S. No.	Date	Topic	Page No.	Teacher's Sign. & Remarks

Discrete Distribution.

1. Binomial Distⁿ
2. Poisson Distⁿ
3. Geometric Distⁿ
4. Negative Binomial Distⁿ
5. Hypergeometric Distⁿ

Continous Distribution

1. Uniform (Rectangular) distⁿ
2. Gamma Distⁿ
3. Exponential Distⁿ
4. Weilbull distⁿ.
5. Normal Distⁿ

Introduction –

We often hear such statements: 'it is likely to rain today'. I have a fair chance of getting admission and " there is an even chance that in tossing a coin the head may come up'. In each case we are not certain of the outcome but we wish to assess the chances of our predictions coming true. The study of probability provides a mathematical frame work for such assertions and is essential in every decision making process.

Principle of Counting:– If an event can happen in n_1 ways and thereafter for each of these events a 2nd event can happen in n_2 ways and for each of these first and 2nd events a third event can happen for n_3 ways and so on, then the number of ways these m event can happen is given by the product $n_1 \cdot n_2 \cdot n_3 \cdots n_m$

Permutation – A permutation of a number of objects is their arrangement in some definite order.

Given three letters a, b, c we can permute them two at a time as bc, cb, ca, ac, ab, ba yielding 6 permutations.

The combinations or grouping are only 3 i.e bc, ca, ab Here the order is immetrial.

e.g. The number of permutation of n different thing taken r at a time is

$n(n-1)(n-2)\cdots(n-r+1)$ which denoted by ^nP_r

Thus $$\boxed{^nP_r = \frac{n!}{(n-r)!}}$$

Permutation with repetitions

The number of permutations of n objects of which n_1 are alike n_2 are alike and n_3 are alike.

$$\boxed{\frac{n!}{n_1!\, n_2!\, n_3!}}$$

Combinations:- The number of Combination of n different objects taken r at a time is denoted by ^nC_r. If we take any one of the Combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements which can be obtained from all the Combinations is

$$\boxed{^nP_r = {}^nC_r\, r!} \qquad {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

→ n Person can be seated in n chairs at a round table in $(n-1)!$

Ex n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.

Sol^n Since n persons can be seated in n chairs at a round table in $(n-1)!$ ways, the exhaustive number of cases $= (n-1)!$

Assuming the two specified persons A and B who sit together as one we get $(n-1)$ persons in all, who can be seated at a round table in $(n-2)!$ ways. Further since A and B can interchange their positions in $2!$ ways, total number of favourable cases of getting A and B together is $(n-2)! \times 2!$

$$\therefore \text{ Required probability} = \frac{(n-2)!\,2!}{(n-1)!}$$

$$= \frac{2}{n-1} \quad \text{Ans}$$

Q. In how many ways can one make a first, 2nd third and 4th choice among 12 firms leasing construction equipment.

Sol^n first choice can be made from any of the 12 firms. thereafter the 2nd choice can be made among the remaining 11 firms. then the third choice can be made from the remaining 10 firms and the fourth choice can be made from the 9 firms.

Thus from the principle of counting, the number of ways in which Ist, 2nd, 3rd and fourth choice can be affected $= 12 \times 11 \times 10 \times 9 = 11880$

Q. find the number of permutations of all the letters of the word (i) Committee (ii) Engineering.

(i) $n = 8+1 = 9$ $\quad n_1 (m,m) = 2$ $\quad$ Committee.

$n_2 (t,t) = 2$

$n_3 (e,e) = 2$

$\therefore$ No of permutations $= \dfrac{n!}{n_1!\, n_2!\, n_3!}$

$= \dfrac{9!}{2!\, 2!\, 2!}$

$= 45360$

(II) $n = 11$

$n_1 (e,e,e) = 3$

$n_2 (n,n,n) = 3$

$n_3 (g,g) = 2$

$n_4 (i,i) = 2$

$\therefore$ No of permutations $= \dfrac{11!}{3!\, 2!\, 2!\, 3!} = 277200$

Q. from six Engineers and five architects a Committee is to be formed having three Engineers and two architects. How many different Committees can be formed if (i) there is no restriction.

(ii) two particular engineers must be included.

(iii) one particular architect must be Excluded.

Solⁿ (i) Number of Committees $= {}^6C_3 \times {}^5C_2 = \frac{6.5.4}{3.2.1} \times \frac{5.4}{2.1} = 200$

(ii) Here we have to choice one Engineer from the remaining four engineers.

$\therefore$ No of Committees $= {}^4C_1 \times {}^5C_2 = 4 \times \frac{5.4}{2.1} = 40$

(iii) Here we have to choose two architects from the remaining four architects.

$\therefore$ No of Committees $= {}^6C_3 \times {}^4C_2 = \frac{6.5.4}{3.2.1} \times \frac{4.3}{2.1} = 120$

Basic Terminology :-

1. **Random Experiment :-** If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a random experiment

Example of random experiment - tossing a coin, throwing a die, selecting a card from a pack.

2. **Outcome :-** The result of a random experiment will be called an outcome.

3. **Trial and Event :-** Any particular performence of a random experiment is called a trial and outcomes or combination of outcomes are termed as events.

for example - If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces, head or tail. Thus tossing of a coin is a random experiment or trial and getting of a head or tail is an event.

4. **Exhaustive event :- (or cases)**

The total number of possible outcomes of a random experiment is known as the exhaustive events or cases.

for example - In tossing of a coin, there are two exhaustive cases head and tail.

In throwing of a die there are 6 exhaustive cases any one of the 6 faces 1, 2 --- 6 may come uppermost.

5. Favourable Events:-

The number of cases favourable to an event in a trial is the number of outcome which entail the happening of the event.

for example:- In a drawing card from a pack of cards the number of cases favourable to drawing of an ace is 4. for drawing a spade is 13 and for drawing a red card is 26

6. Mutually Exclusive Event:-

Events are said to be mutually exclusive or incomplete, if the happening of any one of them can happen simultaneously in the same trial

~~for example; Events are said to be mutually exclusive or incomplete, if the happening of any one of them can happen simultaneously in the same trial~~

~~for example:- In drawing a card from a pack of cards the number of cases favourable to drawing of an ace is 4. for drawing is spade is 13 and for drawing a red card is 26~~

for example:- In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive.

7. Equally likely Events:-

Events are said to be equally likely if there is no reason to expect any one in perference to any other.

for example, in throw of a coin two cases i.e head and tail are equally likely to come.

8. Independent Events:- when the actual happening of one event does not influence in any why the probability of the happening of the other, the two events are called independent events.

in throwing two coins at a time, the outcome of one is independent of the 2nd. But in case a card is drawn from a pack of well shuffled cards and is not replaced them, the 2nd draw of the card is dependent the 2nd draw is then dependent on the first draw. Event.

9. Odds in favour of or against the trial:-

If an experiment can succeed in m ways and fail in n ways, each of these ways being equally likely, then the odds are m to n in favour or n to m against the trail.

Ex If the probability of the horse A winning the race is $\frac{1}{5}$ and the probability of the horse B winning the same race is $\frac{1}{6}$, what is the probability that one of the horses will win the race?

Sol^n Probability of winning of the horse $A = \frac{1}{5}$

Probability of winning of the horse $B = \frac{1}{6}$

$$P(A+B) = P(A) + P(B) = \frac{1}{5} + \frac{1}{6} = \frac{11}{30}$$

Ex A policeman fires four bullets on a dacoit. The probability that the dacoit will be killed by one bullet is 0.6 what is the probability that the dacoit is still alive?

Sol^n Let A_i be the event that the dacoit is not killed by ith bullet. Then,

$$P(A_i) = 1 - 0.6 = 0.4$$

Now, probability that the dacoit is still alive

$$= P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$= P(A_1)\, P(A_2)\, P(A_3)\, P(A_4)$$ [Since all 4 shots are independent]

$$= (0.4)(0.4)(0.4)(0.4)$$

$$= (0.4)^4$$

Ex An urn A contains 2 white and 5 black balls, and an urn B contains 5 black and 6 white balls. Find the probability that at least one ball in white when one ball each is drawn at random from the urns A & B

Solⁿ Probability of drawing a white ball from the urn A $= \frac{2}{7} = p_1$ (say)

Probability of not drawing a white ball from the urn A $= 1-\frac{2}{7} = \frac{5}{7} = q_1$ (say)

Similarly for the urn B, we have

$$p_2 = \frac{6}{11}, \quad q_2 = \frac{5}{11}$$

Therefore, the probability that at least one of the balls in white (the events being independent)

$$= 1-q_1q_2 = 1-\frac{5}{7}\times\frac{5}{11} = \frac{52}{77}$$

Ex Players A and B throw a pair of dice. A wins and B, if he if he throws 6 before B throws 7 before A throws 6. If A begins, show that his chance of winning is $\frac{30}{61}$.

Solⁿ When a pair of dice is thrown the total number of possible cases are 36

The number of possible cases for getting a sum of are (1,5) (2,4), (3,3) (4,2) and (5,1) they are 5 in number

∴ The ~~number~~ probability of A getting the sum $6 = \frac{5}{36}$

The probability of A not getting the sum $6 = \frac{31}{36}$

Similarly, the number of possible cases for getting a sum of 7 are (1,6) (2,5) (3,4) (4,3) (5,2) and (6,1) which are ~~in~~ 6 in number.

$\therefore$ the probability of B getting the sum 7 is $= \frac{6}{36}$

and the probability of B not getting the sum $7 = 1 - \frac{6}{36} = \frac{30}{36}$

If A starts the game: the probability of A wins the game = in Ist time A wins or the 3rd time A wins or 5th time A wins and so on . .

$$= \frac{5}{36} + \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{5}{36} + \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{31}{36} \cdot \frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1 + \frac{31}{36} \cdot \frac{30}{36} + \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{31}{36} \cdot \frac{5}{36} + \cdots\right]$$

$$= \frac{5}{36}\left[1 + \frac{31}{36} \cdot \frac{30}{36} + \left(\frac{31}{36} \cdot \frac{30}{36}\right)^2 + \cdots\right]$$

$$= \frac{5}{36}\left[\frac{1}{1 - \left(\frac{31}{36}\right)\left(\frac{30}{36}\right)}\right] = \frac{30}{61}$$

Note suppose in the question given B starts the game. the probability of A wins the game Ist time B loses and A wins or B loses. A loses B loses and wins so on.

$$= \frac{30}{36} \cdot \frac{5}{36} + \frac{30}{36} \cdot \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{5}{36} + \cdots$$

$$= \frac{30}{36} \cdot \frac{5}{36}\left[1 + \frac{31}{36} \cdot \frac{30}{36} + \left(\frac{31}{60} \cdot \frac{30}{36}\right)^2 - \cdots\right]$$

$$= \frac{30}{36}\left(\frac{5}{36}\right)\left[\frac{1}{1 - \left(\frac{31}{36}\right)\left(\frac{30}{36}\right)}\right]$$

Conditional probability –

Let A and B be two events associated with the same sample space of a random experiment. Then the probability of occurrence of A under the condition that B has already occurred at $P(B) \neq 0$ is called conditional probability, denoted by $P(A/B)$

We define $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$$

where $P(B) \neq 0$

Properties of conditional probability

1. Let A and B be events of a sample space S of an experiment, Then We have

Property – I $P(S/B) = P(B/B) = 1$

We know that

$$P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Also $$P(B/B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Thus $P(S/B) = P(B/B) = 1$

Property – II

If A and B are any two events of a sample space S and F is any event of S such that $P(F) \neq 0$ then

$$\boxed{P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)}$$

Property - 3

$$\boxed{P(E'|F) = 1 - P(E|F)}$$

Ex A pair of dice is rolled, find $P(A|B)$ if

A : 2 appears on atleast one dice

B : Sum of numbers appearing on dice is 6

Sol^n We have

$A = \{(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)\ (1,2)\ (3,2)$
$(4,2)\ (5,2)\ (6,2)\}$

$B = \{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\}$

$A \cap B = \{(2,4)(4,2)\}$

$P(A \cap B) = \frac{2}{36}$

$P(B) = \frac{5}{36}$

Therefore $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$

Bayes theorem:- If $E_1, E_2 - - E_n$ are mutually disjoint events with $P(E_i) \neq 0$ $(i = 1, 2 - - n)$, then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$ we have.

$$\boxed{P(E_i|A) = \frac{P(E_i)\,P(A|E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A|E_i)}}$$

Ex In 2002 there will be three Candidates for the position of principal - Mr Chatterji, Mr. Ayangar and Dr. Singh whose chances of getting the appointment are in the proportion 4 : 2 : 3 respectively. the probability that Mr chatterji is selected would introduce co-education in the College is 0.3 the probobilities of Mr Ayangar and Dr. Singh doing the same are respectively 0.5 and 0.8

(I) what is the probability that there will be co-education in the College in 2003?

(II) If there is coeducation in the College in 2003 what is the probability that Dr. Singh is the principal.

Soln Let us define the following events

A : Introduction of co- education

E_1 Mr chaterji is selected as a principle

E_2 Mr. Ayangar is seclected as a principal

E_3 Dr singh is selected as a principal

$P(E_1) = \frac{4}{9}$ $P(E_2) = \frac{2}{9}$ $P(E_3) = \frac{3}{9}$

$P(A|E_1) = \frac{3}{10}$ $P(A|E_2) = \frac{5}{10}$ and $P(A|E_3) = \frac{8}{10}$

(i) The required probability that there will be Coeducation in the College in 2003 is given by

$$P(A) = P[(A\cap E_1)\cup(A\cap E_2)\cup(A\cap E_3)]$$

$$= P[(A\cap E_1) + (A\cap E_2) + (A\cap E_3)]$$

$$= P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + P(E_3)\,P(A|E_3)$$

$$= \frac{4}{9}\times\frac{3}{10} + \frac{2}{9}\times\frac{5}{10} + \frac{3}{9}\times\frac{8}{10} = \frac{46}{90} = \frac{23}{45}$$

(ii) The required probability is given by bayes Rule by

$$P(E_3|A) = \frac{P(E_3)\,P(A|E_3)}{P(A)}$$

$$= \frac{\frac{3}{9}\times\frac{8}{10}}{\frac{46}{90}} = \frac{24}{46} = \frac{12}{23}$$

Ex A speakes truth 4 out of 5 times. A die is tossed he reports that there is a six what is the chance that actually there was six?

Sol Let us define the following events

E_1 : A speaks truth E_2 A tells a lie

E : A reports a six

We have.

$P(E_1) = \frac{4}{5}$ $P(E_2) = \frac{1}{5}$

$P(E|E_1) = \frac{1}{6}$ $P(E|E_2) = \frac{5}{6}$

The required probability that actually there was six (By Bayes theorem) is

$$P(E_1|E) = \frac{P(E_1)\times P(E|E_1)}{P(E_1)\times P(E|E_1) + P(E_2)\times P(E|E_2)} = \frac{\frac{4}{5}\cdot\frac{1}{6}}{\frac{4}{5}\times\frac{1}{6}+\frac{1}{5}\cdot\frac{5}{6}} = \frac{4}{9}\text{ Ans}$$

Q. from a vessel Containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. from this vessel a ball is drawn and is found to be white. what is the probability that out of four balls transferred 3 are white and 1 is black?

Sol^n Let us define the following events:

E_1 : Transfer of 0 white and 4 black balls.

E_2 : Transfer of 1 white and 3 black balls

E_3 : Transfer of 2 white and 2 black balls

E_4 : Transfer of 3 white and 1 black balls.

(Since the urn contains 3 white balls, more than 3 white balls Can not be transferred from the vessel)

E: Drawing of a white ball from the 2nd vessel.

Then $P(E_1) = \dfrac{^5C_4}{^8C_4} = \dfrac{1}{14}$ $\qquad P(E_2) = \dfrac{^3C_1 \times {^5C_3}}{^8C_4} = \dfrac{3}{7}$

$P(E_3) = \dfrac{^3C_2 \times {^5C_2}}{^8C_4} = \dfrac{3}{7}$ $\qquad P(E_4) = \dfrac{^3C_3 \times {^5C_1}}{^8C_4} = \dfrac{1}{14}$

Also $P(E|E_1) = 0$ $\qquad P(E|E_2) = \dfrac{1}{4}$ $\qquad P(E|E_3) = \dfrac{2}{4}$

and $P(E|E_4) = \dfrac{3}{4}$

Hence By Bayes theorem, the probability that out of four balls transferred. 3 are white and is black is

$$P(E_4|E) = \frac{\frac{1}{14} \times \frac{3}{4}}{\frac{1}{14} \times 0 + \frac{3}{7} \times \frac{1}{4} + \frac{3}{7} \times \frac{1}{2} + \frac{1}{14} \times \frac{3}{4}}$$

$$= \frac{3}{6+12+3} = \frac{1}{7} = 0.14$$

Random Variable –

We are considering a function whose domain is the set of possible outcomes, and whose range is a subset of the set of reals such a function is called a random variable.

intuitively by a random variable (r.v.) we mean a real number X connected with the outcome of a random experiment E. for example, if E consist of two tosses the r.v which is the number of heads (0, 1 or 2)

Out come :	HH	HT	TH	TT
Value of X :	2	1	1	0

Discrete Random Variable –

A random variable X, which can take only a finite number of values in an interval of the domain is called discrete random variable.

Ex

1. Number appearing on top of a die when it is thrown.
2. The number of telephone calls received per day.

Discrete Probability Distribution:-

If a random variable x can assume a discrete set of values say $x_1, x_2 \cdots x_n$ with respect to probabilities $P_1, P_2 \cdots P_n$ such that $P_1+P_2+\cdots P_n = 1$ i.e $\sum_{i=1}^{n} P_i = 1$ then occurrence of values x_i with respective probabilities P_i is called the discrete probability distribution of X.

for example in a throw of a pair of dice the sum (X) is discrete random variable which is an integer between 2 and 12 with probabilities P(X) given as

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

this Constitute a discrete probability distribution.

Probability Mass function:-

If X is a discrete random variable taking at most a countably infinite number of values x_1, x_2 -- then its probabilistic behaviour at each real point is described by a function called the probability mass function (or discrete density function) which is defined below:

Definition— If X is a discrete r.v with distinct Values $x_1, x_2 \ldots x_n \ldots$ then

$$P(x) = \begin{cases} P(X=x_i) = P_i & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \quad i = 1, 2 \ldots \end{cases}$$

is called the probability mass function of r.v X.

The function $p(x)$ satisfies the Condition

(i) $P(x_i) \geq 0$

(ii) $\sum P(x_i) = 1$

Comulative Distribution function (Distribution function)

If X is a r.v. then $P(X \leq x)$ is called the Cumulative distribution function (cdf) or distribution function and is denoted by $F(x)$

$$F(x) = P(X < x)$$

Expectation of a discrete random Variable–

If x is discrete random Variable which assumes the discrete set of values $x_1, x_2 \dots x_n$ with respective probabilities $P_1, P_2 \dots P_n$ then the expectation or Expected value of x is denoted by $E(X)$ and defined as

$$E(X) = x_1P_1 + x_2P_2 + x_3P_3 - \dots + x_nP_n$$

$$= \sum_{i=1}^{n} x_iP_i$$

||ly the expected value of X^2 is defined as

$$E(X^2) = \sum_{i=1}^{n} x_i^2 p_i$$

Variance and standard Deviation of Discrete R.V.

The variable of discrete r.v X is expected value of $(X-\mu)^2$ where μ is mean of Variable X.

$$\text{Var } X = V(x) = E(X-\mu)^2 = E(X^2) - [E(X)]^2$$

The standard deviation (SD) of a random variable x.

$$SD(X) = \sqrt{V(X)} = \sqrt{E(X)^2 - (E(X))^2}$$

Ex (i) A pair of two coins is tossed, what is the expected value?

(ii) A pair of dice is thrown together, find the expected value.

Sol^n (i) Expected value or mean value $= E(X) = \mu$

$$= \sum_{i=1}^{n} p_i x_i$$

In tossing of two coins, probability distribution is represented in tabular form as follows

X	0	1	2
P(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$\therefore E(X) = \frac{1}{4} \times 0 + \frac{1}{2} \times 1 + \frac{1}{4} \times 2 = 1$$

As the probability of getting no head and two head is respectively $\frac{1}{4}; \frac{1}{2}; \frac{1}{4}$

(ii) In a throw of pair of dice the sum (X) is a discrete r.v., which is an integer between 2 and 12 with the probabilities as given below.

X :	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\therefore$ Expected value $= E(X) = \mu$

$$= \frac{1}{36}\cdot 2 + \frac{2}{36}\cdot 3 + \frac{3}{36}\cdot 4 + \cdots + \frac{3}{36}\cdot 10 + \frac{2}{36}\cdot 11 + \frac{1}{36}\cdot 12$$

$$= \frac{252}{36} = 7$$

Ex A bag contains 8 items of which 2 are defective. A man select 3 items at random. Find the expected number of defective items he has drawn.

Sol^n The expected number of defective items can be zero defective, one defective, two defective items. Thus a r.v. may take value 0, 1 and 2

Now $P_1 = P(X=0) = \frac{{}^6C_3 \times {}^2C_0}{{}^8C_3}$

$$= \frac{6!}{3!\times 2!} \times \frac{2!}{2!} \times \frac{3!\times 5!}{8!}$$

$$= \frac{20}{56}$$

$$P_2 = P(X=1) = \frac{{}^6C_2 \times {}^2C_1}{{}^8C_3}$$

$$= \frac{6!}{4!\times 1!} \times \frac{2!}{1\times 1} \times \frac{3!\times 5!}{8!} = \frac{30}{56}$$

$$P_3 = P(X=2) = \frac{{}^6C_1 \times {}^2C_2}{{}^8C_3} = \frac{6!}{5!} \times \frac{2!}{2!} \times \frac{3!\times 5!}{8!}$$

$$= \frac{6}{56}$$

Hence the expected number of defective items drawn is

$$E(X) = P_1x_1 + P_2x_2 + P_3x_3 = \frac{20}{56}\times 0 + \frac{30}{56}\times 1 + \frac{6}{56}\times 2 = \frac{42}{56} = \frac{3}{4} \text{ Ans}$$

Ex A Player tossed two coins. If two heads show he wins ₹ 4. If one head shows he wins ₹ 2 but if two tails show he pays ₹ 3 as Penalty. Calculate the expected value of the game to him.

Solⁿ Here X takes the values 0, 1, 2

Also $P_1 = P(X = \text{Zero head}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P_2 = P(X = \text{one head}) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

$P_3 = P(X = \text{two head}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Also when

$X = 0 \Rightarrow x_1 = ₹ = 3$

$X = 1 \Rightarrow x_2 = ₹ = 2$

$X = 2 \Rightarrow x_3 = ₹ = 4$

We want to find out

$E(X) = P_1 x_1 + P_2 x_2 + P_3 x_3 =$

$= \frac{1}{4}(3) + \frac{1}{2}(2) + \frac{1}{4}(4) = \frac{5}{4} = 1.25$ Ans.

Binomial Distribution:-

$$\boxed{P(r) = {}^{n}C_{r}\, p^{r} q^{n-r}}$$

Properties of a Binomial Distribution.

1. It is discrete distribution which gives the theoretical probabilities.
2. It depends on the parameters p or q the probability of success or failure and n (the number of trails). The parameter n is always a +ve integer.
3. The distribution will be symmetrical if $p = q$
4. The statistics of the Binomial distribution.

   mean $= np$

   Variance $= npq$

   Standard deviation $= \sqrt{npq}$

Conditions for application of Binomial Distribution -

1. The Variable should be discrete
2. The happening of events must be of two alternative it must be either a success or failure.
3. The number of trial n should be finite and small.
4. The trials or events must be independent. The happening of one event must not affective happening of other events.

Ex Ten Coins are thrown simultaneously. find the probability of getting at least seven heads.

Sol when one coin is thrown

the probability of getting a head $= \frac{1}{2}$

$\therefore p = \frac{1}{2}$

Probability of not getting a head $= 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore q = \frac{1}{2}$

Then p (at least 7 heads)

$= p(7 \text{ heads}) + p(8 \text{ heads}) + p(9 \text{ heads}) + p(10 \text{ heads})$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$

$= \frac{1}{2^{10}} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$

$= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024} = \frac{11}{64}$

Ex In a lot of 200 articles 10 are defective. find the probability of (i) No defective article

(ii) One defective article

(iii) at least one defective article

, In a random sample of. 20 articles.

Sol^n The probability of defective article is $\frac{10}{200} = \frac{1}{20}$

$\therefore \; p = \frac{1}{20}$ The probability of non defective article $= \frac{19}{20}$

(I) The probability of non-defective article out of 20

$$= {}^{20}C_0 \, p^0 q^{20} = \left(\frac{19}{20}\right)^{20}$$

(II) The probability of exactly one defective article

$$= {}^{20}C_1 \, p^1 q^{19} = 20 \times \frac{1}{20} \times \left(\frac{19}{20}\right)^{19} = \left(\frac{19}{20}\right)^{19}$$

(III) The probability of at least one will be defective

= 1 − [Probability that none will be defective]

$$= 1 - {}^{20}C_{20} \left(\frac{19}{20}\right)^{20} = 1 - \left(\frac{19}{20}\right)^{19}$$

Example → If on an average, one ship out of 10 is wrecked, find the probability that out of 5 ships expected to arrive the port, at least four will arrive safely.

Sol^n : P. be the probability of a ship arriving safely

$$= 1 - \frac{1}{10} = \frac{9}{10}$$

$$q = 1 - \frac{9}{10} = \frac{1}{10}$$

Probability that at least four ships out of five arrive safely

$$= P(4) + P(5)$$

$$= {}^5C_4 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^4 + {}^5C_5 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^5$$

$$= \left(\frac{9}{10}\right)^4 \left(\frac{14}{10}\right) = \left(\frac{9}{10}\right)^4 \frac{7}{5}$$

$$= 0.91854$$

Q. Six dice are thrown together at a time, the process is repeated 729 times. How many times do you expect at least three dice to have 4 to 6 ?

Sol^n. The chance of getting 4 or 6 with one dice is $\frac{2}{6}$

$$p = \frac{1}{3}$$

$$\text{and } q = 1 - \frac{1}{3} = \frac{2}{3}$$

If one throw of six dice together, we have probability of getting at least 3 dice to have 4 or 6

$$= P(3) + P(4) + P(5) + P(6)$$

$$= {}^6C_3\, p^3 q^3 + {}^6C_4\, p^4 q^2 + {}^6C_5\, p^5 q + {}^6C_6\, p^6$$

$$= 20\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + 15\left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{6}\right)^6$$

$$= \frac{1}{(3)^6}[160 + 60 + 12 + 1] = \frac{233}{3^6}$$

Now the process is repeated 729 times.

$\therefore$ Required ~~result~~ number of times at least 3 dice have 4 or 6 $= 729 \times \frac{233}{3^6} = 233$

Continuous Random Variable:-

A random variable X is said to be continuous if it can take all possible values (integral as well as fractional) between certain limits.

examples of continuous random variable are age, height, weight, etc.

Probability Density function:-

The probability density function of a r.v. X usually denoted by $f_X(x)$ or simply $f(x)$ has the following properties.

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(iii) The probability P(E) given by

$$P(E) = \int_E f(x)\, dx.$$

Cumulative Distribution -

If ~~f(x)~~ X is random variable, then $p(X \leq x)$ is called the cumulative distribution and is denoted by $F(x)$

$$F(x) = p(X \leq x)$$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx.$$

Expection of Random Variable:-

If X is a continuous R.V. then the expectation of the R.V. X as defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

The expected value of X^2 is defined as $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

E(X) is also called mean of X.

Variance and standard deviation of Continuous R.V.

$$Var(X) = V(X) = E(X-\bar{X})^2$$

$$= E(X^2) - [E(X)]^2$$

$$S.D = \sigma = \sqrt{V(X)} = \sqrt{E(X)^2 - [E(X)]^2}$$

Q. A Continuous random Variable X has a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & \text{if } -3 \le x < -1 \\ \frac{1}{16}(6-2x^2) & \text{if } -1 \le x < 1 \\ \frac{1}{16}(3-x)^2 & \text{if } 1 < x \le 3 \\ 0 & \text{else} \end{cases}$$

Verify that f(x) is a density function and also find the mean of the random Variable X.

Sol: Since f(x) is density function

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)dx + \int_{-3}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)$$

$$= \int_{-\infty}^{-1} 0 \cdot dx + \int_{-3}^{-1} \frac{1}{16}(3+x)^2 dx + \int_{-1}^{1} (6-2x^2)\,dx + \int_{1}^{3} \frac{1}{16}(3-x)^2 dx$$

$$+ \int_{3}^{\infty} f(x)dx$$

$$= \frac{1}{16}\int_{-3}^{-1}(3+x)^2\,dx + \int_{-1}^{1}(6-2x^2)\,dx + \frac{1}{16}\int_{1}^{3}(3-x)^2\,dx$$

$$= \frac{1}{16}\left\{\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \left[6x - \frac{2x^3}{3}\right]_{-1}^{1} - \left[\frac{(3-x)^3}{3}\right]_{1}^{3}\right\}$$

$$= \frac{1}{16}\left\{\left[\frac{8}{3} - 0\right] + \left[\left(6-\frac{2}{3}\right) - \left(-6+\frac{2}{3}\right) - \left(0-\frac{8}{3}\right)\right]\right\}.$$

$$= 1$$

$$\Rightarrow \therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Hence $f(x)$ is a density function.

Mean of the random Variable X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(3+x)^2\,dx + \frac{1}{16}\int_{-1}^{1} x(6-2x^2)\,dx + \frac{1}{16}\int_{1}^{3} x(3-x)^2\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(9+x^2+6x)\,dx + 0 + \frac{1}{16}\int_{1}^{3} x(9+x^2-6x)\,dx.$$

[Since the integrand of the 2nd integral is odd function]

$$= \frac{1}{16}\int_{-3}^{-1}(9x+x^3+6x^2)\,dx + \frac{1}{16}\int_{1}^{3}(9x+x^3-6x^2)\,dx$$

$$= \frac{1}{16}\left\{\left[\frac{9x^2}{2} + \frac{x^4}{4} + \frac{6x^3}{3}\right]_{-1}^{-1} + \left[\frac{9x^2}{2} + \frac{x^4}{4} - \frac{6x^3}{3}\right]_{1}^{3}\right\}$$

$$= \frac{1}{16}\left\{\left[\left(\frac{9}{2}-\frac{81}{2}\right) + \left(\frac{1}{4}-\frac{81}{4}\right) + \left(-\frac{6}{3} - \frac{-162}{3}\right)\right] + \left[\left(\frac{81}{2}-\frac{9}{2}\right) + \left(\frac{81}{4}-\frac{1}{4}\right) - \left(\frac{162}{3}-\frac{1}{3}\right)\right]\right\}$$

$$= 0$$

Therefore, the mean of R.V X is Zero

Q. A Continuous random Variable X has

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

represents the density, find the mean and standard deviation of X.

Sol^n if $f(x)$ is density function then it satisfies

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{1}{2}(x+1) + \int_{1}^{\infty} 0 \cdot dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}+1\right) - \left(\frac{(-1)^2}{2} - 1\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{3}{2}\right) - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2} \cdot \frac{4}{2} = 1$$

Hence, $f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{else where} \end{cases}$

is a density function.

Mean of r.v. X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^{1} \frac{x}{2}(x+1)\,dx + \int_{1}^{\infty} 0 \cdot dx.$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2+x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \frac{-1}{3}\right) + \left(\frac{1}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

Therefore the mean of the random variable X is $\frac{1}{3}$

$\therefore$ the variance of the random variable X is

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}(x+1)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^3+x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}+\frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right] = \frac{1}{3}$$

Now $Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

$\therefore$ standard deviation of $X = \frac{\sqrt{2}}{3}$

Ex If the probability density function

$$f(x) = \begin{cases} kx^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{else where.} \end{cases}$$

find the value 'k' and find the probability between ~~the~~ $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Sol^n from the given data

$$f(x) = \begin{cases} kx^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{else where.} \end{cases}$$

If $f(x)$ is a density function; then it satisfy.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow k \cdot \int_{0}^{3} x^3\,dx = 1 \Rightarrow \left[\frac{x^4}{4}\right]_0^3 = 1$$

$$\Rightarrow k\left[\left(\frac{3^4}{4} - 0\right)\right] = 1$$

$$\Rightarrow \frac{81}{4}k = 1$$

$$\boxed{k = \frac{4}{81}}$$

Now $f(x) = \begin{cases} \frac{4}{81}x^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{Else where} \end{cases}$

Now $P\left(\frac{1}{2} \le x \le \frac{3}{2}\right)$

$$P\left(\frac{1}{2} \le x \le \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx.$$

$$= \frac{4}{81} \int_{1/2}^{3/2} x^3\,dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{1/2}^{3/2} = \frac{1}{81}\left[x^4\right]_{1/2}^{3/2}$$

$$= \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right]$$

$$= \frac{1}{81}\left[\frac{80}{16}\right]$$

$$= \frac{5}{81} = 0.0617$$

Q. A Continous random variable has the p.d.f.

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{else where.} \end{cases}$$

find the probabilities that it will take on a value (i) between 1 and 3 and (ii) greater than 0.5

Sol^n (i) $P(1 < X < 3) = \int_1^3 f(x)\,dx = \int_1^3 2e^{-2x}\,dx$

$$= 2\int_1^3 e^{-2x}\,dx.$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

$$= 0.1353 - 0.003478$$

$$= 0.1338.$$

$\therefore \quad P(1<X<3) = 0.1338$

$$P(X>0.5) = \int_{0.5}^{x} f(x)\,dx$$

$$= \int_{0.5}^{2} 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = -\left[e^{-2x}\right]_{0.5}^{\infty}$$

$$= -(0 - e^{-1}) = e^{-1} = 0.3687$$

(ii) $P(X>0.5) = \int_{0.5}^{\infty} f(x)\,dx$

$$= 1 - \int_{-\infty}^{0.5} f(x)\,dx.$$

$$= 1 - \left[\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{0.5} f(x)\,dx\right]$$

$$= 1 - \left[\int_{0}^{0.5} f(x)\,dx\right]$$

$$= 1 - \int_{0}^{0.5} 2e^{-2x}\,dx.$$

$$= 1 - 2\int_{0}^{0.5} e^{-2x}\,dx$$

Discrete Uniform Distribution -

Defn A r.v. X is said to have a discrete Uniform distribution over the range [1, n] if its Probabilit mass function is expressed as follows

$$P(X=x) = \begin{cases} \frac{1}{n} & \text{for } x = 1, 2 \ldots n. \\ 0 & \text{otherwise.} \end{cases}$$

Here n is known as the parameter of the distribution and lies in the set of all +ve integer

Discrete Uniform Distribution also called rectangular distribution.

2. Continous Uniform (rectangular) Distribution -

Defn → A random Variable X is said to have a Continous rectangular (Uniform) distribution over an interval (a, b) i.e $(-\infty < a < b < \infty)$ if its probability mass function is given by

$$f(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Note a and b (a<b) are the two parameters of the distribution the distribution is called uniform distribution on (a, b) since it assumes a Constant Value for all x in (a, b)

→ A Uniform variate X on the interval (a, b) is written as : $X \sim U[a, b]$ or $X \sim R[a, b]$

Ex Subway trains on a certain line run every half hour between mid-night and ~~six~~ six in the morning. What is the probability that a man entering the station at a random time during the period will have to wait at least twenty minutes?

Soln Let the r.v X denote the waiting time (in minutes) for the next train under the assumption that a man arrives at the station at random, X is distributed uniformly on $(0, 30)$ with p.d.f.

$$f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{otherwise} \end{cases}$$

The probability that he has to wait at least 20 minutes is given by

$$P(X \geq 20) = \int_{20}^{30} f(x)\,dx = \frac{1}{30}\int_{20}^{30} 1 \cdot dx = \frac{1}{30}(30-20) = \frac{1}{3}$$

Ex A random variable X has a uniform distribution over $(-3, 3)$, find k for which $P(X > k) = \frac{1}{3}$. Also evaluate $P(X < 2)$ and $P[|X-2| < 2]$

Soln Density of $X = f(x) = \frac{1}{b-a} = \frac{1}{3-(-3)} = \frac{1}{6}$.

$$\therefore \quad P(X > k) = 1 - P(X \leq k) = 1 - \int_{-3}^{k} f(x)\,dx$$

$$= 1 - \frac{1}{6}\int_{-3}^{k} dx = 1 - \frac{1}{6}(k+3) = \frac{1}{3}$$

This gives $k = 1$

(ii) $P(X < 2) = \int_{-3}^{2} f(x)\,dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{5}{6}$

(iii) $P[|X-2| < 2] = P[2-2 < X < 2+2]$

$$= P[0 < x < 4] = \int_0^3 f(x)\,dx = \frac{1}{6}\int_0^3 dx = \frac{1}{2}$$

Poission Distribution-

$$\boxed{P(r) = \frac{e^{-m} \cdot m^r}{r!}}$$

Mean $= \lambda = np = m$

Variance $= \lambda = npq = m$

Condition Under which poisson Distribution is Used-

1. The random variable x should be discrete.
2. The happening of the event must be of two alternative such as sucess and failure, ocurrence and nonoccurance
3. It is applicable in those case where the number of trials n is very large and the probabilities of success p is very small the mean $np = m$ is finite

Ex Suppose a book of 585 pages contains 43 typographical errors. if these errors are randomly distributed throughout the book what is the probability that 10 pages, selected at random, will be free from errors? [Use $e^{-0.735} = 0.4795$]

Sol^n

$$P = \frac{43}{585} = 0.0735 \quad \text{and} \quad n = 10$$

$$\therefore \quad m = np = 10 \times 0.0735 = 0.735$$

Clearly, P is very small and n is large

So, It is a case of poisson distribution.

Let X denote the number of error in 10 pages

$$P(X=r) = \frac{e^{-m} m^r}{r!} = \frac{e^{-0.735} \times (0.735)^r}{r!}$$

$$\therefore P(\text{no error}) = P(X=0) = \frac{e^{-0.735} \times (0.735)^0}{0!} = e^{-0.735}$$

$$= 0.4795$$

Hence the required probability is 0.4795

Ex: Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents are (i) at least one (ii) at most one (Given $e^{-1.8} = 0.16529$)

Sol: The probability function of the poisson distribution is

$$P(X=r) = \frac{e^{-m} m^r}{r!} \qquad r = 0, 1, 2, 3 \ldots$$

Given that $m = 1.8$

(i) $P(X \geq 1)$ $\quad P(X \geq 1) = 1 - P(r < 1)$

$$= 1 - P(r=0)$$

$$= 1 - \frac{e^{-m} m^0}{0!}$$

$$= 1 - \frac{e^{1.8}(1.8)^0}{0!}$$

$$= 1 - e^{-1.8}$$

$$= 1 - 0.16529 = 0.8347$$

(ii) $P(X \leq 1)$: $\quad P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{e^{-m} m^0}{0!} + \frac{e^{-m} m^1}{1!}$$

$$= \frac{e^{-1.8}(1.8)^0}{0!} + \frac{e^{-1.8}(1.8)^1}{1!}$$

$$= e^{-1.8}(1+1.8) = 0.4628$$

Geometric Distribution is the probability distribution of the number X of independent Bernoulli trails performed until a success occurs, where the Bernoulli trials have a constant probability of success p.

Geometric dist^n is applicable to find the probability where we perform an experiment until a success occurs.

Ex 1. Tossing a coin repeatedly until the first head appears.

2. Shot the target until it hits.

3. Give the test until he will pass it.

4. Throwing a die repeatedly until first time a six appears.

Geometric Distribution.

The probability mass function (P.m.f) of G.D is

$$P(x) = \begin{cases} q^{x-1} p & ; x = 1, 2 \ldots \\ 0 & \text{else.} \end{cases}$$

Such that $p + q = 1$, $p \geq 0$

Ex A die is cast until 6 appears. what is the probability that it must be cast more than 5 times?

Sol^n Here probability of getting 6 is $p = \frac{1}{6}$ then $q = \frac{5}{6}$

If X is the number of tosses required for the first success then

$\therefore$ Required prob $= P(X > 5) = 1 - P(X \leq 5)$

$= 1 - [P(1) + P(2) + P(3) + P(4) + P(5)]$

$= 1 - \left[\frac{1}{6} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6}\left(\frac{5}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^3 + \frac{1}{6}\left(\frac{5}{6}\right)^4\right]$

$= 1 - \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4\right]$

$= \cancel{1 - \frac{1}{6}\left(\frac{1}{1-\frac{5}{6}}\right)} = \left(\frac{5}{6}\right)^5$ Ans

Ex Suppose that a trainee solder shoots a target in an independent fashion. if the probability that the target is hit in any shot is 0.7 what is the.

(I) Probability that the target would be hit on 10th attempt. 2

(II) Probability that it takes him less than 4 shots.

(III) Probability that it takes him an even number of shots?

Sol

(i) Required probability.

P of hitting $= 0.7$

q (prob of unhit) $= 1-0.7=0.3$

$\Rightarrow P(X=10) = q^9 p = (0.3)^9 (0.7)$

$= 1.3777 \times 10^{-5}$

(II) Required prob $= P(X<4)$

$\Rightarrow P(X<4) = P(1) + P(2) + P(3)$

$= p + qp + q^2 p$

$= 0.7 + (0.3)(0.7) + (0.3)^2 (0.7)$

$= 0.9730$

(III) Required prob $= P(X = \text{Even})$

$\Rightarrow P(X = \text{Even}) = P(X=2) + P(X=4) + P(X=6) - -$

$= qp + q^3 p + q^5 p + \_\_$

$= qp \underbrace{[1 + q^2 + q^4 + + - - -]}_{G.P.}$

$\frac{a}{1-r}$

$= qp \times \left(\frac{1}{1-q^2}\right)$

$= (0.3)(0.7)\left(\frac{1}{1-(0.3)^2}\right)$

$= 0.2308$

Q. Suppose that the probability for an applicant for a driver's licence to pass the road test on any given attempt is $\frac{2}{3}$. What is the probability that the applicant will pass the road test on the third attempt?

Sol: $P(X=3) = q^2 p$

$= \left(\frac{1}{3}\right)^2 \times \frac{2}{3}$

$= \frac{2}{27}$

Mean of Geometric Distribution $= \frac{1}{p}$

Variance$(X) = \frac{q}{p^2}$

Negative Binomial Distribution:-

This distribution gives the probability that the event occurs for the kth time on the rth trial $(r \geq k)$ If p be the probability of occurence of an event then

$$\boxed{P(k,r) = {}^{r-1}C_{k-1}\, p^k q^{r-k}}$$

It contain two perameter p $(0<p<1)$ and k (a +ve integer) if $k=1$, the -ve binomial distribution reduces to the Geometric Distribution.

Mean $= \frac{r}{p}$ Variance $= \frac{rq}{p^2}$

Hyper-Geometric Distribution:-

When the population is finite and the sampling is done without replacement.

Consider an ~~event~~ urn with N balls, M of which are white and N-M are red. Suppose that we draw a sample of n balls at random (without replacement) from the urn then the probability of getting k white ball out of n (k<n) is.

$$\frac{{}^{M}C_{k} \; {}^{N-M}C_{n-k}}{{}^{N}C_{n}}$$

[Since k white balls can be drawn from 'M' white ball in ${}^{M}C_{k}$ ways and

out of the remaining N-M red ball, (n-k) balls can be ~~drawn~~ ~~in~~ choose in ${}^{N-M}C_{n-k}$ ways

the total number of favourable cases is ${}^{M}C_{k} \times {}^{N-M}C_{n-k}$]

Def A discrete random variable X is said to follow the hypergeometric Distribution with parameters N, M and n if it assumes only non-negative value and its Probability mass function is given by:

$$P(X=k) = P(k: N, M, n) = \begin{cases} \dfrac{{}^{M}C_{k} \times {}^{N-M}C_{n-k}}{{}^{N}C_{n}} & k = 0, 1, 2, 3 \ldots \min(n, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

where N is a +ve integer M is +ve integer not exceeding N and n is a +ve integer that is at most N.

$$\text{Mean of Hypergeometric Distribution} = \frac{nM}{N}$$

$$\text{Variance}(x) = \frac{NM(N-M)(N-n)}{N^2(N-1)}$$

Q. A urn Contain 4 red balls 6 black balls A Sample of 4 balls is selected from the urn without replacement Let X be the number of red balls Contain in the sample than find the probability "dist" for X.

Solⁿ :- Given

Red ball = 4

black ball = 6

$\Rightarrow$ Total = 10 $\Rightarrow$ $N = 10$

4 ball is Selected in sample $\Rightarrow n = 4$

M = Number of red ball = 4

$k = 4$.

Hyper Geometric distribution applicable

$$\therefore P(X=k) = \frac{{}^{M}C_k \; {}^{N-M}C_{n-k}}{{}^{N}C_n} \qquad k = 0, 1, 2, 3, 4$$

$$P(X=k) = \frac{{}^{4}C_k \; {}^{10-4}C_{n-k}}{{}^{10}C_4} \qquad k = 0, 1, 2, 3, 4.$$

1. $P(X=0) = \frac{{}^{4}C_0 \times {}^{6}C_4}{{}^{10}C_4} = 0.07142$

$P(X=1) = \frac{{}^{4}C_1 \times {}^{6}C_3}{{}^{10}C_4} = 0.38095$

$P(X=2) = \frac{{}^{4}C_2 \times {}^{6}C_2}{{}^{10}C_4} = 0.42857$

$P(X=3) = \frac{{}^{4}C_3 \times {}^{6}C_1}{{}^{10}C_4} = 0.1142$

$P(X=4) = \frac{{}^{4}C_4 \times {}^{6}C_0}{{}^{10}C_4}$

$= 0.004761$

Q. A Committee of size 3 is Selected from 4 men and 2 women find the probability distribution for the number of men on the Committee.

Solⁿ $N = 4+2 = 6$ $n = 3$

$k = 4$

$M = 4$

$$P(X=k) = \frac{{}^{M}C_{k}\ {}^{N-M}C_{n-k}}{{}^{N}C_{n}} \quad k = 0, 1, 2, 3, 4$$

$\boxed{\text{Ex Negative Binomial Dist}}$

In a Company 5 % defective component are produced what is the probability that at least 5 Component are to be examined in order to get 3 defective ?

$$\boxed{P(n, r) = {}^{n-1}C_{r-1}\, p^{r} q^{n-r}}$$

Solⁿ Required probability $P(X \geq 5)$

$= 1 - P(X < 5)$

in $P(X=3)$

$n = 3$

$r = 3$

$= 1 - P(X=3) - P(X=4)$

in $P(X=4)$

$n = 4$

$r = 3$

$= \left[1 - {}^{2}C_{2}\, p^{3} q^{0} - {}^{3}C_{2}\, p^{3} q^{1}\right]$

$= 1 - (0.005)^{3}$

$- 3(0.05)^{3}(0.95)^{1}$

$= 0.9995$

Ex A web site contain three identical computer servers. only one is used to operate the site, and the other two are spares that can be activated in case the primary system fails. The probability of a failure in the primary computer (or any activated spare system) from a request for service is 0.0005. Assuming that each request untill failure of all three servers? What is the probability that all three server fail within 5 request?

Sol" Let X denote the number of request untill all three server fail, and let X_1, X_2 and X_3 denote the number of request before a failure of the first, 2nd and 3rd server used,

Probability of failure $p = 0.0005$

$r = 3$

Required Probability

$$P(X \le 5) = P(X=3) + P(X=4) + P(X=5)$$

$$= (0.0005)^3 + {}^3C_2 (0.0005)^3 (0.9995) + {}^4C_2 (0.0005)^3 (0.9995)$$

$$= 1.25 \times 10^{-10} + 3.75 \times 10^{-10} + 7.49 \times 10^{-10}$$

$$= 1.249 \times 10^{-9}$$

HGD

Q. A batch of parts contains 100 parts from a local supplier of tubing and 200 parts from a supplier of tubing in the next state. If four parts are selected randomly and without replacement, what is the probability

(i) they are all from the local supplier.

(ii) what is the probability that two or more parts in the sample are from the local supplier?

(iii) what is the probability that at least one part in the sample is from the local ~~of~~ supplier?

Solⁿ (1)

$$P(x) = \frac{{}^{M}C_{x} \cdot {}^{N-M}C_{n-x}}{{}^{N}C_{n}}$$

(I) Required Probability $P(x=x) = P(x=4)$

$x=4$, $M=100$, $n=4$, $N=300$

$$P(x=4) = \frac{{}^{100}C_4 \times {}^{300-100}C_{4-4}}{{}^{300}C_4} = 0.0119$$

(II) $P(x \geq 2) = P(x=2) + P(x=3) + P(x=4)$

$$= \frac{{}^{100}C_2 \times {}^{200}C_2}{{}^{300}C_4} + \frac{{}^{100}C_3 \times {}^{200}C_1}{{}^{300}C_4} + \frac{{}^{100}C_4 \times {}^{200}C_0}{{}^{300}C_4}$$

$= 0.298 + 0.098 + 0.0119$

$= 0.408$

(iii) $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - \frac{{}^{100}C_0 \times {}^{200}C_4}{{}^{300}C_4} = 0.804$$ Ans.

Exponential Distribution

The random variable X that equal the distance b/w successive counts of a poisson process with mean $\lambda > 0$ is an exponential random variable with parameter λ. the probability density function of X is

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}}$$

Cumulative Dist^n $F(x)$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \\ 1 & x = \infty \end{cases}$$

Mean $\boxed{E(X) = \mu = \frac{1}{\lambda}}$

Variance $\sigma^2 = V(X) = \frac{1}{\lambda^2}$

Ex In a large corporate computer network user log-ons to the system can be modeled as a Poisson Process with a mean of 25 log-ons per hour. what is the probability that there are no log-ons in an interval of 6 minutes?

Sol^n Let X denote the time in hours from the start of the interval untill the first log-on then X has an exponential Distribution with $\lambda = 25$ log-ons per hour.

$\Rightarrow \lambda = 25$

6 minutes = 0.1 hour

$\therefore$ the Probability density function

$$P(X > 0.1) = \int_{0.1}^{\infty} 25 e^{-25x} dx = e^{-25(0.1)}$$

$$= 0.082$$

Erlang and Gamma Distribution :-

The random variable X that equal the interval length until r count occur in a Poisson Process with $\lambda > 0$ has an Erlang random variable with parameter λ and r. The probability density function of X is

$$\boxed{f(x) = \frac{\lambda^{r} x^{r-1} e^{-\lambda x}}{(r-1)!} \quad \text{for } x > 0 \text{ and } r = 1, 2 \ldots}$$

Mean $\boxed{\mu = \frac{r}{\lambda} = E(X)}$

$$\boxed{\sigma^2 = V(X) = \frac{r}{\lambda^2}}$$

Weibull Distribution –

The random variable X with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]}$$

for $x > 0$

is a Weibull random variable with scale parameter $\delta > 0$ and shape parameter $\beta > 0$

$$E(X) = \mu = \delta\,\Gamma\left(1+\frac{1}{\beta}\right)$$

$$V(X) = \sigma^2 = \delta^2\Gamma\left(1+\frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$$

If X has a Weibull Distribution with parameter δ and β then the cumulative Distn function of X is

$$\boxed{F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^{\beta}}}$$

Log Normal Distribution:-

Let we have a normal distribution mean θ and variance ω^2; then $X = e^W$ is a lognormal random variable with probability density function

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} e^{\left[-\frac{(\ln x - \theta)^2}{2\omega^2}\right]} \qquad 0 < x < \infty$$

Mean $\boxed{\begin{aligned} E(X) &= e^{\theta + \omega^2/2} \\ V(X) &= e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right) \end{aligned}}$

Cumulative Distⁿ fⁿ for X

$$F(x) = P[X \le x] = P[e^{(W)} \le x] = P[W \le \ln(x)]$$

$$= P\left[Z \le \frac{\ln(x) - \theta}{\omega}\right]$$

$$= \phi\left[\frac{\ln(x) - \theta}{\omega}\right]$$

for $x > 0$

where Z is a standard normal ~~variate~~ random variable.

Ex (Erlang and Gamma Distribution)

The failures of the Central processor units of large Computer systems are often modeled as a poisson process. Typically, failures are not ~~caused~~ caused by components wearing out, but by more random ~~[illegible]~~ failure of the large number of semiconductor circuits in the units. Assume that the units that fail are immediately repaired, and assume that the mean number of is 0.0001. Let X denote the time until four failure occur in a system. Determine the probability that X exceeds 40,000 hours.

Sol^n $r = 4$ and $\lambda = 0.0001$

$$P(X > 40{,}000) = \int_{40{,}000}^{\infty} f(x)\,dx = \int_{40{,}000}^{\infty} \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}\,dx.$$

Ex (Weibull Dist^n)

The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily modeled as a Weibull random variable with $\beta = \frac{1}{2}$ and $\delta = 5000$ hours. Determine the mean time untill failure. Determine the probability that a bearing lasts 6000 hours ~~Now~~

Sol^n From the expression for the mean

$$E(X) = 5000\,\Gamma\left(1 + \frac{1}{0.5}\right) = 5000\,\Gamma(3) = 5000 \times 2!$$
$$= 10{,}000 \text{ hours.}$$

~~Determine~~ the probability that a bearing lasts 6000 hours. Now

$$P(X > 6000) = 1 - F(6000) = \exp - \left[\left(\frac{6000}{5000}\right)^{1/2}\right]$$
$$= e^{-1.095} = 0.334$$

(Lognormal Distribution)

Ex The lifetime of a semiconductor laser has a lognormal Dist^n with $\theta = 10$ hours and $\omega = 1.5$ hours

(i) What is the probability the lifetime exceeds 10,000 hours?

Determine the mean and deviation of lifetime.

Sol^n From the cumulative distribution function for X

$$P(X > 10{,}000) = 1 - P[e^{(W)} \le 10{,}000]$$

$$= 1 - P[W \le \ln(10{,}000)]$$

$$= \phi\left(\frac{\ln(10{,}000) - 10}{1.5}\right)$$

$$= 1 - \phi(-0.52)$$

$$= 1 - 0.30 = 0.70$$

Now

$$E(X) = e^{\theta + \omega^2/2} = e^{(10 + 1.125)}$$

$$= 67{,}846.3$$

$$V(X) = e^{2\theta + \omega^2}(e^{\omega^2} - 1)$$

$$= e^{(20+2.25)} \cdot [e^{(2.25)} - 1]$$

$$= 39{,}070{,}059{,}886.6$$

So the S.D of X is 197,661.5 hours.

Notice that the standard deviation of lifetime is large relative to the mean.

(ii) what life time is exceeded by 99% of lasers?

Sol^n the question is to determine x such that

$$P(X > x) = 0.99$$

Therefore

$$P(X > x) = P[e^{\omega} > x] = P[\omega > \ln(x)]$$

$$= 1 - \phi\left(\frac{\ln(x) - 10}{1.5}\right) = 0.99$$

$$\Rightarrow 1 - \phi(z) = 0.99 \quad \text{when } z = -233$$

[from table]

$$\Rightarrow \frac{\ln(x) - 10}{1.5} = -2.33$$

and $x = e^{(6.505)} = 668.48$ hours.

Beta Distribution :-

Beta Distribution of first kind

Def^n A r.v X is said to have a beta Dist^n of Ist kind with parameters μ and ν $(\mu > 0, \nu > 0)$ if its p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{B(\mu,\nu)} x^{\mu-1}(1-x)^{\nu-1} & ; (\mu,\nu) > 0 \\ & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $B(\mu,\nu)$ is the Beta function].

$$\boxed{\text{Mean} = \frac{\mu}{\mu+\nu}}$$

Beta Distribution of 2nd kind

A r.v X is said to have a Beta distribution of the 2nd kind with parameter μ and ν $(\mu > 0, \nu > 0)$ if its p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{B(\mu,\nu)} \cdot \frac{x^{\mu-1}}{(1+x)^{\mu+\nu}} & (\mu,\nu) > 0 \\ & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\text{Mean} = \frac{\mu}{\nu-1}, \quad \nu > 1}$$