NEW TOPICS ADDED FROM ACADEMIC SESSION 2021-22 ONWARDS

SECOND SEMESTER

APPLIED MATHEMATICS-II [BS-112]

UNIT - I

Q.1. **If $2\cos\theta = x+\frac{1}{x}$, then prove that** $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}$

Ans. We have given $x+\frac{1}{x} = 2\cos\theta$

$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow x = \cos\theta \pm i\sin\theta$

Let $x = \cos\theta + i\sin\theta$

Consider $\frac{x^{2n}+1}{x^{2n-1}+x}$.

$$= \frac{(\cos\theta + i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$

$$= \frac{\cos 2n\theta + i\sin 2n\theta + 1}{\cos(2n-1)\theta + i\sin(2n-1)\theta + \cos\theta + i\sin\theta}$$

$$= \frac{2\cos^2 n\theta + 2i\sin n\theta\cos n\theta}{2\cos n\theta\cos(n-1)\theta + 2i\sin n\theta\cos(n-1)\theta}$$

$$= \frac{2\cos n\theta[\cos n\theta + i\sin n\theta]}{2\cos(n-1)\theta[\cos n\theta + i\sin n\theta]}$$

$$= \frac{\cos n\theta}{\cos(n-1)\theta}$$

Q.2. **If $a_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, then show that $a_1a_2a_3\ldots$ upto $\infty = -1$** [*IPU, 2007*]

Ans. Consider LHS = $a_1a_2a_3\ldots\infty$

$$= \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\ldots$$

$$= \cos\left(\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{8}+\ldots\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{8}+\ldots\right)$$

$$= \cos\left(\frac{\pi/2}{1-1/2}\right)+i\sin\left(\frac{\pi/2}{1-1/2}\right)$$ [Sum of infinite GP with a = π/2 & r = 1/2]

$= \cos\pi + i\sin\pi$

$= -1 =$ R.H.S

Q.3. Prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos n\left(\dfrac{\pi}{2}-\theta\right)+i\sin n\left(\dfrac{\pi}{2}-\theta\right)$ **[IPU 2007]**

Ans. Let $\alpha=\dfrac{\pi}{2}-\theta\Rightarrow\theta=\dfrac{\pi}{2}-\alpha$

Consider LHS

$$=\left[\frac{1+\sin\left(\frac{\pi}{2}-\alpha\right)+i\cos\left(\frac{\pi}{2}-\alpha\right)}{1+\sin\left(\frac{\pi}{2}-\alpha\right)-i\cos\left(\frac{\pi}{2}-\alpha\right)}\right]^n$$

$$=\left[\frac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}\right]^n$$

$$=\left[\frac{2\cos^2\alpha/2+i2\sin\alpha/2\cos\alpha/2}{2\cos^2\alpha/2-i2\sin\alpha/2\cos\alpha/2}\right]^n$$

$$=\left[\frac{2\cos\alpha/2(\cos\alpha/2+i\sin\alpha/2)}{2\cos\alpha/2(\cos\alpha/2-i\sin\alpha/2)}\right]^n$$

$$=\left[\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)\right]^n$$

$$=\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]^{2n}$$

$$=\cos n\alpha+i\sin n\alpha$$

$$=\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$$

Q.4. If $\sin\alpha+\sin\beta+\sin\gamma=0=\cos\alpha+\cos\beta+\cos\gamma$, Prove that

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

Ans. Let $a=\cos\alpha+i\sin\alpha$ ...(1)

$b=\cos\beta+i\sin\beta$ ...(2)

$c=\cos\gamma+i\sin\gamma$ ...(3)

$\therefore a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$=0+i\,0$

$\therefore a+b+c=0$ ...(4)

$\Rightarrow a+b=-c$

Cubing both sides, we get

$a^3+b^3+3ab(a+b)=-c^3$

$\Rightarrow a^3+b^3-3abc=-c^3$

$\Rightarrow a^3+b^3+c^3=3abc$ ...(5)

By (5), we get

$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\Rightarrow \cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$

$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

Equating real and imaginary parts

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Q.5. Solve the equation $x^7 - x^4 + x^3 - 1 = 0$ and find all the roots of the equation.

Ans. $x^7 - x^4 + x^3 - 1 = 0$

$\Rightarrow x^4(x^3-1) + 1(x^3-1) = 0$

$\Rightarrow (x^4+1)(x^3-1) = 0$

$x^4 + 1 = 0$ and $x^3 - 1 = 0$

Now $x^4 + 1 = 0$

$\Rightarrow x = (-1)^{1/4}$

$= (\cos\pi + i\sin\pi)^{1/4}$

$\Rightarrow x = [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{1/4}$

$\Rightarrow x = \cos(2k\pi+\pi)\frac{1}{4} + i\sin(2k\pi+\pi)\frac{1}{4} \qquad k = 0,1,2,3$

$\Rightarrow x = \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4} \qquad k = 0,1,2,3$

Now $\alpha_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\alpha_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\alpha_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\alpha_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

Let $x^3 - 1 = 0$

$\Rightarrow x = (1)^{1/3}$

$\Rightarrow x = (\cos 0 + i\sin 0)^{1/3}$

$\Rightarrow x = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$

$\Rightarrow x = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$, $k = 0,1,2$

$\alpha_5 = \cos 0 + i\sin 0 = 1$

$\alpha_6 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$\alpha_7 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{-1}{2} \frac{-i\sqrt{3}}{2}$

Q.6. Show that the roots of the equation $(z-1)^5 + z^5 = 0$ are given by

$$z = \frac{1}{2}\left(1 + i\cot\frac{m\pi}{10}\right) \quad m = 1,3,5,7,9$$

Ans. Consider $(z-1)^5 + z^5 = 0$

$$\Rightarrow \left(\frac{z-1}{z}\right)^5 + 1 = 0$$

Let $w = \dfrac{z-1}{z}$

$\Rightarrow w^5 + 1 = 0$

$\Rightarrow w = (-1)^{1/5}$

$\Rightarrow w = [\cos \pi + i \sin \pi]^{1/5}$

$\Rightarrow w = [\cos (2k+1)\pi + i \sin (2k+1)\pi]^{1/5}$ $\quad k = 0,1,2,3,4$

$\Rightarrow w = \cos (2k+1)\dfrac{\pi}{5} + i \sin (2k+1)\dfrac{\pi}{5}$ $\quad k = 0,1,2,3,4$

$\Rightarrow w = \cos \alpha + i \sin \alpha,\ \alpha = (2k+1)\dfrac{\pi}{5}$ ...(1)

As $w = \dfrac{z-1}{z} \Rightarrow z = \dfrac{1}{1-w}$

$$\Rightarrow z = \frac{1}{1-\cos\alpha - i\sin\alpha} \quad \text{(By (1))}$$

$$\Rightarrow z = \frac{1}{2\sin^2\alpha/2 - 2i\sin\alpha/2\cos\alpha/2}$$

$$\Rightarrow z = \frac{1}{2\sin\alpha/2\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)}$$

$$\Rightarrow z = \frac{1}{2\sin\alpha/2}\left(\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}\right)$$

$$= \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)$$

$$= \frac{1}{2}\left(1 + i\cot(2k+1)\frac{\pi}{10}\right),\ k = 0,1,2,3,4$$

$\therefore$ Roots are

$$z = \frac{1}{2}\left(1 + i\cot\frac{m\pi}{10}\right),\ m = 1,3,5,7,9.$$

Q.7. Show that the nth roots of unity are given $1, \lambda, \lambda^2, \ldots \lambda^{n-1}$, where $\lambda = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$ and show that the continued product of all the nth roots is $(-1)^{n+1}$ *[IPU, 2004]*

Ans. Let $x^n = 1$

$\Rightarrow x = (1)^{1/n}$

$\Rightarrow x = (\cos 0 + i \sin 0)^{1/n}$

$$\Rightarrow x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 0,1,\ldots n-1$$

Thus roots are

$$x = \cos 0 + i \sin 0,\ \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

$$\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots\ldots \cos\frac{2(n-1)\pi}{n} + i\sin\frac{(2n-1)\pi}{n}$$

$$\Rightarrow x = 1,\ \cos\frac{2\pi}{n} + i\frac{\sin 2\pi}{n}, \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^2 \ldots \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n-1}$$

$$= 1, \lambda, \lambda^2 .. \lambda^{n-1}$$

where $\lambda = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

Product of values is

$$= (\cos 0 + i\sin 0)\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)$$

$$\left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right) \ldots \left[\cos\frac{2(n-1)\pi}{n} i\sin\frac{2(n-1)\pi}{n}\right]$$

$$\cos\left[0 + \frac{2\pi}{n} + \frac{4\pi}{n} + \ldots + \frac{2(n-1)\pi}{n}\right]$$

$$+ i\sin\left[0 + \frac{2\pi}{n} + \frac{4\pi}{n} + \ldots + 2\frac{(n-1)\pi}{n}\right]$$

$$= \cos\left[\frac{2\pi}{n}(1+2+\ldots(n-1))\right] + i\sin\left[\frac{2\pi}{n}(1+2+\ldots(n-1))\right]$$

$$= \cos\left(\frac{2\pi}{n}.\frac{n(n-1)}{2}\right) + i\sin\left(\frac{2\pi}{n}\frac{n(n-1)}{2}\right)$$

$$= \cos(n-1)\pi + i\sin(n-1)\pi$$

$$= (-1)^{n-1} + 0 = (-1)^{n-1}(-1)^2$$

$$= (-1)^{n+1}$$

Q.8. Find the equation whose roots are $2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$

Ans. Let $y = \cos\theta + i\sin\theta$ such that

$$y + \frac{1}{y} = 2\cos\theta$$

where θ be any of angles

$$\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$$

Now $y^7 = (\cos\theta + i\sin\theta)^7$

$= \cos 7\theta + i\sin 7\theta$

$= -1$

$\Rightarrow y^7 + 1 = 0$

$\Rightarrow (y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

For the factor $y + 1 = 0$

We get $\theta = \pi$ (Not possible)

We reject the factor $(y + 1)$

$\Rightarrow$ We have

$y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$

Dividing the throughout by y^3, we get

$$y^3 - y^2 + y - 1 + \frac{1}{y} - \frac{1}{y^2} + \frac{1}{y^3} = 0$$

$$\Rightarrow \left(y^3 + \frac{1}{y^3}\right) - \left(y^2 + \frac{1}{y^2}\right) + \left(y + \frac{1}{y}\right) - 1 = 0$$

$$\Rightarrow \left\{\left(y + \frac{1}{y}\right)^3 - 3\left(y + \frac{1}{y}\right)\right\} - \left\{\left(y + \frac{1}{y}\right)^2 - 2\right\} + y + \frac{1}{y} - 1 = 0$$

Let $x = y + \dfrac{1}{y} = 2\cos\theta$

$\Rightarrow x^3 - 3x - x^2 + 2 + x - 1 = 0$

$\Rightarrow x^3 - x^2 - 2x + 1 = 0$

Now $\cos\dfrac{13\pi}{7} = \cos\left(2\pi - \dfrac{\pi}{7}\right) = \cos\dfrac{\pi}{7}$

$\cos\dfrac{11\pi}{7} = \cos\left(2\pi - \dfrac{3\pi}{7}\right) = \cos\dfrac{3\pi}{7}$

$\cos\dfrac{9\pi}{7} = \cos\left(2\pi - \dfrac{5\pi}{7}\right) = \cos\dfrac{5\pi}{7}$

Thus roots of equation

$x^3 - x^2 - 2x + 1 = 0$ are

$2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$.

Q.9. Show that

$128\sin^3\theta\cos^5\theta = -\sin 8\theta - 2\sin 6\theta + 2\sin 4\theta + 6\sin 2\theta$

[*IPU, 2004, 2007*]

Ans. Let $x = \cos\theta + i\sin\theta$

then $x^k = (\cos\theta + i\sin\theta)^k$

$= \cos k\theta + i\sin k\theta$

$$\therefore \left.\begin{array}{l} x^k + \dfrac{1}{x^k} = 2\cos k\theta \text{ and} \\ x^k - \dfrac{1}{x^k} = 2i\sin k\theta \end{array}\right\} \text{(A)}$$

Consider $(2\,i\sin\theta)^3 \cdot (2\cos\theta)^5$

$$= \left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^5 \text{ [by (A)]}$$

$$\Rightarrow -256\,i\sin^3\theta\cos^5\theta = \left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^2$$

$$= \left[\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\right]^3\left(x+\frac{1}{x}\right)^2$$

$$= \left(x^2-\frac{1}{x^2}\right)^3\left(x+\frac{1}{x}\right)^2$$

$$= \left(x^6-\frac{1}{x^6}-3x^4.\frac{1}{x^2}+3x^2.\frac{1}{x^4}\right)\left(x^2+\frac{1}{x^2}+2\right)$$

$$= \left(x^8-\frac{1}{x^8}\right)+2\left(x^6-\frac{1}{x^6}\right)+\left(x^4-\frac{1}{x^4}\right)$$

$$-3\left(x^4-\frac{1}{x^4}\right)-6\left(x^2-\frac{1}{x^2}\right)$$

$= 2\,i\sin 8\theta + 4\,i\sin 6\theta + 2\,i\sin 4\theta - 6\,i\sin 4\theta - 12\,i\sin\sin 2\theta$

$= 2\,i\sin\theta + 4\,i\sin 6\theta - 4\,i\sin 4\theta - 12\,i\sin 2\theta$

Dividing both sides by $(-2\,i)$

$\Rightarrow 128\sin^3\theta\cos^5\theta = -\sin 8\theta - 2\sin 6\theta + 2\sin 4\theta + 6\sin 2\theta$

Q.10. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha = e^{i\alpha}$ Prove it.

(i) $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$ (ii) $\phi = \dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$

Ans. As $\tan(\theta + i\,\phi) = \cos\alpha + i\sin\alpha$ ...(1)

By changing i to $(-i)$

$\Rightarrow \tan(\theta - i\,\phi) = \cos\alpha - i\sin\alpha$. ...(2)

(i) Consider $\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$= \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$\Rightarrow \tan 2\theta = \frac{\cos\alpha+i\sin\alpha+\cos\alpha-i\sin\alpha}{1-(\cos\alpha+i\sin\alpha)(\cos\alpha-i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1-(\cos^2\alpha-i^2\sin^2\alpha)}$$

$$= \frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)}$$

$$= \frac{2\cos\alpha}{0} = \alpha = \tan\frac{\pi}{2}$$

$$\Rightarrow 2\theta = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \theta = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

(ii) Now $\tan 2i\,\phi = \tan\,[(\phi + i\phi) - (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = \frac{2i\sin\alpha}{2}$$

$= i \sin \alpha$

$\Rightarrow$ i $\tan h\ 2\phi = i \sin \alpha$

$\Rightarrow \tan h\ 2\phi = \sin \alpha$

$$\Rightarrow \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \sin\alpha$$

$$\Rightarrow \frac{e^{2\phi} + e^{-2\phi}}{e^{2\phi} - e^{-2\phi}} = \frac{1}{\sin\alpha}$$

Apply componendo and dividendo

$$\Rightarrow \frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha}$$

$$\Rightarrow e^{4\phi} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2\cos\alpha/2}$$

$$= \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}$$

$$\Rightarrow e^{2\phi} = \frac{\cos\alpha/2 + \sin\alpha/2}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$= \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}$$

$$\Rightarrow e^{2\phi} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

Take log both sides

$$\log e^{2\phi} = \log \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow 2\phi = \log \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \phi = \frac{1}{2}\log \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Q.11. If $\tan(A + iB) = x + iy$, prove that

(i) $x^2 + y^2 + 2x \cot 2A = 1$

(ii) $x^2 + y^2 - 2y \coth 2B + 1 = 0$

Ans. $\tan(A + iB) = x + iy$ ...(1)

changing i to $(-i)$, we get

$\tan(A - iB) = x - iy$ ...(2)

(i) Consider $2A = (A + iB) + (A - iB)$

$\Rightarrow \tan 2A = \tan[(A + iB) + (A - iB)]$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$\Rightarrow \tan 2A = \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2+y^2)}$$

$$\Rightarrow \frac{1}{\cot 2A} = \frac{2x}{1-(x^2+y^2)}$$

$$\Rightarrow \cot 2A = \frac{1-(x^2+y^2)}{2x}$$

$\Rightarrow 1 - (x^2 + y^2) = 2x \cot 2A.$

$\Rightarrow x^2 + y^2 + 2x \cot 2A = 1$

(ii) Consider $2iB = (A + iB) - (A - iB)$

$\Rightarrow \tan 2iB = \tan[(A + iB) - (A - iB)]$

$$\Rightarrow \tan 2iB = \frac{\tan(A+iB)-\tan(A-iB)}{1+[\tan(A+iB)\tan(A-iB)]}$$

$$\Rightarrow \tan 2iB = \frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+(x^2+y^2)}$$

$$\Rightarrow i \tanh 2B = \frac{2iy}{1+x^2+y^2}$$

$$\Rightarrow \frac{1}{\coth 2B} = \frac{2y}{1+x^2+y^2}$$

$\Rightarrow 2y \cot h2B = 1 + x^2 + y^2$

$\Rightarrow x^2 + y^2 - 2y \cot h2B + 1 = 0$

Q.12. Prove that $\cosh^{-1}x = \log\ (x+\sqrt{x^2-1})$

Ans. Let $\cosh^{-1} x = y$

$\Rightarrow x = \cosh y$

$$\Rightarrow x = \frac{e^y + e^{-y}}{2} = \frac{e^{2y}+1}{2e^y}$$

$\Rightarrow e^{2y} - e^y . 2x + 1 = 0$

This is quadratic equation in e^y

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2-1}$$

Taking positive sign only

$$e^y = x + \sqrt{x^2-1}$$

$$\Rightarrow y = \log\ (x+\sqrt{x^2-1})$$

$$\Rightarrow \cosh^{-1} x = \log\ (x+\sqrt{x^2-1})$$

Q.13. Prove that $\tan\left(i\log\frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2-b^2}$

Ans. Let $a + ib = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

then $a - ib = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

$$\text{LHS} = \tan\left(i\log\frac{a-ib}{a+ib}\right)$$

$$= \tan\left(i\log\frac{re^{-i\theta}}{re^{i\theta}}\right)$$

$= \tan(i \log(e^{-i\theta}. e^{-i\theta}))$

$= \tan(i \log e^{-2i\theta})$

$= \tan(i(-2i\theta))$

$$= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad ...(1)$$

Since $a + ib = r\cos\theta + ri\sin\theta$

Equating real and imaginary parts, we get

$a = r\cos\theta,\ b = r\sin\theta$

$\Rightarrow r = \sqrt{a^2+b^2}$ and $\tan\theta = \dfrac{b}{a}$

By (1), we get

$$\text{LHS} = \frac{2(b/a)}{1-\dfrac{b^2}{a^2}} = \frac{2ab}{a^2-b^2} = \text{RHS}$$

Q.14. Prove that i^i is wholly real and find its principal value. Also show that the values of i^i form a G.P.

Ans. $i^i = e^{i \log_e i}$

$= e^{i(2n\pi i + \log_e i)}$

$= e^{i[2n\pi i + \log_e(\cos\pi/2 + i\sin\pi/2)]}$

$= e^{i(2n\pi i + \log_e e^{i\pi/2})}$

$= e^{i(2n\pi i + i\pi/2)}$

$= e^{i^2(2n\pi + \pi/2)}$

$= e^{-\pi(2n+1/2)}$

$= e^{-(4n+1)\pi/2}$ ...(1)

We get

$i^i = e^{-(4n+1)\pi/2}$ which is wholly real.

The principle value of i^i is $e^{-\pi/2}$ putting n = 0,1,2,... successively in (1), we get the values of of i^i are $e^{-\pi/2}$, $e^{-5\pi/2}$, $e^{-9\pi/2}$, $e^{-13\pi/2}$,... which form a G.P. (with $r = e^{-2\pi}$).

Q.15. Find general value of log (– 3)

Ans. As $-3 = 3(-1) = 3(\cos\pi + i\sin\pi)$

$= 3e^{i\pi}$

Now $\log(-3) = \text{Log}(3e^{i\pi})$

$= 2n\pi i + \log(3e^{i\pi})$

$= 2n\pi i + \log 3 + i\pi$

$= i(2n\pi + \pi) + \log 3.$