this, add a very small resistor in series, for example, $R = 0.00001\ \Omega$.

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## ANSWERS TO IN-PROCESS LEARNING CHECKS

### In-Process Learning Check 1

1. a. $6\angle 90°$ d. $7.21\angle -56.3°$ g. $3.61\angle -123.7°$
   b. $4\angle -90°$ e. $9.43\angle 122.0°$
   c. $4.24\angle 45°$ f. $2.24\angle -63.4°$
2. a. $j4$ d. $3.83 + j3.21$ g. $-1.64 - j0.599$
   b. $3 + j0$ e. $-3 + j5.20$
   c. $-j2$ f. $2.35 - j0.855$
3. $12\angle 40°$
4. $88 - j7$; $-16 + j15$
5. $288\angle -100°$; $2\angle 150°$
6. a. $14.70 + j2.17$ b. $-8.94 + j7.28$
7. $18.0 \sin(\omega t - 56.3°)$

### In-Process Learning Check 2

1. $50 \sin(\omega t + 30°)$ A
2. $50 \sin(\omega t - 60°)$ A
3. $50 \sin(\omega t + 120°)$ A
4. 0.637 H
5. 3.98 $\mu$F
6. a. Voltage and current are in phase. Therefore, R
   b. Current leads by 90°. Therefore, C
   c. Current leads by 90°. Therefore, C
   d. Current lags by 90°. Therefore, L

# 17 Power in AC Circuits

## OBJECTIVES

After studying this chapter, you will be able to

- explain what is meant by active, reactive, and apparent power,
- compute the active power to a load,
- compute the reactive power to a load,
- compute the apparent power to a load,
- construct and use the power triangle to analyze power to complex loads,
- compute power factor,
- explain why equipment is rated in VA instead of watts,
- measure power in single-phase circuits,
- describe why effective resistance differs from geometric resistance,
- describe energy relations in ac circuits,
- use PSpice to study instantaneous power.

## KEY TERMS

Active Power
Apparent Power
Average Power
Effective Resistance
$F_p$
Instantaneous Power
Power Factor Correction
Power Factor
Power Triangle
$Q$
Reactive Power
$S$
Skin Effect
VA
VAR
Wattless Power
Wattmeter

## OUTLINE

## CHAPTER PREVIEW

In Chapter 4, you studied power in dc circuits. In this chapter, we turn our attention to power in ac circuits. In ac circuits, there are additional considerations that are not present with dc. In dc circuits, for example, the only power relationship you encounter is $P = VI$ watts. This is referred to as *real power* or *active power* and is the power that does useful work such as light a lamp, power a heater, run an electric motor, and so on.

In ac circuits, you also encounter this type of power. For ac circuits that contain reactive elements however, (i.e., inductance or capacitance), a second component of power also exists. This component, termed *reactive power*, represents energy that oscillates back and forth throughout the system. For example, during the buildup of current in an inductance, energy flows from the power source to the inductance to create its magnetic field. When the magnetic field collapses, this energy is returned to the circuit. This movement of energy in and out of the inductance constitutes a flow of power. However, since it flows first in one direction, then in the other, it contributes nothing to the average flow of power from the source to the load. For this reason, reactive power is sometimes referred to as *wattless power.* (A similar situation exists regarding power flow to and from the electric field of a capacitor.)

For a circuit that contains resistive as well as reactive elements, some energy is dissipated while the remainder is shuttled back and forth as described above; thus, both active and reactive components of power are present. This combination of real and reactive power is termed *apparent power.*

In this chapter, we look at all three components of power. New ideas that emerge include the concept of power factor, the power triangle, the measurement of power in ac circuits, and the concept of effective resistance.

## PUTTING IT IN PERSPECTIVE

### *Henry Cavendish*

CAVENDISH, AN ENGLISH CHEMIST and physicist born in 1731, is included here not for what he did for the emerging electrical field, but for what he didn't do. A brilliant man, Cavendish was 50 years ahead of his time, and his experiments in electricity preceded and anticipated almost all the major discoveries that came about over the next half century (e.g., he discovered Coulomb's law before Coulomb did). However, Cavendish was interested in research and knowledge purely for its own sake and never bothered to publish most of what he learned, in effect depriving the world of his findings and holding back the development of the field of electricity by many years. Cavendish's work lay unknown for nearly a century before another great scientist, James Clerk Maxwell, had it published. Nowadays, Cavendish is better known for his work in the gravitational field than for his work in the electrical field. One of the amazing things he did was to determine the mass of the earth using the rather primitive technology of his day.

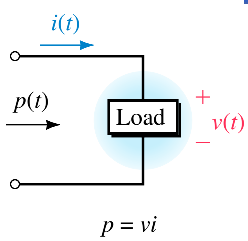



**FIGURE 17–1** Voltage, current, and power references. When $p$ is positive, power is in the direction of the reference arrow.

## 17.1 Introduction

At any given instant, the power to a load is equal to the product of voltage times current (Figure 17–1). This means that if voltage and current vary with time, so will power. This time-varying power is referred to as **instantaneous power** and is given the symbol $p(t)$ or just $p$. Thus,

$$p = vi \quad \text{(watts)} \tag{17–1}$$

Now consider the case of sinusoidal ac. Since voltage and current are positive at various times during their cycle and negative at others, instantaneous power may also be positive at some times and negative at others. This is illustrated in Figure 17–2, where we have multiplied voltage times current point by point to get the power waveform. For example, from $t = 0$ s to $t = t_1$, $v$ and $i$ are both positive; therefore, power is positive. At $t = t_1$, $v = 0$ V and thus $p = 0$ W. From $t_1$ to $t_2$, $i$ is positive and $v$ is negative; therefore, $p$ is negative. From $t_2$ to $t_3$, both $v$ and $i$ are negative; therefore power is positive, and so on. As discussed in Chapter 4, a positive value for $p$ means that power transfer is in the direction of the reference arrow, while a negative value means that it is in the opposite direction. Thus, during positive parts of the power cycle, power flows from the source to the load, while during negative parts, it flows out of the load back into the circuit.

The waveform of Figure 17–2 is the actual power waveform. We will now show that the key aspects of power flow embodied in this waveform can be described in terms of active power, reactive power, and apparent power.

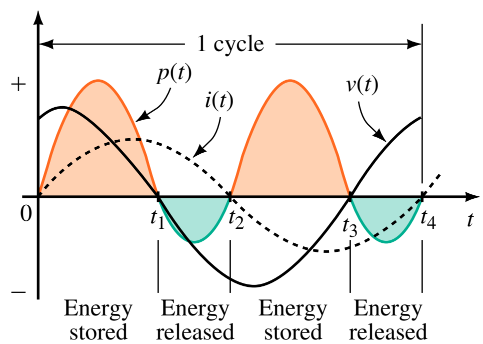



**FIGURE 17–2** Instantaneous power in an ac circuit. Positive $p$ represents power to the load; negative $p$ represents power returned from the load.

### Active Power

Since $p$ represents the power flowing to the load, its average will be the average power to the load. Denote this average by the letter $P$. If $P$ is positive, then, on average, more power flows to the load than is returned from it. (If $P$ is zero, all power sent to the load is returned.) Thus, if $P$ has a positive value, it represents the power that is really dissipated by the load. For this reason, $P$ is called **real power.** In modern terminology, real power is also called **active power.** Thus, *active power is the average value of the instantaneous power, and the terms real power, active power, and average power mean the same* *thing.* (We usually refer to it simply as power.) In this book, we use the terms interchangeably.

### Reactive Power

Consider again Figure 17–2. During the intervals that $p$ is negative, power is being returned from the load. (This can only happen if the load contains reactive elements: $L$ or $C$.) The portion of power that flows into the load then back out is called **reactive power.** Since it first flows one way then the other, *its average value is zero;* thus, reactive power contributes nothing to the average power to the load.

Although reactive power does no useful work, it cannot be ignored. Extra current is required to create reactive power, and this current must be supplied by the source; this also means that conductors, circuit breakers, switches, transformers, and other equipment must be made physically larger to handle the extra current. This increases the cost of a system.

At this point, it should be noted that real power and reactive power do not exist as separate entities. Rather, they are components of the power waveform shown in Figure 17–2. However, as you will see, we are able to conceptually separate them for purposes of analysis.

## 17.2 Power to a Resistive Load

First consider power to a purely resistive load (Figure 17–3). Here, current is in phase with voltage. Assume $i = I_m \sin \omega t$ and $v = V_m \sin \omega t$. Then,

$$p = vi = (V_m \sin \omega t)(I_m \sin \omega t) = V_m I_m \sin^2 \omega t$$

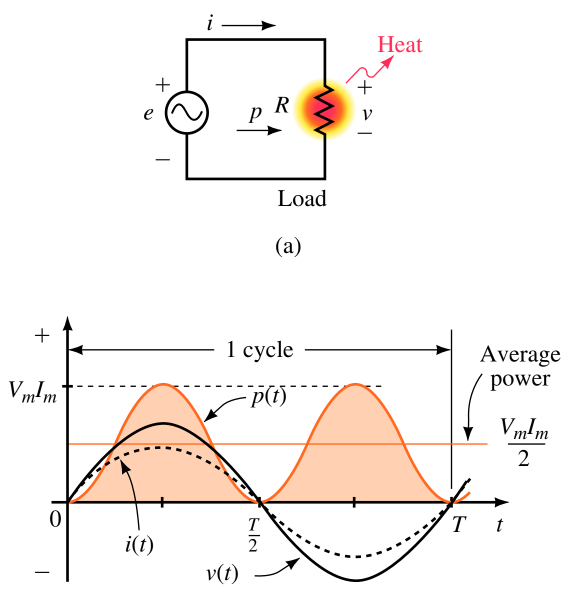



**FIGURE 17–3** Power to a purely resistive load. The peak value of $p$ is $V_m I_m$.

Therefore,

$$p = \frac{V_m I_m}{2}(1 - \cos 2\,\omega t) \tag{17–2}$$

where we have used the trigonometric relationship $\sin^2\omega t = \frac{1}{2}(1 - \cos 2\,\omega t)$ from inside the front cover of the book.

A sketch of $p$ versus time is shown in (b). Note that $p$ is always positive (except where it is momentarily zero). This means that power flows only from the source to the load. Since none is ever returned, all power delivered by the source is absorbed by the load. We therefore conclude that *power to a pure resistance consists of active power only.* Note also that the frequency of the power waveform is double that of the voltage and current waveforms. (This is confirmed by the $2\omega$ in Equation 17–2.)

### Average Power

Inspection of the power waveform of Figure 17–3 shows that its average value lies half way between zero and its peak value of $V_m I_m$. That is,

$$P = V_m I_m/2$$

(You can also get the same result by averaging Equation 17–2 as we did in Chapter 15.) Since $V$ (the magnitude of the rms value of voltage) is $V_m/\sqrt{2}$ and $I$ (the magnitude of the rms value of current) is $I_m/\sqrt{2}$, this can be written as $P = VI$. Thus, average power to a purely resistive load is

$$P = VI \quad \text{(watts)} \tag{17–3}$$

Alternate forms are obtained by substituting $V = IR$ and $I = V/R$ into Equation 17–3. They are

$$P = I^2R \quad \text{(watts)} \tag{17–4}$$

$$= V^2/R \quad \text{(watts)} \tag{17–5}$$

Thus the active power relationships for resistive circuits are the same for ac as for dc.

## 17.3 Power to an Inductive Load

For a purely inductive load as in Figure 17–4(a), current lags voltage by 90°. If we select current as reference, $i = I_m\sin\omega t$ and $v = V_m\sin(\omega t + 90°)$. A sketch of $p$ versus time (obtained by multiplying $v$ times $i$) then looks as shown in (b). Note that during the first quarter-cycle, $p$ is positive and hence power flows to the inductance, while during the second quarter-cycle, $p$ is negative and all power transferred to the inductance during the first quarter-cycle flows back out. Similarly for the third and fourth quarter-cycles. Thus, *the average power to an inductance over a full cycle is zero, i.e., there are no power losses associated with a pure inductance.* Consequently, $P_L = 0$ W and the only power flowing in the circuit is reactive power. This is true in general, that is, *the power that flows into and out of a pure inductance is reactive power only.*

To determine this power, consider again equation 17–1.

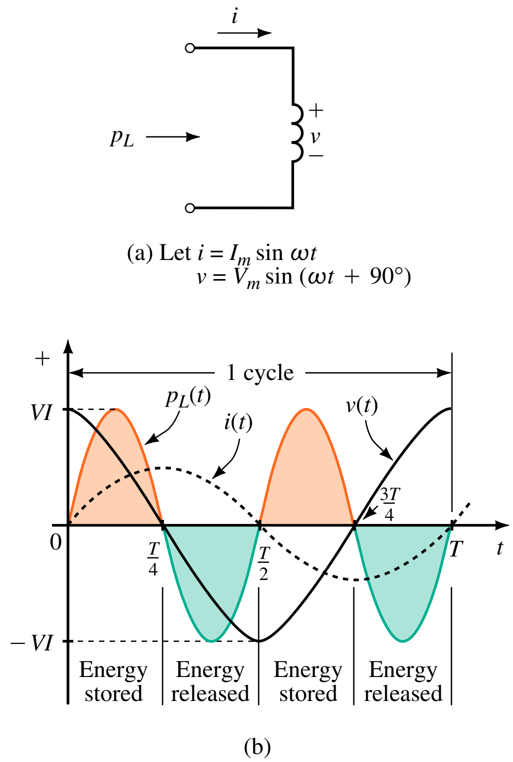



**FIGURE 17–4** Power to a purely inductive load. Energy stored during each quarter-cycle is returned during the next quarter-cycle. Average power is zero.

With $v = V_m\sin(\omega t + 90°)$ and $i = I_m\sin \omega t$, $p_L = vi$ becomes

$$p_L = V_m I_m \sin(\omega t + 90°)\sin \omega t$$

After some trigonometric manipulation, this reduces to

$$p_L = VI \sin 2\omega t \tag{17–6}$$

where $V$ and $I$ are the magnitudes of the rms values of the voltage and current respectively.

The product $VI$ in Equation 17–6 is defined as **reactive power** and is given the symbol $Q_L$. Because it represents "power" that alternately flows into, then out of the inductance, $Q_L$ contributes nothing to the average power to the load and, as noted earlier, is sometimes referred to as wattless power. As you will soon see, however, reactive power is of major concern in the operation of electrical power systems.

Since $Q_L$ is the product of voltage times current, its unit is the volt-amp (VA). To indicate that $Q_L$ represents reactive volt-amps, an "R" is appended to yield a new unit, the *VAR* (*volt-amps reactive*). Thus,

$$Q_L = VI \quad \text{(VAR)} \tag{17–7}$$

Substituting $V = IX_L$ and $I = V/X_L$ yields the following alternate forms:

$$Q_L = I^2 X_L = \frac{V^2}{X_L} \quad \text{(VAR)} \tag{17–8}$$

By convention, $Q_L$ is taken to be positive. Thus, if $I = 4$ A and $X_L = 2\ \Omega$, $Q_L = (4\text{ A})^2(2\ \Omega) = +32$ VAR. Note that the VAR (like the watt) is a scalar quantity with magnitude only and no angle.

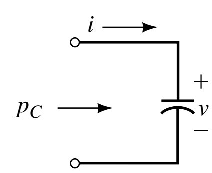
## 17.4 Power to a Capacitive Load

For a purely capacitive load, current leads voltage by 90°. Taking current as reference, $i = I_m \sin \omega t$ and $v = V_m \sin(\omega t - 90°)$. Multiplication of $v$ times $i$ yields the power curve of Figure 17–5. Note that negative and positive loops of the power wave are identical; thus, over a cycle, the power returned to the circuit by the capacitance is exactly equal to that delivered to it by the source. This means that *the average power to a capacitance over a full cycle is zero, i.e., there are no power losses associated with a pure capacitance.* Consequently, $P_C = 0$ W and the only power flowing in the circuit is reactive power. This is true in general, that is, *the power that flows into and out of a pure capacitance is reactive power only.* This reactive power is given by

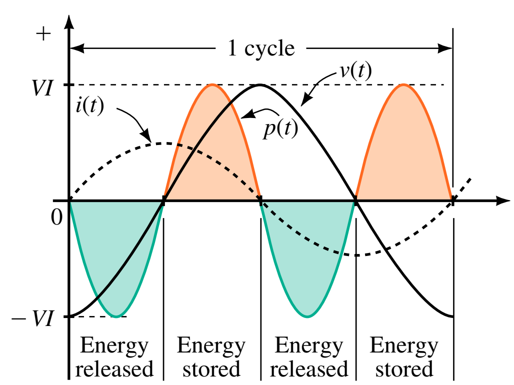
$$p_C = vi = V_m I_m \sin \omega t \sin(\omega t - 90°)$$

which reduces to

$$p_C = -VI \sin 2\omega t \tag{17–9}$$




(a) Let $i = I_m \sin \omega t$
$v = V_m \sin(\omega t - 90°)$




(b)

**FIGURE 17–5** Power to a purely capacitive load. Average power is zero.

where $V$ and $I$ are the magnitudes of the rms values of the voltage and current respectively. Now define the product $VI$ as $Q_C$. This product represents reactive power. That is,

$$Q_C = VI \quad \text{(VAR)} \tag{17–10}$$

Since $V = IX_C$ and $I = V/X_C$, $Q_C$ can also be expressed as

$$Q_C = I^2X_C = \frac{V^2}{X_C} \quad \text{(VAR)} \tag{17–11}$$

By convention, reactive power to capacitance is defined as negative. Thus, if $I = 4$ A and $X_C = 2\ \Omega$, then $I^2X_C = (4\text{ A})^2(2\ \Omega) = 32$ VAR. We can either explicitly show the minus sign as $Q_C = -32$ VAR or imply it by stating that $Q$ represents capacitive vars, i.e. $Q_C = 32$ VAR (cap.).

**EXAMPLE 17–1** For each circuit of Figure 17–6, determine real and reactive power.

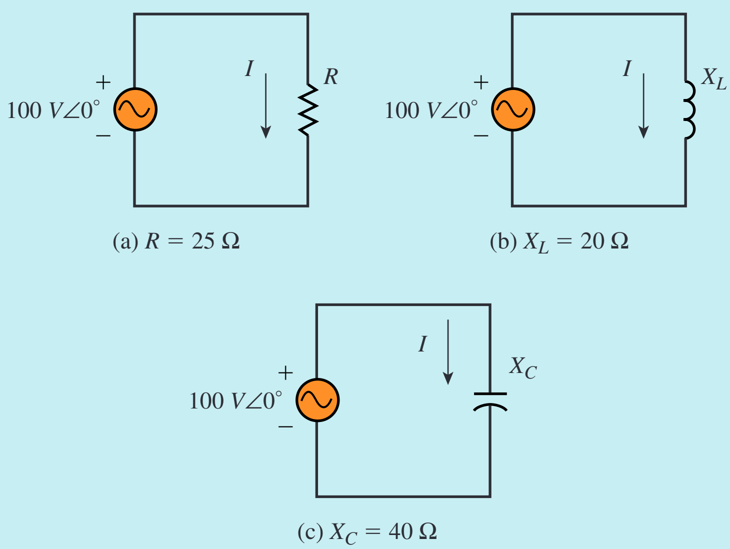



**FIGURE 17–6**

**Solution** Only voltage and current magnitudes are needed.

a. $I = 100\text{ V}/25\ \Omega = 4$ A. $P = VI = (100\text{ V})(4\text{ A}) = 400$ W. $Q = 0$ VAR

b. $I = 100\text{ V}/20\ \Omega = 5$ A. $Q = VI = (100\text{ V})(5\text{ A}) = 500$ VAR (ind.). $P = 0$ W

c. $I = 100\text{ V}/40\ \Omega = 2.5$ A. $Q = VI = (100\text{ V})(2.5\text{ A}) = 250$ VAR (cap.). $P = 0$ W

The answer for (c) can also be expressed as $Q = -250$ VAR.

**PRACTICE PROBLEMS 1**

1. If the power at some instant in Figure 17–1 is $p = -27$ W, in what direction is the power at that instant?
2. For a purely resistive load, $v$ and $i$ are in phase. Given $v = 10 \sin \omega t$ V and $i = 5 \sin \omega t$ A. Using graph paper, carefully plot $v$ and $i$ at 30° intervals. Now multiply the values of $v$ and $i$ at these points and plot the power. (The result should look like Figure 17–3(b).
   a. From the graph, determine the peak power and average power.
   b. Compute power using $P = VI$ and compare to the average value determined in (a).
3. Repeat Example 17–1 using equations 17–4, 17–5, 17–8, and 17–11.

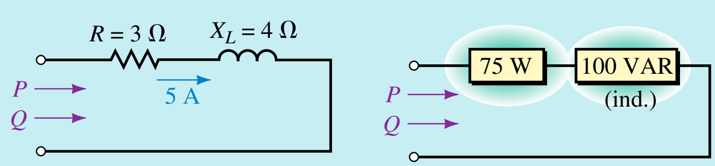
*Answers:*
1. From the load to the source.
2. a. 50 W; 25 W
   b. Same

## 17.5 Power in More Complex Circuits

The relationships described above were developed using the load of Figure 17–1. However, they hold true for every element in a circuit, no matter how complex the circuit or how its elements are interconnected. Further, in any circuit, total real power $P_T$ is found by summing real power to all circuit elements, while total reactive power $Q_T$ is found by summing reactive power, taking into account that inductive $Q$ is positive and capacitive $Q$ is negative.

It is sometimes convenient to show power to circuit elements symbolically as illustrated in the next example.

**EXAMPLE 17–2** For the *RL* circuit of Figure 17–7(a), $I = 5$ A. Determine $P$ and $Q$.




(a) (b) Symbolic representation

**FIGURE 17–7** From the terminals, $P$ and $Q$ are the same for both (a) and (b).

**Solution**

$$P = I^2R = (5\text{ A})^2(3\ \Omega) = 75\text{ W}$$

$$Q = Q_L = I^2X_L = (5\text{ A})^2(4\ \Omega) = 100\text{ VAR (ind.)}$$

These can be represented symbolically as in Figure 17–7(b).

**EXAMPLE 17–3** For the *RC* circuit of Figure 17–8(a), determine *P* and *Q*.

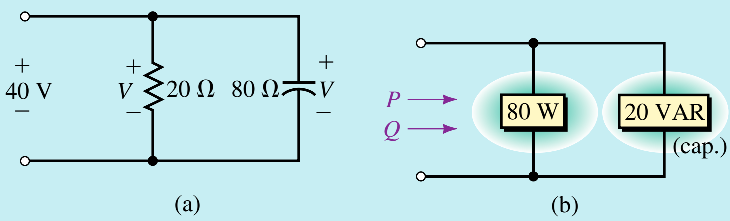



**FIGURE 17–8** From the terminals, *P* and *Q* are the same for both (a) and (b).

**Solution**

$$P = V^2/R = (40\text{ V})^2/(20\ \Omega) = 80\text{ W}$$

$$Q = Q_C = V^2/X_C = (40\text{ V})^2/(80\ \Omega) = 20\text{ VAR (cap.)}$$

These can be represented symbolically as in Figure 17–8(b).

In terms of determining total *P* and *Q*, it does not matter how the circuit or system is connected or what electrical elements it contains. Elements can be connected in series, in parallel, or in series-parallel, for example, and the system can contain electric motors and the like, and total *P* is still found by summing the power to individual elements, while total *Q* is found by algebraically summing their reactive powers.

**EXAMPLE 17–4**

a. For Figure 17–9(a), compute $P_T$ and $Q_T$.

b. Reduce the circuit to its simplest form.

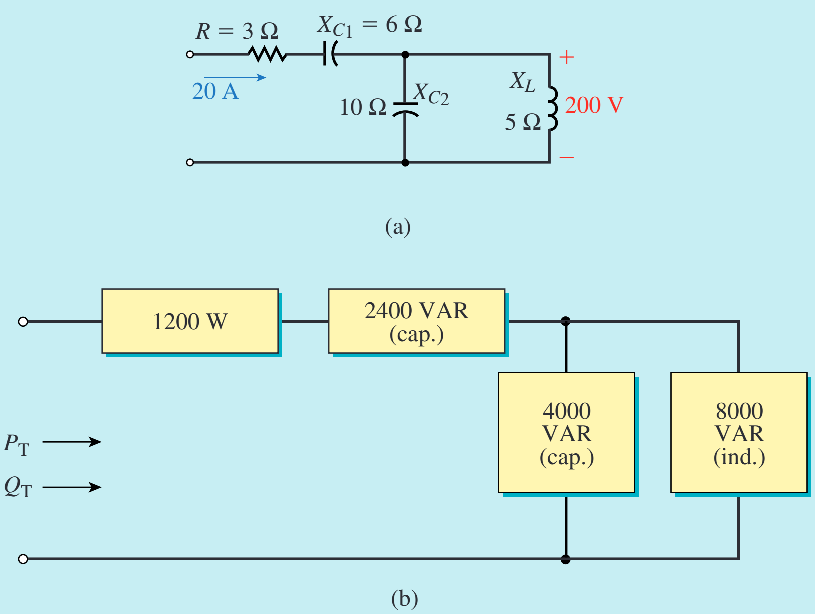
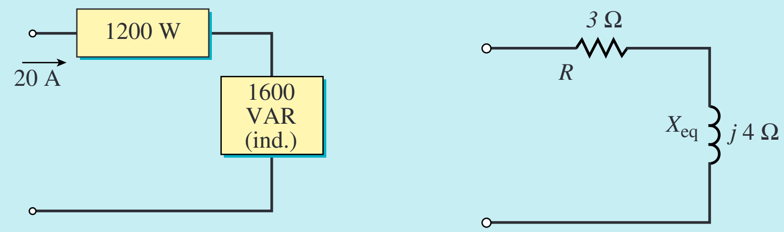



**FIGURE 17–9**




(c) (d)

**FIGURE 17–9** Continued.

**Solution**

a. $P = I^2R = (20\text{ A})^2(3\ \Omega) = 1200\text{ W}$

$$Q_{C_1} = I^2X_{C_1} = (20\text{ A})^2(6\ \Omega) = 2400\text{ VAR (cap.)}$$

$$Q_{C_2} = \frac{V_2^2}{X_{C_2}} = \frac{(200\text{ V})^2}{(10\ \Omega)} = 4000\text{ VAR (cap.)}$$

$$Q_L = \frac{V_2^2}{X_L} = \frac{(200\text{ V})^2}{5\ \Omega} = 8000\text{ VAR (ind.)}$$

These are represented symbolically in part (b). $P_T = 1200$ W and $Q_T = -2400\text{ VAR} - 4000\text{ VAR} + 8000\text{ VAR} = 1600\text{ VAR}$. Thus, the load is net inductive as shown in (c).

b. $Q_T = I^2X_{eq}$. Thus, $X_{eq} = Q_T/I^2 = (1600\text{ VAR})/(20\text{ A})^2 = 4\ \Omega$. Circuit resistance remains unchanged. Thus, the equivalent is as shown in (d).

**PRACTICE PROBLEMS 2**

For the circuit of Figure 17–10, $P_T = 1.9$ kW and $Q_T = 900$ VAR (ind.). Determine $P_2$ and $Q_2$.

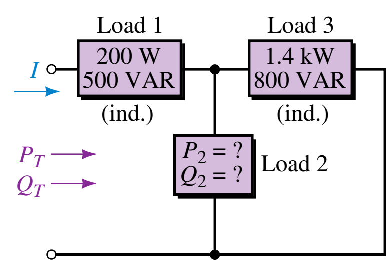



**FIGURE 17–10**

*Answer:* 300 W 400 VAR (cap.)

## 17.6 Apparent Power

When a load has voltage $V$ across it and current $I$ through it as in Figure 17–11, the power that appears to flow to it is $VI$. However, if the load contains both resistance and reactance, this product represents neither real power nor reactive power. Since it appears to represent power, it is called **apparent power.** Apparent power is given the symbol $S$ and has units of volt-amperes (VA). Thus,

$$S = VI \quad \text{(VA)} \tag{17–12}$$

where $V$ and $I$ are the magnitudes of the rms voltage and current respectively. Since $V = IZ$ and $I = V/Z$, $S$ can also be written as

$$S = I^2Z = V^2/Z \quad \text{(VA)} \tag{17–13}$$

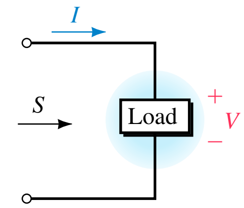



**FIGURE 17–11** Apparent power $S = VI$.

For small equipment (such as found in electronics), VA is a convenient unit. However, for heavy power apparatus (Figure 17–12), it is too small and kVA (kilovolt-amps) is frequently used, where

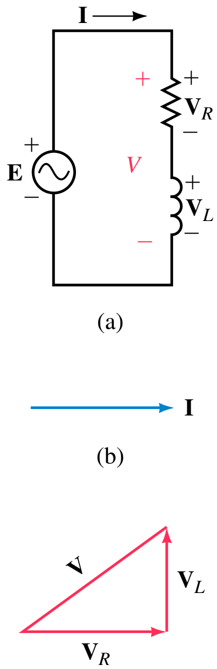
$$S = \frac{VI}{1000} \quad \text{(kVA)} \tag{17–14}$$

In addition to its VA rating, it is common practice to rate electrical apparatus in terms of its operating voltage. Once you know these two, it is easy to determine rated current. For example, a piece of equipment rated at 250 kVA, 4.16 kV has a rated current of $I = S/V = (250 \times 10^3 \text{ VA})/(4.16 \times 10^3 \text{ V}) = 60.1$ A.

**FIGURE 17–12** Power apparatus is rated in apparent power. The transformer shown is a 167-kVA unit. (*Courtesy Carte International Ltd.*)

## 17.7 The Relationship Between *P, Q,* and *S*

Until now, we have treated real, reactive, and apparent power separately. However, they are related by a very simple relationship through the power triangle.



(c)

FIGURE 17–13 Steps in the development of the power triangle.

## The Power Triangle

Consider the series circuit of Figure 17–13(a). Let the current through the circuit be $\mathbf{I} = I\angle 0°$, with phasor representation (b). The voltages across the resistor and inductance are $\mathbf{V}_R$ and $\mathbf{V}_L$ respectively. As noted in Chapter 16, $\mathbf{V}_R$ is in phase with **I,** while $\mathbf{V}_L$ leads it by 90°. Kirchhoff's voltage law applies for ac voltages in phasor form. Thus, $\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L$ as indicated in (c).

The voltage triangle of (c) may be redrawn as in Figure 17–14(a) with magnitudes of $V_R$ and $V_L$ replaced by $IR$ and $IX_L$ respectively. Now multiply all quantities by $I$. This yields sides of $I^2R$, $I^2X_L$, and hypotenuse $VI$ as indicated in (b). Note that these represent $P$, $Q$, and $S$ respectively as indicated in (c). This is called the **power triangle.** From the geometry of this triangle, you can see that

$$S = \sqrt{P^2 + Q_L^2} \tag{17–15}$$

Alternatively, the relationship between $P$, $Q$, and $S$ may be expressed as a complex number:

$$\mathbf{S} = P + jQ_L \tag{17–16a}$$

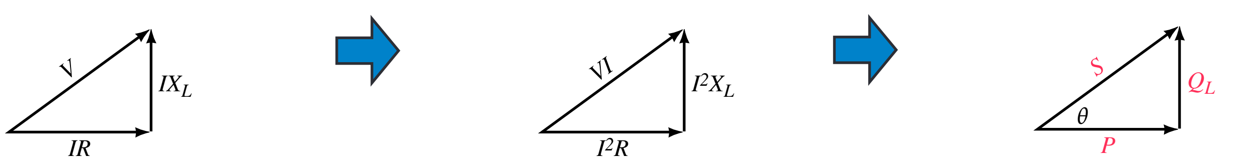
or

$$\mathbf{S} = S\angle\theta \tag{17–16b}$$

If the circuit is capacitive instead of inductive, Equation 17–16a becomes

$$\mathbf{S} = P - jQ_C \tag{17–17}$$

The power triangle in this case has a negative imaginary part as indicated in Figure 17–15.




(a) Magnitudes only shown (b) Multiplied by $I$ (c) Resultant power triangle

FIGURE 17–14 Steps in the development of the power triangle (continued).

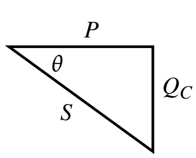



FIGURE 17–15 Power triangle for capacitive case.

The power relationships may be written in generalized forms as

$$\mathbf{S} = \mathbf{P} + \mathbf{Q} \tag{17–18}$$

and

$$\mathbf{S} = \mathbf{VI}^* \tag{17–19}$$

where $\mathbf{P} = P\angle 0°$, $\mathbf{Q}_L = jQ_L$, $\mathbf{Q}_C = -jQ_C$, and $\mathbf{I}^*$ is the conjugate of current **I.** These relationships hold true for all networks regardless of what they contain or how they are configured.

When solving problems involving power, remember that $P$ values can be added to get $P_T$, and $Q$ values to get $Q_T$ (where $Q$ is positive for inductive elements and negative for capacitive). However, apparent power values cannot be added to get $S_T$, i.e., $S_T \neq S_1 + S_2 + \cdots + S_N$. Instead, determine $P_T$ and $Q_T$, then use the power triangle to obtain $S_T$.

**EXAMPLE 17–5** The *P* and *Q* values for a circuit are shown in Figure 17–16(a).

a. Determine the power triangle.

b. Determine the magnitude of the current supplied by the source.

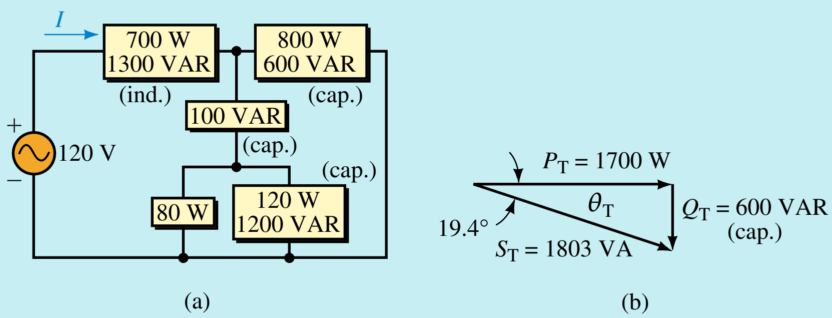



**FIGURE 17–16**

**Solution**

a. $P_T = 700 + 800 + 80 + 120 = 1700$ W

$Q_T = 1300 - 600 - 100 - 1200 = -600$ VAR $= 600$ VAR (cap.)

$\mathbf{S}_T = P_T + jQ_T = 1700 - j600 = 1803\angle -19.4°$ VA

The power triangle is as shown. The load is net capacitive.

b. $I = S_T/E = 1803$ VA/120 V $= 15.0$ A

**EXAMPLE 17–6** A generator supplies power to an electric heater, an inductive element, and a capacitor as in Figure 17–17(a).

a. Find *P* and *Q* for each load.

b. Find total active and reactive power supplied by the generator.

c. Draw the power triangle for the combined loads and determine total apparent power.

d. Find the current supplied by the generator.

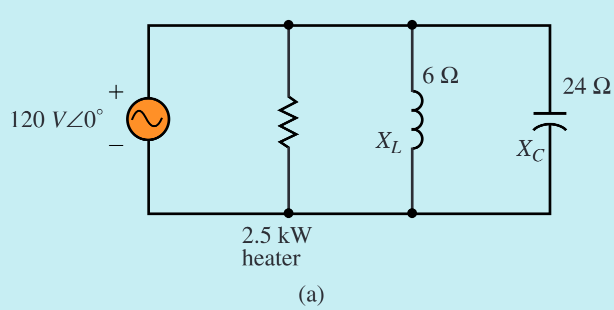
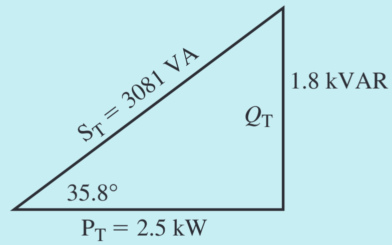



**FIGURE 17–17**




(b)

**FIGURE 17–17** Continued.

**Solution**

a. The components of power are as follows:

Heater: $P_H = 2.5$ kW $\quad Q_H = 0$ VAR

Inductor: $P_L = 0$ W $\quad Q_L = \dfrac{V^2}{X_L} = \dfrac{(120\text{ V})^2}{6\ \Omega} = 2.4$ kVAR (ind.)

Capacitor: $P_C = 0$ W $\quad Q_C = \dfrac{V^2}{X_C} = \dfrac{(120\text{ V})^2}{24\ \Omega} = 600$ VAR (cap.)

b. $P_T = 2.5\text{ kW} + 0\text{ W} + 0\text{ W} = 2.5\text{ kW}$

$Q_T = 0\text{ VAR} + 2.4\text{ kVAR} - 600\text{ VAR} = 1.8\text{ kVAR (ind.)}$

c. The power triangle is sketched as Figure 17–7(b). Both the hypotenuse and the angle can be obtained easily using rectangular to polar conversion. $\mathbf{S}_T = P_T + jQ_T = 2500 + j1800 = 3081\angle 35.8°$. Thus, apparent power is $S_T = 3081$ VA.

d. $I = \dfrac{S_T}{E} = \dfrac{3081\text{ VA}}{120\text{ V}} = 25.7\text{ A}$

## Active and Reactive Power Equations

An examination of the power triangle of Figures 17–14 and 17–15 shows that $P$ and $Q$ may be expressed respectively as

$$P = VI\cos\theta = S\cos\theta \quad \text{(W)} \tag{17–20}$$

and

$$Q = VI\sin\theta = S\sin\theta \quad \text{(VAR)} \tag{17–21}$$

where $V$ and $I$ are the magnitudes of the rms values of the voltage and current respectively and $\theta$ is the angle between them. $P$ is always positive, while $Q$ is positive for inductive circuits and negative for capacitive circuits. Thus, if $V = 120$ volts, $I = 50$ A, and $\theta = 30°$, $P = (120)(50)\cos 30° = 5196$ W and $Q = (120)(50)\sin 30° = 3000$ VAR.

PRACTICE PROBLEMS 3

A 208-V generator supplies power to a group of three loads. Load 1 has an apparent power of 500 VA with $\theta = 36.87°$ (i.e., it is net inductive). Load 2 has an apparent power of 1000 VA and is net capacitive with a power triangle angle of $-53.13°$. Load 3 is purely resistive with power $P_3 = 200$ W. Determine the power triangle for the combined loads and the generator current.

*Answers:* $S_T = 1300$ VA, $\theta_T = -22.6°$, $I = 6.25$ A

## 17.8 Power Factor

The quantity cos $\theta$ in Equation 17–20 is defined as **power factor** and is given the symbol $F_p$. Thus,

$$F_p = \cos\theta \tag{17–22}$$

From Equation 17–20, we see that $F_p$ may be computed as the ratio of real power to apparent power. Thus,

$$\cos\theta = P/S \tag{17–23}$$

Power factor is expressed as a number or as a percent. From Equation 17–23, it is apparent that power factor cannot exceed 1.0 (or 100% if expressed in percent).

The **power factor angle** $\theta$ is of interest. It can be found as

$$\theta = \cos^{-1}(P/S) \tag{17–24}$$

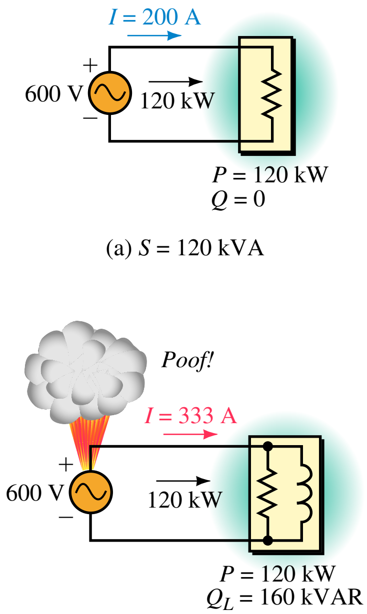
Angle $\theta$ is the angle between voltage and current. For a pure resistance, therefore, $\theta = 0°$. For a pure inductance, $\theta = 90°$; for a pure capacitance, $\theta = -90°$. For a circuit containing both resistance and inductance, $\theta$ will be somewhere between 0° and 90°; for a circuit containing both resistance and capacitance, $\theta$ will be somewhere between 0° and $-90°$.

### Unity, Lagging, and Leading Power Factor

As indicated by Equation 17–23, a load's power factor shows how much of its apparent power is actually real power. For example, for a purely resistive circuit, $\theta = 0°$ and $F_p = \cos 0° = 1.0$. Therefore, $P = VI$ (watts) and all the load's apparent power is real power. This case ($F_p = 1$) is referred to as *unity* power factor.

For a load containing only resistance and inductance, the load current lags voltage. The power factor in this case is described as *lagging.* On the other hand, for a load containing only resistance and capacitance, current leads voltage and the power factor is described as *leading.* Thus, *an inductive circuit has a lagging power factor, while a capacitive circuit has a leading power factor.*

A load with a very poor power factor can draw excessive current. This is discussed next.




(a) $S = 120$ kVA

(b) $S = \sqrt{(120)^2 + (160)^2} = 200$ kVA
The generator is overloaded.

**FIGURE 17–18** Illustrating why electrical apparatus is rated in VA instead of watts. Both loads dissipate 120 kW, but the current rating of generator (b) is exceeded because of the power factor of its load.

## Why Equipment Is Rated in VA

We now examine why electrical apparatus is rated in VA instead of watts. Consider Figure 17–18. Assume that the generator is rated at 600 V, 120 kVA. This means that it is capable of supplying $I = 120$ kVA/600 V $= 200$ A. In (a), the generator is supplying a purely resistive load with 120 kW. Since $S = P$ for a purely resistive load, $S = 120$ kVA and the generator is supplying its rated kVA. In (b), the generator is supplying a load with $P = 120$ kW as before, but $Q = 160$ kVAR. Its apparent power is therefore $S = 200$ kVA, which means that the generator current is $I = 200$ kVA/600 V $= 333.3$ A. Even though it is supplying the same power as in (a), the generator is now greatly overloaded, and damage may result as indicated in (b).

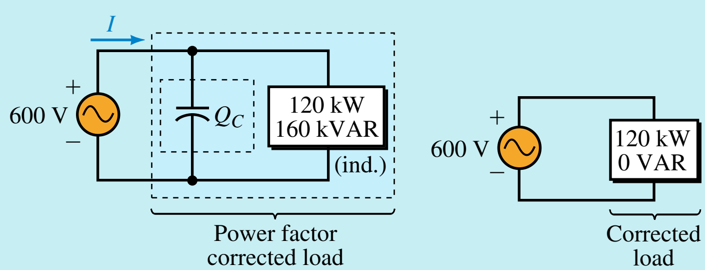
This example illustrates clearly that rating a load or device in terms of power is a poor choice, as its current-carrying capability can be greatly exceeded (even though its power rating is not). Thus, *the size of electrical apparatus (generators, interconnecting wires, transformers, etc.) required to supply a load is governed, not by the load's power requirements, but rather by its VA requirements.*

## Power Factor Correction

The problem shown in Figure 17–18 can be alleviated by cancelling some or all of the reactive component of power by adding reactance of the opposite type to the circuit. This is referred to as **power factor correction.** If you completely cancel the reactive component, the power factor angle is 0° and $F_p = 1$. This is referred to as **unity power factor correction.**

Residential customers are charged solely on the basis of energy used. This is because all residential power factors are essentially the same, and the

**EXAMPLE 17–7** For the circuit of Figure 17–18(b), a capacitance with $Q_C = 160$ kVAR is added in parallel with the load as in Figure 17–19(a). Determine generator current $I$.




(a) Let $Q_C = 160$ kVAR

(b) Load corrected to unity power factor

**FIGURE 17–19** Power factor correction. The parallel capacitor greatly reduces source current.

**Solution** $Q_T = 160$ kVAR $- 160$ kVAR $= 0$. Therefore, $\mathbf{S}_T = 120$ kW $+ j0$ kVAR. Thus, $S_T = 120$ kVA, and $I = 120$ kVA/600 V $= 200$ A. Thus, the generator is no longer overloaded.

power factor effect is simply built into the tariff. Industrial customers, on the other hand, have widely different power factors, and the electrical utility may have to take the power factors of these customers into account.

To illustrate, assume that the loads of Figures 17–18(a) and (b) are two small industrial plants. If the utility based its charge solely on power, both customers would pay the same amount. However, it costs the utility more to supply customer (b) since larger conductors, larger transformers, larger switchgear, and so on are required to handle the larger current. For this reason, industrial customers may pay a penalty if their power factor drops below a prescribed value.

**EXAMPLE 17–8** An industrial client is charged a penalty if the plant power factor drops below 0.85. The equivalent plant loads are as shown in Figure 17–20. The frequency is 60 Hz.

a. Determine $P_T$ and $Q_T$.
b. Determine what value of capacitance (in microfarads) is required to bring the power factor up to 0.85.
c. Determine generator current before and after correction.

**Solution**

a. The components of power are as follows:

Lights: $P = 12\text{ kW}, \qquad Q = 0\text{ kVAR}$

Furnace: $P = I^2R = (150)^2(2.4) = 54\text{ kW}$

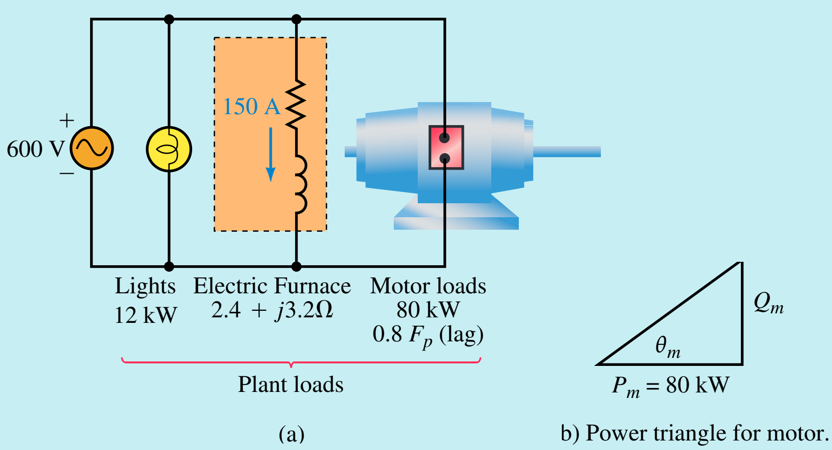
$Q = I^2X = (150)^2(3.2) = 72\text{ kVAR (ind.)}$

Motor: $\theta_m = \cos^{-1}(0.8) = 36.9°$. Thus, from the motor power triangle,

$Q_m = P_m \tan\theta_m = 80 \tan 36.9° = 60\text{ kVAR (ind.)}$

Total: $P_T = 12\text{ kW} + 54\text{ kW} + 80\text{ kW} = 146\text{ kW}$

$Q_T = 0 + 72\text{ kVAR} + 60\text{ kVAR} = 132\text{ kVAR}$




EWB **FIGURE 17–20**

b. The power triangle for the plant is shown in Figure 17–21(a). However, we must correct the power factor to 0.85. Thus we need $\theta' = \cos^{-1}(0.85) = 31.8°$, where $\theta'$ is the power factor angle of the corrected load as indicated in Figure 17–21(b). The maximum reactive power that we can tolerate is thus $Q'_T = P_T \tan\theta' = 146 \tan 31.8° = 90.5$ kVAR.

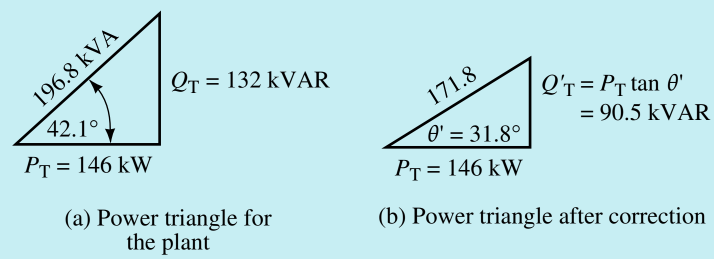



**FIGURE 17–21** Initial and final power triangles. Note that $P_T$ does not change when we correct the power factor.

Now consider Figure 17–22. $Q'_T = Q_C + 132$ kVAR, where $Q'_T = 90.5$ kVAR. Therefore, $Q_C = -41.5$ kVAR $= 41.5$ kVAR (cap.). But $Q_C = V^2/X_C$. Therefore, $X_C = V^2/Q_C = (600)^2/41.5$ kVAR $= 8.67\ \Omega$. But $X_C = 1/\omega C$. Thus a capacitor of

$$C = \frac{1}{\omega X_C} = \frac{1}{(2\pi)(60)(8.67)} = 306\ \mu\text{F}$$

will provide the required correction.

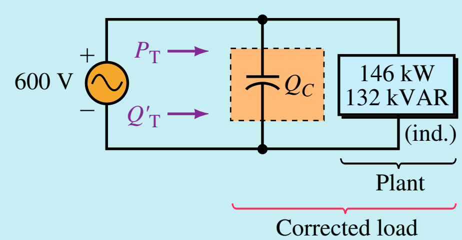



**FIGURE 17–22**

c. For the original circuit Figure 17–21(a), $S_T = 196.8$ kVA. Thus,

$$I = \frac{S_T}{E} = \frac{196.8\text{ kVA}}{600\text{ V}} = 328\text{ A}$$

For the corrected circuit 17–21(b), $S'_T = 171.8$ kVA and

$$I = \frac{171.8\text{ kVA}}{600\text{ V}} = 286\text{ A}$$

Thus, power factor correction has dropped the current by 42 A.

PRACTICE PROBLEMS 4

1. Repeat Example 17–8 except correct the power factor to unity.
2. Due to plant expansion, 102 kW of purely resistive load is added to the plant of Figure 17–20. Determine whether power factor correction is needed to correct the expanded plant to 0.85 $F_p$, or better.

*Answers:* 1. 973 $\mu$F, 243 A. Other answers remain unchanged.
2. $F_p = 0.88$. No correction needed.

In practice, almost all loads (industrial, residential, and commercial) are inductive due to the presence of motors, fluorescent lamp ballasts, and the like. Consequently, you will likely never run into capacitive loads that need power factor correcting.

IN-PROCESS LEARNING CHECK 1

1. Sketch the power triangle for Figure 17–9(c). Using this triangle, determine the magnitude of the applied voltage.
2. For Figure 17–10, assume a source of $E = 240$ volts, $P_2 = 300$ W, and $Q_2 = 400$ VAR (cap.). What is the magnitude of the source current $I$?
3. What is the power factor of each of the circuits of Figure 17–7, 17–8, and 17–9? Indicate whether they are leading or lagging.
4. Consider the circuit of Figure 17–18(b). If $P = 100$ kW and $Q_L = 80$ kVAR, is the source overloaded, assuming it is capable of handling a 120-kVA load?

*(Answers are at the end of the chapter.)*

## 17.9 AC Power Measurement

To measure power in an ac circuit, you need a wattmeter (since the product of voltage times current is not sufficient to determine ac power). Figure 17–23 shows such a meter. It is a digital device that monitors voltage and current and from these, computes and displays power. (You may also encounter older electrodynamometer type wattmeters, i.e., electromechanical analog instruments that use a pivoted pointer to indicate the power reading on a scale, much like the analog meters of Chapter 2. Although their details differ dramatically from the electronic types, the manner in which they are connected in a circuit to measure power is the same. Thus, the measurement techniques described below apply to them as well.)

To help understand power measurement, consider Figure 17–1. Instantaneous load power is the product of load voltage times load current, and average power is the average of this product. One way to implement power measurement is therefore to create a meter with a current sensing circuit, a voltage sensing circuit, a multiplier circuit, and an averaging circuit. Figure 17–24 shows a simplified symbolic representation of such an instrument. Current is passed through its current coil (CC) to create a magnetic field proportional to the current, and a sensor circuit connected across the load voltage reacts with this field to produce an output voltage proportional to the product of instantaneous voltage and current (i.e., proportional to

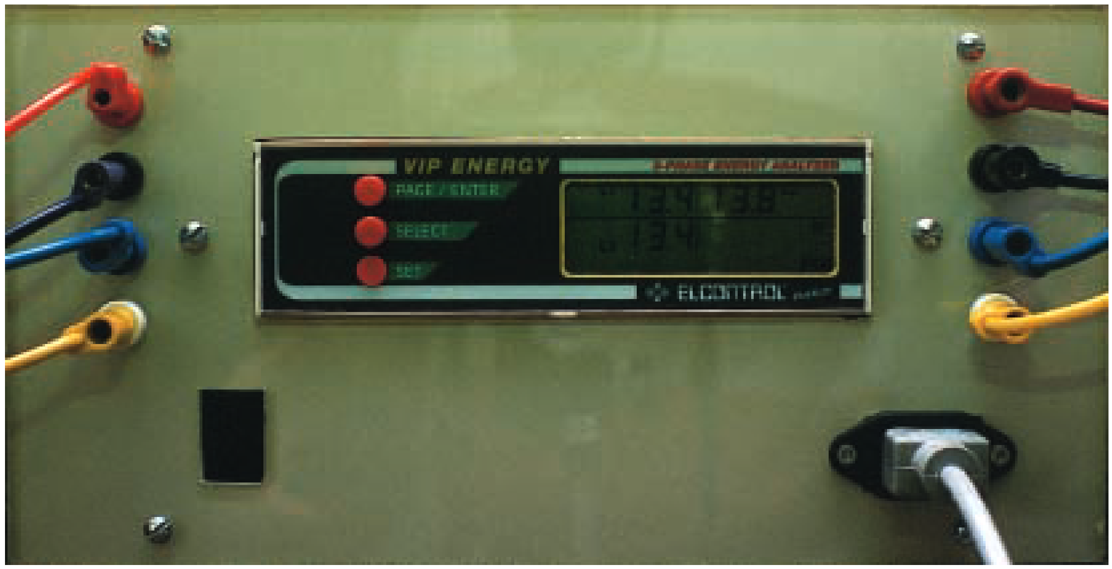



**FIGURE 17–23** Multifunction power/energy meter. It can measure active power (W), reactive power (VARs), apparent power (VA), power factor, energy, and more.

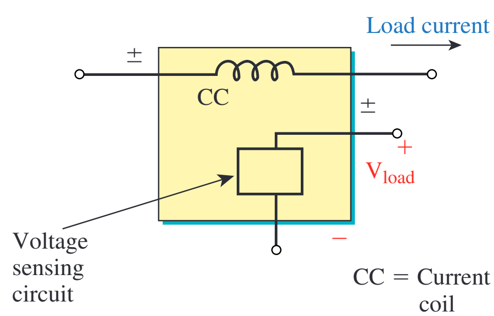



**FIGURE 17–24** Conceptual representation of a wattmeter.

instantaneous power). An averaging circuit averages this voltage and drives a display to indicate average power. (The scheme used by the meter of Figure 17–23 is actually considerably more sophisticated than this because it measures many things besides power—e.g., it measures, VARs, VA, energy, etc. However, the basic idea is conceptually correct.)

Figure 17–25 shows how to connect a wattmeter (whether electronic or electromechanical) into a circuit. Load current passes through its current coil circuit, and load voltage is impressed across its voltage sensing circuit. With this connection, the wattmeter computes and displays the product of the magnitude of the load voltage, the magnitude of the load current, and the cosine of the angle between them, i.e. $V_{load} \cdot I_{load} \cdot \cos \theta_{load}$. Thus, it measures load power. Note the ± marking on the terminals. You usually connect the meter so that load current enters the ± current terminal and the higher potential end of the load is connected to the ± voltage terminal. On many meters, the ± voltage terminal is internally connected so that only three terminals are brought out as in Figure 17–26.

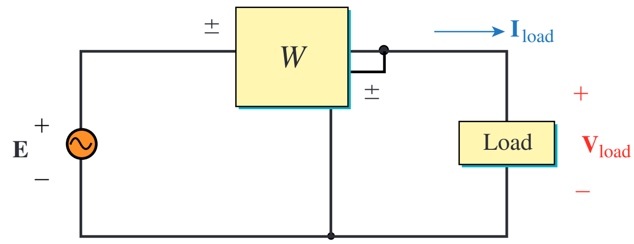



**FIGURE 17–25** Connection of wattmeter.

When power is to be measured in a low power factor circuit, a **low power factor wattmeter** must be used. This is because, for low power factor loads, currents can be very high, even though the power is low. Thus, you can easily exceed the current rating of a standard wattmeter and damage it, even though the power indication on the meter is small.

**EXAMPLE 17–9** For the circuit of Figure 17–25, what does the wattmeter indicate if

a. $\mathbf{V}_{load} = 100\text{ V}\angle 0°$ and $\mathbf{I}_{load} = 15\text{ A}\angle 60°$,
b. $\mathbf{V}_{load} = 100\text{ V}\angle 10°$ and $\mathbf{I}_{load} = 15\text{ A}\angle 30°$?

**Solution**

a. $\theta_{load} = 60°$. Thus, $P = (100)(15)\cos 60° = 750\text{ W}$,
b. $\theta_{load} = 10° - 30° = -20°$. Thus, $P = (100)(15)\cos(-20°) = 1410\text{ W}$.

Note: For (b), since $\cos(-20°) = \cos(+20°)$, it does not matter whether we include the minus sign.

**EXAMPLE 17–10** For Figure 17–26, determine the wattmeter reading.

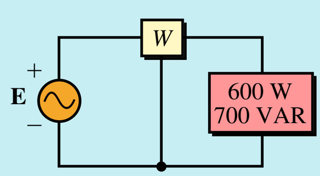



**FIGURE 17–26** This wattmeter has its voltage side ± terminals connected internally.

**Solution** A wattmeter reads only active power. Thus, it indicates 600 W.

It should be noted that the wattmeter reads power only for circuit elements on the load side of the meter. In addition, if the load consists of several elements, it reads the sum of the powers.

PRACTICE PROBLEMS 5

Determine the wattmeter reading for Figure 17–27.

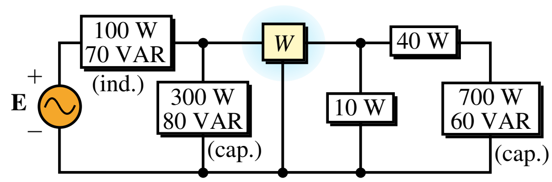



**FIGURE 17–27**

*Answer:* 750 W

## 17.10 Effective Resistance

Up to now, we have assumed that resistance is constant, independent of frequency. However, this is not entirely true. For a number of reasons, the resistance of a circuit to ac is greater than its resistance to dc. While this effect is small at low frequencies, it is very pronounced at high frequencies. AC resistance is known as **effective resistance.**

Before looking at why ac resistance is greater than dc resistance, we need to reexamine the concept of resistance itself. Recall from Chapter 3 that resistance was originally defined as opposition to current, that is, $R = V/I$. (This is ohmic resistance.) Building on this, you learned in Chapter 4 that $P = I^2R$. It is this latter viewpoint that allows us to give meaning to ac resistance. That is, we define ac or effective resistance as

$$R_{\text{eff}} = \frac{P}{I^2} \quad (\Omega) \tag{17–25}$$

where $P$ is dissipated power (as determined by a wattmeter). From this, you can see that anything that affects dissipated power affects resistance. For dc and low-frequency ac, both definitions for $R$, i.e., $R = V/I$ and $R = P/I^2$ yield the same value. However, as frequency increases, other factors cause an increase in resistance. We will now consider some of these.

### Eddy Currents and Hysteresis

The magnetic field surrounding a coil or other circuit carrying ac current varies with time and thus induces voltages in nearby conductive material such as metal equipment cabinets, transformer cores, and so on. The resulting currents (called **eddy currents** because they flow in circular patterns like eddies in a brook) are unwanted and create power losses called **eddy current losses.** Since additional power must be supplied to make up for these losses, $P$ in Equation 17–25 increases, increasing the effective resistance of the coil.

If ferromagnetic material is also present, an additional power loss occurs due to hysteresis effects caused by the magnetic field alternately magnetizing the material in one direction, then the other. Hysteresis and eddy current losses are important even at low frequencies, such as the 60-Hz power system frequency. This is discussed in Chapter 24.

### Skin Effect

Magnetically induced voltages created inside a conductor by its own changing magnetic field force electrons to the periphery of the conductor (Figure 17–28), resulting in a nonuniform distribution of current, with current density greatest near the periphery and smallest in the center. This phenomenon is known as **skin effect.** Because the center of the wire carries little current, its cross-sectional area has effectively been reduced, thus increasing resistance. While skin effect is generally negligible at power line frequencies (except for conductors larger than several hundred thousand circular mils), it is so pronounced at microwave frequencies that the center of a wire carries almost no current. For this reason, hollow conductors are often used instead of solid wires, as shown in Figure 17–28(c).

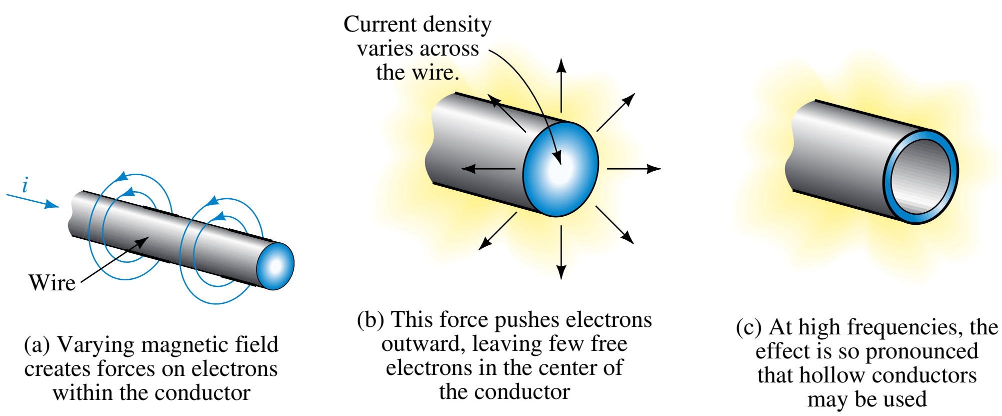



**FIGURE 17–28** Skin effect in ac circuits.

### Radiation Resistance

At high frequencies some of the energy supplied to a circuit escapes as radiated energy. For example, a radio transmitter supplies power to an antenna, where it is converted into radio waves and radiated into space. The resistance effect here is known as **radiation resistance.** This resistance is much higher than simple dc resistance. For example, a TV transmitting antenna may have a resistance of a fraction of an ohm to dc but several hundred ohms effective resistance at its operating frequency.

**FINAL NOTES...**

1. The resistance measured by an ohmmeter is dc resistance.
2. Many of the effects noted above will be treated in detail in your various electronics courses. We will not pursue them further here.

## 17.11 Energy Relationships for AC

Recall, power and energy are related by the equation $p = dw/dt$. Thus, energy can be found by integration as

$$W = \int pdt = \int vidt \qquad \textbf{(17–26)}$$

### Inductance

For an inductance, $v = Ldi/dt$. Substituting this into Equation 17–26, cancelling $dt$, and rearranging terms yields

$$W_L = \int \left(L\frac{di}{dt}\right)idt = L\int idi \qquad \textbf{(17–27)}$$

Recall from Figure 17–4(b), energy flows into an inductor during time interval 0 to $T/4$ and is released during time interval $T/4$ to $T/2$. The process then repeats itself. The energy stored (and subsequently released) can thus be found by integrating power from $t = 0$ to $t = T/4$. Current at $t = 0$ is 0 and current at $t = T/4$ is $I_m$. Using these as our limits of integration, we find

$$W_L = L\int_0^{I_m} i\,di = \frac{1}{2}LI_m^2 = LI^2 \quad \text{(J)} \tag{17–28}$$

where we have used $I = I_m/\sqrt{2}$ to express energy in terms of effective current.

### Capacitance

For a capacitance, $i = Cdv/dt$. Substituting this into Equation 17–26 yields

$$W_C = \int v\left(C\frac{dv}{dt}\right)dt = C\int v\,dv \tag{17–29}$$

Consider Figure 17–5(b). Energy stored can be found by integrating power from $T/4$ to $T/2$. The corresponding limits for voltage are 0 to $V_m$. Thus,

$$W_C = C\int_0^{V_m} v\,dv = \frac{1}{2}CV_m^2 = CV^2 \quad \text{(J)} \tag{17–30}$$

where we have used $V = V_m/\sqrt{2}$. You will use these relationships later.

## 17.12 Circuit Analysis Using Computers

The time-varying relationships between voltage, current and power described earlier in this chapter can be investigated easily using PSpice. To illustrate, consider the circuit of Figure 17–3 with $v = 1.2 \sin \omega t$, $R = 0.8\ \Omega$ and $f = 1000$ Hz. Create the circuit on the screen, including voltage and current markers as in Figure 17–29, then set VSIN parameters to VOFF = **0V,** VAMPL = **1.2V** and FREQ = **1000Hz** as you did in Chapter 16. Click the New Simulation Profile, type **fig17-29** for a name, choose Transient, set TSTOP to **1ms**, then click OK. Run the simulation and the voltage and current waveforms of Figure 17–30 should appear. To plot power (i.e., the product of $vi$), click Trace, then Add Trace, and when the dialog box opens, use the asterisk to create the product

NOTES

1. As of this writing, Electronics Workbench has no simple way to plot current and power; thus, no example is included here. If this changes when EWB is updated, we will add examples and problems to our web site.
2. If you want to include the identification VSIN, 1000Hz, etc. on your schematic as in Figure 17–29, do it from the Property Editor as described in Chapter 16.

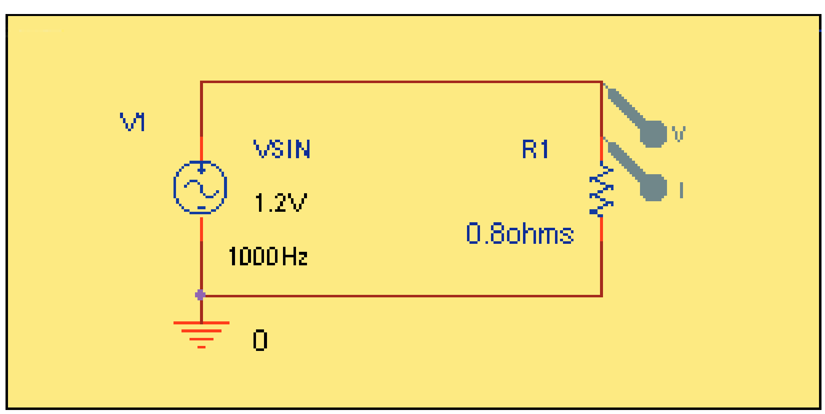



**FIGURE 17–29** Using PSpice to investigate instantaneous power in an ac circuit.

V(R1:1)*I(R1), then click OK. The blue power curve should now appear. Compare to Figure 17–3. Note that all curves agree exactly.

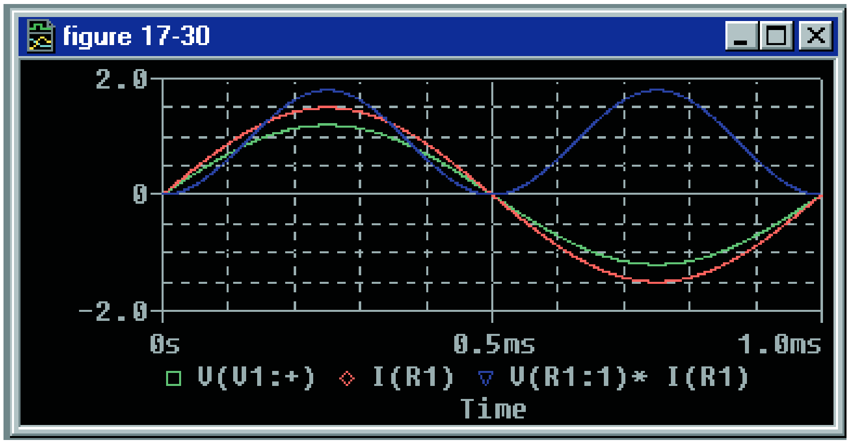



**FIGURE 17–30** Voltage, current, and power waveforms for Figure 17–29.

## PROBLEMS

**17.1–17.5**

1. Note that the power curve of Figure 17–4 is sometimes positive and sometimes negative. What is the significance of this? Between $t = T/4$ and $t = T/2$, what is the direction of power flow?
2. What is real power? What is reactive power? Which power, real or reactive, has an average value of zero?
3. A pair of electric heating elements is shown in Figure 17–31.
   a. Determine the active and reactive power to each.
   b. Determine the active and reactive power delivered by the source.
4. For the circuit of Figure 17–32, determine the active and reactive power to the inductor.
5. If the inductor of Figure 17–32 is replaced by a 40-μF capacitor and source frequency is 60 Hz, what is $Q_C$?
6. Find $R$ and $X_L$ for Figure 17–33.

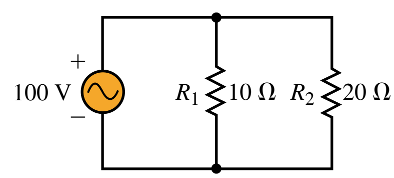



**FIGURE 17–31**

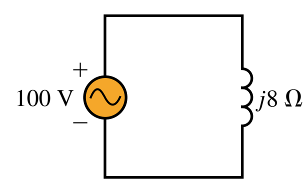



**FIGURE 17–32**

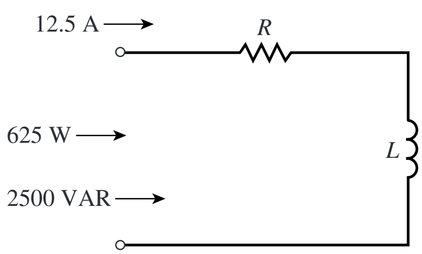



**FIGURE 17–33**

7. For the circuit of Figure 17–34, $f = 100$ Hz. Find
   a. $R$ b. $X_C$ c. $C$

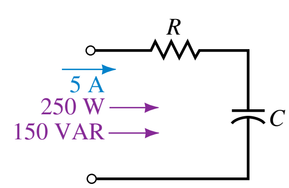



**FIGURE 17–34**

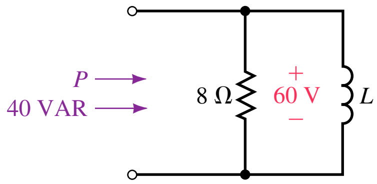



FIGURE 17–35

8. For the circuit of Figure 17–35, $f$ = 10 Hz. Find
   a. $P$ b. $X_L$ c. $L$
9. For Figure 17–36, find $X_C$.

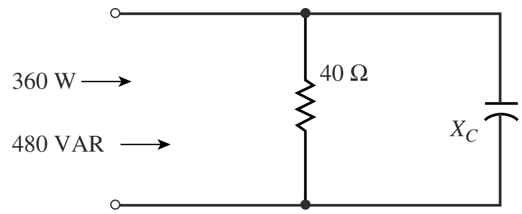



FIGURE 17–36

10. For Figure 17–37, $X_C$ = 42.5 Ω. Find $R$, $P$, and $Q$.
11. Find the total average power and the total reactive power supplied by the source for Figure 17–38.

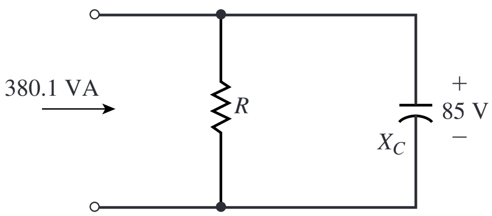



FIGURE 17–37

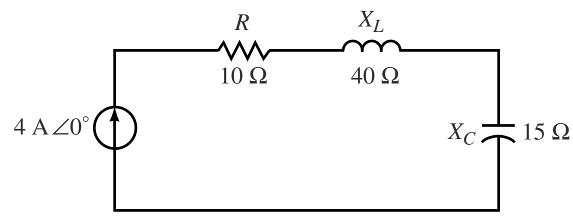



FIGURE 17–38

12. If the source of Figure 17–38 is reversed, what is $P_T$ and $Q_T$? What conclusion can you draw from this?
13. Refer to Figure 17–39. Find $P_2$ and $Q_3$. Is the element in Load 3 inductive or capacitive?

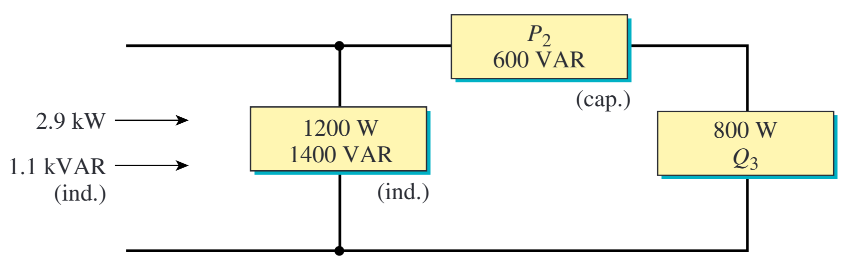



FIGURE 17–39

14. For Figure 17–40, determine $P_T$ and $Q_T$.

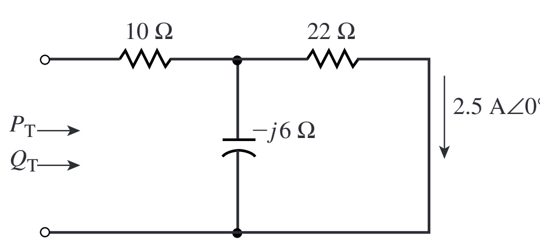



FIGURE 17–40

15. For Figure 17–41, $\omega = 10$ rad/s. Determine

a. $R_T$ b. $R_2$ c. $X_C$ d. $L_{eq}$

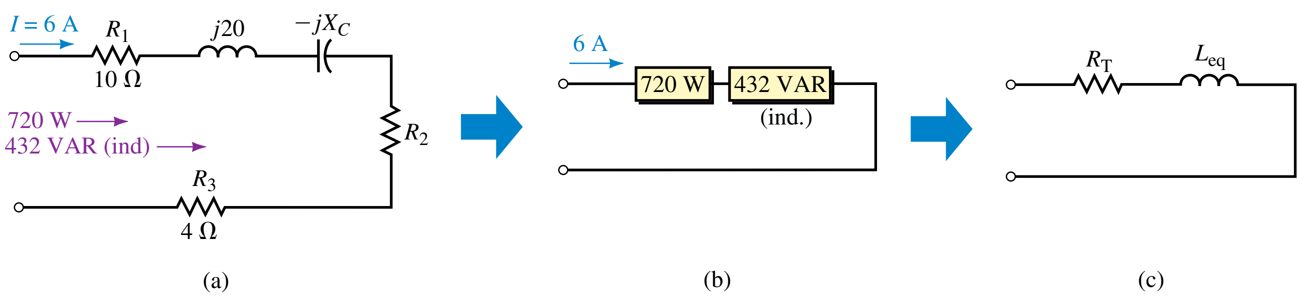



**FIGURE 17–41**

16. For Figure 17–42, determine the total $P_T$ and $Q_T$.

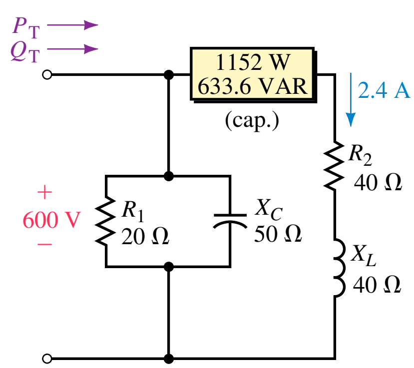



**FIGURE 17–42**

## 17.7 The Relationship Between *P, Q,* and *S*

17. For the circuit of Figure 17–7, draw the power triangle and determine the apparent power.
18. Repeat Problem 17 for Figure 17–8.
19. Ignoring the wattmeter of Figure 17–27, determine the power triangle for the circuit as seen by the source.
20. For the circuit of Figure 17–43, what is the source current?
21. For Figure 17–44, the generator supplies 30 A. What is $R$?

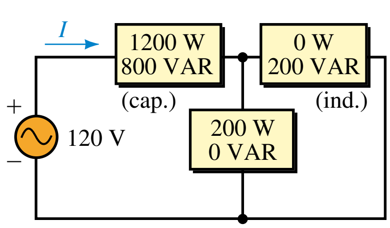



**FIGURE 17–43**

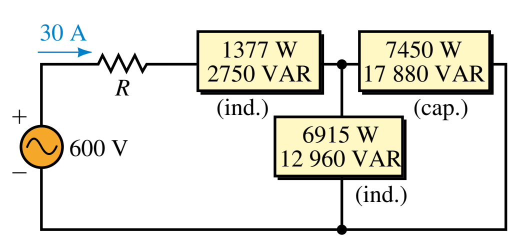



**FIGURE 17–44**

22. Suppose $\mathbf{V} = 100\ \text{V}\angle 60°$ and $\mathbf{I} = 10\text{A}\angle 40°$:
    a. What is $\theta$, the angle between **V** and **I?**
    b. Determine $P$ from $P = VI \cos \theta$.
    c. Determine $Q$ from $Q = VI \sin \theta$.
    d. Sketch the power triangle and from it, determine **S.**
    e. Show that $\mathbf{S} = \mathbf{VI}^*$ gives the same answer as (d).
23. For Figure 17–45, $S_{gen} = 4835$ VA. What is $R$?
24. Refer to the circuit of Figure 17–16:
    a. Determine the apparent power for each box.
    b. Sum the apparent powers that you just computed. Why does the sum not equal $S_T = 1803$ VA as obtained in Example 17–5?

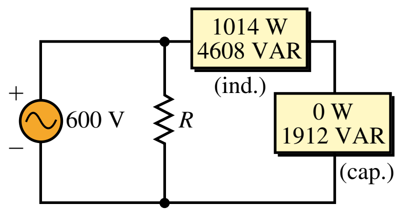



FIGURE 17–45

### 17.8 Power Factor

25. Refer to the circuit of Figure 17–46:
    a. Determine $P_T$, $Q_T$, and $S_T$.
    b. Determine whether the fuse will blow.

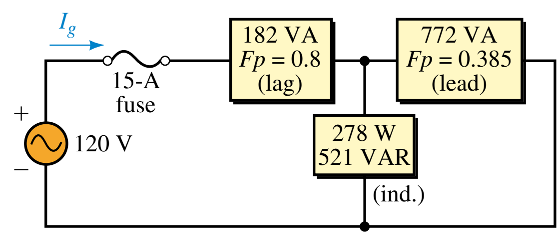



FIGURE 17–46

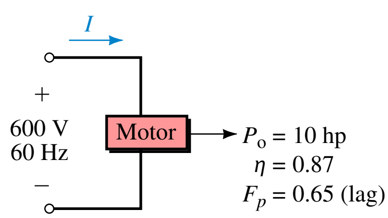
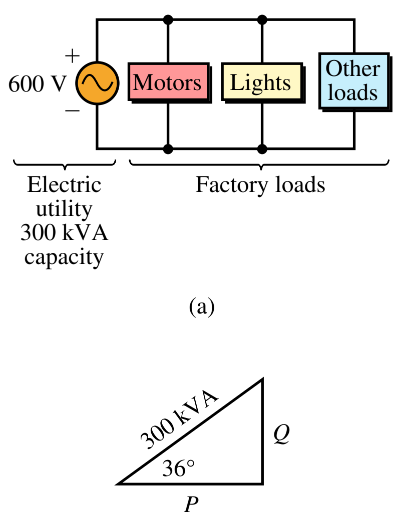



FIGURE 17–47




(b) Factory power triangle

FIGURE 17–48

26. A motor with an efficiency of 87% supplies 10 hp to a load (Figure 17–47). Its power factor is 0.65 (lag).
    a. What is the power input to the motor?
    b. What is the reactive power to the motor?
    c. Draw the motor power triangle. What is the apparent power to the motor?
27. To correct the circuit power factor of Figure 17–47 to unity, a power factor correction capacitor is added.
    a. Show where the capacitor is connected.
    b. Determine its value in microfarads.
28. Consider Figure 17–20. The motor is replaced with a new unit requiring $\mathbf{S}_m = (120 + j35)$ kVA. Everything else remains the same. Find the following:
    a. $P_T$ b. $Q_T$ c. $S_T$
    d. Determine how much kVAR capacitive correction is needed to correct to unity $F_p$.
29. A small electrical utility has a 600-V, 300-kVA capacity. It supplies a factory (Figure 17–48) with the power triangle shown in (b). This fully loads the utility. If a power factor correcting capacitor corrects the load to unity

power factor, how much more power (at unity power factor) can the utility sell to other customers?

### 17.9 AC Power Measurement

30. a. Why does the wattmeter of Figure 17–49 indicate only 1200 watts?
    b. Where would the wattmeter have to be placed to measure power delivered by the source? Sketch the modified circuit.
    c. What would the wattmeter indicate in (b)?

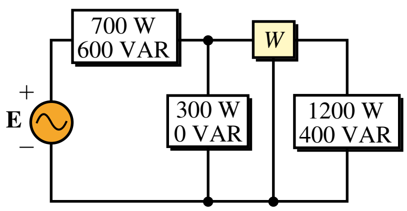



FIGURE 17–49

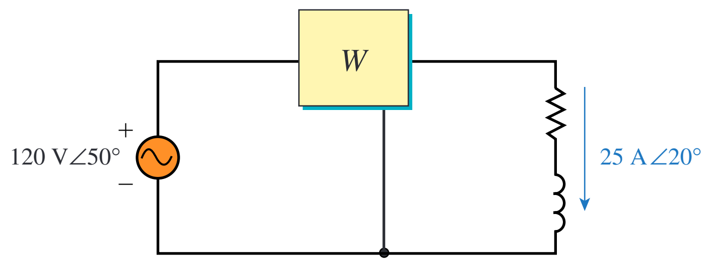



FIGURE 17–50

31. Determine the wattmeter reading for Figure 17–50.
32. Determine the wattmeter reading for Figure 17–51.

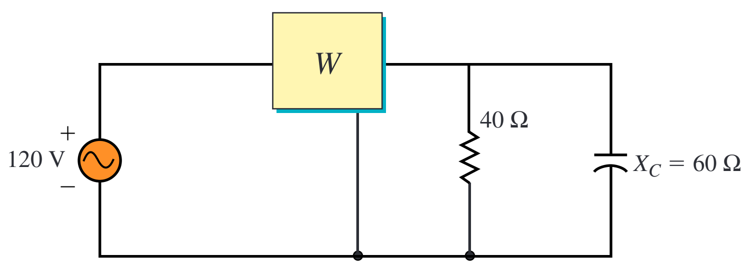



FIGURE 17–51

### 17.10 Effective Resistance

33. Measurements on an iron-core solenoid coil yield the following values: $V = 80$ V, $I = 400$ mA, $P = 25.6$ W, and $R = 140\ \Omega$. (The last measurement was taken with an ohmmeter.) What is the ac resistance of the solenoid coil?

### 17.12 Circuit Analysis Using Computers

34. **PSpice** An inductance $L = 1$ mH has current $i = 4 \sin (2\pi \times 1000)t$. Use PSpice to investigate the power waveform and compare to Figure 17–4. Use current source ISIN (see Note).
35. **PSpice** A 10-$\mu$F capacitor has voltage $v = 10 \sin(\omega t - 90°)$ V. Use PSpice to investigate the power waveform and compare to Figure 17–5. Use voltage source VSIN with $f = 1000$ Hz.

**NOTE...**

PSpice represents current into devices. Thus, when you double click a current source symbol (ISIN, IPWL, etc.) and specify a current waveform, you are specifying the current *into* the source.

36. **PSpice** The voltage waveform of Figure 17–52 is applied to a 200-$\mu$F capacitor.

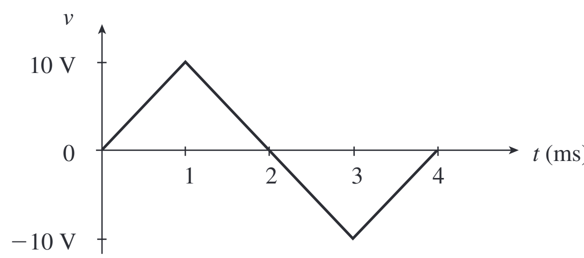



**FIGURE 17–52**

a. Using the principles of Chapter 10, determine the current through the capacitor and sketch. (Also sketch the voltage waveform on your graph.) Multiply the two waveforms to obtain a plot of $p(t)$. Compute power at its max and min points.
b. Use PSpice to verify the results. Use voltage source VPWL. You have to describe the waveform to the source. It has a value of 0 V at $t = 0$, 10 V at $t = 1$ ms, −10 V at $t = 3$ ms, and 0 V at $t = 4$ ms. To set these, double click the source symbol and enter values via the Property Editor as follows: **0** for T1, **0V** for V1, **1ms** for T2, **10V** for V2, etc. Run the simulation and plot voltage, current, and power using the procedure we used to create Figure 17–30. Results should agree with those of (a).

37. **PSpice** Repeat Question 36 for a current waveform identical to Figure 17–52 except that it oscillates between 2 A and −2 A applied to a 2-mH inductor. Use current source IPWL (see Note).

## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

1. 100 V

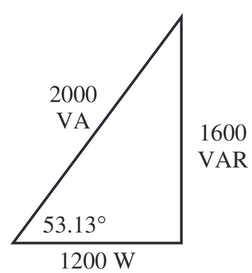



**FIGURE 17–53**

2. 8.76 A
3. Fig. 17–7: 0.6 (lag); Fig. 17–8: 0.97 (lead); Fig. 17–9: 0.6 (lag)
4. Yes. ($S$ = 128 kVA)

# 18 AC Series-Parallel Circuits

## OBJECTIVES

After studying this chapter, you will be able to

- apply Ohm's law to analyze simple series circuits,
- apply the voltage divider rule to determine the voltage across any element in a series circuit,
- apply Kirchhoff's voltage law to verify that the summation of voltages around a closed loop is equal to zero,
- apply Kirchhoff's current law to verify that the summation of currents entering a node is equal to the summation of currents leaving the same node,
- determine unknown voltage, current, and power for any series/parallel circuit,
- determine the series or parallel equivalent of any network consisting of a combination of resistors, inductors, and capacitors.

## KEY TERMS

AC Parallel Circuits
AC Series Circuits
Current Divider Rule
Frequency Effects
Impedance
Kirchhoff's Current Law
Kirchhoff's Voltage Law
Voltage Divider Rule

## OUTLINE

Ohm's Law for AC Circuits
AC Series Circuits
Kirchhoff's Voltage Law and the Voltage Divider Rule
AC Parallel Circuits
Kirchhoff's Current Law and the Current Divider Rule
Series-Parallel Circuits
Frequency Effects
Applications
Circuit Analysis Using Computers

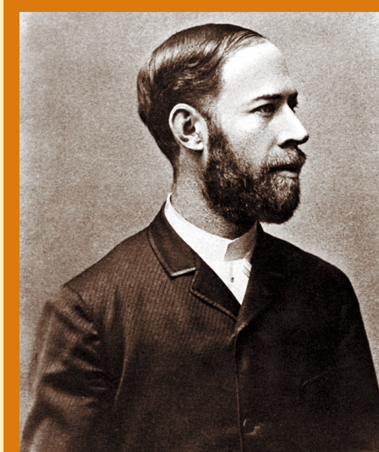
## CHAPTER PREVIEW

In this chapter we examine how simple circuits containing resistors, inductors, and capacitors behave when subjected to sinusoidal voltages and currents. Principally, we find that the rules and laws which were developed for dc circuits will apply equally well for ac circuits. The major difference between solving dc and ac circuits is that analysis of ac circuits requires using vector algebra.

In order to proceed successfully, it is suggested that the student spend time reviewing the important topics covered in dc analysis. These include Ohm's law, the voltage divider rule, Kirchhoff's voltage law, Kirchhoff's current law, and the current divider rule.

You will also find that a brief review of vector algebra will make your understanding of this chapter more productive. In particular, you should be able to add and subtract any number of vector quantities.

## PUTTING IT IN PERSPECTIVE



### *Heinrich Rudolph Hertz*

HEINRICH HERTZ WAS BORN IN HAMBURG, Germany, on February 22, 1857. He is known mainly for his research into the transmission of electromagnetic waves.

Hertz began his career as an assistant to Hermann von Helmholtz in the Berlin Institute physics laboratory. In 1885, he was appointed Professor of Physics at Karlsruhe Polytechnic, where he did much to verify James Clerk Maxwell's theories of electromagnetic waves.

In one of his experiments, Hertz discharged an induction coil with a rectangular loop of wire having a very small gap. When the coil discharged, a spark jumped across the gap. He then placed a second, identical coil close to the first, but with no electrical connection. When the spark jumped across the gap of the first coil, a smaller spark was also induced across the second coil. Today, more elaborate antennas use similar principles to transmit radio signals over vast distances. Through further research, Hertz was able to prove that electromagnetic waves have many of the characteristics of light: they have the same speed as light; they travel in straight lines; they can be reflected and refracted; and they can be polarized.

Hertz's experiments ultimately led to the development of radio communication by such electrical engineers as Guglielmo Marconi and Reginald Fessenden.

Heinrich Hertz died at the age of 36 on January 1, 1894.

## 18.1 Ohm's Law for AC Circuits

This section is a brief review of the relationship between voltage and current for resistors, inductors, and capacitors. Unlike Chapter 16, all phasors are given as rms rather than as peak values. As you saw in Chapter 17, this approach simplifies the calculation of power.

### Resistors

In Chapter 16, we saw that when a resistor is subjected to a sinusoidal voltage as shown in Figure 18–1, the resulting current is also sinusoidal and in phase with the voltage.

The sinusoidal voltage $v = V_m\sin(\omega t + \theta)$ may be written in phasor form as $\mathbf{V} = V\angle\theta$. Whereas the sinusoidal expression gives the instantaneous value of voltage for a waveform having an amplitude of $V_m$ (volts peak), the phasor form has a magnitude which is the effective (or rms) value. The relationship between the magnitude of the phasor and the peak of the sinusoidal voltage is given as

$$V = \frac{V_m}{\sqrt{2}}$$

Because the resistance vector may be expressed as $\mathbf{Z}_R = R\angle 0°$, we evaluate the current phasor as follows:

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{V\angle\theta}{R\angle 0°} = \frac{V}{R}\angle\theta = I\angle\theta$$

**NOTES...**

Although currents and voltages may be shown in either time domain (as sinusoidal quantities) or in phasor domain (as vectors), resistance and reactance are never shown as sinusoidal quantities. The reason for this is that whereas currents and voltages vary as functions of time, resistance and reactance do not.

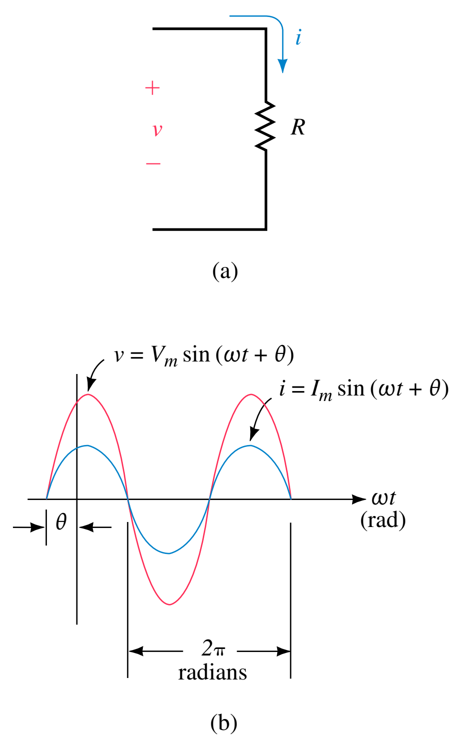



**FIGURE 18–1** Sinusoidal voltage and current for a resistor.

If we wish to convert the current from phasor form to its sinusoidal equivalent in the time domain, we would have $i = I_m \sin(\omega t + \theta)$. Again, the relationship between the magnitude of the phasor and the peak value of the sinusoidal equivalent is given as

$$I = \frac{I_m}{\sqrt{2}}$$

The voltage and current phasors may be shown on a phasor diagram as in Figure 18–2.

Because one phasor is a current and the other is a voltage, the relative lengths of these phasors are purely arbitrary. Regardless of the angle $\theta$, we see that the voltage across and the current through a resistor will always be in phase.

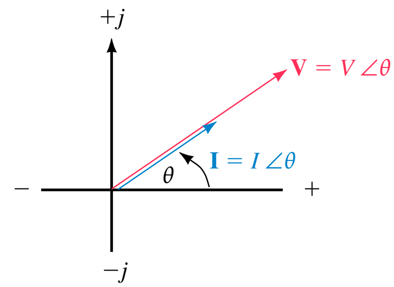



**FIGURE 18–2** Voltage and current phasors for a resistor.

**EXAMPLE 18–1** Refer to the resistor shown in Figure 18–3:

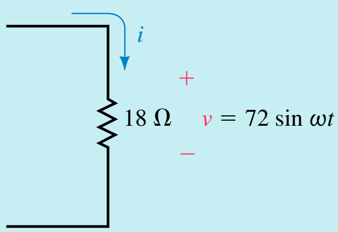



**FIGURE 18–3**

a. Find the sinusoidal current $i$ using phasors.
b. Sketch the sinusoidal waveforms for $v$ and $i$.
c. Sketch the phasor diagram of **V** and **I.**

**Solution:**

a. The phasor form of the voltage is determined as follows:

$$v = 72 \sin \omega t \Leftrightarrow \mathbf{V} = 50.9 \text{ V}\angle 0°$$

From Ohm's law, the current phasor is determined to be

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{50.9 \text{ V}\angle 0°}{18 \ \Omega\angle 0°} = 2.83 \text{ A}\angle 0°$$

which results in the sinusoidal current waveform having an amplitude of

$$I_m = (\sqrt{2})(2.83 \text{ A}) = 4.0 \text{ A}$$

Therefore, the current $i$ will be written as

$$i = 4 \sin \omega t$$

b. The voltage and current waveforms are shown in Figure 18–4.

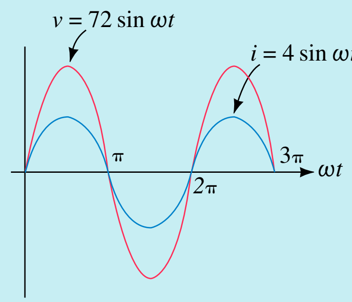



**FIGURE 18–4**

c. Figure 18–5 shows the voltage and current phasors.

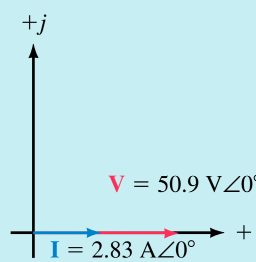



**FIGURE 18–5**

**EXAMPLE 18–2** Refer to the resistor of Figure 18–6:

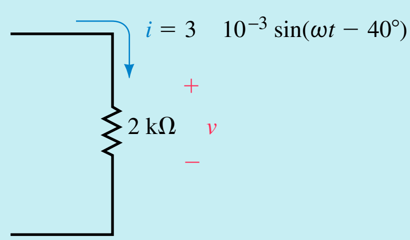



**FIGURE 18–6**

a. Use phasor algebra to find the sinusoidal voltage, *v*.

b. Sketch the sinusoidal waveforms for *v* and *i*.

c. Sketch a phasor diagram showing **V** and **I.**

**Solution**

a. The sinusoidal current has a phasor form as follows:

$$i = 3 \times 10^{-3}\sin(\omega t - 40°) \Leftrightarrow \mathbf{I} = 2.12 \text{ mA}\angle -40°$$

From Ohm's law, the voltage across the 2-kΩ resistor is determined as the phasor product

$$\begin{aligned}\mathbf{V} &= \mathbf{IZ}_R \\ &= (2.12 \text{ mA}\angle -40°)(2 \text{ k}\Omega\angle 0°) \\ &= 4.24 \text{ V}\angle -40°\end{aligned}$$

The amplitude of the sinusoidal voltage is

$$V_m = (\sqrt{2})(4.24 \text{ V}) = 6.0 \text{ V}$$

The voltage may now be written as

$$v = 6.0 \sin(\omega t - 40°)$$

b. Figure 18–7 shows the sinusoidal waveforms for $v$ and $i$.

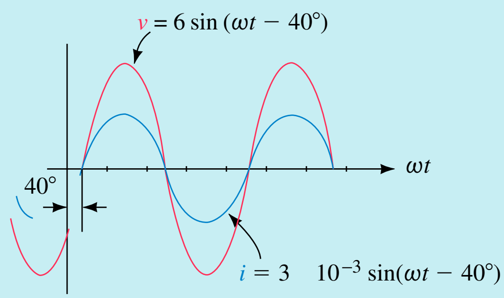



**FIGURE 18–7**

c. The corresponding phasors for the voltage and current are shown in Figure 18–8.

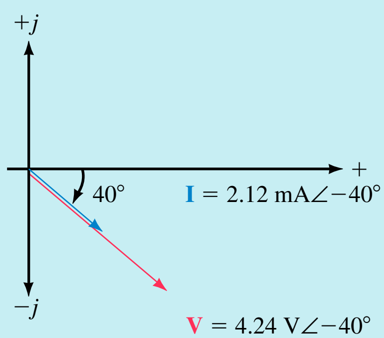



**FIGURE 18–8**

## Inductors

When an inductor is subjected to a sinusoidal current, a sinusoidal voltage is induced across the inductor such that the voltage across the inductor leads the current waveform by exactly 90°. If we know the reactance of an inductor, then from Ohm's law the current in the inductor may be expressed in phasor form as

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{V\angle\theta}{X_L\angle 90°} = \frac{V}{X_L}\angle(\theta - 90°)$$

In vector form, the reactance of the inductor is given as

$$\mathbf{Z}_L = X_L\angle 90°$$

where $X_L = \omega L = 2\pi fL$.

**EXAMPLE 18–3** Consider the inductor shown in Figure 18–9:

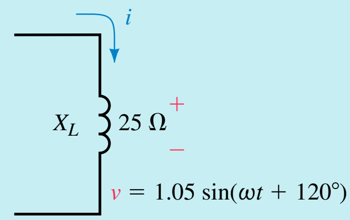



**FIGURE 18–9**

a. Determine the sinusoidal expression for the current $i$ using phasors.
b. Sketch the sinusoidal waveforms for $v$ and $i$.
c. Sketch the phasor diagram showing **V** and **I.**

**Solution:**

a. The phasor form of the voltage is determined as follows:

$$v = 1.05 \sin(\omega t + 120°) \Leftrightarrow \mathbf{V} = 0.742\ \text{V}\angle 120°$$

From Ohm's law, the current phasor is determined to be

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{0.742\ \text{V}\angle 120°}{25\ \Omega\angle 90°} = 29.7\ \text{mA}\angle 30°$$

The amplitude of the sinusoidal current is

$$I_m = (\sqrt{2})(29.7\ \text{mA}) = 42\ \text{mA}$$

The current $i$ is now written as

$$i = 0.042 \sin(\omega t + 30°)$$

b. Figure 18–10 shows the sinusoidal waveforms of the voltage and current.

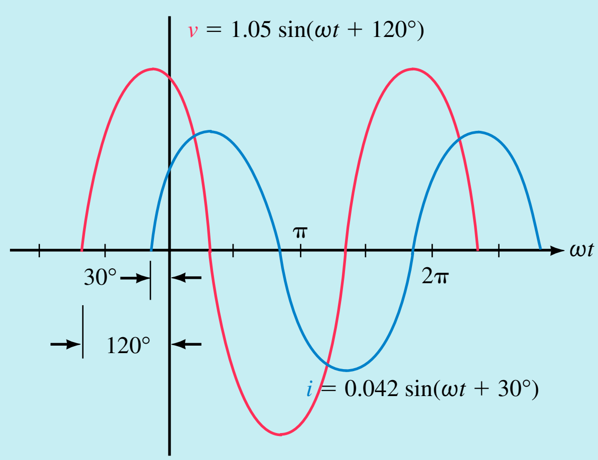



**FIGURE 18–10** Sinusoidal voltage and current for an inductor.

c. The voltage and current phasors are shown in Figure 18–11.

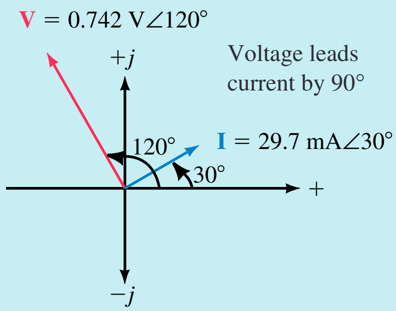



**FIGURE 18–11** Voltage and current phasors for an inductor.

## Capacitors

When a capacitor is subjected to a sinusoidal voltage, a sinusoidal current results. The current through the capacitor leads the voltage by exactly 90°. If we know the reactance of a capacitor, then from Ohm's law the current in the capacitor expressed in phasor form is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_C} = \frac{V\angle\theta}{X_L\angle -90^\circ} = \frac{V}{X_L}\angle(\theta + 90^\circ)$$

In vector form, the reactance of the capacitor is given as

$$\mathbf{Z}_C = X_C\angle -90^\circ$$

where

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

**EXAMPLE 18–4** Consider the capacitor of Figure 18–12.

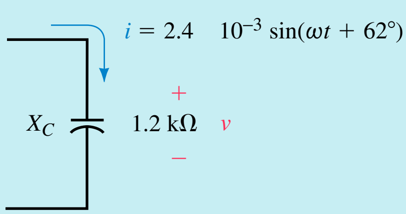



**FIGURE 18–12**

a. Find the voltage $v$ across the capacitor.

b. Sketch the sinusoidal waveforms for $v$ and $i$.

c. Sketch the phasor diagram showing **V** and **I.**

**Solution**

a. Converting the sinusoidal current into its equivalent phasor form gives

$$i = 2.4 \times 10^{-3}\sin(\omega t + 62^\circ) \Leftrightarrow \mathbf{I} = 1.70\ \text{mA}\angle 62^\circ$$

From Ohm's law, the phasor voltage across the capacitor must be

$$\begin{aligned}\mathbf{V} &= \mathbf{IZ}_C \\ &= (1.70\text{ mA}\angle 62°)(1.2\text{ k}\Omega\angle -90°) \\ &= 2.04\text{ V}\angle -28°\end{aligned}$$

The amplitude of the sinusoidal voltage is

$$V_m = (\sqrt{2})(2.04\text{ V}) = 2.88\text{ V}$$

The voltage $v$ is now written as

$$v = 2.88\sin(\omega t - 28°)$$

b. Figure 18–13 shows the waveforms for $v$ and $i$.

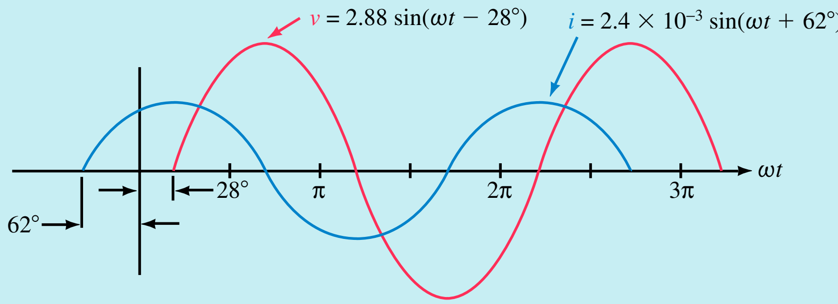



**FIGURE 18–13** Sinusoidal voltage and current for a capacitor.

c. The corresponding phasor diagram for **V** and **I** is shown in Figure 18–14.

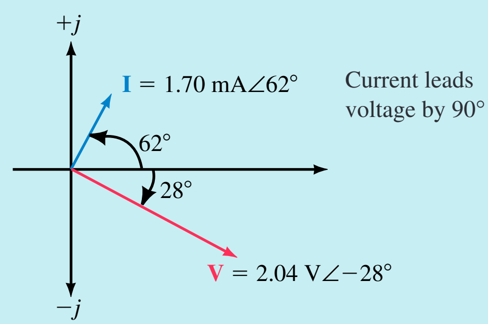



**FIGURE 18–14** Voltage and current phasors for a capacitor.

The relationships between voltage and current, as illustrated in the previous three examples, will always hold for resistors, inductors, and capacitors.

IN-PROCESS LEARNING CHECK 1

1. What is the phase relationship between current and voltage for a resistor?
2. What is the phase relationship between current and voltage for a capacitor?
3. What is the phase relationship between current and voltage for an inductor?

*(Answers are at the end of the chapter.)*

PRACTICE PROBLEMS 1

A voltage source, $\mathbf{E} = 10\ \text{V}\angle 30°$, is applied to an inductive impedance of 50 Ω.

a. Solve for the phasor current, **I.**
b. Sketch the phasor diagram for **E** and **I.**
c. Write the sinusoidal expressions for *e* and *i.*
d. Sketch the sinusoidal expressions for *e* and *i.*

*Answers:*
a. $\mathbf{I} = 0.2\ \text{A}\angle -60°$
c. $e = 14.1 \sin(\omega t + 30°)$
$i = 0.283 \sin(\omega t - 60°)$

PRACTICE PROBLEMS 2

A voltage source, $\mathbf{E} = 10\ \text{V}\angle 30°$, is applied to a capacitive impedance of 20 Ω.

a. Solve for the phasor current, **I.**
b. Sketch the phasor diagram for **E** and **I.**
c. Write the sinusoidal expressions for *e* and *i.*
d. Sketch the sinusoidal expressions for *e* and *i.*

*Answers:*
a. $\mathbf{I} = 0.5\ \text{A}\angle 120°$
c. $e = 14.1 \sin(\omega t + 30°)$
$i = 0.707 \sin(\omega t + 120°)$

## 18.2 AC Series Circuits

When we examined dc circuits we saw that the current everywhere in a series circuit is always constant. This same applies when we have series elements with an ac source. Further, we had seen that the total resistance of a dc series circuit consisting of *n* resistors was determined as the summation

$$R_T = R_1 + R_2 + \cdots + R_n$$

When working with ac circuits we no longer work with only resistance but also with capacitive and inductive reactance. *Impedance is a term used to collectively determine how the resistance, capacitance, and inductance "impede" the current in a circuit.* The symbol for impedance is the letter $Z$ and the unit is the ohm (Ω). Because impedance may be made up of any combination of resistances and reactances, it is written as a vector quantity **Z,** where

$$\mathbf{Z} = Z\angle\theta \quad (\Omega)$$

Each impedance may be represented as a vector on the complex plane, such that the length of the vector is representative of the magnitude of the impedance. The diagram showing one or more impedances is referred to as an **impedance diagram.**

Resistive impedance $\mathbf{Z}_R$ is a vector having a magnitude of $R$ along the positive real axis. Inductive reactance $\mathbf{Z}_L$ is a vector having a magnitude of

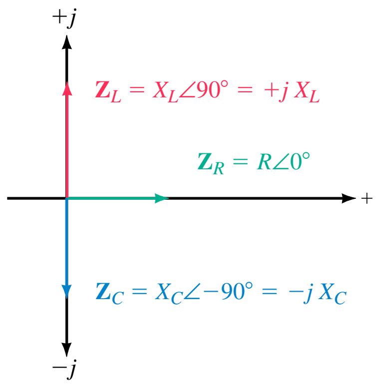



**FIGURE 18–15**

$X_L$ along the positive imaginary axis, while the capacitive reactance $\mathbf{Z}_C$ is a vector having a magnitude of $X_C$ along the negative imaginary axis. Mathematically, each of the vector impedances is written as follows:

$$\mathbf{Z}_R = R\angle 0° = R + j0 = R$$
$$\mathbf{Z}_L = X_L\angle 90° = 0 + jX_L = jX_L$$
$$\mathbf{Z}_C = X_C\angle 90° = 0 - jX_C = -jX_C$$

An impedance diagram showing each of the above impedances is shown in Figure 18–15.

*All impedance vectors will appear in either the first or the fourth quadrants, since the resistive impedance vector is always positive.*

For a series ac circuit consisting of $n$ impedances, as shown in Figure 18–16, the total impedance of the circuit is found as the vector sum

$$\mathbf{Z}_\mathrm{T} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n \qquad \textbf{(18–1)}$$

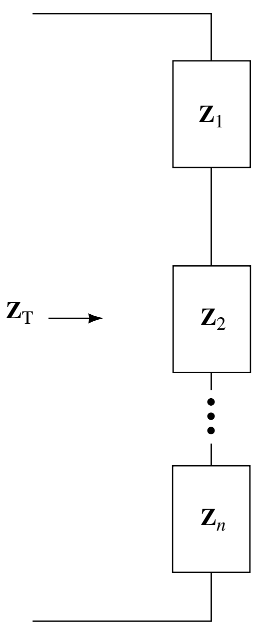



**FIGURE 18–16**

Consider the branch of Figure 18–17.

By applying Equation 18–1, we may determine the total impedance of the circuit as

$$\begin{aligned}\mathbf{Z}_\mathrm{T} &= (3\ \Omega + j0) + (0 + j4\ \Omega) = 3\ \Omega + j4\ \Omega\\ &= 5\ \Omega\angle 53.13°\end{aligned}$$

The above quantities are shown on an impedance diagram as in Figure 18–18.

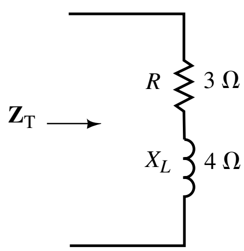



**FIGURE 18–17**

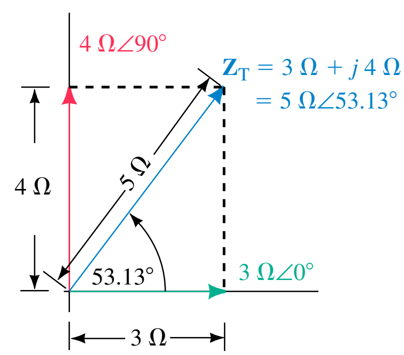



**FIGURE 18–18**

From Figure 18–18 we see that the total impedance of the series elements consists of a real component and an imaginary component. The corresponding total impedance vector may be written in either polar or rectangular form.

The rectangular form of an impedance is written as

$$\mathbf{Z} = R \pm jX$$

If we are given the polar form of the impedance, then we may determine the equivalent rectangular expression from

$$R = Z\cos\theta \qquad \textbf{(18–2)}$$

and

$$X = Z\sin\theta \qquad \textbf{(18–3)}$$

In the rectangular representation for impedance, the resistance term, $R$, is the total of all resistance looking into the network. The reactance term, $X$, is the difference between the total capacitive and inductive reactances. The sign for the imaginary term will be positive if the inductive reactance is greater than the capacitive reactance. In such a case, the impedance vector will appear in the first quadrant of the impedance diagram and is referred to as being an **inductive** impedance. If the capacitive reactance is larger, then the sign for the imaginary term will be negative. In such a case, the impedance vector will appear in the fourth quadrant of the impedance diagram and the impedance is said to be **capacitive.**

The polar form of any impedance will be written in the form

$$\mathbf{Z} = Z\angle\theta$$

The value $Z$ is the magnitude (in ohms) of the impedance vector $\mathbf{Z}$ and is determined as follows:

$$Z = \sqrt{R^2 + X^2} \quad (\Omega) \tag{18–4}$$

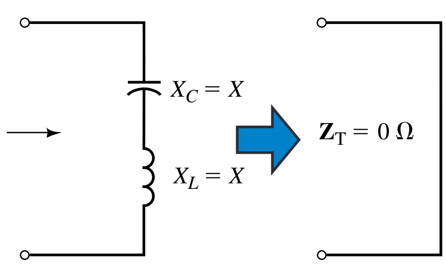
The corresponding angle of the impedance vector is determined as

$$\theta = \pm\tan^{-1}\left(\frac{X}{R}\right) \tag{18–5}$$

Whenever a capacitor and an inductor having equal reactances are placed in series, as shown in Figure 18–19, the equivalent circuit of the two components is a short circuit since the inductive reactance will be exactly balanced by the capacitive reactance.

Any ac circuit having a total impedance with only a real component, is referred to as a **resistive** circuit. In such a case, the impedance vector $\mathbf{Z}_\mathrm{T}$ will be located along the positive real axis of the impedance diagram and the angle of the vector will be 0°. The condition under which series reactances are equal is referred to as "series resonance" and is examined in greater detail in a later chapter.

If the impedance $\mathbf{Z}$ is written in polar form, then the angle $\theta$ will be positive for an inductive impedance and negative for a capacitive impedance. In the event that the circuit is purely reactive, the resulting angle $\theta$ will be either $+90°$ (inductive) or $-90°$ (capacitive). If we reexamine the impedance diagram of Figure 18–18, we conclude that the original circuit is inductive.




$$\mathbf{Z}_\mathrm{T} = jX_L - jX_C = jX - jX = 0\ \Omega$$

FIGURE 18–19

**EXAMPLE 18–5** Consider the network of Figure 18–20.

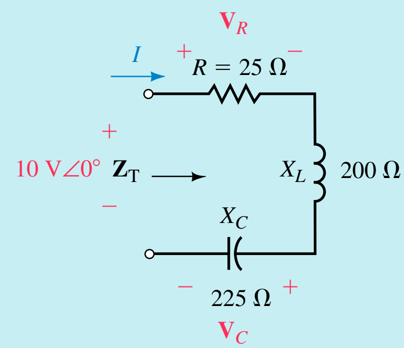



**FIGURE 18–20**

a. Find $\mathbf{Z}_T$.
b. Sketch the impedance diagram for the network and indicate whether the total impedance of the circuit is inductive, capacitive, or resistive.
c. Use Ohm's law to determine **I,** $\mathbf{V}_R$, and $\mathbf{V}_C$.

**Solution**

a. The total impedance is the vector sum

$$\begin{aligned}\mathbf{Z}_T &= 25\ \Omega + j200\ \Omega + (-j225\ \Omega)\\ &= 25\ \Omega - j25\ \Omega\\ &= 35.36\ \Omega\angle -45°\end{aligned}$$

b. The corresponding impedance diagram is shown in Figure 18–21.

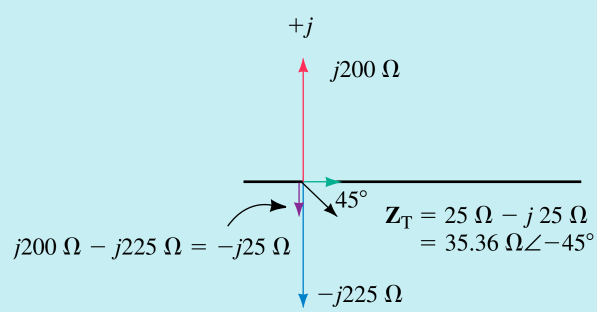



**FIGURE 18–21**

Because the total impedance has a negative reactance term ($-j25\ \Omega$), $\mathbf{Z}_T$ is capacitive.

c. $$\mathbf{I} = \frac{10\ \text{V}\angle 0°}{35.36\ \Omega\angle -45°} = 0.283\ \text{A}\angle 45°$$

$$\mathbf{V}_R = (282.8\ \text{mA}\angle 45°)(25\ \Omega\angle 0°) = 7.07\ \text{V}\angle 45°$$

$$\mathbf{V}_C = (282.8\ \text{mA}\angle 45°)(225\ \Omega\angle -90°) = 63.6\ \text{V}\angle -45°$$

Notice that the magnitude of the voltage across the capacitor is many times larger than the source voltage applied to the circuit. This example illustrates that the voltages across reactive elements must be calculated to ensure that maximum ratings for the components are not exceeded.

**EXAMPLE 18–6** Determine the impedance **Z** which must be within the indicated block of Figure 18–22 if the total impedance of the network is 13 Ω∠22.62°.

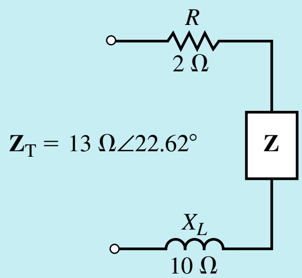



**FIGURE 18–22**

**Solution** Converting the total impedance from polar to rectangular form, we get

$$\mathbf{Z}_T = 13\ \Omega\angle 22.62° \Leftrightarrow 12\ \Omega + j5\ \Omega$$

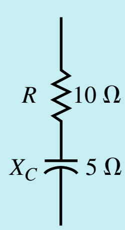
Now, we know that the total impedance is determined from the summation of the individual impedance vectors, namely

$$\mathbf{Z}_T = 2\ \Omega + j10\ \Omega + \mathbf{Z} = 12\ \Omega + j5\ \Omega$$

Therefore, the impedance **Z** is found as

$$\begin{aligned}\mathbf{Z} &= 12\ \Omega + j5\ \Omega - (2\ \Omega + j10\ \Omega)\\ &= 10\ \Omega - j5\ \Omega\\ &= 11.18\ \Omega\angle -26.57°\end{aligned}$$

In its most simple form, the impedance **Z** will consist of a series combination of a 10-Ω resistor and a capacitor having a reactance of 5 Ω. Figure 18–23 shows the elements which may be contained within **Z** to satisfy the given conditions.




$$\begin{aligned}\mathbf{Z} &= 10\ \Omega - j\,5\Omega\\ &= 11.18\ \Omega\ \angle -26.57°\end{aligned}$$

**FIGURE 18–23**

**EXAMPLE 18–7** Find the total impedance for the network of Figure 18–24. Sketch the impedance diagram showing $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_T$.

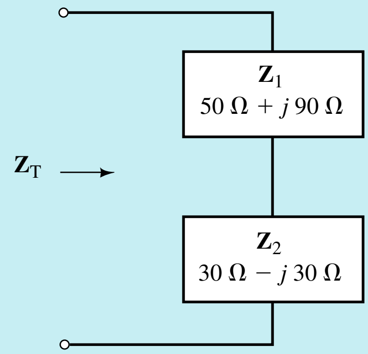



**FIGURE 18–24**

**Solution:**

$$\begin{aligned}\mathbf{Z}_T &= \mathbf{Z}_1 + \mathbf{Z}_2\\ &= (50\ \Omega + j90\ \Omega) + (30\ \Omega - j30\ \Omega)\\ &= (80\ \Omega + j60\ \Omega) = 100\ \Omega\angle 36.87^\circ\end{aligned}$$

The polar forms of the vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are as follows:

$$\begin{aligned}\mathbf{Z}_1 &= 50\ \Omega + j90\ \Omega = 102.96\ \Omega\angle 60.95^\circ\\ \mathbf{Z}_2 &= 30\ \Omega - j30\ \Omega = 42.43\ \Omega\angle -45^\circ\end{aligned}$$

The resulting impedance diagram is shown in Figure 18–25.

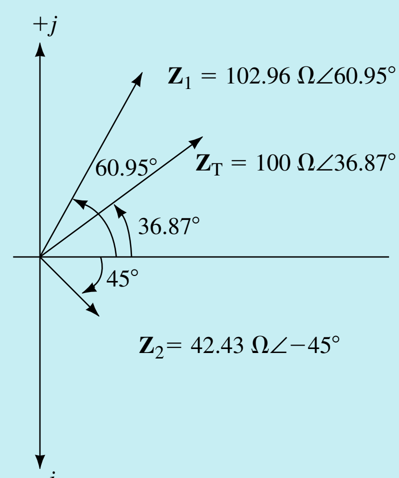



**FIGURE 18–25**

The phase angle $\theta$ for the impedance vector $\mathbf{Z} = Z\angle\theta$ provides the phase angle between the voltage $\mathbf{V}$ across $\mathbf{Z}$ and the current $\mathbf{I}$ through the impedance. For an inductive impedance the voltage will lead the current by $\theta$. If the impedance is capacitive, then the voltage will lag the current by an amount equal to the magnitude of $\theta$.

The phase angle $\theta$ is also useful for determining the average power dissipated by the circuit. In the simple series circuit shown in Figure 18–26, we know that only the resistor will dissipate power.

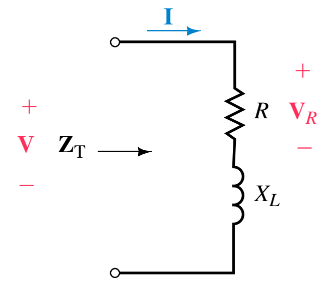



FIGURE 18–26

The average power dissipated by the resistor may be determined as follows:

$$P = V_R I = \frac{V_R^2}{R} = I^2 R \qquad \textbf{(18–6)}$$

Notice that Equation 18–6 uses only the **magnitudes** of the voltage, current, and impedance vectors. *Power is never determined by using phasor products.*

Ohm's law provides the magnitude of the current phasor as

$$I = \frac{V}{Z}$$

Substituting this expression into Equation 18–6, we obtain the expression for power as

$$P = \frac{V^2}{Z^2}R = \frac{V^2}{Z}\left(\frac{R}{Z}\right) \qquad \textbf{(18–7)}$$

From the impedance diagram of Figure 18–27, we see that

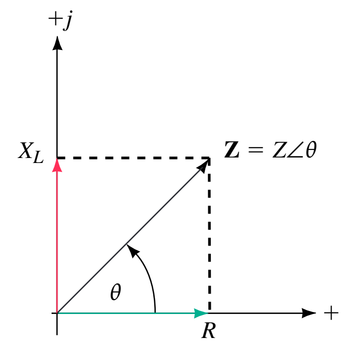



FIGURE 18–27

$$\cos\theta = \frac{R}{Z}$$

The previous chapter had defined the power factor as $F_p = \cos\theta$, where $\theta$ is the angle between the voltage and current phasors. We now see that for a series circuit, the power factor of the circuit can be determined from the magnitudes of resistance and total impedance.

$$F_p = \cos\theta = \frac{R}{Z} \qquad \textbf{(18–8)}$$

The power factor, $F_p$, is said to be **leading** if the current leads the voltage (capacitive circuit) and **lagging** if the current lags the voltage (inductive circuit).

Now substituting the expression for the power factor into Equation 18–7, we express power delivered to the circuit as

$$P = VI\cos\theta$$

Since $V = IZ$, power may be expressed as

$$P = VI\cos\theta = I^2 Z\cos\theta = \frac{V^2}{Z}\cos\theta \qquad \textbf{(18–9)}$$

**EXAMPLE 18–8** Refer to the circuit of Figure 18–28.

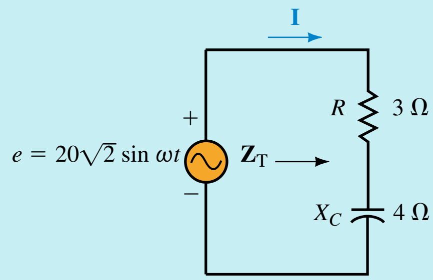



**FIGURE 18–28**

a. Find the impedance $\mathbf{Z}_T$.
b. Calculate the power factor of the circuit.
c. Determine **I.**
d. Sketch the phasor diagram for **E** and **I.**
e. Find the average power delivered to the circuit by the voltage source.
f. Calculate the average power dissipated by both the resistor and the capacitor.

**Solution**

a. $\mathbf{Z}_T = 3\ \Omega - j4\ \Omega = 5\ \Omega\angle -53.13°$
b. $F_p = \cos\theta = 3\ \Omega\backslash 5\ \Omega = 0.6$ (leading)
c. The phasor form of the applied voltage is

$$\mathbf{E} = \frac{(\sqrt{2})(20\text{ V})}{\sqrt{2}}\angle 0° = 20\text{ V}\angle 0°$$

which gives a current of

$$\mathbf{I} = \frac{20\text{ V}\angle 0°}{5\ \Omega\angle -53.13°} = 4.0\text{ A}\angle 53.13°$$

d. The phasor diagram is shown in Figure 18–29.

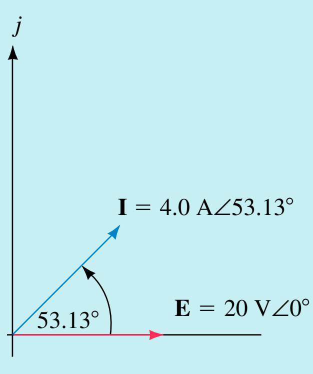



**FIGURE 18–29**

From this phasor diagram, we see that the current phasor for the capacitive circuit leads the voltage phasor by 53.13°.

e. The average power delivered to the circuit by the voltage source is

$$P = (20\text{ V})(4\text{ A})\cos 53.13° = 48.0\text{ W}$$

f. The average power dissipated by the resistor and capacitor will be

$$P_R = (4\text{ A})^2(3\ \Omega)\cos 0° = 48\text{ W}$$
$$P_C = (4\text{ A})^2(4\ \Omega)\cos 90° = 0\text{ W} \quad \text{(as expected!)}$$

Notice that the power factor used in determining the power dissipated by each of the elements is the power factor for that element and not the total power factor for the circuit.

As expected, the summation of powers dissipated by the resistor and capacitor is equal to the total power delivered by the voltage source.

PRACTICE PROBLEMS 3

A circuit consists of a voltage source $\mathbf{E} = 50\text{ V}\angle 25°$ in series with $L = 20$ mH, $C = 50\ \mu$F, and $R = 25\ \Omega$. The circuit operates at an angular frequency of 2 krad/s.

a. Determine the current phasor, **I.**

b. Solve for the power factor of the circuit.

c. Calculate the average power dissipated by the circuit and verify that this is equal to the average power delivered by the source.

d. Use Ohm's law to find $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.

*Answers:*

a. $\mathbf{I} = 1.28\text{ A}\angle -25.19°$

b. $F_p = 0.6402$

c. $P = 41.0$ W

d. $\mathbf{V}_R = 32.0\text{ V}\angle -25.19°$
$\mathbf{V}_C = 12.8\text{ V}\angle -115.19°$
$\mathbf{V}_L = 51.2\text{ V}\angle 64.81°$

## 18.3 Kirchhoff's Voltage Law and the Voltage Divider Rule

When a voltage is applied to impedances in series, as shown in Figure 18–30, Ohm's law may be used to determine the voltage across any impedance as

$$\mathbf{V}_x = \mathbf{I}\mathbf{Z}_x$$

The current in the circuit is

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_\mathrm{T}}$$

Now, by substitution we arrive at the voltage divider rule for any series combination of elements as

$$\mathbf{V}_x = \frac{\mathbf{Z}_x}{\mathbf{Z}_\mathrm{T}}\mathbf{E} \tag{18–10}$$

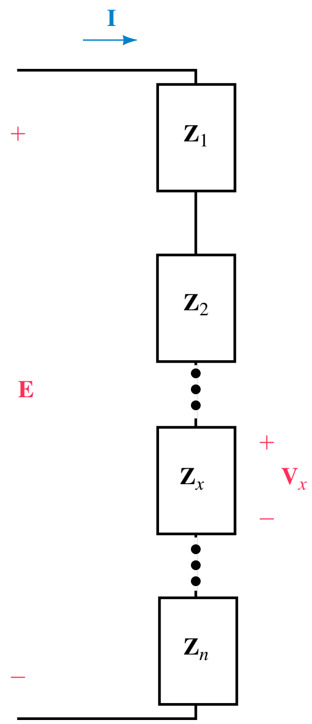



FIGURE 18–30

Equation 18–10 is very similar to the equation for the voltage divider rule in dc circuits. The fundamental differences in solving ac circuits are that we use impedances rather than resistances and that the voltages found are phasors. Because the voltage divider rule involves solving products and quotients of phasors, we generally use the polar form rather than the rectangular form of phasors.

Kirchhoff's voltage law must apply for all circuits whether they are dc or ac circuits. However, because ac circuits have voltages expressed in either sinusoidal or phasor form, Kirchhoff's voltage law for ac circuits may be stated as follows:

*The phasor sum of voltage drops and voltage rises around a closed loop is equal to zero.*

When adding phasor voltages, we find that the summation is generally done more easily in rectangular form rather than the polar form.

**EXAMPLE 18–9** Consider the circuit of Figure 18–31.

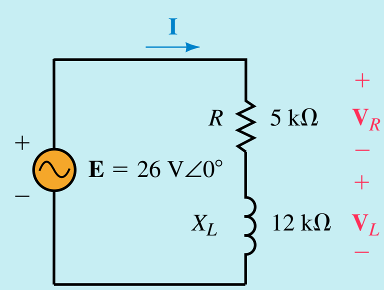



FIGURE 18–31

a. Find $\mathbf{Z}_T$.
b. Determine the voltages $\mathbf{V}_R$ and $\mathbf{V}_L$ using the voltage divider rule.
c. Verify Kirchhoff's voltage law around the closed loop.

**Solution**

a. $\mathbf{Z}_T = 5\text{ k}\Omega + j12\text{ k}\Omega = 13\text{ k}\Omega\angle 67.38^\circ$

b. $$\mathbf{V}_R = \left(\frac{5\text{ k}\Omega\angle 0^\circ}{13\text{ k}\Omega\angle 67.38^\circ}\right)(26\text{ V}\angle 0^\circ) = 10\text{ V}\angle -67.38^\circ$$

$$\mathbf{V}_L = \left(\frac{12\text{ k}\Omega\angle 90^\circ}{13\text{ k}\Omega\angle 67.38^\circ}\right)(26\text{ V}\angle 0^\circ) = 24\text{ V}\angle 22.62^\circ$$

c. Kirchhoff's voltage law around the closed loop will give

$$26\text{ V}\angle 0^\circ - 10\text{ V}\angle -67.38^\circ - 24\text{ V}\angle 22.62^\circ = 0$$
$$(26 + j0) - (3.846 - j9.231) - (22.154 + j9.231) = 0$$
$$(26 - 3.846 - 22.154) + j(0 + 9.231 - 9.231) = 0$$
$$0 + j0 = 0$$

**EXAMPLE 18–10** Consider the circuit of Figure 18–32:

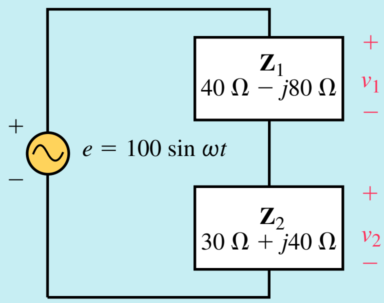



**FIGURE 18–32**

a. Calculate the sinusoidal voltages $v_1$ and $v_2$ using phasors and the voltage divider rule.
b. Sketch the phasor diagram showing $\mathbf{E}$, $\mathbf{V}_1$, and $\mathbf{V}_2$.
c. Sketch the sinusoidal waveforms of $e$, $v_1$, and $v_2$.

**Solution**

a. The phasor form of the voltage source is determined as

$$e = 100 \sin \omega t \Leftrightarrow \mathbf{E} = 70.71\angle\text{V } 0°$$

Applying VDR, we get

$$\mathbf{V}_1 = \left(\frac{40\ \Omega - j80\ \Omega}{(40\ \Omega - j80\ \Omega) + (30\ \Omega + j40\ \Omega)}\right)(70.71\ \text{V}\angle 0°)$$
$$= \left(\frac{89.44\ \Omega\angle -63.43°}{80.62\ \Omega\angle -29.74°}\right)(70.71\ \text{V}\angle 0°)$$
$$= 78.4\ \text{V}\angle -33.69°$$

and

$$\mathbf{V}_2 = \left(\frac{30\ \Omega + j40\ \Omega}{(40\ \Omega - j80\ \Omega) + (30\ \Omega + j40\ \Omega)}\right)(70.71\ \text{V}\angle 0°)$$
$$= \left(\frac{50.00\ \Omega\angle 53.13°}{80.62\ \Omega\angle -29.74°}\right)(70.71\ \text{V}\angle 0°)$$
$$= 43.9\ \text{V}\angle 82.87°$$

The sinusoidal voltages are determined to be

$$v_1 = (\sqrt{2})(78.4)\sin(\omega t - 33.69°)$$
$$= 111 \sin(\omega t - 33.69°)$$

and

$$v_2 = (\sqrt{2})(43.9)\sin(\omega t + 82.87°)$$
$$= 62.0 \sin(\omega t + 82.87°)$$

b. The phasor diagram is shown in Figure 18–33.

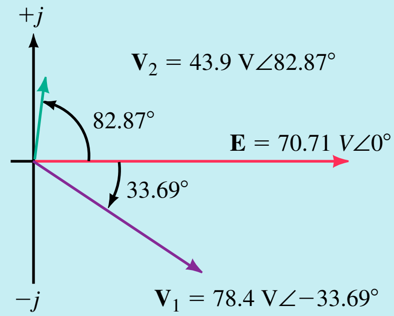



**FIGURE 18–33**

c. The corresponding sinusoidal voltages are shown in Figure 18–34.

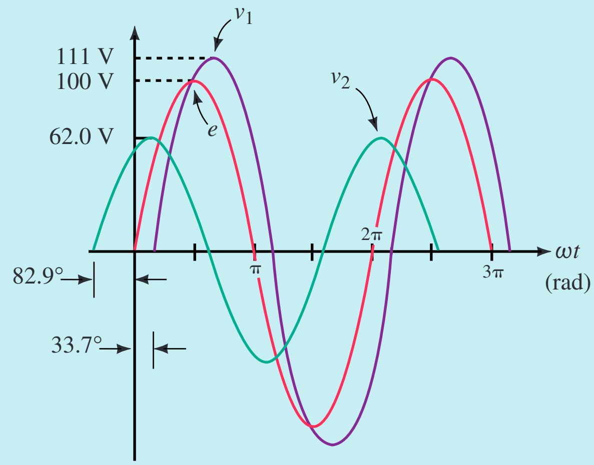
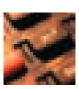


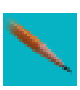

**FIGURE 18–34**



IN-PROCESS LEARNING CHECK 2

1. Express Kirchhoff's voltage law as it applies to ac circuits.
2. What is the fundamental difference between how Kirchhoff's voltage law is used in ac circuits as compared with dc circuits?

*(Answers are at the end of the chapter.)*



PRACTICE PROBLEMS 4

A circuit consists of a voltage source $\mathbf{E} = 50\ \text{V}\angle 25°$ in series with $L = 20$ mH, $C = 50\ \mu$F, and $R = 25\ \Omega$. The circuit operates at an angular frequency of 2 krad/s.

a. Use the voltage divider rule to determine the voltage across each element in the circuit.

b. Verify that Kirchhoff's voltage law applies for the circuit.

*Answers:*

a. $\mathbf{V}_L = 51.2\ \text{V}\angle 64.81°$, $\mathbf{V}_C = 12.8\ \text{V}\angle -115.19°$

$\mathbf{V}_R = 32.0\ \text{V}\angle -25.19°$

b. $51.2\ \text{V}\angle 64.81° + 12.8\ \text{V}\angle -115.19° + 32.0\ \text{V}\angle -25.19° = 50\ \text{V}\angle 25°$

## 18.4 AC Parallel Circuits

The **admittance Y** of any impedance is defined as a vector quantity which is the reciprocal of the impedance **Z.**

Mathematically, admittance is expressed as

$$\mathbf{Y}_{\mathrm{T}} = \frac{1}{\mathbf{Z}_{\mathrm{T}}} = \frac{1}{Z_{\mathrm{T}}\angle\theta} = \left(\frac{1}{Z_{\mathrm{T}}}\right)\angle-\theta = Y_{\mathrm{T}}\angle-\theta \quad \text{(S)} \tag{18–11}$$

where the unit of admittance is the siemens (S).

In particular, we have seen that the admittance of a resistor $R$ is called **conductance** and is given the symbol $\mathbf{Y}_R$. If we consider resistance as a vector quantity, then the corresponding vector form of the conductance is

$$\mathbf{Y}_R = \frac{1}{R\angle 0^\circ} = \frac{1}{R}\angle 0^\circ = G\angle 0^\circ = G + j0 \quad \text{(S)} \tag{18–12}$$

If we determine the admittance of a purely reactive component $X$, the resultant admittance is called the **susceptance** of the component and is assigned the symbol $B$. The unit for susceptance is siemens (S). In order to distinguish between inductive susceptance and capacitive susceptance, we use the subscripts $L$ and $C$ respectively. The vector forms of reactive admittance are given as follows:

$$\mathbf{Y}_L = \frac{1}{X_L\angle 90^\circ} = \frac{1}{X_L}\angle-90^\circ = B_L\angle-90^\circ = 0 - jB_L \quad \text{(S)} \tag{18–13}$$

$$\mathbf{Y}_C = \frac{1}{X_C\angle-90^\circ} = \frac{1}{X_C}\angle 90^\circ = B_C\angle 90^\circ = 0 + jB_C \quad \text{(S)} \tag{18–14}$$

In a manner similar to impedances, admittances may be represented on the complex plane in an **admittance diagram** as shown in Figure 18–35.

The lengths of the various vectors are proportional to the magnitudes of the corresponding admittances. The resistive admittance vector **G** is shown on the positive real axis, whereas the inductive and capacitive admittance vectors $\mathbf{Y}_L$ and $\mathbf{Y}_C$ are shown on the negative and positive imaginary axes respectively.

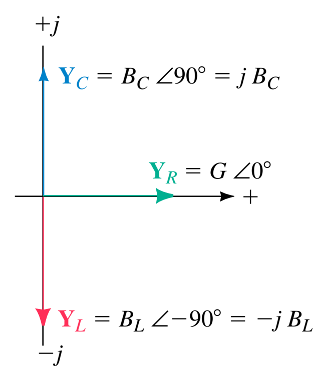



**FIGURE 18–35** Admittance diagram showing conductance ($\mathbf{Y}_R$) and susceptance ($\mathbf{Y}_L$ and $\mathbf{Y}_C$).

**EXAMPLE 18–11** Determine the admittances of the following impedances. Sketch the corresponding admittance diagram.

a. $R = 10\ \Omega$ b. $X_L = 20\ \Omega$ c. $X_C = 40\ \Omega$

**Solutions**

a. $\mathbf{Y}_R = \dfrac{1}{R} = \dfrac{1}{10\ \Omega\angle 0^\circ} = 100\ \text{mS}\angle 0^\circ$

b. $\mathbf{Y}_L = \dfrac{1}{X_L} = \dfrac{1}{20\ \Omega\angle 90^\circ} = 50\ \text{mS}\angle-90^\circ$

c. $\mathbf{Y}_C = \dfrac{1}{X_C} = \dfrac{1}{40\ \Omega\angle-90^\circ} = 25\ \text{mS}\angle 90^\circ$

The admittance diagram is shown in Figure 18–36.

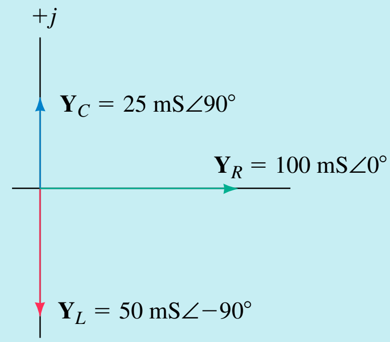



**FIGURE 18–36**

For any network of $n$ admittances as shown in Figure 18–37, the total admittance is the vector sum of the admittances of the network. Mathematically, the total admittance of a network is given as

$$\mathbf{Y}_\text{T} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n \quad \text{(S)} \tag{18–15}$$

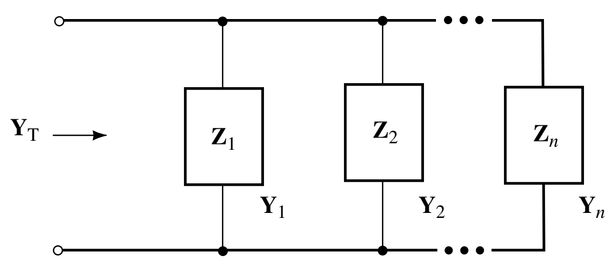



**FIGURE 18–37**

The resultant impedance of a parallel network of $n$ impedances is determined to be

$$\mathbf{Z}_\text{T} = \frac{1}{\mathbf{Y}_\text{T}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n}$$

$$\mathbf{Z}_\text{T} = \frac{1}{\dfrac{1}{\mathbf{Z}_1} + \dfrac{1}{\mathbf{Z}_2} + \cdots + \dfrac{1}{\mathbf{Z}_n}} \quad (\Omega) \tag{18–16}$$

**EXAMPLE 18–12** Find the equivalent admittance and impedance of the network of Figure 18–38. Sketch the admittance diagram.

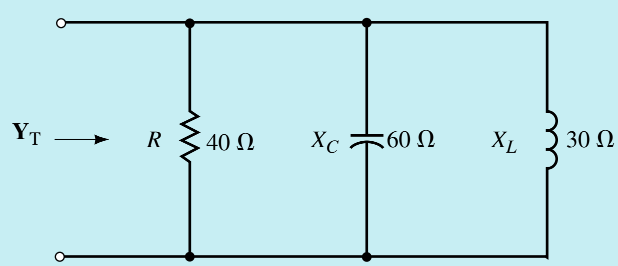



**FIGURE 18–38**

**Solution** The admittances of the various parallel elements are

$$\mathbf{Y}_1 = \frac{1}{40\ \Omega\angle 0^\circ} = 25.0\ \text{mS}\angle 0^\circ = 25.0\ \text{mS} + j0$$

$$\mathbf{Y}_2 = \frac{1}{60\ \Omega\angle -90^\circ} = 16.\overline{6}\ \text{mS}\angle 90^\circ = 0 + j16.\overline{6}\ \text{mS}$$

$$\mathbf{Y}_3 = \frac{1}{30\ \Omega\angle 90^\circ} = 33.\overline{3}\ \text{mS}\angle -90^\circ = 0 - j33.\overline{3}\ \text{mS}$$

The total admittance is determined as

$$\begin{aligned}\mathbf{Y}_\text{T} &= \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3\\ &= 25.0\ \text{mS} + j16.\overline{6}\ \text{mS} + (-j33.\overline{3}\ \text{mS})\\ &= 25.0\ \text{mS} - j16.\overline{6}\ \text{mS}\\ &= 30.0\ \text{mS}\angle -33.69^\circ\end{aligned}$$

This results in a total impedance for the network of

$$\begin{aligned}\mathbf{Z}_\text{T} &= \frac{1}{\mathbf{Y}_\text{T}}\\ &= \frac{1}{30.0\ \text{mS}\angle -33.69^\circ}\\ &= 33.3\ \Omega\angle 33.69^\circ\end{aligned}$$

The admittance diagram is shown in Figure 18–39.

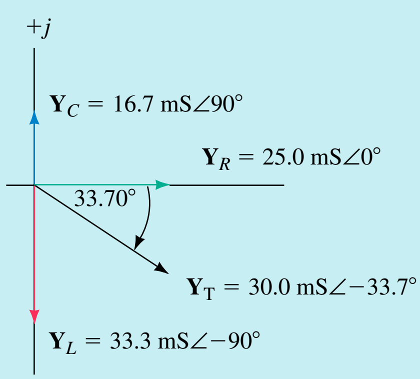



**FIGURE 18–39**

## Two Impedances in Parallel

By applying Equation 18–14 for two impedances, we determine the equivalent impedance of two impedances as

$$\mathbf{Z}_\text{T} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \quad (\Omega) \tag{18–17}$$

From the above expression, we see that for two impedances in parallel, the equivalent impedance is determined as the product of the impedances over the sum. Although the expression for two impedances is very similar to the expression for two resistors in parallel, the difference is that the calculation of impedance involves the use of complex algebra.

**EXAMPLE 18–13** Find the total impedance for the network shown in Figure 18–40.

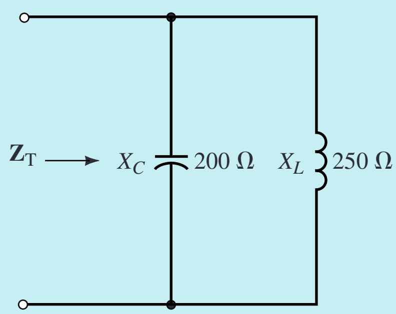



FIGURE 18–40

**Solution**

$$\mathbf{Z}_T = \frac{(200\ \Omega\angle -90°)(250\ \Omega\angle 90°)}{-j200\ \Omega + j250\ \Omega}$$

$$= \frac{50\ \text{k}\Omega\angle 0°}{50\angle 90°} = 1\ \text{k}\Omega\angle -90°$$

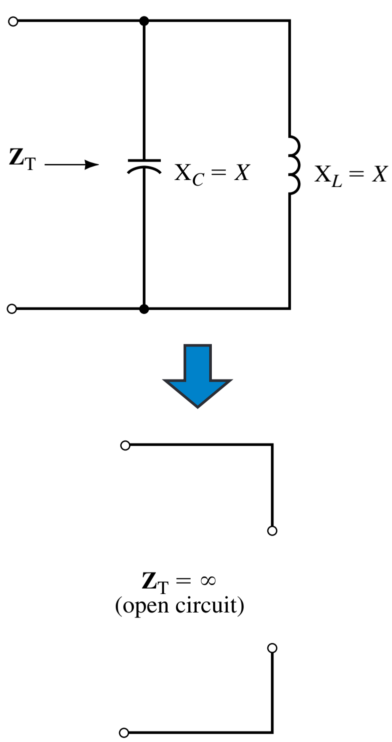



FIGURE 18–41

The previous example illustrates that unlike total parallel resistance, the total impedance of a combination of parallel reactances may be much larger that either of the individual impedances. Indeed, if we are given a parallel combination of equal inductive and capacitive reactances, the total impedance of the combination is equal to infinity (namely an open circuit). Consider the network of Figure 18–41.

The total impedance $\mathbf{Z}_T$ is found as

$$\mathbf{Z}_T = \frac{(X_L\angle 90°)(X_C\angle -90°)}{jX_L - jX_C} = \frac{X^2\angle 0°}{0\angle 0°} = \infty\angle 0°$$

Because the denominator of the above expression is equal to zero, the magnitude of the total impedance will be undefined ($Z = \infty$). The magnitude is undefined and the algebra yields a phase angle $\theta = 0°$, which indicates that the vector lies on the positive real axis of the impedance diagram.

*Whenever a capacitor and an inductor having equal reactances are placed in parallel, the equivalent circuit of the two components is an open circuit.*

The principle of equal parallel reactances will be studied in a later chapter dealing with "resonance."

## Three Impedances in Parallel

Equation 18–16 may be solved for three impedances to give the equivalent impedance as

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3} \quad (\Omega) \tag{18–18}$$

although this is less useful than the general equation.

**EXAMPLE 18–14** Find the equivalent impedance of the network of Figure 18–42.

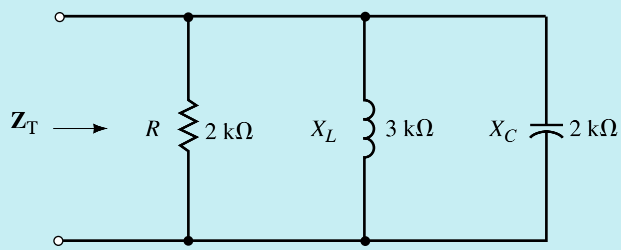



**FIGURE 18–42**

**Solution**

$$\mathbf{Z}_T = \frac{(2\ \text{k}\Omega\angle 0°)(3\ \text{k}\Omega\angle 90°)(2\ \text{k}\Omega\angle -90°)}{(2\ \text{k}\Omega\angle 0°)(3\ \text{k}\Omega\angle 90°) + (2\ \text{k}\Omega\angle 0°)(2\ \text{k}\Omega\angle -90°) + (3\ \text{k}\Omega\angle 90°)(2\ \text{k}\Omega\angle -90°)}$$

$$= \frac{12 \times 10^9\ \Omega\angle 0°}{6 \times 10^6\angle 90° + 4 \times 10^6\angle -90° + 6 \times 10^6\angle 0°}$$

$$= \frac{12 \times 10^9\ \Omega\angle 0°}{6 \times 10^6 + j2 \times 10^6} = \frac{12 \times 10^9\ \Omega\angle 0°}{6.325 \times 10^6\angle 18.43°}$$

$$= 1.90\ \text{k}\Omega\angle -18.43°$$

And so the equivalent impedance of the network is

$$Z_T = 1.80\ \text{k}\Omega - j0.6\ \text{k}\Omega$$

**PRACTICE PROBLEMS 5**

A circuit consists of a current source, $i = 0.030 \sin 500t$, in parallel with $L = 20$ mH, $C = 50\ \mu$F, and $R = 25\ \Omega$.

a. Determine the voltage **V** across the circuit.

b. Solve for the power factor of the circuit.

c. Calculate the average power dissipated by the circuit and verify that this is equal to the power delivered by the source.

d. Use Ohm's law to find the phasor quantities, $\mathbf{I}_R$, $\mathbf{I}_L$, and $\mathbf{I}_C$.

*Answers:*

a. $\mathbf{V} = 0.250\ \text{V}\angle 61.93°$

b. $F_p = 0.4705$

c. $P_R = 41.0\ \text{W} = P_T$

d. $\mathbf{I}_R = 9.98\ \text{mA}\angle 61.98°$

$\mathbf{I}_C = 6.24\ \text{mA}\angle 151.93°$

$\mathbf{I}_L = 25.0\ \text{mA}\angle -28.07°$

**PRACTICE PROBLEMS 6**

A circuit consists of a 2.5-$A_{rms}$ current source connected in parallel with a resistor, an inductor, and a capacitor. The resistor has a value of 10 Ω and dissipates 40 W of power.

a. Calculate the values of $X_L$ and $X_C$ if $X_L = 3X_C$.
b. Determine the magnitudes of current through the inductor and the capacitor.

*Answers:*
a. $X_L = 80\ \Omega$, $X_C = 26.7\ \Omega$
b. $I_L = 0.25$ mA, $I_C = 0.75$ mA

## 18.5 Kirchhoff's Current Law and the Current Divider Rule

The current divider rule for ac circuits has the same form as for dc circuits with the notable exception that currents are expressed as phasors. For a parallel network as shown in Figure 18–43, the current in any branch of the network may be determined using either admittance or impedance.

$$\mathbf{I}_x = \frac{\mathbf{Y}_x}{\mathbf{Y}_\mathrm{T}}\mathbf{I} \quad \text{or} \quad \mathbf{I}_x = \frac{\mathbf{Z}_\mathrm{T}}{\mathbf{Z}_x}\mathbf{I} \tag{18–19}$$

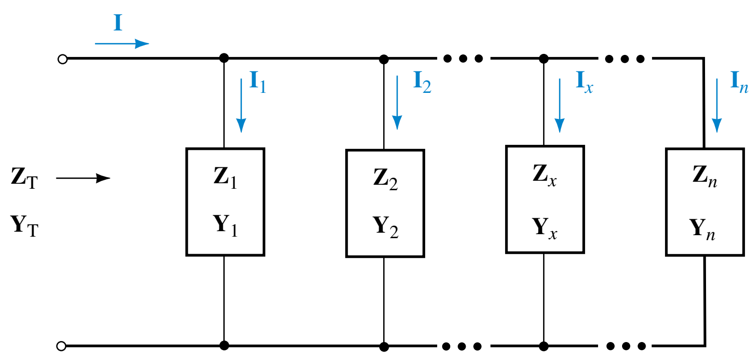



**FIGURE 18–43**

For two branches in parallel the current in either branch is determined from the impedances as

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \tag{18–20}$$

Also, as one would expect, Kirchhoff's current law must apply to any node within an ac circuit. For such circuits, KCL may be stated as follows:

*The summation of current phasors entering and leaving a node is equal to zero.*

**EXAMPLE 18–15** Calculate the current in each of the branches in the network of Figure 18–44.

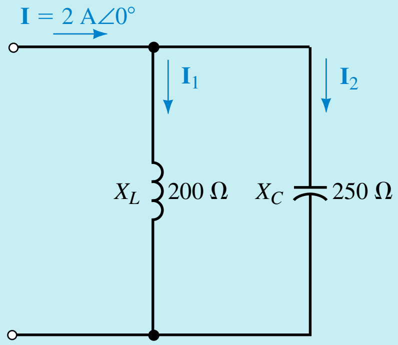



FIGURE 18–44

**Solution**

$$\mathbf{I}_1 = \left(\frac{250\ \Omega\angle-90^\circ}{j200\ \Omega - j250\ \Omega}\right)(2\ \mathrm{A}\angle0^\circ)$$
$$= \left(\frac{250\ \Omega\angle-90^\circ}{50\ \Omega\angle-90^\circ}\right)(2\ \mathrm{A}\angle0^\circ) = 10\ \mathrm{A}\angle0^\circ$$

and

$$\mathbf{I}_2 = \left(\frac{200\ \Omega\angle90^\circ}{j200\ \Omega - j250\ \Omega}\right)(2\ \mathrm{A}\angle0^\circ)$$
$$= \left(\frac{200\ \Omega\angle90^\circ}{50\ \Omega\angle-90^\circ}\right)(2\ \mathrm{A}\angle0^\circ) = 8\ \mathrm{A}\angle180^\circ$$

The above results illustrate that the currents in parallel reactive components may be significantly larger than the applied current. If the current through the component exceeds the maximum current rating of the element, severe damage may occur.

**EXAMPLE 18–16** Refer to the circuit of Figure 18–45:

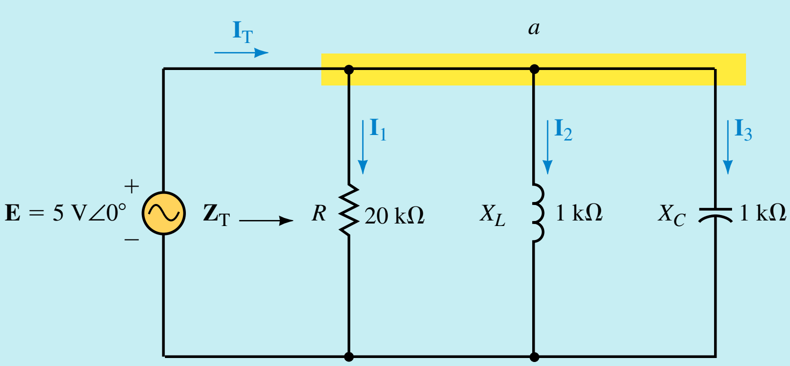



FIGURE 18–45

a. Find the total impedance, $\mathbf{Z}_T$.
b. Determine the supply current, $\mathbf{I}_T$.
c. Calculate $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ using the current divider rule.
d. Verify Kirchhoff's current law at node *a*.

**Solution**

a. Because the inductive and capacitive reactances are in parallel and have the same value, we may replace the combination by an open circuit. Consequently, only the resistor $R$ needs to be considered. As a result

$$\mathbf{Z}_T = 20\ \text{k}\Omega\angle 0°$$

b. $$\mathbf{I}_T = \frac{5\ \text{V}\angle 0°}{20\ \text{k}\Omega\angle 0°} = 250\ \mu\text{A}\angle 0°$$

c. $$\mathbf{I}_1 = \left(\frac{20\ \text{k}\Omega\angle 0°}{20\ \text{k}\Omega\angle 0°}\right)(250\ \mu\text{A}\angle 0°) = 250\ \mu\text{A}\angle 0°$$

$$\mathbf{I}_2 = \left(\frac{20\ \text{k}\Omega\angle 0°}{1\ \text{k}\Omega\angle 90°}\right)(250\ \mu\text{A}\angle 0°) = 5.0\ \text{mA}\angle -90°$$

$$\mathbf{I}_3 = \left(\frac{20\ \text{k}\Omega\angle 0°}{1\ \text{k}\Omega\angle -90°}\right)(250\ \mu\text{A}\angle 0°) = 5.0\ \text{mA}\angle 90°$$

d. Notice that the currents through the inductor and capacitor are 180° out of phase. By adding the current phasors in rectangular form, we have

$$I_T = 250\ \mu\text{A} - j5.0\ \text{A} + j5.0\ \text{A} = 250\ \mu\text{A} + j0 = 250\ \mu\text{A}\angle 0°$$

The above result verifies Kirchhoff's current law at the node.

**IN-PROCESS LEARNING CHECK 3**

1. Express Kirchhoff's current law as it applies to ac circuits.
2. What is the fundamental difference between how Kirchhoff's current law is applied to ac circuits as compared with dc circuits?

*(Answers are at the end of the chapter.)*

**PRACTICE PROBLEMS 7**

a. Use the current divider rule to determine current through each branch in the circuit of Figure 18–46.

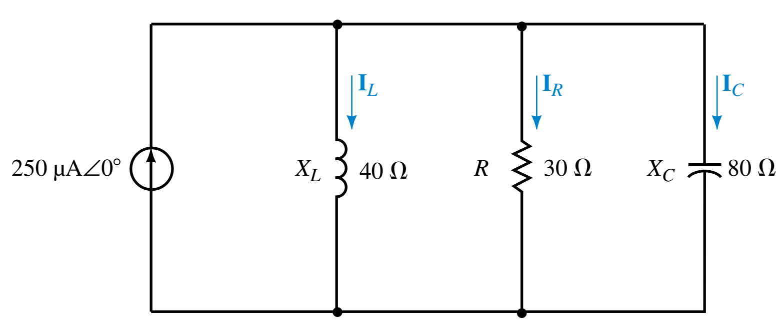



**FIGURE 18–46**

b. Verify that Kirchhoff's current law applies to the circuit of Figure 18–46.

*Answers:*
a. $\mathbf{I}_L = 176\ \mu\text{A}\angle-69.44°$
$\mathbf{I}_R = 234\ \mu\text{A}\angle 20.56°$
$\mathbf{I}_C = 86.8\ \mu\text{A}\angle 110.56°$
b. $\Sigma\mathbf{I}_{\text{out}} = \Sigma\mathbf{I}_{\text{in}} = 250\ \mu\text{A}$

## 18.6 Series-Parallel Circuits

We may now apply the analysis techniques of series and parallel circuits in solving more complicated circuits. As in dc circuits, the analysis of such circuits is simplified by starting with easily recognized combinations. If necessary, the original circuit may be redrawn to make further simplification more apparent. Regardless of the complexity of the circuits, we find that the fundamental rules and laws of circuit analysis must apply in all cases.

Consider the network of Figure 18–47.

We see that the impedances $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are in series. The branch containing this combination is then seen to be in parallel with the impedance $\mathbf{Z}_1$.

The total impedance of the network is expressed as

$$\mathbf{Z}_\text{T} = \mathbf{Z}_1 \parallel (\mathbf{Z}_2 + \mathbf{Z}_3)$$

Solving for $\mathbf{Z}_\text{T}$ gives the following:

$$\begin{aligned}\mathbf{Z}_\text{T} &= (2\ \Omega - j8\ \Omega)\|(2\ \Omega - j5\ \Omega + 6\ \Omega + j7\ \Omega)\\ &= (2\ \Omega - j8\ \Omega)\parallel(8\ \Omega + j2\ \Omega)\\ &= \frac{(2\ \Omega - j8\ \Omega)(8\ \Omega + j2\ \Omega)}{2\ \Omega - j8\ \Omega + 8\ \Omega + j2\ \Omega)}\\ &= \frac{(8.246\ \Omega\angle-75.96°)(8.246\ \Omega\angle 14.04°)}{11.66\ \Omega\angle-30.96°}\\ &= 5.832\ \Omega\angle-30.96° = 5.0\ \Omega - j3.0\ \Omega\end{aligned}$$

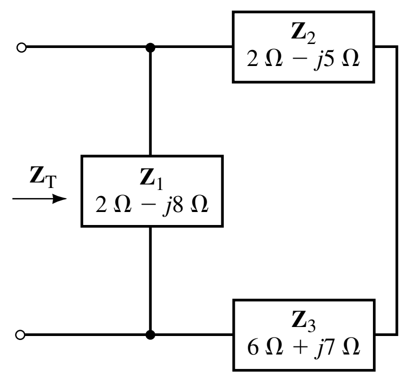


FIGURE 18–47

**EXAMPLE 18–17** Determine the total impedance of the network of Figure 18–48. Express the impedance in both polar form and rectangular form.

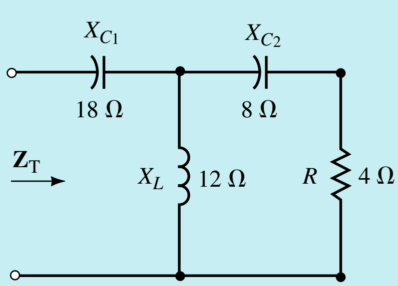
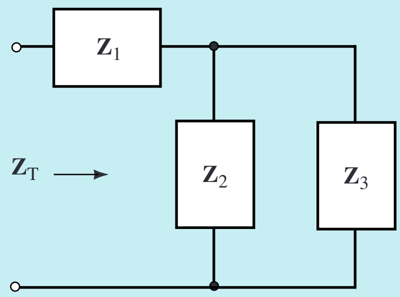


FIGURE 18–48

**Solution** After redrawing and labelling the given circuit, we have the circuit shown in Figure 18–49.




$\mathbf{Z}_1 = -j18\ \Omega$
$\mathbf{Z}_2 = +j12\ \Omega = 12\ \Omega\angle 90°$
$\mathbf{Z}_3 = 4\ \Omega - j8\ \Omega = 8.94\ \Omega\angle -63.43°$

**FIGURE 18–49**

The total impedance is given as

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

where

$$\mathbf{Z}_1 = -j18\ \Omega = 18\Omega\angle -90°$$
$$\mathbf{Z}_2 = j12\ \Omega = 12\Omega\angle 90°$$
$$\mathbf{Z}_3 = 4\ \Omega - j8\ \Omega = 8.94\ \Omega\angle -63.43°$$

We determine the total impedance as

$$\mathbf{Z}_T = -j18\ \Omega + \left[\frac{(12\ \Omega\angle 90°)(8.94\ \Omega\angle -63.43°)}{j12\ \Omega + 4\ \Omega - j8\ \Omega}\right]$$
$$= -j18\ \Omega + \left(\frac{107.3\ \Omega\angle 26.57°}{5.66\angle 45°}\right)$$
$$= -j18\ \Omega + 19.0\ \Omega\angle -18.43°$$
$$= -j18\ \Omega + 18\ \Omega - j6\ \Omega$$
$$= 18\ \Omega - j24\ \Omega = 30\ \Omega\angle -53.13°$$

**EXAMPLE 18–18** Consider the circuit of Figure 18–50:

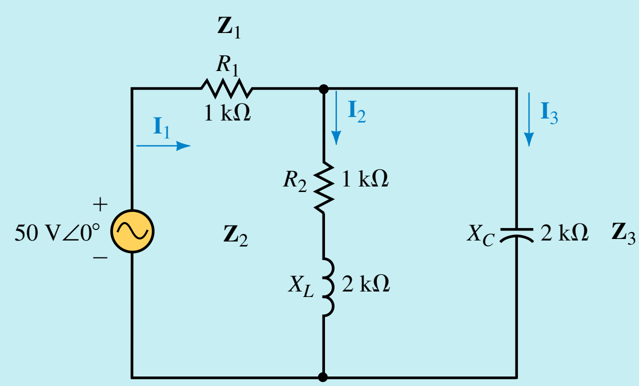



**FIGURE 18–50**

a. Find $\mathbf{Z}_T$.

b. Determine the currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

c. Calculate the total power provided by the voltage source.

d. Determine the average powers $P_1$, $P_2$, and $P_3$ dissipated by each of the impedances. Verify that the average power delivered to the circuit is the same as the power dissipated by the impedances.

**Solution**

a. The total impedance is determined by the combination

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

For the parallel combination we have

$$\begin{aligned}\mathbf{Z}_2 \parallel \mathbf{Z}_3 &= \frac{(1\text{ k}\Omega + j2\text{ k}\Omega)(-j2\text{ k}\Omega)}{1\text{ k}\Omega + j2\text{ k}\Omega - j2\text{ k}\Omega} \\ &= \frac{(2.236\text{ k}\Omega\angle 63.43^\circ)(2\text{ k}\Omega\angle -90^\circ)}{1\text{ k}\Omega\angle 0^\circ} \\ &= 4.472\text{ k}\Omega\angle -26.57^\circ = 4.0\text{ k}\Omega - j2.0\text{ k}\Omega\end{aligned}$$

And so the total impedance is

$$\mathbf{Z}_T = 5\text{ k}\Omega - j2\text{ k}\Omega = 5.385\text{ k}\Omega\angle -21.80^\circ$$

$$\begin{aligned}\mathbf{I}_1 &= \frac{50\text{ V}\angle 0^\circ}{5.385\text{ k}\Omega\angle -21.80^\circ} \\ &= 9.285\text{ mA}\angle 21.80^\circ\end{aligned}$$

Applying the current divider rule, we get

$$\begin{aligned}\mathbf{I}_2 &= \frac{(2\text{ k}\Omega\angle -90^\circ)(9.285\text{ mA}\angle 21.80^\circ)}{1\text{ k}\Omega + j2\text{ k}\Omega - j2\text{ k}\Omega} \\ &= 18.57\text{ mA}\angle -68.20^\circ\end{aligned}$$

and

$$\begin{aligned}\mathbf{I}_3 &= \frac{(1\text{ k}\Omega + j2\text{ k}\Omega)(9.285\text{ mA}\angle 21.80^\circ)}{1\text{ k}\Omega + j2\text{ k}\Omega - j2\text{ k}\Omega} \\ &= \frac{(2.236\text{ k}\Omega\angle 63.43^\circ)(9.285\text{ mA}\angle 21.80^\circ)}{1\text{ k}\Omega\angle 0^\circ} \\ &= 20.761\text{ mA}\angle 85.23^\circ\end{aligned}$$

c.
$$\begin{aligned}P_T &= (50\text{ V})(9.285\text{ mA})\cos 21.80^\circ \\ &= 431.0\text{ mW}\end{aligned}$$

d. Because only the resistors will dissipate power, we may use $P = I^2R$:

$$P_1 = (9.285\text{ mA})^2(1\text{ k}\Omega) = 86.2\text{ mW}$$
$$P_2 = (18.57\text{ mA})^2(1\text{ k}\Omega) = 344.8\text{ mW}$$

Alternatively, the power dissipated by $\mathbf{Z}_2$ may have been determined as $P = I^2Z\cos\theta$:

$$P_2 = (18.57\text{ mA})^2(2.236\text{ k}\Omega)\cos 63.43^\circ = 344.8\text{ mW}$$

Since $\mathbf{Z}_3$ is purely capacitive, it will not dissipate any power:

$$P_3 = 0$$

By combining these powers, the total power dissipated is found:

$$P_T = 86.2 \text{ mW} + 344.8 \text{ mW} + 0 = 431.0 \text{ mW} \quad \text{(checks!)}$$

PRACTICE PROBLEMS 8

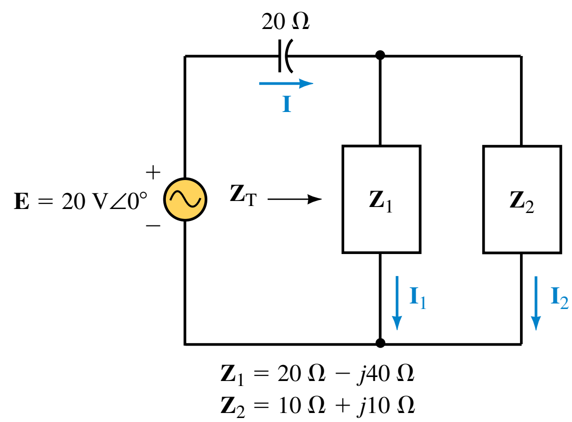



**FIGURE 18–51**

Refer to the circuit of Figure 18–51:

a. Calculate the total impedance, $\mathbf{Z}_T$.

b. Find the current **I.**

c. Use the current divider rule to find $\mathbf{I}_1$ and $\mathbf{I}_2$.

d. Determine the power factor for each impedance, $\mathbf{Z}_1$ and $\mathbf{Z}_2$.

e. Determine the power factor for the circuit.

f. Verify that the total power dissipated by impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is equal to the power delivered by the voltage source.

*Answers:*

a. $\mathbf{Z}_T = 18.9\ \Omega\angle-45°$

b. $\mathbf{I} = 1.06\ \text{A}\angle 45°$

c. $\mathbf{I}_1 = 0.354\ \text{A}\angle 135°$, $\mathbf{I}_2 = 1.12\ \text{A}\angle 26.57°$

d. $F_{P(1)} = 0.4472$ leading, $F_{P(2)} = 0.7071$ lagging

e. $F_P = 0.7071$ leading

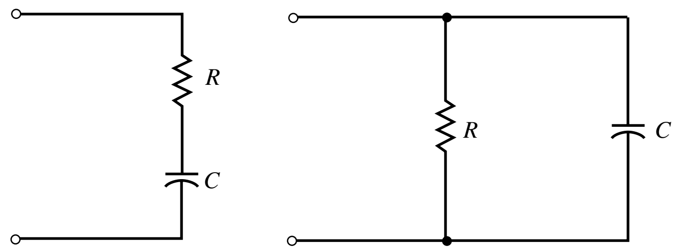
f. $P_T = 15.0$ W, $P_1 = 2.50$ W, $P_2 = 12.5$ W
$P_1 + P_2 = 15.0 \text{ W} = P_T$

## 18.7 Frequency Effects

As we have already seen, the reactance of inductors and capacitors depends on frequency. Consequently, the total impedance of any network having reactive elements is also frequency dependent. Any such circuit would need to be analyzed separately at each frequency of interest. We will examine several fairly simple combinations of resistors, capacitors, and inductors to see how the various circuits operate at different frequencies. Some of the more important combinations will be examined in greater detail in later chapters which deal with resonance and filters.

## *RC* Circuits

As the name implies, *RC* circuits consist of a resistor and a capacitor. The components of an *RC* circuit may be connected either in series or in parallel as shown in Figure 18–52.




(a) Series *RC* circuit (b) Parallel *RC* circuit

**FIGURE 18–52**

Consider the *RC* series circuit of Figure 18–53.
Recall that the capacitive reactance, $X_C$, is given as

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

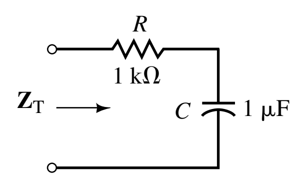



**FIGURE 18–53**

The total impedance of the circuit is a vector quantity expressed as

$$\mathbf{Z}_T = R - j\frac{1}{\omega C} = R + \frac{1}{j\omega C}$$

$$\mathbf{Z}_T = \frac{1 + j\omega RC}{j\omega C} \qquad \textbf{(18–21)}$$

If we define the **cutoff** or **corner frequency** for an *RC* circuit as

$$\omega_c = \frac{1}{RC} = \frac{1}{\tau} \quad \text{(rad/s)} \qquad \textbf{(18–22)}$$

or equivalently as

$$f_c = \frac{1}{2\pi RC} \quad \text{(Hz)} \qquad \textbf{(18–23)}$$

then several important points become evident.

For $\omega \le \omega_c/10$ (or $f \le f_c/10$) Equation 18–21 can be expressed as

$$\mathbf{Z}_T \simeq \frac{1 + j0}{j\omega C} = \frac{1}{j\omega C}$$

and for $\omega \ge 10\omega_c$, the expression of (18–21) can be simplified as

$$\mathbf{Z}_T \simeq \frac{0 + j\omega RC}{j\omega C} = R$$

**TABLE 18–1**

| Angular Frequency, ω (Rad/s) | $X_C$ (Ω) | $Z_T$ (Ω) |
|---|---|---|
| 0 | ∞ | ∞ |
| 1 | 1 M | 1 M |
| 10 | 100 k | 100 k |
| 100 | 10 k | 10.05 k |
| 200 | 5 k | 5.099 k |
| 500 | 2 k | 2.236 k |
| 1000 | 1 k | 1.414 k |
| 2000 | 500 | 1118 |
| 5000 | 200 | 1019 |
| 10 k | 100 | 1005 |
| 100 k | 10 | 1000 |

Solving for the magnitude of the impedance at several angular frequencies, we have the results shown in Table 18–1.

If the magnitude of the impedance $\mathbf{Z}_\mathrm{T}$ is plotted as a function of angular frequency $\omega$, we get the graph of Figure 18–54. Notice that the abscissa and ordinate of the graph are not scaled linearly, but rather logarithmically. This allows for the the display of results over a wide range of frequencies.

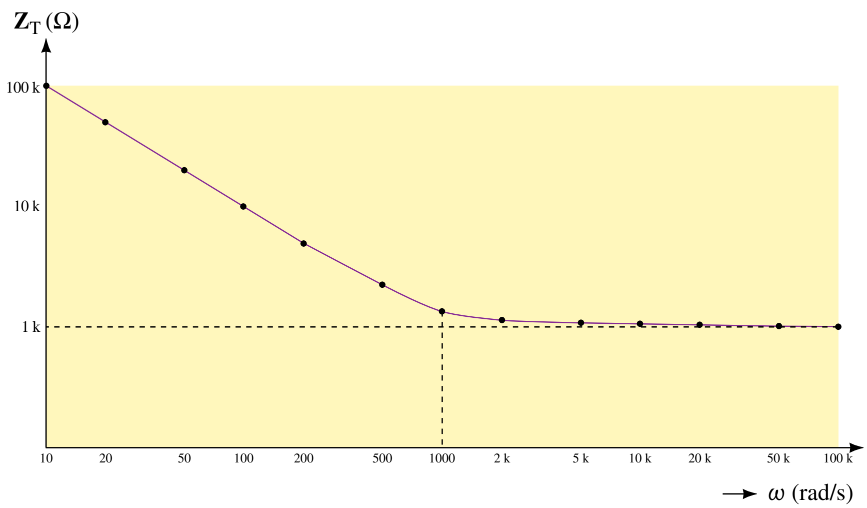



**FIGURE 18–54** Impedance versus angular frequency for the network of Figure 18–53.

The graph illustrates that the reactance of a capacitor is very high (effectively an open circuit) at low frequencies. Consequently, the total impedance of the series circuit will also be very high at low frequencies. Secondly, we notice that as the frequency increases, the reactance decreases. Therefore, as the frequency gets higher, the capacitive reactance has a diminished effect in the circuit. At very high frequencies (typically for $\omega \geq 10\omega_c$), the impedance of the circuit will effectively be $R = 1\ \mathrm{k}\Omega$.

Consider the parallel $RC$ circuit of Figure 18–55. The total impedance, $\mathbf{Z}_\mathrm{T}$, of the circuit is determined as

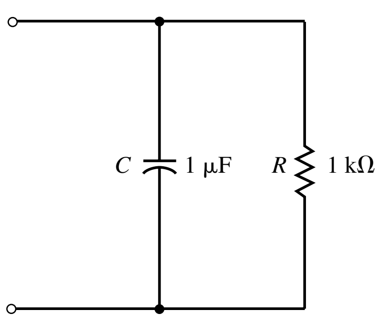



**FIGURE 18–55**

$$\mathbf{Z}_\mathrm{T} = \frac{\mathbf{Z}_R\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{R\left(\dfrac{1}{j\omega C}\right)}{R + \dfrac{1}{j\omega C}} = \frac{\dfrac{R}{j\omega C}}{\dfrac{1 + j\omega RC}{j\omega C}}$$

which may be simplified as

$$\mathbf{Z}_\mathrm{T} = \frac{R}{1 + j\omega RC} \tag{18–24}$$

As before, the cutoff frequency is given by Equation 18–22. Now, by examining the expression of (18–24) for $\omega \leq \omega_c/10$, we have the following result:

$$\mathbf{Z}_T \simeq \frac{R}{1 + j0} = R$$

For $\omega \geq 10\omega_c$, we have

$$\mathbf{Z}_T \simeq \frac{R}{0 + j\omega RC} = \frac{1}{j\omega C}$$

If we solve for the impedance of the circuit in Figure 18–55 at various angular frequencies, we obtain the results of Table 18–2.

Plotting the magnitude of the impedance $\mathbf{Z}_T$ as a function of angular frequency $\omega$, we get the graph of Figure 18–56. Notice that the abscissa and ordinate of the graph are again scaled logarithmically, allowing for the display of results over a wide range of frequencies.

**TABLE 18–2**

| Angular Frequency, ω (Rad/s) | $X_C$ (Ω) | $Z_T$ (Ω) |
|---|---|---|
| 0 | ∞ | 1000 |
| 1 | 1 M | 1000 |
| 10 | 100 k | 1000 |
| 100 | 10 k | 995 |
| 200 | 5 k | 981 |
| 500 | 2 k | 894 |
| 1 k | 1 k | 707 |
| 2 k | 500 | 447 |
| 5 k | 200 | 196 |
| 10 k | 100 | 99.5 |
| 100 k | 10 | 10 |

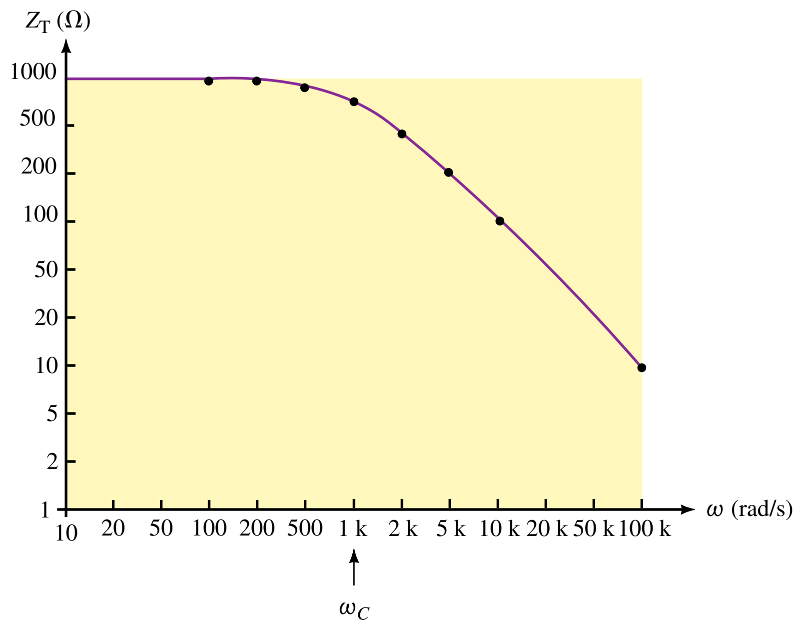



**FIGURE 18–56** Impedance versus angular frequency for the network of Figure 18–55.

The results indicate that at dc ($f$ = 0 Hz) the capacitor, which behaves as an open circuit, will result in a circuit impedance of $R$ = 1 kΩ. As the frequency increases, the capacitor reactance approaches 0 Ω, resulting in a corresponding decrease in circuit impedance.

## *RL* Circuits

*RL* circuits may be analyzed in a manner similar to the analysis of *RC* circuits. Consider the parallel *RL* circuit of Figure 18–57.

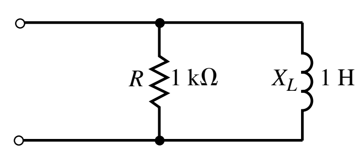



**FIGURE 18–57**

The total impedance of the parallel circuit is found as follows:

$$\begin{aligned}\mathbf{Z}_T &= \frac{\mathbf{Z}_R\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} \\ &= \frac{R(j\omega L)}{R + j\omega L} \\ \mathbf{Z}_T &= \frac{j\omega L}{1 + j\omega\dfrac{L}{R}}\end{aligned} \tag{18–25}$$

If we define the *cutoff* or *corner frequency* for an *RL* circuit as

$$\omega_c = \frac{R}{L} = \frac{1}{\tau} \quad \text{(rad/s)} \tag{18–26}$$

or equivalently as

$$f_c = \frac{R}{2\pi L} \quad \text{(Hz)} \tag{18–27}$$

then several important points become evident.

For $\omega \leq \omega_c/10$ (or $f \leq f_c/10$) Equation 18–25 can be expressed as

$$\mathbf{Z}_T \simeq \frac{j\omega L}{1 + j0} = j\omega L$$

The above result indicates that for low frequencies, the inductor has a very small reactance, resulting in a total impedance which is essentially equal to the inductive reactance.

For $\omega \geq 10\omega_c$, the expression of (18–25) can be simplified as

$$\mathbf{Z}_T \simeq \frac{j\omega L}{0 + j\omega\dfrac{L}{R}} = R$$

The above results indicate that for high frequencies, the impedance of the circuit is essentially equal to the resistance, due to the very high impedance of the inductor.

Evaluating the impedance at several angular frequencies, we have the results of Table 18–3.

When the magnitude of the impedance $\mathbf{Z}_T$ is plotted as a function of angular frequency $\omega$, we get the graph of Figure 18–58.

**TABLE 18–3**

| Angular Frequency, ω (Rad/s) | $X_L$ (Ω) | $Z_T$ (Ω) |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 10 | 10 | 10 |
| 100 | 100 | 99.5 |
| 200 | 200 | 196 |
| 500 | 500 | 447 |
| 1 k | 1 k | 707 |
| 2 k | 2 k | 894 |
| 5 k | 5 k | 981 |
| 10 k | 10 k | 995 |
| 100 k | 100 k | 1000 |

## *RLC* Circuits

When numerous capacitive and inductive components are combined with resistors in series-parallel circuits, the total impedance $\mathbf{Z}_T$ of the circuit may rise and fall several times over the full range of frequencies. The analysis of such complex circuits is outside the scope of this textbook. However, for illustrative purposes we examine the simple series *RLC* circuit of Figure 18–59.

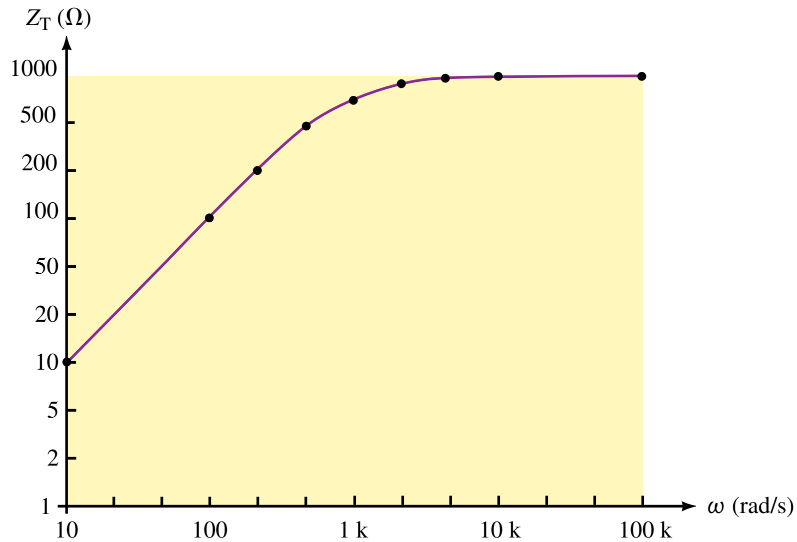



**FIGURE 18–58** Impedance versus angular frequency for the network of Figure 18–57.

The impedance $\mathbf{Z}_T$ at any frequency will be determined as

$$\begin{aligned}\mathbf{Z}_T &= R + jX_L - jX_C \\ &= R + j(X_L - X_C)\end{aligned}$$

At very low frequencies, the inductor will appear as a very low impedance (effectively a short circuit), while the capacitor will appear as a very high impedance (effectively an open circuit). Because the capacitive reactance will be much larger than the inductive reactance, the circuit will have a very large capacitive reactance. This results in a very high circuit impedance, $\mathbf{Z}_T$.

As the frequency increases, the inductive reactance increases, while the capacitive reactance decreases. At some frequency, $f_0$, the inductor and the capacitor will have the same magnitude of reactance. At this frequency, the reactances cancel, resulting in a circuit impedance which is equal to the resistance value.

As the frequency increases still further, the inductive reactance becomes larger than the capacitive reactance. The circuit becomes inductive and the magnitude of the total impedance of the circuit again rises. Figure 18–60 shows how the impedance of a series *RLC* circuit varies with frequency.

The complete analysis of the series *RLC* circuit and the parallel *RLC* circuit is left until we examine the principle of resonance in a later chapter.

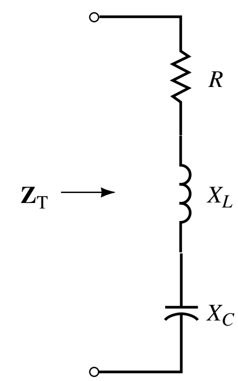



**FIGURE 18–59**

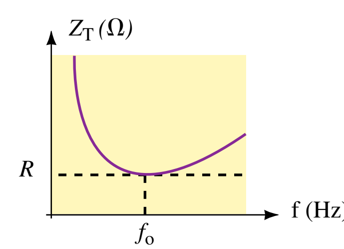



**FIGURE 18–60**

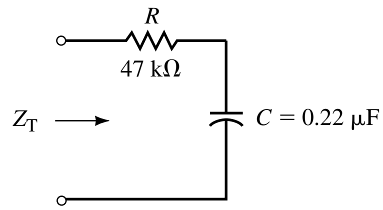
IN-PROCESS LEARNING CHECK 4

1. For a series network consisting of a resistor and a capacitor, what will be the impedance of the network at a frequency of 0 Hz (dc)? What will be the impedance of the network as the frequency approaches infinity?
2. For a parallel network consisting of a resistor and an inductor, what will be the impedance of the network at a frequency of 0 Hz (dc)? What will be the impedance of the network as the frequency approaches infinity?

*(Answers are at the end of the chapter.)*

PRACTICE PROBLEMS 9

FIGURE 18–61




Given the series *RC* network of Figure 18–61, calculate the cutoff frequency in hertz and in radians per second. Sketch the frequency response of $Z_T$ (magnitude) versus angular frequency $\omega$ for the network. Show the magnitude $Z_T$ at $\omega_c/10$, $\omega_c$, and $10\omega_c$.

*Answers:* $\omega_c = 96.7$ rad/s $f_c = 15.4$ Hz
At 0.1 $\omega_c$: $Z_T = 472$ kΩ At $\omega_c$: $Z_T = 66.5$ kΩ At 10 $\omega_c$: $Z_T = 47.2$ kΩ

## 18.8 Applications

As we have seen, we may determine the impedance of any ac circuit as a vector $\mathbf{Z} = R \pm jX$. This means that any ac circuit may now be simplified as a series circuit having a resistance and a reactance, as shown in Figure 18–62.

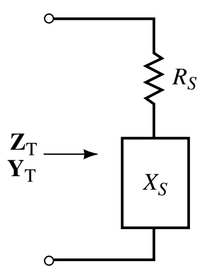



FIGURE 18–62

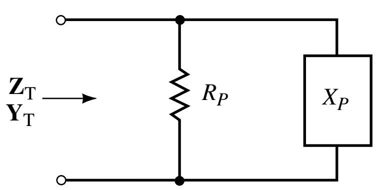



FIGURE 18–63

Additionally, an ac circuit may be represented as an equivalent parallel circuit consisting of a single resistor and a single reactance as shown in Figure 18–63. *Any equivalent circuit will be valid only at the given frequency of operation.*

We will now examine the technique used to convert any series impedance into its parallel equivalent. Suppose that the two circuits of Figure 18–62 and Figure 18–63 are exactly equivalent at some frequency. These circuits can be equivalent only if both circuits have the same total impedance, $\mathbf{Z}_T$, and the same total admittance, $\mathbf{Y}_T$.

From the circuit of Figure 18–62, the total impedance is written as

$$\mathbf{Z}_T = R_S \pm jX_S$$

Therefore, the total admittance of the circuit is

$$\mathbf{Y}_T = \frac{1}{\mathbf{Z}_T} = \frac{1}{R_S \pm jX_S}$$

Multiplying the numerator and denominator by the complex conjugate, we obtain the following:

$$\begin{aligned}\mathbf{Y}_T &= \frac{R_S \mp jX_S}{(R_S \pm jX_S)(R_S \mp jX_S)}\\ &= \frac{R_S \mp jX_S}{R_S^2 + X_S^2}\\ \mathbf{Y}_T &= \frac{R_S}{R_S^2 + X_S^2} \mp j\frac{X_S}{R_S^2 + X_S^2}\end{aligned} \tag{18–28}$$

Now, from the circuit of Figure 18–63, the total admittance of the parallel circuit may be found from the parallel combination of $R_P$ and $X_P$ as

$$\mathbf{Y}_T = \frac{1}{R_P} + \frac{1}{\pm jX_P}$$

which gives

$$\mathbf{Y}_T = \frac{1}{R_P} \mp j\frac{1}{X_P} \tag{18–29}$$

Two vectors can only be equal if both the real components are equal and the imaginary components are equal. Therefore the circuits of Figure 18–62 and Figure 18–63 can only be equivalent if the following conditions are met:

$$R_P = \frac{R_S^2 + X_S^2}{R_S} \tag{18–30}$$

and

$$X_P = \frac{R_S^2 + X_S^2}{X_S} \tag{18–31}$$

In a similar manner, we have the following conversion from a parallel circuit to an equivalent series circuit:

$$R_S = \frac{R_P X_P^2}{R_P^2 + X_P^2} \tag{18–32}$$

and

$$X_S = \frac{R_P^2 X_P}{R_P^2 + X_P^2} \tag{18–33}$$

**EXAMPLE 18–19** A circuit has a total impedance of $\mathbf{Z}_T = 10\ \Omega + j50\ \Omega$. Sketch the equivalent series and parallel circuits.

**Solution** The series circuit will be an inductive circuit having $R_S = 10\ \Omega$ and $X_{LS} = 50\ \Omega$.

The equivalent parallel circuit will also be an inductive circuit having the following values:

$$R_P = \frac{(10\ \Omega)^2 + (50\ \Omega)^2}{10\ \Omega} = 260\ \Omega$$

$$X_{LP} = \frac{(10\ \Omega)^2 + (50\ \Omega)^2}{50\ \Omega} = 52\ \Omega$$

The equivalent series and parallel circuits are shown in Figure 18–64.

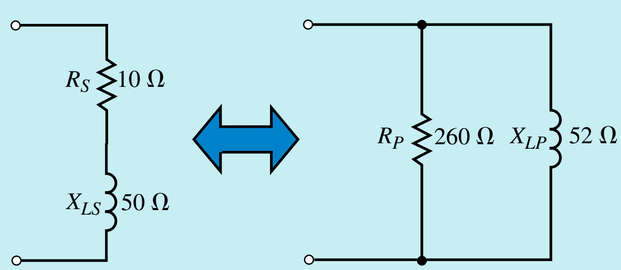



**FIGURE 18–64**

**EXAMPLE 18–20** A circuit has a total admittance of $\mathbf{Y}_T = 0.559\text{ mS}\angle 63.43°$. Sketch the equivalent series and parallel circuits.

**Solution** Because the admittance is written in polar form, we first convert to the rectangular form of the admittance.

$$G_P = (0.559\text{ mS})\cos 63.43° = 0.250\text{ mS} \quad \Leftrightarrow \quad R_P = 4.0\text{ k}\Omega$$
$$B_{CP} = (0.559\text{ mS})\sin 63.43° = 0.500\text{ mS} \quad \Leftrightarrow \quad X_{CP} = 2.0\text{ k}\Omega$$

The equivalent series circuit is found as

$$R_S = \frac{(4\text{ k}\Omega)(2\text{ k}\Omega)^2}{(4\text{ k}\Omega)^2 + (2\text{ k}\Omega)^2} = 0.8\text{ k}\Omega$$

and

$$X_{CS} = \frac{(4\text{ k}\Omega)^2(2\text{ k}\Omega)}{(4\text{ k}\Omega)^2 + (2\text{ k}\Omega)^2} = 1.6\text{ k}\Omega$$

The equivalent circuits are shown in Figure 18–65.

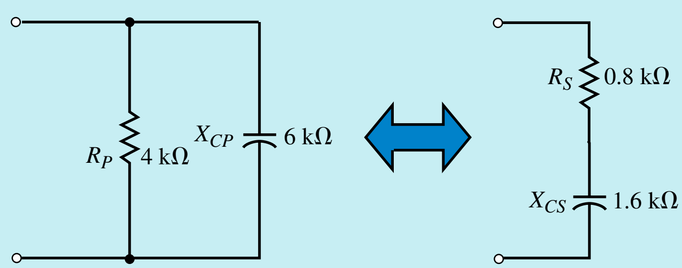



**FIGURE 18–65**

**EXAMPLE 18–21** Refer to the circuit of Figure 18–66.

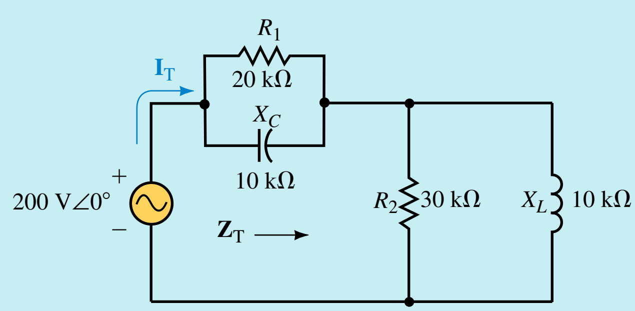



**FIGURE 18–66**

a. Find $\mathbf{Z}_T$.
b. Sketch the equivalent series circuit.
c. Determine $\mathbf{I}_T$.

**Solution**

a. The circuit consists of two parallel networks in series. We apply Equations 18–32 and 18–33 to arrive at equivalent series elements for each of the parallel networks as follows:

$$R_{S_1} = \frac{(20\ \text{k}\Omega)(10\ \text{k}\Omega)^2}{(20\ \text{k}\Omega)^2 + (10\ \text{k}\Omega)^2} = 4\ \text{k}\Omega$$

$$X_{CS} = \frac{(20\ \text{k}\Omega)^2(10\ \text{k}\Omega)}{(20\ \text{k}\Omega)^2 + (10\ \text{k}\Omega)^2} = 8\ \text{k}\Omega$$

and

$$R_{S_2} = \frac{(30\ \text{k}\Omega)(10\ \text{k}\Omega)^2}{(30\ \text{k}\Omega)^2 + (10\ \text{k}\Omega)^2} = 3\ \text{k}\Omega$$

$$X_{LS} = \frac{(30\ \text{k}\Omega)^2(10\ \text{k}\Omega)}{(30\ \text{k}\Omega)^2 + (10\ \text{k}\Omega)^2} = 9\ \text{k}\Omega$$

The equivalent circuits are shown in Figure 18–67.

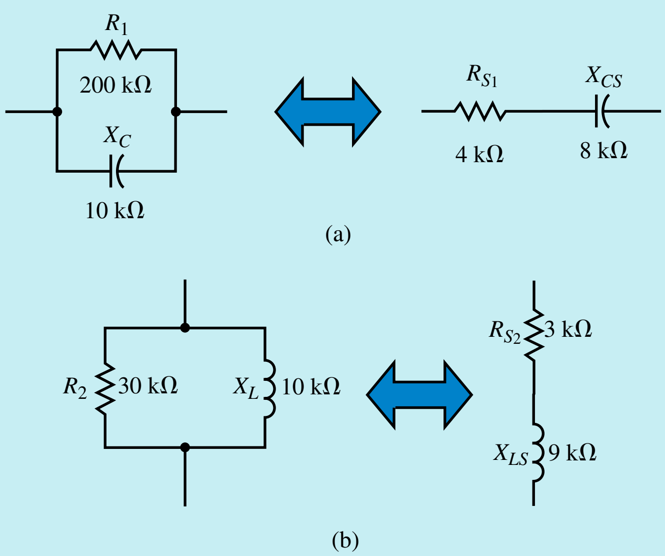



**FIGURE 18–67**

The total impedance of the circuit is found to be

$$\mathbf{Z}_T = (4\ \text{k}\Omega - j8\ \text{k}\Omega) + (3\ \text{k}\Omega + j9\ \text{k}\Omega) = 7\ \text{k}\Omega + j1\ \text{k}\Omega = 7.071\ \text{k}\Omega\angle 8.13°$$

b. Figure 18–68 shows the equivalent series circuit.

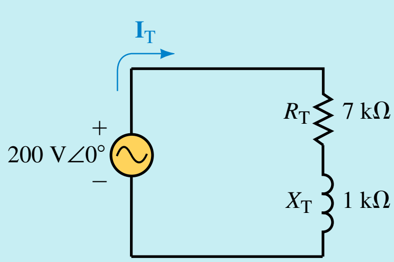



**FIGURE 18–68**

c. $$\mathbf{I}_T = \frac{200\ \text{V}\angle 0^\circ}{7.071\ \text{k}\Omega\angle 8.13^\circ} = 28.3\ \text{mA}\angle -8.13^\circ$$

**IN-PROCESS LEARNING CHECK 5**

An inductor of 10 mH has a series resistance of 5 Ω.

a. Determine the parallel equivalent of the inductor at a frequency of 1 kHz. Sketch the equivalent showing the values of $L_P$ (in henries) and $R_P$.

b. Determine the parallel equivalent of the inductor at a frequency of 1 MHz. Sketch the equivalent showing the values of $L_P$ (in henries) and $R_P$.

c. If the frequency were increased still further, predict what would happen to the values of $L_P$ and $R_P$

*(Answers are at the end of the chapter.)*

**PRACTICE PROBLEMS 10**

A network has an impedance of $\mathbf{Z}_T = 50\ \text{k}\Omega\angle 75^\circ$ at a frequency of 5 kHz.

a. Determine the most simple equivalent series circuit ($L$ and $R$).

b. Determine the most simple equivalent parallel circuit.

*Answers:*

a. $R_S = 12.9\ \text{k}\Omega$, $L_S = 1.54$ H

b. $R_P = 193\ \text{k}\Omega$, $L_P = 1.65$ H

## 18.9 Circuit Analysis Using Computers

ELECTRONICS WORKBENCH

PSpice

### Electronics Workbench

In this section we will use Electronics Workbench to simulate how sinusoidal ac measurements are taken with an oscilloscope. The "measurements" are then interpreted to verify the ac operation of circuits. You will use some of the display features of the software to simplify your work. The following example provides a guide through each step of the procedure.

**EXAMPLE 18–22** Given the circuit of Figure 18–69.

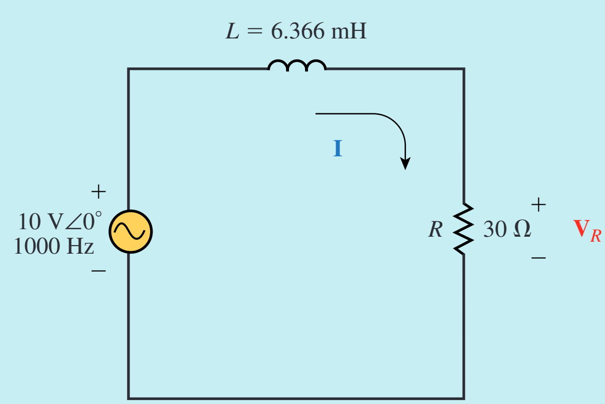



**FIGURE 18–69**

a. Determine the current **I** and the voltage $\mathbf{V}_R$.

b. Use Electronics Workbench to display the resistor voltage $v_R$ and the source voltage $e$. Use the results to verify the results of part (a).

**Solution**

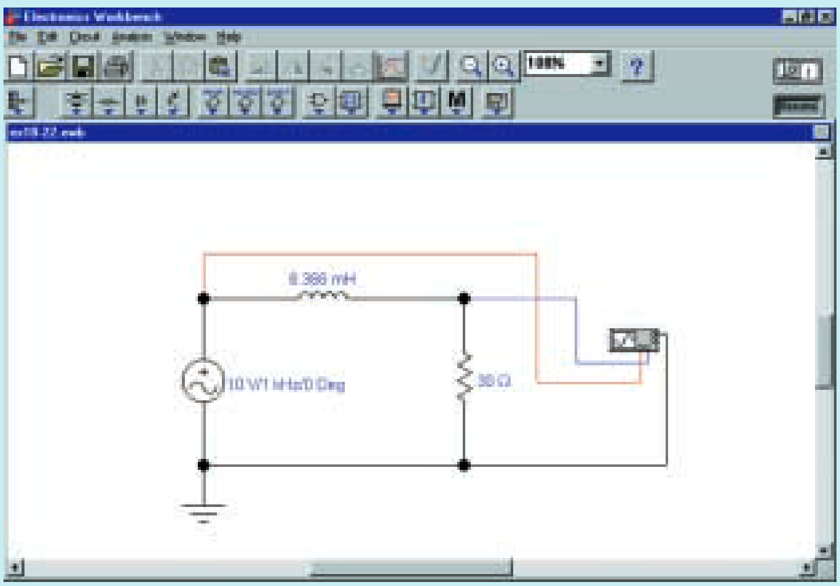
a. $X_L = 2\pi(1000\text{ Hz})(6.366 \times 10^{-3}\text{ H}) = 40\ \Omega$

$$\mathbf{Z} = 30\ \Omega + j40\ \Omega = 50\ \Omega\angle 53.13^\circ$$

$$\mathbf{I} = \frac{10\text{ V}\angle 0^\circ}{50\ \Omega\angle 53.13^\circ} = 0.200\text{ A}\angle -53.13^\circ$$

$$\mathbf{V}_R = (0.200\text{ A}\angle -53.13^\circ)(30\ \Omega) = 6.00\text{ V}\angle -53.13^\circ$$

b. Use the schematic editor to input the circuit shown in Figure 18–70.



EWB **FIGURE 18–70**

The ac voltage source is obtained from the Sources parts bin. The properties of the voltage source are changed by double clicking on the symbol and then selecting the Value tab. Change the values as follows:

Voltage (V): **10 V**
Frequency: **1 kHz**
Phase: **0 Deg**

The oscilloscope is selected from the Instruments parts bin and the settings are changed as follows:

Time base: **0.2 ms/div**
Channel A: **5 V/Div**
Channel B: **5 V/Div**

At this point, the circuit could be simulated and the display will resemble actual lab results. However, we can refine the display to provide information that is more useful. First, click on Analysis/Options. Next click on the Instruments tab and change the following settings for the oscilloscope:

✓Pause after each screen

•Minimum number of time points **1000**

After returning to the main window, click on the Power switch. It is necessary to press the Pause/Resume button once, since the display on the oscilloscope will not immediately show the circuit steady state values, but rather begins from $t = 0$. Consequently the display will show the transient (which in this case will last approximately $5\ L/R = 1$ ms).

Now, we can obtain a more detailed display by clicking on the Display Graphs button. We will use cursors and a grid to help in the analysis. The display will show several cycles of selected voltages. However, we are really only interested in viewing one complete cycle. This is done as follows:

Click on the Properties button. Select the General tab. We enable the cursors by selecting • All traces and ✓Cursors On from the Cursors box. Next, we enable the grid display by selecting Grid On from the Grid box.

The abscissa (time axis) is adjusted to show one period from $t = 4$ ms to $t = 5$ ms by selecting the Bottom Axis tab and adjusting the values as follows:

| | | |
|---|---|---|
| Range | Minimum | **0.004** |
| | Maximum | **0.005** |
| Divisions | Number | **20** |

The resulting display is shown in Figure 18–71.

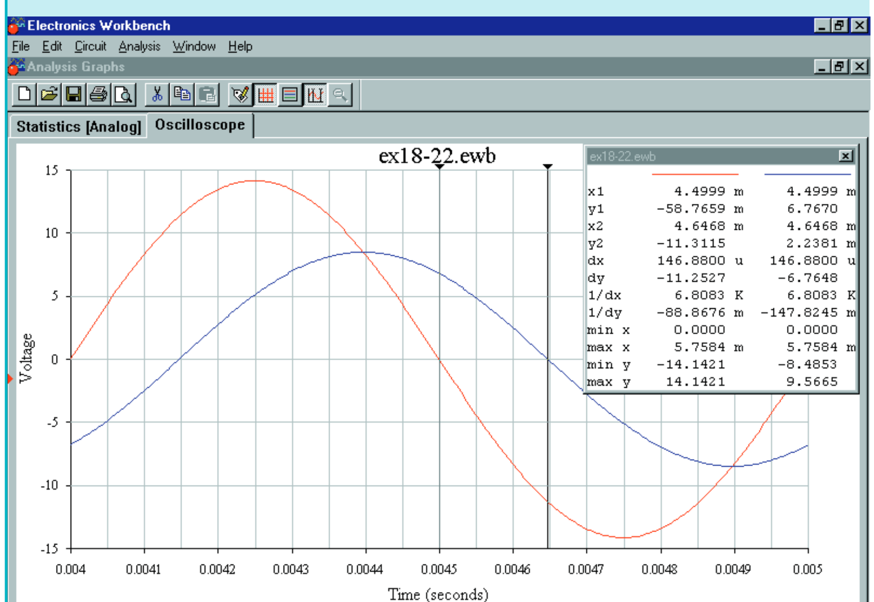



**FIGURE 18–71**

After positioning the cursors and using the display window, we are able to obtain various measurements for the circuit. The phase angle of the resistor voltage with respect to the source voltage is found by using the difference between the cursors in the bottom axis, $dx = 146.88\ \mu\text{s}$. Now we have

$$\theta = \frac{146.88\ \mu\text{s}}{1000\ \mu\text{s}} \times 360° = 52.88°$$

The amplitude of the resistor voltage is 8.49 V, which results in an rms value of 6.00 V. As a result of the measurements we have

$$\mathbf{V}_R = 6.00\ \text{V}\angle-52.88°$$

and

$$\mathbf{I} = \frac{6\ \text{V}\angle-52.88°}{30\ \Omega} = 0.200\ \text{A}\angle-52.88°$$

These values correspond very closely to the theoretical results calculated in part (a) of the example.

## OrCAD PSpice

In the following example we will use the Probe postprocessor of PSpice to show how the impedance of an *RC* circuit changes as a function of frequency. The Probe output will provide a graphical result that is very similar to the frequency responses determined in previous sections of this chapter.

**EXAMPLE 18–23** Refer to the network of Figure 18–72. Use the OrCAD Capture CIS Demo to input the circuit. Run the Probe postprocessor to procide a graphical display of network impedance as a function of frequency from 50 Hz to 500 Hz.

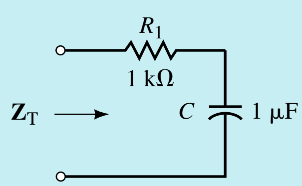



**FIGURE 18–72**

**Solution** Since PSpice is unable to analyze an incomplete circuit, it is necessary to provide a voltage source (and ground) for the circuit of Figure 18–72. The input impedance is not dependent on the actual voltage used, and so we may use any ac voltage source. In this example we arbitrarily select a voltage of 10 V.

- Open the CIS Demo software.
- Open a new project and call it Ch 18 PSpice 1. Ensure that the Analog or Mixed-Signal Circuit Wizard is activated.

- Enter the circuit as show in Figure 18–73. Simply click on the voltage value and change its value to **10V** from the default value of **0V.** (There must be no spaces between the magnitude and the units.

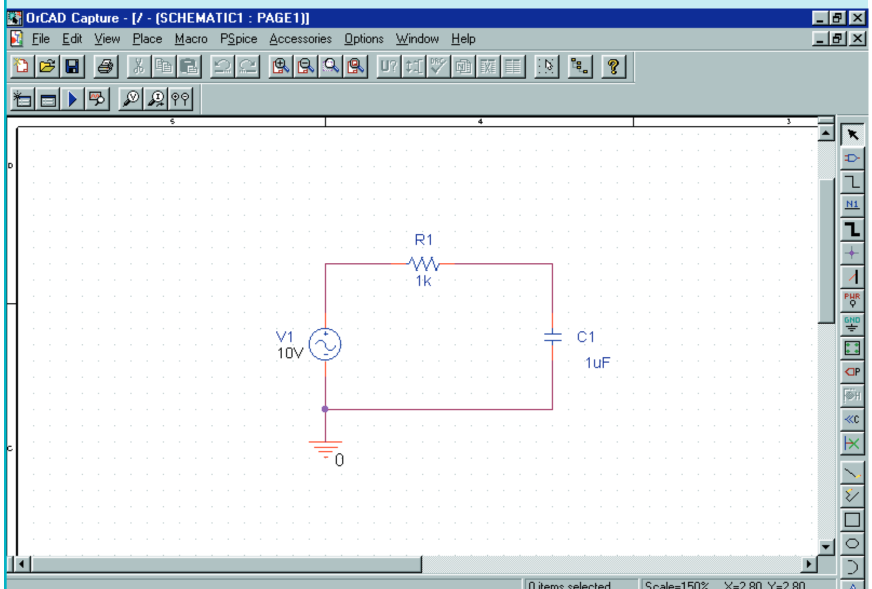



**FIGURE 18–73**

- Click on PSpice, New Simulation Profile and give a name such as Example 18-23 to the new simulation. The Simulation Settings dialog box will open.
- Click on the Analysis tab and select AC Sweep/Noise as the Analysis type. Select General Settings from the Options box.
- Select Linear AC Sweep Type (Quite often, logarithmic frequency sweeps such as Decade, are used). Enter the following values into the appropriate dialog boxes. Start Frequency; **50,** End Frequency: **500,** Total Points: **1001.** Click OK.
- Click on PSpice and Run. The Probe postprocessor will appear on the screen.
- Click on Trace and Add Trace. You may simply click on the appropriate values from the list of variables and use the division symbol to result in impedance. Enter the following expression into the Trace Expression box:

**V(R1:1)/I(R1)**

Notice that the above expression is nothing other than an application of Ohm's Law. The resulting display is shown in Figure 18–74.

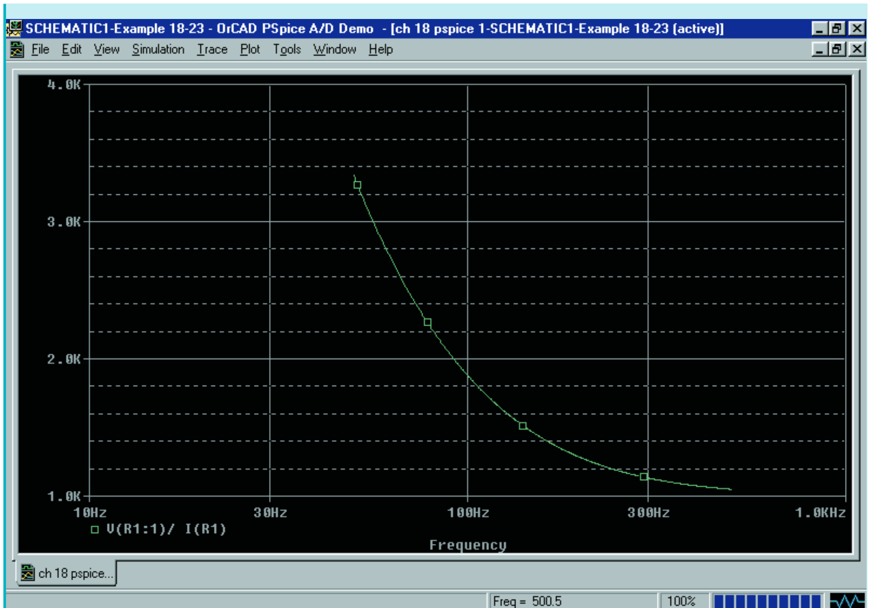



FIGURE 18–74

**EWB** Given the circuit of Figure 18–75.

**PRACTICE PROBLEMS 11**

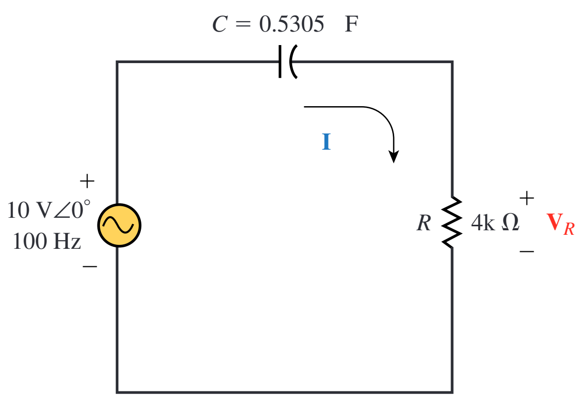



FIGURE 18–75

a. Determine the current **I** and the voltage $\mathbf{V}_R$.

b. Use Electronics Workbench to display the resistor voltage $v_R$ and the source voltage, *e*. Use the results to verify the results of part (a).

*Answers:*

a. $\mathbf{I} = 2.00 \text{ mA}\angle 36.87°$, $\mathbf{V}_R = 8.00 \text{ V}\angle 36.87°$

b. $v_R = 11.3 \sin(\omega t + 36.87°)$, $e = 14.1 \sin \omega t$

PRACTICE PROBLEMS 12

**PSpice** Use OrCAD PSpice to input the circuit of Figure 18–55. Use the Probe postprocessor to obtain a graphical display of the network impedance as a function of frequency from 100 Hz to 2000 Hz.

## PUTTING IT INTO PRACTICE

You are working in a small industrial plant where several small motors are powered by a 60-Hz ac line voltage of 120 Vac. Your supervisor tells you that one of the 2-Hp motors which was recently installed draws too much current when the motor is under full load. You take a current reading and find that the current is 14.4 A. After doing some calculations, you determine that even if the motor is under full load, it shouldn't require that much current.

However, you have an idea. You remember that a motor can be represented as a resistor in series with an inductor. If you could reduce the effect of the inductive reactance of the motor by placing a capacitor across the motor, you should be able to reduce the current since the capacitive reactance will cancel the inductive reactance.

While keeping the motor under load, you place a capacitor into the circuit. Just as you suspected, the current goes down. After using several different values, you observe that the current goes to a minimum of 12.4 A. It is at this value that you have determined that the reactive impedances are exactly balanced. Sketch the complete circuit and determine the value of capacitance that was added into the circuit. Use the information to determine the value of the motor's inductance. (Assume that the motor has an efficiency of 100%.)

## PROBLEMS

### 18.1 Ohm's Law for AC Circuits

1. For the resistor shown in Figure 18–76:
   a. Find the sinusoidal current $i$ using phasors.
   b. Sketch the sinusoidal waveforms for $v$ and $i$.
   c. Sketch the phasor diagram for **V** and **I.**

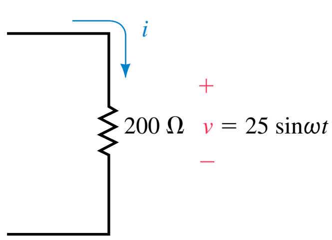



FIGURE 18–76

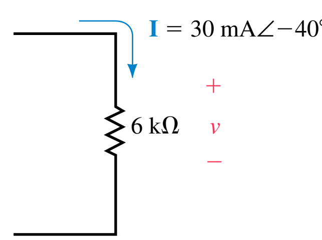



FIGURE 18–77

2. Repeat Problem 1 for the resistor of Figure 18–77.
3. Repeat Problem 1 for the resistor of Figure 18–78.

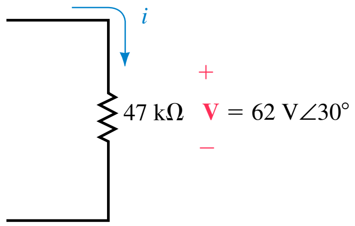



FIGURE 18–78

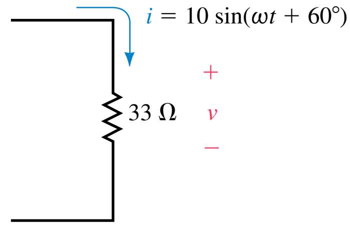



FIGURE 18–79

4. Repeat Problem 1 for the resistor of Figure 18–79.
5. For the component shown in Figure 18–80:
   a. Find the sinusoidal voltage $v$ using phasors.
   b. Sketch the sinusoidal waveforms for $v$ and $i$.
   c. Sketch the phasor diagram for **V** and **I.**

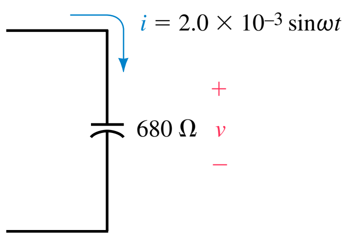



FIGURE 18–80

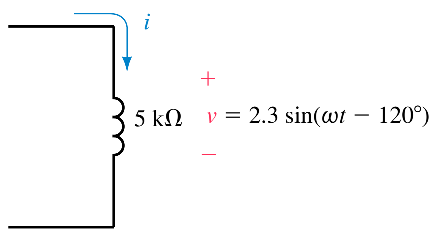



FIGURE 18–81

6. Repeat Problem 5 for the component shown in Figure 18–81.
7. Repeat Problem 5 for the component shown in Figure 18–82.

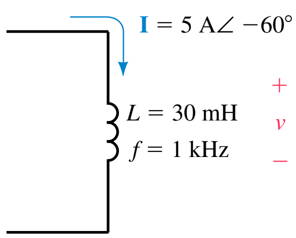



FIGURE 18–82

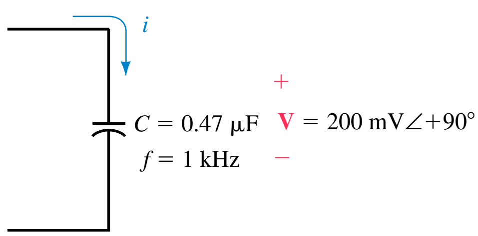



FIGURE 18–83

8. Repeat Problem 5 for the component shown in Figure 18–83.
9. Repeat Problem 5 for the component shown in Figure 18–84.

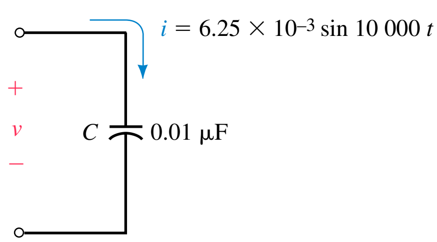



FIGURE 18–84

10. Repeat Problem 5 for the component shown in Figure 18–85.

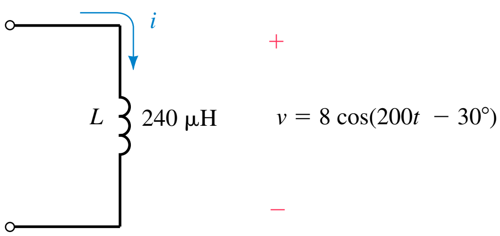



FIGURE 18–85

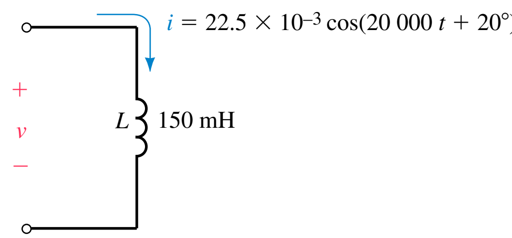



FIGURE 18–86

11. Repeat Problem 5 for the component shown in Figure 18–86.
12. Repeat Problem 5 for the component shown in Figure 18–87.

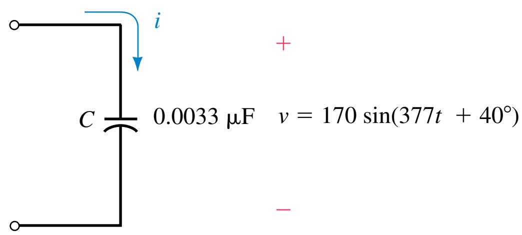



FIGURE 18–87

### 18.2 AC Series Circuits

13. Find the total impedance of each of the networks shown in Figure 18–88.

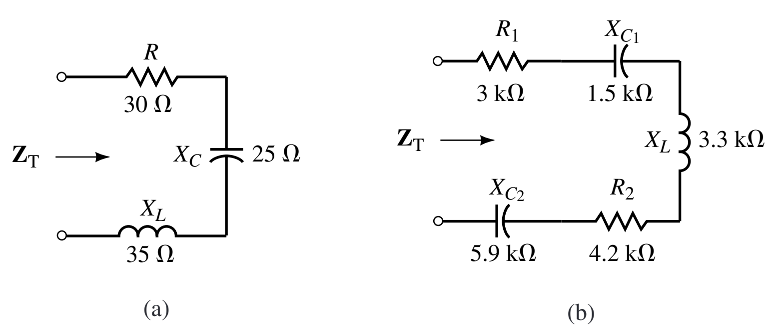



FIGURE 18–88

14. Repeat Problem 13 for the networks of Figure 18–89.
15. Refer to the network of Figure 18–90.
    a. Determine the series impedance **Z** which will result in the given total impedance, $\mathbf{Z}_T$. Express your answer in rectangular and polar form.
    b. Sketch an impedance diagram showing $\mathbf{Z}_T$ and **Z.**
16. Repeat Problem 15 for the network of Figure 18–91.
17. A circuit consisting of two elements has a total impedance of $\mathbf{Z}_T = 2\ \text{k}\Omega\angle 15°$ at a frequency of 18 kHz. Determine the values in ohms, henries, or farads of the unknown elements.

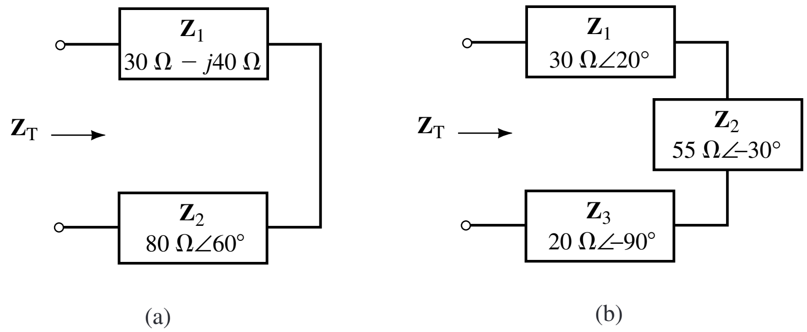



**FIGURE 18–89**

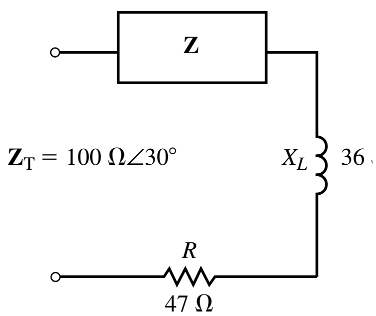



**FIGURE 18–90**

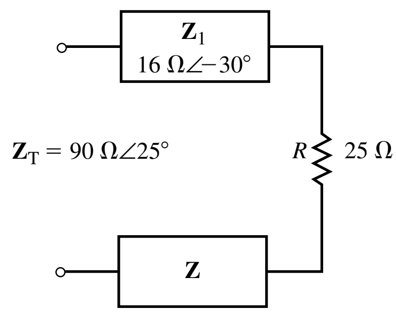



**FIGURE 18–91**

18. A network has a total impedance of $\mathbf{Z}_T = 24.0\ \text{k}\Omega\angle -30°$ at a frequency of 2 kHz. If the network consists of two series elements, determine the values in ohms, henries, or farads of the unknown elements.
19. Given that the network of Figure 18–92 is to operate at a frequency of 1 kHz, what series components $R$ and $L$ (in henries) or $C$ (in farads) must be in the indicated block to result in a total circuit impedance of $\mathbf{Z}_T = 50\ \Omega\angle 60°$?
20. Repeat Problem 19 for a frequency of 2 kHz.
21. Refer to the circuit of Figure 18–93.
    a. Find $\mathbf{Z}_T$, $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.
    b. Sketch the phasor diagram showing $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.

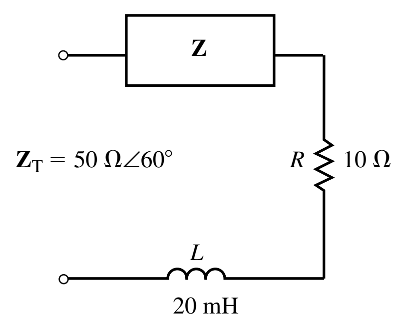



**FIGURE 18–92**

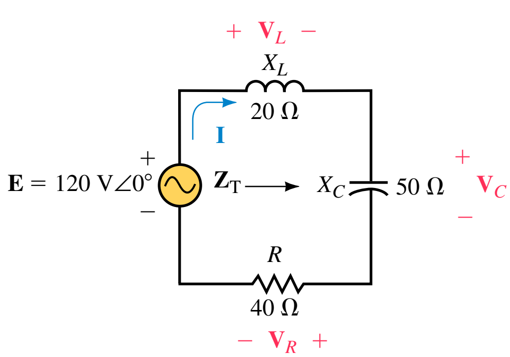



**FIGURE 18–93**

c. Determine the average power dissipated by the resistor.
d. Calculate the average power delivered by the voltage source. Compare the result to (c).

22. Consider the circuit of Figure 18–94.
    a. Find $\mathbf{Z}_\mathrm{T}$, $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.
    b. Sketch the phasor diagram showing $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.
    c. Write the sinusoidal expressions for the current $i$ and the voltages $e$, $v_R$, $v_C$, and $v_L$.
    d. Sketch the sinusoidal current and voltages found in (c).
    e. Determine the average power dissipated by the resistor.
    f. Calculate the average power delivered by the voltage source. Compare the result to (e).

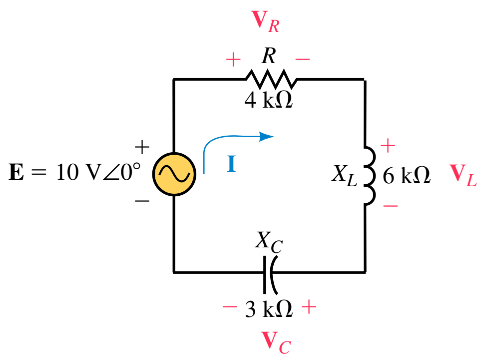



FIGURE 18–94

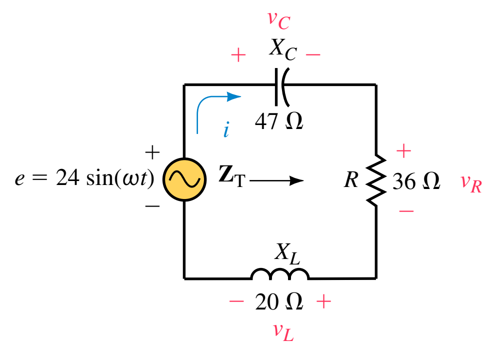



FIGURE 18–95

23. Refer to the circuit of Figure 18–95.
    a. Determine the circuit impedance, $\mathbf{Z}_\mathrm{T}$.
    b. Use phasors to solve for $i$, $v_R$, $v_C$, and $v_L$.
    c. Sketch the phasor diagram showing $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.
    d. Sketch the sinusoidal expressions for the current and voltages found in (b).
    e. Determine the average power dissipated by the resistor.
    f. Calculate the average power delivered by the voltage source. Compare the result to (e).

24. Refer to the circuit of Figure 18–96.
    a. Determine the value of the capacitor reactance, $X_C$, needed so that the resistor in the circuit dissipates a power of 200 mW.
    b. Using the value of $X_C$ from (a), determine the sinusoidal expression for the current $i$ in the circuit.

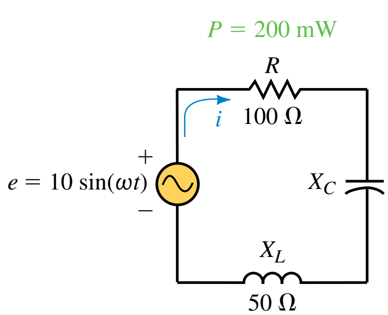



FIGURE 18–96

### 18.3 Kirchhoff's Voltage Law and the Voltage Divider Rule

25. a. Suppose a voltage of $10\text{ V}\angle 0°$ is applied across the network in Figure 18–88a. Use the voltage divider rule to find the voltage appearing across each element.
    b. Verify Kirchhoff's voltage law for each network.

26. a. Suppose a voltage of 240 V$\angle 30°$ is applied across the network in Figure 18–89a. Use the voltage divider rule to find the voltage appearing across each impedance.
    b. Verify Kirchhoff's voltage law for each network.

27. Given the circuit of Figure 18–97:
    a. Find the voltages $\mathbf{V}_C$ and $\mathbf{V}_L$.
    b. Determine the value of $R$.

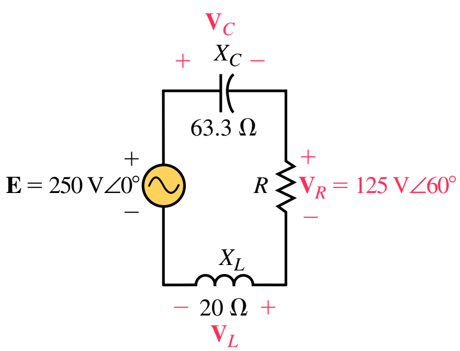



FIGURE 18–97

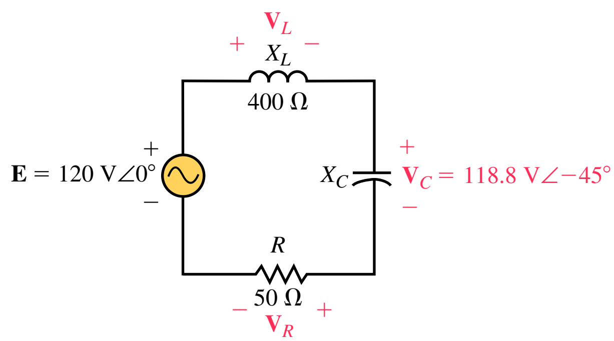



FIGURE 18–98

28. Refer to the circuit of Figure 18–98.
    a. Find the voltages $\mathbf{V}_R$ and $\mathbf{V}_L$.
    b. Determine the value of $X_C$.

29. Refer to the circuit of Figure 18–99:
    a. Find the voltage across $\mathbf{X}_C$.
    b. Use Kirchhoff's voltage law to find the voltage across the unknown impedance.
    c. Calculate the value of the unknown impedance **Z.**
    d. Determine the average power dissipated by the circuit.

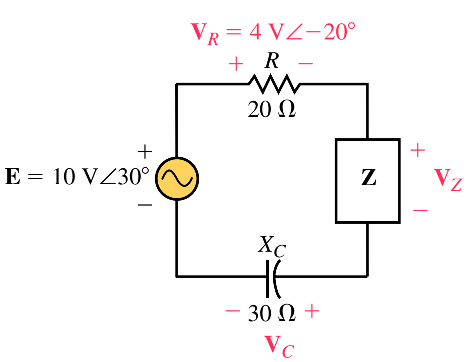



FIGURE 18–99

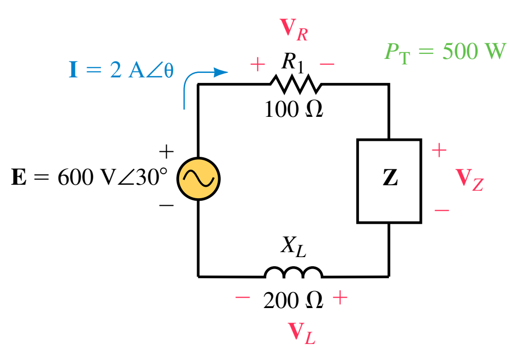



FIGURE 18–100

30. Given that the circuit of Figure 18–100 has a current with a magnitude of 2.0 A and dissipates a total power of 500 W:
    a. Calculate the value of the unknown impedance **Z.** (Hint: Two solutions are possible.)

b. Calculate the phase angle $\theta$ of the current **I.**

c. Find the voltages $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_Z$.

### 18.4 AC Parallel Circuits

31. Determine the input impedance, $\mathbf{Z}_T$, for each of the networks of Figure 18–101.

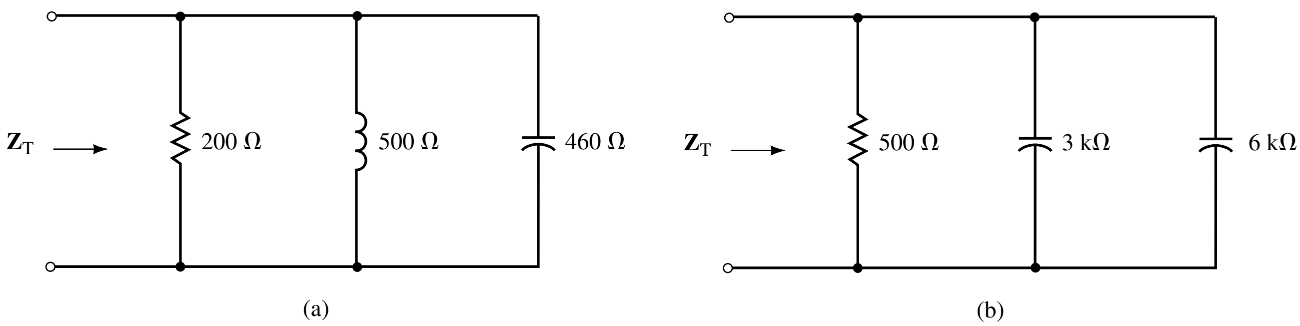



**FIGURE 18–101**

32. Repeat Problem 31 for Figure 18–102.

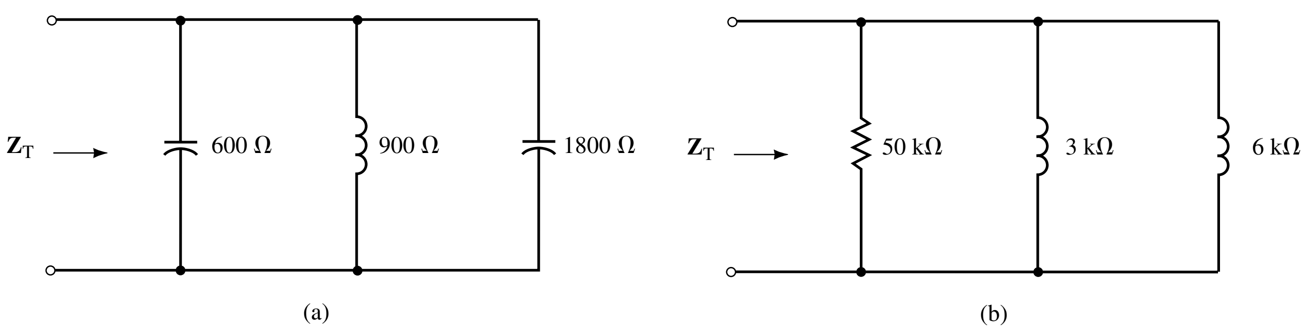



**FIGURE 18–102**

33. Given the circuit of Figure 18–103.

a. Find $\mathbf{Z}_T$, $\mathbf{I}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

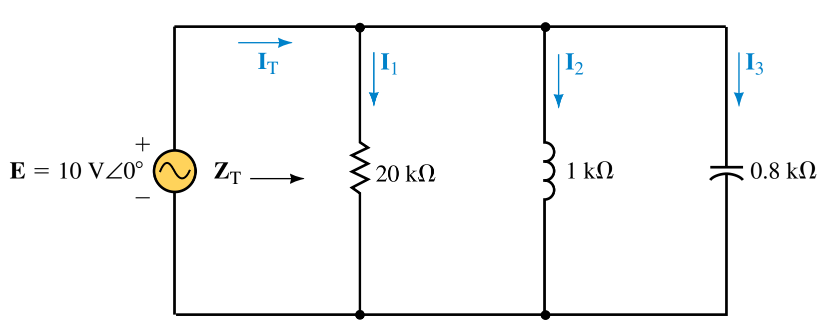



**FIGURE 18–103**

b. Sketch the admittance diagram showing each of the admittances.
c. Sketch the phasor diagram showing **E**, $\mathbf{I}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
d. Determine the average power dissipated by the resistor.
e. Find the power factor of the circuit and calculate the average power delivered by the voltage source. Compare the answer with the result obtained in (d).

34. Refer to the circuit of Figure 18–104.
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
    b. Sketch the admittance diagram for each of the admittances.
    c. Sketch the phasor diagram showing **E**, $\mathbf{I}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
    d. Determine the expressions for the sinusoidal currents $i_T$, $i_1$, $i_2$, and $i_3$.
    e. Sketch the sinusoidal voltage $e$ and current $i_T$.
    f. Determine the average power dissipated by the resistor.
    g. Find the power factor of the circuit and calculate the average power delivered by the voltage source. Compare the answer with the result obtained in (f).

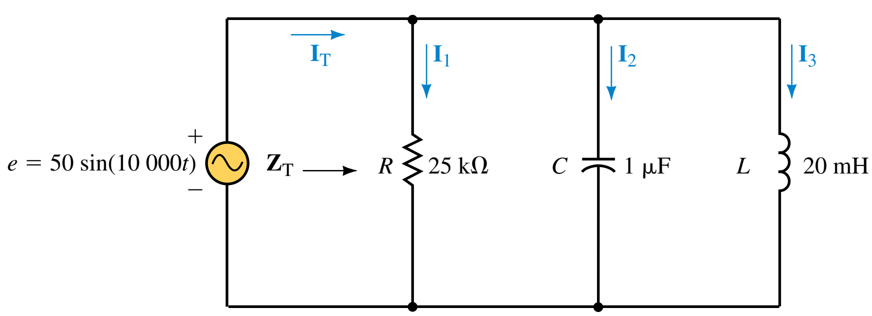



**FIGURE 18–104**

35. Refer to the network of Figure 18–105.
    a. Determine $\mathbf{Z}_T$.
    b. Given the indicated current, use Ohm's law to find the voltage, **V**, across the network.

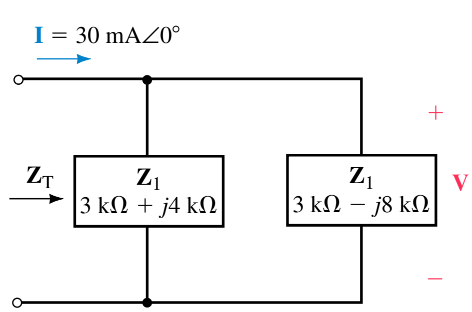



**FIGURE 18–105**

36. Consider the network of Figure 18–106.
    a. Determine $\mathbf{Z}_T$.
    b. Given the indicated current, use Ohm's law to find the voltage, **V,** across the network.
    c. Solve for $\mathbf{I}_2$ and **I.**

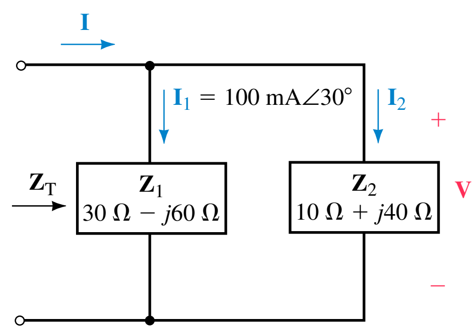



**FIGURE 18–106**

37. Determine the impedance, $\mathbf{Z}_2$, which will result in the total impedance shown in Figure 18–107.

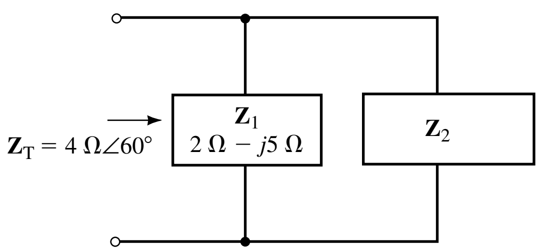



**FIGURE 18–107**

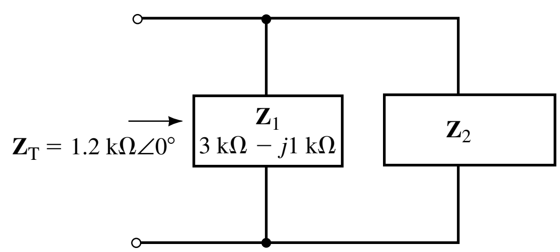



**FIGURE 18–108**

38. Determine the impedance, $\mathbf{Z}_2$, which will result in the total impedance shown in Figure 18–108.

### 18.5 Kirchhoff's Current Law and the Current Divider Rule

39. Solve for the current in each element of the networks in Figure 18–101 if the current applied to each network is $10\ \text{mA}\angle -30°$.
40. Repeat Problem 39 for Figure 18–102.
41. Use the current divider rule to find the current in each of the elements in Figure 18–109. Verify that Kirchhoff's current law applies.
42. Given that $\mathbf{I}_L = 4\ \text{A}\angle 30°$ in the circuit of Figure 18–110, find the currents **I,** $\mathbf{I}_C$, and $\mathbf{I}_R$. Verify that Kirchhoff's current law appplies for this circuit.
43. Suppose that the circuit of Figure 18–111 has a current **I** with a magnitude of 8 A:
    a. Determine the current $\mathbf{I}_R$ through the resistor.
    b. Calculate the value of resistance, *R*.
    c. What is the phase angle of the current **I?**

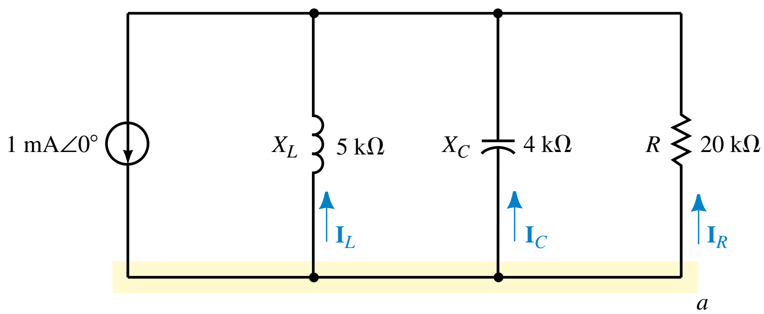



FIGURE 18–109

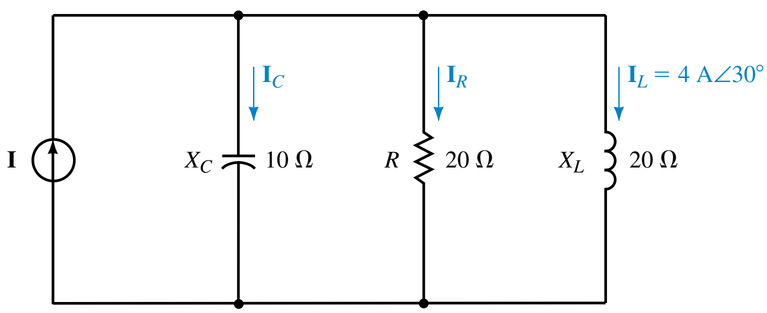



FIGURE 18–110

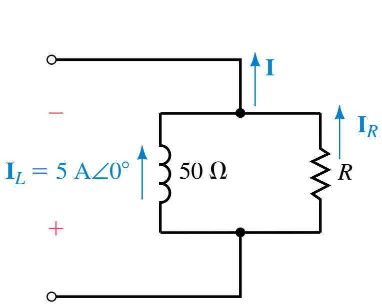



FIGURE 18–111

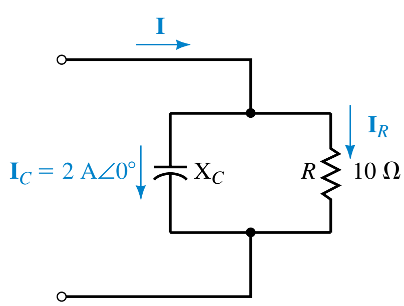



FIGURE 18–112

44. Assume that the circuit of Figure 18–112 has a current **I** with a magnitude of 3 A:
    a. Determine the current $\mathbf{I}_R$ through the resistor.
    b. Calculate the value of capacitive reactance $X_C$.
    c. What is the phase angle of the current **I?**

## 18.6 Series-Parallel Circuits

45. Refer to the circuit of Figure 18–113.
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_L$, $\mathbf{I}_C$, and $\mathbf{I}_R$.
    b. Sketch the phasor diagram showing **E,** $\mathbf{I}_L$, $\mathbf{I}_C$, and $\mathbf{I}_R$.
    c. Calculate the average power dissipated by the resistor.
    d. Use the circuit power factor to calculate the average power delivered by the voltage source. Compare the answer with the results obtained in (c).

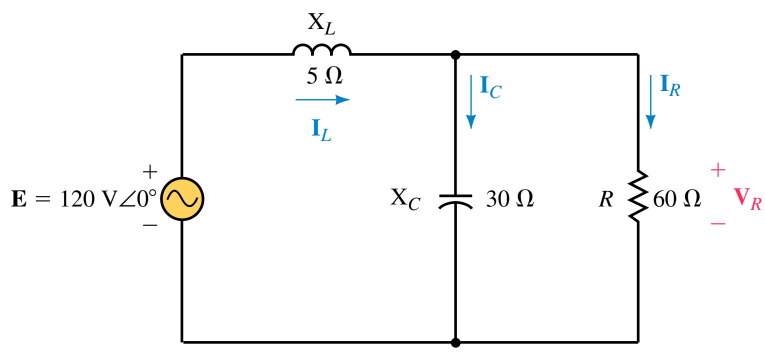



**FIGURE 18–113**

46. Refer to the circuit of Figure 18–114.
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
    b. Sketch the phasor diagram showing **E,** $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
    c. Calculate the average power dissipated by each of the resistors.
    d. Use the circuit power factor to calculate the average power delivered by the voltage source. Compare the answer with the results obtained in (c).

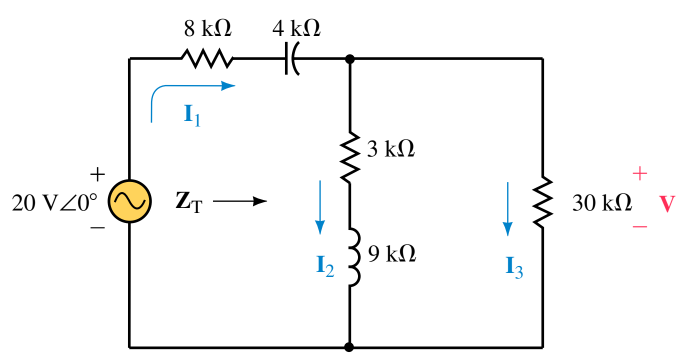



**FIGURE 18–114**

47. Refer to the circuit of Figure 18–115.
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_T$, $\mathbf{I}_1$, and $\mathbf{I}_2$.
    b. Determine the voltage $\mathbf{V}_{ab}$.

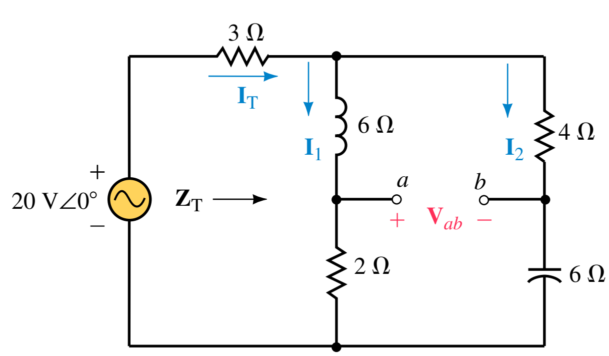



**FIGURE 18–115**

48. Consider the circuit of Figure 18–116.
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_T$, $\mathbf{I}_1$, and $\mathbf{I}_2$.
    b. Determine the voltage **V.**

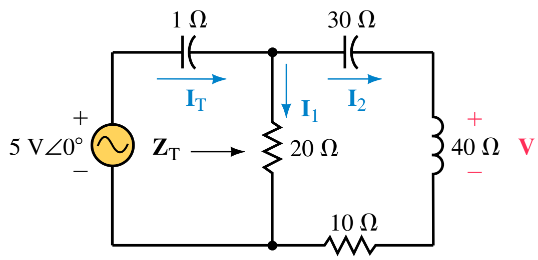



**FIGURE 18–116**

49. Refer to the circuit of Figure 18–117:
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
    b. Determine the voltage **V.**

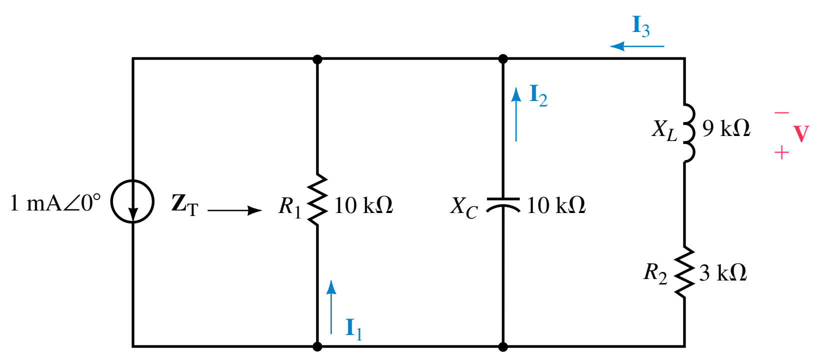



**FIGURE 18–117**

50. Refer to the circuit of Figure 18–118:
    a. Find $\mathbf{Z}_T$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.
    b. Determine the voltage **V.**

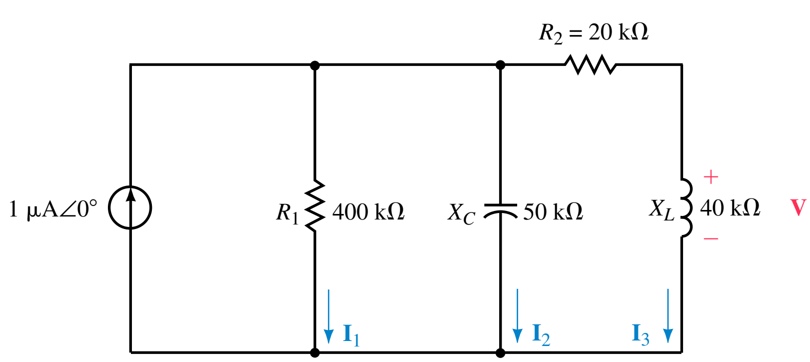



**FIGURE 18–118**

### 18.7 Frequency Effects

51. A 50-kΩ resistor is placed in series with a 0.01-$\mu$F capacitor. Determine the cutoff frequency $\omega_C$ (in rad/s) and sketch the frequency response ($Z_T$ vs. $\omega$) of the network.
52. A 2-mH inductor is placed in parallel with a 2-kΩ resistor. Determine the cutoff frequency $\omega_C$ (in rad/s) and sketch the frequency response ($Z_T$ vs. $\omega$) of the network.
53. A 100-kΩ resistor is placed in parallel with a 0.47-$\mu$F capacitor. Determine the cutoff frequency $f_C$ (in Hz) and sketch the frequency response ($Z_T$ vs. $f$) of the network.
54. A 2.7-kΩ resistor is placed in parallel with a 20-mH inductor. Determine the cutoff frequency $f_C$ (in Hz) and sketch the frequency response ($Z_T$ vs. $f$) of the network.

### 18.8 Applications

55. Convert each each of the networks of Figure 18–119 into an equivalent series network consisting of two elements.

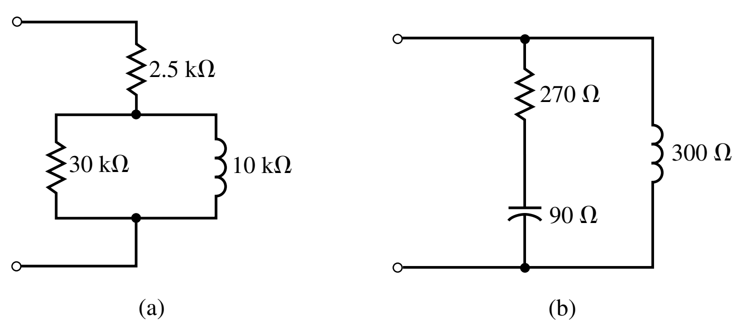



FIGURE 18–119

56. Convert each each of the networks of Figure 18–119 into an equivalent parallel network consisting of two elements.
57. Show that the networks of Figure 18–120 have the same input impedance at frequencies of 1 krad/s and 10 krad/s. (It can be shown that these networks are equivalent at all frequencies.)

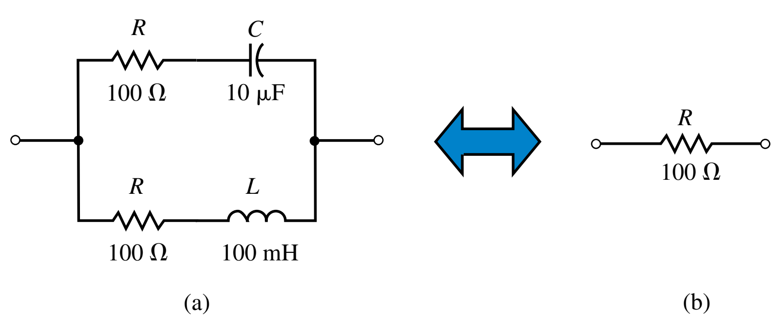



FIGURE 18–120

58. Show that the networks of Figure 18–121 have the same input impedance at frequencies of 5 rad/s and 10 rad/s.

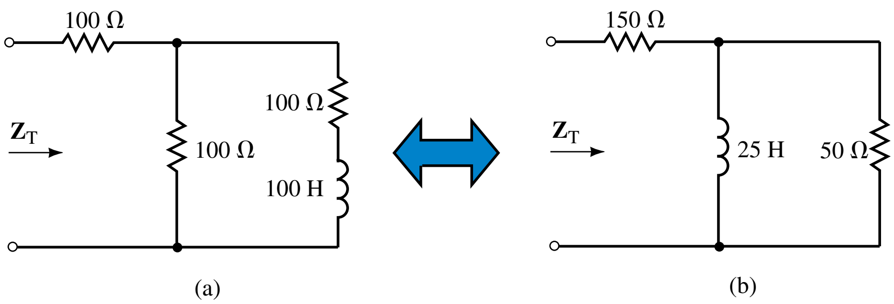



FIGURE 18–121

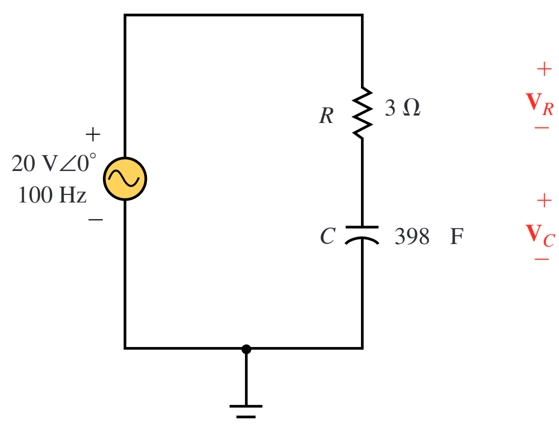
## 18.9 Circuit Analysis Using Computers

59. **EWB** Given the circuit of Figure 18–122:

a. Use Electronics Workbench to simultaneously display the capacitor voltage $v_C$ and $e$. Record your results and determine the phasor voltage $\mathbf{V}_C$.

b. Interchange the positions of the resistor and the capacitor relative to the ground. Use Electronics Workbench to simultaneously display the resistor voltage $v_R$ and $e$. Record your results and determine the phasor voltage $\mathbf{V}_R$.

c. Compare your results to those obtained in Example 18–8.



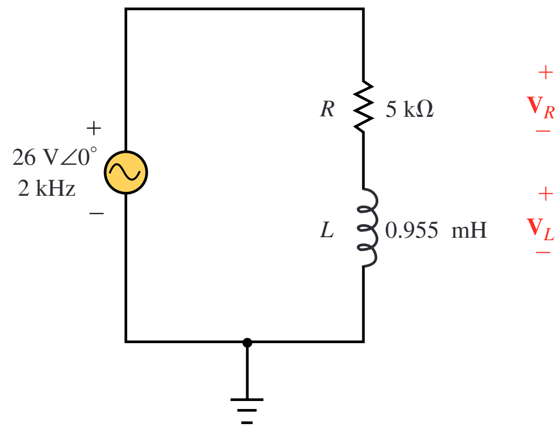

EWB FIGURE 18–122

60. **EWB** Given the circuit of Figure 18–123:

a. Use Electronics Workbench to display the inductor voltage $v_L$ and $e$. Record your results and determine the phasor voltage $\mathbf{V}_L$.

b. Interchange the positions of the resistor and the inductor relative to the ground. Use Electronics Workbench to display the inductor voltage $v_R$ and $e$. Record your results and determine the phasor voltage $\mathbf{V}_R$.

c. Compare your results to those obtained in Example 18–9.




EWB FIGURE 18–123

61. **PSpice** A 50-kΩ resistor is placed in series with a 0.01-μF capacitor. Use OrCAD PSpice to input these components into a circuit. Run the Probe postprocessor to provide a graphical display of the network impedance as a function of frequency from 50 Hz to 500 Hz. Let the frequency sweep logarithmically in octaves.
62. **PSpice** A 2-mH inductor is placed in parallel with a 2-kΩ resistor. Use OrCAD PSpice to input these components into a circuit. Run the Probe postprocessor to provide a graphical display of the network impedance as a function of frequency from 50 kHz to 500 kHz. Let the frequency sweep logarithmically in octaves.
63. **PSpice** A 100-kΩ resistor is placed in parallel with a 0.47-μF capacitor. Use OrCAD PSpice to input these components into a circuit. Run the Probe postprocessor to provide a graphical display of the network impedance as a function of frequency from 0.1 Hz to 10 Hz. Let the frequency sweep logarithmically in octaves.
64. **PSpice** A 2.7-kΩ resistor is placed in parallel with a 20-mH inductor. Use OrCAD PSpice to input these components into a circuit. Run the Probe postprocessor to provide a graphical display of the network impedance as a function of frequency from 100 kHz to 1 MHz. Let the frequency sweep logarithmically in octaves.

## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

1. Current and voltage are in phase.
2. Current leads voltage by 90°.
3. Voltage leads current by 90°.

**In-Process Learning Check 2**

1. The phasor sum of voltage drops and rises around a closed loop is equal to zero.

2. All voltages must be expressed as phasors, i.e., $\mathbf{V} = V\angle\theta = V\cos\theta + jV\sin\theta$.

**In-Process Learning Check 3**

1. The phasor sum of currents entering a node is equal to the phasor sum of currents leaving the same node.
2. All currents must be expressed as phasors, i.e., $\mathbf{I} = I\angle\theta = I\cos\theta + jI\sin\theta$.

**In-Process Learning Check 4**

1. At $f = 0$ Hz, $Z = \infty$ (open circuit)
2. As $f \rightarrow \infty$, $Z = R$

**In-Process Learning Check 5**

1. $R_P = 795\ \Omega$, $L_P = 10.1$ mH
2. $R_P = 790\ \text{M}\Omega$, $L_P = 10.0$ mH

# 19 Methods of AC Analysis

## OBJECTIVES

After studying this chapter, you will be able to

- convert an ac voltage source into its equivalent current source, and conversely, convert a current source into an equivalent voltage source,
- solve for the current or voltage in a circuit having either a dependent current source or a dependent voltage source,
- set up simultaneous linear equations to solve an ac circuit using mesh analysis,
- use complex determinants to find the solutions for a given set of linear equations,
- set up simultaneous linear equations to solve an ac circuit using nodal analysis,
- perform delta-to-wye and wye-to-delta conversions for circuits having reactive elements,
- solve for the balanced condition in a given ac bridge circuit. In particular, you will examine the Maxwell, Hay, and Schering bridges,
- use Electronics Workbench to analyze bridge circuits,
- use PSpice to calculate current and voltage in an ac circuit.

## KEY TERMS

Balanced Bridges
Controlling Elements
Delta-Wye Conversion
Dependent Sources
Hay Bridge
Maxwell Bridge
Mesh Analysis
Nodal Analysis
Schering Bridge
Source Conversion

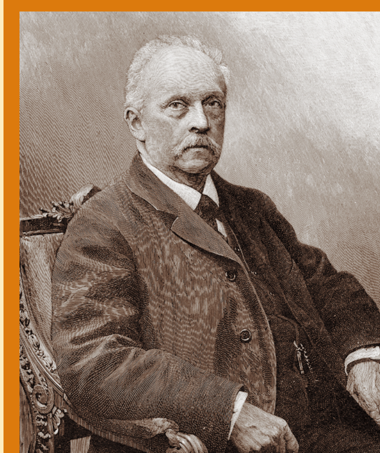
## OUTLINE

## CHAPTER PREVIEW

To this point, we have examined only circuits having a single ac source. In this chapter, we continue our study by analyzing multisource circuits and bridge networks. You will find that most of the techniques used in analyzing ac circuits parallel those of dc circuit analysis. Consequently, a review of Chapter 8 will help you understand the topics of this chapter.

Near the end of the chapter, we examine how computer techniques are used to analyze even the most complex ac circuits. It must be emphasized that although computer techniques are much simpler than using a pencil and calculator, there is virtually no knowledge to be gained by mindlessly entering data into a computer. We use the computer merely as a tool to verify our results and to provide a greater dimension to the analysis of circuits.

## PUTTING IT IN PERSPECTIVE



### *Hermann Ludwig Ferdinand von Helmholtz*

HERMANN HELMHOLTZ WAS BORN IN POTSDAM (near Berlin, Germany) on August 31, 1821. Helmholtz was a leading scientist of the nineteenth century, whose legacy includes contributions in the fields of acoustics, chemistry, mathematics, magnetism, electricity, mechanics, optics, and physiology.

Helmholtz graduated from the Medical Institute in Berlin in 1843 and practiced medicine for five years as a surgeon in the Prussian army. From 1849 to 1871, he served as a professor of physiology at universities in Königsberg, Bonn, and Heidelberg. In 1871, Helmholtz was appointed Professor of Physics at the University of Berlin.

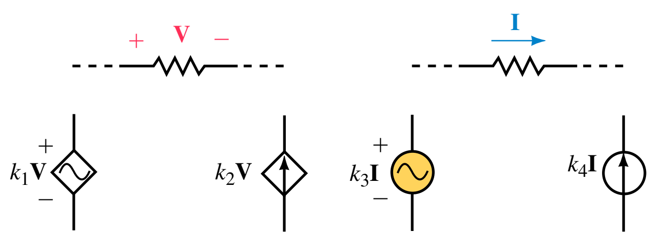
Helmholtz' greatest contributions were as a mathematical physicist, where his work in theoretical and practical physics led to the proof of the Law of Conservation of Energy in his paper *"Über die Erhaltung der Kraft,"* published in 1847. He showed that mechanics, heat, light, electricity, and magnetism were simply manifestations of the same force. His work led to the understanding of electrodynamics (the motion of charge in conductors), and his theory of the electromagnetic properties of light set the groundwork for later scientists to understand how radio waves are propagated.

For his work, the German emperor Kaiser Wilhelm I made Helmholtz a noble in 1883. This great scientist died on September 8, 1894, at the age of 73.

## 19.1 Dependent Sources

The voltage and current sources we have worked with up to now have been **independent sources,** meaning that the voltage or current of the supply was not in any way dependent upon any voltage or current elsewhere in the circuit. In many amplifier circuits, particularly those involving transistors, it is possible to explain the operation of the circuits by replacing the device with an equivalent electronic model. These models often use voltage and current sources which have values dependent upon some internal voltage or current. Such sources are called **dependent sources.** Figure 19–1 compares the symbols for both independent and dependent sources.


(a) Independent sources




(b) Dependent sources

**FIGURE 19–1**

Although the diamond is the accepted symbol for representing dependent sources, many articles and textbooks still use a circle. In this textbook we use both forms of the dependent source to familiarize the student with the various notations. The dependent source has a magnitude and phase angle determined by voltage or current at some internal element multiplied by a constant, *k*. The magnitude of the constant is determined by parameters within the particular model, and the units of the constant correspond to the required quantities in the equation.

**EXAMPLE 19–1** Refer to the resistor shown in Figure 19–2. Determine the voltage $\mathbf{V}_R$ across the resistor given that the controlling voltage has the following values:

a. $\mathbf{V} = 0\text{ V}$.

b. $\mathbf{V} = 5\text{ V}\angle 30°$.

c. $\mathbf{V} = 3\text{ V}\angle -150°$.

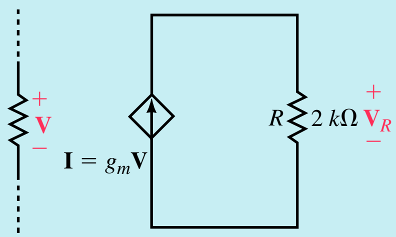



**FIGURE 19–2**

Given: $g_m = 4$ mS

**Solution** Notice that the dependent source of this example has a constant, $g_m$, called the **transconductance.** Here, $g_m = 4$ mS.

a. $\mathbf{I} = (4\text{ mS})(0\text{ V}) = 0$

$\mathbf{V}_R = 0\text{ V}$

b. $\mathbf{I} = (4\text{ mS})(5\text{ V}\angle 30°) = 20\text{ mA}\angle 30°$

$\mathbf{V}_R = (20\text{ mA}\angle 30°)(2\text{ k}\Omega) = 40\text{ V}\angle 30°$

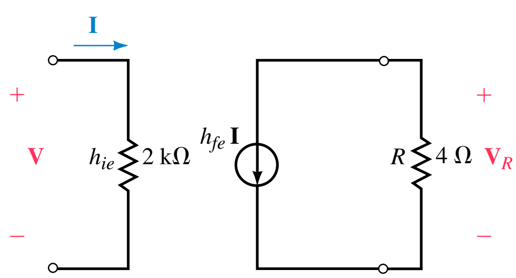
c. $\mathbf{I} = (4\text{ mS})(3\text{ V}\angle -150°) = 12\text{ mA}\angle -150°$

$\mathbf{V}_R = (12\text{ mA}\angle -150°)(2\text{ k}\Omega) = 24\text{ V}\angle -150°$

---

**PRACTICE PROBLEMS 1**

The circuit of Figure 19–3 represents a simplified model of a transistor amplifier.




$h_{fe} = 120$

**FIGURE 19–3**

Determine the voltage $\mathbf{V}_R$ for each of the following applied voltages:

a. $\mathbf{V} = 10\text{ mV}\angle 0°$.

b. $\mathbf{V} = 2\text{ mV}\angle 180°$.

c. $\mathbf{V} = 0.03\text{ V}\angle 90°$.

*Answers:* a. $2.4\text{ mV}\angle 180°$ b. $0.48\text{ mV}\angle 0°$ c. $7.2\text{ mV}\angle -90°$

---

## 19.2 Source Conversion

When working with dc circuits, the analysis of a circuit is often simplified by replacing the source (whether a voltage source or a current source) with its equivalent. The conversion of any ac source is similar to the method used in dc circuit analysis.

A voltage source **E** in series with an impedance **Z** is equivalent to a current source **I** having the same impedance **Z** in parallel. Figure 19–4 shows the equivalent sources.

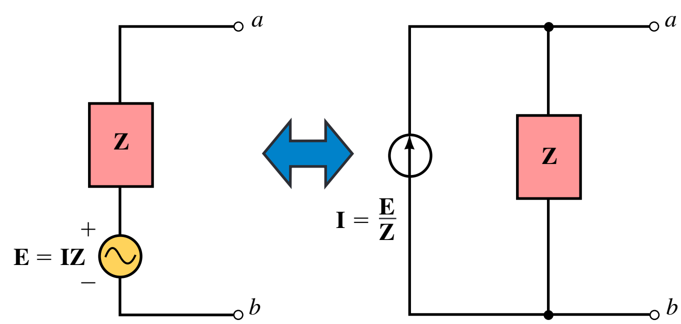



FIGURE 19–4

From Ohm's law, we perform the source conversion as follows:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}}$$

and

$$\mathbf{E} = \mathbf{IZ}$$

It is important to realize that the two circuits of Figure 19–4 are equivalent between points *a* and *b*. This means that any network connected to points *a* and *b* will behave exactly the same regardless of which type of source is used. However, the voltages or currents within the sources will seldom be the same. In order to determine the current through or the voltage across the source impedance, the circuit must be returned to its original state.

**EXAMPLE 19–2** Convert the voltage source of Figure 19–5 into an equivalent current source.

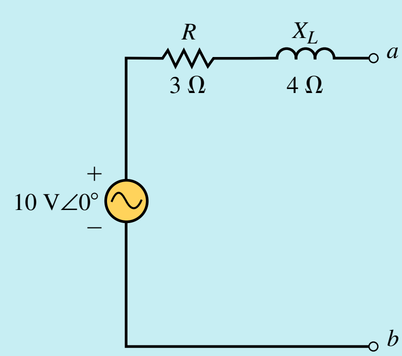



FIGURE 19–5

**Solution**

$$\mathbf{Z}_T = 3\ \Omega + j4\ \Omega = 5\ \Omega\angle 53.13^\circ$$

$$\mathbf{I} = \frac{10\ \text{V}\angle 0^\circ}{5\ \Omega\angle 53.13^\circ} = 2\ \text{A}\angle -53.13^\circ$$

The equivalent current source is shown in Figure 19–6.

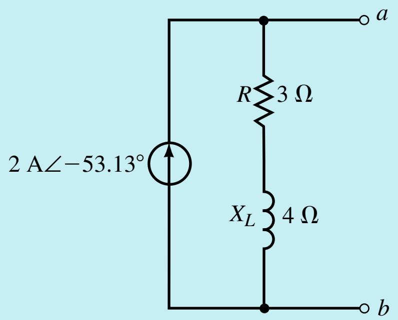



**FIGURE 19–6**

**EXAMPLE 19–3** Convert the current source of Figure 19–7 into an equivalent voltage source.

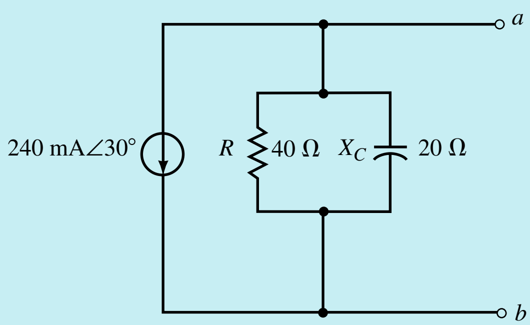



**FIGURE 19–7**

**Solution** The impedance of the parallel combination is determined to be

$$\begin{aligned}\mathbf{Z} &= \frac{(40\ \Omega\angle 0^\circ)(20\ \Omega\ \angle -90^\circ)}{40\ \Omega - j20\ \Omega} \\ &= \frac{800\ \Omega\ \angle -90^\circ}{44.72\angle -26.57^\circ} \\ &= 17.89\ \Omega\angle -63.43^\circ = 8\ \Omega - j16\ \Omega\end{aligned}$$

and so

$$\begin{aligned}\mathbf{E} &= (240\ \text{mA}\angle 30^\circ)(17.89\ \Omega\angle -63.43^\circ) \\ &= 4.29\ \text{V}\angle -33.43^\circ\end{aligned}$$

The resulting equivalent circuit is shown in Figure 19–8.

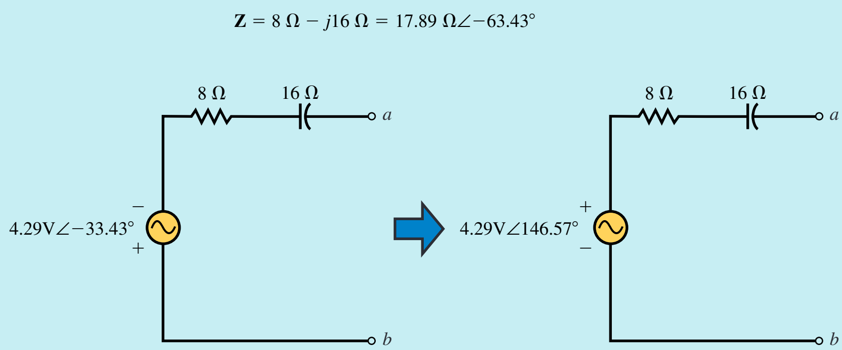



**FIGURE 19–8**

It is possible to use the same procedure to convert a dependent source into its equivalent provided that the controlling element is external to the circuit in which the source appears. *If the controlling element is in the same circuit as the dependent source, this procedure cannot be used.*

**EXAMPLE 19–4** Convert the current source of Figure 19–9 into an equivalent voltage source.

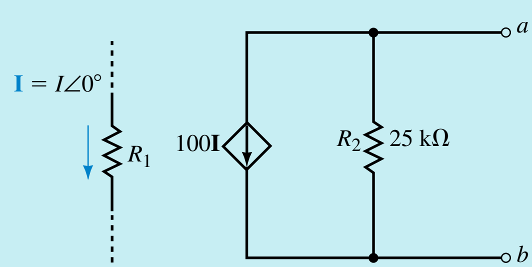



**FIGURE 19–9**

**Solution** In the circuit of Figure 19–9 the controlling element, $R_1$, is in a separate circuit. Therefore, the current source is converted into an equivalent voltage source as follows:

$$
\begin{aligned}
\mathbf{E} &= (100\mathbf{I}_1)(\mathbf{Z}) \\
&= (100\mathbf{I}_1)(25\ \text{k}\Omega\angle 0°) \\
&= (2.5 \times 10^6\ \Omega)\mathbf{I}_1
\end{aligned}
$$

The resulting voltage source is shown in Figure 19–10. Notice that the equivalent voltage source is dependent on the current, **I,** just as the original current source.

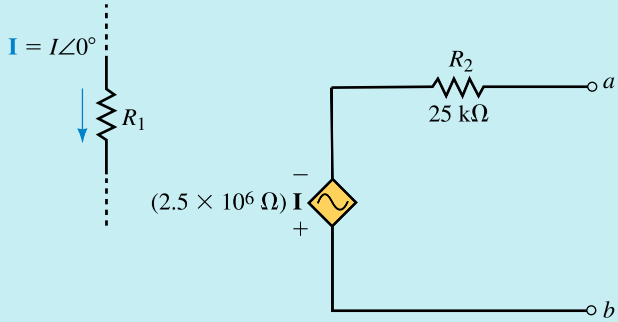



FIGURE 19–10

PRACTICE PROBLEMS 2

Convert the voltage sources of Figure 19–11 into equivalent current sources.

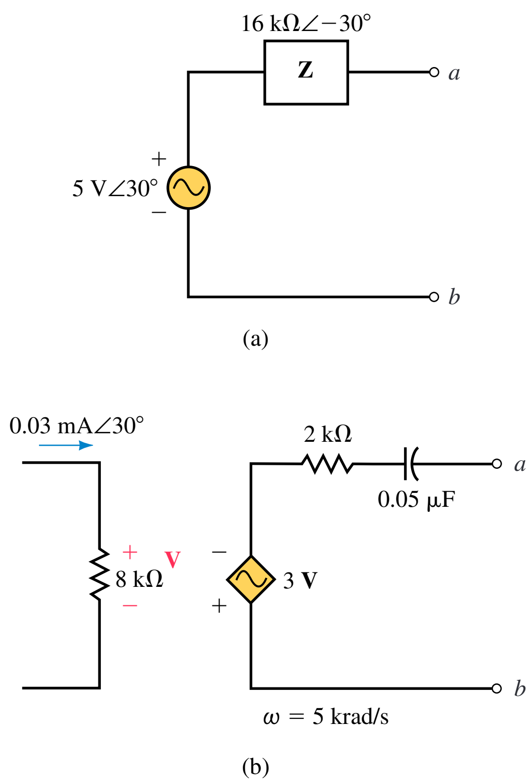



FIGURE 19–11

*Answers:*

a. $\mathbf{I} = 0.3125\ \text{mA}\angle 60°$ (from $b$ to $a$) in parallel with $\mathbf{Z} = 16\ \text{k}\Omega\angle -30°$

b. $\mathbf{I} = 0.161\ \text{mA}\angle 93.43°$ (from $a$ to $b$) in parallel with $\mathbf{Z} = 2\ \text{k}\Omega - j4\ \text{k}\Omega$

IN-PROCESS LEARNING CHECK 1

Given a 40-mA$\angle 0°$ current source in parallel with an impedance, **Z.** Determine the equivalent voltage source for each of the following impedances:

a. $\mathbf{Z} = 25\ \text{k}\Omega\angle 30°$.

b. $\mathbf{Z} = 100\ \Omega\ \angle -90°$.

c. $\mathbf{Z} = 20\ \text{k}\Omega - j16\ \text{k}\Omega$.

*(Answers are at the end of the chapter.)*

## 19.3 Mesh (Loop) Analysis

Mesh analysis allows us to determine each loop current within a circuit, regardless of the number of sources within the circuit. The following steps provide a format which simplifies the process of using mesh analysis:

1. Convert all sinusoidal expressions into equivalent phasor notation. Where necessary, convert current sources into equivalent voltage sources.
2. Redraw the given circuit, simplifying the given impedances wherever possible and labelling the impedances ($\mathbf{Z}_1$, $\mathbf{Z}_2$, etc.).
3. Arbitrarily assign clockwise loop currents to each interior closed loop within a circuit. Show the polarities of all impedances using the assumed current directions. If an impedance is common to two loops, it may be thought to have two simultaneous currents. Although in fact two currents will not occur simultaneously, this maneuver makes the algebraic calculations fairly simple. The actual current through a common impedance is the vector sum of the individual loop currents.
4. Apply Kirchhoff's voltage law to each closed loop in the circuit, writing each equation as follows:

$$\Sigma\ (\mathbf{ZI}) = \Sigma\ \mathbf{E}$$

If the current directions are originally assigned in a clockwise direction, then the resulting linear equations may be simplified to the following format:

$$\begin{aligned}
&\text{Loop 1: } +(\Sigma\mathbf{Z}_1)\mathbf{I}_1 - (\Sigma\mathbf{Z}_{1-2})\mathbf{I}_2 - \cdots -(\Sigma\mathbf{Z}_{1-n})\mathbf{I}_n = (\Sigma\mathbf{E}_1)\\
&\text{Loop 2: } -(\Sigma\mathbf{Z}_{2-1})\mathbf{I}_1 + (\Sigma\mathbf{Z}_2)\mathbf{I}_2 - \cdots -(\Sigma\mathbf{Z}_{2-n})\mathbf{I}_n = (\Sigma\mathbf{E}_2)\\
&\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
&\text{Loop } n\text{: } -(\Sigma\mathbf{Z}_{n-1})\mathbf{I}_1 - (\Sigma\mathbf{Z}_{n-2})\mathbf{I}_2 - \cdots +(\Sigma\mathbf{Z}_n)\mathbf{I}_n = (\Sigma\mathbf{E}_n)
\end{aligned}$$

In the above format, $\Sigma\mathbf{Z}_x$ is the summation of all impedances around loop $x$. The sign in front of all loop impedances will be positive.

$\Sigma\mathbf{Z}_{x-y}$ is the summation of impedances which are common between loop $x$ and loop $y$. If there is no common impedance between two loops, this term is simply set to zero. All common impedance terms in the linear equations are given negative signs.

$\Sigma \mathbf{E}_x$ is the summation of voltage rises in the direction of the assumed current $\mathbf{I}_x$. If a voltage source has a polarity such that it appears as a voltage drop in the assumed current direction, then the voltage is given a negative sign.

5. Solve the resulting simultaneous linear equations using substitution or determinants. If necessary, refer to Appendix B for a review of solving simultaneous linear equations.

**EXAMPLE 19–5** Solve for the loop equations in the circuit of Figure 19–12.

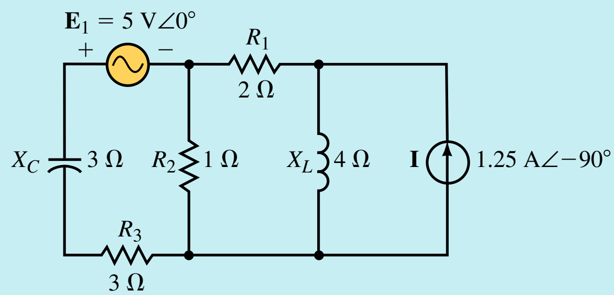
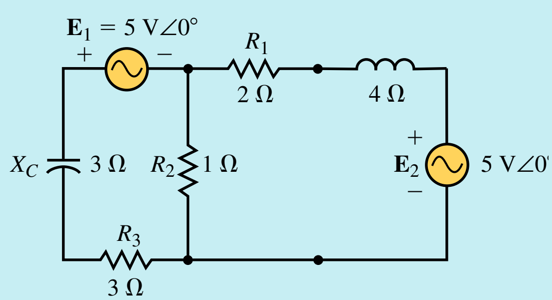



**FIGURE 19–12**

**Solution**

**Step 1:** The current source is first converted into an equivalent voltage source as shown in Figure 19–13.




$$\mathbf{E}_2 = (1.25\text{ A}\angle -90°)(4\ \Omega\angle 90°)$$
$$= 5\text{ V}\angle 0°$$

**FIGURE 19–13**

**Steps 2 and 3:** Next the circuit is redrawn as shown in Figure 19–14. The impedances have been simplified and the loop currents are drawn in a clockwise direction.

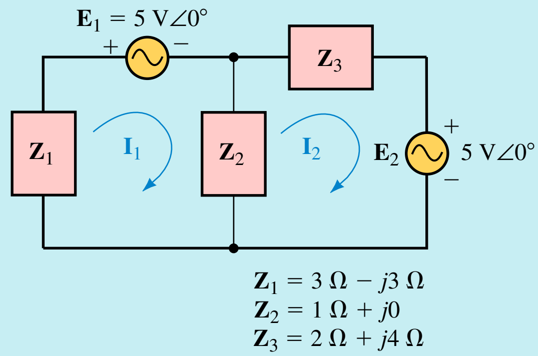



**FIGURE 19–14**

**Step 4:** The loop equations are written as

$$\text{Loop 1: } (\mathbf{Z}_1 + \mathbf{Z}_2)\mathbf{I}_1 - (\mathbf{Z}_2)\mathbf{I}_2 = -\mathbf{E}_1$$
$$\text{Loop 2: } -(\mathbf{Z}_2)\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_2 = -\mathbf{E}_2$$

The solution for the currents may be found by using determinants:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} -\mathbf{E}_1 & -\mathbf{Z}_2 \\ -\mathbf{E}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{Z}_2 \\ -\mathbf{Z}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}$$
$$= \frac{(-\mathbf{E}_1)(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{E}_2\mathbf{Z}_2}{(\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{Z}_2\mathbf{Z}_2}$$
$$= \frac{(-\mathbf{E}_1)(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{E}_2\mathbf{Z}_2}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}$$

and

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{E}_1 \\ -\mathbf{Z}_2 & -\mathbf{E}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{Z}_2 \\ -\mathbf{Z}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}}$$
$$= \frac{-\mathbf{E}_2(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{E}_1\mathbf{Z}_2}{(\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{Z}_2\mathbf{Z}_2}$$
$$= \frac{-\mathbf{E}_2(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{E}_1\mathbf{Z}_2}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}$$

Solving these equations using the actual values of impedances and voltages, we have

$$\mathbf{I}_1 = \frac{-(5)(3 + j4) - (5)(1)}{(3 - j3)(1) + (3 - j3)(2 + j4) + (1)(2 + j4)}$$
$$= \frac{(-15 - j20) - 5}{(3 - j3) + (6 + j6 - j^2 12) + (2 + j4)}$$
$$= \frac{-20 - j20}{23 + j7}$$

$$= \frac{28.28\angle -135°}{24.04\angle 16.93°}$$
$$= 1.18\text{ A}\angle -151.93°$$

and

$$\mathbf{I}_2 = \frac{-(5)(4 - j3) - (5)(1)}{23 + j7}$$
$$= \frac{(-20 + j15) - (5)}{23 + j7}$$
$$= \frac{-25 + j15}{23 + j7}$$
$$= \frac{29.15\angle 149.04°}{24.04\angle 16.93°}$$
$$= 1.21\text{ A}\angle 132.11°$$

**EXAMPLE 19–6** Given the circuit of Figure 19–15, write the loop equations and solve for the loop currents. Determine the voltage, **V**.

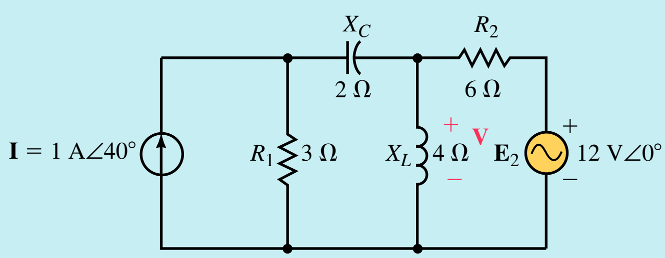



**FIGURE 19–15**

**Solution**

**Step 1:** Converting the current source into an equivalent voltage source gives us the circuit of of Figure 19–16.

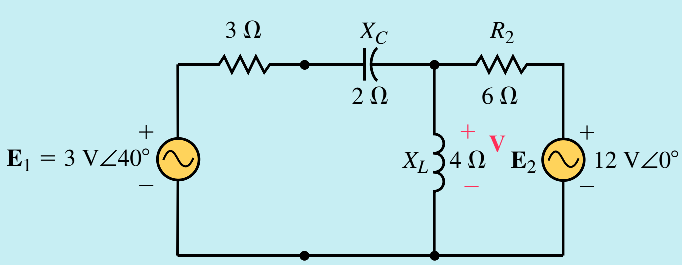



**FIGURE 19–16**

**Steps 2 and 3:** After simplifying the impedances and assigning clockwise loop currents, we have the circuit of Figure 19–17.

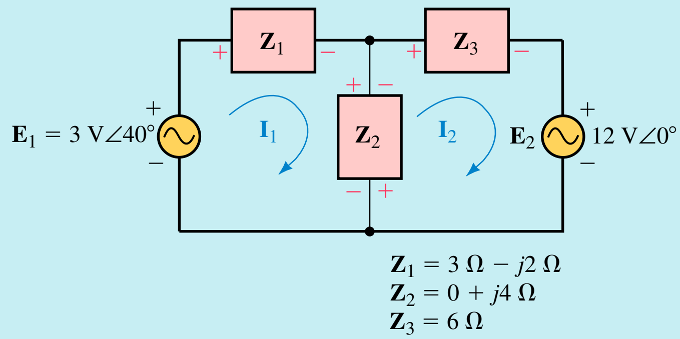



**FIGURE 19–17**

**Step 4:** The loop equations for the circuit of Figure 19–17 are as follows:

$$\text{Loop 1: } (\mathbf{Z}_1 + \mathbf{Z}_2)\mathbf{I}_1 - (\mathbf{Z}_2)\mathbf{I}_2 = \mathbf{E}_1$$
$$\text{Loop 2: } -(\mathbf{Z}_2)\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_2 = -\mathbf{E}_2$$

which, after substituting the impedance values into the expressions, become

$$\text{Loop 1: } (3\ \Omega + j2\ \Omega) - (j4\ \Omega) = 3\text{ V}\angle 40^\circ$$
$$\text{Loop 2: } -(j4\ \Omega) + (6\ \Omega + j4\ \Omega) = -12\text{ V}\angle 0^\circ$$

**Step 5:** Solve for the currents using determinants, where the elements of the determinants are expressed as vectors.

Since the determinant of the denominator is common to both terms we find this value first.

$$\begin{aligned}\mathbf{D} &= \begin{vmatrix} 3 + j2 & -j4 \\ -j4 & 6 + j4 \end{vmatrix} \\ &= (3 + j2)(6 + j4) - (-j4)(-j4) \\ &= 18 + j12 + j12 - 8 + 16 \\ &= 26 + j24 = 35.38\angle 42.71^\circ\end{aligned}$$

Now, the currents are found as

$$\begin{aligned}\mathbf{I}_1 &= \frac{\begin{vmatrix} 3\angle 40^\circ & -j4 \\ -12\angle 0^\circ & 6 + j4 \end{vmatrix}}{\mathbf{D}} \\ &= \frac{(3\angle 40^\circ)(7.211\angle 33.69^\circ) - (12\angle 0^\circ)(4\angle 90^\circ)}{35.38\angle 42.71^\circ} \\ &= \frac{21.633\angle 73.69^\circ - 48\angle 90^\circ}{35.38\angle 42.71^\circ} \\ &= \frac{27.91\angle -77.43^\circ}{35.38\angle 42.71^\circ} \\ &= 0.7887\text{ A}\angle -120.14^\circ\end{aligned}$$

and

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 3 + j2 & 3\angle 40° \\ -j4 & -12\angle 0° \end{vmatrix}}{\mathbf{D}}$$

$$\mathbf{I}_2 = \frac{(3.606\angle 33.69°)(-12\angle 0°) + (3\angle 40°)(4\angle 90°)}{35.38\angle 42.71°}$$

$$= \frac{-43.27\angle 33.69° + 12\angle 130°}{35.38\angle 42.71°}$$

$$= \frac{46.15\angle -161.29°}{35.38\angle 42.71°}$$

$$= 1.304\text{ A}\angle 156.00°$$

The current through the 4-Ω inductive reactance is

$$\mathbf{I} = \mathbf{I}_1 - \mathbf{I}_2$$
$$= (0.7887\text{ A}\angle -120.14°) - (1.304\text{ A}\angle 156.00°)$$
$$= (-0.3960\text{ A} - j0.6821\text{ A}) - (-1.1913\text{ A} + j0.5304\text{ A})$$
$$= 0.795\text{ A} - j1.213\text{ A} = 1.45\text{ A}\angle -56.75°$$

The voltage is now easily found from Ohm's law as

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_L$$
$$= (1.45\text{ A}\angle -56.75°)(4\ \Omega\angle 90°) = 5.80\text{ V}\angle 33.25°$$

**EXAMPLE 19–7** Given the circuit of of Figure 19–18, write the loop equations and show the determinant of the coefficients for the loop equations. Do not solve them.

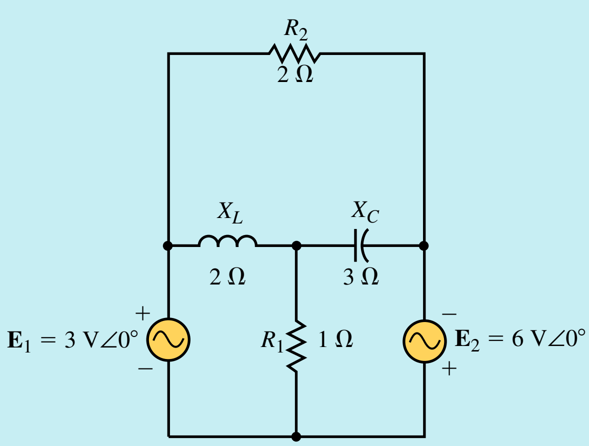



**FIGURE 19–18**

**Solution** The circuit is redrawn in Figure 19–19, showing loop currents and impedances together with the appropriate voltage polarities.

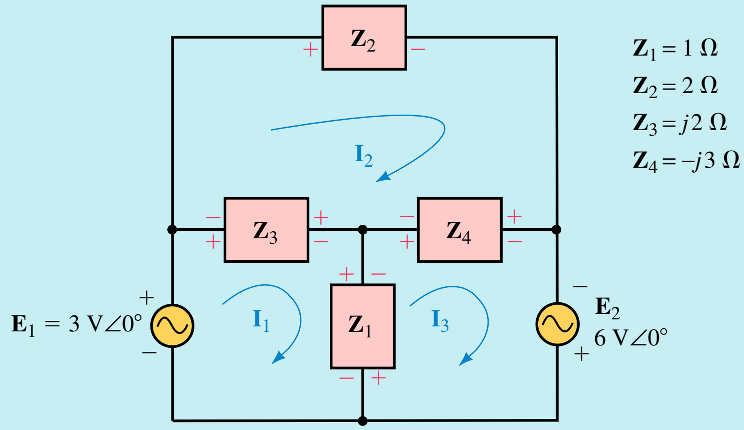



**FIGURE 19–19**

The loop equations may now be written as

$$\text{Loop 1: } (\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{I}_1 - (\mathbf{Z}_3)\mathbf{I}_2 - (\mathbf{Z}_1)\mathbf{I}_3 = \mathbf{E}_1$$
$$\text{Loop 2: } -(\mathbf{Z}_3)\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4)\mathbf{I}_2 - (\mathbf{Z}_4)\mathbf{I}_3 = 0$$
$$\text{Loop 3: } -(\mathbf{Z}_1)\mathbf{I}_1 - (\mathbf{Z}_4)\mathbf{I}_2 + (\mathbf{Z}_1 + \mathbf{Z}_4)\mathbf{I}_3 = \mathbf{E}_2$$

Using the given impedance values, we have

$$\text{Loop 1: } (1\ \Omega + j2\ \Omega)\mathbf{I}_1 - (j2\ \Omega)\mathbf{I}_2 - (1\ \Omega)\mathbf{I}_3 = 3\text{ V}$$
$$\text{Loop 2: } -(j2\ \Omega)\mathbf{I}_1 + (2\ \Omega - j1\ \Omega)\mathbf{I}_2 - (-j3\ \Omega)\mathbf{I}_3 = 0$$
$$\text{Loop 3: } -(1\ \Omega)\mathbf{I}_1 - (-j3\ \Omega)\mathbf{I}_2 + (1\ \Omega - j3\ \Omega)\mathbf{I}_3 = 6\text{ V}$$

Notice that in the above equations, the phase angles ($\theta = 0°$) for the voltages have been omitted. This is because $3\text{ V}\angle 0° = 3\text{ V} + j0\text{ V} = 3\text{ V}$.

The determinant for the coefficients of the loop equations is written as

$$\mathbf{D} = \begin{vmatrix} 1 + j2 & -j2 & -1 \\ -j2 & 2 - j1 & j3 \\ -1 & j3 & 1 - j3 \end{vmatrix}$$

Notice that the above determinant is symmetrical about the principle diagonal. The coefficients in the loop equations were given the required signs (positive for the loop impedance and negative for all common impedances). However, since the coefficients in the determinant may contain imaginary numbers, it is no longer possible to generalize about the signs (+/−) of the various coefficients.

**PRACTICE PROBLEMS 3**

Given the circuit of Figure 19–20, write the mesh equations and solve for the loop currents. Use the results to determine the current **I.**

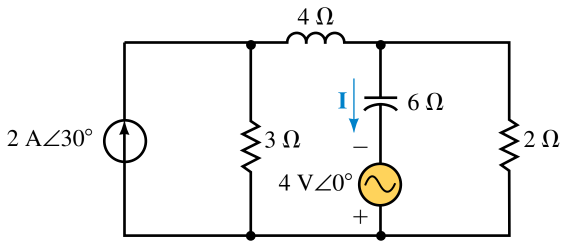



**FIGURE 19–20**

*Answers:* $\mathbf{I}_1 = 1.19\ \text{A}\angle 1.58°$, $\mathbf{I}_2 = 1.28\ \text{A}\angle -46.50°$, $\mathbf{I} = 1.01\ \text{A}\angle 72.15°$

---

IN-PROCESS LEARNING CHECK 2

Briefly list the steps followed in using mesh analysis to solve for the loop currents of a circuit.

*(Answers are at the end of the chapter.)*

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## 19.4 Nodal Analysis

Nodal analysis allows us to calculate all node voltages with respect to an arbitrary reference point in a circuit. The following steps provide a simple format to apply nodal analysis.

1. Convert all sinusoidal expressions into equivalent phasor notation. If necessary, convert voltage sources into equivalent current sources.
2. Redraw the given circuit, simplifying the given impedances wherever possible and relabelling the impedances as admittances ($\mathbf{Y}_1$, $\mathbf{Y}_2$, etc.).
3. Select and label an appropriate reference node. Arbitrarily assign subscripted voltages ($\mathbf{V}_1$, $\mathbf{V}_2$, etc.) to each of the remaining $n$ nodes within the circuit.
4. Indicate assumed current directions through all admittances in the circuit. If an admittance is common to two nodes, it is considered in each of the two node equations.
5. Apply Kirchhoff's current law to each of the $n$ nodes in the circuit, writing each equation as follows:

$$\Sigma(\mathbf{YV}) = \Sigma\mathbf{I}_{\text{sources}}$$

The resulting linear equations may be simplified to the following format:

$$\begin{array}{ll}
\text{Node 1:} & +(\Sigma\mathbf{Y}_1)\mathbf{V}_1 - (\Sigma\mathbf{Y}_{1-2})\mathbf{V}_2 - \cdots -(\Sigma\mathbf{Y}_{1-n})\mathbf{V}_n = (\Sigma\mathbf{I}_1) \\
\text{Node 2:} & -(\Sigma\mathbf{Y}_{2-1})\mathbf{I}_1 + (\Sigma\mathbf{Y}_2)\mathbf{I}_2 - \cdots -(\Sigma\mathbf{Y}_{2-n})\mathbf{I}_n = (\Sigma\mathbf{I}_2) \\
\quad\vdots & \qquad\qquad\qquad\qquad\vdots \\
\text{Node } n\text{:} & -(\Sigma\mathbf{Y}_{n-1})\mathbf{I}_1 - (\Sigma\mathbf{Y}_{n-2})\mathbf{I}_2 - \cdots + (\Sigma\mathbf{Y}_n)\mathbf{I}_n = (\Sigma\mathbf{I}_n)
\end{array}$$

In the above format, $\Sigma\mathbf{Y}_x$ is the summation of all admittances connected to node $x$. The sign in front of all node admittances will be positive.

$\Sigma\mathbf{Y}_{x-y}$ is the summation of common admittances between node $x$ and node $y$. If there are no common admittances between two nodes, this term is simply set to zero. All common admittance terms in the linear equations will have a negative sign.

$\Sigma\mathbf{I}_x$ is the summation of current sources entering node $x$. If a current source leaves the node, the current is given a negative sign.

6. Solve the resulting simultaneous linear equations using substitution or determinants.

**EXAMPLE 19–8** Given the circuit of Figure 19–21, write the nodal equations and solve for the node voltages.

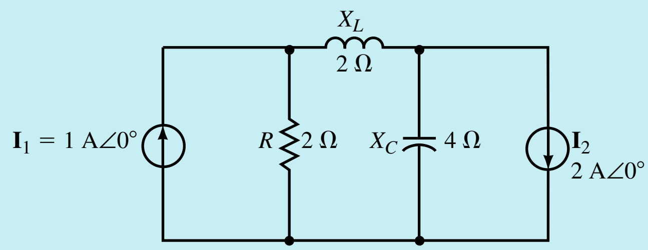



**FIGURE 19–21**

**Solution** The circuit is redrawn in Figure 19–22, showing the nodes and a simplified representation of the admittances.

The nodal equations are written as

$$\text{Node 1: } (\mathbf{Y}_1 + \mathbf{Y}_2)\mathbf{V}_1 - (\mathbf{Y}_2)\mathbf{V}_2 = \mathbf{I}_1$$
$$\text{Node 2: } -(\mathbf{Y}_2)\mathbf{V}_1 + (\mathbf{Y}_2 + \mathbf{Y}_3)\mathbf{V}_2 = -\mathbf{I}_2$$

Using determinants, the following expressions for nodal voltages are obtained:

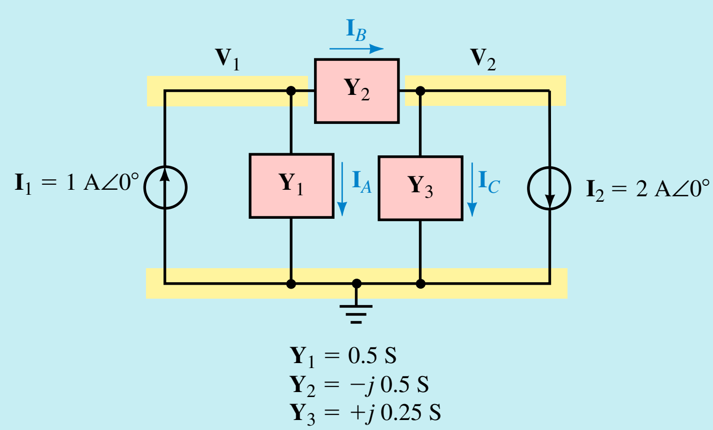



**FIGURE 19–22**

$$\mathbf{V}_1 = \frac{\begin{vmatrix} \mathbf{I}_1 & -\mathbf{Y}_2 \\ -\mathbf{I}_2 & \mathbf{Y}_2 + \mathbf{Y}_3 \end{vmatrix}}{\begin{vmatrix} \mathbf{Y}_1 + \mathbf{Y}_2 & -\mathbf{Y}_2 \\ -\mathbf{Y}_2 & \mathbf{Y}_2 + \mathbf{Y}_3 \end{vmatrix}}$$

$$= \frac{\mathbf{I}_1(\mathbf{Y}_2 + \mathbf{Y}_3) - \mathbf{I}_2\mathbf{Y}_2}{\mathbf{Y}_1\mathbf{Y}_2 + \mathbf{Y}_1\mathbf{Y}_3 + \mathbf{Y}_2\mathbf{Y}_3}$$

and

$$\mathbf{V}_2 = \frac{\begin{vmatrix} \mathbf{Y}_1 + \mathbf{Y}_2 & \mathbf{I}_1 \\ -\mathbf{Y}_2 & -\mathbf{I}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Y}_1 + \mathbf{Y}_2 & -\mathbf{Y}_2 \\ -\mathbf{Y}_2 & \mathbf{Y}_2 + \mathbf{Y}_3 \end{vmatrix}}$$

$$= \frac{-\mathbf{I}_2(\mathbf{Y}_1 + \mathbf{Y}_2) + \mathbf{I}_1\mathbf{Y}_2}{\mathbf{Y}_1\mathbf{Y}_2 + \mathbf{Y}_1\mathbf{Y}_3 + \mathbf{Y}_2\mathbf{Y}_3}$$

Substituting the appropriate values into the above expressions we find the nodal voltages as

$$\mathbf{V}_1 = \frac{1(-j0.5 + j0.25) - (2)(-j0.5)}{(0.5)(-j0.5) + (0.5)(j0.25) + (-j0.5)(j0.25)}$$

$$= \frac{j0.75}{0.125 - j0.125}$$

$$= \frac{0.75\angle 90^\circ}{0.1768\angle -45^\circ}$$

$$= 4.243 \text{ V}\angle 135^\circ$$

and

$$\mathbf{V}_2 = \frac{-2(0.5 - j0.5) + (1)(-j0.5)}{(0.5)(-j0.5) + (0.5)(j0.25) + (-j0.5)(j0.25)}$$

$$= \frac{-1 + j0.5}{0.125 - j0.125}$$

$$= \frac{1.118\angle 153.43^\circ}{0.1768\angle -45^\circ}$$

$$= 6.324 \text{ V}\angle 198.43^\circ$$

$$= 6.324 \text{ V}\angle -161.57^\circ$$

**EXAMPLE 19–9** Use nodal analysis to determine the voltage **V** for the circuit of Figure 19–23. Compare the results to those obtained when the circuit was analyzed using mesh analysis in Example 19–6.

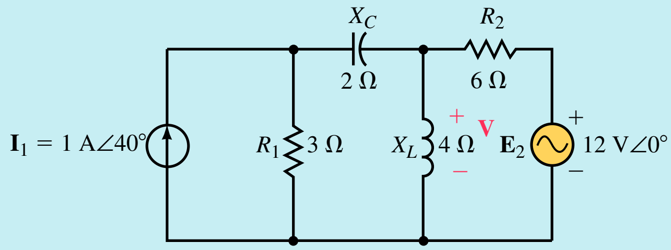
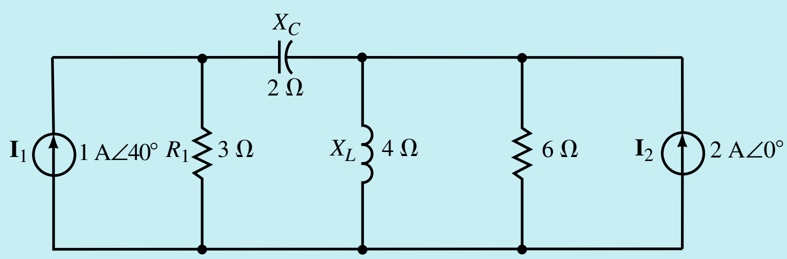



**FIGURE 19–23**

Solution

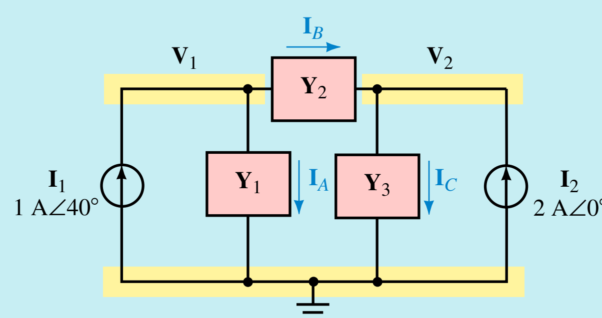
**Step 1:** Convert the voltage source into an equivalent current source as illustrated in Figure 19–24.




$$\mathbf{I}_2 = \frac{\mathbf{E}_2}{6\ \Omega} = \frac{12\ \text{V}\angle 0^\circ}{6\ \Omega} = 2\ \text{A}\angle 0^\circ$$

**FIGURE 19–24**

**Steps 2, 3, and 4:** The reference node is selected to be at the bottom of the circuit and the admittances are simplified as shown in Figure 19–25.




$$\mathbf{Y}_1 = \tfrac{1}{3}\ \text{S} + j\,0$$

$$\mathbf{Y}_2 = 0 + j\tfrac{1}{2}\ \text{S}$$

$$\mathbf{Y}_3 = \tfrac{1}{6}\ \text{S} - j\tfrac{1}{4}\ \text{S}$$

**FIGURE 19–25**

**Step 5:** Applying Kirchhoff's current law to each node, we have the following:

$$\text{Node 1: } \mathbf{I}_A + \mathbf{I}_B = \mathbf{I}_1$$
$$\mathbf{Y}_1\mathbf{V}_1 + \mathbf{Y}_2(\mathbf{V}_1 - \mathbf{V}_2) = \mathbf{I}_1$$
$$(\mathbf{Y}_1 + \mathbf{Y}_2)\mathbf{V}_1 - \mathbf{Y}_2\mathbf{V}_2 = \mathbf{I}_1$$

$$\text{Node 2: } \mathbf{I}_C = \mathbf{I}_B + \mathbf{I}_2$$
$$\mathbf{Y}_3\mathbf{V}_2 = \mathbf{Y}_2(\mathbf{V}_1 - \mathbf{V}_2) + \mathbf{I}_2$$
$$-\mathbf{Y}_2\mathbf{V}_1 + (\mathbf{Y}_2^{\ 2} + \mathbf{Y}_3)\mathbf{V}_2 = \mathbf{I}_2$$

After substituting the appropriate values into the above equations, we have the following simultaneous linear equations:

$$(0.3333\text{ S} + j0.5\text{ S})\mathbf{V}_1 - (j0.5\text{ S})\mathbf{V}_2 = 1\text{ A}\angle 40^\circ$$
$$-(j0.5\text{ S})\mathbf{V}_1 + (0.16667\text{ S} + j0.25\text{ S})\mathbf{V}_2 = 2\text{ A}\angle 0^\circ$$

**Step 6:** We begin by solving for the determinant for the denominator:

$$\mathbf{D} = \begin{vmatrix} 0.3333 + j0.5 & -j0.5 \\ -j0.5 & 0.1667 + j0.25 \end{vmatrix}$$
$$= (0.3333 + j0.5)(0.1667 + j0.25) - (j0.5)(j0.5)$$
$$= 0.0556 + j0.0833 + j0.0833 - 0.125 + 0.25$$
$$= 0.1806 + j0.1667 = 0.2457\angle 42.71^\circ$$

Next we solve the node voltages as

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 1\angle 40^\circ & -j0.5 \\ 2\angle 0^\circ & 0.1667 + j0.25 \end{vmatrix}}{\mathbf{D}}$$
$$= \frac{(1\angle 40^\circ)(0.3005\angle 56.31^\circ) - (2\angle 0^\circ)(0.5\angle -90^\circ)}{\mathbf{D}}$$
$$= \frac{0.3005\angle 96.31^\circ - 1.0\angle -90^\circ}{0.2457\angle 42.71^\circ}$$
$$= \frac{1.299\angle 91.46^\circ}{0.2457\angle 42.71^\circ = 5.29\text{ V}\angle 48.75^\circ}$$

and

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.3333 + j0.5 & 1\angle 40^\circ \\ -j0.5 & 2\angle 0^\circ \end{vmatrix}}{\mathbf{D}}$$
$$= \frac{(0.6009\angle 56.31^\circ)(2\angle 0^\circ) - (0.5\angle -90^\circ)(1\angle 40^\circ)}{\mathbf{D}}$$
$$= \frac{1.2019\angle 56.31^\circ - 0.5\angle -50^\circ}{0.2457\angle 42.71^\circ}$$
$$= \frac{1.426\angle 75.98^\circ}{0.2457\angle 42.71^\circ} = 5.80\text{ V}\angle 33.27^\circ$$

Examining the circuit of Figure 19–23, we see that the voltage **V** is the same as the node voltage $\mathbf{V}_2$. Therefore $\mathbf{V} = 5.80\text{ V}\angle 33.27^\circ$, which is the same result obtained in Example 19–6. (The slight difference in phase angle is the result of rounding error.)

**EXAMPLE 19–10** Given the circuit of Figure 19–26, write the nodal equations expressing all coefficients in rectangular form. Do not solve the equations.

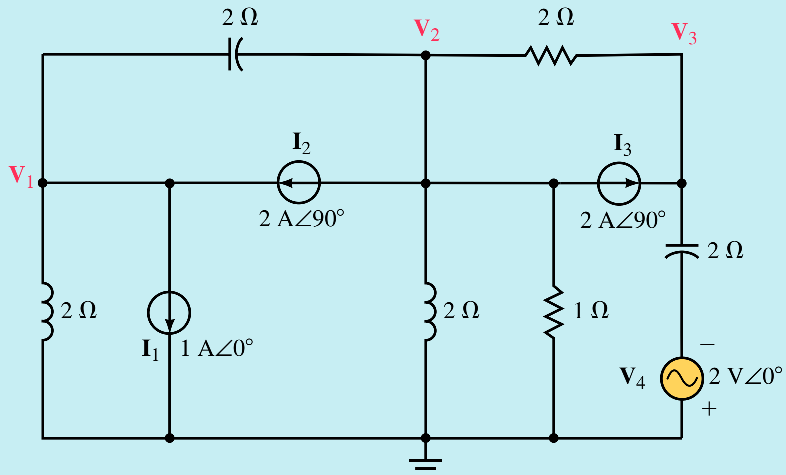



**FIGURE 19–26**

**Solution** As in the previous example, we need to first convert the voltage source into its equivalent current source. The current source will be a phasor $\mathbf{I}_4$, where

$$\mathbf{I}_4 = \frac{\mathbf{V}_4}{\mathbf{X}_C} = \frac{2\text{ V}\angle 0°}{2\ \Omega\angle -90°} = 1.0\text{ A}\angle 90°$$

Figure 19–27 shows the circuit as it appears after the source conversion. Notice that the direction of the current source is downward to correspond with the polarity of the voltage source $\mathbf{V}_4$.

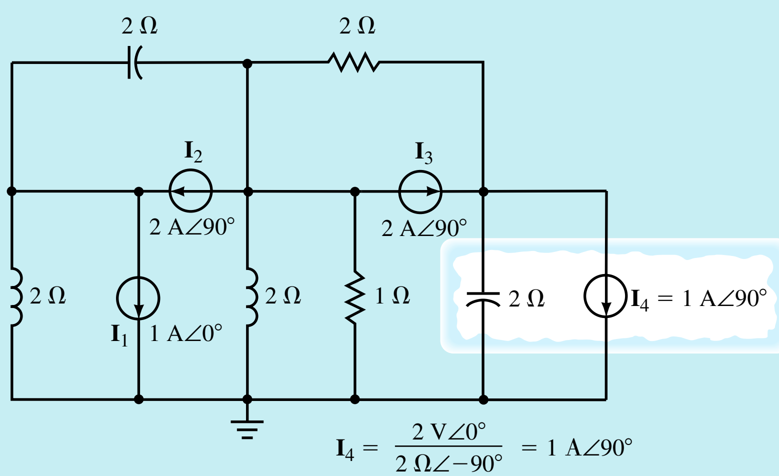



**FIGURE 19–27**

Now, by labelling the nodes and admittances, the circuit may be simplified as shown in Figure 19–28.

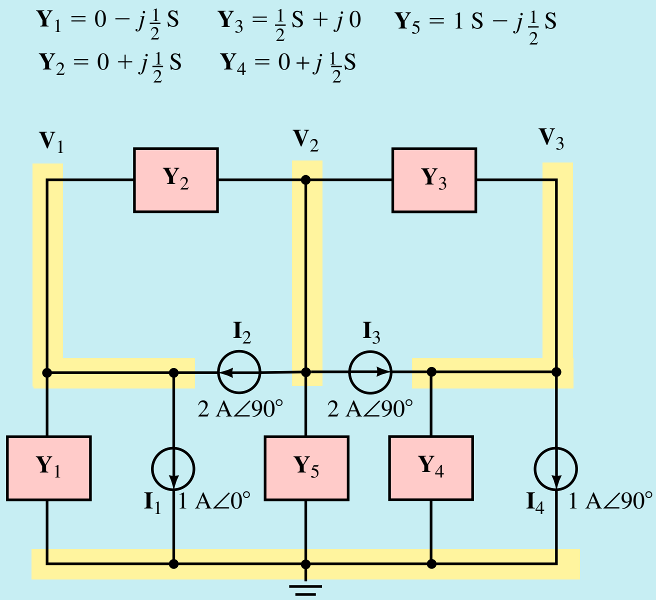



**FIGURE 19–28**

The admittances of Figure 19–28 are as follows:

$$\mathbf{Y}_1 = 0 - j0.5 \text{ S}$$
$$\mathbf{Y}_2 = 0 + j0.5 \text{ S}$$
$$\mathbf{Y}_3 = 0.5 \text{ S} + j0$$
$$\mathbf{Y}_4 = 0 + j0.5 \text{ S}$$
$$\mathbf{Y}_5 = 1.0 \text{ S} - j0.5 \text{ S}$$

Using the assigned admittances, the nodal equations are written as follows:

Node 1: $(\mathbf{Y}_1 + \mathbf{Y}_2)\mathbf{V}_1 - (\mathbf{Y}_2)\mathbf{V}_2 - (0)\mathbf{V}_3 = -\mathbf{I}_1 + \mathbf{I}_2$

Node 2: $-(\mathbf{Y}_2)\mathbf{V}_1 + (\mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_5)\mathbf{V}_2 - (\mathbf{Y}_3)\mathbf{V}_3 = -\mathbf{I}_2 - \mathbf{I}_3$

Node 3: $-(0)\mathbf{V}_1 - (\mathbf{Y}_3)\mathbf{V}_2 + (\mathbf{Y}_3 + \mathbf{Y}_4)\mathbf{V}_3 = \mathbf{I}_3 - \mathbf{I}_4$

By substituting the rectangular form of the admittances and current into the above linear equations, the equations are rewritten as

Node 1: $(-j0.5 + j0.5)\mathbf{V}_1 - (j0.5)\mathbf{V}_2 - (0)\mathbf{V}_3 = -1 + j2$

Node 2: $-(j0.5)\mathbf{V}_1 + (j0.5 + 0.5 + 1 - j0.5)\mathbf{V}_2 - (0.5)\mathbf{V}_3 = -j2 - j2$

Node 3: $-(0)\mathbf{V}_1 - (0.5)\mathbf{V}_2 + (0.5 + j0.5)\mathbf{V}_3 = j2 - j1$

Finally, the nodal equations are simplified as follows:

Node 1: $(0)\mathbf{V}_1 - (j0.5)\mathbf{V}_2 - (0)\mathbf{V}_3 = -1 + j2$

Node 2: $-(j0.5)\mathbf{V}_1 + (1.5)\mathbf{V}_2 - (0.5)\mathbf{V}_3 = -j4$

Node 3: $(0)\mathbf{V}_1 - (0.5)\mathbf{V}_2 + (0.5 + j0.5)\mathbf{V}_3 = j1$

PRACTICE PROBLEMS 4

Given the circuit of Figure 19–29, use nodal analysis to find the voltages $\mathbf{V}_1$ and $\mathbf{V}_2$. Use your results to find the current $\mathbf{I}$.

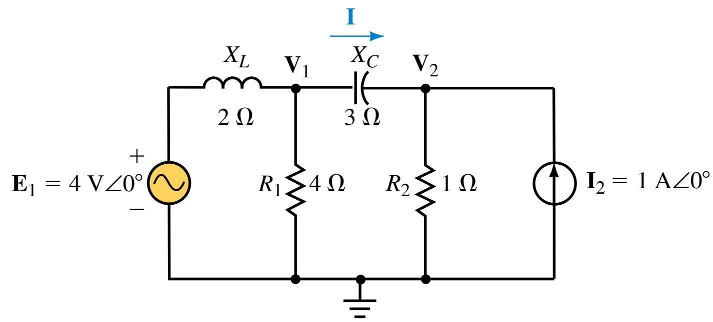



**FIGURE 19–29**

*Answers:* $\mathbf{V}_1 = 4.22\text{ V}\angle -56.89^\circ$, $\mathbf{V}_2 = 2.19\text{ V}\angle 1.01^\circ$, $\mathbf{I} = 1.19\text{ A}\angle 1.85^\circ$

IN-PROCESS LEARNING CHECK 3

Briefly list the steps followed in using nodal analysis to solve for the node voltages of a circuit.

*(Answers are at the end of the chapter.)*

## 19.5 Delta-to-Wye and Wye-to-Delta Conversions

In Chapter 8, we derived the relationships showing the equivalence of "delta" (or "pi") connected resistance to a "wye" (or "tee") configuration.

In a similar manner, impedances connected in a Δ configuration are equivalent to a unique Y configuration. Figure 19–30 shows the equivalent circuits.

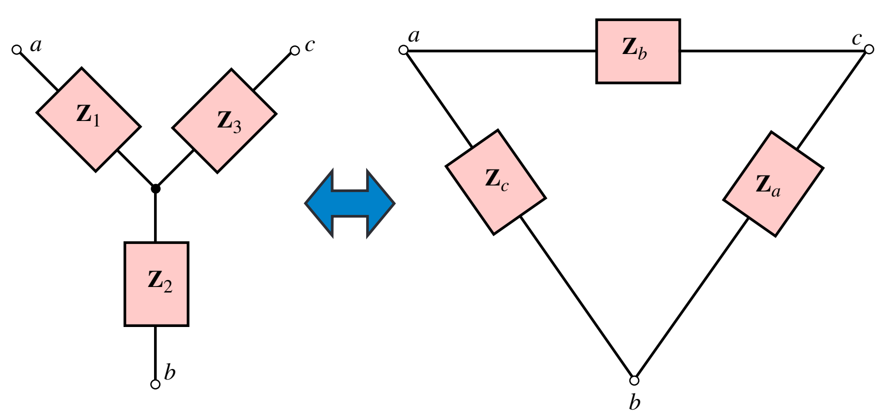



**FIGURE 19–30** Delta-wye equivalence.

A Δ configuration is converted to a Y equivalent by using the following:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} \tag{19–1}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} \tag{19–2}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} \tag{19–3}$$

The above conversion indicates that the impedance in any arm of a Y circuit is determined by taking the product of the two adjacent Δ impedances at this arm and dividing by the summation of the Δ impedances.

If the impedances of the sides of the Δ network are all equal (magnitude and phase angle), the equivalent Y network will have identical impedances, where each impedance is determined as

$$\mathbf{Z}_{\mathrm{Y}} = \frac{\mathbf{Z}_\Delta}{3} \tag{19–4}$$

A Y configuration is converted to a Δ equivalent by using the following:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1} \tag{19–5}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2} \tag{19–6}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_3} \tag{19–7}$$

Any impedance in a "Δ" is determined by summing the possible two-impedance product combinations of the "Y" and then dividing by the impedance found in the opposite branch of the "Y."

If the arms of a "Y" have identical impedances, the equivalent "Δ" will have impedances given as

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_{\mathrm{Y}} \tag{19–8}$$

PRACTICE PROBLEMS 5

**EXAMPLE 19–11** Determine the Y equivalent of the Δ network shown in Figure 19–31.

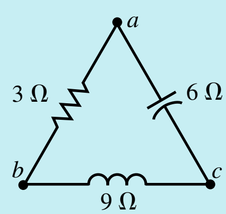



**FIGURE 19–31**

**Solution**

$$\mathbf{Z}_1 = \frac{(3\ \Omega)(-j6\ \Omega)}{3\ \Omega - j6\ \Omega + j9\ \Omega} = \frac{-j18\ \Omega}{3 + j3} = \frac{18\ \Omega\angle -90^\circ}{4.242\angle 45^\circ}$$

$$= 4.242\ \Omega\angle -135^\circ$$

$$= -3.0\ \Omega - j3.0\ \Omega$$

$$\mathbf{Z}_2 = \frac{(3\ \Omega)(j9\ \Omega)}{3\ \Omega - j6\ \Omega + j9\ \Omega} = \frac{j27\ \Omega}{3 + j3} = \frac{27\ \Omega\angle 90^\circ}{4.242\angle 45^\circ}$$
$$= 6.364\ \Omega\angle 45^\circ$$
$$= 4.5\ \Omega + j4.5\ \Omega$$

$$\mathbf{Z}_3 = \frac{(j9\ \Omega)(-j6\ \Omega)}{3\ \Omega - j6\ \Omega + j9\ \Omega} = \frac{54\ \Omega}{3 + j3} = \frac{54\ \Omega\angle 0^\circ}{4.242\angle 45^\circ}$$
$$= 12.73\ \Omega\angle -45^\circ$$
$$= 9.0\ \Omega - j9.0\ \Omega$$

In the above solution, we see that the given Δ network has an equivalent Y network with one arm having a negative resistance. This result indicates that although the Δ circuit has an equivalent Y circuit, the Y circuit cannot actually be constructed from real components since *negative resistors* do not exist (although some active components may demonstrate negative resistance characteristics). If the given conversion is used to simplify a circuit we would treat the impedance $\mathbf{Z}_1 = -3\ \Omega - j3\ \Omega$ as if the resistance actually were a negative value. Figure 19–32 shows the equivalent Y circuit.

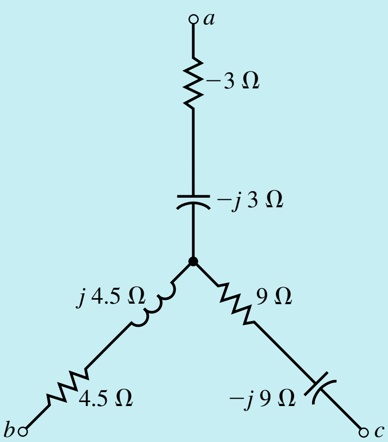



**FIGURE 19–32**

It is left to the student to show that the Y of Figure 19–32 is equivalent to the Δ of Figure 19–31.

**EXAMPLE 19–12** Find the total impedance of the network in Figure 19–33.

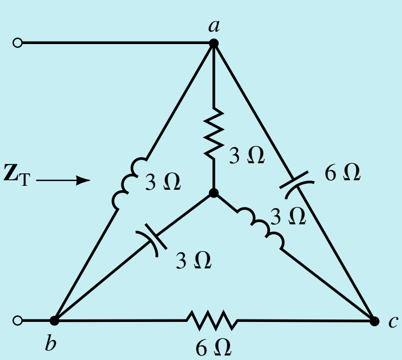



**FIGURE 19–33**

**Solution** If we take a moment to examine this network, we see that the circuit contains both a Δ and a Y. In calculating the total impedance, the solution is easier when we convert the Y to a Δ.

The conversion is shown in Figure 19–34.

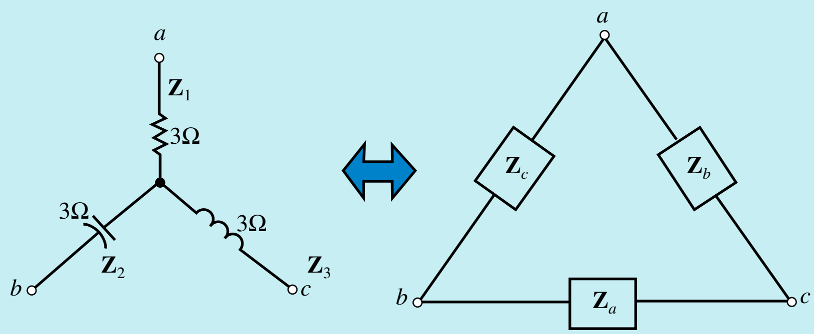



**FIGURE 19–34**

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1}$$

$$= \frac{(3\ \Omega)(j3\ \Omega) + (3\ \Omega)(-j3\ \Omega) + (j3\ \Omega)(-j3\ \Omega)}{3\ \Omega}$$

$$= \frac{-j^2 9\ \Omega}{3} = 3\ \Omega$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2} = \frac{9\ \Omega}{-j3} = j3\ \Omega$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_3} = \frac{9\ \Omega}{j3} = -j3\ \Omega$$

Now, substituting the equivalent Δ into the original network, we have the revised network of Figure 19–35.

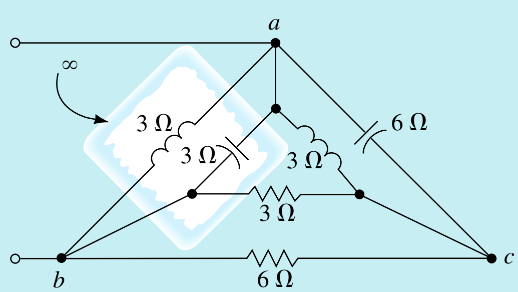



**FIGURE 19–35**

The network of Figure 19–35 shows that the corresponding sides of the Δ are parallel. Because the inductor and the capacitor in the left side of the Δ have the same values, we may replace the parallel combination of these two components with an open circuit. The resulting impedance of the network is now easily determined as

$$\mathbf{Z}_{\mathrm{T}} = 3\ \Omega \| 6\ \Omega + (j3\ \Omega) \| (-j6\ \Omega) = 2\ \Omega + j6\ \Omega$$

A Y network consists of a 60-Ω capacitor, a 180-Ω inductor, and a 540-Ω resistor. Determine the corresponding Δ network.

*Answer:* $\mathbf{Z}_a = -1080\ \Omega + j180\ \Omega$, $\mathbf{Z}_b = 20\ \Omega + j120\ \Omega$, $\mathbf{Z}_c = 360\ \Omega - j60\ \Omega$

IN-PROCESS LEARNING CHECK 4

A Δ network consists of a resistor, inductor, and capacitor, each having an impedance of 150 Ω. Determine the corrresponding Y network.

*(Answers are at the end of the chapter.)*

## 19.6 Bridge Networks

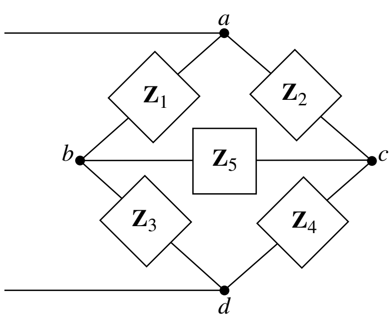



**FIGURE 19–36**

Bridge circuits, similar to the network of Figure 19–36, are used extensively in electronics to measure the values of unknown components.

Recall from Chapter 8 that any bridge circuit is said to be balanced when the current through the branch between the two arms is zero. In a practical circuit, component values of very precise resistors are adjusted until the current through the central element (usually a sensitive galvanometer) is exactly equal to zero. For ac circuits, the condition of a balanced bridge occurs when the impedance vectors of the various arms satisfy the following condition:

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_3} = \frac{\mathbf{Z}_2}{\mathbf{Z}_4} \qquad \textbf{(19–9)}$$

When a balanced bridge occurs in a circuit, the equivalent impedance of the bridge network is easily determined by removing the central impedance and replacing it by either an open or a short circuit. The resulting impedance of the bridge circuit is then found as either of the following:

$$\mathbf{Z}_T = \mathbf{Z}_1 \| \mathbf{Z}_2 + \mathbf{Z}_3 \| \mathbf{Z}_4$$

or

$$\mathbf{Z}_T = (\mathbf{Z}_1 + \mathbf{Z}_3) \| (\mathbf{Z}_2 + \mathbf{Z}_4)$$

If, on the other hand, the bridge is not balanced, then the total impedance must be determined by performing a Δ-to-Y conversion. Alternatively, the circuit may be analyzed by using either mesh analysis or nodal analysis.

**EXAMPLE 19–13** Given that the circuit of Figure 19–37 is a balanced bridge.

a. Calculate the unknown impedance, $\mathbf{Z}_x$.

b. Determine the values of $L_x$ and $R_x$ if the circuit operates at a frequency of 1 kHz.

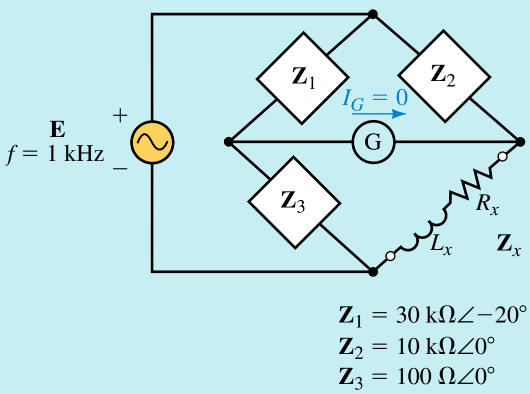



**FIGURE 19–37**

**Solution**

a. The expression for the unknown impedance is determined from Equation 19–9 as

$$\begin{aligned}\mathbf{Z}_x &= \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1}\\ &= \frac{(10\ \text{k}\Omega)(100\ \Omega)}{30\ \text{k}\Omega\angle-20^\circ}\\ &= 33.3\ \Omega\angle 20^\circ\\ &= 31.3 + j11.4\ \Omega\end{aligned}$$

b. From the above result, we have

$$R_x = 31.3\ \Omega$$

and

$$L_x = \frac{X_L}{2\pi f} = \frac{11.4\ \Omega}{2\pi(1000\ \text{Hz})} = 1.81\ \text{mH}$$

We will now consider various forms of bridge circuits which are used in electronic circuits to determine the values of unknown inductors and capacitors. As in resistor bridges, the circuits use variable resistors together with very sensitive galvanometer movements to ensure a balanced condition for the bridge. However, rather than using a dc source to provide current in the circuit, the bridge circuits use ac sources operating at a known frequency (usually 1 kHz). Once the bridge is balanced, the value of unknown inductance or capacitance may be easily determined by obtaining the reading directly from the instrument. Most instruments using bridge circuitry will incorporate several different bridges to enable the measurement of various types of unknown impedances.

## Maxwell Bridge

The **Maxwell bridge,** shown in Figure 19–38, is used to determine the inductance and series resistance of an inductor having a relatively large series resistance (in comparison to $X_L = \omega L$).

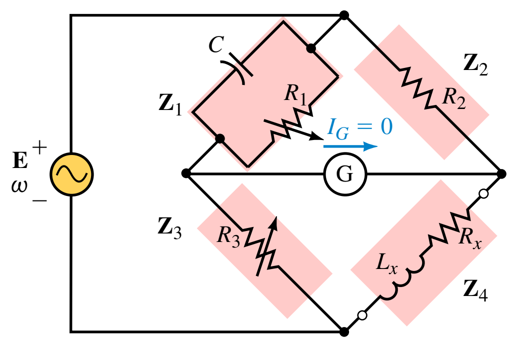



**FIGURE 19–38** Maxwell bridge.

Resistors $R_1$ and $R_3$ are adjusted to provide the balanced condition (when the current through the galvanometer is zero: $I_G = 0$).

When the bridge is balanced, we know that the following condition must apply:

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_3}{\mathbf{Z}_4}$$

If we write the impedances using the rectangular forms, we obtain

$$\frac{\left[\dfrac{(R_1)\left(-j\dfrac{1}{\omega C}\right)}{R_1 - j\dfrac{1}{\omega C}}\right]}{R_2} = \frac{R_3}{R_x + j\omega L_x}$$

$$\frac{\left(-j\dfrac{R_1}{\omega C}\right)}{\left(\dfrac{\omega R_1 C - j1}{\omega C}\right)} = \frac{R_2 R_3}{R_x + j\omega L_x}$$

$$\frac{-jR_1}{\omega C R_1} - j = \frac{R_2 R_3}{R_x + j\omega L_x}$$

$$(-jR_1)(R_x + j\omega L_x) = R_2 R_3(\omega C R_1 - j)$$

$$\omega L_x R_1 - jR_1 R_x = \omega R_1 R_2 R_3 C - jR_2 R_3$$

Now, since two complex numbers can be equal only if their real parts are equal and if their imaginary parts are equal, we must have the following:

$$\omega L_x R_1 = \omega R_1 R_2 R_3 C$$

and

$$R_1 R_x = R_2 R_3$$

Simplifying these expressions, we get the following equations for a Maxwell bridge:

$$L_x = R_2 R_3 C \tag{19–10}$$

and

$$R_x = \frac{R_2 R_3}{R_1} \tag{19–11}$$

## EXAMPLE 19–14

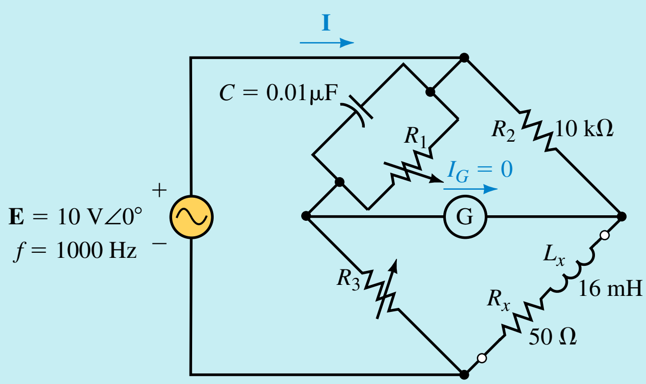



**FIGURE 19–39**

a. Determine the values of $R_1$ and $R_3$ so that the bridge of Figure 19–39 is balanced.
b. Calculate the current **I** when the bridge is balanced.

### Solution

a. Rewriting Equations 19–10 and 19–11 and solving for the unknowns, we have

$$R_3 = \frac{L_x}{R_2 C} = \frac{16\text{ mH}}{(10\text{ k}\Omega)(0.01\ \mu\text{F})} = 160\ \Omega$$

and

$$R_1 = \frac{R_2 R_3}{R_x} = \frac{(10\text{ k}\Omega)(160\ \Omega)}{50\ \Omega} = 32\text{ k}\Omega$$

b. The total impedance is found as

$$\mathbf{Z}_T = (\mathbf{Z}_C \| \mathbf{R}_1 \| \mathbf{R}_2) + [\mathbf{R}_3 \| (\mathbf{R}_x + \mathbf{Z}_{Lx})]$$
$$\mathbf{Z}_T = (-j15.915\text{ k}\Omega)\|32\text{ k}\Omega\|10\text{ k}\Omega + [160\ \Omega\|(50\ \Omega + j100.5\ \Omega)]$$
$$= 6.87\text{ k}\Omega\angle -25.6^\circ + 77.2\ \Omega\angle 38.0^\circ$$
$$= 6.91\text{ k}\Omega\angle -25.0^\circ$$

The resulting circuit current is

$$\mathbf{I} = \frac{10\text{ V}\angle 0^\circ}{6.91\text{ k}\Omega\angle -25^\circ} = 1.45\text{ mA}\angle 25.0^\circ$$

## Hay Bridge

In order to measure the inductance and series resistance of an inductor having a small series resistance, a **Hay bridge** is generally used. The Hay bridge is shown in Figure 19–40.

By applying a method similar to that used to determine the values of the unknown inductance and resistance of the Maxwell bridge, it may be shown that the following equations for the Hay bridge apply:

$$L_x = \frac{R_2 R_3 C}{\omega^2 R_1^2 C^2 + 1} \tag{19–12}$$

and

$$R_x = \frac{\omega^2 R_1 R_2 R_3 C^2}{\omega^2 R_1^2 C^2 + 1} \tag{19–13}$$

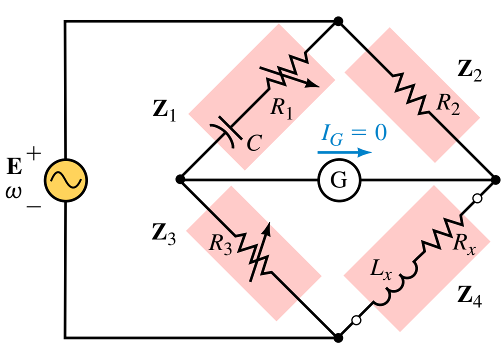



**FIGURE 19–40** Hay bridge.

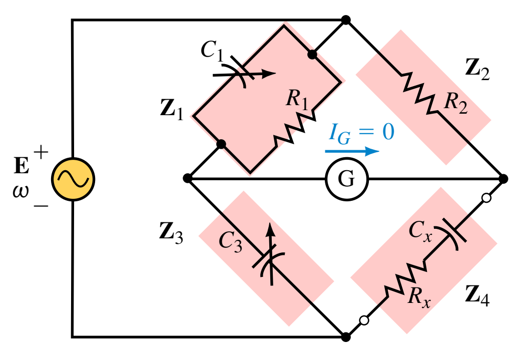



**FIGURE 19–41** Schering bridge.

## Schering Bridge

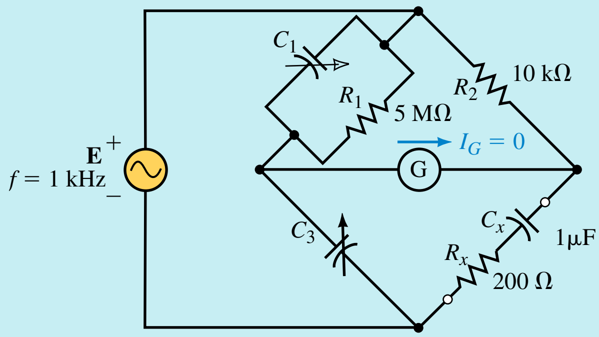
The **Schering bridge,** shown in Figure 19–41, is a circuit used to determine the value of unknown capacitance.

By solving for the balanced bridge condition, we have the following equations for the unknown quantities of the circuit:

$$C_x = \frac{R_1 C_3}{R_2} \tag{19–14}$$

$$R_x = \frac{C_1 R_2}{C_3} \tag{19–15}$$

**EXAMPLE 19–15** Determine the values of $C_1$ and $C_3$ which will result in a balanced bridge for the circuit of Figure 19–42.




EWB **FIGURE 19–42**

**Solution** Rewriting Equations 19–14 and 19–15, we solve for the unknown capacitances as

$$C_3 = \frac{R_2 C_x}{R_1} = \frac{(10\ \text{k}\Omega)(1\ \mu\text{F})}{5\ \text{M}\Omega} = 0.002\ \mu\text{F}$$

and

$$C_1 = \frac{C_3 R_x}{R_2} = \frac{(0.002\ \mu\text{F})(200\ \Omega)}{10\ \text{k}\Omega} = 40\ \text{pF}$$

**PRACTICE PROBLEMS 6**

Determine the values of $R_1$ and $R_3$ so that the bridge of Figure 19–43 is balanced.

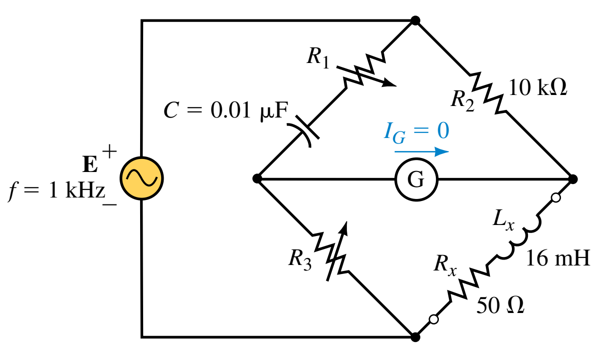



**FIGURE 19–43**

*Answers:* $R_1 = 7916\ \Omega$, $R_3 = 199.6\ \Omega$

ELECTRONICS WORKBENCH

PSpice

## 19.7 Circuit Analysis Using Computers

In some of the examples in this chapter, we analyzed circuits that resulted in as many as three simultaneous linear equations. You have no doubt wondered if there is a less complicated way to solve these circuits without the need for using complex algebra. Computer programs are particularly useful for solving such ac circuits. Both Electronics Workbench and PSpice have individual strengths in the solution of ac circuits. As in previous examples, Electronics Workbench provides an excellent simulation of how measurements are taken in a lab. PSpice, on the other hand, provides voltage and current readings, complete with magnitude and phase angle. The following examples show how these programs are useful for examining the circuits in this chapter.

**EXAMPLE 19–16** Use Electronics Workbench to show that the bridge circuit of Figure 19–44 is balanced.

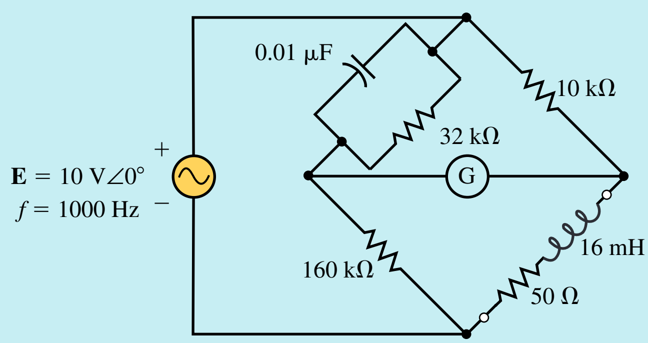
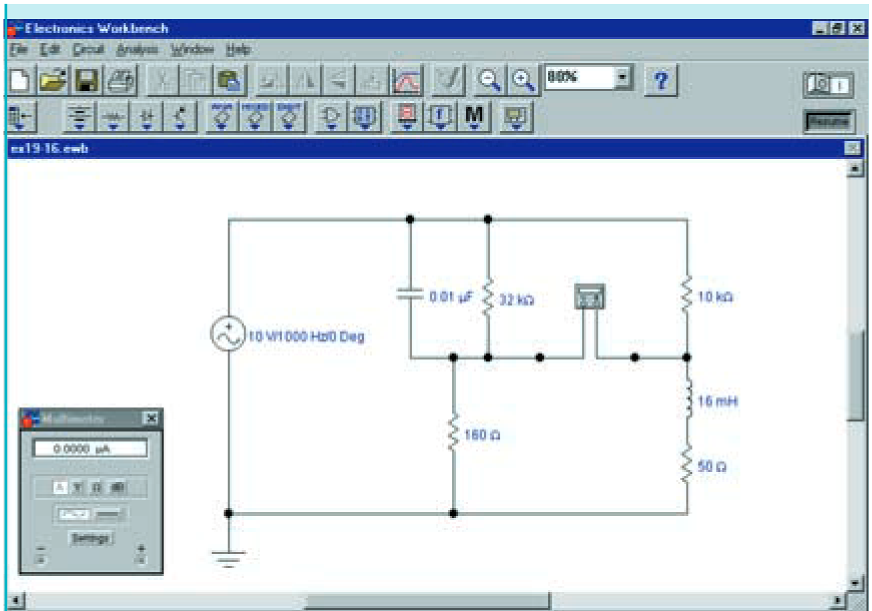



**FIGURE 19–44**

*Solution* Recall that a bridge circuit is balanced when the current through the branch between the two arms of the bridge is equal to zero. In this example, we will use a multimeter set on its ac ammeter range to verify the condition of the circuit. The ammeter is selected by clicking on **A** and it is set to its ac range by clicking on the sinusoidal button. Figure 19–45 shows the circuit connections and the ammeter reading. The results correspond to the conditions that were previously analyzed in Example 19–14. (Note: When using Electronics Workbench, the ammeter may not show exactly zero current in the balanced condition. This is due to the way the program does the calculations. Any current less than 5 $\mu$A is considered to be effectively zero.)




EWB **FIGURE 19–45**

Use Electronics Workbench to verify that the results obtained in Example 19–15 result in a balanced bridge circuit. (Assume that the bridge is balanced if the galvanometer current is less than 5 $\mu$A.)

PRACTICE PROBLEMS 7

## OrCAD PSpice

EXAMPLE 19–17 Use OrCAD Capture CIS to input the circuit of Figure 19–15. Assume that the circuit operates at a frequency of $\omega = 50$ rad/s ($f = 7.958$ Hz). Use PSpice to obtain a printout showing the currents through $X_C$, $R_2$, and $X_L$. Compare the results to those obtained in Example 19–6.

*Solution* Since the reactive components in Figure 19–15 were given as impedance, it is necessary to first determine the corresponding values in henries and farads.

$$L = \frac{4\ \Omega}{50\ \text{rad/s}} = 80\ \text{mH}$$

and

$$C = \frac{1}{(2\ \Omega)(50\ \text{rad/s})} = 10\ \text{mF}$$

Now we are ready to use OrCAD Capture to input the circuit as shown in Figure 19–46. The basic steps are reviewed for you. Use the ac current source,

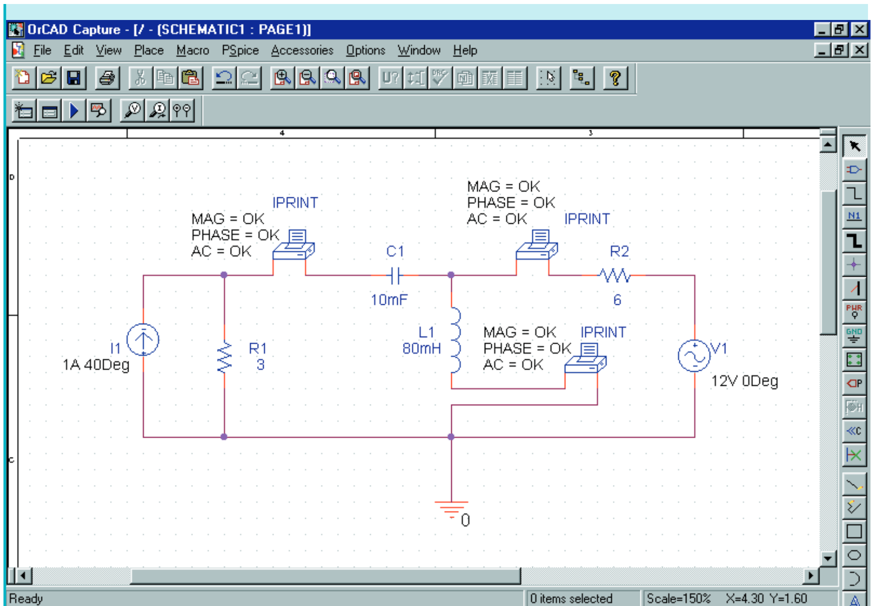



**FIGURE 19–46**

ISRC from the SOURCE library and place one IPRINT part from the SPECIAL library. The resistor, inductor, and capacitor are selected from the ANALOG library and the ground symbol is selected by using the Place ground tool.

Change the value of the current source by double clicking on the part and moving the horizontal scroll bar until you find the field titled AC. Type **1A 40Deg** into this field. A space must be placed between the magnitude and phase angle. Click on Apply. In order for these values to be displayed on the schematic, you must click on the Display button and then Value Only. Click on OK to return to the properties editor and then close the editor by clicking on X.

The IPRINT part is similar to an ammeter and provides a printout of the current magnitude and phase angle. The properties of the IPRINT part are changed by double clicking on the part and scrolling across to show the appropriate fields. Type **OK** in the AC, MAG, and PHASE fields. In order to display the selected fields on the schematic, you must click on the Display button and then select Name and Value after changing each field. Since we need to measure three currents in the circuit, we could follow this procedure two more times. However, an easier method is to click on the IPRINT part and copy the part by using <Ctrl><C> and <Ctrl><V>. Each IPRINT will then have the same properties.

Once the rest of the circuit is completed and wired, click on the New Simulation Profile tool. Give the simulation a name (such as **ac Branch Currents**). Click on the Analysis tab and select AC Sweep/ Noise as the analysis type. Type the following values:

Start Frequency: **7.958Hz**

End Frequency: **7.958Hz**

Total Points: **1**

Since we do not need the Probe postprocessor to run, it is disabled by selecting the Probe Window tab (from the Simulation Settings dialog box). Click on Display Probe window and exit the simulation settings by clicking on OK.

Click on the Run tool. Once PSpice has successfully run, click on the View menu and select the Output File menu item. Scroll through the file until the currents are shown as follows:

```
FREQ         IM(V_PRINT1)IP(V_PRINT1)
 7.958E+00    7.887E-01   -1.201E+02
FREQ         IM(V_PRINT2)IP(V_PRINT2)
 7.958E+00    1.304E+00    1.560E+02
FREQ         IM(V_PRINT3)IP(V_PRINT3)
 7.958E+00    1.450E+00   -5.673E+01
```

The above printout provides: $\mathbf{I}_1 = 0.7887\text{ A}\angle -120.1°$, $\mathbf{I}_2 = 1.304\text{ A}\angle 156.0°$, and $\mathbf{I}_3 = 1.450\text{ A}\angle -56.73°$. These results are consistent with those calculated in Example 19–6.

**PRACTICE PROBLEMS 8**

Use OrCAD PSpice to evaluate the node voltages for the circuit of Figure 19–23. Assume that the circuit operates at an angular frequency of $\omega = 1000$ rad/s ($f = 159.15$ Hz).

## PUTTING IT INTO PRACTICE

The Schering bridge of Figure 19–74 (p. 779) is balanced. In this chapter, you have learned several methods that allow you to find the current anywhere in a circuit. Using any method, determine the current through the galvanometer if the value of $C_x = 0.07\ \mu\text{F}$ (All other values remain unchanged.) Repeat the calculations for a value of $C_x = 0.09\ \mu\text{F}$. Can you make a general statement for current through the galvanometer if $C_x$ is smaller that the value required to balance the bridge? What general statement can be made if the value of $C_x$ is larger than the value in the balanced bridge?

## PROBLEMS

### 19.1 Dependent Sources

1. Refer to the circuit of Figure 19–47.

   Find **V** when the controlling current **I** is the following:

   a. $20\ \mu\text{A}\angle 0°$

   b. $50\ \mu\text{A}\angle -180°$

   c. $60\ \mu\text{A}\angle 60°$

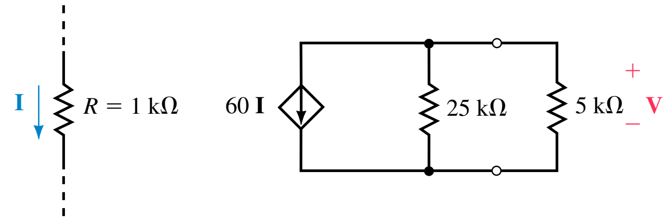



FIGURE 19–47

2. Refer to the circuit of Figure 19–48.
   Find **I** when the controlling voltage, **V**, is the following:
   a. $30\ \text{mV}\angle 0°$
   b. $60\ \text{mV}\angle -180°$
   c. $100\ \text{mV}\angle -30°$

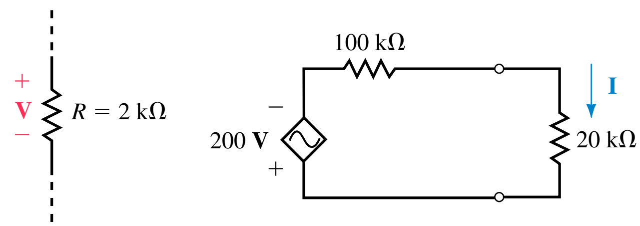



FIGURE 19–48

3. Repeat Problem 1 for the circuit of Figure 19–49.

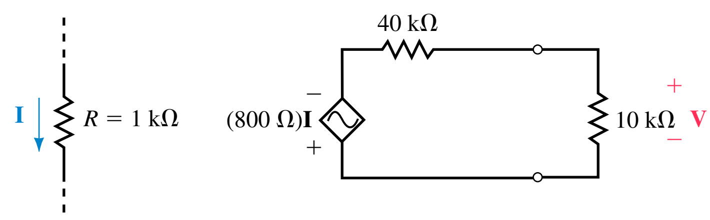



FIGURE 19–49

4. Repeat Problem 2 for the circuit of Figure 19–50.

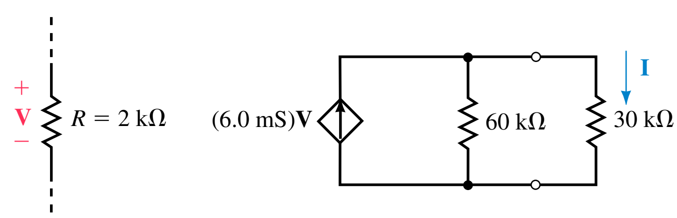



FIGURE 19–50

5. Find the output voltage, $\mathbf{V}_{\text{out}}$, for the circuit of Figure 19–51.

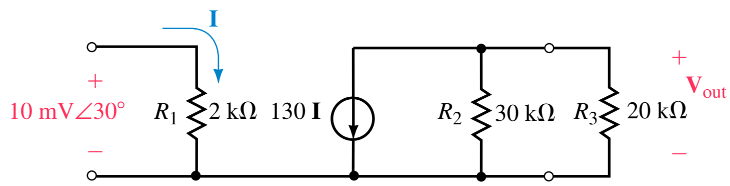



FIGURE 19–51

6. Repeat Problem 5 for the circuit of Figure 19–52.

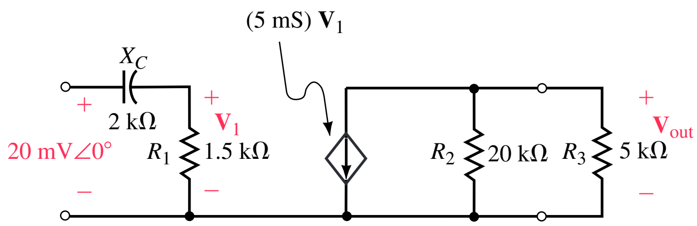



FIGURE 19–52

## 19.2 Source Conversion

7. Given the circuits of Figure 19–53, convert each of the current sources into an equivalent voltage source. Use the resulting circuit to find $\mathbf{V}_L$.

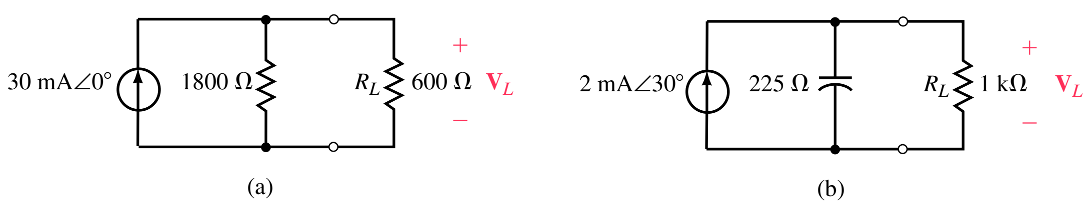



FIGURE 19–53

8. Convert each voltage source of Figure 19–54 into an equivalent current source.

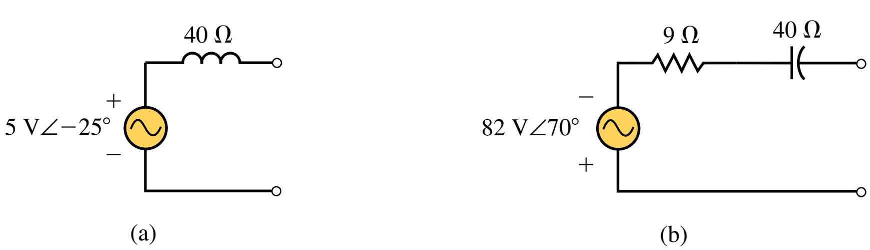



FIGURE 19–54

9. Refer to the circuit of Figure 19–55.
   a. Solve for the voltage, **V**.

b. Convert the current source into an equivalent voltage source and again solve for **V.** Compare to the result obtained in (a).

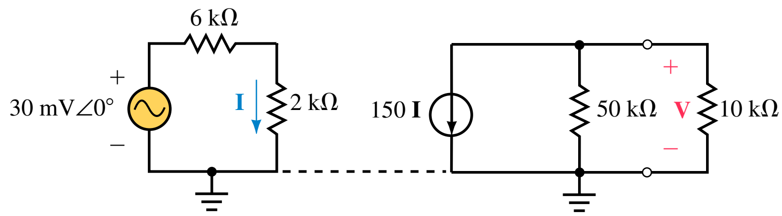



FIGURE 19–55

10. Refer to the circuit of Figure 19–56.
    a. Solve for the voltage, $\mathbf{V}_L$.
    b. Convert the current source into an equivalent voltage source and again solve for $\mathbf{V}_L$.
    c. If $\mathbf{I} = 5\ \mu\text{A}\angle 90°$, what is $\mathbf{V}_L$?

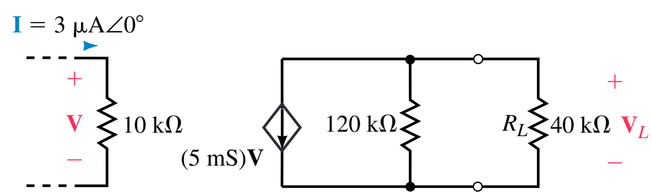



FIGURE 19–56

## 19.3 Mesh (Loop) Analysis

11. Consider the circuit of Figure 19–57.
    a. Write the mesh equations for the circuit.
    b. Solve for the loop currents.
    c. Determine the current **I** through the 4-Ω resistor.

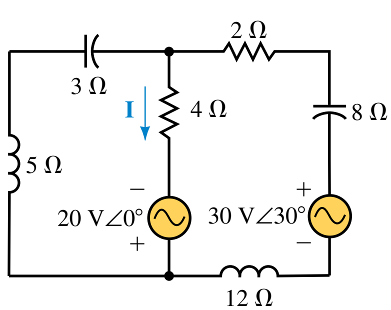



FIGURE 19–57

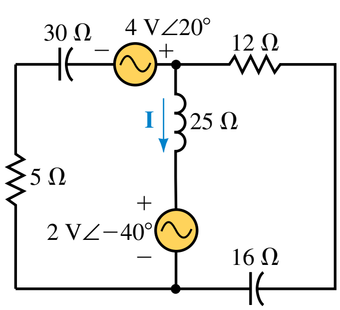



FIGURE 19–58

12. Refer to the circuit of Figure 19–58.
    a. Write the mesh equations for the circuit.
    b. Solve for the loop currents.
    c. Determine the current through the 25-Ω inductor.

13. Refer to the circuit of Figure 19–59.
    a. Simplify the circuit and write the mesh equations.
    b. Solve for the loop currents.
    c. Determine the voltage **V** across the 15-Ω capacitor.

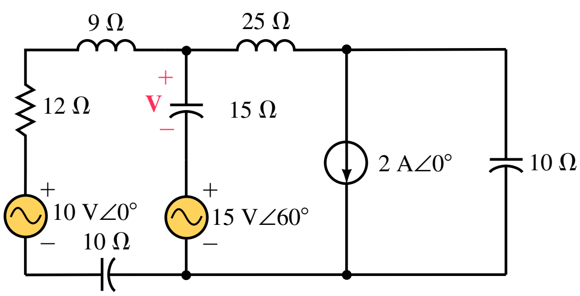



FIGURE 19–59

14. Consider the circuit of Figure 19–60.
    a. Simplify the circuit and write the mesh equations.
    b. Solve for the loop currents.
    c. Determine the voltage **V** across the 2-Ω resistor.
15. Use mesh analysis to find the current **I** and the voltage **V** in the circuit of Figure 19–61.

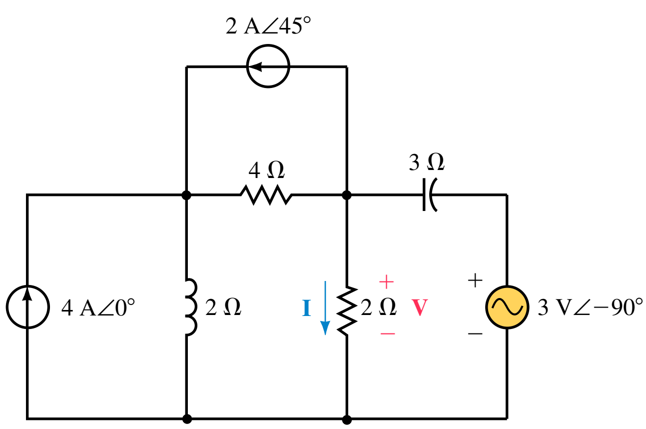



FIGURE 19–60

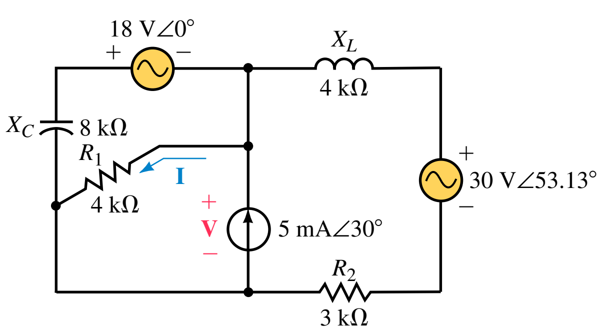



FIGURE 19–61

16. Repeat Problem 15 for the circuit of Figure 19–62.

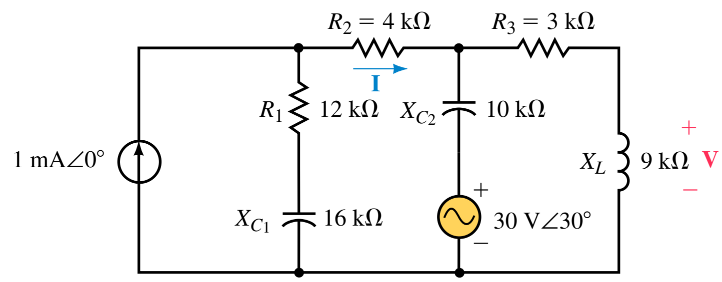



FIGURE 19–62

## 19.4 Nodal Analysis

17. Consider the circuit of Figure 19–63.
    a. Write the nodal equations.
    b. Solve for the node voltages.
    c. Determine the current **I** through the 4-Ω capacitor.

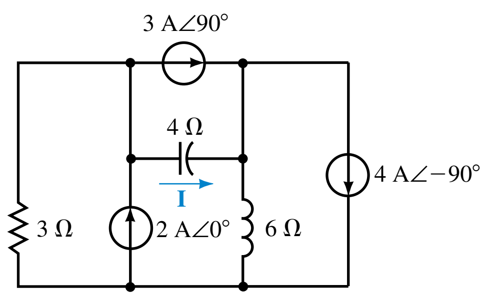



FIGURE 19–63

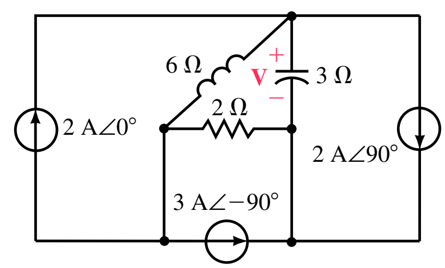



FIGURE 19–64

18. Refer to the circuit of Figure 19–64.
    a. Write the nodal equations.
    b. Solve for the node voltages.
    c. Determine the voltage **V** across the 3-Ω capacitor.
19. a. Simplify the circuit of Figure 19–59, and write the nodal equations.
    b. Solve for the node voltages.
    c. Determine the voltage across the 15-Ω capacitor.
20. a. Simplify the circuit of Figure 19–60, and write the nodal equations.
    b.Solve for the node voltages.
    c. Determine the current through the 2-Ω resistor.
21. Use nodal analysis to determine the node voltages in the circuit of Figure 19–61. Use the results to find the current **I** and the voltage **V.** Compare your answers to those obtained using mesh analysis in Problem 15.
22. Use nodal analysis to determine the node voltages in the circuit of Figure 19–62. Use the results to find the current **I** and the voltage **V.** Compare your answers to those obtained using mesh analysis in Problem 16.

### 19.5 Delta-to-Wye and Wye-to-Delta Conversions

23. Convert each of the Δ networks of Figure 19–65 into an equivalent Y network.
24. Convert each of the Y networks of Figure 19–66 into an equivalent Δ network.

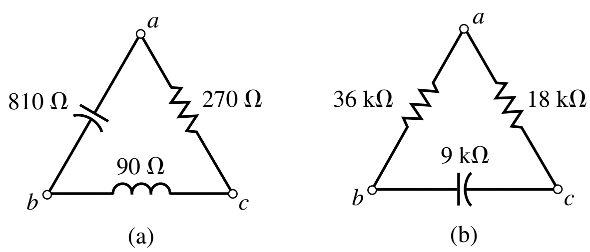



FIGURE 19–65

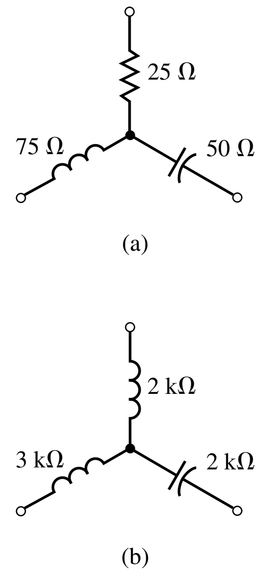



FIGURE 19–66

25. Using Δ→Y or Y→Δ conversion, calculate **I** for the circuit of Figure 19–67.
26. Using Δ→Y or Y→Δ conversion, calculate **I** for the circuit of Figure 19–68.

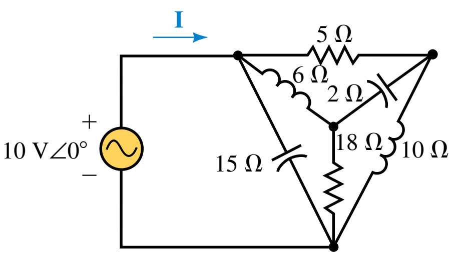



FIGURE 19–67

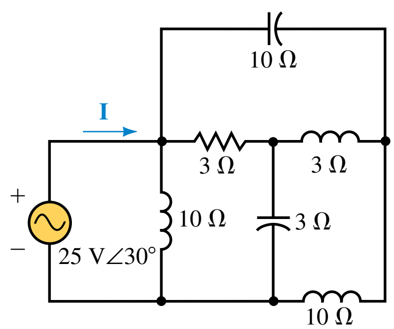



FIGURE 19–68

27. Refer to the circuit of Figure 19–69:
    a. Determine the equivalent impedance, $\mathbf{Z}_T$, of the circuit.
    b. Find the currents **I** and $\mathbf{I}_1$.

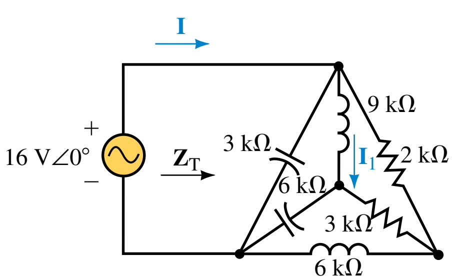



FIGURE 19–69

28. Refer to the circuit of Figure 19–70:
   a. Determine the equivalent impedance, $\mathbf{Z}_T$, of the circuit.
   b. Find the voltages $\mathbf{V}$ and $\mathbf{V}_1$.

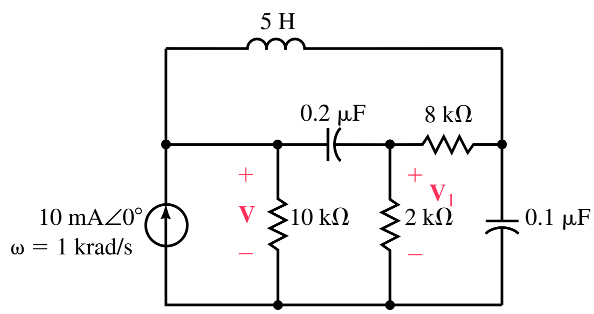



FIGURE 19–70

### 19.6 Bridge Networks

29. Given that the bridge circuit of Figure 19–71 is balanced:
   a. Determine the value of the unknown impedance.
   b. Solve for the current **I.**

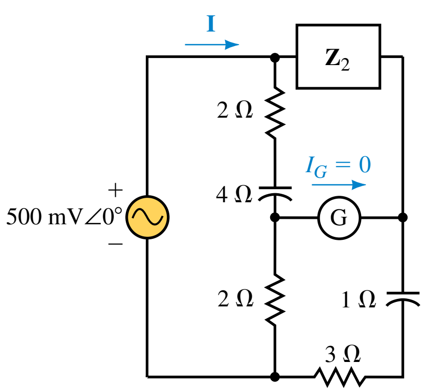



FIGURE 19–71

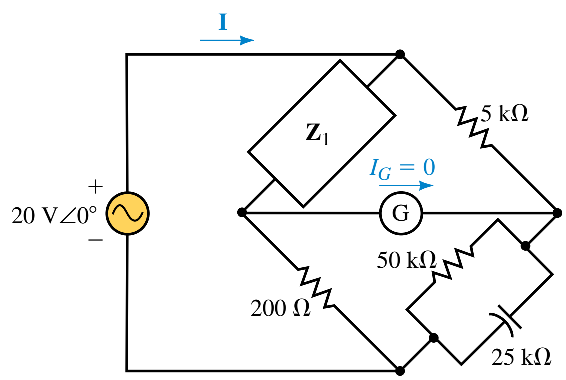
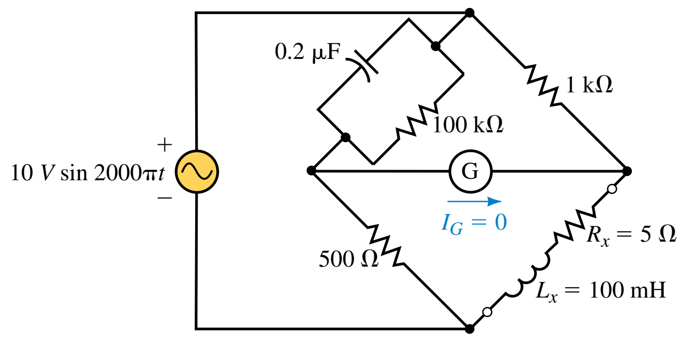
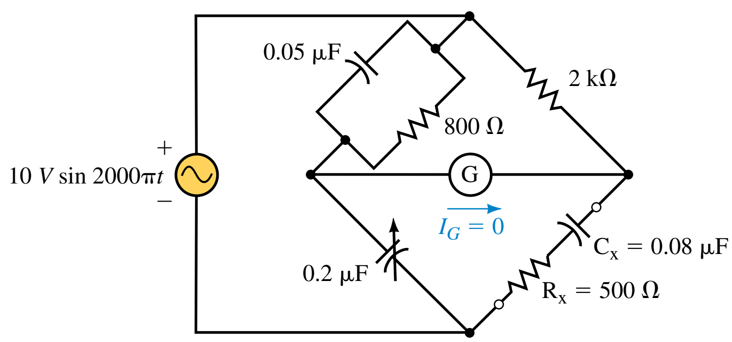



FIGURE 19–72

30. Given that the bridge circuit of Figure 19–72 is balanced:
   a. Determine the value of the unknown impedance.
   b. Solve for the current **I.**
31. Show that the bridge circuit of Figure 19–73 is balanced.
32. Show that the bridge circuit of Figure 19–74 is balanced.
33. Derive Equations 19–14 and 19–15 for the balanced Schering bridge.
34. Derive Equations 19–12 and 19–13 for the balanced Hay bridge.




EWB **FIGURE 19–73**




EWB **FIGURE 19–74**

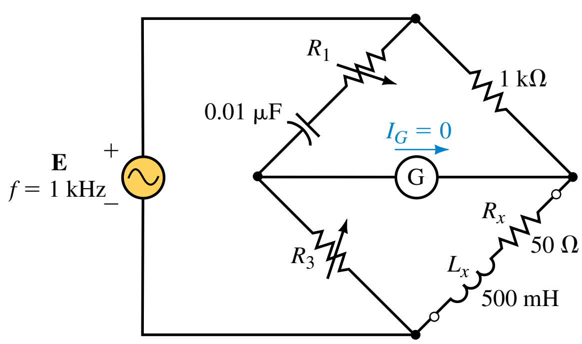



**FIGURE 19–75**

35. Determine the values of the unknown resistors which will result in a balanced bridge for the circuit of Figure 19–75.
36. Determine the values of the unknown capacitors which will result in a balanced bridge for the circuit of Figure 19–76.

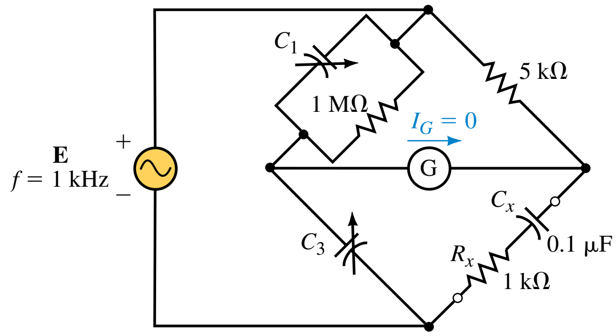



FIGURE 19–76

### 19.7 Circuit Analysis Using Computers

37. **EWB** Use Electronics Workbench to show that the bridge circuit of Figure 19–73 is balanced. (Assume that the bridge is balanced if the galvanometer current is less than 5 $\mu$A.)
38. **EWB** Repeat Problem 37 for the bridge circuit of Figure 19–74.
39. **PSpice** Use the OrCAD Capture to input the file for the circuit of Figure 19–21. Assume that the circuit operates at a frequency $\omega = 2$ krad/s. Use IPRINT and VPRINT to obtain a printout of the node voltages and the current through each element of the circuit.
40. **PSpice** Use the OrCAD Capture to input the file for the circuit of Figure 19–29. Assume that the circuit operates at a frequency $\omega = 1$ krad/s. Use IPRINT and VPRINT to obtain a printout of the node voltages and the current through each element of the circuit.
41. **PSpice** Use the OrCAD Capture to input the file for the circuit of Figure 19–68. Assume that the circuit operates at a frequency $\omega = 20$ rad/s. Use IPRINT to obtain a printout of the current **I.**
42. **PSpice** Use the OrCAD Capture to input the file for the circuit of Figure 19–69. Assume that the circuit operates at a frequency $\omega = 3$ krad/s. Use IPRINT to obtain a printout of the current **I.**

## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

a. $\mathbf{E} = 1000\text{ V}\angle 30°$
b. $\mathbf{E} = 4\text{ V}\angle -90°$
c. $\mathbf{E} = 1024\text{ V}\angle -38.66°$

**In-Process Learning Check 2**

1. Convert current sources to voltage sources.
2. Redraw the circuit.
3. Assign a clockwise current to each loop.
4. Write loop equations using Kirchhoff's voltage law.
5. Solve the resulting simultaneous linear equations to find the loop currents.

**In-Process Learning Check 3**

1. Convert voltage sources to current sources.
2. Redraw the circuit.
3. Label all nodes, including the reference node.
4. Write nodal equations using Kirchhoff's current law.
5. Solve the resulting simultaneous linear equations to find the node voltages.

**In-Process Learning Check 4**

$\mathbf{Z}_1 = 150\ \Omega\angle 90°$ $\mathbf{Z}_2 = 150\ \Omega\angle -90°$ $\mathbf{Z}_3 = 150\ \Omega\angle 0°$

# 20 AC Network Theorems

## OBJECTIVES

After studying this chapter you will be able to

- apply the superposition theorem to determine the voltage across or current through any component in a given circuit,
- determine the Thévenin equivalent of circuits having independent and/or dependent sources,
- determine the Norton equivalent of circuits having independent and/or dependent sources,
- apply the maximum power transfer theorem to determine the load impedance for which maximum power is transferred to the load from a given circuit,
- use PSpice to find the Thévenin and Norton equivalents of circuits having either independent or dependent sources,
- use Electronics Workbench to verify the operation of ac circuits.

## KEY TERMS

Absolute Maximum Power
Maximum Power Transfer
Norton's Theorem
Relative Maximum Power
Superposition Theorem

## OUTLINE

## CHAPTER PREVIEW

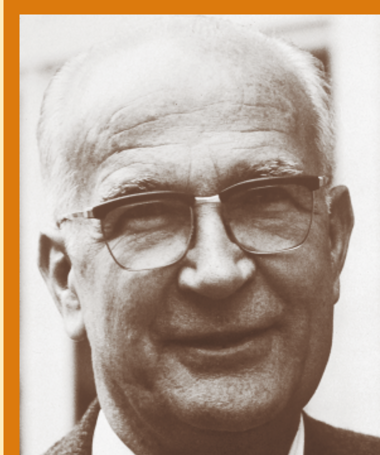
In this chapter we apply the superposition, Thévenin, Norton, and maximum power transfer theorems in the analysis of ac circuits. Although the Millman and reciprocity theorems apply to ac circuits as well as to dc circuits, they are omitted since the applications are virtually identical with those used in analyzing dc circuits.

Many of the techniques used in this chapter are similar to those used in Chapter 9, and as a result, most students will find a brief review of dc theorems useful.

This chapter examines the application of the network theorems by considering both independent and dependent sources. In order to show the distinctions between the methods used in analyzing the various types of sources, the sections are labelled according to the types of sources involved.

An understanding of dependent sources is particularly useful when working with transistor circuits and operational amplifiers. Sections 20.2 and 20.5 are intended to provide the background for analyzing the operation of feedback amplifiers. Your instructor may find that these topics are best left until you cover this topic in a course dealing with such amplifiers. Consequently, the omission of sections 20.2 and 20.5 will not in any way detract from the continuity of the important ideas presented in this chapter.

## PUTTING IT IN PERSPECTIVE



### *William Bradford Shockley*

SHOCKLEY WAS BORN the son of a mining engineer in London, England on February 13, 1910. After graduating from the California Institute of Technology and Massachusetts Institute of Technology, Shockley joined Bell Telephone Laboratories.

With his co-workers, John Bardeen and Walter Brattain, Shockley developed an improved solid-state rectifier using a germanium crystal which had been injected with minute amounts of impurities. Unlike vacuum tubes, the resulting diodes were able to operate at much lower voltages without the need for inefficient heater elements.

In 1948, Shockley combined three layers of germanium to produce a device which was able to not only rectify a signal but to amplify it. Thus was developed the first transistor. Since its humble beginning, the transistor has been improved and decreased in size to the point where now a circuit containing thousands of transistors can easily fit into an area not much bigger than the head of a pin.

The advent of the transistor has permitted the construction of elaborate spacecraft, unprecedented communication, and new forms of energy generation.

Shockley, Bardeen, and Brattain received the 1956 Nobel Prize in Physics for their discovery of the transistor.

## 20.1 Superposition Theorem—Independent Sources

The superposition theorem states the following:

*The voltage across (or current through) an element is determined by summing the voltage (or current) due to each independent source.*

In order to apply this theorem, all sources other than the one being considered are eliminated. As in dc circuits, this is done by replacing current sources with open circuits and by replacing voltage sources with short circuits. The process is repeated until the effects due to all sources have been determined.

Although we generally work with circuits having all sources at the same frequency, occasionally a circuit may operate at more than one frequency at a time. This is particularly true in diode and transistor circuits which use a dc source to set a "bias" (or operating) point and an ac source to provide the signal to be conditioned or amplified. In such cases, the resulting voltages or currents are still determined by applying the superposition theorem. The topic of how to solve circuits operating at several different frequencies simultaneously is covered in Chapter 25.

**NOTES...**

As in dc circuits, the superposition theorem can be applied only to voltage and current; it cannot be used to solve for the total power dissipated by an element. This is because power is not a linear quantity, but rather follows a square-law relationship ($P = V^2/R = I^2R$).

**EXAMPLE 20–1** Determine the current **I** in Figure 20–1 by using the superposition theorem.

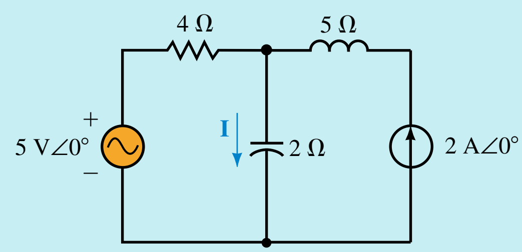



**FIGURE 20–1**

*Solution*

**Current due to the 5 V∠0° voltage source:** Eliminating the current source, we obtain the circuit shown in Figure 20–2.

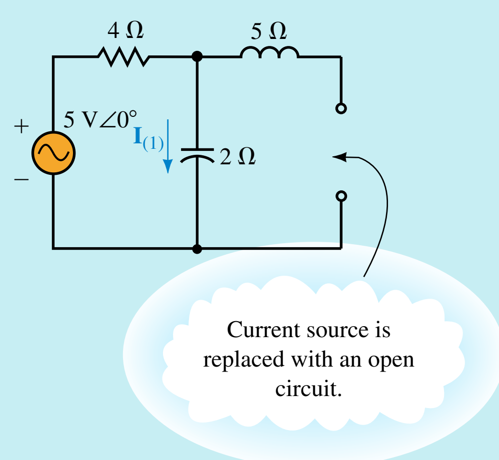



**FIGURE 20–2**

Applying Ohm's law, we have

$$\mathbf{I}_{(1)} = \frac{5\text{ V}\angle 0^\circ}{4 - j2\ \Omega} = \frac{5\text{ V}\angle 0^\circ}{4.472\Omega\angle -26.57^\circ}$$
$$= 1.118\text{ A}\angle 26.57^\circ$$

**Current due to the 2 A∠0° current source:** Eliminating the voltage source, we obtain the circuit shown in Figure 20–3.

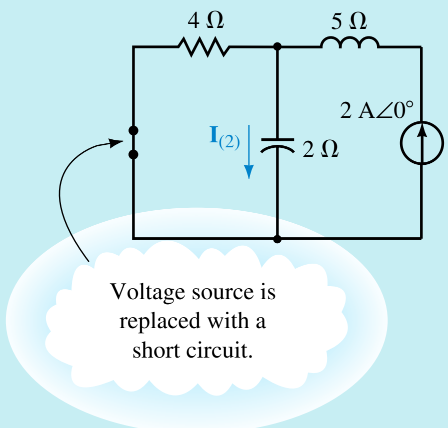



**FIGURE 20–3**

The current $\mathbf{I}_{(2)}$ due to this source is determined by applying the current divider rule:

$$\mathbf{I}_{(2)} = (2\text{ A}\angle 0^\circ)\frac{4\ \Omega\angle 0^\circ}{4\ \Omega - j2\ \Omega}$$
$$= \frac{8\text{ V}\angle 0^\circ}{4.472\ \Omega\angle -26.57^\circ}$$
$$= 1.789\text{ A}\angle 26.57^\circ$$

The total current is determined as the summation of currents $\mathbf{I}_{(1)}$ and $\mathbf{I}_{(2)}$:

$$\begin{aligned}\mathbf{I} &= \mathbf{I}_{(1)} + \mathbf{I}_{(2)}\\ &= 1.118\text{ A}\angle 26.57^\circ + 1.789\text{ A}\angle 26.57^\circ\\ &= (1.0\text{ A} + j0.5\text{ A}) + (1.6\text{ A} + j0.8\text{ A})\\ &= 2.6 + j1.3\text{ A}\\ &= 2.91\text{ A}\angle 26.57^\circ\end{aligned}$$

**EXAMPLE 20–2** Consider the circuit of Figure 20–4:

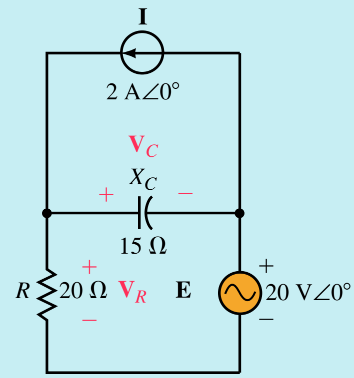



**FIGURE 20–4**

Find the following:

a. $\mathbf{V}_R$ and $\mathbf{V}_C$ using the superposition theorem.
b. Power dissipated by the circuit.
c. Power delivered to the circuit by each of the sources.

**Solution**

a. The superposition theorem may be employed as follows:

**Voltages due to the current source:** Eliminating the voltage source, we obtain the circuit shown in Figure 20–5.

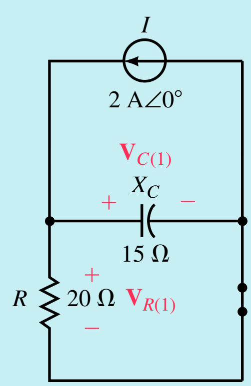



**FIGURE 20–5**

The impedance "seen" by the current source will be the parallel combination of $\mathbf{R}\|\mathbf{Z}_C$.

$$\mathbf{Z}_1 = \frac{(20\ \Omega)(-j15\ \Omega)}{20\ \Omega - j15\ \Omega} = \frac{300\ \Omega\angle -90^\circ}{25\ \Omega\angle -36.87^\circ} = 12\ \Omega\angle -53.13^\circ$$

The voltage $\mathbf{V}_{R(1)}$ is the same as the voltage across the capacitor, $\mathbf{V}_{C(1)}$. Hence,

$$\begin{aligned}\mathbf{V}_{R(1)} &= \mathbf{V}_{C(1)}\\ &= (2\ \mathrm{A}\angle 0^\circ)(12\ \Omega\angle -53.13^\circ)\\ &= 24\ \mathrm{V}\angle -53.13^\circ\end{aligned}$$

**Voltages due to the voltage source:** Eliminating the current source, we have the circuit shown in Figure 20–6.

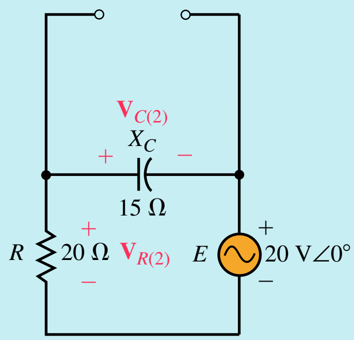



**FIGURE 20–6**

The voltages $\mathbf{V}_{R(2)}$ and $\mathbf{V}_{C(2)}$ are determined by applying the voltage divider rule,

$$\mathbf{V}_{R(2)} = \frac{20\ \Omega\angle 0°}{20\ \Omega - j15\ \Omega}(20\ \text{V}\angle 0°)$$
$$= \frac{400\ \text{V}\angle 0°}{25\ \angle -36.87°} = 16\ \text{V}\angle +36.87°$$

and

$$\mathbf{V}_{C(2)} = \frac{-15\ \Omega\angle -90°}{20\ \Omega - j15\ \Omega}(20\ \text{V}\angle 0°)$$
$$= \frac{300\ \text{V}\angle 90°}{25\ \angle -36.87°} = 12\ \text{V}\angle 126.87°$$

Notice that $\mathbf{V}_{C(2)}$ is assigned to be negative relative to the originally assumed polarity. The negative sign is eliminated from the calculation by adding (or subtracting) 180° from the corresponding calculation.

By applying superposition, we get

$$\begin{aligned}\mathbf{V}_R &= \mathbf{V}_{R(1)} + \mathbf{V}_{R(2)}\\ &= 24\ \text{V}\angle -53.13° + 16\ \text{V}\angle 36.87°\\ &= (14.4\ \text{V} - j19.2\ \text{V}) + (12.8\ \text{V} + j9.6\ \text{V})\\ &= 27.2\ \text{V} - j9.6\ \text{V}\\ &= 28.84\ \text{V}\angle -19.44°\end{aligned}$$

and

$$\begin{aligned}\mathbf{V}_C &= \mathbf{V}_{C(1)} + \mathbf{V}_{C(2)}\\ &= 24\ \text{V}\angle -53.13° + 12\ \text{V}\angle 126.87°\\ &= (14.4\ \text{V} - j19.2\ \text{V}) + (-7.2\ \text{V} + j9.6\ \text{V})\\ &= 7.2\ \text{V} - j9.6\ \text{V}\\ &= 12\ \text{V}\angle -53.13°\end{aligned}$$

b. Since only the resistor will dissipate power, the total power dissipated by the circuit is found as

$$P_{\text{T}} = \frac{(28.84\ \text{V})^2}{20\ \Omega} = 41.60\ \text{W}$$

c. The power delivered to the circuit by the current source is

$$P_1 = V_1 I \cos \theta_1$$

where $\mathbf{V}_1 = \mathbf{V}_C = 12\text{ V}\angle{-53.13°}$ is the voltage across the current source and $\theta_1$ is the phase angle between $\mathbf{V}_1$ and $\mathbf{I}$.

The power delivered by the current source is

$$P_1 = (12\text{ V})(2\text{ A}) \cos 53.13° = 14.4\text{ W}$$

The power delivered to the circuit by the voltage source is similarly determined as

$$P_2 = EI_2 \cos \theta_2$$

where $\mathbf{I}_2$ is the current through the voltage source and $\theta_2$ is the phase angle between $\mathbf{E}$ and $\mathbf{I}_2$.

$$P_2 = (20\text{ V})\left(\frac{28.84\text{ V}}{20\ \Omega}\right) \cos 19.44° = 27.2\text{ W}$$

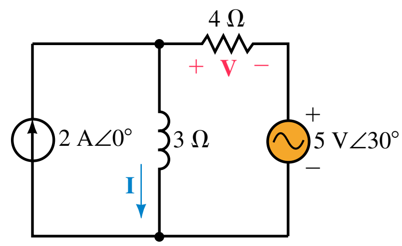
As expected, the total power delivered to the circuit must be the summation

$$P_{\text{T}} = P_1 + P_2 = 41.6\text{ W}$$

**PRACTICE PROBLEMS 1**

Use superposition to find **V** and **I** for the circuit of Figure 20–7.

**FIGURE 20–7**




*Answers:* $\mathbf{I} = 2.52\text{ A}\angle{-25.41°}$, $\mathbf{V} = 4.45\text{ V}\angle 104.18°$

**IN-PROCESS LEARNING CHECK 1**

A 20-Ω resistor is in a circuit having three sinusoidal sources. After analyzing the circuit, it is found that the current through the resistor due to each of the sources is as follows:

$$I_1 = 1.5\text{ A}\angle 20°$$
$$I_2 = 1.0\text{ A}\angle 110°$$
$$I_3 = 2.0\text{ A}\angle 0°$$

a. Use superposition to calculate the resultant current through the resistor.
b. Calculate the power dissipated by the resistor.
c. Show that the power dissipated by the resistor cannot be found by applying superposition, namely, $P_{\text{T}} \neq I_1^2R + I_2^2R + I_3^2R$.

*(Answers are at the end of the chapter.)*

## 20.2 Superposition Theorem—Dependent Sources

Chapter 19 introduced the concept of dependent sources. We now examine ac circuits which are powered by dependent sources. In order to analyze circuits having dependent sources, it is first necessary to determine whether the dependent source is conditional upon a controlling element in its own circuit or whether the controlling element is located in some other circuit.

If the controlling element is external to the circuit under consideration, the method of analysis is the same as for an independent source. However, if the controlling element is in the same circuit, the analysis follows a slightly different stategy. The next two examples show the techniques used to analyze circuits having dependent sources.

**EXAMPLE 20–3** Consider the circuit of Figure 20–8.

a. Determine the general expression for **V** in terms of **I.**
b. Calculate **V** if **I** = 1.0 A∠0°.
c. Calculate **V** if **I** = 0.3 A∠90°.

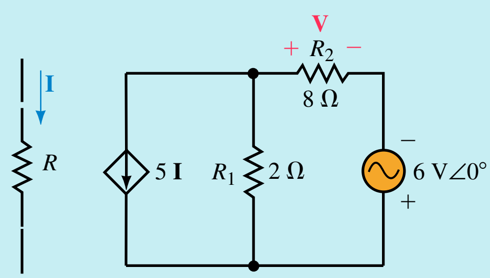



**FIGURE 20–8**

**Solution**

a. Since the current source in the circuit is dependent on current through an element which is located outside of the circuit of interest, the circuit may be analyzed in the same manner as for independent sources.

**Voltage due to the voltage source:** Eliminating the current source, we obtain the circuit shown in Figure 20–9.

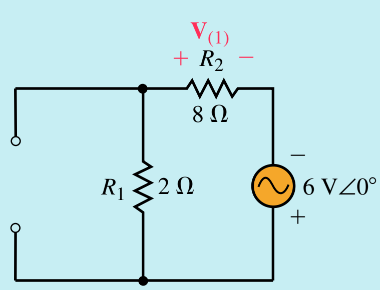



**FIGURE 20–9**

$$\mathbf{V}_{(1)} = \frac{8\ \Omega}{10\ \Omega}(6\ \text{V}\angle 0^\circ) = 4.8\ \text{V}\angle 0^\circ$$

**Voltage due to the current source:** Eliminating the voltage source, we have the circuit shown in Figure 20–10.

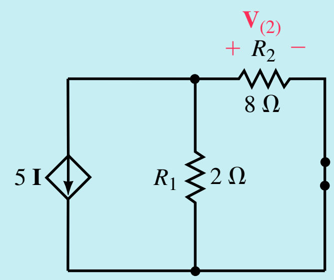



FIGURE 20–10

$$\mathbf{Z}_T = 2\ \Omega \| 8\ \Omega = 1.6\ \Omega\angle 0°$$
$$\mathbf{V}_{(2)} = \mathbf{V}_{Z_T} = -(5\mathbf{I})(1.6\ \Omega\angle 0°) = -8.0\ \Omega\mathbf{I}$$

From superposition, the general expression for voltage is determined to be

$$\begin{aligned}\mathbf{V} &= \mathbf{V}_{(1)} + \mathbf{V}_{(2)}\\ &= 4.8\ \text{V}\angle 0° - 8.0\ \Omega\mathbf{I}\end{aligned}$$

b. If $\mathbf{I} = 1.0\ \text{A}\angle 0°$,

$$\begin{aligned}\mathbf{V} &= 4.8\ \text{V}\angle 0° - (8.0\ \Omega)(1.0\ \text{A}\angle 0°) = -3.2\ \text{V}\\ &= 3.2\ \text{V}\angle 180°\end{aligned}$$

c. If $\mathbf{I} = 0.3\ \text{A}\angle 90°$,

$$\begin{aligned}\mathbf{V} &= 4.8\ \text{V}\angle 0° - (8.0\ \Omega)(0.3\ \text{A}\angle 90°) = 4.8\ \text{V} - j2.4\ \text{V}\\ &= 5.367\ \text{V}\angle -26.57°\end{aligned}$$

EXAMPLE 20–4 Given the circuit of Figure 20–11, calculate the voltage across the 40-Ω resistor.

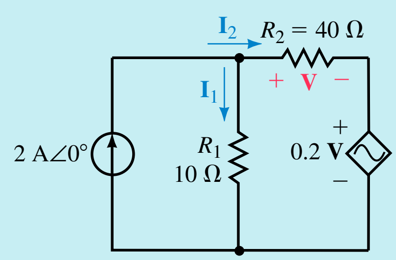



FIGURE 20–11

*Solution* In the circuit of Figure 20–11, the dependent source is controlled by an element located in the circuit. Unlike the sources in the previous examples, the dependent source cannot be eliminated from the circuit since doing so would contradict Kirchhoff's voltage law and/or Kirchhoff's current law.

The circuit must be analyzed by considering all effects simultaneously.

Applying Kirchhoff's current law, we have

$$\mathbf{I}_1 + \mathbf{I}_2 = 2\ \text{A}\angle 0°$$

From Kirchhoff's voltage law, we have

$$(10\ \Omega)\,\mathbf{I}_1 = \mathbf{V} + 0.2\,\mathbf{V} = 1.2\,\mathbf{V}$$
$$\mathbf{I}_1 = 0.12\,\mathbf{V}$$

and,

$$\mathbf{I}_2 = \frac{\mathbf{V}}{40\ \Omega} = 0.025\,\mathbf{V}$$

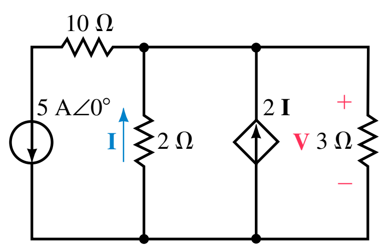
Combining the above expressions, we have

$$0.12\,\mathbf{V} + 0.025\,\mathbf{V} = 2.0\ \text{A}\angle 0^\circ$$
$$0.145\,\mathbf{V} = 2.0\ \text{A}\angle 0^\circ$$
$$\mathbf{V} = 13.79\ \text{V}\angle 0^\circ$$

PRACTICE PROBLEMS 2

Determine the voltage **V** in the circuit of Figure 20–12.

**FIGURE 20–12**




*Answer:* $\mathbf{V} = 2.73\ \text{V}\angle 180^\circ$

## 20.3 Thévenin's Theorem—Independent Sources

Thévenin's theorem is a method which converts any linear bilateral ac circuit into a single ac voltage source in series with an equivalent impedance as shown in Figure 20–13.

The resulting two-terminal network will be equivalent when it is connected to any external branch or component. If the original circuit contains reactive elements, the Thévenin equivalent circuit will be valid only at the frequency at which the reactances were determined. The following method may be used to determine the Thévenin equivalent of an ac circuit having either independent sources or sources which are dependent upon voltage or current in some other circuit. The outlined method may not be used in circuits having dependent sources controlled by voltage or current in the same circuit.

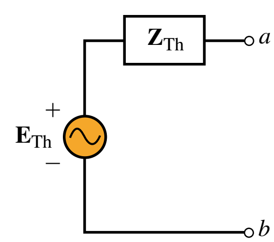



**FIGURE 20–13** Thévenin equivalent circuit.

1. Remove the branch across which the Thévenin equivalent circuit is to be found. Label the resulting two terminals. Although any designation will do, we will use the notations *a* and *b*.
2. Set all sources to zero. As in dc circuits, this is achieved by replacing voltage sources with short circuits and current sources with open circuits.

3. Determine the Thévenin equivalent impedance, $\mathbf{Z}_{\mathrm{Th}}$ by calculating the impedance seen between the open terminals *a* and *b*. Occasionally it may be necessary to redraw the circuit to simplify this process.
4. Replace the sources removed in Step 3 and determine the open-circuit voltage across the terminals *a* and *b*. If any of the sources are expressed in sinusoidal form, it is first necessary to convert these sources into an equivalent phasor form. For circuits having more than one source, it may be necessary to apply the superposition theorem to calculate the open-circuit voltage. Since all voltages will be phasors, the resultant is found by using vector algebra. The open-circuit voltage is the Thévenin voltage, $\mathbf{E}_{\mathrm{Th}}$.
5. Sketch the resulting Thévenin equivalent circuit by including that portion of the circuit removed in Step 1.

**EXAMPLE 20–5** Find the Thévenin equivalent circuit external to $\mathbf{Z}_L$ for the circuit of Figure 20–14.

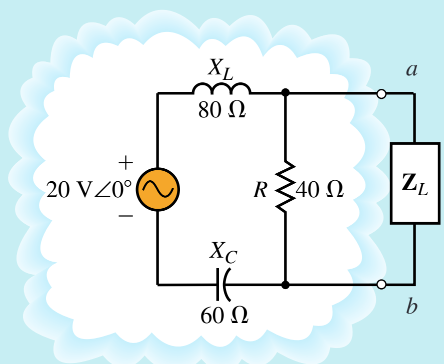



**FIGURE 20–14**

**Solution**

**Steps 1 and 2:** Removing the load impedance $\mathbf{Z}_L$ and setting the voltage source to zero, we have the circuit of Figure 20–15.

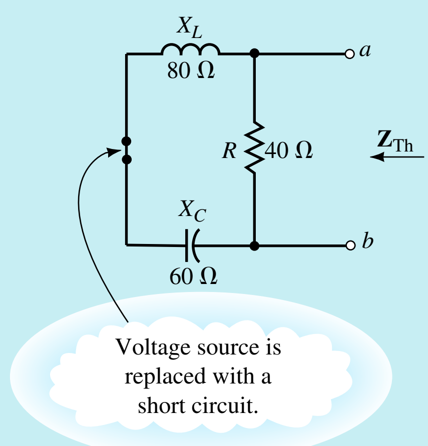



**FIGURE 20–15**

**Step 3:** The Thévenin impedance between terminals *a* and *b* is found as

$$
\begin{aligned}
\mathbf{Z}_{\mathrm{Th}} &= \mathbf{R}\|(\mathbf{Z}_L + \mathbf{Z}_C) \\
&= \frac{(40\ \Omega\angle 0^\circ)(20\ \Omega\angle 90^\circ)}{40\ \Omega + j20\ \Omega} \\
&= \frac{800\ \Omega\angle 90^\circ}{44.72\ \Omega\angle 26.57^\circ} \\
&= 17.89\ \Omega\angle 63.43^\circ \\
&= 8\ \Omega + j16\ \Omega
\end{aligned}
$$

**Step 4:** The Thévenin voltage is found by using the voltage divider rule as shown in the circuit of Figure 20–16.

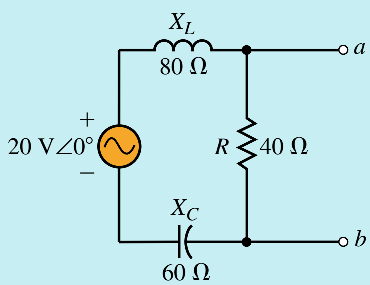



**FIGURE 20–16**

$$
\begin{aligned}
\mathbf{E}_{\mathrm{Th}} = \mathbf{V}_{ab} &= \frac{40\ \Omega\angle 0^\circ}{40\ \Omega + j80\ \Omega - j60\ \Omega}(20\ \mathrm{V}\angle 0^\circ) \\
&= \frac{800\ \mathrm{V}\angle 0^\circ}{44.72\ \Omega\angle 26.57^\circ} \\
&= 17.89\ \mathrm{V}\angle -26.57^\circ
\end{aligned}
$$

**Step 5:** The resultant Thévenin equivalent circuit is shown in Figure 20–17.

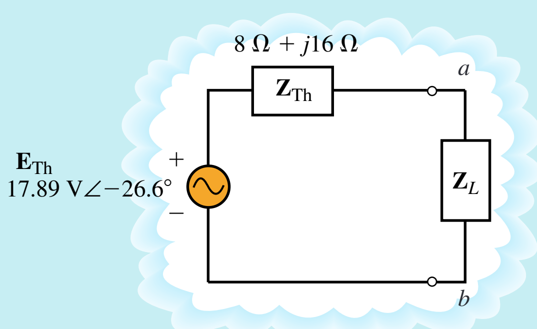



**FIGURE 20–17**

**EXAMPLE 20–6** Determine the Thévenin equivalent circuit external to $\mathbf{Z}_L$ in the circuit in Figure 20–18.

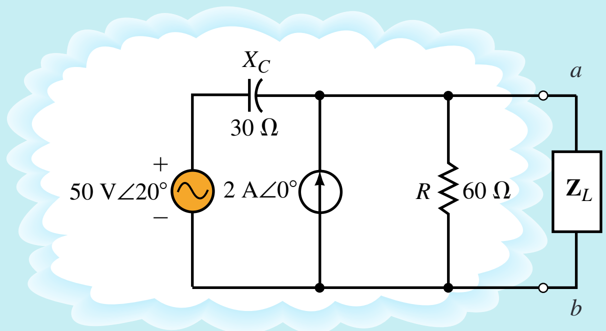



**FIGURE 20–18**

**Solution**

**Step 1:** Removing the branch containing $\mathbf{Z}_L$, we have the circuit of Figure 20–19.

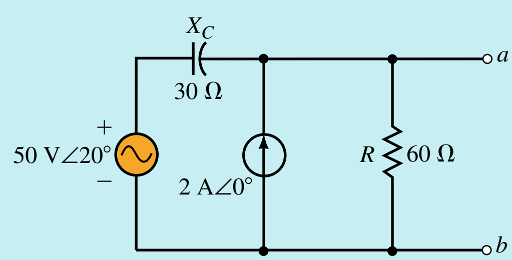



**FIGURE 20–19**

**Step 2:** After setting the voltage and current sources to zero, we have the circuit of Figure 20–20.

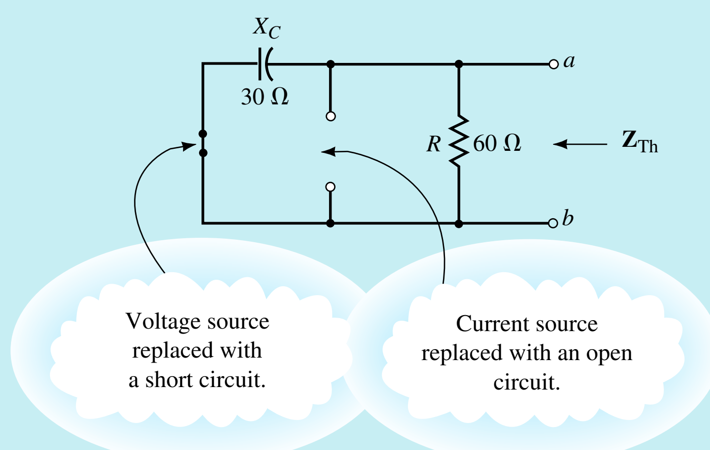



**FIGURE 20–20**

**Step 3:** The Thévenin impedance is determined as

$$\begin{aligned}\mathbf{Z}_{\mathrm{Th}} &= \mathbf{Z}_C \| \mathbf{Z}_R \\ &= \frac{(30\ \Omega\angle -90^\circ)(60\ \Omega\angle 0^\circ)}{60\ \Omega - j30\ \Omega} \\ &= \frac{1800\ \Omega\angle -90^\circ}{67.08\ \Omega\angle -26.57^\circ} \\ &= 26.83\ \Omega\angle -63.43^\circ\end{aligned}$$

**Step 4:** Because the given network consists of two independent sources, we consider the individual effects of each upon the open-circuit voltage. The total effect is then easily determined by applying the superposition theorem. Reinserting only the voltage source into the original circuit, as shown in Figure 20–21, allows us to find the open-circuit voltage, $\mathbf{V}_{ab(1)}$, by applying the voltage divider rule:

$$\begin{aligned}\mathbf{V}_{ab(1)} &= \frac{60\ \Omega}{60\ \Omega - j30\ \Omega}(50\ \mathrm{V}\angle 20^\circ) \\ &= \frac{3000\ \mathrm{V}\angle 20^\circ}{67.08\angle -26.57^\circ} \\ &= 44.72\ \mathrm{V}\angle 46.57^\circ\end{aligned}$$

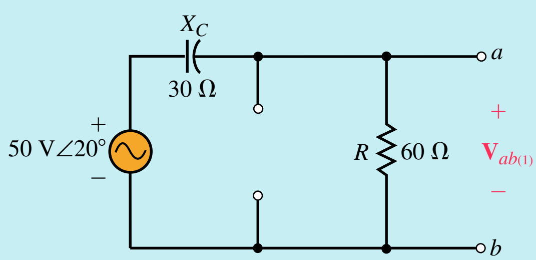



**FIGURE 20–21**

Now, considering only the current source as shown in Figure 20–22, we determine $\mathbf{V}_{ab(2)}$ by Ohm's law:

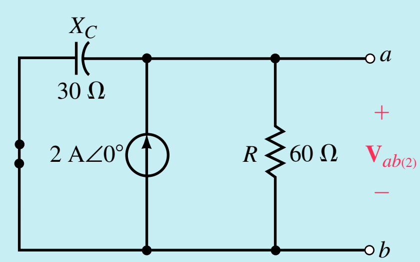



**FIGURE 20–22**

$$\begin{aligned}\mathbf{V}_{ab(2)} &= \frac{(2\ \mathrm{A}\angle 0^\circ)(30\ \Omega\angle -90^\circ)(60\ \Omega\angle 0^\circ)}{60\ \Omega - j30\ \Omega} \\ &= (2\ \mathrm{A}\angle 0^\circ)(26.83\ \Omega\angle -63.43^\circ) \\ &= 53.67\ \mathrm{V}\angle -63.43^\circ\end{aligned}$$

From the superposition theorem, the Thévenin voltage is determined as

$$\begin{aligned}\mathbf{E}_{\mathrm{Th}} &= \mathbf{V}_{ab(1)} + \mathbf{V}_{ab(2)}\\ &= 44.72\ \mathrm{V}\angle 46.57° + 53.67\ \mathrm{V}\angle -63.43°\\ &= (30.74\ \mathrm{V} + j32.48\ \mathrm{V}) + (24.00\ \mathrm{V} - j48.00\ \mathrm{V})\\ &= (54.74\ \mathrm{V} - j15.52\ \mathrm{V}) = 56.90\ \mathrm{V}\angle -15.83°\end{aligned}$$

**Step 5:** The resulting Thévenin equivalent circuit is shown in Figure 20–23.

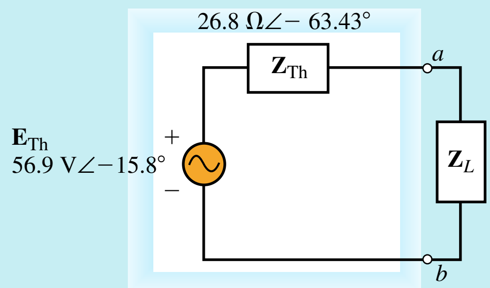



**FIGURE 20–23**

IN-PROCESS LEARNING CHECK 2

Refer to the circuit shown in Figure 20–24 of Practice Problem 3. List the steps that you would use to find the Thévenin equivalent circuit.

*(Answers are at the end of the chapter.)*

PRACTICE PROBLEMS 3

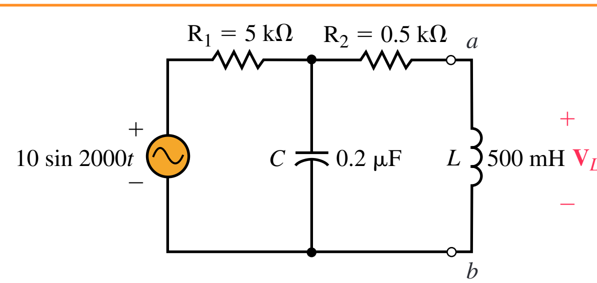



**FIGURE 20–24**

a. Find the Thévenin equivalent circuit external to the inductor in the circuit in Figure 20–24. (Notice that the voltage source is shown as sinusoidal.)

b. Use the Thévenin equivalent circuit to find the phasor output voltage, $\mathbf{V}_L$.

c. Convert the answer of (b) into the equivalent sinusoidal voltage.

*Answers:*

a. $\mathbf{Z}_{\mathrm{Th}} = 1.5\ \mathrm{k\Omega} - j2.0\ \mathrm{k\Omega} = 2.5\ \mathrm{k\Omega}\angle -53.13°$, $\mathbf{E}_{\mathrm{Th}} = 3.16\ \mathrm{V}\angle -63.43°$

b. $\mathbf{V}_L = 1.75\ \mathrm{V}\angle 60.26°$

c. $v_L = 2.48 \sin(2000t + 60.26°)$

## 20.4 Norton's Theorem—Independent Sources

Norton's theorem converts any linear bilateral network into an equivalent circuit consisting of a single current source and a parallel impedance as shown in Figure 20–25.

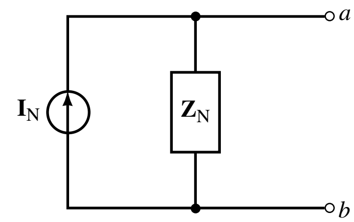



**FIGURE 20–25** Norton equivalent circuit.

Although Norton's equivalent circuit may be determined by first finding the Thévenin equivalent circuit and then performing a source conversion, we generally use the more direct method outlined below. The steps to find the Norton equivalent cirucuit are as follows:

1. Remove the branch across which the Norton equivalent circuit is to be found. Label the resulting two terminals *a* and *b*.
2. Set all sources to zero.
3. Determine the Norton equivalent impedance, $\mathbf{Z}_N$, by calculating the impedance seen between the open terminals *a* and *b*.

   *NOTE: Since the previous steps are identical with those followed for finding the Thévenin equivalent circuit, we conclude that the Norton impedance must be the same as the Thévenin impedance.*
4. Replace the sources removed in Step 3 and determine the current that would occur between terminals *a* and *b* if these terminals were shorted. Any voltages and currents that are given in sinusoidal notation must first be expressed in equivalent phasor notation. If the circuit has more than one source it may be necessary to apply the superposition theorem to calculate the total short-circuit current. Since all currents will be in phasor form, any addition must be done using vector algebra. The resulting current is the Norton current, $I_N$.
5. Sketch the resulting Norton equivalent circuit by inserting that portion of the circuit removed in Step 1.

As mentioned previously, it is possible to find the Norton equivalent circuit from the Thévenin equivalent by simply performing a source conversion. We have already determined that both the Thévenin and Norton impedances are determined in the same way. Consequently, the impedances must be equivalent, and so we have

$$\mathbf{Z}_N = \mathbf{Z}_{Th} \tag{20–1}$$

Now, applying Ohm's law, we determine the Norton current source from the Thévenin voltage and impedance, namely,

$$\mathbf{I}_N = \frac{\mathbf{E}_{Th}}{\mathbf{Z}_{Th}} \tag{20–2}$$

Figure 20–26 shows the equivalent circuits.

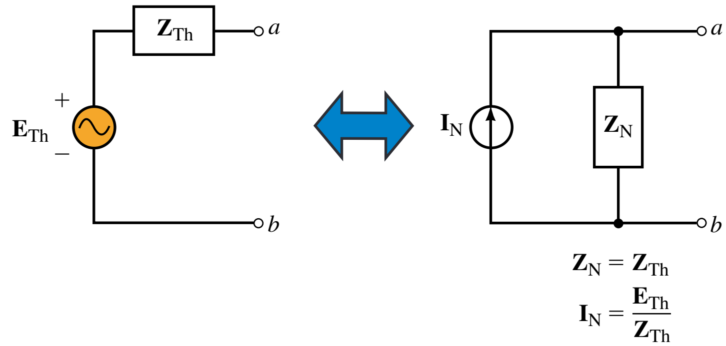



**FIGURE 20–26**

**EXAMPLE 20–7** Given the circuit of Figure 20–27, find the Norton equivalent.

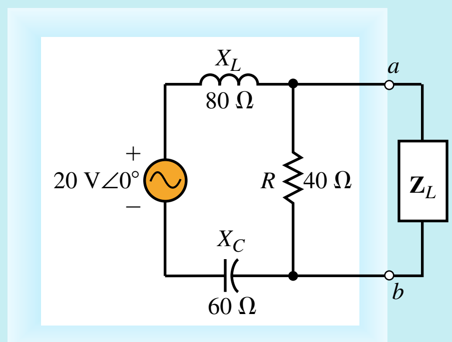



**FIGURE 20–27**

**Solution**

**Steps 1 and 2:** By removing the load impedance, $\mathbf{Z}_L$, and setting the voltage source to zero, we have the network of Figure 20–28.

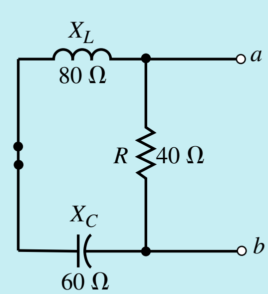



**FIGURE 20–28**

**Step 3:** The Norton impedance may now be determined by evaluating the impedance between terminals *a* and *b*. Hence, we have

$$\begin{aligned}\mathbf{Z}_\mathrm{N} &= \frac{(40\ \Omega\angle 0^\circ)(20\ \Omega\angle 90^\circ)}{40\ \Omega + j20\ \Omega} \\ &= \frac{800\ \Omega\angle 90^\circ}{44.72\angle 26.57^\circ} \\ &= 17.89\ \Omega\angle 63.43^\circ \\ &= 8\ \Omega + j16\ \Omega\end{aligned}$$

**Step 4:** Reinserting the voltage source, as in Figure 20–29, we find the Norton current by calculating the current between the shorted terminals, *a* and *b*.

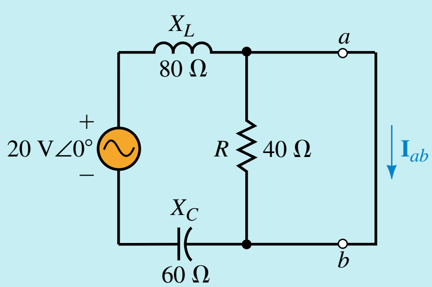



**FIGURE 20–29**

Because the resistor $R = 40\ \Omega$ is shorted, the current is determined by the impedances $X_L$ and $X_C$ as

$$\begin{aligned}\mathbf{I}_\mathrm{N} = \mathbf{I}_{ab} &= \frac{20\ \mathrm{V}\angle 0^\circ}{j80\ \Omega - j60\ \Omega} \\ &= \frac{20\ \mathrm{V}\angle 0^\circ}{20\ \Omega\angle -90^\circ} \\ &= 1.00\ \mathrm{A}\angle -90^\circ\end{aligned}$$

**Step 5:** The resultant Norton equivalent circuit is shown in Figure 20–30.

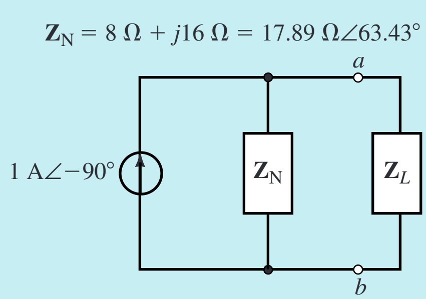



**FIGURE 20–30**

**EXAMPLE 20–8** Find the Norton equivalent circuit external to $R_L$ in the circuit of Figure 20–31. Use the equivalent circuit to calculate the current $\mathbf{I}_L$ when $R_L = 0\ \Omega$, $400\ \Omega$, and $2\ \mathrm{k\Omega}$.

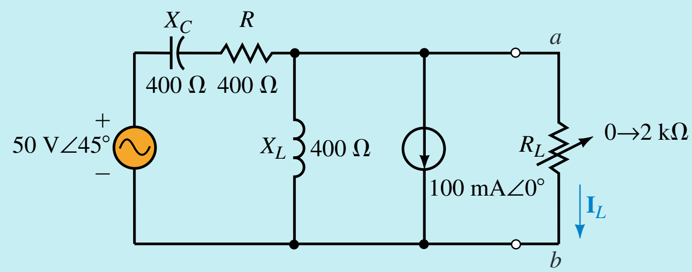



**FIGURE 20–31**

**Solution**

**Steps 1 and 2:** Removing the load resistor and setting the sources to zero, we obtain the network shown in Figure 20–32.

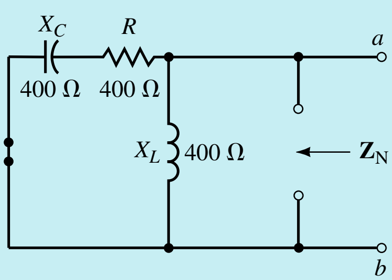



**FIGURE 20–32**

**Step 3:** The Norton impedance is determined as

$$\begin{aligned}\mathbf{Z}_\mathrm{N} &= \frac{(400\ \Omega\angle 90^\circ)(400\ \Omega - j400\ \Omega)}{j400\ \Omega + 400\ \Omega - j400\ \Omega}\\ &= \frac{(400\ \Omega\angle 90^\circ)(565.69\ \Omega\angle -45^\circ)}{400\ \Omega\angle 0^\circ}\\ &= 565.69\ \Omega\angle +45^\circ\end{aligned}$$

**Step 4:** Because the network consists of two sources, we determine the effects due to each source separately and then apply superposition to evaluate the Norton current source.

Reinserting the voltage source into the original network, we see from Figure 20–33 that the short-circuit current between the terminals *a* and *b* is easily found by using Ohm's law.

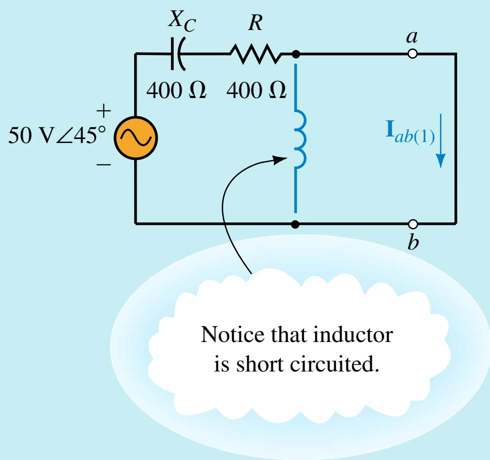



**FIGURE 20–33**

$$\begin{aligned}\mathbf{I}_{ab(1)} &= \frac{50\ \text{V}\angle 45^\circ}{400\ \Omega - j400\ \Omega}\\ &= \frac{50\ \text{V}\angle 45^\circ}{565.69\ \Omega\angle -45^\circ}\\ &= 88.4\ \text{mA}\angle 90^\circ\end{aligned}$$

Since short circuiting the current source effectively removes all impedances, as illustrated in Figure 20–34, the short-circuit current between the terminals $a$ and $b$ is given as follows:

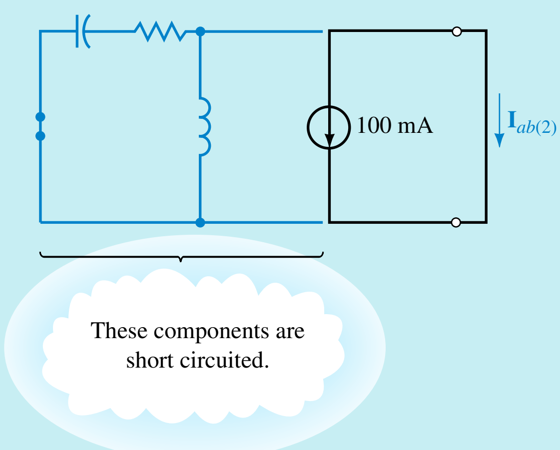



**FIGURE 20–34**

$$\begin{aligned}\mathbf{I}_{ab(2)} &= -100\ \text{mA}\angle 0^\circ\\ &= 100\ \text{mA}\angle 180^\circ\end{aligned}$$

Now applying the superposition theorem, the Norton current is determined as the summation

$$\begin{aligned}\mathbf{I}_N &= \mathbf{I}_{ab(1)} + \mathbf{I}_{ab(2)}\\ &= 88.4\text{ mA}\angle 90^\circ + 100\text{ mA}\angle 180^\circ\\ &= -100\text{ mA} + j88.4\text{ mA}\\ &= 133.5\text{ mA}\angle 138.52^\circ\end{aligned}$$

**Step 5:** The resulting Norton equivalent circuit is shown in Figure 20–35.

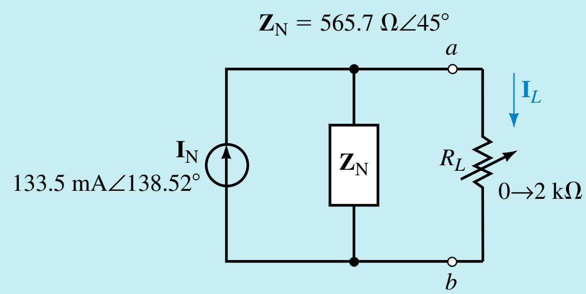



**FIGURE 20–35**

From the above circuit, we express the current through the load, $\mathbf{I}_L$, as

$$\mathbf{I}_L = \frac{\mathbf{Z}_N}{R_L + \mathbf{Z}_N}\mathbf{I}_N$$

$R_L =$ **0 Ω:**

$$\mathbf{I}_L = \mathbf{I}_N = 133.5\text{ mA}\angle 138.52^\circ$$

$R_L =$ **400 Ω:**

$$\begin{aligned}\mathbf{I}_L &= \frac{\mathbf{Z}_N}{R_L + \mathbf{Z}_N}\mathbf{I}_N\\ &= \frac{(565.7\ \Omega\angle 45^\circ)(133.5\text{ mA}\angle 138.52^\circ)}{400\ \Omega + 400\ \Omega + j400\ \Omega}\\ &= \frac{75.24\text{ V}\angle 183.52^\circ}{894.43\ \Omega\angle 26.57^\circ}\\ &= 84.12\text{ mA}\angle 156.95^\circ\end{aligned}$$

$R_L =$ **2 kΩ:**

$$\begin{aligned}\mathbf{I}_L &= \frac{\mathbf{Z}_N}{R_L + \mathbf{Z}_N}\mathbf{I}_N\\ &= \frac{(565.7\ \Omega\angle 45^\circ)(133.5\text{ mA}\angle 138.52^\circ)}{2000\ \Omega + 400\ \Omega + j400\ \Omega}\\ &= \frac{75.24\text{ V}\angle 183.52^\circ}{2433.1\ \Omega\angle 9.46^\circ}\\ &= 30.92\text{ mA}\angle 174.06^\circ\end{aligned}$$

IN-PROCESS LEARNING CHECK 3

Refer to the circuit shown in Figure 20–36 (Practice Problem 4). List the steps that you would use to find the Norton equivalent circuit.

*(Answers are at the end of the chapter.)*

PRACTICE PROBLEMS 4

Find the Norton equivalent circuit external to $R_L$ in the circuit of Figure 20–36. Use the equivalent circuit to find the current $\mathbf{I}_L$.

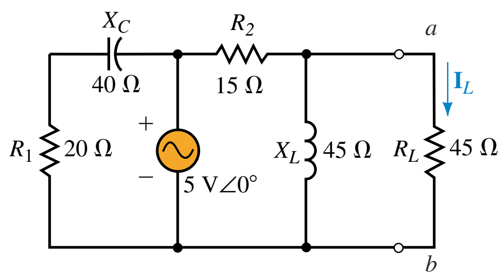



**FIGURE 20–36**

*Answers:* $\mathbf{Z}_N = 13.5\ \Omega + j4.5\ \Omega = 14.23\ \Omega\angle 18.43°$, $\mathbf{I}_N = 0.333\ \text{A}\angle 0°$
$\mathbf{I}_L = 0.0808\ \text{A}\angle 14.03°$

## 20.5 Thévenin's and Norton's Theorem for Dependent Sources

If a circuit contains a dependent source which is controlled by an element outside the circuit of interest, the methods outlined in Sections 20.2 and 20.3 are used to find either the Thévenin or Norton equivalent circuit.

**EXAMPLE 20–9** Given the circuit of Figure 20–37, find the Thévenin equivalent circuit external to $R_L$. If the voltage applied to the resistor $R_1$ is 10 mV, use the Thévenin equivalent circuit to calculate the minimum and maximum voltage across $R_L$.

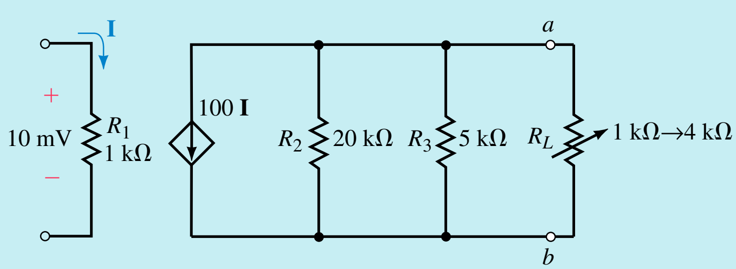



**FIGURE 20–37**

**Solution**

**Step 1:** Removing the load resistor from the circuit and labelling the remaining terminals $a$ and $b$, we have the circuit shown in Figure 20–38.

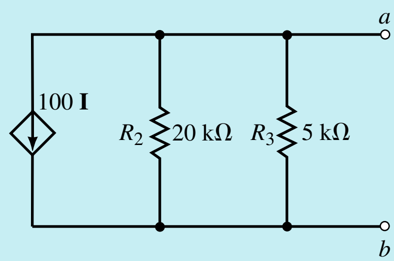



**FIGURE 20–38**

**Steps 2 and 3:** The Thévenin resistance is found by open circuiting the current source and calculating the impedance observed between the terminals $a$ and $b$. Since the circuit is purely resistive, we have

$$\begin{aligned} R_{\text{Th}} &= 20\ \text{k}\Omega \| 5\ \text{k}\Omega \\ &= \frac{(20\ \text{k}\Omega)(5\ \text{k}\Omega)}{20\ \text{k}\Omega + 5\ \text{k}\Omega} \\ &= 4\ \text{k}\Omega \end{aligned}$$

**Step 4:** The open-circuit voltage between the terminals is found to be

$$\begin{aligned} \mathbf{V}_{ab} &= -(100\mathbf{I})(4\ \text{k}\Omega) \\ &= -(4 \times 10^5\ \Omega)\mathbf{I} \end{aligned}$$

As expected, the Thévenin voltage source is dependent upon the current **I.**

**Step 5:** Because the Thévenin voltage is a dependent voltage source we use the appropriate symbol when sketching the equivalent circuit, as shown in Figure 20–39.

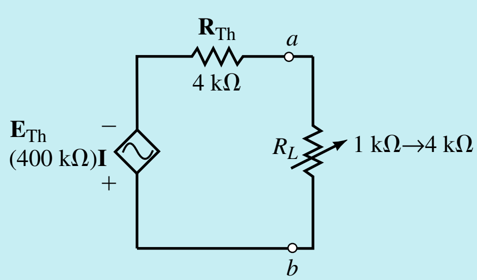



**FIGURE 20–39**

For the given conditions, we have

$$I = \frac{10\ \text{mV}}{1\ \text{k}\Omega} = 10\ \mu\text{A}$$

The voltage across the load is now determined as follows:

$$R_L = \mathbf{1\ k\Omega}: \qquad V_{ab} = -\frac{1\ \text{k}\Omega}{1\ \text{k}\Omega + 4\ \text{k}\Omega}(4 \times 10^5\ \Omega)(10\ \mu\text{A})$$
$$= -0.8\ \text{V}$$

$$R_L = \mathbf{4\ k\Omega}: \qquad V_{ab} = -\frac{4\ \text{k}\Omega}{4\ \text{k}\Omega + 4\ \text{k}\Omega}(4 \times 10^5\ \Omega)(10\ \mu\text{A})$$
$$= -2.0\ \text{V}$$

For an applied voltage of 10 mV, the voltage across the load resistance will vary between 0.8 V and 2.0 V as $R_L$ is adjusted between 1 kΩ and 4 kΩ.

*If a circuit contains one or more dependent sources which are controlled by an element in the circuit being analyzed, all previous methods fail to provide equivalent circuits which correctly model the circuit's behavior.* In order to determine the Thévenin or Norton equivalent circuit of a circuit having a dependent source controlled by a local voltage or current, the following steps must be taken:

1. Remove the branch across which the Norton equivalent circuit is to be found. Label the resulting two terminals *a* and *b*.
2. Calculate the open-circuit voltage (Thévenin voltage) across the two terminals *a* and *b*. Because the circuit contains a dependent source controlled by an element in the circuit, the dependent source may not be set to zero. Its effects must be considered together with the effects of any independent source(s).
3. Determine the short-circuit current (Norton current) that would occur between the terminals. Once again, the dependent source may not be set to zero, but rather must have its effects considered concurrently with the effects of any independent source(s).
4. Determine the Thévenin or Norton impedance by applying Equations 20–1 and 20–2 as follows:

$$\mathbf{Z}_\text{N} = \mathbf{Z}_\text{Th} = \frac{\mathbf{E}_\text{Th}}{\mathbf{I}_\text{N}} \qquad \textbf{(20–3)}$$

5. Sketch the Thévenin or Norton equivalent circuit, as shown previously in Figure 20–26. Ensure that the portion of the network that was removed in Step 1 is reinserted as part of the equivalent circuit.

**EXAMPLE 20–10** For the circuit of Figure 20–40, find the Norton equivalent circuit external to the load resistor, $R_L$.

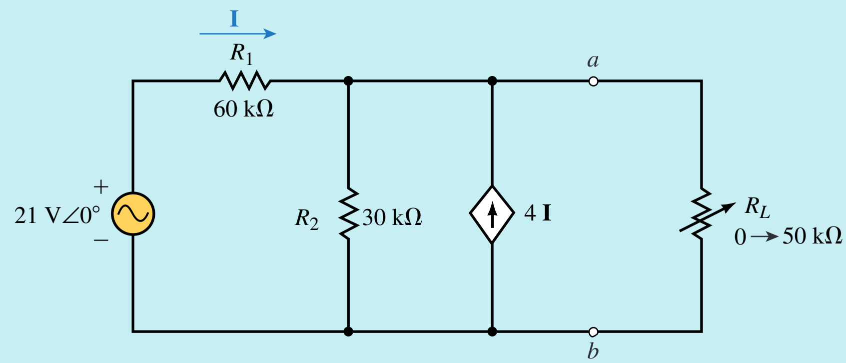



**FIGURE 20–40**

### Solution

**Step 1:** After removing the load resistor from the circuit, we have the network shown in Figure 20–41.

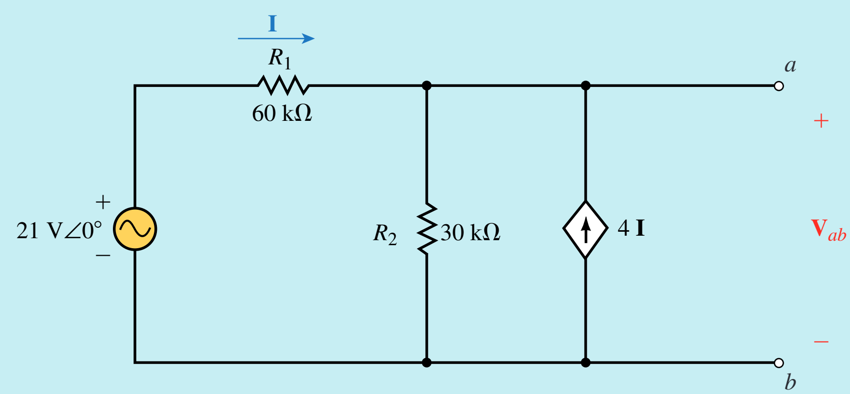



**FIGURE 20–41**

**Step 2:** At first glance, we might look into the open terminals and say that the Norton (or Thévenin) impedance appears to be $60\ \text{k}\Omega \| 30\ \text{k}\Omega = 20\ \text{k}\Omega$. However, we will find that this result is incorrect. The presence of the locally controlled dependent current source makes the analysis of this circuit slightly more complicated than a circuit that contains only an independent source. We know, however, that the basic laws of circuit analysis must apply to all circuits, regardless of the complexity. Applying Kirchhoff's current law at node *a* gives the current through $R_2$ as

$$\mathbf{I}_{R_2} = \mathbf{I} + 4\mathbf{I} = 5\mathbf{I}$$

Now, applying Kirchhoff's voltage law around the closed loop containing the voltage source and the two resistors, gives

$$21\ \text{V}\angle 0° = (60\ \text{k}\Omega)\mathbf{I} + (30\ \text{k}\Omega)(5\mathbf{I}) = 210\ \text{k}\Omega\mathbf{I}$$

which allows us to solve for the current **I** as

$$\mathbf{I} = \frac{21\ \text{V}\angle 0°}{210\ \text{k}\Omega} = 0.100\ \text{mA}\angle 0°$$

Since the open-circuit voltage, $\mathbf{V}_{ab}$ is the same as the voltage across $R_2$, we have

$$\mathbf{E}_{\text{Th}} = \mathbf{V}_{ab} = (60\ \text{k}\Omega)(5)(0.1\ \text{mA}\angle 0°) = 30\ \text{V}\angle 0°$$

**Step 3:** The Norton current source is determined by placing a short-circuit between terminals *a* and *b* as shown in Figure 20–42.

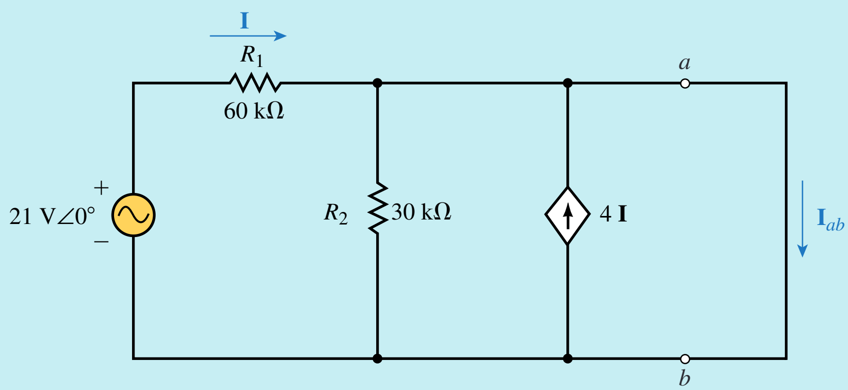



**FIGURE 20–42**

Upon further inspection of this circuit we see that resistor $R_2$ is short-circuited. The simplified circuit is shown in Figure 20–43.

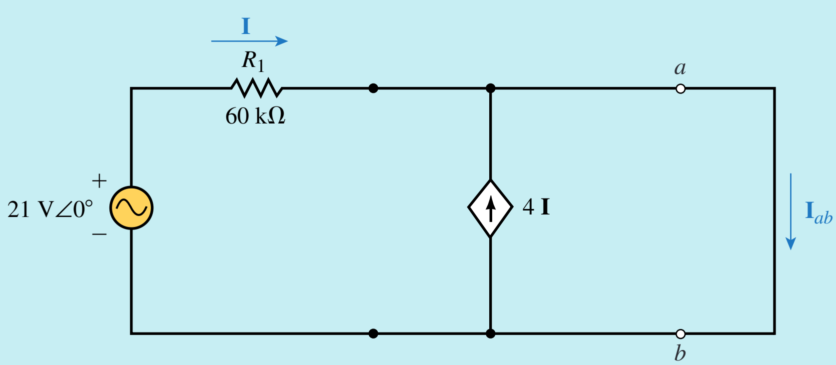



**FIGURE 20–43**

The short-circuit current $\mathbf{I}_{ab}$ is now easily determined by using Kirchhoff's current law at node *a*, and so we have

$$\mathbf{I}_{\text{N}} = \mathbf{I}_{ab} = 5\mathbf{I}$$

From Ohm's law, we have

$$\mathbf{I} = \frac{21\ \text{V}\angle 0°}{60\ \text{k}\Omega} = 0.35\ \text{mA}\angle 0°$$

and so

$$\mathbf{I}_N = 5(0.35\text{ mA}\angle 0°) = 1.75\text{ mA}\angle 0°$$

**Step 4:** The Norton (or Thévenin) impedance is now determined from Ohm's law as

$$\mathbf{Z}_N = \frac{\mathbf{E}_{Th}}{\mathbf{I}_N} = \frac{30\text{ V}\angle 0°}{1.75\text{ mA}\angle 0°} = 17.14\text{ k}\Omega$$

Notice that this impedance is different from the originally assumed 20 kΩ. In general, this condition will occur for most circuits that contain a locally controlled voltage or current source.

**Step 5:** The Norton equivalent circuit is shown in Figure 20–44.

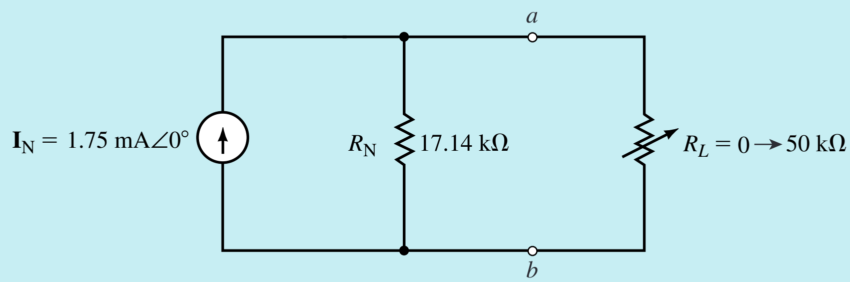



**FIGURE 20–44**

PRACTICE PROBLEMS 5

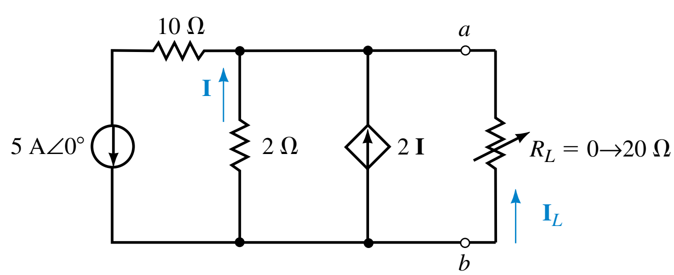



**FIGURE 20–45**

a. Find the Thévenin equivalent circuit external to $R_L$ in the circuit of Figure 20–45.

b. Determine the current $I_L$ when $R_L = 0$ and when $R_L = 20\ \Omega$.

*Answers:*

a. $\mathbf{E}_{Th} = \mathbf{V}_{ab} = -3.33\text{ V}$, $\mathbf{Z}_{Th} = 0.667\ \Omega$

b. For $R_L = 0$: $\mathbf{I}_L = 5.00\text{ A}$ (upward); for $R_L = 20\ \Omega$: $\mathbf{I}_L = 0.161\text{ A}$ (upward)

If a circuit has more than one independent source, it is necessary to determine the open-circuit voltage and short-circuit current due to each independent source while simultaneously considering the effects of the dependent source. The following example illustrates the principle.

**EXAMPLE 20–11** Find the Thévenin and Norton equivalent circuits external to the load resistor in the circuit of Figure 20–46.

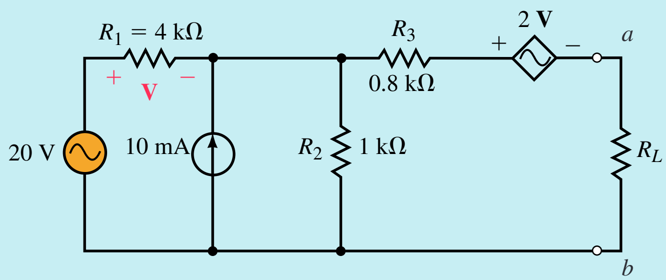



**FIGURE 20–46**

**Solution** There are several methods of solving this circuit. The following approach uses the fewest number of steps.

**Step 1:** Removing the load resistor, we have the circuit shown in Figure 20–47.

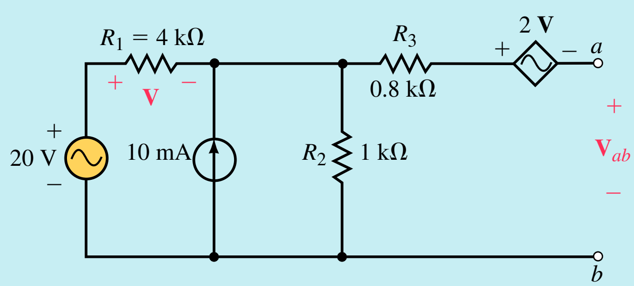



**FIGURE 20–47**

**Step 2:** In order to find the open-circuit voltage, $V_{ab}$ of Figure 20–47, we may isolate the effects due to each independent source and then apply superposition to determine the combined result. However, by converting the current source into an equivalent voltage source, we can determine the open-circuit voltage in one step. Figure 20–48 shows the circuit that results when the current source is converted into an equivalent voltage source.

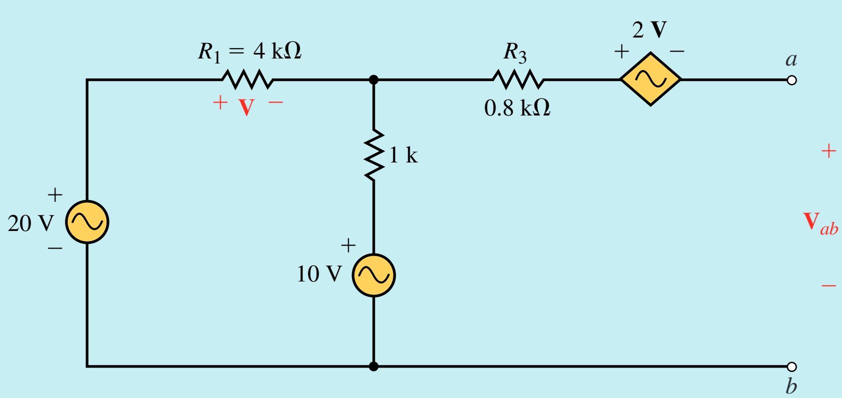



**FIGURE 20–48**

The controlling element ($R_1$) has a voltage, **V**, determined as

$$\mathbf{V} = \left(\frac{4\text{ k}\Omega}{4\text{ k}\Omega + 1\text{ k}\Omega}\right)(20\text{ V} - 10\text{ V})$$
$$= 8\text{ V}$$

which gives a Thévenin (open-circuit) voltage of

$$\mathbf{E}_{\text{Th}} = \mathbf{V}_{ab} = -2(8\text{ V}) + 0\text{ V} - 8\text{ V} + 20\text{ V}$$
$$= -4.0\text{ V}$$

**Step 3:** The short-circuit current is determined by examining the circuit shown in Figure 20–49.

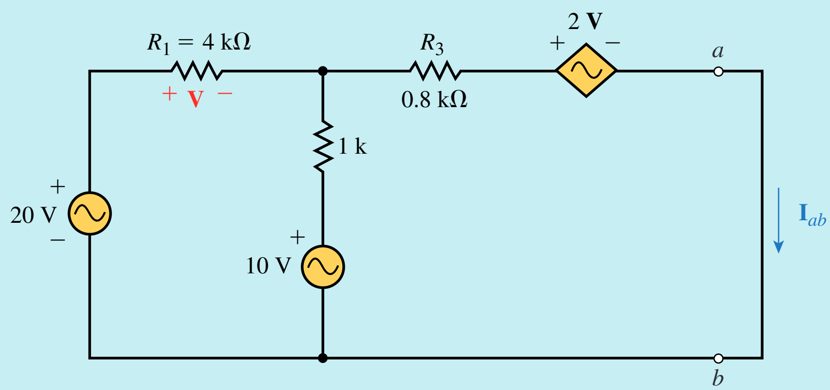



**FIGURE 20–49**

Once again, it is possible to determine the short-circuit current by using superposition. However, upon further reflection, we see that the circuit is easily analyzed using Mesh analysis. Loop currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are assigned in clockwise directions as shown in Figure 20–50.

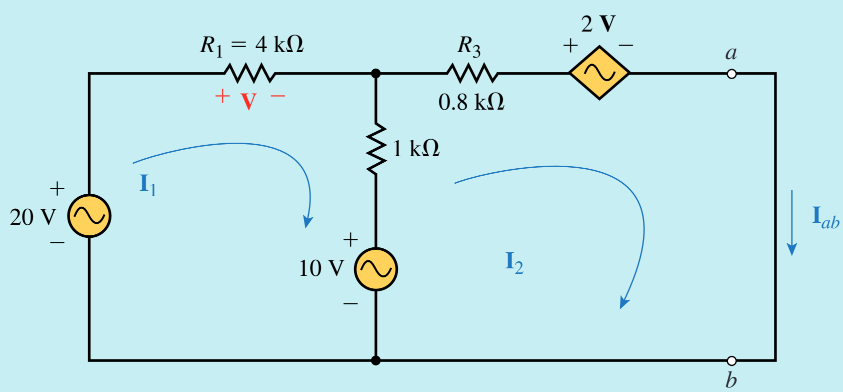



**FIGURE 20–50**

The loop equations are as follows:

Loop 1: $$(5\text{ k}\Omega)\,\mathbf{I}_1 - (1\text{ k}\Omega)\,\mathbf{I}_2 = 10\text{ V}$$

Loop 2: $$-(1\text{ k}\Omega)\,\mathbf{I}_1 + (1.8\text{ k}\Omega)\,\mathbf{I}_2 = 10\text{ V} - 2\,\mathbf{V}$$

Notice that the voltage in the second loop equation is expressed in terms of the controlling voltage across $R_1$. We will not worry about this right now. We may simplify our calculations by ignoring the units in the above equations. It is obvious that if all impedances are expressed in kΩ and all voltages are in volts, then the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ must be in mA. The determinant for the denominator is found to be

$$\mathbf{D} = \begin{vmatrix} 5 & -1 \\ -1 & 1.8 \end{vmatrix} = 9 - 1 = 8$$

The current $\mathbf{I}_1$ is solved by using determinants as follows:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10 & -1 \\ 10 - 2\,\mathbf{V} & 1.8 \end{vmatrix}}{\mathbf{D}} = \frac{18 - (-1)(10 - 2\,\mathbf{V})}{8}$$
$$= 3.5 - 0.25\,\mathbf{V}$$

The above result illustrates that the current $\mathbf{I}_1$ is dependent on the controlling voltage. However, by examining the circuit of Figure 20–50, we see that the controlling voltage depends on the current $\mathbf{I}_1$, and is determined from Ohm's law as

$$\mathbf{V} = (4\text{ k}\Omega)\mathbf{I}_1$$

or more simply as

$$\mathbf{V} = 4\mathbf{I}_1$$

Now, the current $\mathbf{I}_1$ is found as

$$\begin{aligned} \mathbf{I}_1 &= 3.5 - 0.25(4\mathbf{I}_1) \\ 2\mathbf{I}_1 &= 3.5 \\ \mathbf{I}_1 &= 1.75\text{ mA} \end{aligned}$$

which gives $\mathbf{V} = 7.0$ V.

Finally, the short circuit current (which in the circuit of Figure 20–50 is represented by $\mathbf{I}_2$) is found as

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5 & 10 \\ -1 & 10 - 2\,\mathbf{V} \end{vmatrix}}{\mathbf{D}} = \frac{50 - 10\,\mathbf{V} - (-10)}{8}$$
$$= 7.5 - 1.25\,\mathbf{V}$$
$$= 7.5 - 1.25(7.0\text{ V})$$
$$= -1.25\text{ mA}$$

This gives us the Norton current source as

$$I_{\text{N}} = I_{ab} = -1.25\text{ mA}$$

**Step 4:** The Thévenin (or Norton) impedance is determined using Ohm's Law.

$$\mathbf{Z}_{\text{Th}} = \mathbf{Z}_{\text{N}} = \frac{\mathbf{E}_{\text{Th}}}{\mathbf{I}_{\text{N}}} = \frac{-4.0\text{ V}}{-1.25\text{ mA}} = 3.2\text{ k}\Omega$$

The resulting Thévenin equivalent circuit is shown in Figure 20–51 and the Norton equivalent circuit is shown in Figure 20–52.

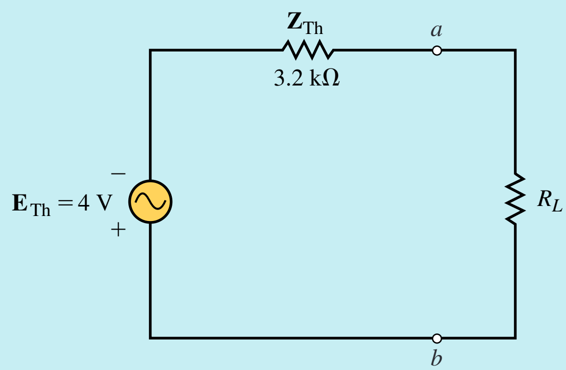



**FIGURE 20–51**

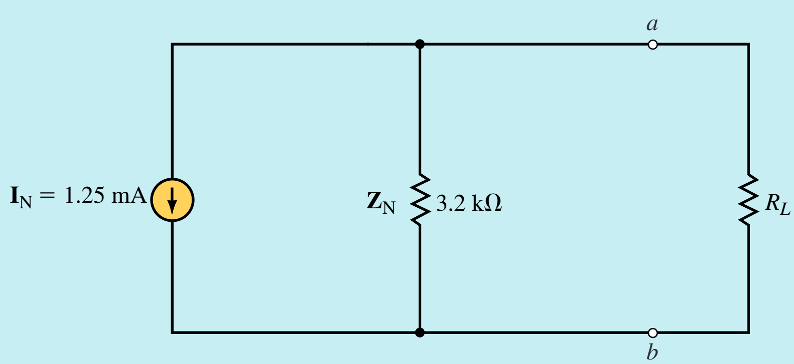



**FIGURE 20–52**

**PRACTICE PROBLEMS 6**

Find the Thévenin and Norton equivalent circuits external to the load resistor in the circuit of Figure 20–53.

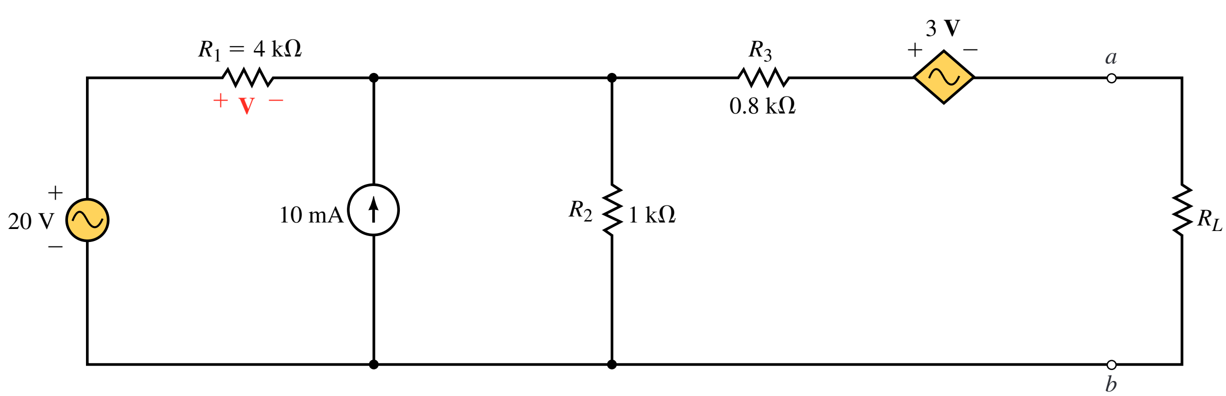



**FIGURE 20–53**

*Answers:*

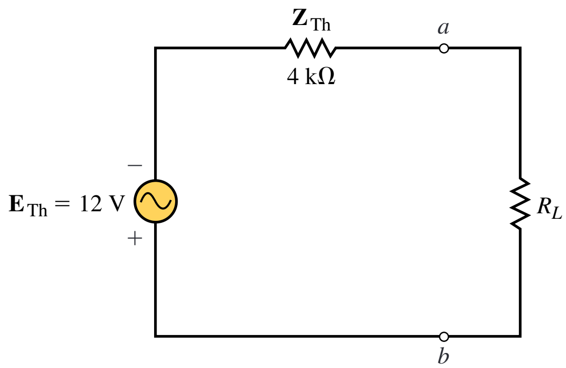



FIGURE 20–54

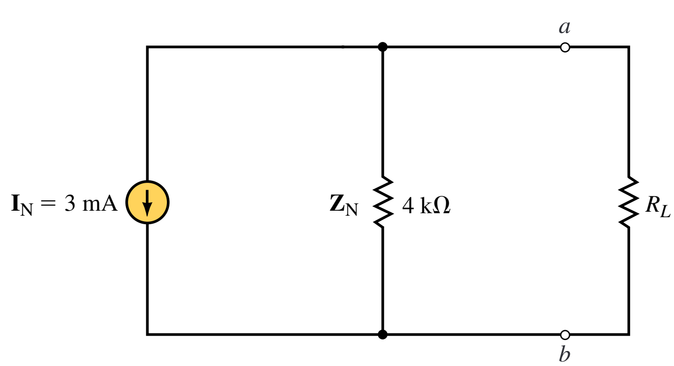



FIGURE 20–55

## 20.6 Maximum Power Transfer Theorem

The maximum power transfer theorem is used to determine the value of load impedance required so that the load receives the maximum amount of power from the circuit. Consider the Thévenin equivalent circuit shown in Figure 20–56.

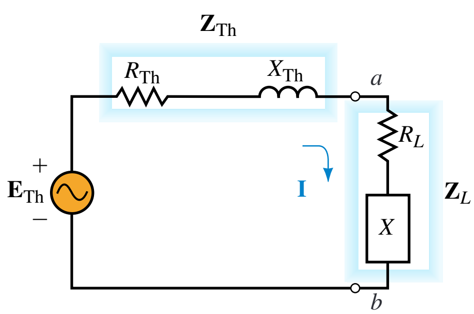



FIGURE 20–56

For any load impedance $\mathbf{Z}_L$ consisting of a resistance and a reactance such that $\mathbf{Z}_L = R_L \pm jX$, the power dissipated by the load will be determined as follows:

$$P_L = I^2R_L$$

$$I = \frac{E_{\text{Th}}}{\sqrt{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} \pm X)^2}}$$

$$P_L = \frac{E_{\text{Th}}{}^2R_L}{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} \pm X)^2}$$

Consider only the reactance portion, $X$, of the load impedance for the moment and neglect the effect of the load resistance. We see that the power dissipated by the load will be maximum when the denominator is kept to a minimum. If the load were to have an impedance such that $jX = -jX_{Th}$, then the power delivered to the load would be given as

$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th} + R_L)^2} \tag{20–4}$$

We recognize that this is the same expression for power as that determined for the Thévenin equivalent of dc circuits in Chapter 9. Recall that maximum power was delivered to the load when

$$R_L = R_{Th}$$

For ac circuits, the maximum power transfer theorem states the following:

*Maximum power will be delivered to a load whenever the load has an impedance which is equal to the complex conjugate of the Thévenin (or Norton) impedance of the equivalent circuit.*

A detailed derivation of the maximum power transfer theorem is provided in Appendix C. The maximum power delivered to the load may be calculated by using Equation 20–4, which is simplified as follows:

$$P_{max} = \frac{E_{Th}^2}{4R_{Th}} \tag{20–5}$$

For a Norton equivalent circuit, the maximum power delivered to a load is determined by substituting $E_{Th} = I_N Z_N$ into the above expression as follows:

$$P_{max} = \frac{I_N^2 Z_N^2}{4R_N} \tag{20–6}$$

**EXAMPLE 20–12** Determine the load impedance $\mathbf{Z}_L$ which will allow maximum power to be delivered to the load in the circuit of Figure 20–57. Find the maximum power.

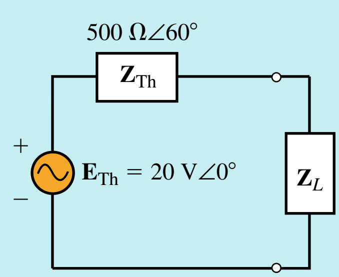



**FIGURE 20–57**

**Solution** Expressing the Thévenin impedance in its rectangular form, we have

$$\mathbf{Z}_{Th} = 500\ \Omega\angle 60° = 250\ \Omega + j433\ \Omega$$

In order to deliver maximum power to the load, the load impedance must be the complex conjugate of the Thévenin impedance. Hence,

$$\mathbf{Z}_L = 250\ \Omega - j433\ \Omega = 500\ \Omega\angle -60°$$

The power delivered to the load is now easily determined by applying Equation 20–5:

$$P_{\max} = \frac{(20\ \text{V})^2}{4(250\ \Omega)} = 400\ \text{mW}$$

PRACTICE PROBLEMS 7

Given the circuit of Figure 20–57, determine the power dissipated by the load if the load impedance is equal to the Thévenin impedance, $\mathbf{Z}_L = 500\ \Omega\angle 60°$. Compare your answer to that obtained in Example 20–12.

*Answer:* $P = 100$ mW, which is less than $P_{\max}$.

Occasionally it is not possible to adjust the reactance portion of a load. In such cases, a **relative maximum power** will be delivered to the load when the load resistance has a value determined as

$$R_L = \sqrt{R_{\text{Th}}^{\ 2} + (X \pm X_{\text{Th}})^2} \qquad \textbf{(20–7)}$$

If the reactance of the Thévenin impedance is of the same type (both capacitive or both inductive) as the reactance in the load, then the reactances are added.

If one reactance is capacitive and the other is inductive, however, then the reactances are subtracted.

To determine the power delivered to the load in such cases, the power will need to be calculated by finding either the voltage across the load or the current through the load. Equations 20–5 and 20–6 will no longer apply, since these equations were based on the premise that the load impedance is the complex conjugate of the Thévenin impedance.

EXAMPLE 20–13 For the circuit of Figure 20–58, determine the value of the load resistor, $R_L$, such that maximum power will be delivered to the load.

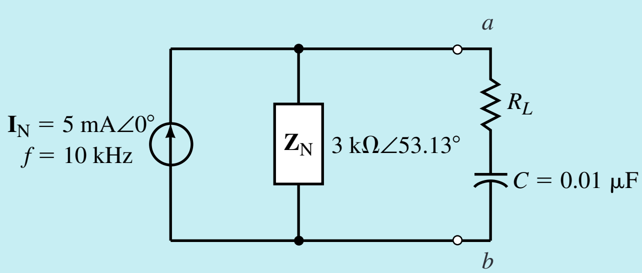



**FIGURE 20–58**

**Solution** Notice that the load impedance consists of a resistor in series with a capacitance of 0.010 $\mu$F. Since the capacitive reactance is determined by the frequency, it is quite likely that the maximum power for this circuit may only be a relative maximum, rather than the absolute maximum. For the **absolute maximum power** to be delivered to the load, the load impedance would need to be

$$\mathbf{Z}_L = 3\text{ k}\Omega\angle-53.13° = 1.80\text{ k}\Omega - j2.40\text{ k}\Omega$$

The reactance of the capacitor at a frequency of 10 kHz is determined to be

$$X_C = \frac{1}{2\pi(10\text{ kHz})(0.010\ \mu\text{F})} = 1.592\text{ k}\Omega$$

Because the capacitive reactance is not equal to the inductive reactance of the Norton impedance, the circuit will not deliver the absolute maximum power to the load. However, relative maximum power will be delivered to the load when

$$\begin{aligned} R_L &= \sqrt{R_{\text{Th}}^2 + (X^2 - X_{\text{Th}})} \\ &= \sqrt{(1.800\text{ k}\Omega)^2 + (1.592\text{ k}\Omega - 2.4\text{ k}\Omega)^2} \\ &= 1.973\text{ k}\Omega \end{aligned}$$

Figure 20–59 shows the circuit with all impedance values.

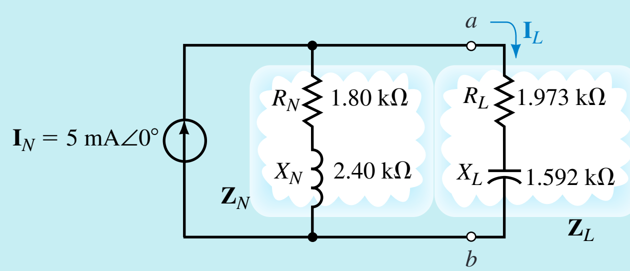



**FIGURE 20–59**

The load current will be

$$\begin{aligned} \mathbf{I}_L &= \frac{\mathbf{Z}_\text{N}}{\mathbf{Z}_\text{N} + \mathbf{Z}_L}\mathbf{I}_\text{N} \\ &= \frac{1.80\text{ k}\Omega + j2.40\text{ k}\Omega}{(1.80\text{ k}\Omega + j2.40\text{ k}\Omega) + (1.973\text{ k}\Omega - j1.592\text{ k}\Omega)}(5\text{ mA}\angle0°) \\ &= \frac{3\text{ k}\Omega\angle53.13°}{3.773\text{ k}\Omega + j0.808\text{ k}\Omega}(5\text{ mA}\angle0°) \\ &= \frac{15.0\text{ V}\angle53.13°}{3.859\text{ k}\Omega\angle12.09°} = 3.887\text{ mA}\angle41.04° \end{aligned}$$

We now determine the power delivered to the load for the given conditions as

$$\begin{aligned} P_L &= I_L^2 R_L \\ &= (3.887\text{ mA})^2(1.973\text{ k}\Omega) = 29.82\text{ mW} \end{aligned}$$

If we had applied Equation 20–6, we would have found the absolute maximum power to be

$$P_{max} = \frac{(5\text{ mA})^2(3.0\text{ k}\Omega)^2}{4(1.8\text{ k}\Omega)} = 31.25\text{ mW}$$

PRACTICE PROBLEMS 8

Refer to the Norton equivalent circuit of Figure 20–60:

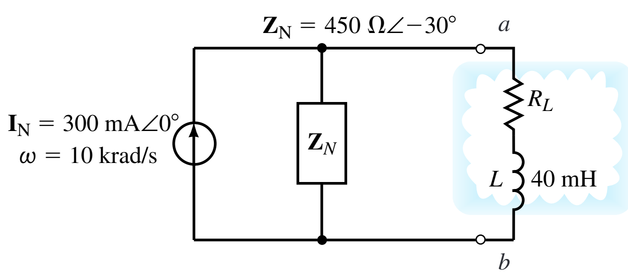



**FIGURE 20–60**

a. Find the value of load resistance, $R_L$, such that the load receives maximum power.
b. Determine the maximum power received by the load for the given conditions.

*Answers:* a. $R_L = 427\ \Omega$, b. $P_L = 11.2$ W

IN-PROCESS LEARNING CHECK 4

Refer to the Thévenin equivalent circuit of Figure 20–61.

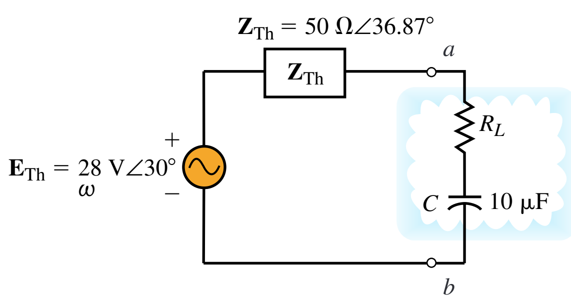


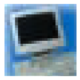

**FIGURE 20–61**

a. Determine the value of the unknown load resistance, $R_L$, which will result in a relative maximum power at an angular frequency of 1 krad/s.
b. Solve for the power dissipated by the load at $\omega_1 = 1$ krad/s.
c. Assuming that the Thévenin impedance remains constant at all frequencies, at what angular frequency, $\omega_2$, will the circuit provide absolute maximum power?
d. Solve for the power dissipated by the load at $\omega_2$.

*(Answers are at the end of the chapter.)*

## 20.7 Circuit Analysis Using Computers

ELECTRONICS WORKBENCH



PSpice

As demonstrated in Chapter 9, circuit analysis programs are very useful in determining the equivalent circuit between specified terminals of a dc circuit. We will use similar methods to obtain the Thévenin and Norton equivalents of ac circuits. Both Electronics Workbench and PSpice are useful in analyzing circuits with dependent sources. As we have already seen, the work in analyzing such a circuit manually is lengthy and very time consuming. In this section, you will learn how to use PSpice to find the Thévenin equivalent of a simple ac circuit. As well, we will use both programs to analyze circuits with dependent sources.

### OrCAD PSpice

The following example shows how PSpice is used to find the Thévenin or Norton equivalent of an ac circuit.

**EXAMPLE 20–14** Use PSpice to determine the Thévenin equivalent of the circuit in Figure 20–18. Assume that the circuit operates at a frequency $\omega = 200$ rad/s ($f = 31.83$ Hz). Compare the result to the solution of Example 20–6.

**Solution** We begin by using OrCAD Capture to input the circuit as shown in Figure 20–62.

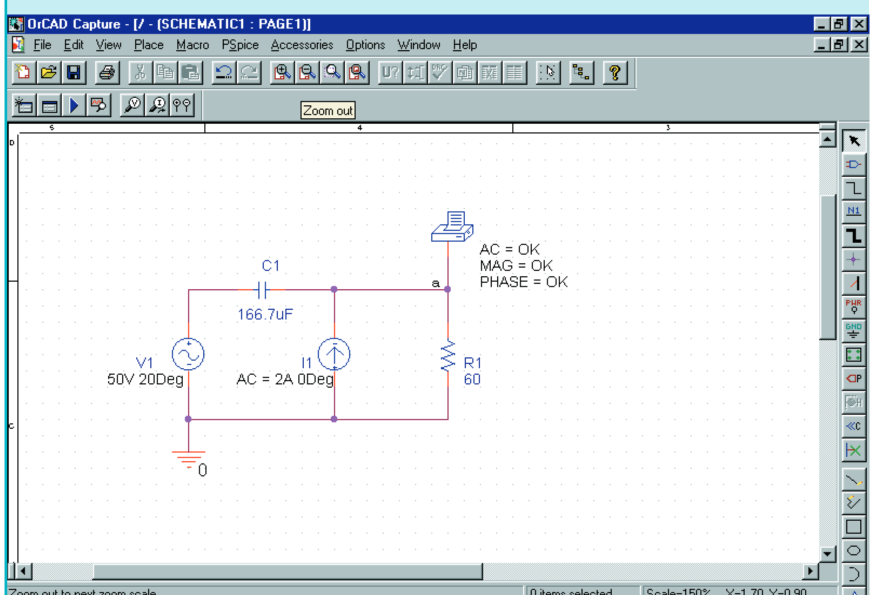



**FIGURE 20–62**

Notice that the load impedance has been removed and the value of capacitance is shown as

$$C = \frac{1}{\omega X_C} = \frac{1}{(200 \text{ rad/s})(30\ \Omega)} = 166.7\ \mu\text{F}$$

In order to adjust for the correct source voltage and current, we select VAC and ISRC from the source library SOURCE.slb of PSpice. The values are set for AC=**50V 20Deg** and AC=**2A 0Deg** respectively. The open-circuit output voltage is displayed using VPRINT1, which can be set to measure magnitude and phase of an ac voltage as follows. Change the properties of VPRINT1 by double clicking on the part. Use the horizontal scroll bar to find the AC, MAG, and PHASE cells. Once you have entered **OK** in each of the cells, click on Apply. Next, click on Display and select Name and Value from the display properties.

Once the circuit is entered, click on the New Simulation Profile tool and set the analysis for AC Sweep/Noise with a linear sweep beginning and ending at a frequency of **31.83Hz** (1 point). As before, it is convenient to disable the Probe postprocessor from the Probe Window tab in the simulation settings box.

As before, it is convenient to disable the Probe postprocessor prior to simulating the design. After simulating the design, the open-circuit voltage is determined by examining the output file of PSpice. The pertinent data from the output file is given as

```
FREQ         VM(a)         VP(a)
 3.183E+01   5.690E+01   -1.583E+01
```

The above result gives $\mathbf{E}_{\text{Th}} = 56.90\ \text{V}\angle -15.83°$. This is the same as the value determined in Example 20–6. Recall that one way of determining the Thévenin (or Norton) impedance is to use Ohm's law, namely

$$\mathbf{Z}_{\text{Th}} = \mathbf{Z}_{\text{N}} = \frac{\mathbf{E}_{\text{Th}}}{\mathbf{I}_{\text{N}}}$$

The Norton current is found by removing the VPRINT1 device from the circuit of Figure 20–62 and inserting an IPRINT device (ammeter) between terminal *a* and ground. The result is shown in Figure 20–63.

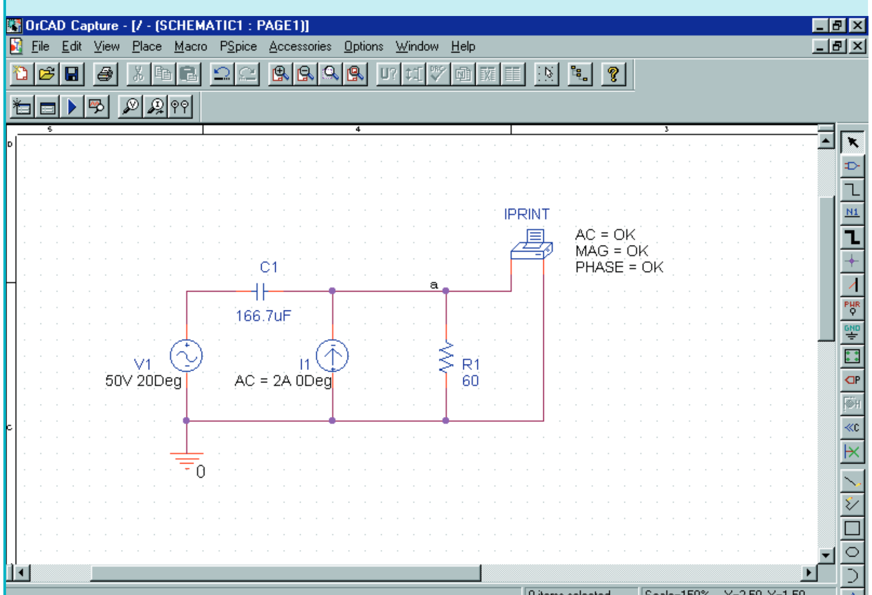



**FIGURE 20–63**

After simulating the design, the short-circuit current is determined by examining the output file of PSpice. The pertinent data from the output file is given as

```
FREQ        IM(V_PRINT2)IP(V_PRINT2)
3.183E+01   2.121E+00   4.761E+01
```

The above result gives $\mathbf{I}_N = 21.21\ \text{A}\angle 47.61°$ and so we calculate the Thévenin impedance as

$$\mathbf{Z}_{\text{Th}} = \frac{56.90\ \text{V}\angle -15.83°}{2.121\ \text{A}\angle 47.61°} = 26.83\Omega\angle -63.44°$$

which is the same value as that obtained in Example 20–6.

In the previous example, it was necessary to determine the Thévenin impedance in two steps, by first solving for the Thévenin voltage and then solving for the Norton current. It is possible to determine the value in one step by using an additional step in the analysis. The following example shows how to determine the Thévenin impedance for a circuit having a dependent source. The same step may also be used for a circuit having independent sources.

The following parts in OrCAD Capture are used to represent dependent sources:

| | |
|---|---|
| Voltage-controlled voltage source: | **E** |
| Current-controlled current source: | **F** |
| Voltage-controlled current source: | **G** |
| Current-controlled voltage source: | **H** |

When using dependent sources, it is necessary to ensure that any voltage-controlled source is placed *across* the controlling voltage and any current-controlled source is placed *in series with* the controlling current. Additionally, each dependent source must have a specified **gain.** This value simply provides the ratio between the output value and the controlling voltage or current. Although the following example shows how to use only one type of dependent source, you will find many similarities between the various sources.

**EXAMPLE 20–15** Use PSpice to find the Thévenin equivalent of the circuit shown in Figure 20–46.

**Solution** OrCAD Capture is used to input the circuit shown in Figure 20–64.

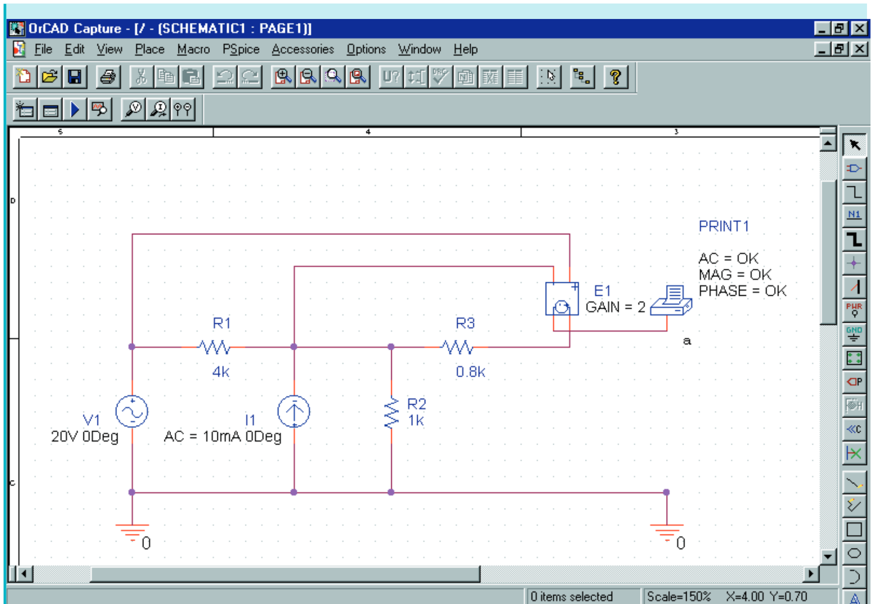



**FIGURE 20–64**

The voltage-controlled voltage source is obtained by clicking on the Place part tool and selecting E from the ANALOG library. Notice the placement of the source. Since $R_1$ is the controlling element, the controlling terminals are placed across this resistor. To adjust the gain of the voltage source, double click on the symbol. Use the scroll bar to find the GAIN cell and type **2.** In order to display this value on the schematic, you will need to click on Display and select Name and Value from the display properties.

As in the previous circuit, you will need to first use a VPRINT1 part to measure the open circuit voltage at terminal *a*. The properties of VPRINT1 are changed by typing **OK** in the AC, MAG, and PHASE cells. Click on Display and select Name and Value for each of the cells. Give the simulation profile a name and run the simulation. The pertinent data in the PSpice output file provides the open-circuit (Thévenin) voltage as follows.

```
FREQ         VM(N00431)   VP(N00431)
1.000E+03    4.000E+00    1.800E+02
```

The VPRINT1 part is then replaced with IPRINT, which is connected between terminal *a* and ground to provide the short-circuit current. Remember to change the appropriate cells using the properties editor. The PSpice output file gives the short-circuit (Norton) current as:

```
FREQ         IM(V_PRINT2) IP(V_PRINT2)
1.000E+03    1.250E-03    1.800E+02
```

The Thévenin impedance is now easily determined as

$$\mathbf{Z}_{\text{Th}} = \frac{4\text{ V}}{1.25\text{ mA}} = 3.2\text{ k}\Omega$$

and the resulting circuit is shown in Figure 20–65.

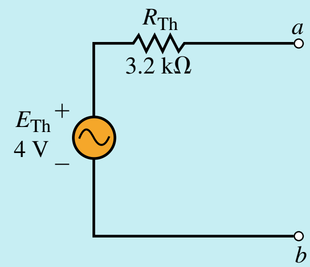


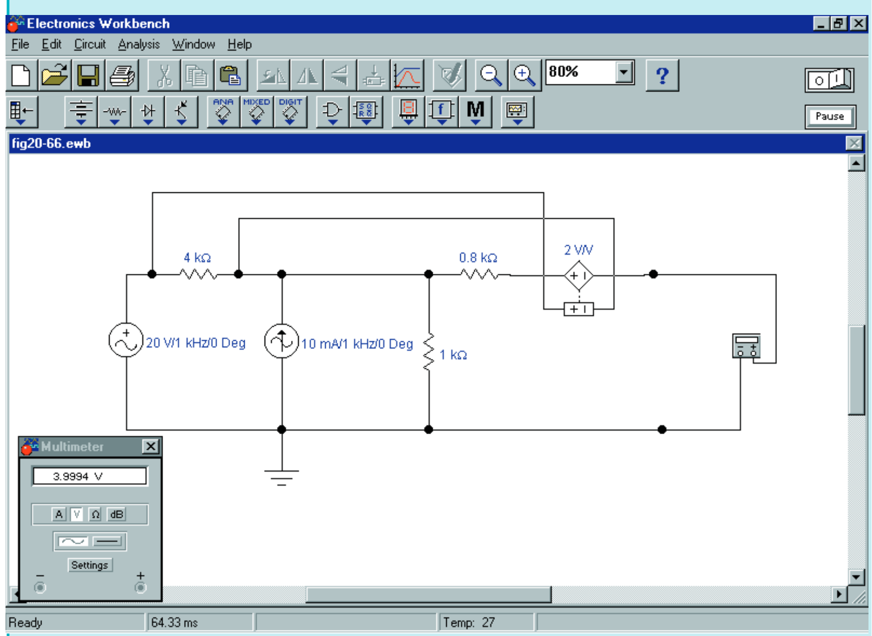

**FIGURE 20–65**

## Electronics Workbench

Electronics Workbench has many similarities to PSpice in its analysis of ac circuits. The following example shows that the results obtained by Electronics Workbench are precisely the same as those obtained from PSpice.

**EXAMPLE 20–16** Use Electronics Workbench to find the Thévenin equivalent of the circuit shown in Figure 20–47. Compare the results to those obtained in Example 20–15.

*Solution* The circuit is entered as shown in Figure 20–66.




EWB **FIGURE 20–66**

We use 1 kHz as the frequency of operation, although any frequency may be used. The Voltage-controlled voltage source is selected from the Sources parts bin and the gain is adjusted by double clicking on the symbol. The Value Tab is selected and the Voltage gain (E): is set to **2 V/V.** The multimeter must be set to measure ac volts. As expected, the reading on the multimeter is 4 V. Current is easily measured by setting the multimeter onto its ac ammeter range as shown in Figure 20–67. The current reading is 1.75 mA. A limitation of Electronics Workbench modeling is that the model does not indicate the phase angle of the voltage or current.

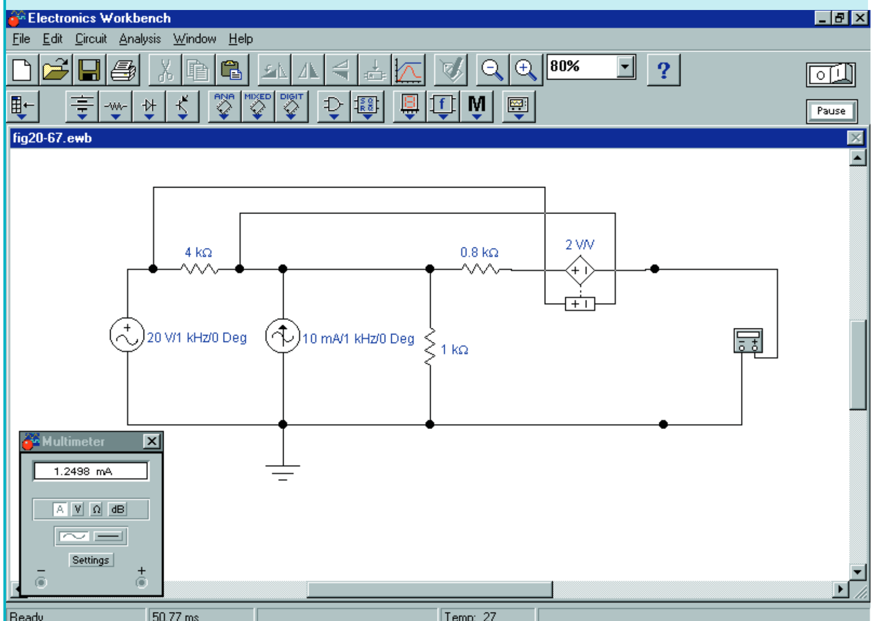



**FIGURE 20–67**

Now, the Thévenin impedance of the circuit is determined using Ohm's law, namely

$$\mathbf{Z}_{\mathrm{Th}} = \frac{4\ \mathrm{V}}{1.75\ \mathrm{mA}} = 3.2\ \mathrm{k}\Omega$$

These results are consistent with the calculations of Example 20–11 and the PSpice results of Example 20–15.

**PRACTICE PROBLEMS 9**

Use PSpice to find the Thévenin equivalent of the circuit shown in Figure 20–45. Compare your answer to that obtained in Practice Problems 5.

*Answer:* $\mathbf{E}_{\mathrm{Th}} = \mathbf{V}_{ab} = -3.33\ \mathrm{V}$, $\mathbf{Z}_{\mathrm{Th}} = 0.667\ \Omega$

**PRACTICE PROBLEMS 10**

Use Electronics Workbench to find the Thévenin equivalent of the circuit shown in Figure 20–45. Compare your answer to that obtained in Practice Problems 5 and Practice Problems 9.

*Answer:* $\mathbf{E}_{\text{Th}} = \mathbf{V}_{ab} = -3.33\text{ V}$, $\mathbf{Z}_{\text{Th}} = 0.667\ \Omega$

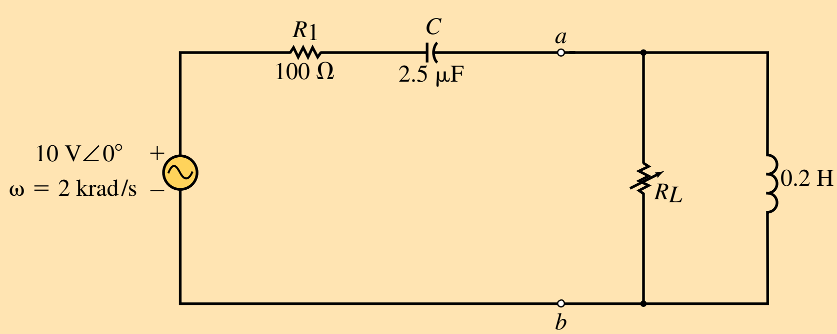
**PRACTICE PROBLEMS 11**

Use PSpice to find the Norton equivalent circuit external to the load resistor in the circuit of Figure 20–31. Assume that the circuit operates at a frequency of 20 kHz. Compare your results to those obtained in Example 20–8. Hint: You will need to place a small resistor (e.g., 1 mΩ) in series with the inductor.

*Answers:* $\mathbf{E}_{\text{Th}} = 7.75\text{ V}\angle -176.5°$ $\mathbf{I}_{\text{N}} = 0.1335\text{ A}\angle 138.5°$ $\mathbf{Z}_{\text{N}} = 566\Omega\angle 45°$
The results are consistent.

## PUTTING IT INTO PRACTICE

In this chapter, you learned how to solve for the required load impedance to enable maximum power transfer to the load. In all cases, you worked with load impedances that were in series with the output terminals. This will not always be the case. The circuit shown in the accompanying figure shows a load that consists of a resistor in parallel with an inductor.




Determine the value of the resistor $R_L$ needed to result in maximum power delivered to the load. Although several methods are possible, you may find that this example lends itself to being solved by using calculus.

## PROBLEMS

### 20.1 Superposition Theorem—Independent Sources

1. Use superposition to determine the current in the indicated branch of the circuit in Figure 20–68.
2. Repeat Problem 1 for the circuit of Figure 20–69.

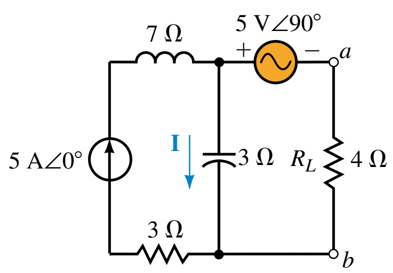



**FIGURE 20–68**

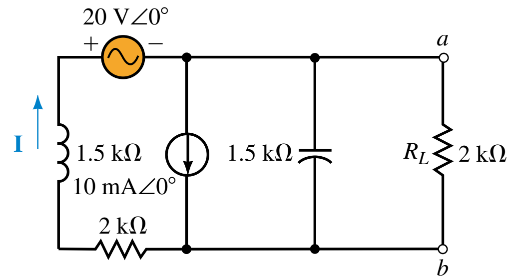



**FIGURE 20–69**

3. Use superposition to determine the voltage $\mathbf{V}_{ab}$ for the circuit of Figure 20–68.
4. Repeat Problem 3 for the circuit of Figure 20–69.
5. Consider the circuit of Figure 20–70.
   a. Use superposition to determine the indicated voltage, **V.**
   b. Show that the power dissipated by the indicated resistor cannot be determined by superposition.

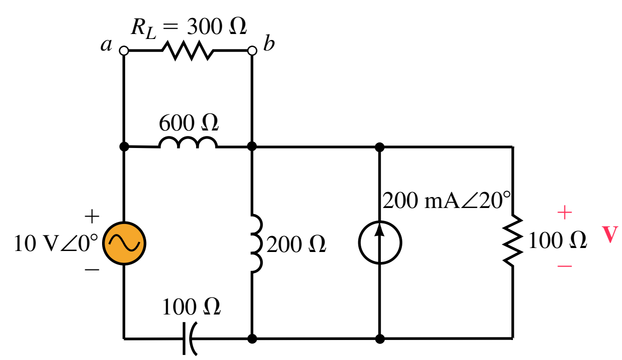



**FIGURE 20–70**

6. Repeat Problem 5 for the circuit of Figure 20–71.

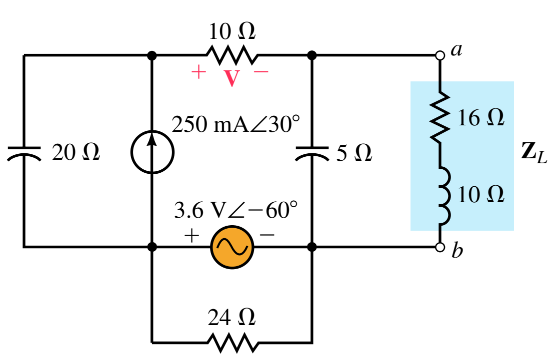



**FIGURE 20–71**

7. Use superposition to determine the current **I** in the circuit of Figure 20–72.

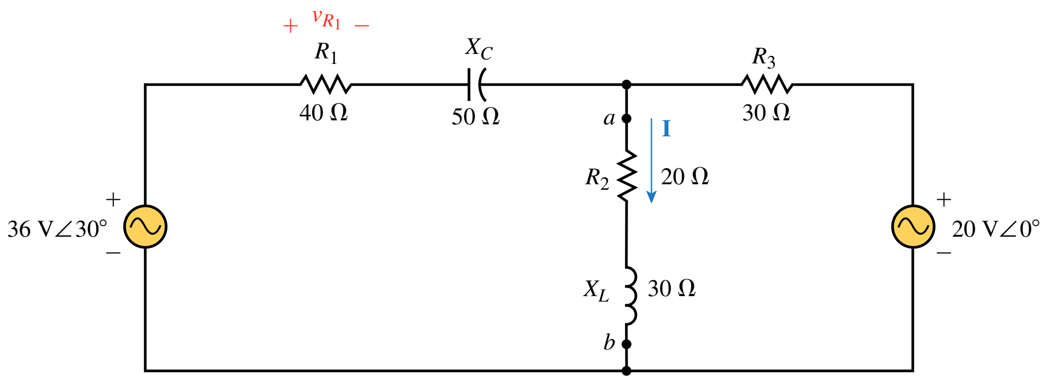



**FIGURE 20–72**

8. Repeat Problem 7 for the circuit of Figure 20–73.

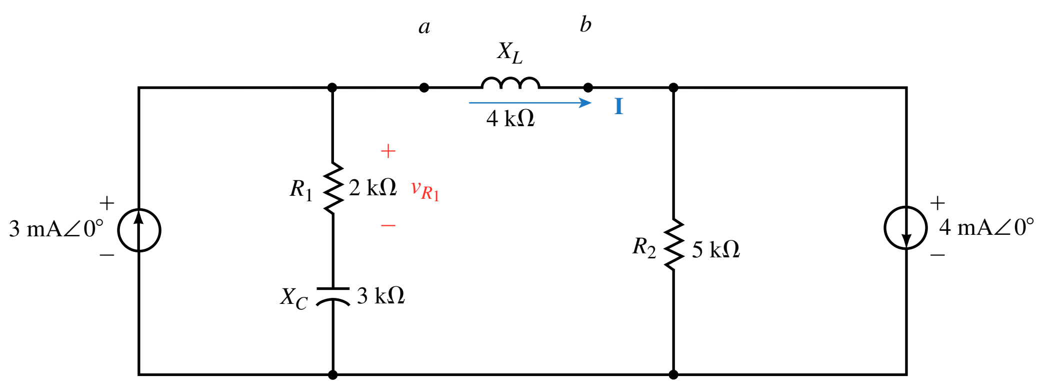



**FIGURE 20–73**

9. Use superposition to determine the sinusoidal voltage, $v_{R_1}$ for the circuit of Figure 20–72.
10. Repeat Problem 9 for the circuit of Figure 20–73.

**20.2 Superposition Theorem—Dependent Sources**

11. Refer to the circuit of Figure 20–74.
    a. Use superposition to find $\mathbf{V}_L$.
    b. If the magnitude of the applied voltage **V** is increased to 200 mV, solve for the resulting $\mathbf{V}_L$.
12. Consider the circuit of Figure 20–75.
    a. Use superposition to find $\mathbf{V}_L$.
    b. If the magnitude of the applied current **I** is decreased to 2 mA, solve for the resulting $\mathbf{V}_L$.

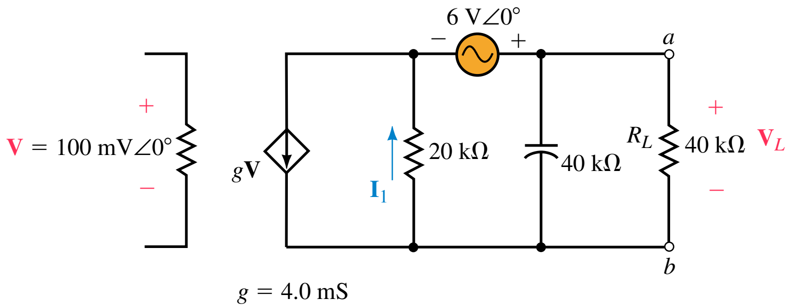



FIGURE 20–74

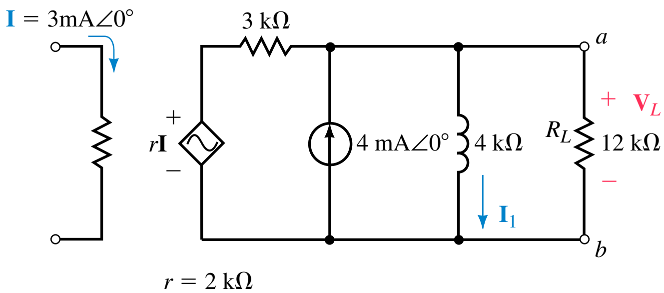



FIGURE 20–75

13. Use superposition to find the current $\mathbf{I}_1$ in the circuit of Figure 20–74.
14. Repeat Problem 13 for the circuit of Figure 20–75.
15. Use superposition to find $\mathbf{V}_L$ in the circuit of Figure 20–76.

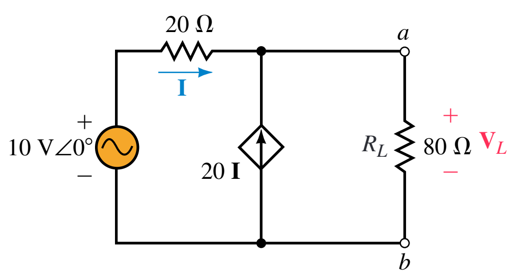



FIGURE 20–76

16. Use superposition to find $\mathbf{V}_L$ in the circuit of Figure 20–77.

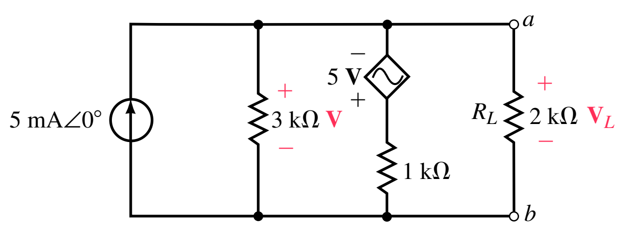
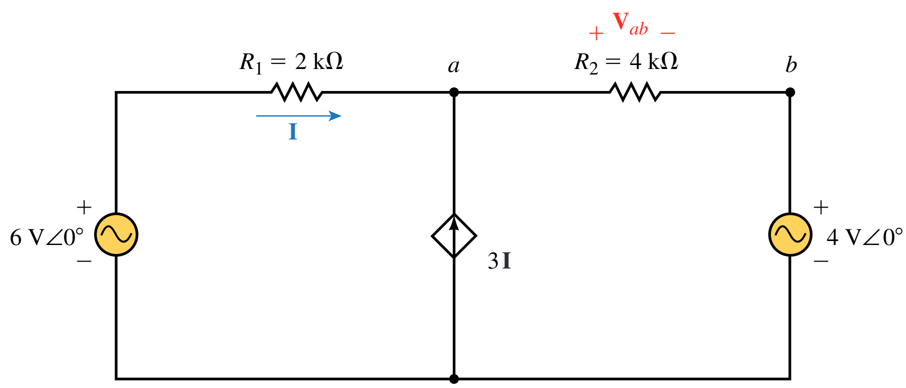


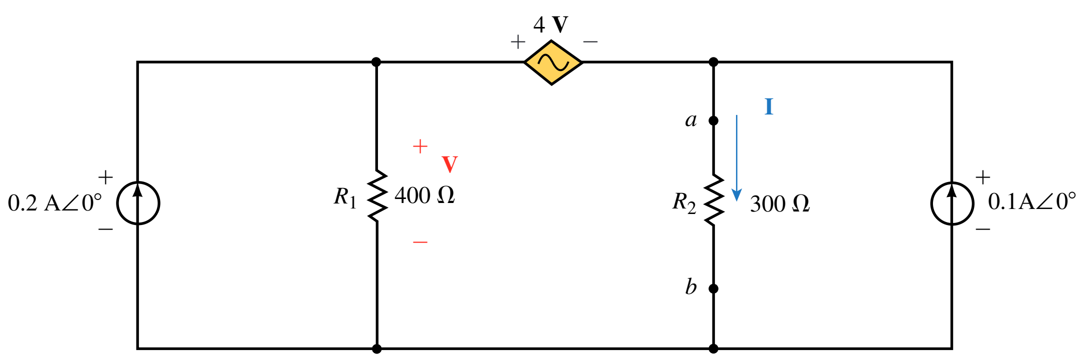

FIGURE 20–77

17. Use superposition to determine the voltage $\mathbf{V}_{ab}$ for the circuit of Figure 20–78.




EWB FIGURE 20–78

18. Use superposition to determine the current **I** for the circuit of Figure 20–79.




EWB FIGURE 20–79

### 20.3 Thévenin's Theorem—Independent Sources

19. Find the Thévenin equivalent circuit external to the load impedance of Figure 20–68.

20. Refer to the circuit of Figure 20–80.

    a. Find the Thévenin equivalent circuit external to the indicated load.

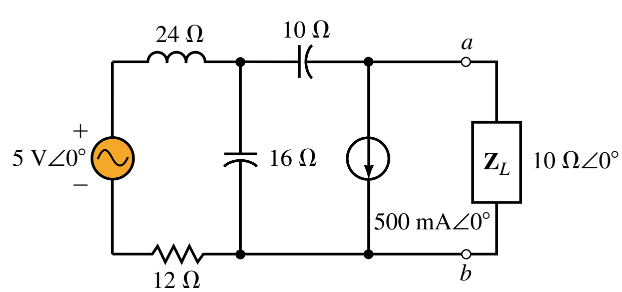
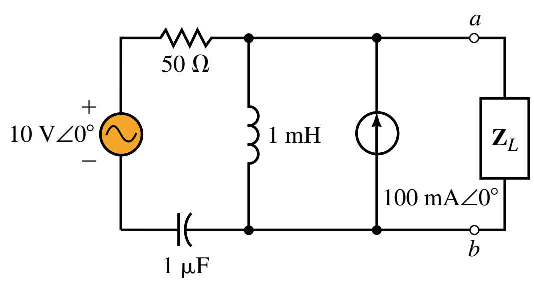



FIGURE 20–80

b. Determine the power dissipated by the load.

21. Refer to the circuit of Figure 20–81.
    a. Find the Thévenin equivalent circuit external to the indicated load at a frequency of 5 kHz.
    b. Determine the power dissipated by the load if $\mathbf{Z}_L = 100\ \Omega\angle 30°$.




EWB **FIGURE 20–81**

22. Repeat Problem 21 for a frequency of 1 kHz.
23. Find the Thévenin equivalent circuit external to $R_L$ in the circuit of Figure 20–72.
24. Repeat Problem 23 for the circuit of Figure 20–69.
25. Repeat Problem 23 for the circuit of Figure 20–70.
26. Find the Thévenin equivalent circuit external to $Z_L$ in the circuit of Figure 20–71.
27. Consider the circuit of Figure 20–82.
    a. Find the Thévenin equivalent circuit external to the indicated load.
    b. Determine the power dissipated by the load if $\mathbf{Z}_L = 20\ \Omega\angle -60°$.

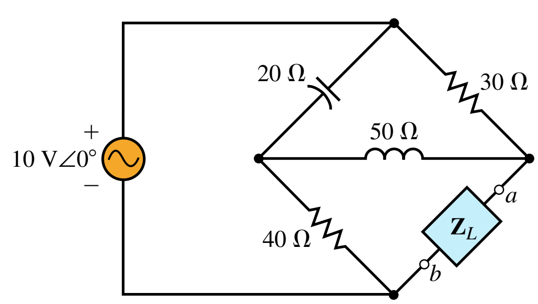



**FIGURE 20–82**

28. Repeat Problem 27 if a 10-Ω resistor is placed in series with the voltage source.

## 20.4 Norton's Theorem—Independent Sources

29. Find the Norton equivalent circuit external to the load impedance of Figure 20–68.

30. Repeat Problem 29 for the circuit of Figure 20–69.
31. a. Using the outlined procedure, find the Norton equivalent circuit external to terminals *a* and *b* in Figure 20–72.
    b. Determine the current through the indicated load.
    c. Find the power dissipated by the load.
32. Repeat Problem 31 for the circuit of Figure 20–73.
33. a. Using the outlined procedure, find the Norton equivalent circuit external to the indicated load impedance (located between terminals *a* and *b*) in Figure 20–70.
    b. Determine the current through the indicated load.
    c. Find the power dissipated by the load.
34. Repeat Problem 33 for the circuit of Figure 20–71.
35. Suppose that the circuit of Figure 20–81 operates at a frequency of 2 kHz.
    a. Find the Norton equivalent circuit external to the load impedance.
    b. If a 30-Ω load resistor is connected between terminals *a* and *b,* find the current through the load.
36. Repeat Problem 35 for a frequency of 8 kHz.

**20.5 Thévenin's and Norton's Theorem for Dependent Sources**

37. a. Find the Thévenin equivalent circuit external to the load impedance in Figure 20–74.
    b. Solve for the current through $R_L$.
    c. Determine the power dissipated by $R_L$.
38. a. Find the Norton equivalent circuit external to the load impedance in Figure 20–75.
    b. Solve for the current through $R_L$.
    c. Determine the power dissipated by $R_L$.
39. Find the Thévenin and Norton equivalent circuits external to the load impedance of Figure 20–76.
40. Find the Thévenin equivalent circuit external to the load impedance of Figure 20–77.

**20.6 Maximum Power Transfer Theorem**

41. Refer to the circuit of Figure 20–83.

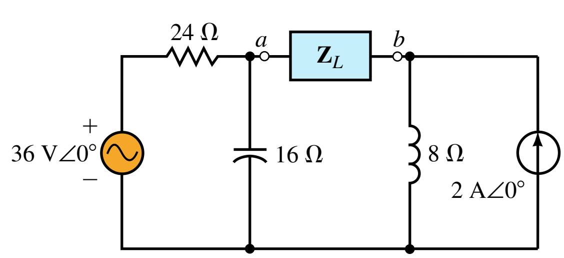



FIGURE 20–83

a. Determine the load impedance, $\mathbf{Z}_L$, needed to ensure that the load receives maximum power.

b. Find the maximum power to the load.

42. Repeat Problem 41 for the circuit of Figure 20–84.

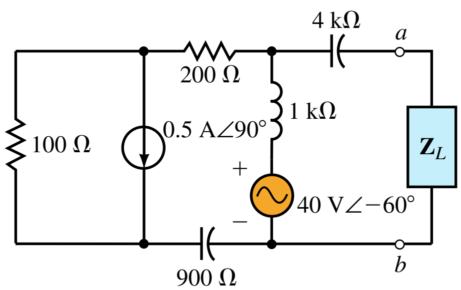



FIGURE 20–84

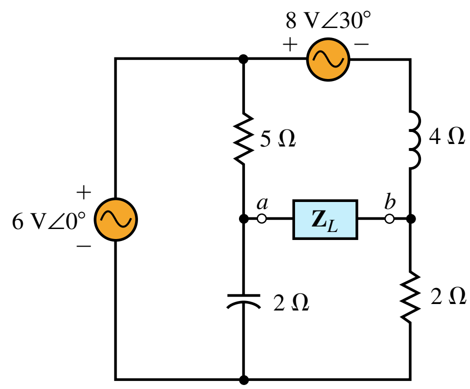



FIGURE 20–85

43. Repeat Problem 41 for the circuit of Figure 20–85.

44. Repeat Problem 41 for the circuit of Figure 20–86.

45. What load impedance is required for the circuit of Figure 20–71 to ensure that the load receives maximum power from the circuit?

46. Determine the load impedance required for the circuit of Figure 20–82 to ensure that the load receives maximum power from the circuit.

47. a. Determine the required load impedance, $\mathbf{Z}_L$, for the circuit of Figure 20–81 to deliver maximum power to the load at a frequency of 5 kHz.

    b. If the load impedance contains a resistor and a 1-$\mu$F capacitor, determine the value of the resistor to result in a relative maximum power transfer.

    c. Solve for the power delivered to the load in (b).

48. a. Determine the required load impedance, $\mathbf{Z}_L$, for the circuit of Figure 20–81 to deliver maximum power to the load at a frequency of 1 kHz.

    b. If the load impedance contains a resistor and a 1-$\mu$F capacitor, determine the value of the resistor to result in a relative maximum power transfer.

    c. Solve for the power delivered to the load in (b).

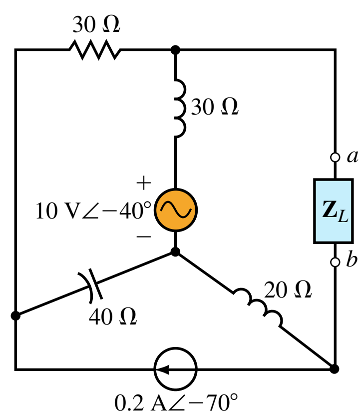



FIGURE 20–86

### 20.7 Circuit Analysis Using Computers

49. **PSpice** Use PSpice to find the Thévenin equivalent circuit external to $R_L$ in the circuit of Figure 20–68. Assume that the circuit operates at a frequency of $\omega = 2000$ rad/s.
    Note: PSpice does not permit a voltage source to have floating terminals. Therefore, a large resistance (e.g., 10 GΩ) must be placed across the output.

50. **PSpice** Repeat Problem 49 for the circuit of Figure 20–69.

51. **PSpice** Use PSpice to find the Norton equivalent circuit external to $R_L$ in the circuit of Figure 20–70. Assume that the circuit operates at a frequency of $\omega = 5000$ rad/s.
Note: PSpice cannot analyze a circuit with a short-circuited inductor. Consequently, it is necessary to place a small resistance (e.g., 1 n$\Omega$) in series with an inductor.
52. **PSpice** Repeat Problem 51 for the circuit of Figure 20–71.
Note: PSpice cannot analyze a circuit with an open-circuited capacitor. Consequently, it is necessary to place a large resistance (e.g., 10 G$\Omega$) in parallel with a capacitor.
53. **PSpice** Use PSpice to find the Thévenin equivalent circuit external to $R_L$ in the circuit of Figure 20–76. Assume that the circuit operates at a frequency of $f = 1000$ Hz.
54. **PSpice** Repeat Problem 53 for the circuit of Figure 20–77.
55. **PSpice** Use PSpice to find the Norton equivalent circuit external to $\mathbf{V}_{ab}$ in the circuit of Figure 20–78. Assume that the circuit operates at a frequency of $f = 1000$ Hz.
56. **PSpice** Repeat Problem 55 for the circuit of Figure 20–79.
57. **EWB** Use Electronics Workbench to find the Thévenin equivalent circuit external to $R_L$ in the circuit of Figure 20–76. Assume that the circuit operates at a frequency of $f = 1000$ Hz.
58. **EWB** Repeat Problem 53 for the circuit of Figure 20–77.
59. **EWB** Use Electronics Workbench to find the Norton equivalent circuit external to $\mathbf{V}_{ab}$ in the circuit of Figure 20–78. Assume that the circuit operates at a frequency of $f = 1000$ Hz.
60. **EWB** Repeat Problem 55 for the circuit of Figure 20–79.

---

## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

a. $\mathbf{I} = 3.39\ \text{A}\angle 25.34°$

b. $P_T = 230.4$ W

c. $P_1 + P_2 + P_3 = 145\ \text{W} \neq P_T = 230.4\ \text{W}$ Superposition does not apply for power.

**In-Process Learning Check 2**

1. Remove the inductor from the circuit. Label the remaining terminals as *a* and *b*.
2. Set the voltage source to zero by removing it from the circuit and replacing it with a short circuit.
3. Determine the values of the impedance using the given frequency. Calculate the Thévenin impedance between terminals *a* and *b*.
4. Convert the voltage source into its equivalent phasor form. Solve for the open-circuit voltage between terminals *a* and *b*.
5. Sketch the resulting Thévenin equivalent circuit.

**In-Process Learning Check 3**

1. Remove the resistor from the circuit. Label the remaining terminals as *a* and *b*.
2. Set the voltage source to zero by removing it from the circuit and replacing it with a short circuit.
3. Calculate the Norton impedance between terminals *a* and *b*.
4. Solve for the short-circuit current between terminals *a* and *b*.
5. Sketch the resulting Norton equivalent circuit.

**In-Process Learning Check 4**

a. $R_L = 80.6\ \Omega$
b. $P_L = 3.25$ W
c. $\omega = 3333$ rad/s
d. $P_L = 4.90$ W

# 21 Resonance

## OBJECTIVES

After studying this chapter, you will be able to

- determine the resonant frequency and bandwidth of a simple series or parallel circuit,
- determine the voltages, currents, and power of elements in a resonant circuit,
- sketch the impedance, current, and power response curves of a series resonant circuit,
- find the quality factor, $Q$, of a resonant circuit and use $Q$ to determine the bandwidth for a given set of conditions,
- explain the dependence of bandwidth on the $L/C$ ratio and on $R$ for both a series and a parallel resonant circuit,
- design a resonant circuit for a given set of parameters,
- convert a series $RL$ network into an equivalent parallel network for a given frequency.

## KEY TERMS

Bandwidth
Damped Oscillations
Half-Power Frequencies
Quality Factor
Parallel Resonance
Selectivity Curve
Series Resonance

## OUTLINE

## CHAPTER PREVIEW

In this chapter, we build upon the knowledge obtained in previous chapters to observe how resonant circuits are able to pass a desired range of frequencies from a signal source to a load. In its most simple form, the **resonant circuit** consists of an inductor and a capacitor together with a voltage or current source. Although the circuit is simple, it is one of the most important circuits used in electronics. As an example, the resonant circuit, in one of its many forms, allows us to select a desired radio or television signal from the vast number of signals that are around us at any time.

Whereas there are various configurations of resonant circuits, they all have several common characteristics. Resonant electronic circuits contain at least one inductor and one capacitor and have a bell-shaped response curve centered at some resonant frequency, $f_r$, as illustrated in Figure 21–1.

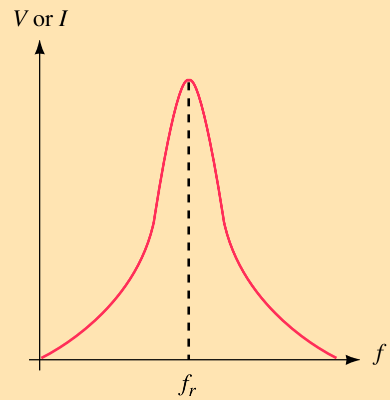


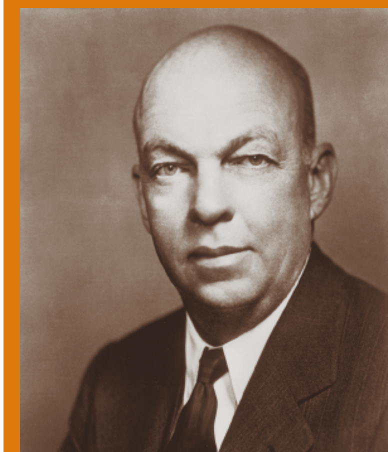

**FIGURE 21–1** Response curve of a resonant circuit.

The response curve of Figure 21–1 indicates that current or voltage will be at a maximum at the resonant frequency, $f_r$. Varying the frequency in either direction results in a reduction of the voltage or current.

If we were to apply variable-frequency sinusoidal signals to a circuit consisting of an inductor and capacitor, we would find that maximum energy will transfer back and forth between the two elements at the resonant frequency. In an ideal *LC* circuit (one containing no resistance), these oscillations would continue unabated even if the signal source were turned off. However, in the practical situation, all circuits have some resistance. As a result, the stored energy will eventually be dissipated by the resistance, resulting in **damped oscillations.** In a manner similar to pushing a child on a swing, the oscillations will continue indefinitely if a small amount of energy is applied to the circuit at exactly the right moment. This phenomenon illustrates the basis of how oscillator circuits operate and therefore provides us with another application of the resonant circuit.

In this chapter, we examine in detail the two main types of resonant circuits: the **series resonant circuit** and the **parallel resonant circuit.**

PUTTING IT IN PERSPECTIVE



### *Edwin Howard Armstrong—Radio Reception*

EDWIN ARMSTRONG WAS BORN in New York City on December 18, 1890. As a young man, he was keenly interested in experiments involving radio transmission and reception.

After earning a degree in electrical engineering at Columbia University, Armstrong used his theoretical background to explain and improve the operation of the triode vacuum tube, which had been invented by Lee de Forest. Edwin Armstrong was able to improve the sensitivity of receivers by using feedback to amplify a signal many times. By increasing the amount of signal feedback, Armstrong also designed and patented a circuit which used the vacuum tube as an oscillator.

Armstrong is best known for conceiving the concept of superheterodyning, in which a high frequency is lowered to a more usable intermediate frequency. Superheterodyning is still used in modern AM and FM receivers and in numerous other electronic circuits such as radar and communication equipment.

Edwin Armstrong was the inventor of FM transmission, which led to greatly improved fidelity in radio transmission.

Although Armstrong was a brilliant engineer, he was an uncompromising person who was involved in numerous lawsuits with Lee de Forest and the communications giant, RCA.

After spending nearly two million dollars in legal battles, Edwin Armstrong jumped to his death from his thirteenth-floor apartment window on January 31, 1954.

## 21.1 Series Resonance

A simple series resonant circuit is constructed by combining an ac source with an inductor, a capacitor, and optionally, a resistor as shown in Figure 21–2a. By combining the generator resistance, $R_G$, with the series resistance, $R_S$, and the resistance of the inductor coil, $R_{coil}$, the circuit may be simplified as illustrated in Figure 21–2b.

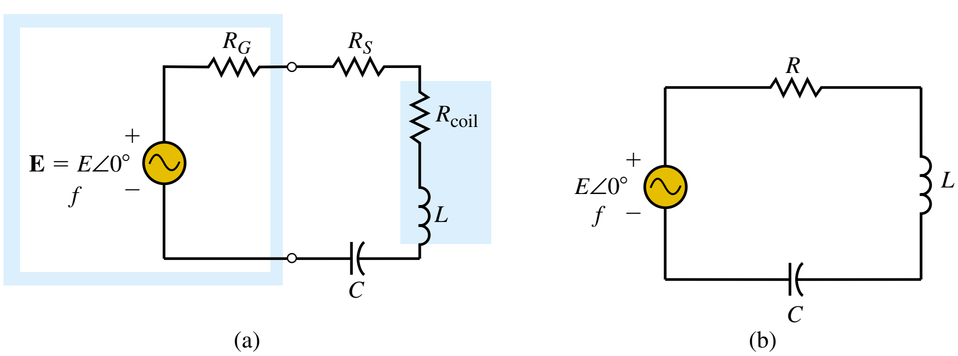



FIGURE 21–2

In this circuit, the total resistance is expressed as

$$R = R_G + R_S + R_{coil}$$

Because the circuit of Figure 21–2 is a series circuit, we calculate the total impedance as follows:

$$\begin{aligned}\mathbf{Z}_{\mathrm{T}} &= R + jX_L - jX_C \\ &= R + j(X_L - X_C)\end{aligned} \tag{21–1}$$

Resonance occurs when the reactance of the circuit is effectively eliminated, resulting in a total impedance that is purely resistive. We know that the reactances of the inductor and capacitor are given as follows:

$$X_L = \omega L = \pi f L \tag{21–2}$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \tag{21–3}$$

Examining Equation 21–1, we see that by setting the reactances of the capacitor and inductor equal to one another, the total impedance, $\mathbf{Z}_{\mathrm{T}}$, is purely resistive since the inductive reactance which is on the positive $j$ axis cancels the capacitive reactance on the negative $j$ axis. The total impedance of the series circuit at resonance is equal to the total circuit resistance, $R$. Hence, at resonance,

$$Z_{\mathrm{T}} = R \tag{21–4}$$

By letting the reactances be equal we are able to determine the series resonance frequency, $\omega_s$ (in radians per second) as follows:

$$\begin{aligned}\omega L &= \frac{1}{\omega C} \\ \omega^2 &= \frac{1}{LC} \\ \omega_s &= \frac{1}{\sqrt{LC}} \quad \text{(rad/s)}\end{aligned} \tag{21–5}$$

Since the calculation of the angular frequency, $\omega$, in radians per second is easier than solving for frequency, $f$, in hertz, we generally express our resonant frequencies in the more simple form. Further calculations of voltage and current will usually be much easier by using $\omega$ rather than $f$. If, however, it becomes necessary to determine a frequency in hertz, recall that the relationship between $\omega$ and $f$ is as follows:

$$\omega = 2\pi f \quad \text{(rad/s)} \tag{21–6}$$

Equation 21–6 is inserted into Equation 21–5 to give the resonant frequency as

$$f_s = \frac{1}{2\pi\sqrt{LC}} \quad \text{(Hz)} \tag{21–7}$$

The subscript $s$ in the above equations indicates that the frequency determined is the series resonant frequency.

At resonance, the total current in the circuit is determined from Ohm's law as

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_{\mathrm{T}}} = \frac{E\angle 0^\circ}{R\angle 0^\circ} = \frac{E}{R}\angle 0^\circ \tag{21–8}$$

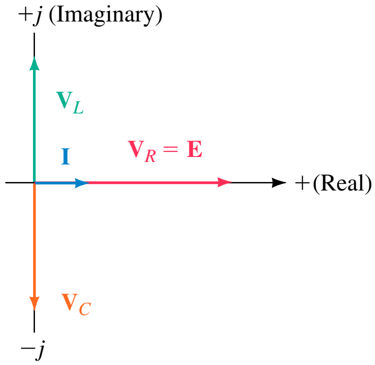



FIGURE 21–3

By again applying Ohm's law, we find the voltage across each of the elements in the circuit as follows:

$$\mathbf{V}_R = IR\angle 0° \quad (21\text{–}9)$$

$$\mathbf{V}_L = IX_L\angle 90° \quad (21\text{–}10)$$

$$\mathbf{V}_C = IX_C\angle -90° \quad (21\text{–}11)$$

The phasor form of the voltages and current is shown in Figure 21–3.

Notice that since the inductive and capacitive reactances have the same magnitude, the voltages across the elements must have the same magnitude but be 180° out of phase.

We determine the average power dissipated by the resistor and the reactive powers of the inductor and capacitor as follows:

$$P_R = I^2R \quad \text{(W)}$$
$$Q_L = I^2X_L \quad \text{(VAR)}$$
$$Q_C = I^2X_C \quad \text{(VAR)}$$

These powers are illustrated graphically in Figure 21–4.

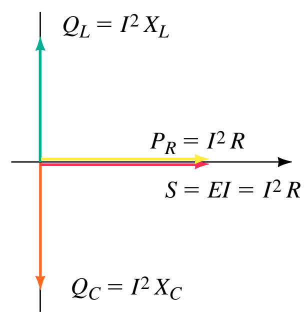



FIGURE 21–4

## 21.2 Quality Factor, $Q$

For any resonant circuit, we define the **quality factor,** $Q$, as the ratio of reactive power to average power, namely,

$$Q = \frac{\text{reactive power}}{\text{average power}} \quad (21\text{–}12)$$

Because the reactive power of the inductor is equal to the reactive power of the capacitor at resonance, we may express $Q$ in terms of either reactive power. Consequently, the above expression is written as follows:

$$Q_s = \frac{I^2X_L}{I^2R}$$

and so we have

$$Q_s = \frac{X_L}{R} = \frac{\omega L}{R} \quad (21\text{–}13)$$

Quite often, the inductor of a given circuit will have a $Q$ expressed in terms of its reactance and internal resistance, as follows:

$$Q_{\text{coil}} = \frac{X_L}{R_{\text{coil}}}$$

If an inductor with a specified $Q_{\text{coil}}$ is included in a circuit, it is necessary to include its effects in the overall calculation of the total circuit $Q$.

We now examine how the $Q$ of a circuit is used in determining other quantities of the circuit. By multiplying both the numerator and denominator of Equation 21–13 by the current, $I$, we have the following:

$$Q_s = \frac{IX_L}{IR} = \frac{V_L}{E} \tag{21–14}$$

Now, since the magnitude of the voltage across the capacitor is equal to the magnitude of the voltage across the inductor at resonance, we see that the voltages across the inductor and capacitor are related to the $Q$ by the following expression:

$$V_C = V_L = Q_sE \quad \text{at resonance} \tag{21–15}$$

*Note:* Since the $Q$ of a resonant circuit is generally significantly larger than 1, we see that the voltage across reactive elements can be many times greater than the applied source voltage. Therefore, it is always necessary to ensure that the reactive elements used in a resonant circuit are able to handle the expected voltages and currents.

**EXAMPLE 21–1** Find the indicated quantities for the circuit of Figure 21–5.

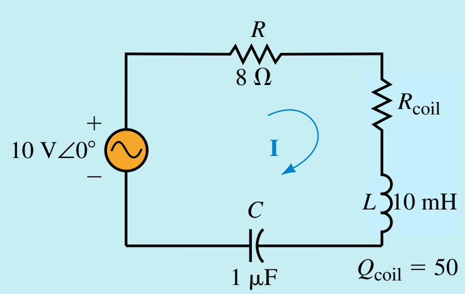



**FIGURE 21–5**

a. Resonant frequency expressed as $\omega$(rad/s) and $f$(Hz).
b. Total impedance at resonance.
c. Current at resonance.
d. $\mathbf{V}_L$ and $\mathbf{V}_C$.
e. Reactive powers, $Q_C$ and $Q_L$.
f. Quality factor of the circuit, $Q_s$.

**Solution**

a.
$$\omega_s = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{(10\text{ mH})(1\,\mu\text{F})}}$$
$$= 10\,000\text{ rad/s}$$
$$f_s = \frac{\omega}{2\pi} = 1592\text{ Hz}$$

b.
$$X_L = \omega L = (10\,000\text{ rad/s})(10\text{ mH}) = 100\ \Omega$$
$$R_{\text{coil}} = \frac{X_L}{Q_{\text{coil}}} = \frac{100\ \Omega}{50} = 2.00\ \Omega$$
$$R_{\text{T}} = R + R_{\text{coil}} = 10.0\ \Omega$$
$$\mathbf{Z}_{\text{T}} = 10\ \Omega\angle 0^\circ$$

c.
$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_{\text{T}}} = \frac{10\text{ V}\angle 0^\circ}{10\ \Omega\angle 0^\circ} = 1.0\text{ A}\angle 0^\circ$$

d.
$$\mathbf{V}_L = (100\ \Omega\angle 90^\circ)(1.0\text{ A}\angle 0^\circ) = 100\text{ V}\angle 90^\circ$$
$$\mathbf{V}_C = (100\ \Omega\angle -90^\circ)(1.0\text{ A}\angle 0^\circ) = 100\text{ V}\angle -90^\circ$$

Notice that the voltage across the reactive elements is ten times greater than the applied signal voltage.

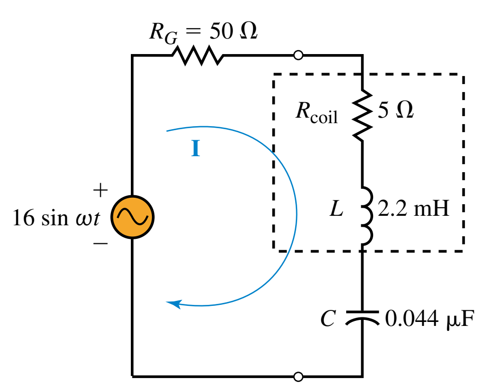
e. Although we use the symbol $Q$ to designate both reactive power and the quality factor, the context of the question generally provides us with a clue as to which meaning to use.

$$Q_L = (1.0\text{ A})^2(100\ \Omega) = 100\text{ VAR}$$
$$Q_C = (1.0\text{ A})^2(100\ \Omega) = 100\text{ VAR}$$

f.
$$Q_s = \frac{Q_L}{P} = \frac{100\text{ VAR}}{10\text{ W}} = 10$$

**PRACTICE PROBLEMS 1**

Consider the circuit of Figure 21–6:




EWB **FIGURE 21–6**

a. Find the resonant frequency expressed as $\omega$(rad/s) and $f$(Hz).
b. Determine the total impedance at resonance.
c. Solve for **I,** $\mathbf{V}_L$, and $\mathbf{V}_C$ at resonance.
d. Calculate reactive powers $Q_C$ and $Q_L$ at resonance.
e. Find the quality factor, $Q_s$, of the circuit.

*Answers:*
a. 102 krad/s, 16.2 kHz b. 55.0 $\Omega\angle 0°$
c. 0.206 A$\angle 0°$, 46.0 V$\angle 90°$, 46.0 V$\angle -90°$ d. 9.46 VAR e. 4.07

---

## 21.3 Impedance of a Series Resonant Circuit

In this section, we examine how the impedance of a series resonant circuit varies as a function of frequency. Because the impedances of inductors and capacitors are dependent upon frequency, the total impedance of a series resonant circuit must similarly vary with frequency. For algebraic simplicity, we use frequency expressed as $\omega$ in radians per second. If it becomes necessary to express the frequency in hertz, the conversion of Equation 21–6 is used.

The total impedance of a simple series resonant circuit is written as

$$\begin{aligned}\mathbf{Z}_\mathrm{T} &= R + j\omega L - j\frac{1}{\omega C} \\ &= R + j\left(\frac{\omega^2 LC - 1}{\omega C}\right)\end{aligned}$$

The magnitude and phase angle of the impedance vector, $\mathbf{Z}_\mathrm{T}$, are expressed as follows:

$$Z_\mathrm{T} = \sqrt{R^2 + \left(\frac{\omega^2 LC - 1}{\omega C}\right)^2} \qquad \textbf{(21–16)}$$

$$\theta = \tan^{-1}\left(\frac{\omega^2 LC - 1}{\omega RC}\right) \qquad \textbf{(21–17)}$$

Examining these equations for various values of frequency, we note that the following conditions will apply:

*When* $\omega = \omega_s$:

$$Z_\mathrm{T} = R$$

and

$$\theta = \tan^{-1} 0 = 0°$$

This result is consistent with the results obtained in the previous section.

*When* $\omega < \omega_s$:

As we decrease $\omega$ from resonance, $Z_\mathrm{T}$ will get larger until $\omega = 0$. At this point, the magnitude of the impedance will be undefined, corresponding to an open circuit. As one might expect, the large impedance occurs because the capacitor behaves like an open circuit at dc.

The angle $\theta$ will occur between of 0° and −90° since the numerator of the argument of the arctangent function will always be negative, corresponding to an angle in the fourth quadrant. Because the angle of the impedance has a negative sign, we conclude that the impedance must appear capacitive in this region.

*When* $\omega > \omega_s$:

As $\omega$ is made larger than resonance, the impedance $Z_T$ will increase due to the increasing reactance of the inductor.

For these values of $\omega$, the angle $\theta$ will always be within 0° and +90° because both the numerator and the denominator of the arctangent function are positive. Because the angle of $\mathbf{Z}_T$ occurs in the first quadrant, the impedance must be inductive.

Sketching the magnitude and phase angle of the impedance $\mathbf{Z}_T$ as a function of angular frequency, we have the curves shown in Figure 21–7.

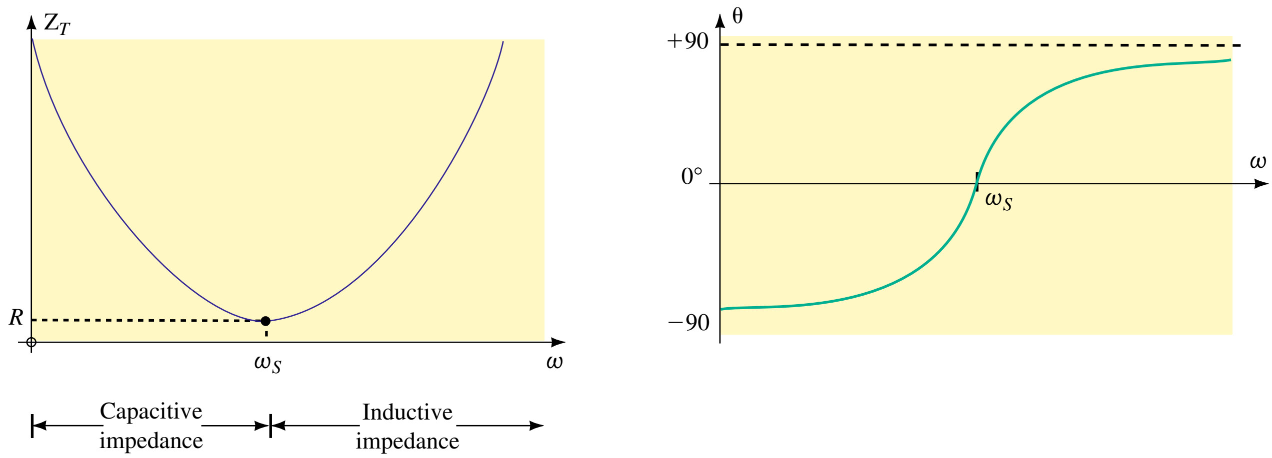



**FIGURE 21–7** Impedance (magnitude and phase angle) versus angular frequency for a series resonant circuit.

## 21.4 Power, Bandwidth, and Selectivity of a Series Resonant Circuit

Due to the changing impedance of the circuit, we conclude that if a constant-amplitude voltage is applied to the series resonant circuit, the current and power of the circuit will not be constant at all frequencies. In this section, we examine how current and power are affected by changing the frequency of the voltage source.

Applying Ohm's law gives the magnitude of the current at resonance as follows:

$$I_{max} = \frac{E}{R} \tag{21–18}$$

For all other frequencies, the magnitude of the current will be less than $I_{max}$ because the impedance is greater than at resonance. Indeed, when the frequency is zero (dc), the current will be zero since the capacitor is effectively an open circuit. On the other hand, at increasingly higher frequencies, the inductor begins to approximate an open circuit, once again causing the current in the circuit to approach zero. The current response curve for a typical series resonant circuit is shown in Figure 21–8.

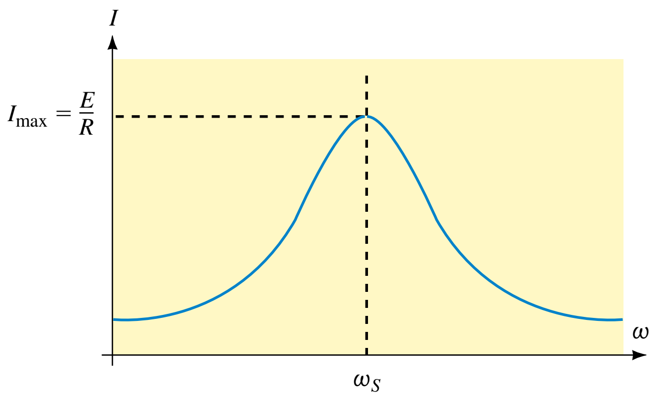



**FIGURE 21–8** Current versus angular frequency for a series resonant circuit.

The total power dissipated by the circuit at any frequency is given as

$$P = I^2R \tag{21–19}$$

Since the current is maximum at resonance, it follows that the power must similarly be maximum at resonance. The maximum power dissipated by the series resonant circuit is therefore given as

$$P_{max} = I^2_{max}R = \frac{E^2}{R} \tag{21–20}$$

The power response of a series resonant circuit has a bell-shaped curve called the **selectivity curve,** which is similar to the current response. Figure 21–9 illustrates the typical selectivity curve.

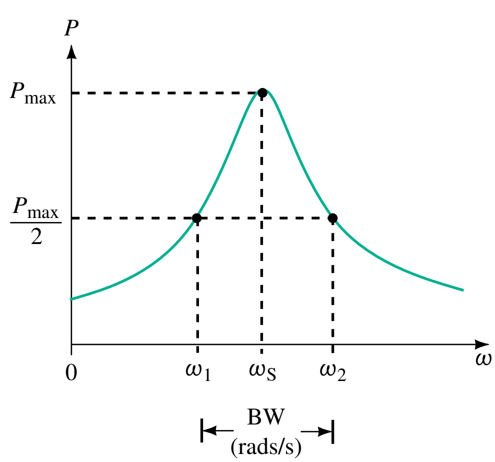



**FIGURE 21–9** Selectivity curve.

**NOTES...**

We see from Figure 21–9 that the selectivity curve is not perfectly symmetrical on both sides of the resonant frequency. As a result, $\omega_s$ is not exactly centered between the half-power frequencies. However, as $Q$ increases, we find that the resonant frequency approaches the midpoint between $\omega_1$ and $\omega_2$. *In general, if* $Q > 10$, *then we assume that the resonant frequency is at the midpoint of half-power frequencies.*

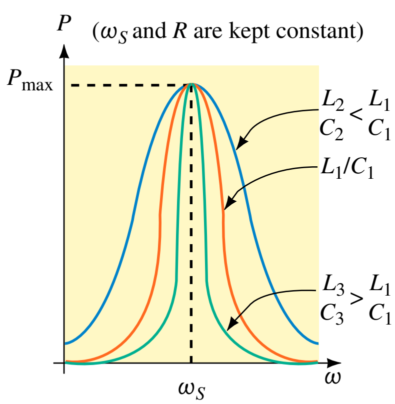



**FIGURE 21–10**

Examining Figure 21–9, we see that only frequencies around $\omega_s$ will permit significant amounts of power to be dissipated by the circuit. We define the **bandwidth,** BW, of the resonant circuit to be the difference between the frequencies at which the circuit delivers half of the maximum power. The frequencies $\omega_1$ and $\omega_2$ are called the **half-power frequencies,** the **cutoff frequencies,** or the **band frequencies.**

If the bandwidth of a circuit is kept very narrow, the circuit is said to have a **high selectivity,** since it is highly selective to signals occuring within a very narrow range of frequencies. On the other hand, if the bandwidth of a circuit is large, the circuit is said to have a **low selectivity.**

The elements of a series resonant circuit determine not only the frequency at which the circuit is resonant, but also the shape (and hence the bandwidth) of the power response curve. Consider a circuit in which the resistance, $R$, and the resonant frequency, $\omega_s$, are held constant. We find that by increasing the ratio of $L/C$, the sides of the power response curve become steeper. This in turn results in a decrease in the bandwidth. Inversely, decreasing the ratio of $L/C$ causes the sides of the curve to become more gradual, resulting in an increased bandwidth. These characteristics are illustrated in Figure 21–10.

If, on the other hand, $L$ and $C$ are kept constant, we find that the bandwidth will decrease as $R$ is decreased and will increase as $R$ is increased. Figure 21–11 shows how the shape of the selectivity curve is dependent upon the value of resistance. A series circuit has the highest selectivity if the resistance of the circuit is kept to a minimum.

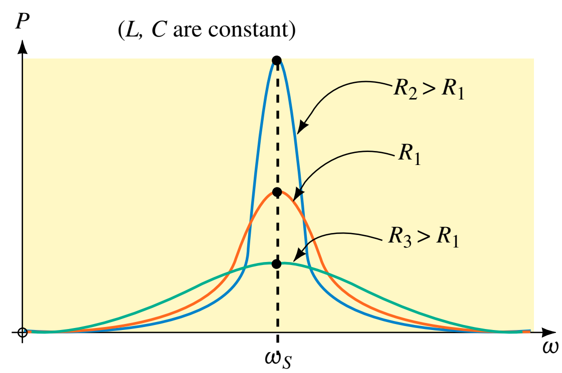



**FIGURE 21–11**

For the series resonant circuit the power at any frequency is determined as

$$P = I^2R$$
$$= \left(\frac{E}{Z_T}\right)^2 R$$

By substituting Equation 21–16 into the above expression, we arrive at the general expression for power as a function of frequency, $\omega$:

$$P = \frac{E^2R}{R^2 + \left(\dfrac{\omega^2LC - 1}{\omega C}\right)^2} \qquad \textbf{(21–21)}$$

At the half-power frequencies, the power must be

$$P_{\text{hpf}} = \frac{E^2}{2R} \tag{21–22}$$

Since the maximum current in the circuit is given as $I_{\text{max}} = E/R$, we see that by manipulating the above expression, the magnitude of current at the half-power frequencies is

$$\begin{aligned} I_{\text{hpf}} &= \sqrt{\frac{P_{\text{hpf}}}{R}} = \sqrt{\frac{E^2}{2R^2}} = \sqrt{\frac{I^2_{\text{max}}}{2}} \\ I_{\text{hpf}} &= \frac{I_{\text{max}}}{\sqrt{2}} \end{aligned} \tag{21–23}$$

The cutoff frequencies are found by evaluating the frequencies at which the power dissipated by the circuit is half of the maximum power. Combining Equations 21–21 and 21–22, we have the following:

$$\begin{aligned} \frac{E^2}{2R} &= \frac{E^2R}{R^2 + \left(\dfrac{\omega^2LC - 1}{\omega C}\right)^2} \\ 2R^2 &= R^2 + \left(\frac{\omega^2LC - 1}{\omega C}\right)^2 \\ \frac{\omega^2LC - 1}{\omega C} &= \pm R \\ \omega^2LC - 1 &= \pm\omega RC \quad \text{(at half-power)} \end{aligned} \tag{21–24}$$

From the selectivity curve for a series circuit, we see that the two half-power points occur on both sides of the resonant angular frequency, $\omega_s$.

When $\omega < \omega_s$, the term $\omega^2LC$ must be less than 1. In this case the solution is determined as follows:

$$\begin{aligned} \omega^2LC - 1 &= -\omega RC \\ \omega^2LC + \omega RC - 1 &= 0 \end{aligned}$$

The solution of this quadratic equation gives the lower half-power frequency as

$$\omega_1 = \frac{-RC + \sqrt{(RC)^2 + 4LC}}{2LC}$$

or

$$\omega_1 = \frac{-R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \tag{21–25}$$

In a similar manner, for $\omega > \omega_s$, the upper half-power frequency is

$$\omega_2 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \tag{21–26}$$

Taking the difference between Equations 21–26 and 21–25, we find the bandwidth of the circuit as

$$\begin{aligned} \text{BW} &= \omega_2 - \omega_1 \\ &= \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} - \left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}\right) \end{aligned}$$

which gives

$$\text{BW} = \frac{R}{L} \quad \text{(rad/s)} \tag{21–27}$$

If the above expression is multiplied by $\omega_s/\omega_s$ we obtain

$$\text{BW} = \frac{\omega_s R}{\omega_s L}$$

and since $Q_s = \omega_s L/R$ we further simplify the bandwidth as

$$\text{BW} = \frac{\omega_s}{Q_s} \quad \text{(rad/s)} \tag{21–28}$$

Because the bandwidth may alternately be expressed in hertz, the above expression is equivalent to having

$$\text{BW} = \frac{f_s}{Q_s} \quad \text{(Hz)} \tag{21–29}$$

**EXAMPLE 21–2** Refer to the circuit of Figure 21–12.

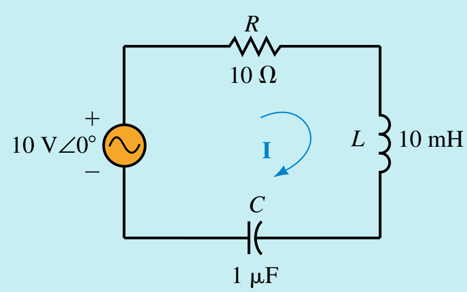



**FIGURE 21–12**

a. Determine the maximum power dissipated by the circuit.
b. Use the results obtained from Example 21–1 to determine the bandwidth of the resonant circuit and to arrive at the approximate half-power frequencies, $\omega_1$ and $\omega_2$.
c. Calculate the actual half-power frequencies, $\omega_1$ and $\omega_2$, from the given component values. Show two decimal places of precision.
d. Solve for the circuit current, **I**, and power dissipated at the lower half-power frequency, $\omega_1$, found in Part (c).

**Solution**

a.
$$P_{\text{max}} = \frac{E^2}{R} = 10.0 \text{ W}$$

b. From Example 21–1, we had the following circuit characteristics:

$$Q_s = 10, \quad \omega_s = 10 \text{ krad/s}$$

The bandwidth of the circuit is determined to be

$$\mathrm{BW} = \omega_s Q_s = 1.0 \text{ krad/s}$$

If the resonant frequency were centered in the bandwidth, then the half-power frequencies occur at approximately

$$\omega_1 = 9.50 \text{ krad/s}$$

and

$$\omega_2 = 10.50 \text{ krad/s}$$

c. $$\omega_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

$$= -\frac{10\ \Omega}{(2)(10\text{ mH})} + \sqrt{\frac{(10\ \Omega)^2}{(4)(10\text{ mH})^2} + \frac{1}{(10\text{ mH})(1\ \mu\text{F})}}$$

$$= -500 + 10\,012.49 = 9512.49 \text{ rad/s} \quad (f_1 = 1514.0 \text{ Hz})$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

$$= 500 + 10\,012.49 = 10\,512.49 \text{ rad/s} \quad (f_2 = 1673.1 \text{ Hz})$$

Notice that the actual half-power frequencies are very nearly equal to the approximate values. For this reason, if $Q \geq 10$, it is often sufficient to calculate the cutoff frequencies by using the easier approach of Part (b).

d. At $\omega_1 = 9.51249$ krad/s, the reactances are as follows:

$$X_L = \omega L = (9.51249 \text{ krad/s})(10 \text{ mH}) = 95.12\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(9.51249 \text{ krad/s})(1\ \mu\text{F})} = 105.12\ \Omega$$

The current is now determined to be

$$\mathbf{I} = \frac{10 \text{ V}\angle 0^\circ}{10\ \Omega + j95.12\ \Omega - j105.12\ \Omega}$$

$$= \frac{10 \text{ V}\angle 0^\circ}{14.14\ \Omega\angle -45^\circ}$$

$$= 0.707 \text{ A}\angle 45^\circ$$

and the power is given as

$$P = I^2 R = (0.707 \text{ A})^2 (10\ \Omega) = 5.0 \text{ W}$$

As expected, we see that the power at the frequency $\omega_1$ is indeed equal to half of the power dissipated by the circuit at resonance.

**EXAMPLE 21–3** Refer to the circuit of Figure 21–13.

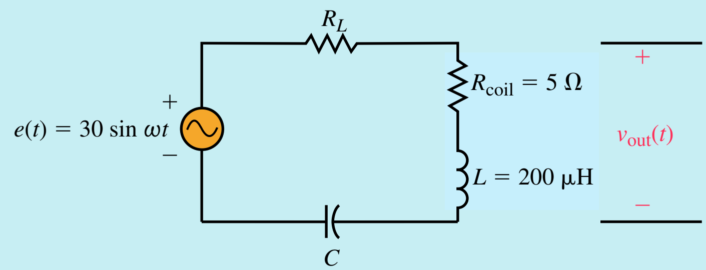



**FIGURE 21–13**

a. Calculate the values of $R_L$ and $C$ for the circuit to have a resonant frequency of 200 kHz and a bandwidth of 16 kHz.

b. Use the designed component values to determine the power dissipated by the circuit at resonance.

c. Solve for $v_{out}(t)$ at resonance.

**Solution**

a. Because the circuit is at resonance, we must have the following conditions:

$$Q_s = \frac{f_s}{\mathrm{BW}} = \frac{200\ \mathrm{kHz}}{16\ \mathrm{kHz}} = 12.5$$

$$X_L = 2\pi f L = 2\pi(200\ \mathrm{kHz})(200\ \mu\mathrm{H}) = 251.3\ \Omega$$

$$R = R_L + R_{coil} = \frac{X_L}{Q_s} = 20.1\ \Omega$$

and so $R_L$ must be

$$R_L = 20.1\ \Omega - 5\ \Omega = 15.1\ \Omega$$

Since $X_C = X_L$, we determine the capacitance as

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(200\ \mathrm{kHz})(251.3\ \Omega)} = 3.17\ \mathrm{nF}\ (\equiv 0.00317\ \mu\mathrm{F})$$

b. The power at resonance is found from Equation 21–20 as

$$P_{max} = \frac{E^2}{R} = \frac{\left(\frac{30\ \mathrm{V}}{\sqrt{2}}\right)^2}{20.1\ \Omega} = 22.4\ \mathrm{W}$$

c. We see from the circuit of Figure 21–13 that the voltage $v_{out}(t)$ may be determined by applying the voltage divider rule to the circuit. However, we must first convert the source voltage from time domain into phasor domain as follows:

$$e(t) = 30 \sin \omega t \Leftrightarrow \mathbf{E} = 21.21\ \text{V}\angle 0°$$

Now, applying the voltage divider rule to the circuit, we have

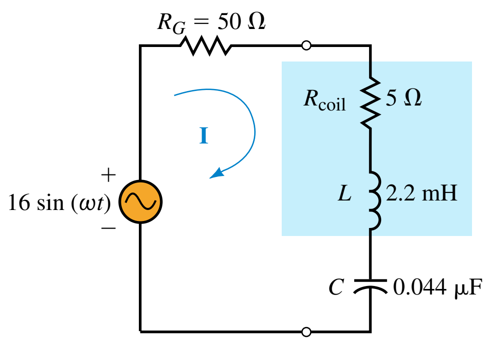
$$\begin{aligned}\mathbf{V}_{out} &= \frac{(R_1 + j\omega L)}{R}\mathbf{E} \\ &= \frac{(5\ \Omega + j251.3\ \Omega)}{20.1\ \Omega} 21.21\ \text{V}\angle 0° \\ &= (251.4\ \Omega\angle 88.86°)(1.056\ \text{A}\angle 0°) \\ &= 265.5\ \text{V}\angle 88.86°\end{aligned}$$

which in time domain is given as

$$v_{out}(t) = 375 \sin(\omega t + 88.86°)$$

PRACTICE PROBLEMS 2

Refer to the circuit of Figure 21–14.




EWB **FIGURE 21–14**

a. Determine the maximum power dissipated by the circuit.
b. Use the results obtained from Practice Problem 1 to determine the bandwidth of the resonant circuit. Solve for the approximate values of the half-power frequencies, $\omega_1$ and $\omega_2$.
c. Calculate the actual half-power frequencies, $\omega_1$ and $\omega_2$, from the given component values. Compare your results to those obtained in Part (b). Briefly explain why there is a discrepancy between the results.
d. Solve for the circuit current, **I,** and power dissipated at the lower half-power frequency, $\omega_1$, found in Part (c).

*Answers:*

a. 2.33 W

b. BW = 25.0 krad/s (3.98 kHz), $\omega_1 \cong 89.1$ krad/s, $\omega_2 \cong 114.1$ krad/s

c. $\omega_1 = 89.9$ krad/s, $\omega_2 \cong 114.9$ krad/s. The approximation assumes that the power-frequency curve is symmetrical around $\omega_S$, which is not quite true.

d. $\mathbf{I} = 0.145\ \text{A}\angle 45°$, $P = 1.16$ W

## IN-PROCESS LEARNING CHECK 1

Refer to the series resonant circuit of Figure 21–15.

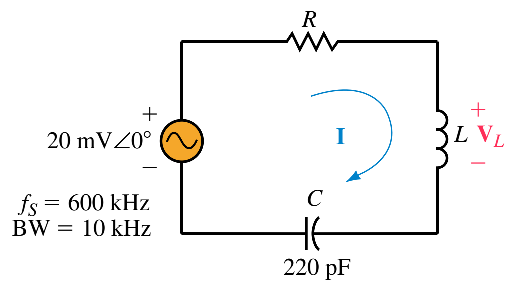



**FIGURE 21–15**

Suppose the circuit has a resonant frequency of 600 kHz and a bandwidth of 10 kHz:

a. Determine the value of inductor $L$ in henries.

b. Calculate the value of resistor $R$ in ohms.

c. Find $\mathbf{I}$, $\mathbf{V}_L$, and power, $P$, at resonance.

d. Find the appoximate values of the half-power frequencies, $f_1$ and $f_2$.

e. Using the results of Part (d), determine the current in the circuit at the lower half-power frequency, $f_1$, and show that the power dissipated by the resistor at this frequency is half the power dissipated at the resonant frequency.

*(Answers are at the end of the chapter.)*

## IN-PROCESS LEARNING CHECK 2

Consider the series resonant circuit of Figure 21–16:

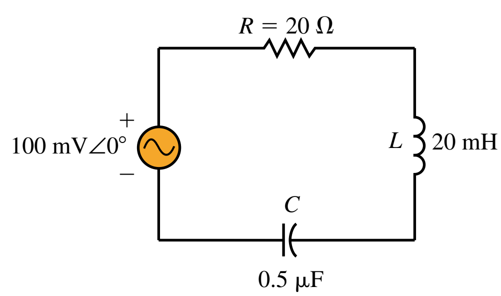



**FIGURE 21–16**

a. Solve for the resonant frequency of the circuit, $\omega_s$, and calculate the power dissipated by the circuit at resonance.

b. Determine $Q$, BW, and the half-power frequencies, $\omega_1$ and $\omega_2$, in radians per second.

c. Sketch the selectivity curve of the circuit, showing $P$ (in watts) versus $\omega$ (in radians per second).

d. Repeat Parts (a) through (c) if the value of resistance is reduced to 10 Ω.

e. Explain briefly how selectivity depends upon the value of resistance in a series resonant circuit.

*(Answers are at the end of the chapter.)*

## 21.5 Series-to-Parallel *RL* and *RC* Conversion

As we have already seen, an inductor will always have some series resistance due to the length of wire used in the coil winding. Even though the resistance of the wire is generally small in comparison with the reactances in the circuit, this resistance may occasionally contribute tremendously to the overall circuit response of a parallel resonant circuit. We begin by converting the series *RL* network as shown in Figure 21–17 into an equivalent parallel *RL* network. It must be emphasized, however, that *the equivalence is only valid at a single frequency, ω.*

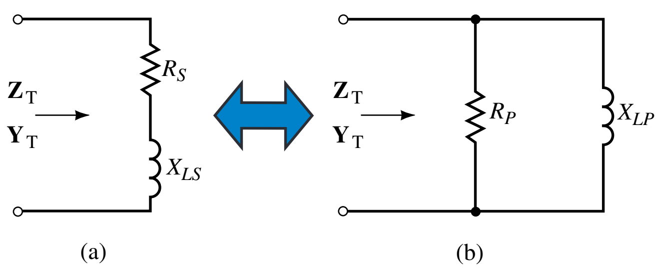



**FIGURE 21–17**

The networks of Figure 21–17 can be equivalent only if they each have the same input impedance, $\mathbf{Z}_T$ (and also the same input admittance, $\mathbf{Y}_T$).

The input impedance of the series network of Figure 21–17(a) is given as

$$\mathbf{Z}_T = R_S + jX_{LS}$$

which gives the input admittance as

$$\mathbf{Y}_T = \frac{1}{\mathbf{Z}_T} = \frac{1}{R_S + jX_{LS}}$$

Multiplying numerator and denominator by the complex conjugate, we have

$$\begin{aligned}\mathbf{Y}_T &= \frac{R_S - jX_{LS}}{(R_S + jX_{LS})(R_S - jX_{LS})}\\ &= \frac{R_S - jX_{LS}}{R_S^2 + X_{LS}^2}\\ &= \frac{R_S}{R_S^2 + X_{LS}^2} - j\frac{X_{LS}}{R_S^2 + X_{LS}^2}\end{aligned} \tag{21–30}$$

From Figure 21–17(b), we see that the input admittance of the parallel network must be

$$\mathbf{Y}_T = G_P - jB_{LP}$$

which may also be written as

$$\mathbf{Y}_T = \frac{1}{R_P} - j\frac{1}{X_{LP}} \tag{21–31}$$

The admittances of Equations 21–30 and 21–31 can only be equal if the real and the imaginary components are equal. As a result, we see that for a given frequency, the following equations enable us to convert a series *RL* network into its equivalent parallel network:

$$R_P = \frac{R_S^2 + X_{LS}^2}{R_S} \tag{21–32}$$

$$X_{LP} = \frac{R_S^2 + X_{LS}^2}{X_{LS}} \tag{21–33}$$

If we were given a parallel *RL* network, it is possible to show that the conversion to an equivalent series network is accomplished by applying the following equations:

$$R_S = \frac{R_P X_{LP}^2}{R_P^2 + X_{LP}^2} \tag{21–34}$$

$$X_{LS} = \frac{R_P^2 X_{LP}}{R_P^2 + X_{LP}^2} \tag{21–35}$$

The derivation of the above equations is left as an exercise for the student.

Equations 21–32 to 21–35 may be simplified by using the quality factor of the coil. Multiplying Equation 21–32 by $R_S/R_S$ and then using Equation 21–13, we have

$$\begin{aligned} R_P &= R_S \frac{R_S^2 + X_{LS}^2}{R_S^2} \\ R_P &= R_S(1 + Q^2) \end{aligned} \tag{21–36}$$

Similarly, Equation 21–33 is simplified as

$$\begin{aligned} X_{LP} &= X_{LS} \frac{R_S^2 + X_{LS}^2}{X_{LS}^2} \\ X_{LP} &= X_{LS}\left(1 + \frac{1}{Q^2}\right) \end{aligned} \tag{21–37}$$

The quality factor of the resulting parallel network must be the same as for the original series network because the reactive and the average powers must be the same. Using the parallel elements, the quality factor is expressed as

$$\begin{aligned} Q &= \frac{X_{LS}}{R_S} \\ &= \frac{\left(\dfrac{R_P^2 X_{LP}}{R_P^2 + X_{LP}^2}\right)}{\left(\dfrac{R_P X_{LP}^2}{R_P^2 + X_{LP}^2}\right)} \\ &= \frac{R_P^2 X_{LP}}{R_P X_{LP}^2} \\ Q &= \frac{R_P}{X_{LP}} \end{aligned} \tag{21–38}$$

**EXAMPLE 21–4** For the series network of Figure 21–18, find the $Q$ of the coil at $\omega = 1000$ rad/s and convert the series *RL* network into its equivalent parallel network. Repeat the above steps for $\omega = 10$ krad/s.

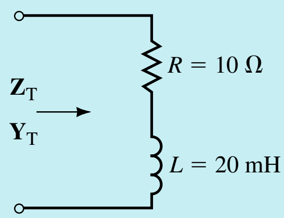



**FIGURE 21–18**

**Solution**

*For $\omega = 1000$ rad/s,*

$$X_L = \omega L = 20\ \Omega$$

$$Q = \frac{X_{LS}}{R_S} = 2.0$$

$$R_P = R_S(1 + Q^2) = 50\ \Omega$$

$$X_{LP} = X_{LS}\left(1 + \frac{1}{Q^2}\right) = 25\ \Omega$$

The resulting parallel network for $\omega = 1000$ rad/s is shown in Figure 21–19.

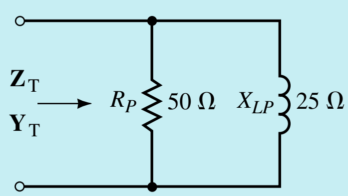



**FIGURE 21–19**

*For $\omega = 10$ krad/s,*

$$X_L = \omega L = 200\ \Omega$$

$$Q = \frac{X_{LS}}{R_S} = 20$$

$$R_P = R_S(1 + Q^2) = 4010\ \Omega$$

$$X_{LP} = X_{LS}\left(1 + \frac{1}{Q^2}\right) = 200.5\ \Omega$$

The resulting parallel network for $\omega = 10$ krad/s is shown in Figure 21–20.

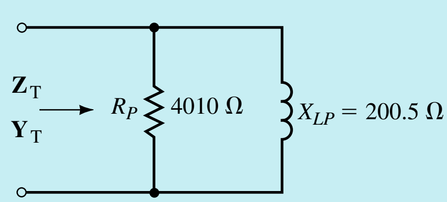



**FIGURE 21–20**

**EXAMPLE 21–5** Find the $Q$ of each of the networks of Figure 21–21 and determine the series equivalent for each.

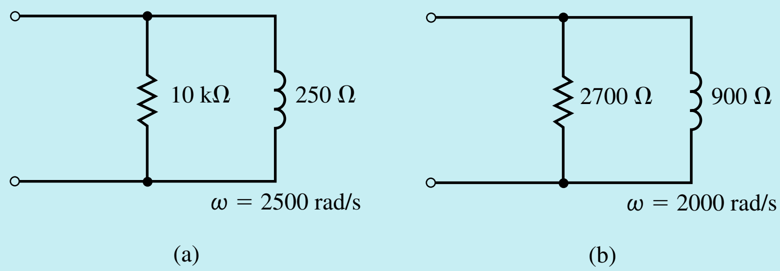



**FIGURE 21–21**

**Solution** For the network of Figure 21–21(a),

$$Q = \frac{R_P}{X_{LP}} = \frac{10\text{ k}\Omega}{250\ \Omega} = 40$$

$$R_S = \frac{R_P}{1 + Q^2} = \frac{10\text{ k}\Omega}{1 + 40^2} = 6.25\ \Omega$$

$$X_{LS} = QR_S = (40)(6.25\ \Omega) = 250\ \Omega$$

$$L = \frac{X_L}{\omega} = \frac{250\ \Omega}{2500\text{ rad/s}} = 0.1\text{ H}$$

For the network of Figure 21–21(b),

$$Q = \frac{R_P}{X_{LP}} = \frac{2700\ \Omega}{900\ \Omega} = 3$$

$$R_S = \frac{R_P}{1 + Q^2} = \frac{2700\ \Omega}{1 + 3^2} = 270\ \Omega$$

$$X_{LS} = QR_S = (3)(270\ \Omega) = 810\ \Omega$$

$$L = \frac{X_L}{\omega} = \frac{810\ \Omega}{2000\text{ rad/s}} = 0.405\text{ H}$$

The resulting equivalent series networks are shown in Figure 21– 22.

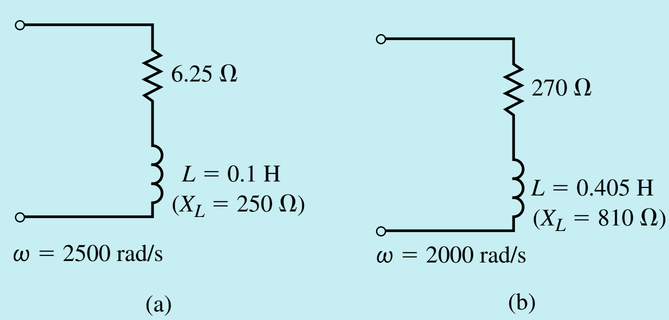



**FIGURE 21–22**

PRACTICE PROBLEMS 3

Refer to the networks of Figure 21–23.

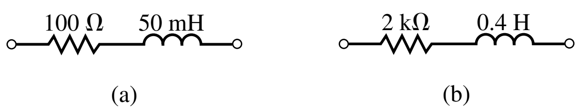



**FIGURE 21–23**

a. Find the quality factors, $Q$, of the networks at $\omega_1 = 5$ krad/s.
b. Use the $Q$ to find the equivalent parallel networks (resistance and reactance) at an angular frequency of $\omega_1 = 5$ krad/s.
c. Repeat Parts (a) and (b) for an angular frequency of $\omega_2 = 25$ krad/s.

*Answers:*

| | | | | |
|---|---|---|---|---|
| a. | $Q_a = 2.5$ | $Q_b = 1.0$ | | |
| b. | Network a: | $R_P = 725\ \Omega$ | $X_{LP} = 290\ \Omega$ | |
| | Network b: | $R_P = 4\ \text{k}\Omega$ | $X_{LP} = 4\ \text{k}\Omega$ | |
| c. | Network a: | $Q_a = 12.5$ | $R_P = 15.725\ \text{k}\Omega$ | $X_{LP} = 1.258\ \text{k}\Omega$ |
| | Network b: | $Q_a = 5$ | $R_P = 52\ \text{k}\Omega$ | $X_{LP} = 10.4\ \text{k}\Omega$ |

The previous examples illustrate two important points which are valid if the $Q$ of the network is large ($Q \geq 10$).

1. The resistance of the parallel network is approximately $Q^2$ larger than the resistance of the series network.
2. The inductive reactances of the series and parallel networks are approximately equal. Hence

$$R_P \cong Q^2 R_S \qquad (Q \geq 10) \tag{21–39}$$

$$X_{LP} \cong X_{LS} \qquad (Q \geq 10) \tag{21–40}$$

Although we have performed conversions between series and parallel *RL* circuits, it is easily shown that if the reactive element is a capacitor, the conversions apply equally well. In all cases, the equations are simply changed by replacing the terms $X_{LS}$ and $X_{LP}$ with $X_{CS}$ and $X_{CP}$ respectively. The $Q$ of the network is determined by the ratios

$$Q = \frac{X_{CS}}{R_S} = \frac{R_P}{X_{CP}} \tag{21–41}$$

PRACTICE PROBLEMS 4

Consider the networks of Figure 21–24:

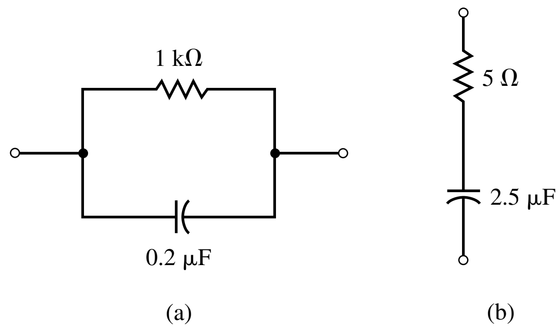



**FIGURE 21–24**

a. Find the $Q$ of each network at a frequency of $f_1 = 1$ kHz.
b. Determine the series equivalent of the network in Figure 21–24(a) and the parallel equivalent of the network in Figure 21–24(b).
c. Repeat Parts (a) and (b) for a frequency of $f_2 = 200$ kHz.

*Answers:*

| | | | | |
|---|---|---|---|---|
| a. | $Q_a = 1.26$ | $Q_b = 12.7$ | | |
| b. | Network a: | $R_S = 388\ \Omega$ | $X_{CS} = 487\ \Omega$ | |
| | Network b: | $R_P = 816\ \Omega$ | $X_{CP} = 64.1\ \Omega$ | |
| c. | Network a: | $Q_a = 251$ | $R_S = 0.0158\ \Omega$ | $X_{CS} = 3.98\ \Omega$ |
| | Network b: | $Q_a = 0.0637$ | $R_P = 5.02\ \Omega$ | $X_{CP} = 78.9\ \Omega$ |

IN-PROCESS LEARNING CHECK 3

Refer to the networks of Figure 21–25:

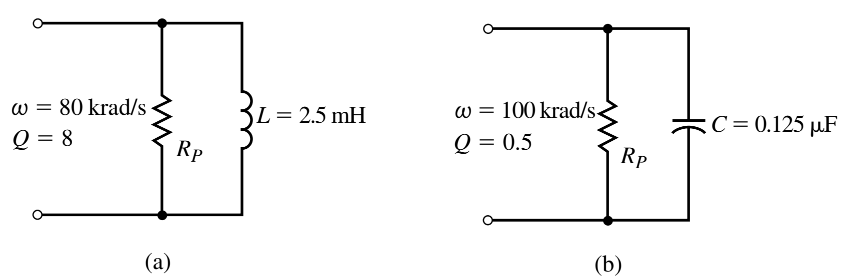



**FIGURE 21–25**

a. Determine the resistance, $R_P$, for each network.
b. Find the equivalent series network by using the quality factor for the given networks.

*(Answers are at the end of the chapter.)*

## 21.6 Parallel Resonance

A simple parallel resonant circuit is illustrated in Figure 21–26. The parallel resonant circuit is best analyzed using a constant-current source, unlike the series resonant circuit which used a constant-voltage source.

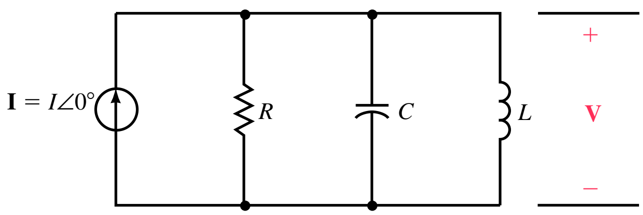



**FIGURE 21–26** Simple parallel resonant circuit.

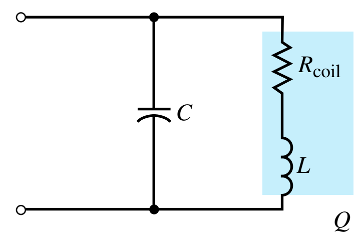



**FIGURE 21–27**

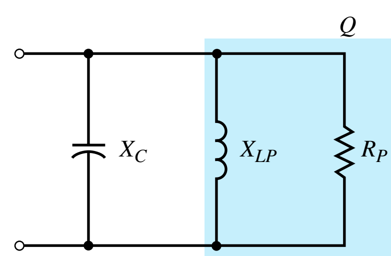



**FIGURE 21–28**

Consider the $LC$ "tank" circuit shown in Figure 21–27. The tank circuit consists of a capacitor in parallel with an inductor. Due to its high $Q$ and frequency response, the tank circuit is used extensively in communications equipment such as AM, FM, and television transmitters and receivers.

The circuit of Figure 21–27 is not exactly a parallel resonant circuit, since the resistance of the coil is in series with the inductance. In order to determine the frequency at which the circuit is purely resistive, we must first convert the series combination of resistance and inductance into an equivalent parallel network. The resulting circuit is shown in Figure 21–28.

At resonance, the capacitive and inductive reactances of the circuit of Figure 21–28 are equal. As we have observed previously, placing equal inductive and capacitive reactances in parallel effectively results in an open circuit at the given frequency. The input impedance of this network at resonance is therefore purely resisitive and given as $Z_T = R_P$. We determine the resonant frequency of a tank circuit by first letting the reactances of the equivalent parallel circuit be equal:

$$X_C = X_{LP}$$

Now, using the component values of the tank circuit, we have

$$X_C = \frac{(R_{\text{coil}})^2 + X_L^2}{X_{LS}}$$

$$\frac{1}{\omega C} = \frac{(R_{\text{coil}})^2 + (\omega L)^2}{\omega L}$$

$$\frac{L}{C} = (R_{\text{coil}})^2 + (\omega L)^2$$

which may be further reduced to

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Factoring $\sqrt{LC}$ from the denominator, we express the parallel resonant frequency as

$$\omega_P = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{(R_{\text{coil}})^2 C}{L}} \qquad \textbf{(21–42)}$$

Notice that if $R_{coil}^2 << L/C$, then the term under the radical is approximately equal to 1.

Consequently, if $L/C \geq 100R_{coil}$, the parallel resonant frequency may be simplified as

$$\omega_p \cong \frac{1}{\sqrt{LC}} \qquad (\text{for } L/C \geq 100R_{coil}) \tag{21–43}$$

Recall that the quality factor, $Q$, of a circuit is defined as the ratio of reactive power to average power for a circuit at resonance. If we consider the parallel resonant circuit of Figure 21–29, we make several important observations.

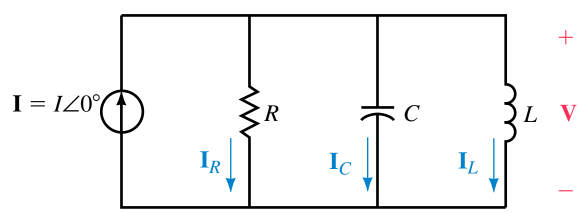



**FIGURE 21–29**

The inductor and capacitor reactances cancel, resulting in a circuit voltage simply determined by Ohm's law as

$$\mathbf{V} = \mathbf{I}R = IR\angle 0°$$

The frequency response of the impedance of the parallel circuit is shown in Figure 21–30.

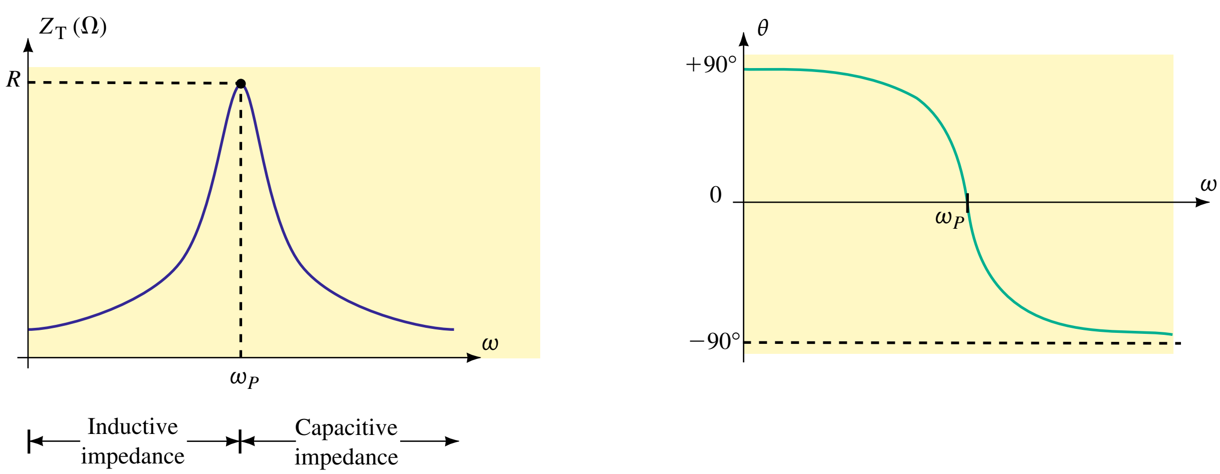



**FIGURE 21–30** Impedance (magnitude and phase angle) versus angular frequency for a parallel resonant circuit.

Notice that the impedance of the entire circuit is maximum at resonance and minimum at the boundary conditions ($\omega = 0$ rad/s and $\omega \rightarrow \infty$). This result is exactly opposite to that observed in series resonant circuits which have minimum impedance at resonance. We also see that for parallel circuits, the impedance will appear inductive for frequencies less than the resonant frequency, $\omega_P$. Inversely, the impedance is capacitive for frequencies greater than $\omega_P$.

The $Q$ of the parallel circuit is determined from the definition as

$$\begin{aligned} Q_P &= \frac{\text{reactive power}}{\text{average power}} \\ &= \frac{V^2/X_L}{V^2/R} \\ Q_P &= \frac{R_L}{X_L} = \frac{R}{X_C} \end{aligned} \qquad \textbf{(21–44)}$$

This is precisely the same result as that obtained when we converted an *RL* series network into its equivalent parallel network. If the resistance of the coil is the only resistance within a circuit, then the circuit $Q$ will be equal to the $Q$ of the coil. However, if the circuit has other sources of resistance, then the additional resistance will reduce the circuit $Q$.

For a parallel *RLC* resonant circuit, the currents in the various elements are found from Ohm's law as follows:

$$\mathbf{I}_R = \frac{\mathbf{V}}{\mathbf{R}} = \mathbf{I} \qquad \textbf{(21–45)}$$

$$\begin{aligned} \mathbf{I}_L &= \frac{\mathbf{V}}{X_L\angle 90°} \\ &= \frac{V}{R/Q_P}\angle -90° \\ &= Q_P I\angle -90° \end{aligned} \qquad \textbf{(21–46)}$$

$$\begin{aligned} \mathbf{I}_C &= \frac{\mathbf{V}}{X_C\angle -90°} \\ &= \frac{V}{R/Q_P}\angle 90° \\ &= Q_P I\angle 90° \end{aligned} \qquad \textbf{(21–47)}$$

At resonance, the currents through the inductor and the capacitor have the same magnitudes but are 180° out of phase. Notice that the magnitude of current in the reactive elements at resonance is $Q$ times greater than the applied source current. Because the $Q$ of a parallel circuit may be very large, we see the importance of choosing elements that are able to handle the expected currents.

In a manner similar to that used in determining the bandwidth of a series resonant circuit, it may be shown that the half-power frequencies of a parallel resonant circuit are

$$\omega_1 = \frac{1}{2RC} - \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \quad \text{(rad/s)} \qquad \textbf{(21–48)}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \quad \text{(rad/s)} \qquad \textbf{(21–49)}$$

The bandwidth is therefore

$$\mathrm{BW} = \omega_2 - \omega_1 = \frac{1}{RC} \quad \text{(rad/s)} \tag{21–50}$$

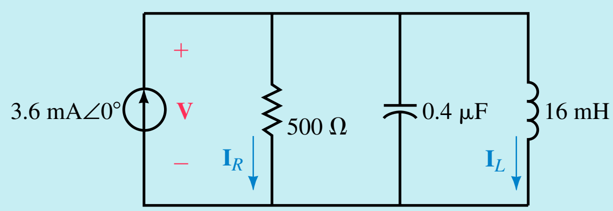
If $Q \geq 10$, then the selectivity curve is very nearly symmetrical around $\omega_P$, resulting in half-power frequencies which are located at $\omega_P \pm \mathrm{BW}/2$.

Multiplying Equation 21–50 by $\omega_P/\omega_P$ results in the following:

$$\mathrm{BW} = \frac{\omega_P}{R(\omega_P C)} = \frac{X_C}{R}\omega_P$$
$$\mathrm{BW} = \frac{\omega_P}{Q_P} \quad \text{(rad/s)} \tag{21–51}$$

Notice that Equation 21–51 is the same for both series and parallel resonant circuits.

**EXAMPLE 21–6** Consider the circuit shown in Figure 21–31.




EWB

**FIGURE 21–31**

a. Determine the resonant frequencies, $\omega_r$(rad/s) and $f_r$(Hz) of the tank circuit.
b. Find the $Q$ of the circuit at resonance.
c. Calculate the voltage across the circuit at resonance.
d. Solve for currents through the inductor and the resistor at resonance.
e. Determine the bandwidth of the circuit in both radians per second and hertz.
f. Sketch the voltage response of the circuit, showing the voltage at the half-power frequencies.
g. Sketch the selectivity curve of the circuit showing $P$(watts) versus $\omega$(rad/s).

**Solution**

a.
$$\omega_P = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(16\ \text{mH})\ (0.4\ \mu\text{F})}} = 12.5\ \text{krad/s}$$
$$f_P = \frac{\omega}{2\pi} = \frac{12.5\ \text{krad/s}}{2\pi} = 1989\ \text{Hz}$$

b.
$$Q_P = \frac{R_P}{\omega L} = \frac{500\ \Omega}{(12.5\ \text{krad/s})\ (16\ \text{mH})} = \frac{500\ \Omega}{200\ \Omega} = 2.5$$

c. At resonance, $\mathbf{V}_C = \mathbf{V}_L = \mathbf{V}_R$, and so

$$\mathbf{V} = \mathbf{IR} = (3.6\text{ mA}\angle 0°)(500\ \Omega\angle 0°) = 1.8\text{ V}\angle 0°$$

d.
$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{Z}_L} = \frac{1.8\text{ V}\angle 0°}{200\ \Omega\angle 90°} = 9.0\text{ mA}\angle -90°$$

$$\mathbf{I}_R = \mathbf{I} = 3.6\text{ mA}\angle 0°$$

e.
$$\text{BW(rad/s)} = \frac{\omega_P}{Q_P} = \frac{12.5\text{ krad/s}}{2.5} = 5\text{ krad/s}$$

$$\text{BW(Hz)} = \frac{\text{BW(rad/s)}}{2\pi} = \frac{5\text{ krad/s}}{2\pi} = 795.8\text{ Hz}$$

f. The half-power frequencies are calculated from Equations 21–48 and 21–49 since the $Q$ of the circuit is less than 10.

$$\begin{aligned}
\omega_1 &= -\frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \\
&= -\frac{1}{0.0004} + \frac{1}{1.6 \times 10^{-7}} + \frac{1}{6.4 \times 10^{-9}} \\
&= -2500 + 12\,748 \\
&= 10\,248\text{ rad/s}
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \\
&= \frac{1}{1.0004} + \sqrt{\frac{1}{1.6 \times 10^{-7}} + \frac{1}{6.4 \times 10^{-9}}} \\
&= 2500 + 12\,748 \\
&= 15\,248\text{ rad/s}
\end{aligned}$$

The resulting voltage response curve is illustrated in Figure 21– 32.

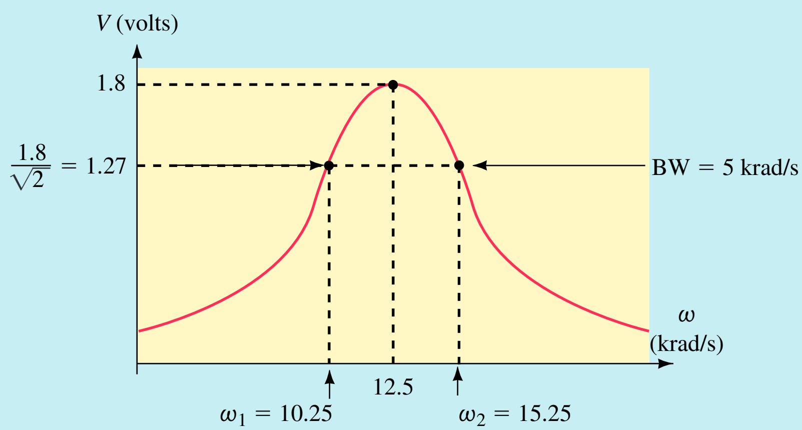



**FIGURE 21–32**

g. The power dissipated by the circuit at resonance is

$$P = \frac{V^2}{R} = \frac{(1.8\text{ V})^2}{500\ \Omega} = 6.48\text{ mW}$$

The selectivity curve is now easily sketched as shown in Figure 21–33.

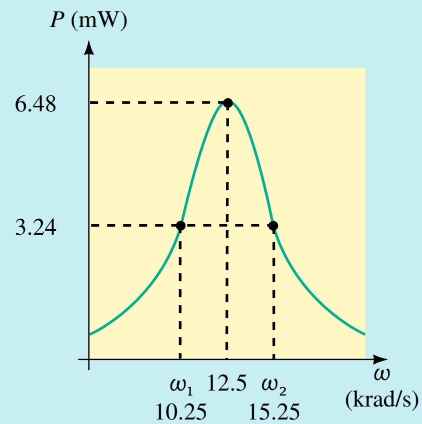
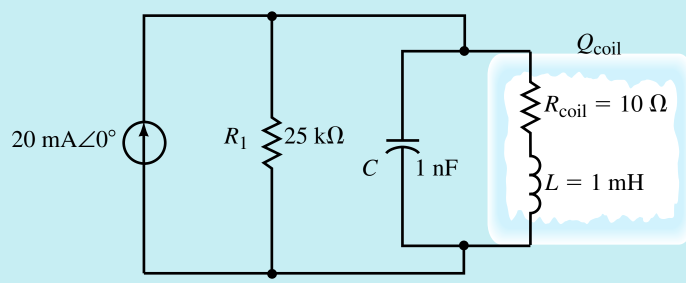



**FIGURE 21–33**

**EXAMPLE 21–7** Consider the circuit of Figure 21–34.




EWB **FIGURE 21–34**

a. Calculate the resonant frequency, $\omega_r$, of the tank circuit.

b. Find the $Q$ of the coil at resonance.

c. Sketch the equivalent parallel circuit.

d. Determine the $Q$ of the entire circuit at resonance.

e. Solve for the voltage across the capacitor at resonance.

f. Find the bandwidth of the circuit in radians per second.

g. Sketch the voltage response of the circuit showing the voltage at the half-power frequencies.

**Solution**

a. Since the ratio $L/C = 1000 \geq 100R_{\text{coil}}$, we use the approximation:

$$\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1\text{ mH})\,(1\text{ nF})}} = 1\text{ Mrad/s}$$

b.
$$Q_{\text{coil}} = \frac{\omega L}{R_{\text{coil}}} = \frac{(1\text{ Mrad/s})\,(1\text{ mH})}{10\ \Omega} = 100$$

c. $$R_P \cong Q^2R_{coil} = (100)^2(10\ \Omega) = 100\ k\Omega$$
$$X_{LP} \cong X_{LS} = \omega L = (1\ Mrad/s)\ (1\ mH) = 1\ k\Omega$$

The circuit of Figure 21–35 shows the circuit with the parallel equivalent of the inductor.

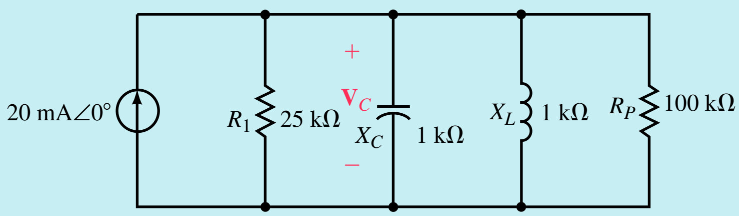



**FIGURE 21–35**

We see that the previous circuit may be further simplified by combining the parallel resistances:

$$R_{eq} = R_1 \| R_P = \frac{(25\ k\Omega)(100\ k\Omega)}{25\ k\Omega + 100\ k\Omega} = 20\ k\Omega$$

The simplified equivalent circuit is shown in Figure 21–36.

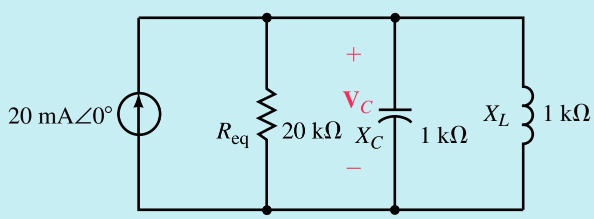



**FIGURE 21–36**

d. $$Q = \frac{R_{eq}}{X_L} = \frac{20\ k\Omega}{1\ k\Omega} = 20$$

e. At resonance,

$$\mathbf{V}_C = \mathbf{I}R_{eq} = (20\ mA\angle 0°)\ (20\ k\Omega) = 400\ V\angle 0°$$

f. $$BW = \frac{\omega_r}{Q} = \frac{1\ Mrad/s}{20} = 50\ krad/s$$

g. The voltage response curve is shown in Figure 21–37. Since $Q \geq 10$, the half-power frequencies will occur at the following angular frequencies:

$$\omega_1 \cong \omega_r - \frac{BW}{2} = 1.0\ Mrad/s - \frac{50\ krad/s}{2} = 0.975\ Mrad/s$$

and

$$\omega_2 \cong \omega_r + \frac{BW}{2} = 1.0\ Mrad/s + \frac{50\ krad/s}{2} = 1.025\ Mrad/s$$

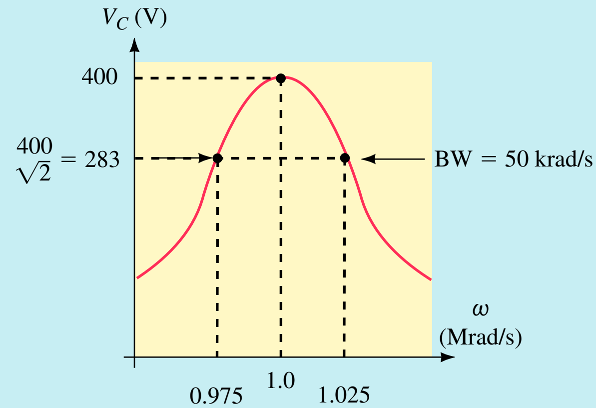



**FIGURE 21–37**

**EXAMPLE 21–8** Determine the values of $R_1$ and $C$ for the resonant tank circuit of Figure 21–38 so that the given conditions are met.

$L = 10$ mH, $R_{coil} = 30\ \Omega$

$f_P = 58$ kHz

BW = 1 kHz

Solve for the current, $\mathbf{I}_L$, through the inductor.

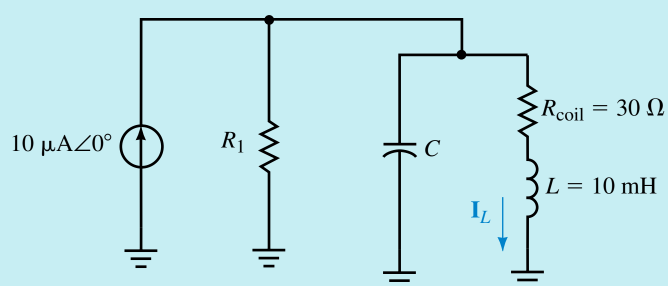



**FIGURE 21–38**

**Solution**

$$Q = \frac{f_P}{\text{BW(Hz)}} = \frac{580 \text{ kHz}}{10 \text{ kHz}} = 58$$

Now, because the frequency expressed in radians per second is more useful than hertz, we convert $f_P$ to $\omega_P$:

$$\omega_P = 2\pi f_P = (2\pi)(58 \text{ kHz}) = 364.4 \text{ krad/s}$$

The capacitance is determined from Equation 21–43 as

$$C = \frac{1}{\omega_P^2 L} = \frac{1}{(364.4 \text{ krad/s})^2(10 \text{ mH})} = 753 \text{ pF}$$

Solving for the $Q$ of the coil permits us to easily convert the series $RL$ network into its equivalent parallel network.

$$Q_{\text{coil}} = \frac{\omega_L}{R_{\text{coil}}}$$

$$= \frac{(364.4\ \text{krad/s})(10\ \text{mH})}{30\ \Omega}$$

$$= \frac{3.644\ \text{k}\Omega}{30\ \Omega} = 121.5$$

$$R_P \cong Q^2_{\text{coil}}R_S = (121.5)^2(30\ \Omega) = 443\ \text{k}\Omega$$

$$X_{LP} \cong X_{LS} = 3644\ \Omega$$

The resulting equivalent parallel circuit is shown in Figure 21–39.

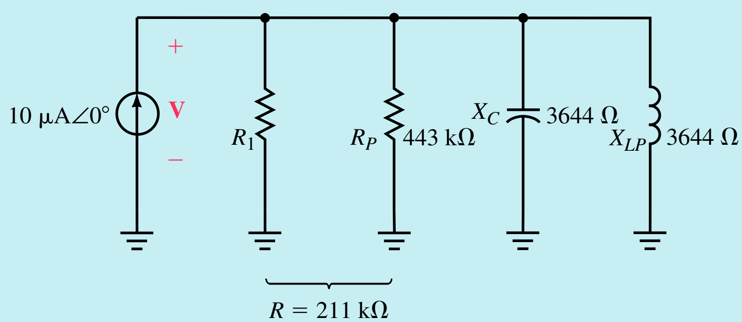



**FIGURE 21–39**

The quality factor, $Q$, is used to determine the total resistance of the circuit as

$$R = QX_C = (58)(3.644\ \text{k}\Omega) = 211\ \text{k}\Omega$$

But

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_P}$$

$$\frac{1}{R_1} = \frac{1}{R} - \frac{1}{R_P} = \frac{1}{211\ \text{k}\Omega} - \frac{1}{443\ \text{k}\Omega} = 2.47\ \mu\text{S}$$

And so

$$R_1 = 405\ \text{k}\Omega$$

The voltage across the circuit is determined to be

$$\mathbf{V} = \mathbf{I}R = (10\ \mu\text{A}\angle 0°)(211\ \text{k}\Omega) = 2.11\ \text{V}\angle 0°$$

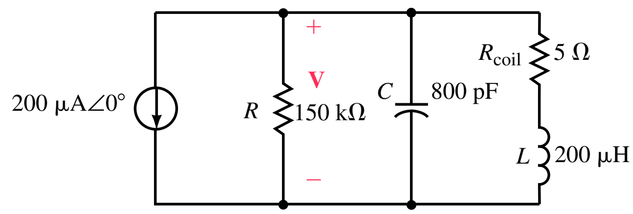
and the current through the inductor is

$$\mathbf{I}_L = \frac{\mathbf{V}}{R_{\text{coil}} + jX_L}$$

$$= \frac{2.11\ \text{V}\angle 0°}{30 + j3644\ \Omega} = \frac{2.11\ \text{V}\angle 0°}{3644\ \Omega\angle 89.95°} = 579\ \mu\text{A}\angle -89.95°$$

**PRACTICE PROBLEMS 5**

Refer to the circuit of Figure 21–40:




EWB **FIGURE 21–40**

a. Determine the resonant frequency and express it in radians per second and in hertz.

b. Calculate the quality factor of the circuit.

c. Solve for the bandwidth.

d. Determine the voltage **V** at resonance.

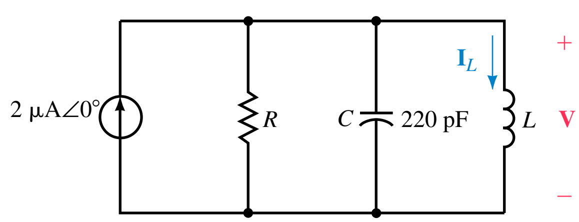
*Answers:*

a. 2.5 Mrad/s (398 kHz) b. 75

c. 33.3 krad/s (5.31 kHz) d. 7.5 V∠180°

**IN-PROCESS LEARNING CHECK 4**

Refer to the parallel resonant circuit of Figure 21–41:




$f_P$ = 800 kHz
BW = 25 kHz

**FIGURE 21–41**

Suppose the circuit has a resonant frequency of 800 kHz and a bandwidth of 25 kHz.

a. Determine the value of the inductor, $L$, in henries.

b. Calculate the value of the resistance, $R$, in ohms.

c. Find **V**, $\mathbf{I}_L$, and power, $P$, at resonance.

d. Find the appoximate values of the half-power frequencies, $f_1$ and $f_2$.

e. Determine the voltage across the circuit at the lower half-power frequency, $f_1$, and show that the power dissipated by the resistor at this frequency is half the power dissipated at the resonant frequency.

*(Answers are at the end of the chapter.)*

## 21.7 Circuit Analysis Using Computers

PSpice is particularly useful in examining the operation of resonant circuits. The ability of the software to provide a visual display of the frequency response is used to evaluate the resonant frequency, maximum current, and bandwidth of a circuit. The *Q* of the given circuit is then easily determined.

### OrCAD PSpice

EXAMPLE 21–9 Use OrCAD PSpice to obtain the frequency response for current in the circuit of Figure 21–12. Use cursors to find the resonant frequency and the bandwidth of the circuit from the observed response. Compare the results to those obtained in Example 21–2.

**Solution**

OrCAD Capture CIS is used to input the circuit as shown in Figure 21–42. For this example, the project is titled **EXAMPLE 21-9.** The voltage source used in this example is VAC and the value is changed to AC=**10V 0Deg.** In order to obtain a plot of the circuit current, use the Current Into Pin tool as shown.

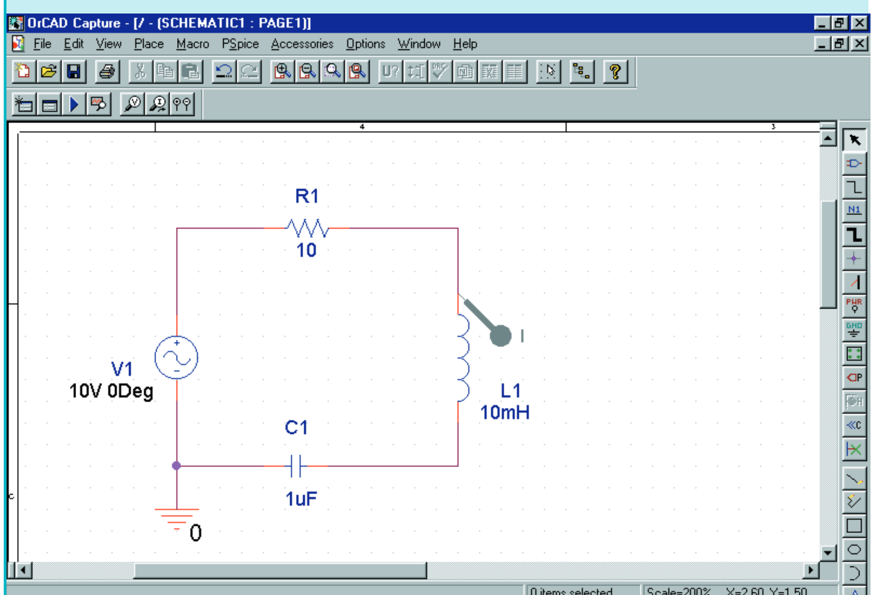



**FIGURE 21–42**

Next, we change the simulation settings by clicking on the New Simulation Profile tool. Give the simulation a name such as **Series Resonance** and click on Create. Once you are in the simulation settings box, click on the Analysis tab and select AC Sweep/Noise as the analysis type. The frequency can be swept either linearly or logarithmically (decade or octave). In this example we select a logarithmic sweep through a decade. In the box titled AC Sweep Type, click on ⊙Logarithmic and select Decade. Type the following values as the settings. Start Frequency: **1kHz,** End Frequency: **10kHz,** and Points/Decade: **10001.** Click OK.

Click on the Run tool. If there are no errors, the PROBE postprocessor will run automatically and display I(L1) as a function of frequency. You will notice that the selectivity curve is largely contained within a narrow range of frequencies. We may zoom into this region as follows. Select the Plot menu, and click on Axis Settings menu item. Click on the X Axis tab and select User Defined Data Range. Change the values to **1kHz** to **3kHz.** Click OK. The resulting display is shown in Figure 21–43.

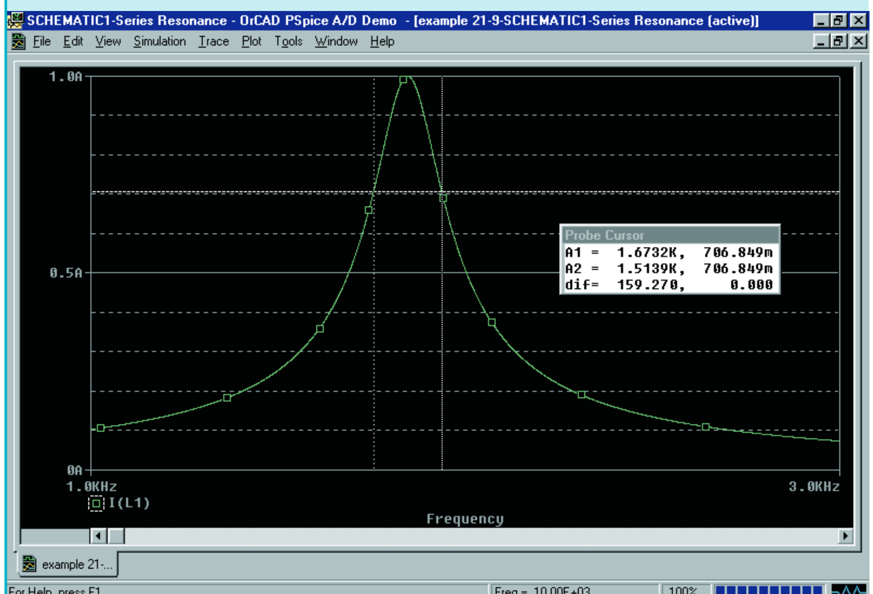



**FIGURE 21–43**

Finally, we may use cursors to provide us with the actual resonant frequency, the maximum current and the half-power frequencies. Cursors are obtained as follows. Click on Trace, Cursor, and Display. The positions of the cursors are adjusted by using either the mouse or the arrow and <Shift> keys. The current at the maximum point of the curve is obtained by clicking on Trace, Cursor, and Max. The dialog box provides the values of both the frequency and the value of current. The bandwidth is determined by determining the frequencies at the half-power points (when the current is 0.707 of the maximum value). We obtain the following results using the cursors:

$$I_{\max} = 1.00\,\text{A}, f_S = 1.591\,\text{kHz}, f_1 = 1.514\,\text{kHz}, f_2 = 1.673\,\text{kHz}, \text{BW} = 0.159\,\text{kHz}.$$

These values correspond very closely to those calculated in Example 21–2.

**EXAMPLE 21–10** Use OrCAD PSpice to obtain the frequency response for the voltage across the parallel resonant circuit of Figure 21–34. Use the PROBE postprocessor to find the resonant frequency, the maximum voltage (at resonance), and the bandwidth of the circuit. Compare the results to those obtained in Example 21–7.

**Solution**

This example is similar to the previous example, with minor exceptions. The OrCAD Capture program is used to enter the circuit as shown in Figure 21–44. The ac current source is found in the SOURCE library as IAC. The value of the ac current source is changed AC=**20mA 0Deg.** The Voltage Level tool is used to provide the voltage simulation for the circuit.

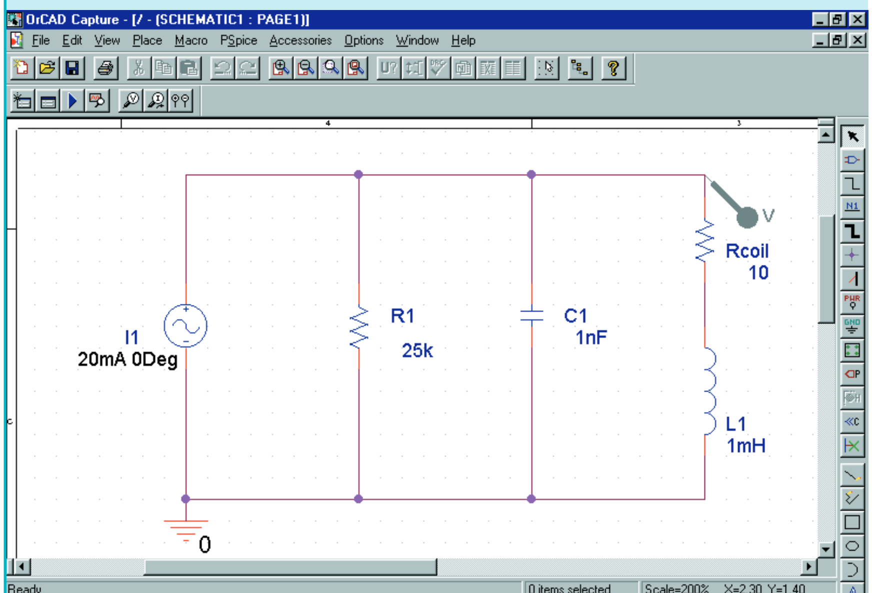



**FIGURE 21–44**

Use the New Simulation Profile tool to set the simulation for a logarithmic ac sweep from 100kHz to 300kHz with a total of 10001 points per decade. Click on the Plot menu to select the Axis Settings. Change the x-axis to indicate a User Defined range from **100kHz** to **300kHz** and change the y-axis to indicate a User Defined range from **0V** to **400V.** The resulting display is shown in Figure 21–45.

As in the previous example, we use cursors to observe that the maximum circuit voltage, $V_{max}$ = 400 V, occurs at the resonance frequency as $f_P$ = 159.2 kHz (1.00 Mrad/s). The half-power frequencies are determined when the output voltage is at 0.707 of the maximum value, namely at $f_1$ = 155.26 kHz (0.796 Mrad/s) and $f_2$ = 163.22 kHz (1.026 Mrad/s). These frequencies give a bandwidth of BW = 7.96 kHz (50.0 krad/s). The above results are the same as those found in Example 21–7.

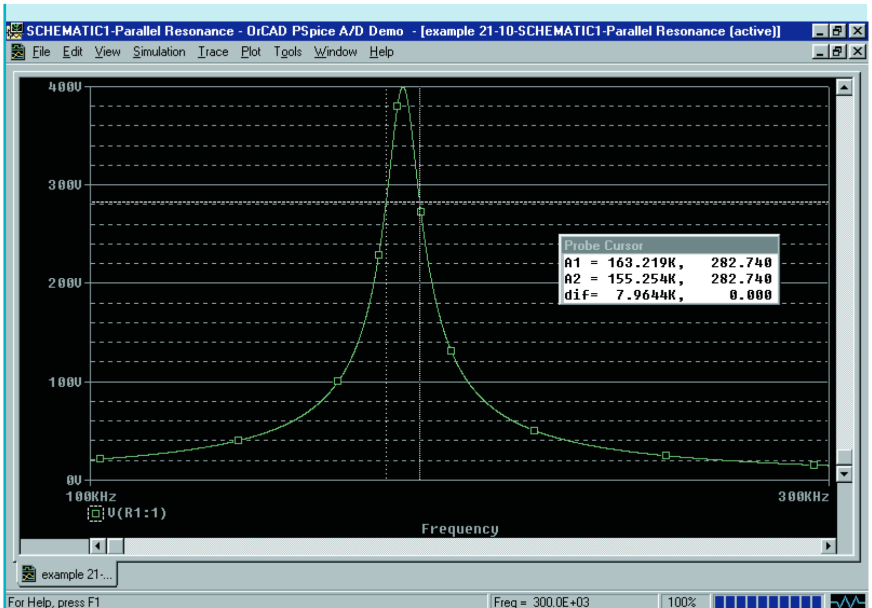



**FIGURE 21–45**

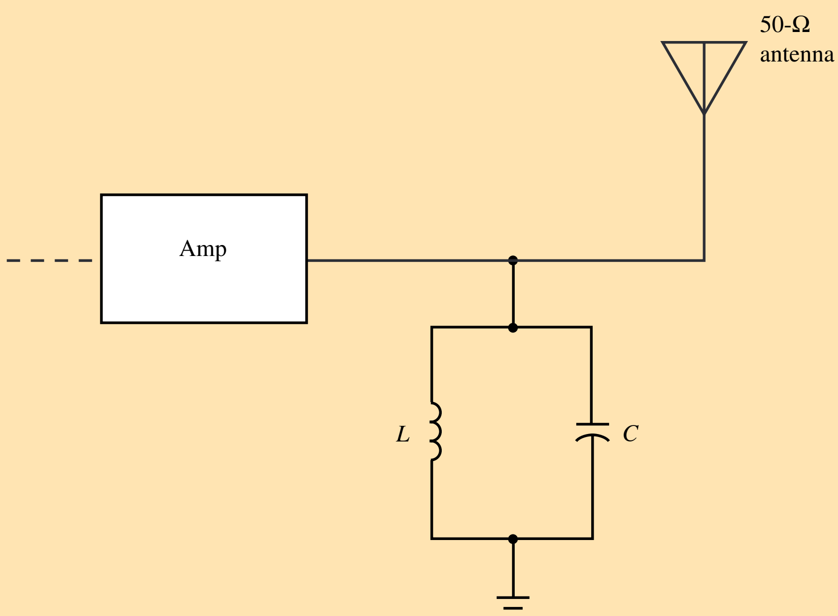
**PRACTICE PROBLEMS 6**

Use OrCAD PSpice to obtain the frequency response of voltage, $V$ versus $f$, for the circuit of Figure 21–40. Use cursors to determine the approximate values of the half-power frequencies and the bandwidth of the circuit. Compare the results to those obtained in Practice Problem 5.

*Answers:*
$V_{max} = 7.50$ V, $f_P = 398$ kHz, $f_1 = 395.3$ kHz , $f_2 = 400.7$ kHz, BW $= 5.31$ kHz

## PUTTING IT INTO PRACTICE

You are the transmitter specialist at an AM commercial radio station, which transmits at a frequency of 990 kHz and an average power of 10 kW. As is the case for all commercial AM stations, the bandwidth for your station is 10 kHz. Your transmitter will radiate the power using a 50-Ω antenna. The accompanying figure shows a simplified block diagram of the output stage of the transmitter. The antenna behaves exactly like a 50-Ω resistor connected between the output of the amplifier and ground.



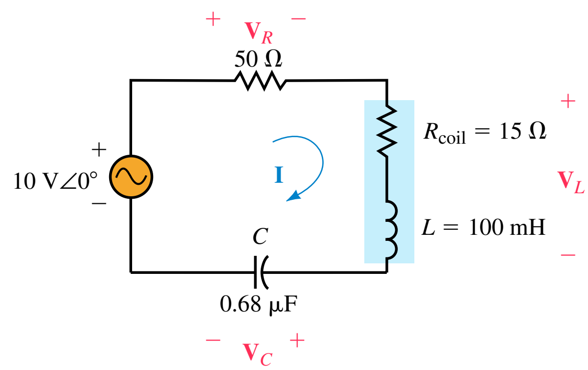

Transmitter stage of a commercial AM radio station.

You have been asked to determine the values of $L$ and $C$ so that the transmitter operates with the given specifications. As part of the calculations, determine the peak current that the inductor must handle and solve for the peak voltage across the capacitor. For your calculations, assume that the transmitted signal is a sinusoidal.

## PROBLEMS

### 21.1 Series Resonance

1. Consider the circuit of Figure 21–46:
   a. Determine the resonant frequency of the circuit in both radians per second and hertz.
   b. Calculate the current, **I**, at resonance.
   c. Solve for the voltages $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$. (Notice that the voltage $\mathbf{V}_L$ includes the voltage dropped across the internal resistance of the coil.)
   d. Determine the power (in watts) dissipated by the inductor. (Hint: The power will not be zero.)




EWB **FIGURE 21–46**

2. Refer to the circuit of Figure 21–47:
   a. Determine the resonant frequency of the circuit in both radians per second and hertz.
   b. Calculate the phasor current, **I**.
   c. Determine the power dissipated by the circuit at resonance.
   d. Calculate the phasor voltages, $\mathbf{V}_L$ and $\mathbf{V}_R$.
   e. Write the sinusoidal form of the voltages $v_L$ and $v_R$.

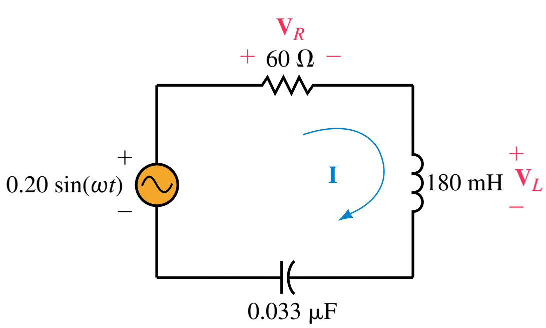



**FIGURE 21–47**

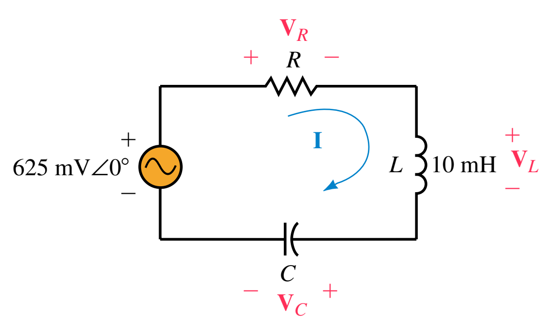



**FIGURE 21–48**

3. Consider the circuit of Figure 21–48:
   a. Determine the values of $R$ and $C$ such that the circuit has a resonant frequency of 25 kHz and an rms current of 25 mA at resonance.
   b. Calculate the power dissipated by the circuit at resonance.
   c. Determine the phasor voltages, $\mathbf{V}_C$, $\mathbf{V}_L$, and $\mathbf{V}_R$.
   d. Write the sinusoidal expressions for the voltages $v_C$, $v_L$, and $v_R$.
4. Refer to the circuit of Figure 21–49.
   a. Determine the capacitance required so that the circuit has a resonant frequency of 100 kHz.
   b. Solve for the phasor quantities **I**, $\mathbf{V}_L$, and $\mathbf{V}_R$.
   c. Find the sinusoidal expressions for $i$, $v_L$, and $v_R$.
   d. Determine the power dissipated by each element in the circuit.

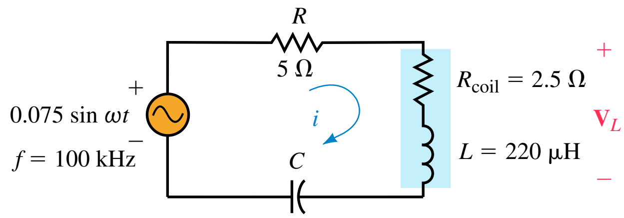



FIGURE 21–49

## 21.2 Quality Factor, $Q$

5. Refer to the circuit of Figure 21–50:
   a. Determine the resonant frequency expressed as $\omega$(rad/s) and $f$(Hz).
   b. Calculate the total impedance, $\mathbf{Z}_T$, at resonance.
   c. Solve for current **I** at resonance.
   d. Solve for $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$ at resonance.
   e. Calculate the power dissipated by the circuit and evaluate the reactive powers, $Q_C$ and $Q_L$.
   f. Find the quality factor, $Q_S$, of the circuit.

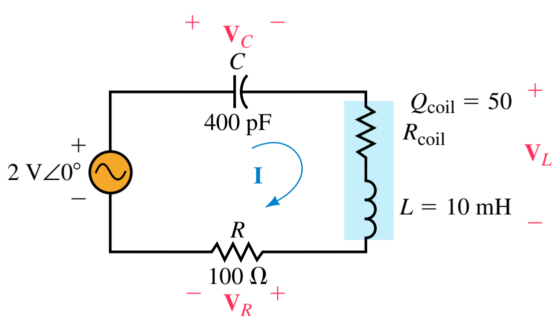



FIGURE 21–50

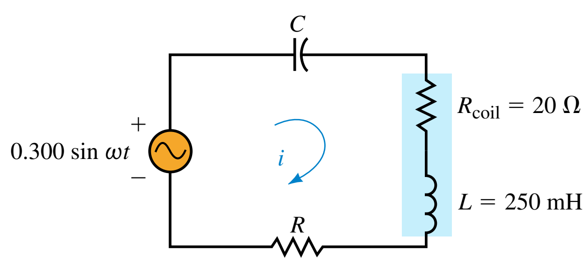



FIGURE 21–51

6. Suppose that the circuit of Figure 21–51 has a resonant frequency of $f_S$ = 2.5 kHz and a quality factor of $Q_S = 10$:
   a. Determine the values of $R$ and $C$.
   b. Solve for the quality factor of inductor, $Q_{\text{coil}}$.
   c. Find $\mathbf{Z}_T$, **I,** $\mathbf{V}_C$, and $\mathbf{V}_R$ at resonance.
   d. Solve for the sinusoidal expression of current $i$ at resonance.
   e. Calcuate the sinusoidal expressions $v_C$ and $v_R$ at resonance.
   f. Calculate the power dissipated by the circuit and determine the reactive powers, $Q_C$ and $Q_L$.
7. Refer to the circuit of Figure 21–52:
   a. Design the circuit to have a resonant frequency of $\omega$ = 50 krad/s and a quality factor $Q_S = 25$.
   b. Calculate the power dissipated by the circuit at the resonant frequency.
   c. Determine the voltage, $\mathbf{V}_L$, across the inductor at resonance.

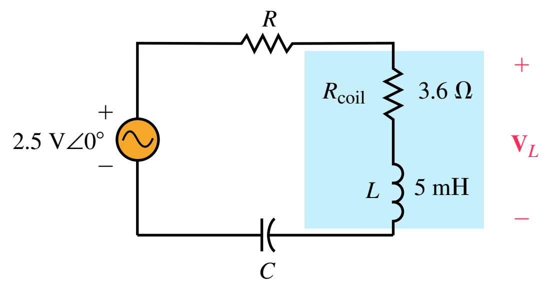



FIGURE 21–52

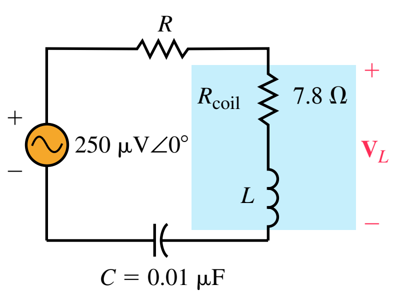



FIGURE 21–53

8. Consider the circuit of Figure 21–53:
   a. Design the circuit to have a resonant frequency of $\omega = 400$ krad/s and a quality factor $Q_S = 10$.
   b. Calculate the power dissipated by the circuit at the resonant frequency.
   c. Determine the voltage, $\mathbf{V}_L$, across the inductor at resonance.

## 21.3 Impedance of a Series Resonant Circuit

9. Refer to the series resonant circuit of Figure 21–54.
   a. Determine the resonant frequency, $\omega_S$.
   b. Solve for the input impedance, $\mathbf{Z}_\mathrm{T} = Z\angle\theta$, of the circuit at frequencies of $0.1\omega_S$, $0.2\omega_S$, $0.5\omega_S$, $\omega_S$, $2\omega_S$, $5\omega_S$, and $10\omega_S$.
   c. Using the results from (b), sketch a graph of $Z$ (magnitude in ohms) versus $\omega$ (in radians per second) and a graph of $\theta$ (in degrees) versus $\omega$ (in radians per second). If possible, use log-log graph paper for the former and semilog graph paper for the latter.
   d. Using your results from (b), determine the magnitude of current at each of the given frequencies.
   e. Use the results from (d) to plot a graph of $I$ (magnitude in amps) versus $\omega$ (in radians per second) on log-log graph paper.

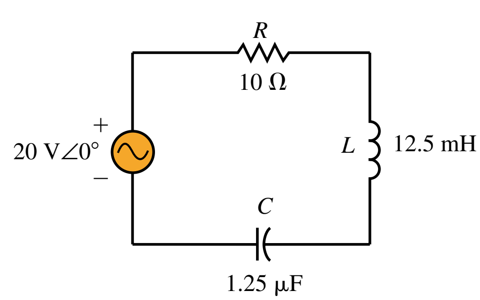



FIGURE 21–54

10. Repeat Problem 9 if the 10-Ω resistor is replaced with a 50-Ω resistor.

## 21.4 Power, Bandwidth, and Selectivity of a Series Resonant Circuit

11. Refer to the circuit of Figure 21–55.
    a. Find $\omega_S$, $Q$, and BW (in radians per second).
    b. Calculate the maximum power dissipated by the circuit.

c. From the results obtained in (a) solve for the approximate half-power frequencies, $\omega_1$ and $\omega_2$.

d. Calculate the actual half-power frequencies, $\omega_1$ and $\omega_2$ using the component values and the appropriate equations.

e. Are the results obtained in (c) and (d) comparable? Explain.

f. Solve for the circuit current, **I**, and power dissipated at the lower half-power frequency, $\omega_1$, determined in (d).

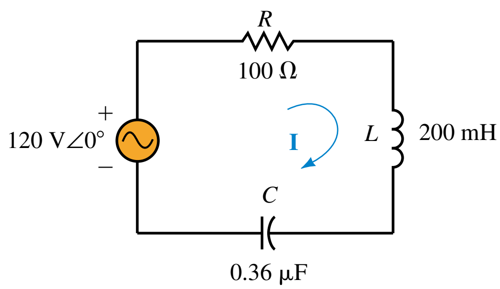



FIGURE 21–55

12. Repeat Problem 11 for the circuit of Figure 21–56.

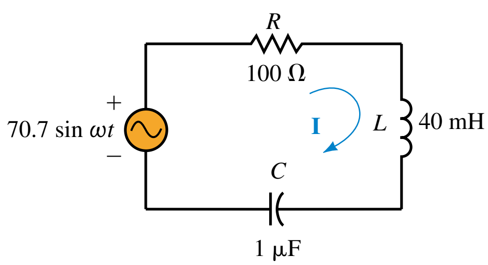



FIGURE 21–56

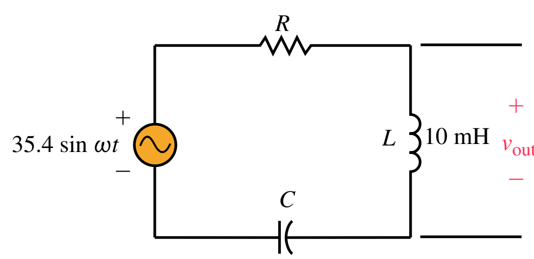



FIGURE 21–57

13. Consider the circuit of Figure 21–57.
    a. Calculate the values of $R$ and $C$ for the circuit to have a resonant frequency of 200 kHz and a bandwidth of 16 kHz.
    b. Use the designed component values to determine the power dissipated by the circuit at resonance.
    c. Solve for $v_{out}$ at resonance.

14. Repeat Problem 13 if the resonant frequency is to be 580 kHz and the bandwidth is 10 kHz.

## 21.5 Series-to-Parallel *RL* and *RC* Conversion

15. Refer to the series networks of Figure 21–58.
    a. Find the $Q$ of each network at $\omega = 1000$ rad/s.
    b. Convert each series $RL$ network into an equivalent parallel network, having $R_P$ and $X_{LP}$ in ohms.
    c. Repeat (a) and (b) for $\omega = 10$ krad/s.

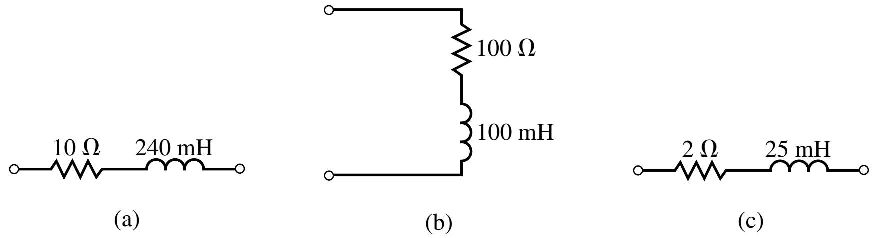



**FIGURE 21–58**

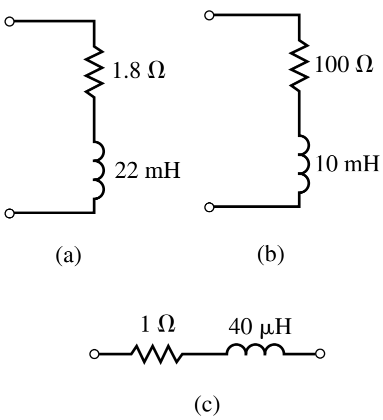



**FIGURE 21–59**

16. Consider the series networks of Figure 21–59.
    a. Find the $Q$ of each coil at $\omega = 20$ krad/s.
    b. Convert each series $RL$ network into an equivalent parallel network consisting of $R_P$ and $X_{LP}$ in ohms.
    c. Repeat (a) and (b) for $\omega = 100$ krad/s.
17. For the series networks of Figure 21–60, find the $Q$ and convert each network into its parallel equivalent.

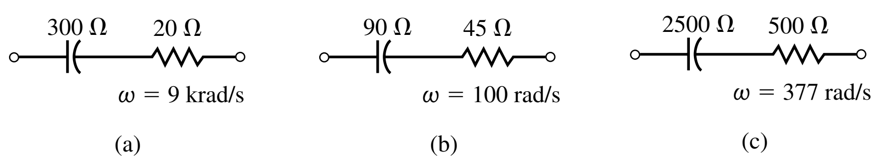



**FIGURE 21–60**

18. Derive Equations 21–34 and 21–35, which enable us to convert a parallel $RL$ network into its series equivalent. (Hint: Begin by determining the expression for the input impedance of the parallel network.)
19. Find the $Q$ of each of the networks of Figure 21–61 and determine the series equivalent of each. Express all component values in ohms.

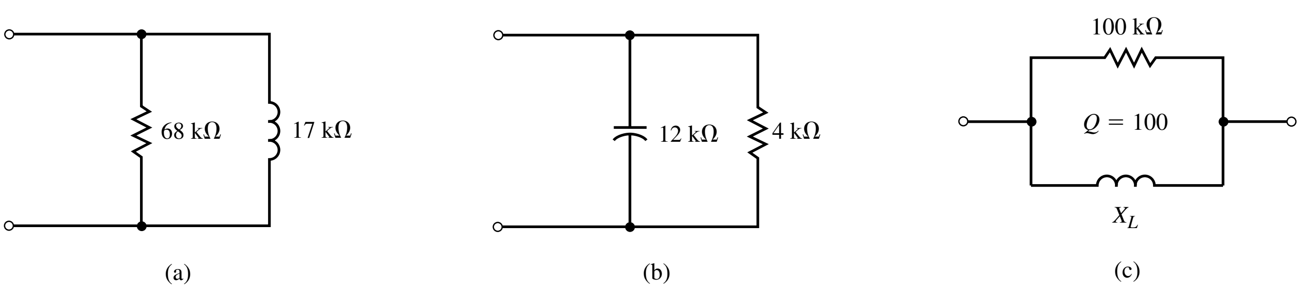



**FIGURE 21–61**

20. Repeat Problem 19 for the networks of Figure 21–62.
21. Determine the values of $L_S$ and $L_P$ in henries, given that the networks of Figure 21–63 are equivalent at a frequency of 250 krad/s.
22. Determine the values of $C_S$ and $C_P$ in farads, given that the networks of Figure 21–64 are equivalent at a frequency of 48 krad/s.

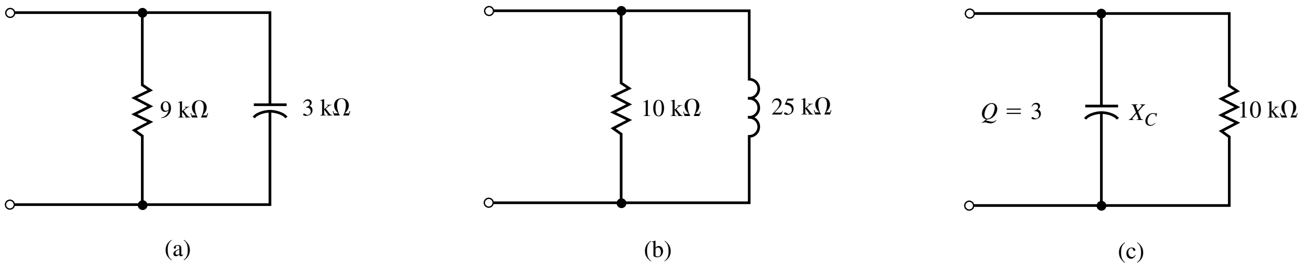



FIGURE 21–62

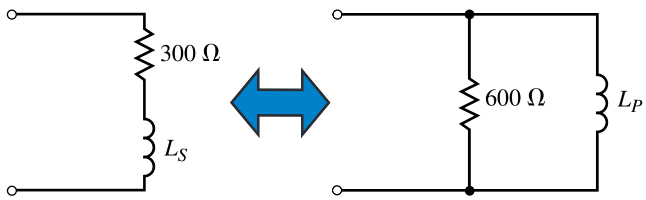



FIGURE 21–63

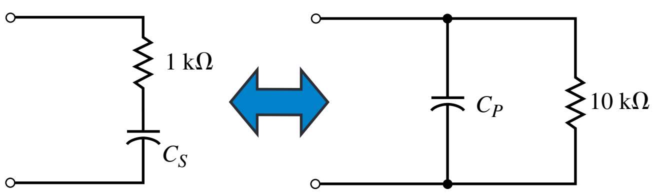



FIGURE 21–64

### 21.6 Parallel Resonance

23. Consider the circuit of Figure 21–65.

a. Determine the resonant frequency, $\omega_P$, in radians per second.

b. Solve for the input impedance, $\mathbf{Z}_T = Z\angle\theta$, of the circuit at frequencies of $0.1\omega_P$, $0.2\omega_P$, $0.5\omega_P$, $\omega_P$, $2\omega_P$, $5\omega_P$, and $10\omega_P$.

c. Using the results obtained in (b), sketch graphs of $Z$ (magnitude in ohms) versus $\omega$ (in radians per second) and $\theta$ (in degrees) versus $\omega$. If possible, use log-log graph paper for the former and semilog for the latter.

d. Using the results from (b), determine the voltage $\mathbf{V}$ at each of the indicated frequencies.

e. Sketch a graph of magnitude $V$ versus $\omega$ on log-log graph paper.

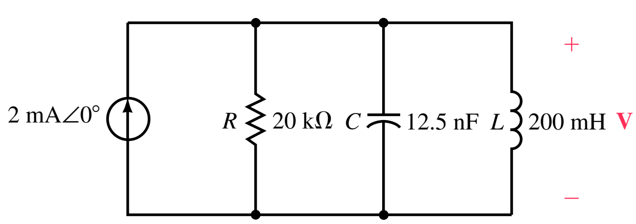



FIGURE 21–65

24. Repeat Problem 23 if the 20-kΩ resistor is replaced with a 40-kΩ resistor.
25. Refer to the circuit shown in Figure 21–66.
    a. Determine the resonant frequencies, $\omega_P$(rad/s) and $f_P$(Hz).
    b. Find the $Q$ of the circuit.
    c. Calculate $\mathbf{V}$, $\mathbf{I}_R$, $\mathbf{I}_L$, and $\mathbf{I}_C$ at resonance.
    d. Determine the power dissipated by the circuit at resonance.
    e. Solve for the bandwidth of the circuit in both radians per second and hertz.
    f. Sketch the voltage response of the circuit, showing the voltage at the half-power frequencies.

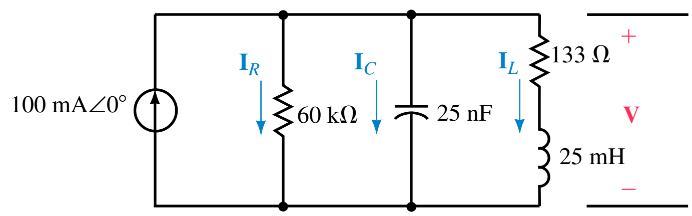



FIGURE 21–66

26. Repeat Problem 25 for the circuit of Figure 21–67.

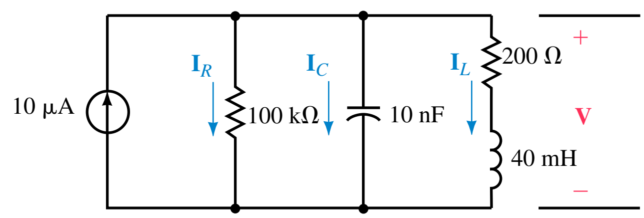



FIGURE 21–67

27. Determine the values of $R_1$ and $C$ for the resonant tank circuit of Figure 21–68 so that the given conditions are met. Solve for current $\mathbf{I}_L$ through the inductor.

    $L = 25$ mH, $R_{coil} = 100\ \Omega$
    $f_P = 50$ kHz
    BW $= 10$ kHz

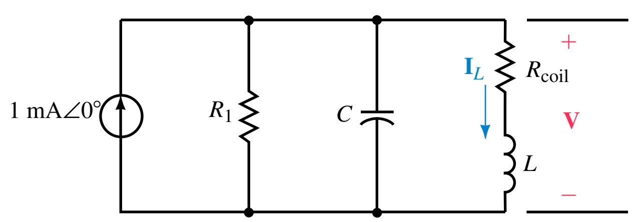



FIGURE 21–68

28. Determine the values of $R_1$ and $C$ for the resonant circuit of Figure 21–68 so that the given conditions are met. Solve for the voltage, **V**, across the circuit.

    $L = 50$ mH, $R_{coil} = 50\ \Omega$

    $\omega_P = 100$ krad/s

    BW $= 10$ krad/s

29. Refer to the circuit of Figure 21–69.
    a. Determine the value of $X_L$ for resonance.
    b. Solve for the $Q$ of the circuit.
    c. If the circuit has a resonant frequency of 2000 rad/s, what is the bandwidth of the circuit?
    d. What must be the values of $C$ and $L$ for the circuit to be resonant at 2000 rad/s?
    e. Calculate the voltage $\mathbf{V}_C$ at resonance.

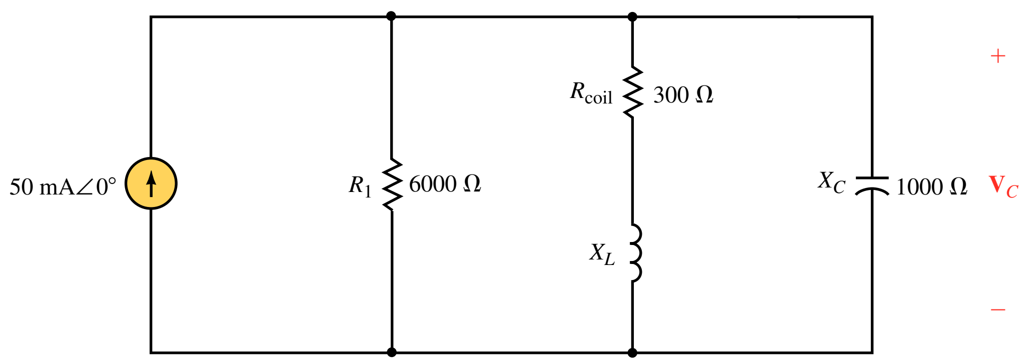



**FIGURE 21–69**

30. Repeat Problem 29 for the circuit of Figure 21–70.

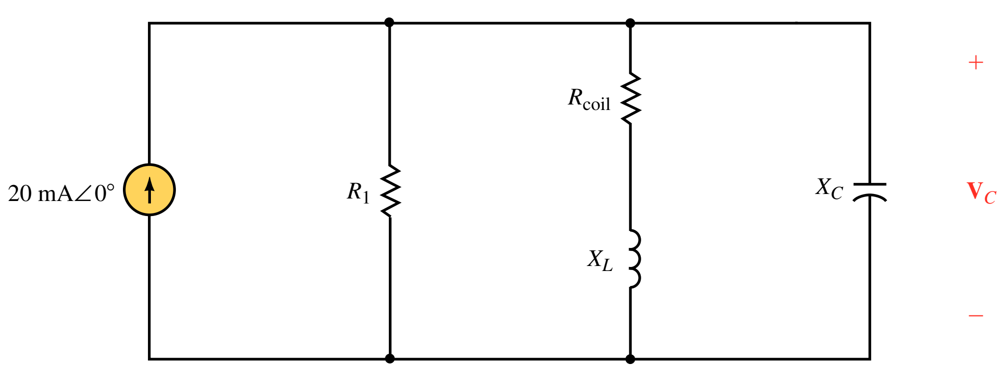



**FIGURE 21–70**

### 21.7 Circuit Analysis Using Computers

31. **PSpice** Use OrCAD PSpice to input the circuit of Figure 21–55. Use the Probe postprocessor to display the response of the inductor voltage as a function of frequency. From the display, determine the maximum rms voltage, the resonant frequency, the half-power frequencies, and the bandwidth. Use the results to determine the quality factor of the circuit.
32. **PSpice** Repeat Problem 31 for the circuit of Figure 21–56.
33. **PSpice** Use OrCAD PSpice to input the circuit of Figure 21–66. Use the Probe postprocessor to display the response of the capacitor voltage as a function of frequency. From the display, determine the maximum rms voltage, the resonant frequency, the half-power frequencies, and the bandwidth. Use the results to determine the quality factor of the circuit.
34. **PSpice** Repeat Problem 33 for the circuit of Figure 21–67.
35. **PSpice** Use the values of $C$ and $L$ determined in Problem 29 to input the circuit of Figure 21–69. Use the Probe postprocessor to display the response of the capacitor voltage as a function of frequency. From the display, determine the maximum rms voltage, the resonant frequency, the half-power frequencies, and the bandwidth. Use the results to determine the quality factor of the circuit.
36. **PSpice** Use the values of $C$ and $L$ determined in Problem 30 to input the circuit of Figure 21–70. Repeat the measurements of Problem 35.

---

## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

a. $L = 320\ \mu\text{H}$

b. $R = 20.1\ \Omega$

c. $\mathbf{I} = 0.995\ \text{mA}\angle 0°$ $\mathbf{V}_L = 1.20\ \text{V}\angle 90°$ $P = 20.0\ \mu\text{W}$

d. $f_1 = 595\ \text{kHz}$ $f_2 = 605\ \text{kHz}$

e. $\mathbf{I} = 0.700\ \text{mA}\angle 45.28°$ $P_1 = 9.85\ \mu\text{W}$ $P_1/P = 0.492 \cong 0.5$

**In-Process Learning Check 2**

a. $\omega_S = 10\ \text{krad/s}$ $P = 500\ \mu\text{W}$

b. $Q = 10$ $\text{BW} = 1\ \text{krad/s}$ $\omega_1 = 9.5\ \text{krad/s}$ $\omega_2 = 10.5\ \text{krad/s}$

d. $\omega_S = 10\ \text{krad/s}$ $P = 1000\ \mu\text{W}$ $Q = 20$ $\text{BW} = 0.5\ \text{krad/s}$
$\omega_1 = 9.75\ \text{krad/s}$ $\omega_2 = 10.25\ \text{krad/s}$

e. As resistance decreases, selectivity increases.

**In-Process Learning Check 3**

a. $X_L = 200\ \Omega$ $R_P = 1600\ \Omega$ $X_C = 80\ \Omega$ $R_P = 40\ \Omega$

b. $X_{LS} = 197\ \Omega$ $R_S = 24.6\ \Omega$ $X_{CS} = 16\ \Omega$ $R_S = 32\ \Omega$

**In-Process Learning Check 4**

a. $L = 180\ \mu\text{H}$

b. $R_p = 28.9\ \text{k}\Omega$

c. $\mathbf{V} = 57.9\ \text{mV}\angle 0°$  $\mathbf{I}_L = 64.0\ \mu\text{A}\angle -90°$  $P = 115\ \text{nW}$

d. $f_1 = 788\ \text{kHz}$  $f_2 = 813\ \text{kHz}$

e. $\mathbf{V} = 41.1\ \text{mV}\angle 44.72°$  $P = 58\ \text{nW}$

# 22 Filters and the Bode Plot

## OBJECTIVES

After studying this chapter you will be able to

- evaluate the power gain and voltage gain of a given system,
- express power gain and voltage gain in decibels,
- express power levels in dBm and voltage levels in dBV and use these levels to determine power gain and voltage gain,
- identify and design simple (first-order) *RL* and *RC* low-pass and high-pass filters and explain the principles of operation of each type of filter,
- write the standard form of a transfer function for a given filter. The circuits which are studied will include band-pass and band-stop as well as low- and high-pass circuits,
- compute $\tau_c$ and use the time constant to determine the cutoff frequency(ies) in both radians per second and hertz for the transfer function of any first-order filter,
- sketch the Bode plot showing the frequency response of voltage gain and phase shift of any first-order filter,
- use PSpice to verify the operation of any first-order filter circuit.

## KEY TERMS

Amplifier
Attenuator
Bode Plots
Cutoff Frequency
Decibels
Filters
Transfer Functions

## OUTLINE

The Decibel
Multistage Systems
Simple *RC* and *RL* Transfer Functions
The Low-Pass Filter
The High-Pass Filter
The Band-Pass Filter
The Band-Reject Filter
Circuit Analysis Using Computers

## CHAPTER PREVIEW

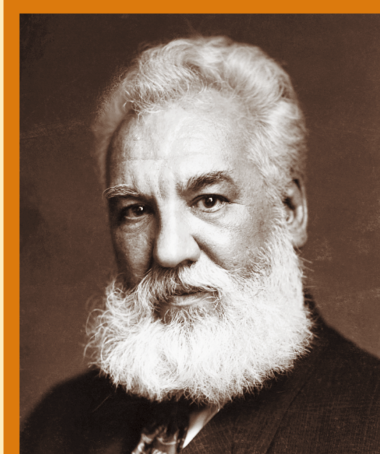
In the previous chapter we examined how *LRC* resonant circuits react to changes in frequency. In this chapter we will continue to study how changes in frequency affect the behavior of other simple circuits. We will analyze simple low-pass, high-pass, band-pass, and band-reject filter circuits. The analysis will compare the amplitude and phase shift of the output signal with respect to the input signal.

As their names imply, low-pass and high-pass filter circuits are able to pass low frequencies and high frequencies while blocking other frequency components. A good understanding of these filters provides a basis for understanding why circuits such as amplifiers and oscilloscopes are not able to pass all signals from their input to their output.

Band-reject filters and band-pass filters are circuits which are similar to resonant circuits with the exception that these circuits are able to reject or pass certain frequencies without the need for *LC* combinations.

The analysis of all filters may be simplified by plotting the output/input voltage relationship on a semilogarithmic graph called a *Bode plot.*

## PUTTING IT IN PERSPECTIVE



### *Alexander Graham Bell*

ALEXANDER GRAHAM BELL was born in Edinburgh, Scotland on March 3, 1847. As a young man, Bell followed his father and grandfather in research dealing with the deaf.

In 1873, Bell was appointed professor of vocal physiology at Boston University. His research was primarily involved in converting sound waves into electrical fluctuations. With the encouragement of Joseph Henry, who had done a great deal of work with inductors, Bell eventually developed the telephone.

In his now-famous accident in which Bell spilled acid on himself, Bell uttered the words “Watson, please come here. I want you.” Watson, who was on another floor, ran to Bell’s assistance.

Although others had worked on the principle of the telephone, Alexander Graham Bell was awarded the patent for the telephone in 1876. The telephone he constructed was a simple device which passed a current through carbon powder. The density of the carbon powder was determined by air fluctuations due to the sound of a person’s voice. When the carbon was compressed, resistivity would decrease, allowing more current.

Bell’s name has been adopted for the decibel, which is the unit used to describe sound intensities and power gain.

Although the invention of the telephone made Bell wealthy, he continued experimenting in electronics, air conditioning, and animal breeding. Bell died at the age of 75 in Baddeck, Nova Scotia on August 2, 1922.

## 22.1 The Decibel

In electronics, we often wish to consider the effects of a circuit without examining the actual operation of the circuit itself. This **black-box** approach is a common technique used to simplify transistor circuits and for depicting integrated circuits which may contain hundreds or even thousands of elements. Consider the system shown in Figure 22–1.

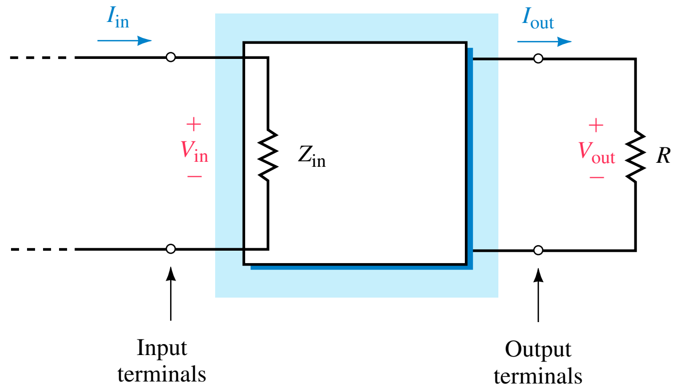



**FIGURE 22–1**

Although the circuit within the box may contain many elements, any source connected to the input terminals will effectively see only the input impedance, $Z_{in}$. Similarly, any load impedance, $R_L$, connected to the output terminals will have voltage and current determined by certain parameters in the circuit. These parameters usually result in an output at the load which is easily predicted for certain conditions.

We now define several terms which are used to analyze any system having two input terminals and two output terminals.

The **power gain,** $A_P$, is defined as the ratio of output signal power to the input signal power:

$$A_P = \frac{P_{out}}{P_{in}} \tag{22–1}$$

We must emphasize that the total output power delivered to any load can never exceed the total input power to a circuit. When we refer to the power gain of a system, we are interested in only the power contained in the ac signal, so we neglect any power due to dc. In many circuits, the ac power will be significantly less than the dc power. However, ac power gains in the order of tens of thousands are quite possible.

The **voltage gain,** $A_v$, is defined as the ratio of output signal voltage to the input signal voltage:

$$A_v = \frac{V_{out}}{V_{in}} \tag{22–2}$$

As mentioned, the power gain of a system may be a very large. For other applications the output power may be much smaller than the input power, resulting in a loss, or **attenuation.** Any circuit in which the output signal power is greater than the input signal power is referred to as an **amplifier.**

Conversely, any circuit in which the output signal power is less than the input signal power is referred to as an **attenuator.**

The ratios expressing power gain or voltage gain may be either very large or very small, making it inconvenient to express the power gain as a simple ratio of two numbers. The **bel,** which is a logarithmic unit named after Alexander Graham Bell, was selected to represent a ten-fold increase or decrease in power. Stated mathematically, the power gain in bels is given as

$$A_{P(\text{bels})} = \log_{10}\frac{P_{\text{out}}}{P_{\text{in}}}$$

Because the bel is an awkwardly large unit, the **decibel** (dB), which is one tenth of a bel, has been adopted as a more acceptable unit for describing the logarithmic change in power levels. One bel contains 10 decibels and so the power gain in decibels is given as

$$A_{P(\text{dB})} = 10\log_{10}\frac{P_{\text{out}}}{P_{\text{in}}} \tag{22–3}$$

If the power level in the system increases from the input to the output, then the gain in dB will be positive. If the power at the output is less than the power at the input, then the power gain will be negative. Notice that if the input and output have the same power levels, then the power gain will be 0 dB, since $\log 1 = 0$.

**EXAMPLE 22–1** An amplifier has the indicated input and output power levels. Determine the power gain both as a ratio and in dB for each of the conditions:

a. $P_{\text{in}} = 1$ mW, $P_{\text{out}} = 100$ W.
b. $P_{\text{in}} = 4\ \mu$W, $P_{\text{out}} = 2\ \mu$W.
c. $P_{\text{in}} = 6$ mW, $P_{\text{out}} = 12$ mW.
d. $P_{\text{in}} = 25$ mW, $P_{\text{out}} = 2.5$ mW.

**Solution**

a.
$$A_P = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{100\text{ W}}{1\text{ mW}} = 100\,000$$
$$A_{P(\text{dB})} = 10\log_{10}(100\,000) = (10)(5) = 50\text{ dB}$$

b.
$$A_P = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{2\ \mu\text{W}}{4\ \mu\text{W}} = 0.5$$
$$A_{P(\text{dB})} = 10\log_{10}(0.5) = (10)(-0.30) = -3.0\text{ dB}$$

c.
$$A_P = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{12\text{ mW}}{6\text{ mW}} = 2$$
$$A_{P(\text{dB})} = 10\log_{10}2 = (10)(.30) = 3.0\text{ dB}$$

d.
$$A_P = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{2.5\text{ mW}}{25\text{ mW}} = 0.10$$
$$A_{P(\text{dB})} = 10\log_{10}0.10 = (10)(-1) = -10\text{ dB}$$

The previous example illustrates that if the power is increased or decreased by a factor of two, the resultant power gain will be either +3 dB or −3 dB, respectively. You will recall that in the previous chapter a similar reference was made when the half-power frequencies of a resonant circuit were referred to as the 3-dB down frequencies.

The voltage gain of a system may also be expressed in dB. In order to derive the expression for voltage gain, we first assume that the input resistance and the load resistance are the same value. Then, using the definition of power gain as given by Equation 22–3, we have the following:

$$\begin{aligned} A_{P(\mathrm{dB})} &= 10 \log_{10}\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}} \\ &= 10 \log_{10}\frac{V_{\mathrm{out}}^2/R}{V_{\mathrm{in}}^2/R} \\ &= 10 \log_{10}\left(\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}\right)^2 \end{aligned}$$

which gives

$$A_{P(\mathrm{dB})} = 20 \log_{10}\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}$$

Because the above expression represents a decibel equivalent of the voltage gain, we simply write the voltage gain in dB as follows:

$$A_{v(\mathrm{dB})} = 20 \log_{10}\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} \tag{22–4}$$

**EXAMPLE 22–2** The amplifier circuit of Figure 22–2 has the given conditions. Calculate the voltage gain and power gain in dB.

$Z_{\mathrm{in}} = 10\ \mathrm{k}\Omega$
$R_L = 600\ \Omega$
$V_{\mathrm{in}} = 20\ \mathrm{mV_{rms}}$
$V_{\mathrm{out}} = 500\ \mathrm{mV_{rms}}$

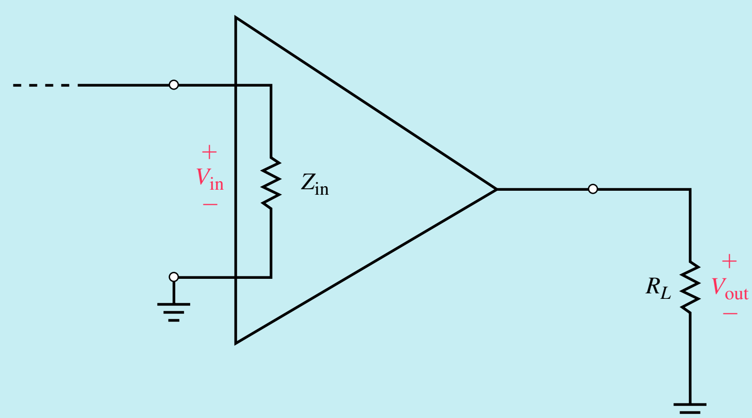



**FIGURE 22–2**

**Solution** The voltage gain of the amplifier is

$$A_v = 20\log_{10}\frac{V_{\text{out}}}{V_{\text{in}}}$$
$$= 20\log_{10}\frac{500\text{ mV}}{20\text{ mV}} = 20\log_{10}25 = 28.0\text{ dB}$$

The signal power available at the input of the amplifier is

$$P_{\text{in}} = \frac{V_{\text{in}}^2}{Z_{\text{in}}} = \frac{(20\text{ mV})^2}{10\text{ k}\Omega} = 0.040\ \mu\text{W}$$

The signal at the output of the amplifier has a power of

$$P_{\text{out}} = \frac{V_{\text{out}}^2}{R_L} = \frac{(500\text{ mV})^2}{600\ \Omega} = 416.7\ \mu\text{W}$$

The power gain of the amplifier is

$$A_P = 10\log_{10}\frac{P_{\text{out}}}{P_{\text{in}}}$$
$$= 10\log_{10}\frac{416.7\ \mu\text{W}}{0.040\ \mu\text{W}}$$
$$= 10\log_{10}(10\,417) = 40.2\text{ dB}$$

**PRACTICE PROBLEMS 1**

Calculate the voltage gain and power gain (in dB) for the amplifier of Figure 22–2, given the following conditions:

$Z_{\text{in}} = 2\text{ k}\Omega$
$R_L = 5\ \Omega$
$V_{\text{in}} = 16\ \mu\text{V}_{\text{rms}}$
$V_{\text{out}} = 32\text{ mV}_{\text{rms}}$

*Answers:* 66.0 dB, 92.0 dB

In order to convert a gain from decibels into a simple ratio of power or voltage, it is necessary to perform the inverse operation of the logarithm; namely, solve the unknown quantity by using the exponential. Recall that the following logarithmic and exponential operations are equivalent:

$$y = \log_b x$$
$$x = b^y$$

Using the above expressions, Equations 22–3 and 22–4 may be used to determine expressions for power and voltage gains as follows:

$$\frac{A_{P(\text{dB})}}{10} = \log_{10}\frac{P_{\text{out}}}{P_{\text{in}}}$$
$$\frac{P_{\text{out}}}{P_{\text{in}}} = 10^{A_{P(\text{dB})}/10} \qquad \textbf{(22–5)}$$

$$\frac{A_{v(\mathrm{dB})}}{20} = \log_{10}\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}$$

$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = 10^{A_{v(\mathrm{dB})}/20} \qquad \text{(22–6)}$$

**EXAMPLE 22–3** Convert the following from decibels to ratios:

a. $A_P = 25$ dB.
b. $A_P = -6$ dB.
c. $A_v = 10$ dB.
d. $A_v = -6$ dB.

**Solution**

a.
$$A_P = \frac{P_{\mathrm{out}}}{P_{\mathrm{in}}} = 10^{A_{P(\mathrm{dB})}/10}$$
$$= 10^{25/10} = 316$$

b.
$$A_P = 10^{A_{P(\mathrm{dB})}/10}$$
$$= 10^{-6/10} = 0.251$$

c.
$$A_v = \frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = 10^{A_{v(\mathrm{dB})}/20}$$
$$= 10^{10/20} = 3.16$$

d.
$$A_v = 10^{A_{v(\mathrm{dB})}/20}$$
$$= 10^{-6/20} = 0.501$$

## Applications of Decibels

Decibels were originally intended as a measure of changes in acoustical levels. The human ear is not a linear instrument; rather, it responds to sounds in a logarithmic fashion. Because of this peculiar phenomenon, a ten-fold increase in sound intensity results in a perceived doubling of sound. This means that if we wish to double the sound heard from a 10-W power amplifier, we must increase the output power to 100 W.

The minimum sound level which may be detected by the human ear is called the **threshold of hearing** and is usually taken to be $I_0 = 1 \times 10^{-12}$ W/m$^2$. Table 22–1 lists approximate sound intensities of several common sounds. The decibel levels are determined from the expression

$$\beta(\mathrm{dB}) = 10\log_{10}\frac{I}{I_0}$$

Some electronic circuits operate with very small power levels. These power levels may be referenced to some arbitrary level and then expressed in decibels in a manner similar to how sound intensities are represented. For

**TABLE 22–1** Intensity Levels of Common Sounds

| Sound | Intensity Level (dB) | Intensity ($W/m^2$) |
|---|---|---|
| threshold of hearing, $I_0$ | 0 | $10^{-12}$ |
| virtual silence | 10 | $10^{-11}$ |
| quiet room | 20 | $10^{-10}$ |
| watch ticking at 1 m | 30 | $10^{-9}$ |
| quiet street | 40 | $10^{-8}$ |
| quiet conversation | 50 | $10^{-7}$ |
| quiet motor at 1 m | 60 | $10^{-6}$ |
| busy traffic | 70 | $10^{-5}$ |
| door slamming | 80 | $10^{-4}$ |
| busy office room | 90 | $10^{-3}$ |
| jackhammer | 100 | $10^{-2}$ |
| motorcycle | 110 | $10^{-1}$ |
| loud indoor rock concert | 120 | 1 |
| threshold of pain | 130 | 10 |

instance, power levels may be referenced to a standard power of 1 mW. In such cases the power level is expressed in dBm and is determined as

$$P_{\text{dBm}} = 10 \log_{10} \frac{P}{1 \text{ mW}} \tag{22–7}$$

If a power level is referenced to a standard of 1 W, then we have

$$P_{\text{dBW}} = 10 \log_{10} \frac{P}{1 \text{ W}} \tag{22–8}$$

**EXAMPLE 22–4** Express the following powers in dBm and in dBW.

a. $P_1 = 0.35\ \mu\text{W}$.

b. $P_2 = 20$ mW.

c. $P_3 = 1000$ W.

d. $P_4 = 1$ pW.

**Solution**

a.

$$P_{1(\text{dBm})} = 10 \log_{10} \frac{0.35\ \mu\text{W}}{1 \text{ mW}} = -34.6 \text{ dBm}$$

$$P_{1(\text{dBW})} = 10 \log_{10} \frac{0.35\ \mu\text{W}}{1 \text{ W}} = -64.6 \text{ dBW}$$

b.

$$P_{2(\text{dBm})} = 10 \log_{10} \frac{20 \text{ mW}}{1 \text{ mW}} = 13.0 \text{ dBm}$$

$$P_{2(\text{dBW})} = 10 \log_{10} \frac{20 \text{ mW}}{1 \text{ W}} = -17.0 \text{ dBW}$$

c.
$$P_{3(\text{dBm})} = 10 \log_{10}\frac{1000\text{ W}}{1\text{ mW}} = 60\text{ dBm}$$

$$P_{3(\text{dBW})} = 10 \log_{10}\frac{1000\text{ W}}{1\text{ W}} = 30\text{ dBW}$$

d.
$$P_{4(\text{dBm})} = 10 \log_{10}\frac{1\text{ pW}}{1\text{ mW}} = -90\text{ dBm}$$

$$P_{4(\text{dBW})} = 10 \log_{10}\frac{1\text{ pW}}{1\text{ W}} = -120\text{ dBW}$$

Many voltmeters have a separate scale calibrated in decibels. In such cases, the voltage expressed in dBV uses 1 $V_{\text{rms}}$ as the reference voltage. In general, any voltage reading may be expressed in dBV as follows:

$$V_{\text{dBV}} = 20 \log_{10}\frac{V_{\text{out}}}{1\text{ V}} \tag{22–9}$$

PRACTICE PROBLEMS 2

Consider the resistors of Figure 22–3:

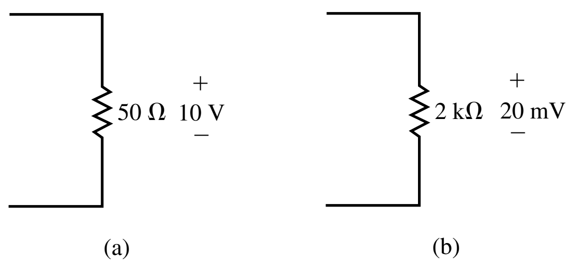



FIGURE 22–3

a. Determine the power levels in dBm and in dBW.
b. Express the voltages in dBV.

*Answers:*
a. 33.0 dBm (3.0 dBW), −37.0 dBm (−67.0 dBW)
b. 20.0 dBV, −34.0 dBV

## 22.2 Multistage Systems

Quite often a system consists of several stages. In order to find the total voltage gain or power gain of the system, we would need to solve for the product of the individual gains. The use of decibels makes the solution of a multistage system easy to find. If the gain of each stage is given in decibels, then the resultant gain is simply determined as the summation of the individual gains.

Consider the system of Figure 22–4, which represents a three-stage system.

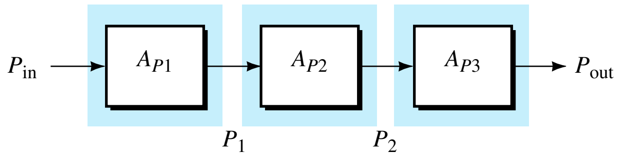



**FIGURE 22–4**

The power at the output of each stage is determined as follows:

$$P_1 = A_{P1} P_{in}$$
$$P_2 = A_{P2} P_1$$
$$P_{out} = A_{P3} P_2$$

The total power gain of the system is found as

$$\begin{aligned} A_{PT} &= \frac{P_{out}}{P_{in}} = \frac{A_{P3} P_2}{P_{in}} \\ &= \frac{A_{P3} (A_{P2} P_1)}{P_{in}} \\ &= \frac{A_{P3} A_{P2} (A_{P1} P_i)}{P_{in}} \\ &= A_{P1} A_{P2} A_{P3} \end{aligned}$$

In general, for $n$ stages the total power gain is found as the product:

$$A_{PT} = A_{P1} A_{P2} \cdots A_{Pn} \tag{22–10}$$

However, if we use logarithms to solve for the gain in decibels, we have the following:

$$\begin{aligned} A_{PT}(\text{dB}) &= 10 \log_{10} A_{PT} \\ &= 10 \log_{10} (A_{P1} A_{P2} \cdots A_{Pn}) \\ A_{PT}(\text{dB}) &= 10 \log_{10} A_{P1} + 10 \log_{10} A_{P2} + \cdots + 10 \log_{10} A_{Pn} \end{aligned}$$

The total power gain in decibels for $n$ stages is determined as the summation of the individual decibel power gains:

$$A_{P(\text{dB})} = A_{P1(\text{dB})} + A_{P2(\text{dB})} + \cdots + A_{Pn(\text{dB})} \tag{22–11}$$

The advantage of using decibels in solving for power gains and power levels is illustrated in the following example.

**EXAMPLE 22–5** The circuit of Figure 22–5 represents the first three stages of a typical AM or FM receiver.

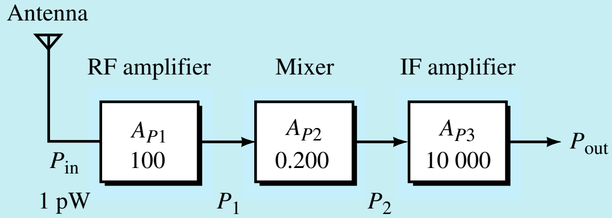



**FIGURE 22–5**

Find the following quantities:

a. $A_{P1(\text{dB})}$, $A_{P2(\text{dB})}$, and $A_{P3(\text{dB})}$.

b. $A_{PT(\text{dB})}$.

c. $P_1$, $P_2$, and $P_{\text{out}}$.

d. $P_{\text{in(dBm)}}$, $P_{1(\text{dBm})}$, $P_{2(\text{dBm})}$, and $P_{\text{out(dBm)}}$.

**Solution**

a.
$$A_{P1(\text{dB})} = 10\log_{10}A_{P1} = 10\log_{10}100 = 20\text{ dB}$$
$$A_{P2(\text{dB})} = \log_{10}A_{P2} = 10\log_{10}0.2 = -7.0\text{ dB}$$
$$A_{P3(\text{dB})} = 10\log_{10}A_{P3} = 10\log_{10}(10\,000) = 40\text{ dB}$$

b.
$$A_{PT(\text{dB})} = A_{P1(\text{dB})} + A_{P2(\text{dB})} + A_{P3(\text{dB})}$$
$$= 20\text{ dB} - 7.0\text{ dB} + 40\text{ dB}$$
$$= 53.0\text{ dB}$$

c.
$$P_1 = A_{P1}P_{\text{in}} = (100)\,(1\text{ pW}) = 100\text{ pW}$$
$$P_2 = A_{P2}P_1 = (100\text{ pW})(0.2) = 20\text{ pW}$$
$$P_{\text{out}} = A_{P3}P_2 = (10\,000)(20\text{ pW}) = 0.20\ \mu\text{W}$$

d.
$$P_{\text{in(dBm)}} = 10\log_{10}\frac{P_{\text{in}}}{1\text{ mW}} = 10\log_{10}\frac{1\text{ pW}}{1\text{ mW}} = -90\text{ dBm}$$
$$P_{1(\text{dBm})} = 10\log_{10}\frac{P_1}{1\text{ mW}} = 10\log_{10}\frac{100\text{ pW}}{1\text{ mW}} = -70\text{ dBm}$$
$$P_{2(\text{dBm})} = 10\log_{10}\frac{P_2}{1\text{ mW}} = 10\log_{10}\frac{20\text{ pW}}{1\text{ mW}} = -77.0\text{ dBm}$$
$$P_{\text{out(dBm)}} = 10\log_{10}\frac{P_{\text{out}}}{1\text{ mW}} = 10\log_{10}\frac{0.20\ \mu\text{W}}{1\text{ mW}} = -37.0\text{ dBm}$$

Notice that the power level (in dBm) at the output of any stage is easily determined as the sum of the input power level (in dBm) and the gain of the stage (in dB). It is for this reason that many communication circuits express power levels in decibels rather than in watts.

**PRACTICE PROBLEMS 3**

Calculate the power level at the output of each of the stages in Figure 22–6.

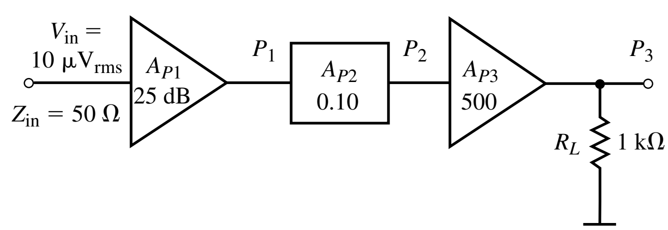



**FIGURE 22–6**

*Answers:* $P_1 = -62$ dBm, $P_2 = -72$ dBm, $P_3 = -45$ dBm

IN-PROCESS LEARNING CHECK 1

1. Given that an amplifier has a power gain of 25 dB, calculate the output power level (in dBm) for the following input characteristics:
   a. $V_{in} = 10\ \text{mV}_{rms}$, $Z_{in} = 50\ \Omega$.
   b. $V_{in} = 10\ \text{mV}_{rms}$, $Z_{in} = 1\ \text{k}\Omega$.
   c. $V_{in} = 400\ \mu\text{V}_{rms}$, $Z_{in} = 200\ \Omega$.
2. Given amplifiers with the following output characteristics, determine the output voltage (in volts rms).
   a. $P_{out} = 8.0\ \text{dBm}$, $R_L = 50\ \Omega$.
   b. $P_{out} = -16.0\ \text{dBm}$, $R_L = 2\ \text{k}\Omega$.
   c. $P_{out} = -16.0\ \text{dBm}$, $R_L = 5\ \text{k}\Omega$.

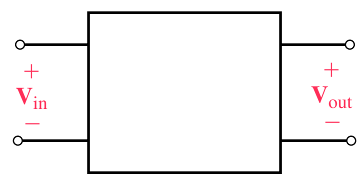
*(Answers are at the end of the chapter.)*

## 22.3 Simple *RC* and *RL* Transfer Functions

Electronic circuits usually operate in a highly predictable fashion. If a certain signal is applied at the input of a system, the output will be determined by the physical characteristics of the circuit. The frequency of the incoming signal is one of the many physical conditions which determine the relationship between a given input signal and the resulting output. Although outside the scope of this text, other conditions that may determine the relationship between the input and output signals of a given circuit are temperature, light, radiation, etc.

For any system subjected to a sinusoidal input voltage, as shown in Figure 22–7, we define the **transfer function** as the ratio of the output voltage phasor to the input voltage phasor for any frequency $\omega$ (in radians per second).

$$\mathbf{TF}(\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = A_v\angle\theta \qquad \textbf{(22–12)}$$




$$\mathbf{TF} = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = A_v\angle\,\theta$$

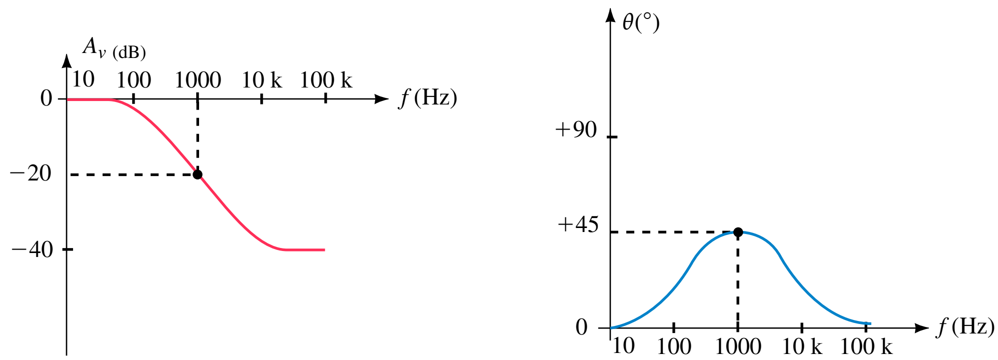
**FIGURE 22–7**

Notice that the definition of transfer function is almost the same as voltage gain. The difference is that the transfer function takes into account both amplitude and phase shift of the voltages, whereas voltage gain is a comparison of only the amplitudes.

From Equation 22–12 we see that the amplitude of the transfer function is in fact the voltage gain. The phase angle, $\theta$, represents the phase shift between the input and output voltage phasors. The angle $\theta$ will be positive if the output leads the input and negative if the output lags the input waveform.

If the elements within the block of Figure 22–7 are resistors, then the output and the input voltages will always be in phase. Also, since resistors have the same value at all frequencies, the voltage gain will remain constant at all frequencies. (This text does not take into account resistance variations due to very high frequencies.) The resulting circuit is called an attenuator since resistance within the block will dissipate some power, thereby reducing (or attenuating) the signal as it passes through the circuit.

If the elements within the block are combinations of resistors, inductors, and capacitors, then the output voltage and phase will depend on frequency since the impedances of inductors and capacitors are frequency dependent. Figure 22–8 illustrates an example of how voltage gain and phase shift of a circuit may change as a function of frequency.




(a) Voltage gain as a function of frequency.

(b) Phase shift as a function of frequency.

**FIGURE 22–8** Frequency response of a circuit.

In order to examine the operation of the circuit over a wide range of frequencies, the abscissa (horizontal axis) is usually shown as a logarithmic scale. The ordinate (vertical axis) is usually shown as a linear scale in decibels or degrees. Such graphs are said to be semilogarithmic since one scale is logarithmic and the other is linear.

By examining the frequency response of a circuit, we are able to determine at a glance the voltage gain (in dB) and phase shift (in degrees) for any sinusoidal input at a given frequency. For example, at a frequency of 1000 Hz, the voltage gain is −20 dB and the phase shift is 45°. This means that the output signal is one tenth as large as the input signal and the output leads the input by 45°.

From the frequency response of Figure 22–8, we see that the circuit which corresponds to this response is able to pass low-frequency signals, while at the same time partially attenuating high-frequency signals. Any circuit which passes a particular range of frequencies while blocking others is referred to as a **filter** circuit. Filter circuits are usually named according to their function, although certain filters are named after their inventors. The filter having the response of Figure 22–8 is referred to as a stepfilter, since the voltage gain occurs between two limits (steps). Other types of filters which are used regularly in electrical and electronic circuits include low-pass, high-pass, band-pass, and band-reject filters.

Although the design of filters is a topic in itself, we will examine a few of the more common types of filters which are used extensively.

## Sketching Transfer Functions

Transfer functions are easily sketched on semilogarithmic graph paper by following a few basic steps. Consider a transfer function given as

$$\mathbf{TF} = \frac{1}{1 + j0.01\omega}$$

Several important characteristics are evident by examining the transfer function above. The magnitude of the transfer function (voltage gain) will be less than 1 for all frequencies other than zero hertz (dc). As the frequency increases, the voltage gain will decrease. For this reason, the circuit which has the above transfer function is called a **low-pass filter.** Finally, we see that the output voltage will lag the input voltage at all frequencies other than at zero hertz.

We define the **cutoff frequency,** $\omega_c$ (also called the **break frequency**), as the frequency at which the imaginary component of $1 + j0.01\omega$ is equal to the real component. This results in a voltage gain which is equal to

$$A_v = \frac{1}{\sqrt{2}} = 0.7071$$

From the transfer function, we see that the cutoff frequency occurs when the angular frequency $\omega = \omega_c = 1/0.01 = 100$ rad/s. At the cutoff frequency, the voltage gain (in dB) is $A_v = -3.0$ dB and the phase shift is $\theta = -45°$. To determine the cutoff frequency in hertz, we simply perform the following operation:

$$f_c = \frac{\omega_c}{2\pi}$$

The voltage gain (in decibels) at any angular frequency is found from the transfer function as

$$A_v(\text{dB}) = -20 \log(\sqrt{1 + (0.01\omega)^2})$$

or

$$A_v(\text{dB}) = -10 \log[1 + (0.01\omega)^2]$$

and the phase shift is given as

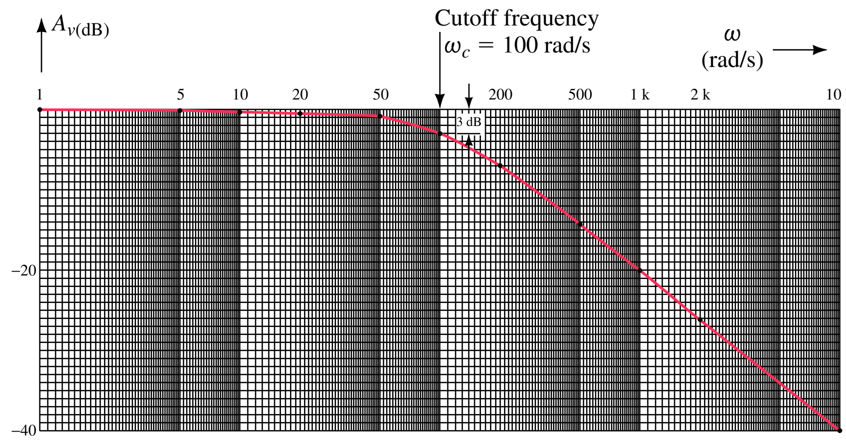
$$\theta = -\arctan(0.01\omega)$$

The frequency response of the transfer function may be sketched by determining the voltage gain and phase shift at various angular frequencies around the cutoff frequency. Table 22–2 provides the voltage gain and phase shift for frequencies between $\omega = 0.01\omega_c$ and $\omega = 100\omega_c$.

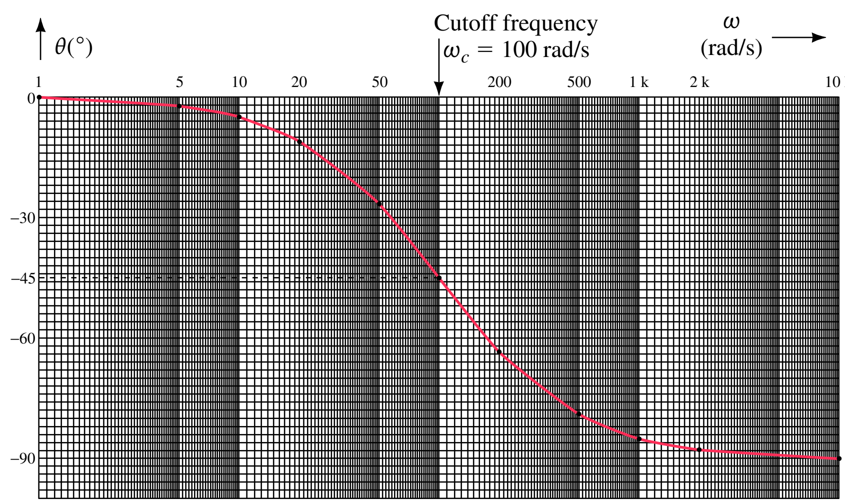
**TABLE 22–2** Frequency Response of a Low-Pass Filter

| **ω (rad/s)** | **TF(ω)** | **$A_v$ (V/V)** | **$A_v$ (db)** | **θ (°)** |
|---|---|---|---|---|
| 1 | $1.00\angle -0.6°$ | 1.000 | 0.00 | $-0.6°$ |
| 5 | $0.999\angle -2.9°$ | 0.999 | 0.00 | $-2.9°$ |
| 10 | $0.995\angle -5.7°$ | 0.995 | $-0.04$ | $-5.7°$ |
| 20 | $0.981\angle -11.3°$ | 0.981 | $-0.17$ | $-11.3°$ |
| 50 | $0.894\angle -26.6°$ | 0.894 | $-0.97$ | $-26.6°$ |
| 100 | $0.707\angle -45°$ | 0.707 | $-3.01$ | $-45.0°$ |
| 200 | $0.447\angle -63.4°$ | 0.447 | $-6.99$ | $-63.4°$ |
| 500 | $0.196\angle -78.9°$ | 0.196 | $-14.2$ | $-78.9°$ |
| 1000 | $0.100\angle -84.3°$ | 0.100 | $-20.0$ | $-84.3°$ |
| 2000 | $0.050\angle -87.1°$ | 0.050 | $-26.0$ | $-87.1°$ |
| 10k | $0.010\angle -89.4°$ | 0.010 | $-40.0$ | $-89.4°$ |

The data of Table 22–2 are plotted in Figure 22–9.




(a) Voltage gain as a function of frequency.




(b) Phase shift as a function of frequency.

**FIGURE 22–9** Frequency response of a low-pass filter circuit.

While the previous method can be used to provide a frequency response for any circuit, it is tedious and very time-consuming. A better method is to approximate the results as follows:

For $\omega << \omega_c$ ($\omega \leq 0.1\omega_c$),

$$A_v \simeq 0 \text{ dB}$$
$$\theta \simeq 0$$

For $\omega >> \omega_c$ ($\omega \geq 10\omega_c$),

$$A_v \simeq -20 \log(0.01\omega)$$
$$\theta \simeq -90°$$

The above information is plotted on semilogarithmic graphs using straight lines to approximate the actual frequency response. Such graphs are called **Bode plots.** The actual response will approach the straight lines for frequencies much less than and much greater than the cutoff frequency. For this reason, the straight lines are also called the **asymptotes** of the transfer function. Figure 22–10 shows the Bode plot for a low-pass filter.

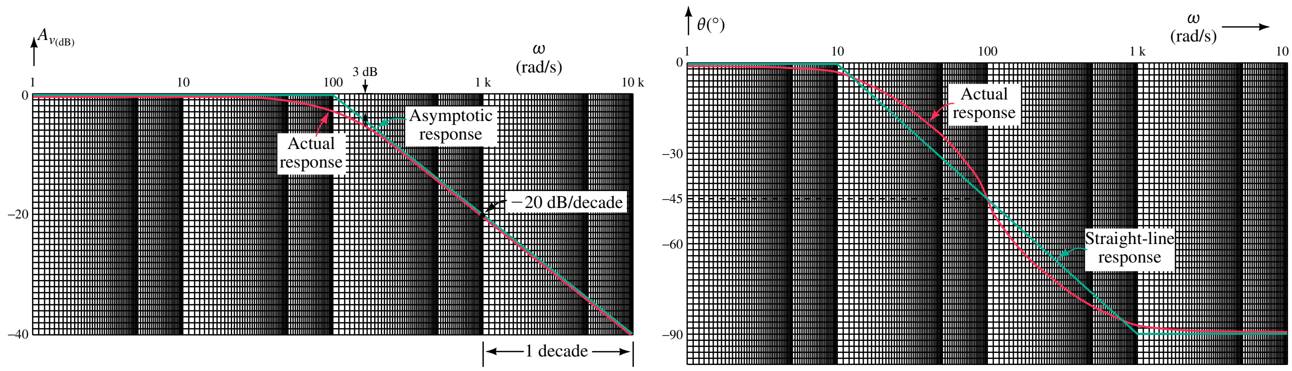



**FIGURE 22–10** Frequency response of a low-pass filter.

The Bode plot illustrates several very important points. Because the abscissa of the Bode plot is not measured linearly, the slope of the asymptotes cannot be expressed in the usual terms of rise over run. A **decade** is defined as the ten-fold increase or decrease in frequency. For instance, increasing a frequency from 10 Hz to 100 Hz represents a decade. Similarly, a ten-fold change in angular frequency is also a decade.

An alternate way of describing a frequency change is by using the term **octave.** An octave is defined as a twofold increase or decrease in frequency. Therefore, a frequency of 400 Hz is one octave higher than a frequency of 200 Hz, while 800 Hz is two octaves higher than 200 Hz.

Both voltage gain and the phase shift have changes which are easily expressed in terms of a decade. The asymptotic response for voltage gain shows that the voltage gain will be approximately 0 dB for frequencies less than $\omega_c$. For frequencies above $\omega_c$, the voltage gain changes at a rate of −20 dB for each decade that the frequency increases (−20 dB/decade). A change of −20 dB/decade is equivalent to having a change of −6 dB/octave.

The straight-line response for phase shift illustrates that for frequencies less than $0.1\omega_c$ the phase shift will be approximately 0°. For frequencies above $10\omega_c$ the phase shift will be essentially constant at −90°. In the region between $0.1\omega_c$ and $10\omega_c$, the phase shift may be approximated to be −45°/decade although it actually follows a curved line in this region.

All transfer functions of first-order filters (the only types of filters covered in this text) can be written as

$$\mathbf{TF} = \frac{(j\omega\tau_{Z_1})(1 + j\omega\tau_{Z_2}) \cdots (1 + j\omega\tau_{Z_n})}{(j\omega\tau_{P_1})(1 + j\omega\tau_{P_2}) \cdots (1 + j\omega\tau_{P_m})}$$

where each of the cutoff frequencies (in radians per second) is found as

$$\omega = \frac{1}{\tau}$$

Since voltage gain is expressed in dB, the Bode plot of any transfer function is determined from the summation of the effects due to the various terms in the transfer function.

**EXAMPLE 22–6** Sketch the straight-line approximation of the following transfer function:

$$\mathbf{TF} = \frac{j0.01\omega}{1 + j0.1\omega}$$

**Solution** The voltage gain in dB is given as

$$A_{v(\mathrm{dB})} = 20\log(0.01\omega) - 20\log\left(\sqrt{1 + (0.1\omega)^2}\right)$$

and the phase shift is given as

$$\theta = 90° - \arctan(0.1\omega)$$

The individual terms of the voltage gain are shown in Figure 22–11(a) and the individual terms of the phase shift are shown in Figure 22–11(b).

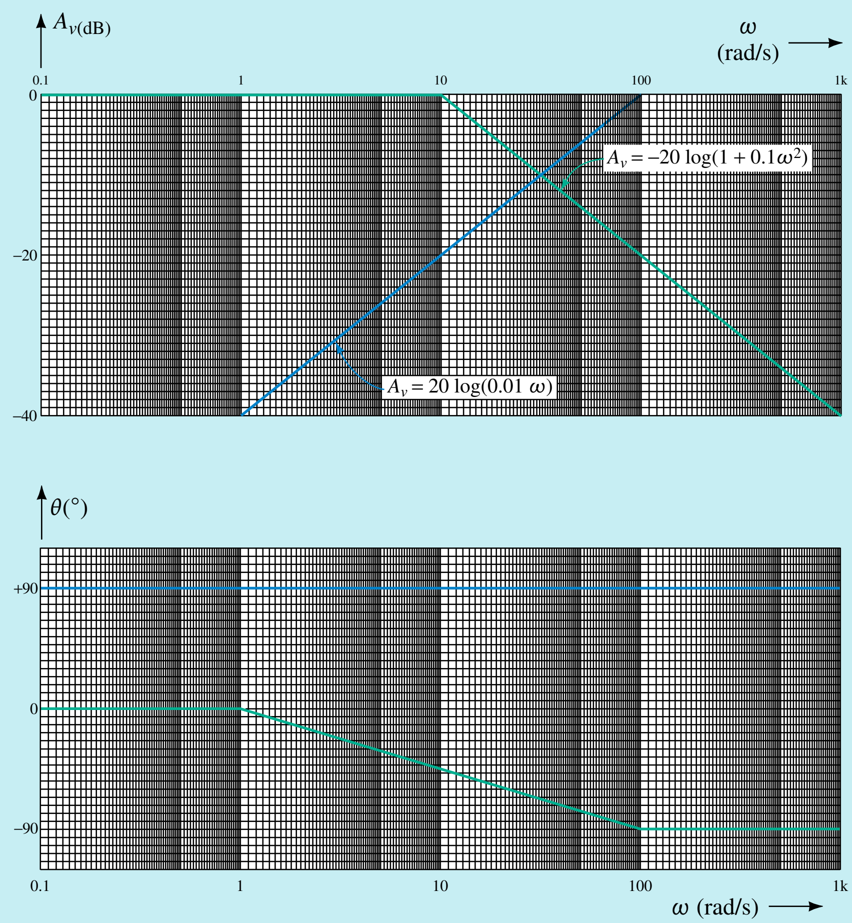



**FIGURE 22–11**

The combined results are shown in Figure 22–12. The total voltage gain at each frequency is determined by summing the gain due to each term of the transfer function. Similarly, the total phase shift at each frequency is the summation of phase shifts due to each of the two terms.

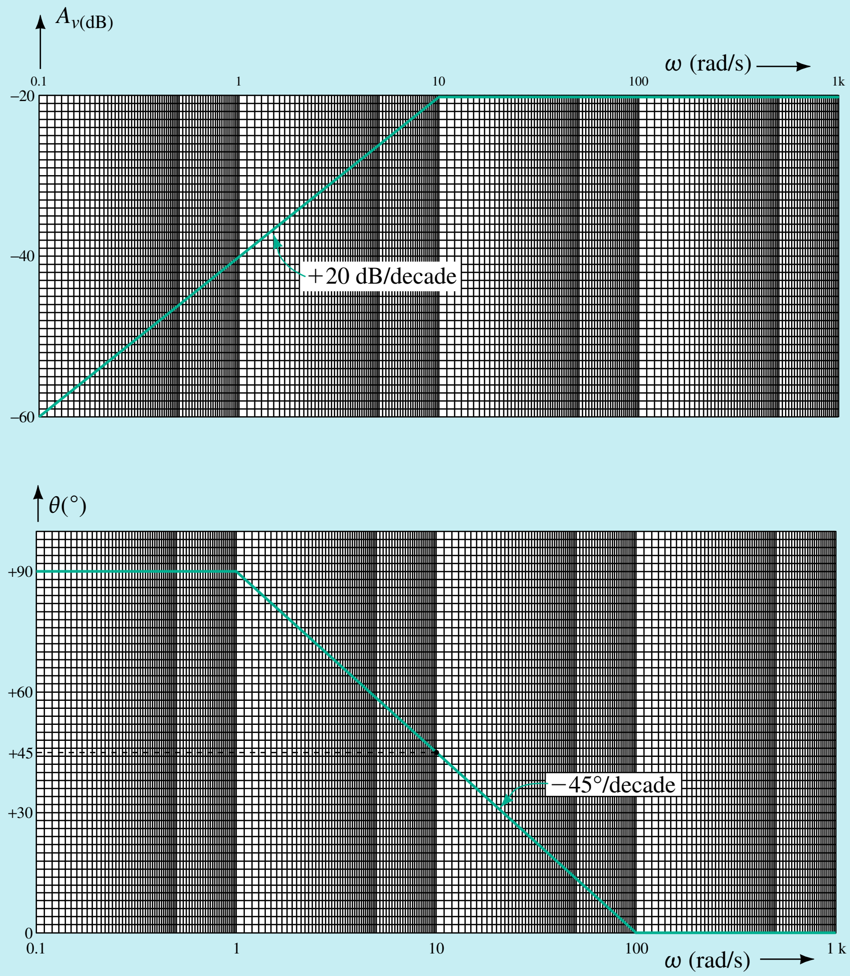



**FIGURE 22–12**

From the Bode plot of Figure 22–12, we conclude that the filter having this response is a high-pass filter with a cutoff frequency of 10 rad/s. The high-frequency gain of this filter is −20 dB.

**PRACTICE PROBLEMS 4**

Sketch the straight-line approximation for each of the transfer functions given.

a. $\mathbf{TF} = 1 + j0.02\omega$

b. $\mathbf{TF} = \dfrac{100}{1 + j0.005\omega}$

c. $\mathbf{TF} = \dfrac{1 + j0.02\omega}{20(1 + j0.002\omega)}$

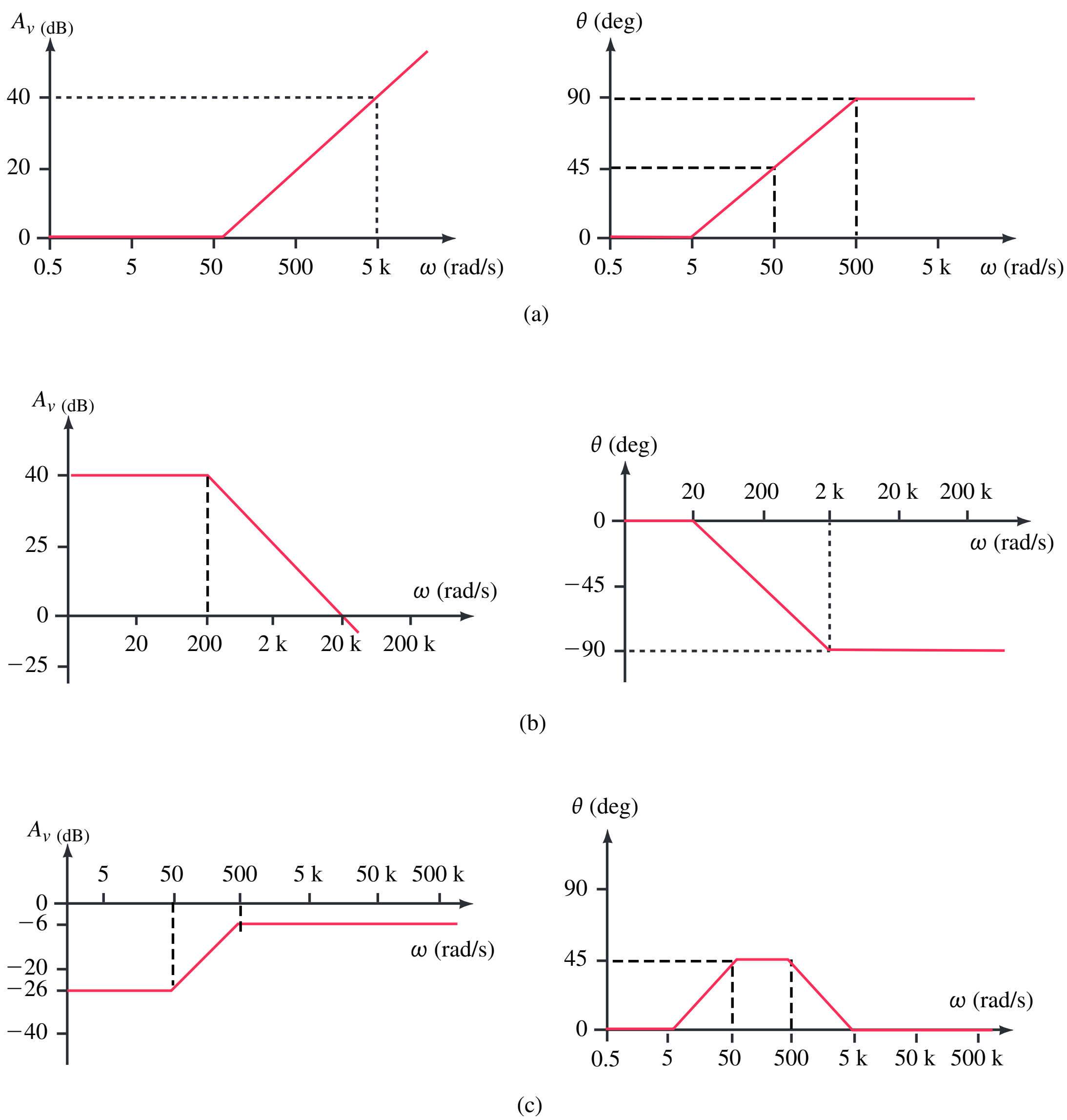



**FIGURE 22–13**

## Writing Transfer Functions

The transfer function of any circuit is found by following a few simple steps. As we have already seen, a properly written transfer function allows us to easily calculate the cutoff frequencies and quickly sketch the corresponding Bode plot for the circuit. The steps are as follows:

1. Determine the boundary conditions for the given circuit by solving for the voltage gain when the frequency is zero (dc) and when the frequency

approaches infinity. The boundary conditions are found by using the following approximations:

At $\omega = 0$, inductors are short circuits,
capacitors are open circuits.

As $\omega \rightarrow \infty$, inductors are open circuits,
capacitors are short circuits.

By using the above approximations, all capacitors and inductors are easily removed from the circuit. The resulting voltage gain is determined for each boundary condition by simply applying the voltage divider rule.

2. Use the voltage divider rule to write the general expression for the transfer function in terms of the frequency, $\omega$. In order to simplify the algebra, all capacitive and inductive reactance vectors are written as follows:

$$\mathbf{Z}_C = \frac{1}{j\omega C}$$

and

$$\mathbf{Z}_L = j\omega L$$

3. Simplify the resulting transfer function so that it is in the following format:

$$\mathbf{TF} = \frac{(j\omega\tau_{Z_1})(1 + j\omega\tau_{Z_2}) \cdots (1 + j\omega\tau_{Z_n})}{(j\omega\tau_{P_1})(1 + j\omega\tau_{P_2}) \cdots (1 + j\omega\tau_{P_m})}$$

Once the function is in this format, it is good practice to verify the boundary conditions found in Step 1. The boundary conditions are determined algebraically by first letting $\omega = 0$ and then solving for the resulting dc voltage gain. Next, we let $\omega \rightarrow \infty$. The various $(1 + j\omega\tau)$ terms of the transfer function may now be approximated as simply $j\omega\tau$, since the imaginary terms will be much larger ($\geq 10$) than the real components. The resultant gain will give the high-frequency gain.

4. Determine the break frequency(ies) at $\omega = 1/\tau$ (in radians per second) where the time constants will be expressed as either $\tau = RC$ or $\tau = L/R$.

5. Sketch the straight-line approximation by separately considering the effects of each term in the transfer function.

6. Sketch the actual response of the circuit from the approximation. The actual voltage gain response will be a smooth, continuous curve which follows the asymptotic curve but which usually has a 3-dB difference at the cutoff frequency(ies). This approximation will not apply if two cutoff frequencies are separated by less than a decade. The actual phase shift response will have the same value as the straight-line approximation at the cutoff frequency. At frequencies one decade above and one decade below the cutoff frequency, the actual phase shift will be 5.71° from the straight-line approximation.

These steps will now be used in analyzing several important types of filters.

## 22.4 The Low-Pass Filter

### The *RC* Low-Pass Filter

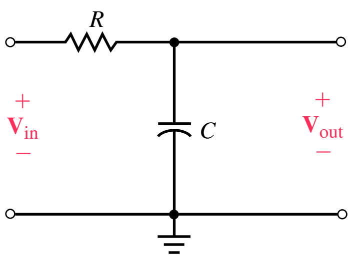



**FIGURE 22–14** *RC* low-pass filter.

The circuit of Figure 22–14 is referred to as a low-pass *RC* filter circuit since it permits low-frequency signals to pass from the input to the output while attenuating high frequency signals.

At low frequencies, the capacitor has a very large reactance. Consequently, at low frequencies the capacitor is essentially an open circuit resulting in the voltage across the capacitor, $\mathbf{V}_{\text{out}}$, to be essentially equal to the applied voltage, $\mathbf{V}_{\text{in}}$.

At high frequencies, the capacitor has a very small reactance, which essentially short circuits the output terminals. The voltage at the output will therefore approach zero as the frequency increases. Although we are able to easily predict what happens at the two extremes of frequency, called the **boundary conditions,** we do not yet know what occurs between the two extremes.

The circuit of Figure 22–14 is easily analyzed by applying the voltage divider rule. Namely,

$$\mathbf{V}_{\text{out}} = \frac{\mathbf{Z}_C}{\mathbf{R} + \mathbf{Z}_C}\mathbf{V}_{\text{in}}$$

In order to simplify the algebra, the reactance of a capacitor is expressed as follows:

$$\mathbf{Z}_C = -j\frac{1}{\omega C} = -j\frac{j}{j\omega C} = \frac{1}{j\omega C} \tag{22–13}$$

The transfer function for the circuit of Figure 22–13 is now evaluated as follows:

$$\mathbf{TF}(\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{\dfrac{1}{j\omega C}}{\dfrac{1 + j\omega RC}{j\omega C}}$$

$$= \frac{1}{1 + j\omega RC}$$

We define the cutoff frequency, $\omega_c$, as the frequency at which the output power is equal to half of the maximum output power (3 dB down from the maximum). This frequency occurs when the output voltage has an amplitude which is 0.7071 of the input voltage. For the *RC* circuit, the cutoff frequency occurs at

$$\omega_c = \frac{1}{\tau} = \frac{1}{RC} \tag{22–14}$$

Then the transfer function is written as

$$\mathbf{TF}(\omega) = \frac{1}{1 + j\dfrac{\omega}{\omega_c}} \tag{22–15}$$

The above transfer function results in the Bode plot shown in Figure 22–15.

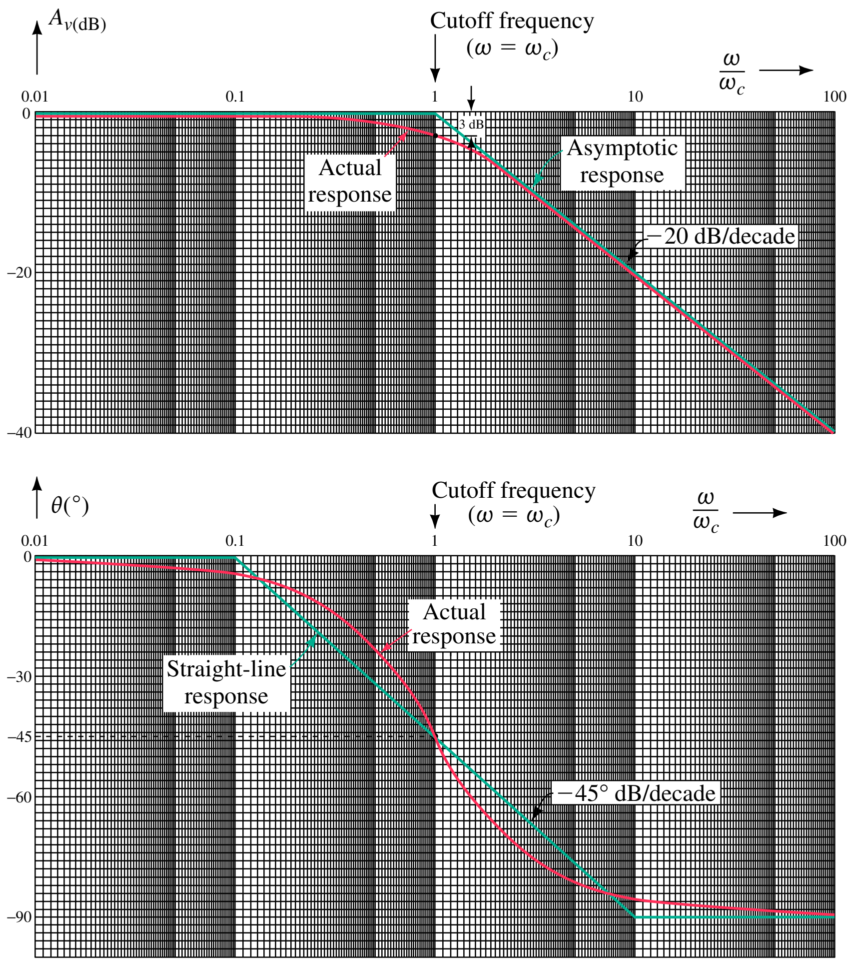



**FIGURE 22–15** Normalized frequency response for an *RC* low-pass filter. (a) Voltage gain response. (b) Phase shift response.

Notice that the abscissas (horizontal axes) of the graphs in Figure 22–15 are shown as a ratio of $\omega/\omega_c$. Such a graph is called a **normalized plot** and eliminates the need to determine the actual cutoff frequency, $\omega_c$. The normalized plot will have the same values for all low-pass *RC* filters. The actual frequency response of the *RC* low-pass filter can be approximated from the straight-line approximation by using the following guidelines:

1. At low frequencies ($\omega/\omega_c \leq 0.1$) the voltage gain is approximately 0 dB with a phase shift of about 0°. This means that the output signal of the filter is very nearly equal to the input signal. The phase shift at $\omega = 0.1\omega_c$ will be 5.71° less than the straight-line approximation.

2. At the cutoff frequency, $\omega_c = 1/RC$ ($f_c = 1/2\pi RC$), the gain of the filter is −3 dB. This means that at the cutoff frequency, the circuit will deliver

half the power that it would deliver at very low frequencies. At the cutoff frequency, the output voltage will lag the input voltage by 45°.

3. As the frequency increases beyond the cutoff frequency, the amplitude of the output signal decreases by a factor of approximately ten for each tenfold increase in frequency; namely the voltage gain is −20 dB per decade. The phase shift at $\omega = 10\omega_c$ will be 5.71° greater than the straight-line approximation, namely at $\theta = -84.29°$. For high frequencies ($\omega/\omega_c \geq 10$), the phase shift between the input and output voltage approaches −90°.

## The *RL* Low-Pass Filter

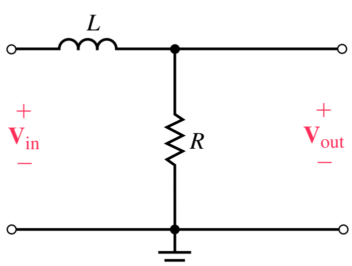



**FIGURE 22–16** *RL* low-pass filter.

A low-pass filter circuit may be made up of a resistor and an inductor as illustrated in Figure 22–16.

In a manner similar to that used for the *RC* low-pass filter, we may write the transfer function for the circuit of Figure 22–16 as follows:

$$\mathbf{TF} = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}$$

$$= \frac{\mathbf{R}}{\mathbf{R} + \mathbf{Z}_L} = \frac{R}{R + j\omega L}$$

Now, dividing the numerator and the denominator by $R$, we have the transfer function expressed as

$$\mathbf{TF} = \frac{1}{1 + j\omega\dfrac{L}{R}}$$

Since the cutoff frequency is found as $\omega_c = 1/\tau$, we have

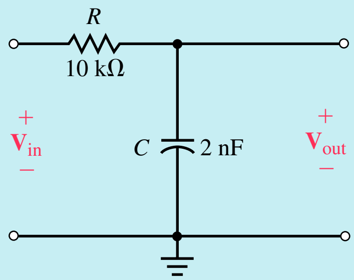
$$\omega_c = \frac{1}{\tau} = \frac{1}{\dfrac{L}{R}} = \frac{R}{L}$$

and so

$$\boxed{\mathbf{TF} = \frac{1}{1 + j\dfrac{\omega}{\omega_c}}} \qquad \textbf{(22–16)}$$

Notice that the transfer function for the *RL* low-pass circuit in Equation 22–16 is identical to the transfer function of an *RC* circuit in Equation 22–15. In each case, the cutoff frequency is determined as the reciprocal of the time constant.

**EXAMPLE 22–7** Sketch the Bode plot showing both the straight-line approximation and the actual response curves for the circuit of Figure 22–17. Show frequencies in hertz.




EWB **FIGURE 22–17**

**Solution** The cutoff frequency (in radians per second) for the circuit occurs at

$$\omega_c = \frac{1}{\tau} = \frac{1}{RC}$$

$$= \frac{1}{(10\ \text{k}\Omega)(2\ \text{nF})} = 50\ \text{krad/s}$$

which gives

$$f_c = \frac{\omega_c}{2\pi} = \frac{50\ \text{krad/s}}{2\pi} = 7.96\ \text{kHz}$$

In order to sketch the Bode plot, we begin with the asymptotes for the voltage gain response. The circuit will have a flat response until $f_c = 7.96$ kHz. Then the gain will drop at a rate of 20 dB for each decade increase in frequency. Therefore, the voltage gain at 79.6 kHz will be $-20$ dB, and at 796 kHz the voltage gain will be $-40$ dB. At the cutoff frequency for the filter, the actual voltage gain response will pass through a point which is 3 dB down from the intersection of the two asymptotes. The frequency response of the voltage gain is shown in Figure 22–18(a).

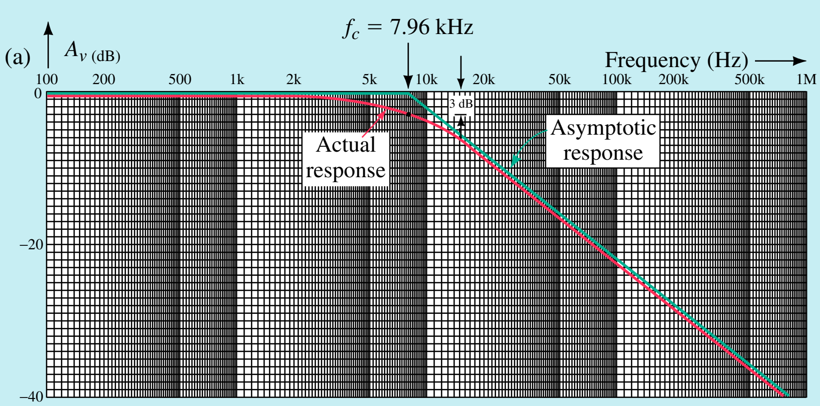



**FIGURE 22–18** *(Continues)*

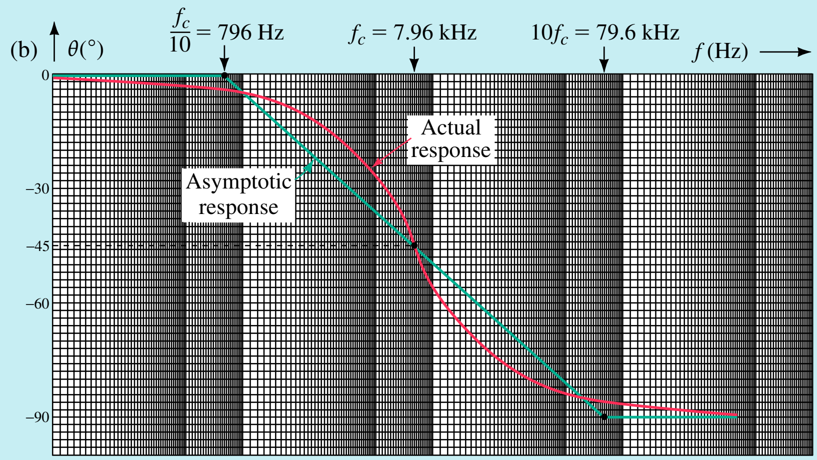
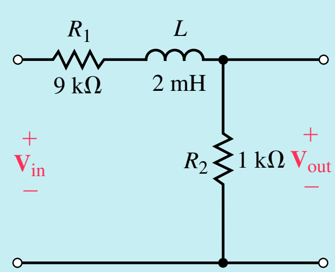



**FIGURE 22–18** *(Continued)*

Next, we sketch the approximate phase shift response. The phase shift at 7.96 kHz will be $-45°$. At a frequency one decade below the cutoff frequency (at 796 Hz) the phase shift will be approximately equal to zero, while at a frequency one decade above the cutoff frequency (at 79.6 kHz) the phase shift will be near the maximum of $-90°$. The actual phase shift response will be a curve which varies slightly from the asymptotic response, as shown in Figure 22–18(b).

**EXAMPLE 22–8** Consider the low-pass circuit of Figure 22–19:




EWB **FIGURE 22–19**

a. Write the transfer function for the circuit.
b. Sketch the frequency response.

**Solution**

a. The transfer function of the circuit is found as

$$\mathbf{TF} = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{R_2}{R_2 + R_1 + j\omega L}$$

which becomes

$$\mathbf{TF} = \frac{R_2}{R_1 + R_2}\left(\frac{1}{1 + j\omega\dfrac{L}{R_1 + R_2}}\right)$$

b. From the transfer function of Part a), we see that the dc gain will no longer be 1 (0 dB) but rather is found as

$$\begin{aligned} A_{v(\mathrm{dc})} &= 20\log\left(\frac{R_2}{R_1 + R_2}\right) \\ &= 20\log\left(\frac{1}{10}\right) \\ &= -20\text{ dB} \end{aligned}$$

The cutoff frequency occurs at

$$\omega_c = \frac{1}{\tau} = \frac{1}{\dfrac{L}{R_1 + R_2}}$$

$$\omega_c = \frac{R_1 + R_2}{L} = \frac{10\text{ k}\Omega}{2\text{ mH}}$$

$$= 5.0\text{ Mrad/s}$$

The resulting Bode plot is shown in Figure 22–20. Notice that the frequency response of the phase shift is precisely the same as for other low-pass filters. However, the response of the voltage gain now starts at $-20$ dB and then drops at a rate of $-20$ dB/decade above the cutoff frequency, $\omega_c = 5$ Mrad/s.

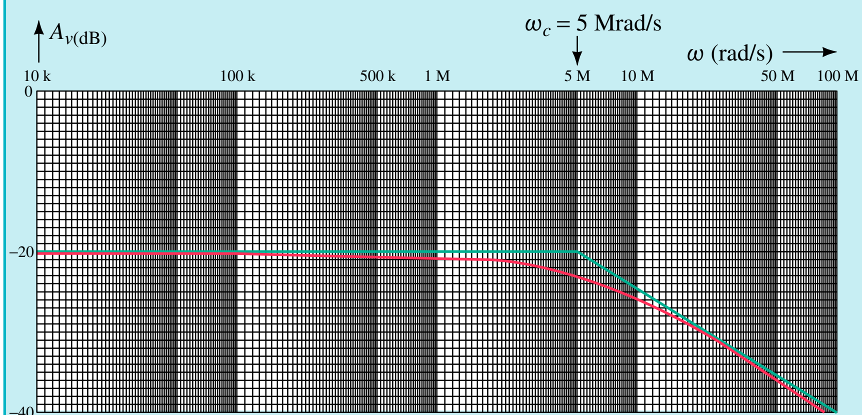



**FIGURE 22–20** *(Continues)*

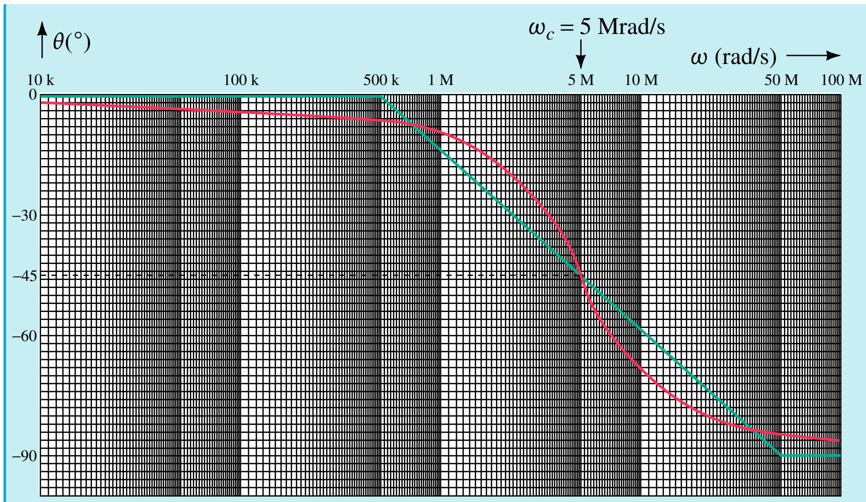
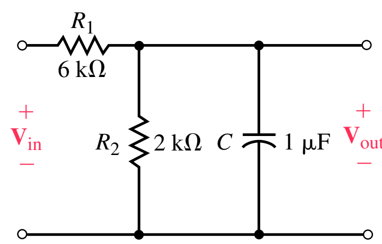



**FIGURE 22–20** *(Continued)*

PRACTICE PROBLEMS 5

Refer to the low-pass circuit of Figure 22–21:

EWB **FIGURE 22–21**




a. Write the transfer function for the circuit.

b. Sketch the frequency response.

*Answers:*

a. $\mathbf{TF}(\omega) = \dfrac{0.25}{1 + j\omega 0.0015}$

b.

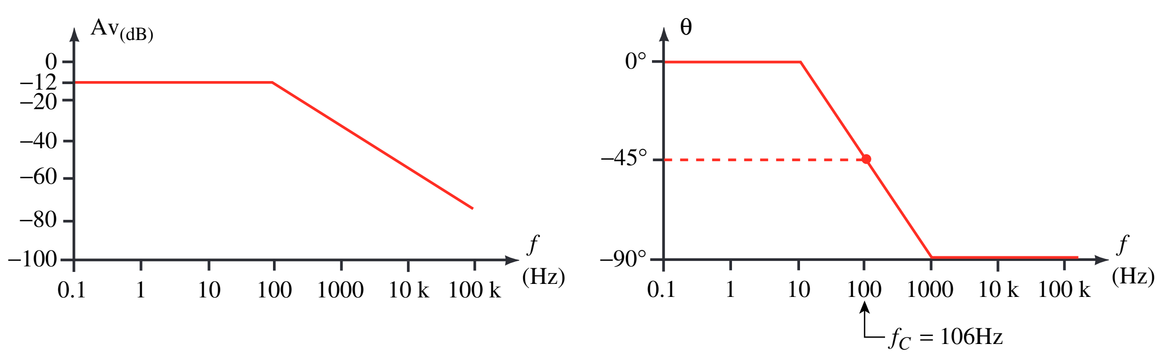



**FIGURE 22–22**

IN-PROCESS LEARNING CHECK 2

a. Design a low-pass *RC* filter to have a cutoff frequency of 30 krad/s. Use a 0.01-$\mu$F capacitor.
b. Design a low-pass *RL* filter to have a cutoff frequency of 20 kHz and a dc gain of $-6$ dB. Use a 10-mH inductor. (Assume that the inductor has no internal resistance.)

*(Answers are at the end of the chapter.)*

## 22.5 The High-Pass Filter

### The *RC* High-Pass Filter

As the name implies, the **high-pass filter** is a circuit which allows high-frequency signals to pass from the input to the output of the circuit while attenuating low-frequency signals. A simple *RC* high-pass filter circuit is illustrated in Figure 22–23.

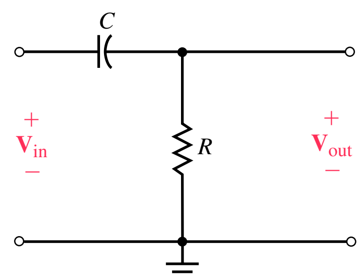



**FIGURE 22–23** *RC* high-pass filter.

At low frequencies, the reactance of the capacitor will be very large, effectively preventing any input signal from passing through to the output. At high frequencies, the capacitive reactance will approach a short-circuit condition, providing a very low impedance path for the signal from the input to the output.

The transfer function of the low-pass filter is determined as follows:

$$\mathbf{TF} = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{\mathbf{R}}{\mathbf{R} + \mathbf{Z}_C}$$

$$= \frac{R}{R + \dfrac{1}{j\omega C}} = \frac{R}{\dfrac{j\omega RC + 1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

Now, if we let $\omega_c = 1/\tau = 1/RC$, we have

$$\mathbf{TF} = \frac{j\dfrac{\omega}{\omega_c}}{1 + j\dfrac{\omega}{\omega_c}} \qquad \textbf{(22–17)}$$

Notice that the expression of Equation 22–17 is very similar to the expression for a low-pass filter, with the exception that there is an additional term in the numerator. Since the transfer function is a complex number which is dependent upon frequency, we may once again find the general expressions for voltage gain and phase shift as functions of frequency, $\omega$.

The voltage gain is found as

$$A_v = \frac{\dfrac{\omega}{\omega_c}}{\sqrt{1 + \left(\dfrac{\omega}{\omega_c}\right)^2}}$$

which, when expressed in decibels becomes

$$A_{v(\mathrm{dB})} = 20 \log \frac{\omega}{\omega_c} - 10 \log\left[1 + \left(\frac{\omega}{\omega_c}\right)^2\right] \tag{22–18}$$

The phase shift of the numerator will be a constant 90°, since the term has only an imaginary component. The overall phase shift of the transfer function is then found as follows:

$$\theta = 90° - \arctan \frac{\omega}{\omega_c} \tag{22–19}$$

In order to sketch the asymptotic response of the voltage gain we need to examine the effect of Equation 22–18 on frequencies around the cutoff frequency, $\omega_c$.

For frequencies $\omega \leq 0.1\omega_c$, the second term of the expression will be essentially equal to zero, and so the voltage gain at low frequencies is approximated as

$$A_{v(\mathrm{dB})} \cong 20 \log \frac{\omega}{\omega_c}$$

If we substitute some arbitrary values of $\omega$ into the above approximation, we arrive at a general statement. For example, by letting $\omega = 0.01\omega_c$, we have the voltage gain as

$$A_v = 20 \log(0.01) = -40 \text{ dB}$$

and by letting $\omega = 0.1\omega_c$, we have

$$A_v = 20 \log(0.1) = -20 \text{ dB}$$

In general, we see that the expression $A_v = 20 \log (\omega/\omega_c)$ may be represented as a straight line on a semilogarithmic graph. The straight line will intersect the 0-dB axis at the cutoff frequency, $\omega_c$, and have a slope of +20 dB/decade.

For frequencies $\omega >> \omega_c$, Equation 22–18 may be expressed as

$$\begin{aligned} A_{v(\mathrm{dB})} &\cong 20 \log \frac{\omega}{\omega_c} - 10 \log\left[\left(\frac{\omega}{\omega_c}\right)^2\right] \\ &= 20 \log \frac{\omega}{\omega_c} - 20 \log \frac{\omega}{\omega_c} \\ &= 0 \text{ dB} \end{aligned}$$

For the particular case when $\omega = \omega_c$, we have

$$A_{v(\mathrm{dB})} = 20 \log 1 - 10 \log 2 = -3.0 \text{ dB}$$

which is exactly the same result that we would expect, since the actual response will be 3 dB down from the asymptotic response.

Examining Equation 22–19 for frequencies $\omega \leq 0.1\omega_c$, we see that the phase shift for the transfer function will be essentially constant at 90°, while for frequencies $\omega \geq 10\omega_c$ the phase shift will be approximately constant at 0°. At $\omega = \omega_c$ we have $\theta = 90° - 45° = 45°$.

Figure 22–24 shows the normalized Bode plot of the high-pass circuit of Figure 22–23.

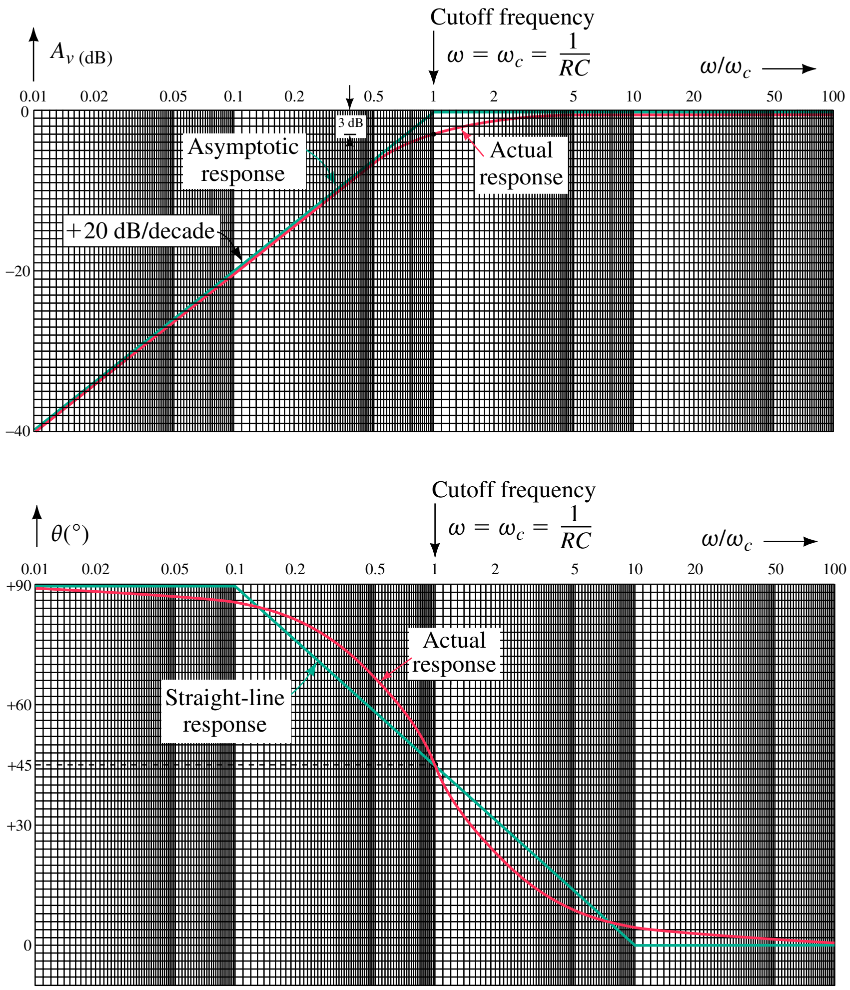



**FIGURE 22–24** Normalized frequency response for an *RC* high-pass filter. (a) Voltage gain response. (b) Phase shift response.

## The *RL* High-Pass Filter

A typical *RL* high-pass filter circuit is shown in Figure 22–25.

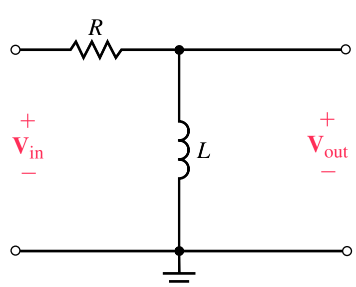



**FIGURE 22–25** *RL* high-pass filter.

At low frequencies, the inductor is effectively a short circuit, which means that the output of the circuit is essentially zero at low frequencies.

Inversely, at high frequencies, the reactance of the inductor approaches infinity and greatly exceeds the resistance, effectively preventing current. The voltage across the inductor is therefore very nearly equal to the applied input voltage signal. The transfer function for the high-pass $RL$ circuit is derived as follows:

$$\mathbf{TF} = \frac{\mathbf{Z}_L}{\mathbf{R} + \mathbf{Z}_L}$$

$$= \frac{j\omega L}{R + j\omega L} = \frac{j\omega \frac{L}{R}}{L + j\omega \frac{L}{R}}$$

Now, letting $\omega_c = 1/\tau = R/L$, we simplify the expression as

$$\mathbf{TF} = \frac{j\omega\tau}{1 + j\omega\tau}$$

The above expression is identical to the transfer function for a high-pass $RC$ filter, with the exception that in this case we have $\tau = L/R$.

**EXAMPLE 22–9** Design the $RL$ high-pass filter circuit of Figure 22–26 to have a cutoff frequency of 40 kHz. (Assume that the inductor has no internal resistance.) Sketch the frequency response of the circuit expressing the frequencies in kilohertz.

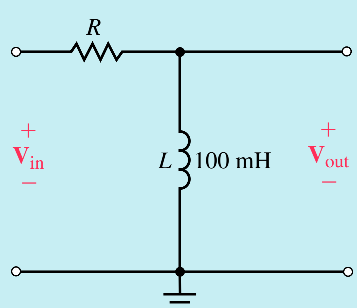



**FIGURE 22–26**

**Solution** The cutoff frequency, $\omega_c$, in radians per second is

$$\omega_c = 2\pi f_c = 2\pi(40\text{ kHz}) = 251.33\text{ krad/s}$$

Now, since $\omega_c = R/L$, we have

$$R = \omega_c L = (251.33\text{ krad/s}(100\text{ mH}) = 25.133\text{ k}\Omega$$

The resulting Bode plot is shown in Figure 22–27.

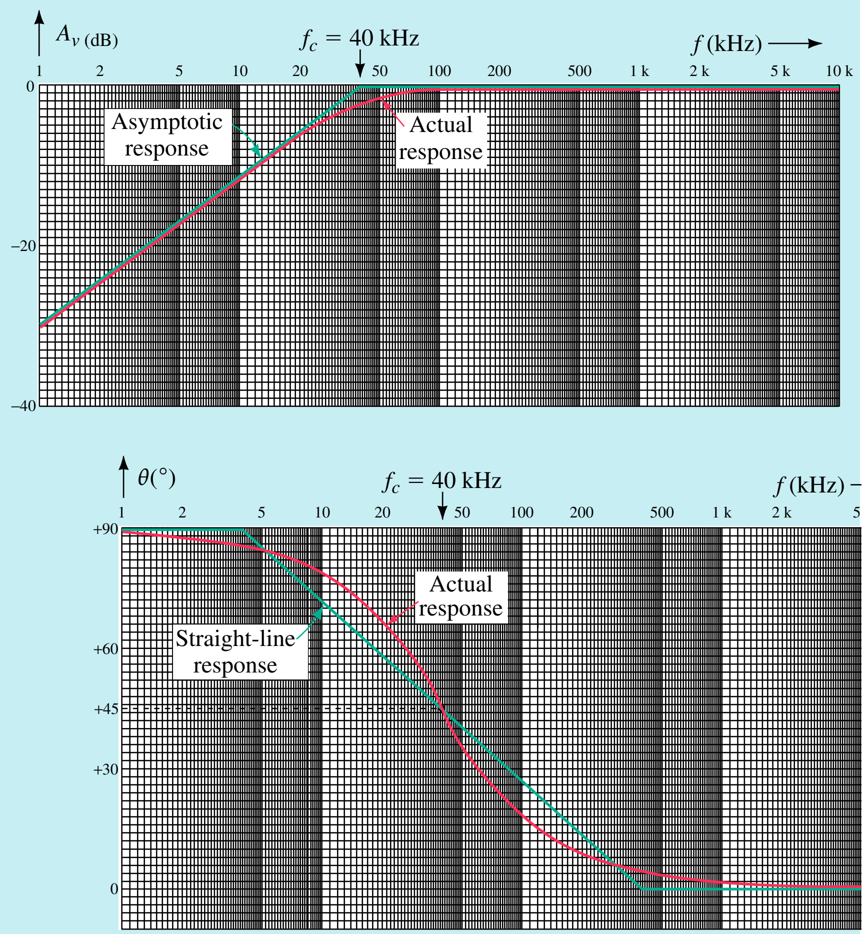
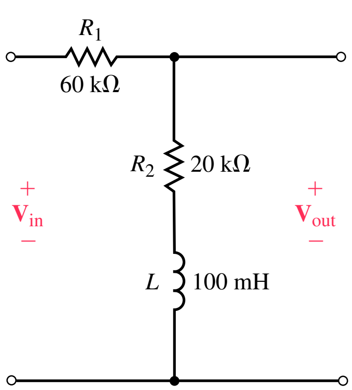



FIGURE 22–27

Consider the high-pass circuit of Figure 22–28:

PRACTICE PROBLEMS 6




EWB FIGURE 22–28

a. Write the transfer function of the filter.

b. Sketch the frequency response of the filter. Show the frequency in radians per second. (Hint: The frequency response of the filter is a step response.)

*Answers:*

a. $$\mathbf{TF}(\omega) = \left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{1 + j\omega\dfrac{L}{R_2}}{1 + j\omega\dfrac{L}{R_1 + R_2}}\right)$$

$f_1 = 31.8$ kHz (200 krad/s), $f_2 = 127$ kHz (800 krad/s)

b.

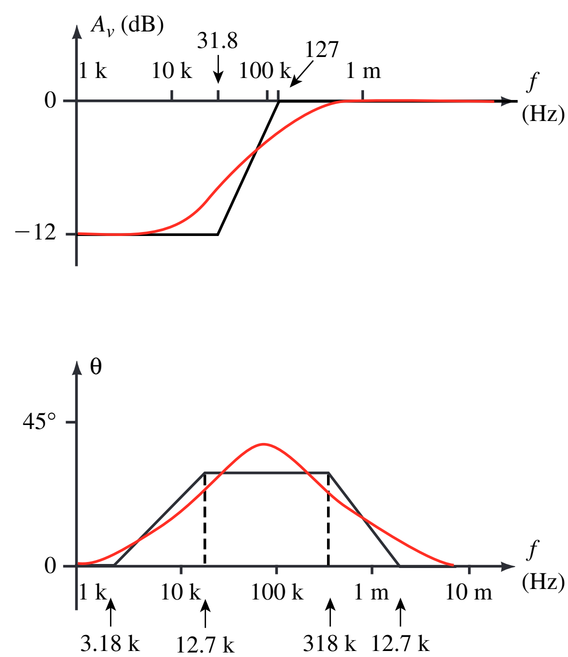



**FIGURE 22–29**

**IN-PROCESS LEARNING CHECK 3**

1. Use a 0.05-$\mu$F capacitor to design a high-pass filter having a cutoff frequency of 25 kHz. Sketch the frequency response of the filter.
2. Use a 25-mH inductor to design a high-pass filter circuit having a cutoff frequency of 80 krad/s and a high-frequency gain of −12 dB. Sketch the frequency response of the filter.

*(Answers are at the end of the chapter.)*

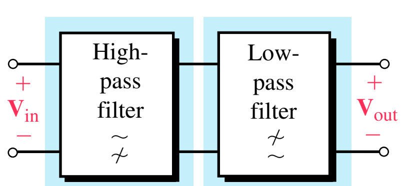



**FIGURE 22–30** Block diagram of a band-pass filter.

## 22.6 The Band-Pass Filter

A **band-pass filter** will permit frequencies within a certain range to pass from the input of a circuit to the output. All frequencies which fall outside the desired range will be attenuated and so will not appear with appreciable power at the output. Such a filter circuit is easily constructed by using a low-pass filter cascaded with a high-pass circuit as illustrated in Figure 22–30.

Although the low-pass and high-pass blocks may consist of various combinations of elements, one possibility is to construct the entire filter network from resistors and capacitors as shown in Figure 22–31.

The bandwidth of the resulting band-pass filter will be approximately equal to the difference between the two cutoff frequencies, namely,

$$\mathrm{BW} \cong \omega_2 - \omega_1 \quad \text{(rad/s)} \tag{22–20}$$

The above approximation will be most valid if the cutoff frequencies of the individual stages are separated by at least one decade.

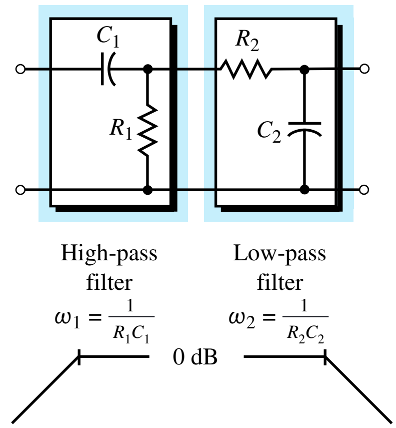



**FIGURE 22–31**

**EXAMPLE 22–10** Write the transfer function for the circuit of Figure 22–32. Sketch the resulting Bode plot and determine the expected bandwidth for the band-pass filter.

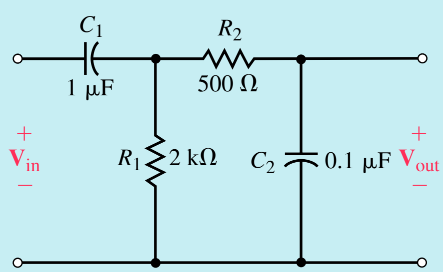



**FIGURE 22–32**

**Solution** While the transfer function for the circuit may be written by using circuit theory, it is easier to recognize that the circuit consists of two stages: one a low-pass stage and the other a high-pass stage. If the cutoff frequencies of each stage are separated by more than one decade, then we may assume that the impedance of one stage will not adversely affect the operation of the other stage. (If this is not the case, the analysis is complicated and is outside the scope of this textbook.) Based on the previous assumption, the transfer function of the first stage is determined as

$$\mathbf{TF}_1 = \frac{\mathbf{V}_1}{\mathbf{V}_{in}} = \frac{j\omega R_1C_1}{1 + j\omega R_1C_1}$$

and for the second stage as

$$\mathbf{TF}_2 = \frac{\mathbf{V}_{out}}{\mathbf{V}_1} = \frac{1}{1 + j\omega R_2C_2}$$

Combining the above results, we have

$$\mathbf{TF} = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{(\mathbf{TF}_2)(\mathbf{V}_1)}{\dfrac{\mathbf{V}_1}{\mathbf{TF}_1}} = \mathbf{TF}_1\mathbf{TF}_2$$

which, when simplified, becomes

$$\mathbf{TF} = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{j\omega\tau_1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)} \tag{22–21}$$

where $\tau_1 = R_1C_1 = 2.0$ ms and $\tau_2 = R_2C_2 = 50$ $\mu$s. The corresponding cutoff frequencies are $\omega_1 = 500$ rad/s and $\omega_2 = 20$ krad/s. The transfer function of Equation 22–21 has three separate terms which, when taken separately, result in the approximate responses illustrated in Figure 22–33.

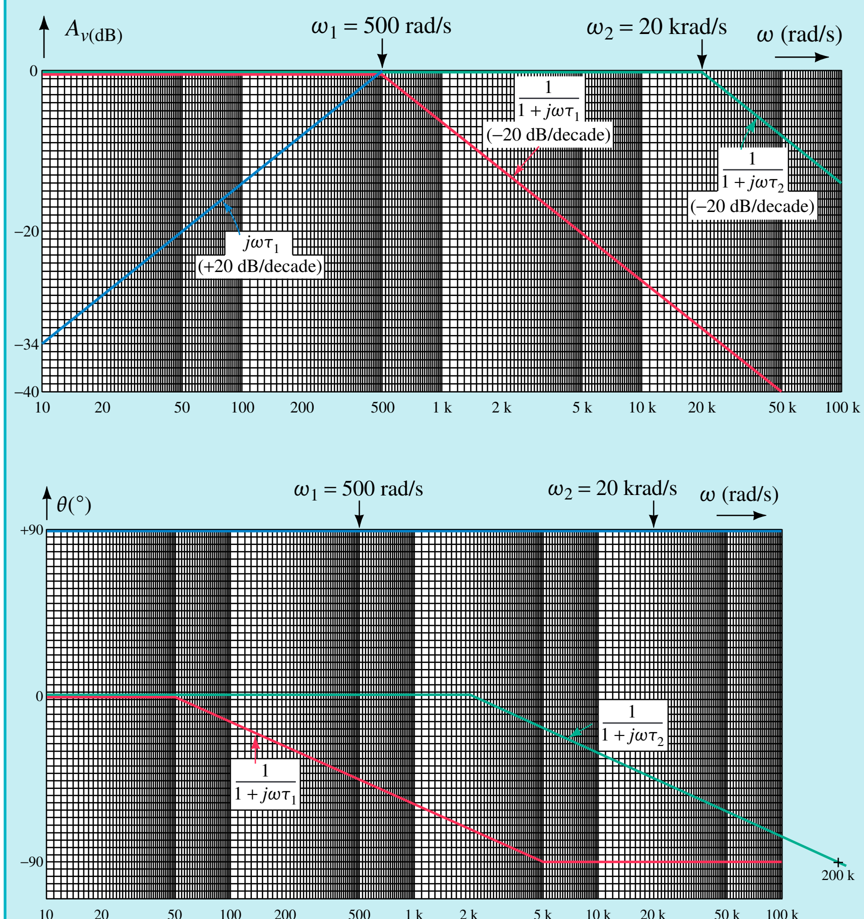



**FIGURE 22–33**

The resulting frequency response is determined by the summation of the individual responses as shown in Figure 22–34.

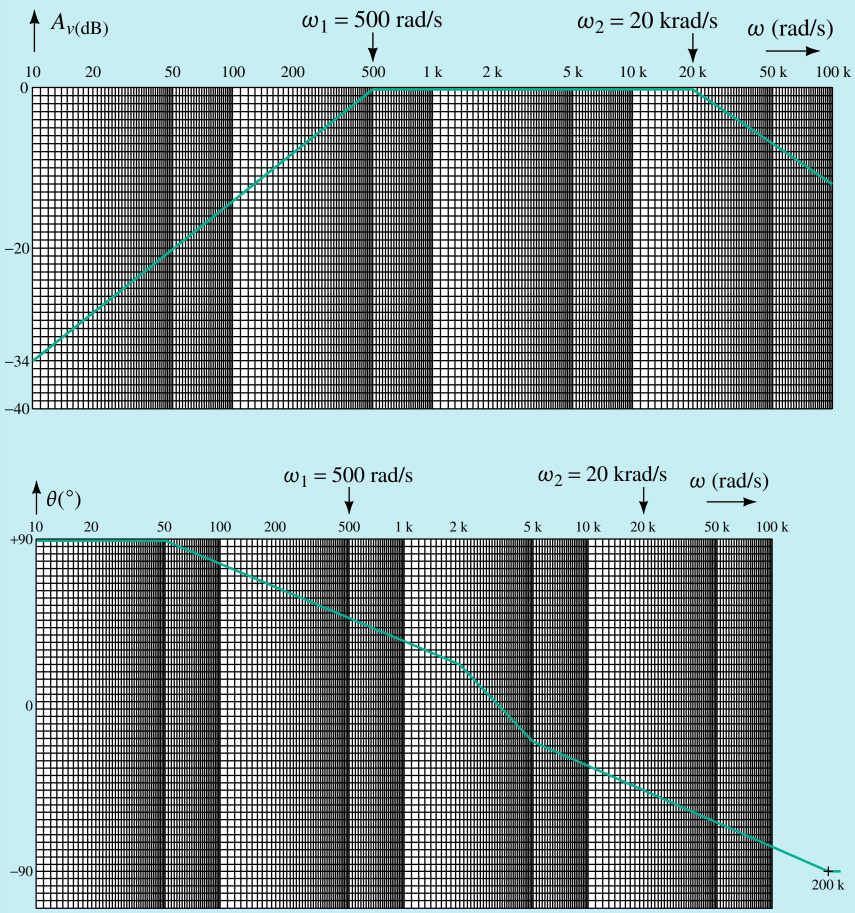



**FIGURE 22–34**

From the Bode plot, we determine that the bandwidth of the resulting filter is

$$\mathrm{BW(rad/s)} = \omega_2 - \omega_1 = 20\ \mathrm{krad/s} - 0.5\ \mathrm{krad/s} = 19.5\ \mathrm{krad/s}$$

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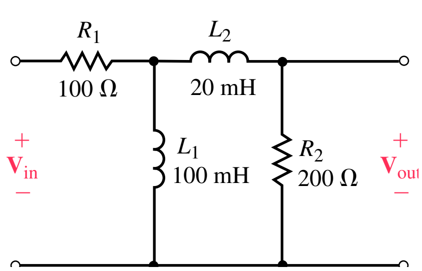
**PRACTICE PROBLEMS 7**

Refer to the band-pass filter of Figure 22–35:




EWB **FIGURE 22–35**

a. Calculate the cutoff frequencies in rad/s and the approximate bandwidth.

b. Sketch the frequency response of the filter.

*Answers:*

a. $\omega_1 = 1.00$ krad/s, $\omega_2 = 10.0$ krad/s, BW = 9.00 krad/s

b.

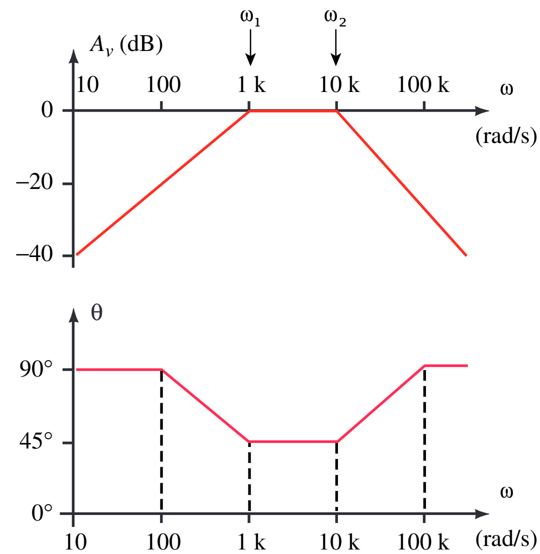



FIGURE 22–36

IN-PROCESS LEARNING CHECK 4

Given a 0.1-$\mu$F capacitor and a 0.04-$\mu$F capacitor, design a band-pass filter having a bandwidth of 30 krad/s and a lower cutoff frequency of 5 krad/s. Sketch the frequency response of the voltage gain and the phase shift.

*(Answers are at the end of the chapter.)*

## 22.7 The Band-Reject Filter

The **band-reject filter** has a response which is opposite to that of the band-pass filter. This filter passes all frequencies with the exception of a narrow band which is greatly attenuated. A band-reject filter constructed of a resistor, inductor, and capacitor is shown in Figure 22–37.

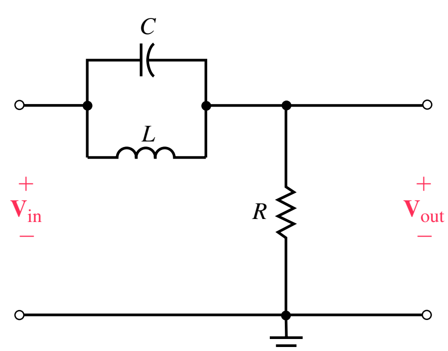



FIGURE 22–37 Notch filter.

Notice that the circuit uses a resonant tank circuit as part of the overall design. As we saw in the previous chapter, the combination of the inductor and the capacitor results in a very high tank impedance at the resonant frequency. Therefore, for any signals occurring at the resonant frequency, the output voltage is effectively zero. Because the filter circuit effectively removes any signal occurring at the resonant frequency, the circuit is often referred to as a **notch filter.** The voltage gain response of the notch filter is shown in Figure 22–38.

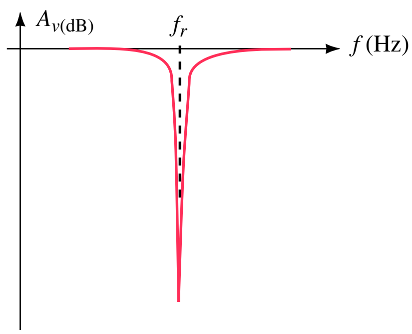



**FIGURE 22–38** Frequency response of voltage gain for a notch filter.

For low-frequency signals, the inductor provides a low-impedance path from the input to the output, allowing these signals to pass from the input and appear across the resistor with minimal attenuation. Conversely, at high frequencies the capacitor provides a low-impedance path from the input to the output. Although the complete analysis of the notch filter is outside the scope of this textbook, the transfer function of the filter is determined by employing the same techniques as those previously developed.

$$\mathbf{TF} = \frac{\mathbf{R}}{\mathbf{R} + \mathbf{Z}_L \| \mathbf{Z}_C}$$

$$= \frac{R}{R + \dfrac{(j\omega L)\left(\dfrac{1}{j\omega C}\right)}{j\omega L + \dfrac{1}{j\omega C}}}$$

$$= \frac{R}{R + \dfrac{j\omega L}{1 - \omega^2 LC}}$$

$$= \frac{R(1 - \omega^2 LC)}{R - \omega^2 RLC + j\omega L}$$

$$\mathbf{TF} = \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega \dfrac{L}{R}} \tag{22–22}$$

Notice that the transfer function for the notch filter is significantly more complicated than for the previous filter circuits. Due to the presence of the complex quadratic in the denominator of the transfer function, this type of filter circuit is called a **second-order filter.** The design of such filters is a separate field in electronics engineering.

Actual filter design often involves using operational amplifiers to provide significant voltage gain in the passband of the filter. In addition, such active filters have the advantage of providing very high input impedance to prevent loading effects. Many excellent textbooks are available for assistance in filter design.

## 22.8 Circuit Analysis Using Computers

ELECTRONICS WORKBENCH

PSpice

Despite the complexity of designing a filter for a particular application, the analysis of the filter is a relatively simple process when done by computer. We have already seen the ease with which PSpice may be used to examine the frequency response of resonant circuits. In this chapter we will again use the Probe postprocessor of PSpice to plot the frequency characteristics of a particular circuit. We find that with some minor adjustments, the program is able to simultaneously plot both the voltage gain (in decibels) and the phase shift (in degrees) of any filter circuit.

Electronics Workbench provides displays that are similar to those obtained in PSpice. The method, though, is somewhat different. Since Electronics Workbench simulates actual lab measurements, an instrument called a Bode plotter is used by the software. As one might expect, the Bode plotter provides a graph of the frequency response (showing both the gain and phase shift) of a circuit even though no such instrument is found in a real electronics lab.

### OrCAD PSpice

EXAMPLE 22–11 Use the PROBE postprocessor of PSpice to view the frequency response from 1 Hz to 100 kHz for the circuit of Figure 22–32. Determine the cutoff frequencies and the bandwidth of the circuit. Compare the results to those obtained in Example 22–10.

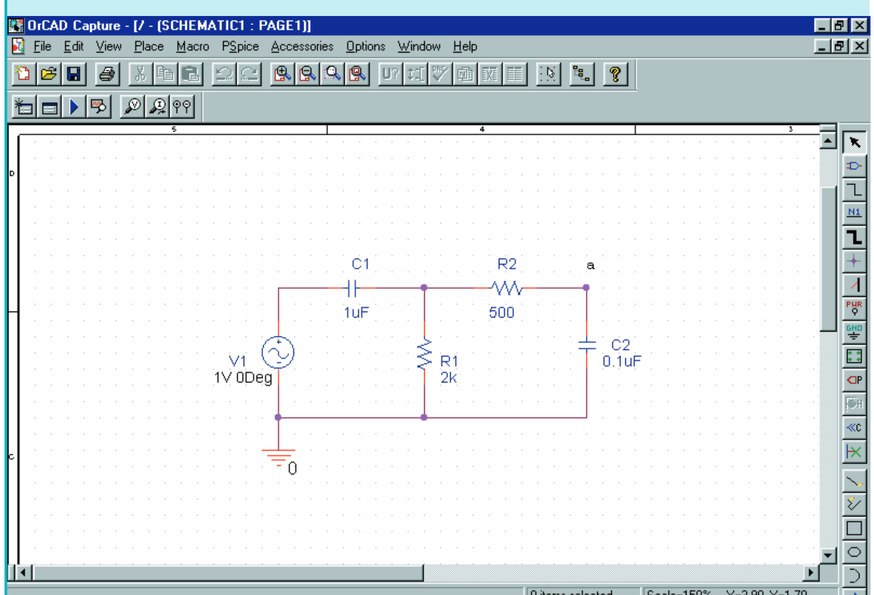



FIGURE 22–39

**Solution** The OrCAD Capture program is used to enter the circuit as shown in Figure 22–39. The analysis is set up to perform an ac sweep from 1 Hz to 100 kHz using 1001 points per decade. The signal generator is set for a magnitude of 1 V and a phase shift of 0°.

In this example, we show both the voltage gain and the phase shift on the same display. Once the PROBE screen is activated, two simultaneous displays are obtained by clicking on Plot/Add Plot to Window. We will use the top display to show voltage gain and the bottom to show the phase shift.

PSpice does not actually calculate the voltage gain in decibels, but rather determines the output voltage level in dBV, referenced to 1 $V_{rms}$. (It is for this reason that we used a supply voltage of 1 V.) The voltage at point *a* of the circuit is obtained by clicking on Trace/Add Trace and then selecting **DB(V(C2:1))** as the Trace Expression. Cursors are obtained by clicking on Tools/Cursor/Display. The maximum of the function is found by clicking on Tools/Cursor/Max. The cursor indicates that the maximum gain for the circuit is −1.02 dB. We find the bandwidth of the circuit (at the −3 dB frequencies) by moving the cursors (arrow keys and <Ctrl> arrow keys) to the frequencies at which the output of the circuit is at −4.02 dB. We determine $f_1 = 0.069$ kHz, $f_2 = 3.67$ kHz, and BW = 3.61 kHz. These results are consistent with those found in Example 22–10.

Finally, we obtain a trace of the phase shift for the circuit as follows. Click anywhere on the bottom plot. Click on Trace/Add Trace and then select **P(V(C2:1))** as the Trace Expression. The range of the ordinate is changed by clicking on the axis, selecting the Y Axis tab, and setting the User defined range for a value of **−90d** to **90d.** The resulting display is shown in Figure 22–40.

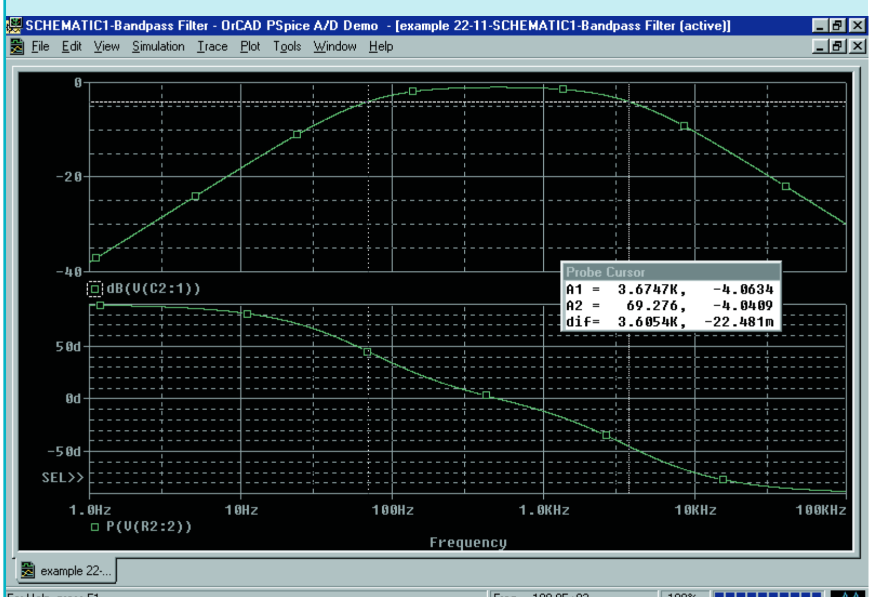



**FIGURE 22–40**

**PRACTICE PROBLEMS 8**

a. Use OrCAD Capture to input the circuit of Figure 22–35.

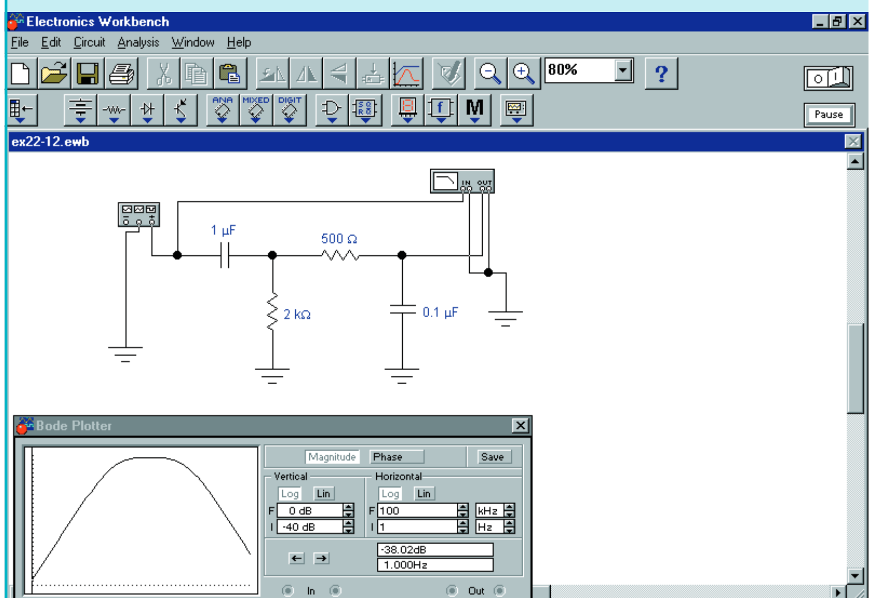
b. Use the Probe postprocessor to observe the frequency response from 1 Hz to 100 kHz.

c. From the display, determine the cutoff frequencies and use the cursors to determine the bandwidth.

d. Compare the results to those obtained in Practice Problem 7.

**EXAMPLE 22–12** Use Electronics Workbench to obtain the frequency response for the circuit of Figure 22–32. Compare the results to those obtained in Example 22–11.

**Solution** In order to perform the required measurements, we need to use the function generator and the Bode plotter, both located in the Instruments parts bin. The circuit is constructed as shown in Figure 22–41.




EWB **FIGURE 22–41**

The Bode plotter is adjusted to provide the desired frequency response by first double clicking on the instrument. Next, we click on the Magnitude button. The Vertical scale is set to log with values between **−40** dB and **0** dB. The Horizontal scale is set to log with values between **1** Hz and **100** kHz. Similarly, the Phase is set to have a Vertical range of **−90**° to **90**°. After clicking the run button, the Bode plotter provides a display of either the voltage gain response or the phase response. However, both displays are shown simultaneously by clicking on the Display Graphs icon. By using the cursor feature, we obtain the same results as those found in Example 22–11. Figure 22–42 shows the frequency response as viewed using the Display Graphs feature.

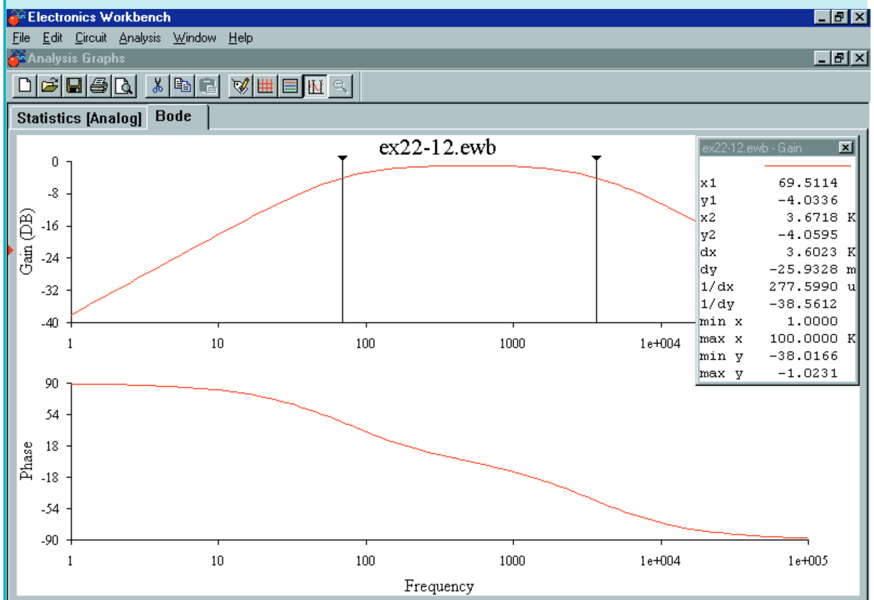



**FIGURE 22–42**

Use Electronics Workbench to obtain the frequency response for the circuit of Figure 22–21. Compare the results to those obtained in Practice Problem 5.

PRACTICE PROBLEMS 9

## PUTTING IT INTO PRACTICE

As a designer for a sound studio, you have been asked to design bandpass filters for a color organ that will be used to provide lighting for a rock concert. The color organ will provide stage lighting that will correspond to the sound level and frequency of the music. You remember from your physics class that the human ear can perceive sounds from 20 Hz to 20 kHz.

The specifications say that the audio frequency spectrum is to be divided into three ranges. Passive *RC* filters will be used to isolate signals for each range. These signals will then be amplified and used to control lights of a particular color. The low-frequency components (20 to 200 Hz) will control blue lights, the mid-frequency components (200 Hz to 2 kHz) will control green lights, and the high-frequency components (2 kHz to 20 kHz) will control red lights.

Although the specifications call for 3 band-pass filters, you realize that you can simplify the design by using a low-pass filter with a break frequency of 200 Hz for the low frequencies and a high-pass filter with a break frequency of 2 kHz for the high frequencies. To simplify your work, you decide to use only 0.5-$\mu$F capacitors for all filters.

Show the design for each of the filters.

## PROBLEMS

### 22.1 The Decibel

1. Refer to the amplifier shown in Figure 22–43. Determine the power gain both as a ratio and in decibels for the following power values.

   a. $P_{in} = 1.2$ mW, $P_{out} = 2.4$ W

   b. $P_{in} = 3.5\ \mu$W, $P_{out} = 700$ mW

   c. $P_{in} = 6.0$ pW, $P_{out} = 12\ \mu$W

   d. $P_{in} = 2.5$ mW, $P_{out} = 1.0$ W

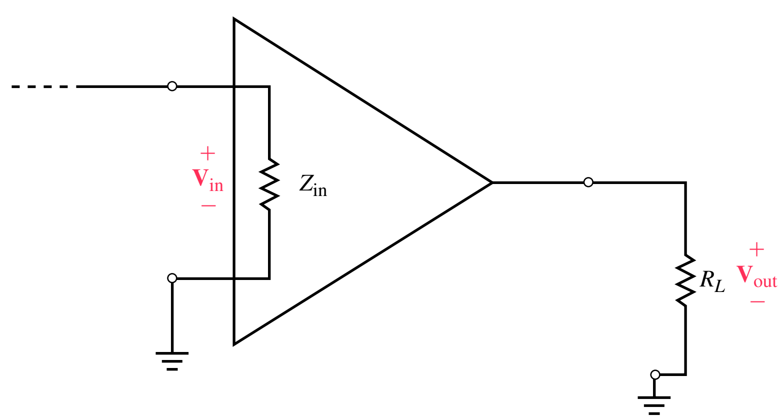



FIGURE 22–43

2. If the amplifier of Figure 22–43 has $Z_{in} = 600\ \Omega$ and $R_L = 2\ k\Omega$, find $P_{in}$, $P_{out}$, and $A_P$(dB) for the following voltage levels:
   a. $V_{in} = 20$ mV, $V_{out} = 100$ mV
   b. $V_{in} = 100\ \mu$V, $V_{out} = 400\ \mu$V
   c. $V_{in} = 320$ mV, $V_{out} = 600$ mV
   d. $V_{in} = 2\ \mu$V, $V_{out} = 8$ V
3. The amplifier of Figure 22–43 has $Z_{in} = 2\ k\Omega$ and $R_L = 10\ \Omega$. Find the voltage gain and power gain both as a ratio and in dB for the following conditions:
   a. $V_{in} = 2$ mV, $P_{out} = 100$ mW
   b. $P_{in} = 16\ \mu$W, $V_{out} = 40$ mV
   c. $V_{in} = 3$ mV, $P_{out} = 60$ mW
   d. $P_{in} = 2$ pW, $V_{out} = 80$ mV
4. The amplifier of Figure 22–43 has an input voltage of $V_{in} = 2$ mV and an output power of $P_{out} = 200$ mW. Find the voltage gain and power gain both as a ratio and in dB for the following conditions:
   a. $Z_{in} = 5\ k\Omega$, $R_L = 2\ k\Omega$
   b. $Z_{in} = 2\ k\Omega$, $R_L = 10\ k\Omega$
   c. $Z_{in} = 300\ k\Omega$, $R_L = 1\ k\Omega$
   d. $Z_{in} = 1\ k\Omega$ $R_L = 1\ k\Omega$
5. The amplifier of Figure 22–43 has an input impedance of 5 k$\Omega$ and a load resistance of 250 $\Omega$. If the power gain of the amplifier is 35 dB, and the input voltage is 250 mV, find $P_{in}$, $P_{out}$, $V_{out}$, $A_v$, and $A_v$(dB).
6. Repeat Problem 5 if the input impedance is increased to 10 k$\Omega$. (All other quantities remain unchanged.)
7. Express the following powers in dBm and in dBW:
   a. $P = 50$ mW
   b. $P = 1$ W
   c. $P = 400$ nW
   d. $P = 250$ pW
8. Express the following powers in dBm and in dBW.
   a. $P = 250$ W
   b. $P = 250$ kW
   c. $P = 540$ nW
   d. $P = 27$ mW
9. Convert the following power levels into watts:
   a. $P = 23.5$ dBm
   b. $P = -45.2$ dBW
   c. $P = -83$ dBm
   d. $P = 33$ dBW
10. Convert the following power levels into watts:
    a. $P = 16$ dBm
    b. $P = -43$ dBW

c. $P = -47.3$ dBm

d. $P = 29$ dBW

11. Express the following rms voltages as voltage levels (in dBV):
    a. 2.00 V
    b. 34.0 mV
    c. 24.0 V
    d. 58.2 $\mu$V
12. Express the following rms voltages as voltage levels (in dBV):
    a. 25 $\mu$V
    b. 90 V
    c. 72.5 mV
    d. 0.84 V
13. Convert the following voltage levels from dBV to rms voltages:
    a. $-2.5$ dBV
    b. 6.0 dBV
    c. $-22.4$ dBV
    d. 10.0 dBV
14. Convert the following voltage levels from dBV to rms voltages:
    a. 20.0 dBV
    b. $-42.0$ dBV
    c. $-6.0$ dBV
    d. 3.0 dBV
15. A sinusoidal waveform is measured as 30.0 $V_{p\text{-}p}$ with an oscilloscope. If this waveform were applied to a voltmeter calibrated to express readings in dBV, what would the voltmeter indicate?
16. A voltmeter shows a reading of 9.20 dBV. What peak-to-peak voltage would be observed on an oscilloscope?

### 22.2 Multistage Systems

17. Calculate the power levels (in dBm) at the output of each of the stages of the system shown in Figure 22–44. Solve for the output power (in watts).

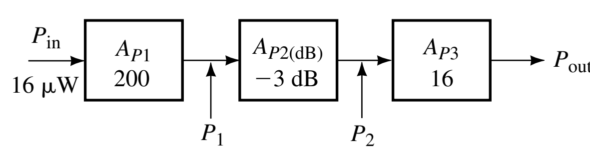



**FIGURE 22–44**

18. Calculate the power levels (in dBm) at the indicated locations of the system shown in Figure 22–45. Solve for the input and output powers (in watts).
19. Given that power $P_2 = 140$ mW as shown in Figure 22–46. Calculate the power levels (in dBm) at each of the indicated locations. Solve for the voltage across the load resistor, $R_L$.

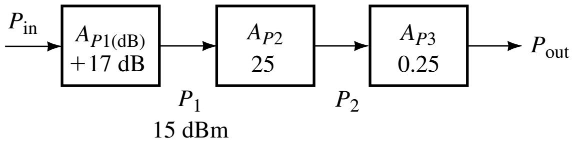



**FIGURE 22–45**

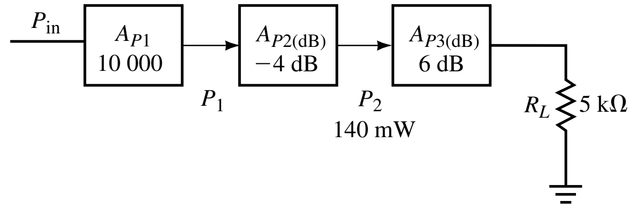



**FIGURE 22–46**

20. Suppose that the system of Figure 22–47 has an output voltage of 2 V:
    a. Determine the power (in watts) at each of the indicated locations.
    b. Solve for the voltage, $V_{in}$, if the input impedance of the first stage is 1.5 kΩ.
    c. Convert $V_{in}$ and $V_L$ into voltage levels (in dBV).
    d. Solve for the voltage gain, $A_v$ (in dB).

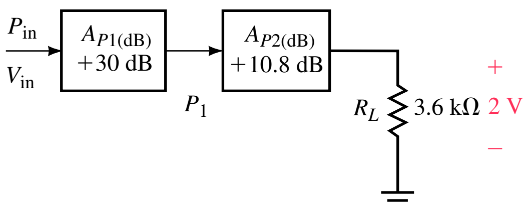



**FIGURE 22–47**

21. A power amplifier (P.A.) with a power gain of 250 has an input impedance of 2.0 kΩ and is used to drive a stereo speaker (output impedance of 8.0 Ω). If the output power is 100 W, determine the following:
    a. Output power level (dBm), input power level (dBm)
    b. Output voltage (rms), input voltage (rms)
    c. Output voltage level (dBV), input voltage level (dBV)
    d. Voltage gain in dB
22. Repeat Problem 21 if the amplifier has a power gain of 400 and $Z_{in}$ = 1.0 kΩ. The power delivered to the 8.0-Ω speaker is 200 W.

### 22.3 Simple *RC* and *RL* Transfer Functions

23. Given the transfer function

$$\mathbf{TF} = \frac{200}{1 + j0.001\omega}$$

    a. Determine the cutoff frequency in radians per second and in hertz.
    b. Sketch the frequency response of the voltage gain and the phase shift responses. Label the abscissa in radians per second.
24. Repeat Problem 23 for the transfer function

$$\mathbf{TF} = \frac{1 + j0.001\omega}{200}$$

25. Repeat Problem 23 for the transfer function

$$\mathbf{TF} = \frac{1 + j0.02\omega}{1 + j0.001\omega}$$

26. Repeat Problem 23 for the transfer function

$$\mathbf{TF} = \frac{1 + j0.04\omega}{(1 + j0.004\omega)(1 + j0.001\omega)}$$

27. Repeat Problem 23 for the transfer function

$$\mathbf{TF}(\omega) = \frac{j0.02\omega}{1 + j0.02\omega}$$

28. Repeat Problem 23 for the transfer function

$$\mathbf{TF}(\omega) = \frac{j0.01\omega}{1 + j0.005\omega}$$

### 22.4 The Low-Pass Filter

29. Use a 4.0-$\mu$F capacitor to design a low-pass filter circuit having a cutoff frequency of 5 krad/s. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
30. Use a 1.0-$\mu$F capacitor to design a low-pass filter with a cutoff frequency of 2500 Hz. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
31. Use a 25-mH inductor to design a low-pass filter with a cutoff frequency of 50 krad/s. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
32. Use a 100-mH inductor (assume $R_{\text{coil}} = 0\ \Omega$) to design a low-pass filter circuit having a cutoff frequency of 15 kHz. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
33. Use a 36-mH inductor to design a low-pass filter having a cutoff frequency of 36 kHz. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
34. Use a 5-$\mu$F capacitor to design a low-pass filter circuit having a cutoff frequency of 100 krad/s. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
35. Refer to the low-pass circuit of Figure 22–48:
    a. Write the transfer function for the circuit.
    b. Sketch the frequency response of the voltage gain and phase shift.
36. Repeat Problem 35 for the circuit of Figure 22–49.

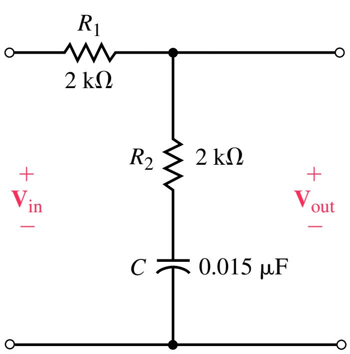
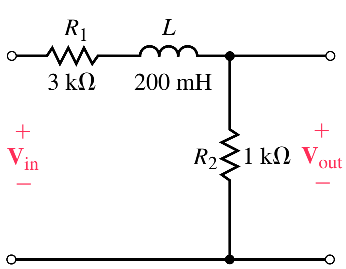



FIGURE 22–48




EWB FIGURE 22–49

### 22.5 The High-Pass Filter

37. Use a 0.05-$\mu$F capacitor to design a high-pass filter to have cutoff frequency of 100 krad/s. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
38. Use a 2.2-nF capacitor to design a high-pass filter to have a cutoff frequency of 5 kHz. Draw a schematic of your design and sketch the frequency response of the voltage gain and phase shift.

39. Use a 2-mH inductor to design a high-pass filter to have a cutoff frequency of 36 krad/s. Draw a schematic of your design and sketch the frequency response of the voltage gain and phase shift.
40. Use a 16-mH inductor to design a high-pass filter circuit having a cutoff frequency of 250 kHz. Draw a schematic of your design and sketch the frequency response of the voltage gain and the phase shift.
41. Refer to the high-pass circuit of Figure 22–50.
    a. Write the transfer function for the circuit.
    b. Sketch the frequency response of the voltage gain and phase shift.

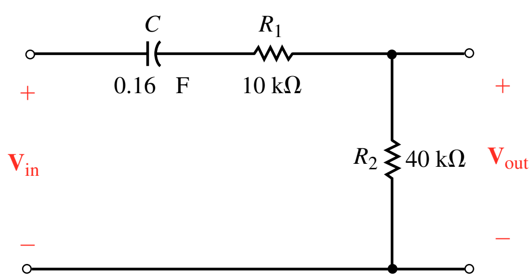



**FIGURE 22–50**

42. Repeat Problem 41 for the high-pass circuit of Figure 22–51.

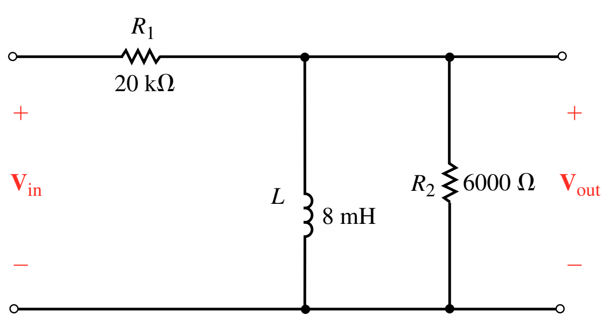



**FIGURE 22–51**

## 22.6 The Band-Pass Filter

43. Refer to the filter of Figure 22–52.
    a. Determine the approximate cutoff frequencies and bandwidth of the filter. (Assume that the two stages of the filter operate independently.)
    b. Sketch the frequency response of the voltage gain and the phase shift.

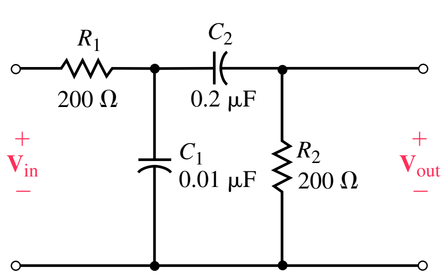



**FIGURE 22–52**

44. Repeat Problem 43 for the circuit of Figure 22–53.

45. a. Use two 0.01-$\mu$F capacitors to design a band-pass filter to have cutoff frequencies of 2 krad/s and 20 krad/s.
    b. Draw your schematic and sketch the frequency response of the voltage gain and the phase shift.
    c. Do you expect that the actual cutoff frequencies will occur at the designed cutoff frequencies? Explain.

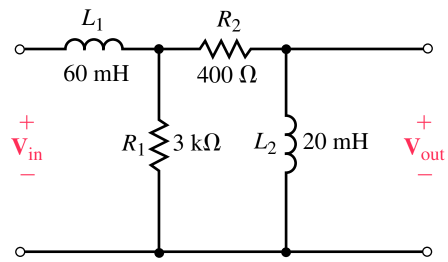
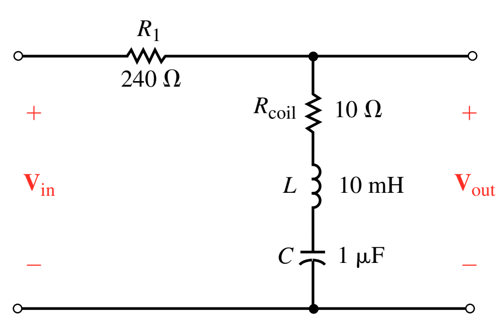



FIGURE 22–53

46. a. Use two 10-mH inductors to design a band-pass filter to have cutoff frequencies of 25 krad/s and 40 krad/s.
    b. Draw your schematic and sketch the frequency response of the voltage gain and the phase shift.
    c. Do you expect that the actual cutoff frequencies will occur at the designed cutoff frequencies? Explain.

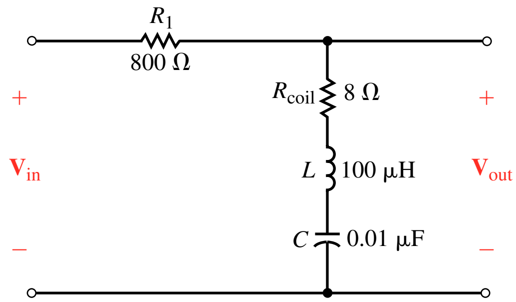
## 22.7 The Band-Reject Filter

47. Given the filter circuit of Figure 22–54:
    a. Determine the "notch" frequency.
    b. Calculate the $Q$ of the circuit.
    c. Solve for the bandwidth and determine the half-power frequencies.
    d. Sketch the voltage gain response of the circuit, showing the level (in dB) at the "notch" frequency.



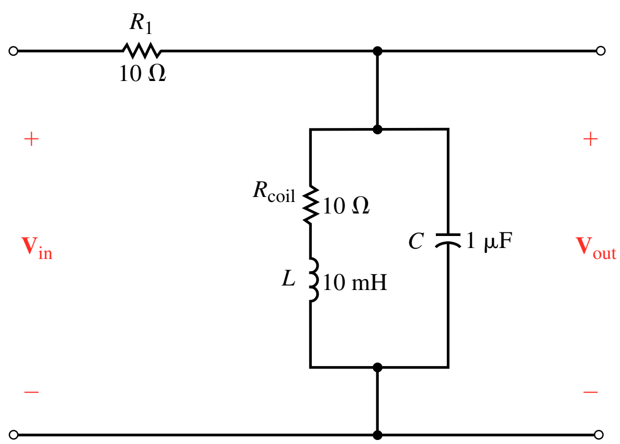

EWB FIGURE 22–54

48. Repeat Problem 47 for the circuit of Figure 22–55.



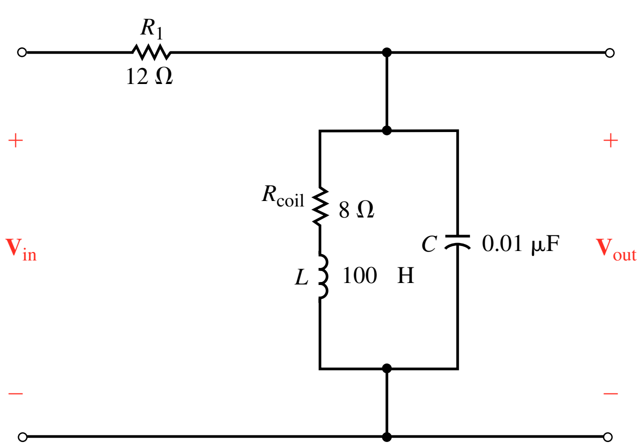

EWB **FIGURE 22–55**

49. Repeat Problem 47 for the circuit of Figure 22–56.




EWB **FIGURE 22–56**

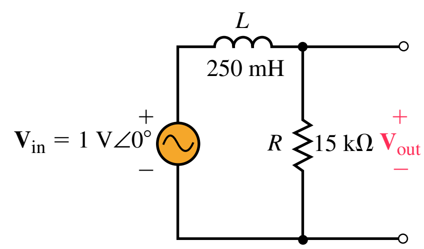
50. Repeat Problem 47 for the circuit of Figure 22–57.




EWB **FIGURE 22–57**

### 22.8 Circuit Analysis Using Computers

51. **PSpice** Use OrCAD Capture to input the circuit of Figure 22–58. Let the circuit sweep through frequencies of 100 Hz to 1 MHz. Use the Probe postprocessor to display the frequency response of voltage gain (in dBV) and phase shift of the circuit.




EWB **FIGURE 22–58**

52. **PSpice** Repeat Problem 51 for the circuit shown in Figure 22–49.
53. **PSpice** Use OrCAD Capture to input the circuit of Figure 22–52. Use the Probe postprocessor to display the frequency response of voltage gain (in dBV) and phase shift of the circuit. Select a suitable range for the frequency sweep and use the cursors to determine the half-power frequencies and the bandwidth of the circuit.
54. **PSpice** Repeat Problem 53 for the circuit shown in Figure 22–53.
55. **PSpice** Repeat Problem 53 for the circuit shown in Figure 22–54.
56. **PSpice** Repeat Problem 53 for the circuit shown in Figure 22–55.
57. **PSpice** Repeat Problem 53 for the circuit shown in Figure 22–56.
58. **PSpice** Repeat Problem 53 for the circuit shown in Figure 22–57.
59. **EWB** Use Electronics Workbench to obtain the frequency response for the circuit of Figure 22–58. Let the circuit sweep through frequencies of 100 Hz to 1 MHz.
60. **EWB** Repeat Problem 59 for the circuit of Figure 22–49.
61. **EWB** Use Electronics Workbench to obtain the frequency response for the circuit shown in Figure 22–54. Select a suitable frequency range and use cursors to determine the "notch" frequency and the bandwidth of the circuit.
62. **EWB** Repeat Problem 61 for the circuit shown in Figure 22–55.
63. **EWB** Repeat Problem 61 for the circuit shown in Figure 22–56.
64. **EWB** Repeat Problem 61 for the circuit shown in Figure 22–57.

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## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

1. a. −1.99 dBm b. −15.0 dBm c. −66.0 dBm
2. a. 0.561 $V_{rms}$ b. 0.224 $V_{rms}$ c. 0.354 $V_{rms}$

**In-Process Learning Check 2**

a. $R = 3333\ \Omega$ in series with $C = 0.01\ \mu\text{F}$ (output across C)

b. $R_1 = 630\ \Omega$ in series with $L = 10$ mH and $R_2 = 630\ \Omega$ (output across $R_2$)

**In-Process Learning Check 3**

1. $\mathbf{TF} = \dfrac{j\omega(6.36 \times 10^{-6})}{1 + j\omega(6.36 \times 10^{-6})}$

   $C = 0.05\ \mu\text{F}$ is in series with $R = 127.3\ \Omega$ (output across $R$)

2. $\mathbf{TF} = \dfrac{j\omega(3.125 \times 10^{-6})}{1 + j\omega(12.5 \times 10^{-6})}$

   $R_1 = 8\ \text{k}\Omega$ is in series with $L = 25\ \text{mH} \| R_2 = 2.67\ \text{k}\Omega$ (output across the parallel combination)

**In-Process Learning Check 4**

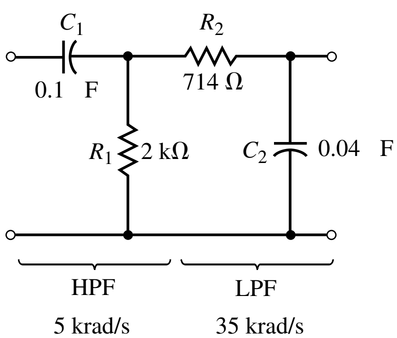



FIGURE 22–59

# 23

# Three-Phase Systems

## OBJECTIVES

After studying this chapter, you will be able to

- describe three-phase voltage generation,
- represent three-phase voltages and currents in phasor form,
- describe standard three-phase load connections,
- analyze balanced three-phase circuits,
- compute active power, reactive power, and apparent power in a three-phase system,
- measure power using the two-wattmeter method and the three-wattmeter method,
- analyze simple, unbalanced three-phase circuits,
- apply Electronics Workbench and PSpice to three-phase problems.

## KEY TERMS

Balanced Systems
Floating
Line Current
Line Voltage
Neutral
Phase Current
Phase Sequence
Phase Voltage
Single-Phase Equivalent
Two-Wattmeter Method
Unbalanced Systems
Watts Ratio Curve

## OUTLINE

Three-Phase Voltage Generation
Basic Three-Phase Circuit Connections
Basic Three-Phase Relationships
Examples
Power in a Balanced System
Measuring Power in Three-Phase Circuits
Unbalanced Loads
Power System Loads
Circuit Analysis Using Computers

## CHAPTER PREVIEW

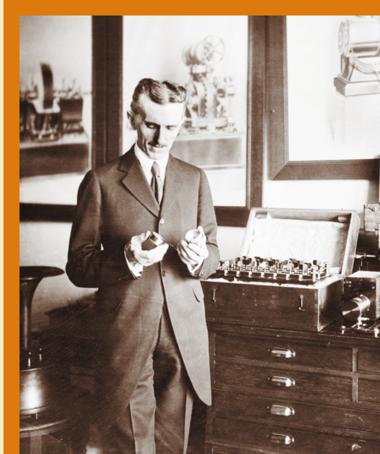
So far, we have looked only at single-phase systems. In this chapter, we consider three-phase systems. (Three-phase systems differ from single-phase systems in that they use a set of three voltages instead of one.) Three-phase systems are used for the generation and transmission of bulk electrical power. All commercial ac power systems, for example, are three-phase systems. Not all loads connected to a three-phase system need be three-phase, however—for example, the electric lights and appliances used in our homes require only single-phase ac. To get single-phase ac from a three-phase system, we simply tap off one of its phases.

Three-phase systems may be **balanced** or **unbalanced.** If a system is balanced, it can be analyzed by considering just one of its phases. (This is because, once you know the solution for one phase, you can write down the solutions to the other two phases with no further computation other than the addition or subtraction of an angle.) This is significant because it makes the analysis of balanced systems only slightly more complex than the analysis of single-phase systems. Since most systems operate close to balance, many practical problems can be dealt with by assuming balance. This is the approach used in practice.

Three-phase systems possess economic and operating advantages over single-phase systems. For example, for the same power output, three-phase generators cost less than single-phase generators, produce uniform power rather than pulsating power, and operate with less vibration and noise.

We begin the chapter with a look at three-phase voltage generation.

## PUTTING IT IN PERSPECTIVE



### *Nikola Tesla*

As noted in Chapter 15, the advent of the commercial electrical power age began with a fierce battle between Thomas A. Edison and George Westinghouse over the use of dc versus ac for the infant electrical power industry. Edison vigorously promoted dc while Westinghouse promoted ac. Tesla settled the argument in favor of ac with his development of the three-phase power system, the induction motor, and other ac devices. Coupled with the creation of a practical power transformer, these developments made long-distance transmission of electrical energy possible and ac became the clear winner.

Tesla was born in Smiljan, Croatia in 1856 and emigrated to the United States in 1884. During part of his career, he was associated with Edison, but the two had a falling out and became bitter rivals. Tesla made many important contributions in the fields of electricity and magnetism (he held over 700 patents), and the SI unit of magnetic flux density (the "tesla") is named after him. Tesla was also primarily responsible for the selection of 60 Hz as the standard power system frequency in North America and much of the world.

## 23.1 Three-Phase Voltage Generation

Three-phase generators have three sets of windings and thus produce three ac voltages instead of one. To get at the idea, consider first the elementary single-phase generator of Figure 23–1. As coil $AA'$ rotates, it produces a sinusoidal waveform $e_{AA'}$ as indicated in (b). This voltage can be represented by phasor $\mathbf{E}_{AA'}$ as shown in (c).

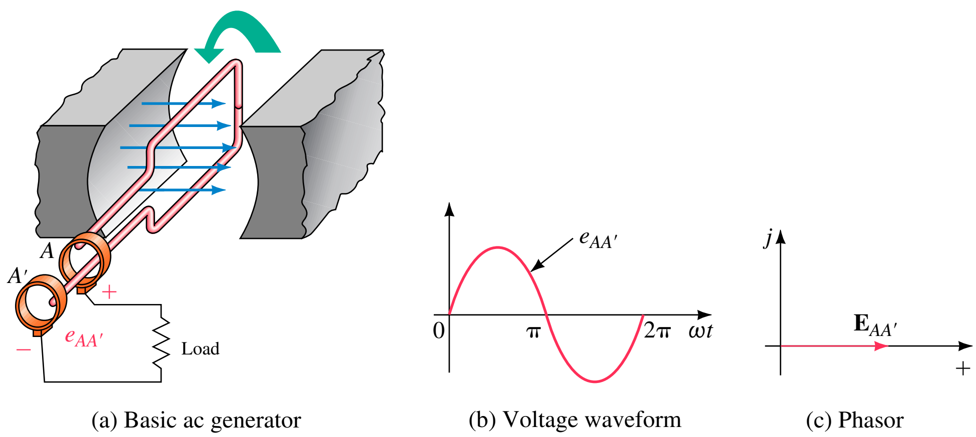



**FIGURE 23–1** A basic single-phase generator.

If two more windings are added as in Figure 23–2, two additional voltages are generated. Since these windings are identical with $AA'$ (except for their position on the rotor), they produce identical voltages. However, since coil $BB'$ is placed 120° behind coil $AA'$, voltage $e_{BB'}$ lags $e_{AA'}$ by 120°; similarly, coil $CC'$, which is placed ahead of coil $AA'$ by 120°, produces voltage $e_{CC'}$ that leads by 120°. Waveforms are shown in (b) and phasors in (c). As indicated, the generated voltages are equal in magnitude and phase displaced

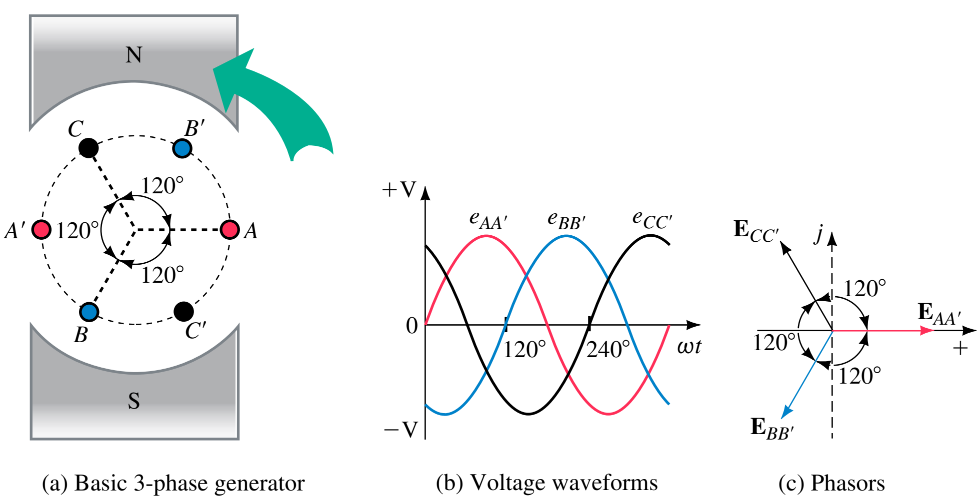



**FIGURE 23–2** Generating three-phase voltages. Three sets of coils are used to produce three balanced voltages.

by 120°. Thus, if $\mathbf{E}_{AA'}$ is at 0°, then $\mathbf{E}_{BB'}$ will be at −120° and $E_{CC'}$ will be at +120°. Assuming an rms value of 120 V and a reference position of 0° for phasor $\mathbf{E}_{AA'}$ for example, yields $\mathbf{E}_{AA'} = 120\text{ V}\angle 0°$, $\mathbf{E}_{BB'} = 120\text{ V}\angle -120°$ and $\mathbf{E}_{CC'} = 120\text{ V}\angle 120°$. Such a set of voltages is said to be balanced. Because of this fixed relationship between balanced voltages, you can, if you know one voltage, easily determine the other two.

PRACTICE PROBLEMS 1

a. If $\mathbf{E}_{AA'} = 277\text{ V}\angle 0°$, what are $\mathbf{E}_{BB'}$ and $\mathbf{E}_{CC'}$?

b. If $\mathbf{E}_{BB'} = 347\text{ V}\angle -120°$, what are $\mathbf{E}_{AA'}$ and $\mathbf{E}_{CC'}$?

c. If $\mathbf{E}_{CC'} = 120\text{ V}\angle 150°$, what are $\mathbf{E}_{AA'}$ and $\mathbf{E}_{BB'}$?

Sketch the phasors for each set.

*Answers:*

a. $\mathbf{E}_{BB'} = 277\text{ V}\angle -120°$; $\mathbf{E}_{CC'} = 277\text{ V}\angle 120°$

b. $\mathbf{E}_{AA'} = 347\text{ V}\angle 0°$; $\mathbf{E}_{CC'} = 347\text{ V}\angle 120°$

c. $\mathbf{E}_{AA'} = 120\text{ V}\angle 30°$; $\mathbf{E}_{BB'} = 120\text{ V}\angle -90°$

## 23.2 Basic Three-Phase Circuit Connections

The generator of Figure 23–2 has three independent windings: $AA'$, $BB'$, and $CC'$. As a first thought, you might try connecting loads using six wires as in Figure 23–3(a). This will work, although it is not a scheme that is used in practice. Nonetheless, some useful insights can be gained from it. To illustrate, assume a voltage of 120 V for each coil and a 12-ohm resistive load. With $\mathbf{E}_{AA'}$ as reference, Ohm's law applied to each circuit yields

$$\mathbf{I}_A = \mathbf{E}_{AA'}/R = 120\text{ V}\angle 0°/12\ \Omega = 10\text{ A}\angle 0°$$
$$\mathbf{I}_B = \mathbf{E}_{BB'}/R = 120\text{ V}\angle -120°/12\ \Omega = 10\text{ A}\angle -120°$$
$$\mathbf{I}_C = \mathbf{E}_{CC'}/R = 120\text{ V}\angle 120°/12\ \Omega = 10\text{ A}\angle 120°$$

These currents form a balanced set, as shown in Figure 23–3(b).

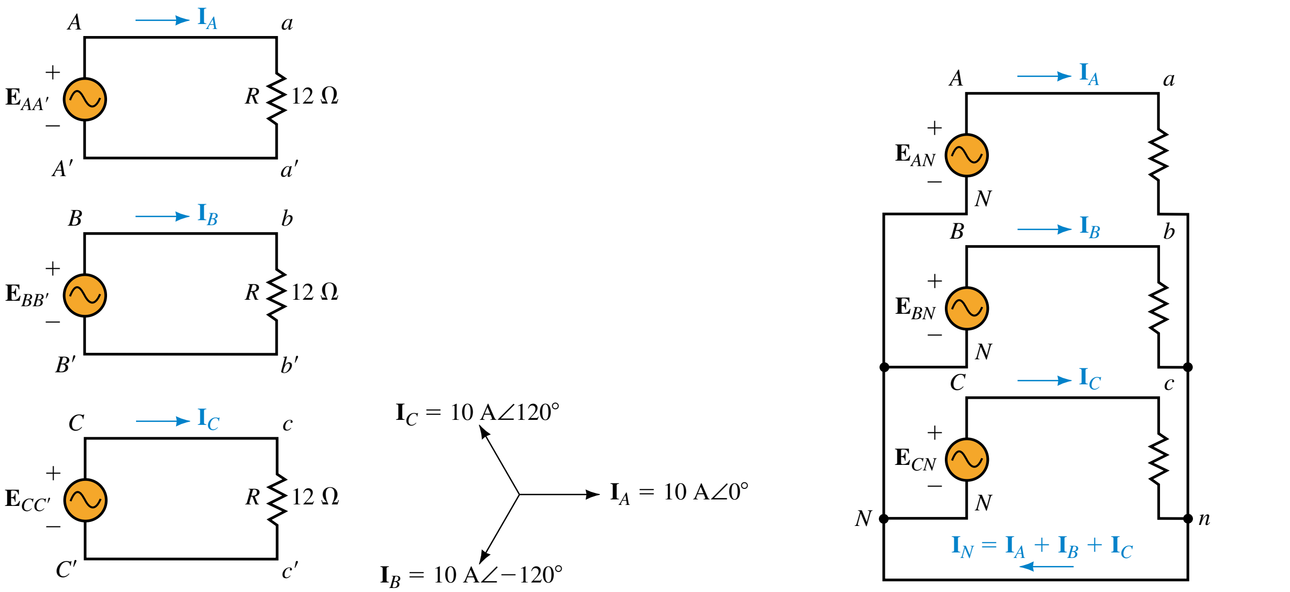
> **NOTES...**
>
> **A Comment on Generator Construction**
>
> Except for small generators, most three-phase generators do not actually use the construction of Figure 23–2. Instead, they use a fixed set of windings and a rotating magnetic field. The design of Figure 23–2 was chosen to illustrate three-phase voltage generation because it is easier to visualize.

### Four-Wire and Three-Wire Systems

Each load in Figure 23–3(a) has its own return wire. What if you replace them with a single wire as in (c)? By Kirchhoff's current law, the current in this wire (which we call the **neutral**) is the phasor sum of $\mathbf{I}_A$, $\mathbf{I}_B$, and $\mathbf{I}_C$. For the balanced 12-ohm load,

$$\mathbf{I}_N = \mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C = 10\text{ A}\angle 0° + 10\text{ A}\angle -120° + 10\text{ A}\angle 120°$$
$$= (10\text{ A} + j0) + (-5\text{ A} - j8.66\text{ A}) + (-5\text{ A} + j8.66\text{ A}) = 0\text{ amps}$$

Thus, the return wire carries no current at all! (This result is always true regardless of load impedance, provided the load is balanced, i.e., all phase impedances are the same.) In practice, power systems are normally operated close to balance. Thus, the return current, while not necessarily zero, will be quite small, and the neutral wire can be made smaller than the other three conductors. This configuration is called a **four-wire system** and is one of the systems used in practice.




(a) First thoughts (b) Currents form a balanced set (c) 4-Wire system. The return wire is called the neutral

EWB **FIGURE 23–3** Evolution of three-phase connections.

The outgoing lines of Figure 23–3(c) are called **line** or **phase conductors.** They are the conductors that you see suspended by insulators on transmission line towers.

## Symbology

Having joined points $A'$, $B'$, and $C'$ in Figure 23–3(c), we now drop the $A'$, $B'$, and $C'$ notation and simply call the common point $N$. The voltages are then renamed $\mathbf{E}_{AN}$, $\mathbf{E}_{BN}$, and $\mathbf{E}_{CN}$. They are known as **line-to-neutral voltages.**

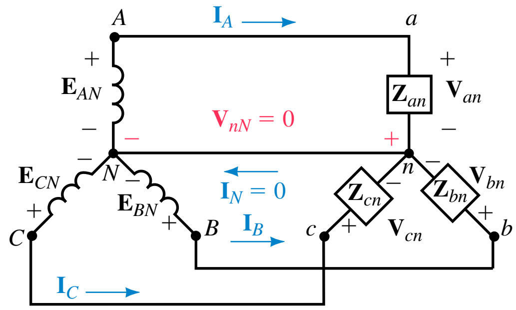
## Standard Representation

Three-phase circuits are not usually drawn as in Figure 23–3. Rather, they are usually drawn as in Figure 23–4. (Figure 23–4(a), for example, shows Figure 23–3(c) redrawn in standard form.) Note that coil symbols are used to represent generator windings rather than the circle symbol that we use for single phase.

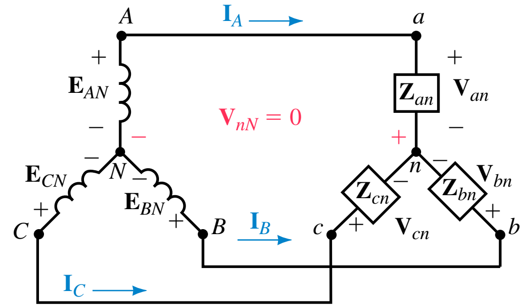
As Figure 23–4(a) shows, the circuit that we have been looking at is a **four-wire, wye-wye (Y-Y) circuit.** A variation, the **three-wire wye-wye circuit,** is shown in (b). Three-wire wye-wye circuits may be used if the load can be guaranteed to remain balanced, since under balanced conditions the neutral conductor carries no current. However, for practical reasons (discussed in Section 23.7), most wye-wye systems use four wires.

## Delta-Connected Generators

Now consider Δ connection of the generator windings. Theoretically, this is possible as indicated in Figure 23–5. However, there are practical difficulties. For example, when generators are loaded, distortions occur in the coil




(a) 4-wire Y-Y system. This is Figure 23-3(c) redrawn in standard form




(b) 3-wire Y-Y system

**FIGURE 23–4** Conventional representation of three-phase circuits. Both are Y-Y systems.

voltages due to magnetic fluxes produced by load currents. In Y-connected generators, these distortions cancel, but in Δ-connected generators, they do not. These distortions create a third harmonic current that circulates within the windings of the Δ-connected generator, lowering its efficiency. (You will learn about third harmonics in Chapter 25). For this and other reasons, Δ-connected generators are seldom used and will not be discussed in this book.

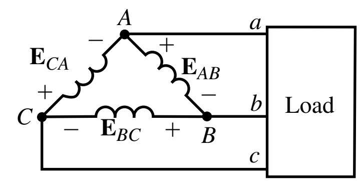



**FIGURE 23–5** A delta-connected generator. For practical reasons, delta generators are seldom used.

## Neutral-Neutral Voltage in a Wye-Wye Circuit

In a balanced Y-Y system, neutral current is zero because line currents sum to zero. As a consequence, the voltage between neutral points is zero. To see why, consider again Figure 23–4(a). Assume that the wire joining points $n$ and $N$ has impedance $\mathbf{Z}_{nN}$. This yields voltage $\mathbf{V}_{nN} = \mathbf{I}_N \times \mathbf{Z}_{nN}$. But since $\mathbf{I}_N = 0$, $\mathbf{V}_{nN} = 0$, regardless of the value of $\mathbf{Z}_{nN}$. Even if the neutral conductor is absent as in (b), $\mathbf{V}_{nN}$ is still zero. Thus, *in a balanced Y-Y system, the voltage between neutral points is zero.*

**EXAMPLE 23–1** Assume the circuits of Figure 23–4(a) and (b) are balanced. If $\mathbf{E}_{AN} = 247\ \text{V}\angle 0°$, what are $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$?

**Solution** In both cases, the voltage $\mathbf{V}_{nN}$ between neutral points is zero. Thus, by KVL, $\mathbf{V}_{an} = \mathbf{E}_{AN} = 247\ \text{V}\angle 0°$. Since the system is balanced, $\mathbf{V}_{bn} = 247\ \text{V}\angle -120$ and $\mathbf{V}_{cn} = 247\ \text{V}\angle 120°$.

## Phase Sequence

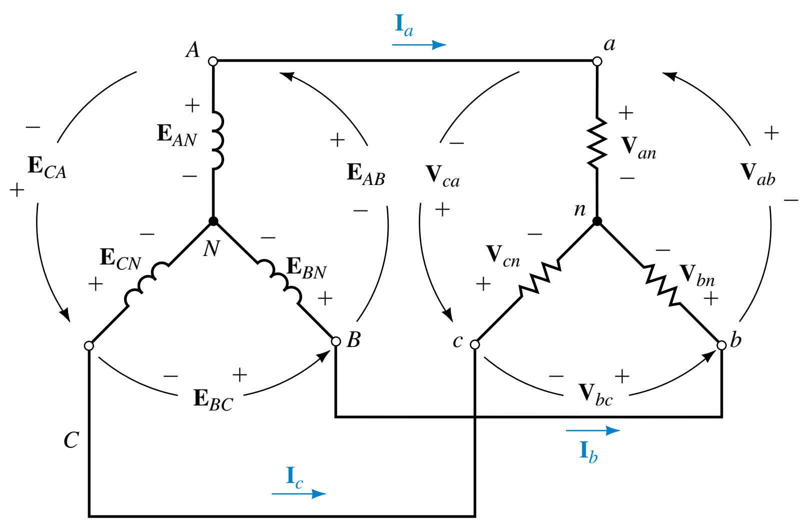
**Phase sequence** refers to the order in which three-phase voltages are generated. Consider again Figure 23–2. As the rotor turns in the counterclockwise direction, voltages are generated in the sequence $e_{AA'}$, $e_{BB'}$, and $e_{CC'}$ as indicated by waveforms (b) and the phasor set (c) and the system is said to have an **ABC phase sequence.** On the other hand, if the direction of rotation were reversed, the sequence would be *ACB*. Sequence *ABC* is called the **positive**

**phase sequence** and is the sequence generated in practice. It is therefore the only sequence considered in this book.

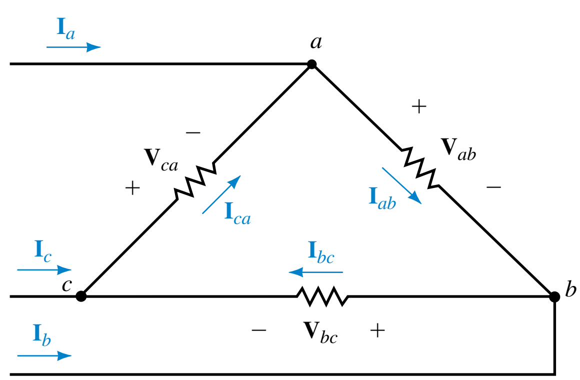
While voltages are generated in the sequence *ABC,* the order of voltages applied to a load depends on how you connect it to the source. For most balanced loads, phase sequence doesn't matter. However, for three-phase motors, the order is important, since if you reverse any pair of wires, the direction of the motor's rotation will reverse.

## 23.3 Basic Three-Phase Relationships

To keep track of voltages and currents, we use the symbols and notations of Figure 23–6. Capital letter subscripts are used at the source and lowercase letters at the load. As usual, *E* is used for source voltage and *V* for voltage drops.




(a) For a Y, phases are defined from line to neutral




(b) For a Δ, phases are defined from line to line

**FIGURE 23–6** Symbols and notation for 3-phase voltages and currents.

## Definitions

**Line** (also called **line-to-line**) voltages are voltages between lines. Thus, $\mathbf{E}_{AB}$, $\mathbf{E}_{BC}$, and $\mathbf{E}_{CA}$ are line-to-line voltages at the generator, while $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are line-to-line voltages at the load.

**Phase voltages** are voltages across phases. For a Y load, phases are defined from line to neutral as indicated in (a); thus, $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are phase voltages for a Y load. For a Δ load, phases are defined from line to line as shown in (b); thus, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are phase voltages for a Δ. As you can see, for a Δ load, phase voltages and line voltages are the same thing. For the generator, $\mathbf{E}_{AN}$, $\mathbf{E}_{BN}$, and $\mathbf{E}_{CN}$ are phase voltages.

**Line currents** are the currents in the line conductors. Only a single subscript is needed. You can use either $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ as in Figure 23–6 or $\mathbf{I}_A$, $\mathbf{I}_B$, and $\mathbf{I}_C$ as in Figure 23–4. (Some authors use double subscripts such as $\mathbf{I}_{Aa}$, but this is unnecessary.)

**Phase currents** are currents through phases. For the Y load Figure 23–6(a), $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ pass through phase impedances and are therefore phase currents. For the Δ load (b), $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ are phase currents. As you can see, for a Y load, phase currents and line currents are the same thing.

**Phase impedances** for a Y load are the impedances from *a-n, b-n,* and *c-n* [Figure 23–6(a)] and are denoted by the symbols $\mathbf{Z}_{an}$, $\mathbf{Z}_{bn}$, and $\mathbf{Z}_{cn}$. For a Δ load (b), phase impedances are $\mathbf{Z}_{ab}$, $\mathbf{Z}_{bc}$, and $\mathbf{Z}_{ca}$. In a balanced load, impedances for all phases are the same, i.e., $\mathbf{Z}_{an} = \mathbf{Z}_{bn} = \mathbf{Z}_{cn}$, etc.

## Line and Phase Voltages for a Wye Circuit

We now need the relationship between line and phase voltages for a Y circuit. Consider Figure 23–7. By KVL, $\mathbf{V}_{ab} - \mathbf{V}_{an} + \mathbf{V}_{bn} = 0$. Thus,

$$\mathbf{V}_{ab} = \mathbf{V}_{an} - \mathbf{V}_{bn} \qquad \textbf{(23–1)}$$

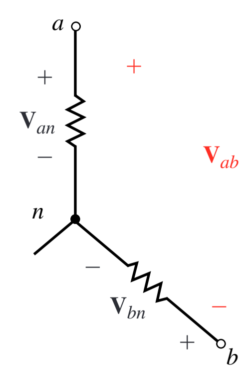



FIGURE 23–7

Now, assume a magnitude $V$ for each phase voltage and take $\mathbf{V}_{an}$ as reference. Thus, $\mathbf{V}_{an} = V\angle 0°$ and $\mathbf{V}_{bn} = V\angle -120°$. Substitute these two into Equation 23–1:

$$\begin{aligned}\mathbf{V}_{ab} &= V\angle 0° - V\angle -120° = V(1 + j0) - V(-0.5 - j0.866)\\ &= V(1.5 + j0.866) = 1.732\ V\angle 30° = \sqrt{3}V\angle 30°\end{aligned}$$

But $\mathbf{V}_{an} = V\angle 0°$. Thus,

$$\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30° \qquad \textbf{(23–2)}$$

Equation 23–2 shows that the magnitude of $\mathbf{V}_{ab}$ is $\sqrt{3}$ times the magnitude of $\mathbf{V}_{an}$ and that $\mathbf{V}_{ab}$ leads $\mathbf{V}_{an}$ by 30°. This is shown in phasor diagram form in Figure 23–8(a). Similar relationships hold for the other two phases. This is shown in (b). Thus, *for a balanced Y system, the magnitude of line-to-line voltage is $\sqrt{3}$ times the magnitude of the phase voltage and each line-to-line voltage leads its corresponding phase voltage by 30°.* From this you can see that *the line-to-line voltages also form a balanced set.* (Although we developed these relationships with $\mathbf{V}_{an}$ in the 0° reference position, they are true regardless of the choice of reference.) They also hold at the source. Thus,

$$\mathbf{E}_{AB} = \sqrt{3}\mathbf{E}_{AN}\angle 30° \qquad \textbf{(23–3)}$$

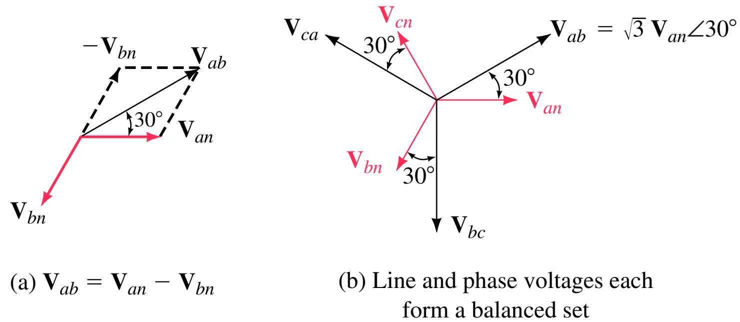



**FIGURE 23–8** Voltages for a balanced Y load. If you know one voltage, you can determine the other five by inspection.

**EXAMPLE 23–2**

a. Given $\mathbf{V}_{an} = 120\text{ V}\angle -45°$. Determine $\mathbf{V}_{ab}$ using Equation 23–2.
b. Verify $\mathbf{V}_{ab}$ by direct substitution of $\mathbf{V}_{an}$ and $\mathbf{V}_{bn}$ into equation 23–1.

**Solution**

a. $\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30° = \sqrt{3}(120\text{ V}\angle -45°)(1\angle 30°) = 207.8\angle -15°$.
b. $\mathbf{V}_{an} = 120\text{ V}\angle -45°$. Thus, $\mathbf{V}_{bn} = 120\text{ V}\angle -165°$.
$\mathbf{V}_{ab} = \mathbf{V}_{an} - \mathbf{V}_{bn} = (120\text{ V}\angle -45°) - (120\text{ V}\angle -165°)$
$= 207.8\text{ V}\angle -15°$ as before.

## Nominal Voltages

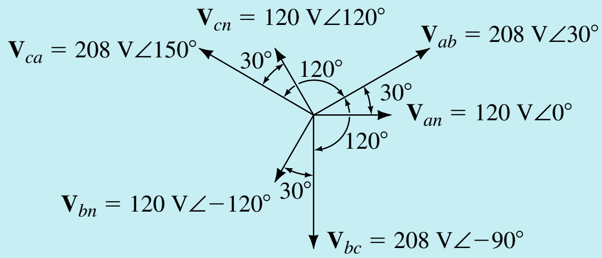
While Example 23–2 yields 207.8 V for line-to-line voltage, we generally round this to 208 V and refer to the system as a 120/208-V system. These are nominal values. Other sets of nominal voltages used in practice are 277/480-V and 347/600-V.

**EXAMPLE 23–3** For the circuits of Figure 23–4, suppose $\mathbf{E}_{AN} = 120\text{ V}\angle 0°$.

a. Determine the phase voltages at the load.
b. Determine the line voltages at the load.
c. Show all voltages on a phasor diagram.

**Solution**

a. $\mathbf{V}_{an} = \mathbf{E}_{AN}$. Thus, $\mathbf{V}_{an} = 120\text{ V}\angle 0°$. Since the system is balanced, $\mathbf{V}_{bn} = 120\text{ V}\angle -120°$ and $\mathbf{V}_{cn} = 120\text{ V}\angle 120°$.
b. $\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30° = \sqrt{3} \times 120\text{ V}\angle(0° + 30°) = 208\text{ V}\angle 30°$. Since line voltages form a balanced set, $\mathbf{V}_{bc} = 208\text{ V}\angle -90°$ and $\mathbf{V}_{ca} = 208\text{ V}\angle 150°$.
c. The phasors are shown in Figure 23–9.




EWB **FIGURE 23–9**

Equations 23–2 and 23–3 permit you to calculate line voltages from phase voltages. Rearranging them yields equation 23–4 which permits you to calculate phase voltage from line voltage.

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}\angle 30°} \qquad \mathbf{E}_{AN} = \frac{\mathbf{E}_{AB}}{\sqrt{3}\angle 30°} \tag{23–4}$$

For example, if $\mathbf{E}_{AB} = 480\ \text{V}\angle 45°$, then

$$\mathbf{E}_{AN} = \frac{\mathbf{E}_{AB}}{\sqrt{3}\angle 30°} = \frac{480\ \text{V}\angle 45°}{\sqrt{3}\angle 30°} = 277\ \text{V}\angle 15°$$

## A Milestone

You have now reached an important milestone. *Given any voltage at a point in a balanced, three-phase Y system, you can, with the aid of Equation 23–2 or 23–4, determine the remaining five voltages by inspection,* i.e., by simply shifting their angles and multiplying or dividing magnitude by $\sqrt{3}$ as appropriate.

PRACTICE PROBLEMS 2

For a balanced Y generator, $\mathbf{E}_{AB} = 480\ \text{V}\angle 20°$.

a. Determine the other two generator line voltages.
b. Determine the generator phase voltages.
c. Sketch the phasors.

*Answers:*
a. $\mathbf{E}_{BC} = 480\ \text{V}\angle -100°$; $\mathbf{E}_{CA} = 480\ \text{V}\angle 140°$
b. $\mathbf{E}_{AN} = 277\ \text{V}\angle -10°$; $\mathbf{E}_{BN} = 277\ \text{V}\angle -130°$; $\mathbf{E}_{CN} = 277\ \text{V}\angle 110°$

## Currents for a Wye Circuit

As you saw earlier, for a Y load, line currents are the same as phase currents. Consider Figure 23–10. As indicated in (b),

$$\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_{an} \tag{23–5}$$

Similarly for $\mathbf{I}_b$ and $\mathbf{I}_c$. Since $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ form a balanced set, *line currents $\boldsymbol{I}_a$, $\boldsymbol{I}_b$, and $\boldsymbol{I}_c$ also form a balanced set.* Thus, if you know one, you can determine the other two by inspection.

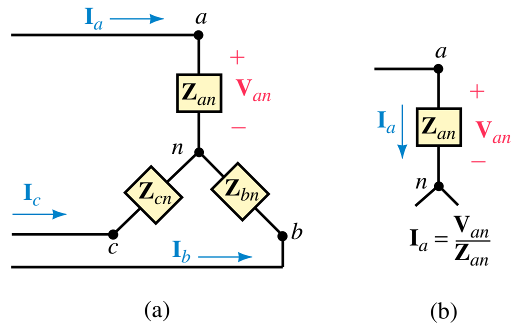
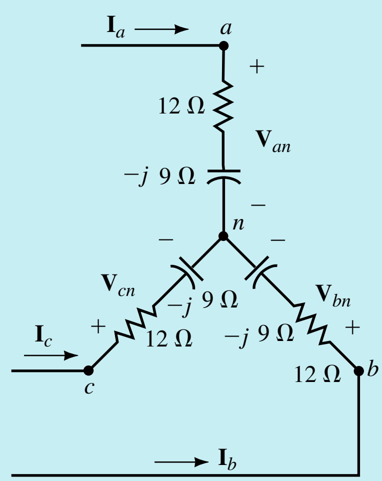



**FIGURE 23–10** Determining currents for a Y load.

**EXAMPLE 23–4** For Figure 23–11, suppose $\mathbf{V}_{an} = 120\text{ V}\angle 0°$.

a. Compute $\mathbf{I}_a$, then determine $\mathbf{I}_b$ and $\mathbf{I}_c$ by inspection.

b. Verify by direct computation.




EWB **FIGURE 23–11**

**Solution**

a. $$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_{an}} = \frac{120\angle 0°}{12 - j9} = \frac{120\angle 0°}{15\angle -36.87°} = 8.0\text{ A}\angle 36.87°$$

$\mathbf{I}_b$ lags $\mathbf{I}_a$ by 120°. Thus, $\mathbf{I}_b = 8\text{ A}\angle -83.13°$.
$\mathbf{I}_c$ leads $\mathbf{I}_a$ by 120°. Thus, $\mathbf{I}_c = 8\text{ A}\angle 156.87°$.

b. Since $\mathbf{V}_{an} = 120\text{ V}\angle 0°$, $\mathbf{V}_{bn} = 120\text{ V}\angle -120°$, and $\mathbf{V}_{cn} = 120\text{ V}\angle 120°$. Thus,

$$\mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_{bn}} = \frac{120\angle -120°}{15\angle -36.87°} = 8.0\text{ A}\angle -83.13°$$

$$\mathbf{I}_c = \frac{\mathbf{V}_{cn}}{\mathbf{Z}_{cn}} = \frac{120\angle 120°}{15\angle -36.87°} = 8.0\text{ A}\angle 156.87°$$

These agree with the results obtained in (a).

**PRACTICE PROBLEMS 3**

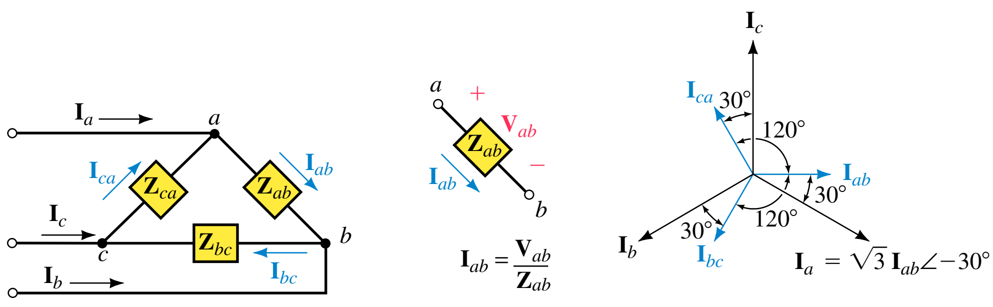
1. If $\mathbf{V}_{ab} = 600\ \text{V}\angle 0°$ for the circuit of Figure 23–11, what are $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$?
2. If $\mathbf{V}_{bc} = 600\ \text{V}\angle -90°$ for the circuit of Figure 23–11, what are $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$?

*Answers:*
1. $\mathbf{I}_a = 23.1\ \text{A}\angle 6.9°$; $\mathbf{I}_b = 23.1\ \text{A}\angle -113.1°$; $\mathbf{I}_c = 23.1\ \text{A}\angle 126.9°$
2. $\mathbf{I}_a = 23.1\ \text{A}\angle 36.9°$; $\mathbf{I}_b = 23.1\ \text{A}\angle -83.1°$; $\mathbf{I}_c = 23.1\ \text{A}\angle 156.9°$

## Line and Phase Currents for a Delta Load

Consider the delta load of Figure 23–12. Phase current $\mathbf{I}_{ab}$ can be found as in (b).

$$\mathbf{I}_{ab} = \mathbf{V}_{ab}/\mathbf{Z}_{ab} \qquad \textbf{(23–6)}$$




(a) $\mathbf{I}_a = \mathbf{I}_{ab} - \mathbf{I}_{ca}$ (b) (c) Line and phase currents each form a balanced set

**FIGURE 23–12** Currents for a balanced Δ load. If you know one current, you can determine the other five by inspection.

Similar relationships hold for $\mathbf{I}_{bc}$ and $\mathbf{I}_{ca}$. Since line voltages are balanced, phase currents are also balanced. Now consider again Figure 23–12(a). KCL at node *a* yields

$$\mathbf{I}_a = \mathbf{I}_{ab} - \mathbf{I}_{ca} \qquad \textbf{(23–7)}$$

After some manipulation, this reduces to

$$\mathbf{I}_a = \sqrt{3}\mathbf{I}_{ab}\angle -30° \qquad \textbf{(23–8)}$$

Thus, the magnitude of $\mathbf{I}_a$ is $\sqrt{3}$ times the magnitude of $\mathbf{I}_{ab}$, and $\mathbf{I}_a$ lags $\mathbf{I}_{ab}$ by 30°. Similarly, for the other two phases. Thus, *in a balanced Δ, the magnitude of the line current is $\sqrt{3}$ times the magnitude of the phase current, and each line current lags its corresponding phase current by 30°*. Since phase currents are balanced, line currents are also balanced. This is shown in (c). To find phase currents from line currents, use

$$\mathbf{I}_{ab} = \frac{\mathbf{I}_a}{\sqrt{3}\angle -30°} \qquad \textbf{(23–9)}$$

## A Second Milestone

You have reached a second milestone. *Given any current in a balanced, three-phase Δ load, you can, with the aid of Equations 23–8 or 23–9, determine all remaining currents by inspection.*

**EXAMPLE 23–5** Suppose $\mathbf{V}_{ab} = 240\text{ V}\angle 15°$ for the circuit of Figure 23–13.

a. Determine the phase currents.

b. Determine the line currents.

c. Sketch the phasor diagram.

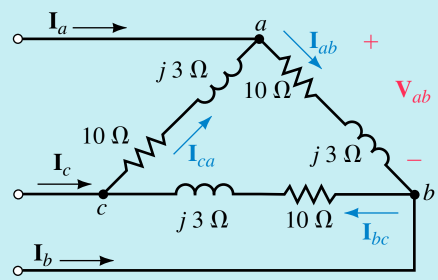



**FIGURE 23–13**

**Solution**

a. $\mathbf{I}_{ab} = \dfrac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \dfrac{240\angle 15°}{10 + j3} = 23.0\text{ A}\angle -1.70°$

Thus,

$$\mathbf{I}_{bc} = 23.0\text{ A}\angle -121.7° \quad \text{and} \quad \mathbf{I}_{ca} = 23.0\text{ A}\angle 118.3°$$

b. $\mathbf{I}_a = \sqrt{3}\mathbf{I}_{ab}\angle -30° = 39.8\text{ A}\angle -31.7°$

Thus,

$$\mathbf{I}_b = 39.8\text{ A}\angle -151.7° \quad \text{and} \quad \mathbf{I}_c = 39.8\text{ A}\angle 88.3°$$

c. Phasors are shown in Figure 23–14.

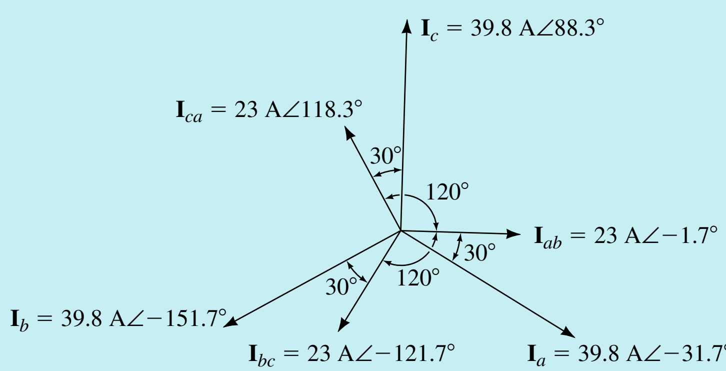



**FIGURE 23–14**

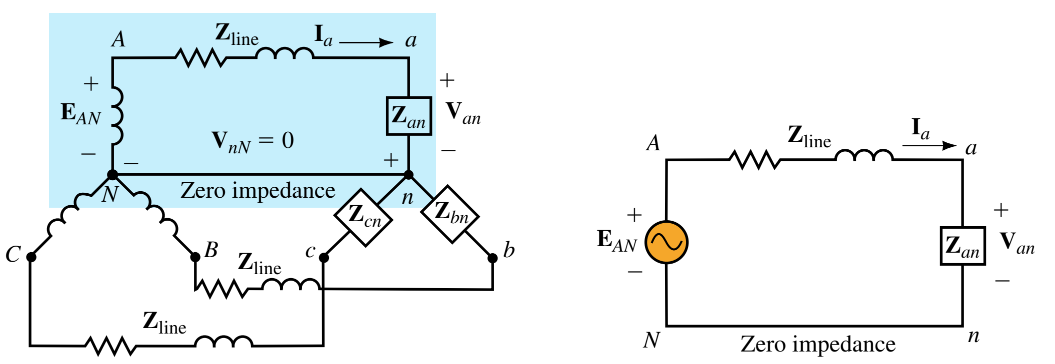
**PRACTICE PROBLEMS 4**

1. For the circuit of Figure 23–13, if $\mathbf{I}_a = 17.32\text{ A}\angle 20°$, determine
   a. $\mathbf{I}_{ab}$
   b. $\mathbf{V}_{ab}$
2. For the circuit of Figure 23–13, if $\mathbf{I}_{bc} = 5\text{ A}\angle -140°$, what is $\mathbf{V}_{ab}$?

*Answers:*
1. a. $10\text{ A}\angle 50°$
   b. $104\text{ V}\angle 66.7°$
2. $52.2\text{ V}\angle -3.30°$

## The Single-Phase Equivalent

By now it should be apparent that if you know the solution for one phase of a balanced system, you effectively know the solution for all three phases. We will now formalize this viewpoint by developing the **single-phase equivalent** approach to solving balanced systems. Consider a Y-Y system with line impedance. The system may be either a three-wire system or a four-wire system with neutral conductor impedance. In either case, since the voltage between neutral points is zero, you can join points $n$ and $N$ with a **zero-impedance conductor** without disturbing voltages or currents elsewhere in the circuit. This is illustrated in Figure 23–15(a). Phase $a$ can now be isolated as in (b). Since $V_{nN} \equiv 0$ as before, the equation describing phase $a$ in circuit (b) is the same as that describing phase $a$ in the original circuit; thus, circuit (b) can be used to solve the original problem. If there are $\Delta$ loads present, convert them to Y loads using the $\Delta$-Y conversion formula for balanced loads: $\mathbf{Z}_Y = \mathbf{Z}_\Delta/3$, from Chapter 19. This procedure is valid regardless of the configuration or complexity of the circuit. We will look at its use in Section 23.4.




(a) Original circuit: $\mathbf{E}_{AN} = \mathbf{I}_a\mathbf{Z}_{\text{line}} + \mathbf{V}_{an}$ (b) Single phase equivalent: $\mathbf{E}_{AN} = \mathbf{I}_a\mathbf{Z}_{\text{line}} + \mathbf{V}_{an}$

**FIGURE 23–15** Reducing a circuit to its single phase equivalent.

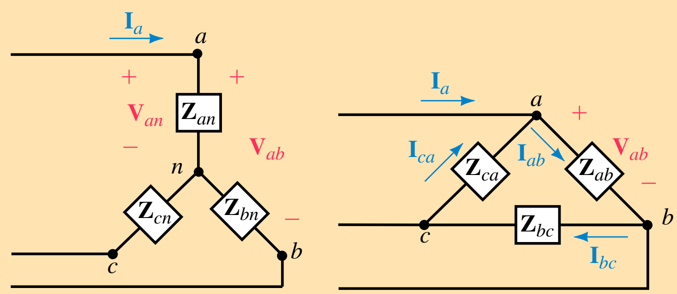
## Selecting a Reference

Before you solve a three-phase problem, you need to select a reference. For Y circuits, we normally choose $\mathbf{E}_{AN}$ or $\mathbf{V}_{an}$; for $\Delta$ circuits, we normally choose $\mathbf{E}_{AB}$ or $\mathbf{V}_{ab}$.

### Summary of Basic Three-Phase Relationships

Table 23–1 summarizes the relationships developed so far. Note that in balanced systems (Y or Δ), *all* voltages and *all* currents are balanced.

**TABLE 23–1** Summary of Relationships (Balanced System). All Voltages and Currents Are Balanced




(a) Y connection

(b) Δ connection

| (a) Y connection | (b) Δ connection |
|---|---|
| $\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30°$ | $\mathbf{I}_a = \sqrt{3}\mathbf{I}_{ab}\angle -30°$ |
| $\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_{an}$ | $\mathbf{I}_{ab} = \mathbf{V}_{ab}/\mathbf{Z}_{ab}$ |
| $\mathbf{Z}_{an} = \mathbf{Z}_{bn} = \mathbf{Z}_{cn}$ | $\mathbf{Z}_{ab} = \mathbf{Z}_{bc} = \mathbf{Z}_{ca}$ |

Generator $\mathbf{E}_{AB} = \sqrt{3}\mathbf{E}_{AN}\angle 30°$

**IN-PROCESS LEARNING CHECK 1**

1. For Figure 23–4(a), if $\mathbf{E}_{AN} = 277\text{ V}\angle -20°$, determine all line and phase voltages, source and load.
2. For Figure 23–4(a), if $\mathbf{V}_{bc} = 208\text{ V}\angle -40°$, determine all line and phase voltages, source and load.
3. For Figure 23–11, if $\mathbf{I}_a = 8.25\text{ A}\angle 35°$, determine $\mathbf{V}_{an}$ and $\mathbf{V}_{ab}$.
4. For Figure 23–13, if $\mathbf{I}_b = 17.32\text{ A}\angle -85°$, determine all voltages.

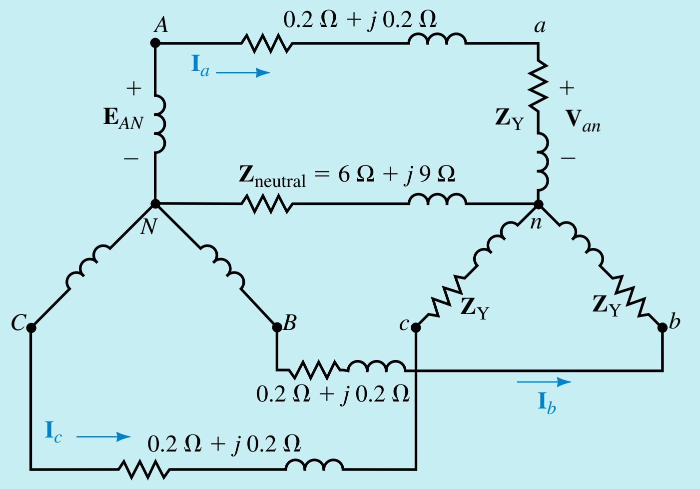
*(Answers are at the end of the chapter.)*

## 23.4 Examples

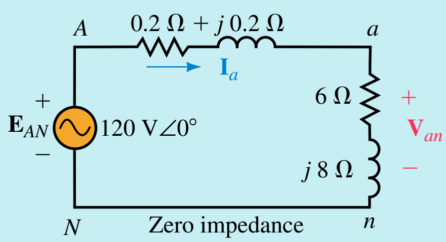
Generally, there are several ways to solve most problems. We usually try to use the simplest approach. Thus, sometimes we use the single-phase equivalent method and sometimes we solve the problem in its three-phase configuration. Generally, if a circuit has line impedance, we use the single-phase equivalent method; otherwise, we may solve it directly.

**EXAMPLE 23–6** For Figure 23–16, $\mathbf{E}_{AN} = 120\text{ V}\angle 0°$.

a. Solve for the line currents.
b. Solve for the phase voltages at the load.
c. Solve for the line voltages at the load.




(a) $\mathbf{Z}_Y = 6\ \Omega + j\,8\ \Omega$




(b) Single phase equivalent. Since the neutral conductor in (a) carries no current, its impedance has no effect on the solution

EWB **FIGURE 23–16** A Y-Y problem.

**Solution**

a. Reduce the circuit to its single-phase equivalent as shown in (b).

$$\mathbf{I}_a = \frac{\mathbf{E}_{AN}}{\mathbf{Z}_T} = \frac{120\angle 0^\circ}{(0.2 + j0.2) + (6 + j8)} = 11.7\ \text{A}\angle{-52.9^\circ}$$

Therefore,

$$\mathbf{I}_b = 11.7\ \text{A}\angle{-172.9^\circ} \quad \text{and} \quad \mathbf{I}_c = 11.7\ \text{A}\angle 67.1^\circ$$

b. $\mathbf{V}_{an} = \mathbf{I}_a \times \mathbf{Z}_{an} = (11.7\angle{-52.9^\circ})(6 + j8) = 117\ \text{V}\angle 0.23^\circ$

Thus,

$$\mathbf{V}_{bn} = 117\ \text{V}\angle{-119.77^\circ} \quad \text{and} \quad \mathbf{V}_{cn} = 117\ \text{V}\angle 120.23^\circ$$

c. $\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30^\circ = \sqrt{3} \times 117\angle(0.23^\circ + 30^\circ) = 202.6\ \text{V}\angle 30.23^\circ$

Thus,

$$\mathbf{V}_{bc} = 202.6\ \text{V}\angle{-89.77^\circ} \quad \text{and} \quad \mathbf{V}_{ca} = 202.6\ \text{V}\angle 150.23^\circ$$

Note the phase shift and voltage drop across the line impedance. Note also that the impedance of the neutral conductor plays no part in the solution, since no current passes through it because the system is balanced.

**PRACTICE PROBLEMS 5**

For the circuit of Figure 23–17, $\mathbf{V}_{an} = 120\ \text{V}\angle 0°$.

a. Find the line currents.

b. Verify that the neutral current is zero.

c. Determine generator voltages $\mathbf{E}_{AN}$ and $\mathbf{E}_{AB}$.

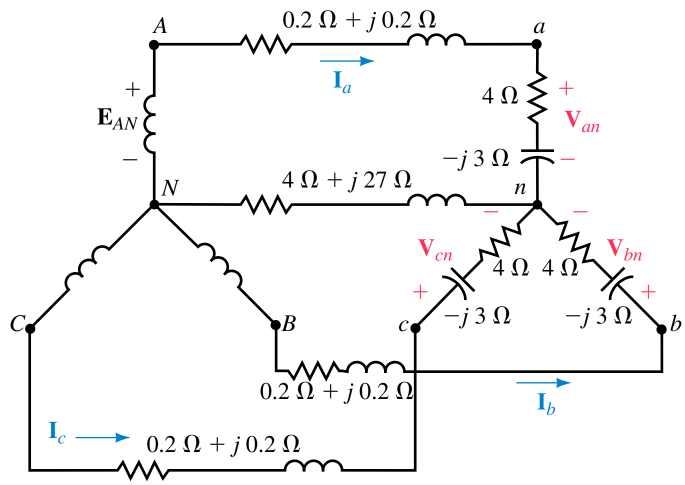



**FIGURE 23–17**

*Answers:*

a. $\mathbf{I}_a = 24\ \text{A}\angle 36.9°$; $\mathbf{I}_b = 24\ \text{A}\angle -83.1°$; $\mathbf{I}_c = 24\ \text{A}\angle 156.9°$

b. $24\ \text{A}\angle 36.9° + 24\ \text{A}\angle -83.1° + 24\ \text{A}\angle 156.9° = 0$

c. $\mathbf{E}_{AN} = 121\ \text{V}\angle 3.18°$; $\mathbf{E}_{AB} = 210\ \text{V}\angle 33.18°$

**EXAMPLE 23–7** For the circuit of Figure 23–18, $\mathbf{E}_{AB} = 208\ \text{V}\angle 30°$.

a. Determine the phase currents.

b. Determine the line currents.

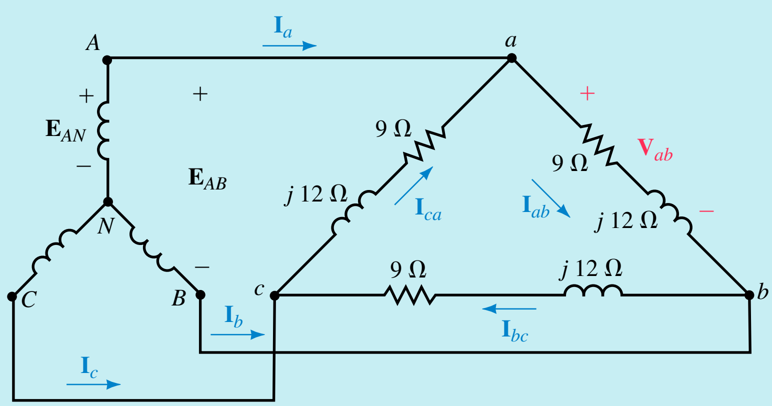



**FIGURE 23–18** A Y-Δ problem.

**Solution**

a. Since this circuit has no line impedance, the load connects directly to the source and $\mathbf{V}_{ab} = \mathbf{E}_{AB} = 208\text{ V}\angle 30°$. Current $\mathbf{I}_{ab}$ can be found as

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{208\angle 30°}{9 + j12} = \frac{208\angle 30°}{15\angle 53.1°} = 13.9\text{ A}\angle -23.13°$$

Thus,

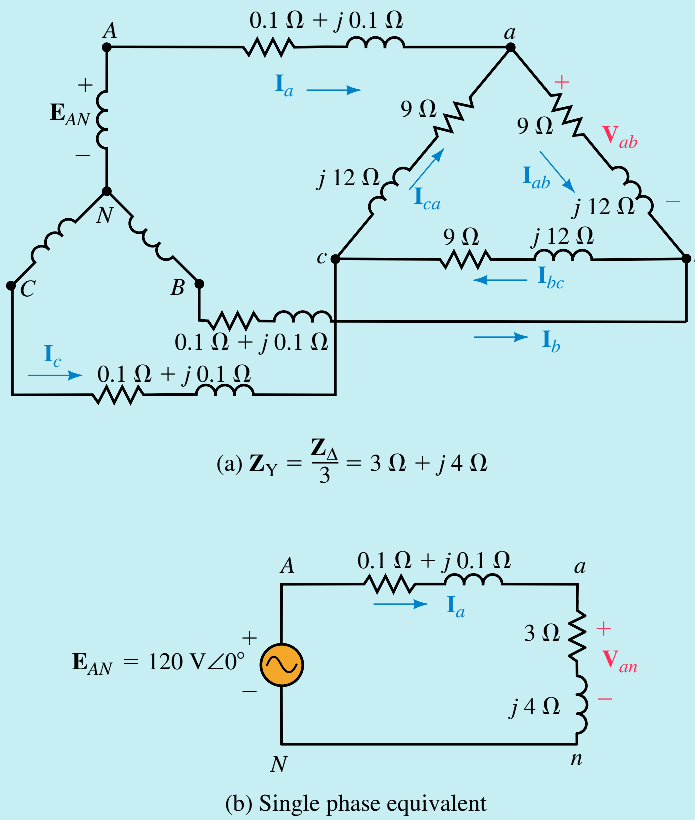
$$\mathbf{I}_{bc} = 13.9\text{ A}\angle -143.13° \text{ and } \mathbf{I}_{ca} = 13.9\text{ A}\angle 96.87°$$

b. $\mathbf{I}_a = \sqrt{3}\mathbf{I}_{ab}\angle -30° = \sqrt{3}(13.9)\angle(-30° - 23.13°) = 24\text{ A}\angle -53.13°$

Thus,

$$\mathbf{I}_b = 24\text{ A}\angle -173.13° \quad \text{and} \quad \mathbf{I}_c = 24\text{ A}\angle 66.87°$$

**EXAMPLE 23–8** For the circuit of Figure 23–19(a), the magnitude of the line voltage at the generator is 208 volts. Solve for the line voltage $\mathbf{V}_{ab}$ at the load.




EWB **FIGURE 23–19** A circuit with line impedances.

**Solution** Since points *A-a* and *B-b* are not directly joined, $\mathbf{V}_{ab} \neq \mathbf{E}_{AB}$ and we cannot solve the circuit as in Example 23–7. Use the single-phase equivalent.

Phase voltage at the source is $208/\sqrt{3} = 120$ V. Choose $\mathbf{E}_{AN}$ as reference: $\mathbf{E}_{AN} = 120\text{ V}\angle 0°$.

$$\mathbf{Z}_{Y} = \mathbf{Z}_{\Delta}/3 = (9 + j12)/3 = 3\ \Omega + j4\ \Omega.$$

The single-phase equivalent is shown in (b). Now use the voltage divider rule to find $\mathbf{V}_{an}$:

$$\mathbf{V}_{an} = \left(\frac{3 + j4}{3.1 + j4.1}\right) \times 120\angle 0° = 117\text{ V}\angle 0.22°$$

Thus,

$$\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30° = \sqrt{3}(117\text{ V})\angle 30.22° = 203\text{ V}\angle 30.22°$$

**EXAMPLE 23–9** For the circuit of Figure 23–19(a), the generator phase voltage is 120 volts. Find the Δ currents.

**Solution** Since the source voltage here is the same as in Example 23–8, load voltage $\mathbf{V}_{ab}$ will also be the same. Thus,

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{203\text{ V}\angle 30.22°}{(9 + j12)\Omega} = 13.5\text{ A}\angle -22.9°$$

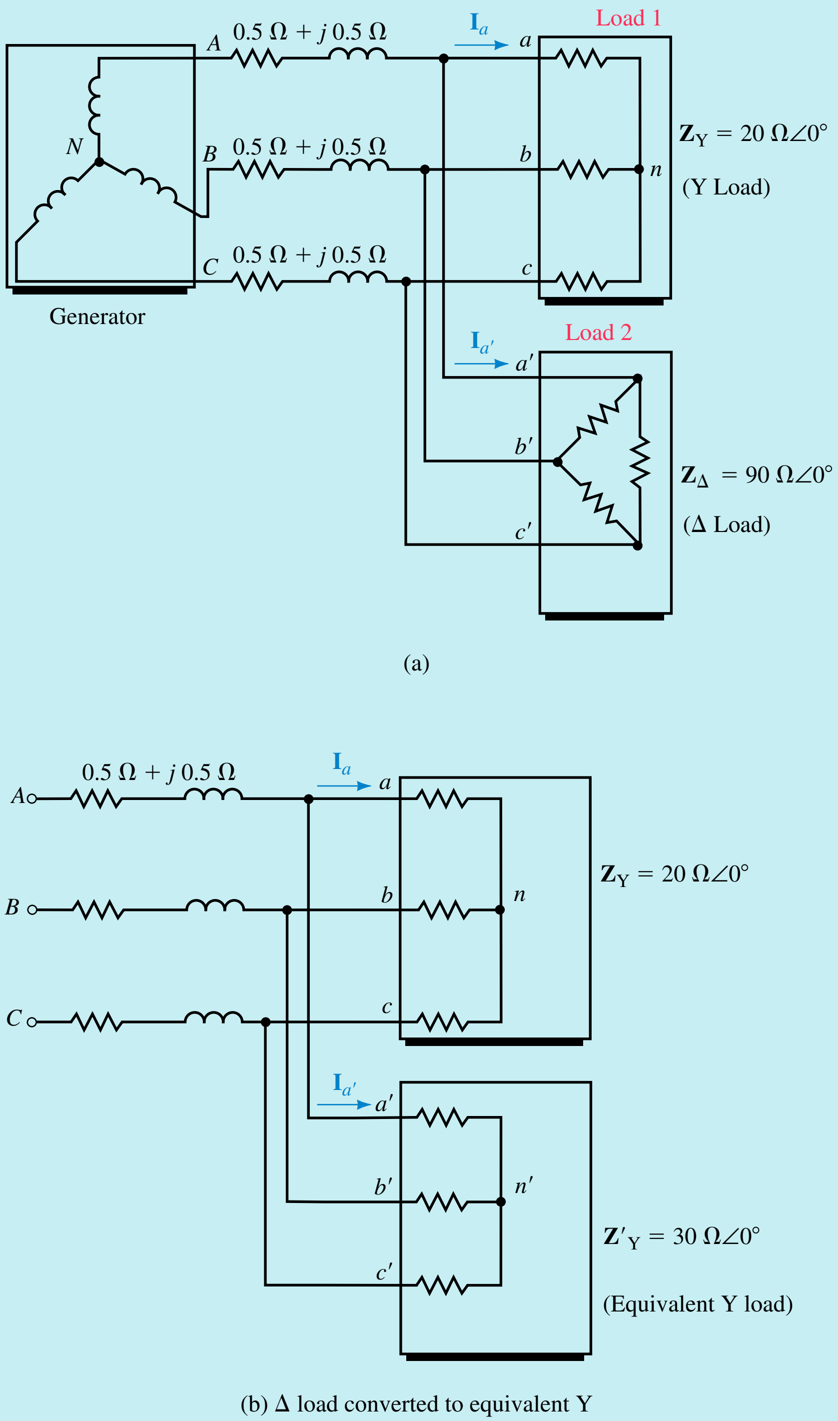
and,

$$\mathbf{I}_{bc} = 13.5\text{ A}\angle -142.9° \quad \text{and} \quad \mathbf{I}_{ca} = 13.5\text{ A}\angle 97.1°$$

**EXAMPLE 23–10** A Y load and a Δ load are connected in parallel as in Figure 23–20(a). Line voltage magnitude at the generator is 208 V.

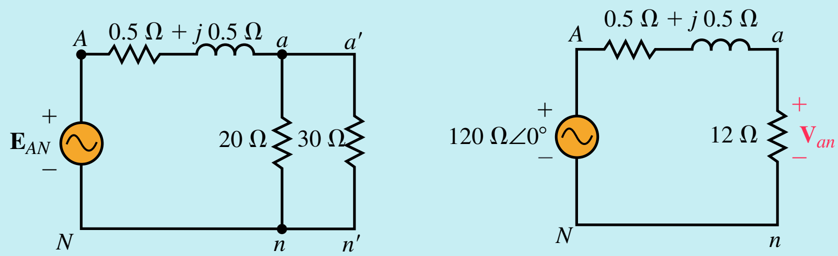
a. Find the phase voltages at the loads.

b. Find the line voltages at the loads.




EWB **FIGURE 23–20**

*Solution* Convert the Δ load to a Y load. Thus, $\mathbf{Z}'_Y = \frac{1}{3}\mathbf{Z}_\Delta = 30\ \Omega\angle 0°$ as in (b). Now join the neutral points *N*, *n*, and *n*′ by a zero-impedance conductor to get the single-phase equivalent, which is shown in Figure 23–21(a). The parallel load resistors can be combined as in (b).




(a) Single phase equivalent of Figure 23-20

(b) Reduced circuit

**FIGURE 23–21**

a. Phase voltage is $208 \text{ V}/\sqrt{3} = 120$ V. Select $\mathbf{E}_{AN}$ as reference; $\mathbf{E}_{AN} = 120 \text{ V}\angle 0°$. Using the voltage divider rule yields

$$\mathbf{V}_{an} = \left(\frac{12}{12.5 + j0.5}\right) \times 120\angle 0° = 115.1 \text{ V}\angle -2.29°$$

Thus,

$$\mathbf{V}_{bn} = 115.1 \text{ V}\angle -122.29° \quad \text{and} \quad \mathbf{V}_{cn} = 115.1 \text{ V}\angle 117.71°$$

b. $\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_{an}\angle 30° = \sqrt{3}(115.1 \text{ V})\angle(-2.29° + 30°) = 199 \text{ V}\angle 27.71°$

Thus,

$$\mathbf{V}_{bc} = 199 \text{ V}\angle -92.29° \quad \text{and} \quad \mathbf{V}_{ca} = 199 \text{ V}\angle 147.71°$$

These are the line voltages for both the Y and Δ loads.

**PRACTICE PROBLEMS 6**

1. Repeat Example 23–7 using the single-phase equivalent.
2. Determine Δ phase currents for the circuit of Figure 23–20(a).

*Answers:*
2. $\mathbf{I}_{a'b'} = 2.22 \text{ A}\angle 27.7°$; $\mathbf{I}_{b'c'} = 2.22 \text{ A}\angle -92.3°$; $\mathbf{I}_{c'a'} = 2.22 \text{ A}\angle 147.7°$

## 23.5 Power in a Balanced System

To find total power in a balanced system, determine power to one phase, then multiply by three. Per phase quantities can be found using the formulas of Chapter 17. Since only magnitudes are involved in many power formulas and calculations and since magnitudes are the same for all three phases, we can use a simplified notation. We will use $V_\phi$ for magnitude of phase voltage, $I_\phi$ for phase current, $V_L$ and $I_L$ for line voltage and line current respectively, and $Z_\phi$ for phase impedance.

## Active Power to a Balanced Wye Load

First, consider a Y load (Figure 23–22). The power to any phase as indicated in (b) is the product of the magnitude of the phase voltage $V_\phi$ times the magnitude of the phase current $I_\phi$ times the cosine of the angle $\theta_\phi$ between them. Since the angle between voltage and current is always the angle of the load impedance, the power per phase is

$$P_\phi = V_\phi I_\phi \cos\theta_\phi \quad \text{(W)} \tag{23–10}$$

where $\theta_\phi$ is the angle of $\mathbf{Z}_\phi$. Total power is

$$P_T = 3P_\phi = 3V_\phi I_\phi \cos\theta_\phi \quad \text{(W)} \tag{23–11}$$

It is also handy to have a formula for power in terms of line quantities. For a Y load, $I_\phi = I_L$ and $V_\phi = V_L/\sqrt{3}$, where $I_L$ is the magnitude of the line current and $V_L$ is the magnitude of the line-to-line voltage. Substituting these relations into Equation 23–11 and noting that $3/\sqrt{3} = \sqrt{3}$ yields

$$P_T = \sqrt{3}V_L I_L \cos\theta_\phi \quad \text{(W)} \tag{23–12}$$

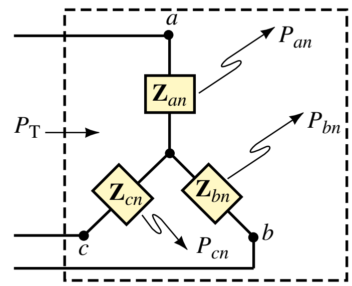
This is a very important formula and one that is widely used. Note carefully, however, that $\theta_\phi$ is the angle of the load impedance and not the angle between $V_L$ and $I_L$.

Power per phase can also be expressed as

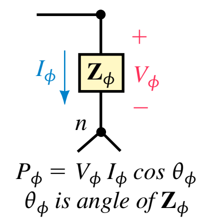
$$P_\phi = I_\phi^2 R_\phi = V_R^2/R_\phi \quad \text{(W)} \tag{23–13}$$

where $R_\phi$ is the resistive component of the phase impedance and $V_R$ is the voltage across it. Total power is thus

$$P_T = 3I_\phi^2 R_\phi = 3V_R^2/R_\phi \quad \text{(W)} \tag{23–14}$$




(a) $P_T = P_{an} + P_{bn} + P_{cn} = 3\,P_\phi$




(b)

**FIGURE 23–22** For a balanced Y, $P_\phi = P_{an} = P_{bn} = P_{cn}$.

## Reactive Power to a Balanced Wye Load

Equivalent expressions for reactive power are

$$Q_\phi = V_\phi I_\phi \sin\theta_\phi \quad \text{(VAR)} \tag{23–15}$$

$$= I_\phi^2 X_\phi = V_X^2/X_\phi \quad \text{(VAR)} \tag{23–16}$$

$$Q_T = \sqrt{3}V_L I_L \sin\theta_\phi \quad \text{(VAR)} \tag{23–17}$$

where $X_\phi$ is the reactive component of $\mathbf{Z}_\phi$ and $V_X$ is the voltage across it.

## Apparent Power

$$S_\phi = V_\phi I_\phi = I_\phi^2 Z_\phi = \frac{V_\phi^2}{Z_\phi} \quad \text{(VA)} \tag{23–18}$$

$$S_T = \sqrt{3}V_L I_L \quad \text{(VA)} \tag{23–19}$$

## Power Factor

$$F_p = \cos\theta_\phi = P_T/S_T = P_\phi/S_\phi \tag{23–20}$$

**EXAMPLE 23–11** For Figure 23–23, the phase voltage is 120 V.

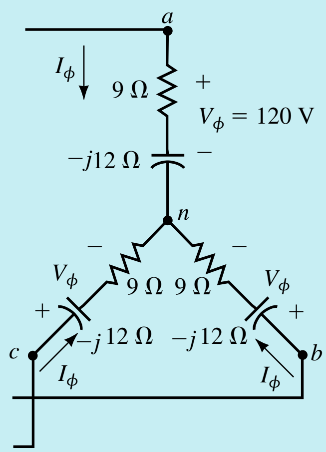



**FIGURE 23–23**

a. Compute active power to each phase and total power using each equation of this section.

b. Repeat (a) for reactive power.

c. Repeat (a) for apparent power.

d. Find the power factor.

**Solution** Since we want to compare answers for the various methods, we will use 207.8 V for the line voltage rather than the nominal value of 208 V to avoid truncation error in our computations.

$$\mathbf{Z}_\phi = 9 - j12 = 15\ \Omega\angle -53.13°. \text{ Thus, } \theta_\phi = -53.13°.$$

$$V_\phi = 120 \text{ V and } I_\phi = V_\phi/Z_\phi = 120 \text{ V}/15\ \Omega = 8.0 \text{ A}.$$

$$V_R = (8 \text{ A})(9\ \Omega) = 72 \text{ V} \quad \text{and} \quad V_X = (8 \text{ A})(12\ \Omega) = 96 \text{ V}$$

a. $P_\phi = V_\phi I_\phi \cos\theta_\phi = (120)(8)\cos(-53.13°) = 576 \text{ W}$

$P_\phi = I_\phi^2 R_\phi = (8^2)(9) = 576 \text{ W}$

$P_\phi = V_R^2/R_\phi = (72)^2/9 = 576 \text{ W}$

$P_T = 3P_\phi = 3(576) = 1728 \text{ W}$

$P_T = \sqrt{3}V_L I_L \cos\theta_\phi = \sqrt{3}(207.8)(8)\cos(-53.13°) = 1728 \text{ W}$

b. $Q_\phi = V_\phi I_\phi \sin\theta_\phi = (120)(8)\sin(-53.13°) = -768 \text{ VAR}$

$= 768 \text{ VAR (cap.)}$

$Q_\phi = I_\phi^2 X_\phi = (8)^2(12) = 768 \text{ VAR (cap.)}$

$Q_\phi = V_X^2/X_\phi = (96)^2/12 = 768 \text{ VAR (cap.)}$

$Q_T = 3Q_\phi = 3(768) = 2304 \text{ VAR (cap.)}$

$Q_T = \sqrt{3}V_L I_L \sin\theta_\phi = \sqrt{3}(207.8)(8)\sin(-53.13°) = -2304 \text{ VAR}$

$= 2304 \text{ VAR (cap.)}$

c. $S_\phi = V_\phi I_\phi = (120)(8) = 960$ VA

$S_T = 3S_\phi = 3(960) = 2880$ VA

$S_T = \sqrt{3}V_L I_L = \sqrt{3}(207.8)(8) = 2880$ VA

Thus, all approaches yield the same answers.

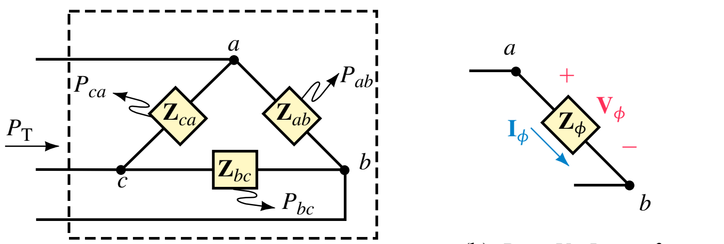
d. The power factor is $F_p = \cos\theta_\phi = \cos 53.13° = 0.6$.

## Power to a Balanced Delta Load

For a Δ load [Figure 23–24(a)],

$$P_\phi = V_\phi I_\phi \cos\theta_\phi \quad \text{(W)} \tag{23–21}$$

where $\theta_\phi$ is the angle of the Δ impedance. Note that this formula is identical with Equation 23–10 for the Y load. Similarly for reactive power, apparent power, and power factor. Thus, all power formulas are the same. Results are tabulated in Table 23–2. *Note: In all of these formulas, $\theta_\phi$ is the angle of the load impedance, i.e., the angle of $\mathbf{Z}_{an}$ for Y loads and $\mathbf{Z}_{ab}$ for Δ loads.*




(a) $P_T = P_{ab} + P_{bc} + P_{ca} = 3\,P_\phi$.

(b) $P_\phi = V_\phi I_\phi \cos\theta_\phi$
$\theta_\phi$ is angle of load impedance

**FIGURE 23–24** For a balanced Δ, $P_\phi = P_{ab} = P_{bc} = P_{ca}$.

**TABLE 23–2** Power Formulas for Balanced Wye and Delta Circuits

| | |
|---|---|
| Active power | $P_\phi = V_\phi I_\phi \cos\theta_\phi = I_\phi^2 R_\phi = \dfrac{V_R^2}{R_\phi}$ |
| | $P_T = \sqrt{3}V_L I_L \cos\theta_\phi$ |
| Reactive power | $Q_\phi = V_\phi I_\phi \sin\theta_\phi = I_\phi^2 X_\phi = \dfrac{V_x^2}{X_\phi}$ |
| | $Q_T = \sqrt{3}V_L I_L \sin\theta_\phi$ |
| Apparent power | $S_\phi = V_\phi I_\phi = I_\phi^2 Z_\phi = \dfrac{V_\phi^2}{Z_\phi}$ |
| | $S_T = \sqrt{3}V_L I_L$ |
| Power factor | $F_p = \cos\theta_\phi = \dfrac{P_T}{S_T} = \dfrac{P_\phi}{S_\phi}$ |
| Power triangle | $\mathbf{S}_T = P_T + jQ_T$ |

**EXAMPLE 23–12** Determine per phase and total power (active, reactive, and apparent) for Figure 23–25. Use $V_\phi = 207.8$ V in order to compare results.

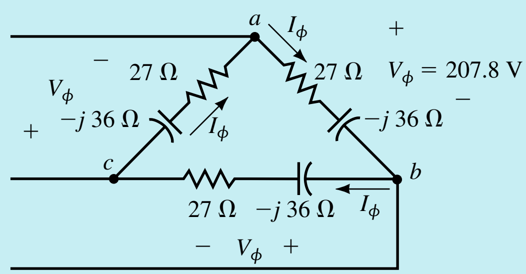



**FIGURE 23–25**

**Solution**

$$\mathbf{Z}_\phi = 27 - j36 = 45\ \Omega\angle -53.13°, \quad \text{so} \quad \theta_\phi = -53.13°$$

$$V_\phi = 207.8\ \text{V} \quad \text{and} \quad I_\phi = V_\phi/Z_\phi = 207.8\ \text{V}/45\ \Omega = 4.62\ \text{A}$$

$$P_\phi = V_\phi I_\phi \cos\theta_\phi = (207.8)(4.62)\cos(-53.13°) = 576\ \text{W}$$

$$Q_\phi = V_\phi I_\phi \sin\theta_\phi = (207.8)(4.62)\sin(-53.13°) = -768\ \text{VAR}$$

$$= 768\ \text{VAR (cap.)}$$

$$S_\phi = V_\phi I_\phi = (207.8)(4.62) = 960\ \text{VA}$$

$$P_\text{T} = 3P_\phi = 3(576) = 1728\ \text{W}$$

$$Q_\text{T} = 3Q_\phi = 3(768) = 2304\ \text{VAR (cap.)}$$

$$S_\text{T} = 3S_\phi = 3(960) = 2880\ \text{VA}$$

Note that the results here are the same as for Example 23–11. This is to be expected since the load of Figure 23–23 is the Y equivalent of the Δ load of Figure 23–25.

**PRACTICE PROBLEMS 7**

Find total active, reactive, and apparent power for the circuit of Example 23–12 using the formulas for $P_\text{T}$, $Q_\text{T}$, and $S_\text{T}$ from Table 23–2.

## Power and the Single-Phase Equivalent

You can also use the single-phase equivalent in power calculations. Here, all single-phase active, reactive, and apparent power formulas apply. The power is of course just the power for one phase.

**EXAMPLE 23–13** The total power to the balanced load of Figure 23–26 is 6912 W. The phase voltage at the load is 120 V. Compute the generator voltage $\mathbf{E}_{AB}$, magnitude and angle.

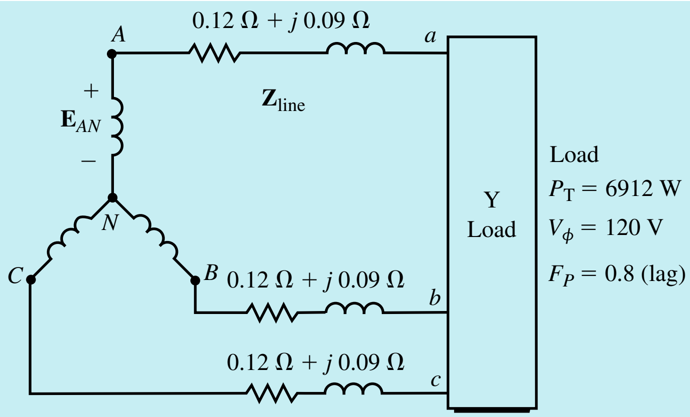



**FIGURE 23–26**

**Solution** Consider the single-phase equivalent in Figure 23–27.

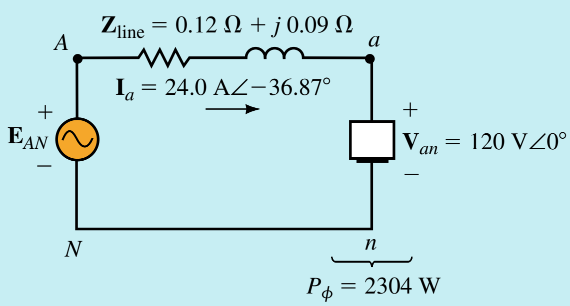



**FIGURE 23–27**

$$P_{an} = P_T/3 = \frac{1}{3}(6912) = 2304 \text{ W}$$
$$V_{an} = 120 \text{ V}$$
$$\theta_{an} = \cos^{-1}(0.8) = 36.87°$$
$$P_{an} = V_{an}I_a\cos\theta_{an}$$

Therefore,

$$I_a = \frac{P_{an}}{V_{an}\cos\theta_{an}} = \frac{2304}{(120)(0.8)} = 24.0 \text{ A}$$

Select $\mathbf{V}_{an}$ as reference. $\mathbf{V}_{an} = 120 \text{ V}\angle 0°$. Thus, $\mathbf{I}_a = 24 \text{ A}\angle -36.87°$ (since power factor was given as lagging).

$$\mathbf{E}_{AN} = \mathbf{I}_a \times \mathbf{Z}_{\text{line}} + \mathbf{V}_{an}$$
$$= (24\angle -36.87)(0.12 + j0.09) + 120\angle 0° = 123.6 \text{ V}\angle 0°$$
$$\mathbf{E}_{AB} = \sqrt{3}\mathbf{E}_{AN}\angle 30° = 214.1 \text{ V}\angle 30°$$

## 23.6 Measuring Power in Three-Phase Circuits

### The Three-Wattmeter Method

Measuring power to a 4-wire Y load requires one wattmeter per phase (i.e., three wattmeters) as in Figure 23–28 (except as noted below). As indicated, wattmeter $W_1$ is connected across voltage $\mathbf{V}_{an}$ and its current is $\mathbf{I}_a$. Thus, its reading is

$$P_1 = V_{an}I_a\cos\theta_{an}$$

which is power to phase *an*. Similarly, $W_2$ indicates power to phase *bn* and $W_3$ to phase *cn*. Loads may be balanced or unbalanced. The total power is

$$P_T = P_1 + P_2 + P_3 \tag{23–22}$$

If the load of Figure 23–28 could be guaranteed to always be balanced, only one wattmeter would be needed. $P_T$ would be 3 times its reading.

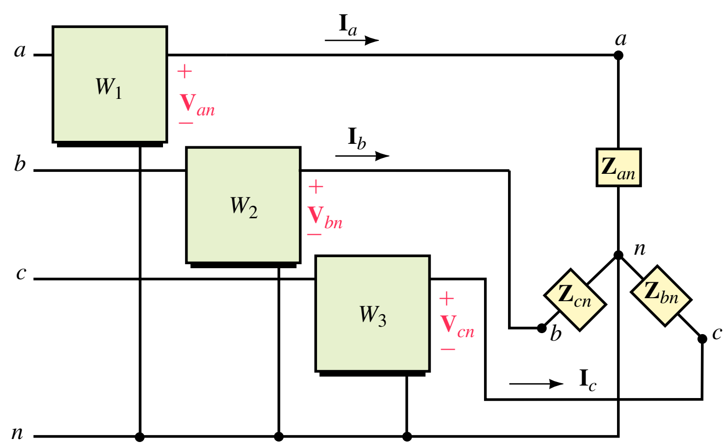



**FIGURE 23–28** Three-wattmeter connection for a 4-wire load.

### The Two-Wattmeter Method

While three wattmeters are required for a four-wire system, for a three-wire system, only two are needed. The connection is shown in Figure 23–29. Loads may be Y or Δ, balanced or unbalanced. The meters may be connected in any pair of lines with the voltage terminals connected to the third line. The total power is the algebraic sum of the meter readings.

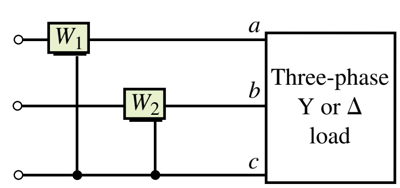



**FIGURE 23–29** Two-wattmeter connection. Load may be balanced or unbalanced.

## Determining Wattmeter Readings

Recall from Chapter 17, the reading of a wattmeter is equal to the product of the magnitude of its voltage, the magnitude of its current, and the cosine of the angle between them. For each meter, you must carefully determine what this angle is. This is illustrated next.

**EXAMPLE 23–14** For Figure 23–30, $\mathbf{V}_{an} = 120\text{ V}\angle 0°$. Compute the readings of each meter, then sum to determine total power. Compare $P_T$ to the $P_T$ found in Example 23–11.

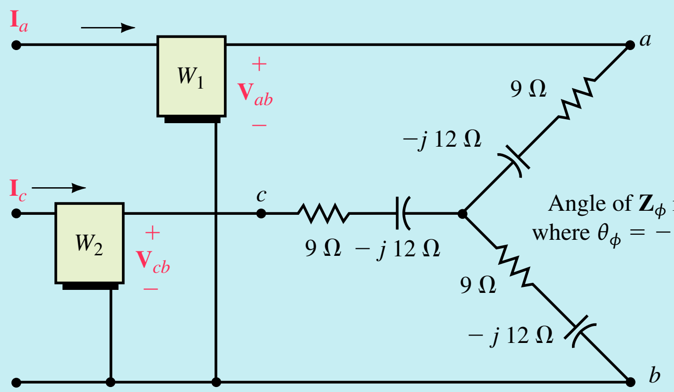



**FIGURE 23–30**

**Solution** $\mathbf{V}_{an} = 120\text{ V}\angle 0°$. Thus, $\mathbf{V}_{ab} = 208\text{ V}\angle 30°$ and $\mathbf{V}_{bc} = 208\text{ V}\angle -90°$. $\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_{an} = 120\text{ V}\angle 0°/(9 - j12)\ \Omega = 8\text{ A}\angle 53.13°$. Thus, $\mathbf{I}_c = 8\text{ A}\angle 173.13°$.

First consider wattmeter 1, Figure 23–31. Note that $W_1$ is connected to terminals *a-b;* thus it has voltage $\mathbf{V}_{ab}$ across it and current $\mathbf{I}_a$ through it. Its reading is therefore $P_1 = V_{ab}I_a\cos\theta_1$, where $\theta_1$ is the angle between $\mathbf{V}_{ab}$ and $\mathbf{I}_a$. $\mathbf{V}_{ab}$ has an angle of 30° and $\mathbf{I}_a$ has an angle of 53.13°. Thus, $\theta_1 = 53.13° - 30° = 23.13°$ and $P_1 = (208)(8)\cos 23.13° = 1530\text{ W}$.

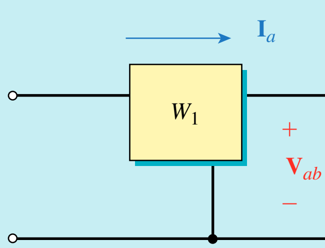



**FIGURE 23–31** $P_1 = V_{ab}I_a\cos\theta_1$ where $\theta_1$ is the angle between $\mathbf{V}_{ab}$ and $\mathbf{I}_a$.

Now consider wattmeter 2, Figure 23–32. Since $W_2$ is connected to terminals *c-b,* the voltage across it is $\mathbf{V}_{cb}$ and the current through it is $\mathbf{I}_c$. But $\mathbf{V}_{cb} = -\mathbf{V}_{bc} = 208\text{ V}\angle 90°$ and $\mathbf{I}_c = 8\text{ A}\angle 173.13°$. The angle between $\mathbf{V}_{cb}$ and

$\mathbf{I}_c$ is thus 173.13° − 90° = 83.13°. Therefore, $P_2 = V_{cb}I_c\cos\theta_2 = (208)(8)\cos 83.13° = 199$ W and $P_T = P_1 + P_2 = 1530 + 199 = 1729$ W. (This agrees well with the answer of 1728 W that we obtained in Example 23–11.) Note that one of the wattmeters reads lower than the other. (This is generally the case for the two-wattmeter method.)

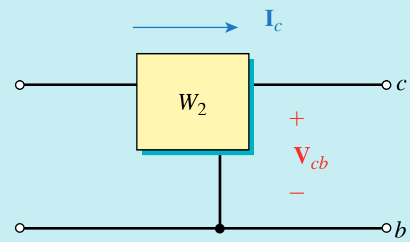



**FIGURE 23–32** $P_2 = V_{cb}\, I_c\cos\theta_2$ where $\theta_2$ is the angle between $\mathbf{V}_{cb}$ and $\mathbf{I}_c$.

**PRACTICE PROBLEMS 8**

Change the load impedances of Figure 23–30 to 15 Ω∠70°. Repeat Example 23–14. (As a check, total power to the load is 985 W. *Hint:* One of the meters reads backwards.)

*Answers:* $P_1 = -289$ W; $P_2 = 1275$ W; $P_1 + P_2 = 986$ W

**NOTES...**

To understand why a wattmeter may read backward, recall that it indicates the product of the magnitude of its voltage times the magnitude of its current times the cosine of the angle between them. This angle is not the angle $\theta_\phi$ of the load impedance; Rather, it can be shown that, for a balanced load, one meter will indicate $V_\phi I_\phi\cos(\theta_\phi - 30°)$ while the other will indicate $V_\phi I_\phi\cos(\theta_\phi + 30°)$. If the magnitude of $(\theta_\phi + 30°)$ or $(\theta_\phi - 30°)$ exceeds 90°, its cosine will be negative and the corresponding meter will read backward.

As Practice Problem 8 shows, the low-reading wattmeter may read backward. (If this happens, reverse either its voltage or current connection to make it read upscale.) Its reading must then be subtracted. Thus, if $P_h$ and $P_\ell$ are the high-reading and low-reading meters respectively,

$$P_T = P_h - P_\ell \qquad \textbf{(23–23)}$$

## Watts Ratio Curve

The power factor for a balanced load can be obtained from the wattmeter readings using a simple curve called the **watts ratio curve,** shown in Figure 23–33.

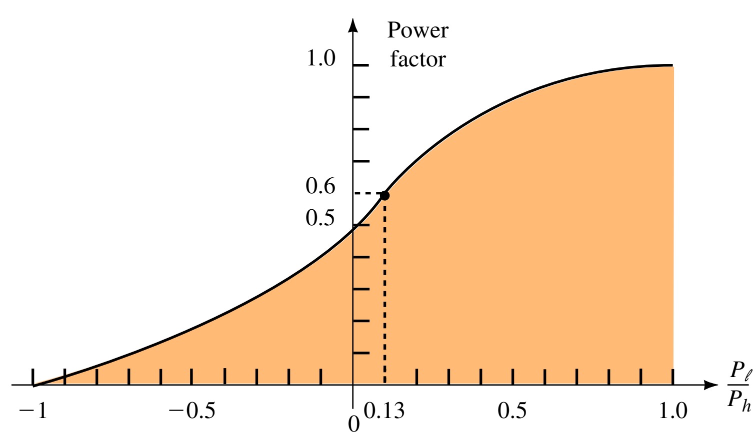



**FIGURE 23–33** Watts ratio curve. Valid for balanced loads only.

**EXAMPLE 23–15** Consider again Figure 23–30.

a. Determine the power factor from the load impedance.
b. Using the meter readings of Example 23–14, determine the power factor from the watts ratio curve.

**Solution**

a. $F_p = \cos \theta_\phi = \cos 53.13° = 0.6.$
b. $P_\ell = 199$ W and $P_h = 1530$ W. Therefore, $P_\ell/P_h = 0.13$. From Figure 23–33, $F_p = 0.6$.

The problem with the watts ratio curve is that values are difficult to determine accurately from the graph. However, it can be shown that

$$\tan \theta_\phi = \sqrt{3}\left(\frac{P_h - P_\ell}{P_h + P_\ell}\right) \qquad \textbf{(23–24)}$$

From this, you can determine $\theta_\phi$ and then you can compute the power factor from $F_p = \cos \theta_\phi$.

## 23.7 Unbalanced Loads

For unbalanced loads, none of the balanced circuit relationships apply. Each problem must therefore be treated as a three-phase problem. We will look at a few examples that can be handled by fundamental circuit techniques such as Kirchhoff's laws and mesh analysis. Source voltages are always balanced.

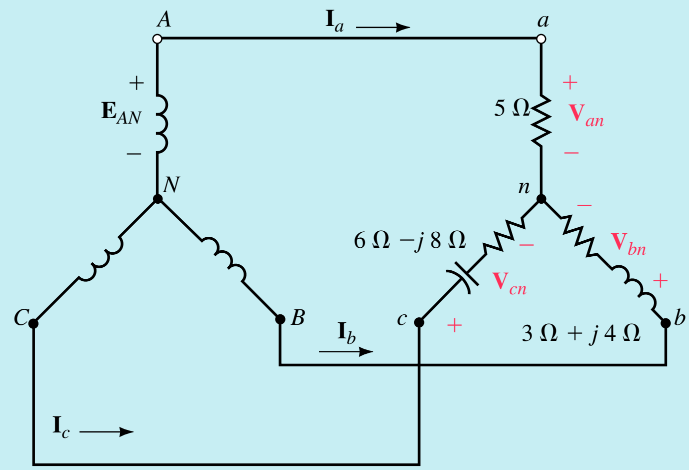
### Unbalanced Wye Loads

Unbalanced four-wire Y systems without line impedance are easily handled using Ohm's law. However, for three-wire systems or four-wire systems with line and neutral impedance, you generally have to use mesh equations or computer methods. One of the problems with unbalanced three-wire Y systems is that you get different voltages across each phase of the load and a voltage between neutral points. This is illustrated next.

**SAFETY NOTE...**

Voltage at a neutral point can be dangerous. For example, in Figure 23–34, if the neutral is grounded at the source, the voltage at the load neutral is *floating* at some potential relative to ground. Since we are accustomed to thinking that neutrals are at ground potential and hence safe to touch, there is a potential safety hazard here.

**EXAMPLE 23–16** For Figure 23–34(a), the generator is balanced with line-to-line voltage of 208 V. Select $\mathbf{E}_{AB}$ as reference and determine line currents and load voltages.




EWB **FIGURE 23–34**

**Solution** Redraw the circuit as shown in Figure 23–35, then use mesh analysis. $\mathbf{E}_{AB} = 208\text{ V}\angle 0°$ and $\mathbf{E}_{BC} = 208\text{ V}\angle -120°$.

$$\textit{Loop 1: } (8 + j4)\mathbf{I}_1 - (3 + j4)\mathbf{I}_2 = 208\text{ V}\angle 0°$$
$$\textit{Loop 2: } -(3 + j4)\mathbf{I}_1 + (9 - j4)\mathbf{I}_2 = 208\text{ V}\angle -120°$$

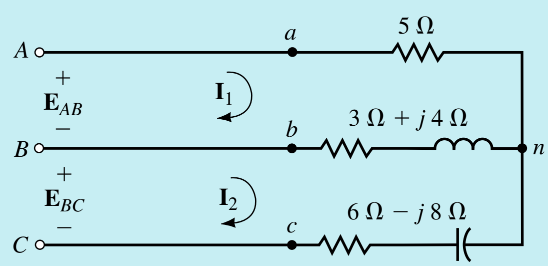



**FIGURE 23–35**

These equations can be solved using standard techniques such as determinants or MathCAD. The solutions are

$$\mathbf{I}_1 = 29.9\text{ A}\angle -26.2° \quad \text{and} \quad \mathbf{I}_2 = 11.8\text{ A}\angle -51.5°$$

$$\text{KCL: } \mathbf{I}_a = \mathbf{I}_1 = 29.9\text{ A}\angle -26.2°$$
$$\mathbf{I}_b = \mathbf{I}_2 - \mathbf{I}_1 = 19.9\text{ A}\angle 168.5°$$
$$\mathbf{I}_c = -\mathbf{I}_2 = 11.8\text{ A}\angle 128.5°$$

$$\mathbf{V}_{an} = \mathbf{I}_a\mathbf{Z}_{an} = (29.9\text{A}\angle -26.2°)\,(5) = 149.5\text{ V}\angle -26.2°$$
$$\mathbf{V}_{bn} = \mathbf{I}_b\mathbf{Z}_{bn} \quad \text{and} \quad \mathbf{V}_{cn} = \mathbf{I}_c\mathbf{Z}_{cn}.$$

Thus,

$$\mathbf{V}_{bn} = 99\text{ V}\angle -138.4° \quad \text{and} \quad \mathbf{V}_{cn} = 118\text{ V}\angle 75.4°$$

PRACTICE PROBLEMS 9

1. Using KVL and the results from Example 23–16, compute the voltage between neutral points $n$ and $N$ of Figure 23–34.
2. For the circuit of Figure 23–36, $\mathbf{E}_{AN} = 120\text{ V}\angle 0°$. Compute the currents, power to each phase, and total power. Hint: This is actually quite a simple problem. Because there is a neutral, source voltages are applied directly to the load and $\mathbf{V}_{an} = \mathbf{E}_{AN}$, etc.

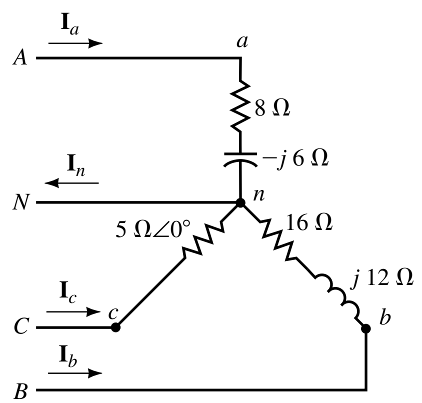



FIGURE 23–36

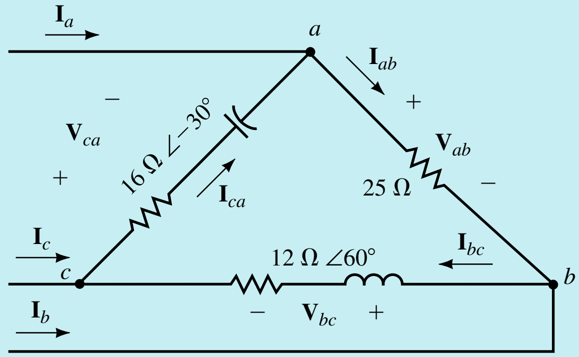
*Answers:*
1. $\mathbf{V}_{nN} = 30.8\text{ V}\angle 168.8°$
2. $\mathbf{I}_a = 12\text{ A}\angle 36.9°$; $\mathbf{I}_b = 6\text{ A}\angle -156.9°$; $\mathbf{I}_c = 24\text{ A}\angle 120°$; $\mathbf{I}_n = 26.8\text{ A}\angle 107.2°$
   $P_a = 1152\text{ W}$; $P_b = 576\text{ W}$; $P_c = 2880\text{ W}$; $P_T = 4608\text{ W}$

## Unbalanced Delta Loads

Systems without line impedance are easily handled since the source voltage is applied directly to the load. However, for systems with line impedance, use mesh equations.

**EXAMPLE 23–17** For the circuit of Figure 23–37, the line voltage is 240 V. Take $\mathbf{V}_{ab}$ as reference and do the following:

a. Determine the phase currents and sketch their phasor diagram.
b. Determine the line currents.
c. Determine the total power to the load.




EWB **FIGURE 23–37**

**Solution**

a. $\mathbf{I}_{ab} = \mathbf{V}_{ab}/\mathbf{Z}_{ab} = (240\ \text{V}\angle 0°)/25\ \Omega = 9.6\ \text{A}\angle 0°$

$\mathbf{I}_{bc} = \mathbf{V}_{bc}/\mathbf{Z}_{bc} = (240\ \text{V}\angle -120°)/(12\ \Omega\angle 60°) = 20\ \text{A}\angle -180°$

$\mathbf{I}_{ca} = \mathbf{V}_{ca}/\mathbf{Z}_{ca} = (240\ \text{V}\angle 120°)/(16\ \Omega\angle -30°) = 15\ \text{A}\angle 150°$

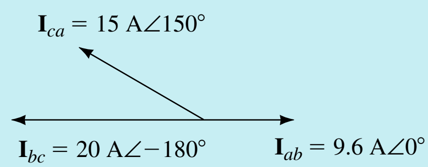



**FIGURE 23–38** Phase currents for the circuit of Figure 23–37.

b. $\mathbf{I}_a = \mathbf{I}_{ab} - \mathbf{I}_{ca} = 9.6\ \text{A}\angle 0° - 15\ \text{A}\angle 150° = 23.8\ \text{A}\angle -18.4°$

$\mathbf{I}_b = \mathbf{I}_{bc} - \mathbf{I}_{ab} = 20\ \text{A}\angle -180° - 9.6\ \text{A}\angle 0° = 29.6\ \text{A}\angle 180°$

$\mathbf{I}_c = \mathbf{I}_{ca} - \mathbf{I}_{bc} = 15\ \text{A}\angle 150° - 20\ \text{A}\angle -180° = 10.3\ \text{A}\angle 46.9°$

c. $P_{ab} = V_{ab}I_{ab}\cos\theta_{ab} = (240)(9.6)\cos 0° = 2304\ \text{W}$

$P_{bc} = V_{bc}I_{bc}\cos\theta_{bc} = (240)(20)\cos 60° = 2400\ \text{W}$

$P_{ca} = V_{ca}I_{ca}\cos\theta_{ca} = (240)(15)\cos 30° = 3118\ \text{W}$

$P_{\text{T}} = P_{ab} + P_{bc} + P_{ca} = 7822\ \text{W}$

**EXAMPLE 23–18** A pair of wattmeters are added to the circuit of Figure 23–37 as illustrated in Figure 23–39. Determine wattmeter readings and compare them to the total power calculated in Example 23–17.

**Solution** $P_1 = V_{ac}I_a\cos\theta_1$, where $\theta_1$ is the angle between $\mathbf{V}_{ac}$ and $\mathbf{I}_a$. From Example 23–17, $\mathbf{I}_a = 23.8\ \text{A}\angle -18.4°$ and $\mathbf{V}_{ac} = -\mathbf{V}_{ca} = 240\ \text{V}\angle -60°$. Thus, $\theta_1 = 60° - 18.4° = 41.6°$. Therefore,

$$P_1 = (240)(23.8)\cos 41.6° = 4271\ \text{W}$$

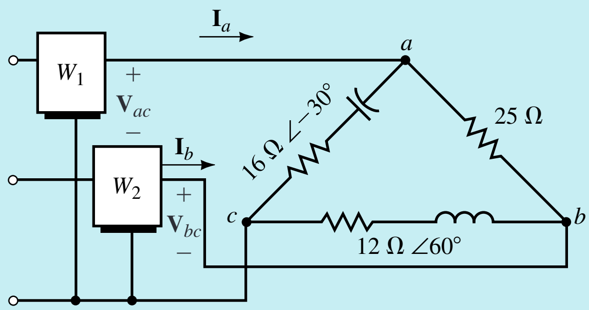



FIGURE 23–39

$P_2 = V_{bc}I_b\cos\theta_2$, where $\theta_2$ is the angle between $\mathbf{V}_{bc}$ and $\mathbf{I}_b$. $\mathbf{V}_{bc} = 240\text{ V}\angle-120°$ and $\mathbf{I}_b = 29.6\text{ A}\angle180°$. Thus, $\theta_2 = 60°$. So

$$P_2 = (240)(29.6)\cos 60° = 3552\text{ W}$$

$P_T = P_1 + P_2 = 7823$ W (compared to our earlier solution of 7822 W).

## 23.8 Power System Loads

Before we leave this chapter, we look briefly at how both single-phase and three-phase loads may be connected to a three-phase system. (This is necessary because residential and business customers require only single- phase power, while industrial customers sometimes require both single-phase and three-phase power.) Figure 23–40 shows how this may be done. (Even this is simplified, as real systems contain transformers. However, the basic principles are correct.) Two points should be noted here.

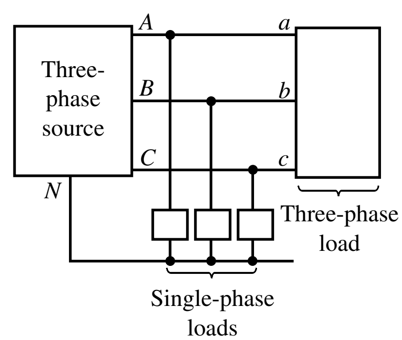



FIGURE 23–40 Single-phase loads are tapped off the three-phase lines.

1. In order to approximately balance the system, the utility tries to connect one-third of its single phase loads to each phase. Three-phase loads generally are balanced.
2. Real loads are seldom expressed in terms of resistance, capacitance, and inductance. Instead, they are described in terms of power, power factor, and so on. This is because most loads consist of electric lights, motors, and the like which are never described in terms of impedance. (For example, you purchase light bulbs as 60-W bulbs, 100-W bulbs, etc., and electric motors as ½ horsepower, etc. You never ask for a 240-ohm light!)

## 23.9 Circuit Analysis Using Computers

PSpice and Electronics Workbench may be used to analyze three-phase systems (balanced or unbalanced, Y or Δ connected). As usual, PSpice yields full phasor solutions, but as of this writing, Workbench yields magnitudes only. Because neither software package allows component placement at an angle, Y and Δ circuits must be drawn with components placed either horizontally or vertically as in Figures 23–42 and 23–45, rather than in their traditional three-phase form. To begin, consider the balanced 4-wire Y circuit of Figure 23–41. First, find the currents so that you have a basis for comparison. Note

$$X_C = 53.05\ \Omega$$

Thus

$$\mathbf{I}_a = \frac{120\ \text{V}\angle 0^\circ}{(30 - j53.05)\ \Omega} = 1.969\ \text{A}\angle 60.51^\circ$$

and

$$\mathbf{I}_b = 1.969\ \text{A}\angle -59.49^\circ$$

and

$$\mathbf{I}_c = 1.969\ \text{A}\angle -179.49^\circ.$$

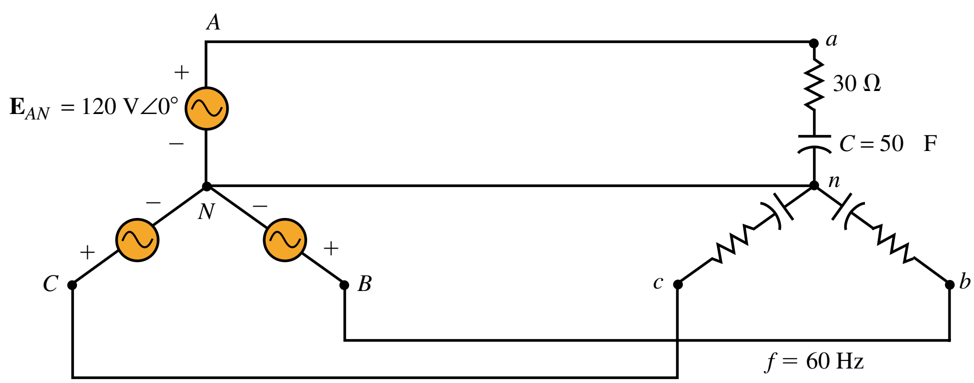



FIGURE 23–41 Balanced system for computer analysis.

## Electronics Workbench

Draw the circuit on the screen as in Figure 23–42. Make sure the voltage sources are oriented with their + ends as shown and that their phase angles are set appropriately. (Workbench does not accept negative angles, thus use 240° instead of −120° for $\mathbf{E}_{BN}$.) Double click the ammeters and set them to

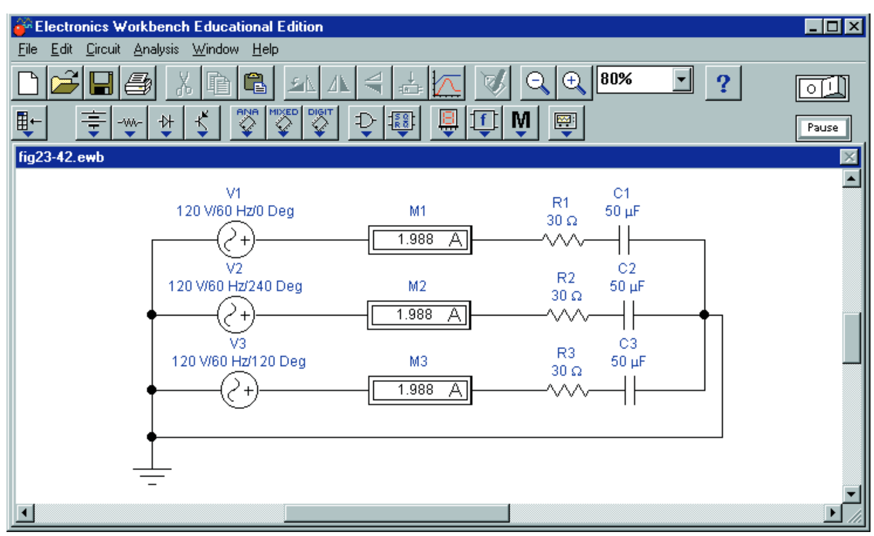



FIGURE 23–42 Solution of the circuit of Figure 23–41 by Electronics Workbench.

AC. Activate the circuit by clicking the power switch. Workbench computes answers of 1.989 for the currents (instead of 1.969 as we calculated above), an error of 0.02 A (approximately 1%).

## OrCAD PSpice

Draw the circuit on the screen as in Figure 23–43 with source + terminals oriented as shown. Double click each source in turn and, in its Property Editor, set its magnitude to 120V with phase angle as indicated. Similarly, double click each IPRINT device and set MAG to **yes,** PHASE to **yes,** and AC to **yes.** Via the New Profile icon, select AC Sweep/Noise, set the Start and End frequencies to **60Hz** and the number of computational points to **1.** Run the simulation, open the Output File (found under View in the results window), then scroll down until you find the answers. (Figure 23–44 shows the answer for current $\mathbf{I}_a$.) Examining the results, you will see that they agree exactly with what we calculated earlier.

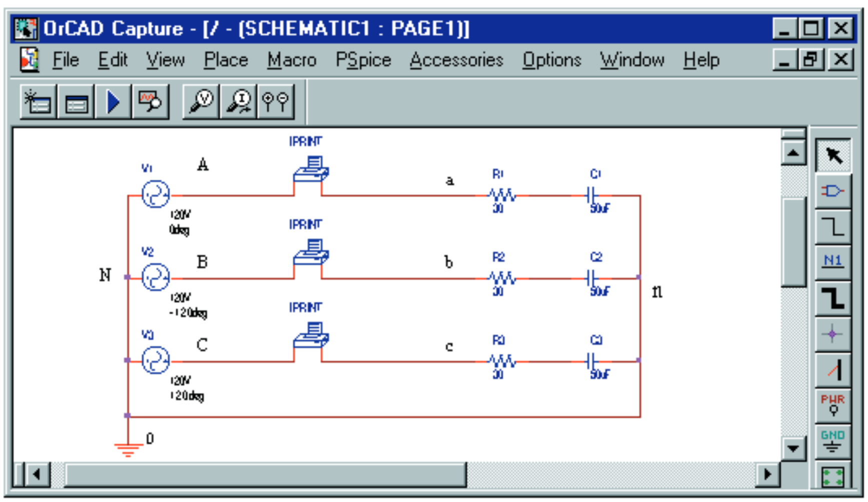



**FIGURE 23–43** Solution of the circuit of Figure 23–41 by PSpice.

| FREQ | IM(V_PRINT1)IP(V_PRINT1) | |
|---|---|---|
| 6.000E+01 | 1.969E+00 | 6.051E+01 |

**FIGURE 23–44** PSpice yields $\mathbf{I}_a = 1.969\ \text{A}\angle 60.51°$ for the circuit of Figure 23–41.

Since the circuit is balanced, you should be able to remove the neutral conductor from between *N-n.* If you do however, you will get errors. This is because PSpice requires a dc path from every node back to the reference, but because capacitors look like open circuits to dc, it sees node *n* as floating. A simple solution is to put a very high value resistor (say 100 kΩ) in the path between *N-n.* Its value is not critical; it just has to be large enough to look like an open circuit. Try it and note that you get the same answers as before.

**FINAL NOTES...**

1. Electronics Workbench and PSpice make no distinction between balanced circuits and unbalanced circuits; to the computer, they are simply circuits. Thus, no special considerations are necessary for unbalanced loads.
2. Delta loads may be drawn as in Figure 23–45.
3. To annotate the diagram as in Figure 23–43 (i.e., with points A, B, C, N, etc. identified), click the Place text icon on the tool palette, key in the text, and place as desired.

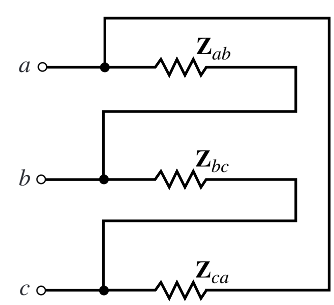


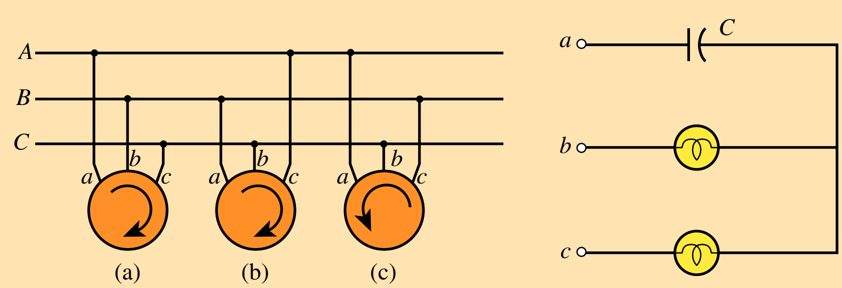

**FIGURE 23–45** Representing a Δ load.

## PUTTING IT INTO PRACTICE

You have been sent to a job site to supervise the installation of a 208-V, three-phase motor. The motor drives a machine, and it is essential that the motor rotate in the correct direction (in this case, clockwise), else the machine may be damaged. You have a drawing that tells you to connect Line *a* of the motor to Line *A* of the three-phase system, Line *b* to Line *B*, etc. However, you find that the three-phase lines have no markings, and you don't know which line is which. Unfortunately, you cannot simply connect the motor and determine which direction it turns for fear of damage.

You think about this for awhile, then you come up with a plan. You know that the direction of rotation of a three-phase motor depends on the phase sequence of the applied voltage, so you make a sketch (sketch A). As indicated in part (a) of that drawing, the motor rotates in the correct direction when *a* is connected to *A*, *b* to *B*, etc. You reason that the phase sequence you need is ...*A*-*B*-*C*-*A*-*B*-*C*... (since the direction of rotation depends only on phase sequence) and that it really doesn't matter to which line *a* is connected as long as the other two are connected such that you provide sequencing in this order to the motor. To convince yourself, you make several more sketches (parts b and c). As indicated, in (b) the sequence is ...*B*-*C*-*A*-*B*-*C*-*A*... (which fits the above pattern) and the motor rotates in the correct direction, but for (c) the motor rotates in the reverse direction. (Show why.)

You also remember reading about a device called a *phase sequence indicator* that allows you to determine phase sequence. It uses lights and a capacitor as in sketch B. To use the device, connect terminal *a* to the three-phase line that you have designated *A* and connect terminals *b* and *c* to the other two lines. The lamp that lights brightly is the one attached to line *B*. After a few calculations, you ask the plant electrician for some standard 120-V 60-W light bulbs and sockets. From your tool kit, you retrieve a 3.9-$\mu$F capacitor (one that is rated for ac operation). You make a slight modification to the schematic of sketch B, solder the parts together, wrap exposed wires with electrical tape (for safety), then connect the device and identify the three-phase lines. You then connect the motor and it rotates in the correct direction. Prepare an analysis to show why the lamp connected to *B* is much brighter than the lamp connected to line *C*. (Note: Don't forget that you made a slight change to the schematic. Hint: If you use a single 60-W lamp in each leg as shown, they will burn out.)




Sketch A

Sketch B

## PROBLEMS

### 23.2 Basic Three-Phase Circuit Connections

1. As long as the loads and voltages of Figure 23–3 are balanced (regardless of their actual value), currents $\mathbf{I}_A$, $\mathbf{I}_B$, and $\mathbf{I}_C$ will sum to zero. To illustrate, change the load impedance from 12 Ω to 15 Ω∠30° and for $\mathbf{E}_{AA'} = 120\text{ V}\angle 0°$, do the following:
   a. Compute currents $\mathbf{I}_A$, $\mathbf{I}_B$, and $\mathbf{I}_C$.
   b. Sum the currents. Does $\mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C = 0$?
2. For Figure 23–3(c), $\mathbf{E}_{AN} = 277\text{ V}\angle -15°$.
   a. What are $\mathbf{E}_{BN}$ and $\mathbf{E}_{CN}$?
   b. If each resistance is 5.54 Ω, compute $\mathbf{I}_A$, $\mathbf{I}_B$, and $\mathbf{I}_C$.
   c. Show that $\mathbf{I}_N = 0$.
3. If the generator of Figure 23–2 is rotated in the clockwise direction, what will be the phase sequence of the generated voltages?

### 23.3 Basic Three-Phase Relationships

4. For the generators of Figure 23–4, $\mathbf{E}_{AN} = 7620\text{ V}\angle -18°$.
   a. What are the phase voltages $\mathbf{E}_{BN}$ and $\mathbf{E}_{CN}$?
   b. Determine the line-to-line voltages.
   c. Sketch their phasor diagram.
5. For the loads of Figure 23–4, $\mathbf{V}_{bc} = 208\text{ V}\angle -75°$.
   a. Determine the line-to-line voltages $\mathbf{V}_{ab}$ and $\mathbf{V}_{ca}$.
   b. Determine the phase voltages.
   c. Sketch their phasor diagram.
6. Repeat Problem 5 if $\mathbf{V}_{ca} = 208\text{ V}\angle 90°$.
7. For the load of Figure 23–46, $\mathbf{V}_{an} = 347\text{ V}\angle 15°$. Determine all line currents. Sketch their phasor diagram.

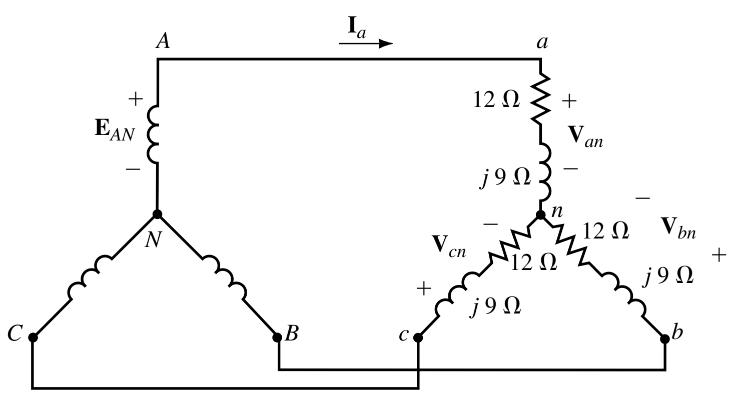



FIGURE 23–46

8. For the load of Figure 23–46, if $\mathbf{I}_a = 7.8\text{ A}\angle -10°$, determine the phase voltages and line voltages. Sketch their phasor diagram.
9. A balanced Y load has impedance $\mathbf{Z}_{an} = 14.7\ \Omega\angle 16°$. If $\mathbf{V}_{cn} = 120\text{ V}\angle 160°$, determine all line currents.

10. For a balanced delta load, $\mathbf{I}_{ab} = 29.3\ \text{A}\angle 43°$. What is $\mathbf{I}_a$?
11. For the circuit of Figure 23–47, $\mathbf{V}_{ab} = 480\ \text{V}\angle 0°$. Find the phase and line currents.

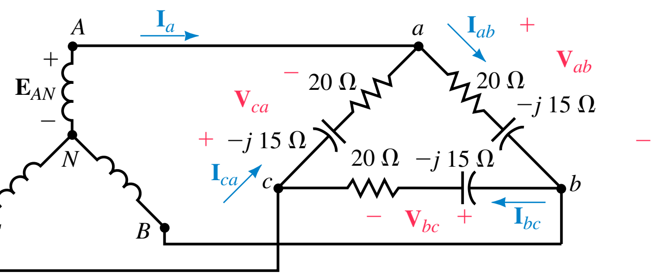



FIGURE 23–47

12. For the circuit of Figure 23–47, if $\mathbf{I}_a = 41.0\ \text{A}\angle -46.7°$, find all the phase currents.
13. For the circuit of Figure 23–47, if $\mathbf{I}_{ab} = 10\ \text{A}\angle -21°$, determine all the line voltages.
14. For the circuit of Figure 23–47, if line current $\mathbf{I}_a = 11.0\ \text{A}\angle 30°$, find all the phase voltages.
15. A balanced Y load has phase impedance of $24\ \Omega\angle 33°$ and line-to-line voltage of 600 V. Take $\mathbf{V}_{an}$ as reference and determine all phase currents.
16. A balanced Δ load has phase impedance of $27\ \Omega\angle -57°$ and phase voltage of 208 V. Take $\mathbf{V}_{ab}$ as reference and determine
    a. Phase currents
    b. Line currents
17. a. For a certain balanced Y load, $\mathbf{V}_{ab} = 208\ \text{V}\angle 30°$, $\mathbf{I}_a = 24\ \text{A}\angle 40°$ and $f =$ 60 Hz. Determine the load ($R$ and $L$ or $C$).
    b. Repeat (a) if $\mathbf{V}_{bc} = 208\ \text{V}\angle -30°$ and $\mathbf{I}_c = 12\ \text{A}\angle 140°$.
18. Consider Figure 23–12(a). Show that $\mathbf{I}_a = \sqrt{3}\,\mathbf{I}_{ab}\angle -30°$.
19. At 60 Hz, a balanced Δ load has current $\mathbf{I}_{bc} = 4.5\ \text{A}\angle -85°$. The line voltage is 240 volts and $\mathbf{V}_{ab}$ is taken as reference.
    a. Find the other phase currents.
    b. Find the line currents.
    c. Find the resistance $R$ and capacitance $C$ of the load.
20. A Y generator with $\mathbf{E}_{AN} = 120\ \text{V}\angle 0°$drives a balanced Δ load. If $\mathbf{I}_a =$ 43.6 $\text{A}\angle -37.5°$, what are the load impedances?

## 23.4 Examples

21. For Figure 23–48, $\mathbf{V}_{an} = 120\ \text{V}\angle 0°$. Draw the single-phase equivalent and:
    a. Find phase voltage $\mathbf{E}_{AN}$, magnitude and angle.
    b. Find line voltage $\mathbf{E}_{AB}$, magnitude and angle.
22. For Figure 23–48, $\mathbf{E}_{AN} = 120\ \text{V}\angle 20°$. Draw the single-phase equivalent and:
    a. Find phase voltage $\mathbf{V}_{an}$, magnitude and angle.
    b. Find line voltage $\mathbf{V}_{ab}$, magnitude and angle.
23. For Figure 23–47, $\mathbf{E}_{AN} = 120\ \text{V}\angle -10°$. Find the line currents using the single-phase equivalent method.

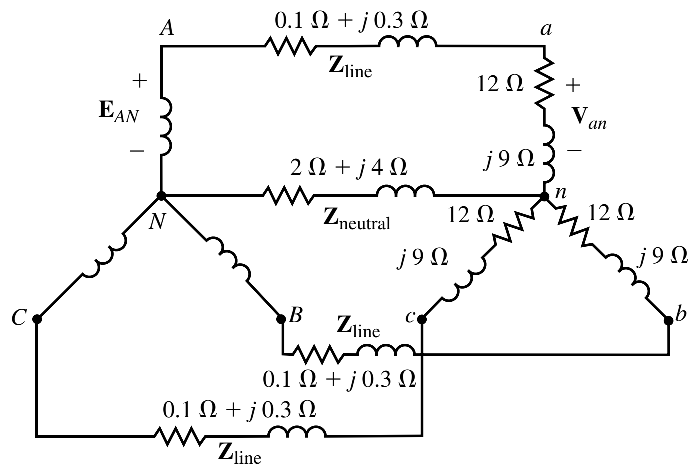



FIGURE 23–48

24. Repeat Problem 23 if $\mathbf{E}_{BN} = 120\ \text{V}\angle -100°$.
25. For Figure 23–47, assume the lines have impedance $\mathbf{Z}_{\text{line}}$ of $0.15\ \Omega + j0.25\ \Omega$ and $\mathbf{E}_{AN} = 120\ \text{V}\angle 0°$. Convert the Δ load to a Y and use the single-phase equivalent to find the line currents.
26. For Problem 25, find the phase currents in the Δ network.
27. For the circuit of Figure 23–48, assume $\mathbf{Z}_{\text{line}} = 0.15\ \Omega + j0.25\ \Omega$ and $\mathbf{V}_{ab} = 600\ \text{V}\angle 30°$. Determine $\mathbf{E}_{AB}$.
28. For Figure 23–49, $\mathbf{Z}_{\text{Y}} = 12\ \Omega + j9\ \Omega$ and $\mathbf{Z}_{\Delta} = 27\ \Omega + j36\ \Omega$. At the Y load, $\mathbf{V}_{an} = 120\ \text{V}\angle 0°$.
    a. Draw the single-phase equivalent.
    b. Find generator voltage $\mathbf{E}_{AN}$.

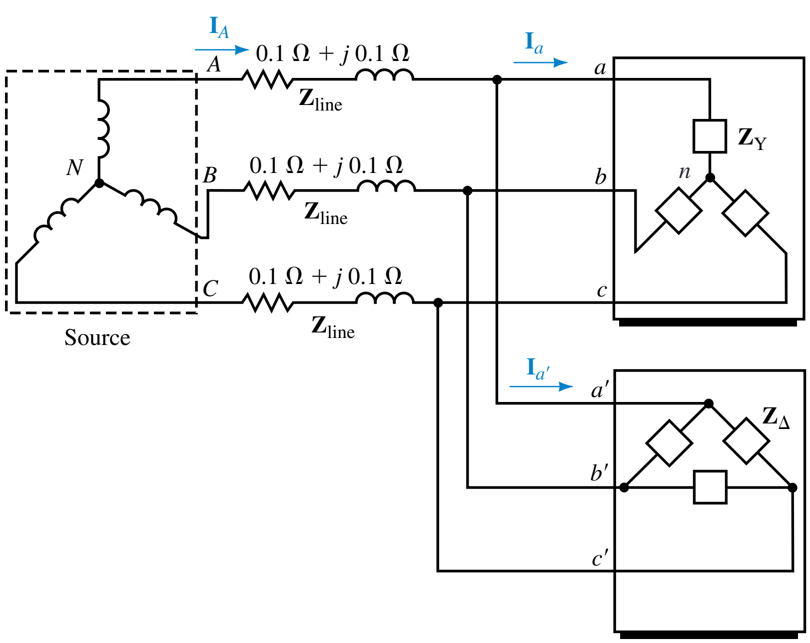



FIGURE 23–49

29. Same as Problem 28 except that the phase voltage at the Δ load is $\mathbf{V}_{a'b'} = 480\ \text{V}\angle 30°$. Find generator voltage $\mathbf{E}_{AB}$, magnitude and angle.
30. For Figure 23–49, $\mathbf{Z}_Y = 12\ \Omega + j9\ \Omega$ and $\mathbf{Z}_\Delta = 36\ \Omega + j27\ \Omega$. Line current $\mathbf{I}_A$ is $46.2\ \text{A}\angle -36.87°$. Find the phase currents for both loads.
31. For Figure 23–49, $\mathbf{Z}_Y = 15\ \Omega + j20\ \Omega$, $\mathbf{Z}_\Delta = 9\ \Omega - j12\ \Omega$ and $\mathbf{I}_{a'b'} = 40\ \text{A}\angle 73.13°$. Find Y phase voltage $\mathbf{V}_{an}$, magnitude and angle.

### 23.5 Power in a Balanced System

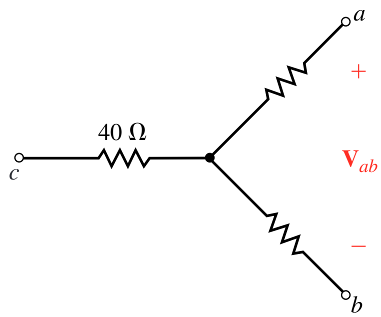



FIGURE 23–50

32. For the balanced load of Figure 23–50, $V_{ab} = 600$ V. Determine per phase and total active, reactive, and apparent power.
33. Repeat Problem 32 for the balanced load of Figure 23–51, given $E_{AN} = 120$ V.
34. For Figure 23–46, $E_{AN} = 120$ volts.
    a. Determine per phase real, reactive, and apparent power.
    b. Multiply per phase quantities by 3 to get total quantities.

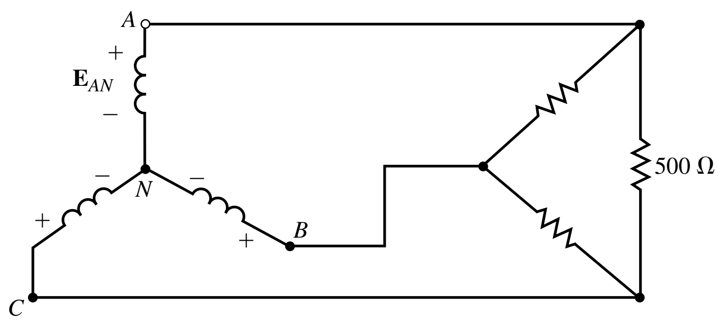



FIGURE 23–51

35. For Figure 23–46, compute the total real, reactive, and apparent power using the $P_T$, $Q_T$, and $S_T$ formulas of Table 23–2. (Use $V_L = 207.8$ V rather than the nominal value of 208 V here.) Compare these to the results of Problem 34.
36. For Figure 23–47, $E_{AB} = 208$ volts.
    a. Determine per phase real, reactive, and apparent power.
    b. Multiply per phase quantities by 3 to get total quantities.
37. For Figure 23–47, $E_{AB} = 208$ V. Compute the total real, reactive, and apparent power using the $P_T$, $Q_T$, and $S_T$ formulas of Table 23–2. Compare to the results of Problem 36.
38. For Figure 23–11, if $V_{an} = 277$ V, determine the total active power, total reactive power, total apparent power, and power factor.
39. For Figure 23–13, if $V_{ab} = 600$ V, determine the total power, total reactive power, total apparent power, and power factor.
40. For Figure 23–17, if $V_{an} = 120$ V, determine the total power, total reactive power, and total apparent power,
    a. supplied to the load.
    b. output by the source.
41. For Figure 23–18, if $V_{ab} = 480$ V, determine the total power, total reactive power, total apparent power, and power factor.
42. For Figure 23–49, let $\mathbf{Z}_{\text{line}} = 0\ \Omega$, $\mathbf{Z}_Y = 20\ \Omega\angle 0°$, $\mathbf{Z}_\Delta = 30\ \Omega\angle 10°$, and $E_{AN} = 120$ V.

a. Find the total real power, reactive power, and apparent power to the Y load.

b. Repeat (a) for the Δ load.

c. Determine total watts, VARs, and VA using the results of (a) and (b).

43. $V_{ab}$ = 208 V for a balanced Y load, $P_T$ = 1200 W, and $Q_T$ = 750 VAR (ind.). Choose $\mathbf{V}_{an}$ as reference and determine $\mathbf{I}_a$. (Use power triangle.)

44. A motor (delivering 100 hp to a load) and a bank of power factor capacitors are connected as in Figure 23–52. The capacitors are rated $Q_C$ = 45 kVAR. Reduce the problem to its single-phase equivalent, then compute the resultant power factor of the system.

45. The capacitors of Figure 23–52 are connected in Y and each has the value $C$ = 120 μF. Compute the resultant power factor. Frequency is 60 Hz.

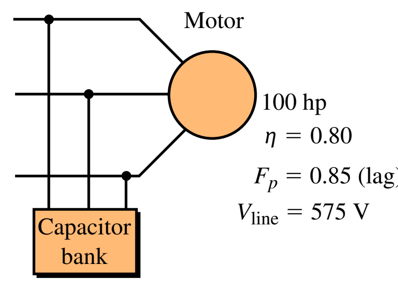



FIGURE 23–52

### 23.6 Measuring Power in Three-Phase Circuits

46. For Figure 23–53:

a. Determine the wattmeter reading.

b. If the load is balanced, what is $P_T$?

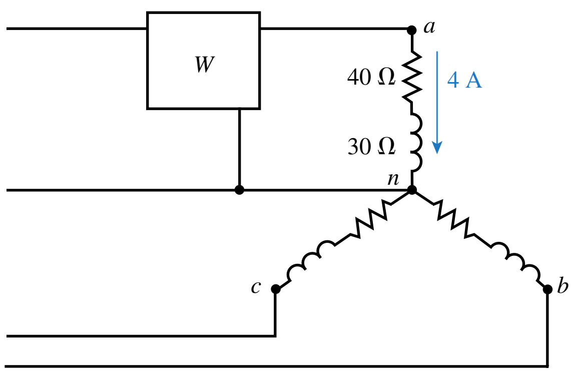



FIGURE 23–53

47. For Figure 23–46, the generator phase voltage is 120 volts.

a. Sketch three wattmeters correctly into the circuit.

b. Compute the reading of each wattmeter.

c. Sum the readings and compare this to the result of 2304 W obtained in Problem 34.

48. Figures 23–29 and 23–30 show two ways that two wattmeters may be connected to measure power in a three-phase, three-wire circuit. There is one more way. Sketch it.

49. For the circuit of Figure 23–54, $\mathbf{V}_{ab} = 208\text{ V}\angle 30°$.

a. Determine the magnitude and angle of the currents.

b. Determine power per phase and total power, $P_T$.

c. Compute the reading of each wattmeter.

d. Sum the meter readings and compare the result to $P_T$ of (b).

50. Two wattmeters measure power to a balanced load. Readings are $P_h$ = 1000 W and $P_\ell$ = −400 W. Determine the power factor of the load from Equation 23–24 and from Figure 23–33. How do they compare?

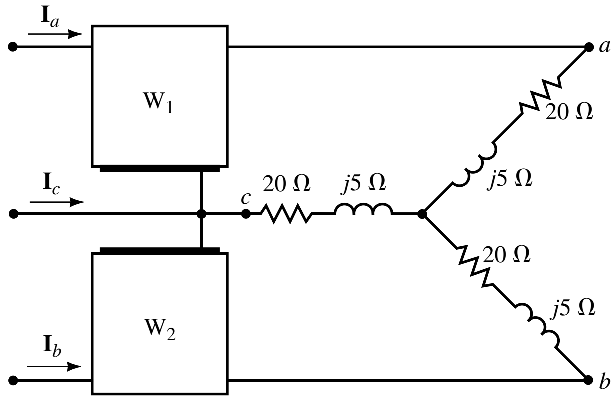



**FIGURE 23–54**

51. Consider the circuit of Figure 23–54.
    a. Compute the power factor from the angle of the phase impedances.
    b. In Problem 49, wattmeter readings $P_h = 1164$ W and $P_\ell = 870$ W were determined. Substitute these values into Equation 23–24 and compute the power factor of the load. Compare the results to (a).
52. For the balanced load of Figure 23–55, $\mathbf{V}_{ab} = 208\text{ V}\angle 0°$.
    a. Compute phase currents and line currents.
    b. Determine power per phase and total power $P_T$.
    c. Compute the reading of each wattmeter, then sum the wattmeter readings and compare to $P_T$ from (b).

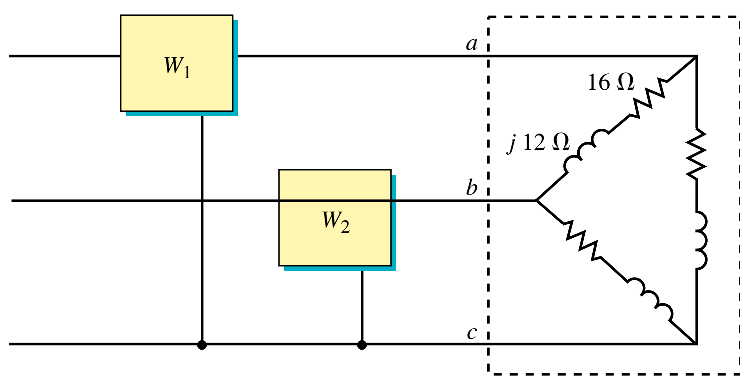



**FIGURE 23–55**

### 23.7 Unbalanced Systems

53. For Figure 23–56, $R_{ab} = 60\ \Omega$, $\mathbf{Z}_{bc} = 80\ \Omega + j60\ \Omega$. Compute
    a. Phase and line currents.
    b. Power to each phase and total power.
54. Repeat Problem 53 if $P_{ab} = 2400$ W and $\mathbf{Z}_{bc} = 50\ \Omega\angle 40°$.
55. For Figure 23–57, compute the following:
    a. Line currents, magnitudes and angles.
    b. The neutral current.

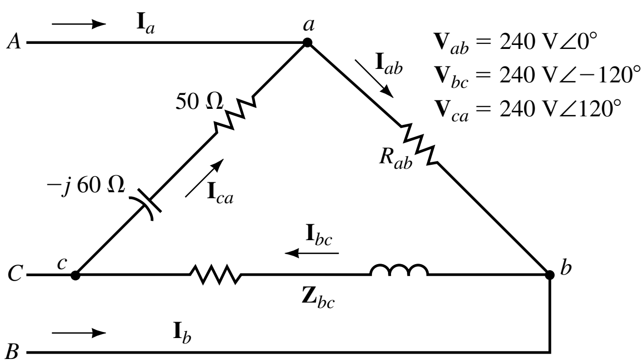



FIGURE 23–56

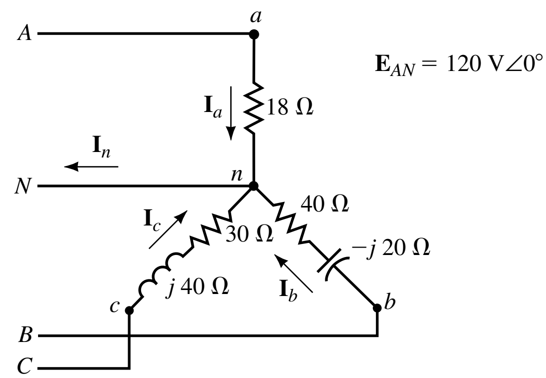



FIGURE 23–57

c. Power to each phase.

d. Total power to the load.

56. Remove the neutral conductor from the circuit of Figure 23–57 and compute the line currents. *Hint:* Use mesh equations.

57. From Problem 56, $\mathbf{I}_a = 1.94\ \text{A}\angle -0.737°$, $\mathbf{I}_b = 4.0\ \text{A}\angle -117.7°$ and $\mathbf{I}_c = 3.57\ \text{A}\angle 91.4°$. Compute the following:

a. Voltages across each phase of the load.

b. Voltage between the neutral of the load and the neutral of the generator.

### 23.9 Circuit Analysis Using Computers

For the following, use either Electronics Workbench or PSpice. With Workbench, you get only magnitude; with PSpice, solve for magnitude and angle. Caution: For PSpice, insert the IPRINT devices so that current enters the positive terminal. Otherwise, the phase angle will be in error by 180°. (See Figure 23–43.)

58. **EWB** or **PSpice** For the balanced system of Figure 23–46, let $\mathbf{E}_{AN} = 347\ \text{V}\angle 15°$, $L = 8.95$ mH, and $f = 160$ Hz. Solve for line currents.

59. **EWB** or **PSpice** For the balanced system of Figure 23–47, let $\mathbf{E}_{AN} = 277\ \text{V}\angle -30°$, $C = 50\ \mu\text{F}$, and $f = 212$ Hz. Solve for phase currents and line currents.

60. **EWB** or **PSpice** Repeat Problem 59 with $C$ replaced with $L = 11.26$ mH.

61. **EWB** or **PSpice** For Figure 23–57, let $L = 40$ mH, $C = 50\ \mu\text{F}$, and $\omega = 1000$ rad/s. Solve for line and neutral currents.

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## ANSWERS TO IN-PROCESS LEARNING CHECKS

### In-Process Learning Check 1

1. $\mathbf{V}_{an} = \mathbf{E}_{AN} = 277\ \text{V}\angle -20°$; $\mathbf{V}_{bn} = \mathbf{E}_{BN} = 277\ \text{V}\angle -140°$; $\mathbf{V}_{cn} = \mathbf{E}_{CN} = 277\ \text{V}\angle 100°$; $\mathbf{V}_{ab} = \mathbf{E}_{AB} = 480\ \text{V}\angle 10°$; $\mathbf{V}_{bc} = \mathbf{E}_{BC} = 480\ \text{V}\angle -110°$; $\mathbf{V}_{ca} = \mathbf{E}_{CA} = 480\ \text{V}\angle 130°$
2. $\mathbf{V}_{an} = \mathbf{E}_{AN} = 120\ \text{V}\angle 50°$; $\mathbf{V}_{bn} = \mathbf{E}_{BN} = 120\ \text{V}\angle -70°$; $\mathbf{V}_{cn} = \mathbf{E}_{CN} = 120\ \text{V}\angle 170°$; $\mathbf{V}_{ab} = \mathbf{E}_{AB} = 208\ \text{V}\angle 80°$; $\mathbf{V}_{bc} = \mathbf{E}_{BC} = 208\ \text{V}\angle -40°$; $\mathbf{V}_{ca} = \mathbf{E}_{CA} = 208\ \text{V}\angle -160°$
3. $\mathbf{V}_{an} = 124\ \text{V}\angle -1.87°$; $\mathbf{V}_{ab} = 214\ \text{V}\angle 28.13°$
4. $\mathbf{V}_{ab} = 104\ \text{V}\angle 81.7°$; $\mathbf{V}_{bc} = 104\ \text{V}\angle -38.3°$; $\mathbf{V}_{ca} = 104\ \text{V}\angle -158.3°$

24

# Transformers and Coupled Circuits

## OBJECTIVES

After studying this chapter, you will be able to

- describe how a transformer couples energy from its primary to its secondary via a changing magnetic field,
- describe basic transformer construction,
- use the dot convention to determine transformer phasing,
- determine voltage and current ratios from the turns ratio for iron-core transformers,
- compute voltage and currents in circuits containing iron-core and air-core transformers,
- use transformers to impedance match loads,
- describe some basic transformer applications,
- determine transformer equivalent circuits,
- compute iron-core transformer efficiency,
- use Electronics Workbench and PSpice to solve circuits with transformers and coupled circuits.

## KEY TERMS

Air-Core Transformer
Autotransformer
Coefficient of Coupling
Copper Loss
Core Loss
Coupled Circuit
Current Ratio
Dot Convention
Ferrite Core
Ideal Transformer
Impedance Matching
Iron Core
Leakage Flux
Loosely Coupled
Magnetizing Current
Mutual Inductance
Open-Circuit Test
Primary
Reflected Impedance
Secondary
Short-Circuit Test
Step-Up/Step-Down
Tightly Coupled
Transformer
Turns Ratio
Voltage Ratio

## OUTLINE

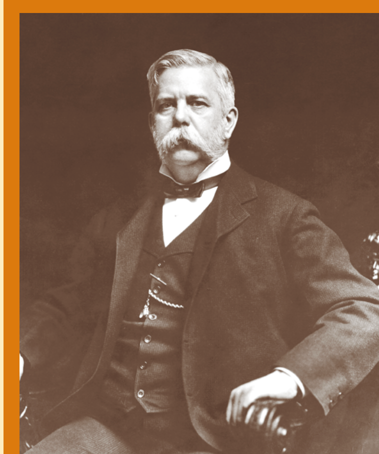
## CHAPTER PREVIEW

In our study of induced voltage in Chapter 13 we found that the changing magnetic field produced by current in one coil induced a voltage in a second coil wound on the same core. A device built to utilize this effect is the **transformer.**

Transformers have many applications. They are used in electrical power systems to step up voltage for long-distance transmission and then to step it down again to a safe level for use in our homes and offices. They are used in electronic equipment power supplies to raise or lower voltages, in audio systems to match speaker loads to amplifiers, in telephone, radio, and TV systems to couple signals, and so on.

In this chapter, we look at transformer fundamentals and the analysis of circuits containing transformers. We discuss transformer action, types of transformers, voltage and current ratios, applications, and so on. Both iron-core and air-core transformers are covered.

## PUTTING IT IN PERSPECTIVE



### *George Westinghouse*

ONE OF THE DEVICES that made possible the commercial ac power system as we know it today is the transformer. Although Westinghouse did not invent the transformer, his acquisition of the transformer patent rights and his manufacturing business helped make him an important player in the battle of dc versus ac in the emerging electrical power industry (see Chapter 15). In concert with Tesla (see Chapter 23), Westinghouse fought vigorously for ac against Edison, who favored dc. In 1893, Westinghouse's company built the Niagara Falls power system using ac, and the battle was over with ac the clear winner. (Ironically, in recent years dc has been resurrected for use in commercial electrical power systems because it is able to transmit power over longer distances than ac. However, this was not possible in Edison's day, and ac was and still is the correct choice for the commercial electrical power system.)

George Westinghouse was born in 1846 in Central Bridge, New York. He made his fortune with the invention of the railway air brake system. He died in 1914 and was elected to the Hall of Fame for Great Americans in 1955.

## 24.1 Introduction

A transformer is a magnetically **coupled circuit,** i.e., a circuit in which the magnetic field produced by time-varying current in one circuit induces voltage in another. To illustrate, a basic iron-core transformer is shown in Figure 24–1. It consists of two coils wound on a common core. Alternating current in one winding establishes a flux which links the other winding and induces a voltage in it. Power thus flows from one circuit to the other via the medium of the magnetic field, with no electrical connection between the two sides. The winding to which we supply power is called the **primary,** while the winding from which we take power is called the **secondary.** Power can flow in either direction, as either winding can be used as the primary or the secondary.

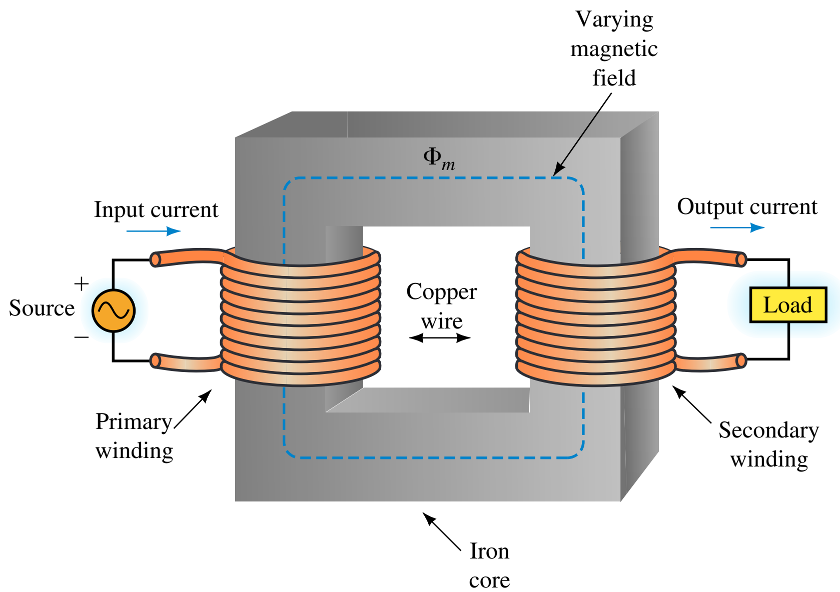



**FIGURE 24–1** Basic iron-core transformer. Energy is transferred from the source to the load via the transformer's magnetic field with no electrical connection between the two sides.

### Transformer Construction

Transformers fall into two broad categories, iron-core and air-core. We begin with **iron-core** types. Iron core transformers are generally used for low frequency applications such as audio- and power-frequency applications. Figures 24–2 and 24–3 show a few examples of iron-core transformers.

Iron (actually a special steel called transformer steel) is used for cores because it increases the coupling between coils by providing an easy path for magnetic flux. Two basic types of iron-core construction are used, the **core type** and the **shell type** (Figure 24–4). In both cases, cores are made from laminations of sheet steel, insulated from each other by thin coatings of ceramic or other material to help minimize eddy current losses (Section 24.6).

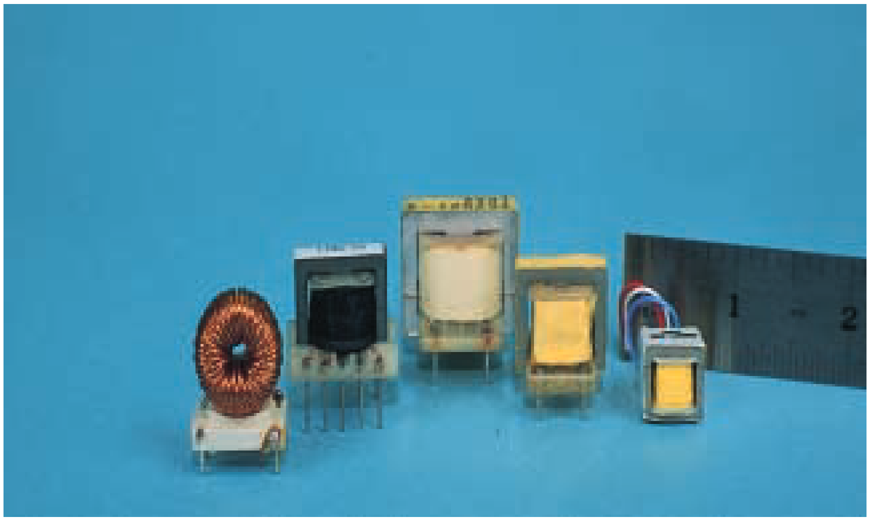


**FIGURE 24–2** Iron-core transformers of the type used in electronic equipment. *(Courtesy Transformer Manufacturers Inc.)*

**FIGURE 24–3** Distribution transformer (cutaway view) of the type used by electric utilities to distribute power to residential and commercial users. The tank is filled with oil to improve insulation and to remove heat from the core and windings. *(Courtesy Carte International Inc.)*

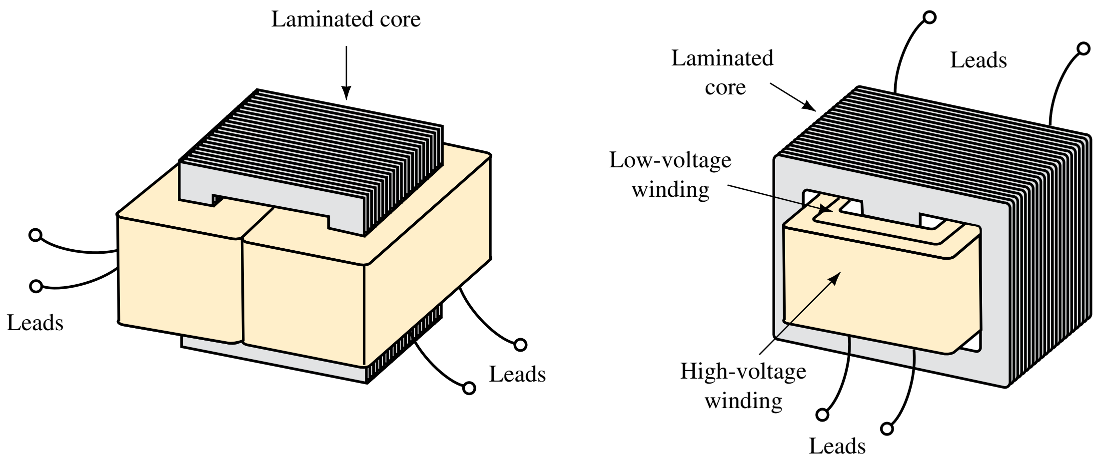



**FIGURE 24–4** For the core type (left), windings are on separate legs, while for the shell type both windings are on the same leg. (Adapted with permission from Perozzo, *Practical Electronics Troubleshooting,* © 1985, Delmar Publishers Inc.)

Iron, however, has considerable power loss due to hysteresis and eddy currents at high frequencies, and is thus not useful as a core material above about 50 kHz. For high-frequency applications (such as in radio circuits), **air-core** and **ferrite-core** types are used. Figure 24–5 shows a ferrite-core device. Ferrite (a magnetic material made from powdered iron oxide) greatly increases coupling between coils (compared with air) while maintaining low losses. Circuit symbols for transformers are shown in Figure 24–6.

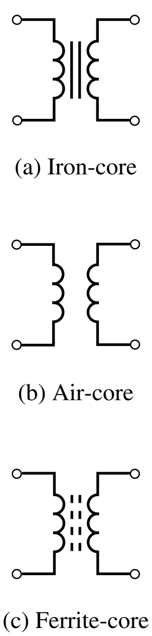



**FIGURE 24–6** Transformer schematic symbols.

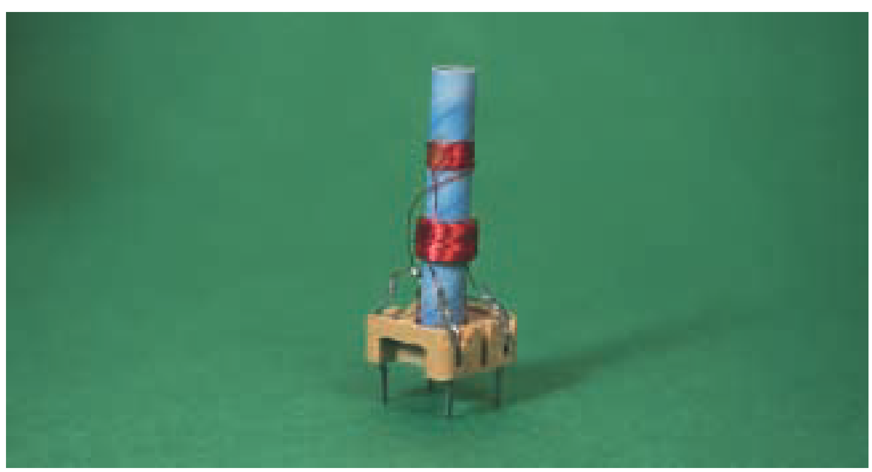
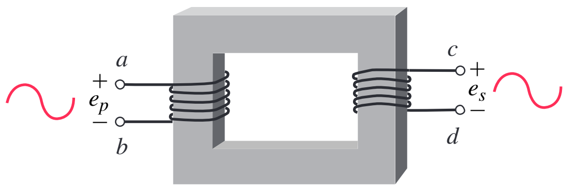


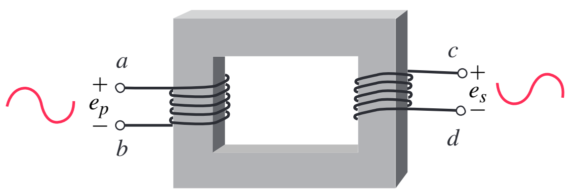
**FIGURE 24–5** A slug-tuned ferrite-core radio frequency transformer. Coupling between coils is varied by a ferrite slug positioned by an adjusting screw.

## Winding Directions

One of the advantages of a transformer is that it may be used to change the polarity of an ac voltage. This is illustrated in Figure 24–7 for a pair of iron-core transformers. For the transformer of (a) the primary and secondary voltages are in phase (for reasons to be discussed later), while for (b) they are 180° out of phase.




(a) 0° phase shift




(b) 180° phase shift

**FIGURE 24–7** The relative direction of the windings determines the phase shift.

### Tightly Coupled and Loosely Coupled Circuits

If most of the flux produced by one coil links the other, the coils are said to be **tightly coupled.** Thus, iron-core transformers are tightly coupled (since close to 100% of the flux is confined to the core and thus links both windings). For air- and ferrite-core transformers, however, much less than 100% of the flux links both windings. They are therefore **loosely coupled.** Since air-core and ferrite-core devices are loosely coupled, the same principles of analysis apply to both and we treat them together in Section 24.9.

### Faraday's Law

All transformer operation is described by Faraday's law. Faraday's law (in SI units) states that the voltage induced in a circuit by a changing magnetic field is equal to the rate at which the flux linking the circuit is changing. When Faraday's law is applied to iron-core and air-core transformers, however, the results that emerge are quite different: Iron-core transformers are found to be characterized by their turns ratios, while air-core transformers are characterized by self- and mutual inductances. We begin with iron-core transformers.

## 24.2 Iron-Core Transformers: The Ideal Model

At first glance, iron-core transformers appear quite difficult to analyze because they have several characteristics such as winding resistance, core loss, and leakage flux that appear difficult to handle. Fortunately, these effects are small and often can be neglected. The result is the **ideal transformer.** Once you know how to analyze an ideal transformer, however, it is relatively easy to go back and add in the nonideal effects. This is the approach we use here.

To idealize a transformer, (1) neglect the resistance of its coils, (2) neglect its core loss, (3) assume all flux is confined to its core, and (4) assume that negligible current is required to establish its core flux. (Well-designed iron-core transformers are very close to this ideal.)

We now apply Faraday's law to the ideal transformer. Before we do this, however, we need to determine flux linkages. The flux linking a winding (as determined in Chapter 13) is the product of the flux that passes through the winding times the number of turns through which it passes. For flux $\Phi$ passing through $N$ turns, flux linkage is $N\Phi$. Thus, for the ideal transformer (Figure 24–8), the primary flux linkage is $N_p\Phi_m$, while the secondary flux linkage is $N_s\Phi_m$, where the subscript "m" indicates mutual flux, i.e., flux that links both windings.

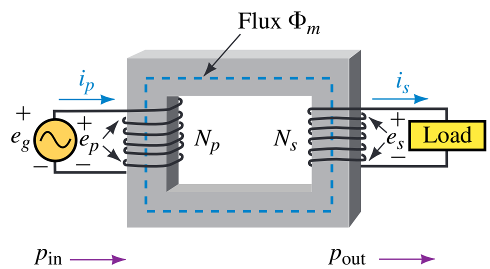



**FIGURE 24–8** Ideal transformer. All flux is confined to the core and links both windings. This is a "tightly-coupled" transformer.

## Voltage Ratio

Now apply Faraday's law. Since the flux linkage equals $N\Phi$ and since $N$ is constant, the induced voltage is equal to $N$ times the rate of change of $\Phi$, i.e., $e = Nd\Phi/dt$. Thus, for the primary,

$$e_p = N_p \frac{d\Phi_m}{dt} \tag{24–1}$$

while for the secondary

$$e_s = N_s \frac{d\Phi_m}{dt} \tag{24–2}$$

Dividing Equation 24–1 by Equation 24–2 and cancelling $d\Phi_m/dt$ yields

$$\frac{e_p}{e_s} = \frac{N_p}{N_s} \tag{24–3}$$

Equation 24–3 states that *the ratio of primary voltage to secondary voltage is equal to the ratio of primary turns to secondary turns.* This ratio is called the **transformation ratio** (or **turns ratio**) and is given the symbol $a$. Thus,

$$a = N_p/N_s \tag{24–4}$$

and

$$e_p/e_s = a \tag{24–5}$$

For example, a transformer with 1000 turns on its primary and 250 turns on its secondary has a turns ratio of 1000/250 = 4. This is referred to as a 4 : 1 ratio.

Since the ratio of two instantaneous sinusoidal voltages is the same as the ratio of their effective values, Equation 24–5 can also be written as

$$E_p/E_s = a \tag{24–6}$$

As noted earlier, $e_p$ and $e_s$ are either in phase or 180° out of phase, depending on the relative direction of coil windings. We can therefore also express the ratio of voltages in terms of phasors as

$$\mathbf{E}_p/\mathbf{E}_s = a \tag{24–7}$$

where the relative polarity (in phase or 180° out of phase) is determined by the direction of the coil windings (Figure 24–7).

## Step-Up and Step-Down Transformers

A **step-up** transformer is one in which the secondary voltage is higher than the primary voltage, while a **step-down** transformer is one in which the secondary voltage is lower. Since $a = E_p/E_s$, a step-up transformer has $a < 1$, while for a step-down transformer, $a > 1$. If $a = 1$, the transformer's turns ratio is **unity** and the secondary voltage is equal to the primary voltage.

**EXAMPLE 24–1** Suppose the transformer of Figure 24–7(a) has 500 turns on its primary and 1000 turns on its secondary.

a. Determine its turns ratio. Is it step-up or step-down?
b. If its primary voltage is $e_p = 25 \sin \omega t$ V, what is its secondary voltage?
c. Sketch the waveforms.

**Solution**

a. The turns ratio is $a = N_p/N_s = 500/1000 = 0.5$. This is a step-up transformer.
b. From Equation 24–5, $e_s = e_p/a = (25 \sin \omega t)/0.5 = 50 \sin \omega t$ V.
c. Primary and secondary voltages are in phase as noted earlier. Figure 24–9 shows the waveforms.

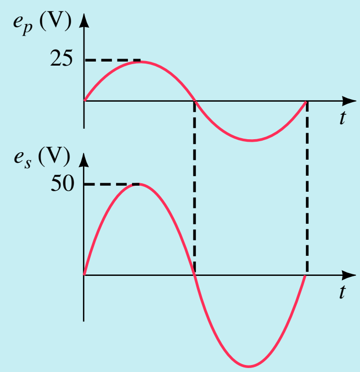



**FIGURE 24–9**

**EXAMPLE 24–2** If the transformers of Figure 24–7 have 600 turns on their primaries and 120 turns on their secondaries, and $\mathbf{E}_p = 120\text{ V}\angle 0°$, what is $\mathbf{E}_s$ for each case?

**Solution** The turns ratio is $a = 600/120 = 5$. For transformer (a), $\mathbf{E}_s$ is in phase with $\mathbf{E}_p$. Therefore, $\mathbf{E}_s = \mathbf{E}_p/5 = (120\text{ V}\angle 0°)/5 = 24\text{ V}\angle 0°$. For transformer (b), $\mathbf{E}_s$ is 180° out of phase with $\mathbf{E}_p$. Therefore, $\mathbf{E}_s = 24\text{ V}\angle 180°$.

**PRACTICE PROBLEMS 1**

Repeat Example 24–1 for the circuit of Figure 24–7(b) if $N_p = 1200$ turns and $N_s = 200$ turns.

*Answer:* $e_s = 4.17 \sin(\omega t + 180°)$

## Current Ratio

Because an ideal transformer has no power loss, its efficiency is 100% and thus power in equals power out. Consider again Figure 24–8. At any instant, $p_{\text{in}} = e_p i_p$ and $p_{\text{out}} = e_s i_s$. Thus,

$$e_p i_p = e_s i_s \tag{24–8}$$

and

$$\frac{i_p}{i_s} = \frac{e_s}{e_p} = \frac{1}{a} \tag{24–9}$$

since $e_s/e_p = 1/a$. (This means that if voltage is stepped up, current is stepped down, and vice versa.) In terms of current phasors and current magnitudes, Equation 24–9 can be written as

$$\frac{\mathbf{I}_p}{\mathbf{I}_s} = \frac{I_p}{I_s} = \frac{1}{a} \tag{24–10}$$

For example, for a transformer with $a = 4$, and $\mathbf{I}_p = 2\text{ A}\angle -20°$, $\mathbf{I}_s = a\mathbf{I}_p = 4(2\text{ A}\angle -20°) = 8\text{ A}\angle -20°$, Figure 24–10.

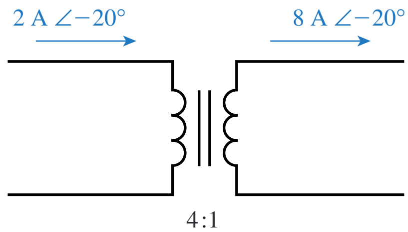



**FIGURE 24–10** Currents are inverse to the turns ratio.

## Polarity of Induced Voltage: The Dot Convention

As noted earlier, an iron core transformer's secondary voltage is either in phase with its primary voltage or 180° out of phase, depending on the relative direction of its windings. We will now demonstrate why.

A simple test, called the **kick test** (sometimes used by electrical workers to determine transformer polarity), can help establish the idea. The basic circuit is shown in Figure 24–11. A switch is used to make and break the circuit (since voltage is induced only while flux is changing).

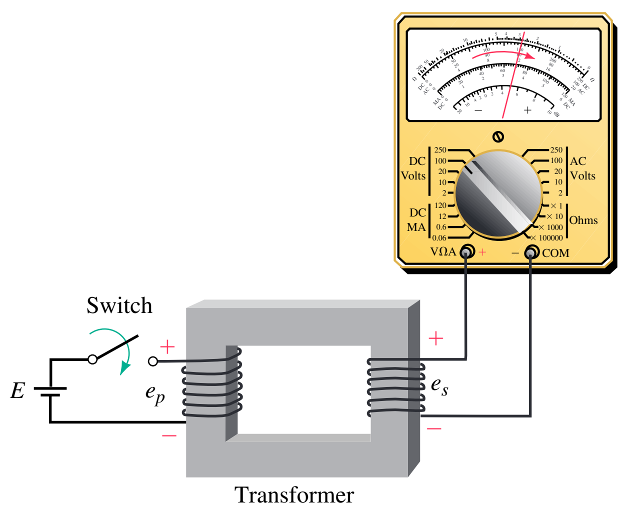



**FIGURE 24–11** The "kick" test. For the winding directions shown, the meter kicks upscale at the instant the switch is closed. (This is the transformer of Figure 24–7a.)

For the winding directions shown, at the instant the switch is closed, the voltmeter needle "kicks" upscale, then settles back to zero. To understand why, we need to consider magnetic fields. Before we start, however, let us place a dot on one of the primary terminals; in this case, we arbitrarily choose the top terminal. Let us also replace the voltmeter by its equivalent resistance (Figure 24–12).

At the instant the switch is closed, the polarity of the dotted primary terminal is positive with respect to the undotted primary terminal (because the + end of the source is directly connected to it). As current in the primary builds, it creates a flux in the upward direction as indicated by the blue arrow (recall the right-hand rule). According to Lenz's law, the effect that results

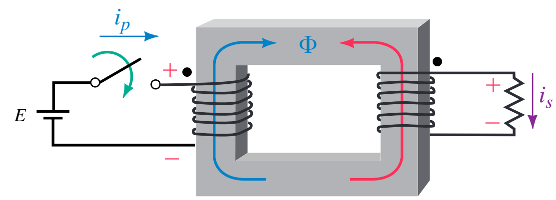



**FIGURE 24–12** Determining dot positions.

must oppose the cause that produced it. The effect is a voltage induced in the secondary winding. The resulting current in the secondary produces a flux which, according to Lenz's law, must *oppose the buildup of the original flux,* i.e., it must be in the direction of the red arrow. Applying the right-hand rule, we see that secondary current must be in the direction indicated by $i_s$. Placing a plus sign at the tail of this arrow shows that the top end of the resistor is positive. This means that the top end of the secondary winding is also positive. Place a dot here. Dotted terminals are called **corresponding** terminals.

As you can see, corresponding terminals are positive (with respect to their companion undotted terminals) at the instant the switch is closed. If you perform a similar analysis at the instant the switch is opened, you will find that both dotted terminals are negative. Thus, *dotted terminals have the same polarity at all instants of time.* What we have developed here is known as the **dot convention for coupled circuits.**

PRACTICAL NOTES...

1. While we developed the dot convention using a switched dc source, it is valid for ac as well. In fact, we will use it mostly for ac.
2. The resistor of Figure 24–12 is required only to work through the physics to establish the secondary voltage polarity. You can now remove it without affecting the resulting dot position.
3. In practice, corresponding terminals may be marked with dots, by color coded wires, or by special letter designations.

EXAMPLE 24–3 Determine the waveform for $e_s$ in the circuit of Figure 24–13(a).

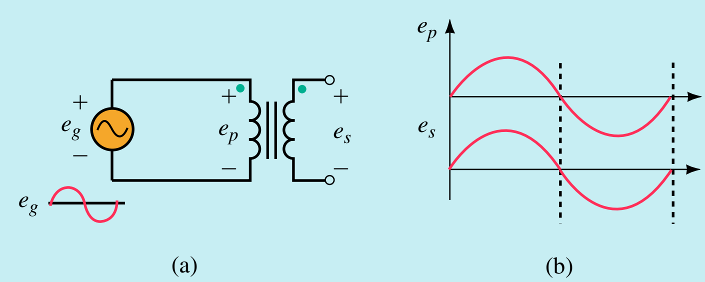



**FIGURE 24–13**

**Solution** Dotted terminals have the same polarity (with respect to their undotted terminals) at all instants. During the first half-cycle, the dotted end of the primary coil is positive. Therefore, the dotted end of the secondary is also positive. During the second half-cycle, both are negative. The polarity marks on $e_s$ mean that we are looking at the polarity of the top end of the secondary coil with respect to its bottom end. Thus, $e_s$ is positive during the first half-cycle and negative during the second half cycle. It is therefore in phase with $e_p$ as indicated in (b). Thus, if $e_p = E_{m_p} \sin \omega t$, then $e_s = E_{m_s} \sin \omega t$.

PRACTICE PROBLEMS 2

1. Determine the equation for $e_s$ in the circuit of Figure 24–14(b).

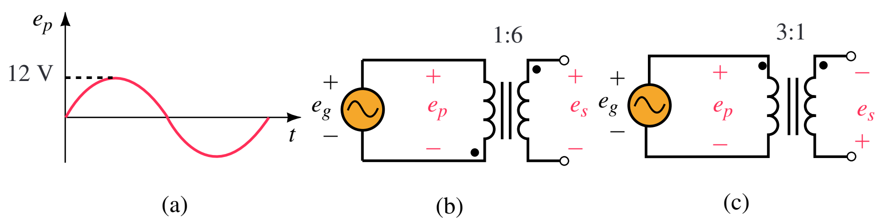



**FIGURE 24–14**

2. Repeat Problem 1 for Figure 24–14(c).
3. If $\mathbf{E}_g = 120\ \text{V}\angle 30°$, determine $\mathbf{E}_s$ for each transformer of Figure 24–14.
4. Where do the dots go on the transformers of Figure 24–7?

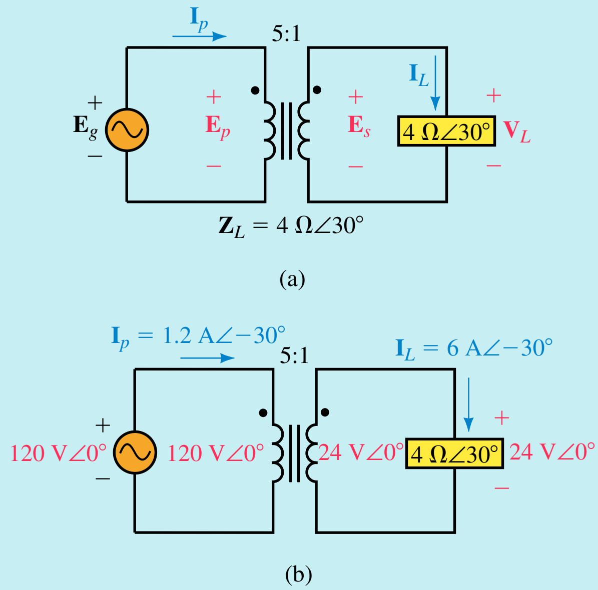
*Answers:*
1. $e_s = 72 \sin(\omega t + 180°)$ V
2. $e_s = 4 \sin(\omega t + 180°)$ V
3. For (b), $\mathbf{E}_s = 720\ \text{V}\angle 180°$; for (c), $\mathbf{E}_s = 40\ \text{V}\angle 180°$.
4. For (a), place dots at *a* and *c*. For (b), place dots at *a* and *d*.

## Analysis of Simple Transformer Circuits

Simple transformer circuits may be analyzed using the relationships described so far, namely $\mathbf{E}_p = a\mathbf{E}_s$, $\mathbf{I}_p = \mathbf{I}_s/a$, and $P_{\text{in}} = P_{\text{out}}$. This is illustrated in Example 24–4. More complex problems require some additional ideas.

**EXAMPLE 24–4** For Figure 24–15(a), $\mathbf{E}_g = 120\ \text{V}\angle 0°$, the turns ratio is 5 : 1, and $\mathbf{Z}_L = 4\ \Omega\angle 30°$. Find

a. the load voltage,
b. the load current,
c. the generator current,
d. the power to the load,
e. the power output by the generator.




EWB **FIGURE 24–15**

**Solution**

a. $\mathbf{E}_p = \mathbf{E}_g = 120\text{ V}\angle 0°$

$\mathbf{V}_L = \mathbf{E}_s = \mathbf{E}_p/a = (120\text{ V}\angle 0°)/5 = 24\text{ V}\angle 0°$

b. $\mathbf{I}_L = \mathbf{V}_L/\mathbf{Z}_L = (24\text{ V}\angle 0°)/(4\ \Omega\angle 30°) = 6\text{ A}\angle -30°$

c. $\mathbf{I}_g = \mathbf{I}_p$. But $\mathbf{I}_p = \mathbf{I}_s/a = \mathbf{I}_L/a$. Thus,

$\mathbf{I}_g = (6\text{ A}\angle -30°)/5 = 1.2\text{ A}\angle -30°$

Values are shown on Figure 24–15(b).

d. $P_L = V_L I_L \cos\theta_L = (24)(6)\cos 30° = 124.7\text{ W}$.

e. $P_g = E_g I_g \cos\theta_g$, where $\theta_g$ is the angle between $\mathbf{E}_g$ and $\mathbf{I}_g$. $\theta_g = 30°$. Thus, $P_g = (120)(1.2)\cos 30° = 124.7\text{ W}$, which agrees with (d) as it must since the transformer is lossless.

**IN-PROCESS LEARNING CHECK 1**

1. A transformer has a turns ratio of 1:8. Is it step-up or step-down? If $E_p$ = 25 V, what is $E_s$?
2. For the transformers of Figure 24–7, if $a = 0.2$ and $\mathbf{E}_s = 600\text{ V}\angle -30°$, what is $\mathbf{E}_p$ for each case?
3. For each of the transformers of Figure 24–16, sketch the secondary voltage, showing both phase and amplitude.
4. For Figure 24–17, determine the position of the missing dot.

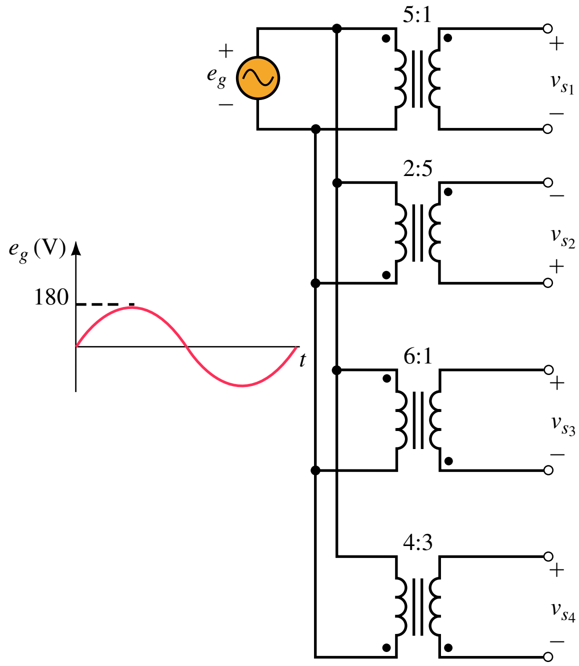



FIGURE 24–16

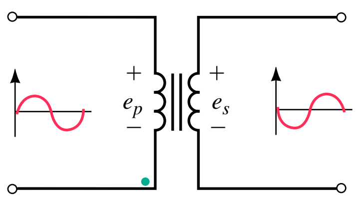
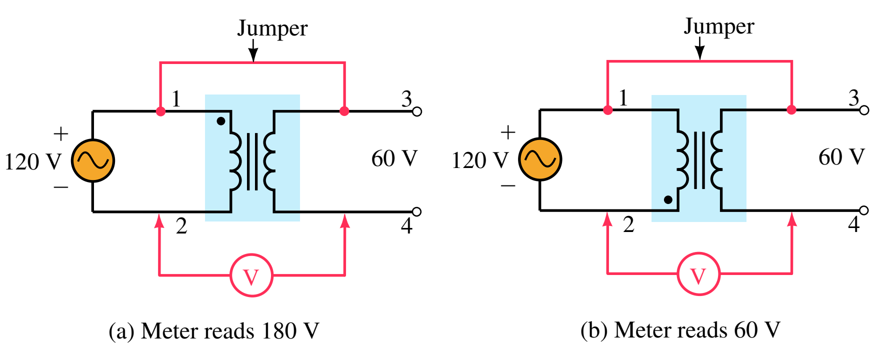



FIGURE 24–17

5. Figure 24–18 shows another way to determine dotted terminals. First, arbitrarily mark one of the primary terminals with a dot. Next, connect a jumper wire and voltmeter as indicated. From the voltmeter readings, you can determine which of the secondary terminals should be dotted. For the two cases indicated, where should the secondary dot be? (*Hint:* Use KVL.)




EWB FIGURE 24–18 Each transformer is rated 120V/60V.

*(Answers are at the end of the chapter.)*

## 24.3 Reflected Impedance

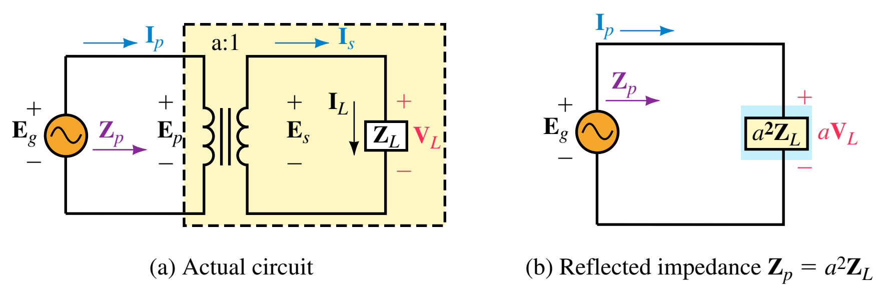
A load impedance $\mathbf{Z}_L$ connected directly to a source is seen by the source as $\mathbf{Z}_L$. However, if a transformer is connected between the source and load as in Figure 24–19(a), the impedance seen by the source is now quite different. Let this equivalent impedance be denoted as $\mathbf{Z}_p$. Then, $\mathbf{I}_p = \mathbf{E}_g/\mathbf{Z}_p$. Rearranging yields $\mathbf{Z}_p = \mathbf{E}_g/\mathbf{I}_p$. But $\mathbf{E}_g = \mathbf{E}_p$, $\mathbf{E}_p = a\mathbf{E}_s$, and $\mathbf{I}_p = \mathbf{I}_s/a$. Thus,

$$\mathbf{Z}_p = \frac{\mathbf{E}_p}{\mathbf{I}_p} = \frac{a\mathbf{E}_s}{\left(\dfrac{\mathbf{I}_s}{a}\right)} = a^2\frac{\mathbf{E}_s}{\mathbf{I}_s} = a^2\frac{\mathbf{V}_L}{\mathbf{I}_L}$$

However $\mathbf{V}_L/\mathbf{I}_L = \mathbf{Z}_L$. Thus,

$$\mathbf{Z}_p = a^2\mathbf{Z}_L \tag{24–11}$$

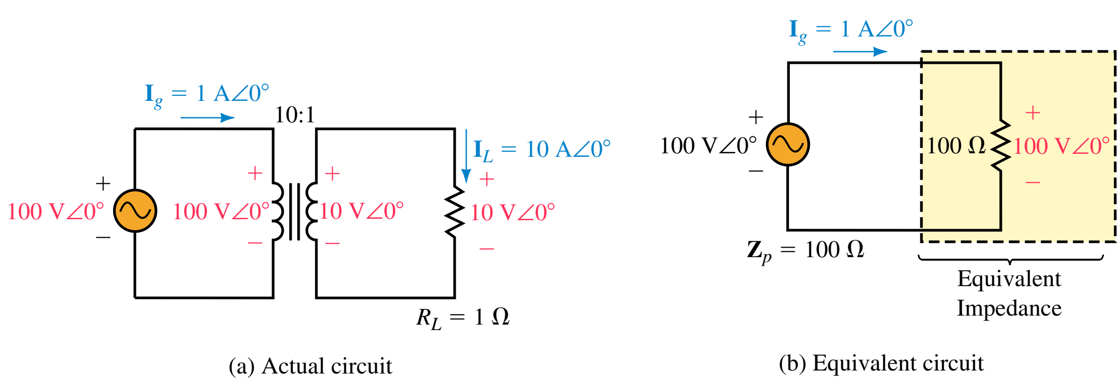
This means that $\mathbf{Z}_L$ now looks to the source like the transformer's turns ratio squared times the load impedance. The term $a^2\mathbf{Z}_L$ is referred to as the load's **reflected impedance.** Note that it retains the load's characteristics, that is, a capacitive load still looks capacitive, an inductive load still looks inductive, and so on.




(a) Actual circuit (b) Reflected impedance $\mathbf{Z}_p = a^2\mathbf{Z}_L$

**FIGURE 24–19** Concept of reflected impedance. From the primary terminals, $\mathbf{Z}_L$ looks like an impedance of $a^2\mathbf{Z}_L$ with voltage $a\mathbf{V}_L$ across it and current $\mathbf{I}_L/a$ through it.

Equation 24–11 shows that a transformer can make a load look larger or smaller, depending on its turns ratio. To illustrate, consider Figure 24–20. If a 1-Ω resistor were connected directly to the source, it would look like a 1-Ω



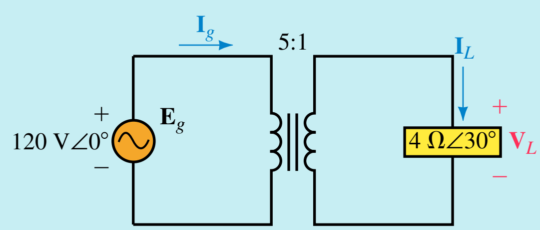

(a) Actual circuit (b) Equivalent circuit

EWB **FIGURE 24–20** The equivalent impedance seen by the source is 100 Ω.

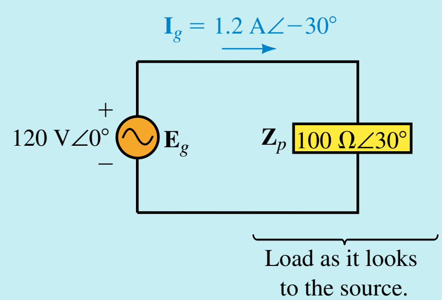
resistor and the generator current would be 100A∠0°. However, when connected to a 10:1 transformer, it looks like a $(10)^2(1\ \Omega) = 100\text{-}\Omega$ resistor and the generator current is only 1 A∠0°.

The concept of reflected impedance is useful in a number of ways. It permits us to match loads to sources (such as amplifiers) as well as provides a better way to solve complex transformer problems.

**EXAMPLE 24–5** Use the reflected impedance idea to solve for primary and secondary currents and the load voltage for the circuit of Figure 24–21(a). (Note: This is the same circuit we solved earlier in Example 24–4. While there is no advantage to the reflected impedance idea over our previous solution for a problem as simple as this, the advantages are considerable for complex problems.)




(a)




(b)

**FIGURE 24–21**

**Solution**

$$\mathbf{Z}_p = a^2\mathbf{Z}_L = (5)^2(4\ \Omega\angle 30°) = 100\ \Omega\angle 30°.$$

The equivalent circuit is shown in (b).

$$\mathbf{I}_g = \mathbf{E}_g/\mathbf{Z}_p = (120\ \text{V}\angle 0°)\ /\ (100\ \Omega\angle 30°) = 1.2\ \text{A}\angle -30°$$
$$\mathbf{I}_L = a\mathbf{I}_p = a\mathbf{I}_g = 5(1.2\ \text{A}\angle -30°) = 6\ \text{A}\angle -30$$
$$\mathbf{V}_L = \mathbf{I}_L\mathbf{Z}_L = (6\ \text{A}\angle -30°)(4\ \Omega\angle 30°) = 24\ \text{V}\angle 0°$$

The answers are the same as in Example 24–4.

## 24.4 Transformer Ratings

Transformers are rated in terms of voltage and apparent power (for reasons discussed in Chapter 17). Rated current can be determined from these ratings. Thus a transformer rated 2400/120 volt, 48 kVA, has a current rating of 48 000 VA/2400 V = 20 A on its 2400-V side and 48 000 VA/120 V = 400 A on its 120-V side (Figure 24–22). This transformer can handle a 48-kVA load, regardless of power factor.

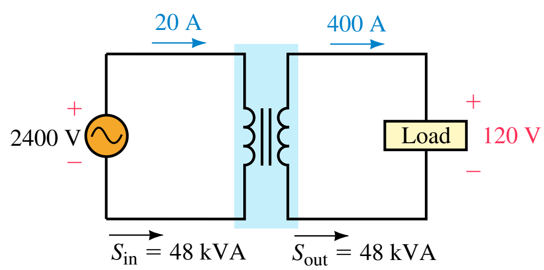



**FIGURE 24–22** Transformers are rated by the amount of apparent power and the voltages that they are designed to handle.

## 24.5 Transformer Applications

### Power Supply Transformers

On electronic equipment, **power supply transformers** are used to convert the incoming 120 Vac to the voltage levels required for internal circuit operation. A variety of commercial transformers are made for this purpose. The transformer of Figure 24–23, for example, has a multi-tapped secondary winding, each tap providing a different output voltage. It is intended for laboratory supplies, test equipment, or experimental power supplies.

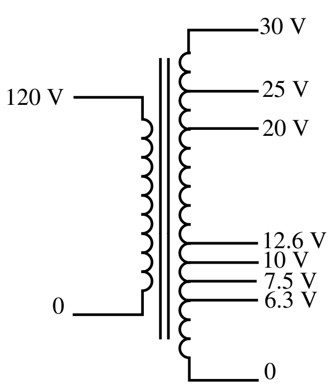



**FIGURE 24–23** A multi-tapped power supply transformer. The secondary is tapped at various voltages.

Figure 24–24 illustrates a typical use of a power supply transformer. First, the incoming line voltage is stepped down, then a rectifier circuit (a circuit that uses diodes to convert ac to dc using a process called rectification) converts the ac to pulsating dc, a filter smooths it, and finally, a voltage regulator (an electronic device used to maintain a constant output voltage) regulates it to the required dc value.

### Transformers in Power Systems

Transformers are one of the key elements that have made commercial ac power systems possible. Transformers are used at generating stations to raise voltage for long-distance transmission. This lowers the transmitted current and hence the $I^2R$ power losses in the transmission line. At the user end, transformers reduce the voltage to a safe level for everyday use. A typical residential connection is shown in Figure 24–25. The taps on the primary permit the electric utility company to compensate for line voltage drops. Transformers located far from substations, for example, have lower input voltages (by a few percent) than those close to substations due to voltage drops on distribution lines. Taps permit the turns ratio to be changed to com-

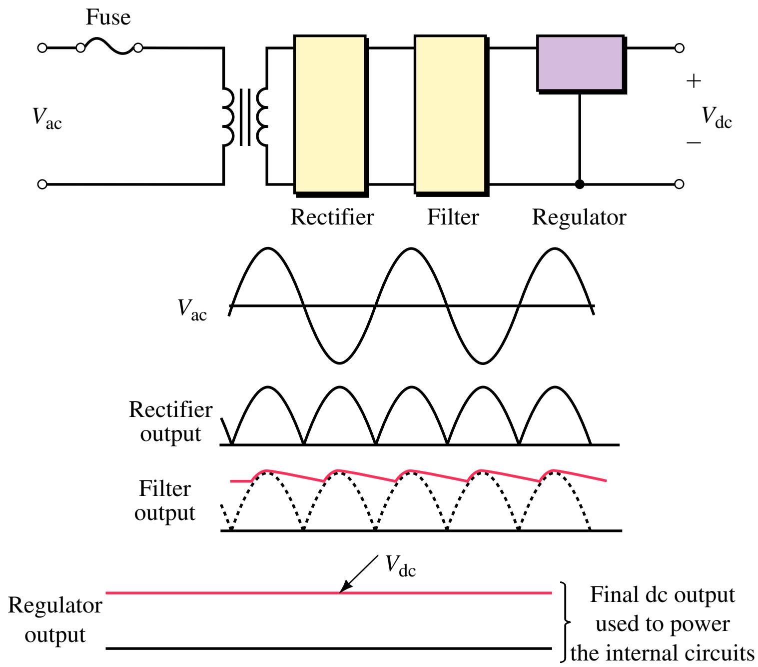



**FIGURE 24–24** A transformer used in a power supply application.

pensate. Note also the split secondary. It permits 120-V and 240-V loads to be supplied from the same transformer.

The transformer of Figure 24–25 is a single-phase unit (since residential customers require only single phase). By connecting its primary from line to neutral (or line to line), the required single-phase input is obtained from a three-phase line.

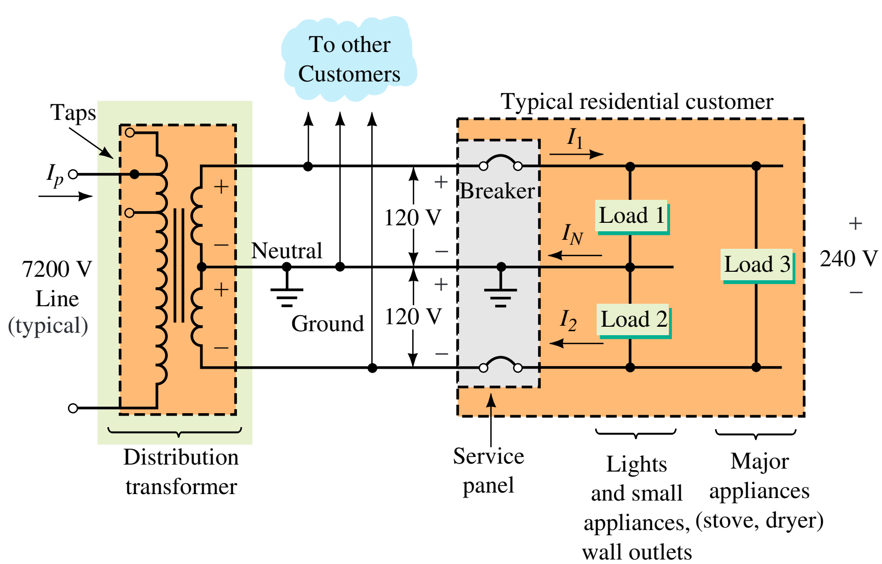
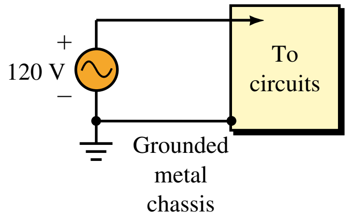
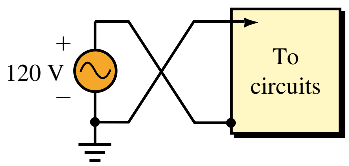



**FIGURE 24–25** Typical distribution transformer supplying residential loads.




(a) Chassis is safe




(b) Chassis is at 120 V

**FIGURE 24–26** If the connections are inadvertently reversed as in (b), you will get a shock if you are grounded and you touch the chassis.

## Isolation Applications

Transformers are sometimes used to isolate equipment for safety or other reasons. If a piece of equipment has its frame or chassis connected to the grounded neutral of Figure 24–25, for example, the connection is perfectly safe as long as the connection is not changed. If, however, connections are inadvertently reversed as in Figure 24–26(b) (due to faulty installation for example), a dangerous situation results. A transformer used as in Figure 24–27 eliminates this danger by ensuring that the chassis is never directly connected to the "hot" wire. Isolation transformers are made for this purpose.

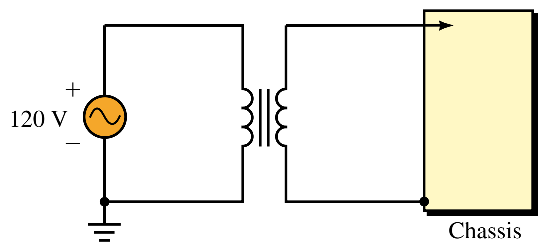



**FIGURE 24–27** Using a transformer for isolation.

## Impedance Matching

As you learned earlier, load impedance $\mathbf{Z}_L$, when viewed through a transformer, has a reflected value of $\mathbf{Z}_p = a^2\mathbf{Z}_L$. This means that a transformer can be used to raise or lower the apparent impedance of a load by choice of turns ratio. This is referred to as **impedance matching.** Impedance matching is sometimes used to match loads to amplifiers to achieve maximum power transfer. If the load and source are not matched, a transformer can be inserted between them as illustrated next.

**EXAMPLE 24–6** Figure 24–28(a) shows the schematic of a multi-tap sound distribution transformer with various taps that permit the matching of speakers to amplifiers. Over their design range, speakers are basically resistive. If the speaker of Figure 24–29(a) has a resistance of 4 Ω, what transformer ratio should be chosen? What is the power to the speaker?

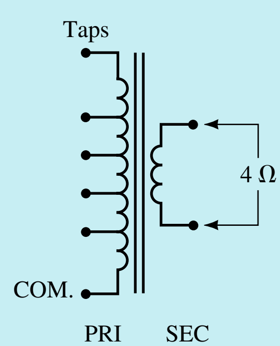
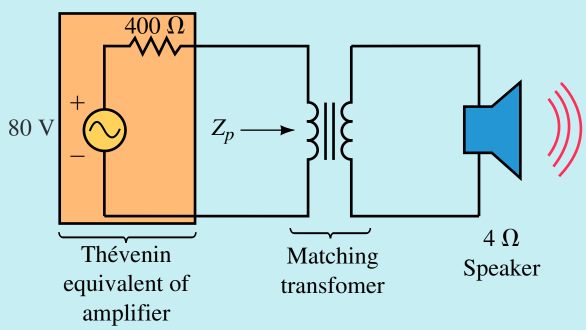



**FIGURE 24–28** A tapped sound distribution transformer.

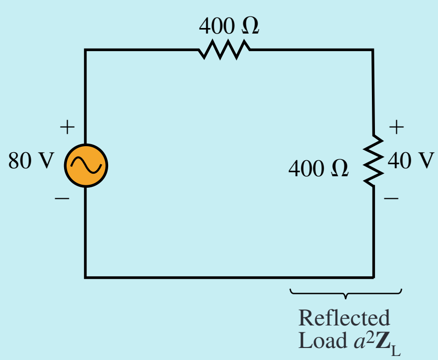
**Solution** Make the reflected resistance of the speaker equal to the internal (Thévenin) resistance of the amplifier. Thus, $Z_p = 400\ \Omega = a^2 Z_L = a^2(4\ \Omega)$. Solving for $a$ yields

$$a = \sqrt{\frac{Z_p}{Z_L}} = \sqrt{\frac{400\ \Omega}{4\ \Omega}} = \sqrt{100} = 10$$

Now consider power. Since $Z_p = 400\ \Omega$, Figure 24–29(b), half the source voltage appears across it. Thus power to $Z_p$ is $(40\text{ V})^2/(400\ \Omega) = 4$ W. Since the transformer is considered lossless, all power is transferred to the speaker. Thus, $P_{\text{speaker}} = 4$ W.




(a) Circuit




(b) Equivalent

**FIGURE 24–29** Matching the 4-Ω speaker for maximum power transfer.

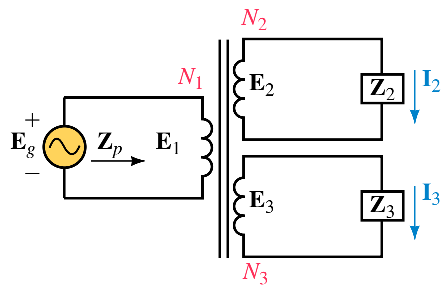
**PRACTICE PROBLEMS 3**

Determine the power to the speaker of Figure 24–29 if the transformer is not present (i.e., the speaker is directly connected to the amplifier). Compare to Example 24–6.

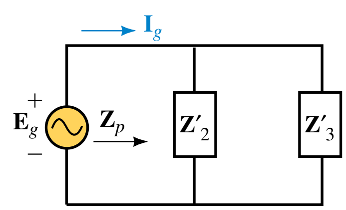
*Answer:* 0.157 W (dramatically lower)

## Transformers with Multiple Secondaries

For a transformer with multiple secondaries (Figure 24–30), each secondary voltage is governed by the appropriate turns ratio, that is, $\mathbf{E}_1/\mathbf{E}_2 = N_1/N_2$ and $\mathbf{E}_1/\mathbf{E}_3 = N_1/N_3$. Loads are reflected in parallel. That is, $\mathbf{Z}'_2 = a_2^{\,2}\mathbf{Z}_2$ and $\mathbf{Z}'_3 = a_3^{\,2}\mathbf{Z}_3$ appear in parallel in the equivalent circuit, (b).




(a) $a_2 = N_1/N_2$
$a_3 = N_1/N_3$




(b) $\mathbf{Z}'_2 = a_2^2\,\mathbf{Z}_2$
$\mathbf{Z}'_3 = a_3^2\,\mathbf{Z}_3$

**FIGURE 24–30** Loads are reflected in parallel.

**EXAMPLE 24–7** For the circuit of Figure 24–31(a),

a. determine the equivalent circuit,
b. determine the generator current,
c. show that apparent power in equals apparent power out.

**Solution**

a. See Figure 24–31(b).

b. $\mathbf{I}_g = \dfrac{\mathbf{E}_g}{\mathbf{Z}'_2} + \dfrac{\mathbf{E}_g}{\mathbf{Z}'_3} = \dfrac{100\angle 0^\circ}{10} + \dfrac{100\angle 0^\circ}{-j10} = 10 + j10 = 14.14\text{ A}\angle 45^\circ$

c. Input: $S_{in} = E_g I_g = (100\text{ V})(14.14\text{ A}) = 1414\text{ VA}$
Output: From Figure 24–31(b), $P_{out} = (100\text{ V})^2/(10\ \Omega) = 1000$ W and $Q_{out} = (100\text{ V})^2/(10\ \Omega) = 1000$ VAR. Thus $S_{out} = \sqrt{P_{out}^{\,2} + Q_{out}^{\,2}} = 1414$ VA which is the same as $S_{in}$.

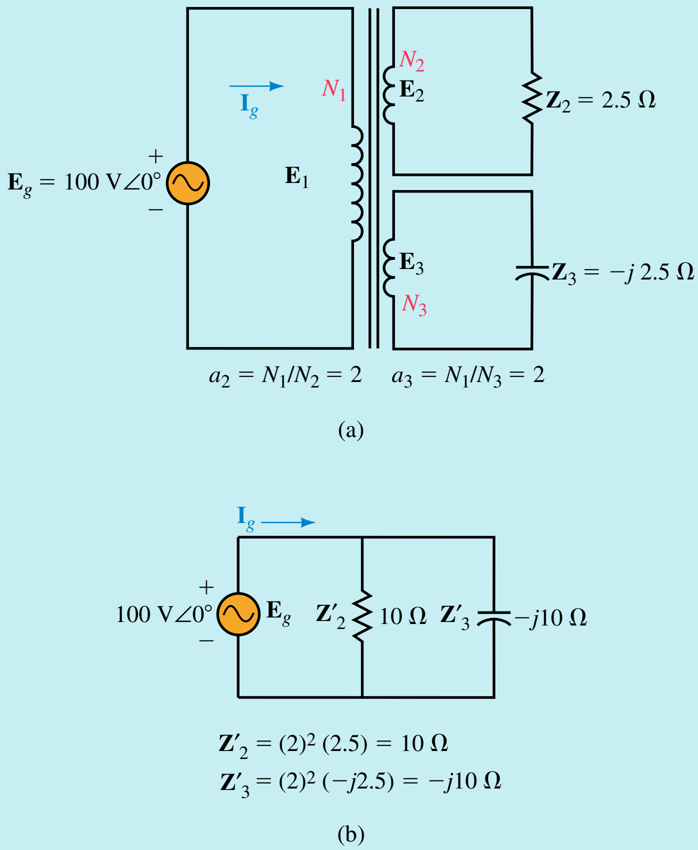



**FIGURE 24–31**

## Autotransformers

An important variation of the transformer is the **autotransformer** (Figure 24–32). Autotransformers are unusual in that their primary circuit is not electrically isolated from their secondary. However, they are smaller and cheaper than conventional transformers for the same load kVA since only part of the load power is transferred inductively. (The remainder is transferred by direct conduction.) Figure 24–32 shows several variations. The transformer of (c) is variable by means of a slider, typically, from 0% to 110%.

For analysis, an autotransformer may be viewed as a standard two-winding transformer reconnected as in Figure 24–33(b). Voltage and current relationships between windings hold as they do for the standard connection. Thus, if you apply rated voltage to the primary winding, you will obtain rated voltage across the secondary winding. Finally, since we are assuming an ideal transformer, apparent power out equals apparent power in.

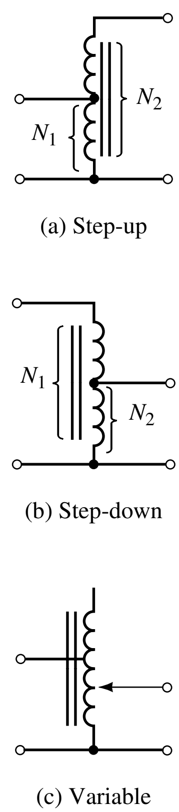
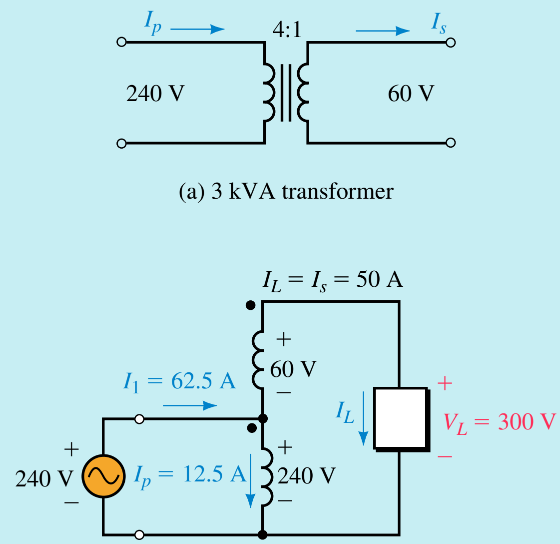



**FIGURE 24–32** Autotransformers.

**EXAMPLE 24–8** A 240/60-V, 3-kVA transformer [Figure 24–33(a)] is reconnected as an autotransformer to supply 300 volts to a load from a 240-V supply [Figure 24–33(b)].

a. Determine the rated primary and secondary currents.
b. Determine the maximum apparent power that can be delivered to the load.
c. Determine the supply current.




(a) 3 kVA transformer

(b) Used as an autotransformer

EWB **FIGURE 24–33**

**Solution**

a. Rated current = rated kVA/rated voltage. Thus,

$$I_p = 3 \text{ kVA}/240 \text{ V} = 12.5 \text{ A} \quad \text{and} \quad I_s = 3 \text{ kVA}/60 \text{ V} = 50 \text{ A}$$

b. Since the 60-V winding is rated at 50 A, the transformer can deliver 50 A to the load [Figure 24–33(b)]. The load voltage is 300 V. Thus,

$$S_L = V_L I_L = (300 \text{ V})(50 \text{ A}) = 15 \text{ kVA}$$

This is five times the rated kVA of the transformer.

c. Apparent power in = apparent power out:

$$240 I_1 = 15 \text{ kVA}.$$

Thus, $I_1 = 15$ kVA/240 V = 62.5 A. Current directions are as shown.

As a check, KCL at the junction of the two coils yields

$$I_1 = I_p + I_L = 12.5 + 50 = 62.5 \text{ A}$$

IN-PROCESS LEARNING CHECK 2

1. For each of the circuits of Figure 24–34, determine the required answer.

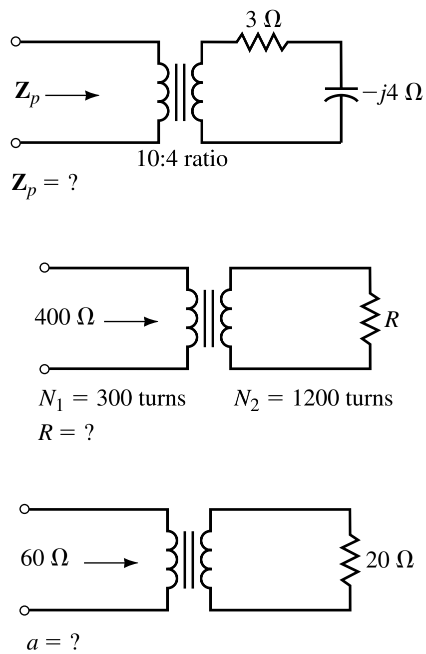



**FIGURE 24–34**

2. For Figure 24–35, if $a = 5$ and $\mathbf{I}_p = 5\ \text{A}\angle{-60°}$, what is $\mathbf{E}_g$?

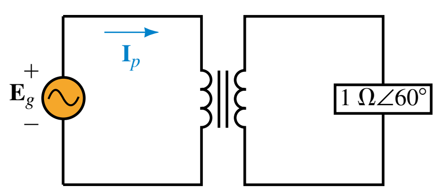



**FIGURE 24–35**

3. For Figure 24–35, if $\mathbf{I}_p = 30\ \text{mA}\angle{-40°}$ and $\mathbf{E}_g = 240\ \text{V}\angle 20°$, what is $a$?
4. a. How many amps can a 24-kVA, 7200/120-V transformer supply to a 120-V unity power factor load? To a 0.75 power factor load?
   b. How many watts can it supply to each load?
5. For the transformer of Figure 24–36, between position 2 and 0 there are 2000 turns. Between taps 1 and 2 there are 200 turns, and between taps 1 and 3 there are 300 turns. What will the output voltage be when the supply is connected to tap 1? To tap 2? To tap 3?
6. For the circuit of Figure 24–37, what is the power delivered to a 4-ohm speaker? What is the power delivered if an 8-ohm speaker is used instead? Why is the power to the 4-ohm speaker larger?

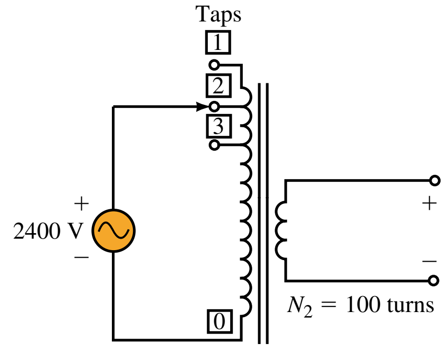



FIGURE 24–36

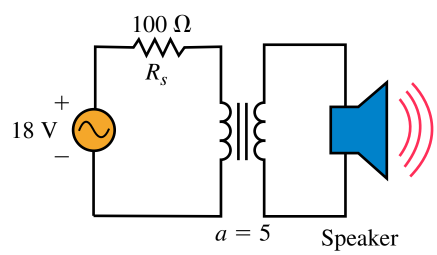



FIGURE 24–37

7. The autotransformer of Figure 24–38 has a 58% tap. The apparent power of the load is 7.2 kVA. Calculate the following:
   a. The load voltage and current.
   b. The source current.
   c. The current in each winding and its direction.

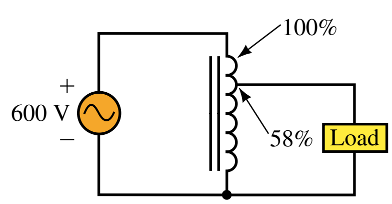



FIGURE 24–38

*(Answers are at the end of the chapter.)*

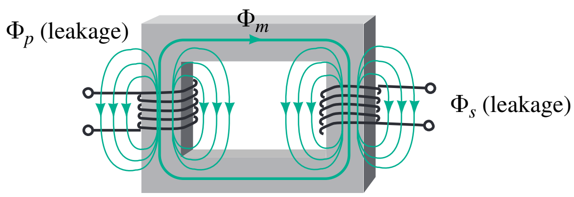
## 24.6 Practical Iron-Core Transformers

In Section 24.2, we idealized the transformer. We now add back the effects that we ignored.

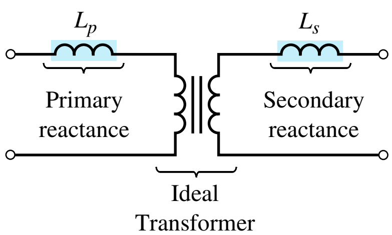
### Leakage Flux

While most flux is confined to the core, a small amount (called **leakage flux**) passes outside the core and through air at each winding as in Figure 24–39(a). The effect of this leakage can be modeled by inductances $L_p$ and $L_s$

as indicated in (b). The remaining flux, the **mutual flux** $\Phi_m$, links both windings and is accounted for by the ideal transformer as previously.




(a) Flux leakage




(b) Equivalent

**FIGURE 24–39** Leakage flux can be modeled by small inductances.

## Winding Resistance

The effect of coil resistance can be approximated by resistances $R_p$ and $R_s$ as shown in Figure 24–40. The effect of these resistances is to cause a slight power loss and hence a reduction in efficiency as well as a small voltage drop. (The power loss associated with coil resistance is called **copper loss** and varies as the square of the load current.)

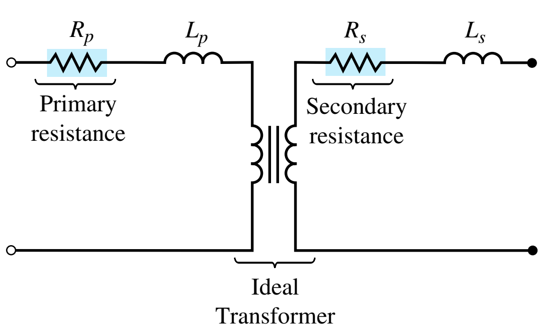



**FIGURE 24–40** Adding the winding resistance to the model.

## Core Loss

Losses occur in the core because of **eddy currents** and **hysteresis.** First, consider eddy currents. Since iron is a conductor, voltage is induced in the core as flux varies. This voltage creates currents that circulate as "eddies" within the core itself. One way to reduce these currents is to break their path of circulation by constructing the core from thin laminations of steel rather than using a solid block of iron. Laminations are insulated from each other by a coat of ceramic, varnish, or similar insulating material. (Although this does not eliminate eddy currents, it greatly reduces them.) Power and audio transformers are built this way (Figure 24–4). Another way to reduce eddy currents is to use powdered iron held together by an insulating binder. Ferrite cores are made like this.

Now consider hysteresis. Because flux constantly reverses, magnetic domains in the core steel constantly reverse as well. This takes energy. However, this energy is minimized by using special grain-oriented transformer steel.

The sum of hysteresis and eddy current loss is called **core loss** or **iron loss.** In a well-designed transformer, it is small, typically one to two percent of the transformer rating. The effect of core loss can be modeled as a resistor, $R_c$ in Figure 24–41. Core losses vary approximately as the square of applied voltage. As long as voltage is constant (which it normally is), core losses remain constant.

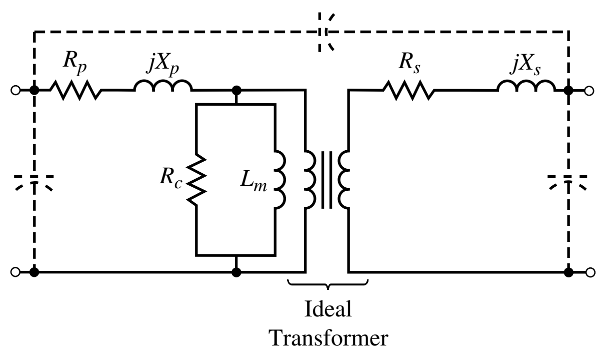



**FIGURE 24–41** Final iron-core transformer equivalent circuit.

## Other Effects

We have also neglected **magnetizing current.** In a real transformer, however, some current is required to magnetize the core. To account for this, add path $L_m$ as shown in Figure 24–41. Stray capacitances also exist between various parts of the transformer. They can be approximated by lumped capacitances as indicated.

## The Full Equivalent

Figure 24–41 shows the final equivalent with all effects incorporated. How good is it? Calculations based on this model agree exceptionally well with measurements made on real transformers. However, the circuit is complex and awkward to use. In practice, therefore, only those elements of the model that affect a given application are actually retained. For example, at power frequencies, the effect of capacitance is negligible and the capacitors are omitted for power frequency analysis.

## Voltage Regulation

Because of the internal impedance of a transformer (Figure 24–41) voltage drops occur inside the transformer. Therefore, the output voltage of a transformer under load is different from its voltage under no load. This change in voltage (expressed as a percentage of full-load voltage) is termed **regulation.** For regulation analysis, parallel branches $R_c$ and $L_m$ and stray capacitance have negligible effect and can be neglected. This yields the simplified circuit of Figure 24–42(a). Even greater simplification is achieved by reflecting secondary impedances into the primary. This yields the circuit of (b). The reflected load voltage is $a\mathbf{V}_L$ and the reflected load current is $\mathbf{I}_L/a$. Regulation calculations are performed using this simplified circuit.

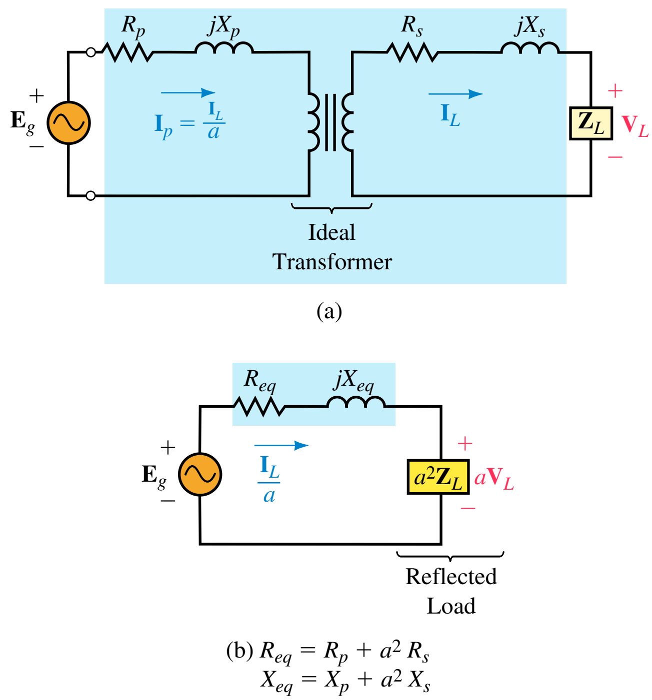


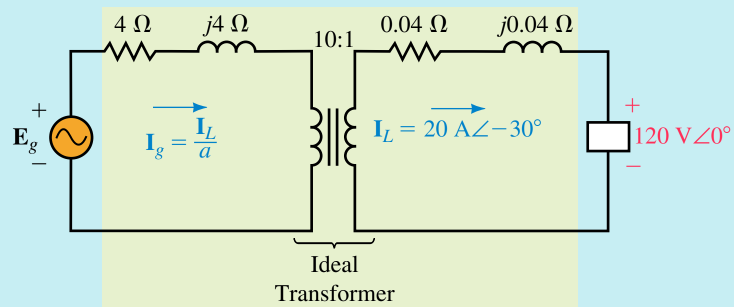

**FIGURE 24–42** Simplifying the equivalent.

**EXAMPLE 24–9** A 10 : 1 transformer has primary and secondary resistance and reactance of 4 Ω + $j$4 Ω and 0.04 Ω + $j$0.04 Ω respectively as in Figure 24–43.

a. Determine its equivalent circuit.

b. If $\mathbf{V}_L$ = 120 V∠0° and $\mathbf{I}_L$ = 20 A∠−30°, what is the supply voltage, $\mathbf{E}_g$?

c. Determine the regulation.




EWB **FIGURE 24–43**

## Solution

a. $R_{eq} = R_p + a^2R_s = 4\ \Omega + (10)^2(0.04\ \Omega) = 8\ \Omega$

$X_{eq} = X_p + a^2X_s = 4\ \Omega + (10)^2(0.04\ \Omega) = 8\ \Omega$

Thus, $\mathbf{Z}_{eq} = 8\ \Omega + j8\ \Omega$ as shown in Figure 24–44.

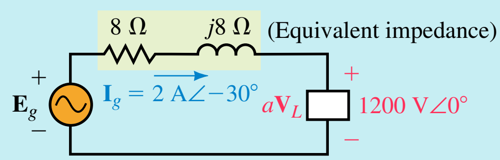



**FIGURE 24–44**

b. $a\mathbf{V}_L = (10)(120\ \text{V}\angle 0°) = 1200\ \text{V}\angle 0°$ and $\mathbf{I}_L/a = (20\ \text{A}\angle -30°)/10 = 2\ \text{A}\angle -30°$. From KVL, $\mathbf{E}_g = (2\ \text{A}\angle -30°)(8\ \Omega + j8) + 1200\ \text{V}\angle 0° = 1222\ \text{V}\angle 0.275°$.

Thus, there is a phase shift of 0.275° across the transformer's internal impedance and a drop of 22 V, requiring that the primary be operated slightly above its rated voltage. (This is normal.)

c. Now consider the no-load condition (Figure 24–45). Let $V_{NL}$ be the no-load voltage. As indicated, $aV_{NL} = 1222$ V. Thus, $V_{NL} = 1222/a = 1222/10 = 122.2$ volts and

$$\text{regulation} = \frac{V_{NL} - V_{FL}}{V_{FL}} \times 100 = \frac{122.2 - 120}{120} \times 100 = 1.83\%$$

Note that only magnitudes are used in determining regulation.

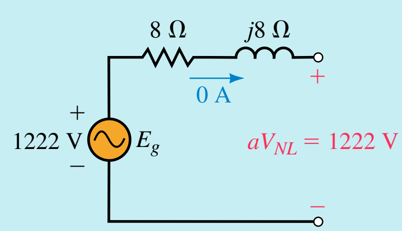



**FIGURE 24–45** No-load equivalent: $aV_{NL} = E_g$.

PRACTICAL NOTES...

1. From Figure 24–45, $a = E_g/V_{NL}$. This means that the turns ratio is the ratio of input voltage to output voltage at no load.
2. The voltage rating of a transformer (such as 1200/120 V) is referred to as its *nominal rating*. The ratio of nominal voltages is the same as the turns ratio. Thus, for a nonloaded transformer, if nominal voltage is applied to the primary, nominal voltage will appear at the secondary.
3. Transformers are normally operated close to their nominal voltages. However, depending on operating conditions, they may be a few percent above or below rated voltage at any given time.

PRACTICE PROBLEMS 4

A transformer used in an electronic power supply has a nominal rating of 120/12 volts and is connected to a 120-Vac source. Its equivalent impedance as seen from the primary is $10\ \Omega + j10\ \Omega$. What is the magnitude of the load voltage if the load is 5 ohms resistive? Determine the regulation.

*Answer:* 11.8 V; 2.04%

### Transformer Efficiency

Efficiency is the ratio of output power to input power.

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} \times 100\% \tag{24–12}$$

But $P_{\text{in}} = P_{\text{out}} + P_{\text{loss}}$. For a transformer, losses are due to $I^2R$ losses in the windings (called copper losses) and losses in the core (called core losses). Thus,

$$\eta = \frac{P_{\text{out}}}{P_{\text{out}} + P_{\text{loss}}} \times 100\% = \frac{P_{\text{out}}}{P_{\text{out}} + P_{\text{copper}} + P_{\text{core}}} \times 100\% \tag{24–13}$$

Large power transformers are exceptionally efficient, of the order of 98 to 99 percent. The efficiencies of smaller transformers are around 95 percent or better.

## 24.7 Transformer Tests

Losses may be determined experimentally using the **short-circuit test** and the **open-circuit test.** (These tests are used mainly with power transformers.) They provide the data needed to determine a transformer's equivalent circuit and to compute its efficiency.

### The Short-Circuit Test

Figure 24–46 shows the test setup for the short-circuit test. Starting at 0 V, gradually increase $E_g$ until the ammeter indicates the rated current. (This occurs at about 5 percent rated input voltage.) Since core losses are proportional to the square of voltage, at 5 percent rated voltage, core losses are negligible. The losses that you measure are therefore only copper losses.

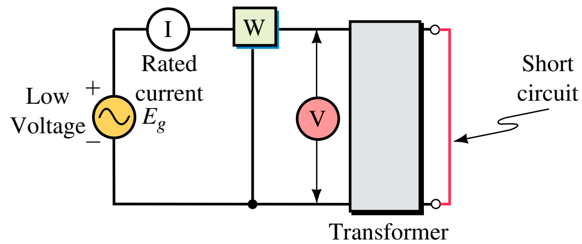



**FIGURE 24–46** Short-circuit test.

**EXAMPLE 24–10** Measurements on the high side of a 240/120-volt, 4.8-kVA transformer yield $E_g = 11.5$ V and $W = 172$ W at the rated current of $I = 4.8$ kVA/240 = 20 A. Determine $\mathbf{Z}_{eq}$.

**Solution** See Figure 24–47. Since $Z_L = 0$, the only impedance in the circuit is $\mathbf{Z}_{eq}$. Thus, $Z_{eq} = E_g/I = 11.5\text{ V}/20\text{ A} = 0.575\ \Omega$. Further, $R_{eq} = W/I^2 = 172\text{ W}/(20\text{ A})^2 = 0.43\ \Omega$. Therefore,

$$X_{eq} = \sqrt{Z_{eq}^2 - R_{eq}^2} = \sqrt{(0.575)^2 - (0.43)^2} = 0.382\ \Omega$$

and $\mathbf{Z}_{eq} = R_{eq} + jX_{eq} = 0.43\ \Omega + j0.382\ \Omega$ as shown in (b).

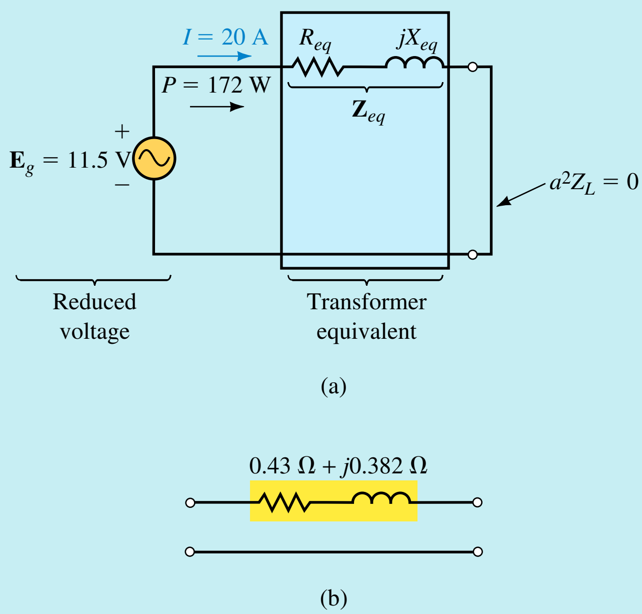



**FIGURE 24–47** Determining the equivalent circuit.

## The Open-Circuit Test

The setup for the open-circuit test is shown in Figure 24–48. Apply the full rated voltage. Since the load current is zero, only exciting current results. Since the exciting current is small, power loss in the winding resistance is negligible, and the power that you measure is just core loss.

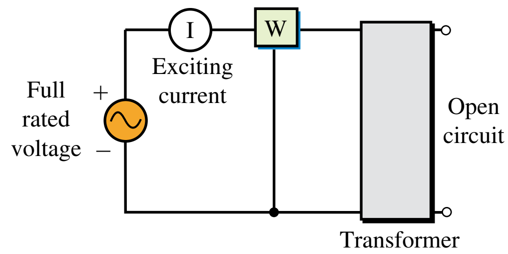



**FIGURE 24–48** Open-circuit test.

**EXAMPLE 24–11** An open-circuit test on the transformer of Example 24–10 yields a core loss of 106 W. Determine this transformer's efficiency when supplying the full, rated VA to a load at unity power factor.

**Solution** Since the transformer is supplying the rated VA, its current is the full, rated current. From the short-circuit test, the copper loss at full rated current is 172 W. Thus,

$$\text{copper loss} = 172\text{ W}$$
$$\text{core loss} = 106\text{ W (Measured above)}$$
$$\text{output} = 4800\text{ W (Rated)}$$
$$\text{input} = \text{output} + \text{losses} = 5078\text{ W}$$

Thus

$$\eta = P_{\text{out}}/P_{\text{in}} = (4800\text{ W}/5078\text{ W}) \times 100 = 94.5\%$$

Copper loss varies as the square of load current. Thus, at half the rated current, the copper loss is $(\frac{1}{2})^2 = \frac{1}{4}$ of its value at full rated current. Core loss remains constant since applied voltage remains constant.

**PRACTICE PROBLEMS 5**

For the transformer of Example 24–11, determine power input and efficiency at half the rated VA output, unity power factor.

*Answer:* 2549 W; 94.2%

## 24.8 Voltage and Frequency Effects

Iron-core transformer characteristics vary with frequency and voltage. To determine why, we start with Faraday's law, $e = Nd\Phi/dt$. Specializing this to the sinusoidal ac case, it can be shown that

$$E_p = 4.44fN_p\Phi_m \qquad \textbf{(24–14)}$$

where $\Phi_m$ is the mutual core flux.

### Effect of Voltage

First, assume constant frequency. Since $\Phi_m = E_p/4.44fN_p$, the core flux is proportional to the applied voltage. Thus, if the applied voltage is increased, the core flux increases. Since magnetizing current is required to produce this flux, magnetizing current must increase as well. An examination of the *B-H* curves of Chapter 12 shows that magnetizing current increases dramatically when flux density rises above the knee of the curve; in fact, the effect is so pronounced that the magnetizing current is no longer negligible, but may exceed the load current! For this reason, power transformers should be operated only at or near their rated voltage.

### Effect of Frequency

Audio transformers must operate over a range of frequencies. Consider again $\Phi_m = E_p/4.44fN_p$. As this indicates, decreasing the frequency increases the core flux and hence the magnetizing current. At low frequencies, this larger current increases internal voltage drops and hence decreases the output voltage as indicated in Figure 24–49. Now consider increasing the frequency. As frequency increases, leakage inductance and shunt capacitance cause the voltage to fall off. To compensate for this, audio transformers are sometimes designed so that their internal capacitances resonate with their inductances to extend the operating range. This is what causes the peaking at the high-frequency end of the curve.

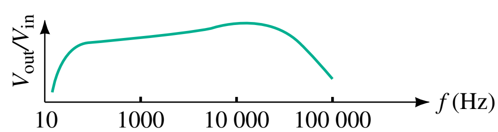



**FIGURE 24–49** Frequency response curve, audio transformer.

IN-PROCESS LEARNING CHECK 3

A transformer with a nominal rating of 240/120 V, 60 Hz, has its load on the 120-V side. Suppose $R_p = 0.4\ \Omega$, $L_p = 1.061$ mH, $R_s = 0.1\ \Omega$, and $L_s = 0.2653$ mH.

a. Determine its equivalent circuit as per Figure 24–42(b).

b. If $\mathbf{E}_g = 240\ \text{V}\angle 0°$ and $\mathbf{Z}_L = 3 + j4\ \Omega$, what is $\mathbf{V}_L$?

c. Compute the regulation.

*(Answers are at the end of the chapter.)*

## 24.9 Loosely Coupled Circuits

We now turn our attention to coupled circuits that do not have iron cores. For such circuits, only a portion of the flux produced by one coil links another and the coils are said to be **loosely coupled.** Loosely coupled circuits cannot be characterized by turns ratios; rather, as you will see, they are characterized by self- and mutual inductances. Air-core transformers, ferrite-core transformers, and general inductive circuit coupling fall into this category. In this section, we develop the main ideas.

## Voltages in Air-Core Coils

To begin, consider the isolated (noncoupled) coil of Figure 24–50. As shown in Chapter 13, the voltage across this coil is given by $v_L = Ldi/dt$, where $i$ is the current through the coil and $L$ is its inductance. Note carefully the polarity of the voltage; the plus sign goes at the tail of the current arrow. Because the coil's voltage is created by its own current, it is called a **self-induced voltage.**

Now consider a pair of coupled coils (Figure 24–51). When coil 1 alone is energized as in (a), it looks just like the isolated coil of Figure 24–50; thus its voltage is

$$v_{11} = L_1 di_1/dt \qquad \text{(self-induced in coil 1)}$$

where $L_1$ is the self-inductance of coil 1 and the subscripts indicate that $v_{11}$ is the voltage across coil 1 due to its own current. Similarly, when coil 2 alone is energized as in (b), its self-induced voltage is

$$v_{22} = L_2 di_2/dt \qquad \text{(self-induced in coil 2)}$$

For both of these self-voltages, note that the plus sign goes at the tail of their respective current arrows.

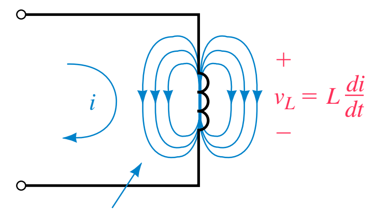
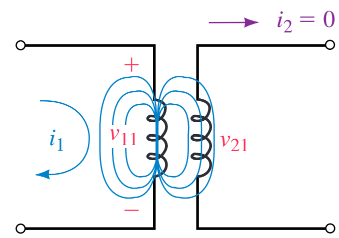
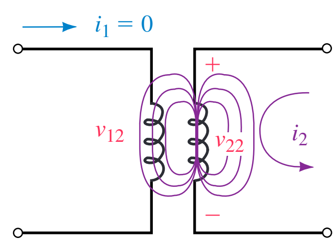



**FIGURE 24–50** Place the plus sign for the self-induced voltage at the tail of the current arrow.




(a) $v_{11}$ is the self-induced voltage in coil 1; $v_{21}$ is the mutual voltage in coil 2




(b) $v_{22}$ is the self-induced voltage in coil 2; $v_{12}$ is the mutual voltage in coil 1

**FIGURE 24–51** Self and mutual voltages.

## Mutual Voltages

Consider again Figure 24–51(a). When coil 1 is energized, some of the flux that it produces links coil 2, inducing voltage $v_{21}$ in coil 2. Since the flux here is due to $i_1$ alone, $v_{21}$ is proportional to the rate of change of $i_1$. Let the constant of proportionality be $M$. Then,

$$v_{21} = M di_1/dt \qquad \text{(mutually induced in coil 2)}$$

$v_{21}$ is the **mutually induced voltage** in coil 2 and $M$ is the **mutual inductance** between the coils. It has units of henries. Similarly, when coil 2 alone is energized as in (b), the voltage induced in coil 1 is

$$v_{12} = M di_2/dt \qquad \text{(mutually induced in coil 1)}$$

When both coils are energized, the voltage of each coil can be found by superposition; *in each coil, the induced voltage is the sum of its self-voltage plus the voltage mutually induced due to the current in the other coil.* The sign of the self term for each coil is straightforward: It is determined by placing a plus sign at the tail of the current arrow for the coil as shown in Figures 24–51(a) and (b). The polarity of the mutual term, however, depends on whether the mutual voltage is additive or subtractive.

## Additive and Subtractive Voltages

Whether self- and mutual voltages add or subtract depends on the direction of currents through the coils relative to their winding directions. This is best described in terms of the dot convention. Consider Figure 24–52(a). Comparing the coils here to Figure 24–12, you can see that their top ends correspond and thus can be marked with dots. Now let currents enter both coils at dotted ends. Using the right-hand rule, you can see that their fluxes add. The total flux linking coil 1 is therefore the *sum* of that produced by $i_1$ and $i_2$;

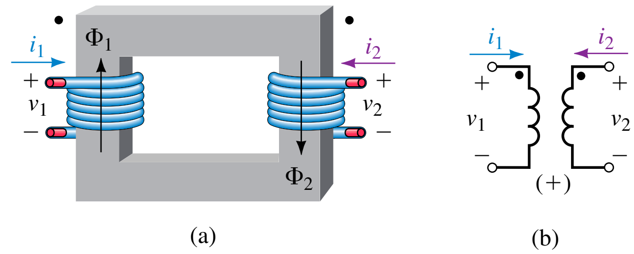



**FIGURE 24–52** When both currents enter dotted terminals, use the + sign for the mutual term in Equation 24–15.

therefore, the voltage across coil 1 is the sum of that produced by $i_1$ and $i_2$. That is,

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \tag{24–15a}$$

Similarly, for coil 2,

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \tag{24–15b}$$

Now consider Figure 24–53. Here, the fluxes oppose and the flux linking each coil is the *difference* between that produced by its own current and that produced by the current of the other coil. Thus, the sign in front of the mutual voltage terms will be negative.

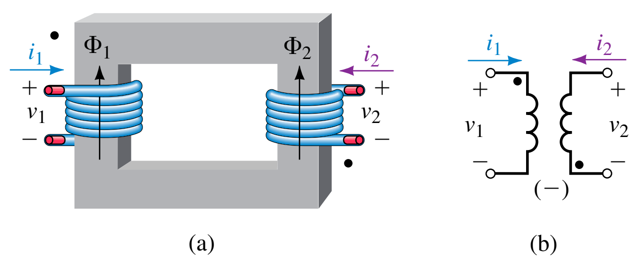



**FIGURE 24–53** When one current enters a dotted terminal and the other enters an undotted terminal, use the − sign for the mutual term in Equation 24–15.

## The Dot Rule

As you can see, the signs of the mutual voltage terms in Equations 24–15 are positive when both currents enter dotted terminals, but negative when one current enters a dotted terminal and the other enters an undotted terminal. Stated another way, *the sign of the mutual voltage is the same as the sign of its self-voltage when both currents enter dotted (or undotted) terminals, but is opposite when one current enters a dotted terminal and the other enters an undotted terminal.* This observation provides us with a procedure for determining voltage polarities in coupled circuits.

1. Assign a direction for currents $i_1$ and $i_2$.
2. Place a plus sign at the tail of the current arrow for each coil to denote the polarity of its self-induced voltage.

3. If both currents enter (or both leave) dotted terminals, make the sign of the mutually induced voltage the same as the sign of the self-induced voltage when you write the equation.

4. If one current enters a dotted terminal and the other leaves, make the sign of the mutually induced voltage opposite to the sign of the self-induced voltage.

**EXAMPLE 24–12** Write equations for $v_1$ and $v_2$ of Figure 24–54(a).

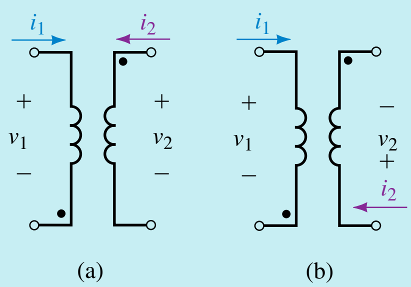



**FIGURE 24–54**

**Solution** Since one current enters an undotted terminal and the other enters a dotted terminal, place a minus sign in front of $M$. Thus,

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$v_2 = -M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

**PRACTICE PROBLEMS 6**

Write equations for $v_1$ and $v_2$ of Figure 24–54(b).

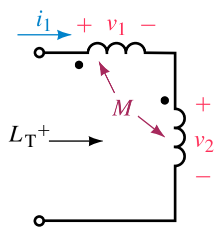
*Answer:* Same as Equation 24–15.

## Coefficient of Coupling

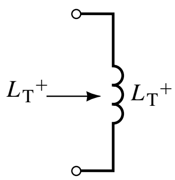
For loosely coupled coils, not all of the flux produced by one coil links the other. To describe the degree of coupling between coils, we introduce a **coefficient of coupling,** $k$. Mathematically, $k$ is defined as the ratio of the flux that links the companion coil to the total flux produced by the energized coil. For iron-core transformers, almost all the flux is confined to the core and links both coils; thus, $k$ is very close to 1. At the other extreme (i.e., isolated coils where no flux linkage occurs), $k = 0$. Thus, $0 \leq k < 1$. Mutual inductance depends on $k$. It can be shown that mutual inductance, self-inductances, and the coefficient of coupling are related by the equation

$$M = k\sqrt{L_1L_2} \tag{24–16}$$

Thus, the larger the coefficient of coupling, the larger the mutual inductance.




(a) $L'_1 = L_1 + M$; $L'_2 = L_2 + M$




(b) $L_T^+ = L_1 + L_2 + 2M$

**FIGURE 24–55** Coils in series with additive mutual coupling.

## Inductors with Mutual Coupling

If a pair of coils are in close proximity, the field of each coil couples the other, resulting in a change in the apparent inductance of each coil. To illustrate, consider Figure 24–55(a), which shows a pair of inductors with self-inductances $L_1$ and $L_2$. If coupling occurs, the effective coil inductances will no longer be $L_1$ and $L_2$. To see why, consider the voltage induced in each winding—it is the sum of the coil's own self-voltage plus the voltage mutually induced from the other coil. Since current is the same for both coils, $v_1 = L_1 di/dt + M di/dt = (L_1 + M)di/dt$, which means that coil 1 has an effective inductance of $L_1' = L_1 + M$. Similarly, $v_2 = (L_2 + M)di/dt$, giving coil 2 an effective inductance of $L_2' = L_2 + M$. The effective inductance of the series combination [Figure 24–55(b)] is then

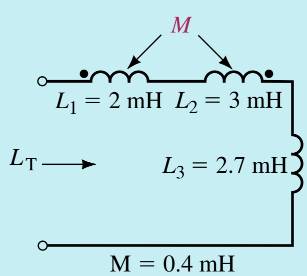
$$L_T^+ = L_1 + L_2 + 2M \quad \text{(henries)} \tag{24–17}$$

If coupling is subtractive as in Figure 24–56, $L_1' = L_1 - M$, $L_2' = L_2 - M$, and

$$L_T^- = L_1 + L_2 - 2M \quad \text{(henries)} \tag{24–18}$$

**EXAMPLE 24–13** Three inductors are connected in series (Figure 24–56). Coils 1 and 2 interact, but coil 3 does not.

**FIGURE 24–56**




a. Determine the effective inductance of each coil.

b. Determine the total inductance of the series connection.

**Solution**

a. $L_1' = L_1 - M = 2\text{ mH} - 0.4\text{ mH} = 1.6\text{ mH}$

$L_2' = L_2 - M = 3\text{ mH} - 0.4\text{ mH} = 2.6\text{ mH}$

$L_1'$ and $L_2'$ are in series with $L_3$. Thus,

b. $L_T = 1.6\text{ mH} + 2.6\text{ mH} + 2.7\text{ mH} = 6.9\text{ mH}$

The same principles apply when more than two coils are coupled. Thus, for the circuit of Figure 24–57, $L_1' = L_1 - M_{12} - M_{31}$, etc.

For two parallel inductors with mutual coupling, the equivalent inductance is

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M} \tag{24–19}$$

If the dots are at the same ends of the coils, use the + sign. For example, if $L_1 = 20$ mH, $L_2 = 5$ mH, and $M = 2$ mH, then $L_{eq} = 4.57$ mH if the dots are both at the same ends of the coils, and $L_{eq} = 3.31$ mH when the dots are at opposite ends.

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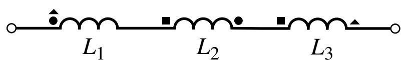
**PRACTICE PROBLEMS 7**

For the circuit of Figure 24–57, different "dot" symbols are used to represent coupling between sets of coils.

a. Determine the effective inductance of each coil.

b. Determine the total inductance of the series connection.




$L_1 = 10$ mH $\quad M_{12} = 2$ mH (Mutual inductance between coils 1 and 2) (●)
$L_2 = 40$ mH $\quad M_{23} = 1$ mH (Mutual inductance between coils 2 and 3) (■)
$L_3 = 20$ mH $\quad M_{31} = 0.6$ mH (Mutual inductance between coils 3 and 1) (▲)

**FIGURE 24–57**

*Answers:*
a. $L_1' = 7.4$ mH; $\quad L_2' = 39$ mH; $\quad L_3' = 20.4$ mH
b. 66.8 mH

---

The effect of unwanted mutual inductance can be minimized by physically separating coils or by orienting their axes at right angles. The latter technique is used where space is limited and coils cannot be spaced widely. While it does not eliminate coupling, it can help minimize its effects.

## 24.10 Magnetically Coupled Circuits with Sinusoidal Excitation

When coupling occurs between various parts of a circuit (whether wanted or not), the foregoing principles apply. However, since it is difficult to continue the analysis in general, we will change over to steady state ac. This will permit us to look for the main ideas. We will use the mesh approach. To use the mesh approach, (1) write mesh equations using KVL, (2) use the dot convention to determine the signs of the induced voltage components, and (3) solve the resulting equations in the usual manner using determinants or a computer program such as MATHCAD.

To specialize to the sinusoidal ac case, convert voltages and currents to phasor form. To do this, recall from Chapter 16 that inductor voltage in phasor form is $\mathbf{V}_L = j\omega L\mathbf{I}$. (This is the phasor equivalent of $v_L = Ldi/dt$, Figure 24–50.) This means that $Ldi/dt$ becomes $j\omega L\mathbf{I}$; in a similar fashion, $Mdi_1/dt \Rightarrow j\omega M\mathbf{I}_1$ and $Mdi_2/dt \Rightarrow j\omega M\mathbf{I}_2$. Thus, in phasor form Equations 24–15 become

$$\mathbf{V}_1 = j\omega L_1\mathbf{I}_1 + j\omega M\mathbf{I}_2$$
$$\mathbf{V}_2 = j\omega M\mathbf{I}_1 + j\omega L_2\mathbf{I}_2$$

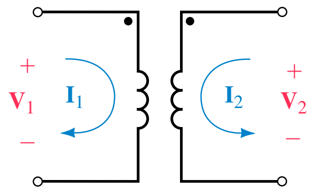



**FIGURE 24–58** Coupled coils with sinusoidal ac excitation.

These equations describe the circuit of Figure 24–58 as you can see if you write KVL for each loop. (Verify this.)

**EXAMPLE 24–14** For Figure 24–59, write the mesh equations and solve for $\mathbf{I}_1$ and $\mathbf{I}_2$. Let $\omega = 100$ rad/s, $L_1 = 0.1$ H, $L_2 = 0.2$ H, $M = 0.08$ H, $R_1 = 15\ \Omega$, and $R_2 = 20\ \Omega$.

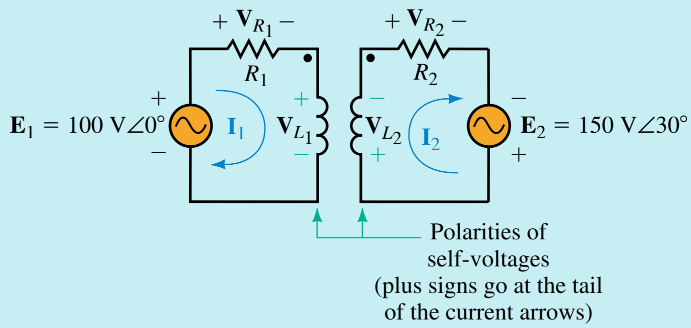



**FIGURE 24–59** Air-core transformer example.

**Solution** $\omega L_1 = (100)(0.1) = 10\ \Omega$, $\omega L_2 = (100)(0.2) = 20\ \Omega$, and $\omega M = (100)(0.08) = 8\ \Omega$. Since one current enters a dotted terminal and the other leaves, the sign of the mutual term is opposite to the sign of the self term. KVL yields

$$\text{Loop 1:}\quad \mathbf{E}_1 - R_1\mathbf{I}_1 - \overbrace{j\omega L_1\mathbf{I}_1 + j\omega M\mathbf{I}_2}^{\text{opposite}} = 0$$

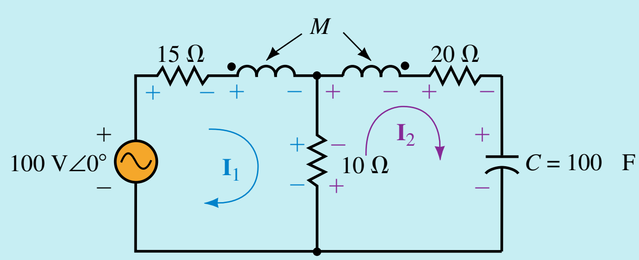
$$\text{Loop 2:}\quad \mathbf{E}_2 - \underbrace{j\omega L_2\mathbf{I}_2 + j\omega M\mathbf{I}_1}_{\text{opposite}} - R_2\mathbf{I}_2 = 0$$

Thus,

$$(15 + j10)\mathbf{I}_1 - j8\mathbf{I}_2 = 100\angle 0°$$
$$-j8\mathbf{I}_1 + (20 + j20)\mathbf{I}_2 = 150\angle 30°$$

These can be solved by the usual methods such as determinants or by computer. The answers are $\mathbf{I}_1 = 6.36\angle -6.57°$ and $\mathbf{I}_2 = 6.54\angle -2.24°$.

**EXAMPLE 24–15** For the circuit of Figure 24–60, determine $\mathbf{I}_1$ and $\mathbf{I}_2$.




$L_1 = 0.1$ H, $L_2 = 0.2$ H, $M = 80$ mH, $\omega = 100$ rad/s

**FIGURE 24–60**

**Solution** $\omega L_1 = 10\ \Omega$, $\omega L_2 = 20\ \Omega$, $\omega M = 8\ \Omega$, and $X_C = 100\ \Omega$.

$$\text{Loop 1:}\quad 100\angle 0° - 15\mathbf{I}_1 - j10\mathbf{I}_1 + j8\mathbf{I}_2 - 10\mathbf{I}_1 + 10\mathbf{I}_2 = 0$$
$$\text{Loop 2:}\quad -10\mathbf{I}_2 + 10\mathbf{I}_1 - j20\mathbf{I}_2 + j8\mathbf{I}_1 - 20\mathbf{I}_2 - (-j100)\mathbf{I}_2 = 0$$

Thus:

$$(25 + j10)\mathbf{I}_1 - (10 + j8)\mathbf{I}_2 = 100\angle 0°$$
$$-(10 + j8)\mathbf{I}_1 + (30 - j80)\mathbf{I}_2 = 0$$

Solution yields $\mathbf{I}_1 = 3.56\ \text{A}\angle -18.6°$ and $\mathbf{I}_2 = 0.534\ \text{A}\angle 89.5°$

PRACTICE PROBLEMS 8

Refer to the circuit of Figure 24–61.

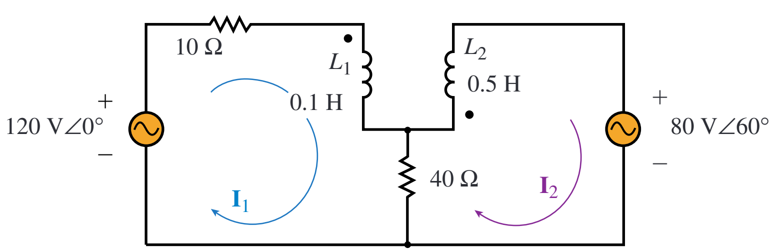



**FIGURE 24–61** $M = 0.12\ \text{H}$, $\omega = 100$ rad/s.

a. Determine the mesh equations.
b. Solve for currents $\mathbf{I}_1$ and $\mathbf{I}_2$.

*Answer:*

a. $(50 + j10)\mathbf{I}_1 - (40 - j12)\mathbf{I}_2 = 120\angle 0°$
$-(40 - j12)\mathbf{I}_1 + (40 + j50)\mathbf{I}_2 = -80\angle 60°$

b. $\mathbf{I}_1 = 1.14\ \text{A}\angle -31.9°$ and $\mathbf{I}_2 = 1.65\ \text{A}\angle -146°$

## 24.11 Coupled Impedance

Earlier we found that an impedance $\mathbf{Z}_L$ on the secondary side of an iron-core transformer is reflected into the primary side as $a^2\mathbf{Z}_L$. A somewhat similar situation occurs in loosely coupled circuits. In this case however, the impedance that you see reflected to the primary side is referred to as **coupled impedance.** To get at the idea, consider Figure 24–62. Writing KVL for each loop yields

$$\text{Loop 1:}\quad \mathbf{E}_g - \mathbf{Z}_1\mathbf{I}_1 - j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 = 0$$
$$\text{Loop 2:}\quad -j\omega L_2\mathbf{I}_2 - j\omega M\mathbf{I}_1 - \mathbf{Z}_2\mathbf{I}_2 - \mathbf{Z}_L\mathbf{L}_2 = 0$$

which reduces to

$$\mathbf{E}_g = \mathbf{Z}_p\mathbf{I}_1 + j\omega M\mathbf{I}_2 \tag{24–20a}$$

$$0 = j\omega M\mathbf{I}_1 + (\mathbf{Z}_s + \mathbf{Z}_L)\mathbf{I}_2 \tag{24–20b}$$

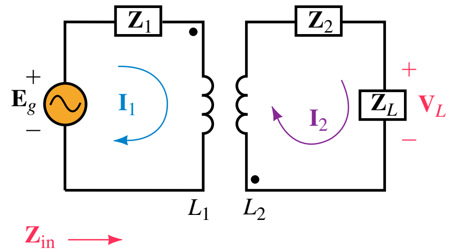



**FIGURE 24–62**

where $\mathbf{Z}_p = \mathbf{Z}_1 + j\omega L_1$ and $\mathbf{Z}_s = \mathbf{Z}_2 + j\omega L_2$. Solving Equation 24–20b for $\mathbf{I}_2$ and substituting this into Equation 24–20a yields, after some manipulation,

$$\mathbf{E}_g = \mathbf{Z}_p\mathbf{I}_1 + \frac{(\omega M)^2}{\mathbf{Z}_s + \mathbf{Z}_L}\mathbf{I}_1$$

Now, divide both sides by $\mathbf{I}_1$, and define $\mathbf{Z}_{in} = \mathbf{E}_g/\mathbf{I}_1$. Thus,

$$\mathbf{Z}_{in} = \mathbf{Z}_p + \frac{(\omega M)^2}{\mathbf{Z}_s + \mathbf{Z}_L} \qquad \textbf{(24–21)}$$

The term $(\omega M)^2/(\mathbf{Z}_s + \mathbf{Z}_L)$, which reflects the secondary impedances into the primary, is the coupled impedance for the circuit. Note that since secondary impedances appear in the denominator, they reflect into the primary with reversed reactive parts. Thus, capacitance in the secondary circuit looks inductive to the source and vice versa.

**EXAMPLE 24–16** For Figure 24–62, let $L_1 = L_2 = 10$ mH, $M = 9$ mH, $\omega = 1000$ rad/s, $\mathbf{Z}_1 = R_1 = 5\ \Omega$, $\mathbf{Z}_2 = 1\ \Omega - j5\ \Omega$, $\mathbf{Z}_L = 1\ \Omega + j20\ \Omega$ and $\mathbf{E}_g = 100\ \text{V}\angle 0°$. Determine $\mathbf{Z}_{in}$ and $\mathbf{I}_1$.

**Solution**

$\omega L_1 = 10\ \Omega$. Thus, $\mathbf{Z}_p = R_1 + j\omega L_1 = 5\ \Omega + j10\ \Omega$.

$\omega L_2 = 10\ \Omega$. Thus, $\mathbf{Z}_s = \mathbf{Z}_2 + j\omega L_2 = (1\ \Omega - j5\ \Omega) + j10\ \Omega = 1\ \Omega + j5\ \Omega$.

$\omega M = 9\ \Omega$ and $\mathbf{Z}_L = 1\ \Omega + j20\ \Omega$. Thus,

$$\mathbf{Z}_{in} = \mathbf{Z}_p + \frac{(\omega M)^2}{\mathbf{Z}_s + \mathbf{Z}_L} = (5 + j10) + \frac{(9)^2}{(1 + j5) + (1 + j20)}$$
$$= 8.58\ \Omega\angle 52.2°$$
$$\mathbf{I}_1 = \mathbf{E}_g/\mathbf{Z}_{in} = (100\angle 0°)/(8.58\angle 52.2°) = 11.7\ \text{A}\angle -52.2°$$

The equivalent circuit is shown in Figure 24–63.

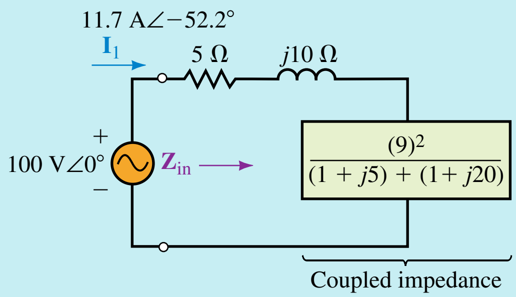



FIGURE 24–63

For Example 24–16, let $R_1 = 10\ \Omega$, $M = 8$ mH, and $\mathbf{Z}_L = (3 - j8)\ \Omega$. Determine $\mathbf{Z}_{in}$ and $\mathbf{I}_1$.

*Answers:* 28.9 Ω∠41.1°; 3.72 A∠−41.1°

PRACTICE PROBLEMS 9

## 24.12 Circuit Analysis Using Computers

Electronics Workbench and PSpice may be used to solve coupled circuits. (PSpice handles both loosely coupled circuits and tightly coupled (iron-core) transformers, but as of this writing, Workbench handles only iron-core devices.) As a first example, let us solve for generator and load currents and the load voltage for the circuit of Figure 24–64. First, manually determine the answers to provide a basis for comparison. Reflecting the load impedance using $a^2\mathbf{Z}_L$ yields the equivalent circuit of Figure 24–65. From this,

$$\mathbf{I}_g = \frac{100\ \text{V}\angle 0^\circ}{200\ \Omega + (200\ \Omega - j265.3\ \Omega)} = 208.4\ \text{mA}\angle 33.5^\circ$$

Thus,

$$\mathbf{I}_L = a\mathbf{I}_g = 416.8\ \text{mA}\angle 33.5^\circ$$

and

$$\mathbf{V}_L = \mathbf{I}_L\mathbf{Z}_L = 34.6\ \text{V}\angle -19.4^\circ.$$

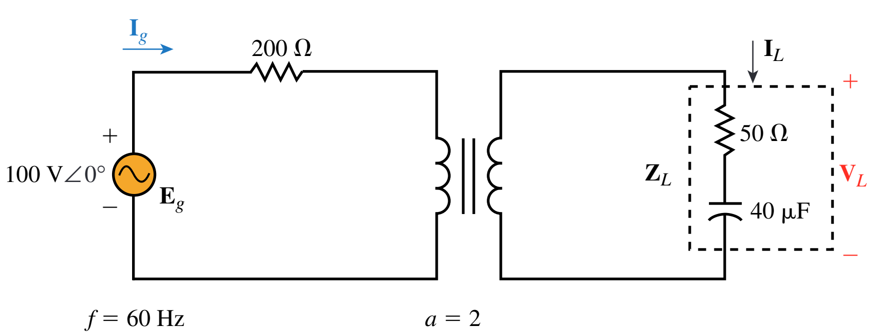



FIGURE 24–64 Iron-core transformer circuit for first Electronics Workbench and PSpice example.

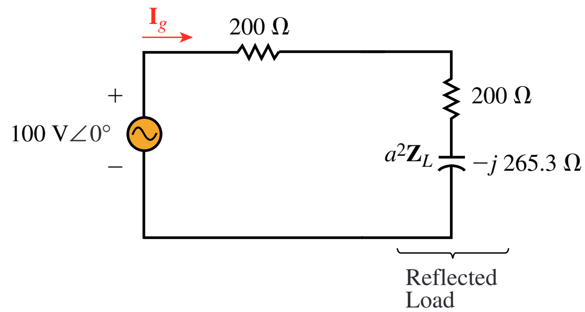



FIGURE 24–65

## Electronics Workbench

Draw the circuit as in Figure 24–66. (The transformer is found in the Basic parts bin. It models the transformer in a straightforward manner using its turns ratio.) Double click the transformer symbol, select Models, default, ideal, click Edit and set Magnetizing Inductance (LM) to a very high value, e.g., **10000 H.** (This is $L_m$ of Figure 24–41, which theoretically for an ideal transformer is infinity.) Set the turns ratio to **2,** leave everything else alone, then click OK, OK. Set all the meters to AC, then activate the circuit. Note the magnitudes of the currents and the load voltage. They agree quite well with the answers computed above.

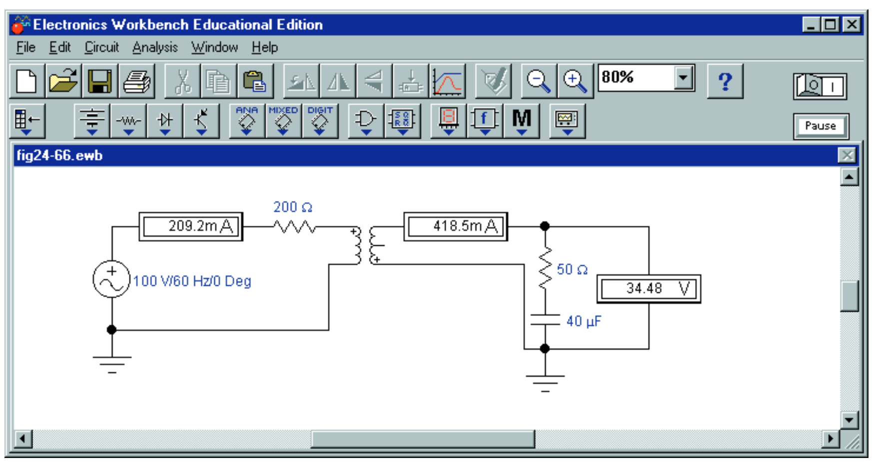
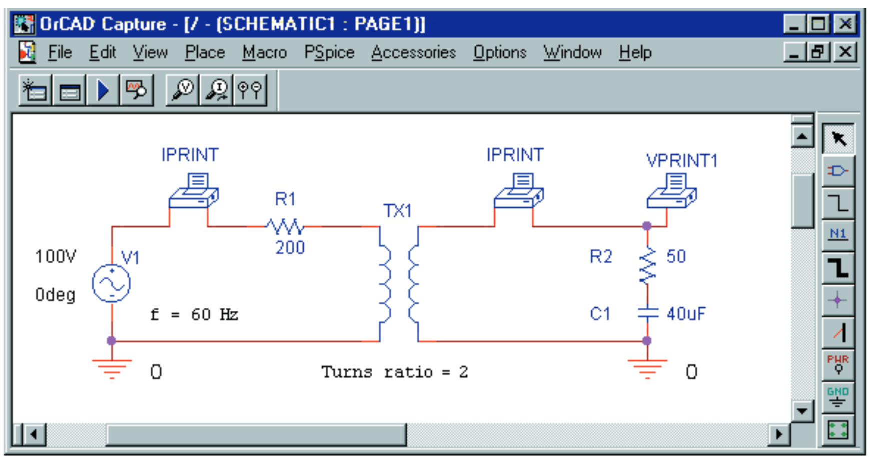



FIGURE 24–66 Electronics Workbench solution for the circuit of Figure 24–64.

## OrCADSpice

First, read Item #2 from “PSpice and EWB Notes for Coupled Circuits” (see page 1022). As indicated, transformer element XFRM_LINEAR may be used to model iron-core transformers based solely on their turns ratios. To do this, set coupling $k = 1$, choose an arbitrarily large value for $L_1$, then let $L_2 = L_1/a^2$ where $a$ is the turns ratio. (The actual values for $L_1$ and $L_2$ are not critical, they simply must be very large.) For example, arbitrarily choose

$L_1$ = 100000 H, then compute $L_2 = 100000/(2^2) = 25000$ H. This sets $a = 2$. Now proceed as follows. Create the circuit of Figure 24–64 on the screen as Figure 24–67. Use source VAC, set up as shown. Double click VPRINT1 and then set AC, MAG, and PHASE to **yes** in the Properties Editor. Repeat for the IPRINT devices. Double click the transformer and set COUPLING to **1**, L1 to **100000H,** and L2 to **25000H.** Select AC Sweep and set Start and End frequencies to **60HZ**. Complete the remainder of the setup, run the simulation, then scroll through the Output File. You should find $\mathbf{I}_g$ = 208.3 mA$\angle$33.6°, $\mathbf{I}_L$ = 416.7 mA$\angle$33.6° and $\mathbf{V}_L$ = 34.6 V$\angle-$19.4°. Note that these agree almost exactly with the computed results.




EWB **FIGURE 24–67** PSpice solution for the circuit of Figure 24–64.

As a final PSpice example, consider the loosely coupled circuit of Figure 24–59, drawn on the screen as Figure 24–68. Use VAC for the sources

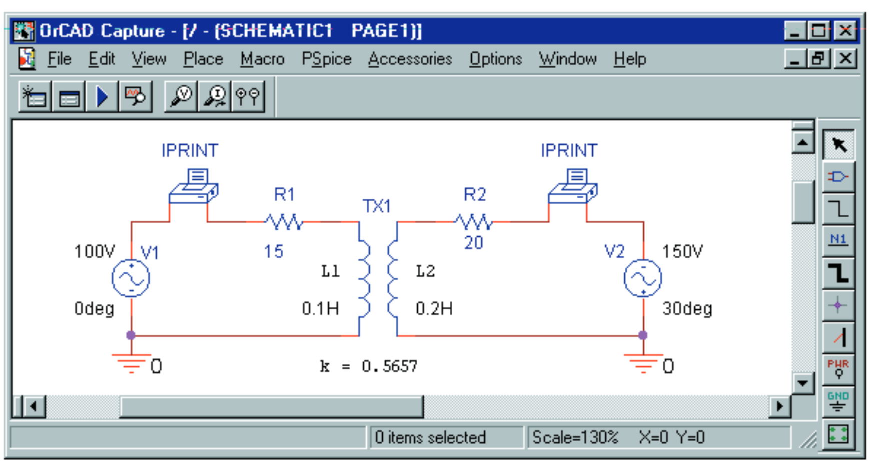



**FIGURE 24–68** PSpice solution for Example 24–14.

and XFRM_LINEAR for the transformer. (Be sure to orient Source 2 as shown.) Compute $k = \dfrac{M}{\sqrt{L_1L_2}} = 0.5657$. Now double click the transformer symbol and set $L_1 =$ **0.1H,** $L_2 =$ **0.2H,** and $k =$ **0.5657.** Set $f =$ **15.9155Hz.** (Remember, PSpice uses $f$, not $\omega$.). Complete the remainder of the setup, then run the simulation. When you scroll the Output File, you will find $\mathbf{I}_1 = 6.36\ \text{A}\angle-6.57°$ and $\mathbf{I}_2 = 6.54\ \text{A}\angle-2.23°$ as determined earlier in Example 24–14.

## PSPICE AND EWB NOTES FOR COUPLED CIRCUITS

1. Electronics Workbench uses the ideal transformer equations $\mathbf{E}_p/\mathbf{E}_s = a$ and $\mathbf{I}_p/\mathbf{I}_s = 1/a$ to model transformers. It also permits you to set winding resistance, leakage flux, and exciting current effects as per Figure 24–41. As of this writing, however, Electronics Workbench cannot easily model loosely coupled circuits. (See the Delmar web site at **www.electronictech.com** for a method.)
2. The PSpice transformer model XFRM_LINEAR is based on self-inductances and the coefficient of coupling and is thus able to handle loosely coupled circuits directly. It is also able to model tightly coupled circuits (such as iron-core transformers). To see how, note that basic theory shows that for an ideal iron-core transformer, $k = 1$ and $L_1$ and $L_2$ are infinite, but their ratio is $L_1/L_2 = a^2$. Thus, to approximate the transformer, just set $L_1$ to an arbitrary very large value, then compute $L_2 = L_1/a^2$. This fixes $a$, permitting you to model iron-core transformers based solely on their turns ratio.
3. The sign of the coefficient of coupling to use with PSpice depends on the dot locations. For example, if dots are on adjacent coil ends (as in Figure 24–59), make $k$ positive; if dots are on opposite ends (Figure 24–62), make $k$ negative.
4. PSpice requires grounds on both sides of a transformer.

## PUTTING IT INTO PRACTICE

A circuit you are building calls for a 3.6-mH inductor. In your parts bin, you find a 1.2-mH and a 2.4-mH inductor. You reason that if you connect them in series, the total inductance will be 3.6 mH. After you build and test the circuit, you find that it is out of spec. After careful reasoning, you become suspicious that mutual coupling between the coils is upsetting operation. You therefore set out to measure this mutual inductance. However, you have a meter that measures only self-inductance. Then an idea hits you. You de-energize the circuit, unsolder the end of one of the inductors and measure total inductance. You get 6.32 mH. What is the mutual inductance?

## PROBLEMS

### 24.1 Introduction

1. For the transformers of Figure 24–69, sketch the missing waveforms.

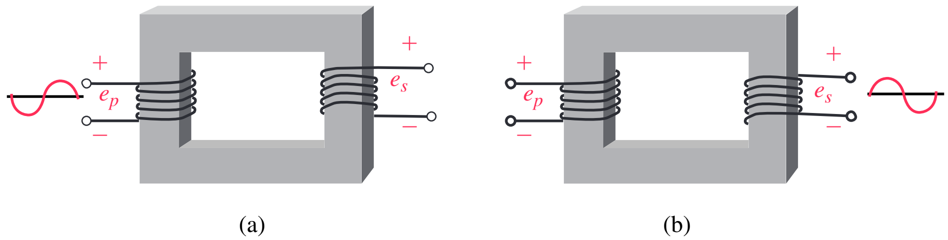



FIGURE 24–69

### 24.2 Iron-Core Transformers: The Ideal Model

2. List the four things that you neglect when you idealize an iron-core transformer.
3. An ideal transformer has $N_p = 1000$ turns and $N_s = 4000$ turns.
   a. Is it step-up or step-down voltage?
   b. If $e_s = 100 \sin \omega t$, what is $e_p$ when wound as in Figure 24–7(a)?
   c. If $E_s = 24$ volts, what is $E_p$?
   d. If $\mathbf{E}_p = 24\ \text{V}\angle 0°$, what is $\mathbf{E}_s$ when wound as in Figure 24–7(a)?
   e. If $\mathbf{E}_p = 800\ \text{V}\angle 0°$, what is $\mathbf{E}_s$ when wound as in Figure 24–7(b)?
4. A 3 : 1 step-down voltage transformer has a secondary current of 6 A. What is its primary current?
5. For Figure 24–70, determine the phase relationship for $v_1$, $v_2$, and $v_3$. Determine the expression for each.

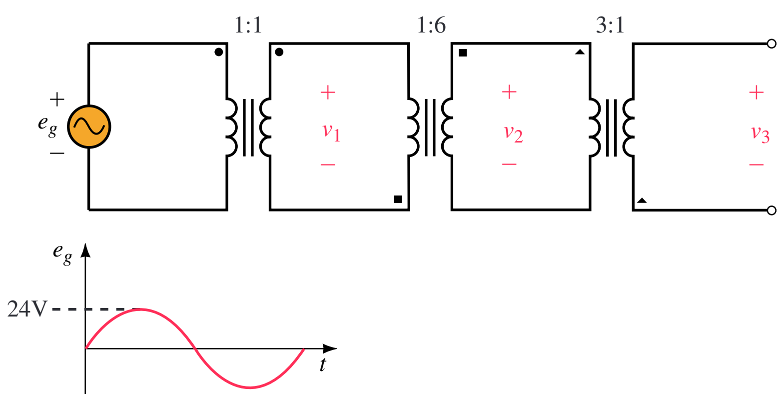



FIGURE 24–70

6. If, for Figure 24–71, $\mathbf{E}_g = 240\ \text{V}\angle 0°$, $a = 2$, and $\mathbf{Z}_L = 8\ \Omega - j6\ \Omega$, determine the following:
   a. $\mathbf{V}_L$ b. $\mathbf{I}_L$ c. $\mathbf{I}_g$

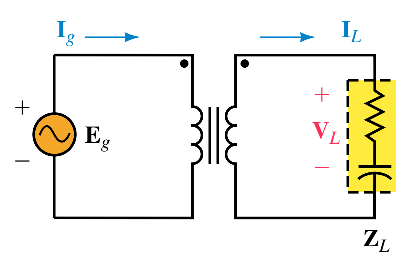



FIGURE 24–71

7. If, for Figure 24–71, $\mathbf{E}_g = 240\ \text{V}\angle 0°$, $a = 0.5$, and $\mathbf{I}_g = 2\ \text{A}\angle 20°$, determine the following:
   a. $\mathbf{I}_L$ b. $\mathbf{V}_L$ c. $\mathbf{Z}_L$
8. If, for Fig. 24–71, $a = 2$, $\mathbf{V}_L = 40\ \text{V}\angle 0°$, and $\mathbf{I}_g = 0.5\ \text{A}\angle 10°$, determine $\mathbf{Z}_L$.
9. If, for Fig. 24–71, $a = 4$, $\mathbf{I}_g = 4\ \text{A}\angle 30°$, and $\mathbf{Z}_L = 6\ \Omega - j8\ \Omega$, determine the following:
   a. $\mathbf{V}_L$ b. $\mathbf{E}_g$
10. If, for the circuit of Figure 24–71, $a = 3$, $\mathbf{I}_L = 4\ \text{A}\angle 25°$, and $\mathbf{Z}_L = 10\ \Omega\angle -5°$, determine the following:
    a. Generator current and voltage.
    b. Power to the load.
    c. Power output by the generator.
    d. Does $P_{\text{out}} = P_{\text{in}}$?

### 24.3 Reflected Impedance

11. For each circuit of Figure 24–72, determine $\mathbf{Z}_p$.

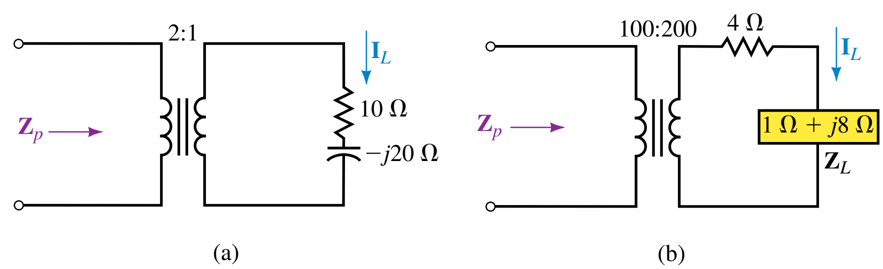



FIGURE 24–72

12. For each circuit of Figure 24–72, if $\mathbf{E}_g = 120\ \text{V}\angle 40°$ is applied, determine the following, using the reflected impedance of Problem 11.
    a. $\mathbf{I}_g$ b. $\mathbf{I}_L$ c. $\mathbf{V}_L$
13. For Figure 24–72(a), what turns ratio is required to make $\mathbf{Z}_p = (62.5 - j125)\ \Omega$?
14. For Figure 24–72(b), what turns ratio is required to make $\mathbf{Z}_p = 84.9\angle 58.0°\ \Omega$?
15. For each circuit of Figure 24–73, determine $\mathbf{Z}_\text{T}$.

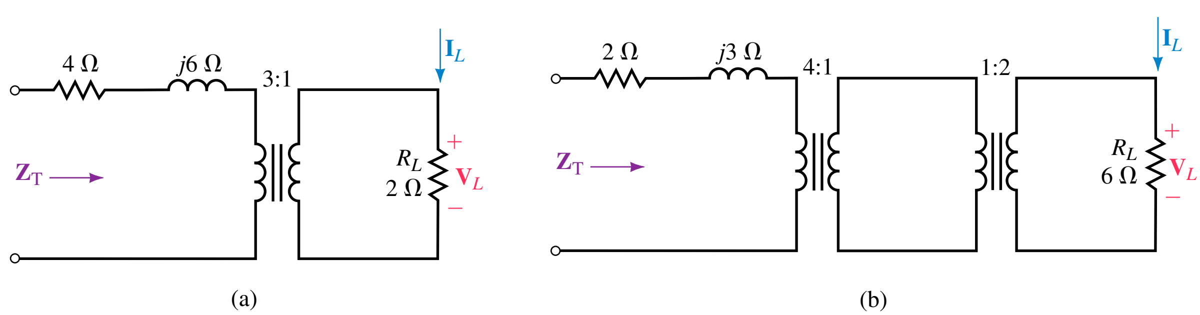



FIGURE 24–73

16. For each circuit of Figure 24–73, if a generator with $\mathbf{E}_g = 120\ \text{V}\angle{-40°}$ is applied, determine the following:

    a. $\mathbf{I}_g$ b. $\mathbf{I}_L$ c. $\mathbf{V}_L$

## 24.4 Transformer Power Ratings

17. A transformer has a rated primary voltage of 7.2 kV, $a = 0.2$, and a secondary rated current of 3 A. What is its kVA rating?
18. Consider a 48 kVA, 1200/120-V transformer.

    a. What is the maximum kVA load that it can handle at $F_p = 0.8$?

    b. What is the maximum power that it can supply to a 0.75 power factor load?

    c. If the transformer supplies 45 kW to a load at 0.6 power factor, is it overloaded? Justify your answer.

## 24.5 Transformer Applications

19. The transformer of Figure 24–25 has a 7200-V primary and center-tapped 240-V secondary. If Load 1 consists of twelve 100-W lamps, Load 2 is a 1500-W heater, and Load 3 is a 2400-W stove with $F_p = 1.0$, determine

    a. $I_1$ b. $I_2$ c. $I_N$ d. $I_p$
20. An amplifier with a Thévenin voltage of 10 V and Thévenin resistance of 128 Ω is connected to an 8-Ω speaker through a 4:1 transformer. Is the load matched? How much power is delivered to the speaker?
21. An amplifier with a Thévenin equivalent of 10 V and $R_{\text{Th}}$ of 25 Ω drives a 4-Ω speaker through a transformer with a turns ratio of $a = 5$. How much power is delivered to the speaker?
22. For Problem 21, what turns ratio do you need to get an output of 1 W to the speaker?
23. For Figure 24–30(a), $a_2 = 2$ and $a_3 = 5$, $\mathbf{Z}_2 = 20\ \Omega\angle 50°$, $\mathbf{Z}_3 = (12 + j4)\ \Omega$ and $\mathbf{E}_g = 120\ \text{V}\angle 0°$. Find each load current and the generator current.
24. It is required to connect a 5-kVA, 120/240-V transformer as an autotransformer to a 120-V source to supply 360 V to a load.

    a. Draw the circuit.

    b. What is the maximum current that the load can draw?

    c. What is the maximum load kVA that can be supplied?

    d. How much current is drawn from the source?

## 24.6 Practical Iron-Core Transformers

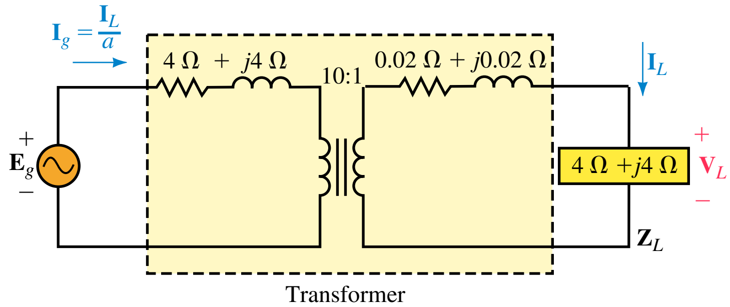
25. For Figure 24–74, $\mathbf{E}_g = 1220\ \text{V}\angle 0°$.

    a. Draw the equivalent circuit,

    b. Determine $\mathbf{I}_g$, $\mathbf{I}_L$, and $\mathbf{V}_L$.
26. For Figure 24–74, if $\mathbf{V}_L = 118\ \text{V}\angle 0°$, draw the equivalent circuit and determine

    a. $\mathbf{I}_L$ b. $\mathbf{I}_g$ c. $\mathbf{E}_g$

    d. no-load voltage e. regulation
27. A transformer delivering $P_{\text{out}} = 48$ kW has a core loss of 280 W and a copper loss of 450 W. What is its efficiency at this load?




EWB **FIGURE 24–74**

## 24.7 Transformer Tests

28. A short-circuit test (Figure 24–46) at rated current yields a wattmeter reading of 96 W, and an open-circuit test (Figure 24–48) yields a core loss of 24 W.
    a. What is the transformer's efficiency when delivering the full, rated output of 5 kVA at unity $F_p$?
    b. What is its efficiency when delivering one quarter the rated kVA at 0.8 $F_p$?

## 24.9 Loosely Coupled Circuits

29. For Figure 24–75,

$$v_1 = L_1\frac{di_1}{dt} \pm M\frac{di_2}{dt}, \qquad v_2 = \pm\frac{Mdi_1}{dt} + L_2\frac{di_2}{dt}$$

For each circuit, indicate whether the sign to use with $M$ is plus or minus.

30. For a set of coils, $L_1 = 250$ mH, $L_2 = 0.4$ H, and $k = 0.85$. What is $M$?
31. For a set of coupled coils, $L_1 = 2$ H, $M = 0.8$ H and the coefficient of coupling is 0.6. Determine $L_2$.
32. For Figure 24–51(a), $L_1 = 25$ mH, $L_2 = 4$ mH, and $M = 0.8$ mH. If $i_1$ changes at a rate of 1200 A/s, what are the primary and secondary induced voltages?
33. Everything the same as Problem 32 except that $i_1 = 10\,e^{-500t}$ A. Find the equations for the primary and secondary voltages. Compute them at $t = 1$ ms.
34. For each circuit of Figure 24–76, determine $L_T$.

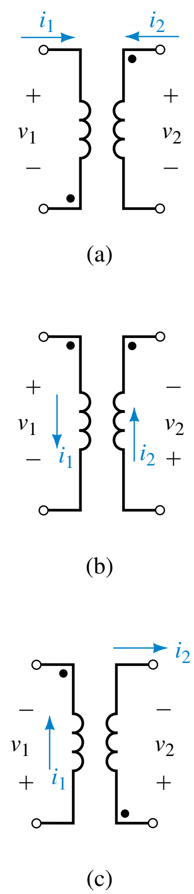



**FIGURE 24–75**

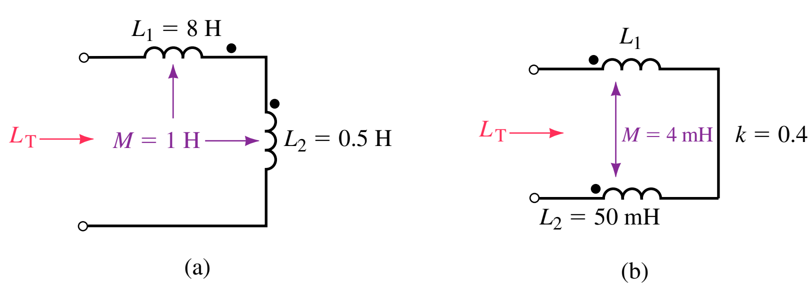



**FIGURE 24–76**

35. For Figure 24–77, determine $L_T$.

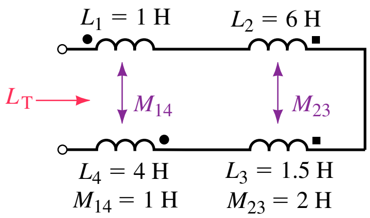



**FIGURE 24–77**

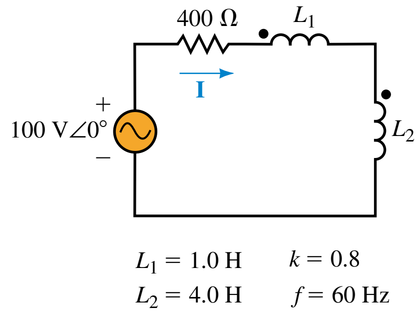



**FIGURE 24–78**

36. For the circuit of Figure 24–78, determine **I.**
37. The inductors of Figure 24–79 are mutually coupled. What is their equivalent inductance? If $f = 60$ Hz, what is the source current?

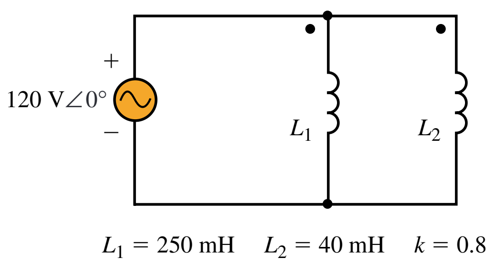



**FIGURE 24–79** Coupled parallel inductors.

### 24.10 Magnetically Coupled Circuits with AC Excitation

38. For Figure 24–59, $R_1 = 10\ \Omega$, $R_2 = 30\ \Omega$, $L_1 = 100$ mH, $L_2 = 200$ mH, $M = 25$ mH, and $f = 31.83$ Hz. Write the mesh equations.
39. For the circuit of Figure 24–80, write mesh equations.

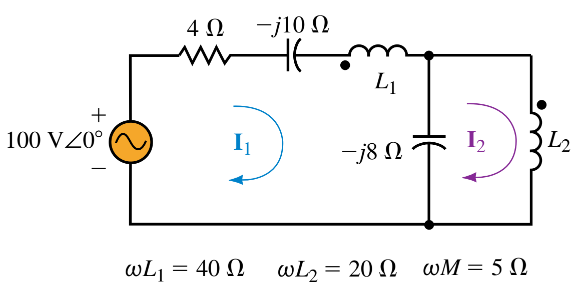
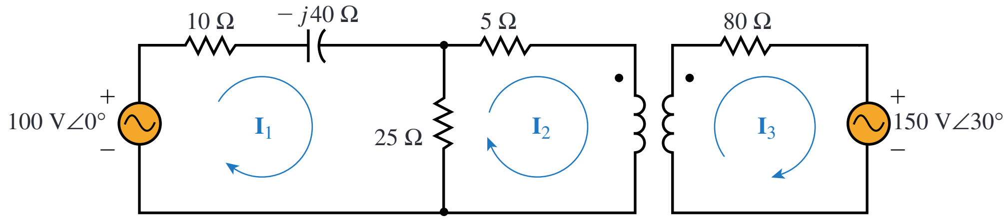


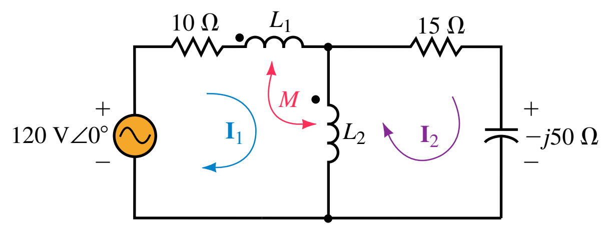

**FIGURE 24–80**

40. Write mesh equations for the circuit of Figure 24–81.
41. Write mesh equations for the circuit of Figure 24–82. (This is a very challenging problem.)




$\omega L_1 = 40\ \Omega$ $\omega L_2 = 30\ \Omega$ $\omega M = 15\ \Omega$

**FIGURE 24–81**

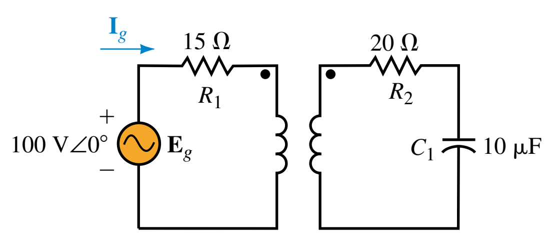



$L_1 = 0.1$ H $M = 0.12$ H
$L_2 = 0.5$ H $\omega = 100$ rad/s

**FIGURE 24–82**

### 24.11 Coupled Impedance

42. For the circuit of Figure 24–83,
    a. determine $\mathbf{Z}_{\text{in}}$
    b. determine $\mathbf{I}_g$.




$L_1 = 0.1$ H; $L_2 = 0.2$ H; $M = 0.08$ H; $f = 60$ Hz

**FIGURE 24–83**

### 24.12 Circuit Analysis Using Computers

Notes: (1) At the time of writing, Electronics Workbench solves for magnitude only. (2) With PSpice, orient the IPRINT devices so that current enters the positive terminal. Otherwise the phase angles will be in error by 180°.

42. **EWB** or **PSpice** An iron-core transformer with a 4:1 turns ratio has a load consisting of a 12-Ω resistor in series with a 250-µF capacitor. The transformer is driven from a 120-V∠0°, 60-Hz source. Use Electronics Workbench or PSpice to determine the source and load currents. Verify the answers by manual computation.
43. **EWB** or **PSpice** Using Workbench or PSpice, solve for the primary and secondary currents and the load voltage for Figure 24–84.
44. **PSpice** Using PSpice, solve for the source current for the coupled parallel inductors of Figure 24–79. Hint: Use XFRM_LINEAR to model the two inductors. You will need two very low-value resistors to avoid creating a source-inductor loop.

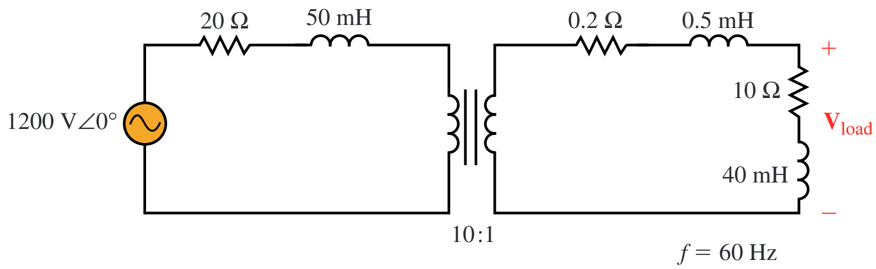



FIGURE 24–84

45. Solve for the currents of Figure 24–60 using PSpice. Compare these to the answers of Example 24–15.
46. Solve for the currents of Figure 24–61 using PSpice. Compare these to the answers of Practice Problem 8.
47. Solve Example 24–16 for current $\mathbf{I}_1$ using PSpice. Compare answers. Hint: If values are given as $X_L$ and $X_C$, you must convert them to $L$ and $C$.

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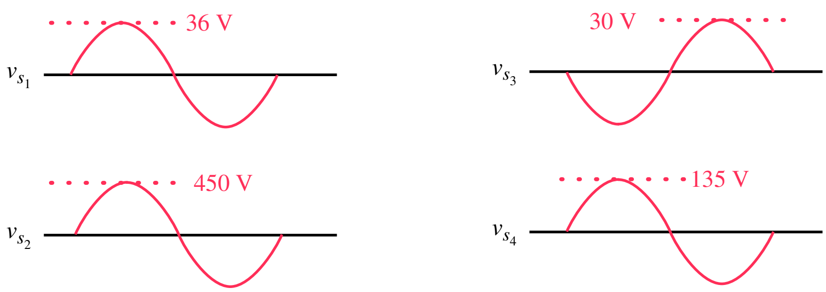
## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

1. Step-up; 200 V
2. a. $120\text{ V}\angle -30°$ b. $120\text{ V}\angle 150°$
3. 




4. Secondary, upper terminal.
5. a. Terminal 4 b. Terminal 4

**In-Process Learning Check 2**

1. $\mathbf{Z}_p = 18.75\ \Omega - j25\ \Omega$; $R = 6400\ \Omega$; $a = 1.73$
2. $125\text{ V}\angle 0°$
3. 89.4
4. a. 200 A; 200 A b. 24 kW; 18 kW
5. Tap 1: 109.1 V; Tap 2: 120 V; Tap 3: 126.3 V
6. 0.81 W; 0.72 W; Maximum power is delivered when $R_s = a^2R_L$.
7. a. 348 V; 20.7 A b. 12 A c. 12 A↓ 8.69 A↑

**In-Process Learning Check 3**

1. a. $\mathbf{Z}_{\text{eq}} = 0.8\ \Omega + j0.8\ \Omega$
   b. $113.6\text{ V}\angle 0.434°$
   c. 5.63%

# 25 Nonsinusoidal Waveforms

## OBJECTIVES

After studying this chapter, you will be able to

- solve for the coefficients of the Fourier series of a simple periodic waveform, using integration,
- use tables to write the Fourier equivalent of any simple periodic waveform,
- sketch the frequency spectrum of a periodic waveform, giving the amplitudes of various harmonics in either volts, watts, or dBm,
- calculate the power dissipated when a complex waveform is applied to a resistive load,
- determine the output of a filter circuit given the frequency spectrum of the input signal and the frequency response of the filter,
- use PSpice to observe the actual response of a filter circuit to a nonsinusoidal input signal.

## KEY TERMS

Fourier Series
Frequency Spectrum
Fundamental Frequency
Distortion
Harmonic Frequency
Spectrum Analyzer

## OUTLINE

Composite Waveforms
Fourier Series
Fourier Series of Common Waveforms
Frequency Spectrum
Circuit Response to a Nonsinusoidal Waveform
Circuit Analysis Using Computers

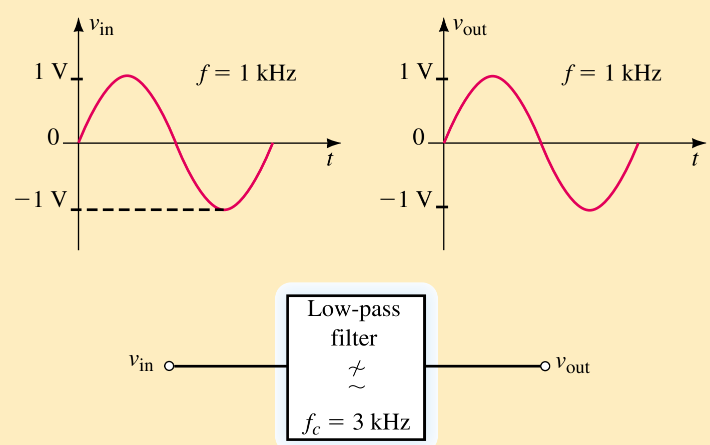



**FIGURE 25–1**

## CHAPTER PREVIEW

In our analysis of ac circuits, we have dealt primarily with sinusoidal waveforms. Although the sinusoidal wave is the most common waveform in electronic circuits, it is by no means the only type of signal used in electronics. In previous chapters we observed how sinusoidal signals were affected by the characteristics of components within a circuit. For instance, if a 1-kHz sinusoid is applied to a low-pass filter circuit having a cutoff frequency of 3 kHz, we know that the signal appearing at the output of the filter will be essentially the same as the signal applied to the input. This effect is illustrated in Figure 25–1.

One would naturally expect that the 1-kHz low-pass filter would allow any other 1-kHz signal to pass from input to the output without being distorted. Unfortunately, this is not the case.

In this chapter we will find that any periodic waveform is composed of numerous sinusoidal waveforms, each of which has a unique amplitude and frequency. As we have already seen, circuits such as the low-pass filter and the resonant tank circuit do not allow all sinusoidal frequencies to pass from the input to the output in the same manner. As a result, the output signal may be dramatically different from the signal applied at the input. For example, if we were to apply a 1-kHz square wave to a low-pass filter having a 3-kHz cutoff, the output would appear as shown in Figure 25–2.

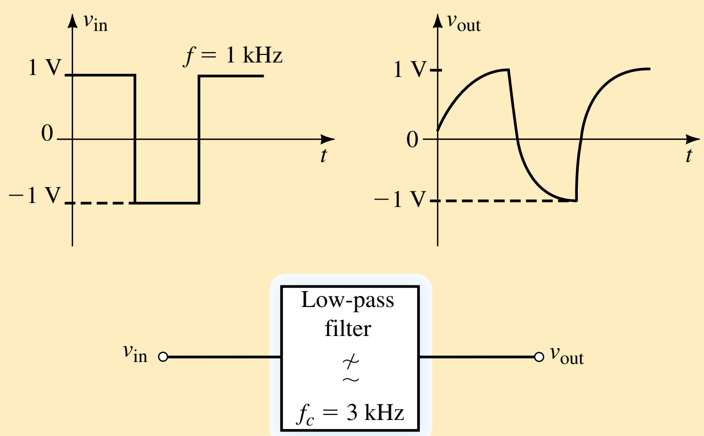



**FIGURE 25–2**

Although the frequency of the square wave is less than the 3-kHz cutoff frequency of the filter, we will find that the square wave has many high-frequency components which are well above the cutoff frequency. It is these components which are affected in the filter and this results in the distortion of the output waveform.

Once a periodic waveform is reduced to the summation of sinusoidal waveforms, it is a fairly simple matter to determine how various frequency components of the original signal will be affected by the circuit. The overall response of the circuit to a particular waveform can then be found.

**PUTTING IT IN PERSPECTIVE**

### *Jean Baptiste Joseph Fourier*

FOURIER WAS BORN IN AUXERRE, Yonne, France on March 21, 1768. As a youth, Fourier reluctantly studied for the priesthood in the monastery of Saint-Benoit-sur Loire. However, his interest was in mathematics. In 1798, Fourier accompanied Napoleon to Egypt, where he was made a governor. After returning to France, Fourier was particularly interested in the study of heat transfer between two points of different temperature. He was appointed joint secretary of the Academy of Sciences in 1822.

In 1807, Fourier announced the discovery of a theorem which made him famous. Fourier's theorem states that any periodic waveform can be written as the summation of a series of simple sinusoidal functions.

Using his theorem, Fourier was able to develop important theories in heat transfer, which were published in 1822 in a book titled *Analytic Theory of Heat.*

Although still used to describe heat transfer, Fourier's theorem is used today to predict how filters and various other electronic circuits operate when subjected to a nonsinusoidal periodic function.

Fourier died in Paris on May 16, 1830 as the result of a fall down the stairs.

## 25.1 Composite Waveforms

Any waveform that is made up of two or more separate waveforms is called a **composite waveform.** Most signals appearing in electronic circuits are comprised of complicated combinations of dc levels and sinusoidal waves. Consider the circuit and signal shown in Figure 25–3.

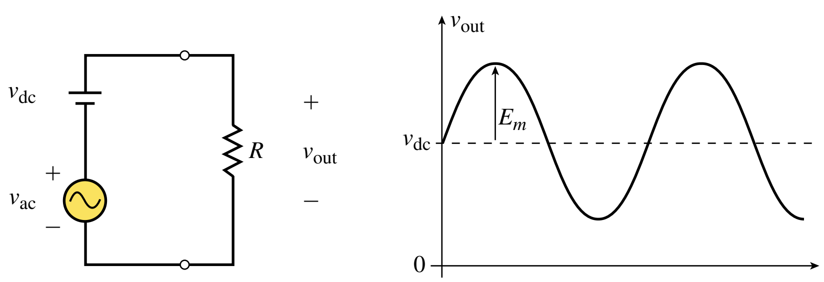



FIGURE 25–3

The voltage appearing across the load is determined by superposition as the combination of the ac source in series with a dc source. The result is a sine wave with a dc offset. As one might expect, when a composite wave is applied to a load resistor, the resulting power is determined by considering the effects of both signals. The rms voltage of the composite waveform is determined as

$$V_{rms} = \sqrt{V_{dc}^2 + V_{ac}^2} \tag{25–1}$$

where $V_{\text{ac}}$ is the rms value of the ac component of waveform and is found as $V_{\text{ac}} = \dfrac{E_m}{\sqrt{2}}$. The power delivered to a load will be determined simply as

$$P_{\text{load}} = \frac{V_{\text{rms}}^2}{R_{\text{load}}}$$

The following example illustrates this principle.

**EXAMPLE 25–1** Determine the power delivered to the load if the waveform of Figure 25–4 is applied to a 500-Ω resistor.

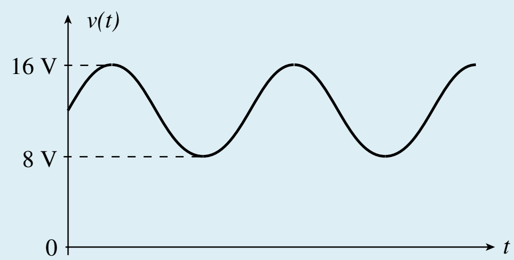



**FIGURE 25–4**

**Solution** By examining the waveform, we see that the average value is $V_{\text{dc}} = 12$ V and the peak value of the sinusoidal is $V_m = 16\text{ V} - 12\text{ V} = 4$ V. The rms value of the sinusoidal waveform is determined to be $V_{\text{ac}} = (0.707)(4\text{ V}) = 2.83$ V. Now we find the rms value of the composite wave as

$$\begin{aligned} V_{\text{rms}} &= \sqrt{(12\text{ V})^2 + (2.83\text{ V})^2} \\ &= \sqrt{152\text{ V}^2} \\ &= 12.3\text{ V} \end{aligned}$$

and so the power delivered to the load is

$$P_{\text{load}} = \frac{(12.3\text{ V})^2}{500\ \Omega} = 0.304\text{ W}$$

**PRACTICE PROBLEMS 1**

Determine the power delivered to the load if the waveform of Figure 25–5 is applied to a 200-Ω resistor.

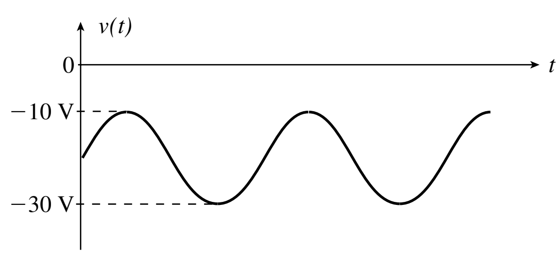



**FIGURE 25–5**

*Answer:* 2.25 W

## 25.2 Fourier Series

In 1826, Baron Jean Baptiste Joseph Fourier developed a branch of mathematics which is used to express any periodic waveform as an infinite series of sinusoidal waveforms. Although it seems as though we are turning a simple waveform into a more complicated form, you will find that the resulting expression actually simplifies the analysis of many circuits which respond differently to signals of various frequencies. Using Fourier analysis, any periodic waveform can be written as a summation of sinusoidal waveforms as follows:

$$\begin{aligned} f(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots + a_n \cos n\omega t + \cdots \\ + b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots + b_n \sin n\omega t + \cdots \end{aligned} \tag{25–2}$$

The coefficients of the individual terms of the Fourier series are found by integrating the original function over one complete period. The coefficients are determined as

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)\, dt \tag{25–3}$$

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos n\omega t\, dt \tag{25–4}$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin n\omega t\, dt \tag{25–5}$$

Notice that Equation 25–2 indicates that the Fourier series of a periodic function may contain both a sine and a cosine component at each frequency. These individual components can be combined to give a single sinusoidal expression as follows:

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= a_n \sin(nx + 90°) + b_n \sin nx \\ &= c_n \sin(nx + \theta) \end{aligned}$$

where

$$c_n = \sqrt{a_n^2 + b_n^2} \tag{25–6}$$

and

$$\theta = \tan^{-1}\left(\frac{a_n}{b_n}\right) \tag{25–7}$$

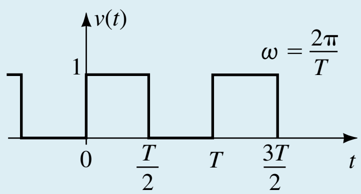
Therefore, the Fourier equivalent of any periodic waveform may be simplified as follows:

$$f(t) = a_0 + c_1 \sin(\omega t + \theta_1) + c_2 \sin(2\omega t + \theta_2) + \cdots$$

The $a_0$ term is a constant which corresponds to the average value of the periodic waveform and the $c_n$ coefficients give the amplitudes of the various sinusoidal terms. Notice that the first sinusoidal term ($n = 1$) has the same frequency as the original waveform. This component is referred to as the **fundamental frequency** of the given waveform. All other frequencies are integer multiples of the fundamental frequency and are called the **harmonic**

**frequencies.** When $n = 2$, the resulting term is called the second harmonic; when $n = 3$, we have the third harmonic, etc. Using Equations 25–3 to 25–7, it is possible to derive the Fourier series for any periodic function.

EXAMPLE 25–2 Write the Fourier series for the pulse waveform shown in Figure 25–6.




$$v(t) = \begin{cases} 1: 0 < t < \dfrac{T}{2} \\ 0: \dfrac{T}{2} < t < T \end{cases}$$

**FIGURE 25–6**

Solution The various coefficients are calculated by integrating as follows:

$$a_0 = \frac{1}{T}\int_0^{T/2}(1)\,dt + \frac{1}{T}\int_{T/2}^{T}(0)\,dt = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^{T/2}(1)\cos n\omega t\,dt + \frac{2}{T}\int_{T/2}^{T}(0)\,dt$$

$$= \frac{2}{T}\left[\left(\frac{1}{n\omega}\right)\sin n\omega t\right]_0^{T/2}$$

$$= \frac{1}{n\pi}\sin n\pi = 0$$

Notice that all $a_n = 0$ since $\sin n\pi = 0$ for all $n$.

$$b_1 = \frac{2}{T}\int_0^{T/2}(1)\sin\omega t\,dt + \frac{2}{T}\int_{T/2}^{T}(0)\,dt$$

$$= \frac{2}{T}\left[-\left(\frac{1}{\omega}\right)\cos\omega t\right]_0^{T/2}$$

$$= -\frac{1}{\pi}\left[\cos\left(\frac{2\pi t}{T}\right)\right]_0^{T/2}$$

$$= -\frac{1}{\pi}[(-1) - (1)] = \frac{2}{\pi}$$

$$b_2 = \frac{2}{T}\int_0^{T/2}(1)\sin 2\omega t\,dt + \frac{2}{T}\int_{T/2}^{T}(0)\,dt$$

$$= \frac{2}{T}\left[-\left(\frac{1}{2\omega}\right)\cos 2\omega t\right]_0^{T/2}$$

$$= -\frac{1}{2\pi}\left[\cos\left(\frac{4\pi t}{T}\right)\right]_0^{T/2}$$

$$= -\frac{1}{2\pi}[(1) - (1)] = 0$$

$$b_3 = \frac{2}{T}\int_0^{T/2}(1)\sin 3\omega t\,dt + \frac{2}{T}\int_{T/2}^{T}(0)\,dt$$

$$= \frac{2}{T}\left[-\left(\frac{1}{3\omega}\right)\cos 3\omega t\right]_0^{T/2}$$

$$= -\frac{1}{3\pi}\left[\cos\left(\frac{6\pi t}{T}\right)\right]_0^{T/2}$$

$$= -\frac{1}{3\pi}[(-1) - (1)] = \frac{2}{3\pi}$$

For all odd values of $n$, we have $b_n = 2/n\pi$ since $\cos n\pi = -1$. Even values of $n$ give $b_n = 0$ since $\cos n\pi = 1$.

The general expression of the Fourier series for the given pulse wave is therefore written as

$$v(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{1}^{\infty}\frac{\sin n\omega t}{n} \qquad n = 1, 3, 5, \ldots \tag{25–8}$$

By examining the general expression for the pulse wave of Figure 25–6, several important characteristics are observed. The Fourier series confirms that the wave shown has an average value of $a_0 = 0.5$. In addition, the pulse wave has only odd harmonics. In other words, a pulse wave having a frequency of 1 kHz would have harmonic components occurring at 3 kHz, 5 kHz, etc. Although the given wave consists of an infinite number of sinusoidal components, the amplitudes of successive terms decrease as $n$ increases.

If we were to consider only the first four nonzero frequency components of the pulse wave, we would have the following expression:

$$v(t) = 0.5 + \frac{2}{\pi}\sin\omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \frac{2}{7\pi}\sin 7\omega t \tag{25–9}$$

The graphical representation of the above expression is shown in Figure 25–7.

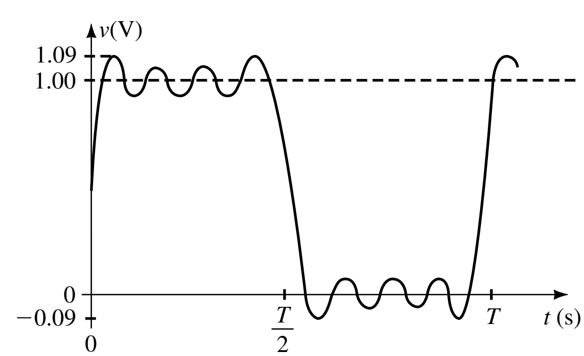



FIGURE 25–7

Although this waveform is not identical with the given pulse wave, we see that the first four nonzero harmonics provide a reasonable approximation of the original waveform.

Derivations of Fourier series for certain waveforms are simplified due to symmetry that occurs in the wave form. We will examine three types of symmetry; even, odd, and half-wave symmetry. Each type of symmetry results in consistent patterns in the Fourier series. The waveforms of Figure 25–8 are symmetrical around the vertical axis and are said to have **even symmetry** (or **cosine symmetry**).

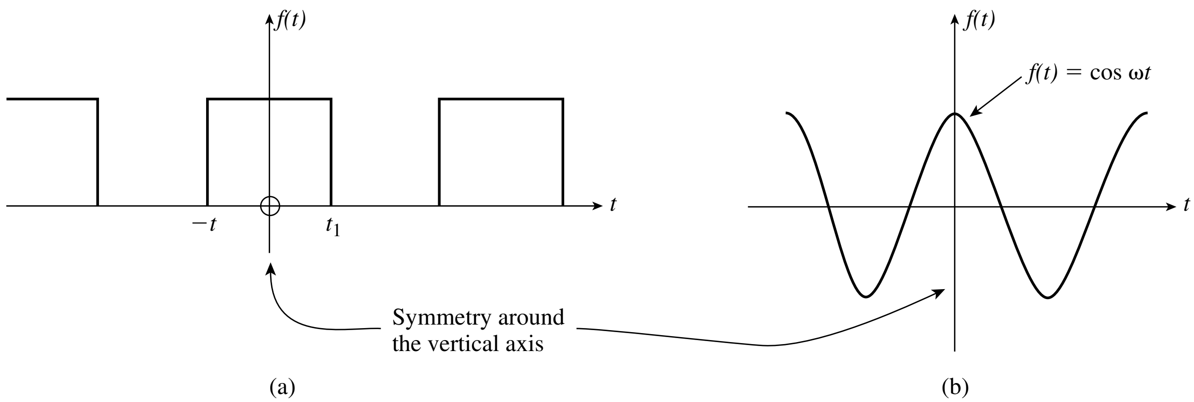



FIGURE 25–8 Even symmetry (cosine symmetry).

Waveforms having even symmetry will always have the form

$$f(-t) = f(t) \qquad \text{(Even symmetry)} \tag{25–10}$$

When the Fourier series of an even symmetry waveform is written, it will only contain cosine ($a_n$) terms and possibly an $a_0$ term. All sine ($b_n$) terms will be zero.

If the portion of the waveform to the right of the vertical axis in each signal of Figure 25–9 is rotated 180°, we find that it will exactly overlap the portion of the waveform to the left of the axis. Such waveforms are said to have **odd symmetry.**

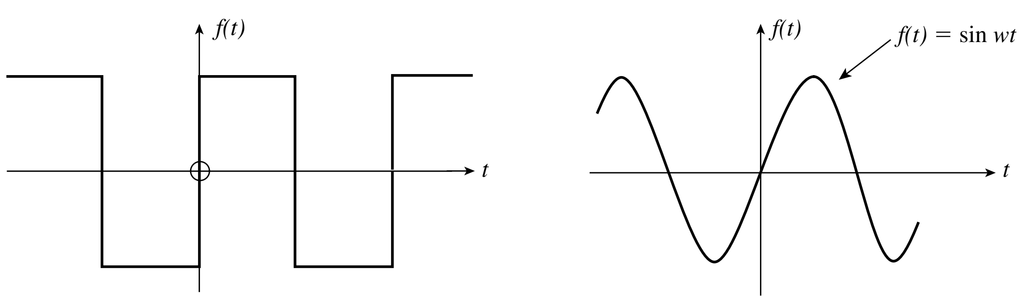



FIGURE 25–9 Odd symmetry (sine symmetry).

Waveforms having odd symmetry will always have the form

$$f(-t) = -f(t) \qquad \text{(Odd symmetry)} \tag{25–11}$$

When the Fourier series of an odd symmetry waveform is written, it will only contain sine ($b_n$) terms and possibly an $a_0$ term. All cosine ($a_n$) terms will be zero.

If the portion of the waveform below the horizontal axis in Figure 25–10 is the mirror image of the portion above the axis, the waveform is said to **half-wave symmetry.**

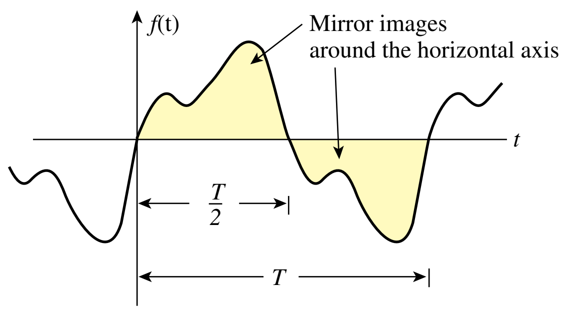



**FIGURE 25–10** Half-wave symmetry.

Waveforms having half-wave symmetry will always have the form

$$f(t + T) = -f(t) \qquad \text{(Half-wave symmetry)} \tag{25–12}$$

When the Fourier series of a half-wave symmetry waveform is written, it will have only odd harmonics and possibly an $a_0$ term. All even harmonic terms will be zero.

If we refer back to the waveform of Figure 25–6, we see that it has both odd symmetry and half-wave symmetry. Using the above rules, we would expect to find only sine terms and odd harmonics. Indeed, we see that Equation 25–9 has these conditions.

## PRACTICE PROBLEMS 2

Consider the ramp function shown in Figure 25–11.

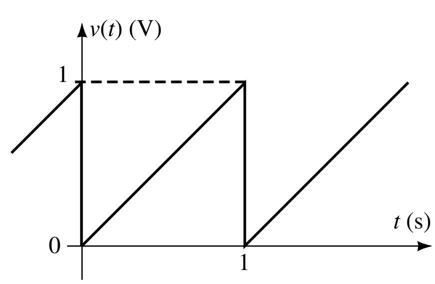



**FIGURE 25–11**

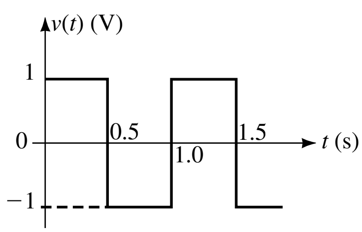
a. Does this wave form show symmetry?
b. Use calculus to determine the Fourier expression for $v(t)$.
c. Verify that the $a_0$ term of the Fourier series is equal to the average value of the waveform.
d. From the Fourier expression, is the ramp function made up of odd harmonics, even harmonics, or all harmonic components? Briefly justify your answer.

*Answers:*
a. Odd symmetry
b. $v(t) = 0.5 - \frac{1}{\pi}\sin(2\pi t) - \frac{1}{2\pi}\sin(4\pi t) - \frac{1}{3\pi}\sin(6\pi t) \cdots$
c. $a_0 = 0.5 \text{ V}$
d. All harmonic components are present since the function does not have half-wave symmetry.

IN-PROCESS LEARNING CHECK 1

Without using calculus, determine a method of rewriting the expression of Equation 25–9 to represent a square wave having an amplitude of 1 V as illustrated in Figure 25–12. *Hint:* Notice that the square wave is similar to the pulse waveform with the exception that its average value is zero and that the peak-to-peak value is twice that of the pulse wave.

FIGURE 25–12

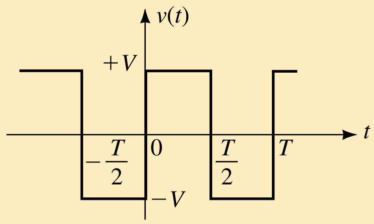



*(Answers are at the end of the chapter.)*

## 25.3 Fourier Series of Common Waveforms

All periodic waveforms can be converted into their Fourier equivalent series by using integration as shown in Section 25.2. Integration of common waveforms is time-consuming and prone to error. A more simple approach is to use tables such as Table 25–1, which gives the Fourier series of several common waveforms encountered in electrical circuits.

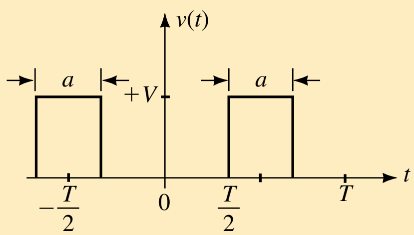
**TABLE 25–1** Fourier Equivalents of Common Waveforms ($\omega = 2\pi/T$)




$$V(t) = \begin{cases} -V, & -\dfrac{T}{2} < t < 0 \\ +V, & 0 < t < \dfrac{T}{2} \end{cases}$$

**FIGURE 25–13**

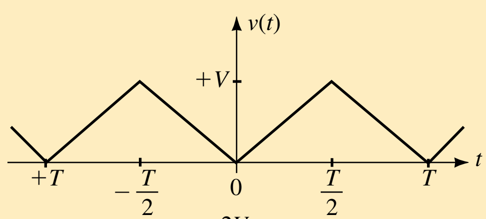
$$v(t) = \frac{4V}{\pi}\left(\sum_{1}^{\infty} \frac{\sin n\omega t}{n}\right) \qquad n = 1, 3, 5, \ldots$$




$$V(t) = \begin{cases} 0, & -\dfrac{T}{2} + a/2 < t < \dfrac{T}{2} - a/2 \\ \text{V}, & \dfrac{T}{2} - a/2 < t < \dfrac{T}{2} + a/2 \end{cases}$$

**FIGURE 25–14**

$$v(t) = \frac{aV}{T} + \frac{2V}{\pi}\left(\sum_{1}^{\infty} (-1)^n \frac{\sin\left(\dfrac{\pi n a}{T}\right)\cos n\omega t}{n}\right) \qquad n = 1, 2, 3, \ldots$$

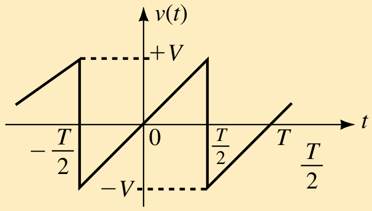



$$V(t) = \begin{cases} -\dfrac{2Vt}{T}, & -\dfrac{T}{2} < t < 0 \\ \dfrac{2Vt}{T}, & 0 < t < \dfrac{T}{2} \end{cases}$$

**FIGURE 25–15**

$$v(t) = \frac{V}{2} - \frac{4V}{\pi^2}\left(\sum_{1}^{\infty} \frac{\cos n\omega t}{n^2}\right) \qquad n = 1, 3, 5, \ldots$$

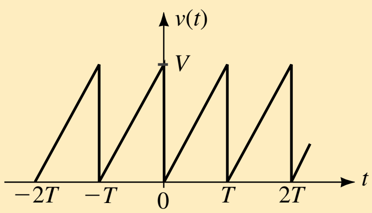
**TABLE 25–1** Fourier Equivalents of Common Waveforms ($\omega = 2\pi/T$) *(continued)*




$$V(t) = \frac{2\,Vt}{T},\ -\frac{T}{2} < t < \frac{T}{2}$$

**FIGURE 25–16**

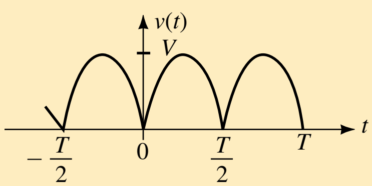
$$v(t) = -\frac{2V}{\pi}\left(\sum_{1}^{\infty}(-1)^n \frac{\sin n\omega t}{n}\right) \qquad n = 1, 2, 3, \ldots$$




$$V(t) = \frac{Vt}{T},\ 0 < t < T$$

**FIGURE 25–17**

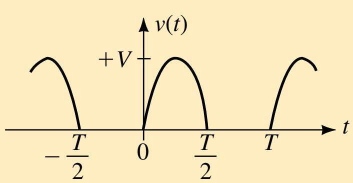
$$v(t) = \frac{V}{2} - \frac{V}{\pi}\left(\sum_{1}^{\infty}\frac{\sin n\omega t}{n}\right) \qquad n = 1, 2, 3, \ldots$$




$$V(t) = \left|V \sin \omega t\right|,\ -\frac{T}{2} < t < \frac{T}{2}$$

**FIGURE 25–18**

$$v(t) = \frac{2V}{\pi} - \frac{4V}{\pi}\left(\frac{\cos 2\omega t}{1 \cdot 3} + \frac{\cos 4\omega t}{3 \cdot 5} + \frac{\cos 6\omega t}{5 \cdot 7} + \cdots\right)$$




$$v(t) = \begin{cases} 0,\ -\frac{T}{2} < t < 0 \\ V \sin \omega t,\ 0 < t < \frac{T}{2} \end{cases}$$

**FIGURE 25–19**

$$v(t) = \frac{V}{\pi} + \frac{V}{2}\sin \omega t - \frac{2V}{\pi}\left(\frac{\cos 2\omega t}{1 \cdot 3} + \frac{\cos 4\omega t}{3 \cdot 5} + \frac{\cos 6\omega t}{5 \cdot 7} + \cdots\right)$$

The following example illustrates how a given waveform is converted into its Fourier series equivalent.

**EXAMPLE 25–3** Use Table 25–1 to determine the Fourier series for the ramp function of Figure 25–20.

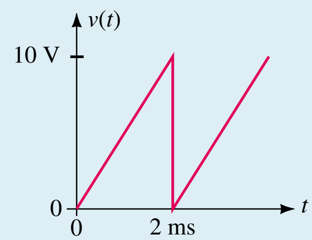



**FIGURE 25–20**

**Solution** The amplitude of the waveform is 10 V and the angular frequency of the fundamental is $\omega = 2\pi/(2 \text{ ms}) = 1000\pi$ rad/s. The resulting series is determined from Table 25–1 as

$$\begin{aligned} v(t) &= \frac{10}{2} - \frac{10}{\pi}\sin 1000\pi t - \frac{10}{2\pi}\sin 2000\pi t \\ &\quad - \frac{10}{3\pi}\sin 3000\pi t - \frac{10}{4\pi}\sin 4000\pi t - \cdots \\ &= 5 - 3.18 \sin 1000\pi t - 1.59 \sin 2000\pi t \\ &\quad - 1.06 \sin 3000\pi t - 0.80 \sin 4000\pi t - \cdots \end{aligned}$$

If a given waveform is similar to one of the types shown in Table 25–1 but is shifted along the time axis, it is necessary to include a phase shift with each of the sinusoidal terms. The phase shift is determined as follows:

1. Determine the period of the given waveform.
2. Compare the given waveform with the figures appearing in Table 25–1 and select which of the waveforms in the table best describes the given wave.
3. Determine whether the given waveform leads or lags the selected figure of Table 25–1. Calculate the amount of the phase shift, as a fraction, *t,* of the total period. Since one complete cycle is equivalent to 360°, the phase shift is determined as

$$\phi = \frac{t}{T} \times 360°$$

4. Write the resulting Fourier expression for the given waveform. If the given waveform leads the selected figure of Table 25–1, then add the angle $\phi$ to each term. If the given waveform lags the selected figure, then subtract the angle $\phi$ from each term.

**EXAMPLE 25–4** Write the Fourier expression for the first four nonzero sinusoidal terms of the waveform shown in Figure 25–21.

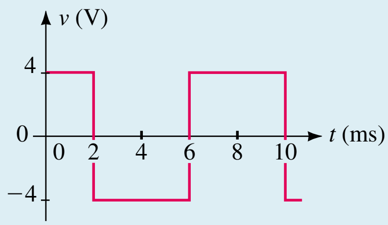



**FIGURE 25–21**

**Solution** **Step 1:** The period of the given waveform is $T = 8.0$ ms, which gives a frequency of $f = 125$ Hz or an angular frequency of $\omega = 250\pi$ rad/s.

**Step 2:** From Table 25–1, we see that the given waveform is similar to the square wave of Figure 25–13.

**Step 3:** The waveform of Figure 25–21 leads the square wave of Figure 25–13 by an amount equivalent to $t = 2$ ms. This corresponds to a phase shift of

$$\phi = \frac{2\text{ ms}}{8\text{ ms}} \times 360° = 90°$$

**Step 4:** The Fourier expression for the first four terms of the waveform of Figure 25–21 is now written as follows:

$$v(t) = \frac{4(4)}{\pi}\sin(250\pi t + 90°) + \frac{4(4)}{3\pi}\sin[3(250\pi t + 90°)]$$
$$+ \frac{4(4)}{5\pi}\sin[5(250\pi t + 90°)] + \frac{4(4)}{7\pi}\sin[7(250\pi t + 90°)]$$

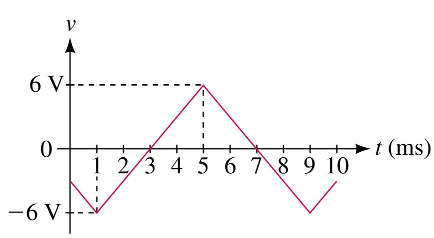
The above expression may be left as the summation of sine waves. However, since the cosine wave leads the sine wave by 90°, the expression may be simplified as a summation of cosine waves without any phase shift. This results in the following:

$$v(t) = 5.09\cos 250\pi t - 1.70\cos 750\pi t + 1.02\cos 1250\pi t - 0.73\cos 1750\,\pi t$$

**PRACTICE PROBLEMS 3**

Write the Fourier expression for the first four nonzero sinusoidal terms of the waveform shown in Figure 25–22. Express each term as a sine wave rather than as a cosine wave.

**FIGURE 25–22**




*Answer:* $v(t) = \frac{48}{\pi^2}\sin(250\pi t - 135°) + \frac{48}{3^2\pi^2}\sin(750\pi t + 135°)$

$+ \frac{48}{5^2\pi^2}\sin(1250\pi t + 45°) + \frac{48}{7^2\pi^2}\sin(1750\pi t - 45°)$

The waveforms of Table 25–1 provide most of the commonly observed waveforms. Occasionally, however, a particular waveform consists of a combination of several simple waveforms. In such a case, it is generally easiest if we first redraw the original waveform as the summation of two or more recognizable waveforms. Then the Fourier series of each of the individual component waves is determined. Finally, the resultant is expressed as the summation of the two series.

**EXAMPLE 25–5** Write the first four nonzero sinusoidal terms of the Fourier series for the waveform of Figure 25–23.

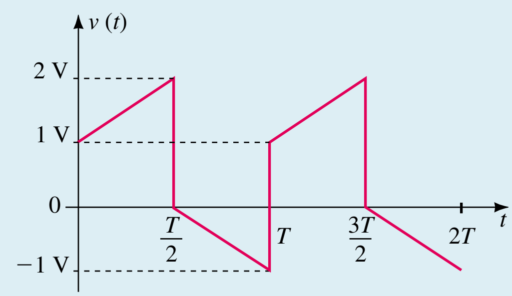



**FIGURE 25–23**

**Solution** The waveform of Figure 25–23 is made up of a combination of waves as illustrated in Figure 25–24.

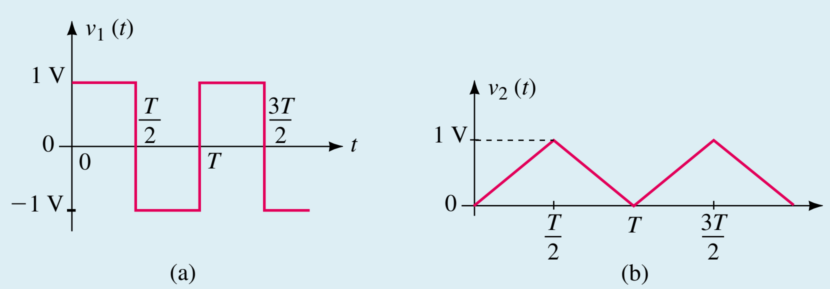



**FIGURE 25–24**

The Fourier series of each of the waveforms is determined from Table 25–1 as

$$v_1(t) = \frac{4}{\pi}\sin \omega t + \frac{4}{3\pi}\sin 3\omega t + \frac{4}{5\pi}\sin 5\omega t + \cdots$$

and

$$v_2(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos \omega t - \frac{4}{3^2\pi^2}\cos 3\omega t - \frac{4}{5^2\pi^2}\cos 5\omega t - \cdots$$

When these series are added algebraically, we get

$$
\begin{aligned}
v(t) &= v_1(t) + v_2(t) \\
&= 0.5 + 1.27 \sin \omega t - 0.41 \cos \omega t \\
&\quad + 0.42 \sin 3\omega t - 0.05 \cos 3\omega t \\
&\quad + 0.25 \sin 5\omega t - 0.02 \cos 5\omega t \\
&\quad + 0.18 \sin 7\omega t - 0.01 \cos 7\omega t \\
&\quad\ \vdots
\end{aligned}
$$

The above series may be further simplified by using Equations 25–6 and 25–7 to provide a single coefficient and phase shift for each frequency. The resultant waveform is written as

$$
\begin{aligned}
v(t) = 0.5 &+ 1.34 \sin(\omega t - 17.7°) + 0.43 \sin(3\omega t - 6.1°) \\
&+ 0.26 \sin(5\omega t - 3.6°) + 0.18 \sin(7\omega t - 2.6°)
\end{aligned}
$$

**PRACTICE PROBLEMS 4**

A composite waveform is made up of the summation of the waveforms illustrated in Figure 25–25.

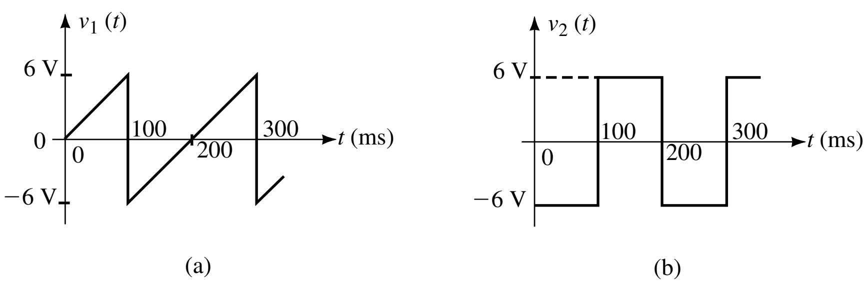



**FIGURE 25–25**

a. Sketch the composite waveform showing all voltage levels and time values.
b. Write the Fourier expression for the resultant waveform, $v(t)$.

*Answers:*

a.

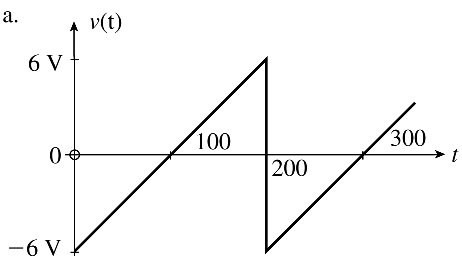


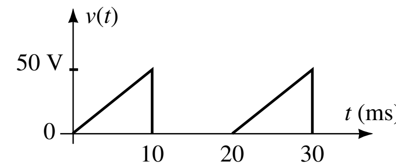

**FIGURE 25–26**

b. $v(t) = \frac{12}{\pi}\sin(10\pi t) - \frac{36}{2\pi}\sin(20\pi t) + \frac{12}{3\pi}\sin(30\pi t) - \cdots$

**IN-PROCESS LEARNING CHECK 2**

Consider the composite waveform of Figure 25–27:

**FIGURE 25–27**

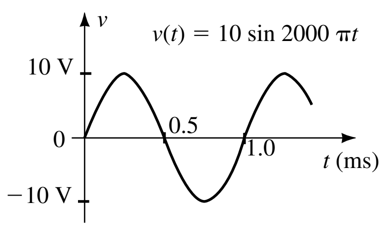



a. Separate the given waveform into two recognizable waveforms appearing in Table 25–1.
b. Use Table 25–1 to write the Fourier series for each of the component waveforms of (a).
c. Combine the results to determine the Fourier series for $v(t)$.

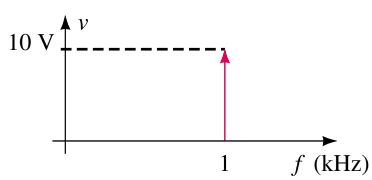
*(Answers are at the end of the chapter.)*

## 25.4 Frequency Spectrum

Most waveforms that we have observed were generally shown as a function of time. However, they may also be shown as a function of frequency. In such cases, the amplitude of each harmonic is indicated at the appropriate frequency. Figure 25–28 shows the display of a 1-kHz sine wave in both the time domain and the frequency domain, while Figure 25–29 shows the corresponding displays for a pulse waveform.

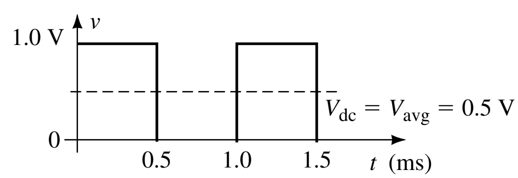



(a) Time-domain display of a 1-kHz sine wave.

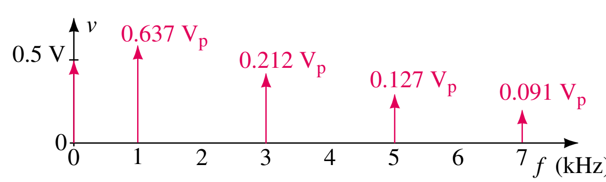



(b) Frequency-domain display of a 1-kHz sine wave.

**FIGURE 25–28**




(a) Time domain display of a 1-kHz pulse wave.




(b) Frequency domain display of a 1-kHz pulse wave.

**FIGURE 25–29**

The frequency spectrum of the pulse wave shows the average value (or dc value) of the wave at a frequency of 0 kHz and illustrates the absence of even harmonics. Notice that the amplitude of successive harmonic components decreases fairly quickly.

The rms voltage of the composite waveform of Figure 25–29 is determined by considering the rms value of each frequency. The resultant rms voltage is found as

$$V_{\mathrm{rms}} = \sqrt{V_{\mathrm{dc}}^2 + V_1^2 + V_2^2 + V_3^2 + \cdots}$$

where each voltage, $V_1$, $V_2$, etc., represents the rms value of the corresponding harmonic component. Using the first five nonzero terms, we would find the rms value of the pulse-wave of Figure 25–29 to be

$$V_{\mathrm{rms}} = \sqrt{(0.500)^2 + \left(\frac{0.637}{\sqrt{2}}\right)^2 + \left(\frac{0.212}{\sqrt{2}}\right)^2 + \left(\frac{0.127}{\sqrt{2}}\right)^2 + \left(\frac{0.091}{\sqrt{2}}\right)^2}$$
$$= 0.698\ \mathrm{V}$$

This value is only slightly smaller than the actual value of $V_{\mathrm{rms}} = 0.707$ V.

If the pulse wave were applied to a resistive element, power would be dissipated as if each frequency component had been applied independently. The total power can be determined by the summation of the individual contributions of each frequency. In order to calculate the power dissipated at each sinusoidal frequency, we need to first convert the voltages into rms values. The frequency spectrum may then be represented in terms of power rather than as voltage.

**EXAMPLE 25–6** Determine the total power dissipated by a 50-Ω resistor if the pulse waveform of Figure 25–29 is applied to the resistor. Consider the dc component and the first four nonzero harmonics. Indicate the power levels (in watts) on a frequency distribution curve.

**Solution** The power dissipated by the dc component is determined as

$$P_0 = \frac{V_0^2}{R_L} = \frac{(0.5\ \mathrm{V})^2}{50\ \Omega} = 5.0\ \mathrm{mW}$$

The power dissipated by any resistor subjected to a sinusoidal frequency is determined as

$$P = \frac{V_{\mathrm{rms}}^2}{R_L} = \frac{\left(\frac{V_P}{\sqrt{2}}\right)^2}{R_L} = \frac{V_P^2}{2R_L}$$

For the pulse wave of Figure 25–29, the power due to each of the first four nonzero sinusoidal components is found as follows:

$$P_1 = \frac{\left(\frac{2\ \mathrm{V}}{\pi}\right)^2}{(2)(50\ \Omega)} = 4.05\ \mathrm{mW}$$

$$P_3 = \frac{\left(\dfrac{2\text{ V}}{3\pi}\right)^2}{(2)(50\ \Omega)} = 0.45\text{ mW}$$

$$P_5 = \frac{\left(\dfrac{2\text{ V}}{5\pi}\right)^2}{(2)(50\ \Omega)} = 0.16\text{ mW}$$

$$P_7 = \frac{\left(\dfrac{2\text{ V}}{7\pi}\right)^2}{(2)(50\ \Omega)} = 0.08\text{ mW}$$

Figure 25–30 shows the power levels (in milliwatts) as a function of frequency.

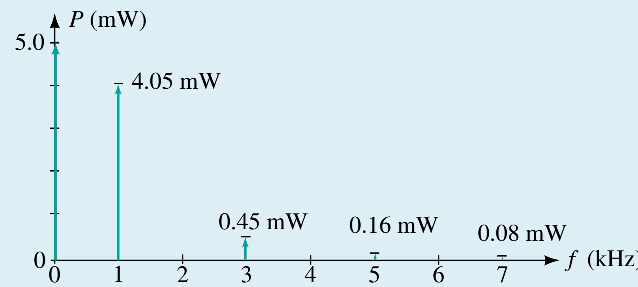



**FIGURE 25–30**

Using only the dc component and the first four nonzero harmonics, the total power dissipated by the resistor is $P_T = 9.74$ mW. From Chapter 15, the actual rms voltage of the pulse waveform is found to be

$$V_{rms} = \sqrt{\frac{(1\text{ V})^2(0.5\text{ ms})}{1.0\text{ ms}}} = 0.707\text{ V}$$

Therefore, using the rms voltage, the power dissipated by the resistor is found as

$$P = \frac{(0.707\text{ V})^2}{50\ \Omega} = 10.0\text{ mW}$$

Although the pulse waveform has power contained in components with frequencies above the seventh harmonic, we see that more than 97% of the total power of a pulse waveform is contained in only the first seven harmonics.

Power levels and frequencies of the various harmonics of a periodic waveform may be measured with an instrument called a **spectrum analyzer,** shown in Figure 25–31.

Some spectrum analyzers are able to display either voltage levels or power levels in the frequency domain, while most others display only power levels (in dBm). When displaying power levels, the spectrum analyzer usually uses a reference 50-Ω load. Figure 25–32 shows the display of a 1.0-V pulse waveform as it would appear on a typical spectrum analyzer.


FIGURE 25–31 Spectrum analyzer *(Courtesy of Tektronix Inc.)*

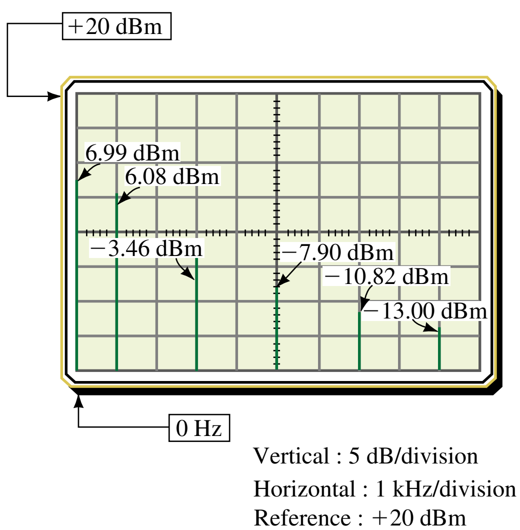



FIGURE 25–32 Spectrum analyzer display of a 1-kHz pulse wave having a peak of 1 V.

Notice that the spectrum analyzer has a reference of +20 dBm and that this reference is shown at the top of the display rather than at the bottom. The vertical scale of the instrument is measured in decibels, where each vertical division corresponds to 5 dB. The horizontal axis for the spectrum analyzer is scaled in hertz, where each division in Figure 25–32 corresponds to 1 kHz.

PRACTICAL NOTES...

Spectrum analyzers are very sensitive instuments. As a result, great care must be taken to ensure that the input power never exceeds the rated maximum. When in doubt, it is best to insert an extra attenuator to lower the amount of power entering the spectrum analyzer.

**EXAMPLE 25–7** A spectrum analyzer with a 50-Ω input is used to display the power levels in dBm of the Fourier series components of the ramp waveform shown in Figure 25–33.

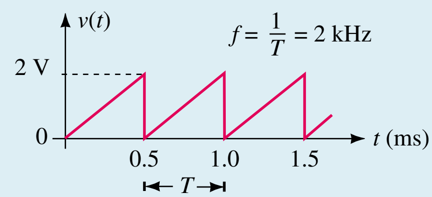



**FIGURE 25–33**

Determine the voltage and power levels of the various components and sketch the resultant display as it would appear on a spectrum analyzer. Assume that the spectrum analyzer has the same vertical and horizontal settings as those shown in Figure 25–32.

**Solution** The Fourier series of the given waveform is determined from Table 25–1 as

$$v(t) = \frac{2}{2} - \frac{2}{\pi}\sin \omega t - \frac{2}{2\pi}\sin 2\omega t - \frac{2}{3\pi}\sin 3\omega t - \cdots$$

Because the fundamental frequency occurs at $f = 2$ kHz, we see that harmonic frequencies will occur at 4 kHz, 6 kHz, etc. However, because the spectrum analyzer is able to display only up to 10 kHz, we need go no further.

The dc component will have an average value of $v_0 = 1.0$ V, as expected. The rms values of the harmonic sinusoidal waveforms are determined as

$$V_{\text{rms}} = \frac{V_p}{\sqrt{2}}$$

which gives the following:

$$V_{1(\text{rms})} = \frac{2}{\pi\sqrt{2}} = 0.450 \text{ V}$$

$$V_{2(\text{rms})} = \frac{2}{2\pi\sqrt{2}} = 0.225 \text{ V}$$

$$V_{3(\text{rms})} = \frac{2}{3\pi\sqrt{2}} = 0.150 \text{ V}$$

$$V_{4(\text{rms})} = \frac{2}{4\pi\sqrt{2}} = 0.113\ \text{V}$$

$$V_{5(\text{rms})} = \frac{2}{5\pi\sqrt{2}} = 0.090\ \text{V}$$

The above rms voltages are used to calculate the powers (and power levels in dBm) of the various harmonic components.

$$P_0 = \frac{(1.0\ \text{V})^2}{50\ \Omega} = 20.0\ \text{mW} \equiv 10\log\frac{20\ \text{mW}}{1\ \text{mW}} = 13.0\ \text{dBm}$$

$$P_1 = \frac{(0.450\ \text{V})^2}{50\ \Omega} = 4.05\ \text{mW} \equiv 10\log\frac{4.05\ \text{mW}}{1\ \text{mW}} = 6.08\ \text{dBm}$$

$$P_2 = \frac{(0.225\ \text{V})^2}{50\ \Omega} = 1.01\ \text{mW} \equiv 10\log\frac{1.01\ \text{mW}}{1\ \text{mW}} = 0.04\ \text{dBm}$$

$$P_3 = \frac{(0.150\ \text{V})^2}{50\ \Omega} = 0.45\ \text{mW} \equiv 10\log\frac{0.45\ \text{mW}}{1\ \text{mW}} = -3.5\ \text{dBm}$$

$$P_4 = \frac{(0.113\ \text{V})^2}{50\ \Omega} = 0.25\ \text{mW} \equiv 10\log\frac{0.25\ \text{mW}}{1\ \text{mW}} = -6.0\ \text{dBm}$$

$$P_5 = \frac{(0.090\ \text{V})^2}{50\ \Omega} = 0.16\ \text{mW} \equiv 10\log\frac{0.16\ \text{mW}}{1\ \text{mW}} = -7.9\ \text{dBm}$$

A spectrum analyzer would indicate a display similar to that shown in Figure 25–34.

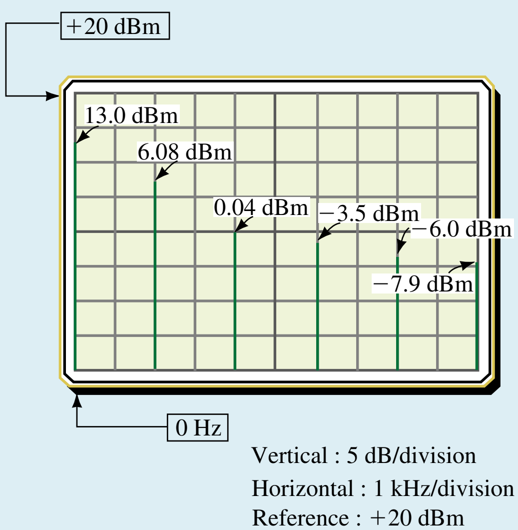



**FIGURE 25–34**

The sawtooth waveform of Figure 25–35 is applied to a 50-Ω spectrum analyzer. Sketch the display that would be observed, assuming that the spectrum analyzer has the same vertical and horizontal settings as those shown in Figure 25–32.

**PRACTICE PROBLEMS 5**

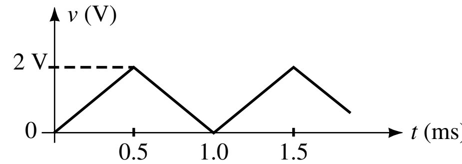



FIGURE 25–35

*Answers:* $P_{dc} = +13.0$ dBm, $P_{1kHz} = +8.2$ dBm, $P_{3kHz} = -10.9$ dBm, $P_{5kHz} = -19.8$ dBm. (All other components are less than $-20$ dBm and so will not appear.)

## 25.5 Circuit Response to a Nonsinusoidal Waveform

We have determined that all periodic, nonsinusoidal waveforms are comprised of numerous sinusoidal components together with a dc component. In Chapter 22 we observed how various frequencies were affected when they were applied to a given filter. We will now examine how the frequency components of a waveform will be modified when applied to the input of a given filter.

Consider what happens when a pulse waveform is applied to a bandpass filter tuned to the third harmonic, as shown in Figure 25–36.

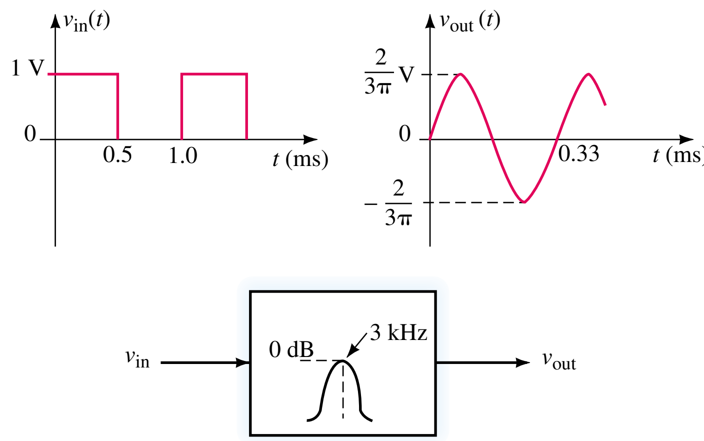



FIGURE 25–36

Since the bandpass filter is tuned to the third harmonic, only the frequency component corresponding to this harmonic will be passed from the input of the filter to the output. Further, because the filter has a voltage gain of 0 dB at the center frequency, the amplitude of the resulting sinusoidal output will be the same voltage level as the amplitude of the original third harmonic. All other frequencies, including the dc component, will be attenuated by the filter, and so are effectively eliminated from the output.

This method is used extensively in electronic circuits to provide frequency multiplication, since any distorted waveform will be rich in harmonics. The desired frequency component is easily extracted by using a tuned filter circuit. Although any integer multiplication is theoretically possible, most frequency multiplier circuits are either frequency doublers or frequency triplers since higher-order harmonics have much lower amplitudes.

In order to determine the resulting waveform after it passes through any other filter, it is necessary to determine the amplitude and phase shift of numerous harmonic components.

**EXAMPLE 25–8** The circuit of Figure 25–37 has the frequency response shown in Figure 25–38.

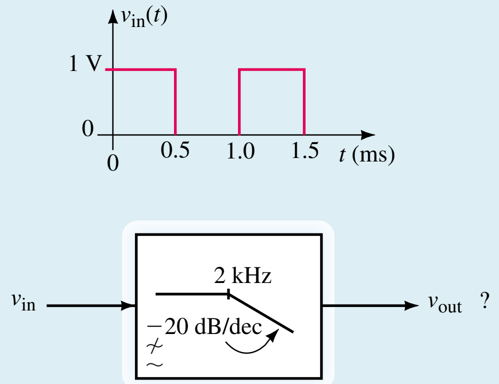



**FIGURE 25–37**

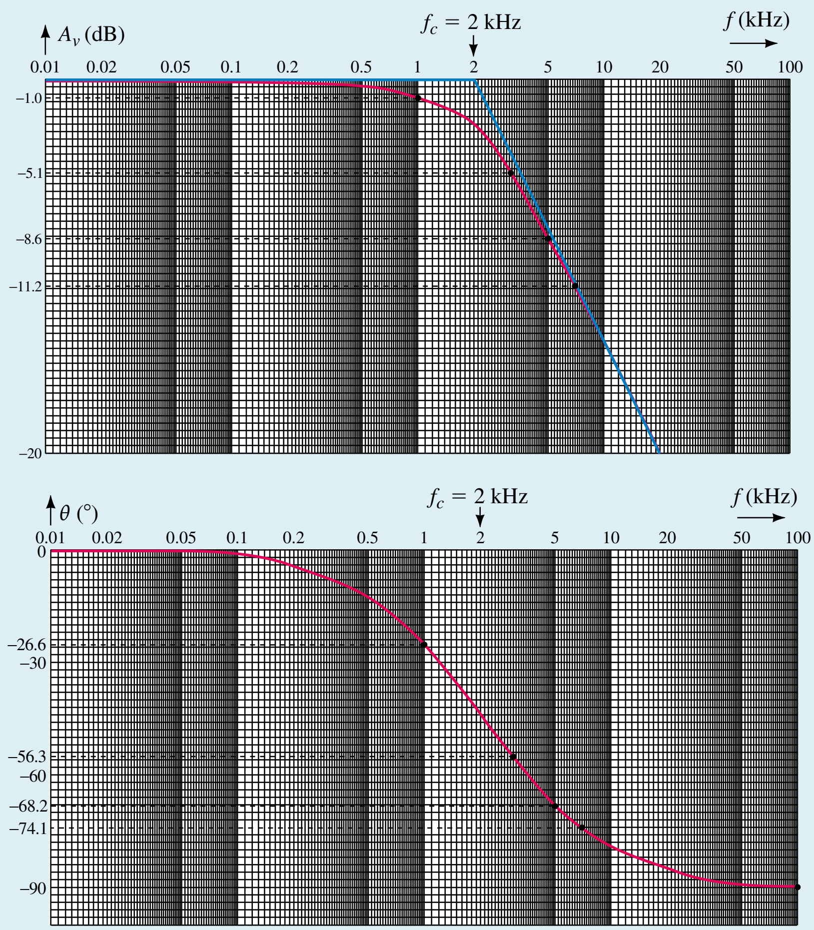



**FIGURE 25–38**

a. Determine the dc component at the output of the low-pass filter.
b. Calculate the amplitude and corresponding phase shift of the first four non-zero sinusoidal output components.

**Solution**

a. From previous examples, we have determined that the given waveform is expressed by the following Fourier series:

$$v(t) = 0.5 + \frac{2}{\pi}\sin \omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \frac{2}{7\pi}\sin 7\omega t$$

Since the circuit is a low-pass filter, we know that the dc component will pass from the input to the output without being attenuated. Therefore

$$V_{0(\text{out})} = V_{0(\text{in})} = 0.5\ \text{Vdc}$$

b. By examining the frequency response of Figure 25–36, we see that all sinusoidal components will be attenuated and phase shifted. From the graphs we have the following:

$$\begin{array}{lll}
1\ \text{kHz:} & A_{V1} = -1.0\ \text{dB}, & \Delta\theta_1 = -26.6^\circ \\
3\ \text{kHz:} & A_{V3} = -5.1\ \text{dB}, & \Delta\theta_3 = -56.3^\circ \\
5\ \text{kHz:} & A_{V5} = -8.6\ \text{dB}, & \Delta\theta_5 = -68.2^\circ \\
7\ \text{kHz:} & A_{V7} = -11.2\ \text{dB}, & \Delta\theta_7 = -74.1^\circ
\end{array}$$

The amplitudes of the various harmonics at the output of the filter are determined as follows:

$$V_{1(\text{out})} = \left(\frac{2}{\pi}\right)10^{-1.0/20} = 0.567\ \text{V}_\text{p}$$

$$V_{3(\text{out})} = \left(\frac{2}{3\pi}\right)10^{-5.1/20} = 0.118\ \text{V}_\text{p}$$

$$V_{5(\text{out})} = \left(\frac{2}{5\pi}\right)10^{-8.6/20} = 0.047\ \text{V}_\text{p}$$

$$V_{7(\text{out})} = \left(\frac{2}{7\pi}\right)10^{-11.2/20} = 0.025\ \text{V}_\text{p}$$

The Fourier series of the output waveform is now approximated as

$$\begin{aligned} v(t) = 0.5 &+ 0.567 \sin(\omega t - 26.6^\circ) + 0.118 \sin(3\,\omega t - 56.3^\circ) \\ &+ 0.047 \sin(5\,\omega t - 68.2^\circ) + 0.025 \sin(7\,\omega t - 74.1^\circ) \end{aligned}$$

Various computer-aided design (CAD) and mathematical applications programs are able to generate a time-domain display of a waveform from a mathematical expression. When the above waveform is plotted in the time domain, it appears as illustrated in Figure 25–39.

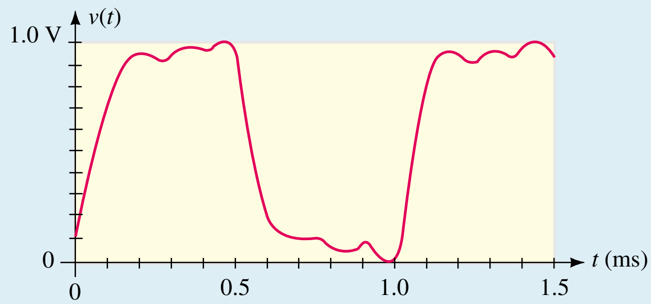



**FIGURE 25–39**

PRACTICE PROBLEMS 6

The voltage waveform of Figure 25–40 is applied to a high-pass filter circuit having the frequency response shown in Figure 25–41.

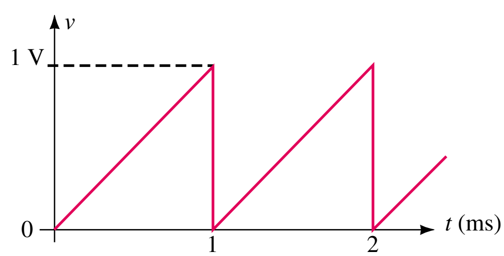
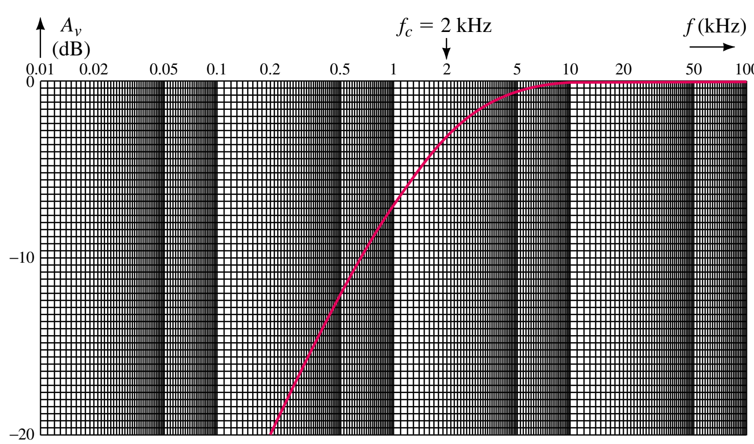
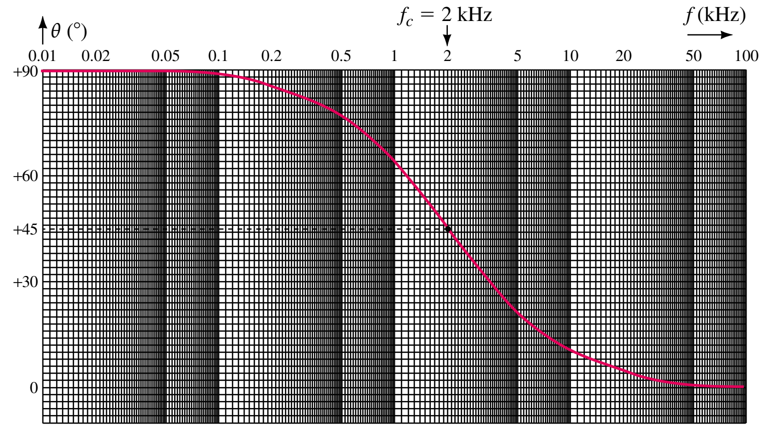



**FIGURE 25–40**




(a) Voltage gain response for a high-pass filter

**FIGURE 25–41** *(Continues)*




(b) Phase shift response for a high-pass filter

**FIGURE 25–41** *(Continued)*

a. Determine the dc component at the output of the high-pass filter.

b. Calculate the amplitude and corresponding phase shift of the first four nonzero sinusoidal output components.

*Answers:*

a. zero

b. 1 kHz: 0.142 $V_P$ $-116°$
2 kHz: 0.113 $V_P$ $-135°$
3 kHz: 0.088 $V_P$ $-147°$
4 kHz: 0.071 $V_P$ $-154°$

## 25.6 Circuit Analysis Using Computers

PSpice can be used to help visualize the frequency spectrum at the input and the output of a given circuit. By comparing the input and the output, we are able to observe how a given circuit distorts the waveform due to attenuation and phase shift of the various frequency components.

In the following example, we use a low-pass filter that has a cutoff frequency of 3 kHz. We will observe the effects of the filter on a 1-V pulse wave. In order to complete the required analysis, it is necessary to set up PSpice correctly.

**EXAMPLE 25–9** Use PSpice to find the Fourier series for both the input and the output waveforms for the circuit shown in Figure 25–42. Use the Probe postprocessor to obtain both a time-domain and a frequency-domain display of the input and output waveforms.

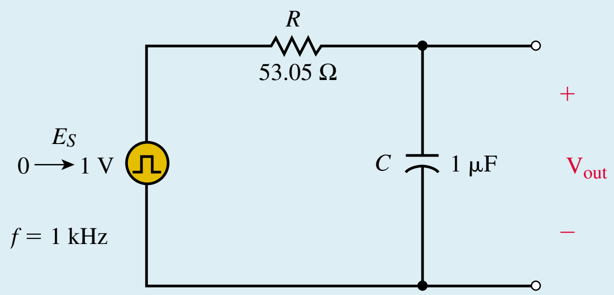



**FIGURE 25–42**

**Solution** The circuit is input as shown in Figure 25–43.

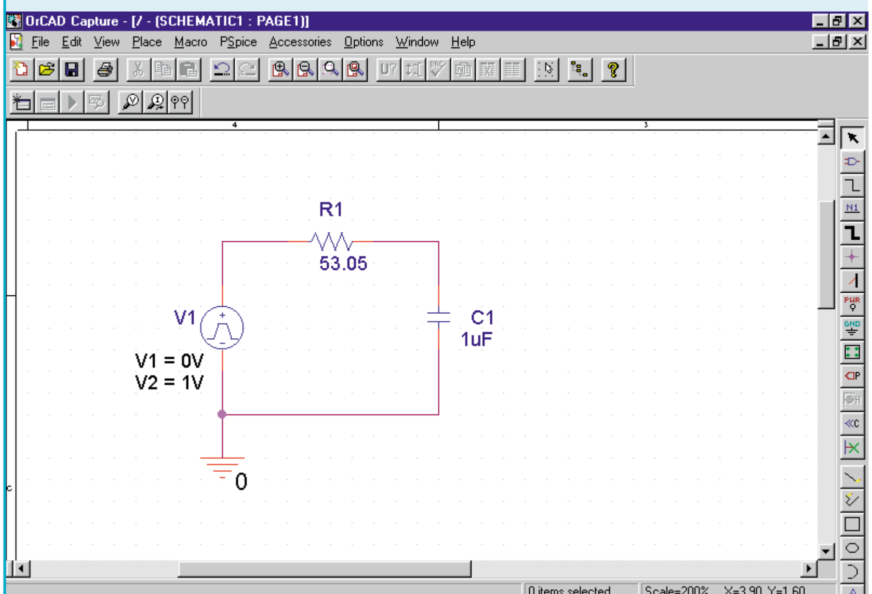



**FIGURE 25–43**

The voltage source is a pulse generator and is obtained from the SOURCE library by calling for VPULSE. The properties for the pulse generator are set as follows. V1=**0V,** V2=**1V,** TD=**0,** TR=**0.01us,** TF=**0.01us,** PW=**0.5ms,** PER=**1.0ms.**

We begin the analysis by first setting the simulation settings for Time Domain (Transient) analysis. Run to time is set for **2.0ms** and the Maximum step size is set for **2us.** Now the analysis can be run.

Once in the Probe window, we will simultaneously display both the input and the output waveforms. Click on Trace and Add Trace. Enter **V(V1:+), V(C1:1)** in the Trace Expression dialog box. You will see the time-domain output as shown in Figure 25–44.

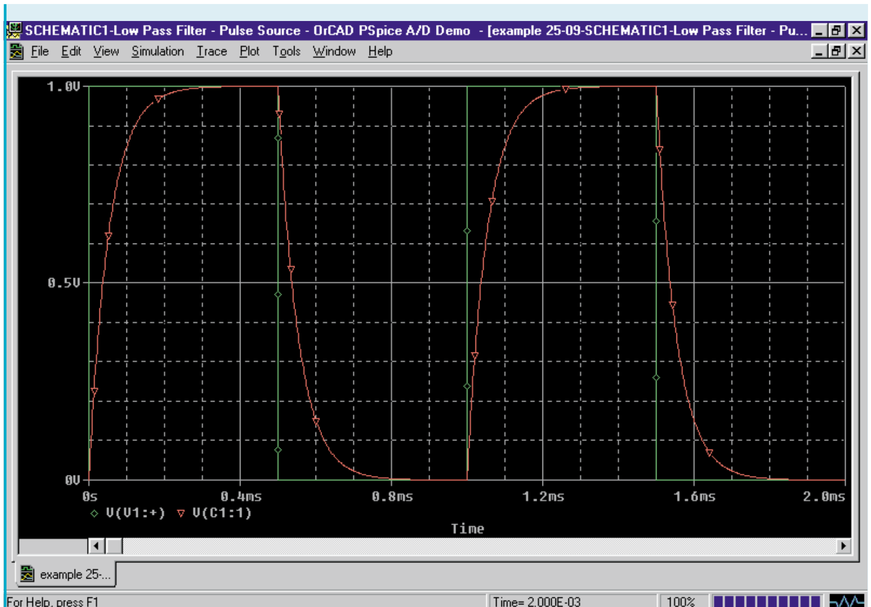



**FIGURE 25–44**

In order to obtain the frequency-domain display, simply click on Trace and Fourier. You will need to adjust the range of the abscissa by clicking on Plot and Axis Settings. Click on the X Axis tab and change the range from 0Hz to 10kHz. The display will appear as shown in Figure 25–45.

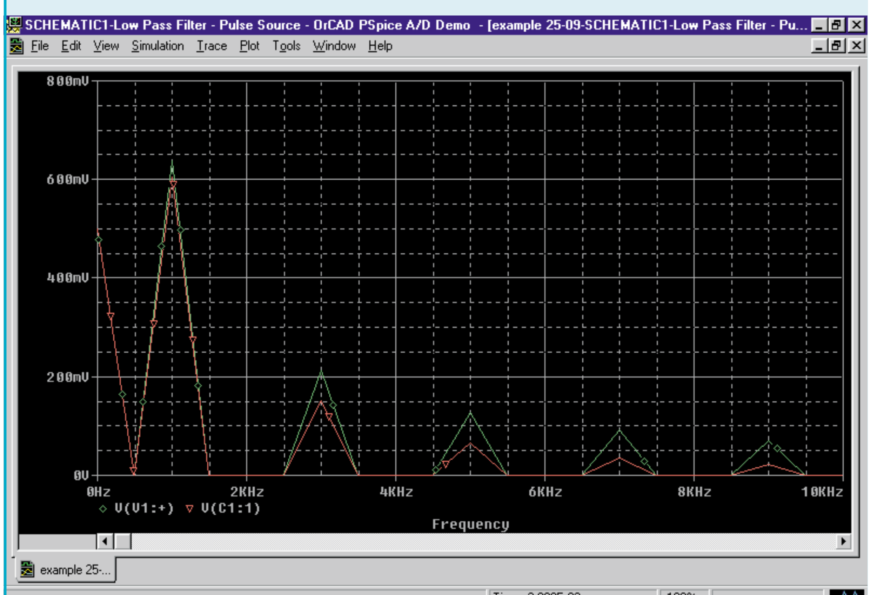



**FIGURE 25–45**

Notice that the output voltage of the third harmonic (3 kHz) is approximately 0.15 V, while the input voltage of the same harmonic is approximately 0.21 V. As expected, this represents approximately 3 dB of attenuation between the input and the output at the cutoff frequency.

## PUTTING IT INTO PRACTICE

One method of building a frequency multiplier circuit is to generate a signal which is "rich" in harmonics. A full-wave rectifier is a circuit that converts a sine wave (which consists of only one frequency component) into one which appears as shown in Figure 25–17. As we see, the waveform at the output of the full-wave rectifier is composed of an infinite number of harmonic components. By applying this signal to a narrow–band-pass filter, it is possible to select any one of the components. The resulting output will be a pure sine wave at the desired frequency.

If a sine wave with an amplitude of 10 V is applied to a full-wave rectifier, what will be the amplitude and frequency at the output of a passive filter tuned to the third harmonic? Assume that there are no losses in the full-wave rectifier or in the filter circuit.

## PROBLEMS

### 25.1 Composite Waveforms

1. a. Determine the rms voltage of the waveform shown in Figure 25–46.
   b. If this waveform is applied to a 50-Ω resistor, how much power will be dissipated by the resistor?

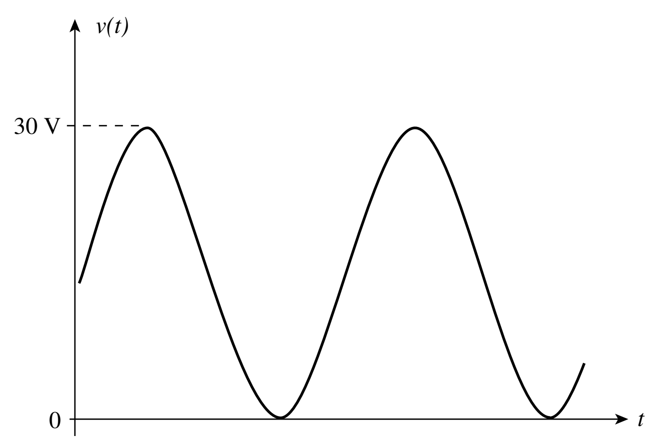



FIGURE 25–46

2. Repeat Problem 1 if the waveform of Figure 25–47 is applied to a 250-Ω resistor.

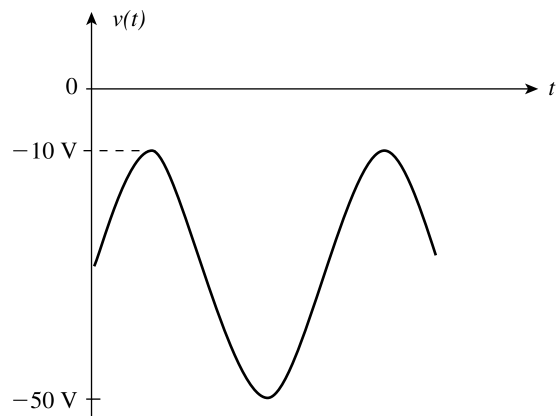



FIGURE 25–47

3. Repeat Problem 1 if the waveform of Figure 25–48 is applied to a 2.5-kΩ resistor.

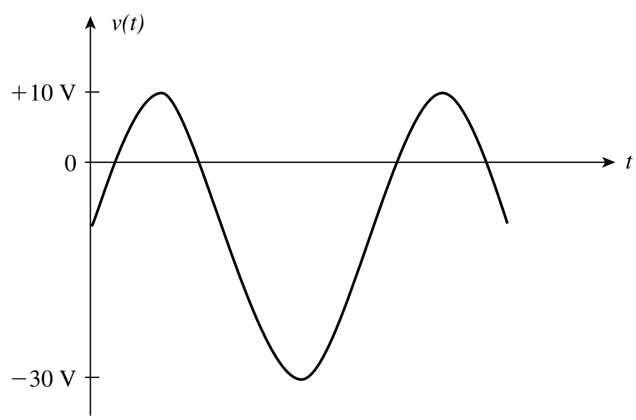



FIGURE 25–48

4. Repeat Problem 1 if the waveform of Figure 25–49 is applied to a 10-kΩ resistor.

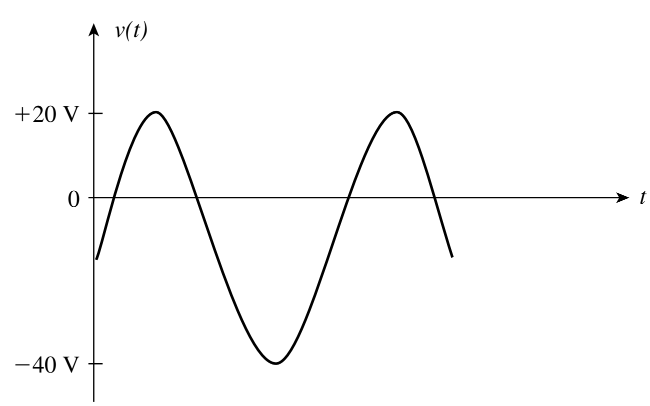



FIGURE 25–49

## 25.2 Fourier Series

5. Use calculus to derive the Fourier series for the waveform shown in Figure 25–50.

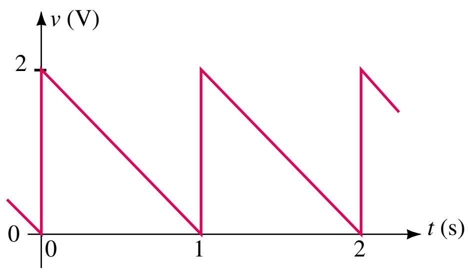



FIGURE 25–50

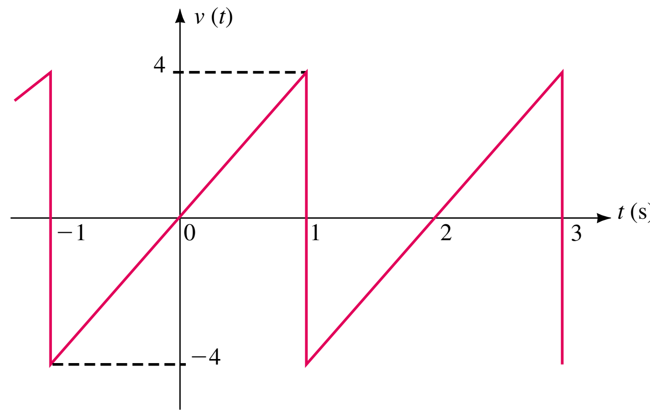



FIGURE 25–51

6. Repeat Problem 5 for the waveform shown in Figure 25–51.

## 25.3 Fourier Series of Common Waveforms

7. Use Table 25–1 to determine the Fourier series for the waveform of Figure 25–52.

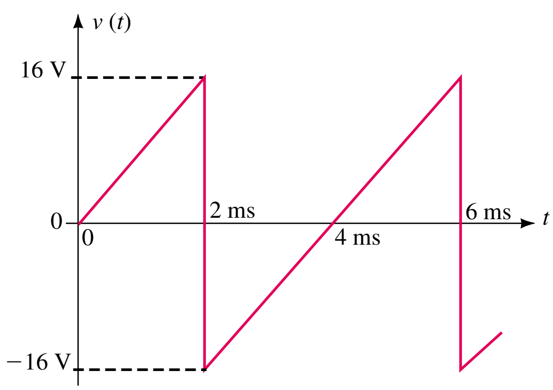



FIGURE 25–52

8. Repeat Problem 7 for the waveform of Figure 25–53.

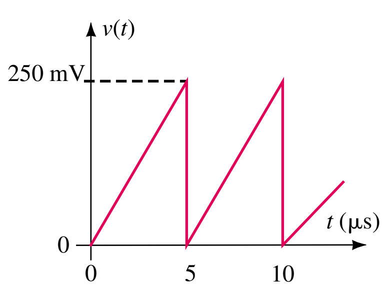



FIGURE 25–53

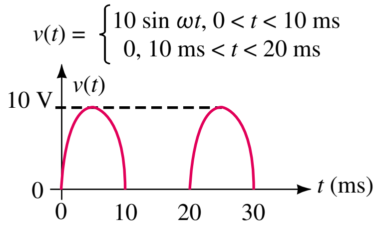



FIGURE 25–54

9. Repeat Problem 7 for the waveform of Figure 25–54.
10. Repeat Problem 7 for the waveform of Figure 25–55.

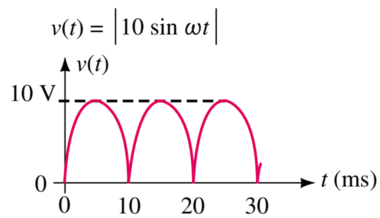



FIGURE 25–55

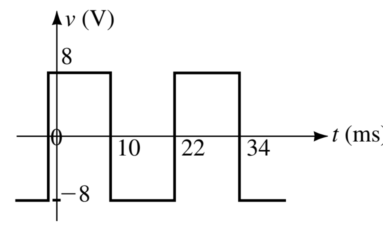



FIGURE 25–56

11. Write the expression including the first four sinusoidal terms of the Fourier series for the waveform of Figure 25–56.
12. Repeat Problem 11 for the waveform of Figure 25–57.

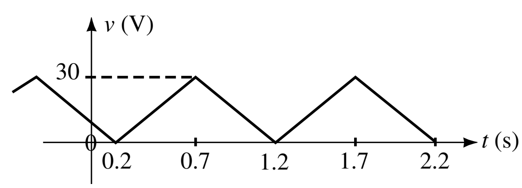



FIGURE 25–57

13. A composite waveform is made up of the two periodic waves shown in Figure 25–58.
    a. Sketch the resulting waveform.
    b. Write the Fourier series of the given waveforms.
    c. Determine the Fourier series of the resultant.

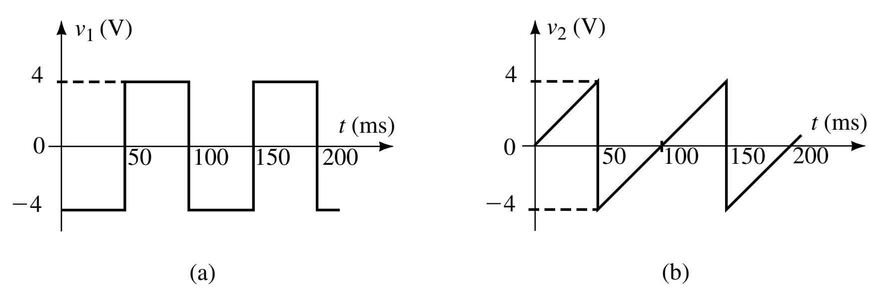



FIGURE 25–58

14. Repeat Problem 13 for the periodic waveforms shown in Figure 25–59.

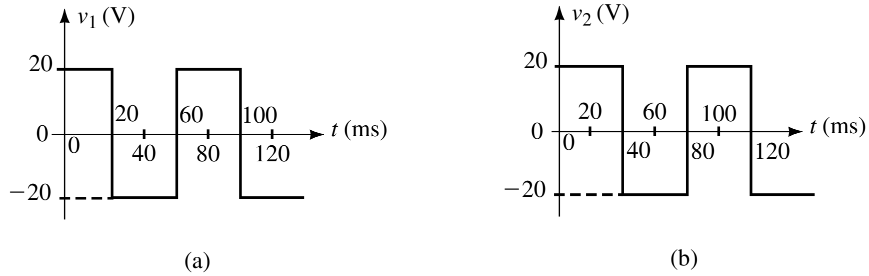



FIGURE 25–59

15. A composite waveform is made up of the two periodic waves shown in Figure 25–60.
    a. Sketch the resulting waveform.
    b. Solve for the dc value of the resultant.
    c. Write the Fourier series of the given waveforms.
    d. Determine the Fourier series of the resultant.

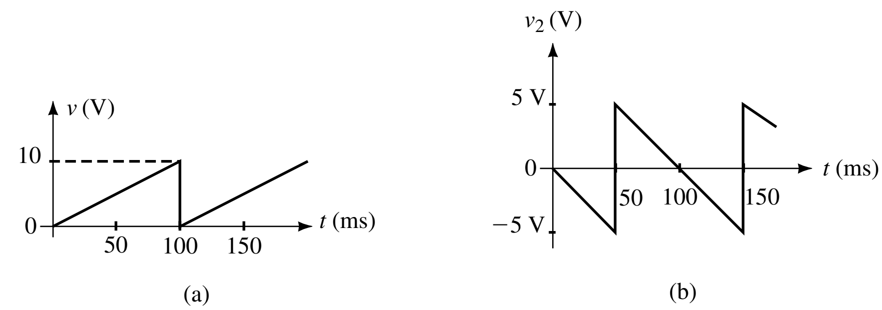



FIGURE 25–60

16. Repeat Problem 15 for the waveforms shown in Figure 25–61.

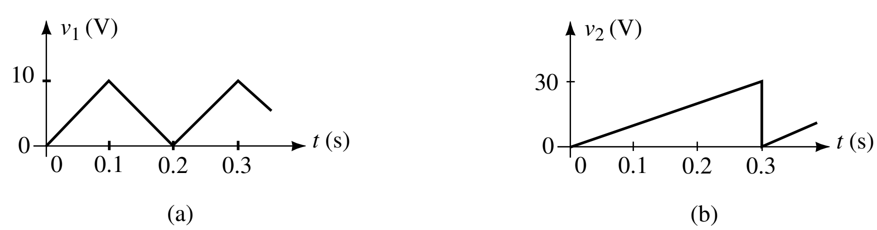



FIGURE 25–61

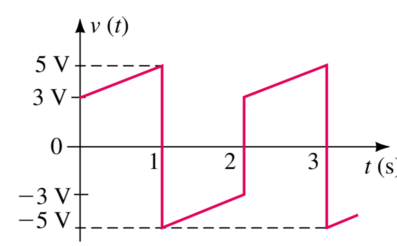



FIGURE 25–62

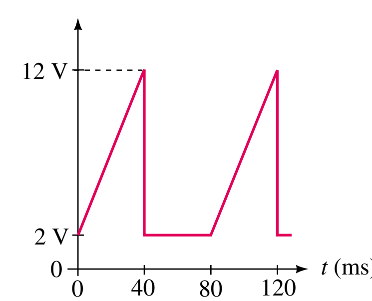



FIGURE 25–63

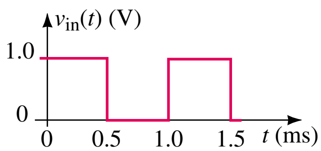
17. The waveform of Figure 25–62 is made up of two fundamental waveforms from Table 25–1. Sketch the two waveforms and determine the Fourier series of the composite wave.
18. The waveform of Figure 25–63 is made up of a dc voltage combined with two fundamental waveforms from Table 25–1. Determine the dc voltage and sketch the two waveforms. Determine the Fourier series of the composite wave.

### 25.4 Frequency Spectrum

19. Determine the total power dissipated by a 50-Ω resistor, if the voltage waveform of Figure 25–52 is applied to the resistor. Consider the dc component and the first four nonzero harmonics. Indicate the power levels (in watts) on a frequency distribution curve.
20. Repeat Problem 19 for the waveform of Figure 25–53.
21. A spectrum analyzer with a 50-Ω input is used to measure the power levels in dBm of the Fourier series components of the waveform shown in Figure 25–54. Determine the power levels (in dBm) of the dc component and the first four nonzero harmonic components. Sketch the resultant display as it would appear on a spectrum analyzer.
22. Repeat Problem 21 if the waveform of Figure 25–55 is applied to the input of the spectrum analyzer.

### 25.5 Circuit Response to a Nonsinusoidal Waveform

23. The circuit of Figure 25–64 has the frequency response shown in Figure 25–65.
    a. Determine the dc component at the output of the filter.



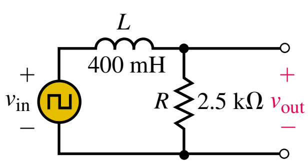




FIGURE 25–64

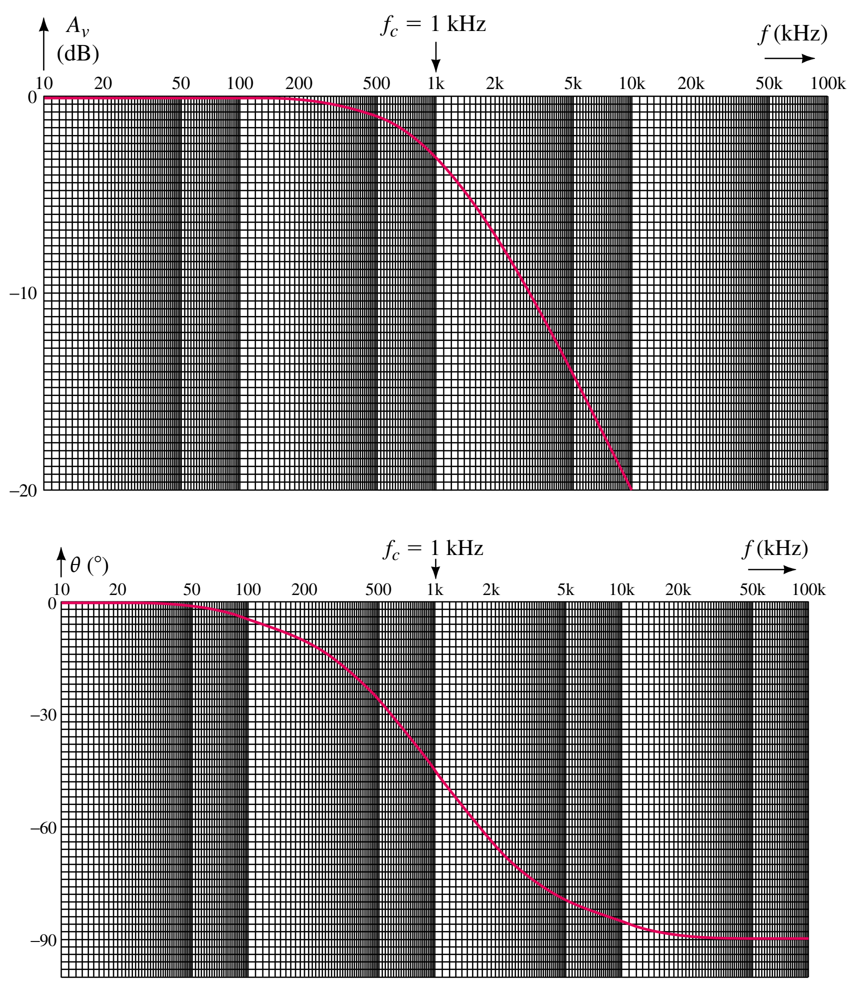



FIGURE 25–65

b. Calculate the amplitude and corresponding phase shift of the first four nonzero sinusoidal output components.

24. Repeat Problem 33 for Figures 25–66 and 25–67.

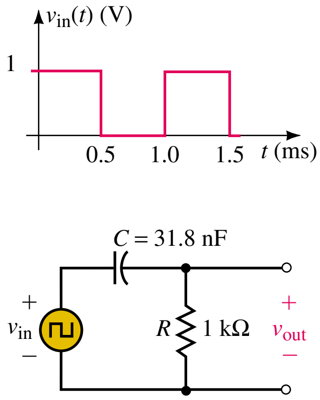



FIGURE 25–66

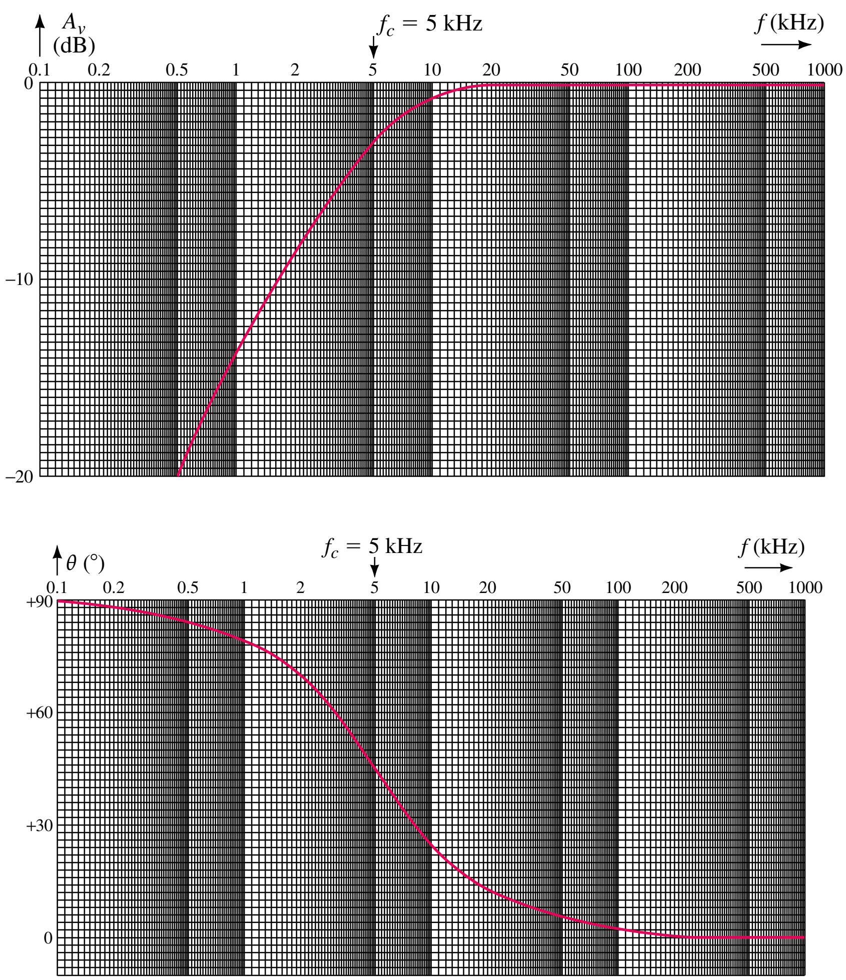



FIGURE 25–67

### 25.6 Circuit Analysis Using Computers

25. **PSpice** Use PSpice to find the Fourier series for both the input and output waveforms for the circuit of Figure 25–64. Use the Probe postprocessor to obtain both a time-domain and frequency-domain display of the output waveform. Compare your results to those obtained in Problem 23.
26. **PSpice** Repeat Problem 25 for the circuit of Figure 25–66. Compare your results to those obtained in Problem 24.

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## ANSWERS TO IN-PROCESS LEARNING CHECKS

**In-Process Learning Check 1**

The waveform of Figure 25–12 has an average value of zero. Therefore, $a_0 = 0$. The waveform also has a peak-to-peak value which is double that of Figure 25–6, which means that the amplitude of each harmonic must be doubled.

$$v(t) = \frac{4}{\pi}\sin \omega t + \frac{4}{3\pi}\sin 3\omega t + \frac{4}{5\pi}\sin 5\omega t + \cdots$$

**In-Process Learning Check 2**

a. Figure 25–15 with $V_{\mathrm{m}} = 25$ V and $T = 20$ ms
b. Figure 25–16 with $V_{\mathrm{m}} = 25$ V and $T = 20$ ms
c. $v(t) = 12.5 + 18.9 \sin(100\pi t - 32.48°) - 7.96 \sin(200\pi t) + 5.42 \sin(300\pi t - 11.98°) - 3.98 \sin(400\pi t)$

# OrCAD–PSpice A/D

**APPENDIX A**

OrCAD PSpice (formerly MicroSim PSpice) is a circuit simulation program designed to analyze electrical and electronic problems using a graphical representation of the circuit. Anyone who has worked with previous versions of the PSpice software will find many similarities between those versions and the latest release. The applications used in this textbook will help make the transition a little easier. For those who have never worked with PSpice, this appendix provides the required steps to guide you through the sometimes-complicated world of software application. A detailed description of the software is provided in the *OrCAD Capture User's Guide*. For an evaluation CD of OrCAD, contact OrCAD at their web site, www.orcad.com. The software consists of several parts, of which we use only the schematic capture and PSpice simulation systems.

## A.1 Getting Started

You will need to install the software from the CD-ROM. The instructions provided by OrCAD will guide you through this process. Since the installation cannot proceed successfully with virus detection software running, it is necessary to disable this software during the installation.

After you have installed the OrCAD software, the Capture program can be started as follows:

1. Click the Start menu on the Windows desktop,
2. Select the OrCAD Demo sub-folder from the Program item,
3. Click on Capture CIS Demo. The Capture *session frame* will open.

Your schematic design and processing will all be done in this frame. To start a new project in the OrCAD Capture session frame, click on the File menu, New menu item, and Project item. A New Project box will appear as shown in Figure A–1.

You will need to give your project a name such as **Project 1** and indicate the location (in which subdirectory) the project is to be saved. In order to perform a PSpice simulation, it is necessary to select the Analog or Mixed-Signal Circuit Wizard in the same box. Click OK.

## A.2 Entering a Simple Circuit with OrCAD Capture

Although each circuit simulation uses different components, the illustration that follows shows some of the common commands that are used to construct a schematic, assign values to the parts and run the PSpice simulation. The PSpice examples throughout the textbook show various applications of