



UNIT 1: Function of Complex Variables - Complex Analysis I

- **Introduction:** A number whose square is -1 is called an imaginary or a complex quantity and is denoted by i (iota).

$$i = \sqrt{-1}$$

$$i^2 = -1$$

- A complex number is written as, $Z = x + iy$ ($x, y \in \mathbb{R}$).

where,

x is real part of Z , $\operatorname{Re}(Z) = x$
 y is imaginary part of Z , $\operatorname{Im}(Z) = y$

1) a) Geometric Representation of Complex Numbers :-

$$Z = x + iy$$

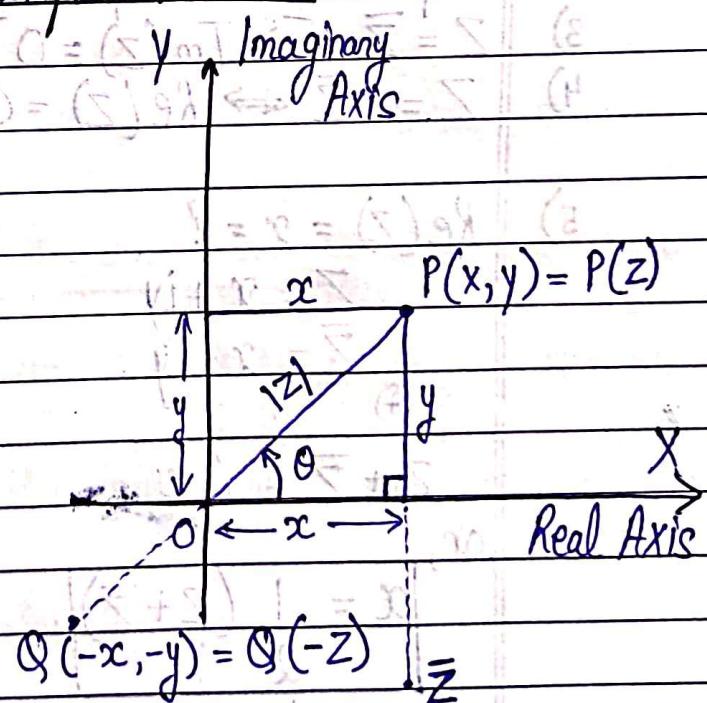
and,

$$\text{conjugate of } Z = \bar{Z} = x - iy.$$

$$P(Z) = x + iy$$

and,

$$Q(-Z) = -x - iy.$$



b) Modulus of a Complex Number :-

$$|Z| = \sqrt{x^2 + y^2}$$

c) Argument of a Complex Number :-

- ' θ ' is called argument or amplitude of Z .
- If argument lies in the range of,

$$-\pi < \theta \leq \pi$$

then, ' θ ' is called principal argument.
- General Argument is written as,

$$\begin{matrix} \text{General} \\ \text{Argument} \end{matrix} = \begin{matrix} \text{Principal} \\ \text{Argument} \end{matrix} + 2k\pi \quad (k \in \mathbb{Z})$$

d) Few Properties :-

1) $(\bar{\bar{z}}) = z$

2) $z_1 = z_2 \iff \bar{z}_1 = \bar{z}_2$

3) $z = \bar{z} \iff \operatorname{Im}(z) = 0$ (Purely Real)

4) $z = -\bar{z} \iff \operatorname{Re}(z) = 0$ (Purely Imaginary)

5) $\operatorname{Re}(z) = x = ?$

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\textcircled{+} \quad \textcircled{+}$$

$$z + \bar{z} = 2x$$

6) $\operatorname{Im}(z) = y = ?$

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\textcircled{-} \quad \textcircled{-}$$

$$z - \bar{z} = 2iy$$

or,

$$x = \frac{1}{2} (z + \bar{z})$$

or,

$$y = \frac{1}{2i} (z - \bar{z})$$

e) Polar Form of a Complex Number:-

Cartesian, $P(x, y) \rightarrow$ Polar form, $Z(r, \theta)$.

$$\Rightarrow Z = r[\cos\theta + i\sin\theta]$$

where,

$$r = \sqrt{x^2 + y^2}$$

and,

$$\theta = \tan^{-1} \left| \frac{y}{x} \right|$$

(NOTE: 'θ' will depend on coordinate axes/values of x & y)

Note: (TRICK to determine 'θ') :-

$$\pi - \theta$$

$$\theta$$

Quadrant Principal Argument

$$1) I \rightarrow \theta$$

$$2) II \rightarrow \pi - \theta$$

$$3) III \rightarrow -(\pi - \theta)$$

$$4) IV \rightarrow -\theta$$

2) De-Moivre's Theorem :-

$$\text{Let, } Z = r(\cos\theta + i\sin\theta)$$

Then,

$$Z^n = r^n (\cos n\theta + i\sin n\theta)$$

where,

n is any positive or negative integer or a real rational number,

$$n \in \mathbb{I} \text{ and } n \in \mathbb{Q}$$

2.1) Roots of a Complex Number :-

In general, $\cos\theta = \cos(2k\pi + \theta)$, where $k \in \mathbb{I}$
 $\sin\theta = \sin(2k\pi + \theta)$, where $k \in \mathbb{I}$.

y

O

x

g

θ

y

$$\therefore (\cos\theta + i \sin\theta)^{1/n} = [\cos(2K\pi + \theta) + i \sin(2K\pi + \theta)]^{1/n}$$

$$= \cos\left(\frac{2K\pi + \theta}{n}\right) + i \sin\left(\frac{2K\pi + \theta}{n}\right)$$

where, $K = 0, 1, 2, 3, \dots, (n-1)$.

Note: 1) $1 = \cos 0 + i \sin 0$

2) $-1 = \cos \pi + i \sin \pi$

3) $i = \cos \pi/2 + i \sin \pi/2$

4) $-i = \cos \pi/2 - i \sin \pi/2$

} (Standard numbers).

5) $(\cos x + i \sin x)(\cos y + i \sin y) = \cos(x+y) + i \sin(x+y)$

6) $(\cos x + i \sin x)(\cos y - i \sin y) = \cos(x-y) + i \sin(x-y)$

7) $(\cos x - i \sin x)(\cos y + i \sin y) = \cos(x-y) - i \sin(x-y)$

8) $(\cos x - i \sin x)(\cos y - i \sin y) = \cos(x+y) - i \sin(x+y)$

3) (i) Function of Complex Variables :-

$$\Rightarrow f(z) = u + iv = \underbrace{u(x,y)}_{\text{Real part}} + i \underbrace{v(x,y)}_{\text{Imaginary part}}$$

or,

$$w = u + iv \quad (\text{Real part}) \quad (\text{Imaginary part})$$

Here,

$f(z)$ is a function of complex variable and is denoted by

(ii) Continuity of function $f(z)$:-

\Rightarrow A function $w = f(z)$ is said to be continuous at $z = z_0$ if,

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

(iii) Differentiability of function $f(z)$:-

\Rightarrow Let, $f(z)$ be a singed value function of the variable z , then

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

- Provided that the limit exists and is independent of path along $\Delta z \rightarrow 0$.

Singed Value Function: Any function that gives only one output value for one input value is a singed value function. Examples are, $f(x) = x$. For $x = n$, $n \in \mathbb{R}$, there will be always one value of $f(x)$...

4) Analytic Function : (Regular Functions OR Holomorphic Functions)

\Rightarrow A singed valued function $f(z)$ which is differentiable at $z = z_0$ is said to be analytic function at the point $z = z_0$ (Point z_0 is said non-singular point or regular).

Singular Point \rightarrow The point at which the function is not differentiable is called a singular point of the function.

\Rightarrow The Necessary Condition of $f(z)$ to be analytic; (at all points in a region R).

$$\textcircled{1} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \rightarrow \text{Provided } \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \text{ exist}$$

$$\textcircled{2} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \rightarrow \text{C-R Equations.} \\ \text{(Cauchy - Riemann Equations)}$$

* Sufficient Condition for $f(z)$ to be Analytic,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \rightarrow (1)$$

$$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \rightarrow (2)$$

* C-R Equations in Polar Form :-

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and}, \quad \frac{\partial u}{\partial \theta} = r - \frac{\partial v}{\partial r}$$

* Derivative of $w = f(z)$ in Polar form :-

$$i) \frac{\partial w}{\partial z} = (\cos\theta - i\sin\theta) \frac{\partial w}{\partial r} \quad \text{--- } ①$$

$$ii) \frac{\partial w}{\partial z} = -i(\cos\theta - i\sin\theta) \frac{\partial w}{\partial \theta} \quad \text{--- } ②$$

4.1) Harmonic Functions :-

• Any function of x and y (of first and second order) if satisfies Laplace equation is called Harmonic function.

* For example,

$$(For 'u') \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$(For 'v') \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

} Laplace Equation.

* Laplace Equations and Conjugates :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

- How to find 'v' if 'u' is given \rightarrow

If, $u(x, y)$ is given. Then,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

and, $f(z)$ is analytic i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So,

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \quad (\text{exact differential form})$$

(For ' $u(x, y)$ '')

Similarly,

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad (\text{For ' $v(x, y)$ '})$$

4.2) Milne's Thomson Method \rightarrow

If $u(x, y)$ is given,

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz$$

Here,

$$\frac{\partial u}{\partial x} = \phi_1(x, y), \quad \frac{\partial u}{\partial y} = \phi_2(x, y)$$

If $v(x, y)$ is given,

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)] dz$$

Here,

$$\frac{\partial v}{\partial y} = \psi_1(x, y), \quad \frac{\partial v}{\partial x} = \psi_2(x, y)$$

5) (i) Exponential Function of Complex Variable \rightarrow

\Rightarrow Exponential function of complex variable $Z = x + iy$ is written as, e^Z or $\exp Z$ and, defined as

$$\Rightarrow e^Z = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \dots + \frac{Z^n}{n!} + \dots \infty$$

\Rightarrow PROPERTIES \rightarrow

(1) \rightarrow Exponential form of $Z = x e^{i\theta}$

$$\Rightarrow Z = x + iy = x(\cos\theta + i\sin\theta) = x e^{i\theta}$$

(2) \rightarrow e^Z is periodic function having imaginary period $2\pi i$

(3) \rightarrow e^Z is not zero for any value of Z .

(ii) Euler's formula \rightarrow

If θ be real or complex;

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$OR \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

\rightarrow Adding ① and ②,

$$\Rightarrow e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

or,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

→ Similarly, subtracting eq. -② from eq. -①,

$$\sin\theta = \frac{1}{2i} (e^{i\theta} + e^{-i\theta})$$

or,

$$\boxed{\sin\theta = \frac{1}{2i} (e^{i\theta} + e^{-i\theta})}$$

And,

$$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

(iii) Hyperbolic Functions → For values of x (real or complex), the quantity, $e^x + e^{-x}$ is called hyperbolic cosine of x

and is written as $\cosh x$.

And,

$$\sinh x = e^x - e^{-x}$$

* Relation between hyperbolic and circular functions :-

We know that,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Putting $\theta = ix$,

$$\begin{aligned}\sin ix &= \frac{e^{ix} - e^{-ix}}{2} = \frac{e^x - e^{-x}}{2} \times i \\ &= \frac{-i^2(e^x - e^{-x})}{2i} = i \left(\frac{e^x - e^{-x}}{2} \right) \\ &= i \sinh x.\end{aligned}$$

$$\sin ix = i \sinh x$$

Similarly,

$$\cosh ix = \cos h x$$

$$\tan ix = i \tanh x$$

- To separate the real and imaginary part of $\sin(x+iy)$,

$$\begin{aligned} \Rightarrow \sin(x+iy) &= \sin x \cdot \cos iy + \cos x \cdot \sin iy \\ &= \sin x \underbrace{(\cos y + i \sin y)}_{\text{Real part}} + i \cos x \underbrace{\sin y}_{\text{Imaginary part}}. \end{aligned}$$

- To separate the real and imaginary part of $\tan(x+iy)$

$$\begin{aligned} \Rightarrow \tan(x+iy) &= \frac{\sin(x+iy)}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)} \times \frac{2}{2} \\ &= \frac{2 \sin(x+iy) \cdot \cos(x-iy)}{2 \cos(x+iy) \cdot \cos(x-iy)} \\ &= \frac{\sin(x+iy+x-iy) + \sin(x+iy-x+iy)}{\cos(x+iy+x-iy) + \cos(x+iy-x+iy)} \\ &= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)} \end{aligned}$$

$$\Rightarrow \tan(x+iy) = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$$

On separating,

$$\rightarrow \text{Real part} = \frac{\sin 2x}{\cos 2x + \cosh(2y)}$$

$$\rightarrow \text{Imaginary part} = i \cdot \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

(iv) Logarithm of Complex Numbers :

$$\Rightarrow z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\Rightarrow x + iy = re^{i\theta}$$

or,

$$\Rightarrow \log(x+iy) = \log(re^{i\theta}) = \log r + i\theta$$

$$\therefore r = \sqrt{x^2 + y^2} \text{ and, } \tan^{-1}\left(\frac{y}{x}\right) = \theta$$

$$\Rightarrow \log(x+iy) = \frac{1}{2} \log(r^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

$\log(x+iy) \rightarrow$ Principal value.

$\underline{\log(x+iy)} \rightarrow$ General value. $(n \in \mathbb{N})$

$$\therefore \underline{\log(x+iy)} = \frac{1}{2} \log(r^2) + i \left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi \right]$$

6) Line Integral in the Complex Plane :-

$$(S) \Rightarrow \int_C f(z) dz = \int_C (u+iv)(dx+idy) : (C \rightarrow \text{closed integral})$$

Here,

$$\Rightarrow Z = x + iy$$

$$\Rightarrow f(z) = u + iv = u(x, y) + iv(x, y)$$

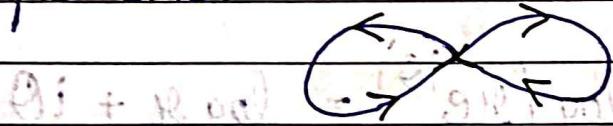
and,

$$\Rightarrow dz = dx + i dy$$

7) Cauchy's Integral Theorem and Cauchy's Integral Formula :-

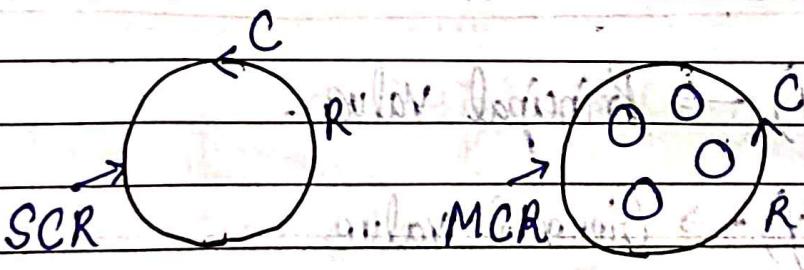
\Rightarrow A simple closed curve is a curve that does not cross itself

(a) Multiple curve: A curve which crosses itself is called a multiple curve.



(b) Simply connected region: A simply connected region is one which has no pole (holes).

(c) Multiple or Multiply connected region: A multiple connected region is one which has more than one poles (holes).



Note: We can convert multiply connected region into simply connected region with the help of cross-cuts.

7.1) Cauchy's Integral Theorem statement:

\Rightarrow If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C , then;

$$\int_C f(z) \cdot dz = 0$$

$$\Rightarrow \int_C F(z) \cdot dz = \int_C (u+iv)(dx+idy) \quad z = x+iy. \\ \text{So,} \\ f(z) = u+iv$$

$$= \int_C u \cdot dx + iu \cdot dy + iv \cdot dx - v \cdot dy$$

$$= \int_C (u \cdot dx - v \cdot dy) + i(u \cdot dy + v \cdot dx)$$

$$= \int_C [(u \cdot dx - v \cdot dy) + i(v \cdot dx + u \cdot dy)]$$

$$= \int_C (u \cdot dx - v \cdot dy) + i \int_C (v \cdot dx + u \cdot dy)$$

Note: (Green's Theorem) \rightarrow

$$\int_C \phi \cdot dx + \psi \cdot dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx \cdot dy.$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx \cdot dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \cdot dy.$$

$\because f(z)$ is analytic, i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \right) dx \cdot dy + i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \right) dx \cdot dy$$

$$\Rightarrow \int_{C_1} F(z) \cdot dz = 0 + i(0) = 0 \quad (\text{Hence proved}).$$

7.2) Extension of Cauchy's theorem to multiple connected regions

$$\Rightarrow \int_C f(z) \cdot dz = \int_{C_1} f(z) \cdot dz + \int_{C_2} f(z) \cdot dz + \int_{C_3} f(z) \cdot dz = 0$$

7.3) Cauchy's Integral formula :-

Statement: If $f(z)$ is analytic function within and on a closed curve C and if 'a' is any point within C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$

7.4) Cauchy's Integral formula for the derivative of an analytic function:-

\Rightarrow If a function $f(z)$ is analytic in a region R , then its derivative at any point $z=a$, if R is also analytic in R ; and is given as;

$$f^n(a) = n! \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

If $n=1$;

$$f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz.$$

$$\text{If } n=2, f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz.$$

8) Expansion of functions :-

① Taylor series

② MacLaurin series

③ Laurent's series

8.1) Taylor Series: If a function $f(z)$ is analytic inside a circle C with centre at $z=a$ then it can be expanded in the series,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} f^n(a) + \dots$$

\Rightarrow If putting $a=0$,

$$\Rightarrow f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0) + \dots$$

This series is called MacLaurin series.

8.2) Laurent's Series :-

If $f(z)$ is analytic in annular region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 \geq r_2$) and centre at ' a ', then for all z in R , we have;

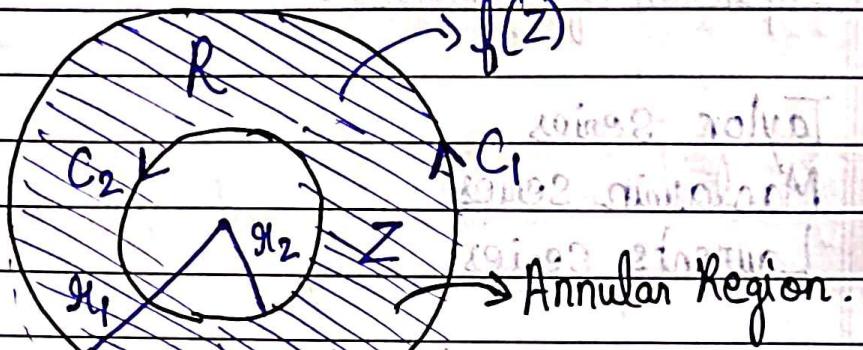
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

General term ————— Principal term —————

where,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt, \quad n=0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{-n+1}} dt, \quad n=1, 2, 3, \dots$$



9) Power Series: A series in powers of $(z-z_0)$ is called power series.

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1 (z-z_0) + a_2 (z-z_0)^2 + \dots + a_n (z-z_0)^n$$

$$\dots + a_n (z-z_0)^n$$

Note: If centre of series, i.e., $z_0 = 0$, then;

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

9.1) Radius of Convergence of Power Series:-

→ Let the power series, $\sum a_n z^n$, by Cauchy theorem on limits.

→ Radius of convergence (R) is given by,

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

10) Zero of Analytic Function :-

→ A zero of analytic function $f(z)$ is the value of z for which $f(z) = 0$, i.e., $f(z) = z - 2$
So for $z=2$, $f(z) = 0$.

11) Singular Point :-

→ A point at which a function $f(z)$ is not analytic is known a singular point or singularity of the function.

Let, $f(z) = \frac{1}{z-2}$ has a singular point at $z=2$.

Isolated Singular Point

→ If $z=a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z=a$, then $z=a$ is said to be an isolated singularity of function $f(z)$, otherwise it is called non-isolated singular point.

Example: ① $f(z) = \frac{1}{(z-1)(z-3)}$

Here, $z=1, 3$ are two isolated singular points.

② $f(z) = \frac{\sin(\pi/z)}{z}$ here, $\sin \pi/z = 0$
 $\Rightarrow \pi/z = n\pi$.

Here, $z = \frac{1}{n}$ ($n=0, 1, 2, 3, \dots, \infty$) $\therefore z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ ($n \in \mathbb{N}$)

- Pole of order 'm' :-

→ By Laurent's series,

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$\Leftrightarrow f(z) = a_0 + a_1 (z-z_0) + \dots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots +$$

① $\rightarrow \frac{b_m}{(z-z_0)^m} + \dots + \frac{b_{m+1}}{(z-z_0)^{m+1}} + \dots$

→ If the number of terms of negative powers in ① is infinite then, $z=z_0$ is called an essential singularity point of $f(z)$.

- Removable Singularity :-

Note → If Laurent's series does not contain negative power of $(z-z_0)$ then, $z=a$ is called a removable singularity.

Example: $f(z) = \frac{\sin(z-a)}{(z-a)}$ no. of terms of $(z-a)$ in the denominator is finite so it is removable.

Expansion of $\sin z$ -

$$\Rightarrow f(z) = \frac{1}{(z-a)} \left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \dots \right]$$

$$\Rightarrow f(z) = 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \dots \infty$$

→ Now, there is no singularity.

Note : ① If $\lim_{z \rightarrow a} f(z) = \text{Exists and finite.}$, then $z=a$ is a removable singular point.

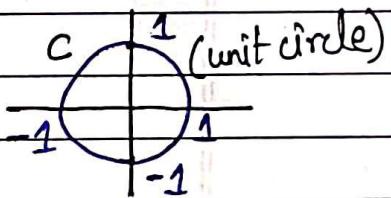
② If $\lim_{z \rightarrow a} f(z)$ does not exists, then $z=a$ is an essential singular point.

③ If $\lim_{z \rightarrow a} f(z)$ is infinite then $f(z)$ has pole at $z=a$

12) Evaluation of real definite integrals by Contour Integration

1) Integration round the unit circle of the type,

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) \cdot d\theta .$$



here, $\int_C \rightarrow \text{Unit circle.}$

To change $\theta \rightarrow 'Z'$. we have to substitute $Z = e^{i\theta}$.

$$\Rightarrow Z = e^{i\theta}$$

$$\Rightarrow dZ = i e^{i\theta} \cdot d\theta .$$

$$\Rightarrow dZ = iZ \cdot d\theta \quad \Rightarrow d\theta = \frac{dZ}{iZ} .$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \cos\theta = \frac{Z + Z^{-1}}{2} = \frac{Z^2 + 1}{2Z}$$

and,

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{Z - Z^{-1}}{2i} = \frac{Z^2 - 1}{2iZ} .$$