Spline Interpolation

Linear spline interpolation is same as linear piecewise interpolation.

(Spline Interpolation)

Def^n A spline function of degree n with knots (nodes) $x_i = i = 0, 1, \dots n$ is a function $F(x)$ satisfying the properties.

(i) $F(x_i) = f(x_i)$ $i = 0, 1, \dots n$.

(ii) On each subinterval $[[x_i - 1], x_i]$, $1 \le i \le n$, $F(x)$ is a polynomial of degree n.

(iii) $F(x)$ and its first $(n-1)$ derivatives are continuous on (a, b)

Quadratic spline interpolation —

A quadratic spline satisfies the following properties

(i) $F(x_i) = f(x_i)$ $i = 0, 1, \dots n$

(ii) On each subinterval $[x_{i-1}, x_i]$ $1 \le i \le n$ $F(x)$ is a 2nd degree polynomial, except in the first or the last interval

(iii) $F(x)$ and $F'(x_i)$ are continuous on (a, b)

Ex Given the data

x	0	1	2	3
$f(x)$	1	2	33	244

Fit quadratic splines with $M(0) = f''(0) = 0$. Hence, find an estimate of $f(2.5)$

Sol^n We write the spline approximation as

$$P_1(x) = a_1 x_2 + b_1 x + c_1 \quad 0 \le x \le 1$$

$$P_2(x) = a_2 x_2 + b_2 x + c_2 \quad 1 \le x \le 2$$

$$P_3(x) = a_3 y_2 + b_3 x + c_3 \quad 2 \le x \le 3$$

Since $M(0) = f''(0) = 0$ we get $a_1 = 0$

Substituting $x_0 = 0$, $x_1 = 2$, $x_2 = 2$, $x_3 = 3$

$f_0 = 1$ $f_1 = 2$ $f_2 = 33$ $f_3 = 244$

in equation

$$\left.\begin{aligned} b_1 x_0 + c_1 &= f_0 \\ b_1 x_1 + c_1 &= f_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} a_2 x_1^2 + b_2 x_1 + c_2 &= f_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 &= f_2 \\ 2a_2 x_1 + b_2 &= 2a_1 x_1 + b_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} a_3 x_2^2 + b_3 x_2 + c_3 &= f_2 \\ 2a_3 x_2 + b_3 &= 2a_2 x_3 + b_2 \\ a_3 x_3^2 + b_3 x_3 + c_3 &= f_3 \end{aligned}\right\}$$

We obtain.

$$\left.\begin{aligned} b_1(0) + c_1 &= 1 \\ b_i + c_i &= 2 \end{aligned}\right\}$$

$$\left.\begin{aligned} a_2 + b_2 + c_2 &= 2 \\ 4a_2 + 2b_2 + c_2 &= 33 \\ 2a_2 + b_2 &= 2a_1 + b_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} 4a_3 + 2b_2 + c_2 &= 2 \\ 4a_2 + 3b_3 + c_3 &= 244 \\ 4a_3 + b_3 &= 4a_2 + b_2 \end{aligned}\right\}$$

Solving the first system we get

$c_1 = 1 \qquad b_1 = 1$

the Second system becomes

$$a_2 + b_2 + c_2 = 2$$
$$4a_2 + 2b_2 + c_2 = 33$$
$$2a_2 + b_2 = 1$$

The Sol^n to this system is

$a_2 = 30, \quad b_2 = -59 \qquad c_2 = 31$

The third system becomes

$$4a_3 + 2b_3 + c_3 = 33$$
$$9a_3 + 3b_3 + c_3 = 244$$
$$4a_3 + b_3 = 61$$

The Solution to this system is

$a_3 = 150 \qquad b_3 = -539 \qquad c_3 = 511$

Therefore, the quadratic splines in the corresponding intervals can be written as

$$P_1(x) = x+1 \qquad 0 \le x < 1$$

$$P_2(x) = 30x^2 - 59x + 31 \qquad 1 \le x \le 2$$

$$P_3(x) = 150x^2 - 539x + 511 \qquad 2 \le x \le 3$$

An estimate at 2.5 is

$$f(2.5) = P_3(2.5) = 150(2.5)^2 - 539(2.5) + 511$$

$$f(2.5) = 101$$

Cubic spline Interpolation:-

A Cubic spline satisfies the following properties

(i) $F(x_i) = f_i \quad i = 0, 1, -- n$

(ii) on Each subinterval $[x_{i-1}, x_i]$ $1 \le i \le n$, $F(x)$ is a third degree polynomial.

(iii) $F(x)$, $F'(x)$ and $F''(x)$ are continous on (a, b)

Working steps:-

(1) $M_0 = M_n = 0$ (natural splines)

(ii) $M_0 = M_n$, $M_1 = M_{n+1}$ $f_0 = f_n$, $f_1 = f_{n+1}$

$h_1 = h_{n+1}$

(A spline satisfying these conditions is called a periodic spline)

(iii) For a non-periodic spline we use the conditions.

$F'(a) = f'(a) = f_0'$ and

$F'(b) = f'(b) = f_n'$

and using Eq^n

$$2M_0 + M_1 = \frac{6}{h_i}\left(\frac{f_i - f_0}{h_1} f_0'\right)$$

$$M_{n-1} + 2M_n = \frac{6}{h_n}\left(f_n' - \frac{f_n - f_{n-1}}{h_n}\right)$$

(iv) for equispaced knots $h_i = h$ for all i,

$$F(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + 4(x - x_{i-1})^3 M_1\right]$$
$$+ \frac{1}{h}(x_i - x)\left(f_{i-1} - \frac{h^2}{6}M_{i-1}\right)$$
$$+ \frac{1}{h}(x - x_{i-1})\left(f_i - \frac{h^2}{6}M_i\right)$$

and $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1})$

obtain the Cubic spline approximation for the function defined by the data.

x	0	1	2	3
$f(x)$	1	2	33	244

with $M(0) = 0$, $M(3) = 0$ Hence find estimate of $f(2.5)$

Sol" Since the points are equispaced with $h = 1$

We obtain from

$$M_{i-1} + 4M_i + M_{i+1} = 6(f_{i+1} - 2f_i + f_{i-1}) \quad i = 1, 2$$

Therefore $M_0 + 4M_1 + M_2 = 6(f_2 - 2f_1 + f_0)$

$$M_1 + 4M_2 + M_3 = 6(f_3 - 2f_2 + f_1)$$

Using $M_0 = 0$ $M_3 = 0$ and the given function values We get

$$4M_1 + M_2 = 6(33 - 4 + 1) = 180$$

$$M_1 + 4M_2 = 6(244 - 66 + 2) = 1080$$

which gives

$$M_1 = -24$$

$$M_2 = 276$$

Cubic splines in the corresponding interval are obtained as follows. on $[0, 1]$

$$F(x) = \frac{1}{6}\left[(1-x)^3 M_0 + (x-0)^3 M_1\right] + (1-x)\left(f_0 - \frac{1}{6}M_0\right)$$

$$+ (x-0)\left(f_1 - \frac{1}{6}M_1\right)$$

$= \frac{1}{6} x^3 (-24) + (1-x) + x \left[2 - \frac{1}{6}(-24)\right]$

$= -4x^3 + 5x + 1$

on $[1, 2]$

$F(x) = \frac{1}{x}\left[(2-x)^3 M_1 + (x-1)^3 M_2\right] + (2-x)\left(f_1 - \frac{1}{6} M_i\right)$

$+ (x-1)\left(f_2 - \frac{1}{6} M_2\right)$

$= \frac{1}{6}\left[(2-x)^3(-24) + (x-1)^3(276)\right]$

$+ (2-x)\left[2 - \frac{1}{6}(-24)\right]$

$+ (x-1)\left[33 - \frac{1}{6}(276)\right]$

$= \frac{1}{6}\left[(8 - 12x + 6x^2 - x^3)(-24)\right.$

$\left. + (x^3 - 3x^2 + 3x - 1)(276)\right]$

$+ 6(2-x) - 13(x-1)$

$= 50x^3 - 162x^2 + 167x - 53$

on $[2, 3]$

$F(x) = \frac{1}{6}\left[(3-x)^3 M_2 + (x-2)^2 M_3\right]$

$+ (3-x)\left\{f_2 - \frac{1}{6} M_2\right) + (x-2)\left(f_3 - \frac{1}{6} M_3\right)$

$= \frac{1}{6}\left[(27 - 27x + 9x^2 - x^3)(276)\right]$

$+ (3-x)\left[33 - \frac{1}{6}(276)\right] + (x-2)[244]$

$= -46x^3 + 414x^2 - 985 + 715$

As estimate at 2.5 is

$f(2.5) = P_3(2.5) = -46(2.5)^3 + 414(2.5)^2$

$- 985(2.5) + 715$

$f(2.5) = 121.25$

Ex find $S(1.6)$ by Cubic spline Method for the given data if $M_0 = 0 = M_4 = 0$

x	0	$\pi/4$	$\pi/2$	$3/4\pi$	π
y	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0

Sol^n In this case the system of Eq^n

$M_{j-1} + 4M_j + M_{j+1} = \frac{6}{h^2}\left[y_{j+1} - 2y_j + y_{j-1}\right]$

where $j = 1, 2, 3$ .. we get

$4M_1 + M_2 = -4$ $\quad [\because M_0 = 0,\ M_4 = 0]$

$M_1 + 4M_2 + 4M_3 = -5.699$

On solving we get

$M_1 = -0.7440$ $\quad M_2 = -1.053$

$M_3 = -0.7440$

for the interval $[x_0, x_1]$ i.e $[0, \frac{\pi}{4}]$, the spline is given by

$$P_1(x) = \frac{1}{6h}\left[(x_1-x_0)^3 M_0 + (x-x_0)^3 M_1\right]$$
$$+ \frac{1}{h}[x_1-x]\left[y_0 - \frac{h^2}{6}M_0\right]$$
$$+ \frac{1}{h}[x-x_0]\left[y_1 - \frac{h^2}{6}M_1\right]$$

Putting $x_0 = 0$ $\quad x_1 = \frac{\pi}{4}$ $\quad M_0 = 0$

$M_1 = -0.7440$

$y_1 = \frac{1}{\sqrt{2}}$

we get

$$P_1(x) = \frac{4}{\pi}\left[-0.124x^3 + 0.7836x\right]$$

and $S_1\left(\frac{\pi}{6}\right) = 0.4998$ It means for the better approximation the interval should be small.

x. find whether the following function are splines or not.

) $f(x) = \begin{cases} x^2 - x + 1 & 1 \le x \le 2 \\ 3x - 3 & 2 \le x \le 3 \end{cases}$

i) $f(x) = \begin{cases} -x^2 - 2x^3 & -1 \le x \le 0 \\ -x^2 + 2x^3 & 0 \le x \le 1 \end{cases}$

$f(x) = \begin{cases} -x^2 - 2x^3 & -1 \le x \le 0 \\ x^2 + 2x^3 & 0 \le x \le 1 \end{cases}$