Unit 4, PSLP Notes

Bachelor Of Technology (Guru Gobind Singh Indraprastha University)

Linear Programming Problem (L.P.P.)

Linear programming deals with the optimization (maximization or minimization) of linear functions subject to linear constraints.

Thus, a general l.p.p. can be stated as follows:

Find $x_1, x_2, \ldots, x_n$ which optimize the linear function

$$Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad \text{——} (1)$$

s.t.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \;(\leq = \geq)\; b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \;(\leq = \geq)\; b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \;(\leq = \geq)\; b_m \end{array}\right] \text{(constraints or restrictions)}$$

and non negative restrictions

$$x_j \geq 0, \; j = 1, 2, \ldots n$$

where all a_{ij}'s, b_i's and c_j's are constants and x_j's are variables.

Graphical method of solving a L.P.P. If the objective function Z is a function of two variables only then the problem can be solved by this method.

In this method we proceed as follows:

Step 1 First of all we consider the constraints as equalities.

Step 2 We draw the lines in the plane corresponding to each equation (obtained in step 1) and non-negative restrictions.

Step 3 In this step, we find the convex region for the values of the variables, which is the region bounded by the lines drawn in step 2.

Find the convex region bounded by the lines drawn.

Step 4 Determine the vertices of the convex region and find the value of the objective function at each vertex. The vertex which gives the optimal (maximum or minimum) value of the objective function gives the desired optimal solution to the problem.

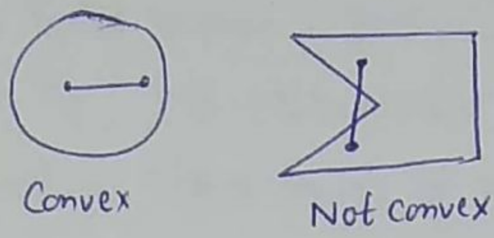
Find the optimal (max. or min.) value of Z by finding the values at the vertices of convex region or by the method of parallel lines

Or

Draw the dotted line through the origin representing the objective function with $Z = 0$. As Z is increased from zero, this line moves to the right remaining parallel to itself. We go on sliding this line (parallel to itself), till it is farthest away (for maximization) and nearest (for minimization) from the origin and passes through only one vertex of the convex region. This is the vertex where the max. value (or min. value) of Z is attained.

Note: When we apply graphical method, make all b's +ive (if any is −ive. then multiply both sides by −1 and change the inequality.

Convex region A region or a set of points is said to be convex if the line joining any two of its points lies completely in the region.



Formulation of the problem

Que

A company manufactures two types of cloth, using three different colours of wool. One yard length of type A cloth requires 4 oz of red wool, 5 oz of green wool and 3 oz of yellow wool. One yard length of type B cloth requires 5 oz of red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacturer is 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool. The manufacturer can make a profit of ₹ 5 on one yard of type A cloth and ₹ 3 on one yard of type B cloth. Find the best combination of the quantities of type A and type B cloth which gives him maximum profit by solving the l.p.p. graphically.

Red wool   Green wool   Yellow wool

Sol Let the manufacturer decide to produce

x_1 yards of type A cloth

and x_2 yards of type B cloth.

Then the total income in rupees, from these units of cloth is given by $Z = 5x_1 + 3x_2$

To produce these units of two types of cloth, he requires

red wool $= 4x_1 + 5x_2$ oz

green wool $= 5x_1 + 2x_2$ oz

yellow wool $= 3x_1 + 8x_2$ oz

Since the manufacturer does not have more than 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool,

$$\therefore \quad 4x_1 + 5x_2 \leq 1000$$
$$5x_1 + 2x_2 \leq 1000$$
$$3x_1 + 8x_2 \leq 1200$$
$$x_1, x_2 \geq 0$$

Step 1 First consider the constraints as equalities

$4x_1 + 5x_2 = 1000$ (0,200), (250,0)

$5x_1 + 2x_2 = 1000$ (0,500), (200,0)

$3x_1 + 8x_2 = 1200$ (0,150), (400,0)

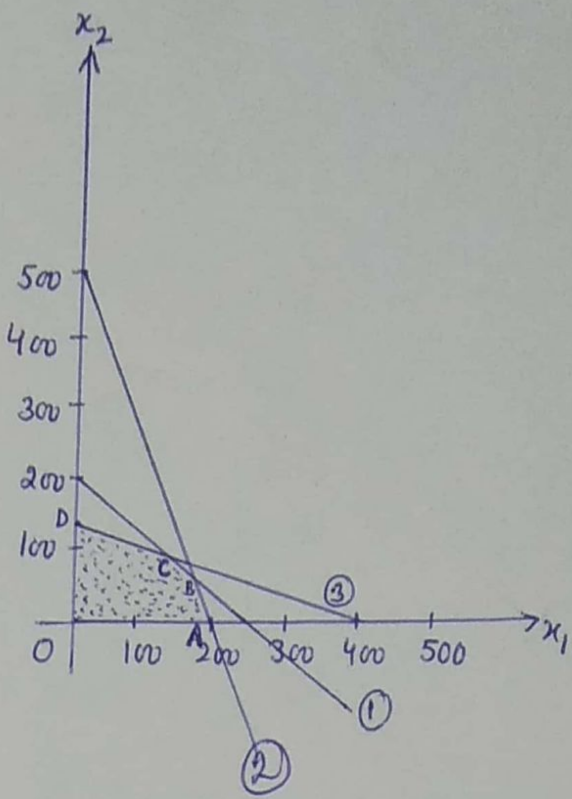
Step 2 We draw the above lines in two dimensional plane.

(See on next page)

Step 3 The shaded region OABC in the figure is the convex region.

Step 4

	Vertex	$Z = 5x_1+3x_2$
O	$(0,0)$	0
A	$(200,0)$	1000
B	$\left(\frac{3000}{17}, \frac{1000}{17}\right)$	$\frac{18000}{17} = \boxed{1058.8}$
C	$\left(\frac{2000}{17}, \frac{1800}{17}\right)$	$\frac{15400}{17} = 905.8$
D	$(0,150)$	450



Here Z is max. at B.

Hence the sol. is

$x_1 = \frac{3000}{17}$, $x_2 = \frac{1000}{17}$, Max. Z = 1058.8
= 176.5 yards = 58.8 yards ↓ max. profit

For ① & ② (B point)

$x_1 - 3x_2 = 0$

$x_1 = 3x_2$

For ① & ③ (C point)

$4x_1 + 5x_2 = 1000$

$x_1 - 3x_2 = -200$

$4x_1 - 12x_2 = -800$

$\therefore 17x_2 = 1800$

$x_2 = \frac{1800}{17}$

Que Max. $Z = 3x_1 + 4x_2$

s.t. $x_1 - x_2 \leq -1$

$-x_1 + x_2 \leq 0$

and $x_1, x_2 \geq 0$

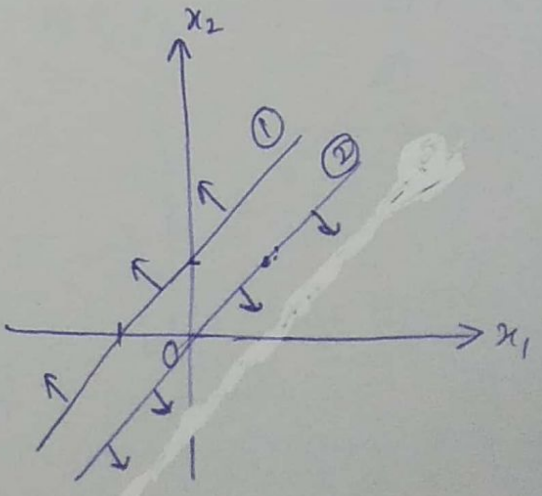
Sol The given problem can be written as;

Max. $Z = 3x_1 + 4x_2$

s.t. $-x_1 + x_2 \geq 1$ (0,1) (−1,0)

$-x_1 + x_2 \leq 0$ (0,0) (1,1)

and $x_1, x_2 \geq 0$



Step 3 There does not exist any convex region which satisfy the given conditions. Thus the problem has no solution.

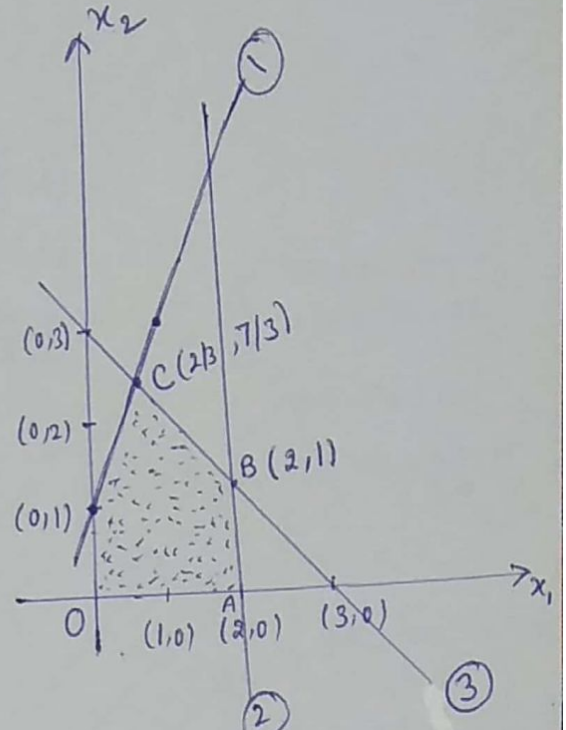
Que Max. $Z = x_1 + x_2$

s.t. $-2x_1 + x_2 \leq 1$ $(0,1)$ $(1,3)$

$x_1 \leq 2$

$x_1 + x_2 \leq 3$ $(0,3)$, $(3,0)$

and $x_1, x_2 \geq 0$



Step 4

	Vertex	$Z = x_1 + x_2$
O	(0,0)	0
A	(2,0)	2
B	(2,1)	$\boxed{3}$
C	(2/3, 7/3)	$\boxed{3}$
D	(0,1)	1

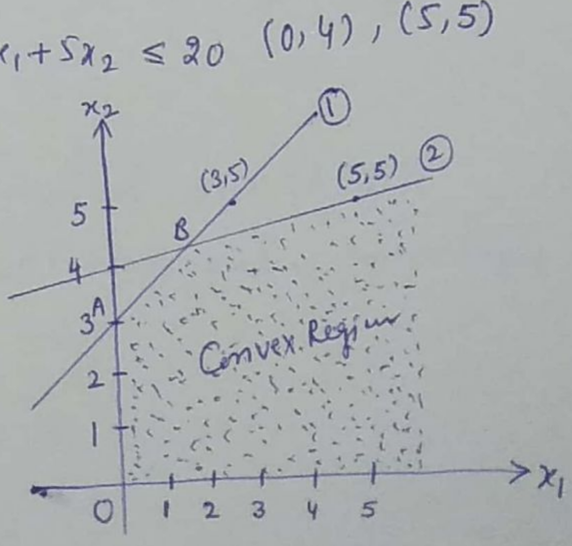
Here Z is max. and equal at B and C, So Z is max. and equal at line BC.
Thus, Z has infinite no. of solutions lying on BC.

Que Max. $Z = 3x_1 + 2x_2$ $(0,3)$, $(3,5)$

s.t. $-2x_1 + 3x_2 \leq 9$

$x_1 - 5x_2 \geq -20$ i.e., $-x_1 + 5x_2 \leq 20$ $(0,4)$, $(5,5)$

and $x_1, x_2 \geq 0$



Sol **Step 3** Here solution space is unbounded. The vertices of the convex ~~feasible~~ region (in the finite plane) are O(0,0), A(0,3) and $B\left(\frac{15}{7}, \frac{31}{7}\right)$

and corresponding values of Z are 0, 6 and $\frac{107}{7}$ resp.

But there are points in this convex region for which Z will have much higher values. Hence the problem has an unbounded ~~solutio~~ solution.

Que Min $Z = 3x_1 + 2x_2$

s.t. $3x_1 + x_2 \geq 40$ $(0,40), (\frac{40}{3}, 0)$

$x_1 + 2.5x_2 \geq 22$ $(7,6), (22,0)$

$x_1 + x_2 \geq \frac{40}{3}$ $(0, \frac{40}{3}), (\frac{40}{3}, 0)$

$x_1, x_2 \geq 0$

$3x_1 + x_2 = 40$

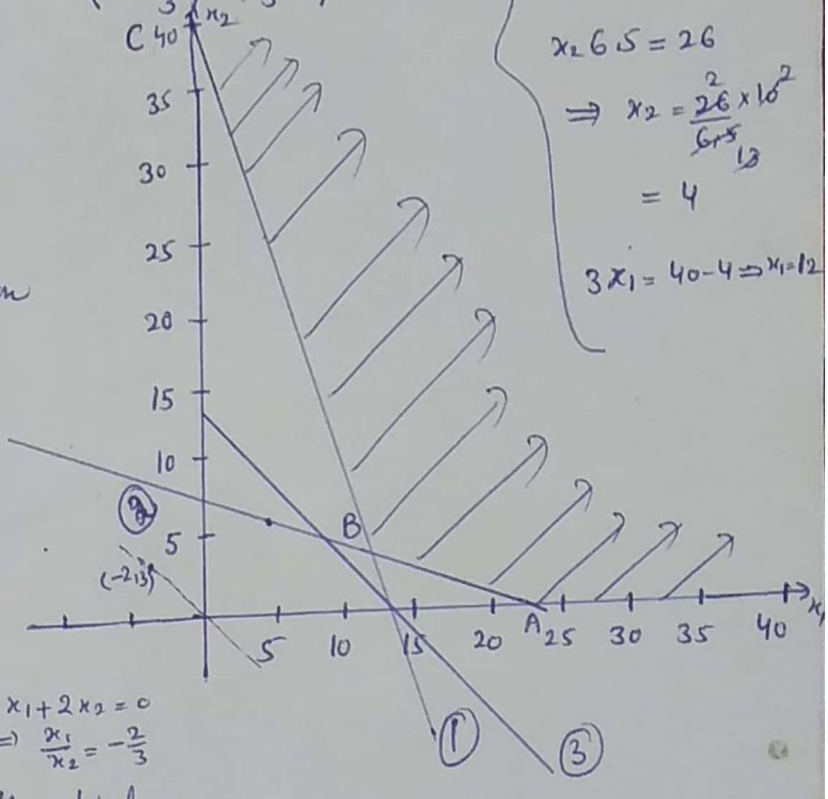
$x_1 + 2.5x_2 = 22$

$x_2 6.5 = 26$

$\Rightarrow x_2 = \frac{26 \times 10}{65} = 4$

$3x_1 = 40 - 4 \Rightarrow x_1 = 12$

Sol Step 4



Vertices of the convex region ABC are

A (22, 0), B (12, 4) and C (0, 40).

$\therefore$ Z(A) = 66

Z(B) = 44

Z(C) = 80

Thus min. value of Z is 44 which is at B (12, 4)

Here the convex region is unbounded but at all other points of the region Z will have value more than at point B.

Hence Min Z = 44 at $x_1 = 12, x_2 = 4$.

Linear Programming Problem: A general l.p.p. can be stated as follows:

Find $x_1, x_2, \dots x_n$ which optimize the linear function

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \text{———(1)} \quad \text{(Objective function)}$$

s.t.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & \ (\leq = \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & \ (\leq = \geq)\ b_2 \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & \ (\leq = \geq)\ b_m \end{aligned}\right\} \text{(Constraints or restrictions)}$$

and $x_j \geq 0$, $j = 1, 2, \dots n$ (non-negative restrictions)

where all a_{ij}'s, b_is and c_js are constants.

Matrix form of a L.P.P. Optimize $Z = CX$

s.t. $AX \ (\leq = \geq)\ b$

and $X \geq 0$

where $A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is the matrix of coefficients.

$C = [c_1 \ c_2 \dots c_n]$ is a row vector, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Some Definitions

(1) **Basic solution (B.S.)** Consider a system $AX = b$ of m equations in n unknowns. A solution obtained by setting any $(n-m)$ variables to zero is called a basic solution, provided the determinant of the coefficients of the remaining m variables is not zero. Here m variables are called basic variables and $(n-m)$ variables are called non-basic variables.

<u>Remark</u> Maximum number of possible basic solutions = ${}^{n}C_{m}$

Basic solutions are of two types:

(a) <u>Non-degenerate B.S</u> A basic solution is called non-degenerate B.S. if none of the basic variables is zero. In ~~other words all the m basic variables are non-zero~~.

(b) <u>Degenerate B.S</u> A basic solution is called degenerate B.S. if one or more of the basic variables are zero.

(2) <u>Feasible solution (F.S.)</u>: A F.S. to a L.P.P. is the set of values of the variables which satisfies the set of constraints and the non-negative restrictions of the problem.

(3) <u>Optimum (or optimal) solution</u> A F.S. to a L.P.P. is said to be optimum (or optimal) solution if it also optimizes the objective function Z of the problem.

(4) <u>Basic Feasible Solution (B.F.S.)</u> A F.S. to a L.P.P. which is also basic is called a B.F.S.

(5) <u>Optimal basic feasible solution</u> A B.F.S. to a L.P.P. is said to be optimal B.F.S. if it also optimizes the objective function Z.

(6) <u>Basic variables</u> The non-zero variables in a B.F.S. are called basic variables.

(7) <u>Non-degenerate B.F.S.</u> A B.F.S. of a L.P.P. is said to be non-degenerate B.F.S. if none of the basic variables is zero.

(8) <u>Degenerate B.F.S.</u> A B.F.S. of a L.P.P. is said to be degenerate B.F.S. if at least one of the basic variables is zero.

Que Find all the basic solutions of the following system

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

and prove that they are non-degenerate.

Sol Here no. of constraints = 2 and no. of variables = 3

So this problem can have atmost $^3C_2 = 3$ basic solutions in which $3-2 = 1$ variable is zero in each case.

Non-basic variable	Reduced equations	Basic variables
(1) $x_1 = 0$	$\left.\begin{array}{l} 2x_2 + x_3 = 4 \\ x_2 + 5x_3 = 5 \end{array}\right\}$ Solving	$x_2 = 5/3$, $x_3 = 2/3$ and $\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} \neq 0$

$\therefore (0, 5/3, 2/3)$ is a B.S. of the given problem.

(2) $x_2 = 0$	$\left.\begin{array}{l} x_1 + x_3 = 4 \\ 2x_1 + 5x_3 = 5 \end{array}\right\}$ Solving	$x_1 = 5$, $x_3 = -1$ and $\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \neq 0$

$\therefore (5, 0, -1)$ is a B.S. of the given problem.

(3) $x_3 = 0$	$\left.\begin{array}{l} x_1 + 2x_2 = 4 \\ 2x_1 + x_2 = 5 \end{array}\right\}$ Solving	$x_1 = 2$, $x_2 = 1$ and $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \neq 0$

$\therefore (2, 1, 0)$ is also a B.S. of the given problem.

Hence the basic solutions of the given problem are $(0, 5/3, 2/3)$, $(5, 0, -1)$ and $(2, 1, 0)$.

Here we see none of the basic variables is zero. So they are non-degenerate B.S.

Que Is the F.S. $x_1 = 1, x_2 = 0, x_3 = 1$ and $Z = 6$ is basic of the following L.P.P.

$$\text{Min } Z = 2x_1 + 3x_2 + 4x_3$$
$$\text{s.t. } x_1 + x_2 + x_3 = 2$$
$$x_1 - x_2 + x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

Sol Clearly $x_1 = 1, x_2 = 0, x_3 = 1$ satisfy the given equations and the value of $3-2 = 1$ variable is zero i.e., $x_2 = 0$ may be considered as non-basic variable.

$\therefore x_2 = 0 \Rightarrow \left.\begin{array}{l} x_1 + x_3 = 2 \\ x_1 + x_3 = \end{array}\right\} \Rightarrow x_1 = 1, x_3 = 1$ but $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ i.e., the given sol. is not basic.

Canonical and Standard forms of L.P.P.

Before using any method to get the solution of the problem, we present it in a suitable form. Here we explain two forms:

Canonical form The general l.p.p. can always be expressed in the form:

$$\text{Max. } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad \Big\} \text{ Objective function}$$

$$\text{s.t.} \quad \left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}\right\} \text{ constraints}$$

$$x_1, x_2, \ldots, x_n \ge 0 \quad (\text{no restriction on } b_i\text{'s})$$

which is known as canonical form of l.p.p.

Standard form The general l.p.p. can also be expressed in the form:

$$\text{Max. } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$x_1, x_2, \ldots, x_n \ge 0 \quad (\text{Here } b_i\text{'s} \ge 0)$$

which is known as standard form of l.p.p.

Remark

(1) In both the forms objective function should be of maximization type (if it is a problem of minimization then change it into maximization, see example given on next page)

(2) All constraints are of ($\le$ type) in canonical form while all constraints are expressed as equations.

(3) All variables are non-negative in both the forms.

(4) Right hand side of each constraint is non-negative in standard form while there is no restriction in canonical form R.H.S.

Slack and surplus variables

If some constraint is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

then to convert it in equality we use slack variable s_1 as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 = b_1$$

where $s_1 \geq 0$

Similarly if constraint is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

then to convert it in equality we use surplus variable s_1 as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - s_1 = b_1$$

where $s_1 \geq 0$

Remark Any L.P.P. can be expressed in the standard form

(1) by using slack and surplus variables to make inequalies as equalities

(2) if the problem is of minimization i.e, Min $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, then we can convert it into maximization as

$$\text{Max. } Z' (= -Z) = -c_1x_1 - c_2x_2 - \cdots - c_nx_n$$

(3) If non-negativity restriction is not satisfied for some variable, say it is not given $x_i \geq 0$, then we can write $x_i = x_i' - x_i''$, where $x_i', x_i'' \geq 0$

Que Convert the following L.P.P. to the standard form:

$$\text{Max. } Z = 3x_1 + 5x_2 + 7x_3$$

s.t.
$$6x_1 - 4x_2 \leq 5$$
$$3x_1 + 2x_2 + 5x_3 \geq 11$$
$$4x_1 + 3x_2 - 2x_3 \leq 2$$
$$x_1, x_2 \geq 0$$

Sol As x_3 is unrestricted, let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$.

$\therefore$ Introducing the slack or surplus variables, the problem in standard form becomes:

$$\text{Max. } Z = 3x_1 + 5x_2 + 7x_3' - 7x_3''$$

$$\begin{aligned} \text{s.t. } \quad & 6x_1 - 4x_2 + s_1 = 5 \\ & 3x_1 + 2x_2 + 5x_3' - 5x_3'' - s_2 = 11 \\ & 4x_1 + 3x_2 - 2x_3' + 2x_3'' + s_3 = 2 \\ & x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0 \end{aligned}$$

Simplex Method If an L.P.P. contains more than two variables then we cannot use the graphical method. In that case, we express the given L.P.P. in standard form by using slack or surplus variable, by conversion of minimization problem into maximization and if any variable is unrestricted then write it as difference of two variable as discussed on previous page. If introduction of slack/surplus variables provide the initial B.F.S., then we can solve it by simplex method. We discuss the method in following problem:

Que 1

$$\text{Max. } Z = 5x_1 + 3x_2$$

$$\begin{aligned} \text{s.t. } \quad & 3x_1 + 5x_2 \leq 15 \\ & 5x_1 + 2x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Sol The given problem may be reformulated as:

$$\text{Max. } Z = 5x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

$$\begin{aligned} \text{s.t. } \quad & 3x_1 + 5x_2 + s_1 = 15 \\ & 5x_1 + 2x_2 + s_2 = 10 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

	B	C_B	C_j / X_B	5 / X_1	3 / X_2	0 / S_1	0 / S_2	Mini ratio X_B/X_1
	S_1	0	15	3	5	1	0	5
	S_2	0	10	key element $\boxed{5}$	2	0	1	2 → outgoing vector
	$Z = C_B X_B = 0$		Δ_j	−5 ↑entering vector	−3	0	0	X_B/X_2
$R_3 \to R_1 - \frac{3}{5}R_2$	S_1	0	9	0	$\boxed{19/5}$	1	−3/5	45/19 →
$R_4 \to \frac{1}{5}R_2$	X_1	5	2	1	2/5	0	1/5	5
	$Z = 10$		Δ_j	0	−1	0	1	
$R_5 \to \frac{5}{19}R_3$	X_2	3	45/19	0	1	5/19	−3/19	
$R_6 \to R_4 - \frac{2}{5}R_5$	X_1	5	20/19	1	0	−2/19	5/19	
	$Z = \frac{235}{19}$		Δ_j	0	0	5/19	16/19	

Since all $\Delta_j \geq 0$, So the optimal B.F.S. is

$$\boxed{x_1 = 20/19, \quad x_2 = 45/19, \quad \text{Max. } Z = \frac{235}{19}}$$

Remark While making the simplex table we take

(1) $Z = C_B X_B$ (2) $\Delta_j = C_B X_j - c_j$

(3) entering vector is corresponding to most negative value of Δ_j

(4) To find Min ratio we divide elements of X_B by the positive corresponding elements of entering vector. ~~while they are non-negative or zero. to~~ If they are negative or zero then leave the corresponding entry.

(5) Outgoing vector is corresponding to min. ratio.

(6) key element is the intersection of incoming and outgoing vectors.

(7) We make the key element as 1 and other elements 0 in the same column by using row operations.

Stopping criteria and final solution

(i) If $\Delta_j \geq 0$ for each j, the solution under test is optimal.

(a) If none of $\Delta_j < 0$ but any non-basic $\Delta_j = 0$, then alternative optimal solutions will exist.

(b) If all $\Delta_j > 0$ for non-basic variables, then solution under test is unique optimal solution.

(ii) If at least one $\Delta_j < 0$, the solution under test is not optimal. So we go to the next step.

(iii) If corresponding to most negative Δ_j, all the elements in the column X_j are negative or zero then the solution under test will be unbounded.

For Big-M-Method

If the optimality condition i.e. $\Delta_j \geq 0$ for each j is satisfied but one or more artificial variable is still in the basis, then we say that the problem has no feasible solution.

Que Solve the following L.P.P. by simplex method :

$$\text{Min. } Z = x_1 - 3x_2 + 3x_3$$

s.t. $3x_1 - x_2 + 2x_3 \leq 7$

$2x_1 + 4x_2 \geq -12$ i.e., $-2x_1 - 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Sol The given problem may be reformulated as

$$\text{Max. } Z' = -\text{Min } Z = -x_1 + 3x_2 - 3x_3 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

s.t. $3x_1 - x_2 + 2x_3 + S_1 = 7$

$-2x_1 - 4x_2 + 0 \cdot x_3 + 0 \cdot S_1 + S_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + 0 \cdot S_1 + 0 \cdot S_2 + S_3 = 10$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

		C_j	-1	3	-3	0	0	0	Min. ratio	
B	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B/X_2	
S_1	0	7	3	-1	2	1	0	0	—	
S_2	0	12	-2	-4	0	0	1	0	—	
S_3	0	10	-4	$\boxed{3}$	8	0	0	1	$\rightarrow$	
$Z' = 0$		Δ_j	1	-3 ↑	3	0	0	0	X_B/X_1	
S_1	0	31/3	$\boxed{5/3}$	0	14/3	1	0	1/3	$\rightarrow$	$R_4 \rightarrow R_1 + R_6$
S_2	0	76/3	$-22/3$	0	32/3	0	1	4/3	—	$R_5 \rightarrow R_2 + 4R_6$
X_2	3	10/3	$-4/3$	1	8/3	0	0	1/3	—	$R_6 \rightarrow \left(\frac{1}{3}\right) R_3$
$Z' = 10$		Δ_j	-3 ↑	0	11	0	0	1		
X_1	-1	31/5	1	0	14/5	3/5	0	1/5		$R_7 \rightarrow \left(\frac{3}{5}\right) R_4$
S_2	0	354/5	0	0	156/5	22/5	1	14/5		$R_8 \rightarrow R_5 + \frac{22}{3} R_7$
X_2	3	58/5	0	1	32/5	4/5	0	3/5		$R_9 \rightarrow R_6 + \frac{4}{3} R_7$
$Z' = 143/5$		Δ_j	0	0	97/5	9/5	0	8/5		

Since all $\Delta_j \geq 0$, So the optimal B.F.S. is $x_1 = \frac{31}{5}$, $x_2 = \frac{58}{5}$, $x_3 = 0$

and $\text{Min } Z = -\text{Max. } Z' = -143/5$.

Que Solve the following problem

$$\text{Maximize } Z = 107x_1 + x_2 + 2x_3$$

$$\text{s.t. } 14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$$

$$3x_1 - x_2 - x_3 \leq 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Sol The given problem may be reformulated as

$$\text{Max. } Z = 107x_1 + x_2 + 2x_3 + 0\cdot x_4 + 0\cdot S_1 + 0\cdot S_2$$

$$\text{s.t. } \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = \frac{7}{3}$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 + S_1 = 5$$

$$3x_1 - x_2 - x_3 + S_2 = 0$$

$$x_1, x_2, x_3, x_4, S_1, S_2 \geq 0$$

		C_j	107	1	2	0	0	0	Min. ratio
B	C_B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	X_B/X_1
X_4	0	7/3	14/3	1/3	−2	1	0	0	1/2
S_1	0	5	16	1/2	−6	0	1	0	5/16
S_2	0	0	$\boxed{3}$	−1	−1	0	0	1	0 →
Z = 0		Δ_j	−107 ↑	−1	−2	0	0	0	
X_4	0	7/3	0	17/9	−4/9	1	0	−14/9	$R_4 \to R_1 - \frac{14}{3}R_6$
S_1	0	5	0	35/6	−2/3	0	1	−16/3	$R_5 \to R_2 - 16R_6$
X_1	107	0	1	−1/3	−1/3	0	0	1/3	$R_6 \to \frac{1}{3}R_3$
Z = 0		Δ_j	0	−110/3	−113/3 ↑	0	0		

Here corresponding to most −ive Δ_j all the elements in the corresponding column X_3 are −ive. So the solution under test will be unbounded.

* Take Problems related to Big M Method and Two-phase method ...

Duality

Standard forms

primal

(i) $\text{Max. } Z_p = cX$

s.t $AX \leq b$

$X \geq 0$

Its dual is $\text{Min } Z_d = bY$

s.t. $A'Y \geq c$

$Y \geq 0$

(ii) $\text{Min } Z_p = cX$

s.t. $AX \geq b$

$X \geq 0$

Its dual is

$\text{Max. } Z_d = bY$

s.t. $A'Y \leq c$

$Y \geq 0$

Note Dual of the dual of a given primal is the primal itself.

Que 1 Write the dual of the following problems

(a) $\text{Max. } Z = x_1 + 2x_2 - x_3$

s.t. $2x_1 - 3x_2 + 4x_3 \leq 5$

$2x_1 - 2x_2 \leq 6$

$3x_1 - x_3 \geq 4$

and $x_1, x_2, x_3 \geq 0$

(b) $\text{Min. } Z = 2x_1 + 2x_2 + 4x_3$

s.t. $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

and $x_1, x_2, x_3 \geq 0$

(c) $\text{Max. } Z = x_1 + 3x_2$

s.t. $3x_1 + 2x_2 \leq 6$

$3x_1 + x_2 = 4$

and $x_1, x_2 \geq 0$

(d) $\text{Min } Z = x_1 + x_2 + x_3$

s.t. $x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

Sol 1(a) The given problem can be written in the standard primal form as

$\text{Max. } Z_p = x_1 + 2x_2 - x_3$

s.t. $2x_1 - 3x_2 + 4x_3 \leq 5$

$2x_1 - 2x_2 + 0 \cdot x_3 \leq 6$

$-3x_1 - 0 \cdot x_2 + x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

$\therefore$ Its dual is $\text{Min } Z_d = 5y_1 + 6y_2 - 4y_3$

s.t. $2y_1 + 2y_2 - 3y_3 \geq 1$

$-3y_1 - 2y_2 \geq 2$

$4y_1 + y_3 \geq -1$

$y_1, y_2, y_3 \geq 0$

(2)

(b) Standard primal form is

$$\text{Min } Z_p = 2x_1 + 2x_2 + 4x_3$$

s.t. $2x_1 + 3x_2 + 5x_3 \geq 2$

$-3x_1 - x_2 - 7x_3 \geq -3$

$-x_1 - 4x_2 - 6x_3 \geq -5$

$x_1, x_2, x_3 \geq 0$

$\therefore$ Its dual is

$$\text{Max. } Z_d = 2y_1 - 3y_2 - 5y_3$$

s.t. $2y_1 - 3y_2 - y_3 \leq 2$

$3y_1 - y_2 - 4y_3 \leq 2$

$5y_1 - 7y_2 - 6y_3 \leq 4$

and $y_1, y_2, y_3 \geq 0$

(c) Standard primal form is

$$\text{Max. } Z_p = x_1 + 3x_2$$

s.t. $3x_1 + 2x_2 \leq 6$

$3x_1 + x_2 \leq 4$

$-3x_1 - x_2 \leq -4$

and $x_1, x_2 \geq 0$

$\therefore$ Its dual is $\text{Min } Z_d = 6y_1 + 4y_2 - 4y_3$

s.t. $3y_1 + 3y_2 - 3y_3 \geq 1$

$2y_1 + y_2 - y_3 \geq 3$

$y_1, y_2, y_3 \geq 0$

This can also be written as

$$\text{Min } Z_d = 6z_1 + 4z_2$$

s.t. $3z_1 + 3z_2 \geq 1$

$2z_1 + z_2 \geq 3$

~~z_1, z_2~~ $z_1 \geq 0$, z_2 unrestricted in sign.

(3)

(d) Since x_3 is unrestricted in sign,

$\therefore$ we can assume $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$

$\therefore$ The given problem is

$$\text{Min } Z = x_1 + x_2 + x_3' - x_3''$$

$$\text{s.t. } x_1 - 3x_2 + 4x_3' - 4x_3'' = 5$$

$$x_1 - 2x_2 \leq 3$$

$$2x_2 - x_3' + x_3'' \geq 4$$

$$x_1, x_2, x_3', x_3'' \geq 0$$

$\therefore$ The standard primal form is

$$\text{Min } Z_p = x_1 + x_2 + x_3' - x_3''$$

$$\text{s.t. } x_1 - 3x_2 + 4x_3' - 4x_3'' \geq 5$$

$$-x_1 + 3x_2 - 4x_3' + 4x_3'' \geq -5$$

$$-x_1 + 2x_2 + 0 \cdot x_3' + 0 \cdot x_3'' \geq -3$$

$$0 \cdot x_1 + 2x_2 - x_3' + x_3'' \geq 4$$

$$x_1, x_2, x_3', x_3'' \geq 0$$

$\therefore$ Its dual is

$$\text{Max. } Z_d = 5(y_1 - y_2) - 3y_3 + 4y_4$$

$$\text{s.t } (y_1 - y_2) - y_3 \leq 1$$

$$-3(y_1 - y_2) + 2y_3 + 2y_4 \leq 1$$

$$\left.\begin{aligned} 4(y_1 - y_2) - y_4 &\leq 1 \\ -4(y_1 - y_2) + y_4 &\leq -1 \end{aligned}\right\} \Rightarrow 4(y_1 - y_2) - y_4 = 1$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Considering $y_1 - y_2 = z_1$, $y_3 = z_2$, $y_4 = z_3$, it can be written as

$$\text{Max. } Z_d = 5z_1 - 3z_2 + 4z_3$$

$$\text{s.t } z_1 - z_2 \leq 1$$

$$-3z_1 + 2z_2 + 2z_3 \leq 1$$

$$4z_1 - z_3 = 1$$

$z_2, z_3 \geq 0$, z_1 is unrestricted in sign

EndTerm (2016) (6 m)

Que Write the dual of the following problem

$$\text{Min } Z = 2x_1 + 3x_2 + 4x_3$$

$$\text{s.t. } 2x_1 + 3x_2 + 5x_3 = 2$$

$$3x_1 + x_2 + 7x_3 \leq 3$$

$$x_1 + 4x_2 + 6x_3 = 5$$

$x_2, x_3 \geq 0$ and x_1 unrestricted.

Sol Since x_1 is unrestricted in sign, therefore we can assume $x_1 = x_1' - x_1''$; $x_1', x_1'' \geq 0$

$\therefore$ The given problem in standard primal form is

$$\text{Min } Z = 2x_1' - 2x_1'' + 3x_2 + 4x_3$$

$$\text{s.t. } 2x_1' - 2x_1'' + 3x_2 + 5x_3 \geq 2$$

$$-2x_1' + 2x_1'' - 3x_2 - 5x_3 \geq -2$$

$$-3x_1' + 3x_1'' - x_2 - 7x_3 \geq -3$$

$$x_1' - x_1'' + 4x_2 + 6x_3 \geq 5$$

$$-x_1' + x_1'' - 4x_2 - 6x_3 \geq -5$$

$$x_1', x_1'', x_2, x_3 \geq 0$$

Its dual is,

$$\text{Max } Z_d = 2(y_1 - y_2) - 3y_3 + 5(y_4 - y_5)$$

$$\text{s.t. } 2(y_1 - y_2) - 3y_3 + (y_4 - y_5) \leq 2$$

$$-2(y_1 - y_2) + 3y_3 - (y_4 - y_5) \leq -2$$

$$3(y_1 - y_2) - y_3 + 4(y_4 - y_5) \leq 3$$

$$5(y_1 - y_2) - 7y_3 + 6(y_4 - y_5) \leq 4$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

A

$$\text{Max } Z_d = 2W_1 - 3W_2 + 5W_3$$

$$\text{s.t. } 2W_1 - 3W_2 + W_3 = 2$$

$$3W_1 - W_2 + 4W_3 \leq 3$$

$$5W_1 - 7W_2 + 6W_3 \leq 4$$

$W_2 \geq 0$ and W_1, W_3 are unrestricted in sign.

Que Use duality to solve the following L.P. problem

$$\text{Max. } Z = 3x_1 + x_2$$

s.t. $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \geq 2$

and $x_1, x_2 \geq 0$

The given problem can be written in the standard primal form as $\text{Max } Z_p = 3x_1 + x_2$

s.t. $x_1 + x_2 \leq 1$

$-2x_1 - 3x_2 \leq -2$

and $x_1, x_2 \geq 0$

The dual of the given problem is

$$\text{Min } Z_d = y_1 - 2y_2$$

s.t. $y_1 - 2y_2 \geq 3$

$y_1 - 3y_2 \geq 1$

$y_1, y_2 \geq 0$

Changing the objective function into maximization form, introducing surplus variables $s_1 \geq 0$, $s_2 \geq 0$ and artificial variables $a_1 \geq 0$, $a_2 \geq 0$ the problem becomes

$$\text{Max. } Z'_d = -Z_d = -y_1 + 2y_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2$$

s.t. $y_1 - 2y_2 - s_1 + a_1 = 3$

$y_1 - 3y_2 - s_2 + a_2 = 1$

$y_1, y_2, s_1, s_2, a_1, a_2 \geq 0$

Now, we apply Big M-Method

	B	C_B	C_j / X_B	-1 / y_1	2 / y_2	0 / S_1	0 / S_2	$-M$ / A_1	$-M$ / A_2	Min. ratio
										X_B/y_1
	A_1	$-M$	3	1	-2	-1	0	1	0	3
	A_2	$-M$	1	$\boxed{1}$	-3	0	-1	0	1	$1 \rightarrow$
	$Z'_d = -4M$		Δ_j	$-2M+1$ ↑	$5M-2$	M	M	0	0	X_B/y_2
$R_3 \rightarrow R_1 - R_2$	A_1	$-M$	2	0	$\boxed{1}$	-1	1	1	-1	$\rightarrow$
	y_1	-1	1	1	-3	0	-1	0	1	
	$Z'_d = -2M-1$		Δ_j	0	$-M+1$ ↑	M	$-M+1$	0	$2M-1$	
	y_2	2	2	0	1	-1	1	1	-1	
$R_6 \rightarrow R_4 + 3R_3$	y_1	-1	7	1	0	-3	2	3	-2	
	$Z'_d = -3$		Δ_j	0	0	1	0	$-1+M$	M	

From the table we see that coefficient of M in each $\Delta_j \geq 0$. So the solution to the dual problem given in above table is an optimal solution.

$\therefore$ Optimal sol. of the dual problem is

$$y_1 = 7,\ y_2 = 2,\quad \text{Max.}\ Z'_d = -3 \Rightarrow \text{Min.}\ Z_d = 3$$

Solution of the original problem (primal)

From the final simplex table for the solution of dual, we have the solution of the primal problem as

$$x_1 = \Delta_3 = 1$$

$$x_2 = \Delta_4 = 0$$

$$\text{Max.}\ Z_p = \text{Min}\ Z_d = 3.$$

Dual Simplex method This method is used only when X_B contains some negative terms, ~~in the constraints~~.

Thus all the problems of Big-M-method and two-phase method may be solved by this method.

In this method we choose outgoing variable corresponding to most -ive value of X_B from the basis and then entering variable by choosing minimum $\left(\frac{-\Delta_k}{x_{ik}}, x_{ik} < 0\right)$. After this the key element is reduce to 1 (unity) and other elements below and above this element to zero.

Que Solve the following L.P.P. by the dual simplex method.

(1) Mini $Z = 2x_1 + x_2$

s.t. $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

and $x_1, x_2 \geq 0$

[A $x_1 = 3/5$, $x_2 = 6/5$, $Z = 12/5$]

(2) Max. $Z = -2x_1 - 2x_2 - 4x_3$

s.t. $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

and $x_1, x_2, x_3 \geq 0$

[A $x_1 = 0$, $x_2 = 2/3$, $x_3 = 0$, $Z = -4/3$]

Sol (1) The given problem may be reformulated as

$$\text{Max. } Z' = -Z = -2x_1 - x_2$$

s.t. $-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 - 2x_2 \leq -3$

$x_1, x_2 \geq 0$

By introducing slack variables s_1, s_2, s_3 it can be written as

$$\text{Max. } Z' = -Z = -2x_1 - x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

s.t. $-3x_1 - x_2 + s_1 = -3$

$-4x_1 - 3x_2 + s_2 = -6$

$-x_1 - 2x_2 + s_3 = -3$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

7

B	C_B	C_j / X_B	-2 / X_1	-1 / X_2	0 / S_1	0 / S_2	0 / S_3	
S_1	0	-3	-3	-1	1	0	0	
S_2	0	-6	-4	$\boxed{-3}$	0	1	0	→ Outgoing
S_3	0	-3	-1	-2	0	0	1	
$Z' = C_B X_B = 0$		Δ_j	2	1 ↑	0	0	0	
S_1	0	-1	$\boxed{-5/3}$	0	1	$-1/3$	0	→
X_2	-1	2	$4/3$	1	0	$-1/3$	0	
S_3	0	1	$5/3$	0	0	$-2/3$	1	
$Z' = -2$		Δ_j	$2/3$ ↑	0	0	$1/3$	0	
X_1	-2	$3/5$	1	0	$-3/5$	$1/5$	0	
X_2	-1	$6/5$	0	1	$4/5$	$-3/5$	0	
S_3	0	0	0	0	1	-1	1	
$Z' = -12/5$		Δ_j	0	0	$2/5$	$1/5$	0	

$\min\left[\frac{-2}{-4}, \frac{-1}{-3}\right] = \min\left[\frac{1}{2}, \frac{1}{3}\right] = \frac{1}{3}$

$\min\left[\frac{-2/3}{-5/3}, \frac{-1/3}{-1/3}\right] = \min\left[\frac{2}{5}, 1\right] = \frac{2}{5}$

Since all ~~$\Delta_j \geq 0$~~, elements of column X_B are +ive and all $\Delta_j \geq 0$ So the optimal B.F.S. is

$x_1 = 3/5$, $x_2 = 6/5$, $\text{Min } Z = -Z' = 12/5$.

Que Using dual simplex method solve following l.p.p. (6.5m)

$$\text{Max. } Z = -3x_1 - 2x_2$$

s.t.

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$

where $x_1, x_2 \geq 0$.

[End Term May-June 2016]

[6.5 m]

Sol The given primal can be written as

$$\text{Max. } Z = -3x_1 - 2x_2$$

s.t.

$$-x_1 - x_2 + s_1 = -1$$
$$x_1 + x_2 + s_2 = 7$$
$$-x_1 - 2x_2 + s_3 = -10$$
$$x_2 + s_4 = 3$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

	B	C_B	C_j / X_B	X_1 (−3)	X_2 (−2)	S_1 (0)	S_2 (0)	S_3 (0)	S_4 (0)	
	S_1	0	−1	−1	−1	1	0	0	0	
	S_2	0	7	1	1	0	1	0	0	
$\text{Min}\left(\frac{-3}{-1}, \frac{-2}{-2}\right)$ ↑	S_3	0	−10	−1	$\boxed{-2}$	0	0	1	0	→ Outgoing
	S_4	0	3	0	1	0	0	0	1	
	$Z = 0$		Δ_j	3	2 ↑	0	0	0	0	
$R_5 \to R_1 + R_7$	S_1	0	4	−1/2	0	1	0	−1/2	0	
$R_6 \to R_2 - R_7$	S_2	0	2	1/2	0	0	1	1/2	0	
$R_7 \to \left(-\frac{1}{2}\right) R_3$	X_2	−2	5	1/2	1	0	0	−1/2	0	
$R_8 \to R_4 - R_7$	S_4	0	−2	$\boxed{-1/2}$	0	0	0	1/2	1	→ Outgoing
	$Z = -10$		Δ_j	↑ 2	0	0	0	1	0	
$R_9 \to R_5 - R_8$	S_1	0	6	0	0	1	0	−1	−1	
$R_{10} \to R_6 + R_8$	S_2	0	0	0	0	0	1	1	1	
$R_{11} \to R_7 + R_8$	X_2	−2	3	0	1	0	0	0	1	
$R_{12} \to (-2) R_8$	X_1	−3	4	1	0	0	0	−1	−2	
	$Z = -18$		Δ_j	0	0	0	0	3	4	

Since all elements in column X_B are ≥ 0 and all $\Delta_j \geq 0$

$\therefore$ The optimal B.F.S is given by

$x_1 = 4$

$x_2 = 3$

$\text{Max } Z = -18$

Working rule for solving an assignment problem

Hungarian Assignment Method

(1) First we examine the problem should be balanced i.e, the given matrix should be square and if it is not ~~a square~~ ~~matrix convert~~ balanced then first convert it into balanced problem by adding fictitious row or column accordingly.

(2) Subtracting the smallest element of each row from every element of that row.

(3) Subtract the smallest element of each column from every element of that column.

(4) Now give assignment in usual way.

(5) If each row and column have assignment then the matrix give optimal sol. otherwise draw minimum no. of lines and proceed further.

All the procedure is clear from the following question step by step:-

Que1 Solve the following minimal assignment problems.

Subordinates

Task	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Sol **Step 1** Subtracting the smallest element of each row from every every element of that row, we have

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

Table 1

②

Step 2 Subtracting smallest element of each column from every element of that column, we have the following matrix

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 0 & 14 & 9 & 3 \\ B & 9 & 20 & 0 & 22 \\ C & 23 & 0 & 3 & 0 \\ D & 9 & 12 & 14 & 0 \\ \hline \end{array}$$

Table-2

Step 3 Now, we give the zero assignment in our usual manner and get the following matrix

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & \boxed{0} & 14 & 9 & 3 \\ B & 9 & 20 & \boxed{0} & 22 \\ C & 23 & \boxed{0} & 3 & \cancel{0} \\ D & 9 & 12 & 14 & \boxed{0} \\ \hline \end{array}$$

Table-3

Since in table-3 every row and every column have one assignment, so we have the optimal assignment as

$A \to I,\ B \to III,\ C \to II,\ D \to IV$

and Min cost $= 8 + 4 + 19 + 10 = 41$ } Ans

Que 2 Solve the following minimal assignment problem:

Man

Task	I	II	III	IV	V
A	1	3	2	3	6
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4

01. Step 1 Subtracting the smallest element of each row from every element of that row, the reduced matrix is

	I	II	III	IV	V
A	0	2	1	2	5
B	1	3	2	0	4
C	2	3	0	1	3
D	2	0	3	1	1
E	0	4	5	4	3

Table-1

Step 2 Subtracting the smallest element of each column from every element of that column, we have the reduced matrix as

	I	II	III	IV	V
A	0	2	1	2	4
B	1	3	2	0	3
C	2	3	0	1	2
D	2	0	3	1	0
E	0	4	5	4	2

Table-2

Step 3 Now, we give the zero assignment in our usual manner and get the following matrix

	I	II	III	IV	V	
A	$\boxed{0}$	2	1	2	4	✓ 3
B	1	3	2	$\boxed{0}$	3	
C	2	3	$\boxed{0}$	1	2	
D	2	$\boxed{0}$	3	1	~~0~~	
E	~~0~~	4	5	4	2	✓ 1
	2					

Table-3

Since there is no assignment in row 5 and column 5, so we go to the next step.

Step 4 In this step we draw the minimum number of lines in usual manner. These lines are 4. (See in the above table)

Step 5 Now subtracting the smallest uncovered element 1 from all uncovered elements and adding it to the elements which lie at the intersection of lines and leaving other elements unchanged, we get the following matrix:

	I	II	III	IV	V	
A	~~0~~	1	~~0~~	1	3	✓ 1
B	2	3	2	[0]	3	
C	3	3	[0]	1	2	✓ 4
D	3	[0]	3	1	~~0~~	
E	[0]	3	4	3	1	✓ 5
	2		3			

Table-4

Step 6 Again repeating the step 3 we make the zero assignment in usual way. Since there is no assignment in row 1 and column 5, so we go to the next step.

Step 7 In this step we again draw the minimum no. of lines in usual manner in the Table-4. These lines are 4.

Step 8 Again repeating the step 5, we get the following matrix

	I	II	III	IV	V
A	[0]	~~0~~	~~0~~	~~0~~	2
B	3	3	3	[0]	3
C	3	2	[0]	~~0~~	1
D	4	[0]	4	1	~~0~~
E	0	2	4	2	[0]

Step 9 Repeating the step 3 we make the zero assignments and get the following optimal assignments (in the above table)

$A \to I,\ B \to IV,\ C \to III,\ D \to II,\ E \to V$

& Min Cost $z = 1+1+3+1+4 = 10$ Ans

Problem of maximization solved by assignments (Also it is unbalanced)

Que The owner of a small machine shop has four machinists available to assign jobs for the day. Five jobs are offered with expected profit (in Rs.) for each machinist on each job as follows: Job

Machinist	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Find by using the assignment method, the assignment of machinist to jobs that will result in a maximum profit. Which job should be declined?

Sol This is a maximization unbalanced assignment problem. So, we convert this maximization problem to minimization problem multiplying by −1 to each element of the given matrix and add one fictitious machinist 5 to make it a square matrix. Thus the resulting matrix is as follows:

	A	B	C	D	E
1	−62	−78	−50	−101	−82
2	−71	−84	−61	−73	−59
3	−87	−92	−111	−71	−81
4	−48	−64	−87	−77	−80
5	0	0	0	0	0

Now, we follow step by step (as in other questions) and get the table

	A	B	C	D	E	
1	39	23	51	$\boxed{0}$	19	
2	13	$\boxed{0}$	23	11	25	
3	24	19	$\boxed{0}$	40	30	✓ 3
4	39	23	$\cancel{0}$	10	7	✓ 1
5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	
			2			

	A	B	C	D	E
1	39	23	58	$\boxed{0}$	19
2	13	$\boxed{0}$	30	11	25
3	17	12	$\boxed{0}$	33	23
4	32	16	$\cancel{0}$	3	$\boxed{0}$
5	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$	$\cancel{0}$

Thus the optimal assignment is

$1 \to D,\ 2 \to B,\ 3 \to C,\ 4 \to E$ and $\text{Max. } Z = -\text{Mini } Z^* = 101 + 84 + 111 + 80$
$= \text{Rs. } 376$

A

Procedure for drawing minimum no. of lines

When the matrix does not contain assignment in every row and every column then we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. For this we proceed as follows:

(i) Mark (✓) all rows for which assignment have not been made.

(ii) Mark (✓) column which have zeros in marked rows.

(iii) Mark (✓) rows (not already marked) which have assignment in marked columns.

(iv) Repeat step (ii) and (iii) until the chain of marking ends.

(v) Draw minimum number of lines through unmarked rows and through marked columns to cover all the zeros.

This procedure will yield the minimum number of lines (equal to the number of assignments in the maximal assignment obtained in the previous step) that will pass through all zeroes.

Transportation Problem

Dr. Nisha Gupta

(Note that here Optimal sol. is not necessarily basic)

Some important definitions:-

1. **A feasible solution:-** A feasible solution to a transportation problem is a set of non-negative individual allocations ($x_{ij} \geq 0$) which satisfies the row and column sum restrictions.

2. **Basic feasible solution:-** A feasible solution of a m by n transportation problem is said to be a basic feasible solution if the total number of positive allocations x_{ij} is exactly equal to $m+n-1$; i.e. one less than the sum of the number of rows and columns.

3. **Optimal solution:-** A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

4. **Non-degenerate basic feasible solution:-** A feasible solution of m by n transportation problem is said to be non-degenerate basic feasible solution if

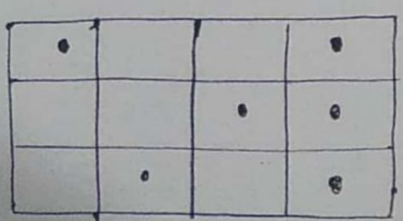
(i) Total number of positive allocations is exactly equal to $(m+n-1)$.

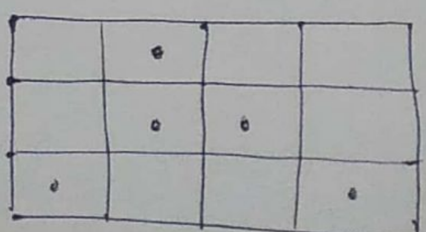
and (ii) These allocations are in independent positions.

In other words, if a F.S. involves exactly $(m+n-1)$ independent individual positive allocations, then it is known as non-degenerate B.F.S., otherwise it is said to be degenerate B.F.S.

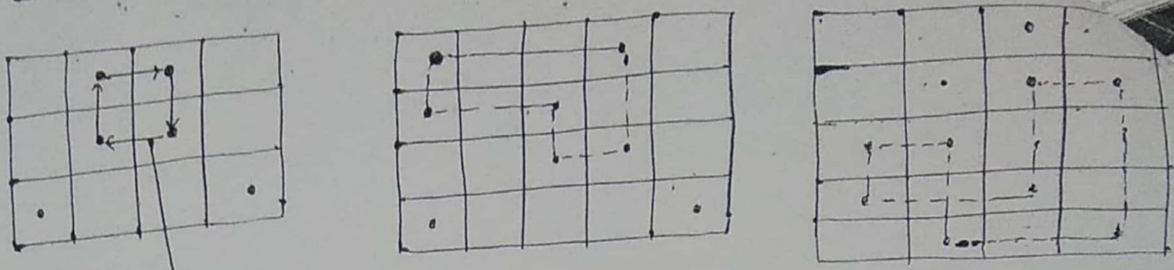
Independent and dependent positions:- By independent positions of the allocation we mean that it is always impossible to form any closed circuit (loop) by joining these allocations by horizontal and vertical lines only.

Examples of independent positions:-





Examples of Non-independent positions



Closed circuit (loop)

Solution of a transportation problem:- The solution (optimal) of a transportation problem consists of the following two steps.

Step 1:- To find an initial basic feasible solution.

Step 2:- To obtain an optimal solution by making successive improvements to initial basic feasible solution until no further decrease in the transportation cost is possible.

To find an initial ~~basic~~ Feasible Solution:- There are three simple method for finding the initial feasible solution of the given transportation problem.

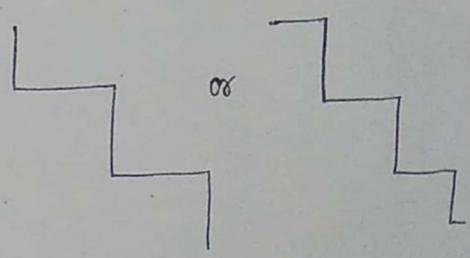
Method I:- (North-West Corner Rule)

Note:- We always get a non-degenerate B.F.S. by ~~to~~ north-west corner rule.

To

From	W_1	W_2	W_3	
F_1	5 (2)	(7)	(4)	5
F_2	2 (3)	6 (3)	(1)	8
F_3	(5)	3 (4)	4 (7)	7
F_4	(1)	(6)	14 (2)	14
	7	9	18	34

Note:-
Here we follow the rule



The total transportation cost by this method
$= 5\times2 + 2\times3 + 6\times3 + 3\times4 + 4\times7 + 14\times2$
$=$ Rs. 102

Method II:- Lowest Cost Entry Method or Method of matrix minima

To

From	W_1	W_2	W_3	
F_1	(2)	2 (7)	3 (4)	5
F_2	(3)	(3)	8 (1)	8
F_3	(5)	7 (4)	(7)	7
F_4	7 (1)	(6)	7 (2)	14
	7	9	18	

The total transportation cost by this method $= 2\times7 + 3\times4 + 8\times1 + 7\times4 + 7\times1 + 7\times2$
$=$ Rs. 83

Note:- (i) In lowest cost entry method we allocate the lowest cost entry cell as much as possible. If such cell of lowest cost is not unique, we select the cell where we can allocate more amount. Again we examine the remaining cells and follow the same procedure.

(ii) The initial b. feasible solution obtained by this method usually gives a lower transportation cost than that ~~obt~~ obtained in north west corner rule.

Method 3 (Unit Cost-Penalty Method) :- (or Vogels Approximation Method)

To

From	W_1	W_2	W_3	Available	Penalties			
F_1	5 (2)	(7)	(4)	5	(2)←			
F_2	(3)	(3)	8 (1)	8	(2)	(2)←		
F_3	~~2~~ (5)	7 (4)	(7)	7	(1)	(1)	(1)	(1)
F_4	2 (1)	2 (6)	10 (2)	14	(1)	(1)	(1)	(5)←
Demand	7	9	18					
Penalties	(1)	(1)	(1)					
	(2)	(1)	(1)					
	(4)	(2)	(5) ↑					
	(4)	(2)						

$\therefore$ The total transportation cost to this F.S.

$$= 5\times 2 + 8\times 1 + 7\times 4 + 2\times 1 + 2\times 6 + 10\times 2 = \text{Rs. } 80$$

Note:- (i) In Unit Cost-Penalty method we write the differences of the smallest and the second smallest costs ~~in~~ to find the penalties. Now we select the row or column for which the penalty is the largest and allocate the maximum possible amount to the cell with lowest cost in that particular row or column. If there are more than one largest penalty rows or columns, then we may select any one of them.

(ii) It is important to note that the Vogel's method which takes more time as compared with the other methods 1 and 2, gives the better initial F.S. So we early reach to the optimal

Optimality test :- After getting the initial F.S. of a transportation problem, we test this solution for optimality i.e; we check whether the solution thus obtained minimizes the total transportation cost or not. It is important to note that the optimality test is applicable to a F.S. consisting of $(m+n-1)$ allocations in independent position solution.

Computational procedure of optimality test :- (MODI Method)

After getting the initial B.F.S. of a transportation problem, we test solution for optimality as follows :-

(1) For a B.F.S. we determine a set of $(m+n)$ numbers

u_i, $i = 1, 2, \ldots m$

v_j, $j = 1, 2, \ldots n$

s.t., for each occupied cell, (r, s) → (where allocations take place)

$$C_{rs} = u_r + v_s$$

For this, generally we choose that u_i or $v_j = 0$ for which the corresponding row or column have the maximum numbers of individual allocations then the rest $(m+n-1)$ of them can easily be solved algebraically from the relation $C_{rs} = u_r + v_s$, for occupied cells.

(2) Then we calculate d_{ij} for each unoccupied cell (i, j) by using the formula $d_{ij} = c_{ij} - (u_i + v_j)$

(i) If all $d_{ij} > 0$, then the solution under test is optimal and unique.

(ii) If all $d_{ij} \geq 0$, with at least one $d_{ij} = 0$, then the solution under test is optimal and alternative optimal solution exists

(iii) If at least one $d_{ij} < 0$, then the sol. is not optimal.

In this case, we form a new B.F.S.. In the next B.F.S. we give maximum allocation to that cell for which d_{ij} is most negative, by making an occupied cell empty.

Then repeat the above steps starting from (1) with new B.F.S.

Que: Solve the following transportation problem ? :-

From \ To	W_1	W_2	W_3	W_4	Available units
A	6	1	9	3	70
B	11	5	2	8	55
C	10	12	4	7	90
Required units	85	35	50	45	215

Sol: By Vogel's method we find an initial B.F.S. as :-

	W_1	W_2	W_3	W_4		Penalties
A	70(6)	(1)	(9)	(3)	70	(2)
B	(11)	35(5)	20(2)	(8)	55	(3) (3) (6) ←
C	15(10)	(12)	30(4)	45(7)	90	(3) (3) (3) (3) (3).
	85	35	50	45		

Penalties	W_1	W_2	W_3	W_4
	(4) ↑	(4)	(2)	(4)
	(1)	(7) ↑	(2)	(1)
	(1)		(2)	(1)
	(10)		(4)	(7)

Total transportation cost

$= 70\times6 + 35\times5 + 20\times2 + 15\times10 + 30\times4 + 45\times7 =$

Since here $(4+3-1) = 6$, positive allocations are in the above table, so the above sol. is B.F.S. Now we check the optimality of this sol. by making the following table:-

Here we make the following table with the notation

(C_{ij}) Allocation	for occupied cells

(C_{ij}) (u_i+v_j) (d_{ij})	for unoccupied cells

where $d_{ij} = C_{ij} - (u_i+v_j)$

Here $d_{12} = -2 < 0$ so we give max. allocation to this cell from an occupied cell and make the necessary changes in other allocations

					u_i
	(6) $70-\theta$	(1) θ (3) (−2)	(9) (0) (9)	(3) (3) (0)	−4
	(11) (8) (3)	(5) $35-\theta$	(2) $20+\theta$	(8) (5) (3)	−2
	(10) $15+\theta$	(12) (7) (5)	(4) $30-\theta$	(7) 45	0
v_j	10	7	4	7	

Here $\theta = \text{Min}(70, 35, 30) = 30$

Now we make the new table:-

					u_i
	(6) 40	(1) 30	(9) (−2) (11)	(3) (3) (0)	−4
	(11) (10) (1)	(5) 5	(2) 50	(8) (7) (1)	0
	(10) 45	(12) (5) (7)	(4) (2) (2)	(7) 45	0
v_j	10	5	2	7	

Since all $d_{ij} \geq 0$, therefore the sol. given by above table is optimal. Also $d_{14} = 0$ implies alternative optimal solution exists.

∴ An optimum basic feasible is obtained as:-

From source A transport 40 and 30 units to destination W_1 and W_2 respectively.

From source B transport 5 and 50 units to W_2 and W_3 respective

From source C " 45 and 45 unit to W_1 and W_4 respective

∴ The transportation cost

$= 6\times40 + 1\times30 + 5\times5 + 2\times50 + 10\times45 + 7\times45$

$= 240 + 30 + 25 + 100 + 450 + 315$

$=$ Rs. 1160/-

Unbalanced transportation problem:-

Que: Solve the following transportation problem:-

Plant	W_1	W_2	W_3	W_4	Availability
F_1	11	20	7	8	50
F_2	21	16	10	12	40
F_3	8	12	18	9	70
Requirement	30	25	35	40	

Sol: Here total availability = 160

and total requirement = 130

Since total availability ≠ total requirement, therefore this transportation problem is unbalanced. So we convert it to a balanced transportation problem by introducing a fictitious destination W_5 with requirement 30, s.t. the cost of transportation from sources to this destination are zero.

∴ The balanced transportation problem has an initial feasible solution as (by Vogel's approximation method)

	W_1	W_2	W_3	W_4	W_5		Penalties
F_1	(11)	(20)	25(7)	25(8)	(0)	50	(7) (1) (1) (1) (1)
F_2	(21)	(16)	10(10)	(12)	30(0)	40	(10)← (2) (2) (2) (2)
F_3	30(8)	25(12)	(18)	15(9)	(0)	70	(8) (1) (1) (9)←
	30	25	35	40	30	160	
	(3)	(4)	(3)	(1)	(0)		
	(3)	(4)↑	(3)	(1)			
	(3)←↑		(3)	(1)			
			(3)	(1)			
			(3)	(4)↑			

Here are $(5+3-1) = 7$ positive allocations in the above table so the above sol. is B.F.S. Now we check for optimality of the solution as:-

	W_1	W_2	W_3	W_4	W_5	u_i
F_1	(11) (7) (4)	(20) (11) (9)	(7) 25	(8) 25	(0) (-3) (3)	-1
F_2	(21) (10) (11)	(16) (14) (2)	(10) 10	(12) (11) (1)	(0) 30	2
F_3	(8) 30	(12) 25	(18) (8) (10)	(9) 15	(0) (-2) (2)	0
v_j	8	12	8	9	-2	

Since all $d_{ij} \geq 0$, therefore the above sol. is optimal B.F.S..

Thus the optimal solution is

Transport from F_1 to W_3 and W_4, 25 units each.

Transport from F_2 to W_3 ~~and~~ 10 units.

Transport from F_3 to W_1, W_2 and W_4 ~~to~~, 30, 25 and 15 units respectively.

Total transportation cost $= 7\times25 + 8\times25 + 10\times10 + 8\times30 + 12\times25 + 9\times15$

$= \text{Rs. } 1150$ Ans

Que:- Solve the following transportation problem:-

		To		
	W_1	W_2	W_3	Availability
F_1	26	32	28	6
From F_2	19	27	16	9
F_3	39	21	32	7
F_4	18	24	23	5
Requirement	8	7	9	

Sol:- Here total availability $= 6+9+7+5 = 27$

and total requirement $= 8+7+9 = 24$

Since total availability $\neq$ total requirement, therefore this transportation problem is unbalanced. So we convert it to a balanced transportation problem by introducing a fictitious destination W_4 with requirement 3 s.t. the cost of transportation from sources to this destination are zero.

$\therefore$ By Vogel's approximation method, the balanced transportation problem has an initial feasible solution as:-

		W_1	W_2	W_3	W_4		Penalties
	F_1	3(26)	(32)	(28)	3(0)	6	(26)← (2) (2) (26)←
	F_2	(19)	(27)	9(16)	Δ(0)	9	(16) (3) (3)
From	F_3	(39)	7(21)	(32)	Δ'(0)	7	(21) (11)←
	F_4	5(18)	(24)	(23)	(0)	5	(18) (5) (5) (18)
		8	7	9	3	27	
Penalties		(1)	(3)	(7)	(0)		
		(1)	(3)	(7)			
		(1)		(7)↑			
		(8)					

Note: ~~Gered~~ Generally we give Δ's to lowest cost entry cells.

In the above table we have 5 positive allocations in independent positions which is ~~[(4+4-1)=7]~~ 2 less than $(4+4-1)=7$. Hence this solution is degenerate solution.

Now to resolve this degeneracy we allocate a very small amount Δ to the cell ~~(2,4) getting 7 all~~ (2,4) and Δ' to the cell (3,4) ~~getting~~ and get 7 allocations at independent positions.

To test solution for optimality

		To W_1	W_2	W_3	W_4	u_i
	F_1	(26) 3−θ	(32) (21) (11)	(28) (16) (12)	(0) 3+θ	0
	F_2	(19) (26) θ (−7)	(27) (21) (6)	(16) 9	(0) Δ−θ	0
From	F_3	(39) (26) (13)	(21) 7	(32) (16) (16)	(0) Δ'	0
	F_4	(18) 5	(24) (13) (11)	(23) (8) (15)	(0) (−8) (8)	−8
	v_j	26	21	16	0	

Since $d_{21} < 0$, therefore this solution is not optimal. So we give maximum allocation to the cell (2,1) and make the necessary changes in other allocations.

Here $\theta = \text{Mini}\{3, \Delta\} = \Delta$.

So enter the nonbasic cell (2, 1) and leave the basic cell (2, 4). Then we get the following table:-

To

	W_1	W_2	W_3	W_4	u_i
F_1	(26) 3	(32) (21) (11)	(28) (23) (5)	(0) (0) 3	0
F_2	(19) Δ	(27) (14) (13)	(16) 9	(0) (-7) (7)	-7
From F_3	(39) (26) (13)	(21) 7	(32) (23) (9)	(0) Δ'	0
F_4	(18) 5	(24) (13) (11)	(23) (15) (8)	(0) (-8) (8)	-8
v_j	26	21	23	0	

~~Since~~ Since all $d_{ij} \geq 0$, therefore the solution given by above table is optimal.

$\therefore$ The optimal solution is :-

From source F_1 transport 3 units to destinations W_1 and ~~W_4~~, each.

From source F_2 transport 9 units to destination W_3.

From source F_3 transport 7 units to destination W_2.

From source F_4 transport 5 units to destination W_1.

<u>Total transportation cost :-</u>

$= 26 \times 3 + 16 \times 9 + 21 \times 7 + 18 \times 5$

$= 78 + 144 + 147 + 90 = 459$ Ans.

<u>Note:-</u> (i) If Δ, Δ' are in the same row, $\Delta < \Delta'$ when Δ is to the left of Δ'.

(ii) If Δ, Δ' are in the same column, $\Delta < \Delta'$ when Δ is above Δ'.

(iii) $x_{ij} \pm \Delta = x_{ij}$

(iv) $0 + \Delta = \Delta = \Delta + 0$

(v) $0 < \Delta < x_{ij} \; \forall \; x_{ij} > 0$