Unit II

Interpolation

Introduction –

Suppose we are given the following values of $y = f(x)$ for a set of values of x:

$x : x_0 \quad x_1 \quad x_2 \ldots \ldots x_n$

$y : y_0 \quad y_1 \quad y_2 \ldots \ldots y_n$

The process of finding the values of y corresponding to any value of $x = x_i$ b/w x_0 & x_n is called interpolation.

Assumptions of Interpolation –

1) There are no sudden jump or falls in the values during the period under consideration.
2) The rise and fall in the values should be uniform.
3) When we apply calculus of finite differences, we assume that the given set of observations are capable of being expressed in a polynomial form.

Errors in polynomial Interpolation –

If the fn $f(x)$ is known explicitly, the value of y corresponding to any value of x can easily be found. If the fn $f(x)$ is not known, it is required to find a simpler fn. say $\varphi(x)$ such that $f(x)$ & $\varphi(x)$ agree at the set of tabulated points. Such a process is called Interpolation. If $\varphi(x)$ is a polynomial, then the process is called polynomial interpolation & $\varphi(x)$ is called interpolating polynomial.

Let the fn $y(x)$ defined by $(n+1)$ points (x_i, y_i) $i = 0, 1, \ldots n$ be continuous & differentiable $(n+1)$ times & let $y(x)$ be approximated by a polynomial $\varphi_n(x)$ of degree not exceeding n such that

$$\varphi_n(x_i) = y_i;\ i = 0, 1, 2 \ldots n$$

Now the problem lies in finding the accuracy of this approximation, if we use $\varphi_n(x)$ to obtain approximate value of $y(x)$ at some points other than those given.

Since the expression $y(x) - \varphi_n(x)$ vanishes for $x = x_0, x_1, \ldots x_n$ we put

$$y(x) - \varphi_n(x) = L(x-x_0)(x-x_1)\ldots(x-x_n) \quad (1)$$

where

$$= L\Pi_{n+1}(x)$$

L is to be determined such that (1) holds for any intermediate value of x say x' where $x_0 < x' < x_n$

Clearly $$L = \frac{y(x') - \varphi_n(x')}{\Pi_{n+1}(x')} \quad (2)$$

Construct a fn. $F(x) = y(x) - \varphi_n(x) - L\Pi_{n+1}(x)$ — (3)

where L is given by (2)

It is clear that

$$F(x_0) = F(x_1) = \ldots = F(x_n) = F(x') = 0$$

ie $F(x)$ vanishes $(n+2)$ times in interval $[x_0, x_n]$ Consequently, by repeated application of Rolle's theorem, $F'(x)$ must vanish $(n+1)$ times, $F''(x)$ must vanish n times in the interval $[x_0, x_n]$.

Particularly $F^{n+1}(x)$ must vanish once in $[x_0, x_n]$

Let this point be

$$x = \xi \qquad x_0 < \xi < x_n$$

Differentiating (3) $(n+1)$ time w.r.t x & put $x = \xi$, we get

Finite differences.

The Calculus of finite differences deals with the changes that take place in the value of the function (dependent variable), due to finite changes in the independent variable. Through this we also study the relations that exist b/w the values assumed by the function, whenever the independent variable changes by finite jumps whether equal or unequal. On the other hand, in infinitesimal calculus, we study those changes of the function which occur when the independent variable changes continously in a given interval. Now we will study the variations in the function when the independent variable changes by equal intervals.

Finite Differences— Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0+h, x_0+2h, \ldots\ldots x_0+nh$ giving $y = y_0, y_1, \ldots\ldots y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three type of differences are found useful:

① **Forward differences**— The differences $y_1 - y_0;\ y_2 - y_1; \ldots\ y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \ldots \Delta y_{n-1}$, respectively are called the first forward differences where Δ is the forward difference operator. Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

[1st forward difference is $\Delta y_r = y_{r+1} - y_r$]

In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$

defines the pth forward differences.

These differences are systematically set out as follows in what is called a forward difference Table.

value of x	value of y	1st differ.	2nd differ.	3rd differ.	4th differ.	5th differ
x_0	y_0					
		Δy_0				
x_0+h	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_0+2h	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_0+3h	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_0+4h	y_4		$\Delta^2 y_3$			
		Δy_4.				
x_0+5h.	y_5					

x is called the argument & y the function or the entry. y_0, the first entry is called the leading term & $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the leading differences.

Note ①: Any higher order forward difference can be expressed in terms of the entries. we have

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$$
$$= \Delta y_2 - 2\Delta y_1 + \Delta y_0$$
$$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

$= y_2 - 2y_1 + y_0.$

The coefficients occuring on the right hand side being the binomial coefficients, we have in general,

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^n y_0.$$

Note ②: The operator Δ obeys the distributive, commutative & index laws. i.e

(i) $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

(ii) $\Delta[cf(x)] = c\Delta f(x)$; c is a constant.

(iii) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$, m, n are +ve constant integers.

(iv) $\Delta[f(x).g(x)] \neq \Delta f(x).\Delta g(x)$.

② <u>Backward differences</u> – The differences $y_1 - y_0, y_2 - y_1, \ldots y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots \nabla y_n$, respectively, are called the first backward differences where ∇ is the backward difference operator. Similarly we define higher order backward differences. Thus, we have

$$\nabla y_r = y_r - y_{r-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \quad \text{etc.}$$

The backward difference table is.

Value of x	Value of y	1st differ.	2nd differ.	3rd differ.	4th differ.	5th differ.
x_0	y_0					
		∇y_1				
x_0+h	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_0+2h	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_0+3h	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_0+4h	y_4		$\nabla^2 y_5$			
		∇y_5				
x_0+5h	y_5					

③ <u>Central differences</u> — The central differences operator δ is defined by the relation

$$y_1 - y_0 = \delta y_{1/2} \,;\quad y_2 - y_1 = \delta y_{3/2} \,;\ \ldots\ldots \,;\ y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$$

similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

$$\delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2 \ \ldots\ldots$$

$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ \& so on.}$$

These differences are shown in the following:

Value of x	Value of y	1st differ.	2nd differ.	3rd differ.	4th differ	5th differ.
x_0	y_0					
		$\delta y_{1/2}$				
x_0+h	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_0+2h	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_0+3h	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_0+4h	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_0+5h	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix & those of even order for only integral values of the suffix. To find the mean of adjacent values in the same column of differences. We denote this mean by μ. Thus

$$\mu\delta y_1 = \frac{1}{2}(\delta y_{1/2} + \delta y_{3/2})$$

$$\mu\delta^2 y_{3/2} = \frac{1}{2}(\delta^2 y_1 + \delta^2 y_2)$$

Note: It is only the notation which changes & not the differences. e.g:

$$y_1 - y_0 = \nabla y_0 = \Delta y_1 = \delta y_{1/2}.$$

eg: Evaluate (i) $\Delta \tan^{-1} x$ (ii) $\Delta(e^x \log 2x)$

Solu: (i) $\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right] = \tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$$

(ii) $\Delta(e^x \log 2x) = e^{x+h}\log 2(x+h) - e^x \log 2x$

$$= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x$$

$$= e^{x+h}\log\left(\frac{x+h}{x}\right) + (e^{x+h} - e^x)\log 2x$$

$$= e^x\left[e^h \log\left(\frac{x+h}{x}\right) + (e^h - 1)\log 2x\right].$$

$$= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + (e^h - 1)\log 2x\right].$$

eg: Form a table of differences for the fn $f(x) = x^3 + 5x - 7$; for $x = -1, 0, 1, 2, 3, 4, 5$.

continue the table to obtain $f(6)$.

Solu: $f(-1) = -13$ $\quad f(3) = 35$

$f(0) = -7$ $\quad f(4) = 77$

$f(1) = -1$ $\quad f(5) = 143$.

$f(2) = 11$

x	y	1st diff	2nd diff	3rd diff	4th diff	5th diff.
-1	-13					
		6				
			0			
				6		
0	-7				0	
		6				
						0
			6			
1	-1					
				6		
		12				
					0	
			12			
2	11					
				6		0
		24				
					0	
			18			
3	35					
		42		6		
			24			
4	77					
		66				
5	143					

① Newton's forward Interpolation formula –

Let the function $y = f(x)$ take the values $y_0, y_1 \dots y_n$ corresponding to the values $x_0, x_1, \dots x_n$ of x. Let these values of x be equispaced such that $x_i = x_0 + ih$; $i = 0, 1, 2 \dots$

Assuming $y(x)$ to be a polynomial of nth degree in x such that $y(x_0) = y_0$; $y_0(x_1) = y_1$; $y(x_2) = y_2$; .... $y(x_n) = y_n$.

we can write.

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad \text{— ①}$$

Putting $x = x_0, x_1, x_2 \dots x_n$ successively in ①

$$y_0 = a_0 \quad \text{—①}$$

$$y_1 = a_0 + a_1(x_1 - x_0) \quad \text{—②}$$

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \text{ —③} \text{ \& so on}$$

from these, we find that

$$a_0 = y_0$$

$$\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$$

$$\therefore a_1 = \frac{\Delta y_0}{h}$$

from ③

$$y_2 - a_0 - a_1(x_2 - x_0) = a_2 \times 2h \times h$$

$$y_2 - y_0 \mp \frac{\Delta y_0}{h} \times 2h = 2h^2 a_2$$

$$(y_2 - y_1) + (y_1 - y_0) \mp 2\Delta y_0 = 2h^2 a_2$$

$$\Delta y_1 + \Delta y_0 - 2\Delta y_0 = 2h^2 a_2$$

$$a_2 = \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

Also $\Delta y_1 = y_2 - y_1 = a_2(x_2 - x_0)(x_2 - x_1) \pm a_1(x_2 - x_0) - a_1(x_1 - x_0)$

$$= a_2(x_2 - x_0)(x_2 - x_1) + a_1(x_2 - x_1)$$

$$= a_1 h + a_2(2h)(h) = a_1 h + 2h^2 a_2.$$

Putting value of a_1 in this eq.

$$\Delta y_1 = a_1 h + 2h^2 a_2 = \frac{\Delta y_0}{h} h + 2h^2 a_2$$

$$\therefore 2h^2 a_2 = \Delta y_1 - \Delta y_0$$

$$a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!\,h^2}\Delta^2 y_0$$

$$a_3 = \frac{1}{3!\,h^3}\Delta^3 y_0 \quad \text{\& so on.}$$

Substituting these values of a_i's in ①, we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x-x_0)(x-x_1) + \dots\dots + \dots\dots \qquad —②$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$x - x_0 = ph,$$

$$x - x_1 = x - x_0 - (x_1 - x_0) = ph - h = (p-1)h$$

$$x - x_2 = (x - x_1) - (x_2 - x_1) = (p-1)h - h$$
$$= (p-2)h \text{ etc.}$$

Hence, writing $y(x) = y(x_0 + ph) = y_p$, ② becomes

$$y(x) = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots +$$

$$+ \frac{p(p-1)\dots(p-(n-1))}{n!}\Delta^n y_0 \qquad —③$$

It is called Newton's forward interpolation formula as ③ contains y_0 & the forward differences of y_0.

e.g: Using Newton's forward formula, find the value of $f(1.6)$ if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

Solⁿ: The difference table is as under.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
1	3.49				
		1.33			
1.4	4.82		−0.19		
		1.14		−0.41	
1.8	5.96		−0.6		
		0.54			
2.2	6.5				

$\therefore\ p = \dfrac{x - x_0}{h} = \dfrac{1.6 - 1.4}{0.4} = 0.5$

$$y(1.6) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0.$$

$$= 4.82 + 0.5 \times 1.14 + \frac{0.5(0.5-1)}{2!} \times (-0.6)$$ -0.5

$$= 4.82 + 0.57 + \frac{(-0.25)}{2!}(-0.6)$$

$$= 4.82 + 0.57 + \frac{0.15}{2!} = 4.82 + 0.57 + 0.075$$

$$= 5.465.$$

e.g: Estimate the values of $f(22)$ & $f(42)$ from the following data

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204.

Solu: The difference table is

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
20	354					
		-22				
			-19			
25	332			29		
		-41			-37	
30	291		+10			45.
		-31		-8		
35	260		2.		8	
		-29		0		
40	231		2.			
		-27				
45	204					

(i) To find $f(22)$. ⑨

$x_0 = 20$ $\quad h = 5$ $\quad \Delta = 22$ $\quad \Delta^3 = 29$ $\quad \Delta^5 = 45$.

$x = 22$ $\quad y_0 = 354$ $\quad \Delta^2 = -19$ $\quad \Delta^4 = -37$ $\quad p = \frac{22-20}{5} = 0.4$

$$f(22) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0$$

$$= 354 + 0.4 \times -22 + \frac{0.4(0.4-1)(-19)}{2!} + \frac{0.4(0.4-1)(0.4-2)\times 2}{3!}$$

$-0.6 \times -1.6 \times -2.6$ -3.6

$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} \times (-37) + \frac{(0.4)(0.4-1)(0.4-(0.4-3)(0.4-4)}{5!} \times 4$$

$$= 354 - 8.8 + 2.28 + 1.856 + 1.5392 + 1.34784$$

$$= 352.22 \simeq 352.$$

(ii) To find $f(42)$.

$x_0 = 40$ $\quad h = 5$

$y_0 = 231$ $\quad p = \frac{42-40}{5} = 0.4$

$\Delta = -27$.

$$f(42) = y_0 + p\Delta y_0$$
$$= 231 + 0.4 \times (-27)$$
$$= 231 - 10.8 = 220.2.$$

we can also use Newton's backward Interpolation formula to solve part (2) i.e to calculate $f(42)$.

② Newton's backward Interpolation formula – Let the function $y=f(x)$ take the values $y_0, y_1, y_2 \dots$ corresponding to the values $x_0, x_0+h, x_0+2h, \dots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n+ph) = E^p f(x_n) = (1-\nabla)^{-p} y_n \qquad (\because E^{-1} = (1-\nabla))$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \dots\right] y_n$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots \quad -①$$

It is called Newton's backward interpolation formula as ① contains y_n & backward differences of y_n.

Note: Newton's forward Inter. formula is used for interpolating the values of y near the beginning of a set of tabulated values & Newton's backward Inter. formula is used for interpolating the values of y near the end of a set of tabulated values.

e.g: The area A of a circle of diameter d is given for the following values:

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

Calculate area of a circle of diameter 105.

Solu: The difference table is given below

$x_n = 100$

$x = 105$

$$p = \frac{x - x_n}{h} = \frac{105-100}{5} = 1.$$

$$\nabla y_n = 766, \quad \nabla^2 y_n = 40, \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 4$$

d	A	Δ	Δ^2	Δ^3	Δ^4
80	5026				
		648	40		
85	5674			-2	
		688			4.
			38		
90	6362			2	
		726			
95	7088		40		
		766			
100	7854.				

$$\therefore\ y(105) = y(100) + p\nabla y(100) + \frac{p(p+1)}{2!}\nabla^2 y(100) + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y(100)$$

$$= 7854 + 1\times 766 + \frac{\cancel{1\times 2}}{\cancel{2!}}\times 40 + \frac{\cancel{1\times 2\times 3}}{\cancel{3!}}\times 2 + \frac{\cancel{1.2.3.4}}{\cancel{4!}}\times 4$$

$$= 7854 + 766 + 40 + 2 + 4 = 8666.$$

e.g : construct Newton's backward Interpolation poly-
-nomial for the data

x :	1	2	3	4	5
y :	1	-1	1	-1	1

The Newton's backward Interpolation poly can be obtained using difference table.

	x	y	Δ	Δ^2	Δ^3	Δ^4
	1	1	-2			
				4		
	2	-1			-8	16.
			2			
		1		-4		
y_0	3		-2		8	
	4	-1		4		
			2			
	5	1				

The Newton's backward formula is

$$y(p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

Here $y_n = 1.$

$x_n = 5$

$p = \frac{x-5}{1} = (x-5).$

$$y(x) = 1 + (x-5).2 + \frac{(x-5)(x-4)\cancel{4}^{2}}{\cancel{2!}} + \frac{(x-5)(x-4)(x-3)\times 8}{3!} + \frac{(x-5)(x-4)(x-3)(x-2)}{4!}\times 16$$

$$= 1 + 2x - 10 + 2(x^2-9x+20) + \frac{4}{3}(x^2-9x+20)(x-3) + \frac{2}{3}(x^2-9x+20)(x^2-5x+6)$$

$$= 1 + 2x - 10 + 2x^2 - 18x + 40 + \frac{4}{3}(x^3 - 3x^2 - 9x^2 + 27 + 20x - 60) + \frac{2}{3}(x^4 - 5x^3 + 6x^2 - 9x^3 + 45x^2 - 54 + 20x^2 - 100x + 120)$$

$$= 1 + 2x - 10 + 2x^2 - 18x + 40 + \frac{4}{3}x^3 - 4x^2 - 12x^2 + 36 + \frac{80}{3}x - 80 + \frac{2}{3}x^4 - \frac{10}{3}x^3 + 4x^2 - 6x^3 + 30x^2 - 36 + \frac{40}{3}x^2 - \frac{200}{3}x + 80$$

$y = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31.$ is the req. polynomial

③ Newton's Divided difference interpolation formula –

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$. Then from the definition of the divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}.$$

$$\therefore \quad y = y_0 + (x - x_0)[x, x_0] \quad — ①$$

Again $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}.$$

which gives $$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in ①, we get

$$y = y_0 + (x - x_0)\big[[x_0, x_1] + (x - x_1)[x, x_0, x_1]\big]$$

$$= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]. \quad — ②$$

Also

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives

$$[x, x_0, x_1] = (x - x_2)[x, x_0, x_1, x_2] + [x_0, x_1, x_2]$$

substituting in ②

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)\{(x - x_2)[x, x_0, x_1, x_2] + [x_0, x_1, x_2]\}.$$

$$= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] + (x - x_0)(x - x_1)[x_0, x_1, x_2].$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots + (x - x_0)(x - x_1)\ldots(x - x_n)[x, x_0, x_1, \ldots, x_n] \quad — ③$$

which is called Newton's general interpolation formula with divided difference

e.g: Use Newton's divided difference formula to find $f(x)$ from the following data.

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

Solu: The divided difference table is

	x	$f(x)$	1st div. differ.	2nd div. differ.	3rd div. diff.	4th div differ
x_0	0	y_0 1				
			13			
x_1	1	y_1 14		-6		
			1		1	
x_2	2	y_2 15		-2		0
			-5		1	
x_3	4	y_3 5		2		0
			1		1	
x_4	5	y_4 6		6		
			13			
x_5	6	y_5 19				

$[x_0 x_1] = 13$

$[x_1 x_2] = 1$

$[x_0 x_1 x_2] = -6$

$[x_1 x_2 x_3] = -2$

$$f(x) = f(x_0) + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3)[x, x_0, x_1, x_2, x_3].$$

$$= 1 + (x-0)\times 13 + (x-0)(x-1)\times -6 + (x-0)(x-1)(x-2)\times 1$$

$$= 1 + (x)\times 13 + (x^2-x)\times -6 + x(x^2-3x+2)$$

$$= 1 + 13x - 6x^2 + 6x + x^3 - 3x^2 + 2x$$

$$= x^3 - 9x^2 + 21x + 1.$$

e.g: Using Newton's divided difference formula, evaluate $f(8)$ for the data given as

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

The difference table is

x	$f(x)$	Ist	IInd	3rd	4th
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

$294 + (x-7)\times 202 + (x-7)(x-10)\times 27 + (x-7)(x-10)(x-11)\times 1$

$f(8) = 294 + 202 + 27\times(-2) + (-2)(-3)\times 1$

$= 294 + 202 - 54 + 6$

$= 502 - 54 = 448.$

$\therefore f(x) = f(x_0) + [x_0, x_1](x-x_0) + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \ldots$

$= 48 + (x-4)[52] + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)\times 1$

$= 48 + (x-4)\times 52 + (x^2-9x+20)\times 15 + (x^3 - 9x^2 + 20x - 7x^2 + 63x - 140)$

$= 48 + 52x - 208 + 15x^2 - 135x + 300 + x^3 - 9x^2 + 20x - 7x^2 + 63x - 140$

$f(x) = x^3 - x^2$ is the polynomial

$f(8) = (8)^3 - (8)^2 = 448.$

(4) <u>Lagrange's Interpolation formula for unequal differences</u>–

If $y = f(x)$ be a function which takes the values (x_0, y_0) $(x_1, y_1), \ldots (x_n, y_n)$. Since there are $(n+1)$ pairs of values of x & y $\therefore$ we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form.

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n) + a_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n) + \ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}). \quad \text{—(1)}$$

Putting $x = x_0, y = y_0$ in this eq, we get

$$y_0 = a_0 (x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)$$

$$\therefore a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

Similarly putting $x = x_1, y = y_1$ we get

$$a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

Proceeding the same way, we get $a_2, a_3 \dots a_n$.

Putting these values of a_i's in the eq ①, we get

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1$$

$$+ \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

This is known as Lagrange's Interpolation formula.

Note ①: It is easy to remember, but quite cubersome to apply.

②: This formula can also be used to split the given function into partial fraction. If we divide both sides of ① by $(x-x_0)(x-x_1)\dots(x-x_n)$, we get

$$\frac{f(x)}{(x-x_0)(x-x_1)\dots(x-x_n)} = \frac{y_0}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \cdot \frac{1}{x-x_0} +$$

$$\frac{y_1}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} \cdot \frac{1}{x-x_1} +$$

$$\dots + \frac{y_n}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} \frac{1}{x-x_n}.$$

① —

given data find $y(10)$, using Lagrange's Interpolation formula.

x : 5 6 9 11 (x_0, x_1, x_2, x_3)

y : 12 13 14 16 (y_0, y_1, y_2, y_3)

Solu :
$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{\overbrace{(x-6)(x-9)}^{(x^2-15x+54)}(x-11)}{(-1)(-4)(-6)} \times 12 + \frac{\overbrace{(x-5)(x-9)}^{(x^2-14x+45)}(x-11)}{1 \times -3 \times -5} \times 13$$

$$+ \frac{\overbrace{(x-5)(x-6)}^{(x^2-11x+30)}(x-11)}{4 \times 3 \times -2} \times 14 + \frac{\overbrace{(x-5)(x-6)}^{(x^2-11x+30)}(x-9)}{6 \times 5 \times 2} \times 16$$

$$= \frac{\overbrace{(x-6)(x-9)(x-11)}^{x^3-26x^2+219x-594}}{-2} + \frac{\overbrace{(x-5)(x-9)(x-11)}^{x^3-25x^2+199x-495}}{} \times \frac{13}{15} +$$

$$-\frac{7}{12}\overbrace{(x-5)(x-6)(x-11)}^{x^3-22x^2+151x-330} + \frac{4}{15}\overbrace{(x-5)(x-6)(x-9)}^{x^3-20x^2+129x-270}$$

This is the polynomial. Now we will find $y(10)$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{-2} + \frac{(10-5)(10-9)(10-11)}{15} \times 13 -$$

$$\frac{7}{12}(10-5)(10-6)(10-11) + \frac{4}{15}(10-5)(10-6)(10-$$

$$= \frac{\cancel{4}^{2} \times 1 \times (-1)}{\cancel{-2}} + \frac{\cancel{5} \times 1 \times (-1)}{\cancel{15}_3} \times 13 - \frac{7}{\cancel{12}_3} \times 5 \times 4 \times (-1) +$$

$$\frac{4}{\cancel{15}_3}(\cancel{5}) \times 4 \times 1.$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{6-13+35+16}{3} = \frac{44}{3} = 14.6$$

−11 −15 +165 54 +99 +30 +154 −45 104 +121 −30 +135 +45 45 13