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Applications of Partial Differential Equations

17.1. INTRODUCTION

Many physical and engineering problems when formulated in the mathematical language give rise to partial differential equations. Besides these, partial differential equations also play an important role in the theory of Elasticity, Hydraulics etc.

Since the general solution of a partial differential equation in a region R contains arbitrary constants or arbitrary functions, the unique solution of a partial differential equation corresponding to a physical problem will satisfy certain other conditions at the boundary of the region R. These are known as *boundary conditions*. When these conditions are specified for the time $t = 0$, they are known as *initial conditions*. A partial differential equation together with boundary conditions constitutes a *boundary value problem.*

In the applications of ordinary linear differential equations, we first find the general solution and then determine the arbitrary constants from the initial values. But the same method is not applicable to problems involving partial differential equations. Most of the boundary value problems involving linear partial differential equations can be solved by the method of separation of variables. In this method, right from the beginning, we try to find the particular solutions of the partial differential equation which satisfy all or some of the boundary conditions and then adjust them till the remaining conditions are also satisfied. A combination of these particular solutions gives the solution of the problem.

Fourier series is a powerful aid in determining the arbitrary functions.

17.2. METHOD OF SEPARATION OF VARIABLES

In this method, we assume the solution to be the product of two functions, each of which involves only one of the variables. The following examples explain the method.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the equation* $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, *given* $u(x, 0) = 6e^{-3x}$.

Sol. Here u is a function of x and t.

Let $\qquad u = XT \qquad \ldots(1)$

where X is a function of x only and T is a function of t only, be a solution of the given equation.

Then $\qquad \frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = XT'$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad X'T = (2T' + T)X$$

Separating the variables, we get $\qquad \frac{X'}{X} = \frac{2T' + T}{T} \qquad \ldots(2)$

Since x and t are independent variables, as t varies x remains constant, so that the L.H.S. and hence the R.H.S. is constant. Therefore, equation (2) can hold only when each side is equal to the some constant, say k.

$\therefore \qquad \frac{X'}{X} = k \quad$ i.e. $\quad \log X = kx + \log c_1$

or $\qquad \log \frac{X}{c_1} = kx \quad \text{or} \quad X = c_1 e^{kx} \qquad \ldots(3)$

and $\qquad \frac{2T' + T}{T} = k \quad$ i.e., $\quad \frac{T'}{T} = \frac{1}{2}(k - 1)$

or $\qquad \log T = \frac{1}{2}(k - 1)t + \log c_2 \quad \text{or} \quad \log \frac{T}{c_2} = \frac{1}{2}(k - 1)t$

or $\qquad T = c_2\, e^{1/2(k-1)t} \qquad \ldots(4)$

From (1), (3) and (4), we have $\quad u = u(x, t) = c_1 e^{kx} \cdot c_2 e^{1/2(k-1)t}$

Since $\qquad u(x, 0) = 6e^{-3x} \qquad$ (given)

$\therefore \qquad c_1 c_2 e^{kx} = 6e^{-3x}$

$\Rightarrow \qquad c_1 c_2 = 6 \quad \text{and} \quad k = -3$

$\therefore$ The unique solution of the given equation is

$$u = 6e^{-3x} \cdot e^{-2t} \quad \text{i.e.,} \quad u = 6e^{-(3x + 2t)}$$

Example 2. *Solve the equation* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ *by the method of separation of variables.*

Sol. Here z is a function of x and y

Let $\qquad z = XY \qquad \ldots(1)$

where X is a function of x only and Y is a function of y only, be a solution of the given equation.

Then $\qquad \frac{\partial z}{\partial x} = X'Y, \quad \frac{\partial z}{\partial y} = XY' \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} = X''Y.$

Substituting in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \quad \text{or} \quad (X'' - 2X')Y + XY' = 0$$

Separating the variables, we get $\qquad \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} \qquad \ldots(2)$

Since x and y are independent variables, equation (2) can hold only when each side is equal to some constant, say k.

$\therefore \qquad \frac{X'' - 2X'}{X} = k \quad \text{or} \quad X'' - 2X' - kX = 0 \qquad \ldots(3)$

and $\qquad -\frac{Y'}{Y} = k \quad \text{or} \quad Y' + kY = 0 \qquad \ldots(4)$

These are ordinary differential equations.

The *boundary conditions*, which the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ has to satisfy are :

$\left.\begin{array}{l}(i)\ y = 0, \text{ when } x = 0 \\ (ii)\ y = 0, \text{ when } x = l\end{array}\right]$. These should be satisfied for every value of t.

If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the *initial conditions* are :

(*i*) $y = f(x)$, when $t = 0$ (*ii*) $\frac{\partial y}{\partial t} = 0$, when $t = 0$.

17.4. SOLUTION OF THE WAVE EQUATION

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let $y = XT$ ...(2)

where X is a function of x only and T is a function of t only, be a solution of (1)

Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

Substituting in (1), we have $XT'' = c^2X''T$

Separating the variables, we get $\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T}$ ...(3)

Now, the L.H.S. of (3) is a function of x only and the R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary linear differential equations

$$X''k - X = 0 \quad \text{and} \quad T'' - kc^2T = 0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3e^{cpt} + c_4e^{-cpt}$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px$$
$$T = c_3 \cos cpt + c_4 \sin cpt$$

(*iii*) *When k = 0*

$$X = c_1x + c_2 \quad T = c_3t + c_4$$

Thus, the various possible solutions of the wave equation (1) are :

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt})$$
$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$
$$y = (c_1x + c_2)(c_3t + c_4)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with a problem on vibrations, y must be a periodic function of x and t. Therefore, the solution must involve trigonometric terms.

Accordingly $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ ...(5)

is the only suitable solution of the wave equation and it corresponds to $k = -p^2$.

Now, applying boundary conditions that

$y = 0$, when $x = 0$

and $y = 0$, when $x = l$, we get

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \qquad ...(6)$$

and $$0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt) \qquad ...(7)$$

From (6), we have $c_1 = 0$ and equation (7) reduces to

$$c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$$

which is satisfied when $\sin pl = 0$ or $pl = n\pi$ or $p = \frac{n\pi}{l}$, where $n = 1, 2, 3, ...$

∴ A solution of the wave equation satisfying the boundary conditions is

$$y = c_2\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$
$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

on replacing $c_2 c_3$ by a_n and $c_2 c_4$ by b_n.

Adding up the solutions for different values of n, we get

$$y = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \qquad ...(8)$$

is also a solution.

Now, applying the initial conditions

$y = f(x)$ and $\frac{\partial y}{\partial t} = 0$, when $t = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \qquad ...(9)$$

and $$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l} \qquad ...(10)$$

Since equation (9) represents Fourier series for $f(x)$, we have

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \qquad ...(11)$$

From (10), $b_n = 0$, for all n.

Hence (8) reduces to $y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ ...(12)

where a_n is given by (11) when $f(x)$ *i.e.*, $y(x, 0)$ is known.

Also $$y\left(\frac{l}{2}, t\right) = \frac{9a}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \cos \frac{2m\pi ct}{l} \sin m\pi$$

$= 0$, since $\sin m\pi = 0$

$\Rightarrow$ The displacement of the mid-point of the string is zero for all values of t.

Thus the mid-point of the string is always at rest.

Example 3. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.*

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$

As proved in Art. 17.4, we have

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \qquad \text{...(1)}$$

Since the string was at rest initially, $y(x, 0) = 0$

$\therefore$ From (1), $$0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \Rightarrow a_n = 0$$

$\therefore$ $$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(2)}$$

and $$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

But $$\frac{\partial y}{\partial t} = \lambda x(l - x), \quad \text{when } t = 0$$

$\therefore$ $$\lambda x(l - x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \quad \frac{\pi c}{l} nb_n = \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l} \left[x(l - x)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right) \right]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3} (1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases}$$

i.e., $$\frac{8\lambda l^2}{\pi^3(2m - 1)^3}, \text{ taking } n = 2m - 1$$

$$\Rightarrow \quad b_n = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4}$$

$\therefore$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi ct}{l} \sin \frac{(2m - 1)\pi x}{l}.$$

TEST YOUR KNOWLEDGE

1. Solve completely the equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\dfrac{\partial y(x, 0)}{\partial t} = 0$, $0 < x < l$.
2. Find the deflection $y(x, t)$ of the vibrating string of length π and ends fixed, corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$, given $c^2 = 1$.
3. Solve the boundary value problem
$$\frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2}, y(0, t) = y(5, t) = 0, y(x, 0) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x)$$
if (*i*) $f(x) = 5 \sin \pi x$ (*ii*) $f(x) = 3 \sin 2\pi x - 2 \sin 5\pi x$.
4. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \left(\dfrac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $y(x, t)$.
5. A tightly stretched string of length l and fixed at both ends is plucked at $x = \dfrac{l}{3}$ and assumes initially the shape of a triangle of height h. Find the displacement $y(x, t)$ after the string is released from rest.
6. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = \mu x (l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.
7. A string is stretched between the fixed points $(0, 0)$ and $(l, 0)$ and released at rest from the initial deflection given by
$$f(x) = \begin{cases} \dfrac{2k}{l} x, & \text{when } 0 < x < \dfrac{l}{2} \\ \dfrac{2k}{l} (l - x), & \text{when } \dfrac{l}{2} < x < l \end{cases}$$
Find the deflection of the string at any time t.
8. The ends of a tightly stretched string of length l are fixed at $x = 0$ and $x = l$. The string is at rest with the point $x = b$ drawn aside through a small distance d and released at time $t = 0$. Show that
$$y(x, t) = \frac{2dl^2}{\pi^2 b(l - b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}.$$

9. A string of length l is initially at rest in equilibrium position and each of its points is given the velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$$

Find the displacement $y(x, t)$.

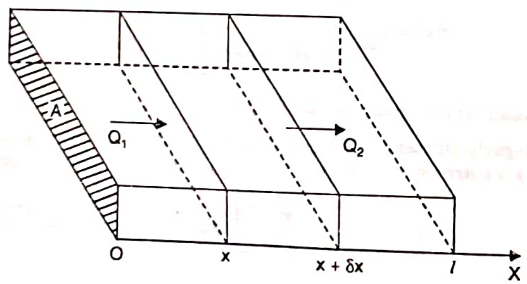
Answers

1. $y(x, t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$, where $a_n = \frac{2}{l} \int_0^l f(x) . \sin \frac{n\pi x}{l}\, dx$
2. $y(x, t) = k (\cos t \sin x - \cos 2t \sin 2x)$
3. (i) $y = \frac{5}{2\pi} \sin \pi x \sin 2\pi t$ (ii) $y = \frac{3}{4\pi} \sin 2\pi x \sin 4\pi t - \frac{1}{5\pi} \sin 5\pi x \sin 10\pi t$
4. $y(x, t) = \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}\right)$
5. $y(x, t) = \frac{9h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$
6. $y = \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi ct}{l}$
7. $y(x, t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$
9. $y(x, t) = \frac{bl}{12c\pi}\left[9 \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}\right]$.

17.5. ONE-DIMENSIONAL HEAT FLOW

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Consider the flow of heat by conduction in a uniform bar. It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as origin and the direction of flow as the positive x-axis. The temperature u at any point of the bar depends on the distance x of the point from one end and the time t. Also, the temperature of all points of any cross-section is the same.



The amount of heat crossing any section of the bar per second depends on the area A of the cross-section, the conductivity K of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$ i.e., rate of change of temperature w.r.t. distance normal to the area.

∴ Q_1, the quantity of heat flowing into the section at a distance x

$$= -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ per sec.}$$

(the negative sign on the right is attached because as x increases, u decreases).

Q_2, the quantity of heat flowing out of the section at a distance $x + \delta x$

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per sec.}$$

Hence the amount of heat retained by the slab with thickness δx is

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per sec} \quad ...(1)$$

But the rate of increase of heat in the slab $= S\rho A\, \delta x \frac{\partial u}{\partial t}$ ...(2)

where S is the specific heat and ρ, the density of the material.

∴ From (1) and (2), $S\rho A\, \delta x \frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$

or

$$S\rho \frac{\partial u}{\partial t} = K\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking the limit as $\delta x \to 0$, we have

$$S\rho \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} = \frac{K}{S\rho} \frac{\partial^2 u}{\partial x^2}$$

or

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{K}{S\rho}$$

is known as diffusivity of the material of the bar.

17.6. SOLUTION OF THE HEAT EQUATION

The heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(1)

Let $u = XT$ ...(2)

where X is a function of x only and T is a function of t only be a solution of (1).

Then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$

Substituting in (1), we have $XT' = c^2X''T$

Separating the variables, we get $\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}$ ...(3)

Now, the L.H.S. of (3) is a function of x only and the R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{dT}{dt} - kc^2T = 0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px} + c_2e^{-px}, \; T = c_3e^{c^2p^2t}$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px, \; T = c_3e^{-c^2p^2t}$$

(*iii*) *When k = 0*

$$X = c_1x + c_2, \; T = c_3.$$

Thus the various possible solutions of the heat equation (1) are :

$$u = (c_1e^{px} + c_2e^{-px}).c_3e^{c^2p^2t}$$

$$u = (c_1 \cos px + c_2 \sin px) \,.\, c_3e^{-c^2p^2t}$$

$$u = (c_1x + c_2)c_3.$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px)\, e^{-c^2p^2t}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function u(x, t).*

Sol. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

As proved in Art. 17.6, we have

$$u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-c^2p^2t} \qquad ...(1)$$

Since the ends $x = 0$ and $x = l$ are cooled to 0°C and kept at that temperature throughout, the boundary conditions are $u(0, t) = u(l, t) = 0$ for all t

Also $u(x, 0) = u_0$ is the initial condition.

Since $u(0, t) = 0$, we have from (1), $0 = c_1\, e^{-c^2p^2t} \;\Rightarrow\; c_1 = 0$

$\therefore$ From (1),
$$u(x, t) = c_2 \sin px.\, e^{-c^2p^2t} \qquad ...(2)$$

Since $u(l, t) = 0$, we have from (2), $0 = c_2 \sin pl \,.\, e^{-c^2p^2t}$

$\Rightarrow$ $\sin pl = 0 \;\Rightarrow\; pl = n\pi$

$\therefore$ $p = \frac{n\pi}{l}$, n being an integer

Solution (2) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} . e^{-\frac{c^2n^2\pi^2t}{l^2}}$ on replacing c_2 by b_n.

The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, \ldots\ldots$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . e^{-\frac{c^2n^2\pi^2t}{l^2}} \qquad ...(3)$$

Since $u(x, 0) = u_0$, we have $u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

which is half-range sine series for u_0.

$$\therefore \quad b_n = \frac{2}{l}\int_0^l u_0 \sin \frac{n\pi x}{l}\, dx = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Hence the temperature function

$$u(x, t) = \frac{4u_0}{\pi} \sum_{n=1, 3, 5, \ldots} \frac{1}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l^2}}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2(2n-1)^2\pi^2t}{l^2}}$$

Example 2. *(a) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

(b) Find also the temperature if the change consists of raising the temperature of A to 20° C and reducing that of B to 80° C.

Sol. (*a*) The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (steady state condition). When the temperature u depends only upon x and not on t, (1) reduces to $\frac{\partial^2 u}{\partial x^2} = 0$

Its general solution is $u = ax + b$ ...(2)

where a, b are arbitrary constants.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, we get from (2), $b = 0$ and $a = \dfrac{100}{l}$

$\therefore$ The *initial condition* is expressed by $\quad u(x, 0) = \dfrac{100}{l}x$

Also the *boundary conditions* for the subsequent flow are $u(0, t) = 0$ and $u(l, t) = 0$ for all values of t.

Proceeding as in Example 1, the most general solution of (1) satisfying the boundary conditions is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{\frac{-c^2n^2\pi^2t}{l^2}} \qquad \ldots(3)$$

Since $u(x, 0) = \dfrac{100}{l}x$, we have $\dfrac{100}{l}x = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

which is half-range sine series for $\dfrac{100}{l}x$.

$$\therefore \qquad b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(-\frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi\right] = -\frac{200}{n\pi}(-1)^n = \frac{200}{n\pi}(-1)^{n+1}$$

Hence the temperature function

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2n^2\pi^2t}{l^2}}.$$

(*b*) Here the *initial condition* remains the same as in part (*a*), *i.e.*, $u(x, 0) = \dfrac{100}{l}x$ and the *boundary conditions* are $u(0, t) = 20$ for all values of t.

$u(l, t) = 80$ for all values of t.

Since the boundary values are non-zero, we modify the procedure. We break up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad \ldots(1)$$

where $u_s(x)$ is a solution of (1) involving x only and satisfying the boundary conditions ; $u_t(x, t)$ is then a function defined by (4). Thus $u_s(x)$ is a steady state solution of the form (2) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of time.

Since $u_s(0) = 20$ and $u_s(l) = 80$, we have from (2)

$$20 = b \quad \text{and} \quad 80 = al + b$$

$$b = 20 \quad \text{and} \quad a = \frac{60}{l}$$

and
$$u_s(x) = \frac{60}{l}x + 20 \qquad \ldots(5)$$

Putting $x = 0$ in (4), we have $u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0$ ...(6)

Putting $x = l$ in (4), we have $\quad u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0$ ...(7)

Also $\quad u_t(x, 0) = u(x, 0) - u_s(x)$

$$= \frac{100}{l}x - \left(\frac{60}{l}x + 20\right) = \frac{40}{l}x - 20 \qquad \ldots(8)$$

Hence (6) and (7) give the boundary conditions and (7) gives the initial condition relative to the transient solution $u_t(x, t)$. Since the boundary values given by (6) and (7) are both zero, therefore as in part (*a*), we have

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{c^2n^2\pi^2t}{l^2}} \qquad \ldots(9)$$

where b_n is given by $\quad b_n = \dfrac{2}{l}\displaystyle\int_0^l \left(\frac{40}{l}x - 20\right)\sin \frac{n\pi x}{l}\, dx = \frac{-40}{n\pi}(1 + \cos n\pi)$

$$= \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\dfrac{80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

Hence (9) becomes

$$u_t(x, t) = -\frac{80}{\pi}\sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n}\sin \frac{n\pi x}{l} e^{\frac{c^2n^2\pi^2t}{l^2}}$$

$$= -\frac{40}{\pi}\sum_{m=1}^{\infty} \frac{1}{m}\sin \frac{2m\pi x}{l} e^{\frac{4c^2m^2\pi^2t}{l^2}} \qquad \ldots(10) \quad (\text{taking } n = 2m)$$

Combining (5) and (10), the required solution is

$$u(x, t) = 20 + \frac{60}{l}x - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin \frac{2m\pi x}{l} e^{\frac{4c^2m^2\pi^2t}{l^2}}.$$

TEST YOUR KNOWLEDGE

1. Determine the solution of one-dimensional heat equation $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$, where the boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$ $(t > 0)$ and the initial condition $u(x, 0) = x$, l being the length of the bar.
2. A bar with insulated sides is initially at temperature 0°C throughout. The end $x = 0$ is kept at 0°C and heat is suddenly applied at the end $x = l$ so that $\dfrac{\partial u}{\partial x} = A$ for $x = l$, where A is a constant. Find the temperature function $u(x, t)$.

3. A homogeneous rod of conducting material of length 100 cm has its ends kept as zero temperature and the temperature initially is $u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$

Find the temperature $u(x, t)$ at any time.

4. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = L$ are kept at zero temperature and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} k, & \text{when } 0 < x < \frac{1}{2}L \\ 0, & \text{when } \frac{1}{2}L < x < L \end{cases}.$$

5. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$, $u(1, t) = 0$, where $0 < x < 1$, $t > 0$.

6. Find the solution of $\dfrac{\partial V}{\partial t} = k\dfrac{\partial^2 V}{\partial x^2}$, having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

7. Show that the solution of the differential equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, subject to the conditions

(i) u not infinite for $t \to \infty$, (ii) $\dfrac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$,

(iii) $u = lx - x^2$ for $t = 0$, between $x = 0$ and $x = l$, is $u = \dfrac{1}{6}l^2 - \dfrac{l^2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos\dfrac{2n\pi x}{l}\, e^{-4n^2\pi^2 kt/l^2}$.

8. Solve $\dfrac{\partial \theta}{\partial t} = k\dfrac{\partial^2 \theta}{\partial x^2}$, such that

(i) θ is finite, when $t \to \infty$, (ii) $\dfrac{\partial \theta}{\partial x} = 0$ when $x = 0$ and $\theta = 0$ when $x = l$ for all t,

(iii) $\theta = \theta_0$, when $t = 0$ for all values of x between 0 and l.

9. Find a solution of the heat conduction equation $\dfrac{\partial u}{\partial t} = \alpha\dfrac{\partial^2 u}{\partial x^2}$ such that

(i) u is finite when $t \to \infty$, (ii) $u = 100$ when $x = 0$ or π for all values of t,

(iii) $u = 0$ when $t = 0$ for all values of x between 0 and π.

(Here the initially ice-cold rod has its ends in boiling water.)

10. A bar 10 cm long, with insulated sides, has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature at A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t.

11. The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady state prevails. The temperatures of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.

Answers

1. $u(x, t) = -\dfrac{2l}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos n\pi}{n}\sin\dfrac{n\pi x}{l}\, e^{(-c^2n^2\pi^2 t)/l^2}$

2. $u(x, t) = Ax + \dfrac{8Al}{\pi^2}\displaystyle\sum_{m=1}^{\infty}\dfrac{(-1)^m}{(2m-1)^2}\sin\dfrac{(2m-1)\pi x}{l}\, e^{[-(2m-1)c^2\pi^2 t]/4l^2}$

3. $u(x, t) = \dfrac{400}{\pi^2}\displaystyle\sum_{m=0}^{\infty}\dfrac{(-1)^m}{(2m+1)^2}\sin\dfrac{(2m+1)\pi x}{100}\, e^{-[(2m+1)c\pi/100]^2 t}$

4. $u(x, t) = \dfrac{4k}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\sin^2\left(\dfrac{n\pi}{4}\right)\sin\dfrac{n\pi x}{L}\, e^{(-c^2n^2\pi^2)t/L^2}$

5. $u(x, t) = 3\displaystyle\sum_{n=1}^{\infty}\sin n\pi x\, e^{-n^2\pi^2 t}$

6. $V = V_0 e^{-\sqrt{\left(\frac{n}{2k}\right)}x}\sin\left[nt - \sqrt{\left(\dfrac{n}{2k}\right)}x\right]$

8. $\theta = \dfrac{4\theta_0}{\pi}\left[e^{-\left(\frac{\pi}{2l}\right)^2 kt}\cos\dfrac{\pi x}{2l} - \dfrac{1}{3}e^{-\left(\frac{3\pi}{2l}\right)^2 kt}\cos\dfrac{3\pi x}{2l} + \dfrac{1}{5}e^{-\left(\frac{5\pi}{2l}\right)^2 kt}\cos\dfrac{5\pi x}{2l} - \ldots\right]$

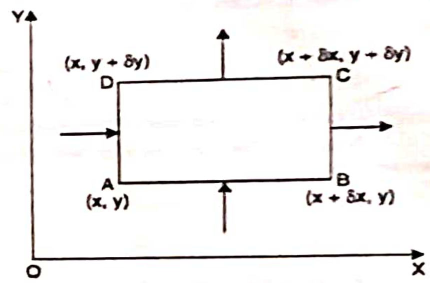
9. $u = 100 - \dfrac{400}{\pi}\displaystyle\sum_{m=1}^{\infty}\dfrac{\sin(2m-1)x}{2m-1}\, e^{-(2m-1)^2\alpha t}$

10. $u(x, t) = 90 - 3x - \dfrac{80}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{5}\, e^{-c^2n^2\pi^2 t/25}$

11. $u(x, t) = 40 + x - \dfrac{20}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}(2\cos n\pi + 1)\sin\dfrac{n\pi x}{20}\, e^{-\frac{c^2n^2\pi^2 t}{400}}$.

17.7. TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate, in the XOY plane. If the temperature at any point is independent of the z-coordinate and depends on x, y and t only, then the flow is called two dimensional and the heat flow lies in the plane XOY only and is zero along the normal to the plane XOY.



Take a rectangular element of the plate with sides δx and δy and thickness α. As discussed in the one-dimensional heat flow along a bar, the quantity of heat that enters the plate per second from the sides AB and AD is given by

$$-k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y \quad \text{and} \quad -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

respectively and that which flows out through the sides CD and BC per second is

$$-k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \quad \text{and} \quad -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \quad \text{respectively.}$$

Therefore, the total gain of heat by the rectangular plate ABCD per second

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad ...(1)$$

The rate of gain of heat by the plate is also given by

$$s\rho\delta x\delta y\frac{\partial u}{\partial t} \quad ...(2)$$

where s = specific heat and ρ = density of the metal plate.

Equating (1) and (2), we have

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = s\rho\delta x\delta y\frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\delta x\delta y$ and taking the limit as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=s\rho\frac{\partial u}{\partial t}$$

or

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}, \text{ where } c^2=\frac{k}{s\rho} \quad ...(3)$$

Equation (3) gives the temperature distribution of the plate in the transient state.

Note 1. In the steady state, u is independent of t, so that $\frac{\partial u}{\partial t}=0$ and the above equation reduces to

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad ...(4)$$

which is known as **Laplace's Equation in two dimensions.**

Note 2. The equation of heat flow in a solid (Three-dimensional heat flow) can similarly be derived as

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t}$$

In the steady state, it reduces to $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

which is **Laplace's Equation in three-dimensions.**

17.8. SOLUTION OF LAPLACE'S EQUATION IN TWO DIMENSIONS

Laplace's equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad ...(1)$$

Let $u = XY$ ...(2)

where X is a function of x only and Y is a function of y only, be a solution of (1).

Then $\frac{\partial^2 u}{\partial x^2}=X''Y$ and $\frac{\partial^2 u}{\partial y^2}=XY''$

Substituting in (1), we have $X''Y+XY''=0$ or $\frac{X''}{X}=-\frac{Y''}{Y}$ ...(3)

Now the L.H.S. of (3) is a function of x only and the R.H.S. is a function of y only. Since x and y are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2}-kX=0 \quad \text{and} \quad \frac{d^2Y}{dy^2}+kY=0 \quad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px}+c_2e^{-px},\ Y = c_3\cos py + c_4\sin py$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1\cos px + c_2\sin px,\ Y = c_3e^{py}+c_4e^{-py}$$

(*iii*) *When* $k = 0$

$$X = c_1x + c_2,\ Y = c_3y + c_4$$

Thus the various possible solutions of Laplace's equation (1) are :

$$u = (c_1e^{px}+c_2e^{-px})(c_3\cos py + c_4\sin py)$$
$$u = (c_1\cos px + c_2\sin px)(c_3e^{py}+c_4e^{-py})$$
$$u = (c_1x+c_2)(c_3y+c_4)$$

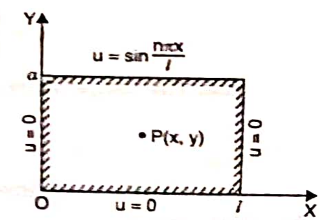
Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *which satisfies the conditions :*

$$u(0, y) = u(l, y) = u(x, 0) = 0 \text{ and } u(x, a) = \sin\frac{n\pi x}{l}.$$

Sol. The given equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ...(1)



The three possible solutions of eqn. (1) are

$$u = (c_1e^{px}+c_2e^{-px})(c_3\cos py + c_4\sin py) \quad ...(2)$$
$$u = (c_1\cos px + c_2\sin px)(c_3e^{py}+c_4e^{-py}) \quad ...(3)$$
$$u = (c_1x+c_2)(c_3y+c_4) \quad ...(4)$$

Keeping in view the given boundary conditions, the only possible solution is (3)

$\therefore \quad u(x, y) = (c_1\cos px + c_2\sin px)(c_3e^{py}+c_4e^{-py})$ ...(5)

Since $u(0, y) = 0 \quad \therefore \quad 0 = c_1(c_3e^{py}+c_4e^{-py})$

$\Rightarrow \quad c_1 = 0$

Equation (5) reduces to $u(x, y) = c_2\sin px(c_3e^{py}+c_4e^{-py})$ ...(6)

Since $u(l, y) = 0 \quad \therefore \quad 0 = c_2\sin pl(c_3e^{py}+c_4e^{-py})$

$$\Rightarrow \quad \sin pl = 0 \quad i.e., \quad pl = n\pi \quad \text{or} \quad p = \frac{n\pi}{l}, n \text{ being an integer}$$

Also $\quad u(x, 0) = 0 \quad \therefore \quad 0 = c_2 \sin \frac{n\pi x}{l}(c_3 + c_4)$

$\Rightarrow \quad c_3 + c_4 = 0 \quad \therefore \quad c_4 = -c_3$

$\therefore$ Equation (6) becomes $\quad u(x, y) = c_2 \sin \frac{n\pi x}{l}\left(c_3 e^{\frac{n\pi y}{l}} - c_3 e^{\frac{-n\pi y}{l}}\right)$

Replacing c_2c_3 by b_n, we have $\quad u(x, y) = b_n \sin \frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}\right)$

$$= 2b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \qquad \text{...(7)}$$

Putting $y = a$, we have $\quad u(x, a) = \sin \frac{n\pi x}{l} = 2b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi a}{l}$

$$\Rightarrow \quad 2b_n \sinh \frac{n\pi a}{l} = 1 \quad \text{or} \quad b_n = \frac{1}{2 \sinh \frac{n\pi a}{l}}$$

Hence (7) reduces to $\quad u(x, y) = \sin \frac{n\pi x}{l}\left[\frac{\sinh \frac{n\pi y}{l}}{\sinh \frac{n\pi a}{l}}\right]$

which is the required solution of (1).

Example 2. *A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by*

$$u = 20x \quad \text{for} \quad 0 \le x \le 5$$

and

$$u = 20(10 - x) \quad \text{for} \quad 5 \le x \le 10$$

and the two long edges $x = 0, x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{800}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}\cdot e^{-\frac{(2n-1)\pi y}{10}}.$$

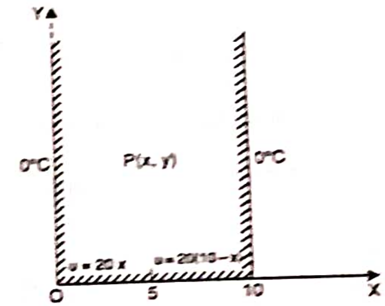
Sol. The temperature $u(x, y)$ at any point P(x, y) satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \text{...(1)}$$

The boundary conditions are $\quad u(0, y) = 0$
for all values of $y \ge 0$ ...(2)

$u(10, y) = 0$
for all values of $y \ge 0$ ...(3)

$u(x, \infty) = 0$
in $0 \le x \le 10$ ...(4)



$$u(x, 0) = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10 - x), & 5 \le x \le 10 \end{cases} \qquad \text{...(5)}$$

Now the three possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \qquad \text{...(6)}$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \text{...(7)}$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad \text{...(8)}$$

The solution (6) cannot satisfy the condition (2) since, we get $u \ne 0$ for $x = 0$, for all values of y. The solution (8) cannot satisfy the condition (4). Thus the only possible solution is (7).

$\therefore \quad u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ ...(9)

Since $\quad u(0, y) = 0 \quad \therefore \quad 0 = c_1(c_3 e^{py} + c_4 e^{-py}) \quad \Rightarrow \quad c_1 = 0$

$\therefore$ Equation (9) reduces to $\quad u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py})$ ...(10)

Since $\quad u(10, y) = 0 \quad \therefore \quad 0 = c_2 \sin 10p\,(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \quad \sin 10p = 0 \quad i.e., \quad 10p = n\pi \quad \text{or} \quad p = \frac{n\pi}{10}$, n being an integer.

Also $\quad u(x, \infty) = 0 \quad \therefore \quad c_3 = 0$

Hence from (10) a solution satisfying (2), (3) and (4) is $u(x, y) = c_2c_4 \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}}$

Replacing c_2c_4 by b_n, the most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \cdot e^{-\frac{n\pi y}{10}} \qquad \text{...(11)}$$

Putting $y = 0$, we have $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$

$$\therefore \quad b_n = \frac{2}{10}\int_0^{10} u(x, 0) \sin \frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[\int_0^5 20x \sin \frac{n\pi x}{10}\,dx + \int_5^{10} 20(10 - x) \sin \frac{n\pi x}{10}\,dx\right]$$

$$= 4\left[x\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (1)\left(-\frac{\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^5 + 4\left[(10 - x)\left(-\frac{\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(-\frac{\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_5^{10}$$

$$= 4\left[-\frac{50}{n\pi}\cos \frac{n\pi}{2} + \left(\frac{10}{n\pi}\right)^2 \sin \frac{n\pi}{2}\right] + 4\left[\frac{50}{n\pi}\cos \frac{n\pi}{2} + \left(\frac{10}{n\pi}\right)^2 \sin \frac{n\pi}{2}\right] = \frac{800}{n^2\pi^2}\sin \frac{n\pi}{2}$$

$$= \begin{cases} \dfrac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Hence from (11), the required solution is

$$u(x, y) = \frac{800}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \qquad \text{(Replacing } n \text{ by } 2n-1\text{)}$$

$$= \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} e^{-\frac{(2n-1)\pi y}{10}}$$

or

$$u = \frac{800}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} e^{-\frac{(2n-1)\pi y}{10}}$$

$$\Big| \because \ (-1)^{n+1} = (-1)^{n-1} \cdot (-1)^2 = (-1)^{n-1}.$$

TEST YOUR KNOWLEDGE

1. A long rectangular plate of width a cm with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\pi y/a} \sin \frac{n\pi x}{a}.$$

2. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin \frac{\pi x}{8}, \ 0 < x < 8$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plate is given by $u(x, y) = 100\, e^{-\pi y/8} \sin \dfrac{\pi x}{8}$.

3. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, given that $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ within the rectangle $0 \le x \le a$, $0 \le y \le b$ given that

$$u(0, y) = u(a, y) = u(x, b) = 0 \quad \text{and} \quad u(x, 0) = x(a - x).$$

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$, when $0 < x < 20$, while other three edges are kept at 0°C. Find the steady state temperature in the plate.

6. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.

7. A rectangular plate has sides a and b. Let the side of length a be taken along OX and that of length b along OY and the other sides along $x = a$ and $y = b$. The sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the steady-state temperature at any point (x, y).

8. The temperature u is maintained at 0°C along three edges of a square plate of length 100 cm and the fourth edge is maintained at 100°C until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi} \left[\frac{1}{\cosh \frac{\pi}{2}} - \frac{1}{3 \cosh \frac{3\pi}{2}} + \frac{1}{5 \cosh \frac{5\pi}{2}} - \ldots\ldots \right].$$

Answers

3. $u(x, y) = -\dfrac{8}{\pi} \displaystyle\sum_{n=1,3,5,\ldots}^{\infty} \dfrac{\sin nx \sinh n(\pi - y)}{n(n^2 - 4) \sinh n\pi}$

4. $u(x, y) = \dfrac{8a^2}{\pi^3} \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^3} \sin \dfrac{(2n+1)\pi x}{a} \times \dfrac{\sinh \frac{(2n+1)\pi}{a}(b - y)}{\sinh \frac{(2n+1)\pi b}{a}}$

5. $u(x, y) = \dfrac{3200}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{\sin \frac{(2n-1)\pi x}{20} \sinh \frac{(n-1)\pi y}{20}}{(2n-1)^3 \sinh (2n-1)\pi}$

6. $u(x, y) = \dfrac{4u_0}{\pi} \left[e^{-y} \sin x + \dfrac{1}{3} e^{-3y} \sin 3x + \dfrac{1}{5} e^{-5y} \sin 5x + \ldots\ldots \right]$

7. $u(x, y) = u_0 \cos \dfrac{\pi x}{a} \cosh \dfrac{\pi}{a}(b - y) \operatorname{sech} \dfrac{\pi b}{a}$.

17.9. VIBRATING MEMBRANE—TWO-DIMENSIONAL WAVE EQUATION

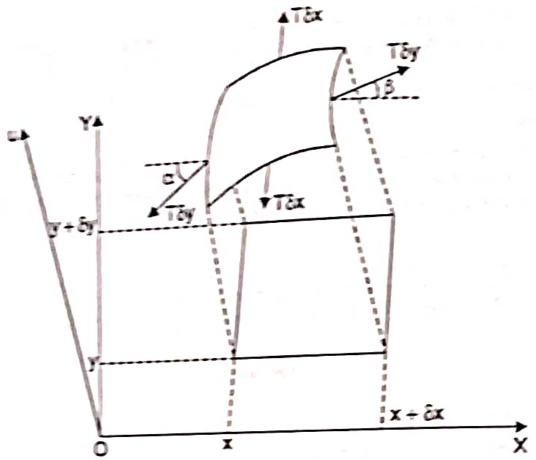
We shall now obtain the equation for the vibrations of a tightly stretched membrane (such as the membrane of a drum). We shall assume that the membrane is uniform and the tension in it per unit length is the same at every point in all directions. Let T be the tension per unit length and m be the mass of the membrane per unit area.

Consider the forces on an element $\delta x \delta y$ of the membrane. Due to its displacement u, perpendicular to the xy-plane, the forces $T\delta y$ (tangential to the membrane) on its opposite edges of length δy act at angles α and β to the horizontal. So their vertical component

$= (T\delta y) \sin \beta - (T\delta y) \sin \alpha = T\delta y(\tan \beta - \tan \alpha)$ approximately, since α and β are small

$$= T\delta y \left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x \right] = T\delta y \delta x \left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} \right]$$

$= T\delta x \delta y \dfrac{\partial^2 u}{\partial x^2}$ upto a first order of approximation.



Similarly, the forces $T\delta x$ acting on the edges of length δx have the vertical component $T\delta x \delta y \dfrac{\partial^2 u}{\partial y^2}$.

Hence the equation of motion of the element is

$$(m\delta x \delta y)\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\delta x \delta y$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \text{where } c^2 = \frac{T}{m}.$$

This is the wave equation in two dimensions.

17.10. SOLUTION OF THE TWO-DIMENSIONAL WAVE EQUATION

The two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ...(1)$$

Let $\quad u = XYT \quad ...(2)$

where X is a function of x only, Y is a function of y only and T is a function of t only, be a solution of (1).

Then $\quad \dfrac{\partial^2 u}{\partial t^2} = XYT'', \dfrac{\partial^2 u}{\partial x^2} = X''YT$ and $\dfrac{\partial^2 u}{\partial y^2} = XY''T$

Substituting in (1), we have $\dfrac{1}{c^2} XYT'' = X''YT + XY''T$

Dividing by XYT, we have $\quad \dfrac{1}{c^2}\dfrac{T''}{T} = \dfrac{X''}{X} + \dfrac{Y''}{Y} \quad ...(3)$

This will be true only when each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0$$

and

$$\frac{d^2T}{dt^2} + (k^2 + l^2)c^2T = 0$$

The solutions of these equations are respectively

$$X = c_1 \cos kx + c_2 \sin kx$$
$$Y = c_3 \cos ly + c_4 \sin ly$$

and

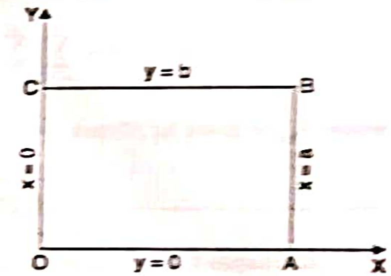
$$T = c_5 \cos \sqrt{(k^2 + l^2)}\, ct + c_6 \sin \sqrt{(k^2 + l^2)}\, ct$$

Hence from (2), a solution of (1) is

$$u(x, y, t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)$$
$$[c_5 \cos \sqrt{(k^2 + l^2)}\, ct + c_6 \sin \sqrt{(k^2 - l^2)}\, ct] \quad ...(4)$$

Now let us suppose that the membrane is rectangular and is stretched between the lines

$$x = 0, x = a, y = 0, y = b.$$



Then the boundary conditions are :

(i) $u = 0$, when $x = 0$

(ii) $u = 0$, when $x = a$

(iii) $u = 0$, when $y = 0$

(iv) $u = 0$, when $y = b$

for all t.

Applying the condition (i), we have

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)}\, ct + c_6 \sin \sqrt{(k^2 + l^2)}\, ct]$$

i.e., $\quad c_1 = 0$

Putting $c_1 = 0$ in (3) and applying the condition (ii), we have $\sin ka = 0$ or $k = \dfrac{m\pi}{a}$, where m is an integer.

Similarly, applying the conditions (iii) and (iv), we get

$$c_3 = 0 \quad \text{or} \quad l = \frac{n\pi}{b}, \text{ where } n \text{ is an integer.}$$

Therefore, the solution (3) becomes

$$u(x, y, t) = c_2c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (c_5 \cos pt + c_6 \sin pt)$$

where $\quad p = \pi c \sqrt{\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}}$

Replacing the arbitrary constants suitably, we can write the general solution as

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \quad ...(5)$$

Now suppose the membrane starts from rest from the initial position $u = f(x, y)$ i.e., $u(x, y, 0) = f(x, y)$.

Then applying the condition : $\frac{\partial u}{\partial t} = 0$ when $t = 0$, we get $B_{mn} = 0$.

Also using the condition : $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad ...(6)$$

This is a double Fourier series. Multiplying both sides by $\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one becomes zero. Thus, we get

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dy\, dx = \frac{ab}{4} A_{mn}$$

i.e.,

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{a}\, dy\, dx \quad ...(7)$$

Hence, from (5), the required solution is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where A_{mn} is given by (7) and $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos (\sqrt{5}\, \pi ct)$.*

Sol. Here we have to solve the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

with boundary conditions $u(0, y, t) = u(l, y, t) = u(x, 0, t) = u(x, 1, t) = 0$

and the initial conditions $u(x, y, 0) = f(x, y) = k \sin 2\pi x \sin \pi y$

$$\frac{\partial u}{\partial t} = 0 \text{ when } t = 0$$

Proceeding as in Art. 17.10, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y \cos pt \quad ...(1)$$

Since $a = b = 1$ where $p = \pi c \sqrt{m^2 + n^2}$

and

$$A_{mn} = \frac{4}{1 \times 1} \int_0^1 \int_0^1 k \sin 2\pi x \sin \pi y \sin m\pi x \sin n\pi y\, dy dx$$

$$= 4k \int_0^1 \sin m\pi x \sin 2\pi x\, dx \int_0^1 \sin n\pi y \sin \pi y\, dy$$

$$= 0 \text{ for } m \neq 2 \text{ or } n \neq 1$$

$$\therefore \quad A_{21} = 4k \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = k \quad \text{and} \quad p = \pi c \sqrt{(2)^2 + (1)^2} = \sqrt{5}\, \pi c$$

Hence solution (1) reduces to $u(x, y, t) = k \sin 2\pi x \sin \pi y \cos (\sqrt{5}\, \pi ct)$.

Example 2. *Find the deflection $u(x, y, t)$ of a rectangular membrane $(0 \le x \le a, 0 \le y \le b)$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = xy(a - x)(b - y)$.*

Sol. As in example 1, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \quad ...(1)$$

where $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

and

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b xy(a - x)(b - y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dy dx$$

$$= \frac{4}{ab} \int_0^a x(a - x) \sin \frac{m\pi x}{a}\, dx \int_0^b y(b - y) \sin \frac{n\pi y}{b}\, dy$$

$$= \frac{4}{ab} \left[\left\{ x(a - x) \left(-\frac{\cos \frac{m\pi x}{a}}{\frac{m\pi}{a}} \right) - (a - 2x) \left(-\frac{\sin \frac{m\pi x}{a}}{\left(\frac{m\pi}{a}\right)^2} \right) + (-2) \left(\frac{\cos \frac{m\pi x}{a}}{\left(\frac{m\pi}{a}\right)^3} \right) \right\}_0^a \left\{ y(b - y) \left(-\frac{\cos \frac{n\pi y}{b}}{\frac{n\pi}{b}} \right) - (b - 2y) \left(-\frac{\sin \frac{n\pi y}{b}}{\left(\frac{n\pi}{b}\right)^2} \right) + (-2) \left(\frac{\cos \frac{n\pi y}{b}}{\left(\frac{n\pi}{b}\right)^3} \right) \right\}_0^b \right]$$

$$= \frac{4}{ab} \left[\left(\frac{-2a^3}{m^3\pi^3} \cos m\pi + \frac{2a^3}{m^3\pi^3} \right) \left(-\frac{2b^3}{n^3\pi^3} \cos n\pi + \frac{2b^3}{n^3\pi^2} \right) \right]$$

$$= \frac{4}{ab} \cdot \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} [1 - (-1)^m][1 - (-1)^n]$$

$$= \begin{cases} 0, & \text{if } m \text{ or } n \text{ is even} \\ \frac{64a^2b^2}{m^3n^3\pi^6}, & \text{if both } m, n \text{ are odd} \end{cases}$$

Hence the deflection of rectangular membrane is given by (1) where

$$A_{mn} = \frac{64a^2b^2}{m^3n^3\pi^6} \text{ both } m, n \text{ odd} \quad \text{and} \quad p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

TEST YOUR KNOWLEDGE

1. Find the deflection $u(x, y, t)$ of a square membrane with $a = b = c = 1$, if the initial velocity is zero and the initial deflection $f(x, y) = k \sin \pi x \sin 2\pi y$.

17.12. LAPLACE'S EQUATION

We know that the two dimensional heat flow equation in steady state reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

which is *Laplace's equation in two dimensions.*

Also the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(2)}$$

which is *Laplace's equation in three dimensions.*

Laplace's equation has wide applications in physics and engineering. The theory of its solutions is called the *potential theory* and its solutions are called *harmonic functions*.

The solution of Laplace's equation, subject to certain boundary conditions, is simplified by a proper choice of coordinate system.

If the problem involves *rectangular boundaries*, we prefer to take Laplace's equation in *cartesian coordinates* given by (1) and (2).

If the problem involves circular boundaries, we prefer to take Laplace's equation in polar coordinates given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(3)$$

This equation can be obtained from (1) by putting $x = r\cos\theta$, $y = r\sin\theta$, thus changing the independent variables from (x, y) to (r, θ).

If the problem involves *cylindrical boundaries*, we prefer to take Laplace's equation in *cylindrical coordinates* given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(4)$$

This equation can be obtained from (2) by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, thus changing the independent variables (x, y, z) to (r, θ, z).

If the problem involves *spherical boundaries*, we prefer to take Laplace's equation in *spherical coordinates* given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad ...(5)$$

This equation can be obtained from (2) by putting,

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,$$

thus changing the independent variables (x, y, z) to (r, θ, ϕ).

17.13. SOLUTIONS OF LAPLACE'S EQUATION

(a) Solution of Laplace's Equation in Two-dimensional Cartesian Form

We have already discussed the solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(b) Solution of Laplace's Equation in Polar Coordinates

Laplace's equation in polar coordinates is $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$ ...(1)

Let $u(r, \theta) = R(r)F(\theta)$ or simply $u = RF$ ...(2)

where R is a function of r only and F is a function of θ only, be a solution of (1).

Substituting it in (1), we get $r^2R''F + rR'F + RF'' = 0$

or $(r^2R'' + rR')F + RF'' = 0$

Separating the variables, $\frac{r^2R'' + rR'}{R} = -\frac{F''}{F}$ = constant = k (say)

Thus, we get ordinary differential equations

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - kR = 0 \qquad ...(3)$$

and $$\frac{d^2F}{d\theta^2} + kF = 0 \qquad ...(4)$$

Now (3) is a homogeneous linear differential equation.

Putting $r = e^z$, (3) reduces to $\frac{d^2R}{dz^2} - kR = 0$ ...(5)

Solving (5) and (4), we get

(i) *When k is positive and* $= p^2$, say

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},$$
$$F = c_3\cos p\theta + c_4\sin p\theta.$$

(ii) *When k is negative and* $= -p^2$, say

$$R = c_1\cos pz + c_2\sin pz = c_1\cos(p\log r) + c_2\sin(p\log r)$$
$$F = c_3e^{p\theta} + c_4e^{-p\theta}$$

(iii) *When k = 0*

$$R = c_1z + c_2 = c_1\log r + c_2$$
$$F = c_3\theta + c_4$$

Thus the three possible solutions of (1) are

$$u = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta) \qquad ...(6)$$
$$u = [c_1\cos(p\log r) + c_2\sin(p\log r)](c_3e^{p\theta} + c_4e^{-p\theta}) \qquad ...(7)$$
$$u = (c_1\log r + c_2)(c_3\theta + c_4) \qquad ...(8)$$

Of these solutions, we choose the one which is consistent with the physical nature of the problem.

Note. Usually we require a solution extending up to the origin.

Since u must be finite at the origin, we reject solutions (7) and (8). Also from (6), $c_2 = 0$.

∴ In this case, the solution may be written as

$$u = (A\cos p\theta + B\sin p\theta)r^p$$

The general solution will consist of a sum of similar terms with different (arbitrary) values of A, B and p.

(c) Solution of Laplace's Equation in Three-dimensional Cartesian Form

Laplace's equation in three-dimensional cartesian form is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

Let $u(x, y, z) = X(x)\,Y(y)\,Z(z)$ or simply $u = XYZ$ ...(2)

be a solution of (1).

Substituting it in (1), we get $X''YZ + XY''Z + XYZ'' = 0$

Dividing by XYZ, we get $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$

or $$\frac{1}{X}\cdot\frac{d^2X}{dx^2} + \frac{1}{Y}\cdot\frac{d^2Y}{dy^2} + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0 \qquad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

Since x, y, z are independent, this is possible only when F_1, F_2, F_3 are constants. Assuming these constants to be k^2, l^2 and $-(k^2+l^2)$ respectively, (3) gives rise to the following equations :

$$\frac{d^2X}{dx^2}-k^2X=0,\quad \frac{d^2Y}{dy^2}-l^2Y=0,\quad \frac{d^2Z}{dz^2}+(k^2+l^2)Z=0$$

Their solutions are $\quad X=c_1e^{kx}+c_2e^{-kx},\quad Y=c_3e^{ly}+c_4e^{-ly}$

$$Z=c_5\cos\sqrt{(k^2+l^2)}\,z+c_6\sin\sqrt{(k^2+l^2)}\,z$$

Hence a solution of (1) is

$$u=(c_1e^{kx}+c_2e^{-kx})(c_3e^{ly}+c_4e^{-ly})\,[c_5\cos\sqrt{(k^2+l^2)}\,z+c_6\sin\sqrt{(k^2+l^2)}\,z]$$

Since the three constants could have been taken as $-k^2, -l^2$ and k^2+l^2, an alternative solution of (1) is

$$u=(c_1\cos kx+c_2\sin kx)\,(c_3\cos ly+c_4\sin ly)\,[c_5e^{\sqrt{k^2+l^2}\,z}+c_6e^{-\sqrt{k^2+l^2}\,z}]$$

The choice of the constants and hence the general solution depends on the given initial and boundary conditions.

(d) Solution of Laplace's Equation in Cylindrical Coordinates

Laplace's equation in cylindrical coordinates is

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\partial^2u}{\partial z^2}=0 \qquad ...(1)$$

Let $\quad u(r,\theta,z)=R(r)F(\theta)Z(z)$

or $\quad$ simply $\quad u=RFZ \qquad ...(2)$

be a solution of (1).

Substituting it in (1), we get $\quad R''FZ+\frac{1}{r}R'FZ+\frac{1}{r^2}RF''Z+RFZ''=0$

Dividing by RFZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2F}\frac{d^2F}{d\theta^2}+\frac{1}{z}\frac{d^2Z}{dz^2}=0 \qquad ...(3)$$

Assuming $\quad \frac{d^2F}{d\theta^2}=-n^2F$ and $\frac{d^2Z}{dz^2}=k^2Z \qquad ...(4)$

Equation (3) reduces to $\quad \frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)-\frac{n^2}{r^2}+k^2=0$

or

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(k^2r^2-n^2)R=0$$

This is Bessel's equation. Its solution is $\quad R=c_1J_n(kr)+c_2Y_n(kr)$

The solutions of equations (3) are $F=c_3\cos n\theta+c_4\sin n\theta,\quad Z=c_5e^{kz}+c_6e^{-kz}$

Hence a solution of (1) is $\quad u=[c_1J_n(kr)+c_2Y_n(kr)]\,(c_3\cos n\theta+c_4\sin n\theta)(c_5e^{kz}+c_6e^{-kz})$

which is known as a *cylindrical harmonic*.

(e) Solution of Laplace's Equation in Spherical Coordinates

Laplace's equation in spherical coordinates is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0 \qquad ...(1)$$

Let $\quad u(r,\theta,\phi)=R(r)G(\theta)H(\phi)$

or $\quad$ simply $u=RGH \qquad ...(2)$

be a solution of (1).

Substituting it in (1), we get

$$R''GH+\frac{2}{r}R'GH+\frac{1}{r^2}RG''H+\frac{\cot\theta}{r^2}RG'H+\frac{1}{r^2\sin^2\theta}RGH''=0$$

Dividing by RGH, we get

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)+\frac{1}{G}\left(\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}\right)+\frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2}=0 \qquad ...(3)$$

Putting $\quad \frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \qquad ...(4)$

and $\quad \frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2 \qquad ...(5)$

Equation (3) reduces to $\quad \frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\operatorname{cosec}^2\theta]G=0$

This is associated Legendre's equation and its solution is

$$G=c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)$$

The solution of (5) is $\quad H=c_3\cos m\phi+c_4\sin m\phi$

To solve (4), assume that $R=r^k$ so that

$$k(k-1)+2k=n(n+1)\quad\text{or}\quad(k^2-n^2)+(k-n)=0$$

or $\quad (k-n)(k+n+1)=0 \quad \therefore\ k=n\ \text{ or }\ -n-1$

Thus $\quad R=c_5r^n+c_6r^{-n-1}$

Hence the general solution of (1) is

$$u=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)]\,(c_3\cos m\phi+c_4\sin m\phi)(c_5r^n+c_6r^{-n-1})$$

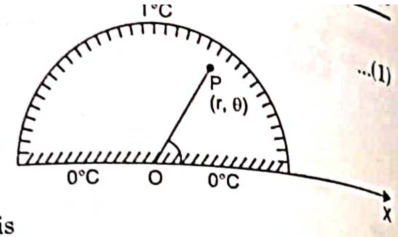
Any solution of (1) is known as a *spherical harmonic*.

Example. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r,\theta)=\frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta.$$

Sol. Take the centre of the circle as the pole and the bounding diameter as the initial line. Let the steady state temperature at any point $P(r,\theta)$ be $u(r,\theta)$, so that u satisfies the equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \qquad ...(1)$$



The boundary conditions are

$$u(r, 0) = 0 \quad \text{in} \quad 0 \le r \le a \quad ...(2)$$

$$u(r, \pi) = 0 \quad \text{in} \quad 0 \le r \le a \quad ...(3)$$

and $$u(a, \theta) = T \qquad ...(4)$$

By Art. 17.13 (*b*), the appropriate solution of (1) is

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \qquad ...(5)$$

Applying condition (2), $u(r, 0) = (c_1 r^p + c_2 r^{-p})c_3 = 0$

$\therefore$ $c_3 = 0$ and equation (5) becomes

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p})c_4 \sin p\theta \qquad ...(6)$$

Applying condition (3), $u(r, \pi) = (c_1 r^p + c_2 r^{-p})c_4 \sin p\pi = 0$

or $\sin p\pi = 0 \quad \therefore \quad p = n$, where n is an integer.

Hence equation (6) reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n})c_4 \sin n\theta \qquad ...(7)$

Since $u = 0$ when $r = 0$, $\therefore$ $c_2 = 0$

and equation (7) becomes $u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$

$\therefore$ The most general solution of (1) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(8)$$

Applying condition (4), $u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta = T$

or $$T = \sum_{n=1}^{\infty} B_n \sin n\theta, \quad \text{where } B_n = b_n a^n$$

and $$B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta \, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi)$$

$\therefore$ $$b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1-\cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4T}{n\pi a^n}, & \text{if } n \text{ is odd} \end{cases}$$

Hence from (8), we have

$$u(r, \theta) = \frac{4T}{\pi}\left[\frac{(r/a)}{1}\sin\theta + \frac{(r/a)^3}{3}\sin 3\theta + \frac{(r/a)^5}{5}\sin 5\theta + \ldots\right]$$

$$= \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta.$$

TEST YOUR KNOWLEDGE

1. Show that the steady state temperature distribution in a semi-circular plate of radius a whose bounding diameter is kept at 0°C, while the circumference is kept at 60°C is given by

$$u(r, \theta) = \frac{240}{\pi}\sum_{n=1}^{\infty}\frac{(r/a)^{2n-1}}{2n-1}\cdot\sin(2n-1)\theta.$$

2. A semi-circular plate of radius a has its circumference kept at temperature $u(a, 0) = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Assuming the surfaces of the plate to be insulated, show that the steady state temperature distribution of the plate is given by

$$u(r, \theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n-1}\cdot\frac{\sin(2n-1)\theta}{(2n-1)^3}.$$

3. The bounding diameter of a semi-circular plate of radius a is kept at 0°C and the temperature along the semi-circular boundary is given by

$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 < \theta < \dfrac{\pi}{2} \\ 50(\pi - \theta), & \text{when } \dfrac{\pi}{2} < \theta < \pi \end{cases}.$$

Show that the steady state temperature distribution is given by

$$u(r, \theta) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta.$$

4. Show that the temperature distribution function $F(r, \theta)$ inside the circle $|z| = 1$, when

$$F(\theta) = \begin{cases} T, & 0 \le \theta \le \pi \\ -T, & \pi \le \theta \le 2\pi \end{cases}$$

on its circumference, is given by $F(r, \theta) = T\left[1 - \frac{2}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right)\right].$

5. Solve the equation $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0$

with the boundary conditions $\frac{\partial v}{\partial r} = 0, r = a$; $\frac{\partial v}{\partial r} \to u_0\cos\theta, r \to \infty$ where u_0 is a constant.

Answer

5. $v = u_0\left(1 + \frac{a^3}{2r^3}\right)r\cos\theta.$