Doolittle method :-

We write the given matrix $A = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Ex Solve the following set of equation by using the Doolittle method.

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Sol$^n$ Let $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Thus we have $u_{11} = 2$, $u_{12} = 3$, $u_{13} = 1$

$l_{21} u_{11} = 1 \Rightarrow l_{21} = \frac{1}{2}$ $\quad$ $l_{31} u_{11} = 3$

$l_{31} = \frac{3}{2}$

$l_{21} u_{12} + u_{22} = 2 \Rightarrow u_{22} = \frac{1}{2}$

$l_{21} u_{13} + u_{23} = 3 \Rightarrow u_{23} = \frac{5}{2}$

$l_{31} u_{12} + l_{32} u_{22} = 1 \Rightarrow l_{32} = -7$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 2 \Rightarrow u_{33} = 18$

Hence we can now write $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$

further the given system can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad \text{— (1)}$$

where $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{— (2)}$$

Now Relation (1) gives $\boxed{y_1 = 9}$

$(y_1/2) + y_2 = 6$

i.e $\boxed{y_2 = \frac{3}{2}}$

$\frac{3}{2} y_1 - 7y_2 + y_3 = 8$

$\Rightarrow \boxed{y_3 = 5}$

Now Using (2) solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$\Rightarrow 2x + 3y + z = 9$, $\frac{y}{2} + \frac{5z}{2} = \frac{3}{2}$ and $18z = 5$

$\boxed{x = \frac{35}{18}}$, $\boxed{y = \frac{29}{18}}$ and $\boxed{z = \frac{5}{18}}$

Ex Apply Cholesky's method to find the inverse of matrix

$$4x_1 + 2x_2 + 6x_3 = 16$$
$$2x_1 + 82x_2 + 39x_3 = 206$$
$$6x_1 + 39x_2 + 26x_3 = 113$$

Solⁿ $AX = B$

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 82 & 39 \\ 6 & 39 & 26 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 16 \\ 206 \\ 113 \end{bmatrix}$$

Note that the coefficient matrix A is symmetric and positive definite and hence It can be written as

$$LL^T = A$$

Let $$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Therefore $$LL^T = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} \begin{bmatrix} 4 & 2 & 6 \\ 2 & 82 & 39 \\ 6 & 39 & 26 \end{bmatrix}$$

Comparing both sides, we get the following system of Eqⁿ

$l_{11}^2 = 4$ or $l_{11} = 2$

$l_{11} l_{21} = 2$ or $l_{21} = 1$

$l_{11} l_{31} = 6$ or $l_{31} = 3$

$l_{21}^2 + l_{22}^2 = 82$ or $l_{22} = (82-1)^{1/2} = 9$

$l_{31} l_{21} + l_{32} l_{22} = 39$ or $l_{32} = \frac{1}{l_{22}}(39 - l_{31} l_{21}) = 4$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 26$ or $l_{33} = (26 - l_{31}^2 - l_{32}^2) = 1$

Therefore $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

Now the system of Equation $LZ = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 206 \\ 113 \end{bmatrix}$$

The solution of these Equation is

$z_1 = 8.0$ $z_2 = 22.0$ $z_3 = 1.0$

$\boxed{z_1 = 8 \quad z_2 = 22 \quad z_3 = 1}$

Now from the Equation

$L^T X = Z$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 9 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 22 \\ 1 \end{bmatrix}$$

$2x_1 + x_2 + 3x_3 = 8$

$9x_2 + 4x_3 = 22$

$x_3 = 1$

Solution to these Equation is ~~[illegible]~~,

$x_3 = 1$, $x_2 = 2$ and $x_1 = 1.5$

Hence the Solution $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 1.0$

## Cholesky Method :-

~~The matrix A can be written~~ as

~~$A = LL^T$~~

The Cholesky method is used to solve a system of linear eqⁿ $Ax=b$ if the coefficient matrix A is symmetric and positive definite.

then A can be written as a product of lower triangular matrix and its transpose. That is,

$$A = LL^t$$

**Ex** Solve
$$25x + 15y - 5z = 35$$
$$15x + 18y + 0z = 33$$
$$-5x + 0y + 11z = 6$$

Now Converting given eqⁿ into matrix form.

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

Now $A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$

Here matrix is symmetric positive definite

So cholesky decomposition is possible.

$$A = L L^T$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{31} l_{11} & l_{31} l_{21} + l_{32} l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing both sides, we get the following system of Equations

$$l_{11}^2 = 25 \Rightarrow \boxed{l_{11} = 5}$$

$$l_{11} l_{21} = 15 \Rightarrow 5 l_{21} = 15 \Rightarrow \boxed{l_{21} = 3}$$

$$l_{21}^2 + l_{22}^2 = 18 = l_{22}^2 = 18 - 9 \Rightarrow l_{22}^2 = 9 \Rightarrow \boxed{l_{22} = 3}$$

$$l_{31} l_{11} = -5 \Rightarrow l_{31} = \frac{-5}{5} \Rightarrow \boxed{l_{31} = -1}$$

$$l_{21} l_{31} + l_{22} l_{32} = 0 \Rightarrow 3 \times (-1) + 3 l_{32} = 0 \Rightarrow 3 l_{32} = 3$$

$$\boxed{l_{32} = 1}$$

$$\cancel{l_{33} \rightarrow l_{31} l_{21} + l_{33} l_{22} = 0}$$

$$\cancel{(-1)(3) + l_{33}(3) = 0}$$

$$\cancel{\Rightarrow 3 l_{33} = 3}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 11$$

$$1 + 1 + l_{33}^2 = 11$$

$$l_{33}^2 = 9$$

$$\boxed{l_{33} = 3}$$

So $$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$LL^T = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

Now $AX = B$ and $A = LL^T \Rightarrow LL^TX = B$.

Let $L^TX = y$ then $Ly = B$

$$\Rightarrow \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

$5y_1 = 35$ — (I)

$3y_1 + 3y_2 = 33$ — (II)

$-y_1 + y_2 + 3y_3 = 6$ — (III)

Now use forward substitution method

$5y_1 = 35$

$y_1 = \frac{35}{5} \Rightarrow \boxed{y_1 = 7}$

from (II)

$3y_1 + 3y_2 = 33$

$3(7) + 3y_2 = 33 \Rightarrow 21 + 3y_2 = 33$

$3y_2 = 33 - 21$

$y_2 = \frac{12}{3} \Rightarrow \boxed{y_2 = 4}$

from III

$-y_1 + y_2 + 3y_3 = 6$

$-7 + 4 + 3y_3 = 6$

$-3 + 3y_3 = 6$

$+3y_3 = 6 + 3 = 9$

$y_3 = \frac{9}{3} \Rightarrow \boxed{y_3 = 3}$

Now, $L^T X = Y$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$5x + 3y - z = 7 \quad — (4)$

$3y + z = 4 \quad — (5)$

$3z = 3 \quad — (6)$

Now use back substitution method

from (6)

$3z = 3$

$\Rightarrow z = \frac{3}{3} = 1$

from (5)

$3y + z = 4$

$\Rightarrow 3y + 1 = 4$

$3y + 1 = 4$

$3y + \cancel{4} = \cancel{3}4 \Rightarrow 3y = 3 \Rightarrow \boxed{y = 3}$

from (4)

$5x + 3y + z = 7$

$\Rightarrow 5x + 3(1) - 1 = 7$

$\Rightarrow 5x + 2 = 7$

$\Rightarrow 5x = 7 - 2 = 5$

$\Rightarrow x = \frac{5}{5} = 1$

Solⁿ by cholesky Decomposition method is

$\boxed{x = 1}$, $\boxed{y = 1}$, $\boxed{z = 1}$

Ex Solve the following system of equation by cholesky method

$$4x_1 + 2x_2 + 6x_3 = 16$$
$$2x_1 + 82x_2 + 39x_3 = 206$$
$$6x_1 + 39x_2 + 26x_3 = 113$$

Solⁿ The given system of Eqⁿ can be written as

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 82 & 39 \\ 6 & 39 & 26 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 16 \\ 206 \\ 113 \end{bmatrix}$$

A is symmetric

$A = LL^T$

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 82 & 39 \\ 6 & 39 & 26 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing both sides, we get the following system of Eqⁿ

$l_{11}^2 = 4$ $\qquad l_{11} = 2$

$l_{11}l_{21} = 2$ $\qquad l_{21} = 1$

$l_{11}l_{31} = 6$ $\qquad l_{31} = 3$

$l_{21}^2 + l_{22}^2 = 82$ $\qquad l_{22} = (82 - 1)^{1/2} = 9$

$l_{31}l_{21} + l_{32}l_{22} = 39$ $\qquad \boxed{l_{32} = 4}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 26$ $\qquad \boxed{l_{33} = 1}$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

Now the system of

Eqⁿ $LV = b$.

$2v_1 = 16$

$v_1 + 9v_2 = 206$

$3v_1 + 4v_2 + v_3 = 113$

The Solution to these Eqⁿ is

$v_1 = 8$ $\quad v_2 = 22$ $\quad v_3 = 1$

Now from the Eqⁿ $L^T X = V$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 9 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 22 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $2x_1 + x_2 + 3x_3 = 8$

$9x_2 + 4x_3 = 22$

$x_3 = 1$

$9x_2 + 4 = 22 \Rightarrow 9x_2 = 22 - 4 = 18$

$\boxed{x_2 = 2}$

and $2x_1 + 2 + 3(1) = 8 \Rightarrow \boxed{x_1 = 1.5}$

Hence the solution is $x_1 = 1.5$ $\quad x_2 = 2;\ x_3 = 1$.

Ex. Solve the following equation using Cholesky method.

1. $x + 2y + 3z = 5$

$2x + 8y + 22z = 6$

$3x + 22y + 82z = 10$

Ans $x = 2, y = 3, z = -1$

2. $x + y + z = 3$

$x - y + z = 4$

$x + y$ [redacted] $z = 5$

Ans $x = 11/2$, $y = 1/$[illegible], $z = -1$

/ 2



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix} = IA^{-1}$$

The adjacent matrix of $I_{3\times3}$ gives $A^{-1}$.

### (II) Gauss-Jordon Method

Let $A$ be a non-singular matrix of order $n$ then

$$A = I_n A \qquad \text{... (i)}$$

We know that an elementary row transformation on the product of two matrices can be affected by subjecting the prefactor to the same elementary row transformation. We shall use elementary row transformation on (*i*) so that L.H.S. reduces to $I_n$ and R.H.S. (after applying corresponding elementary row transformation on the pre-factor $I_n$) takes the shape.

$$I_n = BA \qquad \text{... (ii)}$$

then the matrix $B$ will be the required inverse.

**Method of Find $A^{-1}$ using Elementary Operations when $A$ is a 3 × 3 Matrix**

**Step I.** Make $a_{11} = 1$ either by operating $R_1 \to \frac{1}{a_{11}} R_1$ (if $a_{11} \neq 0$) or by operating $R_1 \leftrightarrow R_2$ or $R_1 \leftrightarrow R_3$ and then $R_1 \to \frac{1}{a_{11}} R_1$ (if needed).

**Step II.** Operate $R_2 \to R_2 - a_{21} R_1$ and $R_3 \to R_3 - a_{31} R_1$ to obtain $a_{21} = a_{31} = 0$.

**Step III.** Operate $R_2 \to \frac{1}{a_{22}} R_2$ (if $a_{22} \neq 0$) or $R_2 \leftrightarrow R_3$ followed by $R_2 \to \frac{1}{a_{22}} R_2$ (if needed) to make $a_{22} = 1$.

**Step IV.** Operate $R_3 \to R_3 - a_{32}R_2$ to make $a_{32} = 0$.

**Step V.** Operate $R_3 \to \frac{1}{a_{33}} R_3$ to make $a_{33} = 1$.

**Step VI.** Operate $R_2 \to R_2 - a_{23}R_3$ and $R_1 \to R_1 - a_{13}R_3$ to make $a_{23} = a_{13} = 0$.

**Step VII.** Operate $R_1 \to R_1 - a_{12} R_2$ to make $a_{12} = 0$

## SOLVED EXAMPLES

**Example 4.2.** By Gauss-Jordon method, find the inverse of the matrix

$$A = \begin{bmatrix} 8 & 4 & -3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Solution:** We have

$$A = IA$$

$$\therefore \quad \begin{bmatrix} 8 & 4 & -3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operation $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 8 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 8R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -12 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Operate $R_2 \to -\frac{1}{3} R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & -12 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/3 & 2/3 \\ 1 & 0 & -8 \end{bmatrix} A$$

Operate $R_3 \to R_3 + 12R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/3 & 2/3 \\ 1 & -4 & 0 \end{bmatrix} A$$

Operate $R_3 \to -\frac{1}{7} R_3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/3 & 2/3 \\ -1/7 & 4/7 & 0 \end{bmatrix} A$$

Operate $R_2 \to -R_2 - \frac{1}{3} R_3$; $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & -\frac{4}{7} & 1 \\ \frac{1}{21} & -\frac{11}{21} & \frac{2}{3} \\ -\frac{1}{7} & \frac{4}{7} & 0 \end{bmatrix} A$$

Operate $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{21} & \frac{10}{21} & -\frac{1}{3} \\ \frac{1}{21} & -\frac{11}{21} & \frac{2}{3} \\ -\frac{1}{7} & \frac{4}{7} & 0 \end{bmatrix} A$$

Thus we get $I = BA$

$$A^{-1} = B = \begin{bmatrix} \frac{1}{21} & \frac{10}{21} & -\frac{1}{3} \\ \frac{1}{21} & -\frac{11}{21} & \frac{2}{3} \\ -\frac{1}{7} & \frac{4}{7} & 0 \end{bmatrix} A$$

**Example 4.3. Find the inverse of the following matrix using Gauss-Jordon method:** $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

**Solution: We have** $A = IA$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \to \frac{1}{3} R_1$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to - R_2$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operate $R_3 \to - 3R_3$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A$$

Operate $R_2 \to R_2 + \frac{4}{3}R_3$, $R_1 \to R_1 \frac{4}{3}R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

Operate $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

Thus we get $I = BA$

$$A^{-1} = B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

**Example 4.4. Compute the inverse of the following matrix by using Gauss method:** $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

**Solution:** We have $A = IA$

$$\therefore \quad \begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 + 2R_1$, $R_3 \to R_4 + 5R_1$ then $R_1 \to -R_1$

$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 2 & 0 & 1 \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \to R_1 + 2R_2$, $R_2 \to \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 \to R_3 - 7R_2$, $R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{-7}{2} & 1 & -2 & 0 \\ \frac{-1}{2} & 0 & -1 & 1 \end{bmatrix} A$$

**Operate $R_3 \to R_3 + R_4$, $R_2 \to R_2 - R_4$**

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ \frac{-1}{2} & 0 & -1 & 1 \end{bmatrix} A$$

Operate $R_1 \to R_1 + R_3$, $R_4 \to 2R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Thus we get $$I = BA$$

$$\therefore \quad A^{-1} = B = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$$

## EXERCISE 4.1

Find the inverse of the following matrices by (*i*) Gauss Elimination Method (*ii*) Gauss-Jordon Method.

1. $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $\left[\textbf{Ans.} \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}\right]$

2. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ $\left[\textbf{Ans.} \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}\right]$

3. $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ $\left[\textbf{Ans.} \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}\right]$