Runge - kutta Method

Taylor's Series method of solving differential Equations numerically involves lot of labour in finding out the higher order derivatives.

R-K Method then was introduced and does not require the calculation of higher order derivatives and also gives greater accuracy.

R-K Method only requires the functional values at some selected points and agree with Taylor's Series solution up to term in h^r where r differs from method to method and is known as the order of that method.

(i) First Order Runge-kutta method.

Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

By Euler's method we know that

$$\boxed{y_1 = y_0 + h\,(f(x_0, y_0))}$$

Euler's method is the Runge-kutta method of first order.

(II) 2nd order Runge - kutta Method :-

$$\boxed{y_1 = y_0 + \frac{1}{2}(k_1+k_2)}$$ where $k_1 = h f(x_0, y_0)$

$$k_2 = hf(x_0+h, y_0+k_1)$$

the modified Euler's method is the Runge-kutta method of 2nd order.

(III) Third order R-K Method :-

$$y_1 = y_0 + \frac{1}{6}(k_1+4k_2+k_3)$$

where

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = h f(x_0, y_0+k')$$

$$k' = h f(x_0, h, y_0+k_1)$$

(iv) fourth order R-K Method

$$y_1 = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$k_1 = h f(x_0 y_0)$$

$$k_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$k_4 = h f(x_0+h, y_0+k_3)$$

Use Runge's method to find an approximate value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + y \quad \text{and } y = 1 \text{ when } x = 0$$

Here we have

$$x_0 = 0 \quad y_0 = 1 \quad h = 0.2 \quad f(x_0, y_0) = 1$$

$$\therefore \quad k_1 = h f(x_0, y_0) = (0.2)(1) = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2) f(0.1, 1.1) = 0.240$$

$$k' = h f(x_0 h, y_0 + k) = 0.2 f(0.2, 1.2)$$

$$= 0.280$$

and $k_3 = h f(x_0, h + y_0 + k') = (0.2 f(0.1, 1.28)$

$$= 0.296$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 4k_2 + k_3]$$

$$= \frac{1}{6}[0.200 + 0.960 + 0.296]$$

$$= 2.2426$$

Hence the required approximation value of

$$y = y_0 + k = 1 + 0.2426 = 1.2426.$$

Ex Solve $10y' = x^2+y^2$, $y(0)=1$ to evaluate $y(0.2)$ and $y(0.4)$ by fourth order R-K method.

Sol^n The given equation can be written as

$$y' = \frac{dy}{dx} = \frac{x^2+y^2}{10}$$

We have $f(x,y) = \dfrac{x^2+y^2}{10}$

To find $y(0.2)$

Here $x_0 = 0$, $y_0 = 1$ $h = 0.2$

$k_1 = h f(x_0, y_0) = 0.2 \times \frac{1}{10} = 0.02$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.2 f(0.1, 1.0103)$

$= 0.0206$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2 f(0.1, 1.0103)$

$= 0.0206$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.0206)$

$= 0.0216$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4)$

$= \frac{1}{6}[0.02 + 2 \times 0.0206 + 2 \times 0.0206 + 0.0216]$

$= 0.0207$

$y(0.2) = y_0 + k$

$y(0.2) = 0.0207 \qquad h = 0.2.$

find $y(0.4)$

Here $\quad x_1 = 0.2 \quad y_1 = 1.0207, \quad h = 0.2$

$k_1 = h f(x_1, y_1) = 0.2\, f(0.2, 1.0207)$

$= 0.0216$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= 0.2\, f(0.3, 1.0322) = 0.0231$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$= 0.2\, f(0.3, 1.0322) = 0.0231$

$k_4 = h f(x_1 + h, y_1 + k_3)$

$= 0.2\, f(0.4, 1.0438)$

$= 0.0250$

$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}(0.0216 + 2 \times 0.0231 + 2 \times 0.0231 + 0.0250)$

$= 0.0232$

$\therefore y(0.4) = y_1 + k = 1.0207 + 0.0232$

$= 1.0439$

Q. Using R-K method of forth order, solve,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} \quad \text{with} \quad y(0) = 1 \quad \text{at} \quad x = 0.2, 0.4$$

Sol^n We have

$$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$$

To find $y(0.2)$

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.2$$

Here $k_1 = h f(x_0, y_0) = 0.2 f(0,1) = 0.2000$

$$k_2 = h f(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1)$$

$$= 0.2 f(0.1, 1.0103)$$

$$= 0.19672$$

$$k_3 = h f(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2)$$

$$= 0.2 f(0.1, 1.09836)$$

$$= 0.1967$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= 0.2 f(0.2, 1.1967)$$

$$= 0.1891$$

$\therefore$ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$= 0.19599$

Hence

$$y(0.2) = y_0 + k = 1.196$$

To find $y(0.4)$

$x_1 = 0.2 \qquad y_1 = 1196 \qquad y = 0.2$

$$k_1 = h\,f(x_1, y_1) = 0.1891$$

$$k_2 = h\,f(x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}k_1)$$

$$= 0.2\,f(0.3,\ 1.2906) = 0.1795$$

$$k_3 = h\,f(x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}k_2)$$

$$= 0.2\,f(0.3,\ 1.2858)$$

$$= 0.1793$$

$$k_4 = h\,f(x_1 + h,\ y_1 + k_3)$$

$$= 0.2\,f(0.4,\ 1.3753)$$

$$= 0.1688$$

$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688]$$

$$= 0.1792$$

$$y(0.4) = y_1 + k = 1.196 + 0.1792$$

$$= 1.3752$$

$\frac{dy}{dx} = xy^{1/3}$, $y(1)=1$ find $y(1.2)$ using

R-K Method

$f(x,y) = xy^{1/3}$ $x_0 = 1$, $y_0 = 1$, and $h = 0.1$

$k_1 = h\, f(x_0, y_0)$

$= 0.1\,(1)\,(1)^{1/3} = 0.1$

$k_2 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$

$= 0.1\left(1+\frac{0.1}{2}\right)\left(1+\frac{0.1}{2}\right)^{1/3}$

$= 0.106722$

$k_3 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$

$= 0.1\left(1+\frac{0.1}{2}\right)\left(1+\frac{0.106722}{2}\right)^{1/3}$

$= 0.106835$

$k_4 = h\, f(x_0 + h,\ y_0 + k_3)$

$= 0.1\left[(1+0.1)\left(1+\frac{0.106835}{2}\right)^{1/3}\right]$

$= 0.111925$

$\therefore$ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1 + 2 \times 0.106722 + 2 \times 0.106835 + 0.111925)$

$= 0.106506$

$\therefore\ y(1.1) = y_0 + k = 1.106506$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = 1.106506$

$k_1 = h f(x_1, y_1)$

$= 0.1 (0.1) (1.106506)^{1/3}$

$= 0.010343$

$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= 0.1 \left(0.1 + \frac{0.1}{2}\right) \left(1.106506 + \frac{0.010343}{2}\right)^{1/3}$

$= 0.015539$

$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.1 \left(0.1 + \frac{0.1}{2}\right) \left(1.106506 + \frac{0.015539}{2}\right)^{1/3}$

$= 0.015551$

$k_4 = h f(x_1 + h, y_1 + k_3)$

$= 0.1 (0.1 + 0.1) (1.106506 + 0.015551)^{1/3}$

$= 0.020783$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.010343 + 2 \times 0.015539 + 2 \times 0.015551 + 0.020783)$$

$$= 0.015551$$

$$y_2 = y_1 + k = 1.106506 + 0.015551$$

$$y_2 = 1.122057$$

Using R-K method of 4th order, solve

$\frac{dy}{dx} = 3x + y^2$ with $y(1) = 1.2$ at $x = 1.1$

Find an approximate value of y when $x = 0.8$ for the particular solution of $\frac{dy}{dx} = \sqrt{x+y}$ satisfying $y = 0.41$ when $x = 0.40$ using R-K method.