

QUESTIONS :-

(Applied Maths - II)

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(UNIT - 1)

{Complex Analysis - I}

Q.1)Find $|z|$ and $\arg(z)$ when,

$$z = \frac{(2-3i)(1+i)}{(2+i)}$$

Sol:

$$z = \frac{-5i-1}{2+i} \times \frac{2-i}{2-i} = \frac{(1+5i)(i-2)}{5} = -\frac{7}{5} - \frac{9}{5}i$$

So,

$$|z| = \sqrt{\left(\frac{-7}{5}\right)^2 + \left(\frac{-9}{5}\right)^2} = \sqrt{\frac{130}{25}} = \frac{\sqrt{130}}{5}$$

and,

$$\arg(z) = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{-9}{8} \times \frac{-8}{7} \right|$$

For 3rd Quadrant,

$$\Rightarrow \arg(z) = -(\pi - \theta)$$

$$\therefore \arg(z) = -\left(\pi - \tan^{-1}\left(\frac{9}{7}\right)\right).$$

Q.2)Find $\arg(z)$ if, $z = \frac{(3+4i)(2-i)}{(2+3i)^2}$ Sol:

Concept same as above,

$$\Rightarrow z = \frac{-5}{169} (-2+29i) = \frac{10}{169} - \frac{145}{169}i$$

∴ It lies in Quadrant IV,

$$\arg(z) = -\theta.$$

So,

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-145}{169} \times \frac{169}{10}\right) = \tan^{-1}\left(\frac{-145}{10}\right).$$

$$\text{Therefore, } \arg(z) = \tan^{-1}\left(\frac{145}{10}\right).$$

Q.3)

Prove that,

$$\frac{(\cos\theta + i \sin\theta)^4}{(\sin\theta + i \cos\theta)^5} = \sin 9\theta - i \cos 9\theta.$$

Sol:

On LHS,

$$\Rightarrow (\cos\theta + i \sin\theta)^4$$

$$[\cos(90-\theta) + i \sin(90-\theta)]^5$$

$$\Rightarrow (\cos\theta + i \sin\theta)^4 \cdot [\cos(90-\theta) + i \sin(90-\theta)]$$

$$\Rightarrow (\cos 4\theta + i \sin 4\theta) \cdot (\cos(450-5\theta) - i \sin(450-5\theta))$$

$$\Rightarrow (\cos 4\theta + i \sin 4\theta) \cdot (\sin 5\theta - i \cos 5\theta)$$

$$\Rightarrow \sin 5\theta \cdot \cos 4\theta + \sin 4\theta \cdot \cos 5\theta - \cos 4\theta \cdot \cos 5\theta i + \sin 4\theta \cdot \sin 5\theta i$$

$$\Rightarrow \sin(5\theta + 4\theta) + i[-(\cos 4\theta \cdot \cos 5\theta - \sin 4\theta \cdot \sin 5\theta)]$$

$$\Rightarrow \sin 9\theta - i \cdot \cos(4\theta + 5\theta)$$

$$\Rightarrow \sin 9\theta - i \cos 9\theta = \text{RHS.}$$

Q.4)

If $x + \frac{1}{x} = 2 \cos\theta$, then prove that,

$$\frac{x^n + 1}{x^n} = 2 \cos n\theta.$$

Sol:

$$\Rightarrow x^2 + 1 = 2 \cos\theta \cdot x.$$

$$\Rightarrow x^2 - 2 \cos\theta \cdot x + 1 = 0.$$

$$\Rightarrow x = \frac{2 \cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} = \cos\theta \pm i \sin\theta.$$

$$\Rightarrow x^n = (\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta.$$

So,

$$\Rightarrow x^n + 1 = (\cos n\theta \pm i\sin n\theta) + (\cos n\theta \mp i\sin n\theta)$$

$$= 2\cos n\theta \pm i\sin n\theta \mp i\sin n\theta$$

$$\therefore x^n + x^{-n} = 2\cos n\theta = \text{RHS}.$$

Q.5) Prove that,

$$(1+i)^n + (1-i)^n = 2^{\left(\frac{n}{2}+1\right)} \cdot \cos \frac{n\pi}{4}$$

Sol: (Use De - Moivre's Theorem on LHS...).

Q.6) If ω is a complex fourth root of unity, show that

$$1 + \omega + \omega^2 + \omega^3 = 0.$$

Sol: Let, $\Rightarrow x^4 = 1 \Rightarrow x = (1)^{1/4} = (\cos 0 + i\sin 0)^{1/4}$.
 Generalising, since x has four roots,
 $x = \cos \frac{2k\pi}{4} + i\sin \frac{2k\pi}{4} \quad (k \in \mathbb{Z}).$

For $K=0$,

$$\Rightarrow x_0 = \cos 0 + i\sin 0 = 1.$$

For $K=1$,

$$\Rightarrow x_1 = \cos \frac{2\pi}{4} + i\sin \frac{2\pi}{4} = \omega = 0 + ix = i.$$

For $K=2$,

$$\Rightarrow x_2 = \cos \frac{2(2\pi)}{4} + i\sin \frac{2(2\pi)}{4} = \left(\cos 2\pi + i\sin 2\pi\right)^2$$

$$\therefore x_2 = \omega^2 = (0+i)^2 = i^2 = -1.$$

$$\Rightarrow x_3 = \cos 3\left(\frac{2\pi}{4}\right) + i \sin 3\left(\frac{2\pi}{4}\right)$$

$$= \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4}\right)^3 = \omega^3 = -i$$

On LHS,

$$\Rightarrow 1 + \omega + \omega^2 + \omega^3$$

$$= x + x - x - i = 0 = \text{RHS}.$$

Q.7 Solve, $x^4 + x^3 + x^2 + x + 1 = 0$

Sol: * Multiply both sides by $(x-1)$,

$$\Rightarrow (x-1)(x^4 + x^3 + x^2 + x + 1) = 0$$

$$\Rightarrow x^5 + x^4 + x^3 + x^2 + x - x^4 - x^3 - x^2 - x - 1 = 0$$

$$\Rightarrow x^5 - 1 = 0$$

$$\Rightarrow x^5 = 1 \Rightarrow x = (1)^{1/5}$$

Or,

$$\Rightarrow x = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}$$

For $k=0$,

$$\Rightarrow x_0 = \cos 0 + i \sin 0 = 1$$

For $k=1$,

$$\Rightarrow x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \alpha$$

For $k=2$,

$$\Rightarrow x_2 = \cos \frac{2(2\pi)}{5} + i \sin \frac{2(2\pi)}{5} = \alpha^2$$

For $K=3$,

$$\Rightarrow z_3 = \cos 3\left(\frac{2\pi}{5}\right) + i \sin 3\left(\frac{2\pi}{5}\right) = \alpha^3$$

For $K=4$,

$$\Rightarrow z_4 = \cos 4\left(\frac{2\pi}{5}\right) + i \sin 4\left(\frac{2\pi}{5}\right) = \alpha^4$$

Here,

$$\Rightarrow \alpha = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \frac{(-1+\sqrt{5})}{4} + i \frac{(10+2\sqrt{5})}{4}$$

Similarly; α^2 , α^3 and α^4 can be calculated;

$$\therefore z = 1, \alpha, \alpha^2, \alpha^3, \alpha^4$$

Q.8) Solve, $z^6 + z^4 - z^2 - 1 = 0$.

$$\text{Sol: } z^6 + z^4 - (z^2 + 1) = 0$$

$$\Rightarrow z^4(z^2 + 1) - (z^2 + 1) = 0$$

$$\Rightarrow (z^2 + 1)(z^4 - 1) = 0$$

i.e.,

$$\Rightarrow z^2 + 1 = 0 \Rightarrow z = (-1)^{1/2} = [\cos \pi + i \sin \pi]^{1/2}$$

$$\Rightarrow z = \left(\cos \left(\frac{2K\pi + \pi}{2} \right) + i \sin \left(\frac{2K\pi + \pi}{2} \right) \right).$$

For $K=0$,

$$\Rightarrow z_1 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i$$

For $K=1$,

$$\Rightarrow z_2 = \cos \left(\frac{\pi + \pi}{2} \right) + i \sin \left(\frac{\pi + \pi}{2} \right) = -i$$

And,

$$\Rightarrow z^4 = 1 \Rightarrow z = (1)^{1/4} = (\cos 0 + i \sin 0)^{1/4}$$

$$\Rightarrow Z = \cos \frac{2K\pi}{4} + i \sin \frac{2K\pi}{4} \quad (K \in \mathbb{I})$$

For $K=0, 1, 2, 3$,

$$\Rightarrow Z_3 = \cos 0 + i \sin 0 = 1.$$

$$\Rightarrow Z_4 = \cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} = i$$

$$\Rightarrow Z_5 = \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right)^2 = i^2 = -1.$$

$$\Rightarrow Z_6 = \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right)^3 = i^3 = -i.$$

\therefore All six roots of this equation are, (Distinct Roots)

$$Z = 1, i, -i, -1, -i, i \rightarrow \{1, -1, i, -i\}$$

Q.9) a) Represent $\frac{1+i}{1-i}$ in the form of $r(\cos\theta + i \sin\theta)$?

$$\text{Sol: } Z = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = 0+i.$$

$$\text{So, } r = \sqrt{(0)^2 + (1)^2} = 1$$

$$\Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(\frac{1}{0} \right) = \frac{\pi}{2}$$

In Polar form,

$$\Rightarrow Z = r(\cos\theta + i \sin\theta)$$

$$\therefore Z = 1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

b) If $Z = 1-i$, Analytically find Z^8 ?

Sol: $r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$.

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left|\frac{-1}{1}\right| = \frac{\pi}{4}.$$

$\therefore Z$ lies in Quadrant IV, then
 $\arg(Z) = -\theta = -\frac{\pi}{4}$.

$$\therefore Z = \sqrt{2} \left[\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right]$$

$$\Rightarrow Z^8 = (\sqrt{2})^8 \left(\cos \frac{8\pi}{4} - i \sin \frac{8\pi}{4} \right) = 16 (1-0) = \boxed{16}.$$

c) Express $-1 + \sqrt{-3}$ in polar form?

Sol: $Z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$

So,

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\arg(Z) = \pi - \theta = \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Hence,

$$\Rightarrow Z = 2 \left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right]$$

(Q.10) Test the analyticity of function $w = \sin z$ and hence, derive that,

$$\frac{d}{dz}(\sin z) = \cos z.$$

Sol: Given function will be analytic if it satisfies C-R equation

$$\Rightarrow w = \sin z = \sin(x+iy)$$

or,

$$\Rightarrow w = \sin x \cdot \cos iy + \sin iy \cdot \cos x.$$

Note : $\sin ix = i \sinh x$; $\cos ix = \cosh x$

$$\Rightarrow w = \sin x (\cosh y) + (i \sinh y) \cdot \cos x.$$

or,

$$\Rightarrow u + iv = (\sin x, \cosh y) + i(\sinh y, \cos x)$$

So,

$$u(x, y) = \sin x \cdot \cosh y.$$

$$v(x, y) = \cos x \cdot \sinh y.$$

Now,

$$\Rightarrow \frac{\partial u}{\partial x} = \cos x \cdot \cosh y, \quad \frac{\partial v}{\partial y} = \cos x \cdot \cosh y.$$

$$\Rightarrow \frac{\partial u}{\partial y} = \sin x \cdot \sinh y, \quad \frac{\partial v}{\partial x} = -\sin x \cdot \sinh y.$$

Since,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, function $f(z)$ is analytic.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

$$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y.$$

$$= \cos x \cdot \cos iy - i \sin x \cdot \sin iy \quad (\text{using above properties})$$

$$= \cos x \cdot \cos iy - \sin x \cdot \sin iy.$$

$$= \cos(x+iy) \quad (\text{from trigonometry})$$

$$\Rightarrow f'(z) = \cos z.$$

Therefore, $\boxed{\frac{d}{dz}(\sin z) = \cos z}$. Hence proved.

Q.11) Show that the function $f(z) = z|z|$ is not analytic?

Sol: Given, $f(z) = z|z|$

If, $z = x + iy$ then, $|z| = \sqrt{x^2 + y^2}$.

$$\therefore f(z) = w = (x + iy)\sqrt{x^2 + y^2} = x\sqrt{x^2 + y^2} + y\sqrt{x^2 + y^2}i$$

i.e.,

$$u = x\sqrt{x^2 + y^2}, \quad v = y\sqrt{x^2 + y^2}$$

For C-R equations,

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\sqrt{x^2 + y^2} + x \cdot 1}{2\sqrt{x^2 + y^2}} \quad (\partial x) = \frac{\sqrt{x^2 + y^2} + x^2}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{x \cdot 1}{2\sqrt{x^2 + y^2}} \quad (\partial y) = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{y \cdot 1}{2\sqrt{x^2 + y^2}} \quad (\partial x) = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\sqrt{x^2 + y^2} + y^2}{\sqrt{x^2 + y^2}}$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{and}, \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \quad \left. \begin{array}{l} \text{C-R conditions.} \\ \hline \end{array} \right\}$$

Therefore,

given function $f(z) = z|z|$ is not analytic.

Q.12) Find the value of C_1 and C_2 such that the function,

$$f(z) = x^2 + C_1 y^2 - 2xy + i(C_2 x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$?

Sol:

Given,

$$u = x^2 + C_1 y^2 - 2xy; \quad v = C_2 x^2 - y^2 + 2xy.$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x - 2y, \quad \Rightarrow \frac{\partial v}{\partial x} = 2C_2 x + 2y$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2C_1 y - 2x, \quad \Rightarrow \frac{\partial v}{\partial y} = -2y + 2x.$$

From C-R equations,

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow 2x - 2y = 2x - 2y \quad (\text{LHS} = \text{RHS})$$

and,

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow 2C_1 y - 2x = -(2C_2 x + 2y)$$

$$\Rightarrow C_1 y - x = -(C_2 x + y) = -C_2 x - y.$$

or,

$$\Rightarrow C_1 y + C_2 x = x - y.$$

$$\Rightarrow C_1 y + y + C_2 x - x = 0$$

or,

$$\Rightarrow x(C_2 - 1) + y(C_1 + 1) = 0 = x(0) + y(0).$$

On comparing,

$$\Rightarrow C_2 - 1 = 0 \Rightarrow C_2 = 1$$

$$\Rightarrow C_1 + 1 = 0 \Rightarrow C_1 = -1$$

$$\therefore f(z) = x^2 + y^2 - 2xy + i(2xy - x^2 - y^2)$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= (2x - 2y) + i(2y - 2x)$$

$$= 2(x-y) - 2(x-y)i$$

or,

$$f'(z) = 2(x-y)(1-i)$$

Q.13) Show that the function $f(z) = \log z$ where, $z = x+iy$ and $z \neq 0$, is a analytic function? Also find $f'(z)$.

Sol: $z = x+iy$.

$$\Rightarrow f(z) = \log z = \log(x+iy)$$

or,

$$\Rightarrow u+iv = \frac{1}{2} \log(x^2+y^2) + i \tan^{-1}\left(\frac{y}{x}\right) \quad (\text{Direct Formula})$$

On comparing,

$$u = \frac{1}{2} \log(x^2+y^2), \quad v = \tan^{-1}\left(\frac{y}{x}\right).$$

$$\Rightarrow \frac{\partial u}{\partial x} = u_x = \frac{1}{2(x^2+y^2)} \times (2x) = \frac{x}{x^2+y^2}.$$

$$\Rightarrow \frac{\partial u}{\partial y} = u_y = \frac{1}{2(x^2+y^2)} \times (2y) = \frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial v}{\partial x} = v_x = \frac{x^2}{x^2+y^2} \times y \times -\frac{1}{x^2} = \frac{-y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial v}{\partial y} = v_y = \frac{x^2}{x^2+y^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

From C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

\therefore Both equation's hold true for the function $f(z)$. So, $f(z)$ is a analytic function.

Now,

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = \frac{x}{x^2+y^2} + i \left(\frac{-y}{x^2+y^2} \right)$$

Therefore,

$$\Rightarrow f'(z) = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$$

Q.14) Verify if the function $f(z)$ is analytic or not at $f(0)=0$ for $z \neq 0$.

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4} \quad (z \neq 0)$$

[IMPORTANT]

Sol: $f(z) = \frac{xy^2(x+iy)}{x^2+y^4} = \frac{x^2y^2}{x^2+y^4} + i \frac{xy^3}{x^2+y^4}$

So,

$$u(x, y) = \frac{x^2y^2}{x^2+y^4}, \quad v(x, y) = \frac{xy^3}{x^2+y^4}$$

\therefore Both functions $u(x, y)$ and $v(x, y)$ are implicit functions and, checking at origin $(0, 0)$ first....

For,

$f(0)=0$; from first derivative principle,

$$\Rightarrow \frac{\partial u}{\partial x} = \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h}$$

$$\Rightarrow \left. \frac{\partial u}{\partial x} \right|_{x=0} = \lim_{h \rightarrow 0} \frac{u(0+h, y) - u(0, y)}{h}$$

Similarly at $(0, 0) \rightarrow (x, y) = (0, 0)$;

$$\Rightarrow \left. \frac{\partial u}{\partial x} \right|_{(0,0)} = \lim_{h \rightarrow 0} \frac{u(0+h, 0) - u(0, 0)}{h}$$

$$\Rightarrow u(0+h, 0) = \lim_{h \rightarrow 0} \frac{(0+h)^2 \times (0)^2}{(0+h)^2 + (0)^4} = 0$$

also,

$$\Rightarrow u(0,0) = \frac{0^2 \times 0^2}{0^2 + 0^4} = 0.$$

$$\therefore \frac{\partial u}{\partial x} = \lim_{h \rightarrow 0} \frac{u(0+h, 0) - u(0,0)}{h} = 0.$$

Now,

$$\Rightarrow \frac{\partial u}{\partial y} = \lim_{k \rightarrow 0} \frac{u(x, y+k) - u(x, y)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{u(x, 0+k) - u(x, 0)}{k} \quad (x, y = 0, 0)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{0 - 0}{k} = 0.$$

Similarly, $\frac{\partial v}{\partial x} = 0$ and, $\frac{\partial v}{\partial y} = 0$

\therefore Function $f(z)$ is analytic at $f(0) = 0$ and holds C-R equations at origin.

Now,

$$\Rightarrow f'(z) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z}$$

$$\Rightarrow f'(z) = \lim_{z \rightarrow 0} \frac{xy^2(x+iy) - 0}{x^2 + y^4} \quad (\because z = x+iy).$$

* Concept used here : We will substitute y by (mx) , i.e., as $y = mx$ (straight line). If, $f'(z)$ is independent of the term ' m ' ($m \rightarrow$ slope) then we can say $f'(z)$ is independent of the path and has a one unique value resulting, $f(z)$ being a ANALYTIC function.

But, if $f'(z)$ is dependent of ' m ' and is not independent then, $f(z)$ is not a ANALYTIC function.

Using general curve,
 $y = mx$ (Straight line at origin)

So,

$$z = x + iy = x + i(mx) = x(1 + im).$$

If,

$$\lim_{z \rightarrow 0} f(z) \Rightarrow z \rightarrow 0 \text{ then } x \rightarrow 0.$$

Therefore,

$$\Rightarrow f'(z) = \lim_{x \rightarrow 0} \frac{x(mx)^2(x + i(mx))}{x^2 + (mx)^4} - 0$$

$$\Rightarrow f'(z) = \lim_{x \rightarrow 0} \frac{x^3 m^2 \cdot x(1+im)}{x^2 + x^4 m^4}$$

$$\Rightarrow f'(z) = \lim_{x \rightarrow 0} \frac{x^{5/2} m^2}{x^2(1+x^2 m^4)} = \lim_{x \rightarrow 0} \frac{x m^2}{1+x^2 m^4}$$

$$\Rightarrow f'(z) = \frac{0 \times m^2}{1+0^2 \times m^4} = \frac{0}{1+0} = 0 \quad \underline{\underline{=}}$$

$\therefore f'(z)$ has a unique value = 0, i.e., it is independent of the term 'm'.

Therefore,

$f(z)$ is A ANALYTIC function.

Q.15)

Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at origin, although C-R Equations are satisfied?

Sol:

Given function $f(z)$ is not a single-value function which already fails the first condition of analyticity.

$$\Rightarrow f(z) = \sqrt{|xy|} \Rightarrow u + iv = \sqrt{|xy|} + i(0)$$

So,

$$u = \sqrt{|xy|} \text{ and, } v = 0$$

At origin, C-R equations are satisfied.

But,

$$\Rightarrow f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} \quad (\text{from first principle})$$

$$\Rightarrow f'(0) = \lim_{z \rightarrow 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

For a general curve,

$$z \rightarrow 0, x \rightarrow 0 \text{ for } z = x + iy = x(1+im)$$

$$\Rightarrow f'(0) = \lim_{x \rightarrow 0} \frac{\sqrt{|x \cdot mx|} - 0}{x + i(mx)}$$

$$\therefore f'(0) = \lim_{x \rightarrow 0} \frac{x\sqrt{1m^2}}{x(1+im)} = \frac{\sqrt{m}}{1+im} \quad (m > 0)$$

\therefore Function $f(z)$ is NOT analytic, since $f'(0)$ is dependent on 'm'

Q.16) Show that an analytic function with constant modulus is constant?

Sol: Let, $f(z) = u + iv$ be an analytic function.

$$\Rightarrow |f(z)| = |u + iv| = \sqrt{u^2 + v^2}$$

or,

$$\Rightarrow C = u^2 + v^2 \quad \text{--- (1)} \quad (C: \text{constant})$$

Given,

$f(z)$ is analytic, i.e., it satisfies C-R equations,

$$\rightarrow \text{Given equation: } u^2 + v^2 = 0^2 \quad \text{--- (1)}$$

• Partially differentiating eq - (1) w.r.t x and y ,

$$\Rightarrow 2u \cdot \frac{\partial u}{\partial x} + 2v \cdot \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} = 0 \quad \text{--- (2)}$$

and,

$$\Rightarrow u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial v}{\partial y} = 0 \quad \text{--- (3)}$$

\rightarrow From equation (2),

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} + v \left(-\frac{\partial u}{\partial y} \right) = 0.$$

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} - v \cdot \frac{\partial u}{\partial y} = 0$$

\rightarrow From equation (3),

$$\Rightarrow u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial u}{\partial x} = 0$$

• Eliminating $\frac{\partial u}{\partial y}$,

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} - v \left(-v \cdot \frac{\partial u}{\partial x} \right) = 0.$$

$$\Rightarrow u \cdot \frac{\partial u}{\partial x} + v^2 \cdot \frac{\partial u}{\partial x} = 0.$$

$$\Rightarrow \left(\frac{u^2 + v^2}{u} \right) \frac{\partial u}{\partial x} = 0$$

$$\therefore \frac{\partial u}{\partial x} = 0$$

On integration, $u = \text{Constant}$

From equation (2),
 $\rightarrow u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} = 0$.

$\rightarrow u \cdot \frac{\partial v}{\partial y} + v \cdot \frac{\partial v}{\partial x} = 0 \quad \left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right)$.

similarly,
 $\rightarrow u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial v}{\partial y} = 0 \quad (\text{equation - (3)})$

$\rightarrow u \cdot \left(-\frac{\partial v}{\partial x} \right) + v \cdot \frac{\partial v}{\partial y} = 0 \quad \left(\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$.

$\rightarrow -u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} = 0$.

- Eliminating $\frac{\partial v}{\partial x}$ from both equations,

$$\Rightarrow u \cdot \frac{\partial v}{\partial y} + v \left(\frac{v \cdot \frac{\partial v}{\partial y}}{u} \right) = 0$$

$$\Rightarrow u \cdot \frac{\partial v}{\partial y} + v^2 \cdot \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \left(\frac{u^2 + v^2}{u} \right) \cdot \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = 0$$

On integration,

$$V = \text{Constant}$$

$$\therefore f(z) = u + iv = \text{Constant} + i(\text{Constant}) = K = \text{Constant}$$

(Q.17) Show that the function $u = \log \sqrt{x^2 + y^2}$ is harmonic.
 Also, find its harmonic conjugate?

Sol: $u = \frac{1}{2} \log(x^2 + y^2)^2 = \frac{1}{2} \log(x^2 + y^2)$

We know that,

a. given function will be harmonic if it satisfies the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

So,

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{+y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

\therefore Given function $u(x, y)$ is a harmonic function. (Laplace equation is satisfied)

Here, $v = ?$

$$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$$

Using C-R equations,

$$\Rightarrow dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy.$$

$$\Rightarrow dv = \left(-\frac{y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$$

\therefore It is a form of exact differential equation and is exact D.E. so,

$$\Rightarrow u(x, y) = \int M(x, y) dx, \text{ where } M = -\frac{y}{x^2+y^2}$$

$$\Rightarrow U(x, y) = \int -y \frac{dx}{x^2 + y^2} = -y \int \frac{dx}{x^2 + y^2}$$

$$\Rightarrow U(x, y) = -y \times \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) + C.$$

$$\therefore U(x, y) = -\tan^{-1}\left(\frac{x}{y}\right) + C.$$

also,

$$\Rightarrow F(y) = \int N(y) \cdot dy \quad [\text{Here, } N(y) \text{ should be independent term of } x]$$

$$\therefore F(y) = \int 0 \cdot dy = 0.$$

Hence,

$$\Rightarrow V = U(x, y) + F(y)$$

$$\Rightarrow V = -\tan^{-1}\left(\frac{x}{y}\right) + 0 + C.$$

$$\therefore V = -\tan^{-1}\left(\frac{x}{y}\right) + C$$

$$\therefore f(z) = u + iv$$

$$\Rightarrow f(z) = \frac{1}{2} \log(x^2 + y^2) - i \tan^{-1}\left(\frac{x}{y}\right) + C \quad \text{Final Answer}$$

Q.18) Show that the function $u(x, y) = x^3 - 3xy^2$ is a harmonic function. Find the corresponding analytic function $f(z)$. Also express $f(z)$ in terms of variable 'z' only?

Sol: $u(x, y) = x^3 - 3xy^2.$

So,

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x.$$

$$\Rightarrow \frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + (-6x) = 0$$

$\therefore u(x, y)$ satisfies the Laplace equation, i.e., the function is harmonic.

Now, $v = ?$

$$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow dv = \left(-\frac{\partial u}{\partial y} \right) dx + \left(\frac{\partial u}{\partial x} \right) dy$$

$$\Rightarrow dv = 6xy \cdot dx + 3(x^2 - y^2) dy \text{ (exact D.E.)}$$

$$M(x, y) = 6xy ; N(y) = -3y^2$$

$$\Rightarrow u(x, y) = \int M(x, y) \cdot dx$$

$$\Rightarrow u(x, y) = \int 6xy \cdot dx = 6y \int x \cdot dx = 3yx^2 + C$$

$$\Rightarrow F(y) = \int N(y) \cdot dy = -3 \int y^2 \cdot dy = -y^3$$

Hence,

$$\Rightarrow u(x, y) + F(y) = v(x, y)$$

$$\therefore v(x, y) = 3yx^2 - y^3 + C$$

$$\therefore f(z) = u + iv$$

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) + C$$

(Analytic function)

From direct identity,

$$\Rightarrow f(z) = (x + iy)^3 = z^3 \quad ((a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2)$$

Q.19)

Construct the analytic function $f(z)$ of which the real part is $e^x \cos y$?

Sol:

Here, $u = e^x \cos y$.

So,

$$\Rightarrow \frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \phi_1(x, y) = e^x \cos y$$

$$\text{and, } \Rightarrow \phi_1(z, 0) = e^z \cdot \cos 0 = e^z.$$

$$\Rightarrow \frac{\partial u}{\partial y} = -e^x \sin y \Rightarrow \phi_2(x, y) = -e^x \sin y$$

$$\Rightarrow \phi_2(z, 0) = -e^z \cdot \sin 0 = 0.$$

$\therefore u(x, y)$ is given,

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz + C.$$

$$f(z) = \int (e^z - 0) dz + C$$

$$\therefore f(z) = e^z + C. \quad (C: \text{constant}).$$

Q.20)

Construct the analytic function whose imaginary part is,

$$v = \log(x^2 + y^2) + x - 2y.$$

Sol:

Here, $v = \log(x^2 + y^2) + x - 2y$.

So,

$$\Rightarrow \psi_1 = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \{ \log(x^2 + y^2) + x - 2y \}$$

$$\Rightarrow \psi_1(x, y) = \frac{\partial y}{x^2 + y^2} - 2$$

$$\Rightarrow \psi_1(z, 0) = 0 - 2 = -2.$$

$$\Rightarrow \Psi_2 = \frac{\partial V}{\partial x} = \frac{2x}{x^2+y^2} + 1.$$

$$\Rightarrow \Psi_2(x, y) = \frac{2x}{x^2+y^2} + 1.$$

$$\Rightarrow \Psi_2(z, 0) = \frac{2z}{z^2+0} + 1 = \frac{2z}{z^2} + 1 = \frac{2}{z} + 1$$

$$\therefore f(z) = \int [\Psi_1(z, 0) + i\Psi_2(z, 0)] dz + C.$$

$$= \int (-2 + i(2z)) dz + C$$

$$\Rightarrow f(z) = -2z + i(2 \log z + z) + C.$$

Q.21) If $f(z) = u + iv$ is analytic function of z in any domain, then show that,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] |f(z)|^2 = 4 |f'(z)|^2.$$

Sol: Here, $f(z) = u + iv$

$$\Rightarrow |f(z)| = (u^2 + v^2)^{1/2}.$$

or,

$$\Rightarrow |f(z)|^2 = u^2 + v^2 = \phi$$

On LHS,

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

here,

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} (u^2 + v^2)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2 \left[u \cdot \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 \right] + v \cdot \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x} \right)^2$$

Again,

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2 \left[u \cdot \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y} \right)^2 \right] + v \cdot \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y} \right)^2$$

Now LHS,

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$= 2 \left[u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2$$

$\because f(z)$ is Analytic, so $f'(z)$ is also harmonic and it satisfies the Laplace equation. So,

$$= 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right]$$

$$\left(\because \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right)$$

- By C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$= 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right]$$

$$= 2 \left[2 \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right) \right] = 4 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right]$$

$$= 4 \cdot |f'(z)|^2 = \text{RHS.}$$

On RHS,

$$\Rightarrow f(z) = u + iv$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\Rightarrow |f'(z)| = \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right)^{1/2}$$

So,

$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2$$

Or,

$$\Rightarrow 4 |f'(z)|^2 = \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \times 4 = \text{RHS}$$

Q.22) If $f(z)$ is a harmonic function of z , show that

$$\left[\frac{\partial}{\partial x} |f(z)| \right]^2 + \left[\frac{\partial}{\partial y} |f(z)| \right]^2 = |f'(z)|^2$$

Sol:

$$\text{Let, } f(z) = u + iv$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

So,

$$\Rightarrow |f(z)| = (u^2 + v^2)^{1/2}$$

and,

$$\Rightarrow |f'(z)| = \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right)^{1/2}$$

or,

$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \text{RHS}$$

Now on LHS,

$$\Rightarrow \frac{\partial}{\partial x} \left[(u^2 + v^2)^{1/2} \right] = \frac{\partial}{\partial x} (\sqrt{\phi}) \quad [\phi = u^2 + v^2]$$

$$= \frac{1}{2\sqrt{\phi}} \cdot \frac{\partial \phi}{\partial x} = \frac{1}{2\sqrt{\phi}} \left[u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} \right] \\ = \frac{1}{\sqrt{\phi}} \left(u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} \right) \quad \text{--- (1)}$$

and,

$$= \frac{\partial}{\partial y} (\sqrt{\phi}) = \frac{1}{\sqrt{\phi}} \left(u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial v}{\partial y} \right) \quad \text{--- (2)}$$

→ From (1) and (2),

$$= \frac{1}{\phi} \left\{ \left[u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} \right]^2 + \left[u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial v}{\partial y} \right]^2 \right\}.$$

→ From C-R equations,

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$= \frac{1}{\phi} \left[\left[u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial x} \right]^2 + \left[-u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial u}{\partial x} \right]^2 \right].$$

$$= \frac{1}{\phi} \left\{ u^2 \left(\frac{\partial u}{\partial x} \right)^2 + v^2 \left(\frac{\partial v}{\partial x} \right)^2 + u^2 \left(\frac{\partial v}{\partial x} \right)^2 + v^2 \left(\frac{\partial u}{\partial x} \right)^2 \right\}$$

$$= \frac{1}{\phi} \left[(u^2 + v^2) \left(\frac{\partial u}{\partial x} \right)^2 + (u^2 + v^2) \left(\frac{\partial v}{\partial x} \right)^2 \right]$$

$$= \frac{1}{u^2 + v^2} \times (u^2 + v^2) \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right).$$

$$= \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = \text{LHS}.$$

$\therefore \text{LHS} = \text{RHS}$ (hence proved).

Q23)

Find the value of z for which e^z is real?

Sol:

We know that,

$$\rightarrow e^z = e^{x+iy}$$

$$\Rightarrow e^z = e^x \cdot e^{iy}$$

$$\Rightarrow e^z = e^x [\cos y + i \sin y]$$

$$\Rightarrow e^z = e^x \cancel{e^{iy}}$$

$$\Rightarrow e^z = e^x (\cos y + i \sin y)$$

If e^z is real,

$$\Rightarrow e^x \sin y = 0$$

$$\Rightarrow \sin y = 0$$

$$\Rightarrow y = n\pi, n \in \mathbb{N} \text{ & } n \in \mathbb{Z}^-$$

$$\therefore z = x + i(n\pi) = [x + n\pi i]$$

Q24)

Show that, $\exp\left(\pm i\frac{\pi}{2}\right) = \pm i$

Sol:

We know that,

$$\exp\left(\pm i\frac{\pi}{2}\right) = \cos \frac{\pi}{2} \pm i \sin \frac{\pi}{2}$$

$$= (0 \pm i(1)) = \pm i$$

Q25)

Find all values of z which satisfies $e^z = 1+i$.

Sol:

$$e^z = 1+i$$

$$\Rightarrow \exp(x+iy) = 1+i$$

$$\Rightarrow \exp(x) \cdot \exp(iy) = 1+i$$

$$\Rightarrow e^x [\cos y + i \sin y] = 1+i$$

$$\Rightarrow e^x \cos y + i e^x \sin y = 1+i$$

So,

$$e^x \cos y = 1 \quad \text{--- (1)}, \quad e^x \sin y = 1 \quad \text{--- (2)}$$

→ On squaring ① and ② and adding,

$$\Rightarrow e^{2x} [\cos^2 y + \sin^2 y] = 2$$

$$\Rightarrow e^{2x} = 2 \Rightarrow 2x = \ln 2 \Rightarrow x = \frac{1}{2} \ln 2$$

→ Divide ①/②,

$$\Rightarrow \frac{e^x \sin y}{e^x \cos y} = \frac{1}{\sqrt{2}} \Rightarrow \tan y = \frac{1}{\sqrt{2}} \Rightarrow y = \frac{\pi}{4} + 2n\pi$$

$$\therefore z = x + iy = \frac{1}{2} \ln 2 \pm i \left(\frac{\pi}{4} + 2n\pi \right)$$

Q26) To separate the real and imaginary part of $\sin h(x+iy)$?

Sol: We know that,

$$\Rightarrow \sin ix = i \sinh x$$

$$\Rightarrow \sinh ix = i \sin ix$$

$$\Rightarrow \sin h(x+iy) = i \sin i(x+iy) = i \cdot \sin(ix-y)$$

$$= i x + i [\sin ix \cos y - \cos ix \sin y]$$

$$= \frac{i}{i^2} [i \sinh x \cos y - \cosh x \sin y]$$

$$= (-\sinh x \cos y - i \cosh x \sin y)$$

$$= \sinh x \cos y + i \cosh x \sin y$$

Real part

Imaginary part

(Q27) If $\tan(\theta + i\phi) = \sin(x+iy)$, then prove that
 $\cot hy \cdot \sinh 2\phi = \cot x \cdot \sin 2\theta$.

Sol:

Given,

$$\sin(x+iy) = \tan(\theta+i\phi).$$

$$\Rightarrow \sin x \cdot \cos iy + \cos x \cdot \sin iy = \frac{2\sin(\theta+i\phi)}{2\cos(\theta+i\phi)} \times \frac{\cos(\theta-i\phi)}{\cos(\theta-i\phi)}$$

$$\Rightarrow \sin x \cdot \cosh y + \cos x \cdot i \sinh y = \frac{\sin 2\theta + \sin 2i\phi}{\cos 2\theta + \cos 2i\phi}.$$

$$\Rightarrow \underbrace{\sin x \cdot \cosh y}_{R.P} + i \underbrace{\cos x \cdot \sinh y}_{I.P.} = \frac{\sin 2\theta}{\cos 2\theta + \cos 2i\phi} + \frac{\sin 2i\phi}{\cos 2\theta + \cos 2i\phi}$$

\rightarrow On comparing real and imaginary parts. I.M.

$$\Rightarrow \sin x \cdot \cosh y = \frac{\sin 2\theta}{\cos 2\theta + \cos 2i\phi} \quad \text{--- (1)}$$

$$\Rightarrow \cos x \cdot \sinh y = \frac{i \sin 2i\phi}{\cos 2\theta + \cos 2i\phi} = \frac{i \sin 2\theta}{\cos 2\theta + \cos 2i\phi} \quad \text{--- (2)}.$$

\Rightarrow Divide eq - (2) by eq - (1),

$$\Rightarrow \tan x \cdot \cot hy = \frac{\sin 2\theta}{\cos 2\theta + \cos 2i\phi} \cdot \frac{\cos 2\theta + \cos 2i\phi}{i \sin 2\theta} = \frac{\sin 2\theta}{i \sin 2\theta} = -i$$

$$\Rightarrow \sin 2i\phi \cdot \cot hy = \cot x \cdot \sin 2\theta.$$

or,

$$\Rightarrow [\cot hy \cdot \sinh 2\phi] = \cot x \cdot \sin 2\theta$$

Hence proved.

(Q28)

Separate into real and imaginary parts : $\log(3+4i)$

Sol:

We know that,

$$\log(x+iy) = \frac{1}{2} \log(x^2+y^2) + i \left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi \right].$$

$$= \frac{1}{2} \log(3^2+4^2) + i \left(\tan^{-1}\left(\frac{4}{3}\right) \right).$$

$$\therefore \log(x+iy) = \log 5 + i \left[\tan^{-1}\left(\frac{4}{3}\right) + 2n\pi \right]$$

• Real part = $\log 5$, Imaginary part = $\tan^{-1}\left(\frac{4}{3}\right) + 2n\pi$.

(Q29)

Show that,

$$\log(-\log i) = \log \frac{\pi}{2} - i \frac{\pi}{2}.$$

Sol:

On LHS,

$$\log i = \log(0+i) = \frac{1}{2} \log(0^2+1^2) + i \frac{\pi}{2} = i \frac{\pi}{2}$$

now,

$$\log\left(-\frac{i\pi}{2}\right) = \log\left(0 - i\frac{\pi}{2}\right)$$

$$= \frac{1}{2} \log\left[0^2 + \left(\frac{\pi}{2}\right)^2\right] + i\left(\frac{\pi}{2}\right)$$

$$= \left[\log \frac{\pi}{2} - i \frac{\pi}{2} \right] = RHS$$

(Q30)

Separate real and imaginary part of i^i ?

Sol:

$$i^i = e^{i \log i}$$

$$= e^{i \log i} = e^{i \left[\frac{0+ii\pi}{2} \right]} = e^{i \cdot \frac{\pi}{2}}$$

Hence, i^i is a real number.

Sol:

We know that,

$$\log(x+iy) = \frac{1}{2} \log(x^2+y^2) + i \left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi \right]$$

$$= \frac{1}{2} \log(3^2+4^2) + i \left[\tan^{-1}\left(\frac{4}{3}\right) \right]$$

$$\therefore \log(x+iy) = \log 5 + i \left[\tan^{-1}\left(\frac{4}{3}\right) + 2n\pi \right]$$

• Real part = $-\log 5$, Imaginary part = $\tan^{-1}\left(\frac{4}{3}\right) + 2n\pi$

(Q29)

Show that,

$$\log(-\log i) = \log \frac{\pi}{2} - i\frac{\pi}{2}$$

Sol:

On LHS,

$$\log i = \log(0+i) = \frac{1}{2} \log(0^2+1^2) + i\frac{\pi}{2} = i\frac{\pi}{2}$$

now,

$$\log\left(-\frac{i\pi}{2}\right) = \log\left(0 - i\frac{\pi}{2}\right)$$

$$= \log\left[\sqrt{0^2 + \left(\frac{\pi}{2}\right)^2}\right] + i\left(-\frac{\pi}{2}\right)$$

$$= \log\frac{\pi}{2} - i\frac{\pi}{2} = \text{RHS}$$

(Q30)

Separate real and imaginary part of i^i ?

Sol:

$$i^i = e^{i \log i}$$

$$= e^{i \log i} = e^{i [0+i\frac{\pi}{2}]} = e^{i\frac{\pi}{2}}$$

Hence, i^i is a real number.

Q31) Separate real and imaginary parts of :-

$$\log[(1+i)\log i]$$

Sol: $\log \left[\frac{(1+i)i\pi}{2} \right] = \log \left(\frac{i\pi - \pi}{2} \right)$

or,
 $\Rightarrow \log \left[\frac{-\pi + i\pi}{2} \right]$

$$R = -1 \log \left(\frac{\pi^2}{4} \right) + i \tan^{-1} \left[\tan \left(\frac{\pi}{4} \right) \right]$$

$$= \log \left(\frac{\pi}{\sqrt{2}} \right) + i \frac{3\pi}{4}$$

Q32) Show that, $\log(1+e^{i\theta}) = \log(2\cos\theta) + i\theta$

if $\theta \in (-\pi, \pi)$?

Sol: $\log(1+e^{i\theta}) = \log(1+\cos\theta + i\sin\theta)$

$$= \frac{1}{2} \log [1 + \cos^2\theta + 2\cos\theta + \sin^2\theta] + i \tan^{-1} \left(\frac{\sin\theta}{1+\cos\theta} \right)$$

$$= \frac{1}{2} \log (2(1+\cos\theta)) + i \tan^{-1} \left(\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} \right)$$

$$= \frac{1}{2} \log (2 \times 2\cos^2\theta) + i \tan^{-1} \left(\frac{1-\cos\theta}{\sin\theta} \right)$$

$$= \frac{1}{2} \log \left[\frac{2\cos\theta}{2} \right]^2 + i \tan^{-1} \left(\frac{\tan\theta}{2} \right)$$

$$= \log \left(\frac{2\cos\theta}{2} \right) + i\theta$$

Hence proved

Q33

Prove that,

$$\sin(\log i^i) = -1.$$

Sol:

$$\sin \left[\log \left(e^{-\pi/2} \right) \right] \quad \text{--- (se) } 1 - (\because i^i = e^{-\pi/2})$$

$$= \sin \left[-\frac{\pi}{2} \times \log e \right]$$

$$= \sin \left(-\frac{\pi}{2} \right) = -\frac{\sin \pi}{2} = -\underline{\underline{1}} = \text{RHS}$$

Q34)

Evaluate the integral, $I = \int_C \operatorname{Re}(z^2) dz$ from $z=0$ to $2+4i$ along the parabola $y=x^2$?

Sol:

$$z = x+iy \Rightarrow z^2 = (x+iy)^2$$

So,

$$\Rightarrow z^2 = (x^2 - y^2) + 2xyi$$

$$\Rightarrow \operatorname{Re}(z^2) = x^2 - y^2.$$

$$\Rightarrow I = \int_C \operatorname{Re}(z^2) dz = \int_C (x^2 - y^2) \cdot (dx + idy)$$

$$\Rightarrow I = \int_{x=0}^{x=2} (x^2 - x^4) (dx + i2x dx) \quad \begin{array}{l} \text{Im} \\ \uparrow \\ (2, 4) \\ \nearrow \\ z = 2+4i \\ \downarrow \\ (0, 0) \\ \text{Z=0} \end{array} \quad \begin{array}{l} \rightarrow \text{Re} \\ \swarrow \end{array}$$

$$\left(\because y = x^2 \Rightarrow dy = 2x dx \right)$$

$$\Rightarrow I = \int_0^2 (x^2 - x^4) (1 + 2xi) dx$$

$$\Rightarrow I = \int_0^2 (x^2 - 2x^3 i - x^4 - 2x^5 i) dx$$

i.e.,

$$\Rightarrow I = \left[\frac{x^3}{3} + 2ix^4 - \frac{x^5}{5} - 2ix^6 \right]_0^2$$

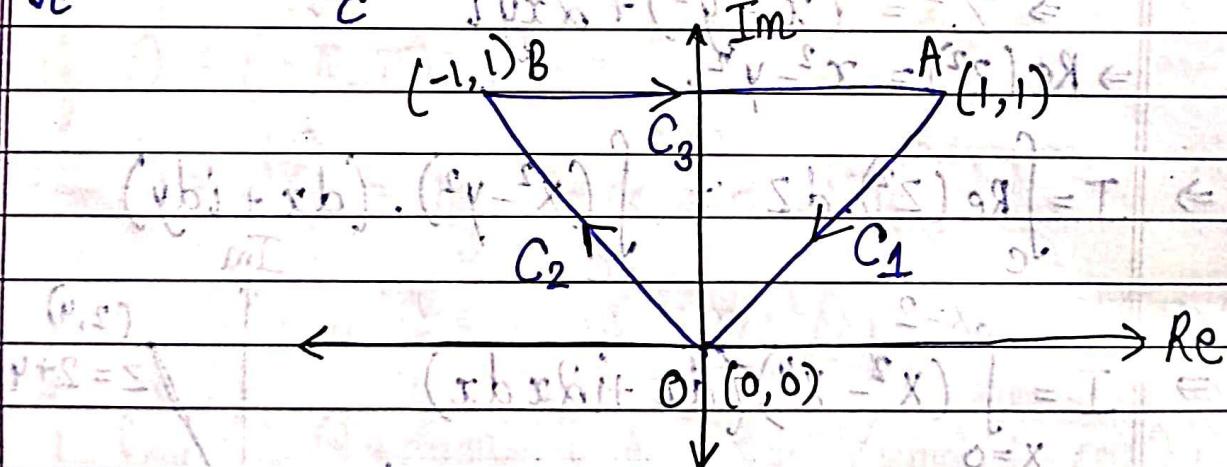
$$\Rightarrow I = \frac{8}{3} + i(8) - \frac{1}{5}(32) - i(64)$$

$$\therefore I = \left(\frac{8}{3} - \frac{32}{5} \right) + i \left(8 - \frac{64}{3} \right)$$

$$\therefore I = -\frac{56}{15} + \frac{-40i}{3}$$

Q35) Evaluate the line integral $\int_C z^2 \cdot dz$ where C is the boundary of a triangle with vertices $0, 1+i, -1+i$ clockwise.

Sol: $\int_C z^2 \cdot dz = \int_C (x^2 - y^2 + 2ixy)(dx + idy)$



$$\Rightarrow \int_C z^2 \cdot dz = \sum_{i=1}^3 \int_{C_i} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$\Rightarrow \int_{C_1} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_{x=1}^0 (x^2 - x^2 + 2x^2i)(dx + idx)$$

$$= \int_1^0 (2x^2 i)(1+i) dx = \int_1^0 (2x^2 i - 2x^2) dx$$

$$= 2 \int_1^0 x^2 i dx - 2 \int_1^0 x^2 dx$$

$$= 2 \left[\frac{x^3}{3} i \right]_1^0 - 2 \left[\frac{x^3}{3} \right]_1^0 = i \frac{2}{3} (-1) + 2 = -\frac{2i}{3} + \frac{2}{3}$$

$$\Rightarrow \int_{C_2} (x^2 - y^2 + 2ixy)(dx + idy) \quad \text{For OB, } (0,0) \rightarrow (1,1)$$

$$\Rightarrow \int_0^{-1} (x^2 - y^2 - 2x^2 i)(1+i) dx \quad \text{y} = 1 - x$$

$$= \int_0^{-1} (-2x^2 i)(1-i) dx$$

$$= 2 \left[\int_0^1 (x^2 i)(i-1) dx \right] = 2 \left[\int_0^1 -x^2 dx - \int_0^1 x^2 i dx \right]$$

$$= 2 \left[-\frac{1}{3} \left[x^3 \right]_0^1 - \frac{1}{3} \left[x^3 i \right]_0^1 \right]$$

$$= -\frac{2}{3} - \frac{2}{3} i$$

$$\Rightarrow \int_{C_3} (x^2 - y^2 + 2ixy)(dx + idy)$$

"here, 'y' is constant so 'y' doesn't change $\Rightarrow y = 1$ ".

$$\Rightarrow \int_{-1}^1 (x^2 - 1 + 2xi) dx = \left[\frac{x^3}{3} - x + x^2 \right]_{-1}^1$$

$$= \frac{1}{3} - 1 + 1 - \left(-\frac{1}{3} + 1 + 1 \right) = \frac{1}{3} - \frac{5}{3} = -\frac{4}{3}$$

\therefore On adding all values,

$$= \cancel{\frac{2}{3}} - \cancel{\frac{2}{3}} + \left(\cancel{\frac{+2}{3}} \right) + \cancel{\frac{2}{3}} - \cancel{\frac{4}{3}} = 0$$

(Q36) Evaluate $\int_C (z-a)^n dz$, where C is the circle with centre ' a ' and radius r . Also discuss the case when $n = -1$?

Sol: We know that, the equation of circle is $\Rightarrow |z-a| = r$, where a is centre and r is radius of circle.

$$\Rightarrow |z-a| = r$$

$$\Rightarrow z-a = e^{i\theta} r$$

$$\Rightarrow dz = e^{i\theta} (ir) d\theta = ire^{i\theta} \cdot d\theta$$

$$\Rightarrow \int_C (z-a)^n dz = \int_C r^n \cdot e^{in\theta} \cdot ire^{i\theta} \cdot d\theta$$

$$= \int_0^{2\pi} r^{n+1} e^{i(n+1)\theta} \cdot i d\theta$$

$$= \int_0^{2\pi} (ir^{n+1}) \cdot e^{i\theta(n+1)} \cdot d\theta$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \cdot d\theta$$

$$= ir^{n+1} \left[-\frac{1}{i(n+1)} e^{i(n+1)\theta} \right]_0^{2\pi}$$

$$= \frac{e^{n+1}}{n+1} \left[\cos(n+1)2\pi + i \sin(n+1)2\pi - 1 \right]$$

$$= \frac{e^{n+1}}{n+1} (1 + 0 - 1)$$

$$= 0$$

$$\cos n\pi = (-1)^n$$

$$\sin n\pi = 0$$

Q37) Find $\int_C \frac{1}{(z-a)} dz$, where C is simple closed curve and the point $z=a$ is inside.

Sol: $\because \int_C \frac{f(z)}{(z-a)} dz = 2\pi i f(a)$. Put denominator = 0
 Here a is lies outside, $\Rightarrow z-a=0 \Rightarrow z=a$
 $\Rightarrow \int_C \frac{1}{z-a} dz = 2\pi i f(a)$
 $= 2\pi i [f(z)]_{z=a}$
 $= [2\pi i] . (f(z) = 1)$

Q38) Solve the integral, $\int_C \frac{\cos \pi z}{z-1} dz$, (where C is the

circle $|z|=3$? Given, $|z-a|=r \rightarrow$ centre $a \rightarrow$ centre

Sol: For pole,
 $\Rightarrow z-1=0$ Given, $|z-0|=r \rightarrow$ radius.

$$\Rightarrow [z=1].$$

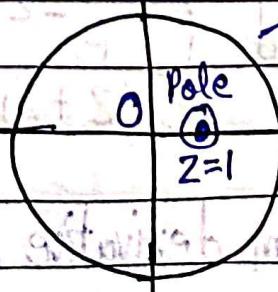
i.e., pole lies inside C.

Centre = 0

Radius = 3

\Rightarrow By Cauchy integral formula,

$$\Rightarrow f(a) \cdot 2\pi i = \int_C \frac{f(z)}{z-a} dz$$

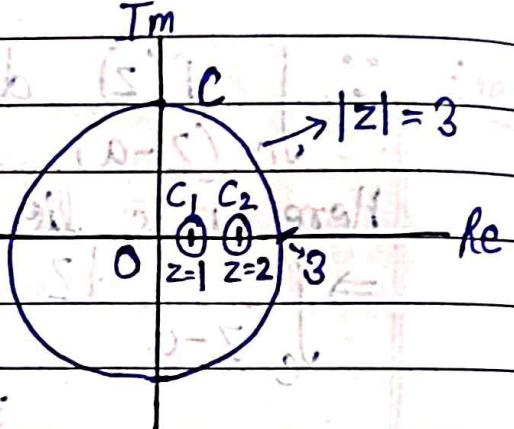


$$\Rightarrow \int_C \frac{\cos \pi z}{z-1} dz = 2\pi i [f(z)]_{z=a} = 2\pi i [\cos \pi z]_{z=1}$$

$$= 2\pi i \times \cos \pi(1) \\ = -2\pi i.$$

Q39) Find $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where C is $|z|=3$?

Ans: Poles are $z=1$ and $z=2$, i.e., both poles lie inside C .



$$\Rightarrow \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \int_{C_1} + \int_{C_2}$$

$$= \int_{C_1} \frac{(\cos \pi z^2)}{(z-2)} dz + \int_{C_2} \frac{(\cos \pi z^2)}{(z-1)} dz.$$

$$= 2\pi i [\cos \pi z^2]_{z=1} + 2\pi i [\cos \pi z^2]_{z=2}.$$

$$= 2\pi i (\cos \pi) + 2\pi i (\cos 4\pi) \quad \begin{cases} \cos \pi = -1 \\ \cos 4\pi = +1 \end{cases}$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

Q40) Find $\int_C \frac{e^{-z}}{(z+2)^5} dz$, where C is the circle $|z|=3$?

Ans: From derivative formula,

$$\Rightarrow f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz. \quad (\text{formula})$$

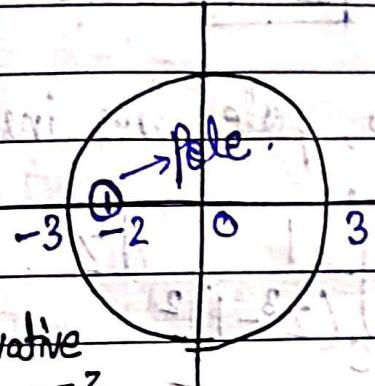
For $n=4$,

$$\Rightarrow f^4(a) = \frac{4!}{2\pi i} \int_C \frac{f(z)}{(z-a)^5} dz.$$

\Rightarrow Here, pole $z = -2$ (order 5)

\therefore Pole lies inside C ,

$$\Rightarrow \int_C \frac{e^{-z}}{(z+2)^5} dz$$



$$= \frac{2\pi i}{4!} \cdot f^4(e^{-2})$$

fourth derivative
of $e^{-z} = e^{-z}$

$$= \frac{2\pi i}{24} \times [e^{-z}]_{z=-2}$$

$$= \frac{2\pi i}{24} \times e^{2^2}$$

$$= \boxed{\frac{\pi i e^2}{12}}$$

Q41) Find $\int_C \frac{e^{iz}}{z+3i} dz$, where C is the circle $|z+3i|=1$.

Ans: Pole, $z+3i=0$

$$\Rightarrow z = -3i$$

i.e., pole lies inside C .



$$\Rightarrow \int_C \frac{e^{iz}}{z+3i} dz = 2\pi i [e^{iz}]_{z=-3i}$$

$$= 2\pi i (e^{i(-3i)})$$

$$= \boxed{2\pi i e^3}$$



Q42). Integrate $(z^3 - 1)^{-2}$ around the circle, $|z-1| = 1$.

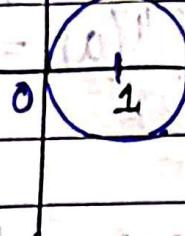
Sol: To find, $\int_C \frac{dz}{(z^3 - 1)^2}$

Pole is at,

$$\Rightarrow z^3 - 1 = 0$$

$$\Rightarrow (z-1)(z^2 + z + 1) = 0$$

$$\Rightarrow [z=1] \quad [z^2 + z + 1 \neq 0]$$



So, pole lies inside C,

$$\Rightarrow \int_C \frac{1}{(z^3 - 1)^2} dz = \int_C \frac{1}{(z^2 + z + 1)(z-1)} dz.$$

Using Cauchy's integral formula,

$$\begin{aligned} \Rightarrow \int_C \frac{1}{(z^2 + z + 1)(z-1)} dz &= 2\pi i \left[(z^2 + z + 1)^{-1} \right]_{z=1} \\ &= 2\pi i \left[\frac{1}{1+1+1} \right] \\ &= \boxed{\frac{2\pi i}{3}}. \end{aligned}$$

Q43) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is $|z|=3$.

Q44) $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$, where C is $|z-i|=2$.

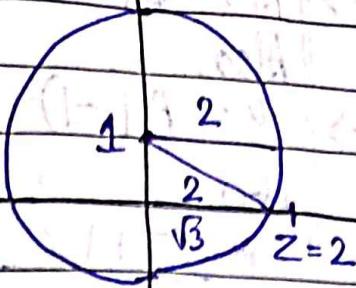
$$\Rightarrow (z+1)^2(z-2) = 0$$

$$\Rightarrow z=2 \text{ and } z=-1, -1$$

$$= -1 \text{ (order 2)}$$

Here, $z=2$ pole lies outside C ,
and $z=-1$ pole lies inside C .

Answer = $-\frac{2}{9}\pi i$.



Q45 $\int_C \frac{z}{z^2+1} dz$, where C is $|z+1|=2$. (Assig.)

Ans: $\rightarrow |z^2+1|=2$

$$\rightarrow |x^2-y^2+2xyi| = 2|x+iy|.$$

$$\rightarrow \sqrt{(x^2-y^2+1)^2+(2xy)^2} = 2\sqrt{x^2+y^2}.$$

$$\rightarrow (x^2-y^2+1)^2+(2xy)^2 = 4(x^2+y^2) \quad \dots \text{Solve yourself.}$$

Q46 Expand e^z about a ?

Ans: From Taylor series,
 $\Rightarrow f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \dots + \frac{(z-a)^n}{n!}f^{(n)}(a)$

$$\Rightarrow e^z = e^a + (z-a)e^z + \frac{(z-a)^2}{2!}e^z + \dots + \frac{(z-a)^n}{n!}(e^z)$$

Q47 Expand $f(z) = \frac{z+1}{(z-3)(z-4)}$ about $z=2$ upto first 4 terms?

Ans: By partial fractions,

$$\Rightarrow \frac{z+1}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$$

$$\Rightarrow z+1 = A(z-4) + B(z-3).$$

So,

$$\Rightarrow 5 = B(4-3) \Rightarrow B=5$$

Also,

$$\Rightarrow 4 = A(-1) \Rightarrow A=-4$$

Now,

$$\Rightarrow \frac{z+1}{(z-3)(z-4)} = \frac{-4}{z-3} + \frac{5}{z-4}$$

Now,

$$\Rightarrow f(z) = \frac{-4}{(z-3)} + \frac{5}{(z-4)}$$

$$\Rightarrow f'(z) = -4 \frac{(-1)}{(z-3)^2} + 5 \frac{(-1)}{(z-4)^2}$$

$$\Rightarrow f'(z) = \frac{4}{(z-3)^2} - \frac{5}{(z-4)^2}$$

and,

$$\Rightarrow f''(z) = 4 \left[-2(z-3)^{-3} \right] - 5 \left[-2(z-4)^{-3} \right]$$

$$\Rightarrow f''(z) = \frac{-8}{(z-3)^3} + \frac{10}{(z-4)^3}$$

$$\Rightarrow f'''(z) = -8(-3)(z-3)^{-4} + 10(-3)(z-4)^{-4}$$

$$\Rightarrow f'''(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4}$$

$$\Rightarrow f'''(z) = \frac{24(-4)}{(z-3)^5} - \frac{30(-4)}{(z-4)^5} = \frac{-96}{(z-3)^5} + \frac{120}{(z-4)^5}$$

\Rightarrow At $z=2$,

$$\Rightarrow f(2) = \frac{2+1}{(-1)(-2)} = \frac{3}{2}$$

$$\Rightarrow f'(2) = \frac{4}{(-1)^2} - \frac{5}{(-2)^2} = \frac{4}{1} - \frac{5}{4} = \frac{11}{4}$$

$$\Rightarrow f''(2) = \frac{-8}{(-1)^3} + \frac{10}{(-2)^3} = \frac{-8}{-8} - \frac{10}{8} = \frac{27}{4}$$

$$\Rightarrow f'''(2) = \frac{24}{(-1)^4} - \frac{30}{(-2)^4} = \frac{24}{16} - \frac{30}{16} = \frac{177}{8}$$

\therefore Series expansion,

$$\Rightarrow f(z) = \frac{3}{2} + \frac{(z-2)11}{4} + \frac{(z-2)^2 \times 27}{4 \times 2} + \frac{(z-2)^3 \times 177}{3 \times 2 \times 8}$$

$$\therefore f(z) = \frac{3}{2} + \frac{11(z-2)}{4} + \frac{27(z-2)^2}{8} + \frac{177(z-2)^3}{48}$$

(Q48) Expand $f(z) = \frac{1}{z^2+4}$ about $z = -i$?

$$\text{Ans: } f(z) = \frac{1}{z^2 - i^2(4)} = \frac{1}{(z+2i)(z-2i)} = \frac{A}{z+2i} + \frac{B}{z-2i}$$

$$\Rightarrow f(z) = \frac{1}{4i} \left(\frac{1}{z-2i} - \frac{1}{z+2i} \right)$$

$$\Rightarrow f'(z) = \frac{1}{4i} \left[\frac{-1}{(z-2i)^2} + \frac{1}{(z+2i)^2} \right]$$

$$\Rightarrow f''(z) = \frac{1}{4i} \left[\frac{2}{(z-2i)^3} - \frac{2}{(z+2i)^3} \right]$$

$$\Rightarrow f'''(z) = \frac{1}{4i} \left[\frac{-6}{(z-2i)^4} + \frac{6}{(z+2i)^4} \right]$$

Now,

$$\Rightarrow f(0) = f(-i) = \frac{1}{(-i)^2 + 4} = \frac{1}{3}$$

$$\Rightarrow f'(-i) = \frac{1}{4i} \left[\frac{-1}{(-3i)^2} + \frac{1}{(i)^2} \right] = \frac{1}{4i} \left(\frac{-1}{-9} - 1 \right)$$

Now,

$$\Rightarrow f'(-i) = \frac{1}{4i} \left(\frac{1}{9} \right) = \frac{10}{36i} = \frac{5}{18i} = -\frac{2}{9i}$$

$$\Rightarrow f''(-i) = \frac{1}{4i} \left(\frac{-2}{(-3i)^3} \right) = \frac{-2}{54} = -\frac{1}{27}$$

$$\therefore f(z) = \frac{1}{3} + (z+i) \frac{2}{9} i + \dots n\text{-terms}$$

Q49) Find the Laurent expansion of $f(z) = \frac{1}{z(z^2-3z+2)}$ in the region $1 \leq |z| < 2$?

$$\text{Ans: } \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

On solving,

$$A = \frac{1}{2}$$

$$B = -1$$

$$C = \frac{1}{2}$$

$$\therefore \frac{1}{z(z-1)(z-2)} = \frac{\frac{1}{2}}{z} + \frac{-1}{z-1} + \frac{\frac{1}{2}}{z-2}$$

$$\text{Note: } ① (1+x)^{-1} = 1-x+x^2-x^3+x^4-x^5+\dots \infty$$

$$② (1-x)^{-1} = 1+x+x^2+x^3+x^4+x^5+\dots \infty$$

$$= \frac{1}{2z} - \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(-2)(1-\frac{1}{2})}$$

$$= \frac{1}{2z} - \frac{1}{z} \left(1 - \frac{1}{z} \right)^{-1} - \frac{1}{4} \left(1 - \frac{1}{2} \right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{z} \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots \right) + \left(-\frac{1}{4} \right)$$

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) \quad \text{Ans.}$$

Q50) Find the Laurent's expansion of $f(z) = \frac{1}{(z+1)(z+3)}$

a) $|z| < 3$

b) $|z| > 3$

c) $|z| < 1$

d) $0 < |z+1| < 2$

Sol: $f(z) = \frac{1}{2} \left(\frac{1}{z+1} - \frac{1}{z+3} \right)$

(a) $|z| < 3$

$$\Rightarrow f(z) = \frac{1}{2} \left[\frac{1}{z(1+\frac{1}{z})} - \frac{1}{2 \cdot 3(1+\frac{z}{3})} \right]$$

$$\Rightarrow f(z) = \frac{1}{2z(1+\frac{1}{z})} - \frac{1}{2 \cdot 3(1+\frac{z}{3})}$$

(d) $0 < |z+1| < 2$

$$\Rightarrow f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2(z+1)} - \frac{1}{2(z+1)+2}$$

$$\Rightarrow f(z) = \frac{1}{2(z+1)} - \frac{1}{2} \left[\frac{1}{2(1+\frac{z+1}{2})} \right]$$

$$\Rightarrow f(z) = \frac{1}{2} (z+1)^{-1} - \frac{1}{4} \left[1 + \left(\frac{z+1}{2} \right) \right]^{-1}$$

$$\therefore f(z) = \frac{1}{2} \left(1 - z + z^2 - z^3 + z^4 - \dots \infty \right) - \frac{1}{4} \left(1 + z + \frac{(z+1)^2}{2} - \frac{(z+1)^3}{3} + \frac{(z+1)^4}{4} - \dots \infty \right)$$

Q51) Find the radius of convergence of the power series,

$$(a) f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Ans: Here, $a_n = \frac{1}{n!}$, $a_{n+1} = \frac{1}{(n+1)!}$

$$\Rightarrow \frac{1}{R} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n!}{(n+1)!} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) = 0$$

$$\Rightarrow \frac{1}{R} = \lim_{n \rightarrow \infty} \left(\frac{n!}{n! (n+1)} \right) = \frac{1}{\infty + 1} = 0$$

Or, $R = \infty$.

$$(b) f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^n}$$

$$(c) f(z) = \sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$$

$$\oint_C f(z) dz = 2\pi i \sum R^+$$

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Q52) Evaluate $\int_C \frac{2z^2 + 5}{(z+2)^2(z^2+4)} dz$, where C is the square with the vertices at $1+i$, $2+i$, $2+2i$, $1+2i$. (Use Cauchy's residue theorem).

Coordinates are :

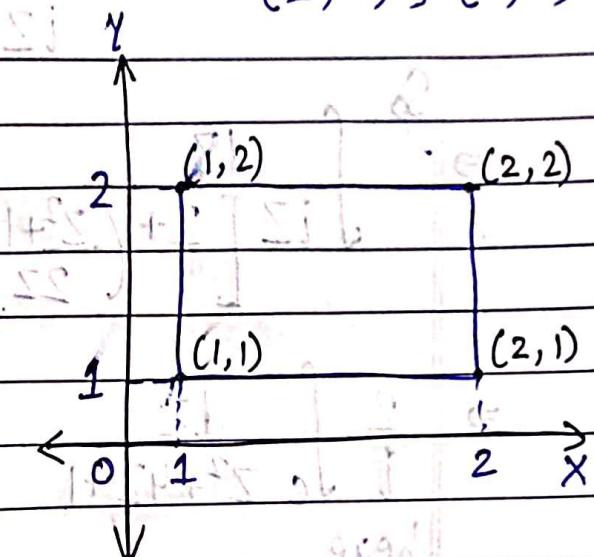
Ans: Given, $I = \int_C \frac{2z^2 + 5}{(z+2)^2(z^2+4)} dz$ (1, 1), (2, 1), (2, 2) & (1, 2).

The pole will be,

$$z = -2$$

Here, no point lies on the real or imaginary axis. So,

$$I = 0$$



Q53) Evaluate $\int_C \frac{z+3}{z^2+2z+5} dz$, where C is $|z+1-i| = 2$.

$$\begin{aligned} \text{Ans: } & \Rightarrow |x+iy+1-i| = 2 \\ & \Rightarrow |(x+1)+i(y-1)| = 2 \\ & \Rightarrow (x+1)^2 + (y-1)^2 = 4 \\ & \Rightarrow x^2 + 1 + 2x + y^2 + 1 - 2y = 4 \\ & \Rightarrow x^2 + y^2 + 2x - 2y - 2 = 0. \end{aligned}$$

So,

$$C \equiv (-1, 1) \text{ and } r = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 1 - (-2)} = 2.$$

Hence,

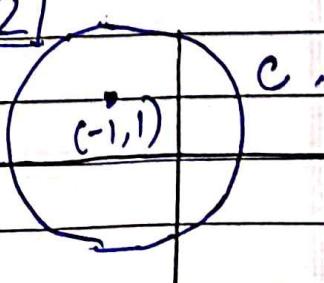
$$C \equiv (-1, 1) \text{ and } r = 2$$

Here,

pole lies outside the circle

i.e.,

$$I = 0$$



Q54) Evaluate: $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$, by contour integration?

Ans: Putting $Z = e^{i\theta}$
 $\Rightarrow dZ = e^{i\theta} \cdot i d\theta$.
 $\Rightarrow dZ = iZ \cdot d\theta$.
 $\Rightarrow d\theta = \frac{dZ}{iZ}$.

So, $\int \frac{dZ}{iZ [2 + (Z^2 + 1)]} = \int_C \frac{2}{Z^2 + 4Z + 1} dZ$

$\Rightarrow \frac{2}{i} \int_C \frac{dZ}{Z^2 + 4Z + 1}$. Here, $C: |Z| = 1$.
 centre = $(0, 0)$; radius = 1.

here,
 pole is $= (-2 + \sqrt{3})$ and not $(-2 - \sqrt{3})$.
 The pole $(-2 + \sqrt{3})$ lies inside the circle C . So,