with the input $f[k] = e^{-k}u[k]$ and the auxiliary condition $y[-1] = 0$. Hint: You will have to determine the auxiliary condition $y[0]$ using the iterative method.

9.5-3 (a)Using the classical method, solve

$$y[k+2] + 3y[k+1] + 2y[k] = f[k+2] + 3f[k+1] + 3f[k]$$

with the input $f[k] = (3)^k$ and the auxiliary conditions $y[0] = 1, y[1] = 3$.
(b) Repeat (a) if the auxiliary conditions are $y[-1] = y[-2] = 1$. Hint: Using the iterative method, determine $y[0]$ and $y[1]$.

9.5-4 Using the classical method, solve

$$y[k] + 2y[k-1] + y[k-2] = 2f[k] - f[k-1]$$

with the input $f[k] = 3^{-k}u[k]$ and the auxiliary conditions $y[0] = 2$ and $y[1] = -\frac{13}{3}$.

9.5-5 Using the classical method, solve

$$(E^2 - E + 0.16)y[k] = Ef[k]$$

with the input $f[k] = (0.2)^k u[k]$ and the auxiliary conditions $y[0] = 1, y[1] = 2$. Hint: The input is a natural mode of the system.

9.5-6 Using the classical method, solve

$$(E^2 - E + 0.16)y[k] = Ef[k]$$

with the input $f[k] = \cos\left(\frac{\pi k}{2} + \frac{\pi}{3}\right)u[k]$ and the initial conditions $y[-1] = y[-2] = 0$. Hint: Find $y[0]$ and $y[1]$ iteratively.

9.6-1 Each of the following equations specifies an LTID system. Determine whether these systems are asymptotically stable, unstable, or marginally stable.

(a) $y[k+2] + 0.6y[k+1] - 0.16y[k] = f[k+1] - 2f[k]$

(b) $(E^2 + 1)(E^2 + E + 1)y[k] = Ef[k]$

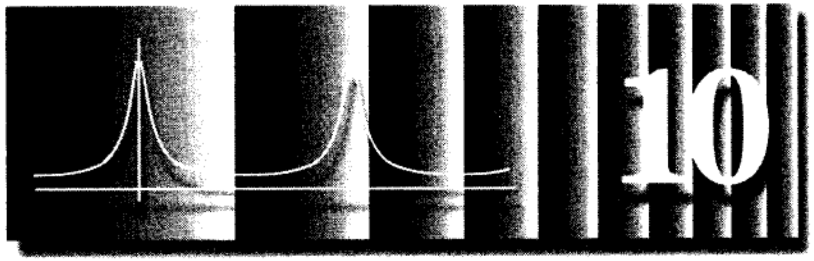
(c) $(E-1)^2(E + \frac{1}{2})y[k] = (E+2)f[k]$

(d) $y[k] + 2y[k-1] + 0.96y[k-2] = 2f[k-1] + 3f[k-3]$

(e) $(E^2 - 1)(E^2 + 1)y[k] = f[k]$

9.6-2 In Sec. 9.6 we showed that for BIBO stability in an LTID system, it is sufficient for its impulse response $h[k]$ to satisfy Eq. (9.75). Show that this is also a necessary condition for the system to be BIBO-stable. In other words, show that if Eq. (9.75) is not satisfied, there exists a bounded input that produces unbounded output.
Hint: Assume that a system exists for which $h[k]$ violates Eq. (9.75), yet its output is bounded for every bounded input. Establish contradiction in this statement by considering an input $f[k]$ defined by $f[k_1 - m] = 1$ when $h[m] > 0$ and $f[k_1 - m] = -1$ when $h[m] < 0$, where k_1 is some fixed integer.

9.6-3 Show that a marginally stable system is BIBO-unstable. Verify your result by considering a system with characteristic roots on the unit circle and show that for the input of the form of the natural mode (which is bounded), the response is unbounded.



Fourier Analysis of Discrete-Time Signals

In Chapters 3, 4, and 6, we studied the ways of representing a continuous-time signal as a sum of sinusoids or exponentials. In this chapter we shall discuss similar development for discrete-time signals. Our approach is parallel to that used for continuous-time signals. We first represent a periodic $f[k]$ as a Fourier series formed by a discrete-time exponential (or sinusoid) and its harmonics. Later we extend this representation to an aperiodic signal $f[k]$ by considering $f[k]$ as a limiting case of a periodic signal with the period approaching infinity.

10.1 Periodic Signal Representation by Discrete-Time Fourier Series (DTFS)

A periodic signal of period N_0 is called an N_0-periodic signal. Figure 8.9 shows an example of a periodic signal of period 6. A continuous-time periodic signal of period T_0 can be represented as a trigonometric Fourier series consisting of a sinusoid of the fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$, and all its harmonics (sinusoids of frequencies that are integral multiples of ω_0). The exponential form of the Fourier series consists of exponentials e^{j0t}, $e^{\pm j\omega_0 t}$, $e^{\pm j2\omega_0 t}$, $e^{\pm j3\omega_0 t}$, ... For a parallel development of the discrete time case, recall that the frequency of a sinusoid of period N_0 is $\Omega_0 = 2\pi/N_0$. Hence, an N_0-periodic discrete-time signal $f[k]$ can be represented by a discrete-time Fourier series with fundamental frequency $\Omega_0 = \frac{2\pi}{N_0}$ and its harmonics. As in the continuous-time case, we may use a trigonometric or an exponential form of the Fourier series. Because of its compactness and ease of mathematical manipulations, the exponential form is preferable to the trigonometric. For this reason we shall bypass the trigonometric form and go directly to the exponential form of the discrete-time Fourier series.

The exponential Fourier series consists of the exponentials e^{j0k}, $e^{\pm j\Omega_0 k}$, $e^{\pm j2\Omega_0 k}$, ..., $e^{\pm jn\Omega_0 k}$, ..., and so on. There would be an infinite number of harmonics, except for the property proved in Sec. 8.2: that discrete-time exponentials whose frequencies are separated by 2π (or integral multiples of 2π) are identical because

$$e^{j(\Omega\pm 2\pi)k} = e^{j\Omega k}e^{\pm 2\pi k} = e^{j\Omega k} \tag{10.1}$$

The consequence of this result is that the rth harmonic is identical to the $(r+N_0)$th harmonic. To demonstrate this, let g_n denote the nth harmonic $e^{jn\Omega_0 k}$. Then

$$g_{r+N_0} = e^{j(r+N_0)\Omega_0 k} = e^{j(r\Omega_0 k+2\pi k)} = e^{jr\Omega_0 k} = g_r \tag{10.2}$$

and

$$g_r = g_{r+N_0} = g_{r+2N_0} = \cdots = g_{r+mN_0} \qquad m,\text{ integer} \tag{10.3}$$

Thus, the first harmonic is identical to the (N_0+1)st harmonic, the second harmonic is identical to the (N_0+2)nd harmonic, and so on. In other words, there are only N_0 independent harmonics, and they range over an interval 2π (because the harmonics are separated by $\Omega_0 = \frac{2\pi}{N_0}$. We may choose these N_0 independent harmonics as $e^{jr\Omega_0 k}$ over $0 \le r \le N_0 - 1$, or over $-1 \le r \le N_0 - 2$, or over $1 \le r \le N_0$, or over any other suitable choice for that matter. Every one of these sets will have the same harmonics, although in different order. Let us take the first choice $(0 \le r \le N_0-1)$. This choice corresponds to exponentials $e^{jr\Omega_0 k}$ for $r = 0, 1, 2, \ldots, N_0 - 1$. The Fourier series for an N_0-periodic signal $f[k]$ consists of only these N_0 harmonics, and can be expressed as

$$f[k] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{10.4}$$

To compute coefficients $\mathcal{D}_r$ in the Fourier series (10.4), we multiply both sides of (10.4) by $e^{-jm\Omega_0 k}$ and sum over k from $k = 0$ to $(N_0 - 1)$.

$$\sum_{k=0}^{N_0-1} f[k]e^{-jm\Omega_0 k} = \sum_{k=0}^{N_0-1}\sum_{r=0}^{N_0-1} \mathcal{D}_r e^{j(r-m)\Omega_0 k} \tag{10.5}$$

The right-hand sum, after interchanging the order of summation, results in

$$\sum_{r=0}^{N_0-1} \mathcal{D}_r \left[\sum_{k=0}^{N_0-1} e^{j(r-m)\Omega_0 k}\right] \tag{10.6}$$

The inner sum, according to Eq. (5.43), is zero for all values of $r \ne m$. It is nonzero with a value N_0 only when $r = m$. This fact means the outside sum has only one term $\mathcal{D}_m N_0$ (corresponding to $r = m$). Therefore, the right-hand side of Eq. (10.5) is equal to $\mathcal{D}_m N_0$, and

$$\sum_{k=0}^{N_0-1} f[k]e^{-jm\Omega_0 k} = \mathcal{D}_m N_0$$

and

$$\mathcal{D}_m = \frac{1}{N_0}\sum_{k=0}^{N_0-1} f[k]e^{-jm\Omega_0 k} \tag{10.7}$$

We now have a discrete-time Fourier series (DTFS) representation of an N_0-periodic signal $f[k]$ as

$$f[k] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 k} \tag{10.8}$$

where

$$\mathcal{D}_r = \frac{1}{N_0}\sum_{k=0}^{N_0-1} f[k]e^{-jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{10.9}$$

Observe that DTFS equations (10.8) and (10.9) are identical (within a scaling constant) to the DFT equations (5.18b) and (5.18a).† Therefore, we can compute the DTFS coefficients using the efficient FFT algorithm.

10.1-1 Fourier Spectra of a Periodic Signal $f[k]$

The Fourier series consists of N_0 components

$$\mathcal{D}_0,\ \mathcal{D}_1 e^{j\Omega_0 k},\ \mathcal{D}_2 e^{j2\Omega_0 k},\ \ldots,\ \mathcal{D}_{N_0-1}e^{j(N_0-1)\Omega_0 k}$$

The frequencies of these components are $0, \Omega_0, 2\Omega_0, \ldots, (N_0 - 1)\Omega_0$ where $\Omega_0 = 2\pi/N_0$. The amount of the rth harmonic is $\mathcal{D}_r$. We can plot this amount $\mathcal{D}_r$ (the Fourier coefficient) as a function of Ω. Such a plot, called the **Fourier spectrum** of $f[k]$, gives us, at a glance, the graphical picture of the amounts of various harmonics of $f[k]$.

In general, the Fourier coefficients $\mathcal{D}_r$ are complex, and they can be represented in the polar form as

$$\mathcal{D}_r = |\mathcal{D}_r|e^{j\angle\mathcal{D}_r} \tag{10.10}$$

The plot of $|\mathcal{D}_r|$ vs. Ω is called the amplitude spectrum and that of $\angle\mathcal{D}_r$ vs. Ω is called the angle (or phase) spectrum. These two plots together are the frequency spectra of $f[k]$. Knowing these spectra, we can reconstruct or synthesize $f[k]$ according to Eq. (10.8). Therefore, the Fourier (or frequency) spectra, which are an alternative way of describing a signal $f[k]$, are in every way equivalent (in terms of the information) to the plot of $f[k]$ as a function of k. The Fourier spectra of a signal constitute the **frequency-domain** description of $f[k]$, in contrast to the time-domain description, where $f[k]$ is specified as a function of time (k).

The results are very similar to the representation of a continuous-time periodic signal by an exponential Fourier series except that, generally, the continuous-time signal spectrum bandwidth is infinite, and consists of an infinite number of exponential components (harmonics). The spectrum of the discrete-time periodic signal, in contrast, is bandlimited and has at most N_0 components.

Periodic Extension of Fourier Spectrum

Note that if $\phi[r]$ is an N_0-periodic function of r, then

$$\sum_{r=0}^{N_0-1}\phi[r] = \sum_{r=\langle N_0\rangle}\phi[r] \tag{10.11}$$

†If we let $f[k] = N_0 f_k$ and $\mathcal{D}_r = F_r$, Eqs. (10.8) and (10.9) are identical to Eqs. (5.18b) and (5.18a), respectively.

where $r = <N_0>$ indicates summation over any N_0 consecutive values of r. This follows because the right-hand side of Eq. (10.11) is the sum of all the N_0 consecutive values of $\phi[r]$. Because $\phi[r]$ is periodic, this sum must be the same regardless of where we start the first term. Now $e^{-jr\Omega_0 k}$ is N_0-periodic because

$$e^{-jr\Omega_0(k+N_0)} = e^{-jr\Omega_0 k}e^{-j2\pi r} = e^{-jr\Omega_0 k}$$

Therefore, if $f[k]$ is N_0-periodic, $f[k]e^{-jr\Omega_0 k}$ is also N_0-periodic. Hence, from Eq. (10.9) it follows that $\mathcal{D}_r$ is also N_0-periodic, as is $\mathcal{D}_r e^{jr\Omega_0 k}$. Now, because of property (10.11), we can express Eqs. (10.8) and (10.9) as

$$f[k] = \sum_{r=<N_0>} \mathcal{D}_r e^{jr\Omega_0 k} \tag{10.12}$$

and

$$\mathcal{D}_r = \frac{1}{N_0} \sum_{k=<N_0>} f[k] e^{-jr\Omega_0 k} \tag{10.13}$$

If we plot $\mathcal{D}_r$ for all values of r (rather than only $0 \le r \le N_0 - 1$), then the spectrum $\mathcal{D}_r$ is N_0-periodic. Moreover, Eq. (10.12) shows that $f[k]$ can be synthesized by not only the N_0 exponentials corresponding to $0 \le r \le N_0 - 1$, but by any successive N_0 exponentials in this spectrum, starting at any value of r (positive or negative). For this reason, it is customary to show the spectrum $\mathcal{D}_r$ for all values of r (not just over the interval $0 \le r \le N_0 - 1$). *Yet we must remember that to synthesize $f[k]$ from this spectrum, we need to add only N_0 consecutive components.*

The spectral components $\mathcal{D}_r$ are separated by the frequency $\Omega_0 = \frac{2\pi}{N_0}$, and there are a total of N_0 components repeating periodically along the Ω axis. Thus, on the frequency scale Ω, $\mathcal{D}_r$ repeats every 2π intervals. Equations (10.12) and (10.13) show that both $f[k]$ and its spectrum $\mathcal{D}_r$ are periodic and both have exactly the same number of components (N_0) over one period. The period of $f[k]$ is N_0 and that of $\mathcal{D}_r$ is 2π radians.

Equation (10.13) shows that $\mathcal{D}_r$ is complex in general, and $\mathcal{D}_{-r}$ is the conjugate of $\mathcal{D}_r$ if $f[k]$ is real. Thus

$$|\mathcal{D}_r| = |\mathcal{D}_{-r}| \qquad \text{and} \qquad \angle\mathcal{D}_r = -\angle\mathcal{D}_{-r} \tag{10.14}$$

so that the amplitude spectrum $|\mathcal{D}_r|$ is an even function , and $\angle\mathcal{D}_r$ is an odd function of r (or Ω). All these concepts will be clarified by the examples to follow.

Example 10.1

Find the discrete-time Fourier series (DTFS) for $f[k] = \sin 0.1\pi k$ (Fig. 10.1a). Sketch the amplitude and phase spectra.

In this case the sinusoid $\sin 0.1\pi k$ is periodic because $\frac{\Omega}{2\pi} = \frac{1}{20}$ is a rational number and the period N_0 is [see Eq. (8.9b)]

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) = m\left(\frac{2\pi}{0.1\pi}\right) = 20m$$

The smallest value of m that makes $20m$ an integer is $m = 1$. Therefore, the period $N_0 = 20$, so that $\Omega_0 = \frac{2\pi}{N_0} = 0.1\pi$, and from Eq. (10.12)

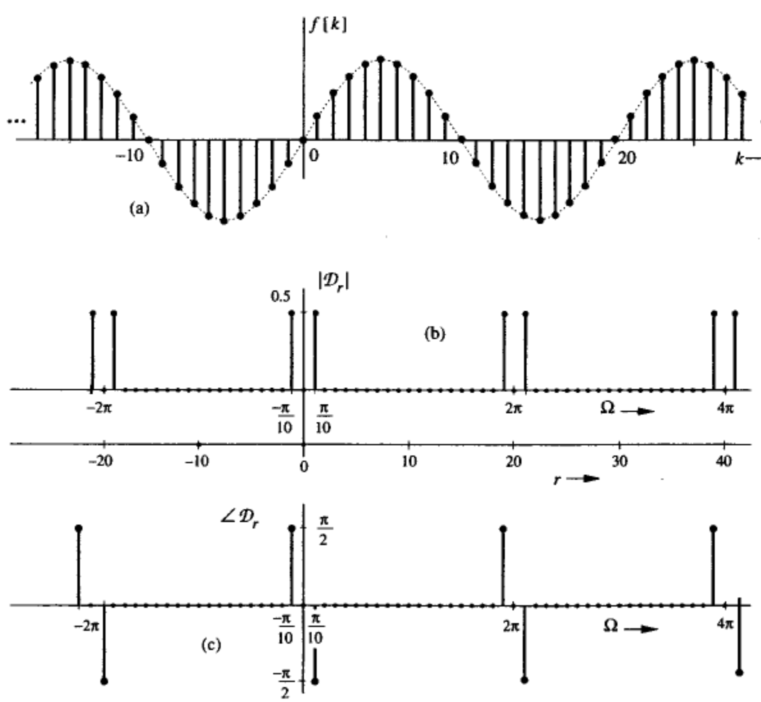


Fig. 10.1 Discrete-time sinusoid $\sin 0.1\pi k$ and its Fourier spectra.

$$f[k] = \sum_{r=<20>} \mathcal{D}_r e^{j0.1\pi rk}$$

where the sum is performed over any 20 consecutive values of r. We shall select the range $-10 \le r < 10$ (values of r from -10 to 9). This choice corresponds to synthesizing $f[k]$ using the spectral components in the fundamental frequency range ($-\pi \le \Omega < \pi$). Thus,

$$f[k] = \sum_{r=-10}^{9} \mathcal{D}_r e^{j0.1\pi rk}$$

where, according to Eq. (10.13),

$$\begin{aligned}
\mathcal{D}_r &= \frac{1}{20}\sum_{k=-10}^{9} \sin 0.1\pi k\, e^{-j0.1\pi rk} \\
&= \frac{1}{20}\sum_{k=-10}^{9} \frac{1}{2j}\left(e^{j0.1\pi k} - e^{-j0.1\pi k}\right) e^{-j0.1\pi rk} \\
&= \frac{1}{40j}\left[\sum_{k=-10}^{9} e^{j0.1\pi k(1-r)} - \sum_{k=-10}^{9} e^{-j0.1\pi k(1+r)}\right]
\end{aligned}$$

In these sums, r takes on all values between -10 and 9. From Eq. (5.43) it follows that the first sum on the right-hand side is zero for all values of r except $r = 1$, when the sum is equal to $N_0 = 20$. Similarly, the second sum is zero for all values of r except $r = -1$, when it is equal to $N_0 = 20$. Therefore

$$\mathcal{D}_1 = \frac{1}{2j} \quad \text{and} \quad \mathcal{D}_{-1} = -\frac{1}{2j}$$

and all other coefficients are zero. The corresponding Fourier series is given by

$$f[k] = \sin 0.1\pi k = \tfrac{1}{2j}\left(e^{j0.1\pi k} - e^{-j0.1\pi k}\right) \tag{10.15}$$

Here the fundamental frequency $\Omega_0 = 0.1\pi$, and there are only two nonzero components:

$$\mathcal{D}_1 = \tfrac{1}{2j} = \tfrac{1}{2}e^{-j\pi/2}$$

$$\mathcal{D}_{-1} = -\tfrac{1}{2j} = \tfrac{1}{2}e^{j\pi/2}$$

Therefore

$$|\mathcal{D}_1| = |\mathcal{D}_{-1}| = \tfrac{1}{2}$$

$$\angle\mathcal{D}_1 = -\tfrac{\pi}{2} \quad \text{and} \quad \angle\mathcal{D}_{-1} = \tfrac{\pi}{2}$$

Figures 10.1b and c shows the sketch of $\mathcal{D}_r$ for the interval $(-10 \le r < 10)$. According to Eq. (10.15a), there are only two components corresponding to $r = 1$ and -1. The remaining 18 coefficients are zero. The rth component $\mathcal{D}_r$ is the amplitude of the frequency $r\Omega_0 = 0.1r\pi$. Therefore, the frequency interval corresponding to $-10 \le r < 10$ is $-\pi \le \Omega < \pi$, as depicted in Figs. 10.1b and c. This spectrum in the interval $-10 \le r < 10$ (or $-\pi \le \Omega < \pi$) is sufficient to specify the frequency-domain description (Fourier series), and we can synthesize $f[k]$ by adding these spectral components. Because of the periodicity property discussed in Sec. 10.1-1, the spectrum $\mathcal{D}_r$ is a periodic function of r with period $N_0 = 20$. For this reason, we repeat the spectrum with period $N_0 = 20$ (or $\Omega = 2\pi$), as illustrated in Figs. 10.1b and c, which are periodic extensions of the spectrum in the range $-10 \le r < 10$. Observe that the amplitude spectrum is an even function and the angle or phase spectrum is an odd function of r (or Ω) as expected.

The result (10.15) is a trigonometric identity, and could have been obtained immediately without the formality of finding the Fourier coefficients. We have intentionally chosen this trivial example to introduce the reader gently to the new concept of the discrete-time Fourier series and its periodic nature. The Fourier series is a way of expressing a periodic signal $f[k]$ in terms of exponentials of the form $e^{jr\Omega_0 k}$ and its harmonics. The result in Eq. (10.15) is merely a statement of the (obvious) fact that $\sin 0.1\pi k$ can be expressed as a sum of two exponentials $e^{j0.1\pi k}$ and $e^{-j0.1\pi k}$.

Because of the periodicity property of the discrete-time exponentials $e^{jr\Omega_0 k}$, the Fourier series components can be selected in any range of length $N_0 = 20$ (or $\Omega = 2\pi$). For example, if we select the frequency range $0 \le \Omega < 2\pi$ (or $0 \le r < 20$), we obtain the Fourier series as

$$f[k] = \sin 0.1\pi k = \frac{1}{2j}\left(e^{j0.1\pi k} - e^{-j1.9\pi k}\right) \tag{10.16}$$

This series is equivalent to that in Eq. (10.15) because, as seen in Sec. 8.2, the two exponentials $e^{j1.9\pi k}$ and $e^{-j0.1\pi k}$ are identical.

We could have selected the spectrum over any other range of width $\Omega = 2\pi$ in Figs. 10.1b and c as a valid discrete-time Fourier series. The reader may satisfy himself by proving that such a spectrum starting anywhere (and of width $\Omega = 2\pi$) is equivalent to the same two components on the right-hand side of Eq. (10.15). ■

△ **Exercise E10.1**

From the spectrum in Fig. 10.1 write the Fourier series corresponding to the interval $-10 \ge r > -30$ (or $-\pi \ge \Omega > -3\pi$). Show that this Fourier is equivalent to that in Eq. (10.15). ▽

△ **Exercise E10.2**

Find the period and the DTFS for

$$f[k] = 4\cos 0.2\pi k + 6\sin 0.5\pi k$$

over the interval $0 \le r \le 19$. Use Eq. (10.9) to compute $\mathcal{D}_r$.
Answer:

$$N_0 = 20, \quad \text{and} \quad f[k] = 2e^{j0.2\pi k} + (3e^{-j\pi/2})e^{j0.5\pi k} + (3e^{j\pi/2})e^{j1.5\pi k} + 2e^{j1.8\pi k} \quad ▽$$

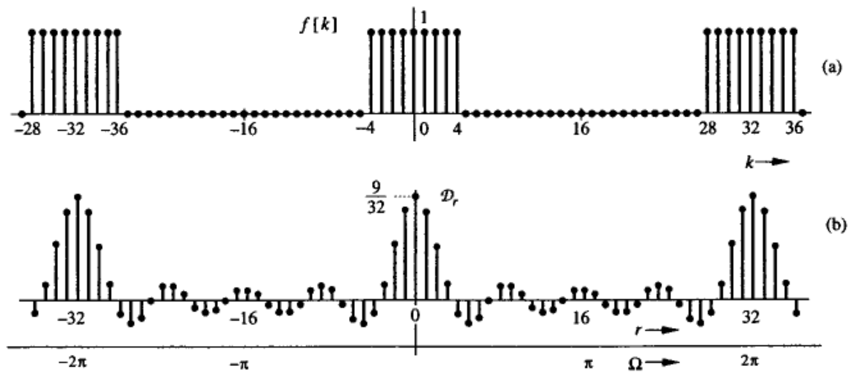


Fig. 10.2 Periodic sampled gate pulse and its Fourier spectrum.

■ **Example 10.2**

Find the discrete-time Fourier series for the periodic sampled gate function shown in Fig. 10.2a.

In this case $N_0 = 32$ and $\Omega_0 = \frac{2\pi}{32} = \frac{\pi}{16}$. Therefore

$$f[k] = \sum_{r=\langle 32 \rangle} \mathcal{D}_r e^{jr\frac{\pi}{16}k} \tag{10.17}$$

where

$$\mathcal{D}_r = \frac{1}{32}\sum_{k=\langle 32 \rangle} f[k]e^{-jr\frac{\pi}{16}k} \tag{10.18}$$

For our convenience, we shall choose the interval $-16 \le k \le 15$ for the summation (10.18), although any other interval of the same width (32 points) would give the same result.

$$\mathcal{D}_r = \frac{1}{32}\sum_{k=-16}^{15} f[k]e^{-jr\frac{\pi}{16}k}$$

Now $f[k] = 1$ for $-4 \le k \le 4$ and is zero for all other values of k. Therefore

$$\mathcal{D}_r = \frac{1}{32}\sum_{k=-4}^{4} e^{-jr\frac{\pi}{16}k}$$

This is a geometric progression with a common ratio $e^{-j\frac{\pi}{16}r}$. Therefore,[see Sec. (B.7-4)]

$$\begin{aligned}
\mathcal{D}_r &= \frac{1}{32}\left[\frac{e^{-j\frac{5\pi r}{16}} - e^{j\frac{4\pi r}{16}}}{e^{-j\frac{\pi r}{16}} - 1}\right] \\
&= \left(\frac{1}{32}\right)\frac{e^{-j\frac{0.5\pi r}{16}}\left[e^{-j\frac{4.5\pi r}{16}} - e^{j\frac{4.5\pi r}{16}}\right]}{e^{-j\frac{0.5\pi r}{16}}\left[e^{-j\frac{0.5\pi r}{16}} - e^{j\frac{0.5\pi r}{16}}\right]} \\
&= \left(\frac{1}{32}\right)\frac{\sin\left(\frac{4.5\pi r}{16}\right)}{\sin\left(\frac{0.5\pi r}{16}\right)} \\
&= \left(\frac{1}{32}\right)\frac{\sin(4.5r\Omega_0)}{\sin(0.5r\Omega_0)} \qquad \Omega_0 = \frac{\pi}{16} \qquad (10.19)
\end{aligned}$$

This spectrum (with its periodic extension) is depicted in Fig. 10.2b.† ■

⊙ **Computer Example C10.1**
Do Example 10.2 using MATLAB.

```
N0=32;k=0:N0-1;
f=[ones(1,5) zeros(1,23) ones(1,4)];
Fr=1/32*fft(f);
r=k;
stem(k,Fr),grid ⊙
```

10.2 Aperiodic Signal representation by Fourier Integral

In Sec. 10.1 we succeeded in representing periodic signals as a sum of (everlasting) exponentials. In this section we extend this representation to aperiodic signals. The procedure is identical conceptually to that in Chapter 4 used for continuous-time signals.

Applying a limiting process, we now show that aperiodic signals $f[k]$ can be expressed as a continuous sum (integral) of everlasting exponentials. To represent an aperiodic signal $f[k]$ such as the one illustrated in Fig. 10.3a by everlasting exponential signals, let us construct a new periodic signal $f_{N_0}[k]$ formed by repeating the signal $f[k]$ every N_0 units , as shown in Fig. 10.3b. The period N_0 is made long enough to avoid overlap between the repeating cycles ($N_0 \geq 2N+1$). The periodic signal $f_{N_0}[k]$ can be represented by an exponential Fourier series. If we let $N_0 \to \infty$, the signal $f[k]$ repeats after an infinite interval, and therefore

$$\lim_{N_0\to\infty} f_{N_0}[k] = f[k]$$

†In this example we have used the same equations as those for DFT in Example C5.2, with a minor difference. In the present example, the values of $f[k]$ at $k = 4$ and -4 are taken as 1, whereas in Example 5.3 these values are 0.5. This is the reason for the slight difference in spectra in Fig. 10.2b and Fig. 5.16d. Unlike continuous-time signals, discrete-time signals can have no discontinuity.

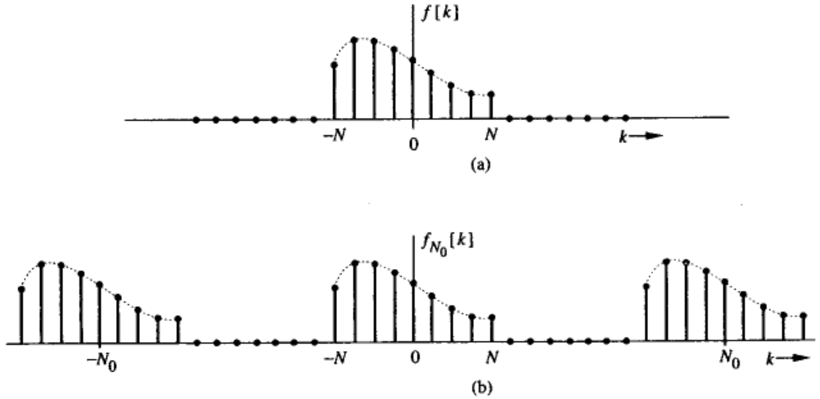


Fig. 10.3 Generation of a periodic signal by periodic extension of a signal $f[k]$.

Thus, the Fourier series representing $f_{N_0}[k]$ will also represent $f[k]$ in the limit $N_0 \to \infty$. The exponential Fourier series for $f_{N_0}[k]$ is given by

$$f_{N_0}[k] = \sum_{r=<N_0>} \mathcal{D}_r e^{jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0} \qquad (10.20)$$

where

$$\mathcal{D}_r = \frac{1}{N_0}\sum_{k=-\infty}^{\infty} f[k]e^{-jr\Omega_0 k} \qquad (10.21)$$

The limits for the sum on the right-hand side of Eq. (10.21) should be from $-N$ to N. But because $f[k] = 0$ for $|k| > N$, it does not matter if the limits are taken from $-\infty$ to ∞.

It is interesting to see how the nature of the spectrum changes as N_0 increases. To understand this fascinating behavior, let us define $F(\Omega)$, a continuous function of Ω, as

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \qquad (10.22)$$

From this definition and Eq. (10.21), we have

$$\mathcal{D}_r = \frac{1}{N_0}F(r\Omega_0) \qquad (10.23)$$

This result shows that the Fourier coefficients $\mathcal{D}_r$ are $(1/N_0)$ times the samples of $F(\Omega)$ taken every Ω_0 rad/s.† Therefore, $(1/N_0)F(\Omega)$ is the envelope for the coefficients $\mathcal{D}_r$. We now let $N_0 \to \infty$ by doubling N_0 repeatedly. Doubling N_0 halves the fundamental frequency Ω_0 so the spacing between successive spectral components (harmonics) is halved, and there are now twice as many components (samples) in

†For the sake of simplicity we assume $\mathcal{D}_r$ and therefore $F(\Omega)$ to be real. The argument, however, is also valid for complex $\mathcal{D}_r$ [or $F(\Omega)$].

the spectrum. At the same time, by doubling N_0, the envelope of the coefficients $\mathcal{D}_r$ is halved, as seen from Eq. (10.23). If we continue this process of doubling N_0 repeatedly, the number of components doubles in each step; the spectrum progressively becomes denser while its magnitude $\mathcal{D}_r$ becomes smaller. Note, however, that the relative shape of the envelope remains the same [proportional to $F(\Omega)$ in Eq. (10.22)]. In the limit, as $N_0 \to \infty$, the fundamental frequency $\Omega_0 \to 0$, and $\mathcal{D}_r \to 0$. The separation between successive harmonics, which is Ω_0, is approaching zero (infinitesimal), and the spectrum becomes so dense that it appears continuous. But as the number of harmonics increases indefinitely, the harmonic amplitudes $\mathcal{D}_r$ become vanishingly small (infinitesimal). We have a strange situation of having **nothing of everything**. This phenomenon is already discussed in Chapter 4, where we showed that these are the classic characteristics of a familiar phenomenon (the density function).

Let us see what happens mathematically as the period $N_0 \to \infty$. According to Eq. (10.22)

$$F(r\Omega_0) = \sum_{k=-\infty}^{\infty} f[k]e^{-jr\Omega_0 k} \tag{10.24}$$

Using Eqs. (10.23) and (10.21), we can express Eq. (10.20) as

$$f_{N_0}[k] = \frac{1}{N_0} \sum_{r=<N_0>} F(r\Omega_0)e^{jr\Omega_0 k} \tag{10.25a}$$

$$= \sum_{r=<N_0>} F(r\Omega_0)e^{jr\Omega_0 k}\left(\frac{\Omega_0}{2\pi}\right) \tag{10.25b}$$

In the limit as $N_0 \to \infty$, $\Omega_0 \to 0$ and $f_{N_0}[k] \to f[k]$. Therefore

$$f[k] = \lim_{\Omega_0 \to 0} \sum_{r=<N_0>} \left[\frac{F(r\Omega_0)\Omega_0}{2\pi}\right] e^{jr\Omega_0 k} \tag{10.26}$$

As $N_0 \to 0$, Ω_0 becomes infinitesimal ($\Omega_0 \to 0$). For this reason it will be appropriate to replace Ω_0 with an infinitesimal notation $\Delta\Omega$:

$$\Delta\Omega = \frac{2\pi}{N_0} \tag{10.27}$$

Equation (10.26) can be expressed as

$$f[k] = \lim_{\Delta\Omega \to 0} \sum_{r=<N_0>} \left[\frac{F(r\Delta\Omega)\Delta\Omega}{2\pi}\right] e^{jr\Delta\Omega k} \tag{10.28}$$

$$= \lim_{\Delta\Omega \to 0} \frac{1}{2\pi} \sum_{r=<N_0>} F(r\Delta\Omega)e^{jr\Delta\Omega k}\Delta\Omega \tag{10.29}$$

The range $r =< N_0 >$ implies the interval of N_0 number of harmonics, which is $N_0\Delta\Omega = 2\pi$ according to Eq. (10.27). In the limit, the right-hand side of Eq. (10.29) becomes the integral

$$f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega)e^{jk\Omega}\, d\Omega \tag{10.30}$$

where $\int_{2\pi}$ indicates integration over any continuous interval of 2π. The spectrum $F(\Omega)$ is given by [Eq. (10.22)]

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \tag{10.31}$$

The integral on the right-hand side of Eq. (10.30) is called the **Fourier integral**. We have now succeeded in representing an aperiodic signal $f[k]$ by a Fourier integral (rather than a Fourier series). This integral is basically a Fourier series (in the limit) with fundamental frequency $\Delta\Omega \to 0$, as seen in Eq. (10.28). The amount of the exponential $e^{jr\Delta\Omega k}$ is $F(r\Delta\Omega)\Delta\Omega/2\pi$. Thus, the function $F(\Omega)$ given by Eq. (10.31) acts as a spectral function, which indicates the relative amounts of various exponential components of $f[k]$.

We call $F(\Omega)$ the (direct) discrete-time Fourier transform (DTFT) of $f[k]$, and $f[k]$ the inverse discrete-time Fourier transform (IDTFT) of $F(\Omega)$. This can be represented as

$$F(\Omega) = \mathcal{F}\{f[k]\} \qquad \text{and} \qquad f[k] = \mathcal{F}^{-1}\{F(\Omega)\}$$

The same information is conveyed by the statement that $f[k]$ and $F(\Omega)$ are a (discrete-time) Fourier transform pair. Symbolically, this is expressed as

$$f[k] \Longleftrightarrow F(\Omega)$$

The Fourier transform $F(\Omega)$ is the frequency-domain description of $f[k]$.

10.2-1 Nature of Fourier Spectra

We now discuss several important features of the discrete-time Fourier transform and the spectra associated with it.

The Fourier Spectra are Continuous Functions of Ω.

It is helpful to keep in mind that the Fourier integral in Eq. (10.30) is basically a Fourier series with fundamental frequency $\Delta\Omega$ approaching zero [Eq. (10.28)]. Therefore, most of the discussion and properties of Fourier series apply to the Fourier transform as well. The successive harmonics are separated by the fundamental frequency $\Delta\Omega$, which approaches zero. This fact makes the spectra continuous functions of Ω.

The Fourier Spectra are Periodic Functions of Ω with Period 2π

According to Eq. (10.31) it follows that

$$F(\Omega + 2\pi) = \sum_{k=-\infty}^{\infty} f[k]e^{-j(\Omega+2\pi)k} = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}e^{-j2\pi k} = F(\Omega) \tag{10.32}$$

Clearly, the spectrum $F(\Omega)$ is a continuous and periodic function of Ω with period 2π. We must remember, however, that to synthesize $f[k]$, we need to use the spectrum over a frequency interval of only 2π, starting at any value of Ω [see Eq. (10.30)].

As a matter of convenience, we shall choose this interval to be the fundamental frequency range $(-\pi, \pi)$. It is, therefore, not necessary to show discrete-time-signal spectra beyond the fundamental range, although we often do so.

Conjugate Symmetry of $F(\Omega)$ for real $f[k]$

From Eq. (10.31), we obtain

$$F(-\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{j\Omega k}$$

The right-hand side of this equation is the conjugate of the right-hand side of Eq. (10.31) for real $f[k]$. Therefore, for real $f[k]$, $F(\Omega)$ and $F(-\Omega)$ are conjugates; that is, $F(-\Omega) = F^*(\Omega)$.

Since $F(\Omega)$ is generally complex, we have both amplitude and angle (or phase) spectra

$$F(\Omega) = |F(\Omega)|e^{j\angle F(\Omega)} \tag{10.33}$$

Because of conjugate symmetry of $F(\Omega)$, it follows that

$$|F(\Omega)| = |F(-\Omega)| \tag{10.34a}$$

$$\angle F(\Omega) = -\angle F(-\Omega) \tag{10.34b}$$

Therefore, the amplitude spectrum $|F(\Omega)|$ is an even function of Ω and the phase spectrum $\angle F(\Omega)$ is an odd function of Ω for real $f[k]$.

Linearity of the DTFT

According to Eq. (10.31), it follows that if

$$f_1[k] \Longleftrightarrow F_1(\Omega) \qquad \text{and} \qquad f_2[k] \Longleftrightarrow F_2(\Omega)$$

then

$$a_1f_1[k] + a_2f_2[k] \Longleftrightarrow a_1F_1(\Omega) + a_2F_2(\Omega) \tag{10.35}$$

Existence of the DTFT

Because $|e^{-jk\Omega}| = 1$, from Eq. (10.31), it follows that the existence of $F(\Omega)$ is guaranteed if $f[k]$ is absolutely summable; that is,

$$\sum_{k=-\infty}^{\infty} |f[k]| < \infty \tag{10.36}$$

This condition is sufficient but not necessary for the existence of $F(\Omega)$. For instance, the signal $f[k] = \sin k/k$ violates the condition (10.36), but does have DTFT (see Example 10.6).

Physical Appreciation of the Discrete-Time Fourier Transform

In understanding any aspect of the Fourier transform, we should remember that Fourier representation is a way of expressing a signal $f[k]$ as a sum of everlasting exponentials (or sinusoids). The Fourier spectrum of a signal indicates the relative amplitudes and phases of the exponentials (or sinusoids) required to synthesize $f[k]$. If $f[k]$ is periodic, then its Fourier spectrum has finite amplitudes and exists at discrete frequencies (Ω_0 and its multiples). Such a spectrum is easy to visualize, but the spectrum of an aperiodic signal is not easy to visualize because it is continuous. The physical meaning of the continuous spectrum is fully explained in Sec. 4.1-1.

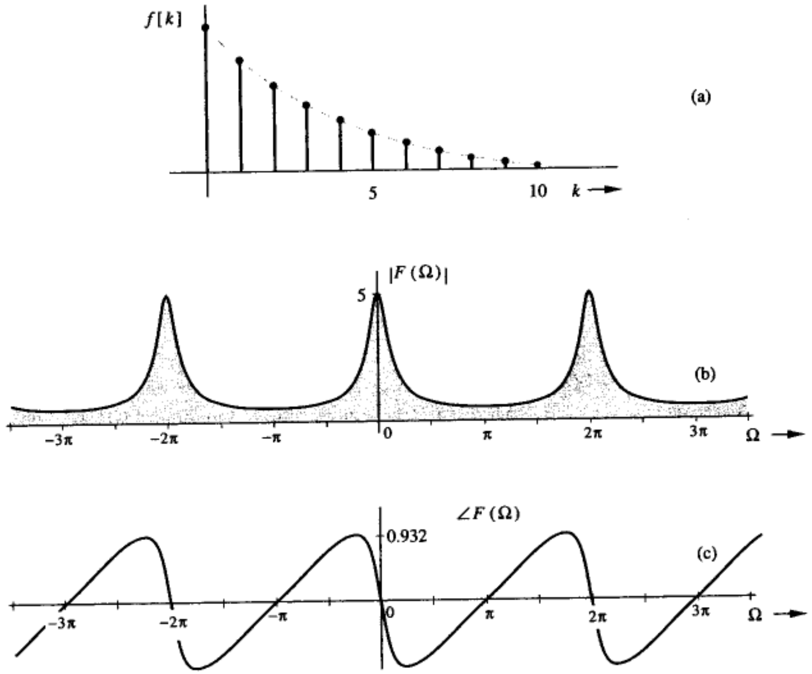


Fig. 10.4 Exponential $\gamma^k u[k]$ and its frequency spectra.

■ **Example 10.3**

Find the DTFT of $f[k] = \gamma^k u[k]$.

$$F(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k$$

This is a geometric progression with a common ratio $\gamma e^{-j\Omega}$. Therefore, [see Sec. (B.7-4)]

$$F(\Omega) = \frac{1}{1-\gamma e^{-j\Omega}}$$

provided that $|\gamma e^{-j\Omega}| < 1$. But because $|e^{-j\Omega}| = 1$, this condition implies $|\gamma| < 1$. Therefore

$$F(\Omega) = \frac{1}{1-\gamma e^{-j\Omega}} \qquad |\gamma| < 1 \tag{10.37}$$

If $|\gamma| > 1$, $F(\Omega)$ does not converge. This result is in conformity with condition (10.36). From Eq. (10.37)

$$F(\Omega) = \frac{1}{1-\gamma\cos\Omega + j\gamma\sin\Omega} \tag{10.38}$$

so that

$$|F(\Omega)| = \frac{1}{\sqrt{(1-\gamma\cos\Omega)^2 + (\gamma\sin\Omega)^2}} \tag{10.39a}$$

$$= \frac{1}{\sqrt{1+\gamma^2-2\gamma\cos\Omega}}$$

$$\angle F(\Omega) = -\tan^{-1}\left[\frac{\gamma\sin\Omega}{1-\gamma\cos\Omega}\right] \tag{10.39b}$$

Figure 10.4 shows $f[k] = \gamma^k u[k]$ and its spectra for $\gamma = 0.8$. Observe that the frequency spectra are continuous and periodic functions of Ω with the period 2π. As explained earlier, we need to use the spectrum only over the frequency interval of 2π. We often select this interval to be the fundamental frequency range $(-\pi, \pi)$.

The amplitude spectrum $|F(\Omega)|$ is an even function and the phase spectrum $\angle F(\Omega)$ is an odd function of Ω. ■

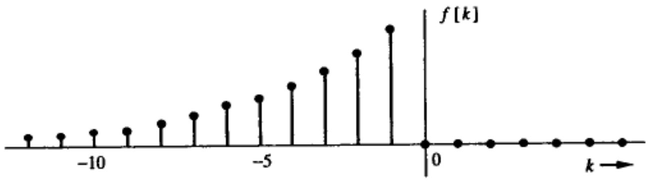


Fig. 10.5 Exponential $\gamma^k u[-(k+1)]$.

■ **Example 10.4**

Find the DTFT of $\gamma^k u[-(k+1)]$ depicted in Fig. 10.5.

$$F(\Omega) = \sum_{k=-\infty}^{\infty} \gamma^k u[-(k+1)]e^{-j\Omega k} = \sum_{k=-1}^{-\infty} (\gamma e^{-j\Omega})^k = \sum_{k=-1}^{-\infty} (\frac{1}{\gamma}e^{j\Omega})^{-k}$$

Setting $k = -m$ yields

$$f[k] = \sum_{m=1}^{\infty} (\frac{1}{\gamma}e^{j\Omega})^m = \frac{1}{\gamma}e^{j\Omega} + \left(\frac{1}{\gamma}e^{j\Omega}\right)^2 + \left(\frac{1}{\gamma}e^{j\Omega}\right)^3 + \cdots$$

This is a geometric series with a common ratio $e^{j\Omega}/\gamma$. Therefore, from Sec. B.7-4,

$$F(\Omega) = \frac{1}{\gamma e^{-j\Omega} - 1} \qquad |\gamma| > 1 \tag{10.40}$$

$$= \frac{1}{(\gamma\cos\Omega - 1) - j\gamma\sin\Omega}$$

Therefore

$$|F(\Omega)| = \frac{1}{\sqrt{1+\gamma^2-2\gamma\cos\Omega}}$$

$$\angle F(\Omega) = \tan^{-1}\left[\frac{\gamma\sin\Omega}{\gamma\cos\Omega - 1}\right] \tag{10.41}$$

The Fourier transform (and the frequency spectra) for this signal is identical to that of $f[k] = \gamma^k u[k]$. Yet there is no ambiguity in determining the IDTFT of $F(\Omega) = \frac{1}{\gamma e^{-j\Omega}-1}$ because of the restrictions on the value of γ in each case. If $|\gamma| < 1$, then the inverse transform is $f[k] = \gamma^k u[k]$. If $|\gamma| > 1$, it is $f[k] = \gamma^k[-(k+1)]$. ■

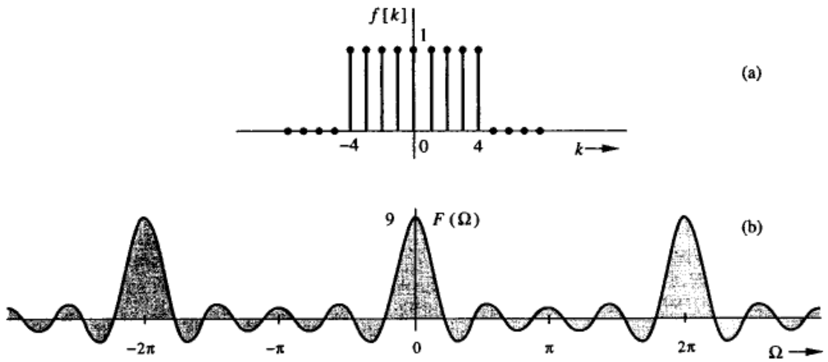


Fig. 10.6 Discrete-time gate pulse and its Fourier spectrum.

■ **Example 10.5**

Find the DTFT of the discrete-time rectangular pulse illustrated in Fig. 10.6a. This pulse is also known as the 9-point rectangular window function.

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}$$

$$= \sum_{k=-\frac{M-1}{2}}^{\frac{M-1}{2}} (e^{-j\Omega})^k \qquad M = 9 \tag{10.42}$$

This is a geometric progression with a common ratio $e^{-j\Omega}$ and [see Sec. B.7-4]

$$
\begin{aligned}
F(\Omega) &= \frac{e^{-j\frac{M+1}{2}\Omega} - e^{j\frac{M-1}{2}\Omega}}{e^{-j\Omega} - 1} \\
&= \frac{e^{-j\Omega/2}\left(e^{-j\frac{M}{2}\Omega} - e^{j\frac{M}{2}\Omega}\right)}{e^{-j\Omega/2}(e^{-j\Omega/2} - e^{j\Omega/2})} \\
&= \frac{\sin\left(\frac{M}{2}\Omega\right)}{\sin(0.5\Omega)} \qquad (10.43) \\
&= \frac{\sin(4.5\Omega)}{\sin(0.5\Omega)} \qquad \text{for } M = 9 \qquad (10.44)
\end{aligned}
$$

Figure 10.6b shows the spectrum $F(\Omega)$ for $M = 9$.

Note that the spectrum $\mathcal{D}_r$ in Fig. 10.2b [Eq. (10.19)] is a sampled version of $F(\Omega)$ in Fig. 10.6b [Eq. (10.44)]:

$$
\mathcal{D}_r = \frac{1}{32}F(r\Omega_0) \qquad \Omega_0 = \frac{\pi}{16}
$$

Therefore, $F(\Omega)$ in Fig. 10.6b is the envelope of $\mathcal{D}_r$ (within a multiplicative constant 32) in Fig. 10.2b. The reason for this behavior is discussed later in Sec. 10.6. ■

⊙ **Computer Example C10.2**
Do Example 10.5 using MATLAB.

```
N0=512;
f=[ones(1,5) zeros(1,N0-9) ones(1,4)];
F=fft(f);
r=0:N0-1;
W=r.*2*pi/512;
plot(W,F);
xlabel('W');ylabel('F(W)');grid on;
```
⊙

■ **Example 10.6**

Find the inverse DTFT of the rectangular pulse spectrum $F(\Omega) = \text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$ with $\Omega_c = \frac{\pi}{4}$ and repeating at the intervals of 2π, as shown in Fig. 10.7a.

According to Eq. (10.30)

$$
\begin{aligned}
f[k] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\Omega)e^{jk\Omega}\,d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{jk\Omega}\,d\Omega \\
&= \frac{1}{j2\pi k}e^{jk\Omega}\Big|_{-\Omega_c}^{\Omega_c} \\
&= \frac{\sin(\Omega_c k)}{\pi k} \\
&= \frac{\Omega_c}{\pi}\,\text{sinc}(\Omega_c k) \qquad (10.45)
\end{aligned}
$$

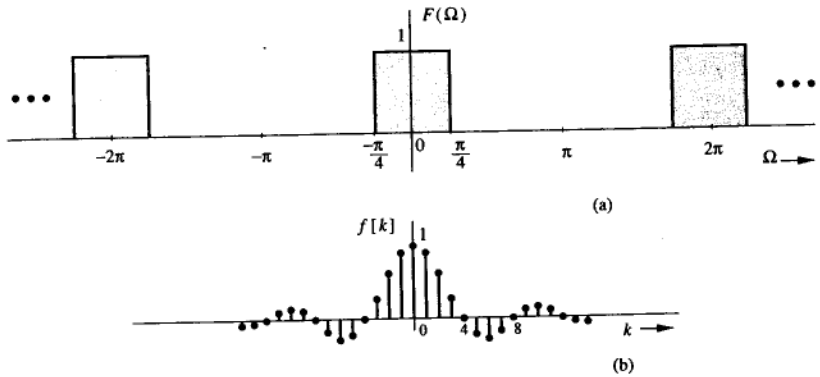


Fig. 10.7 Inverse Discrete-time Fourier transform of a periodic gate spectrum.

The signal $f[k]$ is depicted in Fig. 10.7b (for the case $\Omega_c = \pi/4$). ■

△ **Exercise E10.3**

Find and sketch the amplitude and phase spectra of the DTFT of the signal $f[k] = \gamma^{|k|}$ with $|\gamma| < 1$.

Answer:

$$
F(\Omega) = \frac{1-\gamma^2}{1 - 2\gamma\cos\Omega + \gamma^2} \qquad \nabla
$$

10.3 Properties of the DTFT

The *linearity property* [Eq. (10.35)] of DTFT has been already discussed. Other useful properties of the DTFT are as follows:

Time and Frequency Inversion

$$
f[-k] \Longleftrightarrow F(-\Omega) \tag{10.46}
$$

From Eq. (10.31), the DTFT of $f[-k]$ is

$$
\text{DTFT}\{f[-k]\} = \sum_{k=-\infty}^{\infty} f[-k]e^{-j\Omega k} = \sum_{m=-\infty}^{\infty} f[m]e^{j\Omega m} = F(-\Omega)
$$

Multiplication by k: Frequency Differentiation

$$
kf[k] \Longleftrightarrow j\frac{dF(\Omega)}{d\Omega} \tag{10.47}
$$

The result follows immediately by differentiating both sides of Eq. (10.31) with respect to Ω.

Time-Shifting Property

If

$$f[k] \Longleftrightarrow F(\Omega)$$

then

$$f[k-k_0] \Longleftrightarrow F(\Omega)e^{-jk_0\Omega} \qquad k_0 \text{ an integer} \tag{10.48}$$

This property can be proved by direct substitution in the equation defining the direct transform. From Eq. (10.31) we obtain

$$\begin{aligned} f[k-k_0] \Longleftrightarrow \sum_{k=-\infty}^{\infty} f[k-k_0]e^{-j\Omega k} &= \sum_{m=-\infty}^{\infty} f[m]e^{-j\Omega[m+k_0]} \\ &= e^{-j\Omega k_0}\sum_{k=-\infty}^{\infty} f[m]e^{-j\Omega m} = e^{-jk_0\Omega}F(\Omega) \end{aligned}$$

This result shows that *delaying a signal by k_0 units does not change its amplitude spectrum. The phase spectrum, however, is changed by $-k_0\Omega$.* This added phase is a linear function of Ω with slope $-k_0$.

Physical Explanation of the Linear Phase

Time delay in a signal causes a linear phase shift in its spectrum. The heuristic explanation of this result is exactly parallel to that for continuous-time signals given in Sec. 4.3-4 (see Fig. 4.20).

Frequency-Shifting Property

If

$$f[k] \Longleftrightarrow F(\Omega)$$

then

$$f[k]e^{jk\Omega_s} \Longleftrightarrow F(\Omega-\Omega_s) \tag{10.49}$$

This property is the dual of the time-shifting property. To prove this property, we have from Eq. (10.31)

$$f[k]e^{jk\Omega_s} \Longleftrightarrow \sum_{k=-\infty}^{\infty} f[k]e^{jk\Omega_s}e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f[k]e^{-j[\Omega-\Omega_s]} = F[\Omega-\Omega_s]$$

Time and Frequency Convolution Property

If

$$f_1[k] \Longleftrightarrow F_1(\Omega) \quad \text{and} \quad f_2[k] \Longleftrightarrow F_2(\Omega)$$

then

$$f_1[k] * f_2[k] \Longleftrightarrow F_1(\Omega)F_2(\Omega) \tag{10.50a}$$

and

$$f_1[k]f_2[k] \Longleftrightarrow \frac{1}{2\pi}F_1(\Omega) * F_2(\Omega) \tag{10.50b}$$

where

$$f_1[k] * f_2[k] = \sum_{m=-\infty}^{\infty} f_1[m]f_2[k-m]$$

and

$$F_1(\Omega) * F_2(\Omega) = \int_{2\pi} F_1(u)F_2(\Omega-u)\,du$$

The time convolution property is proved in Chapter 11 [Eq. (11.18)]. All we have to do is replace z with $e^{j\Omega}$. To prove the frequency-convolution property (10.50b), we have

$$f_1[k]f_2[k] \Longleftrightarrow \sum_{k=-\infty}^{\infty} f_1[k]f_2[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f_2[k]\left[\frac{1}{2\pi}\int_{2\pi} F_1(u)e^{-jku}\,du\right]e^{-j\Omega k}$$

Interchanging the order of summation and integration, we obtain

$$f_1[k]f_2[k] \Longleftrightarrow \frac{1}{2\pi}\int_{2\pi} F_1(u)\left[\sum_{k=-\infty}^{\infty} f_2[k]e^{-j(\Omega-u)k}\right]du = \frac{1}{2\pi}\int_{2\pi} F_1(u)F_2(\Omega-u)\,du$$

Parseval's Theorem

If

$$f[k] \Longleftrightarrow F(\Omega)$$

then E_f, the energy of $f[k]$, is given by

$$E_f = \sum_{k=-\infty}^{\infty} |f[k]|^2 = \frac{1}{2\pi}\int_{2\pi} |F(\Omega)|^2\,d\Omega \tag{10.51}$$

In order to prove this property, we have from Eq. (10.31)

$$F^*(-\Omega) = \sum_{k=-\infty}^{\infty} f^*[k]e^{-j\Omega k} \tag{10.52a}$$

This result shows that

$$f^*[k] \Longleftrightarrow F^*(-\Omega) \tag{10.52b}$$

Now

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |f[k]|^2 = \sum_{k=-\infty}^{\infty} f^*[k]f[k] &= \sum_{k=-\infty}^{\infty} f^*[k]\left[\frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}\,d\Omega\right] \\ &= \frac{1}{2\pi}\int_{2\pi} F(\Omega)\left[\sum_{k=-\infty}^{\infty} f^*[k]e^{j\Omega k}\right] \\ &= \frac{1}{2\pi}\int_{2\pi} F(\Omega)F^*(\Omega)\,d\Omega = \frac{1}{2\pi}\int_{2\pi} |F(\Omega)|^2\,d\Omega \end{aligned}$$

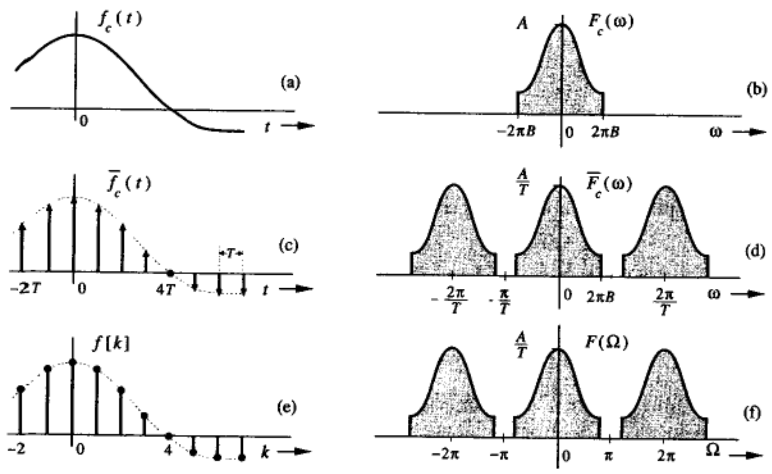


Fig. 10.8 Connection between the DTFT and the Fourier transform.

10.4 DTFT Connection with the Continuous-Time Fourier Transform

Consider a continuous-time signal $f_c(t)$ (Fig. 10.8a) with the Fourier transform $F_c(\omega)$. This signal may or may not be bandlimited. For convenience, we shall assume the signal to be bandlimited to B Hz (Fig. 10.8b). This signal is sampled with a sampling interval T. The sampling rate may or may not be above the Nyquist rate. Again, for convenience, we shall assume that the sampling rate is at least equal to the Nyquist rate; that is, $T \leq 1/2B$. The sampled signal $\overline{f}_c(t)$ (Fig. 10.8c) can be expressed as

$$\overline{f}_c(t) = \sum_{k=-\infty}^{\infty} f_c(kT)\delta(t-kT)$$

The continuous-time Fourier transform of the above equation yields

$$\overline{F}_c(\omega) = \sum_{k=-\infty}^{\infty} f_c(kT)e^{-jkT\omega} \tag{10.53}$$

In Sec. 5.1 (Fig. 5.1e), we have shown that $\overline{F}_c(\omega)$ is $F_c(\omega)/T$ repeating periodically with a period $\omega_s = 2\pi/T$, as illustrated in Fig. 10.8d. Let us construct a discrete-time signal $f[k]$ such that its kth element value is equal to the value of the kth sample of $f_c(t)$, as depicted in Fig. 10.8e; that is,

$$f[k] = f_c(kT)$$

Now, $F(\Omega)$, the DTFT of $f[k]$, is given by

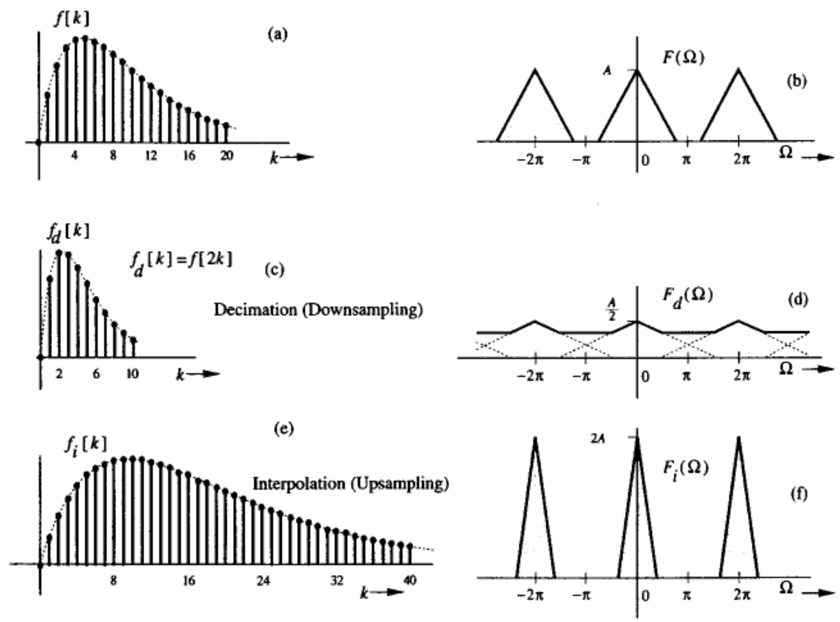


Fig. 10.9 Spectra of the decimated and interpolated signals.

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-jk\Omega} = \sum_{k=-\infty}^{\infty} f_c(kT)e^{-jk\Omega} \tag{10.54}$$

Comparison of (10.54) with (10.53) shows that

$$F(\Omega) = \overline{F}_c\left(\frac{\Omega}{T}\right) \tag{10.55}$$

Thus, $F(\Omega)$ can be obtained from $\overline{F}_c(\omega)$ by replacing ω with Ω/T. Therefore, $F(\Omega)$ is identical to $\overline{F}_c(\omega)$ frequency-scaled by a factor T, as shown in Fig. 10.8f.

In the above derivation, we did not have to use the assumption that $f_c(t)$ is bandlimited or that $f_c(t)$ is sampled at a rate at least equal to its Nyquist rate. If the signal were not bandlimited, the only difference in our discussion would be in the sketches in Fig. 10.8. For instance, $F_c(\omega)$ in Fig. 10.8b would not be bandlimited. This fact would cause overlapping (aliasing) of repeating cycles of $F_c(\omega)$ in Figs. 10.8d and f. A similar thing happens when the signal $f_c(t)$ is sampled below the Nyquist rate.†

10.4-1 DTFT of Decimated and Interpolated Signals

We can use Eq. (10.55) to find the DTFT of decimated and interpolated sig-

†In case the signal is not bandlimited and/ot the sampling rate is below its Nyquist rate, the samples $f_c(kT)$ can be interpreted as the Nyquist samples of the inverse Fourier transform of the first cycle (centered at $\omega = 0$) of $T\overline{F}_c(\omega)$.

nals, which are explained in Fig. 8.17. Consider a signal $f[k]$ and its DTFT $F(\Omega)$, as illustrated in Figs. 10.9a and b. Figure 10.9c shows the decimated signal $f[2k]$. If $f[k]$ is considered to be the sample sequence of a continuous-time signal $f_c(t)$, then $f[2k]$ is the sample sequence of $f_c(2t)$, whose Fourier transform is given by $\frac{1}{2}F_c(\omega/2)$ according to Eq. (4.34).† As seen in Eq. (10.55), $F(\Omega)$ is $F_c(\Omega/T)$ repeating periodically with period 2π, and the DTFT of $f[2k]$ is $\frac{1}{2}F_c(\Omega/2T)$ repeating periodically with period 2π. Note that $F(\Omega/2T)$ is $F(\Omega/T)$ time-expanded by factor 2, as shown in Figs. 10.9b and d. If we use mth-order decimation; that is, if we select every mth element in the sequence, the resulting decimated sequence will be $f[mk]$. Using the above argument, it follows that in the fundamental frequency range, the DTFT of $f[mk]$ will be $1/m$ times $F(\Omega/m)$, and it repeats periodically with period 2π. If m is too large, so that the first cycle of $F(\Omega/m)$ goes beyond π, the successive cycles of $F(\Omega/m)$ will overlap, as illustrated in Fig. 10.9d.

Now consider the interpolated signal $f_i[k]$ in Fig. 10.9c. This signal is $f[k/2]$ (obtained by expanding $f[k]$ by factor 2), with the alternate (missing) points filled by interpolated values obtained by ideal lowpass filtering. If the envelope of $f[k]$ is $f_c(t)$, then the envelope of $f_i[k]$ is $f_c(t/2)$, whose Fourier transform is $2F_c(2\omega)$. Thus, in the fundamental frequency range, the spectrum of $f_i[k]$ is 2 times $F(\Omega)$ compressed by factor 2 along the frequency axis, and periodically repeating with period 2π as depicted in Fig. 10.9f. Using a similar argument, we can generalize this result for a time-expanded signal $f[k/m]$ with missing values filled by ideal interpolation. In this case, the spectrum in the fundamental frequency range will be m times $F(m\Omega)$ [$F(\Omega)$ frequency-compressed by factor m] repeating periodically with period 2π.

10.5 Discrete-Time Linear System Analysis by DTFT

Consider a linear time-invariant discrete-time system with the unit impulse response $h[k]$. We shall find the (zero-state) system response $y[k]$ for the input $f[k]$. Because

$$y[k] = f[k] * h[k] \tag{10.56}$$

According to Eq. (10.50a) it follows that

$$Y(\Omega) = F(\Omega)H(\Omega) \tag{10.57}$$

where $F(\Omega)$, $Y(\Omega)$, and $H(\Omega)$ are DTFTs of $f[k]$, $y[k]$, and $h[k]$, respectively; that is,

$$f[k] \Longleftrightarrow F(\Omega), \quad y[k] \Longleftrightarrow Y(\Omega), \quad \text{and} \quad h[k] \Longleftrightarrow H(\Omega)$$

This result is similar to that obtained for continuous-time systems.

The Frequency Response of an LTID System

Equation (9.57a) states that the response to an everlasting exponential input z^k of an LTID system with transfer function $H[z]$ is $H[z]z^k$. If we let $z = e^{j\Omega}$, then

†Here, $F_c(\omega)$ should be interpreted as the first cycle (centered at $\omega = 0$) of $T\overline{F}_c(\omega)$

$z^k = e^{j\Omega k}$, and the response to an everlasting exponential input $e^{j\Omega k}$ of an LTID system with transfer function $H[z]$ is $H[e^{j\Omega}]e^{j\Omega k}$. This result can be represented as the input-output pair with a directed arrow notation as usual

$$e^{j\Omega k} \Longrightarrow H[e^{j\Omega}]e^{j\Omega k}$$

Also, according to Eq. (9.57b) with $z = e^{j\Omega}$, it follows that

$$H[e^{j\Omega}] = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

Observe that the right-hand side of the above equation is $H(\Omega)$, the DTFT of $h[k]$. Therefore

$$H[e^{j\Omega}] = H(\Omega)$$

and

$$e^{j\Omega k} \Longrightarrow H(\Omega)e^{j\Omega k}$$

Clearly $H(\Omega)$ is the LTID system frequency response. Following the argument in Sec. 7.1, the amplitude response of the system is $|H(\Omega)|$, and the phase response is $\angle H(\Omega)$. We shall discuss this topic again in greater details in Chapter 12.

Equation (10.57) states that the frequency spectrum of the output signal is the product of the frequency spectrum of the input signal and the frequency response of the system. From Eq. (10.57), we have

$$|Y(\Omega)| = |F(\Omega)|\,|H(\Omega)| \tag{10.58}$$

and

$$\angle Y(\Omega) = \angle F(\Omega) + \angle H(\Omega) \tag{10.59}$$

This result shows that the output amplitude spectrum is the product of the input amplitude spectrum and the amplitude response of the system. The output phase spectrum is the sum of the input phase spectrum and the phase response of the system.

We can also interpret Eq. (10.57) in terms of the frequency-domain viewpoint, which sees a system in terms of its frequency response (system response to various exponential or sinusoidal components). Frequency-domain views a signal as a sum of various exponential or sinusoidal components. Transmission of a signal through a (linear) system is viewed as transmission of various exponential or sinusoidal components of the input signal through the system. This concept can be understood by displaying the input-output relationships by a directed arrow as follows:

$$e^{j\Omega k} \Longrightarrow H(\Omega)e^{j\Omega k} \qquad \text{the system response to } e^{j\Omega k} \text{ is } H(\Omega)e^{j\Omega k}$$

$$f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}\,d\Omega \qquad \text{shows } f[k] \text{ as a sum of everlasting exponential components}$$

and from Eq. (10.57)

$$y[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)H(\Omega)e^{j\Omega k}\,d\Omega \qquad y[k] \text{ as a sum of responses to all input components}$$

which is precisely the relationship in Eq. (10.57). Thus, $F(\Omega)$ is the input spectrum and $Y(\Omega)$, the output spectrum (of the exponential components), is $F(\Omega)H(\Omega)$.

■ **Example 10.7**

For a system with unit impulse response $h[k] = (0.5)^k u[k]$, determine the (zero-state) response $y[k]$ for the input $f[k] = (0.8)^k u[k]$.

According to Eq. (10.57)

$$Y(\Omega) = F(\Omega)H(\Omega)$$

From the results in Eq. (10.37), we obtain

$$F(\Omega) = \frac{1}{1-0.8e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega}-0.8} \tag{10.60}$$

Also, $H(\Omega)$ is the DTFT of $(0.5)^k u[k]$, which is obtained from Eq. (10.37) by substituting $\gamma = 0.5$:

$$H(\Omega) = \frac{1}{1-0.5e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega}-0.5} \tag{10.61}$$

Therefore

$$Y(\Omega) = \frac{e^{j2\Omega}}{(e^{j\Omega}-0.5)(e^{j\Omega}-0.8)}$$

We can express the right-hand side as a sum of two first-order terms (modified partial fraction expansion as discussed in Sec. B.5-5) as follows:†

$$\begin{aligned}\frac{Y(\Omega)}{e^{j\Omega}} &= \frac{e^{j\Omega}}{(e^{j\Omega}-0.5)(e^{j\Omega}-0.8)} \\ &= \frac{-\frac{5}{3}}{e^{j\Omega}-0.5} + \frac{\frac{8}{3}}{e^{j\Omega}-0.8}\end{aligned}$$

Consequently,

$$\begin{aligned}Y(\Omega) &= -\left(\frac{5}{3}\right)\frac{e^{j\Omega}}{e^{j\Omega}-0.5} + \left(\frac{8}{3}\right)\frac{e^{j\Omega}}{e^{j\Omega}-0.8} \\ &= -\left(\frac{5}{3}\right)\frac{1}{1-0.5e^{-j\Omega}} + \left(\frac{8}{3}\right)\frac{1}{1-0.8e^{-j\Omega}}\end{aligned}$$

According to Eq. (10.37), the inverse DTFT of this equation is

$$y[k] = \left[-\tfrac{5}{3}(0.5)^k + \tfrac{8}{3}(0.8)^k\right] u[k]$$

This example demonstrates the procedure of determining an LTID system response using DTFT. It is similar to the method of Fourier transform in analysis of LTIC systems. As in the case of Fourier transform, this method can be used only if the system is asymptotically stable and if the input signal is DTF-transformable. We shall not belabor this method further because it is clumsier and more restricted than the z-transform method discussed in the next chapter. In the z-transform, we generalize the frequency variable $j\Omega$ to $\sigma + j\Omega$ so that the resulting exponentials can grow or decay with k. This procedure is

†Here $Y(\Omega)$ is a function of variable $e^{j\Omega}$. Hence, $x = e^{j\Omega}$ for the purpose of comparison with the expression in Sec. B.5-5.

similar to what we did in the continuous-time case by generalizing the frequency variable $j\omega$ to $s = \sigma + j\omega$ (from Fourier to Laplace transform) . ■

10.6 Signal processing by DFT and FFT

In this section, we use **DFT** (developed in Chapter 5) as a tool, which allows us to utilize a digital computer for digital signal processing. This signal processing includes spectral analysis of digital signals and LTID system analysis. By spectral analysis, we mean determining the discrete time Fourier series (DTFS) of periodic signals and determining $F(\Omega)$ from $f[k]$ (and vice versa) for aperiodic signals. As a tool for LTID system analysis, DFT can be used as a software oriented solution to digital filtering. DFT can be implemented on a digital computer by an efficient algorithm, the *fast Fourier transform (FFT)* also discussed in Chapter 5. The DFT (using FFT) is truly the workhorse of modern digital signal processing.

10.6-1 Computation of Discrete-Time Fourier Series (DTFS)

The discrete-time Fourier series (DTFS) equations (10.8) and (10.9) are identical to the DFT equations (5.18b) and (5.18a) within a scaling constant N_0. If we let $f[k] = N_0 f_k$ and $\mathcal{D}_r = F_r$ in Eqs. (10.9) and (10.8), we obtain

$$F_r = \sum_{k=0}^{N_0-1} f_k e^{-jr\Omega_0 k}$$

$$f_k = \frac{1}{N_0}\sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

This is precisely the DFT pair in Eqs. (5.18). For instance, to compute the DTFS for the periodic signal in Fig. 10.2a, we use the values of $f_k = f[k]/N_0$ as

$$f_k = \begin{cases} \frac{1}{32} & 0 \le k \le 4 \quad \text{and} \quad 28 \le k \le 31 \\ 0 & 5 \le k \le 27 \end{cases}$$

We use these values in the FFT algorithm discussed in Sec. 5.2-2 to obtain F_r, which is the same as $\mathcal{D}_r$.

10.6-2 Computation of Direct and Inverse DTFT

Spectral analysis of digital signals requires determination of DTFT and IDTFT [determining $F(\Omega)$ from $f[k]$ and vice versa]. This determination could be accomplished by using the DTFT equations [Eqs. (10.30) and (10.31)] directly on a digital computer. However, there are two difficulties in implementation of these equations on a digital computer.

1. Equation (10.31) involves summing an infinite number of terms, which is not possible because it requires infinite computer time.
2. Equation (10.30) requires integration which can only be performed approximately on a computer because a computer approximates an integral by a sum.

The first problem can be surmounted either by restricting the analysis only to a finite length $f[k]$ or by truncating $f[k]$ by a suitable window. The error because

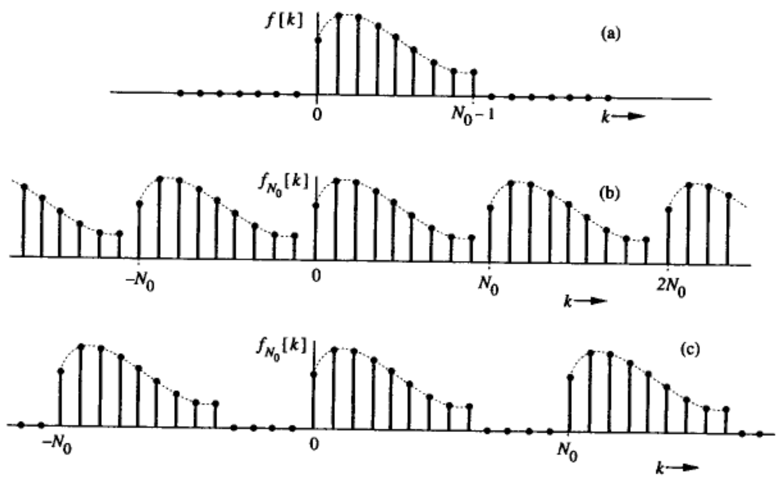


Fig. 10.10 DFT computation of a finite length signal.

of windowing may be reduced by using a wider and a tapered window. We shall see that if $f[k]$ has a finite duration, the samples of $F(\Omega)$ can be computed using a finite sum (rather than an integral). This solves the second problem. Moreover, $f[k]$ is uniquely determined from these samples of $F(\Omega)$.

In order to derive appropriate relationships, consider the signal $f[k]$ starting at $k = 0$, and with a finite length N_0, as shown in Fig. 10.10a. Let us construct a periodic signal $f_{N_0}[k]$ by repeating $f[k]$ periodically at intervals of N_0, as illustrated in Fig. 10.10b. We can represent the periodic signal $f_{N_0}[k]$ by the discrete-time Fourier series (DTFS) as [see Eqs. (10.8) and (10.9)]

$$f_{N_0}[k] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{10.62}$$

where

$$\mathcal{D}_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k} \tag{10.63}$$

In Eq. (10.63) we used the fact that $f_{N_0}[k] = f[k]$ for $k = 0, 1, 2, \ldots, N_0 - 1$.

By definition, the DTFT of $f[k]$ is

$$F(\Omega) = \sum_{k=0}^{N_0-1} f[k] e^{-j\Omega k} \tag{10.64}$$

According to Eqs. (10.63) and (10.64), it follows that $N_0\mathcal{D}_r$ is $F(r\Omega_0)$, the rth sample of $F(\Omega)$. For convenience, we denote this sample by F_r. Thus

$$N_0\mathcal{D}_r = F(r\Omega_0) = F_r \tag{10.65}$$

Therefore, $N_0\mathcal{D}_r$ are the samples of $F(\Omega)$ taken uniformly at the frequency intervals of Ω_0. But because $\Omega_0 = 2\pi/N_0$, there are exactly N_0 number of these samples of $F(\Omega)$ over the fundamental frequency interval of 2π. According to Eqs. (10.62) and (10.65), it follows that

$$f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k} \tag{10.66}$$

where [from Eqs. (10.63) and (10.65)]

$$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k} \tag{10.67}$$

Because $f_{N_0}[k] = f[k]$ for $k = 0, 1, 2, \ldots, (N_0 - 1)$, we can express the Eq. (10.66) as

$$f[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k} \qquad k = 0, 1, 2, \ldots, N_0 - 1 \tag{10.68}$$

Moreover, we need to determine F_r, the samples of the DTFT, only over the interval $0 \le \Omega < 2\pi$. Therefore, Eq. (10.67) can be expressed as

$$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k} \qquad r = 0, 1, 2, \ldots, N_0 - 1 \tag{10.69}$$

Equations (10.68) and (10.69) are precisely the DFT pair derived in Eqs. (5.18a) and (5.18b). These equations relate F_r [the samples of $F(\Omega)$] to $f[k]$ and vice versa. Here F_r is the DFT of $f[k]$ and $f[k]$ is the IDFT (inverse DFT) of F_r. This relationship is also denoted by the bidirectional arrow notation of a transform as

$$f[k] \Longleftrightarrow F_r$$

To repeat, *F_r, the DFT of an N_0-point sequence $f[k]$, is a set of uniform samples of its DTFT $F(\Omega)$ taken at frequency intervals of $\Omega_0 = \frac{2\pi}{N_0}$.* The sequence $f[k]$ and its DFT F_r are related by Eqs. (10.68) and (10.69). Observe that there are N_0 elements in $f[k]$. Also, there are exactly N_0 elements in F_r (over the frequency range 2π). The DFT relationships are finite sums and can be readily computed on a digital computer using the efficient fast Fourier transform (FFT) algorithm.

Properties of DFT

We list some of the important properties of DFT proved in Chapter 5. from the preceding discussion, it follows that these properties of DFT also apply to DTFT samples of a finite length $f[k]$.

1. **Linearity:** If $f[k] \Longleftrightarrow F_r$ and $g[k] \Longleftrightarrow G_r$, then

$$a_1 f[k] + a_2 g[k] \Longleftrightarrow a_1 F_r + a_2 G_r \tag{10.70}$$

2. **Conjugate Symmetry:** For real $f[k]$

$$F_{N_0-r} = F_r^* \tag{10.71}$$

There is a conjugate symmetry about $N_0/2$, which enables us to determine roughly half the values of F_r from the other half of the values, when $f[k]$ is real. For instance, in a 7-point DFT, $F_6 = F_1^*$, $F_5 = F_2^*$ and $F_4 = F_3^*$. In an 8-point DFT, $F_7 = F_1^*$, $F_6 = F_2^*$, $F_5 = F_3^*$, and so on.

3. **Time Shifting (Circular Shifting):**

$$f[k-n] \Longleftrightarrow F_r e^{-jr\Omega_0 n} \tag{10.72}$$

4. **Frequency Shifting:**

$$f[k]e^{jk\Omega_0 m} \Longleftrightarrow F_{r-m} \tag{10.73}$$

5. **Circular (or Periodic) Convolution:**

$$f[k] \circledast g[k] \Longleftrightarrow F_r G_r \tag{10.74a}$$

and

$$f[k]g[k] \Longleftrightarrow \frac{1}{N_0} F_r \circledast G_r \tag{10.74b}$$

where the circular (or periodic) convolution of two N_0-point periodic sequences $f[k]$ and $g[k]$ is defined as

$$f[k] \circledast g[k] = \sum_{n=0}^{N_0-1} f[n]g[k-n] = \sum_{n=0}^{N_0-1} g[n]f[k-n] \tag{10.75}$$

Caution in Interpreting DFT and IDFT

Equations (10.68) and (10.69) allow us to compute samples of DTFT and IDTFT for a finite length signal on a digital computer. To avoid certain pitfalls, we must understand clearly the nature of functions synthesized by the sums on the right-hand side of these equations. According to Eq. (10.66), it follows that the sum on the right-hand side of Eq. (10.68) is $f_{N_0}[k]$, which is a periodic signal of which $f[k]$ is the first cycle. Similarly, the sum on the right-hand side of Eq. (10.69) is periodic. This is because $F_r = N_0 \mathcal{D}_r$, which is periodic. Therefore, both the DFT equations are periodic. We require only part of these results (over one cycle) to compute the samples of $F(\Omega)$ from $f[k]$ and vice versa. That is why we placed the restriction that k or $r = 0, 1, 2, \ldots, N_0 - 1$ in Eqs. (10.68) and (10.69).

Signal $f[k]$ can start at any value of k.

In deriving the above results, we assumed that the signal $f[k]$ starts at $k = 0$. This restriction, fortunately, is not necessary. We now show that this procedure can be applied to $f[k]$ starting at any instant. Recall that the DFT found by this procedure is actually the DFT of $f_{N_0}[k]$, which is a periodic extension of $f[k]$ with period N_0. In other words, $f_{N_0}[k]$ can be generated from $f[k]$ by placing $f[k]$ and reproduction thereof end to end ad infinitum. Consider now the signal $f[k]$ in Fig. 10.6 in which $f[k]$ begins at $k = -4$. The periodic extension of this signal is depicted in Fig. 10.2a for $N_0 = 32$. A careful glance at this figure shows that this periodic signal can be constructed by any segment of length 32 and repeating it periodically by placing it end to end ad infinitum. We may choose a segment over the range $k = -16$ to 15 or a segment over the range $k = 0$ to 31, or any other segment of length 32. The reader should satisfy himself that the periodic extension of any such segment yields the same periodic signal $f_{N_0}[k]$. Therefore, the DFT corresponding to the periodic signal $f_{N_0}[k]$ is the DFT of any of its segment of N_0 length starting at any point. So the signal $f[k]$ may start at any point. All we need is to construct a periodic signal $f_{N_0}[k]$ which is a periodic extension of $f[k]$, then compute the DFT according to Eq. (10.68) using the sample values in the range $k = 0$ to $N_0 - 1$. For instance, the signal $f[k]$ in Fig. 10.6 does not start at $k = 0$. But we can construct its periodic extension $f_{N_0}[k]$, as shown in Fig. 10.2a, and use the values for $k = 0, 1, 2, \ldots, 31$ in Eq. (10.69) to compute F_r. These values are

$$f[k] = \begin{cases} 1 & 0 \le k \le 4 \quad \text{and} \quad 28 \le k \le 31 \\ 0 & 5 \le k \le 27 \end{cases}$$

Hence, according to Eq. 10.69

$$F_0 = \sum_{k=0}^{N_0-1} f[k] = 9$$

and

$$F_1 = e^{-j\Omega_0} + e^{-j2\Omega_0} + e^{-j3\Omega_0} + e^{-j4\Omega_0} + e^{-j28\Omega_0} + e^{-j29\Omega_0} + e^{-j30\Omega_0} + e^{-j31\Omega_0}$$

Because $\Omega_0 = \frac{\pi}{16}$, we recognize that $e^{-j31\Omega_0} = e^{j\Omega_0}$, $e^{-j30\Omega_0} = e^{j2\Omega_0}$, and so on. Hence

$$F_1 = 1 + 2(\cos\Omega_0 + \cos 2\Omega_0 + \cos 3\Omega_0 + \cos 4\Omega_0) = 7.8865$$

Note that $F_0 = 9$ is the first sample of $F(\Omega)$, $F_1 = F(\frac{\pi}{16})$ is the second sample, and so on. The samples are spaced $\pi/16$ radians apart, giving a total of 32 samples in the fundamental frequency range. The reader may confirm these values from from Eq. (10.44).

■ **Example 10.8**

Find the DFT of a 3-point signal $f[k]$ illustrated in Fig. 10.11a.

Figure 10.11a shows $f[k]$ (solid line) and $f_{N_0}[k]$ obtained by periodic extension of $f[k]$ (shown by dotted lines). In this case

$$N_0 = 3 \quad \text{and} \quad \Omega_0 = \frac{2\pi}{3}$$

From Eq. (10.69), we obtain

$$F_r = \sum_{k=0}^{2} f[k]e^{-jr(\frac{2\pi}{3})k} = 3 + 2e^{-jr\frac{2\pi}{3}} + 3e^{-jr\frac{4\pi}{3}}$$

Therefore

$$F_0 = 3 + 2 + 3 = 8$$

$$F_1 = 3 + 2\,e^{-j\frac{2\pi}{3}} + 3\,e^{-j\frac{4\pi}{3}} = 3 + \left(-1 - j\sqrt{3}\right) + \left(-\frac{3}{2} + j\frac{3\sqrt{3}}{2}\right) = \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = e^{j\frac{\pi}{3}}$$

$$F_2 = 3 + 2\,e^{-j\frac{4\pi}{3}} + 3\,e^{-j\frac{8\pi}{3}} = 3 + \left(-1 + j\sqrt{3}\right) + \left(-\frac{3}{2} - j\frac{3\sqrt{3}}{2}\right) = \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = e^{-j\frac{\pi}{3}}$$

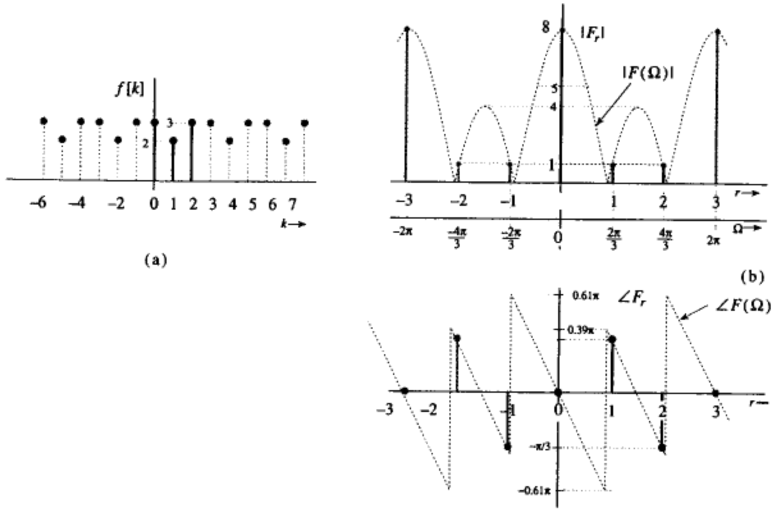


Fig. 10.11 Computation of DFT of a signal $f[k]$.

The magnitudes and angles of F_r are shown in the Table below.

r	0	1	2
$\lvert F_r\rvert$	8	1	1
$\angle F_r$	0	$\frac{\pi}{3}$	$-\frac{\pi}{3}$

Observe that $F_2 = F_1^*$ (the conjugate symmetry property). Because $f[k]$ is a 3-point sequence, F_r (its DFT) is also a 3-point sequence, which repeats periodically. Figure 10.11b shows F_r and $\angle F_r$. Recall that the DFT gives only the sample values of $F(\Omega)$. We want to know if DFT has enough samples to give a reasonably good idea of DTFT. The DTFT for $f[k]$ is given by [Eq. (10.31)]

$$F(\Omega) = \sum_{k=0}^{2} f[k]e^{-j\Omega k} = 3 + 2e^{-j\Omega} + 3e^{-j2\Omega} = e^{-j\Omega}[2 + 3e^{j\Omega} + 3e^{-j\Omega}] = e^{-j\Omega}(2 + 6\cos\Omega)$$

The amplitude and phase spectra are given by

$$|F(\Omega)| = |2 + 6\cos\Omega|$$

$$\angle F(\Omega) = \begin{cases} -\Omega & \text{when } (2 + 6\cos\Omega) > 0 \\ \pi - \Omega & \text{when } (2 + 6\cos\Omega) < 0 \end{cases}$$

Figure 10.11b shows $|F(\Omega)|$ and $\angle F(\Omega)$ (dotted). Observe that DFT values are exactly equal to DTFT values at the sampling frequencies; there is no approximation. This is always true of DFT of a finite length $f[k]$. However, if $f[k]$ is obtained by truncating or windowing a longer sequence, we shall see that the DFT gives only approximate sample values of the DTFT.

The DFT in this example has too few points to give a reasonable picture of the DTFT. The peak of DTFT appearing between the second and the third sample (between $r = 1$ and 2), for instance, is missed by the DFT. The two valleys of the DTFT are also missed. We definitely need more points in the DFT for an acceptable resolution. This goal is accomplished by **zero padding**, explained below. ■

Use of Zero Padding

The DFT yields the samples of DTFT at the frequency intervals of $\Omega_0 = 2\pi/N_0$, where Ω_0 is the frequency resolution. Seeing the DTFT through DFT is like viewing $F(\Omega)$ through a *picket fence*. Only the spectral components at the sampled frequencies (which are integral multiples of Ω_0) will be visible. But frequency components lying in between will be hidden behind the picket fence. If DFT has too few points, major peaks and valleys of $F(\Omega)$ existing between the DFT points (sampled frequencies) will not be seen, thus giving an erroneous view of the spectrum $F(\Omega)$. This is precisely the case in Example 10.8. Actually, using the interpolation formula, it is possible to compute any number of values of DTFT from the DFT. But having to use the interpolation formula really defeats the purpose of DFT. We therefore seek to reduce Ω_0 so that the number of samples is increased for a better view of the DTFT.

Because $\Omega_0 = 2\pi/N_0$, we can reduce Ω_0 by increasing N_0, the length of $f[k]$. For a finite length sequence, the only way to increase N_0 is by appending sufficient number of zero valued points to $f[k]$. This procedure of zero padding is depicted in Fig. 10.10c. Recall that N_0 is the period of $f_{N_0}[k]$, which is formed by periodic repetition of $f[k]$. By appending sufficient number of zeros to $f[k]$, as illustrated in Fig. 10.10c, we can increase the period N_0 as much as we wish, thereby increasing the number of points of the DFT. Recall that the number of points of both $f[k]$ as well as F_r are identical (N_0). We shall rework example 10.8 using zero padding to increase the number of samples of DTFT.

■ Example 10.9

Do Example 10.8 by padding three zeros to $f[k]$.

Figure 10.12a shows the zero padded $f[k]$, and the corresponding $f_{N_0}[k]$. Now $f[k]$ is a 6-point sequence. Hence,

$$N_0 = 6, \qquad \text{and} \qquad \Omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

and from Eq. (10.69), we obtain

$$F_r = \sum_{k=0}^{5} f[k]e^{-jr(\frac{\pi}{3})k} = 3 + 2e^{-jr\frac{\pi}{3}} + 3e^{-jr\frac{2\pi}{3}}$$

Therefore

$$F_0 = 3 + 2 + 3 = 8$$

$$F_1 = 3 + 2\,e^{-j\frac{\pi}{3}} + 3\,e^{-j\frac{2\pi}{3}} = 3 + (1 - j\sqrt{3}) + \left(-\frac{3}{2} - j\frac{3\sqrt{3}}{2}\right) = \left(\frac{5}{2} - j\frac{5\sqrt{3}}{2}\right) = 5e^{-j\frac{\pi}{3}}$$

In the same way, we find

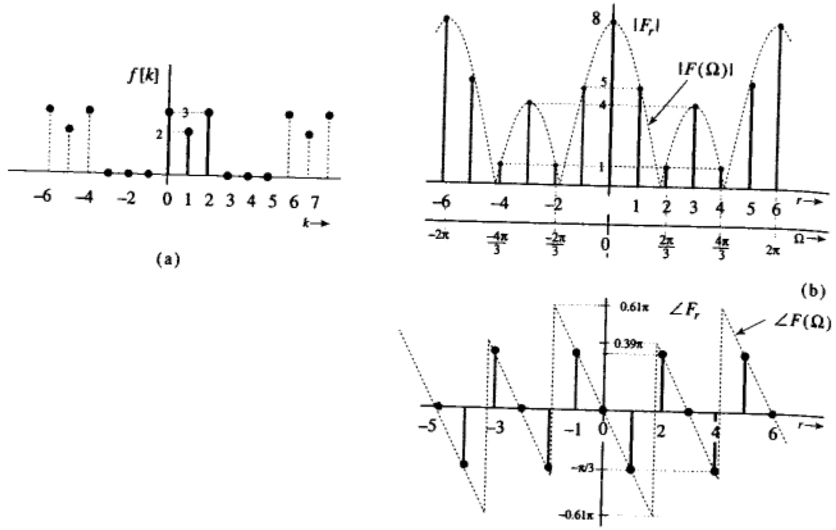


Fig. 10.12 DFT of a signal with zero padding.

$$F_2 = 3 + 2e^{-j\frac{2\pi}{3}} + 3e^{-j\frac{4\pi}{3}} = e^{j\frac{\pi}{3}}$$
$$F_3 = 3 + 2e^{-j\pi} + 3e^{-j2\pi} = 4$$
$$F_4 = 3 + 2e^{-j\frac{4\pi}{3}} + 3e^{-j\frac{8\pi}{3}} = e^{-j\frac{\pi}{3}}$$
$$F_5 = 3 + 2e^{-j\frac{5\pi}{3}} + 3e^{-j\frac{10\pi}{3}} = 5e^{j\frac{\pi}{3}}$$

The magnitude and angles of F_r are shown in the Table below

r	0	1	2	3	4	5
$\lvert F_r\rvert$	8	5	1	4	1	5
$\angle F_r$	0	$-\frac{\pi}{3}$	$\frac{\pi}{3}$	0	$-\frac{\pi}{3}$	$\frac{\pi}{3}$

Observe that $F_5 = F_1^*$ and $F_4 = F_2^*$ as expected from the conjugate symmetry property. Figure 10.12b shows the plots of $|F_r|$ and $\angle F_r$. Observe that we now have a 6-point DFT, which provides 6 samples of the DTFT spaced at the frequency interval of $\pi/3$ (in contrast to $2\pi/3$ spacing in Example 10.8). The samples corresponding to $r = 0$, 2, and 4 in Example 10.9 are identical to the samples corresponding to $r = 0$, 1, and 2 in Example 10.8. The DFT spectrum in Fig. 10.12b contains all the three samples appearing in Fig. 10.11b plus 3 more samples in between. Clearly, the zero padding allows us a better assessment of the DTFT. But even in this case, the valleys of the $F(\Omega)$ are missed by this (6-point) DFT.

Computer Example C10.3

Use MATLAB to do Example 10.8 to yield 32 sample values of DTFT.
Because the signal length is 3, we need to pad 29 zeros to the signal.


```
N0=32;k=0:N-1;
f=[3 2 3 zeros(1,N0-3)];
Fr=fft(f)
r=k;
subplot(3,1,1)
stem(k,f);
xlabel('k');ylabel('f[k]');
subplot(3,1,2)
stem(r,abs(Fr))
xlabel('r');ylabel('Fr');
subplot(3,1,3);
stem(r,angle(Fr))
xlabel('r');ylabel('angle Fr');grid on;
```

Example 10.10

For the signal $f[k]$ in Example 10.8, we need a frequency resolution $\Omega_0 = \pi/6$ for a reasonable view of $F(\Omega)$. Determine the number of zeros needed to be padded, and write the expression for computing the DFT.

For a resolution of $\Omega_0 = \pi/6$, the padded length N_0 of the signal is

$$N_0 = \frac{2\pi}{\Omega_0} = \frac{2\pi}{\pi/6} = 12$$

The padded length required is 12. Therefore we need to pad 9 zeros to $f[k]$. In this case

$$F_r = \sum_{k=0}^{11} f[k]e^{-jr(\frac{\pi}{6})k}$$

However, the last 9 samples of $f[k]$ are zero. Therefore, the above equation reduces to

$$F_r = \sum_{k=0}^{2} f[k]e^{-jr(\frac{\pi}{6})k} \quad \blacksquare$$

△ **Exercise E10.4**

A 3-point signal $f[k]$ is specified by $f[0] = 2$, $f[-1] = f[1] = 1$, and $f[k] = 0$ for all other k. Show that the DFT of this signal is $F_0 = 4$, $F_1 = 1$, and $F_2 = 1$. Find $F(\Omega)$, the DTFT of this signal, and verify that the DFT is equal to the samples of the DTFT at intervals of $\omega_0 = 2\pi/N_0 = 2\pi/3$. ▽

△ **Exercise E10.5**

Show that the 8-point DFT of the signal $f[k]$ in Exercise E10.4 is

$$F_0 = 4,\ F_1 = 3.4142,\ F_2 = 2,\ F_3 = 0.5858,\ F_4 = 0,\ F_5 = 0.5858,\ F_6 = 2,\ F_7 = 3.4142$$

Observe the conjugate symmetry of F_r about $r = N_0/2 = 4$. ▽

Practical Choice of N_0 The value of N_0 is determined by the desired resolution Ω_0. However, there is another consideration in selecting a value of N_0. If we are using the FFT algorithm to compute the DFT, then for efficient computation of

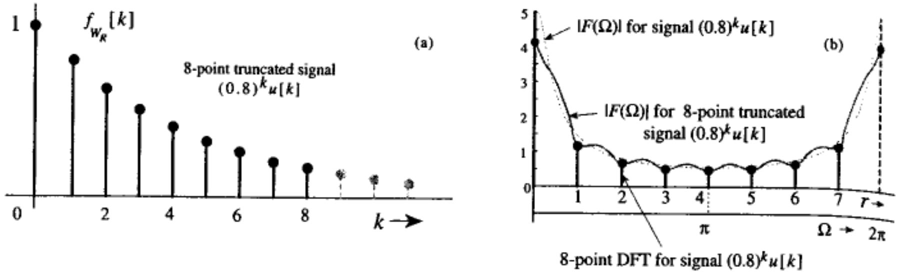


Fig. 10.13 DFT computations for $f[k] = (0.8)^k u[k]$ using an 8-point rectangular window.

DFT, N_0 should be a power of 2. Hence, we often pad sufficient numbers of zeros to ensure this requirement.

Effect of Signal truncation

So far we have considered only finite length sequences. For such sequences, we can readily find the N_0-point DFT, where N_0 is at least equal to the length of the sequence. How do we handle signals of infinite length? It is practically impossible to process infinite length sequences, because they generally require an infinite number of computations. Fortunately, every practical signal $f[k]$ must decay with k (because of a finite energy requirement), and such signal becomes negligibly small beyond $k \geq N_0$ for some suitable value of N_0. For instance, the signal $f[k] = (0.6)^k$ has infinite length. However, $f[k] \leq 0.00028$ at $k \geq 16$. Hence, we may truncate this signal beyond $k = 16$ (or even a little earlier). Straightforward signal truncation in this manner amounts to using a rectangular window with a unit weight for the data in the range $0 \leq k \leq N_0$, and zero weight for the data beyond $k = N_0$. Such a truncation results in Gibbs phenomenon with consequent oscillations in the spectrum of the truncated signal as demonstrated in the following example.

■ **Example 10.11**

The signal $f[k] = (0.8)^k u[k]$ has infinite length. Find the DFT of this signal using an 8-point rectangular window.

The DTFT of this signal obtained in Eq. (10.37) is

$$F(\Omega) = \frac{1}{1 - 0.8e^{-j\Omega}} \qquad \text{and} \qquad |F(\Omega)| = \frac{1}{\sqrt{1.64 - 1.6\cos\Omega}}$$

The 8-point rectangular window function is

$$w_R[k] = \begin{cases} 1 & 0 \leq k \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

The windowed signal is

$$f_{w_R}[k] = \begin{cases} (0.8)^k & 0 \leq k \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

The DFT of the windowed signal is obtained using the FFT algorithm. Figure 10.13a shows the windowed signal and Fig. 10.13b shows the corresponding 8-point DFT. The dotted curve shows the DTFT amplitude $|F(\Omega)|$ of the complete (untruncated) signal $(0.8)^k u[k]$ for comparison. The unbroken oscillating curve is the plot of the DTFT amplitude for the truncated (8-point) signal. The oscillations are because of the Gibbs phenomenon arising from the rectangular window. Observe the interesting fact that the 8 DFT values computed from the truncated signal are exactly equal to the 8 samples of the DTFT of the complete signal.

10.6-3 Discrete Time Filtering (Convolution) Using DFT

The DFT is useful not only in the computation of direct and inverse Fourier transforms, but also in other applications such as filtering, convolution, and correlation. Use of the efficient FFT algorithm makes DFT particularly appealing. We generally think of filtering in terms of some hardware-oriented solution (using summers, multipliers, and delay elements). However, filtering also has a software-oriented solution [a computer algorithm that yields the filtered output $y(t)$ for a given input $f(t)$]. Such filtering can be conveniently accomplished by using the DFT.

Filtering can be accomplished either in the frequency domain or in the time domain. In the frequency domain approach, for a given input $f[k]$, we are required to find the output $y[k]$ of a filter with a given transfer function $H(\Omega)$. In the time domain approach, for a given input $f[k]$, we are required to find the output $y[k]$ of a filter with a given impulse response $h[k]$. In the time domain, the output is obtained by (linear) convolution of $f[k]$ with $h[k]$. In the frequency domain, the output is obtained as an IDFT of $F_r H_r$ with $F_r = F(r\Omega_0)$ and $H_r = H(r\Omega_0)$. Because the frequency domain method appears as a substep of the time domain method, we shall consider here only the time domain (convolution) method.

We would like to perform linear convolution required in filtering operation using DFT (utilizing FFT algorithm) because of its computational efficiency. However, DFT can be used to evaluate only the circular convolution, not the linear convolution. Fortunately, linear convolution can be made equivalent to circular convolution by suitably padding the two sequences with zeros. This statement, introduced in Chapter 5, will now be proved.

When is Linear Convolution Equivalent to Circular Convolution?

The *circular (or periodic) convolution*, is explained in Sec. 5.2-1. In circular convolution, both sequences to be convolved are N_0-periodic. If $f[k]$ and $g[k]$ are both N_0-periodic, their periodic (or circular) convolution $c[k]$ is defined as

$$c[k] = f[k] \circledast g[k] = \sum_{m=0}^{N_0-1} f[m]g[k-m] \tag{10.76}$$

Note that the circular convolution differs from the regular (linear) convolution by the fact that the summation is over one period (starting at any point). In the linear convolution, the summation is from $-\infty$ to ∞. The result of a periodic convolution is also an N_0-periodic sequence.

Suppose we wish to convolve two finite length sequences $f[k]$ and $h[k]$ of length N_f and N_h, respectively. Let $y[k]$ be the (linear) convolution of these sequences; that is,

$$y[k] = f[k] * h[k] \tag{10.77}$$

From the width property of the convolution (Sec. 9.4), the length of $y[k]$ is $(N_f + N_h - 1)$. Let the DTFT of the sequences $f[k]$, $h[k]$, and $y[k]$ be $F(\Omega)$, $H(\Omega)$, and $Y(\Omega)$, respectively. Then, from Eq. (10.50a), we have

$$Y(\Omega) = F(\Omega)H(\Omega) \tag{10.78}$$

We know that the DFT of an N_0-point sequence is the set of uniform samples of its DTFT at frequency interval $\Omega_0 = 2\pi/N_0$. Therefore, if Y_r is the N_0-point DFT of $y[k]$, then

$$Y_r = Y(r\Omega_0) \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{10.79}$$

According to Eq. (10.78) it follows that

$$Y_r = F_r H_r \tag{10.80}$$

where F_r and H_r are the rth samples of $F(\Omega)$ and $H(\Omega)$, respectively. For Eq. (10.80) to be valid, F_r and H_r must be compatible for multiplication. In other words, both must be N_0-point sequences if Y_r is an N_0-point sequence. But we know that $f[k]$, $h[k]$, and $y[k]$ are N_f, N_h, and $N_0 = (N_f + N_h - 1)$-point sequences. Hence, we must pad $N_h - 1$ zeros to $f[k]$ and pad $N_f - 1$ zeros to $h[k]$ to ensure that F_r, H_r, and Y_r are all $N_0 = (N_f + N_h - 1)$-point sequences. Once we compute F_r and H_r (after suitably zero-padding $f[k]$ and $h[k]$), we take the IDFT of $F_r H_r$ to obtain $y[k]$. However, we have shown in Chapter 5 [Eq. (5.32a)] that $F_r H_r$ is the DFT of a circular convolution of $f[k]$ and $h[k]$; that is $f[k] \circledast h[k] \Longleftrightarrow F_r H_r$. This result means $y[k]$ is equal to the circular convolution of (suitably padded) $f[k]$ and $h[k]$. Thus, $y[k]$ is a periodic sequence whose first period is the linear convolution of (unpadded) $f[k]$ and $h[k]$. To summarize, $y[k]$, which is the linear convolution of $f[k]$ and $h[k]$, is also equal to the circular convolution of suitably padded $f[k]$ and $h[k]$. This is an extremely important result. The system response is given by the linear convolution of $f[k]$ and $h[k]$. But the preceding result allows us to compute this convolution as if it were the circular convolution of (suitably padded) $f[k]$ and $h[k]$. This, in turn, allows us to use DFT to perform the computations.

The procedure to find the (linear) convolution of $f[k]$ and $h[k]$, whose lengths are N_f and N_h, respectively, can be summarized in four steps as follows:

1. Pad $N_h - 1$ zeros to $f[k]$ and $N_f - 1$ zeros to $h[k]$.
2. Find F_r and H_r, the DFTs of the zero-padded sequences $f[k]$ and $h[k]$.
3. Multiply F_r by H_r to obtain Y_r.
4. The desired convolution $y[k]$ is the IDFT of Y_r.

Filtering In the Frequency Domain If we are given the filter transfer function $H(\Omega)$, we know H_r. We compute F_r, the DFT of $f[k]$. Then follow the steps 3 and 4 to obtain the output $y[k]$.

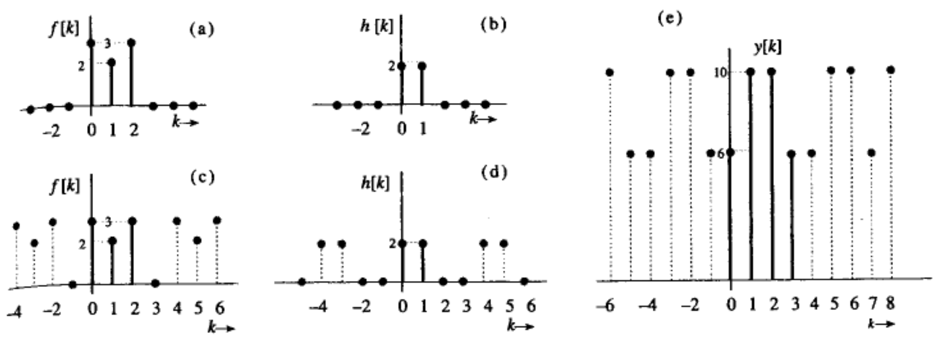


Fig. 10.14 Linear convolution of signals $f[k]$ and $h[k]$ using DFT.

■ Example 10.12

Find $y[k]$, the output of an LTID filter with impulse response $h[k]$ to an input $f[k]$ illustrated in Figs. 10.14a and b.

We obtain the convolution $y[k]$ in four steps listed above.

1. In this case $N_f = 3$ and $N_h = 2$. Therefore, we pad 1 zero to $f[k]$ and 2 zeros to $h[k]$, as depicted in Figs. 10.14c and d, respectively.
2. Now, $N_0 = 4$ and $\Omega_0 = \pi/2$. The DFTs F_r and H_r of the zero-padded sequences $f[k]$ and $h[k]$ are given by

$$F_r = \sum_{k=0}^{2} f[k]e^{-jr\Omega_0 k} = \sum_{k=0}^{2} f[k]e^{-jr\frac{\pi}{2}k}$$
$$= 3 + 2e^{-j\frac{\pi}{2}r} + 3e^{-j\pi r}$$

$$H_r = \sum_{k=0}^{1} h[k]e^{-jr\Omega_0 k} = \sum_{k=0}^{1} h[k]e^{-jr\frac{\pi}{2}k}$$
$$= 2 + 2e^{-j\frac{\pi}{2}r}$$

Substituting $r = 0, 1, 2, 3$, we obtain

$F_0 = 8$	$F_1 = -2j$	$F_2 = 4$	$F_3 = 2j$
$H_0 = 4$	$H_1 = 2\sqrt{2}e^{-j\frac{\pi}{4}}$	$H_2 = 0$	$H_3 = 2\sqrt{2}e^{j\frac{\pi}{4}}$

3. Multiply F_r by H_r to obtain Y_r. This step yields

$$Y_0 = 32 \qquad Y_1 = 4\sqrt{2}e^{-j\frac{3\pi}{4}} \qquad Y_2 = 0 \qquad Y_3 = 4\sqrt{2}e^{j\frac{3\pi}{4}}$$

4. The desired convolution is the IDFT of Y_r, given by

$$y[k] = \frac{1}{4}\sum_{r=0}^{3} Y_r e^{jr\Omega_0 k} = \frac{1}{4}\sum_{r=0}^{3} Y_r e^{jr\frac{\pi}{2}k}$$
$$= \frac{1}{4}\left(Y_0 + Y_1 e^{j\frac{\pi}{2}k} + Y_2 e^{j\pi k} + Y_3 e^{j\frac{3\pi}{2}k}\right)$$

Substitution of $r = 0, 1, 2, 3$ in this equation yields

$$y[0] = 6 \qquad y[1] = 10 \qquad y[2] = 10 \qquad y[3] = 6$$

Figure 10.14e shows $y[k]$ and its periodic extension (dotted). The IDFT yields the periodic signal. We need only the first cycle. ■

Computer Example C10.4

Use MATLAB to do Example 10.12. Find the answer by direct convolution as well as by using DFT.

In performing the convolution via DFT, we shall use the command 'fft(f,L)', which gives the FFT of a sequence f with sufficient zeros padded to make its length equal to L.

```
f=[3 2 3];
h=[2 2];
L=length(f)+length(h)-1;
k=0:1:L-1;
% Linear convolution: Direct approach
y1=conv(f,h);
subplot(2,1,2)
stem(k,y1)
% Linear Convolution:via DFT
FE=fft(f,L);
HE=fft(h,L);
y2=ifft(FE.*HE);
subplot(2,1,1);
stem(k,y2)
```

Computer Example C10.5

Use MATLAB to convolve $(0.8)^k u[k]$ and $(0.5)^k u[k]$. Find the answer by direct convolution and using DFT.

Both the signals have infinite duration. Hence, we must truncate them beyond some suitable value of k, where both functions become negligible. For this purpose $k = 32$ is a reasonable choice.

```
R=32;
m=0:1:R-1;
f=(0.8).^m;
h=(0.5).^m;
L=length(f)+length(h)-1;
k=0:1:L-1;
% Linear convolution: Direct approach
y1=conv(f,h);
subplot(2,1,2)
stem(k,y1)
% Linear Convolution:via DFT
FE=fft(f,L);
HE=fft(h,L);
y2=ifft(FE.*HE);
k=0:1:L-1;
subplot(2,1,1);stem(k,y2)
stem(k,y2)
```


△ **Exercise E10.6**

Show that the circular convolution of two 3-point sequences 3, 2, 3, and 1, 2, 3 , (both starting at $k = 0$) is also a 3-point sequence 15, 17, 16. Find the answer by using DFT, and verify your answer using the graphical method explained in Fig. 5.17. Using the sliding tape method (Fig. 9.4), show that the linear convolution of these sequences is a 5-point sequence 3, 8, 16, 12, 9. Indicate how you will obtain the linear convolution of these two sequences using circular convolution. ▽

△ **Exercise E10.7**

The input $f[k]$ of an LTID system is a 4-point sequence 1, 1, 1, 1, and $h[k]$, the impulse response of the system, is a 3-point sequence 1, 2, 1. Both sequences start at $k = 0$. Using DFT, show that the output $y[k]$ is a 6-point sequence 1, 3, 4, 4, 3, 1 starting at $k = 0$. Verify your answer by deriving the output as a linear convolution of the two sequences using the sliding tape method. ▽

Efficacy of DFT in Convolution Computation

Using the sliding tape algorithm discussed in Chapter 9, we can perform the convolution in Example 10.12 with a mere 6 (real) multiplications and 2 additions. The DFT method discussed in this section appears much too laborious, and the use of DFT for convolution may seem questionable. Recall, however, our discussion in Sec. 5.3, which showed that the use of FFT algorithm to compute DFT reduces the number of computations dramatically, especially for large N_0. For small length sequences, the direct convolution method, such as the sliding tape method is faster than the DFT method. But for long sequences, the DFT method using the FFT algorithm is much faster and far more efficient.† The method of finding convolution using the fast Fourier transform (FFT) is known as the *fast convolution*.

Block Filtering or Convolution

In practice, the length of an input signal may be very large, whereas a computer processing such a signal may have a limited memory. To process such long sequences, we can section the input signal into blocks of a length small enough to be processed by a given computer, and then add the outputs resulting from all the input blocks. This procedure can be used because the operation is linear. Such a procedure would be desirable even if the processing computer had an unlimited memory. For longer inputs, we must wait a long time before the input is fed to the computer (before it even starts processing), and then an even longer time for processing the large amount of data. Consequently, there is a long delay in the output. Sectioning the input allows the output to have a smaller processing delay. We shall now discuss two such method of block processing, **overlap and add** method and **overlap and save** method. Either of the methods requires the same number of computations; hence, which method is used is a matter of choice.

Overlap and Add Method

Consider a filtering operation, where the output $y[k]$ is the convolution of the input $f[k]$ and a FIR filter impulse response $h[k]$ of length M. In practice, usually

†It can be shown that in convolution of two sequences, each of length N_0, the number of computations required is on the order of N_0^2, whereas for DFT, using FFT algorithm, the number of computations required is only on the order of $N_0 \log_2 N_0$.

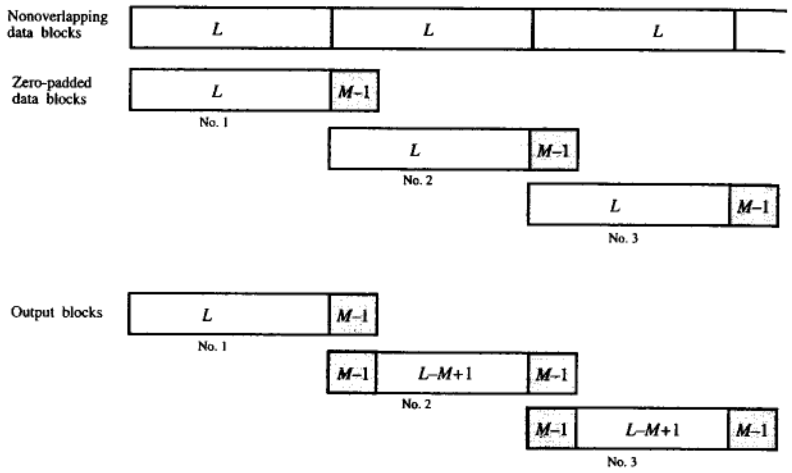


Fig. 10.15 Overlap and add method of block filtering.

M is much smaller than the length of the input. Figure 10.15 illustrates a long input sequence sectioned into nonoverlapping blocks of a manageable length L. The number inside each block indicates its length. Let us assume that $L \gg M$. We now process each block of the input data in sequence. To be able to use the circular convolution for performing the linear convolution, we need to pad $M - 1$ zeros at the end of each data block.‡ Figure 10.15 shows each data block augmented by $M - 1$ zeros. The augmented portion of the block is shown shaded. Observe that the augmented (zero-padded) blocks of the input, each of length $L + M - 1$, now overlap.

The output sequence corresponding to each block also has a length $L + M - 1$ (recall that the length of the circular convolution of two sequences, each of length $L+M-1$, is also $L+M-1$). The output sequences, therefore, also overlap, as shown in Fig. 10.15. The total output is given by the sum of all these overlapping output blocks of length $L + M - 1$. The contents of the two successive blocks are added wherever they overlap. This method is known as the *overlap and add* method.

■ Example 10.13

Using overlap and add method of block filtering, find the response $y[k]$ of an LTID system, whose impulse response $h[k]$ and the input $f[k]$ are shown in Fig. 10.16.

The output $y[k]$ is a linear convolution of $f[k]$ and $h[k]$. Let us use $L = 3$ for the block convolution. Also $M = 2$. Hence, we need to break the input sequence in blocks of 3 digits and pad each block with $M - 1 = 1$ zero, as depicted in Fig. 10.16. We convolve each of these blocks with $h[k]$ using DFT, as demonstrated in Example 10.11.

In this case, $N_0 = 4$ and $\Omega_0 = \pi/4$. The DFTs F_r and H_r of the zero-padded sequences $f[k]$ and $h[k]$ are given by

$$F_r = \sum_{k=0}^{2} f[k]e^{-jr\frac{\pi}{2}k} \qquad \text{and} \qquad H_r = \sum_{k=0}^{1} h[k]e^{-jr\frac{\pi}{2}k}$$

‡We also pad $h[k]$ with $L - 1$ zeros so that the length of the padded $h[k]$ is $L + M - 1$.

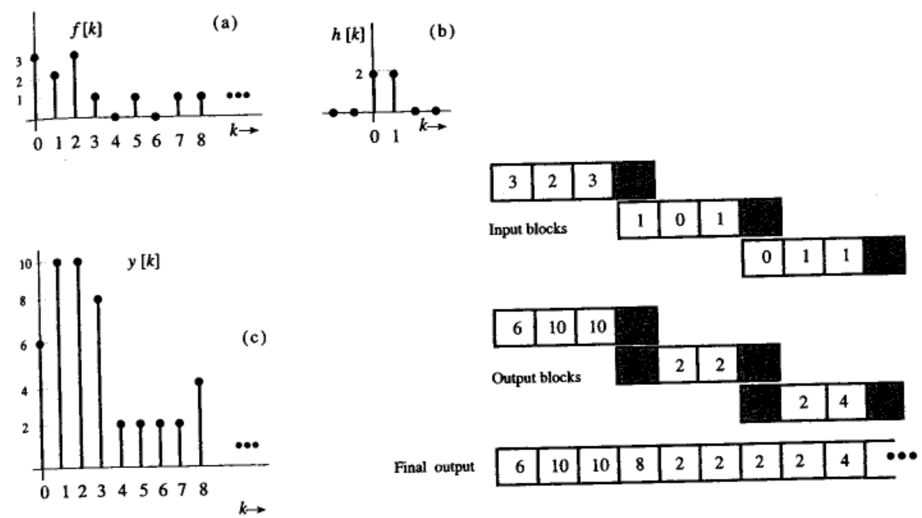


Fig. 10.16 An Example of overlap and add method of block filtering.

Also, $Y_r = F_rH_r$. We compute the values of F_r, H_r, and Y_r using these equations for each block:

For the first block,

$$F_r = 3 + 2e^{-j\frac{\pi}{2}r} + 3e^{-j\pi r}, \qquad H_r = 2 + 2e^{-j\frac{\pi}{2}r} \qquad \text{and} \qquad Y_r = F_rH_r$$

Also

$$y[k] = \frac{1}{4}\sum_{r=0}^{3} Y_r e^{jr\frac{\pi}{2}k} = \frac{1}{4}\left(Y_0 + Y_1e^{j\frac{\pi}{2}k} + Y_2e^{j\pi k} + Y_3e^{j\frac{3\pi}{2}k}\right)$$

Substituting $r = 0, 1, 2, 3$, we obtain

$$\begin{array}{llll}
F_0 = 8 & F_1 = -2j & F_2 = 4 & F_3 = 2j \\
H_0 = 4 & H_1 = 2\sqrt{2}e^{-j\frac{\pi}{4}} & H_2 = 0 & H_3 = 2\sqrt{2}e^{j\frac{\pi}{4}} \\
Y_0 = 32 & Y_1 = 4\sqrt{2}e^{-j\frac{3\pi}{4}} & Y_2 = 0 & Y_3 = 4\sqrt{2}e^{j\frac{3\pi}{4}} \\
y[0] = 6 & y[1] = 10 & y[2] = 10 & y[3] = 6
\end{array}$$

Using the same procedure for the second block, we obtain

$$y[0] = 2, \quad y[1] = 2 \quad y[2] = 2, \quad y[3] = 2$$

For the third block, we obtain

$$y[0] = 0, \quad y[1] = 2 \quad y[2] = 4, \quad y[3] = 2$$

Figure 10.16 shows the overlapping input and output blocks, and the convolution sequence obtained by adding the output blocks.

The procedure using DFT given here is much more laborious than the direct convolution by the sliding tape method (Sec. 9.4). The reason is that we did not use FFT algorithm to compute DFT here. In this example, with a rather small N_0, even FFT will

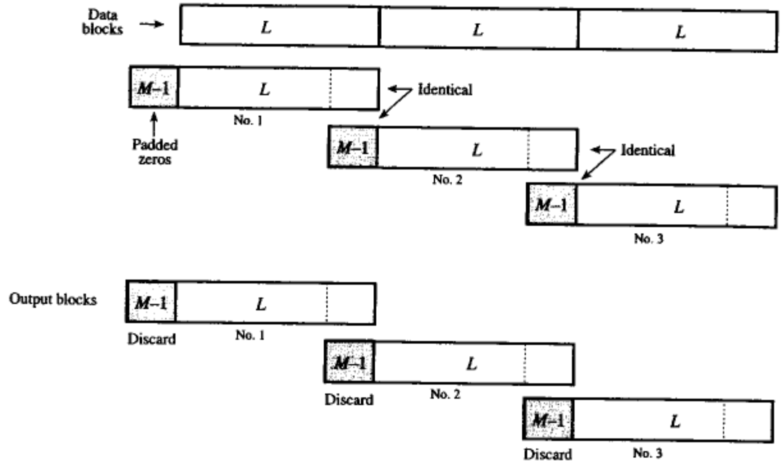


Fig. 10.17 Overlap and Save method of block filtering.

have more number of computations than the direct convolution. In practice, we generally deal with much larger values of N_0, where use of DFT (utilizing FFT algorithm) pays off. It will be informative for the reader to find linear convolution (using the sliding tape method) of one unpadded data block with unpadded $h[k]$. Next, find the circular convolution of the same data block padded with 1 zero and $h[k]$ padded with 2 zeros. Use the graphical method of circular convolution illustrated in Fig. 5.17. Verify that you get the same answer in both cases. ■

Overlap and Save Method

In the *overlap and save* method also, the input sequence is sectioned into nonoverlapping blocks of a manageable length L. As before, each block is augmented with $M-1$ data points. But unlike the previous method, this method places augmented points at the beginning of each block. The augmented $M-1$ data points of a block are the same as the last $M-1$ points of the previous block so that the last $M-1$ data points of each block also appear as the first $M-1$ data points of the succeeding block. The exception is the first block, where the first $M-1$ data points are taken as zeros, as shown in Fig. 10.17. We now convolve each of these blocks with $h[k]$ (padded by $L-1$ zeros). As before, DFT is used to perform convolution.

The output sequence corresponding to each block also has a length $L+M-1$. We discard the first $M-1$ data points and save the last L data points from each output block, as depicted in Fig. 10.17. The total output is given by combining all the saved blocks in sequence. This method is known as the *overlap and save* method. More details of this method can be found in the literature.[1]

■ **Example 10.14**

Using overlap and save method of block filtering, find the response $y[k]$ of an LTID system, whose impulse response $h[k]$ and the input are illustrated in Fig. 10.16.

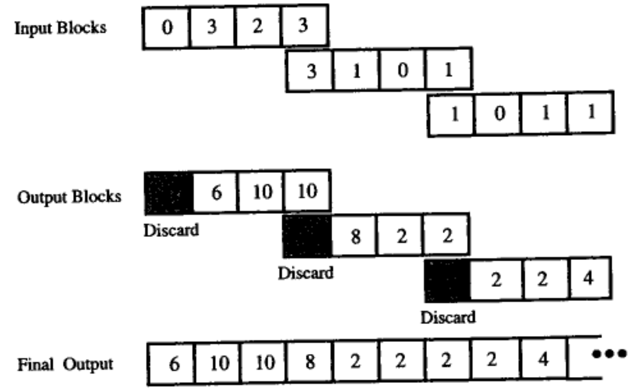


Fig. 10.18 An Example of overlap and save method of block filtering.

We follow the procedure given in Fig. 10.17 to section the input data as shown in Fig. 10.18. Note that the first $M-1=1$ data point of the first block is padded (zero) and the last $M-1=1$ point of each block also appears as the first point of the next block. Each block of length $L-M+1=4$ is now convolved (using DFT) with $h[k]$ (padded with two zeros). The DFT procedure is already explained in Example 10.13. We shall omit the details here. the resulting output blocks are depicted in Fig. 10.18. The first $M-1=1$ point of each output block is discarded. The total output is given by combining all the saved blocks in sequence. ■

⊙ **Computer Example C10.6**

Use MATLAB to do example 10.13 (overlap and add method).

Here, we use the MATLAB command 'fftfilt(h,f,M)' to perform convolution using overlap and add method with blocks of length M. This m-file is available in *Signal Processing Toolbox*.

```
f=[3 2 3 1 0 1 0 1 1];
h=[2 2];
L=length(f)+length(h)-1;
k=0:1:L-2;
y=fftfilt(h,f,3);
stem(k,real(y));
```
⊙

△ **Exercise E10.8**

The input $f[k]$ of an LTID system is a sequence 1, 0, −1, 2, ..., and $h[k]$, the impulse response of the system is a 3-point sequence 3, 2, 3. Both sequences start at $k=0$. Using block convolution with $L=2$, show that the output is 3, 2, 0, 4, ... Derive your answer using both methods of block filtering. ▽

10.7 Generalization of the DTFT to the Z-Transform

LTID systems can be analyzed using DTFT. This method, however, has the following limitations:

1. Existence of the DTFT is guaranteed only for absolutely summable signals [see

Eq. (10.36)]. The DTFT does not exist for exponentially growing signals. This means the DTFT method can be applied only for a limited class of inputs.

2. Moreover, this method can be applied only to asymptotically stable systems; it cannot be used for unstable or even marginally stable systems.

These are serious limitations in the study of LTID system analysis. Actually it is the first limitation that is also the cause of the second limitation. Because DTFT is incapable of handling growing signals, it is incapable of handling unstable or marginally stable systems.† Our goal is, therefore, to extend the concept of DTFT so that it can handle exponentially growing signals.

We may wonder what causes this limitation on DTFT so that it is incapable of handling exponentially growing signals. Recall that in DTFT, we are synthesizing an arbitrary signal $f[k]$ using sinusoids or exponentials of the form $e^{j\Omega k}$. These signals are sinusoids with constant amplitudes. They are incapable of synthesizing exponentially growing signals no matter how many such components we add. Our hope, therefore, lies in trying to synthesize $f[k]$ using exponentially growing sinusoids or exponentials. This goal can be accomplished by generalizing the frequency variable $j\Omega$ to $\sigma + j\Omega$; that is, by using exponentials of the form $e^{(\sigma+j\Omega)}$ instead of exponentials $e^{j\Omega}$. The procedure is almost identical to that used in extending the Fourier transform to the Laplace transform in Sec. 6.1. The intuitive argument is identical to that discussed in Sec. 6.1, and the reader may wish to review it to refresh his memory. Here we shall go straight to the analytical development.

As in the case of Fourier to Laplace, it is desirable to use the notation $F(j\Omega)$ instead of $F(\Omega)$ for the DTFT in order to unify the DTFT and the generalized transform (z-transform). Thus,

$$F(j\Omega) = \sum_{k=-\infty}^{\infty} f[k]\, e^{-j\Omega k} \tag{10.81}$$

and

$$f[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(j\Omega)\, e^{j\Omega k}\, d\Omega \tag{10.82}$$

Consider now the DTFT of $f[k]\, e^{-\sigma k}$ (σ real)

$$\text{DTFT}\,[f[k]\, e^{-\sigma k}] = \sum_{k=-\infty}^{\infty} f[k]\, e^{-\sigma k}\, e^{-j\Omega k} \tag{10.83}$$

$$= \sum_{k=-\infty}^{\infty} f[k]\, e^{-(\sigma+j\Omega)k} \tag{10.84}$$

It follows from Eq. (10.81) that the above sum is $F(\sigma + j\Omega)$. Thus

$$\text{DTFT}\,[f[k]\, e^{-\sigma k}] = \sum_{k=-\infty}^{\infty} f[k]\, e^{-(\sigma+j\Omega)k} = F(\sigma + j\Omega) \tag{10.85}$$

†Recall that the output of an unstable system grows exponentially. Also, the output of a marginally stable system to characteristic mode input grows with time.

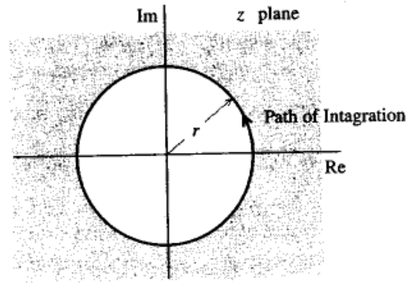


Fig. 10.19 Contour of integration for the z-transform.

Hence, the inverse DTFT of $F(\sigma + j\Omega)$ is $f[k]\, e^{-\sigma k}$. Therefore

$$f[k]\, e^{-\sigma k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\sigma + j\Omega)\, e^{j\Omega k}\, d\Omega \tag{10.86}$$

Multiplying both sides of the above equation by $e^{\sigma k}$ yields

$$f[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\sigma + j\Omega)\, e^{(\sigma+j\Omega)k}\, d\Omega \tag{10.87}$$

Let us define a new variable z as

$$z = e^{\sigma+j\Omega} \qquad \text{so that} \qquad \ln z = \sigma + j\Omega \qquad \text{and} \qquad \frac{1}{z}\, dz = j\, d\Omega \tag{10.88}$$

Because $z = e^{\sigma+j\Omega}$ is complex, we can express it as $z = r\, e^{j\Omega}$, where $r = e^{\sigma}$. Thus, z lies on a circle of radius r, and as Ω varies from $-\pi$ to π, z circumambulates along this circle, completing exactly one rotation in counterclockwise direction, as illustrated in Fig. 10.19. Changing to variable z in Eq. (10.87) yields

$$f[k] = \frac{1}{2\pi j} \oint F(\ln z)\, z^{k-1}\, dz \tag{10.89a}$$

and from Eq. (10.85) we obtain

$$F(\ln z) = \sum_{k=-\infty}^{\infty} f[k]\, z^{-k} \tag{10.89b}$$

where the integral $\oint$ indicates a contour integral around a circle of radius r in counterclockwise direction.

The above two equations are the desired extensions. They are, however, in a clumsy form. For the sake of convenience, we make a notational change by noting that $F(\ln z)$ is a function of z. Let us denote it by a simpler notation $F[z]$. Thus, Eqs. (10.89) become

$$f[k] = \frac{1}{2\pi j} \oint F[z]\, z^{k-1}\, dz \tag{10.90}$$

and

$$F[z] = \sum_{k=-\infty}^{\infty} f[k]\, z^{-k} \tag{10.91}$$

This is the (bilateral) z-transform pair. Equation (10.90) expresses $f[k]$ as a continuous sum of exponentials of the form $z^k = e^{(\sigma+j\Omega)k} = r^k e^{j\Omega k}$. Thus, by selecting a proper value for r (or σ), we can make the exponential grow (or decay) at any exponential rate we desire.

If we let $\sigma = 0$, we have $z = e^{j\Omega}$ and

$$F[z] = F(\ln z) = F(j\Omega) = F(\Omega) \tag{10.92}$$

Thus, the familiar DTFT is just a special case of the z-transform $F[z]$ obtained by letting $z = e^{j\Omega}$.

10.8 Summary

This chapter deals with analysis and processing of discrete-time signals. For analysis, our approach is parallel to that used in continuous-time signals. We first represent a periodic $f[k]$ as a Fourier series formed by a discrete-time exponential and its harmonics. Later we extend this representation to an aperiodic signal $f[k]$ by considering $f[k]$ as a limiting case of a periodic signal with period approaching infinity. Periodic signals are represented by discrete-time Fourier series (DTFS); aperiodic signals are represented by the discrete-time Fourier transform (DTFT). The development, although similar to that of continuous-time signals, also reveals some significant differences. The basic difference in the two cases arises because a continuous-time exponential $e^{j\omega t}$ has a unique waveform for every value of ω in the range $-\infty$ to ∞. In contrast, a discrete-time exponential $e^{j\Omega k}$ has a unique waveform only for values of Ω in a continuous interval of 2π. Therefore, if Ω_0 is the fundamental frequency, then at most $\frac{2\pi}{\Omega_0}$ number of exponentials in the Fourier series are independent. Consequently, the discrete-time exponential Fourier series has only $N_0 = \frac{2\pi}{\Omega_0}$ terms.

The discrete-time Fourier transform (DTFT) of an aperiodic signal is a continuous function of Ω and is periodic with period 2π. We can synthesize $F(\Omega)$ from its spectral components in any band of width 2π. Linear time-invariant discrete-time (LTID) systems can be analyzed using DTFT if the input signals are DTF-transformable and if the system is stable. Analysis of unstable (or marginally stable) systems and/or exponentially growing inputs can be performed by z-transform, which is a generalized DTFT. The relationship of DTFT to z-transform is similar to that of the Fourier transform to the Laplace transform. Whereas the z-transform is superior to DTFT for analysis of LTID systems, DTFT is preferable in signal analysis.

If $H(\Omega)$ is the DTFT of the system's impulse response $h[k]$, then $|H(\Omega)|$ is the amplitude response, and $\angle H(\Omega)$ is the phase response of the system. Moreover, if $F(\Omega)$ and $Y(\Omega)$ are the DTFTs of the input $f[k]$ and the corresponding output $y[k]$, then $Y(\Omega) = H(\Omega)F(\Omega)$. Therefore the output spectrum is the product of the input spectrum and the system's frequency response.

The numerical computations in modern digital signal processing can be conveniently performed with the discrete Fourier transform (DFT) introduced in Chapter

5. The DFT computations can be very efficiently executed by using the fast Fourier transform (FFT) algorithm. The DFT is indeed the workhorse of modern digital signal processing. The discrete-time Fourier transform (DTFT) and the inverse discrete-time Fourier transform (IDTFT) can be computed using the DFT. For an N_0-point signal $f[k]$, its DFT yields exactly N_0 samples of $F(\Omega)$ at frequency intervals of $2\pi/N_0$. We can obtain a larger number of samples of $F(\Omega)$ by padding sufficient number of zero valued samples to $f[k]$. The N_0-point DFT of $f[k]$ gives exact values of the DTFT samples if $f[k]$ has a finite length N_0. If the length of $f[k]$ is infinite, we need to truncate $f[k]$ using the appropriate window function.

Because of the convolution property, we can compute convolution of two signals $f[k]$ and $h[k]$ using DFT. For this purpose, we need to pad both the signals by a suitable number of zeros so as to make the linear convolution of the two signals identical to the circular (or periodic) convolution of the padded signals. Large blocks of data may be processed by sectioning the data into smaller blocks and processing such smaller blocks in sequence. Such a procedure requires smaller memory and reduces the processing time.

References

1. Mitra, S.K., *Digital Signal processing: A Computer Based Approach*, McGraw-Hill, New York, 1998.

Problems

10.1-1 Find the discrete-time Fourier series (DTFS) and sketch their spectra $|\mathcal{D}_r|$ and $\angle\mathcal{D}_r$ for $0 \le r \le N_0 - 1$ for the following periodic signal:

$$f[k] = 4\cos 2.4\pi k + 2\sin 3.2\pi k$$

Hint: Reduce frequencies to the fundamental range $(0 \le \Omega \le 2\pi)$. The fundamental frequency Ω_0 is the largest number of which the frequencies appearing in the Fourier series are integral multiples.

10.1-2 Repeat Prob. 10.1-1 if $f[k] = \cos 2.2\pi k \cos 3.3\pi k$.

10.1-3 Repeat Prob. 10.1-1 if $f[k] = 2\cos 3.2\pi(k-3)$.

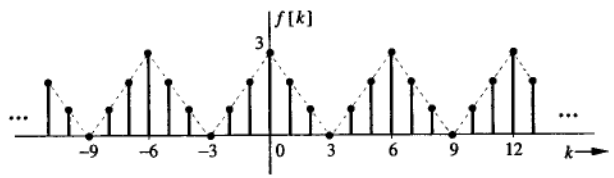


Fig. P10.1-4

10.1-4 Find the discrete-time Fourier series and the corresponding amplitude and phase spectra for the $f[k]$ shown in Fig. P10.1-4.

10.1-5 Repeat Prob. 10.1-4 for the $f[k]$ depicted in Fig. P10.1-5.

10.1-6 Repeat Prob. 10.1-4 for the $f[k]$ illustrated in Fig. P10.1-6.

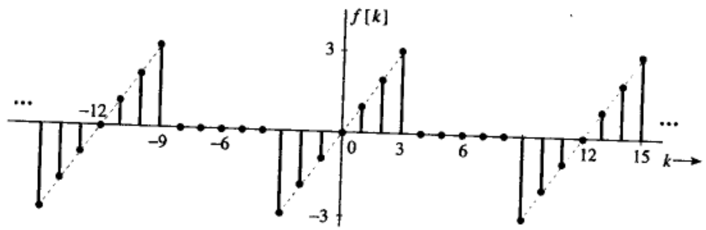


Fig. P10.1-5

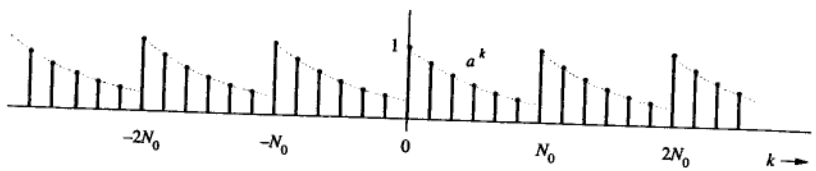


Fig. P10.1-6

10.1-7 A signal $f[k]$ is approximated in terms of another signal $x[k]$ over an interval ($N_1 \le k \le N_1$) as

$$f[k] \approx cx[k] \qquad N_1 \le k \le N_2$$

(a) Show that for the best approximation that minimizes the energy of the error signal $e[k] = f[k] - cx[k]$ over the same interval

$$c = \frac{1}{E_x} \sum_{k=N_1}^{N_2} f[k]x^*[k]$$

(b) If $c = 0$, the discrete-time signals $f[k]$ and $x[k]$ are said to be orthogonal over the interval ($N_1 \le k \le N_2$). Use this observation to define the orthogonality of discrete-time signals.
(c) Show that the set of signals $e^{jr\Omega_0 k}$ for $r = 0, 1, 2, 3, \ldots, N_0 - 1$ is orthogonal over an interval ($0 \le k \le N_0 - 1$). Hence, find the exponential Fourier series (DTFS) using the result in part **(a)**.
Hint: Recall that if w is complex, then $|w|^2 = ww^*$.

10.1-8 An N_0-periodic signal $f[k]$ is represented by its DTFS as in Eq. (10.8). Prove Parseval's theorem (for DTFS), which states that

$$\frac{1}{N_0} \sum_{k=<N_0>} |f[k]|^2 = \sum_{r=<N_0>} |\mathcal{D}_r|^2$$

Earlier [Eq. (10.51)], we obtained the Pareseval's theorem for DTFT.

Hint: If w is complex, then $|w|^2 = ww^*$, and use Eq. (5.43).

10.2-1 For the following signals, find the DTFT directly, using the definition in Eq. (10.31).

(a) $f[k] = \delta[k]$ **(b)** $\delta[k - k_0]$ **(c)** $a^k u[k-1]$ $|a| < 1$

(d) $f[k] = a^k u[k+1]$ $|a| < 1$.

In each case, sketch the signal and its amplitude spectrum. Sketch phase spectra for parts **(a)** and **(b)** only.

10.2-2 Find the DTFT for the signals shown in Fig. P8.2-9 (Chapter 8).

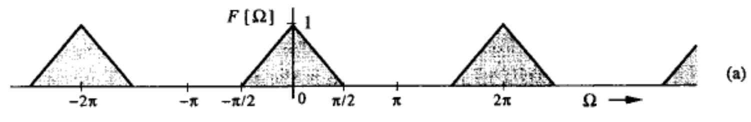


Fig. P10.2-3

10.2-3 Find the inverse DTFT for the spectrum depicted in Fig. P10.2-3.

10.3-1 Using the time-shifting property and the results in Examples 10.3 and 10.5, find the DTFT of **(a)** $a^k\{u[k] - u[k-10]\}$ **(b)** $u[k] - u[k-9]$.

10.3-2 Using appropriate properties and the result in Example 10.3, find the DTFT of **(a)** $(k+1)a^k u[k]$ ($|a| < 1$) **(b)** $a^k \cos \Omega_0 k\, u[k]$.

10.4-1 For the spectrum $F(\Omega)$ in Fig. P10.2-3
(a) Find and sketch its IDTFT $f[k]$.
(b) Sketch $f[2k]$, $f[4k]$, and find their DTFTs.
(c) Sketch $f[k/2]$ and fill in the alternate missing samples using ideal interpolation (upsampling by a factor 2). Find the DTFT of the resulting interpolated (upsampled) signal $f_i[k]$.

10.5-1 Using the DTFT method, find the zero-state response $y[k]$ of a causal system with frequency response

$$H(\Omega) = \frac{e^{j\Omega} + 0.32}{e^{j2\Omega} + e^{j\Omega} + 0.16}$$

and the input

$$f[k] = (-0.5)^k u[k]$$

10.5-2 Repeat Prob. 10.5-1 if

$$H(\Omega) = \frac{e^{j\Omega} - 0.5}{(e^{j\Omega} + 0.5)(e^{j\Omega} - 1)}$$

and

$$f[k] = 3^{-(k+1)} u[k]$$

10.5-3 Repeat Prob. 10.5-1 if

$$H(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 0.5}$$

and

$$f[k] = 0.8^k u[k] + 2(2)^k u[-(k+1)]$$

10.6-1 Find the DFT of a 3-point signal $f[k]$ specified by $f[-1] = f[0] = 3$, $f[1] = 2$ and $f[k] = 0$ otherwise. Now determine $F(\Omega)$, the DTFT of $f[k]$, and verify that DFT values are the samples of $F(\Omega)$.

(b) Show that the 3-point DFT of this signal is identical to that of the signal $f[k]$ in Fig. 10.11a. Can you explain why? Does this mean the DTFTs of the two signals are also identical? Determine the DTFTs of the two signals and see if they are identical (for all values of Ω).

(c) Find the 8-point DFT of $f[k]$.

10.6-2 (a) Find the 4-point and 8-point DFT of a 4-point signal specified by the sequence 1, 2, 2, 1 starting at $k = 0$.

(b) Find $F(\Omega)$, the DTFT of $f[k]$, and verify the DFT values from $F(\Omega)$.

10.6-3 (a)Find the DFT of the signal $f[k] = \delta[k]$. Find also $F(\Omega)$, the DTFT of $\delta[k]$, and verify the DFT values from $F(\Omega)$. Note that this is a 1-point signal ($N_0 = 1$).

(b) Show that the DFT of $f[k] = \delta[k-m]$ is the same as the DFT of $\delta[k]$ for any integral value of m. Explain this behavior.

(c) Repeat part (a) for the N_0-point DFT (found by padding $N_0 - 1$ zeros to $\delta[k]$). Explain this DFT from $F(\Omega)$ found in part (a).

10.6-4 (a) Find the DFT of the N_0-point signal $f[k] = u[k] - u[k - N_0]$. Find $F(\Omega)$, the DTFT of $f[k]$ and verify that the DFT values are the uniform samples of DTFT at frequency intervals of $\Omega_0 = 2\pi/N_0$.

(b) Is the DTFT found in part (a) an adequate frequency-domain description of $f[k]$. If not, what needs to be done to obtain a reasonably adequate DFT?

10.6-5 (a) Find the 5-point and 8-point DFT of the signal $f[k]$ illustrated in Fig. P8.2-9d.

(b) Find $F(\Omega)$, the DTFT of $f[k]$, and verify the DFT values from $F(\Omega)$.

Fig. P10.6-6

10.6-6 (a) Using the graphical method shown in Fig. 5.17, find the circular convolution of the sequences $f[k]$ and $g[k]$, depicted in Fig. P10.6-6.

(b) Using the sliding tape method (Fig. 9.4), find the linear convolution of the first cycles (over the range $0 \leq k \leq 3$) of the sequences $f[k]$ and $g[k]$. Is the result same as that found in part (a)?

(c) The circular convolution can be made equivalent to the linear convolution by suitably padding (the first cycle of) the sequences $f[k]$ and $g[k]$ with zeros. How many zeros do you need to pad to $f[k]$ and $h[k]$? After suitably padding these sequences, perform the circular convolution using the graphical method illustrated in Fig. 5.17, and show that it is equivalent to the linear convolution found in part (b).

(d) Find the circular convolution of $f[k]$ and $g[k]$ obtained in part (a) using DFT.

(e) Find the linear convolution of $f[k]$ and $g[k]$ obtained in part (b) using DFT.

Fig. P10.6-7

10.6-7 Find the output of a system with impulse response $h[k]$ and the input $f[k]$ shown in Fig. P10.6-7 by the following methods: (i)using linear convolution of $f[k]$ and $g[k]$ by sliding tape method (ii) using circular convolution of suitably padded $f[k]$ and $g[k]$ using the graphical method, depicted in Fig. 5.17 (iii) using DFT.

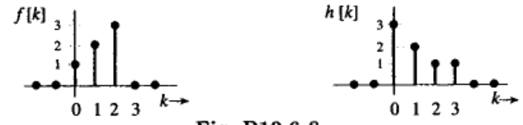


Fig. P10.6-8

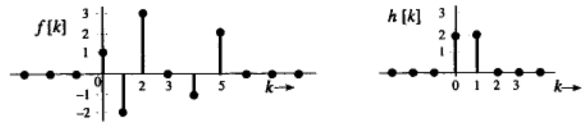
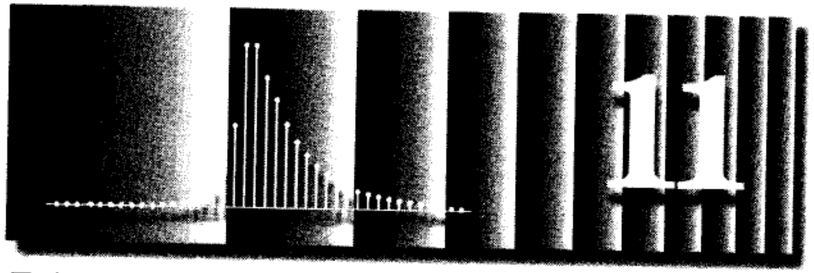


Fig. P10.6-9

10.6-8 (a) Repeat Prob. 10.6-7 for the signals $f[k]$ and $h[k]$ illustrated in Fig. P10.6-8.

10.6-9 Using both the methods of block filtering (overlap and add, and overlap and save), find the output of a filter with impulse response $h[k]$ and the input $f[k]$ shown in Fig. P10.6-9. Take $L = 3$. Verify that the linear convolution of the input sequence $f[k] = \{1,\ -2,\ 3,\ 0,\ -1,\ 2, \ldots\}$ with $h[k] = \{2,\ 2\}$ gives the same output as that found by the block filtering methods.

10.6-10 Find the 16-point IDFT of $F(\Omega)$ in Fig. 10.7a. Do the values of IDFT agree with the values of $f[k]$ found in Eq. 10.45. If not, why not?



Discrete-Time System Analysis Using the Z-Transform

The counterpart of the Laplace transform for discrete-time systems is the z-transform. The Laplace transform converts integro-differential equations into algebraic equations. In the same way, the z-transforms changes difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems. The z-transform method of analysis of discrete-time systems parallels the Laplace transform method of analysis of continuous-time systems, with some minor differences. In fact, we shall see that *the z-transform is the Laplace transform in disguise.*

The behavior of discrete-time systems (with some differences) is similar to that of continuous-time systems. The frequency-domain analysis of discrete-time systems is based on the fact (proved in Sec. 9.4-2) that the response of a linear time-invariant discrete-time (LTID) system to an everlasting exponential z^k is also the same exponential (within a multiplicative constant), given by $H[z]z^k$. We then express an input $f[k]$ as a sum of (everlasting) exponentials of the form z^k. The system response to $f[k]$ is then found as a sum of the system's responses to all these exponential components. The tool which allows us to represent an arbitrary input $f[k]$ as a sum of (everlasting) exponentials of the form z^k is the z-transform.

11.1 The Z-Transform

In the last Chapter, we extended the discrete-time Fourier transform to derive the pair of equations defining the z-transform as

$$F[z] \equiv \sum_{k=-\infty}^{\infty} f[k]z^{-k} \tag{11.1}$$

$$f[k] = \frac{1}{2\pi j}\oint F[z]z^{k-1}\,dz \tag{11.2}$$

where the symbol $\oint$ indicates an integration in counterclockwise direction around a closed path in the complex plane (see Fig. 11.1). As in the case of the Laplace transform, we need not worry about this integral at this point because inverse z-transforms of many signals of engineering interest can be found in a z-transform Table. The direct and inverse z-transforms can be expressed symbolically as

$$F[z] = \mathcal{Z}\{f[k]\} \qquad \text{and} \qquad f[k] = \mathcal{Z}^{-1}\{F[z]\}$$

or simply as

$$f[k] \Longleftrightarrow F[z]$$

Note that

$$\mathcal{Z}^{-1}[\mathcal{Z}\{f[k]\}] = f[k] \qquad \text{and} \qquad \mathcal{Z}\left[\mathcal{Z}^{-1}\{F[z]\}\right] = F[z]$$

Following the earlier argument, we can find an LTID system response to an input $f[k]$ using the steps as follows:

$$z^k \Longrightarrow H[z]z^k \qquad \text{the system response to } z^k \text{ is } H[z]z^k$$

$$f[k] = \frac{1}{2\pi j}\oint F[z]z^{k-1}\,dz \quad \text{shows } f[k] \text{ as a sum of everlasting exponential components}$$

and

$$\begin{aligned} y[k] &= \frac{1}{2\pi j}\oint F[z]H[z]z^{k-1}\,dz \quad \text{shows } y[k] \text{ as a sum of responses to exponential components} \\ &= \frac{1}{2\pi j}\oint Y[z]z^{k-1}\,dz \end{aligned}$$

where

$$Y[z] = F[z]H[z]$$

In conclusion, we have shown that for an LTID system with transfer function $H[z]$, if the input and the output are $f[k]$ and $y[k]$, respectively, and if

$$f[k] \Longleftrightarrow F[z] \qquad y[k] \Longleftrightarrow Y[z]$$

then

$$Y[z] = F[z]H[z]$$

We shall derive this result more formally later.

Linearity of the $\mathcal{Z}$-Transform

Like the Laplace transform, the z-transform is a linear operator. If

$$f_1[k] \Longleftrightarrow F_1[z] \quad \text{and} \quad f_2[k] \Longleftrightarrow F_2[z]$$

then

$$a_1f_1[k] + a_2f_2[k] \Longleftrightarrow a_1F_1[z] + a_2F_2[z] \tag{11.3}$$

The proof is trivial and follows from the definition of the z-transform. This result can be extended to finite sums.

The Unilateral $\mathcal{Z}$-Transform

For the same reasons discussed in Chapter 6, we first start with a simpler version of the z-transform, the **unilateral z-transform**, that is restricted only to the analysis of causal systems with causal inputs (signals starting at $k = 0$). The more general *bilateral z-transform* is discussed later in Sec. 11.7. In the unilateral case, the signals are restricted to be causal; that is, they start at $k = 0$. The definition of the unilateral transform is the same as that of the bilateral [Eq. (11.1)] except that the limits of the sum are from 0 to ∞

$$F[z] \equiv \sum_{k=0}^{\infty} f[k]z^{-k} \tag{11.4}$$

where z is complex in general. The expression for the inverse z-transform in Eq. (11.2) remains valid for the unilateral case also.

The Region of Convergence of F[z]

The sum in Eq. (11.1) [or (11.4)] defining the direct z-transform $F[z]$ may not converge (exist) for all values of z. The values of z (the region in the complex plane) for which the sum in Eq. (11.1) converges (or exists) is called the **region of convergence** (or **region of existence**) of $F[z]$. This concept will become clear in the following example.

■ **Example 11.1**

Find the z-transform and the corresponding region of convergence for the signal $\gamma^k u[k]$.

By definition

$$F[z] = \sum_{k=0}^{\infty} \gamma^k u[k] z^{-k}$$

Since $u[k] = 1$ for all $k \geq 0$,

$$F[z] = \sum_{k=0}^{\infty} \left(\frac{\gamma}{z}\right)^k$$

$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots + \cdots \tag{11.5}$$

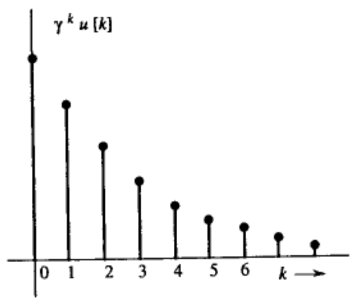
It is helpful to remember the following well-known geometric progression and its sum:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad \text{if} \quad |x| < 1 \tag{11.6}$$

Use of Eq. (11.6) in Eq. (11.5) yields

$$F[z] = \frac{1}{1 - \frac{\gamma}{z}} \qquad \left|\frac{\gamma}{z}\right| < 1$$

$$= \frac{z}{z - \gamma} \qquad |z| > |\gamma| \tag{11.7}$$



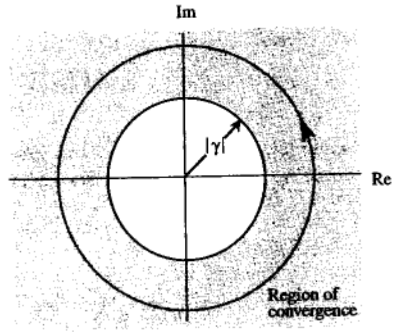


Fig. 11.1 $\gamma^k u[k]$ and the region of convergence of its z-transform.

Observe that $F[z]$ exists only for $|z| > |\gamma|$. For $|z| < |\gamma|$, the sum in Eq. (11.5) does not converge; it goes to infinity. Therefore, the region of convergence (or existence) of $F[z]$ is the shaded region outside the circle of radius $|\gamma|$, centered at the origin, in the z-plane, as depicted in Fig. 11.1b. ■

The region of convergence is required for evaluating $f[k]$ from $F[z]$, according to Eq. (11.2). The integral in Eq. (11.2) is a contour integral implying integration in a counterclockwise direction along a closed path centered at the origin and satisfying the condition $|z| > |\gamma|$. Thus, any circular path centered at the origin and with a radius greater than $|\gamma|$ (Fig. 11.1b) will suffice. We can show that the integral in Eq. (11.2) along any such path (with a radius greater than $|\gamma|$) yields the same result, namely $f[k]$. Such integration in the complex plane requires a background in the theory of functions of complex variables. We can avoid this integration by compiling a table of z-transforms (Table 11.1), where z-transform pairs are tabulated for a variety of signals. To find the inverse z-transform of say, $z/(z-\gamma)$, instead of using the complex integration in (11.2), we consult the table and find the inverse z-transform of $z/(z-\gamma)$ as $\gamma^k u[k]$. Although the table given here is rather short, it comprises the functions of most practical interest.

The bilateral z-transform is defined by Eq. (11.1) with the limits of the right-hand sum from $-\infty$ to ∞ instead of from 0 to ∞. The situation of the z-transform regarding the uniqueness of the inverse transform is parallel to that of the Laplace transform. For the bilateral case, the inverse z-transform is not unique unless the region of convergence is specified. For the unilateral case, the inverse transform is unique; the region of convergence need not be specified to determine the inverse z-transform. For this reason, we shall ignore the region of convergence in the unilateral z-transform Table 11.1.

Existence of the $\mathcal{Z}$-Transform

By definition

$$F[z] = \sum_{k=0}^{\infty} f[k]z^{-k} = \sum_{k=0}^{\infty} \frac{f[k]}{z^k}$$

The existence of the z-transform is guaranteed if

$$|F[z]| \le \sum_{k=0}^{\infty} \frac{|f[k]|}{|z|^k} < \infty$$

for some $|z|$. Any signal $f[k]$ that grows no faster than an exponential signal ${r_0}^k$, for some r_0, satisfies this condition. Thus, if

$$|f[k]| \le {r_0}^k \qquad \text{for some } r_0 \tag{11.8}$$

then

$$|F[z]| \le \sum_{k=0}^{\infty} \left(\frac{r_0}{|z|}\right)^k = \frac{1}{1-\frac{r_0}{|z|}} \qquad |z| > r_0$$

Therefore, $F[z]$ exists for $|z| > r_0$. All practical signals satisfy (11.8) and are therefore z-transformable. Some signal models (e.g. γ^{k^2}) which grow faster than the exponential signal ${r_0}^k$ (for any r_0) do not satisfy (11.8) and therefore are not z-transformable. Fortunately, such signals are of little practical or theoretical interest.

■ **Example 11.2**

Find the z-transforms of **(a)** $\delta[k]$ **(b)** $u[k]$ **(c)** $\cos \beta k\, u[k]$ **(d)** signal shown in Fig. 11.2.

Recall that by definition

$$F[z] = \sum_{k=0}^{\infty} f[k] z^{-k}$$

$$= f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \frac{f[3]}{z^3} + \cdots \tag{11.9}$$

(a) For $f[k] = \delta[k]$, $f[0] = 1$ and $f[2] = f[3] = f[4] = \cdots = 0$. Therefore

$$\delta[k] \Longleftrightarrow 1 \qquad \text{for all } z \tag{11.10}$$

(b) For $f[k] = u[k]$, $f[0] = f[1] = f[3] = \cdots = 1$. Therefore

$$F[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

From Eq. (11.6) it follows that

$$F[z] = \frac{1}{1-\frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1} \qquad |z| > 1$$

Therefore

$$u[k] \Longleftrightarrow \frac{z}{z-1} \qquad |z| > 1 \tag{11.11}$$

(c) Recall that $\cos \beta k = \left(e^{j\beta k} + e^{-j\beta k}\right)/2$. Moreover, according to Eq. (11.7),

$$e^{\pm j\beta k} u[k] \Longleftrightarrow \frac{z}{z-e^{\pm j\beta}} \qquad |z| > |e^{\pm j\beta}| = 1$$

Therefore

$$F[z] = \frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right]$$

$$= \frac{z(z-\cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1$$

(d) Here $f[0] = f[1] = f[2] = f[3] = f[4] = 1$ and $f[5] = f[6] = \cdots = 0$. Therefore, according to Eq. (11.9)

$$F[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$

$$= \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

We can also express this result in a closed form by summing the geometric progression on the right-hand side of the above equation, using the formula in Sec. B.7-4. Here the common ratio $r = \frac{1}{z}$, $M = 0$, and $N = 4$, so that

$$F[z] = \frac{(\frac{1}{z})^5 - (\frac{1}{z})^0}{\frac{1}{z} - 1} = \frac{z}{z-1}(1 - z^{-5}) \quad ■$$

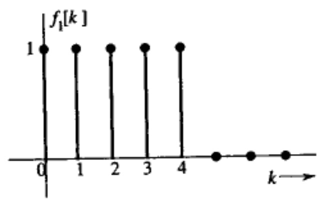


Fig. 11.2

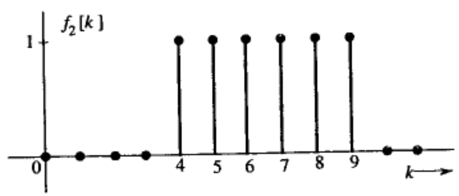


Fig. 11.3

△ **Exercise E11.1**

(a) Find the z-transform of a signal shown in Fig. 11.3. **(b)** Using Pair 12a (Table 11.1) find the z-transform of $f[k] = 20.65(\sqrt{2})^k \cos\left(\frac{\pi}{4}k - 1.415\right) u[k]$.

Answers: **(a)** $F[z] = \dfrac{z^5 + z^4 + +z^3 + z^2 + z + 1}{z^9}$ or $\dfrac{z}{z-1}(z^{-4} - z^{-10})$

(b) $\dfrac{z(3.2z + 17.2)}{z^2 - 2z + 2}$ ▽

11.1-1 Finding the Inverse Transform

As in the Laplace transform, we shall avoid the integration in the complex plane required to find the inverse z-transform [Eq. (11.2)] by using the (unilateral) transform Table. Many of the transforms $F[z]$ of practical interest are rational functions (ratio of polynomials in z). Such functions can be expressed as a sum of simpler functions using partial fraction expansion. This method works because for every transformable $f[k]$ defined for $k \ge 0$, there is a corresponding unique $F[z]$ defined for $|z| > r_0$ (where r_0 is some constant), and vice versa.

Table 11.1: (Unilateral) z-Transform Pairs

	$f[k]$	$F[z]$
1	$\delta[k-j]$	z^{-j}
2	$u[k]$	$\frac{z}{z-1}$
3	$ku[k]$	$\frac{z}{(z-1)^2}$
4	$k^2u[k]$	$\frac{z(z+1)}{(z-1)^3}$
5	$k^3u[k]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^{k-1}u[k-1]$	$\frac{1}{z-\gamma}$
7	$\gamma^k u[k]$	$\frac{z}{z-\gamma}$
8	$k\gamma^k u[k]$	$\frac{\gamma z}{(z-\gamma)^2}$
9	$k^2\gamma^k u[k]$	$\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\frac{k(k-1)(k-2)\cdots(k-m+1)}{\gamma^m m!}\gamma^k u[k]$	$\frac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^k \cos \beta k\, u[k]$	$\frac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^k \sin \beta k\, u[k]$	$\frac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^k\cos(\beta k+\theta)u[k]$	$\frac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^k\cos(\beta k+\theta)u[k]$ $\quad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\frac{(0.5re^{j\theta})z}{z-\gamma}+\frac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^k\cos(\beta k+\theta)u[k]$ $r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$ $\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert},\ \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$	$\frac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

■ **Example 11.3**

Find the inverse z-transform of

(a) $\frac{8z-19}{(z-2)(z-3)}$ **(b)** $\frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$ **(c)** $\frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

(a) Expanding $F[z]$ into partial fractions yields

$$F[z]=\frac{8z-19}{(z-2)(z-3)}=\frac{3}{z-2}+\frac{5}{z-3}$$

From Table 11.1, Pair 6, we obtain

$$f[k]=\left[3(2)^{k-1}+5(3)^{k-1}\right]u[k-1] \tag{11.12a}$$

If we expand rational $F[z]$ into partial fractions directly, we shall always obtain an answer that is multiplied by $u[k-1]$ because of the nature of Pair 6 in Table 11.1. This form is rather awkward as well as inconvenient. We prefer the form that is multiplied by $u[k]$ rather than $u[k-1]$. A glance at Table 11.1 shows that the z-transform of every signal that is multiplied by $u[k]$ has a factor z in the numerator. This observation suggests that we expand $F[z]$ into *modified partial fractions*, where each term has a factor z in the numerator. This goal can be accomplished by expanding $F[z]/z$ into partial fractions and then multiplying both sides by z. We shall demonstrate this procedure by reworking part **(a)** in Example 11.3. For this case

$$\frac{F[z]}{z}=\frac{8z-19}{z(z-2)(z-3)}$$
$$=\frac{(-19/6)}{z}+\frac{(3/2)}{z-2}+\frac{(5/3)}{z-3}$$

Multiplying both sides by z yields

$$F[z]=-\frac{19}{6}+\frac{3}{2}\left(\frac{z}{z-2}\right)+\frac{5}{3}\left(\frac{z}{z-3}\right)$$

From Pairs 1 and 7 in Table 11.1, it follows that

$$f[k]=-\tfrac{19}{6}\delta[k]+\left[\tfrac{3}{2}(2)^k+\tfrac{5}{3}(3)^k\right]u[k] \tag{11.12b}$$

The reader can verify that this answer is equivalent to that in Eq. (11.12a) by computing $f[k]$ in both cases for $k=0,1,2,3,\cdots$, and then comparing the results. The form in Eq. (11.12b) is more convenient than that in Eq. (11.12a). For this reason, we shall always expand $F[z]/z$ rather than $F[z]$ into partial fractions and then multiply both sides by z to obtain modified partial fractions of $F[z]$, which have a factor z in the numerator.

(b)
$$F[z]=\frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$$

and

$$\frac{F[z]}{z}=\frac{2z^2-11z+12}{(z-1)(z-2)^3}$$
$$=\frac{k}{z-1}+\frac{a_0}{(z-2)^3}+\frac{a_1}{(z-2)^2}+\frac{a_2}{(z-2)}$$

where

$$k = \left.\frac{2z^2 - 11z + 12}{\text{[redacted]}(z-2)^3}\right|_{z=1} = -3$$

$$a_0 = \left.\frac{2z^2 - 11z + 12}{(z-1)\text{[redacted]}}\right|_{z=2} = -2$$

Therefore

$$\frac{F[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)} \tag{11.13}$$

We can determine a_1 and a_2 by clearing fractions or by using the short cuts discussed in Sec. B.5-3. For example, to determine a_2, we multiply both sides of Eq. (11.13) by z and let $z \to \infty$. This yields

$$0 = -3 - 0 + 0 + a_2 \Longrightarrow a_2 = 3$$

This result leaves only one unknown, a_1, which is readily determined by letting z take any convenient value, say $z = 0$, on both sides of Eq. (11.13). This step yields

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2}$$

Multiplying both sides by 8 yields

$$12 = 24 + 2 + 2a_1 - 12 \Longrightarrow a_1 = -1$$

Therefore

$$\frac{F[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

and

$$F[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

Now the use of Table 11.1, Pairs 7 and 10, yields

$$\begin{aligned} f[k] &= \left[-3 - 2\frac{k(k-1)}{8}(2)^k - \frac{k}{2}(2)^k + 3(2)^k\right]u[k] \\ &= -[3 + \tfrac{1}{4}(k^2 + k - 12)2^k]u[k] \end{aligned}$$

(c) Complex Poles

$$F[z] = \frac{2z(3z+17)}{(z-1)(z^2 - 6z + 25)} = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

Poles of $F[z]$ are 1, $3 + j4$, and $3 - j4$. Whenever there are complex conjugate poles, the problem can be worked out in two ways. In the first method we expand $F[z]$ into (modified) first-order partial fractions. In the second method, rather than obtaining one factor corresponding to each complex conjugate pole, we obtain quadratic factors corresponding to each pair of complex conjugate poles. This procedure is explained below.

Method of First-Order Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

We find the partial fraction of $F[z]/z$ using the Heaviside "cover-up" method:

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

and

$$F[z] = 2\frac{z}{z-1} + (1.6e^{-j2.246})\frac{z}{z-3-j4} + (1.6e^{j2.246})\frac{z}{z-3+j4}$$

The inverse transform of the first term on the right-hand side is $2u[k]$. The inverse transform of the remaining two terms (complex conjugate poles) can be obtained from Pair 12b (Table 11.1) by identifying $\frac{r}{2} = 1.6$, $\theta = -2.246$ rad., $\gamma = 3 + j4 = 5e^{j0.927}$, so that $|\gamma| = 5$, $\beta = 0.927$. Therefore

$$f[k] = \left[2 + 3.2(5)^k \cos(0.927k - 2.246)\right] u[k]$$

Method of Quadratic Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

Multiplying both sides by z and letting $z \to \infty$, we find

$$0 = 2 + A \Longrightarrow A = -2$$

and

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B, we let z take any convenient value, say $z = 0$. This step yields

$$\frac{-34}{25} = -2 + \frac{B}{25}$$

Multiplying both sides by 25 yields

$$-34 = -50 + B \Longrightarrow B = 16$$

Therefore

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

and

$$F[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

We now use Pair 12c where we identify $A = -2$, $B = 16$, $|\gamma| = 5$, $a = -3$. Therefore

$$r = \sqrt{\frac{100+256-192}{25-9}} = 3.2, \quad \beta = \cos^{-1}(\tfrac{3}{5}) = 0.927\,\text{rad., and}$$

$$\theta = \tan^{-1}(\tfrac{-10}{-8}) = -2.246\,\text{rad., so that}$$

$$f[k] = \left[2 + 3.2(5)^k \cos(0.927k - 2.246)\right] u[k] \qquad \blacksquare$$

The procedure for finding partial fractions using MATLAB was demonstrated in chapter 6. The same program can be used in this case, except that we have to find the modified partial fractions here. This goal is readily accomplished by dividing $F[z]$ by z and then taking the partial fractions. We shall demonstrate this procedure with an example.

Computer Example C11.1
Solve Example 11.3a using MATLAB.

```
num=[8 -19]; den=[conv([1 -2],[1 -3]) 0];
[r, p, k]= residue(num,den)
% We could also express den=[1 -5 6 0]
r =
        1.6667
        1.5000
       -3.1667
p =
        3
        2
        0
k =
        []
```

Hence,

$$F[z] = -3.1667 + \frac{1.5z}{z-2} + \frac{1.6667z}{z-3}$$

Exercise E11.2
Find the inverse z-transform of the following functions:

(a) $\dfrac{z(2z-1)}{(z-1)(z+0.5)}$ (b) $\dfrac{1}{(z-1)(z+0.5)}$

(c) $\dfrac{9}{(z+2)(z-0.5)^2}$ (d) $\dfrac{5z(z-1)}{z^2-1.6z+0.8}$

Answer: (a) $\left[\frac{2}{3}+\frac{4}{3}(-0.5)^k\right]u[k]$ (b) $-2\delta[k]+\left[\frac{2}{3}+\frac{4}{3}(-0.5)^k\right]u[k]$

(c) $18\delta[k]-[0.72(-2)^k+17.28(0.5)^k-14.4k(0.5)^k]u[k]$

(d) $\frac{5\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\right)^k\cos(0.464k+0.464)u[k]$. Hint: $\sqrt{0.8}=\frac{2}{\sqrt{5}}$.

Inverse Transform by Expansion of F[z] in Power Series of z^{-1}

By definition

$$\begin{aligned} F[z] &= \sum_{k=0}^{\infty} f[k]z^{-k} \\ &= f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \frac{f[3]}{z^3} + \cdots \\ &= f[0]z^0 + f[1]z^{-1} + f[2]z^{-2} + f[3]z^{-3} + \cdots \end{aligned}$$

This result is a power series in z^{-1}. Therefore, if we can expand $F[z]$ into a power series in z^{-1}, the coefficients of this power series can be identified as $f[0]$, $f[1]$, $f[2]$,

$f[3]$, $\cdots$, and so on. A rational $F[z]$ can be expanded into a power series of z^{-1} by dividing its numerator by the denominator. Consider, for example,

$$\begin{aligned} F[z] &= \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} \\ &= \frac{7z^3-2z^2}{z^3-1.7z^2+0.8z-0.1} \end{aligned}$$

To obtain a series expansion in powers of z^{-1}, we divide the numerator by the denominator as follows:

$$\begin{array}{r|l} & \underline{7+9.9z^{-1}+11.23z^{-2}+11.87z^{-3}+\cdots} \\ z^3-1.7z^2+0.8z-0.1 & 7z^3-2z^2 \\ & \underline{7z^3-11.9z^2+5.60z-0.7} \\ & \qquad 9.9z^2-5.60z+0.7 \\ & \qquad \underline{9.9z^2-16.83z+7.92-0.99z^{-1}} \\ & \qquad\qquad 11.23z-7.22+0.99z^{-1} \\ & \qquad\qquad \underline{11.23z-19.09+8.98z^{-1}} \\ & \qquad\qquad\qquad 11.87-7.99z^{-1} \end{array}$$

Thus

$$F[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

Therefore

$$f[0]=7, \; f[1]=9.9, \; f[2]=11.23, \; f[3]=11.87, \; \cdots, \text{ and so on.}$$

We give here a simple MATLAB program to find the first N terms of the inverse z-transform.

Computer Example C11.2
Using MATLAB, find the first 10 values ($f[0]$ through $f[9]$) of the inverse z-transform of $F[z]$ in the above example.

```
num=[7 -2 0 0]; den=[1 -1.7 0.8 -0.1];
f=dimpulse(num, den, 10)
% We could also write den=conv(conv([1 -0.2],[1 -0.5]),[1 -1])
f =
      7.0000
      9.9000
      11.2300
      11.8710
      12.1867
      12.3436
      12.4218
      12.4609
      12.4805
      12.4902
```

Although this procedure yields $f[k]$ directly, it does not provide a closed-form solution. For this reason, it is not very useful unless we want to know only the first few terms of the sequence $f[k]$.

△ **Exercise E11.3**

Using long division to find the power series in z^{-1}, show that the inverse z-transform of $z/(z-0.5)$ is $(0.5)^k u[k]$ or $(2)^{-k}u[k]$. ▽

Relationship Between h[k] and H[z]

For an LTID system, if $h[k]$ is its unit impulse response, then in Eq. (9.57b) we defined $H[z]$, the system transfer function, as

$$H[z] = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

For causal systems, the limits on the sum are from $k = 0$ to ∞. This equation shows that the transfer function $H[z]$ is the z-transform of the impulse response $h[k]$ of an LTID system; that is

$$h[k] \Longleftrightarrow H[z] \tag{11.14}$$

This important result relates the impulse response $h[k]$, which is a time-domain specification of a system, to $H[z]$, which is a frequency-domain specification of a system. The result is parallel to that for LTIC systems.

△ **Exercise E11.4**

Redo Exercise E9.5 by taking the inverse z-transform of $H[z]$. ▽

11.2 Some properties of the Z-Transform

The z-transform properties are useful in the derivation of z-transforms of many functions and also in the solution of linear difference equations with constant coefficients. Here we consider a few important properties of the z-transform.

Right Shift (Delay)

If

$$f[k]u[k] \Longleftrightarrow F[z]$$

then

$$f[k-1]u[k-1] \Longleftrightarrow \frac{1}{z}F[z] \tag{11.15a}$$

and

$$f[k-m]u[k-m] \Longleftrightarrow \frac{1}{z^m}F[z] \tag{11.15b}$$

and

$$f[k-1]u[k] \Longleftrightarrow \frac{1}{z}F[z] + f[-1] \tag{11.16a}$$

Repeated application of this property yields

$$\begin{aligned} f[k-2]u[k] &\Longleftrightarrow \frac{1}{z}\left[\frac{1}{z}F[z] + f[-1]\right] + f[-2] \\ &= \frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2] \end{aligned} \tag{11.16b}$$

and

$$f[k-m]u[k] \Longleftrightarrow z^{-m}F[z] + z^{-m}\sum_{k=1}^{m} f[-k]z^k \tag{11.16c}$$

Proof:

$$\mathcal{Z}\{f[k-m]u[k-m]\} = \sum_{k=0}^{\infty} f[k-m]u[k-m]z^{-k}$$

Recall that $f[k-m]u[k-m] = 0$ for $k < m$, so that the limits on the summation on the right-hand side can be taken from $k = m$ to ∞. Therefore

$$\begin{aligned} \mathcal{Z}\{f[k-m]u[k-m]\} &= \sum_{k=m}^{\infty} f[k-m]z^{-k} \\ &= \sum_{r=0}^{\infty} f[r]z^{-(r+m)} \\ &= \frac{1}{z^m}\sum_{r=0}^{\infty} f[r]z^{-r} \\ &= \frac{1}{z^m}F[z] \end{aligned}$$

To prove Eq. (11.16c), we have

$$\begin{aligned} \mathcal{Z}\{f[k-m]u[k]\} &= \sum_{k=0}^{\infty} f[k-m]z^{-k} = \sum_{r=-m}^{\infty} f[r]z^{-(r+m)} \\ &= z^{-m}\left[\sum_{r=-m}^{-1} f[r]z^{-r} + \sum_{r=0}^{\infty} f[r]z^{-r}\right] \\ &= z^{-m}\sum_{k=1}^{m} f[-k]z^k + z^{-m}F[z] \end{aligned}$$

Left Shift (Advance)

If

$$f[k]u[k] \Longleftrightarrow F[z]$$

then

$$f[k+1]u[k] \Longleftrightarrow zF[z] - zf[0] \tag{11.17a}$$

Repeated application of this property yields

$$\begin{aligned} f[k+2]u[k] &\Longleftrightarrow z\{z(F[z]-zf[0])-f[1]\} \\ &= z^2F[z]-z^2f[0]-zf[1] \end{aligned} \tag{11.17b}$$

and

$$f[k+m]u[k] \Longleftrightarrow z^mF[z]-z^m\sum_{k=0}^{m-1}f[k]z^{-k} \tag{11.17c}$$

Proof: By definition

$$\begin{aligned} \mathcal{Z}\{f[k+m]u[k]\} &= \sum_{k=0}^{\infty}f[k+m]z^{-k} \\ &= \sum_{r=m}^{\infty}f[r]z^{-(r-m)} \\ &= z^m\sum_{r=m}^{\infty}f[r]z^{-r} \\ &= z^m\left[\sum_{r=0}^{\infty}f[r]z^{-r}-\sum_{r=0}^{m-1}f[r]z^{-r}\right] \\ &= z^mF[z]-z^m\sum_{r=0}^{m-1}f[r]z^{-r} \end{aligned}$$

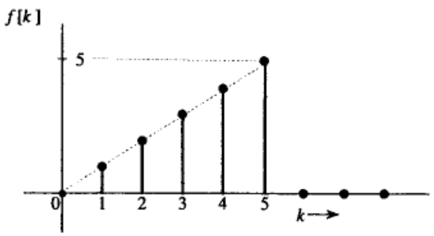


Fig. 11.4 Signal for Example 11.4.

■ **Example 11.4**

Find the z-transform of the signal $f[k]$ depicted in Fig. 11.4.

The signal $f[k]$ can be expressed as a product of k and a gate pulse $u[k]-u[k-6]$. Therefore

$$\begin{aligned} f[k] &= k\{u[k]-u[k-6]\} \\ &= ku[k]-ku[k-6] \end{aligned}$$

We cannot find the z-transform of $ku[k-6]$ directly by using the right-shift property [Eq. (11.15b)]. So we rearrange it in terms of $(k-6)u[k-6]$ as follows:

$$f[k] = ku[k]-[(k-6)u[k-6]+6u[k-6]]$$

We can now find the z-transform of the bracketed term by using the right-shift property [Eq. (11.15b)]. Because $u[k] \Longleftrightarrow \frac{z}{z-1}$

$$u[k-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{z-1} = \frac{1}{z^5(z-1)}$$

Also, because $ku[k] \Longleftrightarrow \frac{z}{(z-1)^2}$

$$(k-6)u[k-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}$$

Therefore

$$\begin{aligned} F[z] &= \frac{z}{(z-1)^2}-\frac{1}{z^5(z-1)^2}-\frac{6}{z^5(z-1)} \\ &= \frac{z^6-6z+5}{z^5(z-1)^2} \quad ■ \end{aligned}$$

△ **Exercise E11.5**

Using only the fact that $u[k] \Longleftrightarrow \frac{z}{z-1}$ and the right-shift property [Eq. (11.15)], find the z-transforms of the signals in Figs. 11.2 and 11.3. The answers are given in Example 11.2d and Exercise E11.1a. ▽

Convolution

The time convolution property and the frequency convolution property state that if

$$f_1[k] \Longleftrightarrow F_1[z] \quad \text{and} \quad f_2[k] \Longleftrightarrow F_2[z],$$

then (**time convolution**)

$$f_1[k]*f_2[k] \Longleftrightarrow F_1[z]F_2[z] \tag{11.18}$$

and (**frequency convolution**)

$$f_1[k]f_2[k] \Longleftrightarrow \frac{1}{2\pi j}\oint F_1[u]F_2\left[\frac{z}{u}\right]u^{-1}\,du \tag{11.19}$$

Proof: These properties apply to causal as well as noncausal sequences. For this reason, we shall prove them for the more general case of noncausal sequences, where the convolution sum ranges from $-\infty$ to ∞. To prove the time convolution, we have

$$\begin{aligned} \mathcal{Z}\{f_1[k]*f_2[k]\} &= \mathcal{Z}\left[\sum_{m=-\infty}^{\infty}f_1[m]f_2[k-m]\right] \\ &= \sum_{k=-\infty}^{\infty}z^{-k}\sum_{m=-\infty}^{\infty}f_1[m]f_2[k-m] \end{aligned}$$

Interchanging the order of summation,

$$\begin{aligned}\mathcal{Z}[f_1[k]*f_2[k]] &= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{k=-\infty}^{\infty} f_2[k-m]z^{-k} \\ &= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{r=-\infty}^{\infty} f_2[r]z^{-(r+m)} \\ &= \sum_{m=-\infty}^{\infty} f_1[m]z^{-m} \sum_{r=-\infty}^{\infty} f_2[r]z^{-r} \\ &= F_1[z]F_2[z]\end{aligned}$$

To prove the frequency convolution, we start with

$$\begin{aligned}\mathcal{Z}\{f_1[k]f_2[k]\} &= \sum_{k=-\infty}^{\infty} f_1[k]f_2[k]z^{-k} \\ &= \frac{1}{2\pi j}\sum_{k=-\infty}^{\infty} f_2[k]z^{-k} \oint F_1[u]u^{k-1}\,du\end{aligned}$$

Interchanging the order of summation and integration

$$\begin{aligned}\mathcal{Z}[f_1[k]f_2[k]] &= \frac{1}{2\pi j}\oint F_1[u]\left[\sum_{k=-\infty}^{\infty} f_2[k]\left(\frac{z}{u}\right)^{-k}\right]u^{-1}\,du \\ &= \frac{1}{2\pi j}\oint F_1[u]F_2\left[\frac{z}{u}\right]u^{-1}\,du\end{aligned}$$

LTID System Response

It is interesting to apply the time convolution property to the LTID input-output equation $y[k] = f[k]*h[k]$. In Eq. (11.14), we have shown that $h[k] \Longleftrightarrow H[z]$. Hence, according to Eq. (11.18), it follows that

$$Y[z] = F[z]H[z] \tag{11.20}$$

Earlier in the chapter, we derived this important result using informal arguments.

Multiplication by γ^k

If

$$f[k]u[k] \Longleftrightarrow F[z]$$

then

$$\gamma^k f[k]u[k] \Longleftrightarrow F\left[\frac{z}{\gamma}\right] \tag{11.21}$$

Proof:

$$\mathcal{Z}\{\gamma^k f[k]u[k]\} = \sum_{k=0}^{\infty}\gamma^k f[k]z^{-k} = \sum_{k=0}^{\infty} f[k]\left(\frac{z}{\gamma}\right)^{-k} = F\left[\frac{z}{\gamma}\right]$$

△ **Exercise E11.6**

Using Eq. (11.21), derive Pairs 7 and 8 in Table 11.1 from Pairs 2 and 3, respectively. ▽

Multiplication by k (Scaling in the z-Domain

If

$$f[k]u[k] \Longleftrightarrow F[z]$$

then

$$kf[k]u[k] \Longleftrightarrow -z\frac{d}{dz}F[z] \tag{11.22}$$

Proof:

$$\begin{aligned}-z\frac{d}{dz}F[z] &= -z\frac{d}{dz}\sum_{k=0}^{\infty} f[k]z^{-k} = -z\sum_{k=0}^{\infty} -kf[k]z^{-k-1} \\ &= \sum_{k=0}^{\infty} kf[k]z^{-k} = \mathcal{Z}\{kf[k]u[k]\}\end{aligned}$$

△ **Exercise E11.7**

Using Eq. (11.22), derive Pairs 3 and 4 in Table 11.1 from Pair 2. Similarly, derive Pairs 8 and 9 from Pair 7. ▽

Initial and Final Value

For a causal $f[k]$,

$$f[0] = \lim_{z\to\infty} F[z] \tag{11.23a}$$

This result follows immediately from Eq. (11.9)

We can also show that if $(z-1)F(z)$ has no poles outside the unit circle, then

$$\lim_{N\to\infty} f(N) = \lim_{z\to 1}(z-1)F(z) \tag{11.23b}$$

11.3 Z-Transform Solution of Linear Difference Equations

The time-shifting (left- or right-shift) property has set the stage for solving linear difference equations with constant coefficients. As in the case of the Laplace transform with differential equations, the z-transform converts difference equations into algebraic equations which are readily solved to find the solution in the z-domain. Taking the inverse z-transform of the z-domain solution yields the desired time-domain solution. The following examples demonstrate the procedure.

■ **Example 11.5**

Solve

$$y[k+2] - 5y[k+1] + 6y[k] = 3f[k+1] + 5f[k] \tag{11.24}$$

if the initial conditions are $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$, and the input $f[k] = (2)^{-k}u[k]$.

Table 11.2

$\mathcal{Z}$- Transform Operations

Operation	$f[k]$	$F[z]$
Addition	$f_1[k]+f_2[k]$	$F_1[z]+F_2[z]$
Scalar multiplication	$af[k]$	$aF[z]$
Right-shift	$f[k-m]u[k-m]$	$\frac{1}{z^m}F[z]$
	$f[k-m]u[k]$	$\frac{1}{z^m}F[z]+\frac{1}{z^m}\sum_{k=1}^{m}f[-k]z^k$
	$f[k-1]u[k]$	$\frac{1}{z}F[z]+f[-1]$
	$f[k-2]u[k]$	$\frac{1}{z^2}F[z]+\frac{1}{z}f[-1]+f[-2]$
	$f[k-3]u[k]$	$\frac{1}{z^3}F[z]+\frac{1}{z^2}f[-1]+\frac{1}{z}f[-2]+f[-3]$
Left-shift	$f[k+m]u[k]$	$z^mF[z]-z^m\sum_{k=0}^{m-1}f[k]z^{-k}$
	$f[k+1]u[k]$	$zF[z]-zf[0]$
	$f[k+2]u[k]$	$z^2F[z]-z^2f[0]-zf[1]$
	$f[k+3]u[k]$	$z^3F[z]-z^3f[0]-z^2f[1]-zf[2]$
Multiplication by γ^k	$\gamma^k f[k]u[k]$	$F\left[\frac{z}{\gamma}\right]$
Multiplication by k	$kf[k]u[k]$	$-z\frac{d}{dz}F[z]$
Time Convolution	$f_1[k]*f_2[k]$	$F_1[z]F_2[z]$
Frequency Convolution	$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1[u]F_2\left[\frac{z}{u}\right]u^{-1}\,du$
Initial value	$f[0]$	$\lim_{z\to\infty}F[z]$
Final value	$\lim_{N\to\infty}f[N]$	$\lim_{z\to1}(z-1)F[z]$ poles of $(z-1)F[z]$ inside the unit circle.

As we shall see, difference equations can be solved by using the right-shift or the left-shift property. Because the difference equation (11.24) is in advance-operator form, the use of the left-shift property in Eqs. (11.17a) and (11.17b) may seem appropriate for its solution. Unfortunately, as seen from Eqs. (11.17a) and (11.17b), these properties require a knowledge of auxiliary conditions $y[0], y[1], \cdots, y[n-1]$ rather than of the initial conditions $y[-1], y[-2], \cdots, y[-n]$, which are generally given. This difficulty can be overcome by expressing the difference equation (11.24) in delay operator form (obtained by replacing k with $k-2$) and then using the right-shift property.† Equation (11.24) in delay operator form is

$$y[k]-5y[k-1]+6y[k-2]=3f[k-1]+5f[k-2] \tag{11.25}$$

We now use the right-shift property to take the z-transform of this equation. But before proceeding, we must be clear about the meaning of a term like $y[k-1]$. Does it mean $y[k-1]u[k-1]$ or $y[k-1]u[k]$? The answer becomes clear when we recognize that the use of the unilateral transform implies that we are considering the situation for $k \geq 0$, and that every signal in Eq. (11.25) must be counted from $k=0$. Therefore, the term $y[k-j]$ means $y[k-j]u[k]$. Remember also that although we are considering the situation for $k \geq 0$, $y[k]$ is present even before $k=0$ (in the form of initial conditions). Now

$$y[k]u[k] \Longleftrightarrow Y[z]$$

$$y[k-1]u[k] \Longleftrightarrow \frac{1}{z}Y[z]+y[-1]=\frac{1}{z}Y[z]+\frac{11}{6}$$

$$y[k-2]u[k] \Longleftrightarrow \frac{1}{z^2}Y[z]+\frac{1}{z}y[-1]+y[-2]=\frac{1}{z^2}Y[z]+\frac{11}{6z}+\frac{37}{36}$$

Also

$$f[k]=(2)^{-k}u[k]=(2^{-1})^ku[k]=(0.5)^ku[k] \Longleftrightarrow \frac{z}{z-0.5}$$

$$f[k-1]u[k] \Longleftrightarrow \frac{1}{z}F[z]+f[-1]=\frac{1}{z}\frac{z}{z-0.5}+0=\frac{1}{z-0.5}$$

$$f[k-2]u[k] \Longleftrightarrow \frac{1}{z^2}F[z]+\frac{1}{z}f[-1]+f[-2]=\frac{1}{z^2}F[z]+0+0=\frac{1}{z(z-0.5)}$$

Note that for causal input $f[k]$,

$$f[-1]=f[-2]=\cdots=f[-n]=0$$

Hence

$$f[k-r]u[k] \Longleftrightarrow \frac{1}{z^r}F[z]$$

Taking the z-transform of Eq. (11.25) and substituting the above results, we obtain

$$Y[z]-5\left[\frac{1}{z}Y[z]+\frac{11}{6}\right]+6\left[\frac{1}{z^2}Y[z]+\frac{11}{6z}+\frac{37}{36}\right]=\frac{3}{z-0.5}+\frac{5}{z(z-0.5)} \tag{11.26a}$$

or

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y[z]-\left(3-\frac{11}{z}\right)=\frac{3}{z-0.5}+\frac{5}{z(z-0.5)} \tag{11.26b}$$

†Another approach is to find $y[0], y[1], y[2], \cdots, y[n-1]$ from $y[-1], y[-2], \cdots, y[-n]$ iteratively, as in Sec. 9.1-1, and then apply the left-shift property to Eq. (11.24)

and

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y[z]=\left(3-\frac{11}{z}\right)+\frac{3z+5}{z(z-0.5)}$$
$$=\frac{3z^2-9.5z+10.5}{z(z-0.5)}$$

Multiplication of both sides by z^2 yields

$$\left(z^2-5z+6\right)Y[z]=\frac{z\left(3z^2-9.5z+10.5\right)}{(z-0.5)}$$

so that

$$Y[z]=\frac{z(3z^2-9.5z+10.5)}{(z-0.5)(z^2-5z+6)} \tag{11.27}$$

and

$$\frac{Y[z]}{z}=\frac{3z^2-9.5z+10.5}{(z-0.5)(z-2)(z-3)}$$
$$=\frac{(26/15)}{z-0.5}-\frac{(7/3)}{z-2}+\frac{(18/5)}{z-3}$$

Therefore

$$Y[z]=\frac{26}{15}\left(\frac{z}{z-0.5}\right)-\frac{7}{3}\left(\frac{z}{z-2}\right)+\frac{18}{5}\left(\frac{z}{z-3}\right)$$

and

$$y[k]=\left[\frac{26}{15}(0.5)^k-\frac{7}{3}(2)^k+\frac{18}{5}(3)^k\right]u[k] \tag{11.28}$$

■

This example demonstrates the ease with which linear difference equations with constant coefficients can be solved by z-transform. This method is general; it can be used to solve a single difference equation or a set of simultaneous difference equations of any order as long as the equations are linear with constant coefficients.

Comment

Sometimes auxiliary conditions $y[0], y[1], \cdots, y[n-1]$ (instead of initial conditions $y[-1], y[-2], \cdots, y[-n]$) are given to solve a difference equation. In this case, the equation can be solved by expressing it in the advance operator form and then using the left-shift property (see Exercise E11.9 below).

△ **Exercise E11.8**
Solve the equation below if the initial conditions are $y[-1]=2$, $y[-2]=0$, and the input $f[k]=u[k]$:

$$y[k+2]-\tfrac{5}{6}y[k+1]+\tfrac{1}{6}y[k]=5f[k+1]-f[k]$$

Answer: $y[k]=\left[12-15(\tfrac{1}{2})^k+\tfrac{14}{3}(\tfrac{1}{3})^k\right]u[k]$ ▽

△ **Exercise E11.9**
Solve the following equation if the auxiliary conditions are $y[0]=1$, $y[1]=2$, and the input $f[k]=u[k]$:

$$y[k+2]+3y[k+1]+2y[k]=f[k+1]+3f[k]$$

Answer: $y[k]=\left[\tfrac{2}{3}+2(-1)^k-\tfrac{5}{3}(-2)^k\right]u[k]$ ▽

Zero-Input and Zero-State Components

In Example 11.5 we found the total solution of the difference equation. It is relatively easy to separate the solution into zero-input and zero-state components. All we have to do is to separate the response into terms arising from the input and terms arising from initial conditions. We can separate the response in Eq. (11.26b) as follows:

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y[z]-\underbrace{\left(3-\frac{11}{z}\right)}_{\text{initial condition terms}}=\underbrace{\frac{3}{z-0.5}+\frac{5}{z(z-0.5)}}_{\text{terms arising from input}} \tag{11.29}$$

Therefore

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y[z]=\underbrace{\left(3-\frac{11}{z}\right)}_{\text{initial condition terms}}+\underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

Multiplying both sides by z^2 yields

$$\left(z^2-5z+6\right)Y[z]=\underbrace{z(3z-11)}_{\text{initial condition terms}}+\underbrace{\frac{z(3z+5)}{z-0.5}}_{\text{input terms}}$$

and

$$Y[z]=\underbrace{\frac{z(3z-11)}{z^2-5z+6}}_{\text{zero-input response}}+\underbrace{\frac{z(3z+5)}{(z-0.5)(z^2-5z+6)}}_{\text{zero-state response}} \tag{11.30}$$

We expand both terms on the right-hand side into modified partial fractions to yield

$$Y[z]=\underbrace{\left[5\left(\frac{z}{z-2}\right)-2\left(\frac{z}{z-3}\right)\right]}_{\text{zero-input}}+\underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right)-\frac{22}{3}\left(\frac{z}{z-2}\right)+\frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero-state}}$$

and

$$y[k]=\left[\underbrace{5(2)^k-2(3)^k}_{\text{zero-input}}\underbrace{-\tfrac{22}{3}(2)^k+\tfrac{28}{5}(3)^k+\tfrac{26}{15}(0.5)^k}_{\text{zero-state}}\right]u[k]$$
$$=\left[-\tfrac{7}{3}(2)^k+\tfrac{18}{5}(3)^k+\tfrac{26}{15}(0.5)^k\right]u[k]$$

a conclusion, which agrees with the result in Eq. (11.28).

△ **Exercise E11.10**
Solve

$$y[k+2]-\tfrac{5}{6}y[k+1]+\tfrac{1}{6}y[k]=5f[k+1]-f[k]$$

if the initial conditions are $y[-1]=2$, $y[-2]=0$, and the input $f[k]=u[k]$. Separate the response into zero-input and zero-state components.

Answer:

$$y[k] = \left\{ \underbrace{[3(\tfrac{1}{2})^k - \tfrac{4}{3}(\tfrac{1}{3})^k]}_{\text{zero-input}} + \underbrace{[12 - 18(\tfrac{1}{2})^k + 6(\tfrac{1}{3})^k]}_{\text{zero-state}} \right\} u[k]$$
$$= \left[12 - 15(\tfrac{1}{2})^k + \tfrac{14}{3}(\tfrac{1}{3})^k\right] u[k] \quad \triangledown$$

11.3-1 Zero-State Response of LTID Systems: The Transfer Function

Consider an nth-order LTID system specified by the difference equation

$$Q[E]y[k] = P[E]f[k] \tag{11.31a}$$

or

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0)y[k] = (b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0)f[k] \tag{11.31b}$$

or

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1y[k+1] + a_0y[k] = b_nf[k+n] + \cdots + b_1f[k+1] + b_0f[k] \tag{11.31c}$$

We now derive the general expression for the zero-state response; that is, the system response to input $f[k]$ when all the initial conditions $y[-1] = y[-2] = \cdots = y[-n] = 0$ (zero state). The input $f[k]$ is assumed to be causal so that $f[-1] = f[-2] = \cdots = f[-n] = 0$.

Equation (11.31c) can be expressed in the delay operator form as

$$y[k] + a_{n-1}y[k-1] + \cdots + a_0y[k-n] = b_nf[k] + b_{n-1}f[k-1] + \cdots + b_0f[k-n] \tag{11.31d}$$

Because $y[-r] = f[-r] = 0$ for $r = 1, 2, \ldots, n$

$$y[k-m]u[k] \Longleftrightarrow \frac{1}{z^m}Y[z]$$
$$f[k-m]u[k] \Longleftrightarrow \frac{1}{z^m}F[z] \qquad m = 1, 2, \ldots, n$$

Now the z-transform of Eq. (11.31d) is given by

$$\left(1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right)Y[z] = \left(b_n + \frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \cdots + \frac{b_0}{z^n}\right)F[z]$$

Multiplication of both sides by z^n yields

$$(z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0)Y[z] = (b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0)F[z]$$

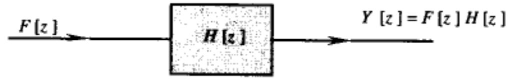


Fig. 11.5 The transformed representation of an LTID system.

Therefore

$$Y[z] = \left(\frac{b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0}\right)F[z] \tag{11.32}$$
$$= \frac{P[z]}{Q[z]}F[z] \tag{11.33}$$

We have shown in Eq. (11.20) that $Y[z] = F[z]H[z]$. Hence, it follows that

$$H[z] = \frac{P[z]}{Q[z]} = \frac{b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0} \tag{11.34}$$

As in the case of LTIC systems, this result leads to an alternative definition of the LTID system transfer function as the ratio of $Y[z]$ to $F[z]$ (assuming all initial conditions zero).

$$H[z] \equiv \frac{Y[z]}{F[z]} = \frac{\mathcal{Z}[\text{zero-state response}]}{\mathcal{Z}[\text{input}]} \tag{11.35}$$

Because $Y[z]$, the z-transform of the zero-state response $y[k]$, is the product of $F[z]$ and $H[z]$, we can represent an LTID system in the frequency domain by a block diagram, as illustrated in Fig. 11.5. Just as in continuous-time systems, we can represent discrete-time systems in the transformed manner by representing all signals by their z-transforms and all system components (or elements) by their transfer functions.

Observe that the denominator of $H[z]$ is $Q[z]$, the characteristic polynomial of the system. *Therefore the poles of $H[z]$ are the characteristic roots of the system.* Consequently, the system stability criterion can be stated in terms of the poles of the transfer function of an LTID system as follows:

1. An LTID system is asymptotically stable if and only if all the poles of its transfer function $H[z]$ lie inside a unit circle (centered at the origin) in the complex plane. The poles may be repeated or unrepeated.

2. An LTID system is unstable if and only if either one or both of the following conditions exist: (i) at least one pole of $H[z]$ is outside the unit circle; (ii) there are repeated poles of $H[z]$ on the unit circle.

3. An LTID system is marginally stable if and only if there are no poles of $H[z]$ outside the unit circle, and there are some unrepeated poles on the unit circle.

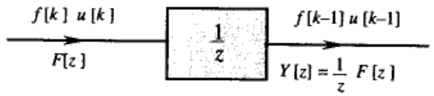


Fig. 11.6 Ideal unit delay and its transfer function.

■ **Example 11.6: The Transfer Function of a Unit Delay**

Show that the transfer function of a unit delay is $1/z$.

If the input to the unit delay is $f[k]u[k]$, then its output (Fig. 11.6) is given by

$$y[k] = f[k-1]u[k-1]$$

The z-transform of this equation yields [see Eq. (11.15a)]

$$\begin{aligned} Y[z] &= \frac{1}{z}F[z] \\ &= H[z]F[z] \end{aligned}$$

It follows that the transfer function of the unit delay is

$$H[z] = \frac{1}{z} \tag{11.36}$$

■

■ **Example 11.7**

Find the response $y[k]$ of an LTID system described by the difference equation

$$y[k+2] + y[k+1] + 0.16y[k] = f[k+1] + 0.32f[k]$$

or

$$(E^2 + E + 0.16)y[k] = (E + 0.32)f[k]$$

for the input $f[k] = (-2)^{-k}u[k]$ and with all the initial conditions zero (system in zero state initially).

From the difference equation we find

$$H[z] = \frac{P[z]}{Q[z]} = \frac{z + 0.32}{z^2 + z + 0.16}$$

For the input $f[k] = (-2)^{-k}u[k] = [(-2)^{-1}]^k u(k) = (-0.5)^k u[k]$

$$F[z] = \frac{z}{z + 0.5}$$

and

$$Y[z] = F[z]H[z] = \frac{z(z + 0.32)}{(z^2 + z + 0.16)(z + 0.5)}$$

Therefore

$$\begin{aligned} \frac{Y[z]}{z} &= \frac{(z + 0.32)}{(z^2 + z + 0.16)(z + 0.5)} = \frac{(z + 0.32)}{(z + 0.2)(z + 0.8)(z + 0.5)} \\ &= \frac{2/3}{z + 0.2} - \frac{8/3}{z + 0.8} + \frac{2}{z + 0.5} \end{aligned} \tag{11.37}$$

so that

$$Y[z] = \frac{2}{3}\left(\frac{z}{z + 0.2}\right) - \frac{8}{3}\left(\frac{z}{z + 0.8}\right) + 2\left(\frac{z}{z + 0.5}\right) \tag{11.38}$$

and

$$y[k] = \left[\tfrac{2}{3}(-0.2)^k - \tfrac{8}{3}(-0.8)^k + 2(-0.5)^k\right] u[k]$$ ■

⊙ **Computer Example C11.3**

Solve Example 11.7 using MATLAB. Plot $y[k]$ for $0 \le k \le 10$.

```
k=0:10;
b=[0 1 0.32];
a=[1 1 0.16];
f=(-2).^(-k);
y=filter(b,a,f);
stem(k,y)
xlabel('k');ylabel('y[k]')
```
⊙

△ **Exercise E11.11**

A discrete-time system is described by the following transfer function:

$$H[z] = \frac{z - 0.5}{(z + 0.5)(z - 1)}$$

(a) Find the system response to input $f[k] = 3^{-(k+1)}u[k]$ if all initial conditions are zero. **(b)** Write the difference equation relating the output $y[k]$ to input $f[k]$ for this system.

Answers: **(a)** $y[k] = \frac{1}{3}\left[\frac{1}{2} - 0.8(-0.5)^k + 0.3\left(\frac{1}{3}\right)^k\right] u[k]$

(b) $y[k+2] - 0.5y[k+1] - 0.5y[k] = f[k+1] - 0.5f[k]$ ▽

11.4 System Realization

We now discuss ways to realize an nth-order discrete-time system described by a transfer function

$$H[z] = \frac{b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \tag{11.39}$$

This transfer function is identical to the general nth-order continuous-time transfer function $H(s)$ in Eq. (6.70) with s replaced by z. It is reasonable to believe that the realization of $H[z]$ in (11.39) would be identical to that of $H(s)$ with s replaced by z. Fortunately this happens to be the case. In realizations of $H(s)$ the basic element used was an integrator with transfer function $1/s$. In realizations of $H[z]$ the basic element is unit delay with transfer function $1/z$. Therefore, all the realizations of $H(s)$ studied in Sec. 6.6 are also the realizations of $H[z]$ if we replace integrators by unit delays. To demonstrate this point, consider a realization of a third-order transfer function.

$$H[z] = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0} \tag{11.40}$$

Figure 11.7 shows Fig. 6.21 with all the integrators (with transfer function $1/s$) replaced with unit delays (with transfer function $1/z$). We shall now show that this

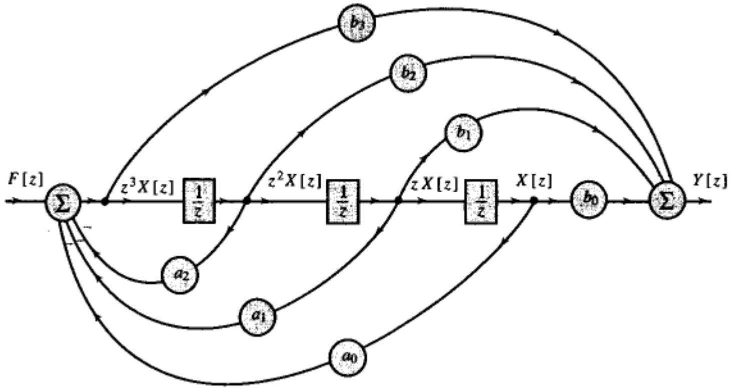


Fig. 11.7 A canonical realization of $H[z]$.

realization indeed represents $H[z]$ in Eq. (11.40). Let the signal at the output of the third delay be $X[z]$. Consequently, signals at the inputs of the second and the first delay are $zX[z]$ and $z^2X[z]$. The first summer output $z^3X[z]$ is equal to the sum of the four inputs to that summer. Therefore

$$z^3X[z] = -a_2z^2X[z] - a_1zX[z] - a_0X[z] + F[z]$$

so that

$$(z^3 + a_2z^2 + a_1z + a_0)X[z] = F[z] \tag{11.41}$$

Moreover, $Y[z]$, the output of the second summer, is equal to the sum of four signals to that summer. Therefore

$$Y[z] = (b_3z^3 + b_2z^2 + b_1z + b_0)X[z] \tag{11.42}$$

From Eqs. (11.41) and (11.42), it follows that

$$\frac{Y[z]}{F[z]} = \frac{b_3z^3 + b_2z^2 + b_1z + b_0}{z^3 + a_2z^2 + a_1z + a_0}$$

This result shows that Fig. 11.7 is indeed a realization of $H[z]$ in Eq. (11.40). Similarly, the cascade and parallel realizations of the continuous-time case are directly applicable to discrete-time systems, with integrators replaced by unit delays. The second canonical realization developed in Appendix 6.1 also applies to discrete-time case with $1/s$ replaced by $1/z$.

■ Example 11.8

Realize the following transfer functions, using only the cascade form for part **a** and using only the parallel form for part **b**.

(a) $H[z] = \dfrac{4z+28}{z^2+6z+5}$ **(b)** $H[z] = \dfrac{7z^2+37z+51}{(z+2)(z+3)^2}$

Identical transfer functions for continuous-time systems are realized in Figs. 6.27 and 6.28. ■

△ **Exercise E11.12**

Give the canonical realization of the following transfer functions. **(a)** $\dfrac{2}{z+5}$ **(b)** $\dfrac{z+8}{z+5}$ **(c)** $\dfrac{z}{z+5}$ **(d)** $\dfrac{2z+3}{z^2+6z+25}$

Answer: See Example 6.18. Replace $1/s$ by $1/z$ and make appropriate changes in coefficients. ▽

11.5 Connection between the Laplace and the Z-Transform

We now show that discrete-time systems also can be analyzed using the Laplace transform. In fact, we shall see that *the z-transform is the Laplace transform in disguise* and that discrete-time systems can be analyzed as if they were continuous-time systems.

So far we have considered the discrete-time signal as a sequence of numbers and not as an electrical signal (voltage or current). Similarly, we have considered a discrete-time system as a mechanism that processes a sequence of numbers (input) to yield another sequence of numbers (output). The system was built by using delays (along with adders and multipliers) that delay sequences of numbers, not electrical signals (voltages or currents). A digital computer is a perfect example: every signal is a sequence of numbers, and the processing involves delaying sequences of numbers (along with addition and multiplication).

Consider a discrete-time system with transfer function $H[z]$ and an input $f[k]$, as shown in Fig. 11.8a. We can think of (or generate, for that matter) a corresponding continuous-time signal $\bar{f}(t)$ consisting of impulses spaced T seconds apart. Let the kth impulse of strength be $f[k]$ as depicted in Fig. 11.8b. Thus

$$\bar{f}(t) = \sum_{k=0}^{\infty} f[k]\delta(t-kT) \tag{11.43}$$

Figure 11.8 shows $f[k]$ and corresponding $\bar{f}(t)$. Let us now consider a system identical in structure to the discrete-time system with transfer function $H[z]$, except that the delays in $H[z]$ are replaced by elements that delay continuous-time signals (such as voltages or currents). If a continuous-time impulse $\delta(t)$ is applied to such a delay of T seconds, the output will be $\delta(t-T)$. The continuous-time transfer function of such a delay is e^{-sT} [see Eq. (6.54)]. Hence, the delay elements with transfer function $1/z$ in the realization of $H[z]$ will be replaced by the delay elements with transfer function e^{-sT} in the realization of the corresponding $\bar{H}(s)$. This step is the same as z being replaced by e^{sT}. Therefore, the transfer function of this system is $H[z]$ with z replaced by e^{sT}. Thus $\bar{H}(s) = H[e^{sT}]$. Now whatever operations are performed by the discrete-time system $H[z]$ on $f[k]$ (Fig. 11.8a) are also performed by the corresponding continuous-time system $H[e^{sT}]$ on the impulse sequence $\bar{f}(t)$

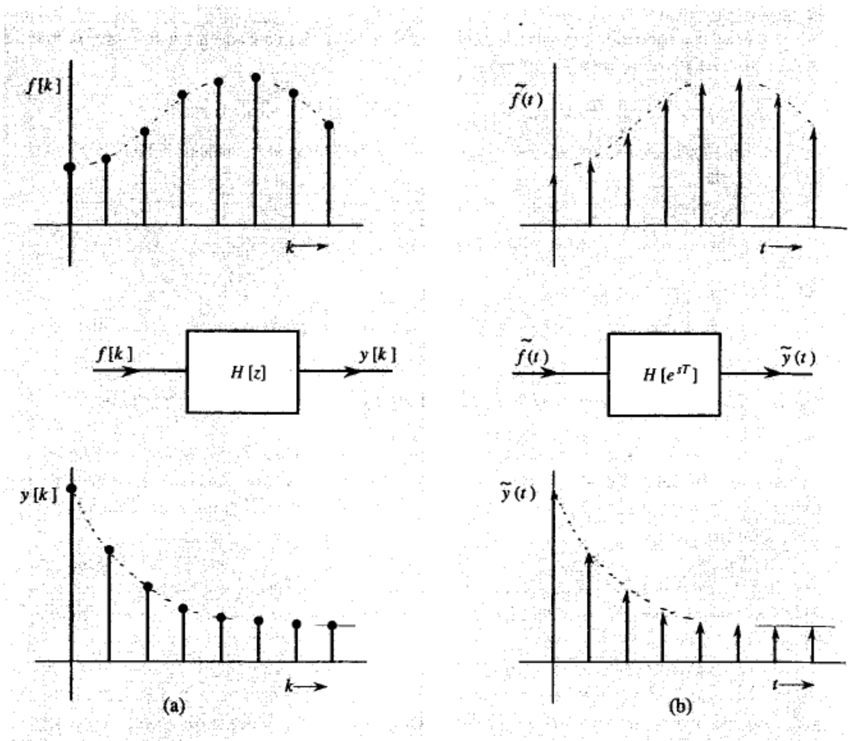


Fig. 11.8 Connection between the Laplace and z-transform.

(Fig. 11.8b). The delaying of a sequence in $H[z]$ would amount to the delaying of an impulse train in $H[e^{sT}]$. The case of adding and multiplying operations is similar. In other words, one-to-one correspondence of the two systems is preserved in every aspect. Therefore, if $y[k]$ is the output of the discrete-time system in Fig. 11.8a, then $\tilde{y}(t)$, the output of the continuous-time system in Fig. 11.8b, would be a sequence of impulses whose kth impulse strength is $y[k]$. Thus

$$\tilde{y}(t) = \sum_{k=0}^{\infty} y[k]\delta(t - kT) \tag{11.44}$$

The system in Fig. 11.8b, being a continuous-time system, can be analyzed by using the Laplace transform. If

$$\tilde{f}(t) \Longleftrightarrow \tilde{F}(s) \quad \text{and} \quad \tilde{y}(t) \Longleftrightarrow \tilde{Y}(s)$$

then

$$\tilde{Y}(s) = H[e^{sT}]\tilde{F}(s) \tag{11.45}$$

Also

$$\tilde{F}(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} f[k]\delta(t - kT)\right]$$

Now, because the Laplace transform of $\delta(t - kT)$ is e^{-skT}

$$\tilde{F}(s) = \sum_{k=0}^{\infty} f[k]e^{-skT} \tag{11.46}$$

Similarly

$$\tilde{Y}(s) = \sum_{k=0}^{\infty} y[k]e^{-skT} \tag{11.47}$$

Substitution of Eqs. (11.46)and (11.47) in Eq. (11.45) yields

$$\sum_{k=0}^{\infty} y[k]e^{-skT} = H[e^{sT}]\left[\sum_{k=0}^{\infty} f[k]e^{-skT}\right]$$

By introducing a new variable $z = e^{sT}$, this equation can be expressed as

$$\sum_{k=0}^{\infty} y[k]z^{-k} = H[z]\sum_{k=0}^{\infty} f[k]z^{-k}$$

or

$$Y[z] = H[z]F[z]$$

where

$$F[z] = \sum_{k=0}^{\infty} f[k]z^{-k} \quad \text{and} \quad Y[z] = \sum_{k=0}^{\infty} y[k]z^{-k}$$

It is clear from this discussion that the z-transform can be considered as a Laplace transform with a change of variable $z = e^{sT}$ or $s = (1/T)\ln z$. On the other hand, we may consider the z-transform as an independent transform in its own right. Note that the transformation $z = e^{sT}$ transforms the imaginary axis in the s-plane ($s = j\omega$) into a unit circle in the z-plane ($z = e^{sT} = e^{j\omega T}$, or $|z| = 1$). The LHP and RHP in the s-plane map into the inside and the outside, respectively, of the unit circle in the z-plane.

11.6 Sampled-data (Hybrid) Systems

Sampled-data systems are hybrid systems consisting of discrete-time as well as continuous-time subsystems. Consider, for example, a fire control system. In this case, the problem is to search and track a moving target and fire a projectile for a direct hit. The data obtained from the search and tracking radar is discrete-time data because of a scanning operation, which results in sampling of azimuth, elevation, and the target velocity. This data is now fed to a digital (discrete-time) processor, which performs extensive computations. The computer output is then fed to a continuous-time plant, such as a gun mount, which accordingly positions

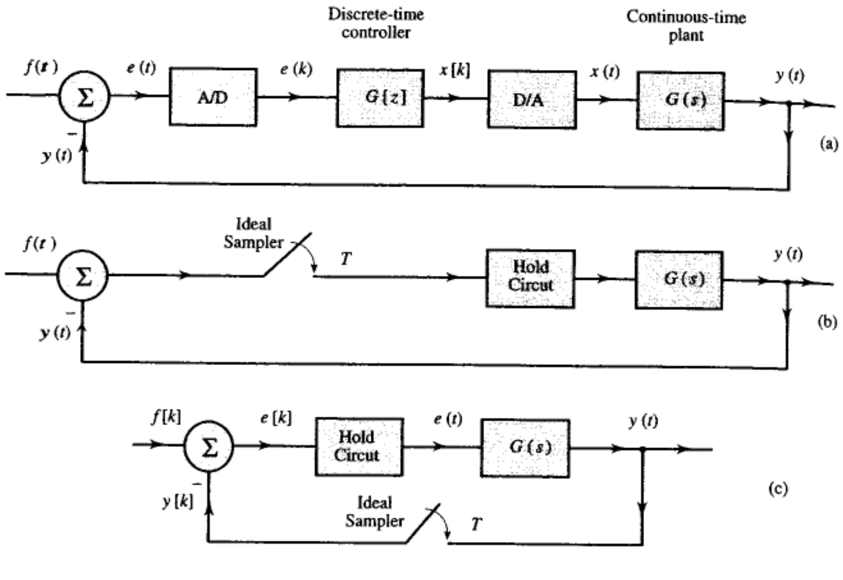


Fig. 11.9 Typical sampled-data systems.

itself at a certain position and fires. Another example is attitude-control problem in a spacecraft, where the information about the actual spacecraft attitude is fed back to a digital processor, which generates corrective input to be applied to the spacecraft, which is a continuous-time system. In automatic periodic quality check in production line, the discrete-data obtained from the periodic check, after some digital processing, generates the corrective input to be applied to a continuous-time plant. In complex control systems, use of digital processor as a controller or a compensator for continuous-time plants is growing rapidly.

In time-sharing systems, where, for economic reasons, certain facilities are shared by several systems, the signals are, by nature, discrete-time or sampled. In regulator type control systems, where an output variable must be maintained at a constant value, the external disturbance and plant parameters variations are usually so slow that continuous monitoring (or feedback) is unnecessary. It is adequate to sample the output periodically and then feed back this discrete-data. In such cases, feedback transducers, data-processing facilities and possibly long and expensive feedback communication facilities can be shared among several control systems.

Figure 11.9 shows some typical sampled-data systems. Figure 11.9a contains a digital processor, whereas in Fig. 11.9b, the sampled signal is directly applied to D/A converter (the hold circuit) without further digital processing. Figure 11.9c shows a practical system, where the input signal itself is a discrete-time signal $f[k]$, and the sampler is in the feedback path. This system is equivalent to that in Fig. 11.9b. How do we analyze such hybrid systems, where continuous-time and discrete-time signals intermingle? An effective strategy in such a situation is to

relate the samples of the output to those of the input. But, this procedure yields information about the output only at sampling instants. We can overcome this difficulty by taking the samples at instants in between samples using the **modified z-transform** as explained later.

In sampled-data systems, the discrete-time signals are often obtained as a result of sampling continuous-time signals. These samples are narrow pulses, which may be considered as impulses, provided the pulse width is small compared to the system time constant. Thus, in the following discussion, a discrete-time signal, when it appears in conjunction with a continuous-time system, is a sequence of impulses rather than a sequence of numbers. Hence, a discrete-time signal $f[k]$ can also be considered as continuous-time signal $f(t)$, where

$$f(t) = \sum_k f[k]\delta(t - kT)$$

Observe an interesting fact: in this representation a discrete-time unit impulse $\delta[k]$ *is* the continuous-time unit impulse $\delta(t)$. Thus, at the input of a discrete-time processor, a discrete-time signal $f[k]$ is just a sequence of numbers. But at the input of a continuous-time system, $f[k]$ is a sequence of impulses. There are appropriate converters at the interface of discrete-time and continuous-time systems to carry out signal conversion to appropriate forms.

To begin with, consider a basic continuous-time system (Fig. 11.10a) with transfer function $H(s)$. The input $f(t)$ is sampled and the sampled signal $f[k]$ is applied to the input of $H(s)$. Although $y(t)$, the output of this system, is continuous, we shall endeavor to find the values of $y(t)$ only at the discrete instants $t = kT$. Such an analysis is relatively simple using the method of z-transform. For this purpose, we consider as if the output is sampled by an hypothetical sampler shown dotted in Fig. 11.10a. Now, we shall relate the input samples $f[k]$ and the output samples $y[k]$. Let $h[k]$ be the unit impulse response relating the output samples to the input samples. In other words, $y[k] = h[k] * f[k]$. Recall also that an unit impulse $\delta[k]$ is $\delta(t)$ when considered in conjunction with a continuous-time system. Hence, $h[k]$, the unit impulse response is the sampled version of the system's unit impulse response $h(t)$. Thus,

$$h[k] = h(kT)$$

where T is the sampling interval. For instance, if $H(s) = \frac{1}{s-\lambda}$, then $h(t) = e^{\lambda t}$ and

$$h[k] = e^{\lambda kT}$$

Therefore, the equivalent discrete-time transfer function $H[z]$ of this system is given by

$$\begin{aligned} H[z] &= \mathcal{Z}\{h[k]\} \\ &= \mathcal{Z}[e^{\lambda kT}] \\ &= \frac{z}{z - e^{\lambda T}} \end{aligned} \tag{11.49}$$

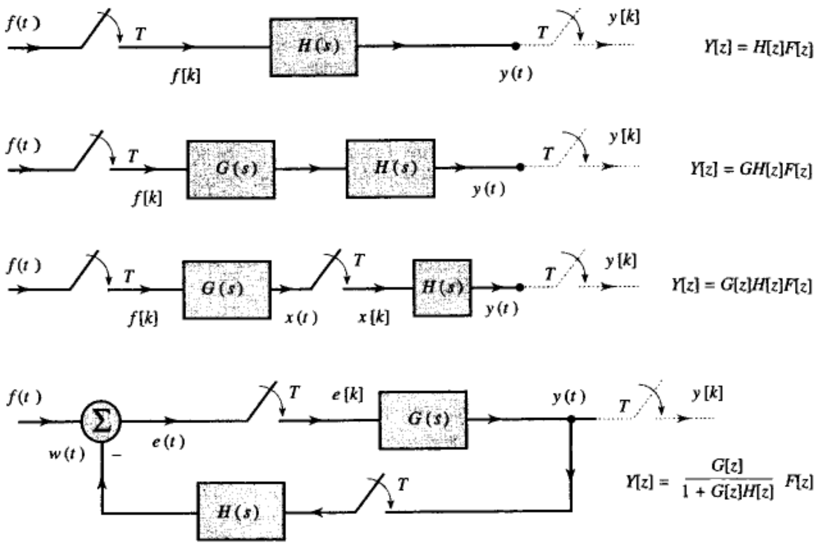


Fig. 11.10 Computing the output in hybrid or sampled-data systems.

Thus, $H[z]$ is the discrete-time transfer function of $H(s) = \frac{1}{s+\lambda}$ that relates $y[k]$ (the output samples) to the discrete-time input $f[k]$.†

If we have two systems with transfer functions $G(s)$ and $H(s)$ in cascade (Fig. 11.10b), the equivalent transfer $T[z] \neq G[z]H[z]$, but is $GH[z]$, where $G[z]$, $H[z]$ and $GH[z]$ correspond to discrete-time transfer functions of $G(s)$, $H(s)$ and $G(s)H(s)$, respectively. For instance, if

$$G(s) = \frac{1}{s+2} \qquad \text{and} \qquad H(s) = \frac{1}{s}$$

Then, according to Eq. (11.49)

$$G[z] = \frac{z}{z - e^{-2T}} \qquad \text{and} \qquad H[z] = \frac{z}{z-1}$$

However, the continuous-time system transfer function is $G(s)H(s)$, where

$$G(s)H(s) = \frac{1}{s(s+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

And from Eq. (11.49)

†Using this procedure, we have listed $H(s)$ and corresponding $H[z]$ in Table 12.1 in Chapter 12. In this Table, $H[z]$ is multiplied with a scaling factor T, which results in $H[z] = \frac{Tz}{z-e^{\lambda T}}$. For the purpose of the sampled data application, the extra factor T should be ignored throughout in Table 12.1.

$$T[z] = \frac{1}{2}\left(\frac{z}{z-1} - \frac{z}{z-e^{-2T}}\right) \neq G[z]H[z]$$

In this case, we use the notation $GH[z]$ for $T[z]$. Thus, $GH[z] \neq G[z]H[z]$, but is the discrete-time transfer function which corresponds to $G(s)H(s)$.

For the system in Fig. 11.10c,

$$Y[z] = H[z]X[z] = H[z]G[z]F[z] \qquad \text{so that} \qquad T[z] = G[z]H[z]$$

For the system in Fig. 11.10d,

$$E[z] = F[z] - W[z]$$

Moreover,

$$\begin{aligned} W[z] &= H[z]Y[z] \\ Y[z] &= G[z]E[z] \\ &= G[z]\,(F[z] - W[z]) \\ &= G[z](F[z] - H[z]Y[z]) \end{aligned}$$

Hence

$$Y[z] = \frac{G[z]}{1 + G[z]H[z]}F[z]$$

Consequently

$$T[z] = \frac{G[z]}{1 + G[z]H[z]}$$

■ **Example 11.9**

Find the output samples $y[k]$ for the sampled-data system illustrated in Fig. 11.11a when the input is a unit step function $u(t)$, the sampling interval $T = 0.5$ second and

$$G_c[z] = \frac{z}{z-1} \qquad \text{and} \qquad G(s) = \frac{1}{s+4}$$

This system has a discrete-time controller and a continuous-time plant.† To find the transfer function of this system, we observe that

$$Y[z] = G[z]X[z], \qquad X[z] = G_c[z]E[z], \qquad \text{and} \qquad E[z] = F[z] - Y[z]$$

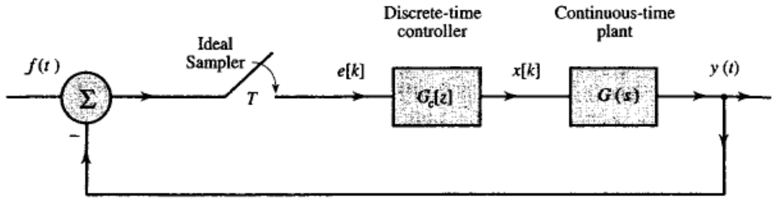
Hence

$$Y[z] = G_c[z]G[z](F[z] - Y[z])$$

and the system transfer function $T[z]$ is

$$T[z] = \frac{Y[z]}{F[z]} = \frac{G_c[z]G[z]}{1 + G_c[z]G[z]}$$

†The block diagram in Fig. 11.11a does not show the appropriate converters required at the interface of discrete-time and continuous-time systems; these are implied. Thus, the output of the sampler, which consists of impulse sequence, is converted into sequence of numbers to act as the input to the discrete-time controller. Similarly, the output of a discrete-time controller, which is a sequence of numbers, is converted to a sequence of impulses to act as an input to the plant.



(a)

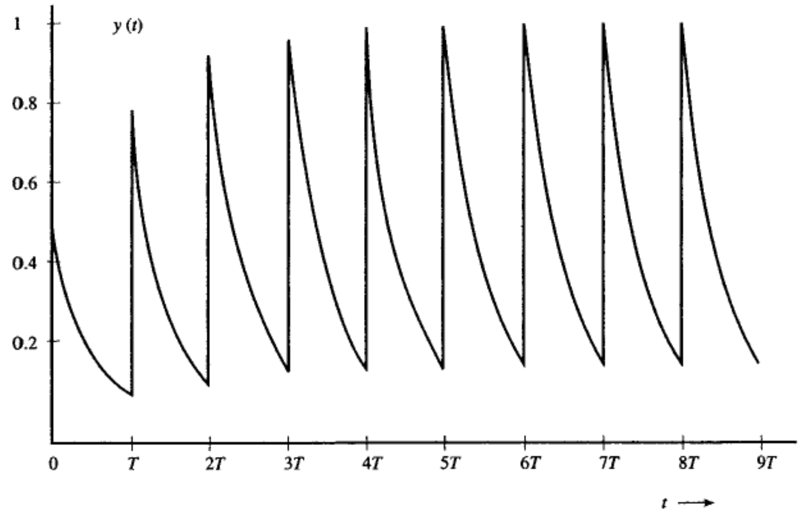


Fig. 11.11 The sampled-data system in Example 11.9.

For $G(s) = 1/(s+4)$ and $T = 0.5$, we find, from Eq. (11.49), $G[z] = \frac{z}{z-e^{-2}} = \frac{z}{z-0.1353}$. Also $G_c[z] = z/(z-1)$. Substitution of these expressions in $T[z]$ yields

$$T[z] = \frac{z^2}{(z-0.394)(z-0.174)}$$

The output $Y[z]$ is given by $Y[z] = T[z]F[z]$. For the step input $f(t) = u(t)$, the corresponding sampled signal is $u[k]$ so that $F[z] = z/(z-1)$. Hence,

$$Y[z] = T[z]F[z] = \frac{z^3}{(z-1)(z-0.394)(z-0.174)}$$

and

$$\begin{aligned}\frac{Y[z]}{z} &= \frac{z^2}{(z-1)(z-0.394)(z-0.174)} \\ &= \frac{1}{z-1} - \frac{0.583}{z-0.394} + \frac{0.083}{z-0.174}\end{aligned}$$

Hence

$$Y[z] = \frac{z}{z-1} - \frac{0.583z}{z-0.394} + \frac{0.083z}{z-0.174}$$

and

$$y[k] = 1 - 0.583(0.394)^k + 0.083(0.174)^k$$

This response is depicted in Fig. 11.11. ■

11.6-1 Response Between Sampling Instants: The Modified Z-Transform

The above analysis yields the output only at sampling instants. We can readily find the response between successive sampling instants by using the *modified z-transform*. This goal can be accomplished by considering the response at another set of sampling instants $t = (k+\mu)T$, where $0 < \mu < 1$.

Consider the system in Fig. 11.10a with $H(s) = 1/(s-\lambda)$. The impulse response is $h(t) = e^{\lambda t}$ and its samples at instants $t = (k+\mu)T$ are

$$h(t, \mu) = e^{\lambda(k+\mu)T} = e^{\lambda\mu T}\left[e^{\lambda kT}\right] \tag{11.50a}$$

The corresponding z-transfer function is

$$H[z, \mu] = e^{\lambda\mu T}\frac{z}{z-e^{\lambda T}} = \frac{ze^{\lambda\mu T}}{z-e^{\lambda T}} \tag{11.50b}$$

In this manner, We can prepare a table of modified z-transform. When we use $H[z, \mu]$ instead of $H[z]$ in our analysis, we obtain the response at instants $t = (k+\mu)T$. By using different values of μ in the range 0 to T, we can obtain the complete response $y(t)$.

■ **Example 11.10**

Find the output $y(t)$ for all t in Example 11.9.

In Example 11.9, we found the response $y[k]$ only at the sampling instants. To find the output values between sampling instants, we use the modified z-transform. The procedure is the same as before, except that we use modified z-transform corresponding to continuous-time systems and signals. For the system $G(s) = 1/s + 4$ with $T = 0.5$, the modified z-transform [Eq. (11.50b) with $\lambda = -4$, and $T = 0.5$] is

$$H[z, \mu] = e^{\lambda\mu T}\frac{z}{z-e^{-2}} = e^{-2\mu}\frac{z}{z-0.1353}$$

Moreover to find the modified z-transform corresponding to $f(t) = u(t)$ [$\lambda = 0$ in Eq. (11.50a)], we have $F[z, \mu] = z/(z-1)$. Substitution of these expressions in those found in Example 11.9, we obtain

$$Y[z, \mu] = e^{-2\mu}\left[\frac{z}{z-1} - \frac{0.583z}{z-0.394} + \frac{0.083z}{z-0.174}\right]$$

From Eqs. (11.50), we obtain the inverse (modified) z-transform of this equation as

$$y[(k+\mu)T] = e^{-2\mu}\left[1 - 0.583(0.394)^k + 0.083(-0.174)^k\right] \qquad 0 < \mu < 1$$

The complete response is also shown in Fig. 11.11. ■

Design of Sampled-Data Systems

As with continuous-time control systems, sampled-data systems are designed to meet certain transient (PO, t_r, t_s, etc.) and steady-state specifications. The design procedure follows along the lines similar to those used for continuous-time systems. We begin with a general second-order system. the relationship between closed-loop pole locations and the corresponding transient parameters PO, t_r, $t_s, \ldots$ are determined. Hence, for a given transient specifications, an acceptable region in the z-plane where the dominant poles of the closed-loop transfer function $T[z]$ should lie is determined. Next, we sketch the root locus for the system. The rules for sketching the root locus are the same as those for continuous-time systems. If the root locus passes through the acceptable region, the transient specifications can be met by simple adjustment of the gain K. If not, we must use a compensator, which will steer the root locus in the acceptable region.

11.7 The Bilateral Z-Transform

Situations involving noncausal signals or systems cannot be handled by the (unilateral) z-transform discussed so far. Such cases can be analyzed by the **bilateral** (or two-sided) z-transform defined by

$$F[z] \equiv \sum_{k=-\infty}^{\infty} f[k] z^{-k}$$

The inverse z-transform is given by

$$f[k] = \frac{1}{2\pi j} \oint F[z] z^{k-1}\, dz$$

These equations define the bilateral z-transform. The unilateral z-transform discussed so far is a special case, where the input signals are restricted to be causal. Restricting signals in this way results in considerable simplification in the region of convergence. Earlier, we showed that

$$\gamma^k u[k] \Longleftrightarrow \frac{z}{z-\gamma} \qquad |z| > |\gamma| \tag{11.51}$$

In contrast, the z-transform of the signal $-\gamma^k u[-(k+1)]$, illustrated in Fig. 11.12a, is

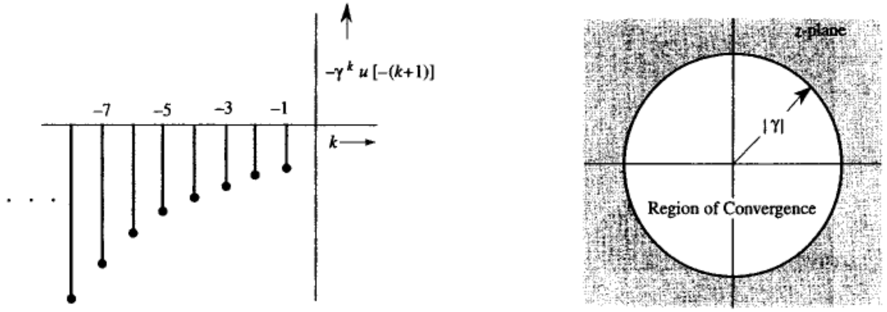


Fig. 11.12 $-\gamma^k u[-(k+1)]$ and the region of convergence of its z-transform.

$$\begin{aligned}
\mathcal{Z}\left\{-\gamma^k u[-(k+1)]\right\} &= \sum_{-\infty}^{-1} -\gamma^k z^{-k} = \sum_{-\infty}^{-1} -\left(\frac{\gamma}{z}\right)^k \\
&= -\left[\frac{z}{\gamma} + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right] \\
&= 1 - \left[1 + \frac{z}{\gamma} + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right] \\
&= 1 - \frac{1}{1 - \frac{z}{\gamma}} \qquad \left|\frac{z}{\gamma}\right| < 1 \\
&= \frac{z}{z-\gamma} \qquad |z| < |\gamma|
\end{aligned}$$

Therefore

$$\mathcal{Z}\left\{-\gamma^k u[-(k+1)]\right\} = \frac{z}{z-\gamma} \qquad |z| < |\gamma| \tag{11.52}$$

A comparison of Eqs. (11.51) with (11.52) shows that the z-transform of $\gamma^k u[k]$ is identical to that of $-\gamma^k u[-(k+1)]$. The regions of convergence, however, are different. In the former case, $F[z]$ converges for $|z| > |\gamma|$; in the latter, $F[z]$ converges for $|z| < |\gamma|$ (see Fig. 11.12b). Clearly, the inverse transform of $F[z]$ is not unique unless the region of convergence is specified. If we restrict all our signals to be causal, however, this ambiguity does not arise. The inverse transform of $z/(z-\gamma)$ is $\gamma^k u[k]$ even without specifying the region of convergence. Thus, in the unilateral transform, we can ignore the region of convergence in determining the inverse z-transform of $F[z]$.

■ **Example 11.11**

Determine the z-transform of

$$\begin{aligned}
f[k] &= (0.9)^k u[k] + (1.2)^k u[-(k+1)] \\
&= f_1[k] + f_2[k]
\end{aligned}$$

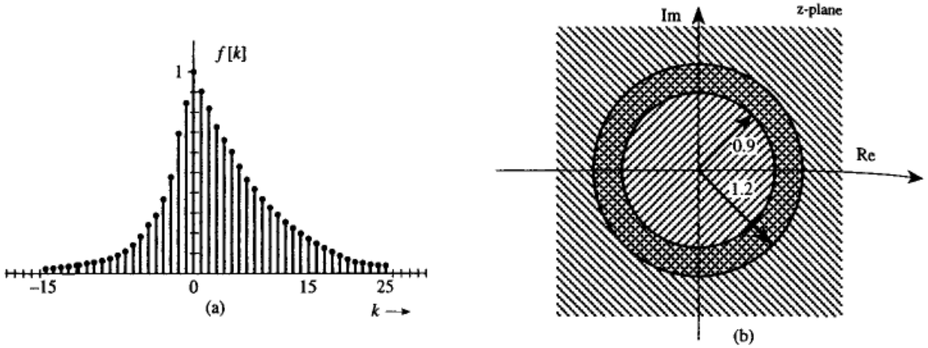


Fig. 11.13 Signal $f[k]$ for Example 11.11.

From the results in Eqs. (11.51) and (11.52), we have

$$F_1[z] = \frac{z}{z-0.9} \qquad |z| > 0.9$$

$$F_2[z] = \frac{-z}{z-1.2} \qquad |z| < 1.2$$

The common region where both $F_1[z]$ and $F_2[z]$ converge is $0.9 < |z| < 1.2$ (Fig. 11.13a). Hence

$$\begin{aligned} F[z] &= F_1[z] + F_2[z] \\ &= \frac{z}{z-0.9} - \frac{z}{z-1.2} \\ &= \frac{-0.3z}{(z-0.9)(z-1.2)} \qquad 0.9 < |z| < 1.2 \end{aligned} \tag{11.53}$$

The sequence $f[k]$ and the region of convergence of $F[z]$ are depicted in Fig. 11.13. ■

■ **Example 11.12**

Find the inverse z-transform of

$$F[z] = \frac{-z(z+0.4)}{(z-0.8)(z-2)}$$

if the region of convergence is **(a)** $|z| > 2$ **(b)** $|z| < 0.8$ **(c)** $0.8 < |z| < 2$.

(a)

$$\begin{aligned} \frac{F[z]}{z} &= \frac{-(z+0.4)}{(z-0.8)(z-2)} \\ &= \frac{1}{z-0.8} - \frac{2}{z-2} \end{aligned}$$

and

$$F[z] = \frac{z}{z-0.8} - 2\frac{z}{z-2}$$

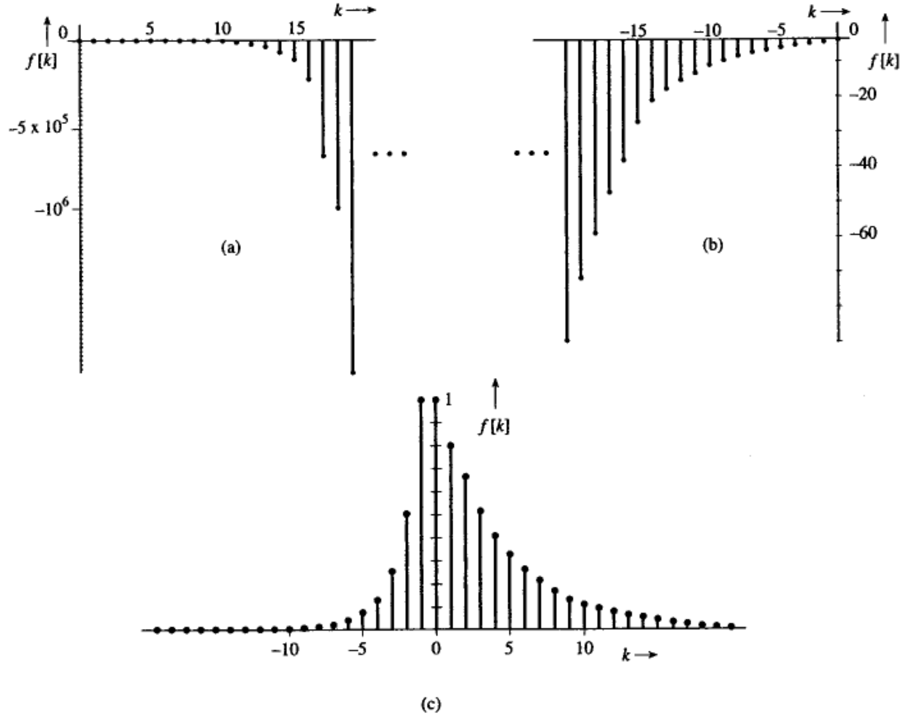


Fig. 11.14 Three possible inverse transforms of $F[z]$ in Example 11.12.

Since the region of convergence is $|z| > 2$, both terms correspond to causal sequences and

$$f[k] = \left[(0.8)^k - 2(2)^k\right] u[k]$$

This sequence appears in Fig. 11.14a.

(b) In this case, $|z| < 0.8$, which is less than the magnitudes of both poles. Hence, both terms correspond to anticausal sequences, and

$$f[k] = \left[-(0.8)^k + 2(2)^k\right] u\left[-(k+1)\right]$$

This sequence appears in Fig. 11.14b.

(c) In this case, $0.8 < |z| < 2$; the part of $F[z]$ corresponding to the pole at 0.8 is a causal sequence, and the part corresponding to the pole at 2 is an anticausal sequence:

$$f[k] = (0.8)^k u[k] + 2(2)^k u[-(k+1)]$$

This sequence appears in Fig. 11.14c. ■

△ **Exercise E11.13**
Find the inverse z-transform of

$$F[z] = \frac{z}{z^2 + \frac{5}{6}z + \frac{1}{6}} \qquad \frac{1}{2} > |z| > \frac{1}{3}$$

Answer: $6(-\frac{1}{3})^k u[k] + 6(-\frac{1}{2})^k u[-(k+1)]$ ▽

Inverse Transform by Expansion of F[z] in Power Series of z

We have

$$F[z] = \sum_k f[k] z^{-k}$$

For an anticausal sequence, which exists only for $k \leq -1$, this equation becomes

$$F[z] = f[-1]z + f[-2]z^2 + f[-3]z^3 + \cdots$$

We can find the inverse z-transform of $F[z]$ by dividing the numerator polynomial with the denominator polynomial, both in ascending powers of z, to obtain a polynomial in ascending powers of z. Thus, to find the inverse transform of $z/(z-0.5)$ (when the region of convergence is $|z| < 0.5$), we divide z with $-0.5+z$ to obtain $-2z - 4z^2 - 8z^3 - \cdots$. Hence, $f[-1] = -2$, $f[-2] = -4$, $f[-3] = -8$ and so on.

11.7-1 Analysis of LTID Systems Using the Bilateral $\mathcal{Z}$-Transform

Because the bilateral z-transform can handle noncausal signals, we can analyze noncausal linear systems using this transform. The zero-state response $y[k]$ is given by

$$y[k] = \mathcal{Z}^{-1}\{F[z]H[z]\}$$

provided that $F[z]H[z]$ exists. The region of convergence of $F[z]H[z]$ is the region where both $F[z]$ and $H[z]$ exist, a fact which means that the region is common to the convergence of both $F[z]$ and $H[z]$.

■ **Example 11.13**
For a causal system specified by the transfer function

$$H[z] = \frac{z}{z-0.5}$$

find the zero-state response to input

$$f[k] = (0.8)^k u[k] + 2(2)^k u[-(k+1)]$$

The z-transform of this signal is found from Example 11.12 (part **c**) as

$$F[z] = \frac{-z(z+0.4)}{(z-0.8)(z-2)} \qquad 0.8 < |z| < 2$$

Therefore

$$Y[z] = F[z]H[z] = \frac{-z^2(z+0.4)}{(z-0.5)(z-0.8)(z-2)}$$

Since the system is causal, the region of convergence of $H[z]$ is $|z| > 0.5$. The region of convergence of $F[z]$ is $0.8 < |z| < 2$. The common region of convergence for $F[z]$ and $H[z]$ is $0.8 < |z| < 2$. Therefore

$$Y[z] = \frac{-z^2(z+0.4)}{(z-0.5)(z-0.8)(z-2)} \qquad 0.8 < |z| < 2$$

Expanding $Y[z]$ into modified partial fractions yields

$$Y[z] = -\frac{z}{z-0.5} + \frac{8}{3}\left(\frac{z}{z-0.8}\right) - \frac{8}{3}\left(\frac{z}{z-2}\right) \qquad 0.8 < |z| < 2$$

The poles at 0.5 and 0.8 are enclosed within the ring of convergence and therefore correspond to the causal part, and the pole at 2 is outside the ring of convergence and corresponds to the anticausal part of $Y[z]$. Therefore

$$y[k] = \left[-(0.5)^k + \tfrac{8}{3}(0.8)^k\right] u[k] + \tfrac{8}{3}(2)^k u[-(k+1)] \qquad ■$$

■ **Example 11.14**
For the system in Example 11.13 find the zero-state response to input

$$f[k] = \underbrace{(0.8)^k u[k]}_{f_1[k]} + \underbrace{(0.6)^k u[-(k+1)]}_{f_2[k]}$$

The z-transforms of the causal and anticausal components $f_1[k]$ and $f_2[k]$ of the output are

$$F_1[z] = \frac{z}{z-0.8} \qquad |z| > 0.8$$

$$F_2[z] = \frac{-z}{z-0.6} \qquad |z| < 0.6$$

Observe that a common region of convergence for $F_1[z]$ and $F_2[z]$ does not exist. Therefore $F[z]$ does not exist. In such a case we take advantage of the superposition principle and find $y_1[k]$ and $y_2[k]$, the system responses to $f_1[k]$ and $f_2[k]$, separately. The desired response $y[k]$ is the sum of $y_1[k]$ and $y_2[k]$. Now

$$H[z] = \frac{z}{z-0.5} \qquad |z| > 0.5$$

$$Y_1[z] = F_1[z]H[z] = \frac{z^2}{(z-0.5)(z-0.8)} \qquad |z| > 0.8$$

$$Y_2[z] = F_2[z]H[z] = \frac{-z^2}{(z-0.5)(z-0.6)} \qquad 0.5 < |z| < 0.6$$

Expanding $Y_1[z]$ and $Y_2[z]$ into modified partial fractions yields

$$Y_1[z] = -\frac{5}{3}\left(\frac{z}{z-0.5}\right) + \frac{8}{3}\left(\frac{z}{z-0.8}\right) \qquad |z| > 0.8$$

$$Y_2[z] = 5\left(\frac{z}{z-0.5}\right) - 6\left(\frac{z}{z-0.6}\right) \qquad 0.5 < |z| < 0.6$$

Therefore

$$y_1[k] = \left[-\tfrac{5}{3}(0.5)^k + \tfrac{8}{3}(0.8)^k\right] u[k]$$

$$y_2[k] = 5(0.5)^k u[k] + 6(0.6)^k u[-(k+1)]$$

and

$$\begin{aligned} y[k] &= y_1[k] + y_2[k] \\ &= \left[\tfrac{10}{3}(0.5)^k + \tfrac{8}{3}(0.8)^k\right] u[k] + 6(0.6)^k u[-(k+1)] \qquad \blacksquare \end{aligned}$$

△ **Exercise E11.14**

For a causal system in Example 11.13, find the zero-state response to input

$$f[k] = \left(\tfrac{1}{4}\right)^k u[k] + 5(3)^k u[-(k+1)]$$

Answer: $\left[-(\tfrac{1}{4})^k + 3(\tfrac{1}{2})^k\right] u[k] + 6(3)^k u[-(k+1)]$ ▽

11.8 Summary

In this chapter we discuss the analysis of linear, time-invariant, discrete-time (LTID) systems by z-transform. The z-transform is an extension of the DTFT with the frequency variable $j\Omega$ generalized to $\sigma + j\Omega$. Such an extension allows us to synthesize discrete-time signals by using exponentially growing (discrete-time) sinusoids. The relationship of the z-transform to the DTFT is identical to that of the Laplace transform to the Fourier. Because of the generalization of the frequency variable, we can analyze all kinds of LTID systems and also handle exponentially growing inputs.

The z-transform changes the difference equations of LTID systems into algebraic equations. Therefore, solving these difference equations reduces to solving algebraic equations.

The transfer function $H[z]$ of an LTID system is equal to the ratio of the z-transform of the output to the z-transform of the input when all initial conditions are zero. Therefore, if $F[z]$ is the z-transform of the input $f[k]$ and $Y[z]$ is the z-transform of the corresponding output $y[k]$ (when all initial conditions are zero), then $Y[z] = H[z]F[z]$. For a system specified by the difference equation $Q[E]y[k] = P[E]f[k]$, the transfer function $H[z] = P[z]/Q[z]$. Moreover, $H[z]$ is the z-transform of the system impulse response $h[k]$. We also showed in Chapter 9 that the system response to an everlasting exponential z^k is $H[z]z^k$.

LTID systems can be realized by scalar multipliers, summers, and time delays. A given transfer function can be synthesized in many different ways. Canonical, cascade and parallel forms of realization are discussed. The realization procedure is identical to that for continuous-time systems.

In Sec. 11.5, we showed that discrete-time systems can be analyzed by the Laplace transform as if they were continuous-time systems. In fact, we showed that the z-transform is the Laplace transform with a change in variable.

In practice, we often have to deal with hybrid systems consisting of discrete-time and continuous-time subsystems. Feedback hybrid systems are also called sampled-data systems. In such systems, we can relate the samples of the output to those of the input. However, the output is generally a continuous-time signal. The output values during the successive sampling intervals can be found by using the modified z-transform.

The majority of the input signals and practical systems are causal. Consequently, we are required to deal with causal signals most of the time. When all signals are restricted to the causal type, the z-transform analysis is greatly simplified; the region of convergence of a signal becomes irrelevant to the analysis process. This special case of z-transform (which is restricted to causal signals) is called the unilateral z-transform. Much of the chapter deals with this transform. Section 11.7 discusses the general variety of the z-transform (bilateral z-transform), which can handle causal and noncausal signals and systems. In the bilateral transform, the inverse transform of $F[z]$ is not unique, but depends on the region of convergence of $F[z]$. Thus, the region of convergence plays a crucial role in the bilateral z-transform.

Problems

11.1-1 Using the definition of the z-transform, show that

(a) $\gamma^{k-1}u[k-1] \Longleftrightarrow \dfrac{1}{z-\gamma}$ **(c)** $\dfrac{\gamma^k}{k!}u[k] \Longleftrightarrow e^{\gamma/z}$

(b) $u[k-m] \Longleftrightarrow \dfrac{z}{z^m(z-1)}$ **(d)** $\dfrac{(\ln\alpha)^k}{k!}u[k] \Longleftrightarrow \alpha^{1/z}$

11.1-2 Using only the z-transform Table 11.1, show that

(a) $2^{k+1}u[k-1] + e^{k-1}u[k] \Longleftrightarrow \frac{4}{z-2} + \frac{z}{e(z-e)}$

(b) $k\gamma^k u[k-1] \Longleftrightarrow \dfrac{\gamma z}{(z-\gamma)^2}$

Hint: Express $u[k-1]$ in terms of $u[k]$.

(c) $\left[2^{-k}\cos\left(\frac{\pi}{3}k\right)\right]u[k-1] \Longleftrightarrow \frac{0.25(z-1)}{z^2-0.5z+0.25}$

Hint: See the hint for part **b**.

(d) $k(k-1)(k-2)2^{k-3}u[k-m] \Longleftrightarrow \frac{6z}{(z-2)^4}$ for m=0, 1, 2, or 3.

Hint: Examine what happens to the function if $u[k-m]$ is replaced by $u[k]$.

11.1-3 Find the inverse z-transform of

(a) $\dfrac{z(z-4)}{z^2-5z+6}$

(b) $\dfrac{z-4}{z^2-5z+6}$

(c) $\dfrac{(e^{-2}-2)z}{(z-e^{-2})(z-2)}$

(d) $\dfrac{z(2z+3)}{(z-1)(z^2-5z+6)}$

(e) $\dfrac{z(-5z+22)}{(z+1)(z-2)^2}$

(f) $\dfrac{z(1.4z+0.08)}{(z-0.2)(z-0.8)^2}$

(g) $\dfrac{z(z-2)}{z^2-z+1}$

(h) $\dfrac{2z^2-0.3z+0.25}{z^2+0.6z+0.25}$

(i) $\dfrac{2z(3z-23)}{(z-1)(z^2-6z+25)}$

(j) $\dfrac{z(3.83z+11.34)}{(z-2)(z^2-5z+25)}$

(k) $\dfrac{z^2(-2z^2+8z-7)}{(z-1)(z-2)^3}$

11.1-4 Find the first three terms of $f[k]$ if

$$F[z] = \frac{2z^3+13z^2+z}{z^3+7z^2+2z+1}$$

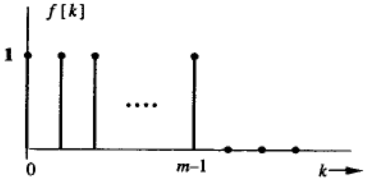


Fig. P11.2-1

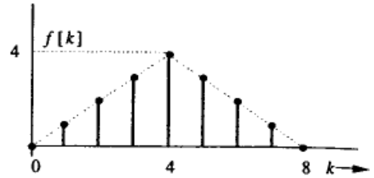


Fig. P11.2-2

Find your answer by expanding $F[z]$ as a power series in z^{-1}.

11.1-5 By expanding

$$F[z] = \frac{\gamma z}{(z-\gamma)^2}$$

as a power series in z^{-1}, show that $f[k] = k\gamma^k u[k]$.

11.2-1 For a discrete-time signal shown in Fig. P11.2-1 show that

$$F[z] = \frac{1-z^{-m}}{1-z^{-1}}$$

11.2-2 Find the z-transform of the signal illustrated in Fig. P11.2-2. Solve this problem in two ways, as in Examples 11.2d and 11.4. Verify that the two answers are equivalent.

11.2-3 Using only the fact that $\gamma^k u[k] \Longleftrightarrow \frac{z}{z-\gamma}$ and properties of the z-transform, find the z-transform of

(a) $k^2\gamma^k u[k]$ (c) $a^k\,[u[k]-u[k-m]]$

(b) $k^3 u[k]$ (d) $ke^{-2k}u[k-m]$

11.2-4 Using only Pair 1 in Table 11.1 and appropriate properties of the z-transform, derive iteratively pairs 2 through 9. In other words, first derive Pair 2. Then use Pair 2 (and Pair 1, if needed) to derive Pair 3, and so on. However, pair 6 should be derived after pair 7.

11.3-1 Solve Prob. 9.4-9 by the z-transform method.

11.3-2 Solve

$$y[k+1] + 2y[k] = f[k+1]$$

with $y[0]=1$ and $f[k] = e^{-(k-1)}u[k]$

11.3-3 Find the output $y[k]$ of an LTID system specified by the equation

$$2y[k+2] - 3y[k+1] + y[k] = 4f[k+2] - 3f[k+1]$$

if the initial conditions are $y[-1]=0$, $y[-2]=1$, and the input $f[k] = (4)^{-k}u[k]$.

11.3-4 Solve Prob. 11.3-3 if instead of initial conditions $y[-1], y[-2]$ you are given the auxiliary conditions $y[0] = \frac{3}{2}$ and $y[1] = \frac{35}{4}$.

11.3-5 Solve

$$4y[k+2] + 4y[k+1] + y[k] = f[k+1]$$

with $y[-1]=0$, $y[-2]=1$, and $f[k]=u[k]$.

11.3-6 Solve

$$y[k+2] - 3y[k+1] + 2y[k] = f[k+1]$$

if $y[-1]=2$, $y[-2]=3$, and $f[k]=(3)^k u[k]$.

11.3-7 Solve

$$y[k+2] - 2y[k+1] + 2y[k] = f[k]$$

with $y[-1]=1$, $y[-2]=0$, and $f[k]=u[k]$.

11.3-8 Solve

$$y[k] + 2y[k-1] + 2y[k-2] = f[k-1] + 2f[k-2]$$

with $y[0]=0, y[1]=1$, and $f[k]=e^k u[k]$.

11.3-9 (a) Find the zero-state response of an LTID system with transfer function

$$H[z] = \frac{z}{(z+0.2)(z-0.8)}$$

and the input $f[k] = e^{(k+1)}u[k]$.

(b) Write the difference equation relating the output $y[k]$ to input $f[k]$.

11.3-10 Repeat Prob. 11.3-9 if $f[k]=u[k]$ and

$$H[z] = \frac{2z+3}{(z-2)(z-3)}$$

11.3-11 Repeat Prob. 11.3-9 if

$$H[z] = \frac{6(5z-1)}{6z^2-5z+1}$$

and the input $f[k]$ is (a) $(4)^{-k}u[k]$ (b) $(4)^{-(k-2)}u[k-2]$ (c) $(4)^{-(k-2)}u[k]$ (d) $(4)^{-k}u[k-2]$.

11.3-12 Repeat Prob. 11.3-9 if $f[k]=u[k]$ and

$$H[z] = \frac{2z-1}{z^2-1.6z+0.8}$$

11.3-13 Find the transfer functions corresponding to each of the systems specified by difference equations in Probs. 11.3-2, 11.3-3, 11.3-5, and 11.3-8.

11.3-14 Find $h[k]$, the unit impulse response of the systems described by the following equations:

(a) $y[k] + 3y[k-1] + 2y[k-2] = f[k] + 3f[k-1] + 3f[k-2]$

(b) $y[k+2] + 2y[k+1] + y[k] = 2f[k+2] - f[k+1]$

(c) $y[k] - y[k-1] + 0.5y[k-2] = f[k] + 2f[k-1]$

11.3-15 Find $h[k]$, the unit impulse response of the systems in Probs. 11.3-9, 11.3-10, and 11.3-12.

11.4-1 Show a canonical, a cascade and a parallel realization of the following transfer functions:

(a) $H[z] = \dfrac{z(3z-1.8)}{z^2-z+0.16}$

(b) $H[z] = \dfrac{5z+2.2}{z^2+z+0.16}$

(c) $H[z] = \dfrac{3.8z-1.1}{(z-0.2)(z^2-0.6z+0.25)}$

11.4-2 Give cascade and parallel realizations of the following transfer functions:

(a) $\dfrac{z(1.6z-1.8)}{(z-0.2)(z^2+z+0.5)}$ (b) $\dfrac{z(2z^2+1.3z+0.96)}{(z+0.5)(z-0.4)^2}$

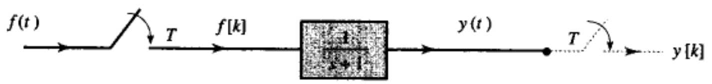


Fig. P11.6-1.

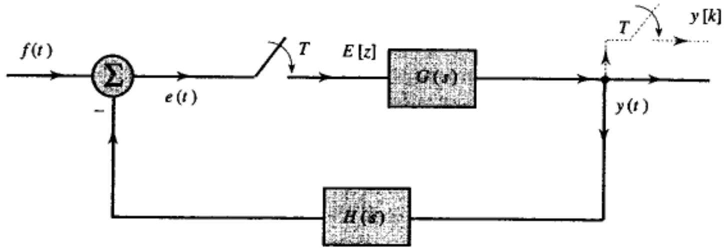


Fig. P11.6-2.

11.4-3 Realize a system whose transfer function is

$$H[z] = \frac{2z^4 + z^3 + 0.8z^2 + 2z + 8}{z^4}$$

11.4-4 Realize a system whose transfer function is given by

$$H[z] = \sum_{k=0}^{6} kz^{-k}$$

11.6-1 Determine $y[k]$, the output samples for the system depicted in Fig. P11.6-1 if the input $f(t) = e^{-2t}u(t)$.

11.6-2 For the sampled-data system in Fig. P11.6-2, show that the z-transfer function is

$$T[z] = \frac{G[z]}{1 + GH[z]}$$

where the transfer function $GH[z]$ corresponds to $G(s)H(s)$ in Table 12.1.

11.6-3 For the sampled-data system in Fig. P11.6-3, show that the output $Y[z]$ is given by

$$Y[z] = \frac{FG[z]}{1 + GH[z]}$$

Note that $FG[z]$ corresponds to the entry for $F(s)G(s)$ in Table 12.1. It is not the same as $F[z]G[z]$. In this case, it is not possible to express the output as $Y[z] = T[z]F[z]$. Consequently, the z-transfer function of such systems does not exist, and their analysis is little more complicated.

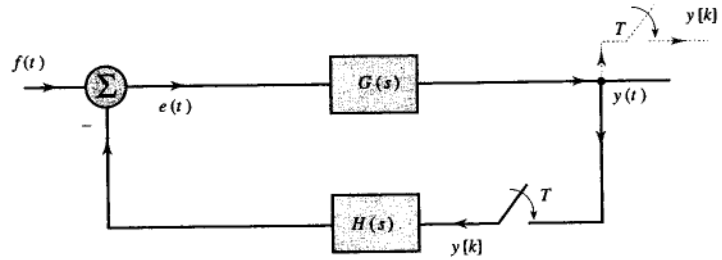


Fig. P11.6-3.

11.7-1 Find the z-transform (if it exists) and the corresponding region of convergence for each of the following signals:

(a) $(0.8)^k u[k] + 2^k u[-(k+1)]$

(b) $2^k u[k] - 3^k u[-(k+1)]$

(c) $(0.8)^k u[k] + (0.9)^k u[-(k+1)]$

(d) $\left[(0.8)^k + 3(0.4)^k\right] u[-(k+1)]$

(e) $\left[(0.8)^k + 3(0.4)^k\right] u[k]$

(f) $(0.8)^k u[k] + 3(0.4)^k u[-(k+1)]$

11.7-2 Find the inverse z-transform of

$$F[z] = \frac{(e^{-2} - 2)z}{(z - e^{-2})(z - 2)}$$

when the region of convergence is

(a) $|z| > 2$ **(b)** $e^{-2} < |z| < 2$ **(c)** $|z| < e^{-2}$.

11.7-3 Determine the zero-state response of a system having a transfer function

$$H[z] = \frac{z}{(z + 0.2)(z - 0.8)} \qquad |z| > 0.8$$

and an input $f[k]$ given by

(a) $f[k] = e^k u[k]$

(b) $f[k] = 2^k u[-(k+1)]$

(c) $f[k] = e^k u[k] + 2^k u[-(k+1)]$

11.7-4 For the system in Problem 11.7-3, determine the zero-state response if the input

$$f[k] = 2^k u[k] + u[-(k+1)]$$

11.7-5 For the system in Problem 11.7-3, determine the zero-state response if the input

$$f[k] = e^{-2k}[-(k+1)]$$