11 Partial Differentiation

FUNCTIONS OF SEVERAL VARIABLES

In engineering problems one frequently comes across a variable quantity which depends for its values on two or more independent variables. For instance, the area of a rectangle depends upon its length and breadth and we say that the area is a function of two independent variables, the length and the breadth.

We use the notation $f(x, y)$ or $F(x, y)$ etc., to denote the value of the function at (x, y) and write $z = f(x, y)$ or $z = F(x, y)$, etc., Also, we may write $z = z(x, y)$ where it should be clearly understood that in this notation z is used in two senses, namely, as a function and as a variable. The concept can be easily extended to the functions of three or more variables. Thus, $\omega = f(x, y, z)$ denotes the value of the function f at (x, y, z), a point in three dimensional space.

PARTIAL DERIVATIVES

Let us extend the notion of ordinary derivative of the function of one variable to the derivative of a function f (or z) of two independent variables x and y. Now, question arises, should we differentiate f with respect to x or y? The answer is simple : treat y as constant while differentiating f with respect to x and treat x as constant while differentiating f with respect to y. This way we define two different derivatives and call them partial derivatives to distinguish them from the ordinary derivative of a function of a single independent variable.

We denote the partial derivative of z (or f) with respect to x by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x \text{ or } z_x \text{ or } \left(\frac{\partial z}{\partial x}\right)_y .$$

and the partial derivative of z (or f) w.r.t. y by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y or z_y or $\left(\frac{\partial z}{\partial y}\right)_x$.

Thus, $$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} = \left(\frac{\partial f}{\partial x}\right)_y = \text{value of } \frac{\partial f}{\partial x} \text{ when } y \text{ is kept constant.}$$

Similarly $$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \left(\frac{\partial f}{\partial y}\right)_x = \text{value of } \frac{\partial f}{\partial y} \text{ when } x \text{ is kept constant.}$$

Geometrically, $\frac{\partial z}{\partial x}$ $\left(\text{or } \frac{\partial f}{\partial x}\right)$ is the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ with a plane parallel to the plane $y = 0$, *i.e.*, *ZOX*- plane.

[GGSIPU II Sem I Term 2011]

Similarly for, $\dfrac{\partial z}{\partial y}$ $\left(\text{or } \dfrac{\partial f}{\partial y}\right)$.

In general, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are functions of both x and y and therefore, we can also obtain the higher order partial derivatives of $f(x, y)$ as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy},$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}, \text{ etc.}$$

In $\dfrac{\partial^2 f}{\partial x\,\partial y}$ we first differentiate f partially with respect to y and then partially with respect to x, It must be noted that $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$, in general, meaning thereby, that the order of partial differentiation is generally immaterial.

Example 11.1. If $z = f(x + ay) + \text{f}(x - ay)$, then show that $z_{yy} = a^2 z_{xx}$.

Solution: From $z = f(x + a y) + \phi(x - a y)$, we get

$$z_x = \frac{\partial z}{\partial x} = f'(x + ay)\frac{\partial}{\partial x}(x + a y) + \phi'(x - a y)\frac{\partial}{\partial x}(x - ay)$$

$$= f'(x + ay) + \phi'(x - ay)$$

where f' and ϕ' mean ordinary derivatives of f and ϕ with respect to $x + ay$ and $x - ay$ respectively.

Similarly, $$z_y = \frac{\partial z}{\partial y} = f'(x + a y)\frac{\partial}{\partial y}(x + a y) + \phi'(x - ay)\frac{\partial}{\partial y}(x - ay)$$

$$= a f'(x + a y) - a\phi'(x - a y)$$

Now differentiating z_x and z_y again partially with respect to x and y respectively, gives

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x} f'(x + a y) + \frac{\partial}{\partial x}\phi'(x - a y)$$

$$= f''\frac{\partial}{\partial x}(x + a y) + \phi''\frac{\partial}{\partial x}(x - a y) = f''(x + ay) + \phi''(x - ay).$$

and $$z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}[a f'(x + a y)] - \frac{\partial}{\partial y}[a\phi'(x - a y)]$$

$$= a f''\frac{\partial}{\partial y}(x + a y) - a\phi''\frac{\partial}{\partial y}(x - a y)$$

$$= a^2 f''(x + a y) + (-a)^2\phi''(x - a y)$$

which clearly implies that $z_{yy} = a^2 z_{xx}$.

Hence Proved.

EXAMPLE 11.2. If $u = \log \dfrac{x^2+y^2}{xy}$, verify that $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$

SOLUTION: We can write $u = \log(x^2+y^2) - \log x - \log y$

Hence $\dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2+y^2} - \dfrac{1}{x}$ and $\dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2+y^2} - \dfrac{1}{y}$

Further $$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{2y}{x^2+y^2} - \frac{1}{y}\right] = \frac{-4xy}{(x^2+y^2)^2} - 0$$

and $$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{2x}{x^2+y^2} - \frac{1}{x}\right] = \frac{-2x.2y}{(x^2+y^2)^2} - 0$$

Thus, $\dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{\partial^2 u}{\partial y\partial x}$ **is verified.**

EXAMPLE 11.3. (*a*) Show that, at a point on the surface $x^x y^y z^z = c$ where $x = y = z$, we have $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{-1}{x\,\log(ex)}$. **[GGSIPU II Sem End Term 2010]**

(*b*) If $z(x+y) = x^2+y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$. **[GGSIPU I Sem End Term January 2011]**

SOLUTION: (*a*) From the given relation $x^x y^y z^z = c$ ...(1)

it is clear that we can take x as a function of y and z, or y as a function of z and x, or z as a function of x and y. As we are to calculate $\dfrac{\partial^2 z}{\partial x\partial y}$ we shall take here z as a function of x and y.

Taking logarithm in (1), gives

$$x\log x + y\log y + z\log z = \log c \qquad \text{... (2)}$$

Differentiating (2) partially with respect to y, gives

$$0 + y\cdot\frac{1}{y} + 1.\log y + z\cdot\frac{1}{z}\frac{\partial z}{\partial y} + \log z\cdot\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z} \qquad \text{... (3)}$$

Similarly, differentiating (2) partially *w.r.t.* x, gives $\dfrac{\partial z}{\partial x} = \dfrac{-(1+\log x)}{1+\log z}$ ... (4)

Next, differentiating (3) partially *w.r.t.* x, we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(1+\log y)\frac{\partial}{\partial x}\frac{1}{(1+\log z)}$$

$$= -(1+\log y)\frac{(-1)}{(1+\log z)^2}\frac{\partial}{\partial x}(1+\log z) = \frac{1+\log y}{(1+\log z)^2}\frac{1}{z}\frac{\partial z}{\partial x}$$

Using (4) here, gives

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1+\log y}{(1+\log z)^2}\,\frac{(-1)(1+\log x)}{z(1+\log z)} = \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

At the point $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-1}{x(1+\log x)} = \frac{-1}{x(\log e+\log x)} = \frac{-1}{x\log(ex)}.$$ **Hence Proved.**

(b) From $Z(x+y) = x^2+y^2$, we get

$$\frac{\partial z}{\partial x}(x+y)+z = 2x \quad \text{and} \quad \frac{\partial z}{\partial y}(x+y)+z = 2y$$

$$\Rightarrow \quad (x+y)\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = 2(x-y) \quad \therefore \quad \frac{\partial z}{\partial x}-\frac{\partial z}{\partial y} = \frac{2(x-y)}{x+y}$$

and $$(x+y)\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right) = 2(x+y)-2z$$

$$\therefore \quad \left[1-\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)\right] = 1-\left[\frac{2(x+y-z)}{x+y}\right] = \frac{-(x+y-2z)}{x+y}$$

$$= -1+\frac{2z}{x+y} = -1+\frac{2(x^2+y^2)}{(x+y)^2} = \frac{x^2+y^2-2xy}{(x+y)^2} = \left(\frac{x-y}{x+y}\right)^2$$

Hence LHS = RHS. **Hence Proved.**

EXAMPLE 11.4. If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} = 1$, show that $u_x^2+u_y^2+u_z^2 = 2(xu_x+yu_y+zu_z)$.

[GGSIPU II Sem End Term 2009]

SOLUTION: Presence of u_x, u_y and u_z indicates that we have to take u as a function of x, y and z.

Differentiating the given relation partially *w.r.t.* x, we get

$$\frac{2x}{a^2+u}-\frac{x^2u_x}{(a^2+u)^2}-\frac{y^2u_x}{(b^2+u)^2}-\frac{z^2u_x}{(c^2+u)^2} = 0$$

or $$u_x = \frac{2x}{K(a^2+u)} \quad \text{where} \quad K = \frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

Similarly, differentiating the given relation partially *w.r.t.* y and z separately, gives

$$u_y = \frac{2y}{K(b^2+u)} \quad \text{and} \quad u_z = \frac{2z}{K(c^2+u)}$$

Therefore, $$u_x^2+u_y^2+u_z^2 = \frac{4}{K^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{4}{K^2}.K = \frac{4}{K}.$$

And $$xu_x+yu_y+zu_z = \frac{2}{K}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

$$= \frac{2}{K}\cdot 1 \quad \text{(using the given relation.)}$$

This establishes that $u_x^2+u_y^2+u_z^2 = 2(xu_x+yu_y+zu_z)$.

Hence Proved.

EXAMPLE 11.5. If $u = \log(x^3 + y^3 - x^2y - y^2x)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2}.$$

SOLUTION: Differentiating the given relation *w.r.t.* x and y separately, we get

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 2xy - y^2}{x^3 + y^3 - x^2y - y^2x}, \quad \frac{\partial u}{\partial y} = \frac{3y^2 - x^2 - 2xy}{x^3 + y^3 - x^2y - y^2x}$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2x^2 + 2y^2 - 4xy}{x^3 + y^3 - xy(x+y)} = \frac{2(x-y)^2}{(x+y)[x^2 + y^2 - xy - xy]} = \frac{2}{x+y}$$

Now

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{2}{x+y} = 2\left[\frac{\partial}{\partial x}\frac{1}{x+y} + \frac{\partial}{\partial y}\frac{1}{x+y}\right]$$

$$= 2\left[\frac{-1}{(x+y)^2} + \frac{-1}{(x+y)^2}\right] = \frac{-4}{(x+y)^2}$$

Hence Proved.

EXAMPLE 11.6. If $\theta = t^n e^{-\frac{r^2}{4t}}$ find the value of n for which

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$

SOLUTION: From the given relation $\theta = t^n e^{-\frac{r^2}{4t}}$ it is clear that θ is a function of r and t.

Taking logarithm on the both sides, we get $\log \theta = n \log t - \dfrac{r^2}{4t}$ ... (1)

Differentiating (1) partially *w.r.t.* r, gives $\dfrac{1}{\theta}\dfrac{\partial \theta}{\partial r} = 0 - \dfrac{2r}{4t}$

... (2)

Hence $r^2 \dfrac{\partial \theta}{\partial r} = -\dfrac{r^3 \theta}{2t}$

Differentiating (2) partially *w.r.t.* r, gives

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{3r^2\theta}{2t} - \frac{r^3}{2t}\frac{\partial \theta}{\partial r} = -\frac{3r^2\theta}{2t} + \frac{r^4\theta}{4t^2} \quad \text{[using (2)]}$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \left(-\frac{3}{2t} + \frac{r^2}{4t^2}\right)\theta$$

Next, differentiating (1) partially *w.r.t.* t, we get

$$\frac{1}{\theta}\frac{\partial \theta}{\partial t} = \frac{n}{t} + \frac{r^2}{4t^2}$$

Since $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$, we get

$$\left(-\frac{3}{2t} + \frac{r^2}{4t^2}\right)\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta$$ which gives $n = -3/2$.

Ans.

EXAMPLE 11.7. If $z = e^{ax+by} f(ax - by)$ show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$

[GGSIPU II Sem End Term 2005]

SOLUTION: Given that $z = e^{ax+by} f(ax - by)$, we have

$$\frac{\partial z}{\partial x} = ae^{ax+by} f(ax-by) + ae^{ax+by} f'(ax-by)$$

and $$\frac{\partial z}{\partial y} = be^{ax+by} f(ax-by) - be^{ax+by} f'(ax-by)$$

Therefore $$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by}[2f(ax-by)] = 2abz.$$

Hence the result.

EXAMPLE 11.8. If $V = (x^2 + y^2 + z^2)^{m/2}$ find the value of $m(\neq 0)$ which will make

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

[GGSIPU I Sem End Term 2003; I Sem End Term 2006]

SOLUTION: Given $V = (x^2 + y^2 + z^2)^{m/2}$ hence $\frac{\partial V}{\partial x} = \frac{m}{2}(x^2+y^2+z^2)^{\frac{m}{2}-1} \cdot 2x$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = mx\left(\frac{m}{2}-1\right)(x^2+y^2+z^2)^{\frac{m}{2}-2} \cdot 2x + m(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$= m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,[(m-2)\,x^2 + (x^2+y^2+z^2)]$$

Similarly for $\frac{\partial^2 V}{\partial y^2}$ and $\frac{\partial^2 V}{\partial z^2}$.

Thus, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,[(m-2)\,(x^2+y^2+z^2) + 3(x^2+y^2+z^2)]$$

$$= m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,(m+1)\,(x^2+y^2+z^2)$$

$$= m(m+1)\,(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \text{ when } m = 0 \text{ or } m = -1$$

Therefore the required non-zero value of m is -1. **Ans.**

EXAMPLE 11.9. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

[GGSIPU I Sem End Term 2004 Reappear]

SOLUTION: Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, we have

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \quad \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$$

Next
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\frac{3}{(x+y+z)}$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

Hence proved.

EXAMPLE 11.10. **(*a*)** If $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$ show that

$$\left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = 1$$

(*b*) If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

[GGSIPU II Sem End Term 2011]

SOLUTION: **(*a*)** From $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$, we can get

$$\left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2} \quad \text{and} \quad \left(\frac{\partial y}{\partial u}\right)_\theta = -\frac{\sin\theta}{u^2}$$

Next, to find $\left(\dfrac{\partial u}{\partial x}\right)_y$ and $\left(\dfrac{\partial u}{\partial y}\right)_x$ we have to express u as a function of x and y.

Eliminating θ in the given relations, we get

$$x^2u^2 + y^2u^2 = 1 \quad \text{or} \quad u^2 = \frac{1}{x^2+y^2}$$

Therefore,
$$2u\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{(x^2+y^2)^2} = -2xu^4$$

or
$$\left(\frac{\partial u}{\partial x}\right)_y = -xu^3 = -u^2\cos\theta$$

Similarly $\left(\frac{\partial u}{\partial y}\right)_x = -yu^3 = -u^2 \sin\theta.$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = \frac{-\cos\theta}{u^2}(-u^2\cos\theta) - \frac{\sin\theta}{u^2}(-u^2\sin\theta) = 1.$$ **Hence Prove**

(*b*) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$ and $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$

Since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$ we have

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta, \quad \frac{\partial \theta}{\partial x} = \frac{-y/x^2}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}.$$

and $$\frac{\partial \theta}{\partial y} = \frac{1/x}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}.$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \cos^2\theta\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\sin^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 - \frac{2\sin\cos\theta}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta}$$

$$+\sin^2\theta\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\cos^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \frac{2\sin\cos\theta}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta}$$

$$= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$ **Hence Proved.**

EXAMPLE 11.11. If $u = f(r)$ *where* $r^2 = x^2 + y^2 + z^2$, show that

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r}f'(r).$$

SOLUTION: Since $u = f(r)$ we have $u_x = f'(r)\frac{\partial r}{\partial x}$

Differentiating the relation $r^2 = x^2 + y^2 + z^2$ partially w.r. t. x, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly, we can have $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}.$

$\therefore \quad u_x = f'(r)\frac{x}{r}.$ Differentiating again *w.r.t* x partially, we get

$$u_{xx} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\left(\frac{1}{r}-\frac{x}{r^2}\cdot\frac{\partial r}{\partial x}\right)+\frac{x^2}{r^2}f''(r) = \frac{f'(r)}{r}-f'(r)\cdot\frac{x}{r^2}\cdot\frac{x}{r}+\frac{x^2}{r^2}f''(r)$$

$$= \frac{f'(r)}{r}+x^2\left(\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right)$$

Similarly, we can get

$$u_{yy} = \frac{f'(r)}{r}+y^2\left(\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right) \quad \text{and} \quad u_{zz} = \frac{f'(r)}{r}+z^2\left(\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right)$$

$$\therefore \quad u_{xx}+u_{yy}+u_{zz} = \frac{3f'(r)}{r}+(x^2+y^2+z^2)\left\{\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right\}$$

$$= \frac{3f'(r)}{r}+r^2\left\{\frac{1}{r^2}f''(r)-\frac{1}{r^3}f'(r)\right\} = f''(r)+\frac{2}{r}f'(r).$$

Hence Proved.

HOMOGENEOUS FUNCTIONS

An expression of the form $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$ in which all the terms are of same degree is called homogeneous function in x and y of degree n. The definition can be extended to functions of three or more variables. The above expression can also be written as

$$x^n [a_0 + a_1 (y/x) + a_2 (y/x)^2 + \dots + a_n (y/x)^n].$$

In this form it can be taken as $x^n f(y/x)$ where $f(y/x)$ is a n^{th} degree polynomial in (y/x). It is important to note here that we are now in a position to generalise $f(y/x)$ to include trigonometric, exponential and other functions.

Thus, $x^n f(y/x)$ defines a homogeneous function in x and y, of degree n, whatever may be the functional nature of f. For example, $x^4 \tan(y/x)$ is homogeneous in x and y of degree 4. Similarly, the expression $\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$ is a homogeneous function of degree $-\dfrac{1}{6}$ because it can be written as

$$\frac{x^{1/3}\left\{1+(y/x)^{1/3}\right\}}{x^{1/2}\left\{1+(y/x)^{1/2}\right\}} = x^{\frac{1}{3}-\frac{1}{2}}\, g(y/x) = x^{-1/6}\, g(y/x).$$

In this connection a very useful theorem follows:

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If u is a homogeneous function in x and y, of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

[GGSIOPU II Sem End Term 2006]

PROOF : As explained above we can write $u = x^n f(y/x)$

or $\quad u = x^n f(t) \quad$ where $\quad t = y/x$.

Then $\quad \dfrac{\partial u}{\partial x} = nx^{n-1} f(t) + x^n f'(t)\dfrac{\partial t}{\partial x} = nx^{n-1} f(t) + x^n f'(t)(-y/x^2)$

$\quad = nx^{n-1} f(t) - x^{n-2}\, y f'(t)$

and $\quad \dfrac{\partial u}{\partial y} = x^n f'(t)\dfrac{\partial t}{\partial y} = x^n f'(t)\dfrac{1}{x} = x^{n-1} f'(t)$

Therefore, $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nx^n f(t) - x^{n-1} y f'(t) + x^{n-1} y f'(t) = nu.$

Some Useful Deductions of Euler's Theorem

I. If u is a homogeneous function in x, y and z of degree n, we can write $\boldsymbol{u = x^n f(y/x, z/x)}$ and then it can be easily proved that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

II. If u is a homogeneous function in x and y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u.$$

...: We already have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$ ...(1)

...entiating (1) partially w.r.t. x and y separately, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}.1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad ...(2)$$

and
$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \quad ...(3)$$

Multiplying (2) by x and (3) by y and adding these, gives

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

Using (1) here and the fact that $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)x\frac{\partial u}{\partial x}+(n-1)y\frac{\partial u}{\partial y}=n(n-1)u$$

II. If z is homogeneous in x and y of degree n, and z is a function of u as $z=f(u)$ then we have

(i) $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}$$

(ii) $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1] \quad \text{where} \quad g(u)=\frac{n\,f(u)}{f'(u)}.$$

PROOF: (i) Since z is homogenous in x and y of degree n, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$ ...(4)

Also, since $z=f(u)$, we have $\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x}$, $\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y}$.

Therefore (4) becomes

$$x\,f'(u)\frac{\partial u}{\partial x}+y\,f'(u)\frac{\partial u}{\partial y}=n\,f(u)$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}=g(u) \quad \text{where} \quad g(u)=\frac{n\,f(u)}{f'(u)}. \quad ...(5)$$

(ii) Differentiating (5) partially w.r.t. x and y separately, gives

$$x\frac{\partial^2 u}{\partial x^2}+1.\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x} \quad ...(6)$$

and
$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=g'(u)\frac{\partial u}{\partial y} \quad ...(7)$$

Multiplying (6) by x and (7) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g'(u)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)(g'(u)-1)=g(u)(g'(u)-1).$$

where $$g(u)=\frac{n\,f(u)}{f'(u)}.$$

EXAMPLE 11.12. If $u=x^4\log\dfrac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}$ evaluate $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$.

SOLUTION : The expression $\dfrac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}$ is homogeneous in x and y of degree 0 therefore $x^4\log\dfrac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}$ is homogeneous in x and y of degree 4.

Applying Euler's theorem, we get $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=4u.$ **Ans.**

EXAMPLE 11.13. If $z=x^4y^2\sin^{-1}\dfrac{x}{y}+\log x-\log y$, show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=6x^4y^2\sin^{-1}\frac{x}{y}.$$

SOLUTION: Let us write $z=u+v$

where $u=x^4y^2\sin^{-1}\dfrac{x}{y}$ and $v=\log x-\log y=\log\left(\dfrac{x}{y}\right)$

Observe here that $\dfrac{x}{y}$ is homogenous of degree 0, so is $\sin^{-1}\dfrac{x}{y}$ and, in turn, u is homogenous in x, y of degree $4+2\ (=6)$. Also $\log\left(\dfrac{x}{y}\right)$ is homogenous in x and y of degree 0, therefore by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=6u \quad\text{and}\quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0.$$

Adding these, gives $x\dfrac{\partial}{\partial x}(u+v)+y\dfrac{\partial}{\partial y}(u+v)=6u+0$

or $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=6x^4y^2\sin^{-1}\dfrac{x}{y}.$ **Hence Proved.**

EXAMPLE 11.14. *(a)* If $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\dfrac{1}{2}\cot u=0$

[GGSIPU II Sem End Term 2005]

(b) If $u=\log\left(\dfrac{x^2+y^2}{x+y}\right)$ prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1.$

Also find the value of $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}\,y^2\dfrac{\partial^2 u}{\partial y^2}$.

[GGSIPU IInd Sem End Term 2010]

SOLUTION: (a) Given that $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ we have $\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$.

Obviously $\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is homogeneous is x and y of degree 1/2 hence $\cos u$ is homogeneous of degree 1/2.

By Euler's theorem $x\dfrac{\partial}{\partial x}\cos u + y\dfrac{\partial}{\partial y}\cos u = \dfrac{1}{2}\cos u$

or $$-x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

Hence the result.

(b) $$u = \log\left(\frac{x^2+y^2}{x+y}\right)$$

$\therefore$ $\dfrac{x^2+y^2}{x+y} = e^u = f(u)$ is a homogeneous function in x and y of degree 1, hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = 1\cdot\frac{e^u}{e^u} = 1.$$ Hence Proved.

Also we know that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y}\, y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u)-1]$

where $g(u) = \dfrac{nf(u)}{f'(u)}$ which is equal to 1 as shown above.

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 1(0-1) = -1.$$

Ans.

EXAMPLE 11.15. **(a)** If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u.$

[GGSIPU II Sem End Term 2007]

(b) If $\tan u = \dfrac{x^3+y^3}{x-y}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \text{and} \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u.$$

[GGSIPU II Sem End Term 2011]

SOLUTION: (a) Given $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$ we have $\sin u = \dfrac{x^2+y^2}{x+y}$

Clearly $\sin u$ is a homogeneous function in x and y of degree one, hence by Euler's theorem

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = 1\sin u$$

or $$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = \sin u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u.$$ **Hence the result.**

(*b*) $\tan u=\dfrac{x^3+y^3}{x-y}=f(u)$ is homogeneous in x and y of degree 2.

$\therefore$ By Euler's theorem, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x\,f(u)}{f'(u)}=\frac{2\tan u}{\sec^2 u}=\sin 2u=g(u)$$

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1]=\sin 2u\,(2\cos 2u-1)=\sin 2u\,[2\,(1-2\sin^2 u)-1]$$

$$=\sin 2u\,(1-4\sin^2 u).$$ **Hence Proved.**

EXAMPLE 11.16. If $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$ **[GGSIPU II Sem End Term 2006 Reappear]**

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u.$$

[GGSIPU I Sem End Term 2003; II Sem End Term 2006; II Sem I Term I 2005; II Sem I Term 2011]

SOLUTION: Give that $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ we have $\tan u=\dfrac{x^3+y^3}{x-y}=f(u)$.

Clearly $\tan u$ is a homogeneous function in x and y of degree 2, hence by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u} = \sin 2u = g(u), \text{ say.}$$

Also by Euler's theorem (Deduction III), we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[.g'(u)-1]$$

$$= \sin 2u\,[2\cos 2u-1] = \sin 4u - \sin 2u.$$ **Hence the result.**

EXAMPLE 11.17. If $u=\sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$

[GGSIPU II Sem End Term 2009; I Sem End Term 2004 (Reappear); II Sem I Term 2006]

SOLUTION: Since $u=\sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ we have $\sin u=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}=f(u)$, say.

Clearly $\sin u$ is homogeneous in x and y of degree 1/2 hence by Euler's theorem we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u = g(u), \text{ say}$$

Also by Euler's theorem (Deduction III), we know that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] = \frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u-1\right] = \frac{1}{4}\tan u(\sec^2 u-2)$$

$$= \frac{1}{4}\frac{\sin u}{\cos u}\left[\frac{1}{\cos^2 u}-2\right] = -\frac{1}{4}\frac{\sin u}{\cos u}\cdot\frac{2\cos^2 u-1}{\cos^2 u}$$

$$= \frac{-1}{4}\frac{\sin u\cos 2u}{\cos^3 u}.$$

Hence the result.

EXAMPLE 11.18. If $u = \sin^{-1}(x^2+y^2)^{1/5}$ prove that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{2}{25}\tan u\,(2\tan^2 u-3).$$

[GGSIPU I Sem II Term 2003]

SOLUTION: Since $u = \sin^{-1}(x^2+y^2)^{1/5}$ we have $\sin u = (x^2+y^2)^{1/5} = f(u)$, Say.

Since $\sin u$ is homogeneous in x and y of degree 2/5, we have, by Euler's theorem

$$xu_x + yu_y = \frac{nf(u)}{f'(u)} = \frac{2}{5}\frac{\sin u}{\cos u} = \frac{2}{5}\tan u = g(u).$$

Again by Euler's theorem (Deduction III), we have

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = g(u)\,[g'(u)-1]$$

$$= \frac{2}{5}\tan u\left[\frac{2}{5}\sec^2 u-1\right] = \frac{2}{25}\tan u\,[2\tan^2 u-3].$$

Hence the result.

EXAMPLE 11.19. If $u = \dfrac{(x^2+y^2)^n}{2n(2n-1)} + x f\left(\dfrac{y}{x}\right)+\phi\left(\dfrac{x}{y}\right)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$.

Hyrugvl

SOLUTION: Let $u = u_1 + u_2 + u_3$ where $u_1 = \dfrac{(x^2+y^2)^n}{2n(2n-1)}$, $u_2 = x f\left(\dfrac{y}{x}\right)$, $u_3 = \phi\left(\dfrac{x}{y}\right)$.

Clearly u_1 is homogeneous in x and y of degree $2n$, u_2 is homogeneous in x and y of degree 1 and u_3 is homogenous in x and y of degree 0. Thus applying Euler's theorem separately to u_1, u_2, u_3 we get

$$x^2\frac{\partial^2 u_1}{\partial x^2}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2} = 2n(2n-1)\,u_1$$

$$x^2\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2} = 1(1-1)\,u_2=0$$

$$x^2 \frac{\partial^2 u_3}{\partial x^2} + 2xy \frac{\partial^2 u_3}{\partial x \partial y} + y^2 \frac{\partial^2 u_3}{\partial y^2} = 0\,(0-1)\, u_3 = 0$$

Adding these, gives

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2n\,(2n-1)\, u_1 + 0 + 0$$

$$= 2n\,(2n-1)\, \frac{(x^2+y^2)^n}{2n\,(2n-1)} = (x^2+y^2)^n.$$

Ans.

EXAMPLE 11.20. If $u = x^n f\left(\frac{y}{x}\right) + y^{-n} \phi\left(\frac{x}{y}\right)$, show that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + x u_x + y u_y = n^2 u.$$

SOLUTION: Let $u = u_1 + u_2$ where

$$u_1 = x^n f\left(\frac{y}{x}\right) \quad \text{and} \quad u_2 = y^{-n} \phi\left(\frac{x}{y}\right).$$

Here u_1 is homogeneous in x and y of degree n while u_2 is homogenous in x and of degree $-n$. Making use of Euler's theorem to u_1 and u_2 separately, we get

$$x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = n u_1, \qquad \text{...(1)}$$

$$x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} = -n u_2 \qquad \text{...(2)}$$

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = n\,(n-1)\, u_1 \qquad \text{...(3)}$$

and

$$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = -n\,(-n-1) u_2 \qquad \text{...(4)}$$

Adding (1), (2), (3) and (4), gives

$$x^2 \frac{\partial^2}{\partial x^2}(u_1+u_2) + 2xy \frac{\partial^2}{\partial x \partial y}(u_1+u_2) + y^2 \frac{\partial^2}{\partial y^2}(u_1+u_2) + x \frac{\partial}{\partial x}(u_1+u_2) + y \frac{\partial}{\partial y}(u_1+u_2)$$

$$= n\,(n-1)\, u_1 + (-n)\,(-n-1)\, u_2 + n u_1 - n u_2$$

or

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u_1\,(n^2 - n + n) + u_2\,(n^2 + n - n) = n^2 u.$$

Hence Proved.

EXAMPLE 11.21. If $u = \operatorname{cosec}^{-1} \sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ evaluate $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$.

SOLUTION: From the given equation we can write

$$\operatorname{cosec} u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}} = z, \text{ say,}$$

or $$z = \sqrt{\frac{x^{1/2}\left(1+(y/x)^{1/2}\right)}{x^{1/3}\left(1+(y/x)^{1/3}\right)}} = x^{1/12}\sqrt{\frac{1+(y/x)^{1/2}}{1+(y/x)^{1/3}}}$$

Thus z is a homogeneous function in x and y of degree $\frac{1}{12}$ and z is a function of u as $z = f(u) = \operatorname{cosec} u$ therefore by Euler's theorem, (Deduction III), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = g(u)\,[\,g'(u) - 1]. \qquad \text{... (1)}$$

where $$g(u) = \frac{n\,f(u)}{f'(u)} = \left(\frac{1}{12}\right)\left(\frac{\operatorname{cosec} u}{-\operatorname{cosec} u \cot u}\right) = -\frac{1}{12}\tan u.$$

As such $$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \left(-\frac{1}{12}\tan u\right)\left(-\frac{1}{12}\sec^2 u - 1\right) = \frac{1}{12}\tan u\left(\frac{13}{12} + \frac{1}{12}\tan^2 u\right).$$

Ans.

EXAMPLE 11.22. If $u = \sin^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u.$

SOLUTION: Here u is not homogeneous function

but $$f(u) = \sin u = \frac{x^3 + y^3 + z^3}{ax + by + cz}$$ is homogeneous in x, y, z of degree 2.

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2f(u)}{f'(u)} = \frac{2\sin u}{\cos u} = 2\tan u.$$

Hence Proved.

EXAMPLE 11.23. If $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, then prove that $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$ and evaluate $x^2 z_{xx} + 2xy\,z_{xy} + y^2 z_{yy}$.

[GGSIPU II Sem I Term 2011]

SOLUTION: Here $\tan^{-1}(y/x)$ and $\tan^{-1}(x/y)$ are both of degree zero and hence behave like constants. Therefore z is homogeneous function in x and y of degree two. Applying Euler's theorem, we get

$$x^2 z_{xx} + 2xy\,z_{xy} + y^2 z_{yy} = 2(2-1)\,z = 2z.$$

Ans.

TOTAL DERIVATIVE

If $u = f(x, y)$ is a function of two independent variables x and y, and x and y are separately functions of a single independent variable t then u can be expressed as a function of t alone and then we can find the ordinary derivative $\frac{du}{dt}$ which is called the *total derivative* of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Let us be interested in finding $\frac{du}{dt}$ without actually substituting the values of x and y in terms of t in $f(x, y)$. We derive below the relation between the total derivative $\frac{du}{dt}$ and the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. When t is given an increment δt, suppose x and y get increments δx and δy respectively and this, in turn, causes u to get an increment of δu. In other words, when t becomes $t + \delta t$ let x become $x + \delta x$ and y become $y + \delta y$ and consequently, u becomes $u + \delta u$. Thus, δx and δy both tend to 0 as δt tends to 0 and then by definition,

$$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \lim_{\delta t \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \left(\frac{dx}{dt}\right) + \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \left(\frac{dy}{dt}\right)$$

$$= \lim_{\delta x \to 0} \frac{\delta f(x, y+\delta y)}{\delta x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{since } u = f(x, y).$$

Thus, we have $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \dots (1)$$

and in terms of differentials this result can be better written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad \dots (2)$$

This du is called the total derivative of u.

This is a very important result and will be used very frequently in partial differentiation.

On extending this result to functions of three variables, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

or $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz.$$

Some Deductions

If $f(x, y) = 0$ is an implicit relation between x and y, *i.e.*, y is an implicit function of x, then we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Therefore, $\frac{dy}{dx}$ in terms of partial derivatives of f w.r.t. x and y, is given by

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$$

In this case we can also calculate $\frac{d^2y}{dx^2}$ in terms of partial derivatives. But before that let us introduce the following conventional notations:

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x} = r, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y} = t, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = s.$$

Thus, we can write $\frac{dy}{dx} = \frac{-p}{q}$.

Hence $$\frac{d^2y}{dx^2} = -\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}$$

Now, using (1) we can write

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}.1 + \frac{\partial p}{\partial y}.\frac{dy}{dx} = r + s\left(\frac{-p}{q}\right) = \frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\frac{dy}{dx} = s + t\left(\frac{-p}{q}\right) = \frac{qs - pt}{q}$$

Putting the values of $\frac{dp}{dx}$ and $\frac{dq}{dx}$ in (3), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q.\frac{qr - ps}{q} - p.\frac{qs - pt}{q}\right] = -\frac{1}{q^3}\left[q^2r - 2\,pqs + p^2t\right]$$

Thus, we have $$\boldsymbol{\frac{d^2y}{dx^2} = -\frac{[q^2r - 2pqs + p^2t]}{q^3}}$$

CHANGE OF INDEPENDENT VARIABLES

Let $u = f(x, y)$ where $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$ ... (1)

It is frequently necessiated to change the expressions involving $u, x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ to the expressions involving $u, t_1, t_2, \frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}$ etc. In this constext, the earlier result

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$ can now be extended to the following

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

and $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}.$$

Next, if instead of (1) we are given that

$$u = u(t_1, t_2), \quad t_1 = \phi_1(x, y) \quad \text{and} \quad t_2 = \phi_2(x, y).$$

Then the equations of transformation become

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y}.$$

Conversion from cartesian to polar coordinates and *vice versa* are frequently encountered examples. Thus, if $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

and other way round, we have the formulae

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

EXAMPLE 11.24. If $(\cos x)^y - (\sin y)^x = 0$, find $\frac{dy}{dx}$.

SOLUTION: The given relation is $(\cos x)^y = (\sin y)^x$

Taking logarithm on both sides, gives $y\log(\cos x) = x\log(\sin y)$

Now, let us take $f(x, y) = y\log(\cos x) - x\log(\sin y) = 0$

So y can be taken as an imlicit function of x or *vice versa*.

Here $$\frac{\partial f}{\partial x} = \frac{y(-\sin x)}{\cos x} - 1.\log(\sin y) = -(y\tan x + \log\sin y)$$

and $$\frac{\partial f}{\partial y} = \log\cos x\,.\,1 - \frac{x\cos y}{\sin y} = \log\cos x - x\cot y.$$

Therefore, $$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = \frac{(y\tan x + \log\sin y)}{\log\cos x - x\cot y}.$$ **Ans.**

EXAMPLE 11.25. If $x^n + y^n = a^n$, find $\frac{d^2y}{dx^2}$.

SOLUTION: Let $f(x, y) = x^n + y^n - a^n = 0$

then $$\frac{\partial f}{\partial x} = p = n\,x^{n-1}, \quad \frac{\partial f}{\partial y} = q = n\,y^{n-1},$$

$$\frac{\partial^2 f}{\partial x^2} = r = n(n-1)x^{n-2},\ s = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial p}{\partial y} = 0 \text{ and } \frac{\partial^2 f}{\partial y^2} = t = n(n-1)y^{n-2}$$

Since $f(x, y) = 0$ we can take y as an implicit function of x

$$\therefore \quad \frac{d^2 y}{dx^2} = -\frac{1}{q^3}[q^2 r - 2pqs + p^2 t]$$

$$= -\frac{1}{n^3 y^{3n-3}}[n^2 y^{2n-2} \cdot n(n-1)x^{n-2} - 2n^2 x^{n-1} y^{n-1} \cdot 0 + n^2 x^{2n-2} \cdot n(n-1)y^{n-2}]$$

$$= -\frac{n^3(n-1)}{n^3 y^{3n-3}}[x^{n-2} y^{2n-2} + y^{n-2} x^{2n-2}]$$

$$= -\frac{(n-1)}{y^{3n-3}} x^{n-2} y^{n-2}[y^n + x^n] = -\frac{(n-1)x^{n-2} \cdot a^n}{y^{2n-1}}. \quad \textbf{Ans.}$$

EXAMPLE 11.26. If $f(x, y) = 0$ and $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

SOLUTION: Since $f(x, y) = 0$ we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \qquad \therefore \quad \frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

And since $\phi(y, z) = 0$, we have

$$d\phi = \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \qquad \therefore \quad \frac{dz}{dy} = -\frac{\partial \phi}{\partial y} \Big/ \frac{\partial \phi}{\partial z}$$

$$\therefore \quad \frac{dy}{dx} \cdot \frac{dz}{dy} = (-1)^2 \frac{\dfrac{\partial f}{\partial x} \dfrac{\partial \phi}{\partial y}}{\dfrac{\partial f}{\partial y} \dfrac{\partial \phi}{\partial z}} \quad \text{or} \quad \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y}. \qquad \textbf{Hence Proved.}$$

EXAMPLE 11.27. (*a*) If $y^x + x^y = (x+y)^{(x+y)}$ find $\dfrac{dy}{dx}$.

(*b*) If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^3$ find the value of $\dfrac{dz}{dx}$ when $x = y = a$.

[GGSIPU IInd Sem End Term 2010]

SOLUTION: (*a*) Let $f(x, y) = y^x + x^y - (x+y)^{(x+y)}$

$$\therefore \quad \frac{\partial f}{\partial x} = y^x \log y + y \cdot x^{y-1} - (x+y)^{x+y}[1 + \log(x+y)]$$

and
$$\frac{\partial f}{\partial y} = x y^{x-1} + x^y \log x - (x+y)^{x+y}[1 + \log(x+y)]$$

Hence
$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\left[\frac{y^x \log y + y x^{y-1} - (x+y)^{x+y}(1 + \log(x+y))}{x y^{x-1} + x^y \log x - (x+y)^{x+y}(1 + \log(x+y))}\right].$$

(b) Since $z = \sqrt{x^2 + y^2}$ we have, from total derivative concept,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{or} \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \frac{dy}{dx}$$

or
$$\frac{dz}{dx} = \frac{1}{2}(x^2 + y^2)^{-1/2} 2x \cdot 1 + \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2y \cdot \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + y^2}}\left[x + y\frac{dy}{dx}\right]$$

From the relation $x^3 + y^3 + 3axy = 5a^3$ we have

$$\frac{dy}{dx} = \frac{-\frac{\partial}{\partial x}(x^3 + y^3 + 3axy)}{\frac{\partial}{\partial y}(x^3 + y^3 + 3axy)} = \frac{-(3x^2 + 3ay)}{3y^2 + 3ax} = -1 \text{ at } (a, a).$$

$$\therefore \quad \left(\frac{dz}{dx}\right)_{(a,\,a)} = \left[\frac{1}{\sqrt{x^2 + y^2}}\{x + y(-1)\}\right]_{(a,\,a)} = 0. \qquad \textbf{Ans.}$$

EXAMPLE 11.28. If $z = xy\, f(y/x)$ and z is constant, show that

$$\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}.$$

[GGSIPU II Sem I Term 2005]

SOLUTION: Since z is constant we write $xy\, f(y/x) = z = \text{constant}$

$$\therefore \quad \frac{dy}{dx} = \frac{-\frac{\partial}{\partial x}[xyf(y/x)]}{\frac{\partial}{\partial y}[xyf(y/x)]} = \frac{-[yf(y/x) + xyf'(y/x)(-y/x^2)]}{[xf(y/x) + xyf'(y/x)(1/x)]}$$

or
$$\frac{x\frac{dy}{dx}}{y} = \frac{\frac{y}{x}f'(y/x) - f(y/x)}{\frac{y}{x}f'(y/x) + f(y/x)}$$

Applying componendo and dividendo here we get

$$\frac{y + x\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{2\frac{y}{x}f'(y/x)}{2f(y/x)} = \frac{yf'(y/x)}{xf(y/x)}$$

or
$$\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}.$$

Hence the result.

EXAMPLE 11.29. (*a*) If $u = f(x-y, y-z, z-x)$ show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$

(*b*) If $u = (e^{x-y}, e^{y-z}, e^{z-x})$ find the value of $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$.

[GGSIPU II Sem I Term Jan. 2011]

SOLUTION: (*a*) Let $x' = x-y,\ y' = y-z,\ z' = z-x$ then we have $u = f(x', y', z')$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial x}+\frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial x}$$

But $\dfrac{\partial x'}{\partial x}=1,\quad \dfrac{\partial y'}{\partial x}=0$ and $\dfrac{\partial z'}{\partial x}=-1$

Hence $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y}+\frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial y}+\frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial y} = -\frac{\partial u}{\partial x'}+\frac{\partial u}{\partial y'}+0$$

And $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial z}+\frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial z}+\frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial z} = 0-\frac{\partial u}{\partial y'}+\frac{\partial u}{\partial z'}$$

Therefore, $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x'}-\frac{\partial u}{\partial z'}-\frac{\partial u}{\partial x'}+\frac{\partial u}{\partial y'}-\frac{\partial u}{\partial y'}+\frac{\partial u}{\partial z'}=0.$$

(*b*) Putting $X' = x-y,\ Y' = y-z,\ Z' = z-x$ we have $u = (X', Y', Z')$

then $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X'}\cdot\frac{\partial X'}{\partial x}+\frac{\partial u}{\partial Y'}\cdot\frac{\partial Y'}{\partial x}+\frac{\partial u}{\partial Z'}\cdot\frac{\partial Z'}{\partial x}=1\cdot\frac{\partial u}{\partial X'}+0\cdot\frac{\partial u}{\partial Y'}-1\cdot\frac{\partial u}{\partial Z'}$$

and $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X'}\cdot\frac{\partial X'}{\partial y}+\frac{\partial u}{\partial Y'}\cdot\frac{\partial Y'}{\partial y}+\frac{\partial u}{\partial Z'}\cdot\frac{\partial Z'}{\partial y}=-1\cdot\frac{\partial u}{\partial X'}+1\cdot\frac{\partial u}{\partial Y'}+0\cdot\frac{\partial u}{\partial Z'}$$

and $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X'}\cdot\frac{\partial X'}{\partial z}+\frac{\partial u}{\partial Y'}\cdot\frac{\partial Y'}{\partial z}+\frac{\partial u}{\partial Z'}\cdot\frac{\partial Z'}{\partial z}=0\cdot\frac{\partial u}{\partial X'}-1\cdot\frac{\partial u}{\partial Y'}+1\cdot\frac{\partial u}{\partial Z'}$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$ **Ans.**

EXAMPLE 11.30. If $z=f(u,v)$ where $u=x\cos\alpha-y\sin\alpha$ and $v=x\sin\alpha+y\cos\alpha$

show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = u\dfrac{\partial z}{\partial u}+v\dfrac{\partial z}{\partial v}$, α being constant.

SOLUTION: Here $z = f(u, v)$ where u and v are functions of x, y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \cos\alpha\frac{\partial z}{\partial u}+\sin\alpha\frac{\partial z}{\partial v}$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = -\sin\alpha\frac{\partial z}{\partial u}+\cos\alpha\frac{\partial z}{\partial v}$$

Therefore, $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = x\left(\cos\alpha\frac{\partial z}{\partial u}+\sin\alpha\frac{\partial z}{\partial v}\right)+y\left(-\sin\alpha\frac{\partial z}{\partial u}+\cos\alpha\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u}(x\cos\alpha-y\sin\alpha)+\frac{\partial z}{\partial v}(x\sin\alpha+y\cos\alpha) = u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}.$$

Hence Proved.

EXAMPLE 11.31. If $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

[GGSIPU II Sem End Term 2009]

SOLUTION: Here,
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad \text{... (1)}$$

and
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

or
$$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\sin\theta\frac{\partial z}{\partial x} + \cos\theta\frac{\partial z}{\partial y} \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$+\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta - 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

Hence Proved.

EXAMPLE 11.32. Let $z = f(x, y)$ where $x = e^u\cos v$, $y = e^u\sin v$ then show that

(*i*) $y\dfrac{\partial z}{\partial u} + x\dfrac{\partial z}{\partial v} = e^{2u}\dfrac{\partial z}{\partial y}$

(*ii*) $\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\dfrac{\partial f}{\partial u}\right)^2 + \left(\dfrac{\partial f}{\partial v}\right)^2\right].$

SOLUTION: (*i*) We have $z=f(x,y)$ and x and y are functions of u and v,

Hence
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^u\cos v\frac{\partial z}{\partial x} + e^u\sin v\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \qquad \text{...(1)}$$

And
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -e^u\sin v\frac{\partial z}{\partial x} + e^u\cos v\frac{\partial z}{\partial y} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \qquad \text{...(2)}$$

$\therefore$
$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y}$$

$$= (x^2+y^2)\frac{\partial z}{\partial y} = (e^{2u}\cos^2 v + e^{2u}\sin^2 v)\frac{\partial z}{\partial y} = e^{2u}\frac{\partial z}{\partial y}.$$

(*ii*) Squaring and adding (1) and (2), gives

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2 = x^2\left(\frac{\partial z}{\partial x}\right)^2+2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+y^2\left(\frac{\partial z}{\partial y}\right)^2+y^2\left(\frac{\partial z}{\partial x}\right)^2-2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+x^2\left(\frac{\partial z}{\partial y}\right)^2$$

$$= (x^2+y^2)\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = (e^{2u}\cos^2 v+e^{2u}\sin^2 v)\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$= e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right].$$

or $$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right] = \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$ **Hence the result.**

EXAMPLE 11.33. By changing the independent variables u and v to x and y by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$

show that $\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$ transforms into $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$.

[GGSIPU II Sem I Term 2006]

SOLUTION: Since $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$ we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}.$$

Next $$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right).\frac{\partial x}{\partial u}+\frac{\partial}{\partial y}\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u}$$

$$= \left(\cos\alpha\frac{\partial^2 z}{\partial x^2}+\sin\alpha\frac{\partial^2 z}{\partial x\partial y}\right)\cos\alpha+\left(\cos\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin\alpha\frac{\partial^2 z}{\partial y^2}\right)\sin\alpha.$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\sin\alpha\cdot\cos\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

and $$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial x}\left(-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}\right).\frac{\partial y}{\partial v}$$

$$= \left(-\sin\alpha\frac{\partial^2 z}{\partial x^2}+\cos\alpha\frac{\partial^2 z}{\partial x\partial y}\right)(-\sin\alpha)+\left(-\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\cos\alpha\frac{\partial^2 z}{\partial y^2}\right)\cos\alpha$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (\cos^2\alpha + \sin^2\alpha)\frac{\partial^2 z}{\partial x^2} + 0\frac{\partial^2 z}{\partial x\partial y} + (\cos^2\alpha + \sin^2\alpha)\frac{\partial^2 z}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

Hence the result.

EXAMPLE 11.34. If z is a function of x and y, and $u = lx + my$ and $v = ly - mx$, then prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

SOLUTION: Let $z = z(u, v)$ where $u = lx + my$, $v = ly - mx$.

Thus, z is composite function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v} = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)z$$

$$\Rightarrow \quad \frac{\partial}{\partial x} \equiv l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l\frac{\partial}{\partial u}\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right) - m\frac{\partial}{\partial v}\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l^2\frac{\partial^2 z}{\partial u^2} - lm\frac{\partial^2 z}{\partial u\partial v} - lm\frac{\partial^2 z}{\partial v\partial u} + m^2\frac{\partial^2 z}{\partial v^2} \qquad \text{... (1)}$$

$$\text{Next,} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v} = \left(m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}\right)z$$

$$\Rightarrow \quad \frac{\partial}{\partial y} \equiv m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}$$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}\right)\left(m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right) \qquad \text{...(2)}$$

$$= m^2\frac{\partial^2 z}{\partial u^2} + ml\frac{\partial^2 z}{\partial u\partial v} + lm\frac{\partial^2 z}{\partial v\partial u} + l^2\frac{\partial^2 z}{\partial v^2}$$

Adding (1) and (2), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\frac{\partial^2 z}{\partial u^2} + 0 + (m^2 + l^2)\frac{\partial^2 z}{\partial v^2}$$

$$= (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

Hence Proved.

EXAMPLE 11.35. Transform the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ into polar co-ordinates r and θ.

[GGSIPU II Sem End Term 2009; IInd Sem. End Term 2006]

SOLUTION: Since $x = r\cos\theta$, $y = r\sin\theta$ we have $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\dfrac{y}{x}$

We have $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x}+\dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x}$ and $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y}+\dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial y}$.

From $r^2 = x^2 + y^2$, we have $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$.

Similarly from $\theta = \tan^{-1}\dfrac{y}{x}$ we have $\dfrac{\partial \theta}{\partial x} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2+y^2} = \dfrac{-\sin\theta}{r}$

and $\dfrac{\partial \theta}{\partial y} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$.

Therefore $\dfrac{\partial u}{\partial x} = \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial \theta}$ and $\dfrac{\partial u}{\partial y} = \sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial \theta}$.

Thus, in terms of operators we, can write

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \quad\text{and}\quad \frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore,
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\,\partial \theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial \theta\,\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial \theta\,\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}$$

and, $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}\right]$$

$$+ \frac{\cos\theta}{r}\left[\sin\theta\frac{\partial^2 u}{\partial\theta\,\partial r} + \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}$$

Therefore, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\sin^2\theta + \cos^2\theta)\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta + \cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta + \cos^2\theta}{r}\frac{\partial u}{\partial r}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Thus, the Laplace equation in cartesian form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

gets transformed into polar form as $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0.$ **Ans.**

EXAMPLE 11.36. Let f be a composite function of u and v and u and v be functions of x and y given by $u = x^2 - y^2$, $v = 2xy$, then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$

SOLUTION: Given that $f = f(u, v)$ and $u = x^2 - y^2$, $v = 2xy$, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(2y) \quad \text{...(1)}$$

$$\Rightarrow \quad \frac{\partial}{\partial x} \equiv 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)$$

Similarly $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(-2y) + \frac{\partial f}{\partial v}(2x) \quad \text{...(2)}$$

$$\Rightarrow \quad \frac{\partial}{\partial y} \equiv 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right)$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}\right) = 2\left[x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) + 1.\frac{\partial f}{\partial u} + y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)\right]$$

$$= 2\left[x.2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial u}\right) + \frac{\partial f}{\partial u} + y.2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial v}\right)\right]$$

$$= 4\left[x\left(x\frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial u \partial v}\right) + \frac{1}{2}\frac{\partial f}{\partial u} + y\left(x\frac{\partial^2 f}{\partial u \partial v} + y\frac{\partial^2 f}{\partial v^2}\right)\right]$$

$$= 4x^2\frac{\partial^2 f}{\partial u^2} + 8xy\frac{\partial^2 f}{\partial u \partial v} + 4y^2\frac{\partial^2 f}{\partial v^2} + 2\frac{\partial f}{\partial u} \qquad \text{... (3)}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-2y\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v}\right) = -2y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right) - 2.1\frac{\partial f}{\partial u} + 2x\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right)$$

$$= -2y\left(-2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial u}\right) - 2\frac{\partial f}{\partial u} + 2x\left(-2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial v}\right)$$

$$= 4y^2\frac{\partial^2 f}{\partial u^2} - 4xy\frac{\partial^2 f}{\partial u \partial v} - 2\frac{\partial f}{\partial u} - 4xy\frac{\partial^2 f}{\partial u \partial v} + 4x^2\frac{\partial^2 f}{\partial v^2} \qquad \text{...(4)}$$

Adding (3) and (4), gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\frac{\partial^2 f}{\partial u^2} + 0 + 4(y^2 + x^2)\frac{\partial^2 f}{\partial v^2} + 0 = 4(x^2 + y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Hence Proved.

EXAMPLE 11.37. (*a*) By changing the independent variables x and y to u and v by means of the relations $u = x - ay$, $v = x + ay$ show that the relation $a^2\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ transforms into $4a^2\frac{\partial^2 z}{\partial u\,\partial v}$.

(*b*) If $z = f(x, y)$, $x^2 = uv$, $y^2 = u/v$ then change the independent variables to u and v in the equation $x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial xy} + y^2\frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} = 0$.

[GGSIPU II Sem I Term 2010]

SOLUTION: (*a*) $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = -a\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}$$

Next, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x} = \frac{\partial^2 z}{\partial u^2}+\frac{2\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}$$

and $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial u}\left(-a\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(-a\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= -a\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)(-a) - a\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}--\frac{\partial z}{\partial v}\right)(a)$$

$$= a^2\frac{\partial^2 z}{\partial u^2}+a^2\frac{\partial^2 z}{\partial v^2}-2a^2\frac{\partial^2 z}{\partial u\,\partial v}$$

Therefore $$a^2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2} = 4a^2\frac{\partial^2 z}{\partial u\,\partial v}.$$

(b) $x^2 = uv,\ y^2 = u/v \Rightarrow u^2 = x^2 y^2,\ v^2 = x^2/y^2$ or $u = xy,\ v = x/y$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\cdot\frac{\partial z}{\partial u}+\frac{1}{y}\frac{\partial z}{\partial v}$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u}-\frac{x}{y}\frac{\partial z}{\partial v}$$

$\Rightarrow$ $$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y} = 2\frac{x}{y}\frac{\partial z}{\partial v} = 2v\frac{\partial z}{\partial v}.$$ Squaring the operators on both sides

$\therefore$ $$x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = 4v^2\frac{\partial^2 z}{\partial v^2}$$

Also, $$2y\frac{\partial z}{\partial y} = 2u\frac{\partial z}{\partial u}-2v\frac{\partial z}{\partial v}$$

Therefore the equation $$x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial v^2}-2y\frac{\partial z}{\partial y}=0$$ becomes

$$4v^2\frac{\partial^2 z}{\partial v^2}+2u\frac{\partial z}{\partial u}-2v\frac{\partial z}{\partial v}=0.$$ **Ans.**

CHAPTER 12

Applications of Partial Differentiation

ERRORS AND APPROXIMATIONS

Let u be a function of two variables x and y and let δx and δy be small changes made in x and y respectively and the resulting change in u be δu, then

$$\delta u = u\,(x + \delta x, y + \delta y) - u\,(x, y)$$
$$= [u\,(x + \delta x, y + \delta y) - u\,(x, y + \delta y)] + [u\,(x, y + \delta y) - u\,(x, y)]$$
$$= \frac{u(x+\delta x, y+\delta y) - u(x, y+\delta y)}{\delta x}\,\delta x + \frac{u(x, y+\delta y) - u(x,y)}{\delta y}\,\delta y.$$

Since $\displaystyle\lim_{\delta x \to 0} \frac{u(x+\delta x, y+\delta y) - u(x, y+\delta y)}{\delta x} = \frac{\partial u}{\partial x}$

and $\displaystyle\lim_{\delta y \to 0} \frac{u\,(x, y+\delta y) - u(x,y)}{\delta y} = \frac{\partial u}{\partial y}$

we can write $\quad \delta u = \dfrac{\partial u}{\partial x}\,\delta x + \dfrac{\partial u}{\partial y}\,\delta y$ approximately. ...(1)

If δx and δy are considered as small errors in the measurement of x and y then δu, given by the above equation (1), represents the error in the calculated value of u.

EXAMPLE 12.1. Compute an approximate values of $(1.04)^{3.01}$.

[GGSIPU II Sem End Term 2006 Reappear; II Sem I Term 2011]

SOLUTION: Let $f(x, y) = x^y$. Here $x = 1$, $y = 3$ and $\delta x = 0.04$, $\delta y = 0.01$.

Therefore, $\quad \delta f = \dfrac{\partial f}{\partial x}\,\delta x + \dfrac{\partial f}{\partial y}\,\delta y$ approximately.

$$= yx^{y-1}.\delta x + x^y \log x \,.\, \delta y = 3(1^{3-1})\,(0.04) = 0.12$$

$\therefore \quad (1.04)^{3.01} = 1.12$ approx. **Ans.**

EXAMPLE 12.2. Compute the value of $\left[(3.8)^2 + 2\,(2.1)^3\right]^{1/5}$ using the theory of approximation.

SOLUTION: Consider the function $\quad u = (x^2 + 2y^3)^{\frac{1}{5}}$

Let us take $x = 4$ and $y = 2$ so that

$x + \delta x = 3.8$ and $y + \delta y = 2.1 \quad \Rightarrow \quad \delta x = -\,0.2$ and $\delta y = 0{\cdot}1$.

The corresponding error δu in u, is given by

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y = \frac{1}{5}(x^2+2y^3)^{-\frac{4}{5}}.2x\,\delta x + \frac{1}{5}(x^2+2y^3)^{-\frac{4}{5}}.6y^2\,\delta y$$

$$= \frac{1}{5}\left(x^2+2y^3\right)^{\frac{-4}{5}}(2x\,\delta x + 6y^2\,\delta y)$$

$$= \frac{1}{5}\left(4^2+2(2)^3\right)^{-\frac{4}{5}}[2(4)(-0.2)+6(2)^2(0.1)] \quad \text{at } x = 4, y = 2$$

$$= \frac{1}{5}(32)^{-\frac{4}{5}}[-1.6+2.4] = \frac{1}{5}2^{-4}(0.8) = \frac{0.8}{80} = 0.01$$

Also, at $x = 4$, $y = 2$ we have

$$u = (x^2+2y^3)^{\frac{1}{5}} = \left(4^2+2(2^3)\right)^{\frac{1}{5}} = 32^{\frac{1}{5}} = 2$$

$\therefore$ Required approximate value $= u + \delta u = 2 + 0.01 = 2.01$. **Ans.**

EXAMPLE 12.3. If the sides and angles of a triangle ABC vary is such a way that its circum-radius remains constant, prove that $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0$

[GGSIPU II Sem I Term 2006]

SOLUTION: By sine rule in triangle ABC we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \quad \text{(which is constant)}$$

$\Rightarrow$ $a = 2R\sin A,\quad b = 2R\sin B,\quad c = 2R\sin C$

hence $da = 2R\cos A\,dA,\quad db = 2R\cos B\,dB,\quad dc = 2R\cos C\,dc$

or $$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C} = 2R(dA+dB+dC)$$

Since $A + B + C = \pi$ we have $dA + dB + dC = 0$

Therefore $$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0.$$ **Hence Proved.**

EXAMPLE 12.4. The acceleration of a piston is calculated from the formula $f = w^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$ where r and l are constants. If ω and θ suffer changes $\delta\omega$ and $\delta\theta$, then show that the resulting change δf in f, is given by

$$\frac{\delta f}{f} = 2\frac{\delta\omega}{\omega} - \left(\frac{\sin\theta + \frac{2r}{l}\sin 2\theta}{\cos\theta + \frac{r}{l}\cos 2\theta}\right)\delta\theta$$

If $\theta = 30^\circ$ and $\frac{r}{l} = \frac{1}{4}$ and θ and ω were each 1% less, find the corresponding percentage change in f.

SOLUTION: In the given relation, taking logarithm, we get

$$\log f = 2\log \omega + \log r + \log\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$$

$$\therefore \quad \frac{\delta f}{f} = 2\frac{\delta\omega}{\omega} + 0 + \frac{-\sin\theta - \frac{2r}{l}\sin 2\theta}{\cos\theta + \frac{r}{l}\cos 2\theta}\delta\theta$$ hence the first result.

$$\text{or} \quad \frac{\delta f}{f}\times 100 = 2\frac{\delta\omega}{\omega}\times 100 - \frac{\sin\theta + \frac{2r}{l}\sin 2\theta}{\cos\theta + \frac{r}{l}\cos 2\theta}\left(\frac{\delta\theta\times 100}{\theta}\right)\cdot\theta.$$

Taking $\theta = (\pi/6)$, $\frac{r}{l} = \frac{1}{4}$, $\frac{\delta\theta}{\theta}\times 100 = -1$ and $\frac{\delta\omega}{\omega}\times 100 = -1$, we get

$$\frac{\delta f}{f}\times 100 = 2(-1) - \frac{\sin 30^\circ + 2\left(\frac{1}{4}\right)\sin 60^\circ}{\cos 30^\circ + \left(\frac{1}{4}\right)\cos 60^\circ}.(-1)\left(\frac{\pi}{6}\right)$$

$$= -2 + \frac{\frac{1}{2} + \frac{2}{4}\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} + \frac{1}{8}}\cdot\frac{\pi}{6} = -2 + \frac{4 + 2\sqrt{3}}{4\sqrt{3}+1}\cdot\frac{\pi}{6}$$

$= -2 + 0.5 = 1{\cdot}5$ approximately.

$\therefore$ Error in f is 1.5 % approximately. **Ans.**

EXAMPLE 12.5. The angles of a triangle are calculated from the sides a, b, c. If small changes $\delta a, \delta b, \delta c$ are made in measuring the sides a, b, c, show that the change in the angle A, is given by

$$\delta A = \frac{a}{2\Delta}[\delta a - \delta b\cos C - \delta c\cos B]$$ where Δ is the area of the triangle.

Also verify that $\delta A + \delta B + \delta C = 0$.

SOLUTION: We know that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\therefore \quad -\sin A\,\delta A = \frac{2bc(2b\delta b + 2c\delta c - 2a\delta a) - (b^2 + c^2 - a^2)(2b\delta c + 2c\delta b)}{(2bc)^2}$$

$$\text{or} \quad -2\sin A\,\delta A = \frac{1}{(bc)^2}[2b^2c\,\delta b + 2bc^2\,\delta c - 2abc\,\delta a - (b^2 + c^2 - a^2)(b\delta c + c\delta b)$$

$$= \frac{1}{(bc)^2}[\delta b\,(2b^2c - b^2c - c^3 + ca^2) + \delta c\,(2bc^2 - b^3 - bc^2 + ba^2) - 2abc\,\delta a]$$

$$= \frac{1}{(bc)^2}[c\delta b\,(b^2 + a^2 - c^2) + b\delta c\,(c^2 + a^2 - b^2) - 2abc\,\delta a]$$

$$= \frac{\delta b}{b}\left(\frac{b^2 + a^2 - c^2}{bc}\right) + \frac{\delta c}{c}\left(\frac{c^2 + a^2 - b^2}{bc}\right) - \frac{2a\delta a}{bc}$$

Since $\quad \Delta = \frac{1}{2} bc \sin A$

$$\therefore \quad \delta A = -\frac{1.bc}{4\Delta}\left[\frac{\delta b}{b}\frac{a^2+b^2-c^2}{bc}+\frac{\delta c}{c}\frac{c^2+a^2-b^2}{bc}-\frac{2a\,\delta a}{bc}\right]$$

$$= -\frac{1}{4\Delta}\left[\frac{\delta b}{b}\cos C\,.\,2ab+\frac{\delta c}{c}\cos B\,.\,2ac-2a\,\delta a\right]$$

$$\left(\text{since in } \Delta ABC, \quad \cos C = \frac{a^2+b^2-c^2}{2ab} \text{ and } \cos B = \frac{c^2+a^2-b^2}{2ac}\right)$$

$$= -\frac{2a}{4\Delta}[\cos C\,\delta b + \cos B\,\delta c - \delta a]$$

or $\quad \delta A = \frac{a}{2\Delta}[\delta a - \delta b \cos C - \delta c \cos B]$ which is the required result.

Similarly, we can get

$$\delta B = \frac{b}{2\Delta}[\delta b - \delta a \cos C - \delta c \cos A] \quad \text{and} \quad \delta C = \frac{c}{2\Delta}[\delta c - \delta a \cos B - \delta b \cos A]$$

Adding these, we get

$$\delta A + \delta B + \delta C = \frac{1}{2\Delta}[a\delta a - a\delta b \cos C - a\delta c \cos B + b\delta b - b\delta a \cos C - b\delta c \cos A + c\delta c - c\delta a \cos B - c\delta b \cos A]$$

$$= \frac{1}{2\Delta}[\delta a\,(a - b\cos C - c\cos B) + \delta b\,(b - a\cos C - c\cos A) + \delta c\,(c - a\cos B - b\cos A)]$$

$$= \frac{1}{2\Delta}.(0) = 0$$ (using the projection rule for Δ ABC) **Hence Proved.**

EXAMPLE 12.6. The height h and the semi vertical angle α of a cone are measured and from these, A, the total area of the cone including the base, is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in A. Also show that, if $\alpha = \frac{\pi}{6}$, an error of $+1\%$ in h will be compensated by an error of $-0.33°$ in α.

SOLUTION: Let r be the radius of the base and l the slant heihgt of the cone, then $r = h \tan\alpha$ and $l = h \sec\alpha$

$$\therefore \quad \text{Total surface area} = A = \pi rl + \pi r^2 = \pi h^2 \tan\alpha \sec\alpha + \pi h^2 \tan^2\alpha$$

$$= \pi h^2 (\tan\alpha \sec\alpha + \tan^2\alpha)$$

Hence $$\delta A = \frac{\partial A}{\partial h}\delta h + \frac{\partial A}{\partial \alpha}\delta\alpha$$

$$= 2\pi h(\tan\alpha\sec\alpha + \tan^2\alpha)\,\delta h + \pi h^2(\tan^2\alpha\sec\alpha + \sec^3\alpha + 2\tan\alpha\sec^2\alpha)\,\delta\alpha$$

or $$\delta A = 2\pi h \tan\alpha(\sec\alpha + \tan\alpha)\,\delta h + \pi h^2 \sec\alpha(\sec\alpha + \tan\alpha)^2\,\delta\alpha$$

Putting $\quad \alpha = \frac{\pi}{6}$ and $\frac{\delta h}{h} \times 100 = 1$, we get

$$\delta A = \frac{2\pi h^2}{100}\frac{1}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)+\pi h^2\frac{2}{\sqrt{3}}\left(\frac{1}{3}+\frac{4}{3}+\frac{4}{\sqrt{3}}\frac{1}{\sqrt{3}}\right)\delta\alpha$$

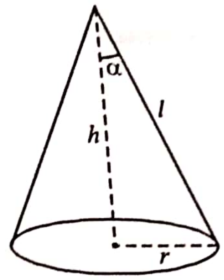
$$= 2\pi h^2 (0.01) + \pi h^2\, 2\sqrt{3}\,\delta\alpha$$

However, as given above $\delta A = 0$, therefore

$$2\pi h^2 (0.01) + 2\sqrt{3}\,\pi h^2\,\delta\alpha = 0$$

or $\quad 0.01 + \sqrt{3}\,\delta\alpha = 0 \quad \Rightarrow \quad \delta\alpha = -\frac{0.01}{\sqrt{3}}$ radians

or $\quad \delta\alpha = -\frac{0.01}{\sqrt{3}}\times\frac{180}{\pi}$ (in degrees) $= -0.33°$



Hence Proved.

EXAMPLE 12.7. In a plane triangle ABC if the sides a and b are kept constant, show that the variations of its angles are given by the relation

$$\frac{dA}{\sqrt{a^2-b^2\sin^2 A}} = \frac{dB}{\sqrt{b^2-a^2\sin^2 B}} = -\frac{dC}{c}.$$

SOLUTION: By the sine formula we have $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $a\sin B = b\sin A$. ...(1)

Taking differentials on both sides, we get $a\cos B\, dB = b\cos A\, dA$

$$\Rightarrow \quad \frac{dA}{a\cos B} = \frac{dB}{b\cos A} = \frac{dA+dB}{a\cos B + b\cos A} \quad \text{(by componendo and dividendo)} \quad ...(2)$$

Next, $\quad a\cos B = a\sqrt{1-\sin^2 B} = \sqrt{a^2-a^2\sin^2 B} = \sqrt{a^2-b^2\sin^2 A}$ using (1),

and $\quad b\cos A = b\sqrt{1-\sin^2 A} = \sqrt{b^2-b^2\sin^2 A} = \sqrt{b^2-a^2\sin^2 B}$. using (1).

Also, by projection rule in Δ ABC we have $a\cos B + b\cos A = c$,

and since $A + B + C = \pi$ we have

$$dA + dB + dC = 0 \quad \text{or} \quad dA + dB = -dC.$$

As such (1) becomes

$$\frac{dA}{\sqrt{a^2-b^2\sin A}} = \frac{dB}{\sqrt{b^2-a^2\sin^2 B}} = \frac{-dC}{c}.$$

Hence Proved.

EXAMPLE 12.8. (*a*) Find the possible percentage error in computing the resistance r from the formulae $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$ if r_1 and r_2 are both in error by 2%.

[GGSIPU IInd Sem End Term 2010]

(*b*) Find the possible percentage error in computing the parallel resistance r of the three resistances r_1, r_2, and r_3 from formula

$$\frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \quad \text{if} \quad r_1, r_2, r_3 \quad \text{are each in error by} \quad +1.2\ \%.$$

SOLUTION : (*a*) From $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$ we have

$$-\frac{1}{r^2}\delta r = -\frac{1}{r_1^2}\delta r_1 - \frac{1}{r_2^2}\delta r_2 \quad \text{or} \quad \frac{\delta r}{r}=\frac{r}{r_1}\frac{\delta r_1}{r_1}+\frac{r}{r_2}\frac{\delta r_2}{r_2}=\frac{r_2}{r_1+r_2}\frac{\delta r_1}{r_1}+\frac{r_1}{r_1+r_2}\frac{\delta r_2}{r_2}$$

$$\therefore \quad \frac{\delta r}{r}\times 100 = \frac{r_2}{r_1+r_2}(2)+\frac{r_2}{r_1+r_2}(2)=2 \quad \textbf{Ans.}$$

(*b*) Using the differentials in $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$, we have

$$-\frac{1}{r^2}\delta r = -\frac{1}{r_1^2}\delta r_1 - \frac{1}{r_2^2}\delta r_2 - \frac{1}{r_3^2}\delta r_3$$

and since $\frac{\delta r_1}{r_1}\times 100 = \frac{\delta r_2}{r_2}\times 100 = \frac{\delta r_3}{r_3}\times 100 = 1.2,$ we get

$$\frac{1}{r^2}\delta r = \frac{1.2}{r_1\times 100}+\frac{1.2}{r_2\times 100}+\frac{1.2}{r_3\times 100}$$

$$= \frac{1.2}{100}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right) = \frac{1.2}{r\times 100}$$

$$\Rightarrow \quad \frac{\delta r}{r}\times 100 = \frac{1.2\,r}{r} = 1.2$$

Therefore, the resultant resistance is also having an error of + 1.2 %. **Ans.**

EXAMPLE 12.9. A balloon in the form of a right cylinder of radius 1.5 m and height 4 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m how much change (%) will be in the volume of balloon?

SOLUTION : $V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3.$

$$\delta V = \pi\, 2r\cdot\delta r\cdot h + \pi r^2 \delta h + \frac{4}{3}\cdot\pi\, 3r^2 \delta r$$

$$\therefore \quad \frac{\delta V}{V} = \frac{\pi r[2h\,\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3} = \frac{2\delta r\cdot h + r\cdot\delta h + 4r\cdot\delta r}{rh+\frac{4}{3}r^2}$$

$$= \frac{2(0.01)(4)+1.5(0.05)+4(1.5)\,0.01}{(1.5)4+\frac{4}{3}(1.5)^2} = \frac{0.08+0.075+0.06}{6+3} = \frac{0.215}{9}$$

and hence percentage error in volume $= \frac{\delta V}{V}\cdot 100 = 2.389\%.$

Ans.

EXAMPLE 12.10. In estimating the number of bricks in a pile which is measured to be $(5m \times 10m \times 5m)$ the count of brick is 100 bricks per m^3. Find the error in the cost when the tape is streched 2% beyond its standard length. The cost of bricks is Rs. 2000 per thousand bricks.

SOLUTION : Volume $V = xyz$ or $\log V = \log x + \log y + \log z$.

$$\Rightarrow \quad \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z} \quad \text{hence} \quad \frac{\delta V}{V} \times 100 = 2 + 2 + 2 = 6$$

or
$$\delta V = \frac{6V}{100} = \frac{6(5 \times 10 \times 5)}{100} = 15m^3.$$

The number of bricks in δV, the error in $V = 15 \times 100 = 1500$.

Therefore, the error in the cost $= \frac{1500 \times 2000}{1000} = \text{Rs. } 3000.$ **Ans.**

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JACOBIANS

Let $u = u(x, y)$ and $v = v(x, y)$ be two continuous functions of the independent variables x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y. Then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u and v with respect to x and y, and is denoted by $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$.

Similarly if u, v, w are functions of x, y, z then Jacobian of u, v, w with respect to x, y, z is defined as

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u, v, w)}{\partial(x, y, z)}.$$

CHAIN RULE FOR JACOBIANS

If u, v are functions of x, y and x, y are themselves functions of r, s then

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(r, s)}$$ **[GGSIPU II Sem I Term 2011]**

Now $$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \end{vmatrix} \quad \dots(1)$$

But since $u = f_1(x, y)$, $v = f_2(x, y)$ and $x = \phi_1(r, s)$, $y = \phi_2(r, s)$, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s}.$$

Therefore (1) becomes $$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} = \frac{\partial(u, v)}{\partial(r, s)}.$$

In general $$\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)} \cdot \frac{\partial(u_1, u_2, \dots, u_n)}{\partial(v_1, v_2, \dots, v_n)} = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(v_1, v_2, \dots, v_n)}.$$

Corollary of Chain Rule

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$$

In the previous result if we replace r, s by u, v this corollary immediately follows:

If $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ then the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ is denoted by J′, that is,

$$J'=\frac{\partial(x,y)}{\partial(u,v)} \quad \text{and we have } JJ'=1.$$

Also, in general, we have

$$\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}\cdot\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}=1$$

JACOBIAN OF IMPLICIT FUNCTIONS

If u and v are implicit functions of the variables x and y, connected by the relations

$$f_1(u,v,x,y)=0 \quad \text{and} \quad f_2(u,v,x,y)=0$$

Differentiating the above relations partially *w.r.t.* x and y separately, we get

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x}=0, \quad \frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y}=0 \qquad \ldots(1)$$

and

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x}=0, \quad \frac{\partial f_2}{\partial y}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}=0 \qquad \ldots(2)$$

Now consider

$$\frac{\partial(f_1,f_2)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y}\\ \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}-\frac{\partial f_1}{\partial x} & -\frac{\partial f_1}{\partial y}\\ -\frac{\partial f_2}{\partial x} & -\frac{\partial f_2}{\partial y}\end{vmatrix}=(-1)^2\frac{\partial(f_1,f_2)}{\partial(x,y)} \qquad \text{[using (1) and (2)]}$$

Thus, we have

$$\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\,\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

For the case when u, v, w are implicit functions of x, y, z, given by the relations

$$f_1(u, v, w, x, y, z) = 0$$
$$f_2(u, v, w, x, y, z) = 0$$
$$f_3(u, v, w, x, y, z) = 0$$

we shall have
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}.$$

EXAMPLE 12.11. If $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$ evaluate $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

SOLUTION: Since $r^2 = x^2 + y^2$ we have $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$

and as $\theta = \tan^{-1}(y/x)$, we have
$$\frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+\dfrac{y^2}{x^2}} = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

and
$$\frac{\partial\theta}{\partial y} = \frac{1/x}{1+\dfrac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}.$$

Therefore
$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial\theta}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \dfrac{-\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}[\cos^2\theta + \sin^2\theta] = \frac{1}{r}$$ **Ans.**

EXAMPLE 12.12. Find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ where $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$.

[GGSIPU II Sem End Term 2009; II Sem End Term 2004 Reappear]

SOLUTION: We know that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)}$

Now
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

and
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Therefore
$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^2.r = 4r^3$$ **Ans.**

N

PARTIAL DERIVATIVES FROM IMPLICIT FUNCTIONS USING JACOBIANS

Suppose u, v are implicit functions of the independent variables x, y connected by the functional relations

$$f_1(u, v, x, y) = 0 \quad \cdots (1)$$

and

$$f_2(u, v, x, y) = 0. \quad \cdots (2)$$

To obtain partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ we partially differentiate (1) and (2) w.r.t. x and y separately and get

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial x}.1 + \frac{\partial f_1}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}.\frac{\partial v}{\partial x} = 0 \\ \frac{\partial f_2}{\partial x}.1 + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} = 0 \end{array}\right\} \quad \cdots (3)$$

and

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial y}.1 + \frac{\partial f_1}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}.\frac{\partial v}{\partial y} = 0 \\ \frac{\partial f_2}{\partial y}.1 + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y} = 0 \end{array}\right\} \quad \cdots (4)$$

Solving (3) for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, we get

$$\frac{\partial u}{\partial x} = -\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = -\frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$$

[GGSIPU IInd Sem. Ist Term 2005]

and

$$\frac{\partial v}{\partial x} = -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial x} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = -\frac{\frac{\partial (f_1, f_2)}{\partial (u, x)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$$

Similarly, solving (4) for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial (f_1, f_2)}{\partial (y, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial (f_1, f_2)}{\partial (u, y)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}.$$

EXAMPLE 12.13. If $u^2 + xv^2 = x + y$, $v^2 + yu^2 = x - y$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$.

SOLUTION: The functional relations are

$$f_1(u, v, x, y) = u^2 + xv^2 - x - y = 0 \quad \text{and} \quad f_2(u, v, x, y) = v^2 + yu^2 - x + y = 0$$

Now

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}} = -\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= -\begin{vmatrix} v^2-1 & 2xv \\ -1 & 2v \end{vmatrix} \Big/ \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} v^2-1 & xv \\ -1 & v \end{vmatrix} \Big/ \begin{vmatrix} u & xv \\ uy & v \end{vmatrix}$$

$$= -\frac{1}{2}\frac{(v^3 - v + xv)}{uv - uvxy} = \frac{-1\,(v^2-1+x)}{2u(1-xy)}.$$

and $$\frac{\partial v}{\partial y} = \frac{-\dfrac{\partial(f_1, f_2)}{\partial(u, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}} = -\begin{vmatrix} 2u & -1 \\ 2yu & 1 \end{vmatrix} \Big/ \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix}$$

$2uv^2+1$

$$= \frac{-1}{2}\begin{vmatrix} u & -1 \\ yu & 1 \end{vmatrix} \Big/ \begin{vmatrix} u & xv \\ uy & v \end{vmatrix} = \frac{-1}{2}\,\frac{(u+yu)}{(uv - xy\,uv)} = \frac{-(1+y)}{2v(1-xy)}.$$ **Ans.**

EXAMPLE 12.14. **(*a*)** If $u = x + y^2,\ v = y + z^2,\ w = z + x^2$, prove that $\dfrac{\partial x}{\partial u} = -(1+8xyz)^{-1}$.

[GGSIPU II Sem I Term 2005]

(*b*) Using Jacobians find $\dfrac{\partial u}{\partial x}$ if $u^2 + xy^2 - xy = 0$ and $u^2 + u\,vx + v^2 = 0$.

[GGSIPU II Sem I Term 2011]

SOLUTION: **(*a*)** We are given three relations in u, v, w, x, y, z as

$$f_1 = u - x - y^2 = 0,$$
$$f_2 = v - y - z^2 = 0$$

and $$f_3 = w - z - x^2 = 0.$$

then $$\frac{\partial x}{\partial u} = \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2 f_3)}{\partial(x, y, z)}} = \frac{\begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix}} = \frac{1}{-1 + 2y(-4xz)}$$

$$= \frac{-1}{1+8xyz}.$$ **Hence Proved.**

(*b*) Since $u^2 = x\,(y - y^2)$ we can find $\dfrac{\partial u}{\partial x}$ without Jacobians also, as follows:

$$2u\frac{\partial u}{\partial x} = y - y^2 \qquad \therefore \qquad \frac{\partial u}{\partial x} = \frac{y(1-y)}{2u} = \frac{u^2}{2ux} = \frac{u}{2x}.$$ **Ans.**

However by the method of Jacobians we have

$$f_1 \equiv u^2 + xy^2 - xy = 0, \quad f_2 \equiv u^2 + uvx + v^2 = 0$$

$$\therefore \quad \frac{\partial(f_1, f_2)}{\partial(x, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} y^2 - y & 0 \\ uv & ux + 2v \end{vmatrix} = -\frac{u^2}{x}(ux + 2v)$$

and $\quad \dfrac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 2u + vx & ux + 2v \end{vmatrix} = 2u\,(ux + 2v)$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1, f_2)}{\partial(x, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}} = \frac{\dfrac{u^2}{x}(ux + 2v)}{2u\,(ux + 2v)} = \frac{u}{2x}.$$ **Ans.**

EXAMPLE 12.15. Show that the functions $u = x + y + z$, $v = x^3 + y^3 + z^3 - 3xyz$, and $w = x^2 + y^2 + z^2 - xy - yz - zx$ are functionally dependent. **[GGSIPU II Sem End Term 2005]** Find the relation between them.

SOLUTION: The functions u, v, w are functionally dependent if $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3(x^2 - yz) & 3(y^2 - xz) & 3(z^2 - xy) \\ 2x - y - z & 2y - z - x & 2z - x - y \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2 - yz & y^2 - x^2 + z(y - x) & z^2 - x^2 + y(z - x) \\ 2x - y - z & 3(y - x) & 3(z - x) \end{vmatrix}$$

$$= 6\begin{vmatrix} (y - x)(x + y + z) & (z - x)(x + y + z) \\ y - x & z - x \end{vmatrix} = 6(y - x)\,(z - x)\,(x + y + z)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Therefore u, v, w are functionally dependent.

Next, $\quad u = x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx) = vw$

Hence $u = vw$ is the desired relation. **Ans.**

JACOBIANS TO DETERMINE FUNCTIONAL DEPENDENCE

Jacobian is also used in determining whether or not two functions are functionally dependent. Two functions $f(x, y)$ and $\phi(x, y)$ are called functionally dependent if they are functions of each other.

Assume that $f(x, y)$ and $\phi(x, y)$ are functionally dependent then there exists a relation of the type $F(f, \phi) = 0$.

Differentiating it partially *w.r.t.* x and y, we get

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial f}\frac{\partial f}{\partial y} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial y} = 0$$

These are homogeneous equations. Their non-trivial solution would exist only when

$$\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial f}{\partial y} & \dfrac{\partial \phi}{\partial y} \end{vmatrix} = 0, \quad i.e. \quad \frac{\partial(f,\phi)}{\partial(x,y)} = 0$$

Hence $f(x, y)$ and $\phi(x, y)$ **are functionally dependent if their Jacobian vanishes identically.** The idea can be easily extended to three functions and, in general, to n functions.

EXAMPLE 12.16. Find the Jacobian of the following transformation
$u = a \cosh x \cos y, \quad v = a \sinh x \sin y.$

SOLUTION: The required Jacobian J is given by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} a \sinh x \cos y & -a \cosh x \sin y \\ a \cosh x \sin y & a \sinh x \cos y \end{vmatrix}$$

$$= a^2 (\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y) = \frac{a^2}{2} [\sinh^2 x (1 + \cos 2y) + \cosh^2 x (1 - \cos 2y)]$$

$$= \frac{a^2}{2} [\sinh^2 x + \cosh^2 x - \cos 2y (\cosh^2 x - \sinh^2 x)] = \frac{a^2}{2} [\cosh 2x - \cos 2y].$$ **Ans.**

EXAMPLE 12.17. If $y_1 = \dfrac{x_2 x_3}{x_1}, \; y_2 = \dfrac{x_3 x_1}{x_2}, \; y_3 = \dfrac{x_1 x_2}{x_3}$, find the value of $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.

[GGSIPU Ist Sem. End Term 2003]

SOLUTION: $$J = \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\dfrac{x_2 x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_3 x_1}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2}\left(\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right) + \frac{x_3}{x_1}\left(\frac{x_1 x_2}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_2 x_3^2}\right) + \frac{x_2}{x_1}\left(\frac{x_3 x_1}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_3 x_2^2}\right)$$

$$= 0 + \frac{x_3}{x_1}\left(\frac{x_1}{x_3} + \frac{x_1}{x_3}\right) + \frac{x_2}{x_1}\left(\frac{x_1}{x_2} + \frac{x_1}{x_2}\right) = 2 + 2 = 4.$$ **Ans.**

EXAMPLE 12.18. For the transformation $x = a(u+v)$, $y = b(u-v)$ and $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$ find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

SOLUTION: From the relations $x = a(u+v)$, $y = b(u-v)$ we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$$

And from the relations $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$, we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix}$$

$$= 4r^3 \begin{vmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix} = 4r^3$$

By chain rule, $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(r,\theta)} = -2ab \,.\, 4r^3 = -8\,abr^3.$ **Ans.**

EXAMPLE 12.19. If $x+y+z = u$, $y+z = uv$, $z = uvw$, find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

SOLUTION: The given relations can be written as

$z = uvw$, $y = uv - z = uv - uvw = uv(1-w)$

and $x = u - (y+z) = u - uv = u(1-v)$.

Therefore, $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} \text{ Applying } R_3 \to R_3 + R_2$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = uv\,[u(1-v) + uv] = u^2 v.$$ **Ans.**

EXAMPLE 12.20. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

[GGSIPU II Sem I Term 2010]

SOLUTION: Let us first calculate the value of

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} yz & z(x-y) & y(x-z) \\ 2x & 2(y-x) & 2(z-x) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2z\,(x - y)\,(z - x) - 2y\,(y - x)\,(x - z) = -\,2\,(x - y)\,(z - x)\,(y - z)$$

Using the fact that $JJ' = 1$, we have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = J' = \frac{1}{J} = \frac{-1}{2(x-y)(y-z)(z-x)}. \qquad \textbf{Ans.}$$

EXAMPLE 12.21. If $u^3 + v + w = x + y^2 + z^2$, $u + v^3 + w = x^2 + y + z^2$ and $u + v + w^3 = x^2 + y^2 + z$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

[GGSIPU IInd Sem. Ist Term 2006; IInd Sem. End Term 2006]

SOLUTION: The given relations can be written as implicit functions as

$$f_1\,(u, v, w, x, y, z) = u^3 + v + w - x - y^2 - z^2 = 0$$
$$f_2\,(u, v, w, x, y, z) = u + v^3 + w - x^2 - y - z^2 = 0$$
$$f_3\,(u, v, w, x, y, z) = u + v + w^3 - x^2 - y^2 - z = 0$$

Now $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$

$$= 3u^2\,(9v^2w^2 - 1) - 1\,(3w^2 - 1) + 1\,(1 - 3v^2)$$
$$= 27\,u^2 v^2 w^2 - 3u^2 - 3v^2 - 3\,w^2 + 2$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$$

$$= -\,1\,(1 - 4yz) + 2y\,(2x - 4\,xz) - 2z\,(4xy - 2x)$$
$$= -\,1 + 4\,(xy + yz + zx) - 16\,xyz$$

Therefore

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \Big/ \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

$$= \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$

Ans.

EXAMPLE 12.22. (*a*) Show that the functions

$f_1(x, y) = x^2 + y^2 + xy = 0$ and

$f_2(x, y) = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = 0$

are functionally dependent.

(*b*) Examine the functional dependence of $u = \dfrac{x-y}{1+xy}$ and $v = \tan^{-1}x - \tan^{-1}y$.

If dependent, find the relation. **[GGSIPU II Sem I Term 2011]**

SOLUTION: (*a*) We have

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} 2x+y & 2y+x \\ 4x^3+6x^2y+6xy^2+2y^3 & 2x^3+6x^2y+6xy^2+4y^3 \end{vmatrix}$$

$$= (2x+y)(2x^3+6x^2y+6xy^2+4y^3) - (2y+x)(4x^3+6x^2y+6xy^2+2y^3)$$

$= 0$ (on simplification).

Hence Proved.

which implies that f_1 and f_2 are functionally dependent.

(*b*) $\dfrac{\partial u}{\partial x} = \dfrac{1+y^2}{(1+xy)^2}$, $\dfrac{\partial u}{\partial y} = \dfrac{-(1+x^2)}{(1+xy)^2}$, $\dfrac{\partial v}{\partial x} = \dfrac{1}{1+x^2}$ and $\dfrac{\partial v}{\partial y} = \dfrac{-1}{1+y^2}$.

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1+y^2}{(1+xy)^2} & \dfrac{-(1+x^2)}{(1+xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{-1}{1+y^2} \end{vmatrix} = \frac{1}{(1+xy)^2}\begin{vmatrix} 1+y^2 & -(1+x^2) \\ \dfrac{1}{1+x^2} & \dfrac{-1}{1+y^2} \end{vmatrix} = 0$$

$\Rightarrow$ u and v are functionally dependent. Clearly the relation between them is $u = \tan v$. **Ans.**

TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Let us first recall the Taylor's expansion of function of one variable as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots \quad \dots (1)$$

where h is small. Consider the extension of above result to function of two independent variables, *i.e.*, to obtain the expansion of $f(x+h, y+k)$ in powers of h and k, both h and k being small. We shall prove that

$$f(x+h,\ y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \dots$$

In $f(x+h,\ y+k)$ first take $x+h$ as constant and expand $f(x+h,\ y+k)$ in powers of k using the relation (1), as

$$f(x+h, y+k) = f(x+h, y) + k\frac{\partial}{\partial y} f(x+h, k) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x+h, y) + \dots \quad \dots (2)$$

Now expanding each term on the R.H.S. of (2) in powers of h by Taylor's theorem of one variable considering y as constant, we have

$$f(x+h, y+k) = \left[f(x,y) + h\frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \dots\right] + k\frac{\partial}{\partial y}\left[f(x,y) + h\frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \dots\right]$$

$$+ \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}\left[f(x,y) + h\frac{\partial f}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \dots\right] + \dots$$

$$= f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial x^2}\right) + \dots$$

This extension of Taylor's theorem is useful when we discuss the maxima and minima of function of two variables. If we put $x = 0$, $y = 0$ and write x for h and y for k we arrive at the extension of *Maclaurin's theorem for two independent variables* as

$$f(x,y) = f(0,0) + x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)}$$

$$+ \frac{1}{2!}\left\{x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + 2xy\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(0,0)} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)}\right\} + \dots$$

EXAMPLE 12.23. Expand $f(x, y) = e^{xy}$ about $(1, 1)$ upto second degree terms.

[GGSIPU IInd Sem. Ist Term 2005, 2010]

SOLUTION:
$$f(x,y) = f(1,1) + \left[(x-1)\frac{\partial f}{\partial x} + (y-1)\frac{\partial f}{\partial y}\right]_{(1,1)}$$

$$+ \frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f}{\partial x^2} + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x\,\partial y} + (y-1)^2\frac{\partial^2 f}{\partial y^2}\right]_{(1,1)} + \dots$$

$$= e + (x-1)(ye^{xy})_{(1,1)} + (y-1)(xe^{xy})_{(1,1)} + \frac{1}{2}[(x-1)^2 e + 2(x-1)(y-1)e + (y-1)^2 e] + \dots$$

$$= e(x+y-1) + \frac{e}{2}(x+y-2)^2.$$ **Ans.**

EXAMPLE 12.24. Find the expansion of $\cos x \cos y$ is powers of x, y upto fourth order terms.

[GGSIPU I Sem End Term 2004 Reappear]

SOLUTION: Expanding the function $f(x, y) = \cos x \cos y$ about the origin by Taylor's theorem and retaining terms upto fourth order, we have

$$f(x,y) = f(0,0) + (x f_x + y f_y) + \frac{1}{2!}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})$$

$$+\frac{1}{3!}[x^3 f_{xxx} + 3x^2 y\, f_{xxy} + 3xy^2\, f_{xyy} + y^3\, f_{yyy}] +$$

$$\frac{1}{4!}[x^4 f_{xxxx} + 4x^3 y f_{xxxy} + 6x^2 y^2 f_{xxyy} + 4xy^3 f_{xyyy} + y^4 f_{yyyy}]$$

where derivatives are taken at the origin.

Now $f(0, 0) = \cos 0 \cos 0 = 1, f_x = -\sin x \cos y = 0$ at $(0, 0), f_y = -\cos x \sin y = 0$ at $(0,0)$

$f_{xx} = -\cos x \cos y = -1$ at $(0, 0)$, $f_{xy} = \sin x \sin y = 0$ at $(0, 0)$,

$f_{yy} = -\cos x \cos y = -1$ at $(0, 0)$.

$f_{xxx} = \sin x \sin y = 0$ at $(0, 0)$, $f_{xxy} = +\cos x \sin y = 0$ at $(0, 0)$,

$f_{xyy} = \sin x \cos y = 0$ at $(0, 0)$

$f_{yyy} = +\cos x \sin y = 0$ at $(0, 0)$, $f_{xxxx} = \cos x \cos y =$ at $(0, 0)$,

$f_{xxxy} = -\sin x \sin y = 0$ at $(0, 0)$, $f_{xxyy} = \cos x \cos y = 1$ at $(0, 0)$,

$f_{xyyy} = -\sin x \sin y = 0$ at $(0, 0)$, $f_{yyyy} = \cos x \cos y = 1$ at $(0, 0)$

$$\therefore \cos x \cos y = 1 + (0.x + 0.y) + \frac{1}{2!}(-1x^2 + 0.2xy - 1y^2) + \frac{1}{3!}[0\, x^3 + 0.3x^2y + 0.3xy^2 + 0.y^3]$$

$$+\frac{1}{4!}[1.x^4 + 0.4x^3y + 1.6x^2y^2 + 0.4xy^3 + 1.y^4]$$

$$= 1 - \frac{x^2}{2!} - \frac{y^2}{2!} + \frac{x^4}{4!} + \frac{6x^2y^2}{4!} + \frac{1}{4!}y^4$$

$$= 1 - \frac{1}{2}(x^2 + y^2) + \frac{1}{24}(x^4 + 6x^2y^2 + y^4) \qquad \textbf{Ans.}$$

ALITER

$$\cos x \cos y = [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots][1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots]$$

$$= 1 - \frac{x^2}{2!} - \frac{y^2}{2!} + \frac{x^4}{4!} + \frac{y^4}{4!} + \frac{x^2y^2}{2!2!} \quad \text{keeping terms upto } 4^{th} \text{ orders}$$

$$= 1 - \frac{1}{2}(x^2 + y^2) + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24}. \qquad \textbf{Ans.}$$

EXAMPLE 12.25. (*a*) Expand $e^x \log(1+y)$ in the neighbourhood of the origin retaining terms upto second degree in x and y. **[GGSIPU II Sem End Term 2006 Reappear]**

(*b*) Obtain the Taylor's linear approximation to the function $f(x, y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ about the point $(-1, 1)$. Also, find the maximum error in the region $x + 1 < 0.1$, $y - 1 < 0.1$ **[GGSIPU II Sem End Term 2011]**

SOLUTION: (*a*) We know that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

and
$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$$

Therefore
$$e^x \log(1+y) = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]\left[y - \frac{y^2}{2} + \frac{y^3}{3} \ldots\right]$$

$$= y - \frac{y^2}{2} + xy \quad \text{retaining terms upto second degree.}$$ **Ans.**

(*b*) $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1.$ $\quad (x_0, y_0) = (-1, 1).$

By Taylor's theorem, linear approximations to $f(x,y)$ is

$$f(x,y) = f(x_0, y_0) + (x - x_0)\left(\frac{\partial f}{\partial x}\right) + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + \ldots$$

$$f(-1,1) = 2 + 1 + 1 - 3 - 4 + 1 = -2.$$

$$\frac{\partial f}{\partial x} = 4x - y + 3 = -2 \text{ at } (-1, 1)$$

$$\frac{\partial f}{\partial y} = -x + 2y - 4 = -1 \text{ at } (-1, 1)$$

Hence $\quad f(x,y) = -2 + (x+1)(-2) + (y-1)(-1)$

The error term $= -2(x+1) - (y-1)$

$\therefore$ Maximum error $= |-2(0.1) - (0.1)| = 0.3$ **Ans.**

EXAMPLE 12.26. Expand $\sin(xy)$ in powers of $(x-1)$ and $(y - \pi/2)$ upto and including the second degree terms. **[GGSIPU I Sem End Term 2003]**

SOLUTION: $f(x, y) = \sin(xy)$, $f_x = y\cos(xy)$, $f_y = x\cos(xy)$,
$f_{xx} = -y^2\sin(xy)$, $f_{xy} = 1\cos(xy) - xy\sin(xy)$, $f_{yy} = -x^2\sin(xy)$

The expansion of $f(x, y)$ about $(1, \pi/2)$ by Taylor's theorem, is

$$f(x, y) = f(1, \pi/2) + [(x-1) f_x + (y - \pi/2) f_y] + \frac{1}{2!}\left[(x-1)^2 f_{xx} + 2(x-1)\left(y - \frac{\pi}{2}\right) f_{xy} + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\right]$$

retaining terms upto second degree and taking derivatives at $(1, \pi/2)$.

Here $f(1, \pi/2) = 1$, $\quad f_x = \dfrac{\pi}{2}\cos\dfrac{\pi}{2}$, $\quad f_y = 1\cos\pi/2$

$$f_{xx} = -\left(\frac{\pi}{2}\right)^2 .1, \quad f_{xy} = 0 - \frac{\pi}{2}.1 = -\frac{\pi}{2}, \quad f_{yy} = -1 \text{ at } (1, \pi/2)$$

$$\therefore \quad \sin(xy) = 1 + [0(x-1) + 0(y-\pi/2)] + \frac{1}{2!}\left[-\frac{\pi^2}{4}(x-1)^2 - 2\left(\frac{-\pi}{2}\right)(x-1)\left(y-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right]$$

$$= 1 - \frac{\pi}{8}(x-1)^2 + \pi(x-1)(y-\pi/2) - (y-\pi/2)^2 \qquad \textbf{Ans.}$$

EXAMPLE 12.27. (*a*) Obtain the Taylor's expansion of the function $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ about (1, 1) upto including second degree terms and compute $f(1.1, 0.9)$.

[GGSIPU II Sem End Term 2009; II Sem I Term 2011]

(*b*) Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ by using Taylor's series.

[GGSIPU IInd Sem End Term 2010]

SOLUTION: (*a*) Let $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$. By Taylor's expansion

$$f(x, y) = f(1, 1) + (x-1)\left(\frac{\partial f}{\partial x}\right)_{(1,1)} + (y-1)\left(\frac{\partial f}{\partial y}\right)_{(1,1)}$$

$$+\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f}{\partial x^2} + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x\,\partial y} + (y-1)^2\frac{\partial^2 f}{\partial y^2}\right]_{(1,1)} + \ldots$$

Hence $\quad \dfrac{\partial f}{\partial x} = \dfrac{-y}{x^2+y^2} = -\dfrac{1}{2}$ at (1, 1), $\quad \dfrac{\partial f}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{1}{2}$ at (1, 1)

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} = \frac{1}{2} \text{ at } (1, 1), \quad \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \equiv -\frac{1}{2} \text{ at } (1, 1)$$

and $\quad \dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{y^2 - x^2}{(x^2+y^2)} \equiv 0$ at (1, 1).

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

Here $\quad x - 1 = 1.1 - 1 = 0.1, \quad y - 1 = 0.9 - 1 = -0.1$

$$\therefore \quad f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2 = 0.7862. \qquad \textbf{Ans.}$$

(*b*) By Taylor's theorem expansion of $f(x, y)$ about the point (a, b), is

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$\frac{1}{3!}\left[(x-a)^3\frac{\partial^3 f}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\partial y} + 3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\partial y^2} + (y-b)^3\frac{\partial^3 f}{\partial y^3}\right] + \ldots$$

where derivatives are to be taken at (a, b).

Here $\quad f(x, y) = x^2y + 3y - 2 \qquad \therefore \quad f(1, -2) = -10,$

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 3, \quad \frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^3 f}{\partial x^3} = 0, \quad \frac{\partial^3 f}{\partial y^3} = 0,$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = 2, \quad \frac{\partial^3 f}{\partial x \partial y^2} = 0.$$

$$\therefore x^2y + 3y - 2 = -10 + \Big[(x-1)(-4) + (y+2)(4) + \frac{1}{2!}\{(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)\}$$

$$+ \frac{1}{6}\big[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)\big] + \ldots$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 4(x-1)(y+2) + (x-1)^2(y+2) \text{ Ans.}$$

MAXIMA AND MINIMA OF A FUNCTION OF TWO OR MORE VARIABLES

We shall simply extend the definition of maxima and minima of functions of one variable to functions of two variables. The function $f(x, y)$ has a maximum value for a certain pair of values of x and y, if this value, say f_{max}, is greater than the values of $f(x, y)$ for all values of x and y in the small neighbourhood of the particular pair of values, *i.e.*, $f(x, y) > f(x+h, y+k)$ where h and k are small. Similary, a minimum value of $f(x, y)$ is defined.

Thus, the quantity $f(x, y) - f(x+h, y+k)$ must retain a constant sign for small variations in h and k. When this constant sign is positive there exists a maximum value of $f(x, y)$ and when it is negative there exists a minimum value of $f(x, y)$. Clearly, if the sign does not remain constant, there will be *neither a maximum nor a minimum* and then the point is called ***saddle point.***

To obtain the criterion for maxima and minima, recall the Taylor's expansion of $f(x+h, y+k)$ as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

or $$f(x+h, y+k) - f(x, y) = (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \ldots \quad \ldots(1)$$

When h and k are small the first term on the R.H.S. of (1) governs the sign of the R.H.S. and hence, for constant sign on the R.H.S., we must have

$$f_x = f_y = 0 \quad \text{or} \quad p = q = 0 \quad \ldots(2)$$

as preliminary condition for the existence of maxima and minima. A point satisfying the conditions in (2), is called a **stationary point or critical point**. Then in that case eqn. (1) becomes

$$f(x+h, y+k) - f(x, y) = \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \ldots \quad \ldots(3)$$

Again since h and k are small, the sign of R.H.S. of (3) will be governed by the sign of its first term. Now, consider

$$h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} = \frac{1}{f_{xx}}\left[h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{xx} f_{yy}\right]$$

$$= \frac{1}{f_{xx}}\left[(h f_{xx} + k f_{xy})^2 + k^2(f_{xx} f_{yy} - f_{xy}^2)\right]$$

If $f_{xx}f_{yy} - f_{xy}^2 \geq 0$ or $rt - s^2 \geq 0$, the above expression in the square brackets is always positive and there will be maxima if $f_{xx}\,(= r)$ is negative and minima if $f_{xx}\,(= r)$ is positive. Whereas, if $f_{xx}f_{yy} - f_{xy}^2 < 0$ the expression in the square brackets can change in sign and therefore there will be neither maxima nor minima. Thus, provided that f_{xx}, f_{yy} and f_{xy} are not all zero, the criteria for the existence of maxima and minima runs as follows:

(*i*) $f_x = f_y = 0$, or $p = q = 0$ the solution of which gives pairs of values of x and y called *critical points or stationary points.*

(*ii*) Compute $f_{xx}\,(= r)$, $f_{yy}\,(= t)$ and $f_{xy}\,(= s)$ for each pair (x, y). If $f_{xx}\, f_{yy} - f_{xy}^2 > 0$, or $rt - s^2 > 0$, there will be a maxima if f_{xx} (or f_{yy}) < 0 and minimum if f_{xx} (or f_{yy}) > 0.

(*iii*) If $f_{xx}\, f_{yy} - f_{xy}^2 < 0$ there will be neither a maxima nor minima and points are called ***saddle points***.

EXAMPLE 12.28. Find the maximum and minimum values of the function $x^3 + y^3 - 3axy$.

SOLUTION: Let $f(x, y) = x^3 + y^3 - 3\,axy$ hence

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3\,ax$$

For stationary points $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$, which gives $x^2 = ay$ and $y^2 = ax$

thus the stationary points are $(0, 0)$ and (a, a).

Next $r = \dfrac{\partial^2 f}{\partial x^2} = 6x$, $s = \dfrac{\partial^2 f}{\partial x\, \partial y} = -3a$ and $t = \dfrac{\partial^2 f}{\partial y^2} = 6y$

At $(0, 0)$, $rt - s^2 = 0 - 9a^2 < 0$

Therefore, at $(0, 0)$ $f(x, y)$ is neither maximum nor minimum.

At (a, a) we have $rt - s^2 = 36a^2 - 9a^2 = 27a^2$ which is positive. Now, the sign of r (or t) decides the existence of maxima or minima at the point (a, a) Since $r = 6a = t$ at (a, a) which is positive hence (a, a) is minima and the minimum value of the function is $a^3 + a^3 - 3a^3 = -a^3$. **Ans.**

EXAMPLE 12.29. Find the dimensions of a rectangular box (without top) with a given volume so that the material used is minimum. **[GGSIPU IInd Sem. End Term 2007]**

SOLUTION: Let the length, breadth and height of the box be x, y and z respectively and V be the given volume, then $V = xyz =$ constant ... (1)

The surface area $S = xy + 2yz + 2zx$ ... (2)

Substituting $z = \dfrac{V}{xy}$ in (1), we get $S = xy + 2(x + y)\dfrac{V}{xy} = xy + \dfrac{2V}{y} + \dfrac{2V}{x}$

Therefore, $\dfrac{\partial S}{\partial x} = y - \dfrac{2V}{x^2}$ and $\dfrac{\partial S}{\partial y} = x - \dfrac{2V}{y^2}$.

At stationary points $\dfrac{\partial S}{\partial x} = 0$, $\dfrac{\partial S}{\partial y} = 0$, *i.e.*, $x^2 y = 2V$ and $xy^2 = 2V$

$\Rightarrow \quad xy\,(x - y) = 0$ which gives $x = 0,\; y = 0,\; x = y.$

$\therefore$ Stationary points are $(0, 0)$ and $\left((2V)^{1/3}, (2V)^{1/3}\right)$.

Next, $\dfrac{\partial^2 S}{\partial x^2} = \dfrac{4V}{x^3}, \quad \dfrac{\partial^2 S}{\partial x \partial y} = 1, \quad \dfrac{\partial^2 S}{\partial y^2} = \dfrac{4V}{y^3}$

At $(0, 0)$ $\dfrac{\partial^2 s}{\partial x^2}$ and $\dfrac{\partial^2 s}{\partial y^2}$ are not defined, therefore, neither maxima nor minima at $(0, 0)$.

At $\left((2V)^{1/3}, (2V)^{1/3}\right)$ $\dfrac{\partial^2 S}{\partial x^2} = 2, \dfrac{\partial^2 S}{\partial y^2} = 2$ hence

$$\frac{\partial^2 S}{\partial x^2}\frac{\partial^2 S}{\partial y^2} - \left(\frac{\partial^2 S}{\partial x \partial y}\right)^2 = 4 - 1 > 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial x^2} > 0$$

hence S has minimum value at $x = (2V)^{1/3},\; y = (2V)^{1/3}$.

At this point $z = \dfrac{V}{xy} = \dfrac{V}{(2V)^{1/3}.(2V)^{1/3}} = \dfrac{V^{1/3}}{2^{2/3}} = \dfrac{x}{2}$.

Therefore, the box should have a square base and the height should be half the length of the base for material to be used, to be least in making the box. **Ans.**

EXAMPLE 12.30. (*a*) Examine the function $f(x, y) = \sin x + \sin y + \sin (x + y)$ for maximum and minimum values. **[GGSIPU II Sem End Term 2005]**

(*b*) In a plane triangle ABC find the maximum value of $\cos A \cos B \cos C$.

[GGSIPU IInd Sem End Term 2010]

SOLUTION: (*a*) $f(x, y) = \sin x + \sin y + \sin (x + y)$

Hence $\dfrac{\partial f}{\partial x} = \cos x + \cos (x + y) = 2 \cos\left(x + \dfrac{y}{2}\right)\cos\dfrac{y}{2}$

and $\dfrac{\partial f}{\partial y} = \cos y + \cos (x + y) = 2 \cos\left(y + \dfrac{x}{2}\right)\cos\dfrac{x}{2}$

For f to be maximum or minimum $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$, that is,

$$\cos\left(x + \frac{y}{2}\right)\cos\frac{y}{2} = 0 \quad \text{and} \quad \cos\left(y + \frac{x}{2}\right)\cos\frac{x}{2} = 0$$

which gives $x = \pi,\quad y = \pi;\quad x = \pi/3,\quad y = \pi/3$

$\therefore$ Critical points are $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, and (π, π) if x and $y \in [0, \pi]$.

Next, $\dfrac{\partial^2 f}{\partial x^2} = r = -\sin x - \sin (x + y), \quad \dfrac{\partial^2 f}{\partial y^2} = t = -\sin y - \sin (x + y)$

and $\dfrac{\partial^2 f}{\partial x \partial y} = s = -\sin (x + y).$

At (π, π), $r = s = t = 0$. Hence saddle point

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = -2\sin\pi/3 = -\sqrt{3}$, $t = -\sqrt{3}$, $s = -\sqrt{3}/2$

$\therefore \quad rt - s^2 = (\sqrt{3})^2 - \frac{3}{4} = \frac{9}{4} > 0$.

Hence $f(x, y)$ is maximum at $(\pi/3, \pi/3)$ and (π, π) is saddle point. **Ans.**

(*b*) In triangle ABC, $A + B + C = \pi$, hence

$$\cos A\cos B\cos C = \cos A\cos B\cos(\pi - A - B) = -\cos A\cos B\cos(A+B)$$

$$= -\frac{1}{2}[\cos(A+B) + \cos(A-B)]\cos(A+B)$$

$$= -\frac{1}{4}[1 + \cos(2A+2B) + \cos 2A + \cos 2B] = f(A, B), \text{ say.}$$

then $\frac{\partial f}{\partial A} = \frac{1}{2}\sin(2A+2B) + \frac{1}{2}\sin 2A$ and $\frac{\partial f}{\partial B} = \frac{1}{2}\sin(2A+2B) + \frac{1}{2}\sin 2B$

For f to be maximum $\frac{\partial f}{\partial A} = 0$, $\frac{\partial f}{\partial B} = 0$ or $\sin(2A+B)\cos B = 0$ and $\sin(2B+A)\cos A = 0$

Now if $\cos B = 0$ then $B = \frac{\pi}{2}$ which gives $\sin A\cos A = 0$ or $A = \frac{\pi}{2}$ which is not possible.

Therefore

$$2A + B = \pi \text{ and } 2B + A = \pi \Rightarrow A = \frac{\pi}{3}, \quad B = \frac{\pi}{3}.$$

Next, $\frac{\partial^2 f}{\partial A^2} = \cos(2A+2B) + \cos 2A$, $\frac{\partial^2 f}{\partial A \partial B} = \cos(2A+2B)$, $\frac{\partial^2 f}{\partial B^2} = \cos(2A+2B) + \cos 2B$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ $\frac{\partial^2 f}{\partial A^2} = -\cos\frac{\pi}{3} - \cos\frac{\pi}{3}$, $\frac{\partial^2 f}{\partial A \partial B} = -\cos\frac{\pi}{3}$, $\frac{\partial^2 f}{\partial B^2} = -\cos\frac{\pi}{3} - \cos\frac{\pi}{3}$

$$\frac{\partial^2 f}{\partial A^2} \cdot \frac{\partial^2 f}{\partial B^2} - \left(\frac{\partial^2 f}{\partial A \partial B}\right)^2 = 4\cos^2\frac{\pi}{3} - \cos^2\frac{\pi}{3} = 3\cos^2\frac{\pi}{3} > 0$$

and $\frac{\partial^2 f}{\partial A^2} < 0$ Hence Maxima at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

and Max. value of $\cos A\cos B\cos C = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. **Ans.**

EXAMPLE 12.31. Find the maximum and minimum values of $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

[GGSIPU I Sem II Term 2003]

SOLUTION: $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Hence $\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$ and $\frac{\partial f}{\partial y} = 6xy - 30y$.

Now for f to have maximum or minimum $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

or $\quad x^2 + y^2 - 10x + 24 = 0$ and $(x-5)y = 0$

$\Rightarrow$ when $y = 0$, $(x-4)(x-6) = 0$. Hence two critical points are $(4, 0)$ and $(6, 0)$

and when $x = 5$ we have $y^2 - 1 = 0$, $y = \pm 1$, hence, we have

two critical points $(5, 1)$ and $(5, -1)$.

Next, $\quad r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$, $\frac{\partial^2 f}{\partial x \partial y} = s = 6y$ and $t = \frac{\partial^2 f}{\partial y^2} = 6x - 30.$

Now at $(4, 0)$, $r = t = -6$ and $s = 0$ hence maxima at $(4, 0)$.

At $(6, 0)$, $\quad r = t = 6$, $s = 0$ hence maxima at $(6, 0)$.

At $(5, 1)$, $\quad r = t = 0$ and $s = 6$ so it is a saddle point.

At $(5, -1)$, $r = t = 0$ and $s = -6$ so it is also a saddle point.

Therefore, max. $f(x, y) = 64 - 16(15) + 228 = 112$

and min. $f(x, y) = 36(6) - 15(36) + 72(6) = 108$

and at $(5, \pm 1)$ we have saddle points. **Ans.**

EXAMPLE 12.32. Examine the function $f(x, y) = x^3 - 3x^2 - 4y^2 + 1$ for maximum and minimum.

[GGSIPU II Sem End Term 2006; II Sem End Term 2005]

SOLUTION: $f(x, y) = x^3 - 3x^2 - 4y^2 + 1$. Here $\frac{\partial f}{\partial x} = 3x^2 - 6x$, $\frac{\partial f}{\delta y} = -8y$

For $f(x, y)$ to have maximum and minimum, $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\delta y}$.

Thus, the critical points are $(0, 0)$ and $(2, 0)$.

Next, $\quad r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$, $\quad s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $\quad t = \frac{\partial^2 f}{\partial y^2} = -8$

At $(0, 0)$, $r < 0$, $t < 0$ and $rt - s^2 = 48 > 0$ hence maxima.

At $(2, 0)$, $r = 6$, $s = 0$, $t = -8$ hence $rt - s^2 < 0$

hence at $(2, 0)$ we have a saddle point. **Ans.**

EXAMPLE 12.33. Discuss the maxima and minima of $x^3y^2(1 - x - y)$.

[GGSIPU II Sem I Term 2006; End Term 2011]

SOLUTION: Given function is $\quad f = x^3y^2(1 - x - y)$.

$$\therefore \quad \frac{\partial f}{\partial x} = -3x^2y^2(x + y - 1) - x^3y^2 = -x^2y^2[4x + 3y - 3]$$

and $$\frac{\partial f}{\partial y} = -2x^3y(x + y - 1) - x^3y^2 = -x^3y[2x + 3y - 2]$$

For f to be maximum or minimum $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$\Rightarrow$ $x = 0$, $y = 0$ and $x = \frac{1}{2}$, $y = \frac{1}{3}$.

Thus the critical points are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Next, $r = \frac{\partial^2 f}{\partial x^2} = -2xy^2(4x+3y-3) - 4x^2y^2$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -2x^2y(4x+3y-3) - 3x^2y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = -x^3(2x+3y-2) - 3x^3y$$

At $(0, 0)$ $r = s = t = 0$ hence neither nor maxima here

and at $\left(\frac{1}{2}, \frac{1}{3}\right)$, $r = -\frac{1}{9}$, $s = \frac{-1}{12}$, $t = \frac{-1}{8}$ hence $rt - s^2 = \frac{1}{72} - \frac{1}{144}$.

Since $rt - s^2 > 0$ and r and t are negative we have a maxima at $\left(\frac{1}{2}, \frac{1}{3}\right)$. **Ans.**

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

[GGSIPU II Sem End Term 2006]

At occasions it would be necessary to find the maxima or minima of a function subject to one or two conditions (or constraints) being satisfied. Suppose we are to maximise or minimise the function $u = f(x, y)$ subject to the condition $g(x, y) = 0$. If we can solve the latter equation for y in terms of x and substitute it in the function $f(x, y)$, then the problem reduces to that of finding the maxima or minima of a function of a single variable x. However, usually it is not possible and we take recourse to Lagrange's method of undetermined multipliers as explained below:

$\frac{dx}{dx}$

Since $u = f(x, y)$ we have $\frac{du}{dx} = \frac{\partial f}{\partial x}.1 + \frac{\partial f}{\partial y}\frac{dy}{dx}$ ... (1)

and from the relation $g(x, y) = 0$ we have $\frac{\partial g}{\partial x}.1 + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0$ ... (2)

From (2) we have $\frac{dy}{dx} = \frac{-\partial g/\partial x}{\partial g/\partial y} = -\frac{g_x}{g_y}$ on the curve $g(x, y) = 0$.

But on $g(x, y) = 0$ the function $u = f(x, y)$ is only a function of x as mentioned above, hence the stationary points are given by $\frac{du}{dx} = 0$, that is,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\left(\frac{-\partial g/\partial x}{\partial g/\partial y}\right) = 0$$

...(3)

...(4)

or $\quad f_x\, g_y - f_y\, g_x = 0$

which can be solved subject to the condition $\quad g(x, y) = 0.$

Algebraically, this is equivalent to finding the stationary points of a function F given by .. (5)

$$F(x, y) = f(x, y) + \lambda g(x, y)$$

where λ is a parameter to be determined.

The stationary points of $F(x, y)$ are given by the equations

$$f_x + \lambda\, g_x = 0 \quad \text{and} \quad f_y + \lambda\, g_y = 0$$

which will have a solution if the condition (3) is satisfied. The maxima or minima can then be obtained by examining the condition in the neighbourhood of the stationary points.

Further, suppose it is required to find the stationary values of a function of three variables, say

$$u = f(x, y, z) \qquad ...(1)$$

subject to conditions $\quad \phi(x, y, z) = 0 \quad$ and $\quad \psi(x, y, z) = 0.$ ... (2)

In this case we construct the function F as

$$F = f + \lambda_1\, \phi + \lambda_2\, \psi \qquad ...(3)$$

where λ_1, λ_2 are called non-zero Lagrange's multipliers.

Next, form the equations $\quad \dfrac{\partial F}{\partial x} = 0, \quad \dfrac{\partial F}{\partial y} = 0, \quad \dfrac{\partial F}{\partial z} = 0$ ... (4)

Eliminating $x, y, z, \lambda_1, \lambda_2$ between (1), (2) and (4) we shall get an equation in u, the roots of which give the stationary values of $u = f(x, y, z)$.

The method would be clear through the following examples.

EXAMPLE 12.34. (*a*) Determine the maxima of the function u given by

$$u = (x + 1)(y + 1)(z + 1)$$

subject to the condition $\quad a^x\, b^y\, c^z = k.$

(*b*) Use Lagrange's method of multipliers to find the smallest and largest value of $x + 2y$ on the circle $x^2 + y^2 = 1.$ **[GGSIPU II Sem End Term 2011]**

SOLUTION: (*a*) From the given function and the condition, we can write

$$\log u = \log(x + 1) + \log(y + 1) + \log(z + 1) \qquad ... (1)$$

and $\quad x \log a + y \log b + z \log c = \log k$ ... (2)

Now, when u is maximum $\log u$ will also be maximum. For finding critical points, we differetiate (1) and (2) and get

$$\frac{dx}{x+1} + \frac{dy}{y+1} + \frac{dz}{z+1} = 0 \qquad ... (3)$$

and $\quad \log a\, dx + \log b\, dy + \log c\, dz = 0$ ... (4)

Multiplying (4) by λ and then adding to (3) and equating to zero the co-efficients of dx, dy, dz we get

$$\frac{1}{x+1} + \lambda \log a = 0, \quad \frac{1}{y+1} + \lambda \log b = 0, \quad \frac{1}{z+1} + \lambda \log c = 0$$

or $\quad 1 + \lambda\, x \log a + \lambda \log a = 0$
$1 + \lambda\, y \log b + \lambda \log b = 0$
$1 + \lambda\, z \log c + \lambda \log c = 0 \qquad \text{...(5)}$

Adding these, we get

$$3 + \lambda\,(x \log a + y \log b + z \log c) + \lambda\,(\log a + \log b + \log c) = 0$$

or $\quad 3 + \lambda \log k + \lambda \log (abc) = 0 \quad \Rightarrow \quad \lambda = \dfrac{-3}{\log (k\, abc)}$.

Substituting this value of λ in (5), we get

$$x + 1 = \frac{-1}{\lambda \log a} = \frac{\log (k\, abc)}{3 \log a},\; y + 1 = \frac{\log (k\, abc)}{3 \log b},\; z + 1 = \frac{\log (k\, abc)}{3 \log c}$$

Thus, the maxima is given by $x = \dfrac{\log\left(k\dfrac{bc}{a^2}\right)}{3 \log a}, \quad y = \dfrac{\log\left(k\dfrac{ca}{b^2}\right)}{3 \log b}, \quad z = \dfrac{\log\left(k\dfrac{ab}{c^2}\right)}{3 \log c}$. **Ans.**

(*b*) $f(x, y) = x + 2y, \quad \phi\,(x, y) = x^2 + y^2 = 1.$

Let $\quad F = x + 2y + \lambda\,(x^2 + y^2 - 1)$ where λ is undetermined multiplier

$$\frac{\partial F}{\partial x} = 1 + 2\lambda x, \quad \frac{\partial F}{\partial y} = 2 + 2\lambda y. \quad \text{For Max. and Min.} \quad \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0$$

$\therefore \quad 1 + 2\lambda x = 0 \quad$ and $\quad 2 + 2\lambda y = 0 \quad$ or $\quad x = \dfrac{-1}{2\lambda}, \quad y = \dfrac{-1}{\lambda}$

Since $\quad x^2 + y^2 = 1 \quad$ we have $\quad \dfrac{1}{4\lambda^2} + \dfrac{1}{\lambda^2} = 1, \quad \therefore \quad \lambda^2 = \dfrac{5}{4}, \quad \lambda = \dfrac{\pm\sqrt{5}}{2}$

Then $\quad x + 2y = -\dfrac{1}{2\lambda} - \dfrac{2}{\lambda} = -\dfrac{5}{2\lambda} \qquad -\dfrac{5}{2} \times \dfrac{2}{\sqrt{5}}$

$$\therefore \quad \boxed{\text{Max} f(x, y) = 5\left(\frac{2}{\sqrt{5}}\right) = 2\sqrt{5} \quad \text{and} \quad \text{Min } f(x, y) = -2\sqrt{5}.}$$ **Ans.**

EXAMPLE 12.35. Find the stationary value of $a^3 x^2 + b^3 y^2 + c^3 z^2$ subject to the fulfilment of the condition $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$.

SOLUTION: Let $\quad u = f(x, y, z) = a^3 x^2 + b^3 y^2 + c^3 z^2 \quad$ and $\quad \phi(x, y, z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1$

Now, consider the function $\quad F = f + \lambda\, \phi \quad$ where λ is an unknown non-zero quantity

or $\quad F = a^3 x^2 + b^3 y^2 + c^3 z^2 + \lambda\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1\right)$

For stationary points we have $\quad \dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z} = 0$

or $\quad 2a^3 x - \dfrac{\lambda}{x^2} = 0, \quad 2b^3 y - \dfrac{\lambda}{y^2} = 0, \quad 2c^3 z - \dfrac{\lambda}{z^2} = 0$

or $\quad 2a^3 x^3 = \lambda, \quad 2b^3 y^3 = \lambda, \quad 2c^3 z^3 = \lambda.$

or $\quad ax = by = cz = \text{K}$, say.

$\Rightarrow \quad x = \dfrac{\text{K}}{a}, \quad y = \dfrac{\text{K}}{b}, \quad z = \dfrac{\text{K}}{c}.$

Substituting these in $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, we get

$$\frac{a}{\text{K}} + \frac{b}{\text{K}} + \frac{c}{\text{K}} = 1 \quad \text{or} \quad \text{K} = a + b + c.$$

Hence the stationary point is given by $x = \dfrac{a+b+c}{a}, \quad y = \dfrac{a+b+c}{b}, \quad z = \dfrac{a+b+c}{c}$

$\therefore$ Stationary value of $f = \dfrac{a^3(a+b+c)^2}{a^2} + \dfrac{b^3(a+b+c)^2}{b^2} + \dfrac{c^3(a+b+c)^2}{c^2}$

$$= (a+b+c)^3. \qquad \textbf{Ans.}$$

EXAMPLE 12.36. Find the volume of the greatest rectangular parallelopiped that can be inscribed inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. **[GGSIPU IInd Sem. End Term 2006]**

SOLUTION: Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ parallel to the co-ordinate axes. The volume V of the parallelopiped is given by $\text{V} = 8\,xyz$ which is to be maximised subject to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Consider the function $\text{F} = 8\,xyz + \lambda\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right)$ where λ is an unknown multiplier.

For stationary values $\dfrac{\partial \text{F}}{\partial x} = \dfrac{\partial \text{F}}{\partial y} = \dfrac{\partial \text{F}}{\partial z} = 0$

or $\quad 8yz + \dfrac{2\lambda x}{a^2} = 0, \quad 8zx + \dfrac{2\lambda y}{b^2} = 0, \quad 8xy + \dfrac{2\lambda z}{c^2} = 0$

Eliminating λ between the above equations, taken in pairs, we get $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$.

Substituting these in the condition $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3} \quad \Rightarrow \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}.$$

When $x = 0$ the parallelopiped becomes a rectangular sheet and, hence, the volume V is 0. Note that as x increases volume also increases.

Hence V is maximum when $x = \dfrac{a}{\sqrt{3}}, \quad y = \dfrac{b}{\sqrt{3}}, \quad z = \dfrac{c}{\sqrt{3}}.$

$\therefore \quad$ Maximum volume $= 8xyz = \dfrac{8abc}{3\sqrt{3}}.$ **Ans.**

EXAMPLE 12.37. Show that the stationary values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ where $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, are roots of equation $\frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} = 0$.

SOLUTION: Let us write $u = f(x, y, z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$, ... (1)

$\phi(x, y, z) = lx + my + nz = 0$. ... (2)

and $\psi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. ... (3)

Now construct the function $F = f + \lambda_1 \phi + \lambda_2 \psi$, ... (4)

where λ_1, λ_2 are non-zero Lagrangian multipliers. For stationary values, we have

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0, \quad \text{that is,}$$

$$\frac{2x}{a^4} + \lambda_1 l + \lambda_2 \frac{2x}{a^2} = 0 \qquad ...(5)$$

$$\frac{2y}{b^4} + \lambda_1 m + \lambda_2 \frac{2y}{b^2} = 0 \qquad ...(6)$$

$$\frac{2z}{c^4} + \lambda_1 n + \lambda_2 \frac{2z}{c^2} = 0 \qquad ...(7)$$

Multiplying (5), (6), (7) by x, y, z respectively and adding, gives

$2u + 0 \cdot \lambda_1 + 2\lambda_2 = 0$ hence $\lambda_2 = -u$

Substituting this value of λ_2 in (5), (6) and (7), we get

$$x\left(\frac{2}{a^4} - \frac{2u}{a^2}\right) = -\lambda_1 l \quad \text{or} \quad x = \frac{-a^4 \lambda_1 l}{2(1 - a^2 u)} \quad \text{and} \quad y = \frac{-b^4 \lambda_1 m}{2(1 - b^2 u)}, \quad z = \frac{-c^4 \lambda_1 n}{2(1 - c^2 u)}.$$

These values must satisfy (2), hence

$$-\frac{\lambda_1 a^4 l^2}{2(1 - a^2 u)} - \frac{\lambda_1 b^4 m^2}{2(1 - b^2 u)} - \frac{\lambda_1 c^4 n^2}{2(1 - c^2 u)} = 0$$

Since $\lambda_1 \neq 0$ we have $\frac{a^4 l^2}{1 - a^2 u} + \frac{b^4 m^2}{1 - b^2 u} + \frac{c^4 n^2}{1 - c^2 u} = 0.$

Hence Proved.

which gives the stationary values of u.

EXAMPLE 12.38. Find the largest and the smallest distances from the origin to the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1 \quad \text{and the plane} \quad z = x + y.$$

SOLUTION: Let $u = x^2 + y^2 + z^2$ be the sequare of the distance of any point on the ellipsoid from the origin. We are to maximise and minimise u subject to the constraints ...(1)

$$\phi_1 = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0 \qquad ...(2)$$

and $\phi_2 = x + y - z = 0$.

Now consider the function $F = u + \lambda_1 \phi_1 + \lambda_2 \phi_2$

$$= x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2 (x + y - z).$$

For critical points we have $\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0.$

Hence $2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0, \quad 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0$ and $2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0$

Above equations when solved for $x, y, z,$ give

$$x = \frac{-2\lambda_2}{\lambda_1 + 4}, \quad y = \frac{-5\lambda_2}{2\lambda_1 + 10}, \quad z = \frac{25\lambda_2}{2\lambda_1 + 50} \quad \text{... (3)}$$

Substituting these in the constraint $x + y - z = 0$ and on dividing by λ_2 ($\neq 0$), we get

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0$$

or $\quad 2(2\lambda_1 + 10)(2\lambda_1 + 50) + 5(\lambda_1 + 4)(2\lambda_1 + 50) + 25(\lambda_1 + 4)(2\lambda_1 + 10) = 0$

or $\quad 17\lambda_1^2 + 245\lambda_1 + 750 = 0 \quad$ or $\quad (17\lambda_1 + 75)(\lambda_1 + 10) = 0$

which gives $\lambda_1 = -10, \ -\frac{75}{17}.$

When $\lambda_1 = -10$ we have, from (3)

$$x = \frac{\lambda_2}{3}, \quad y = \frac{\lambda_2}{2}, \quad z = \frac{5}{6}\lambda_2$$

These values should satisfy (1), hence

$$\lambda_2^2\left(\frac{1}{36} + \frac{1}{20} + \frac{1}{36}\right) = 1 \quad \text{or} \quad \lambda_2 = \pm 6\sqrt{\frac{5}{19}}.$$

This gives two critical points $\left(2\sqrt{\frac{5}{19}}, 3\sqrt{\frac{5}{19}}, 5\sqrt{\frac{5}{19}}\right), \left(-2\sqrt{\frac{5}{19}}, -3\sqrt{\frac{5}{19}}, -5\sqrt{\frac{5}{19}}\right).$

The stationary value of $x^2 + y^2 + z^2$ corresponding to these critical points, is

$$= \frac{(20 + 45 + 125)}{19} = 10.$$

Next, when $\lambda_1 = -\frac{75}{17}$ we have from (3).

$$x = \frac{34}{7}\lambda_2, \quad y = -\frac{17}{4}\lambda_2, \quad z = \frac{17}{28}\lambda_2.$$

Substituting these in (1), gives $\lambda_2 = \pm\frac{140}{17\sqrt{646}}$ which, in turn, gives the critical points as

$$\left(\frac{40}{\sqrt{646}}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right) \text{ and } \left(\frac{-40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$$

The stationary value of $x^2 + y^2 + z^2$ corresponding to these critical points, is

$$= \frac{(1600 + 1225 + 25)}{646} = \frac{75}{17}.$$

Thus, the required maximum value is 10 and the minimum value is $\frac{75}{17}$. **Ans.**

EXAMPLE 12.39. Find a point upon the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find the minimum f.
[GGSIPU I Sem End Term 2004 (Reappear); II Sem End Term 2006 (Reappear)]

SOLUTION: We are to minimise $f = x^2 + y^2 + z^2$ such that $ax + by + cz = p$

We can write $f = x^2 + y^2 + \frac{1}{c^2}(p - ax - by)^2$

$$= \left(1 + \frac{a^2}{c^2}\right)x^2 + \left(1 + \frac{b^2}{c^2}\right)y^2 - \frac{2ap}{c^2}x - \frac{2bp}{c^2}y + \frac{2abxy}{c^2} + \frac{p^2}{c^2}$$

Hence $\frac{\partial f}{\partial x} = 2\left(1 + \frac{a^2}{c^2}\right)x - \frac{2ap}{c^2} + \frac{2aby}{c^2}$ and $\frac{\partial f}{\partial y} = 2\left(1 + \frac{b^2}{c^2}\right)y - \frac{2bp}{c^2} + \frac{2abx}{c^2}$.

For maxima and minima of f, we have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$.

Thus $(a^2 + c^2)\,x + aby = ap$ and $(b^2 + c^2)\,y + abx = bp$

which on solving for x and y, give

$$x = \frac{ap}{a^2 + b^2 + c^2} \quad \text{and} \quad y = \frac{bp}{a^2 + b^2 + c^2}$$

which defines a critical point $\left(\frac{ap}{a^2 + b^2 + c^2}, \frac{bp}{a^2 + b^2 + c^2}\right)$.

Next, $r = \frac{\partial^2 f}{\partial x^2} = 2\left(1 + \frac{a^2}{c^2}\right)$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{2ab}{c^2}$ and $t = \frac{\partial^2 f}{\partial y^2} = 2\left(1 + \frac{b^2}{c^2}\right)$

$$\therefore \quad rt - s^2 = \frac{4}{c^4}(a^2 + c^2)(b^2 + c^2) - \frac{4a^2b^2}{c^4} = \frac{4}{c^4}[c^4 + a^2c^2 + b^2c^2] = \frac{4}{c^2}[a^2 + b^2 + c^2] > 0$$

Here $r > 0$ hence at the critical point function f has minima.

When $x = \frac{ap}{a^2 + b^2 + c^2}$, $y = \frac{bp}{a^2 + b^2 + c^2}$, we have

$$z = \frac{1}{c}[p - ax - by] = \frac{1}{c}\left[p - \frac{a^2 p}{a^2 + b^2 + c^2} - \frac{b^2 p}{a^2 + b^2 + c^2}\right]$$

$$= \frac{p}{c(a^2 + b^2 + c^2)}[a^2 + b^2 + c^2 - a^2 - b^2] = \frac{pc}{a^2 + b^2 + c^2}$$

$$\therefore \quad \text{Min. } f = x^2 + y^2 + z^2 = \frac{p^2}{(a^2 + b^2 + c^2)^2}[a^2 + b^2 + c^2] = \frac{p^2}{a^2 + b^2 + c^2}$$ **Ans.**

EXAMPLE 12.40. The temperature T at any point (x, y, z) of space is given by $T = 400\ xyz^2$ find the highest temperature at the surface of the sphere $x^2 + y^2 + z^2 = 1$.
[GGSIPU II Sem I Term 2005]

SOLUTION: $T = 400\ xyz^2$ where $x^2 + y^2 + z^2 = 1$.

Let $T' = xyz^2 = xy\,(1 - x^2 - y^2)$

$$\frac{\partial T'}{\partial x} = y\,(1 - x^2 - y^2) + xy\,(-2x) = -\,y\,(3x^2 + y^2 - 1)$$

$$\frac{\partial T'}{\partial y} = x(1 - x^2 - y^2) + xy\,(-2y) = -\,x\,(3y^2 + x^2 - 1)$$

For T' to be maximum or minimum $\dfrac{\partial T'}{\partial x} = \dfrac{\partial T'}{\partial y} = 0$.

Thus, $y\,(3x^2 + y^2 - 1) = 0$ and $x\,(3y^2 + x^2 - 1) = 0$.

The above relations when solved for x and y, give

$x = 0,\quad y = 0$ and $x^2 = \dfrac{1}{4}\ y^2 = \dfrac{1}{4}$. Thus the critical points are

$$(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{-1}{2}\right), \left(\frac{-1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{-1}{2}\right).$$

Next, $r = \dfrac{\partial^2 T'}{\partial x^2} = -6xy,\quad t = \dfrac{\partial^2 T'}{\partial y^2} = -6xy$ and $s = \dfrac{\partial^2 T'}{\partial x \partial y} = -3x^2 - 3y^2 + 1.$

At $(0, 0)$, $r = t = 0, s = 1$ so $rt < s^2$. Hence $(0, 0)$ is a saddle point.

At $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{-1}{2}, \dfrac{-1}{2}\right)$, $r = -\dfrac{3}{4},\ t = \dfrac{-3}{4},\ s = \dfrac{-1}{2}$ $\quad\therefore\quad rt - s^2 = \dfrac{5}{16} > 0$ and $r < 0$

and at $\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$, $r = \dfrac{3}{4} = t$ and $s = \dfrac{-1}{2}$ $\quad\therefore\quad rt - s^2 = \dfrac{5}{16} > 0$ and $r > 0$.

Therefore $(0, 0)$ is a saddle point, $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{-1}{2}, \dfrac{-1}{2}\right)$ are maxima and $\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$ and $\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$ are minima.

Maximum $T = 400.\ \dfrac{1}{2}.\dfrac{1}{2}.\left(1 - \dfrac{1}{4} - \dfrac{1}{4}\right) = 100$ **Ans.**

CHAPTER

18

Laplace Transformation and its Applications

To every scientist and engineer Laplace transform is a very versatile tool in finding solutions to initial value problems involving homogeneous and non-homogeneous equations alike. The systems of differential equations, partial differential equations and integral equations when subjected to Laplace transformation get converted into algebraic equations which are relatively much easier to solve.

Definition. Let $f(t)$ be a well defined function of t for all positive values of t, then the definite integral

$\int_0^\infty e^{-st} f(t)\, dt$, **if it exists, is called the *Laplace transform* of** $f(t)$ and we write

$$L(f(t); t \to s) = L(f(t)) = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

where s is a dummy parameter.

[GGSIPU III Sem End Term 2009]

EXISTENCE CONDITION

The Laplace transform of $f(t)$ exists if $f(t)$ is of exponential order, that is, if there exists constants M and a such that

$$|f(t)| \leq Me^{at} \quad \text{for all } t > 0.$$

The function $f(t)$ is sometimes known as **object function** defined for all $t > 0$ and $\bar{f}(s)$ is known as the **resultant image function,** here the parameter should be sufficiently large to make the integral convergent. It is also to be noted that the condition for the existence of $\bar{f}(s)$ is sufficient but not necessary.

[GGSIPU II Sem End Term 2006]

Consequentially, we can also write

$$f(t) = L^{-1}(\bar{f}(s); s \to t) \quad \text{or simply} \quad f(t) = L^{-1}(\bar{f}(s)).$$

Here, **$f(t)$ is called the *inverse Laplace transform* of** $\bar{f}(s)$.

Linearity theorem. Laplace transformation is a linear transformation, that is,

if $L(f_1(t)) = \bar{f}_1(s)$ and $L(f_2(t)) = \bar{f}_2(s)$

then for any constants c_1 and c_2

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s).$$

FIRST SHIFTING THEOREM

[GGSIPU III Sem End Term 2010]

If $L(f(t)) = \bar{f}(s)$ then $L(e^{at} f(t)) = \bar{f}(s-a)$.

Proof: $L(e^{at} f(t)) = \int_0^\infty e^{-st} \{e^{at} f(t)\} dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt$

$= \int_0^\infty e^{-pt} f(t)\, dt = \bar{f}(p)$ where $p = s - a$

$= \bar{f}(s-a)$.

LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

(i) $f(t) = e^{at}, a > 0$

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt \quad \text{where } s > a$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{s-a}[0-1] = \frac{1}{s-a}.$$

Thus $L(e^{at}) = \frac{1}{s-a}$ where $s > a$.

[GGSIPU II Sem End Term 2006 Reappear]

If we take here $a = 0$ we have $L(1) = \frac{1}{s}$.

(ii) $f(t) = \sin at$ or $\cos at$.

We know that $e^{ait} = \cos at + i \sin at$

Taking Laplace Transform on both sides, we get

$L(e^{ait}) = L(\cos at) + iL(\sin at)$ by Linearity property.

By (i) $L(e^{ait}) = \frac{1}{s - ai} = \frac{s + ai}{s^2 + a^2}$

Therefore $L(\cos at) + iL(\sin at) = \frac{s + ai}{s^2 + a^2}$.

$\Rightarrow$ $L(\cos at) = \frac{s}{s^2 + a^2}$ and $L(\sin at) = \frac{a}{s^2 + a^2}$.

(iii) $f(t) = \sinh at$ or $\cosh at$.

[GGSIPU III Sem End Term 2004]

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2} L(e^{at}) - \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2 - a^2}$$

and
$$L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2} L(e^{at}) + \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2(s-a)} + \frac{1}{2(s+a)} = \frac{s}{s^2 - a^2}$$

Thus, $L(\cosh at) = \frac{s}{s^2 - a^2}$ and $L(\sinh at) = \frac{a}{s^2 - a^2}$.

(iv) $f(t) = t^n$ **[GGSIPU III Sem End Term 2004]**

$$L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt, \quad \text{(now putting } st = x)$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{provided } n > -1 \text{ and } s > 0.$$

If n is a positive integer $\Gamma(n+1) = n!$ then $L(t^n) = \dfrac{n!}{s^{n+1}}$.

Thus, $\mathbf{L(t^n)} = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$, **if n is a positive integer.**

(v) Making use of the first shifting theorem we can write a few more results as

$$\mathbf{L(e^{at} t^n) = \frac{\Gamma(n+1)}{(s-a)^{n+1}}},$$

$$\mathbf{L(e^{at}\sin bt) = \frac{b}{(s-a)^2 + b^2}}, \qquad \mathbf{L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2 + b^2}},$$

$$\mathbf{L(e^{at}\sinh bt) = \frac{b}{(s-a)^2 - b^2}}, \qquad \mathbf{L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2 - b^2}}.$$

EXAMPLE 18.1. Find the Laplace transform of

(i) $\sin^2 t$ (ii) $\sin 2t \cos 3t$ (iii) $\cos^3 t$.

SOLUTION: (i) $L(\sin^2 t) = L\left(\dfrac{1 - \cos 2t}{2}\right) = L\left(\dfrac{1}{2}\right) - \dfrac{1}{2} L(\cos 2t)$ iv) sint cost

$$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}.$$

L(sint cost)

$= L(\frac{1}{2}\sin 2t)$

(ii) $L(\sin 2t \cos 3t) = L\left(\dfrac{1}{2}(\sin 5t - \sin t)\right) = \dfrac{1}{2}\cdot\dfrac{5}{s^2+25} - \dfrac{1}{2(s^2+1)}$

$$= \frac{5(s^2+1) - (s^2+25)}{2(s^2+1)(s^2+25)} = \frac{2(s^2-5)}{(s^2+1)(s^2+25)}.$$

$= \frac{1}{2} \times \frac{2}{s^2+2^2}$

(iii) Since $\cos 3t = 4\cos^3 t - 3\cos t$

$= \frac{1}{s^2+4}$

$$\therefore \quad L(\cos^3 t) = L\left(\frac{3\cos t + \cos 3t}{4}\right) = \frac{3}{4}L(\cos t) + \frac{1}{4}L(\cos 3t)$$

$$= \frac{3}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{s}{s^2+9} = \frac{s(3s^2+27+s^2+1)}{4(s^2+1)(s^2+9)} = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}.$$ **Ans.**

EXAMPLE 18.2. Find the Laplace transform of (i) $e^{-2t}(3\cos 4t - 2\sin 5t)$ (ii) $L(\sinh at \cos at)$

[GGSIPU III Sem End Term 2010]

SOLUTION: (i)

$$L\{e^{-2t}(3\cos 4t - 2\sin 5t)\} = 3\,L(e^{-2t}\cos 4t) - 2\,L(e^{-2t}\sin 5t)$$

$$= 3\cdot\frac{s+2}{(s+2)^2+4^2} - 2\cdot\frac{5}{(s+2)^2+5^2}$$

$$= \frac{3(s+2)}{(s+2)^2+16} - \frac{10}{(s+2)^2+25}.$$ **Ans.**

(ii)

$$L(\sinh at \cos at) = L\left[\frac{1}{2}(e^{at}-e^{-at})\cos at\right] = \frac{1}{2}L(e^{at}\cos at) - \frac{1}{2}L(e^{-at}\cos at)$$

$$= \frac{1}{2}\frac{s-a}{(s-a)^2+a^2} - \frac{1}{2}\frac{s+a}{(s+a)^2+a^2} = \frac{1}{2}\left[\frac{s-a}{s^2-2as+2a^2} - \frac{s+a}{s^2+2as+2a^2}\right].$$ **Ans.**

EXAMPLE 18.3. Let

$$f(t) = \frac{t}{a} \text{ for } 0 < t < a$$

$$= 1 \text{ for } t > a$$

Find the Laplace transform of $f(t)$.

SOLUTION:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^a e^{-st}\cdot\frac{t}{a}\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \left[\frac{t}{a}\left(\frac{e^{-st}}{-s}\right)\right]_0^a - \int_0^a \frac{1}{a}\left(\frac{e^{-st}}{-s}\right)dt + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{-e^{-as}}{s} - 0 + \frac{1}{as}\left[\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{s}(0-e^{-as})$$

$$= \frac{1}{as^2}\left[1-e^{-as}\right].$$ **Ans.**

EXAMPLE 18.4. Find the Laplace transform of the function

$$\text{(i)}\quad f(t) = \begin{cases} 2; & 0 < t < \pi \\ 0; & \pi < t < 2\pi \\ \sin t; & t > 2\pi \end{cases}$$

[GGSIPU II Sem End Term 2006]

$$\text{(ii)}\quad f(t) = \begin{cases} 2+t^2, & 0 < t < 2 \\ 6, & 2 < t < 3 \\ 2t-5, & 3 < t < \infty \end{cases}$$

[GGSIPU III Sem End Term 2009]

SOLUTION: (i)

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt = \int_0^\pi 2e^{-st}dt + \int_\pi^{2\pi} 0e^{-st}dt + \int_{2\pi}^\infty \sin t\, e^{-st}dt$$

$$= \frac{2e^{-st}}{-s}\Big|_0^{\pi} + 0 + \left[\frac{e^{-st}}{1+s^2}(-s\sin t - 1\cos t)\right]_{2\pi}^{\infty}$$

$$= \frac{2}{s}(1-e^{-s\pi}) + \frac{e^{-2\pi s}}{1+s^2}(s\sin 2\pi + \cos 2\pi) = \frac{2}{s}(1-e^{-\pi s}) + \frac{e^{-2\pi s}}{1+s^2}$$

(ii) $L(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt$

$$= \int_0^2 e^{-st}(2+t^2)\,dt + \int_2^3 6e^{-st}\,dt + \int_3^{\infty}(2t-5)e^{-st}\,dt$$

$$= \left[(2+t^2)\frac{e^{-st}}{-s}\right]_0^2 - \int_0^2 2t\frac{e^{-st}}{-s}dt + \left[\frac{6e^{-st}}{-s}\right]_2^3 + \left[(2t-5)\frac{e^{-st}}{-s}\right]_3^{\infty} - 2\int_3^{\infty}\frac{e^{-st}}{-s}dt$$

$$= \frac{-6}{s}e^{-2s} + \frac{2}{s} + \frac{2}{s}\left[\frac{te^{-st}}{-s}\right]_0^2 - \int_0^2 \frac{2e^{-st}}{s(-s)}dt - \frac{6}{s}(e^{-3s} - e^{-2s}) + 0 + \frac{e^{-3s}}{s} + \frac{2}{s}\left[\frac{e^{-st}}{-s}\right]_3^{\infty}$$

$$= \frac{-6}{s}e^{-2s} + \frac{2}{s} - \frac{4}{s^2}e^{-2s} + \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^2 + \frac{6}{s}(e^{-2s} - e^{-3s}) + \frac{e^{-3s}}{s} + \frac{2}{s^2}e^{-3s}$$

$$= e^{-2s}\left(-\frac{6}{s} - \frac{4}{s^2} - \frac{2}{s^3} + \frac{6}{s}\right) + e^{-3s}\left(-\frac{6}{s} + \frac{1}{s} + \frac{2}{s^2}\right) + \frac{2}{s} + \frac{2}{s^3}$$

or $\quad L(f(t)) = \frac{2}{s} + \frac{2}{s^3} - e^{-2s}\left(\frac{4}{s^2} + \frac{2}{s^3}\right) + e^{-3s}\left(\frac{2}{s^2} - \frac{5}{s}\right)$ **Ans.**

EXAMPLE 18.5. **(a)** Find the Laplace transform of $f(t)$ defined as

$$f(t) = |t-1| + |t+1|, \quad t \geq 0$$

[GGSIPU III Sem End Term 2005, II Sem End Term 2010]

(b) Find the Laplace transform of the function $f(t) = \begin{cases} 0, & 0 < t < 3 \\ (t-3)^2, & t > 3. \end{cases}$

[GGSIPU II Sem End Term 2011]

SOLUTION: (a) $f(t) = |t-1| + |t+1|, \quad t \geq 0$

$\therefore$ We can write $\quad f(t) = 1 - t + 1 + t = 2 \quad$ for $\quad 0 < t < 1$

$\qquad = t - 1 + t + 1 = 2t \quad$ for $\quad t > 1.$

$$\therefore \quad L(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^1 2e^{-st}\,dt + \int_1^{\infty} 2t\,e^{-st}\,dt$$

$$=\left[2\frac{e^{-st}}{-s}\right]_0^1+\left[2t\frac{e^{-st}}{-s}\right]_1^{\infty}-\int_1^{\infty}2\frac{e^{-st}}{-s}dt$$

$$=\frac{2}{s}(1-e^{-s})+0+\frac{2}{s}e^{-s}+\frac{2}{s^2}e^{-s}=\frac{2}{s}+\frac{2}{s^2}e^{-s}$$

Ans.

Therefore the Laplace transform of $f(t)=\frac{2}{s}+\frac{2}{s^2}e^{-s}$.

(*b*) $$L[f(t)]=\int_0^{\infty}e^{-st}f(t)\,dt=\int_0^3 0\,e^{-st}dt+\int_3^{\infty}(t-3)^2e^{-st}\,dt=0+\left[(t-3)^2\frac{e^{-st}}{-s}\right]_3^{\infty}-\int_3^{\infty}2(t-3)\frac{e^{-st}}{-s}dt$$

$$=0+0+\frac{2}{s}\int_3^{\infty}(t-3)\,e^{-st}\,dt=\frac{2}{s}\left[(t-3)\frac{e^{-st}}{-s}\right]_3^{\infty}-\frac{2}{s}\int_3^{\infty}\frac{1e^{-st}}{-s}dt$$

$$=0+\frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_3^{\infty}=\frac{2}{s^3}(0+e^{-3s})=\frac{2e^{-3s}}{s^3}.$$

Ans.

PROPERTIES OF LAPLACE TRANSFORM

I. Change of Scale.

If $L\{f(t)\}=\bar{f}(s)$ then $L\{f(at)\}=\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

By definition $$L\{f(at)\}=\int_0^{\infty}e^{-st}f(at)\,dt \quad \text{(now putting } at=x\text{)}$$

$$=\int_0^{\infty}e^{-s\frac{x}{a}}f(x)\frac{dx}{a}=\frac{1}{a}\int_0^{\infty}e^{-\left(\frac{s}{a}\right)x}f(x)\,dx$$

$$=\frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

II. Multiplication by *t*.

If $L\{f(t)\}=\bar{f}(s)$ then $L\{tf(t)\}=-\frac{d}{ds}\bar{f}(s)$

We have $\bar{f}(s)=\int_0^{\infty}e^{-st}f(t)\,dt$

Differentiating both sides w.r.t. s under the integral sign, we get

$$\frac{d}{ds}\bar{f}(s)=\int_0^{\infty}\frac{\partial}{\partial s}\left\{e^{-st}f(t)\right\}dt=\int_0^{\infty}-te^{-st}f(t)\,dt=-L\{tf(t)\}$$

Hence $L\{tf(t)\}=-\frac{d}{ds}\bar{f}(s)$

Replacing $f(t)$ by $t\,f(t)$ on both sides, we can get

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\bar{f}(s)$$

In general, $\mathbf{L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}\bar{f}(s)}$

which can be easily verified using the method of Mathematical induction.

***III.* Division by *t*.**

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$ **[GGSIPU III Sem End Term 2010]**

We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$.

Integrating both sides w.r.t. s under the limits s to ∞, we get

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds \quad \text{(changing the order of integration)}$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t)\left[0 - \frac{e^{-st}}{(-t)}\right] dt = \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = L\left\{\frac{1}{t} f(t)\right\}$$

Therefore $\mathbf{L\left\{\dfrac{1}{t} f(t)\right\} = \displaystyle\int_s^\infty \bar{f}(s)\,ds}$

Replacing $f(t)$ by $\frac{1}{t} f(t)$ in the above result, we can get

$$L\left\{\frac{1}{t^2} f(t)\right\} = \int_s^\infty \left[\int_s^\infty \bar{f}(s)\,ds\right] ds.$$

***IV.* Laplace Transform of Derivatives.**

If $L\{f(t)\} = \bar{f}(s)$ and $f'(t)$ is continuous then $\mathbf{L\{f'(t)\} = s\bar{f}(s) - f(0)}$.

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \quad \text{(integrating by parts)}$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty (-s)\,e^{-st} f(t)\,dt$$

$$= \operatorname*{Lt}_{t\to\infty} e^{-st} f(t) - f(0) + s\bar{f}(s)$$

But $\operatorname*{Lt}_{t\to\infty} e^{-st} f(t) = 0$ since $f(t)$ is of exponential order.

Therefore, $L\{f'(t)\} = s\bar{f}(s) - f(0)$.

Replacing $f(t)$ by $f'(t)$ in the above relation, we get

$$L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$$

In general, $L\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$

which can be easily verified using the principle of mathematical induction.

V. Laplace Transform of integrals.

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$.

$$L\left\{\int_0^t f(u)\,du\right\} = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,du\right]dt$$

[integrating by parts and keeping in mind $\frac{d}{dt}\int_0^t f(u)\,du = f(t)$]

$$= \left[\frac{e^{-st}}{-s}\int_0^t f(u)\,du\right]_0^\infty - \int_0^\infty f(t)\cdot\frac{e^{-st}}{-s}\,dt = \frac{1}{s}\bar{f}(s).$$

EXAMPLE 18.6. Show that $L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$ and $L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$.

SOLUTION: Since $L(\sin at) = \frac{a}{s^2+a^2}$ we have

$$L(t\sin at) = -\frac{d}{ds}\frac{a}{s^2+a^2} = -a\frac{d}{ds}(s^2+a^2)^{-1} = (-1)^2 a(s^2+a^2)^{-2}\cdot 2s = \frac{2as}{(s^2+a^2)^2}.$$

Next, since $L(\cos at) = \frac{s}{s^2+a^2}$ we have

$$L(t\cos at) = -\frac{d}{ds}\frac{s}{s^2+a^2} = -\frac{(s^2+a^2)1 - s\cdot 2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Hence Proved.

EXAMPLE 18.7. Find the Laplace transform of

(i) $\sin at \sin bt$ **[GGSIPU II Sem End Term 2007]**

(ii) $t^2 e^{-2t}$ **[GGSIPU II Sem End Term 2005]**

(iii) $\frac{e^{-t}\sin t}{t}$ **[GGSIPU II Sem End Term 2007]**

(iv) Find the Laplace transform of $te^{-t}\sin 3t$. **[GGSIPU II Sem End Term 2011]**

SOLUTION: (i) $\sin at \sin bt = \frac{1}{2}[\cos(a-b)t - \cos(a+b)t]$

$$\therefore \quad L(\sin at \sin bt) = \frac{1}{2}L\cos(a-b)t - \frac{1}{2}L\cos(a+b)t$$

$$= \frac{1}{2}\frac{s}{s^2+(a-b)^2} - \frac{1}{2}\frac{s}{s^2+(a+b)^2}$$

$$= \frac{s}{2}\frac{[s^2+(a+b)^2-s^2-(a-b)^2]}{[s^2+(a-b)^2][s^2+(a+b)^2]}$$

or $$L(\sin at \sin bt) = \frac{2abs}{[s^2+(a-b)^2][s^2+(a+b)^2]}$$ **Ans.**

(*ii*) We know that if $L[f(t)] = \bar{f}(s)$ then $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}\bar{f}(s)$

$$\therefore \quad L(t^2 e^{-2t}) = (-1)^2 \frac{d^2}{ds^2}\frac{1}{s+2} = \frac{2}{(s+2)^3}$$

Thus $$L(t^2 e^{-2t}) = \frac{2}{(s+2)^3}.$$ **Ans.**

(*iii*) We know that $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ $\therefore$ $L(e^{-t}\sin t) = \dfrac{1}{(s+1)^2+1}$

since $$L\left(\frac{1}{t}f(t)\right) = \int_s^\infty \bar{f}(s)\,ds \quad \text{where} \quad L(f(t)) = \bar{f}(s).$$

Then $$L\left[\frac{1}{t}(e^{-t}\sin t)\right] = \int_s^\infty \frac{ds}{(s+1)^2+1} = \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

Therefore $$L\left(\frac{1}{t}e^{-t}\sin t\right) = \cot^{-1}(s+1).$$ **Ans.**

(*iv*) We know that $$L[e^{-t}\sin 3t] = \frac{3}{(s+1)^2+9} = \frac{3}{s^2+2s+10}$$

$$\therefore \quad L[t(e^{-t}\sin 3t] = -\frac{d}{ds}\left(\frac{3}{s^2+2s+10}\right) = \frac{6(s+1)}{[(s+1)^2+9]^2} = \frac{6s+6}{(s^2+2s+10)^2}.$$ **Ans.**

EXAMPLE 18.8. (*i*) Evaluate $L\left(\dfrac{\sin at}{t}\right)$. Does $L\left(\dfrac{\cos at}{t}\right)$ exist

[GGSIPU III Sem End Term 2007]

(*ii*) Using Laplace transform show that $L\displaystyle\int_0^t \frac{\cos at - \cos bt}{t}\,dt = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$

[GGSIPU III Sem End Term 2007]

(*iii*) Show that $L\displaystyle\int_0^t e^t \frac{\sin t}{t}\,dt = \frac{1}{s}\cot^{-1}(s-1)$

[GGSIPU III Sem End Term 2005]

SOLUTION: (*i*) We know that $L(\sin at) = \dfrac{a}{s^2+a^2}$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

Next, we know that $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$\therefore \quad L\left(\frac{\cos at}{t}\right) = \int_s^\infty \frac{s}{s^2 + a^2}\, ds = \frac{1}{2}\left[\log(s^2 + a^2)\right]_s^\infty$$ which is non-existent.

(ii) $L(\cos at - \cos bt) = \dfrac{s}{s^2 + a^2} - \dfrac{s}{s^2 + b^2}$

$$\therefore \quad L\left[\frac{1}{t}(\cos at - \cos bt)\right] = \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2 + a^2) - \frac{1}{2}\log(s^2 + b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2 + a^2}{s^2 + b^2}\right]_s^\infty = 0 - \frac{1}{2}\log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)$$

$$= \frac{1}{2}\log\frac{s^2 + b^2}{s^2 + a^2}$$

Now since $\quad L\left(\int_0^t f(t)\, dt\right) = \dfrac{1}{s}\bar{f}(s)$, we have

$$L\left[\int_0^t \frac{\cos at - \cos bt}{t}\, dt\right] = \frac{1}{2s}\log\left(\frac{s^2 + b^2}{s^2 + a^2}\right).$$ **Ans.**

(iii) We know that $\quad L(e^t \sin t) = \dfrac{1}{(s-1)^2 + 1}$

Hence $\quad L\left(\dfrac{e^t \sin t}{t}\right) = \displaystyle\int_s^\infty \frac{ds}{(s-1)^2 + 1} = \left[\tan^{-1}(s-1)\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1)$$

Therefore $\quad L\left[\displaystyle\int_0^t \frac{e^t \sin t}{t}\, dt\right] = \dfrac{1}{s}\cot^{-1}(s-1)$ **Ans.**

EXAMPLE 18.9. *(i)* Using Laplace transform, evaluate $\displaystyle\int_0^\infty t^3 e^{-t} \sin t\, dt$

[GGSIPU III Sem End Term 2004]

(ii) Evaluate $L(\sin h^3\, 2t)$

[GGSIPU III Sem End Term 2003]

SOLUTION: $L(t^3 \sin t) = (-1)^3 \dfrac{d^3}{ds^3}\dfrac{1}{(s^2 + 1)} = \dfrac{-d^2}{ds^2}\dfrac{(-1)\,2s}{(s^2 + 1)^2} = \dfrac{d^2}{ds^2}\dfrac{2s}{(s^2 + 1)^2}$

$$= \frac{d}{ds}\frac{(s^2+1)^2\,2 - 2s\times 2(s^2+1)2s}{(s^2+1)^4} = \frac{d}{ds}\frac{2(s^2+1)-8s^2}{(s^2+1)^3}$$

$$= \frac{d}{ds}\frac{2-6s^2}{(s^2+1)^3} = \frac{(s^2+1)^3(-12s)-(2-6s^2)\,3\,(s^2+1)^2\times 2s}{(s^2+1)^6}$$

$$= \frac{-12s\,(s^2+1)+12s\,(3s^2-1)}{(s^2+1)^4} = \frac{24s\,(s^2-1)}{(s^2+1)^4}$$

That is, $\displaystyle\int_0^\infty e^{-st}t^3\sin t\,dt = \frac{24s\,(s^2-1)}{(s^2+1)^4}$

Putting here $s = 1$, we get $\displaystyle\int_0^\infty e^{-t}t^3\sin t\,dt = 0.$ **Ans.**

(*ii*) we know that $\sinh 3t = 3\sinh t + 4\sinh^3 t$

or $\quad \sinh^3 t = \dfrac{1}{4}[\sinh 3t - 3\sinh t]$

Therefore $\quad \sinh^3 2t = \dfrac{1}{4}[\sinh 6t - 3\sinh 2t]$

and $\quad L(\sinh^3 2t) = \dfrac{1}{4}L(\sinh 6t) - \dfrac{3}{4}L(\sinh 2t)$

$$= \frac{1}{4}\cdot\frac{6}{s^2-6^2} - \frac{3}{4}\frac{2}{s^2-2^2} = \frac{3}{2}\frac{32}{(s^2-4)(s^2-36)}$$

$\therefore \quad L(\sinh^3 2t) = \dfrac{48}{(s^2-4)(s^2-36)}.$ **Ans.**

EXAMPLE 18.10. Using Laplace transformation, show that $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$

[GGSIPU II Sem End Term 2006]

SOLUTION: We know that $\quad L(\sin at) = \dfrac{a}{s^2+a^2}$

Therefore $\quad L\left(\dfrac{1}{t}\sin at\right) = \displaystyle\int_s^\infty \frac{a}{s^2+a^2}\,ds = a\cdot\frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

But, by definition, $\quad L\left(\dfrac{1}{t}\sin at\right) = \displaystyle\int_0^\infty e^{-st}\frac{1}{t}\sin at\,dt$

Hence we have $\quad \displaystyle\int_0^\infty e^{-st}\frac{\sin at}{t}\,dt = \cot^{-1}\left(\frac{s}{a}\right)$

Taking $a = 1$ and $s = 0$ in the above relation, we get

$$\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}(0) = \frac{\pi}{2}.$$

Hence Proved.

EXAMPLE 18.11. Find the Laplace transform of the function $f(t)$ given by

$$f(t) = \begin{cases} \sin(t-\alpha), & t > \alpha \\ 0 \quad , & t < \alpha \end{cases}$$

SOLUTION: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\alpha e^{-st} \cdot 0\, dt + \int_\alpha^\infty e^{-st} \sin(t-\alpha)\, dt$

$$= 0 + \int_0^\infty e^{-s(u+\alpha)} \sin u\, du \qquad \text{on putting } u = t - \alpha$$

$$= e^{-\alpha s} \int_0^\infty e^{-su} \sin u\, du = e^{-\alpha s} L\{\sin u\} = \frac{e^{-\alpha s}}{s^2+1}.$$

EXAMPLE 18.12. Given that $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$.

SOLUTION: Let $f(t) = 2\sqrt{\frac{t}{\pi}}$ and we are given $L\{f(t)\} = \frac{1}{s^{3/2}} = \bar{f}(s)$.

Then by the property on derivatives, we have

$$L\{f'(t)\} = s\bar{f}(s) - f(0) = s \cdot \frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$$

But $\quad f'(t) = \frac{2}{\sqrt{\pi}} \frac{d}{dt}\sqrt{t} = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi t}}$

Therefore we have $\quad L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$. **Hence Proved**

EXAMPLE 18.13. Find the Laplace transform of

(*i*) $2\, e^t \sin 4t \cos 2t$. $\qquad$ (*ii*) $\frac{(e^{-at} - e^{-bt})}{t}$.

SOLUTION: (*i*) By Linearity theorem

$$L(2\sin 4t \cos 2t) = L(\sin 6t + \sin 2t) = \frac{6}{s^2+6^2} + \frac{2}{s^2+2^2}$$

$\therefore$ By first shifting theorem

$$L(2e^t \sin 4t \cos 2t) = \frac{6}{(s-1)^2+36} + \frac{2}{(s-1)^2+4} = \frac{6}{s^2-2s+37} + \frac{2}{s^2-2s+5}.$$

(*ii*) $\quad L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$

Now, by the property of Laplace transform, we have

$$L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = \left[\log \frac{s+a}{s+b}\right]_s^\infty$$

$$= \underset{s \to \infty}{\text{Lt}} \left[\log \frac{s+a}{s+b}\right] - \log \frac{s+a}{s+b} = \underset{s \to \infty}{\text{Lt}} \log \frac{1+\frac{a}{s}}{1+\frac{b}{s}} + \log \frac{s+b}{s+a}$$

$$= \log\left(\frac{1+0}{1+0}\right) + \log \frac{s+b}{s+a} = \log \frac{s+b}{s+a}$$ **Ans.**

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EXAMPLE 18.14. Evaluate $\int_0^\infty t\, e^{-2t} \cos t\, dt$ using L' transformation.

[GGSIPU II Sem, End Term 2006]

SOLUTION: We know that $\quad L\,(t \cos t) = -\dfrac{d}{ds}\dfrac{s}{s^2+1}$

or $$\int_0^\infty e^{-st}\, t \cos t\, dt = -\frac{d}{ds}\frac{s}{s^2+1} = -\left[\frac{(s^2+1)\,1 - s\cdot 2s}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^2}$$

Putting $s = 2$ on both sides, we get $\displaystyle\int_0^\infty e^{-2t}\, t \cos t\, dt = \frac{2^2-1}{(2^2+1)^2} = \frac{3}{25}.$ **Ans.**

INVERSE LAPLACE TRANSFORM

Consider now the problem — Given $\bar{f}(s)$ we are to find the object function $f(t)$ of which $\bar{f}(s)$ is the Laplace transform and we write $L^{-1}\{\bar{f}(s)\} = f(t)$.

We recall the first shifting theorem as; if $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

This can also be stated as $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$.

It will be quite useful in finding inverse Laplace transform of many functions, In addition to this the method of partial fractions is extremely useful in problems of finding inverse Laplace transform of functions in the form of proper rational fractions.

Let us first list inverse Laplace transforms of some standard functions.

1. $L^{-1}\left(\frac{1}{s}\right) = 1.$

2. $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}, n = 1, 2, 3, \ldots.$

3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$

4. $L^{-1}\frac{1}{(s-a)^{n+1}} = \frac{e^{at}\, t^n}{n!}, n = 1, 2, 3, \ldots.$

5. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$

6. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at.$

7. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at.$

8. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$

9. $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at}\sin bt.$

10. $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt.$

11. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin at.$

12. $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}[\sin at - at\cos at].$

[GGSIPU III Sem End Term 2009]

We derive below the last two results given in the above table.

Since $L(\sin at) = \frac{a}{s^2+a^2}$ we have $L(t\sin at) = -\frac{d}{ds}\frac{a}{s^2+a^2} = \frac{2as}{(s^2+a^2)^2}$

$$\Rightarrow \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t\sin at.$$

Next, $L(t\cos at) = -\frac{d}{ds}\frac{s}{s^2+a^2} = \frac{s^2-a^2}{(s^2+a^2)^2} = \frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}$

$$= \frac{1}{s^2+a^2} - \frac{2a^2}{(s^2+a^2)^2} = L\left(\frac{1}{a}\sin at\right) - \frac{2a^2}{(s^2+a^2)^2}$$

Hence $L^{-1}\frac{2a^2}{(s^2+a^2)^2} = \frac{1}{a}\sin at - t\cos at$

or $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\left[\sin at - at\cos at\right].$

EXAMPLE 18.15. Find the inverse Laplace transform of

(i) $\frac{s+7}{s^2+2s+5}$ (ii) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$. (iii) $\frac{1}{s^2-5s+6}$

[GGSIPU II Sem II Term 20..

SOLUTION: (i) Let $\bar{f}(s)=\frac{s+7}{s^2+2s+5}$. We can write

$$\bar{f}(s)=\frac{s+7}{(s+1)^2+4}=\frac{s+1}{(s+1)^2+2^2}+\frac{6}{(s+1)^2+2^2}$$

$$\therefore \quad f(t)=L^{-1}(\bar{f}(s))=L^{-1}\frac{s+1}{(s+1)^2+2^2}+6\cdot L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$=e^{-t}\cos 2t+6e^{-t}\frac{\sin 2t}{2}=e^{-t}[\cos 2t+3\sin 2t].$$ **Ans.**

(ii) Resolving the given expression into partial fractions

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6}=\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$\Rightarrow \quad 2s^2-6s+5=A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

Putting $s=1,2,3$ successively, we get $A=\frac{1}{2}$, $B=-1$, $C=\frac{5}{2}$

$$\therefore \quad L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]=\frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right)-L^{-1}\left(\frac{1}{s-2}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)=\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}$$ **Ans.**

(iii) $$L^{-1}\left(\frac{1}{s^2-5s+6}\right)=L^{-1}\frac{1}{(s-3)(s-2)}=L^{-1}\left(\frac{1}{s-3}-\frac{1}{s-2}\right)=e^{3t}-e^{2t}.$$ **Ans.**

EXAMPLE 18.16. Find the inverse Laplace transform of the following functions

(i) $\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}$ (ii) $\frac{5s+3}{(s-1)(s^2+2s+5)}$

(iii) $\frac{2s-3}{s^2+4s+13}$.

[GGSIPU II Sem End Term 2011]

SOLUTION: (i) Let $$\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

$$\Rightarrow \quad 6s^3-21s^2+20s-7=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1) \quad ...(1)$$

Putting $s=-1$ and 2 successively in (1), we get $A=2$ and $D=-1$.

Then, comparing the co-efficients of terms of various powers of s on both sides, gives

$B=4$ and $C=3$.

$$\therefore \quad L^{-1}\left(\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}\right)=2L^{-1}\left(\frac{1}{s+1}\right)+4L^{-1}\left(\frac{1}{s-2}\right)+3L^{-1}\left(\frac{1}{(s-2)^2}\right)-L^{-1}\frac{1}{(s-2)^3}$$

$$=2e^{-t}+4e^{2t}+3t\,e^{2t}-\frac{t^2}{2}e^{2t}=2\,e^{-t}+\left(4+3t-\frac{t^2}{2}\right)e^{2t}.$$ **Ans.**

(ii) Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$\Rightarrow$ $$5s+3 = A(s^2+2s+5)+(s-1)(Bs+C)$$

Putting $s = 1$ on both sides, we get $A = 1$ and then comparing the coefficients of terms of various powers of s, gives $B = -1$, $C = 2$.

Then $$L^{-1}\frac{5s+3}{(s-1)(s^2+2s+5)} = L^{-1}\frac{1}{s-1} + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + 3L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t.$$ **Ans.**

(iii) $$\bar{f}(s) = \frac{2s-3}{(s+2)^2+3^2} = \frac{2(s+2)-7}{(s+2)^2+3^2} = 2\frac{(s+2)}{(s+2)^2+3^2} - \frac{7}{(s+2)^2+3^2}$$

$\therefore$ $$L^{-1}\bar{f}(s) = 2L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - \frac{7}{3}\left[\frac{3}{(s+2)^2+3^2}\right] = 2e^{-2t}\cos 3t - \frac{7}{3}e^{-2t}\sin 3t$$

$$= e^{-2t}\left(2\cos 3t - \frac{7}{3}\sin 3t\right).$$ **Ans.**

EXAMPLE 18.17. Find the inverse Laplace transform of

(i) $$\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$$ (ii) $$\frac{s}{s^4+4a^4}.$$

[GGSIPU II Sem End Term 2006; GGSIPU II Sem End Term 2009]

SOLUTION: (i) $$\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{p-4}{(p+5)(p+2)}$$ where $p = s^2 + 2s$

$$= \frac{3}{p+5} - \frac{2}{p+2} = \frac{3}{s^2+2s+5} - \frac{2}{s^2+2s+2} = \frac{3}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+1^2}$$

$\therefore$ $$L^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\} = 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] - 2L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= \frac{3}{2}e^{-t}\sin 2t - 2e^{-t}\sin t = \frac{e^{-t}}{2}[3\sin 2t - 4\sin t]$$ **Ans.**

(ii) $$\frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2)^2-4a^2s^2} = \frac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$$

$$= \frac{1}{4a}\frac{(s^2+2as+2a^2)-(s^2-2as+2a^2)}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$\therefore \quad L^{-1}\frac{s}{s^4+4a^4} = \frac{1}{4a}L^{-1}\frac{1}{(s-a)^2+a^2} - \frac{1}{4a}L^{-1}\frac{1}{(s+a)^2+a^2}$$

$$= \frac{1}{4a}\cdot e^{at}\frac{\sin at}{a} - \frac{1}{4a}\cdot\frac{e^{-at}\sin at}{a} = \frac{\sin at}{4a^2}(e^{at}-e^{-at}) = \frac{1}{2a^2}\sin at \sinh at. \quad \textbf{Ans.}$$

EXAMPLE 18.18. Find the inverse Laplace transform of

(i) $\dfrac{s+4}{s(s-1)(s^2+4)}$ [GGSIPU II Sem End Term 2005]

(ii) $\log\dfrac{s-1}{s}$ [GGSIPU III Sem End Term 2007]

(iii) $\dfrac{3s+5\sqrt{2}}{s^2+8}$ [GGSIPU III Sem End Term 2004]

SOLUTION: (i) Resolving the given expression into partial fractions, we get

$$\frac{s+4}{s(s-1)(s^2+1)} = -\frac{4}{s}+\frac{5}{2(s-1)}+\frac{3s-5}{2(s^2+1)}$$

$$\therefore \quad L^{-1}\left[\frac{s+4}{s(s-1)(s^2+1)}\right] = -4\,L^{-1}\left(\frac{1}{s}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-1}\right)+\frac{3}{2}L^{-1}\left(\frac{s}{s^2+1}\right)-\frac{5}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= -4.1+\frac{5}{2}e^t+\frac{3}{2}\cos t-\frac{5}{2}\sin t$$

$\therefore$ Inverse L^i transform of the given function, is equal to

$$-4+\frac{5}{2}e^t+\frac{3}{2}\cos t-\frac{5}{2}\sin t. \qquad \textbf{Ans.}$$

(ii) Consider $f(t) = 1-e^t$

then $$L(f(t)) = \frac{1}{s}-\frac{1}{s-1} = \bar{f}(s)$$

$$\Rightarrow \quad L\left(\frac{1}{t}f(t)\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds = [\log s-\log(s-1)]_s^\infty$$

$$= \left[\log\frac{s}{s-1}\right]_s^\infty = \log\frac{s-1}{s}.$$

Therefore $L^{-1}\log\dfrac{s-1}{s} = \dfrac{1}{t}(1-e^t)$. **Ans.**

(iii) $$L^{-1}\left(\frac{3s+5\sqrt{2}}{s^2+8}\right) = 3L^{-1}\left(\frac{s}{s^2+8}\right)+5\sqrt{2}\,L^{-1}\left(\frac{1}{s^2+8}\right)$$

$$= 3\cos 2\sqrt{2}\,t + \frac{5\sqrt{2}}{2\sqrt{2}}\sin 2\sqrt{2}\,t$$

Thus $$L^{-1}\left(\frac{3s+5\sqrt{2}}{s^2+8}\right) = 3\cos 2\sqrt{2}\,t + \frac{5}{2}\sin 2\sqrt{2}\,t.$$ **Ans.**

EXAMPLE 18.19. Find the inverse Laplace transform of

(i) $\dfrac{s+2}{s^2-4s+13}$ **[GGSIPU II Sem End Term 2007]**

(ii) $\dfrac{s^2+2s-3}{s(s-3)(s+2)}$ **[GGSIPU II Sem End Term 2006 Reappear]**

(iii) $\dfrac{s^2}{s^4-a^4}$ **[GGSIPU III Sem End Term 2003]**

[GGSIPU II Sem End Term 2009]

SOLUTION: (i) $$L^{-1}\frac{s+2}{s^2-4s+13} = L^{-1}\frac{s+2}{(s-2)^2+3^2}$$

$$= L^{-1}\frac{s-2}{(s-2)^2+3^2} + L^{-1}\frac{4}{(s-2)^2+3^2}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t = \frac{e^{2t}}{3}(3\cos 3t + 4\sin 3t)$$

$$\left\{\text{since } L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2} \text{ and } L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}\right\}$$

Therefore $$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right) = \frac{e^{2t}}{3}(3\cos 3t + 4\sin 3t).$$ **Ans.**

(ii) Resolving the given expression into partial fraction, we get

$$\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{1}{2s}+\frac{4}{5(s-3)}-\frac{3}{10(s+2)}$$

$$\therefore \quad L^{-1}\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{5}L^{-1}\left(\frac{1}{s-3}\right)-\frac{3}{10}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{2}(1)+\frac{4}{5}e^{3t}-\frac{3}{10}e^{-2t}$$

Thus, the $$L^{-1}\frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{1}{2}+\frac{4}{5}e^{3t}-\frac{3}{10}e^{-2t}$$ **Ans.**

(iii) $\dfrac{s^2}{s^4-a^4} = \dfrac{s^2}{(s^2+a^2)(s^2-a^2)} = \dfrac{1}{2}\left(\dfrac{1}{s^2+a^2}+\dfrac{1}{s^2-a^2}\right)$

$\therefore$ $$L^{-1}\left(\frac{s^2}{s^4-a^4}\right) = \frac{1}{2}L^{-1}\frac{1}{s^2+a^2}+\frac{1}{2}L^{-1}\frac{1}{s^2-a^2} = \frac{1}{2a}\sin at+\frac{1}{2a}\sinh at$$

or $$L^{-1}\left(\frac{s^2}{s^4-a^4}\right) = \frac{1}{2a}(\sin at+\sinh at).$$

EXAMPLE 18.20. (*a*) Find the inverse Laplace transform of $\log\dfrac{s+1}{s-1}$.

[GGSIPU II Sem End Term 2006 (Reappear)]

(*b*) Find the inverse laplace transform of $\dfrac{1}{s^3(s^2+1)}$.

[GGSIPU IIst Sem End Term 2010]

SOLUTION: (*a*) Let $f(t) = L^{-1}\left\{\log\left(\dfrac{s+1}{s-1}\right)\right\}$ hence $\bar{f}(s) = \log\left(\dfrac{s+1}{s-1}\right)$

then $$L(tf(t)) = -\frac{d}{ds}\bar{f}(s) = -\frac{d}{ds}\log\frac{s+1}{s-1} = -\frac{d}{ds}\{\log(s+1)-\log(s-1)\}$$

$$= -\left\{\frac{1}{s+1}-\frac{1}{s-1}\right\} = \frac{2}{(s^2-1)}$$

$\therefore$ $$tf(t) = 2L^{-1}\frac{1}{s^2-1} = 2\sinh t$$

Therefore $f(t) = \dfrac{2}{t}\sinh t$. **Ans.**

(*b*) $$\frac{1}{s^3(s^2+1)} = \frac{1}{s}\cdot\frac{1}{s^2(s^2+1)} = \frac{1}{s}\left[\frac{1}{s^2}-\frac{1}{s^2+1}\right]$$

By the property $L\displaystyle\int_0^t f(t)\,dt = \frac{1}{s}\bar{f}(s)$ and since

$$L^{-1}\left[\frac{1}{s^2}-\frac{1}{s^2+1}\right] = t-\sin t,$$ we have

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t(t-\sin t)\,dt = \left[\frac{t^2}{2}+\cos t\right]_0^t$$

$$= \frac{t^2}{2}+\cos t-1$$ **Ans.**

EXAMPLE 18.21. Find the inverse Laplace transform of

(*i*) $\dfrac{s}{(s^2-a^2)^2}$ (*ii*) $\dfrac{1}{(s^2-a^2)^2}$ (*iii*) $\dfrac{s}{s^4+s^2+1}$.

[GGSIPU III Sem End Term 2010]

SOLUTION: (*i*) We know that $\mathrm{L}\,(\sinh at) = \dfrac{a}{s^2-a^2}$

hence $\mathrm{L}\,(t\sinh at) = -\dfrac{d}{ds}\dfrac{a}{s^2-a^2} = -a\dfrac{(-1)\cdot 2s}{(s^2-a^2)^2} = \dfrac{2as}{(s^2-a^2)^2}$

$\Rightarrow$ $\mathrm{L}^{-1}\dfrac{s}{(s^2-a^2)^2} = \dfrac{t\sinh at}{2a}$. **Ans.**

(*ii*) We know that $\mathrm{L}\,(\cosh at) = \dfrac{s}{s^2-a^2}$

hence $\mathrm{L}\,(t\cosh at) = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-a^2}\right) = -\dfrac{\{(s^2-a^2)\cdot 1 - s\cdot 2s\}}{(s^2-a^2)^2} = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

$$= \frac{s^2-a^2+2a^2}{(s^2-a^2)^2} = \frac{1}{s^2-a^2} + \frac{2a^2}{(s^2-a^2)^2}$$

$\Rightarrow$ $t\cosh at = \mathrm{L}^{-1}\dfrac{1}{s^2-a^2} + 2a^2\,\mathrm{L}^{-1}\dfrac{1}{(s^2-a^2)^2} = \dfrac{1}{a}\sinh at + 2a^2\,\mathrm{L}^{-1}\dfrac{1}{(s^2-a^2)^2}$

$\therefore$ $\mathrm{L}^{-1}\dfrac{1}{(s^2-a^2)^2} = \dfrac{1}{2a^2}\left[\dfrac{-1}{a}\sinh at + t\cosh at\right] = \dfrac{at\cosh at - \sinh at}{2a^3}$. **Ans.**

(*iii*) $\dfrac{s}{s^4+s^2+1} = \dfrac{s}{(s^2+1)^2-s^2} = \dfrac{s}{(s^2+s+1)(s^2-s+1)}$

$$= \frac{1}{2}\left\{\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right\} = \frac{1}{2\left\{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}} - \frac{1}{2\left\{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}$$

Therefore $\mathrm{L}^{-1}\left(\dfrac{s}{s^4+s^2+1}\right) = \dfrac{1}{2}\left[e^{t/2}\dfrac{1}{\left(\frac{\sqrt{3}}{2}\right)}\sin\left(\dfrac{\sqrt{3}\,t}{2}\right) - e^{-t/2}\dfrac{1}{\left(\frac{\sqrt{3}}{2}\right)}\sin\left(\dfrac{\sqrt{3}\,t}{2}\right)\right]$

$$= \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right) = \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)\sinh\frac{t}{2}.$$ **Ans.**

EXAMPLE 18.22. Find (*i*) $\mathrm{L}^{-1}\log\left(1+\dfrac{1}{s^2}\right)$ (*ii*) $\mathrm{L}^{-1}\,[\tan^{-1}(s-1)]$.

SOLUTION: (*i*) Let $\bar{f}(s) = \log\left(1+\dfrac{1}{s^2}\right) = \log(s^2+1) - 2\log s$

$\therefore$ $\dfrac{d}{ds}\bar{f}(s) = \dfrac{2s}{s^2+1} - \dfrac{2}{s}$.

$$L^{-1}\left(\frac{2s}{s^2+1}\right) - L^{-1}\left(\frac{2}{s}\right) = 2\cos t - 2$$

Hence $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\frac{2s}{s^2+1} - L^{-1}\left(\frac{2}{s}\right) = 2\cos t - 2$

But $L^{-1}\frac{d}{ds}\bar{f}(s) = -t\,f(t)$ therefore $-t\,f(t) = 2\cos t - 2$

$\Rightarrow$ $f(t) = \frac{2(1-\cos t)}{t}$ which is the required function. **Ans.**

(*ii*) Let $f(t) = L^{-1}\tan^{-1}(s-1)$ then $\bar{f}(s) = \tan^{-1}(s-1)$

$\therefore$ $\frac{d}{ds}\bar{f}(s) = \frac{1}{(s-1)^2+1}$

$\Rightarrow$ $L^{-1}\frac{d}{ds}\bar{f}(s) = L^{-1}\frac{1}{(s-1)^2+1} = e^t L^{-1}\frac{1}{s^2+1} = e^t \sin t$

But $L^{-1}\frac{d}{ds}\bar{f}(s) = -tf(t)$

hence $-tf(t) = e^t \sin t$ or $f(t) = -\frac{1}{t}e^t \sin t$, which is the required function. **Ans.**

EXAMPLE 18.23. Find $L^{-1}\left[s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s\right]$.

SOLUTION: Let $L(f(t)) = F(s) = s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s. = s\log s - \frac{s}{2}\log(s^2+1) + \cot^{-1}s.$

$$F'(s) = 1\cdot\log s + s\cdot\frac{1}{s} - \frac{s}{2}\cdot\frac{2s}{s^2+1} - \frac{1}{2}\log(s^2+1) - \frac{1}{s^2+1}$$

$$= \log s + 1 - \frac{s^2}{s^2+1} - \frac{1}{2}\log(s^2+1) - \frac{1}{s^2+1} = \log s - \frac{1}{2}\log(s^2+1)$$

Therefore $F''(s) = \frac{1}{s} - \frac{1}{2}\cdot\frac{2s}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$

$\therefore$ $L^{-1}\{F''(s)\} = 1 - \cos t$

But $L^{-1}\{F''(s)\} = (-1)^2 t^2 f(t)$

$\Rightarrow$ $t^2 f(t) = 1 - \cos t$ or $f(t) = \frac{1-\cos t}{t^2}$. **Ans.**

CONVOLUTION THEOREM

The function $\int_0^t f_1(u)\,f_2(t-u)\,du$ is called the *convolution* of the functions f_1 and f_2 and is denoted by $f_1 * f_2$. Thus, $L(f_1 * f_2) = \bar{f}_1(s)\,\bar{f}_2(s)$.
It is easy to verify that $f_1 * f_2 = f_2 * f_1$

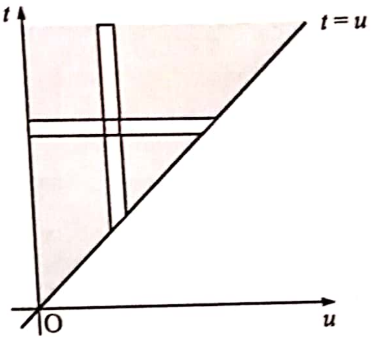
Let $f_1(t)$ and $f_2(t)$ be two functions of t and $L\{f_1(t)\} = \bar{f}_1(s)$ and $L\{f_2(t)\} = \bar{f}_2(s)$ then the theorem states that

$$L^{-1}\{\bar{f}_1(s)\,\bar{f}_2(s)\} = \int_0^t f_1(u)\,f_2(t-u)\,du = \int_0^t f_2(u)\,f_1(t-u)\,du.$$

PROOF : By definition

$$L\left\{\int_0^t f_1(u)\, f_2(t-u)\, du\right\} = \int_0^\infty e^{-st}\left[\int_0^t f_1(u)\, f_2(t-u)\, du\right] dt = \int_0^\infty \int_0^t e^{-st} f_1(u)\, f_2(t-u)\, du\, dt$$

[Now changing the order of integration, (see the figure)]



$$= \int_0^\infty \int_u^\infty e^{-st} f_1(u)\, f_2(t-u)\, dt\, du$$

$$= \int_0^\infty \left[\int_u^\infty e^{-st} f_2(t-u)\, dt\right] f_1(u)\, du$$

(Now putting $t - u = y$ in the inner integral)

$$= \int_0^\infty \left[\int_0^\infty e^{-s(u+y)} f_2(y)\, dy\right] f_1(u)\, du$$

$$= \int_0^\infty e^{-su} f_1(u)\, du \int_0^\infty e^{-sy} f_2(y)\, dy = \bar{f}_1(s)\, \bar{f}_2(s)$$

$$\Rightarrow \qquad L^{-1}\{\bar{f}_1(s)\, \bar{f}_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

EXAMPLE 18.24. Use convolution to find (*i*) $L^{-1}\dfrac{1}{(s^2+a^2)^2}$. (*ii*) $L^{-1}\dfrac{s}{(s^2+a^2)^3}$

[GGSIPU III Sem End Term 2010]

SOLUTION: (*i*) Since $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{1}{a}\sin at$ and using convolution, we get

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{a}\sin a(t-u)\, du$$

$$= \frac{1}{2a^2}\int_0^t [\cos a(2u-t) - \cos at]\, du = \frac{1}{2a^2}\left[\frac{1}{2a}\sin a(2u-t) - u\cos at\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at - t\cos at + \frac{1}{2a}\sin at\right] = \frac{1}{2a^3}[\sin at - at\cos at].$$ **Ans.**

(*ii*) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\,\dfrac{1}{(s^2+a^2)}\right]$. Since $L\left(\dfrac{t}{2a}\sin at\right) = \dfrac{s}{(s^2+a^2)^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$.

Applying convolution theorem we get

$$= \int_0^t \frac{u}{2a}\sin au \cdot \frac{1}{a}\sin a(t-u)\, du = \frac{1}{2a^2}\int_0^t u\sin au\sin a(t-u)\, du$$

$$= \frac{1}{4a^2}\int_0^t u[\cos(2au - at) - \cos at]\, dt$$

$$= \frac{1}{4a^2}\int_0^t u\cos(2au - at)\,du - \frac{1}{4a^2}\left[\frac{u^2}{2}\cos at\right]_0^t$$

$$= \frac{1}{4a^2}\left[\left\{u\,\frac{\sin(2au-at)}{2a}\right\}_0^t - \int_0^t \frac{1\cdot\sin(2au-at)}{2a}\,du\right] - \frac{t^2}{8a^2}\cos at$$

$$= \frac{1}{4a^2}\left[\frac{t}{2a}\sin at + \frac{1}{4a^2}\{\cos(2au-at)\}_0^t\right] - \frac{t^2}{8a^2}\cos at$$

$$= \frac{t}{8a^3}\sin at + \frac{1}{16a^4}(\cos at - \cos at) - \frac{t^2}{8a^2}\cos at$$

$$= \frac{t}{8a^3}(\sin at - at\cos at).$$ **Ans.**

EXAMPLE 18.25. (*a*) Employ convolution theorem to find

(*i*) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$ (*ii*) $L^{-1}\left[\dfrac{1}{s\sqrt{s+4}}\right]$.

[GGSIPU III Sem End Term 2008]

(*b*) Applying convolution theorem to show that

$$\int_0^t \sin u\cos(t-u)\,du = \frac{1}{2}t\sin t$$ **[GGSIPU III Sem End Term 2010]**

SOLUTION: (*a*) (*i*) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{1}{s^2+a^2}\cdot\dfrac{s}{s^2+a^2}\right]$ (now using convolution theorem)

$$= \int_0^t \frac{1}{a}\sin au\cdot\cos a(t-u)\,du = \frac{1}{2a}\int_0^t [\sin\{au + a(t-u)\} + \sin\{au - a(t-u)\}]\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin at + \sin a(2u-t)\}\,du = \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos a(2u-t)\right]_{u=0}^t$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\cos at - 0 + \frac{1}{2a}\cos at\right] = \frac{t}{2a}\sin at.$$ **Ans.**

(*ii*) Since $L^{-1}\left(\dfrac{1}{s}\right) = 1$, $L^{-1}\left(\dfrac{1}{\sqrt{s}}\right) = \dfrac{t^{-1/2}}{\lceil 1/2} = \dfrac{1}{\sqrt{\pi t}}$ $\therefore$ $L^{-1}\left(\dfrac{1}{\sqrt{s+4}}\right) = \dfrac{e^{-4t}}{\sqrt{\pi t}}$

therefore, by convolution theorem, we have

$$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right] = \int_0^t \frac{e^{-4u}}{\sqrt{\pi u}}\cdot 1\,du = \frac{1}{\sqrt{\pi}}\int_0^t u^{-\frac{1}{2}}e^{-4u}\,du$$ **Ans.**

(*b*) By convolution theorem we have

$$\int_0^t f_1(u)\, f_2\,(t-u)\, du \;= L^{-1}\,[\bar{f}_1(s)\bar{f}_2(s)]$$

Here $f_1(u) = \sin u$ and $f_2(u) = \cos u$ and $\bar{f}_1(s) = \dfrac{1}{s^2+1}$ and $\bar{f}_2(s) = \dfrac{s}{s^2+1}$

$$\therefore \quad \int_0^t (\sin u \cos\,(t-u)\, du = L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right) = L^{-1}\,\frac{s}{(s^2+1)^2}\Bigg) = \frac{1}{2}\,t\sin t\,.$$

Hence Proved.

EXAMPLE 18.26. Use convolution theorem to evaluate the Laplace transform of

(*i*) $\dfrac{s^2}{(s^2+a^2)\,(s^2+b^2)}$ **[GGSIPU II Sem End Term 2007; II Sem End Term 2010]**

(*ii*) $\dfrac{s^2}{(s^2+w^2)^2}$ **[GGSIPU II Sem End Term 2006]**

(*iii*) $\dfrac{1}{s(s+1)\,(s+3)}$ **[GGSIPU II Sem End Term 2011]**

SOLUTION: (*i*) By convolution theorem we have

$$L^{-1}\left[\bar{f}_1(s)\,\bar{f}_2(s)\right] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

where $\bar{f}_1(s) = L(\bar{f}_1(t))$ and $\bar{f}_2(s) = L(f_2(t))$

Since $L(\cos at) = \dfrac{s}{s^2+a^2}$ and $L(\cos bt) = \dfrac{s}{s^2+b^2}$, we have

$$L^{-1}\left[\frac{s}{s^2+a^2}\times\frac{s}{s^2+b^2}\right] = \int_0^t \cos au \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}]\,du$$

$$= \left[\frac{1}{2_{(a-b)}}\sin\{(a-b)\,u+bt\} + \frac{1}{2(a+b)}\sin\{(a+b)\,u-bt\}\right]_0^t$$

$$= \frac{1}{2(a-b)}\,(\sin at - \sin bt) + \frac{1}{2(a+b)}\,(\sin at + \sin bt)$$

$$= \frac{1}{2(a^2-b^2)}\,[(a+b)\,(\sin at - \sin bt) + (a-b)\,(\sin at + \sin bt)]$$

$$= \frac{1}{2(a^2-b^2)}\,[2a\sin at - 2b\sin bt]$$

or $$L^{-1}\left[\frac{s^2}{(s^2+a^2)\,(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$ **Ans.**

$\Gamma\tfrac{1}{2} = \sqrt{\pi}$

(ii) $L^{-1}\left[\dfrac{s^2}{(s^2+w^2)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+w^2}\times\dfrac{s}{s^2+w^2}\right] = \int_0^t \cos wu \cos w(t-u)\, du$

$$= \frac{1}{2}\int_0^t [\cos wt + \cos w(t-2u)]\, du = \frac{1}{2}\left[u\cos wt - \frac{1}{2w}\sin w(t-2u)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos wt + \frac{1}{w}\sin wt\right]$$

Therefore $L^{-1}\left[\dfrac{s^2}{(s^2+w^2)^2}\right] = \dfrac{t}{2}\cos wt + \dfrac{1}{2w}\sin wt$. **Ans.**

(iii) Applying the convolution theorem $L^{-1}[f(s)\, g(s) = \int_0^t f(u)\, g(t-u)\, du$ we get

$$L^{-1}\frac{1}{(s+1)(s+2)} = \int_0^t e^{-u}\, e^{-2(t-u)} du = \int_0^t e^{(u-2t)} du = \left[e^{u-2t}\right]_0^t = e^{-t} - e^{-2t}.$$

Next, $L^{-1}\dfrac{1}{s}\left\{\dfrac{1}{(s+1)(s+2)}\right\} = \int_0^t \left[1\left\{-e^{-(t-u)} - e^{-2(t-u)}\right\}\right] dt$

$$= \left[e^{-(t-u)} - \frac{1}{2}e^{-2(t-u)}\right]_0^t = 1 - \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$ **Ans.**

LAPLACE TRANSFORM OF A PERIODIC FUNCTION

[GGSIPU III Sem End Term 2010]

Let $f(t)$ be a periodic function with period T, that is, $f(t+T) = f(t)$, then by definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \ldots$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + \ldots$$

In I_2 putting $t = T + u$ so that $dt = du$, we get

$$I_2 = \int_T^{2T} e^{-st} f(t)\,dt = \int_0^T e^{-s(T+u)} f(T+u)\,du$$

$$= e^{-sT}\int_0^T e^{-su} f(u)\,du \quad [\text{since } f(T+u) = f(u)]$$

$$= e^{-sT} I_1 \quad \text{where } I_1 = \int_0^T e^{-st} f(t)\,dt.$$

Similarly, in I_3 putting $t = 2T + u$ so that $dt = du$, we get

$$I_3 = \int_{2T}^{3T} e^{-st} f(t)\,dt = \int_0^T e^{-s(2T+u)} f(2T+u)\,du$$

$$= e^{-2sT}\int_0^T e^{-su} f(u)\,du \quad [\text{since } f(2T+u) = f(T+u) = f(u)]$$

$$= e^{-2sT} I_1$$

Proceeding the same way, we get

$$I_4 = e^{-3sT} I_1, \quad I_5 = e^{-4sT} I_1, \quad \text{and so on.}$$

Therefore, $L\{f(t)\} = I_1 + e^{-sT} I_1 + e^{-2sT} I_1 + e^{-3sT} I_1 + \ldots$.

$= \dfrac{I_1}{1-e^{-sT}}$ as a sum of an infinite geometric series with common ratio less than 1.

Hence $$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$

Example 18.27. If $f(t) = t^2$ for $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$, find $L(f(t))$.

[GGSIPU III Sem End Term 2003]

Solution: If $f(t)$ is a periodic function with period T, then

$$L(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Here $f(t) = t^2$ is periodic with period 2, hence

$$L(f(t)) = \frac{\int_0^2 t^2 e^{-st}\,dt}{1-e^{-2s}} = \frac{1}{1-e^{-2s}}\left[\left\{t^2\frac{e^{-st}}{-s}\right\}_0^2 - \int_0^2 \frac{2te^{-st}}{-s}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} + \frac{2}{s}\left\{\frac{te^{-st}}{-s}\right\}_0^2 - \frac{2}{s}\int_0^2 \frac{1e^{-st}}{-s}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} - \frac{4}{s^2}e^{-2s} + \frac{2}{s^2}\left\{\frac{e^{-st}}{-s}\right\}_0^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} - \frac{4}{s^2}e^{-2s} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}\right]$$

$$= \frac{-2}{1-e^{2s}}\left[\frac{2}{s} + \frac{2}{s^2} + \frac{1}{s^3} - \frac{e^{2s}}{s^3}\right]$$

or $$L(f(t)) = \frac{2}{s^3(1-e^{2s})}\,[2s(1+s) + 1 - e^{2s}].$$ **Ans.**

EXAMPLE 18.28. (*i*) For the periodic function $f(t)$ of period 4, defined by

$$f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}, \quad \text{find} \quad L(f(t)).$$

[GGSIPU III Sem End Term 2007]

(*ii*) Find the Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$$

[GGSIPU III Sem End Term 2006]

SOLUTION: (*i*) Given $f(t+4) = f(t)$ and $f(t) = 3t, \quad 0 < t < 2$
$= 6, \quad 2 < t < 4.$

We know that for a T-periodic function $f(t)$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$\therefore \quad L(f(t)) = \frac{1}{1-e^{-4s}}\int_0^4 e^{-st} f(t)\,dt = \frac{1}{1-e^{-4s}}\left[\int_0^2 3t\,e^{-st}\,dt + \int_2^4 6e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\left\{3\frac{t\,e^{-st}}{-s}\right\}_0^2 - \int_0^2 3\frac{e^{-st}\,dt}{-s} + 6\left\{\frac{e^{-st}}{-s}\right\}_2^4\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{-6}{s}e^{-2s} + \frac{3}{s}\left\{\frac{e^{-st}}{-s}\right\}_0^2 - \frac{6}{s}\{e^{-4s} - e^{-2s}\}\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{-6}{s}e^{-2s} + \frac{3}{s^2}(1-e^{-2s}) - \frac{6}{s}e^{-4s} + \frac{6}{s}e^{-2s}\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3}{s^2}(1-e^{-2s}) - \frac{6}{s}e^{-4s}\right] = \frac{3}{s^2(1+e^{-2s})} - \frac{6e^{-4s}}{s(1-e^{-4s})} \qquad \textbf{Ans.}$$

(ii) $f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c \end{cases}$ which is $2c$ – periodic function

$$\therefore \quad Lf(t) = \frac{1}{1-e^{-2cs}}\int_0^{2c} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2cs}}\left[\int_0^c t\,e^{-st}dt + \int_c^{2c}(2c-t)\,e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2cs}}\left[\left\{t\frac{e^{-st}}{-s}\right\}_0^c - \int_0^c 1\frac{e^{-st}dt}{-s} + \left\{(2c-t)\frac{e^{-st}}{-s}\right\}_c^{2c} + \int_c^{2c}\frac{e^{-st}}{-s}dt\right]$$

$$= \frac{1}{1-e^{-2cs}}\left[-\frac{c}{s}e^{-cs} - \frac{1}{s^2}(e^{-cs}-1) + \frac{c}{s}e^{-cs} + \frac{1}{s^2}(e^{-2cs} - e^{-cs})\right]$$

$$= \frac{1}{s^2(1-e^{-2cs})}\left[1 - 2e^{-cs} + e^{-2cs}\right] = \frac{1-e^{-cs}}{s^2(1+e^{-cs})}$$

Therefore $\quad L(f(t)) = \dfrac{1-e^{-cs}}{s^2(1+e^{-cs})}$. **Ans.**

EXAMPLE 18.29. Find the Laplace transform of a periodic function $f(t)$ given by

$$f(t) = \begin{cases} 1, & 0 < t < L \\ -1, & L < t < 2L \end{cases}.$$

SOLUTION: The function $f(t)$ is 2L-periodic, therefore

$$L\{f(t)\} = \frac{1}{1-e^{-2sL}}\int_0^{2L} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2sL}}\left[\int_0^L e^{-st}\cdot 1\,dt + \int_L^{2L} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2sL}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^L - \left[\frac{e^{-st}}{-s}\right]_L^{2L}\right\}$$

$$= \frac{1}{s(1-e^{-2sL})}\left[1 - 2e^{-sL} + e^{-2sL}\right] = \frac{(1-e^{-sL})^2}{s(1-e^{-sL})(1+e^{-sL})}$$

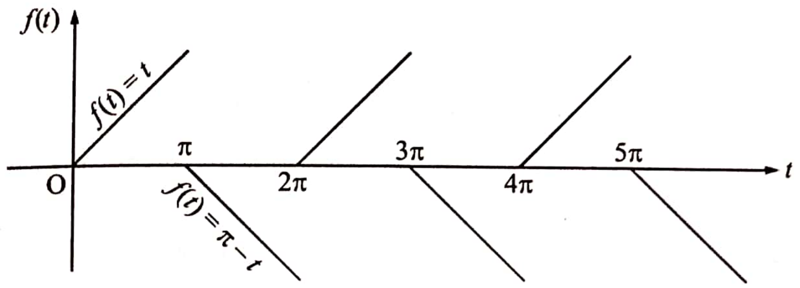
$$= \frac{1-e^{-sL}}{s(1+e^{-sL})} = \frac{1}{s}\frac{e^{sL/2} - e^{-sL/2}}{e^{sL/2} + e^{-sL/2}} = \frac{1}{s}\tanh\left(\frac{sL}{2}\right). \qquad \textbf{Ans.}$$

EXAMPLE 18.30. Plot the 2π-periodic function $f(t)$, given by

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

and find its Laplace transform. **[GGSIPU II Sem End Term 2006]**

SOLUTION: Graph of the function $f(t)$ against t is as shown in the adjoining figure



Since $f(t)$ is a periodic function with period 2π, we have

$$L\{f(t)\} = \frac{1}{1-e^{-2s\pi}} \int_0^{2\pi} e^{-st} f(t)\, dt = \frac{1}{1-e^{-2s\pi}} \left[\int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t)\, dt \right]$$

$$= \frac{1}{1-e^{-2s\pi}} \left\{ \left[t\, \frac{e^{-st}}{-s} \right]_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{e^{-st}}{-s}\, dt + \left[(\pi - t) \frac{e^{-st}}{-s} \right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \frac{e^{-st}(-1)}{-s}\, dt \right\}$$

$$= \frac{1}{1-e^{-2s\pi}} \left\{ -\frac{\pi}{s} e^{-s\pi} + \left[\frac{e^{-st}}{-s^2} \right]_0^{\pi} + \frac{\pi}{s} e^{-2\pi s} + \left[\frac{e^{-st}}{s^2} \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1-e^{-2s\pi}} \left\{ \frac{\pi}{s} (e^{-2\pi s} - e^{-\pi s}) + \frac{1}{s^2} (1 - 2e^{-\pi s} + e^{-2\pi s}) \right\}$$

$$= \frac{1}{(1-e^{-\pi s})(1+e^{-\pi s})} \left[-\frac{\pi}{s} e^{-\pi s} (1 - e^{-\pi s}) + \frac{1}{s^2} (1 - e^{-\pi s})^2 \right]$$

$$= \frac{1}{1+e^{-\pi s}} \left[-\frac{\pi}{s} e^{-\pi s} + \frac{1}{s^2} (1 - e^{-\pi s}) \right] = \frac{1}{s^2 (1 + e^{-\pi s})} \left[1 - (1 + \pi s) e^{-\pi s} \right] \qquad \textbf{Ans.}$$

APPLICATIONS TO SOLVE DIFFERENTIAL EQUATIONS

In the methods discussed earlier for solving the differential equations we have been finding the complementary function, then particular integral and then evaluating the arbitrary constants with the help of given initial conditions. The Laplace transform method is much shorter specially for linear differential equations with constant coefficients. In this method, we first find the Laplace transform of both sides of differential equation. The result is a subsidiary algebraic equation in $\bar{y}(s)$ where $\bar{y}$ is the Laplace transform of $y(t)$. It is then solved for $\bar{y}$ and finally we obtain the inverse transform which represents y in terms of t as solution of the differential equation.

We shall illustrate the use of Laplace transform technique to the following three types of differential equations :

(*a*) Linear differential equations with constant co-efficients.

(*b*) Linear differential equations with variable co-efficients.

(*c*) Simultaneous ordinary differential equations.

EXAMPLE 18.31. (*a*) Use Laplace transform to solve

$$\frac{d^2y}{dx^2}+\frac{4dy}{dx}+8y=1$$

given that $y=0$ and $\frac{dy}{dx}=1$ at $x=0$.

(*b*) Find the solution of the initial value problem $y''+4y'+4y=12t^2e^{-2t}$, $y(0)=2$ and $y'(0)=1$. **[GGSIPU II Sem End Term 2011]**

SOLUTION: (*a*) Taking Laplace transform on both sides of the given equation, we get

$$s^2\bar{y}-sy(0)-y'(0)+4\{s\bar{y}-y(0)\}+8\bar{y}=\frac{1}{s} \quad \text{where } \bar{y}=L(y(x);\ x\to s)$$

or $$\bar{y}(s^2+4s+8) = sy(0)+y'(0)+4y(0)+\frac{1}{s}=\frac{1}{s}+1 \quad \text{as } y(0)=0 \text{ and } y'(0)=1.$$

$$\bar{y}=\frac{s+1}{s(s^2+4s+8)}=\frac{1}{8s}+\frac{-\frac{1}{8}s+\frac{1}{2}}{s^2+4s+8} \quad \text{(on resolving into partial fractions)}$$

$\therefore$ $$y=\frac{1}{8}L^{-1}\left(\frac{1}{s}\right)-\frac{1}{8}L^{-1}\frac{s-4}{s^2+4s+8}$$

$$=\frac{1}{8}\cdot 1-\frac{1}{8}L^{-1}\frac{s+2-6}{(s+2)^2+4}=\frac{1}{8}-\frac{1}{8}L^{-1}\frac{s+2}{(s+2)^2+2^2}+\frac{3}{4}L^{-1}\frac{1}{(s+2)^2+2^2}$$

$$=\frac{1}{8}-\frac{1}{8}e^{-2x}\cos 2x+\frac{3}{8}e^{-2x}\sin 2x$$

or $$y=\frac{1}{8}-\frac{e^{-2x}}{8}(\cos 2x-3\sin 2x).$$ **Ans.**

(*b*) Taking Laplace transform on both sides of the given eequation we get

$$s^2\bar{y}-sy(0)-y'(0)+4[s\bar{y}-y(0)]+4\bar{y}=12L(t^2e^{-2t})=12\cdot\frac{2}{(s+2)^3}.$$

or $$(s^2+4s+4)\bar{y}-2s-1-8=\frac{24}{(s+2)^3}$$

or $$(s+2)^2\bar{y}=2s+9+\frac{24}{(s+2)^3}$$

or $$\bar{y}=\frac{2s+4+5}{(s+2)^2}+\frac{24}{(s+2)^5}=\frac{2}{s+2}+\frac{5}{(s+2)^2}+\frac{24}{(s+2)^5}$$

$\Rightarrow$ $$y(t)=L^{-1}\left[\frac{2}{s+2}+\frac{5}{(s+2)^2}+\frac{24}{(s+2)^5}\right]=2e^{-2t}+5te^{-2t}+24t^4e^{-2t}\cdot 4!$$

$$=e^{-2t}[2+5t+24^2t^4]$$ **Ans.**

EXAMPLE 18.32. Find the general solution of the equation $\dfrac{d^2x}{dt^2}+9x = \cos 2t.$

[GGSIPU III Sem End Term 2009]

SOLUTION: Let the initial conditions be $x(0) = c_1$ and $x'(0) = c_2$.

Taking L′ transform on both sides of the given equation, we get

$$s^2\bar{x} - sx(0) - x'(0) + 9\bar{x} = \frac{s}{s^2+2^2} \quad \text{where} \quad \bar{x} = L\{x(t)\}.$$

or $$(s^2+9)\,\bar{x} = c_1 s + c_2 + \frac{s}{s^2+4}$$

or $$\bar{x} = c_1\frac{s}{s^2+9} + \frac{c_2}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$= c_1\frac{s}{s^2+9} + \frac{c_2}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{s}{s^2+9} \text{ (on resolving into partial fractions)}$$

Therefore $$x(t) = \left(c_1 - \frac{1}{5}\right)L^{-1}\left(\frac{s}{s^2+9}\right) + c_2\,L^{-1}\left(\frac{1}{s^2+9}\right) + \frac{1}{5}L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= \left(c_1 - \frac{1}{5}\right)\cos 3t + \frac{c_2}{3}\sin 3t + \frac{1}{5}\cos 2t = c'_1\cos 3t + c'_2\sin 3t + \frac{1}{5}\cos 2t$$

which is the required general solution of the given equation, where c'_1 and c'_2 are arbitrary constants.

Ans.

EXAMPLE 18.33. A particle moves in a line so that its displacement x from a fixed point O at any time t, is given by $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 80 \sin 5t.$

If initially particle is at rest at $x = 0$, find its displacement at any time t.

SOLUTION: Taking Laplace transform on both sides of the given equation, we obtain

$$s^2\,\bar{x} - sx(0) - x'(0) + 4[s\bar{x} - x(0)] + 5\bar{x} = \frac{80(5)}{s^2+(5)^2}$$

From the given initial conditions we have $x(0) = x'(0) = 0$, hence

$$\bar{x}(s^2+4s+5) = \frac{400}{s^2+25} \quad \text{or} \quad \bar{x} = \frac{400}{(s^2+25)(s^2+4s+5)}$$

Resolving into partial fractions, we get

$$\bar{x} = \frac{-2s-10}{s^2+25} + \frac{2s+18}{s^2+4s+5}$$

Now taking the inverse Laplace transform on both sides, we get

$$x(t) = -2\cos 5t - \frac{10}{5}\sin 5t + 2L^{-1}\frac{s+2+7}{(s+2)^2+1}$$

$$= -2\cos 5t - 2\sin 5t + 2e^{-2t}(\cos t + 7\sin t).$$

Therefore, the distance x, covered at any time t, is given by

$$x = -2(\cos 5t + \sin 5t) + 2e^{-2t}(\cos t + 7\sin t).$$ **Ans.**

EXAMPLE 18.34. Solve $\frac{d^4x}{dt^4} - a^4x = 0$ where a is constant, using Laplace transform, given that $x = 1, x' = x'' = x''' = 0$ at $t = 0$.

SOLUTION: Taking Laplace transform on both sides of the given equation, we get

$$s^4\,\bar{x} - s^3\,x(0) - s^2\,x'(0) - s\,x''(0) - x'''(0) - a^4\,\bar{x} = 0 \quad \text{where} \quad \bar{x} = L\{x(t)\}$$

Using the given initial conditions, we get

$$s^4\,\bar{x} - s^3 - a^4\,\bar{x} = 0 \quad \text{or} \quad \bar{x} = \frac{s^3}{s^4 - a^4} = \frac{s^3}{(s^2 + a^2)(s - a)(s + a)}$$

Resolving into partial fractions, gives

$$\bar{x} = \frac{1}{4(s-a)} + \frac{1}{4(s+a)} + \frac{1}{2}\frac{s}{s^2 + a^2}$$

Taking inverse transform on both sides, gives

$$x(t) = \frac{1}{4}e^{at} + \frac{1}{4}e^{-at} + \frac{1}{2}\cos at \quad \text{or} \quad x(t) = \frac{1}{2}[\cosh at + \cos at] \qquad \textbf{Ans.}$$

which is the required solution.

EXAMPLE 18.35. Solve the intial value problem $y'' + ay' - 2a^2y = 0$, $y(0) = 6$, $y'(0) = 0$.

[GGSIPU II Sem End Term 2006]

SOLUTION: Taking Laplace transform on both sides of $y'' + ay' - 2a^2y = 0$, we get

$$s^2\,\bar{y} - sy(0) - y'(0) + a[s\,\bar{y} - y(0)] - 2a^2\,\bar{y} = 0 \text{ where } \bar{y} = L(y)$$

or $\quad (s^2 + as - 2a^2)\,\bar{y} = 6(s + a)$ since $y(0) = 6$ and $y'(0) = 0$, given.

or $$\bar{y} = \frac{6(s+a)}{(s-a)(s+2a)} = \frac{4}{s-a} + \frac{2}{s+2a}$$

Taking inverse Laplace transform here, we get $\quad y(x) = 4e^{ax} + 2\,e^{-2ax}$

which is the required solution of the given equation. **Ans.**

EXAMPLE 18.36. Solve, using Laplace transform technique, the differential equation

(*i*) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}$ where $y(0) = 1$, $y'(0) = -1$

[GGSIPU II Sem End Term 2007]

(*ii*) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$.

[GGSIPU II Sem End Term 2006 Reappear; II Sem End Term 2005]

SOLUTION: (*i*) Taking Laplace transform on both sides of the given equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}, \quad \text{we get}$$

$$[s^2\,\bar{y} - sy(0) - y'(0)] - 3[s\,\bar{y} - y(0)] + 2\,\bar{y} = \frac{4}{s^2} + \frac{1}{s-3} \quad \text{where } \bar{y} = L(y(x)).$$

Since $y(0) = 1$ and $y'(0) = -1$ we have

$$(s^2 - 3s + 2)\bar{y} - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$$

or $$(s^2 - 3s + 2)\ \bar{y} = \frac{4}{s^2} + \frac{1}{s-3} + s - 4 = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2\,(s-3)}$$

or $$\bar{y} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} = \frac{2}{s^2} + \frac{3}{s} - \frac{1}{2(s-1)} - \frac{2}{s-2} + \frac{1}{2(s-3)}$$

(On resolving into partial fractions)

Taking inverse Laplace transform on both sides, we get

$$y = 2x + 3 - \frac{1}{2}e^{x} - 2e^{2x} + \frac{1}{2}e^{3x}$$

which is the required solution. **Ans.**

(*ii*) Taking Laplace transform on both sides of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, we get

$$[s^2 . \bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s+1)^2 + 1} \quad \text{where } \bar{y} = L(y(t)).$$

or $$\bar{y}(s^2 + 2s + 5) = \frac{1}{s^2 + 2s + 2} + 1, \text{ since } \quad y(0) = 0, \quad y'(0) = 1$$

or $$\bar{y} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{1}{3(s^2 + 2s + 2)} + \frac{2}{3(s^2 + 2s + 5)}$$

$$= \frac{1}{3[(s+1)^2 + 1]} + \frac{2}{3[(s+1)^2 + 4]}$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\sin 2t = \frac{e^{-t}}{3}(\sin t + 2\sin 2t)$$

Thus the required solution is $y = \frac{e^{-t}}{3}(\sin t + 2\sin 2t)$. **Ans.**

EXAMPLE 18.37. Using Laplace transform solve the equation

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0 \text{ under the conditions } y(0) = 1,\ y'(0) = 2,\ y''(0) = 2.$$

[GGSIPU III Sem End Term 2004 – 7.5 Marks)

SOLUTION: Taking Laplace transform on both sides of the equation, we get

$$s^3\bar{y} - s^2y(0) - sy'(0) - y''(0) + 2[s^2\ \bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$

Since $y(0) = 1, \quad y'(0) = 2, y''(0) = 2,$ we get

$$\bar{y}(s^3 + 2s^2 - s - 2) - s^2 - 2s - 2 - 2s - 4 + 1 = 0$$

$$= \frac{5}{3(s-1)} - \frac{1}{s+1} + \frac{1}{3(s+2)}$$

or $\quad \bar{y} = \dfrac{s^2+4s+5}{s^3+2s^2-s-2} = \dfrac{s^2+4s+5}{(s-1)(s+1)(s+2)}$

On taking inverse Laplace transform, we get

$$y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$

which is the required solution. **Ans.**

EXAMPLE 18.38. Using Laplace transform solve the equation

(*i*) $y'' - 3y' + 2y = 4e^{2t}$ under the conditions $y(0) = -3$ and $y'(0) = 5$.
[GGSIPU III Sem End Term 200]

(*ii*) $y'' + y = 6\cos 2t$ under the conditions $y(0) = 3,\ y'(0) = 1$
[GGSIPU III Sem End Term 200]

SOLUTION: (*i*) Taking Laplace transform on both sides of $y'' - 3y' + 2y = 4e^{2t}$ gives

$$s^2\bar{y} - sy(0) - y'(0) - 3[s\bar{y} - y(0)] + 2\bar{y} = \frac{4}{s-2}$$

or $\quad (s^2 - 3s + 2)\,\bar{y} = \dfrac{4}{s-2} - 3s + 5 + 9 = \dfrac{4 + (14-3s)(s-2)}{(s-2)}$

or $\quad \bar{y} = \dfrac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} = \dfrac{-7}{s-1} + \dfrac{4}{(s-2)^2} + \dfrac{4}{s-2}$

Inverting the Laplace transform on the above relation, gives

$$y = -7e^{t} + 4te^{2t} + 4e^{2t}$$

which is the required relation. **Ans.**

(*ii*) Taking Laplace transform on both sides of $y'' + y = 6\cos 2t$, gives

$$s^2\bar{y} - sy(0) - y'(0) + \bar{y} = \frac{6s}{s^2+4}$$

or $\quad (s^2+1)\,\bar{y} = \dfrac{6s}{s^2+4} + 3s + 1 = \dfrac{3s^3 + 18s + s^2 + 4}{s^2+4}$

or $\quad \bar{y} = \dfrac{3s^3 + s^2 + 18s + 4}{(s^2+1)(s^2+4)} = \dfrac{5s+1}{s^2+1} - \dfrac{2s}{s^2+4}$

$\therefore \quad y = L^{-1}\left(\dfrac{5s}{s^2+1} + \dfrac{1}{s^2+1} - \dfrac{2s}{s^2+4}\right)$

$\quad = 5\cos t + \sin t - 2\cos 2t.$

Ans.

EXAMPLE 18.39. Solve the following differential equation with variable coefficients

$$t\frac{d^2x}{dt^2}-(t+2)\frac{dx}{dt}+3x=t$$

given that $x(0)=0$ and $x(2)=9$.

SOLUTION: Taking Laplace transform on both sides of the given equation, we get

$$L\{tx''\}-L(tx')-2L(x')+3L(x)=L(t-1) \qquad ...(1)$$

Here $L(x)=\bar{x}$, $L(x')=s\bar{x}-x(0)$,

$$L(tx')=-\frac{d}{ds}L(x')=-\frac{d}{ds}[s\bar{x}-x(0)]$$

and $$L(tx'')=-\frac{d}{ds}L(x'')=-\frac{d}{ds}[s^2\bar{x}-sx(0)-x'(0)]$$

Using the initial conditions and taking $x'(0)=k$, (1) becomes

$$-\frac{d}{ds}[s^2\bar{x}(s)-k]+\frac{d}{ds}[s\bar{x}-0]-2[s\bar{x}-0]+3\bar{x}=\frac{1}{s^2}-\frac{1}{s}$$

or $$-s^2\frac{d}{ds}\bar{x}-2s\bar{x}+s\frac{d}{ds}\bar{x}+\bar{x}-2s\bar{x}+3\bar{x}=\frac{1}{s^2}-\frac{1}{s}$$

or $$(-s^2+s)\frac{d}{ds}\bar{x}+4(1-s)\bar{x}=\frac{1-s}{s^2} \qquad \text{or} \qquad \frac{d}{ds}\bar{x}+\frac{4}{s}\bar{x}=\frac{1}{s^3}$$

which is linear differential equation of first order in $\bar{x}$.

Integrating factor $=e^{\int\frac{4}{s}ds}=e^{4\log s}=s^4$

$\therefore$ Solution is $$s^4\bar{x}=\int\frac{1}{s^3}s^4\,ds=\frac{s^2}{2}+c$$

or $$\bar{x}=\frac{1}{2s^2}+\frac{c}{s^4}$$ where c is an arbitrary constant.

Taking inverse transform on both sides, gives $x(t)=\frac{t}{2}+\frac{ct^3}{3!}$

Using the given condition $x(2)=9$, we get $c=6$

$\therefore$ The required solution is $x(t)=\frac{t}{2}+t^3$. **Ans.**

EXAMPLE 18.40. Solve the simultaneous equations

$$\frac{dx}{dt}+\frac{dy}{dt}+x=-e^{-t}$$
$$\frac{dx}{dt}+2\frac{dy}{dt}+2x+2y=0$$

given that $x(0)=-1$, $y(0)=1$.

SOLUTION: Taking Laplace transform on both sides of both the given equations, we get

$$s\bar{x}-x(0)+s\bar{y}-y(0)+\bar{x}=-\frac{1}{s+1}$$

and $$s\bar{x}-x(0)+2[s\bar{y}-y(0)]+2\bar{x}+2\bar{y}=0$$

Using here $x(0) = -1$ and $y(0) = 1$, we get

$$(s+1)\,\bar{x} + s\bar{y} = -\frac{1}{s+1} \quad \text{and} \quad (s+2)\,\bar{x} + 2(s+1)\bar{y} = 1$$

Solving these simultaneous equations for $\bar{x}$ and $\bar{y}$, gives

$$\bar{x} = -\frac{s+2}{s^2+2s+2} \quad \text{and} \quad \bar{y} = \frac{s^2+3s+3}{(s^2+2s+2)(s+1)}$$

or $$\bar{x} = -\left[\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right] \quad \text{and} \quad \bar{y} = \frac{1}{s+1} + \frac{1}{(s+1)^2+1}$$

Taking inverse Laplace transform on the above relations, we get

$x(t) = -e^{-t}(\cos t + \sin t)$ and $y(t) = e^{-t}(1 + \sin t)$. **Ans.**

EXAMPLE 18.41. Solve the following simultaneous equations

$$\frac{dx}{dt} + x + 3\int_0^t y\,dt = \cos t + 3\sin t$$

$$2\frac{dx}{dt} + 3\frac{dy}{dt} + 6y = 0$$

subject to the conditions $x = -3$, $y = 2$ at $t = 0$.

SOLUTION: Taking Laplace transform on both sides of the above equations, we get

$$s\bar{x} - x(0) + \bar{x} + 3\cdot\frac{1}{s}\bar{y} = \frac{s}{s^2+1} + \frac{3}{s^2+1}$$

and $$2[s\bar{x} - x(0)] + 3[s\bar{y} - y(0)] + 6\bar{y} = 0$$

$$(s+1)\,\bar{x} + \frac{3}{s}\,\bar{y} = \frac{s+3}{s^2+1} - 3 \quad \text{and} \quad 2s\,\bar{x} + (3s+6)\bar{y} = 0$$

Solving these simultaneous equations system for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = -\frac{(s+2)(3s-1)}{(s+3)(s^2+1)} \quad \text{and} \quad \bar{y} = -\frac{s(3s-1)}{(s+3)(s^2+1)}$$

Resolving into partial fractions, gives

$$\bar{x} = -\frac{1}{s+3} + \frac{1-2s}{s^2+1} \quad \text{and} \quad \bar{y} = \frac{2}{s+3} - \frac{2}{3}\cdot\frac{1}{s^2+1}$$

Now taking Laplace inverse transformation, we get

$$x(t) = e^{-3t} + \sin t - 2\cos t, \quad y(t) = 2e^{-3t} - \frac{2}{3}\sin t.$$

which is required solution of given system of simultaneous equations. **Ans.**

EXAMPLE 18.42. Solve the simultaneous equations

$$(D^2 - 3)x - 4y = 0, \quad x + (D^2 + 1)y = 0 \quad \text{where } D \equiv \frac{d}{dt} \text{ and } t > 0,$$

given that $x = y = \dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} = 2$ at $t = 0$.

[GGSIPU II Sem End Term 2006 Reappear]

breaking part :-)

SOLUTION: Taking Laplace transform on both sides of the given equations, we get

$$s^2\bar{x} - sx(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0$$

and $\bar{x} + s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 0$ where $\bar{x} = L(x(t)),\ \bar{y} = L(y(t))$.

Using $x(0) = y(0) = y'(0) = 0$ and $x'(0) = 2$, we get

$$(s^2 - 3)\,\bar{x} - 4\,\bar{y} = 2 \qquad ...(1)$$

$$\bar{x} + (s^2 + 1)\,\bar{y} = 0 \qquad ...(2)$$

Eliminating $\bar{x}$ in (1) and (2), we get

$$[(s^2 + 1)(s^2 - 3) + 4]\ \bar{y} = -2$$

or

$$\bar{y} = \frac{-2}{s^4 - 2s^2 + 1} = \frac{-2}{(s^2 - 1)^2} = \frac{-2}{(s-1)^2 (s+1)^2}$$

$$= \frac{-1}{2(s-1)^2} - \frac{1}{2(s+1)^2} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

Taking inverse Laplace transform, gives

$$y(t) = \frac{-1}{2}e^t t - \frac{1}{2}e^{-t} t - \frac{1}{2}e^{-t} + \frac{1}{2}e^t$$

or $$y = \sinh t - t \sinh t = (1 - t) \sinh t$$

or $$y = \sinh t - t \cosh t$$

Putting this value of y in the given equation $x + (D^2 + 1)\,y = 0$, we get

$$x = -y - D^2 (\sinh t - t \cosh t)$$

$$= -\sinh t + t \cosh t - \sinh t + D\,(t \sinh t + \cosh t)$$

$$= -2 \sinh t + t \cosh t + (t \cosh t + \sinh t + \sinh t)$$

$$= 2t \cosh t$$

$\therefore$ The soloution is $x = 2t \cosh t$, $y = \sinh t - t \cosh t$. **Ans.**

LAPLACE TRANSFORM OF SOME SPECIAL FUNCTIONS

Sometimes we are to find the solution of a differential equation of a physical system which is acted upon by

(*i*) a periodic force or a periodic voltage,

(*ii*) an impulsive force or a voltage acting instantaneously at a certain time or a concentrated load acting at that time,

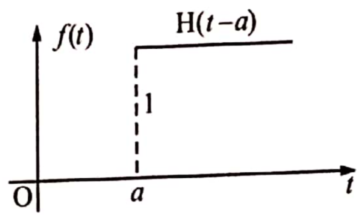
(*iii*) a force acting on a part of the system or a voltage acting for a finite interval of time.

Let us study such functions and their transforms.

HEAVISIDE UNIT STEP FUNCTION

It is an extremely useful and simplest discontinuous function, denoted by $U(t-a)$ or by $H(t-a)$ and is defined as

$$U(t-a) \text{ [or } H(t-a)] = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a. \end{cases}$$



In particular if $a = 0$ we have the unit step function defined as

$$U(t) \text{ [or } H(t)] = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0. \end{cases}$$

Now let us find the Laplace transform of $U(t-a)$.

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\, U(t-a)\, dt = \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt = 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{1}{s} e^{-as}.$$

Thus, $L\{U(t-a)\} = \dfrac{e^{-as}}{s}$.

In particular, when $a = 0$ we have $L\{U(t)\} = \dfrac{1}{s}$.

SECOND SHIFTING THEOREM

This is a property in which the Laplace transform of the function $f(t-a)\, U(t-a)$ is connected to that of $f(t)$. **[GGSIPU III Sem End Term 2009]**

$$L\{f(t-a)\, U(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, U(t-a)\, dt = \int_0^a e^{-st} \cdot f(t-a) \cdot 0\, dt + \int_a^\infty e^{-st} f(t-a) \cdot 1\, dt$$

$$= 0 + \int_0^\infty e^{-s(a+x)} f(x)\, dx \quad \text{(on putting } t - a = x)$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)\, dx = e^{-as} \bar{f}(s).$$

Thus, we have $\quad L\{f(t-a)\, U(t-a)\} = e^{-as} \bar{f}(s)$.

In particular, if $a = 0$ we have $L\{f(t)\, U(t)\} = \bar{f}(s)$.

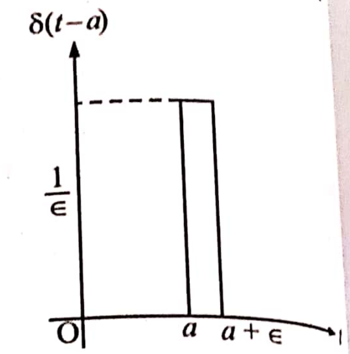
UNIT IMPULSE FUNCTION (OR DIRAC-DELTA FUNCTION)

The concept of a very large force acting for a very short time frequently comes into being in mechanics. The unit impulse function is the limiting form of the function

$$\delta(t-a) = \frac{1}{\epsilon}, \quad a < t < a+\epsilon$$

$$= 0 \quad \text{otherwise, where} \quad \epsilon \to 0.$$

From the adjoing figure it is clear that smaller the ϵ, more the height of strip because the area of the strip is always unity.



The above fact can be notionally stated as

$$\delta(t-a) = \infty \quad \text{for} \quad t = a$$

$$= 0 \quad \text{for} \quad t \neq a$$

such that $\int_0^\infty \delta(t-a)\,dt = 1$ for $a > 0$.

Laplace Transform of the Dirac-delta function is given by

$$L\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = \int_0^a e^{-st}\cdot 0\,dt + \int_a^{a+\epsilon} e^{-st}\frac{1}{\epsilon}\,dt + \int_\epsilon^\infty e^{-st}\,0\,dt$$

$$= \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,dt = \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon} = \frac{1}{s\epsilon}\left[-e^{-s(a+\epsilon)} + e^{-as}\right] = \frac{e^{-as}}{s}\left[\frac{1-e^{-s\epsilon}}{\epsilon}\right]$$

Taking limit as $\epsilon \to 0$ we have

$$L\{\delta(t-a)\} = \frac{e^{-as}}{s}\lim_{\epsilon\to 0}\frac{1-e^{-s\epsilon}}{\epsilon} = \frac{e^{-as}}{s}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[1-\left\{1-s\epsilon+\frac{s^2\epsilon^2}{2!}-\cdots\right\}\right]$$

$$= \frac{e^{-as}}{s}\lim_{\epsilon\to 0}\left[s-\frac{s\epsilon^2}{2!}+\frac{s^2\epsilon^3}{3!}-\cdots\right]$$

$$= \frac{e^{-as}}{s}\cdot s = e^{-as}$$

Thus, $L\{\delta(t-a)\} = e^{-as}$. **[GGSIPU II Sem End Term 2006]**

In particular when $a = 0$ we have

$$L\{\delta(t)\} = 1.$$

AN IMPORTANT PROPERTY — LAPLACE TRANSFORM OF $f(t)\,\delta(t-a)$

$$L\{f(t)\,\delta(t-a)\} = \int_0^\infty e^{-st} f(t)\,\delta(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t)\,0\,dt + \int_a^{a+\epsilon} e^{-st} f(t)\frac{1}{\epsilon}\,dt + \int_{a+\epsilon}^\infty e^{-st} f(t)\,0\,dt$$

$$= \frac{1}{\epsilon}\int_a^{a+\epsilon} e^{-st} f(t)\,dt = \frac{f(a)}{\epsilon}\int_a^{a+\epsilon} e^{-st}\,dt \text{ \{assuming } f(a+\epsilon)=f(a)\}$$

$$= \frac{f(a)}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon} = f(a)\left(\frac{e^{-as}-e^{-as-s\epsilon}}{s\epsilon}\right) = f(a)\,e^{-as}\left[\frac{1-e^{-s\epsilon}}{s\epsilon}\right]$$

$$= f(a)\,e^{-as}\cdot 1. \quad \text{(On taking limit as } \epsilon \to 0)$$

Thus, $\quad L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

EXAMPLE 18.43. Find the Laplace transform of the function

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

[GGSIPU II Sem End Term 2010]

SOLUTION: $f(t) = (t-1)[u(t-1)-u(t-2)] + (3-t)[U(t-2)-U(t-3)]$

$$= (t-1)\,U(t-1) - 2(t-2)\,U(t-2) + (t-3)\,U(t-3)$$

$\therefore \quad L\{f(t)\} = L[(t-1)\,U(t-1)] - 2L[(t-2)\,U(t-2)] + L[(t-3)\,U(t-3)]$

$$= e^{-s} L(t) - 2e^{-2s} L(t) + e^{-3s} L(t)$$

$$= (e^{-s} - 2e^{-2s} + e^{-3s})\frac{1}{s^2} = \frac{e^{-s}}{s^2}[1 - 2e^{-s} + e^{-2s}] = \frac{e^{-s}}{s^2}(1-e^{-s})^2.$$ **Ans.**

EXAMPLE 18.44. Find the Laplace transform of

(i) $t^2 \cup (t-3)$

(ii) $(\sin 2t) \cup (t-\pi)$

(iii) $e^{-3t} \cup (t-2)$.

[GGSIPU II Sem End Term 2005; III Sem End Term 2007; III Sem End Term 2003

SOLUTION: (i) We know that if $L(f(t)) = \bar{f}(s)$ then

$$L[f(t-a) \cup (t-a)] = e^{-as}\,\bar{f}(s)$$

Now $\quad t^2 \cup (t-3) = \{(t-3)^2 + 6(t-3) + 9\} \cup (t-3)$

$\therefore \quad L[t^2 \cup (t-3) = L(t-3)^2 \cup (t-3) + 6\,L(t-3) \cup (t-3) + 9\,L[\,U(t-3)]$

$$= e^{-3s}\frac{2!}{s^3} + 6\,e^{-3s}\frac{1}{s^2} + 9\frac{e^{-3s}}{s} = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Therefore the Laplace transform of $t^2 \cup (t-3) = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$ **Ans.**

(ii) $\sin 2t \cup (t-\pi) = \sin 2(t-\pi+\pi) \cup (t-\pi) = \sin 2(t-\pi) \cup (t-\pi)$

If $f(t) = \sin 2t$, we have $\bar{f}(s) = \dfrac{2}{s^2+2^2}$

Now using $L[f(t-a) \cup (t-a)] = e^{-as}\,\bar{f}(s)$, we have

$$L[\sin 2(t-\pi) \cup (t-\pi)] = e^{-\pi s}\frac{2}{s^2+4}.$$

Therefore, $L[\sin 2t \cup (t-\pi)] = 2\dfrac{e^{-\pi s}}{s^2+4}.$ **Ans.**

(*iii*) We know that $L[f(t-a) \cup (t-a)] = e^{-as}\bar{f}(s)$

Now $\quad e^{-3t} \cup (t-2) = e^{-3(t-2)} \times e^{-6} \cup (t-2)$

$\therefore \quad L[e^{-3t} \cup (t-2)] = e^{-6}\, L[e^{-3(t-2)} \cup (t-2)] = e^{-6} \times e^{-2s}\, \dfrac{1}{s+3}$

or $\quad L[e^{-3t} \cup (t-2)] = \dfrac{e^{-2(s+3)}}{s+3}$ **Ans.**

EXAMPLE 18.45. Solve $\dfrac{d^2y}{dx^2} + 4y = \cup\,(x-2)$ where $\cup$ is the unit step function and $y(0) = 0$ and $y'(0) = 1$. **[GGSIPU III Sem End Term 2006]**

SOLUTION: Taking Laplace transform on both sides of $y'' + 4y = \cup\,(x-2)$, we get

$$s^2\bar{y} - sy(0) - y'(0) + 4\bar{y} = \frac{e^{-2s}}{s}$$

or $$\bar{y}(s^2+4) = 1 + \frac{e^{-2s}}{s} \quad (\text{as } y(0) = 0 \text{ and } y'(0) = 1)$$

or $$\bar{y} = \frac{1}{s^2+4} + \frac{e^{-2s}}{s(s^2+4)}$$

Therefore $$y = L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left[\frac{e^{-2s}}{s(s^2+4)}\right]$$

Here $$L^{-1}\frac{1}{s^2+4} = \frac{1}{2}\sin 2t$$

and $$L^{-1}\frac{1}{s(s^2+4)} = \int_0^t \frac{1}{2}\sin 2t\, dt = \left[\frac{-1}{4}\cos 2t\right]_0^t = \frac{1}{4}(1-\cos 2t) = \frac{1}{2}\sin^2 t$$

Further, using $L[f(t-a) \cup (t-a)] = \bar{f}(s)e^{-as}$, we have

$$L^{-1}\left[\frac{e^{-2s}}{s(s^2+4)}\right] = \frac{1}{2}\sin^2(t-2) \cup (t-2)$$

Thus $$y = \frac{1}{2}\sin 2t + \frac{1}{2}\sin^2(t-2) \cup (t-2)$$

which is the required solution.

Ans.

EXAMPLE 18.46. Find $L^{-1}\dfrac{e^{-3s}}{s^2+8s+25}$.

SOLUTION: Let us write $L^{-1}\dfrac{e^{-3s}}{s^2+8s+25} = L^{-1}[e^{-3s}\bar{f}(s)]$

where $\bar{f}(s) = \dfrac{1}{s^2+8s+25} = \dfrac{1}{(s+4)^2+3^2} = L\left(e^{-4t}\dfrac{1}{3}\sin 3t\right) = L[f(t)]$

$\therefore \quad f(t) = \dfrac{e^{-4t}}{3}\sin 3t \qquad$ then $\qquad f(t-3) = \dfrac{1}{3}e^{-4(t-3)}\sin 3(t-3)$

We know that $\quad L[f(t-3)\,U(t-3)] = e^{-3s}\bar{f}(s)$

$\therefore \quad L^{-1}\dfrac{e^{-3s}}{s^2+8s+25} = \dfrac{1}{3}e^{-4(t-3)}\cdot\sin(3t-9)\,U(t-3).$ **Ans.**

EXAMPLE 18.47. Find the inverse Laplace transform of (*i*) $\dfrac{e^{-\pi s}}{s^2+4}$. (*ii*) $\dfrac{(3s+1)}{s^2(s^2+4)}e^{-3s}$

[GGSIPU III Sem End Term 2009]

SOLUTION: (*i*) We know that $L^{-1}\dfrac{1}{s^2+4} = \dfrac{1}{2}\sin 2t$

therefore $\quad L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+4}\right] = \dfrac{1}{2}\sin 2(t-\pi)\,U(t-\pi)$

$$= \begin{cases} 0, & \text{for } 0 < t < \pi \\ \dfrac{1}{2}\sin 2t, & \text{for } t > \pi \end{cases}$$ **Ans.**

(*ii*) Let $\bar{f}(s) = \dfrac{3s+1}{s^2(s^2+4)} = \dfrac{(3s+1)}{4}\left[\dfrac{1}{s^2} - \dfrac{1}{s^2+4}\right]$

$$= \frac{3}{4s} + \frac{1}{4s^2} - \frac{3s}{4(s^2+4)} - \frac{1}{4(s^2+4)}$$

then $\quad f(t) = L^{-1}(\bar{f}(s)) = \dfrac{3}{4} + \dfrac{t}{4} - \dfrac{3}{4}\cos 2t - \dfrac{1}{8}\sin 2t$

Now since $L(f(t-a) \cup (t-a)\} = e^{-as}\bar{f}(s)$

$\therefore \quad L^{-1}\left[\dfrac{(3s+1)e^{-3s}}{s^2(s^2+4)}\right] = \left[\dfrac{3+t-3}{4} - \dfrac{3}{4}\cos 2(t-3) - \dfrac{1}{8}\sin 2(t-3)\right] \cup (t-3)$

$$= \frac{1}{8}\left[-\sin^2(t-3) - t\cos 2(t-3) + 2t\right] \cup (t-3)$$ **Ans.**

Example 18.48. Find (*i*) $L[t\,U(t-4) - t^3\delta(t-2)]$ (*ii*) $L[\sin t\, U(t-\pi)]$.

Solution: (*i*) For $t\,U(t-4)$, we write

$$t = (t-4) + 4 = f(t-4) \quad \text{where} \quad f(t) = t+4$$

$$\therefore \quad L\{f(t)\} = \bar{f}(s) = \frac{1}{s^2} + \frac{4}{s}$$

Hence $L\{t\,U(t-4)\} = L\{f(t-4)\,U(t-4)\} = e^{-4s}\bar{f}(s) = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$

Next, for $t^3\delta(t-2)$, we consider $f(t) = t^3 \quad \therefore \quad f(2) = 8,$

$$\therefore \quad L\{t^3\delta(t-2)\} = f(2)e^{-2s} = 8e^{-2s}$$

Therefore, $L[t\,U(t-4) - t^3\delta(t-2)] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - 8e^{-2s}$. **Ans.**

(*ii*) For $L[\sin t\, U(t-\pi)]$ we can write

$$\sin t = \sin(t - \pi + \pi) = -\sin(t-\pi)$$

$$\therefore \quad L[\sin t\, U(t-\pi)] = -L[\sin(t-\pi)\,U(t-\pi)]$$

$$= -[e^{-as}L(\sin t)]_{a=\pi} = -e^{-\pi s}\frac{1}{s^2+1}.$$

Example 18.49. Solve $\dfrac{d^2x}{dt^2} + 4x = \phi(t)$ with $x(0) = x'(0) = 0$,

where
$$\phi(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ \sin t & \text{when } \pi < t < 2\pi \\ 0 & \text{when } t > 2\pi. \end{cases}$$

Solution: The function $\phi(t)$ in terms of unit step function can be written as

$$\phi(t) = \sin t\,[U(t-\pi) - U(t-2\pi)]$$

$$= \sin t\; U(t-\pi) - \sin t \cdot U(t-2\pi)$$

$$= \sin(t-\pi+\pi)\,U(t-\pi) - \sin(t-2\pi+2\pi)\,U(t-2\pi)$$

$$= -\sin(t-\pi)\,U(t-\pi) - \sin(t-2\pi)\,U(t-2\pi)$$

Taking Laplace transform on both sides of the given differetial equation, we get

$$s^2\bar{x}(s) - sx(0) - x'(0) + 4\bar{x} = L[\phi(t)] = -e^{-\pi s}\frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}$$

Since $x(0) = x'(0) = 0$, we have

$$\bar{x} = -\frac{1}{(s^2+1)(s^2+4)}(e^{-\pi s} + e^{-2\pi s}) = -\frac{1}{3}\left[\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)(e^{-\pi s} + e^{-2\pi s})\right]$$

Therefore, $x(t) = \frac{-1}{3} L^{-1}\left[\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)e^{-\pi s}\right] - \frac{1}{3} L^{-1}\left[\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)e^{-2\pi s}\right]$.

But $L^{-1}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) = \sin t - \frac{1}{2}\sin 2t$

$$\therefore x(t) = -\frac{1}{3}\left[\left\{\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi)\right\} U(t-\pi) + \left\{\sin(t-2\pi) - \frac{1}{2}\sin 2(t-2\pi)\right\} U(t-2\pi)\right]$$

$$= \left(\frac{1}{6}\sin 2t + \frac{1}{3}\sin t\right) U(t-\pi) + \left(\frac{1}{6}\sin 2t - \frac{1}{3}\sin t\right) U(t-2\pi)$$

Here, when $0 < t < \pi$, $U(t-\pi) = U(t-2\pi) = 0$,

when $\pi < t < 2\pi$, $U(t-\pi) = 1$, $U(t-2\pi) = 0$,

and when $t > 2\pi$, $U(t-\pi) = 1$, $U(t-2\pi) = 1$

Therefore the solution of the given equation, is

$x = 0$, for $0 < t < \pi$

$= \frac{1}{6}\sin 2t + \frac{1}{3}\sin t$, for $\pi < t < 2\pi$

$= \frac{1}{3}\sin 2t$, for $t > 2\pi$. **Ans.**

EXAMPLE 18.50. In an electrical circuit containing inductance L, resistance R and *e.m.f.* E $(= E_0)$ the rate of change of current at anytime t, is given by

$$L\frac{di}{dt} + Ri = E(t).$$

If the switch is connected at $t = 0$ and disconnected at $t = a$, find the current i at any time t.

SOLUTION: As per the given conditions

$i = 0$ at $t = 0$ and $E(t) = \begin{cases} E_0 & \text{for } 0 < t < a \\ 0 & \text{for } t > a \end{cases}$

Taking Laplace transform on both sides of the given equation and denoting $L(i(t))$ by $\bar{i}$, we get

$$[s\bar{i} - i(0)]L + R\bar{i} = \int_0^\infty e^{-st} E(t)\, dt = \int_0^a e^{-st} E_0\, dt + \int_a^\infty e^{-st} \cdot 0\, dt$$

$$= E_0\left[\frac{e^{-st}}{-s}\right]_0^a = \frac{E_0}{s}(1 - e^{-as})$$

or $(Ls + R)\bar{i} = \frac{E_0}{s}(1 - e^{-as})$ as $i(0) = 0$

$$\therefore \quad \bar{i} = \frac{E_0}{s(Ls+R)} - \frac{E_0}{s(Ls+R)} e^{-as}$$

Now, taking inverse transform on both sides, gives

$$i = L^{-1}\left\{\frac{E_0}{s(Ls+R)}\right\} - L^{-1}\left\{\frac{E_0 e^{-as}}{s(Ls+R)}\right\}.$$

Here $L^{-1}\left\{\frac{E_0}{s(Ls+R)}\right\} = L^{-1}\left\{\frac{E_0}{L}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right)\right\} = \frac{E_0}{L}[1 - e^{-\frac{Rt}{L}}]$

and $L^{-1}\left[\frac{E_0}{s(Ls+R)} e^{-as}\right] = \frac{E_0}{L}\left[1 - e^{-\frac{R(t-a)}{L}}\right] U(t-a)$

Therefore, $i = \frac{E_0}{L}\left(1 - e^{-\frac{Rt}{L}}\right) - \frac{E_0}{L}\left(1 - e^{-\frac{R(t-a)}{L}}\right) U(t-a)$

Thus, when $0 < t < a$, $\quad i = \frac{E_0}{L}\left(1 - e^{-\frac{Rt}{L}}\right)$

and when $t \geq a$, $\quad i = \frac{E_0}{L}\left(1 - e^{-\frac{Rt}{L}}\right) - \frac{E_0}{L}\left(1 - e^{-\frac{R(t-a)}{L}}\right)$

$$= \frac{E_0}{L} e^{-\frac{Rt}{L}}\left(e^{\frac{Ra}{L}} - 1\right)$$

Ans.

PROBLEMS ON DEFLECTION OF A LOADED BEAM

While dealing with the problems of deflection of a beam which is subjected to a concentrated load or partially loaded, we have the differential equation.

$$EI \frac{d^4 y}{dx^4} = \omega(x)$$

where $\omega(x)$ is the transverse load intensity at a distance x from one end taken as origin and the end conditions are as follows :

(*a*) On the clamped end : $y = 0$ and $\frac{dy}{dx} = 0$ as the slopes are zero.

(*b*) On the supported end : $y = \frac{d^2 y}{dx^2} = 0$ as the bending moment is zero.

(*c*) On the free end : $\frac{d^2 y}{dx^2} = \frac{d^3 y}{dx^3} = 0$ as bending ...