CHAPTER 3

Circuit Analysis by Classical Method

Inside this Chapter

- Introduction
- Initial Conditions in Circuits
- Differential Equations
- Various Responses
- Transient Response of series R-L-C circuit having DC excitation
- Transient Response of series R-L circuit having DC excitation
- Transient Response of series R-C circuit having DC excitation
- Transient response of series R-L circuit having sinusoidal excitation
- Transient response of series R-C circuit having sinusoidal excitation

3.1. INTRODUCTION

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transition period during which the branch currents and element voltages change from their former values to new ones. This period is called the *transient*. After the transient has passed, the circuit is said to be in the *steady state*.

Now, the linear differential equation that describes the circuit will have two parts to its solution, the complementary function corresponds to the transient and the particular solution corresponds to the steady-state.

The *v-i* relation for an inductor or a capacitor is a differential. A circuit containing an inductor L or a capacitor C, and resistors will have current and voltage variables given by differential equations of the same form. It is a linear first order differential equation, with constant coefficients when the values of R, L and C are constant. L and C are storage elements. Circuits have two storage elements like one L and one C are referred to as second order circuits.

Therefore, the series or parallel combinations of R and L or R and C are first order circuits, and RLC in series and RLC in parallel are typical second order circuits.

The circuit changes are assumed to occur at time $t = 0$, and represented by a switch. The switch may be supposed to close/$_{on}$ and open/$_{off}$ at $t = 0$ as shown in figure 3.1 (*a*) or (*b*) respectively, for convenience, it is defined that:

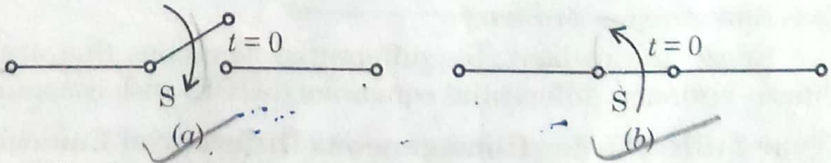


Fig. 3.1. *Switch S is (a) closed at t = 0, (b) opened at t = 0.*

$t = 0^-$, the instant prior to $t = 0$ and

$t = 0^+$, the instant immediately after switching.

Switching on or off an element or source in a circuit at $t = 0$ will not disturb the storage element so that $i_L(0^-) = i_L(0^+)$ and $v_c(0^-) = v_c(0^+)$. This provides a basis for constructing an equivalent circuit for a charged capacitor voltage (V_0) and current (I_0) carrying inductor.

$i_L(0^-) = i_L(0^+)$ & $V_C(0^-) = V_C(0^+)$

The voltage-current relationships of the three circuit elements R, L and C are given in Table 3.1.

Table 3.1: Relationship for the Parameters

Parameter	Basic Relationship	Voltage-current Relationships	Energy
R $\left(G=\frac{1}{R}\right)$	$v(t) = R\,i(t)$	$v_R(t) = R \cdot i_R(t)$ $i_R(t) = G\,v_R(t)$	$W_R(t) = \int_{-\infty}^{t} v_R(t)\,i_R(t)\,dt$
L	$\Psi(t) = Li(t)$	$v_L(t) = L\frac{di_L(t)}{dt}$ $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt$ or $i_L(t) = \frac{1}{L}\int_{0}^{t} v_L(t)\,dt + i_L(0^+)$	$W_L(t) = \frac{1}{2}Li_L^2(t)$
C	$q(t) = Cv(t)$	$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(t)\,dt$ or $v_c(t) = \frac{1}{C}\int_{0}^{t} i_c(t)\,dt + v_c(0^+)$ $i_c(t) = C\frac{dv_c(t)}{dt}$	$W_C(t) = \frac{1}{2}Cv_c^2(t)$

3.2. DIFFERENTIAL EQUATIONS

To study the transients in electric circuits, it is necessary to be familiar with the mathematical concept of differential equations and the solution techniques.

The order of the differential equation represents the highest derivative involved and is equal to the number of energy storing elements. As the differential equation contains no partial derivatives, it is considered as ordinary.

Hence, in this book, the differential equations that are formed for transient analysis will be linear, ordinary, differential equations (LODE) with constant coefficients.

Type I (First Order Homogeneous Differential Equation)

$$\frac{dy(t)}{dt} + Py(t) = 0 \quad \Rightarrow i/p = 0$$

order - no. of energy storing elements

(where P is any constant.)

$$\frac{dy(t)}{y(t)} = -P\,dt$$

$C\frac{dV(t)}{dt}$

On integrating,

$$ln\,y(t) = -Pt + K'$$

$dy(t) = -Py(t)dt$

log A + log B
log AB
log(A·B)

Take
$$K' = \ln K$$
$$\ln y(t) = \ln e^{-Pt} + \ln K = \ln(K e^{-Pt})$$
with the equation in this form, the antilogarithm may be taken to give a general solution
$$\boxed{y(t) = K e^{-Pt}}$$ for $\boxed{\frac{dy(t)}{dt} + P(y(t)) = 0}$

where K is a constant.

If the constant K is evaluated, the solution is a particular solution.

total solution $\boxed{Ke^{-Pt}}$

Type II (First Order Non-homogeneous Differential Equation)
$$\boxed{\frac{dy(t)}{dt} + Py(t) = Q}$$
where P is a constant and Q may be a function of independent variable t or a constant. The equation is not altered if every term is multiplied by the e^{Pt}. *i.e.*
$$e^{Pt}.\frac{dy(t)}{dt} + e^{Pt}.Py(t) = Q\,e^{Pt}$$
since $d(x{\cdot}y) = x\,dy + y\,dx$, so left hand side of above equation is equal to $\frac{d}{dt}[y(t){\cdot}e^{Pt}]$. Thus we have
$$\frac{d}{dt}[y(t){\cdot}e^{Pt}] = Q\,e^{Pt}$$

$y(t) = e^{-Pt}\int Qe^{Pt}dt + Ke^{-Pt}$

Thus equation may be integrated to give
$$y(t){\cdot}e^{Pt} = \int Qe^{Pt}dt + K$$

P.I. → steady state

or
$$\boxed{y(t) = e^{-Pt}\int Q.e^{Pt}dt + K.e^{-Pt}}$$

C.F. CF → transient state

where K is a constant.

P.I.

The first term of above solution is known as the *particular integral*; the second is known as the *complementary function*. Note that the particular integral does not contain the arbitrary constant, (K) and the complementary function does not depend on the forcing function Q.

If Q is a constant, then
$$y(t) = e^{-Pt}.Q.\frac{e^{Pt}}{p} + K\,e^{-Pt})$$
$$\boxed{y(t) = \frac{Q}{P} + K\,e^{-Pt}}$$

if Q is a constant then $y(t) = \frac{Q}{P} + Ke^{-Pt}$

Type III (Second Order Differential Equation)
$$A\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + Cy(t) = 0$$
The general solution of the above second order differential equation is
$$y(t) = K_1\,e^{p_1 t} + K_2\,e^{p_2 t}$$

$y(t) = K_1e^{p_1 t} + K_2e^{p_2 t}$

where K_1 and K_2 are constants.

And, p_1 and p_2 are the roots of the quadratic equation
$$A\,p^2 + Bp + C = 0, \text{ and are given by}$$
$$p_1, p_2 = -\frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$
If $p_1 = p_2$, *i.e.*, roots of the quadratic equation are repeated, then the general solution of the given second order differential equation is given as
$$\boxed{y(t) = K_1{\cdot}e^{p_1 t} + K_2{\cdot}t{\cdot}e^{p_1 t}}$$

if $p_1 = p_2$

Scanned by CamScanner

3.3. INITIAL CONDITIONS IN CIRCUITS

We need initial (or boundary) conditions to evaluate the arbitrary constants in the general solution of differential equations. The number of initial conditions required is equal to the order of the differential equation for a unique solution.

Procedure to Evaluate the Initial Conditions

There is no set procedure to follow in determining the initial conditions. First step is to draw the equivalent circuit at $t = 0^+$ based on the element given in Table 3.2. The next step is to evaluate the initial values of voltages and currents of all the branches. After this, the derivatives at $t = 0^+$ are evaluated.

Initial values of current and voltage may be found directly from a study of the network schematic. For each element in the network, we must determine just what will happen when the switching action takes place. From this analysis, a new schematic of an equivalent network for $t = 0^+$ may be constructed according to these rules:

(i) Replace all inductors with open circuits or with current generators having the value of current flowing at $t = 0^+$.

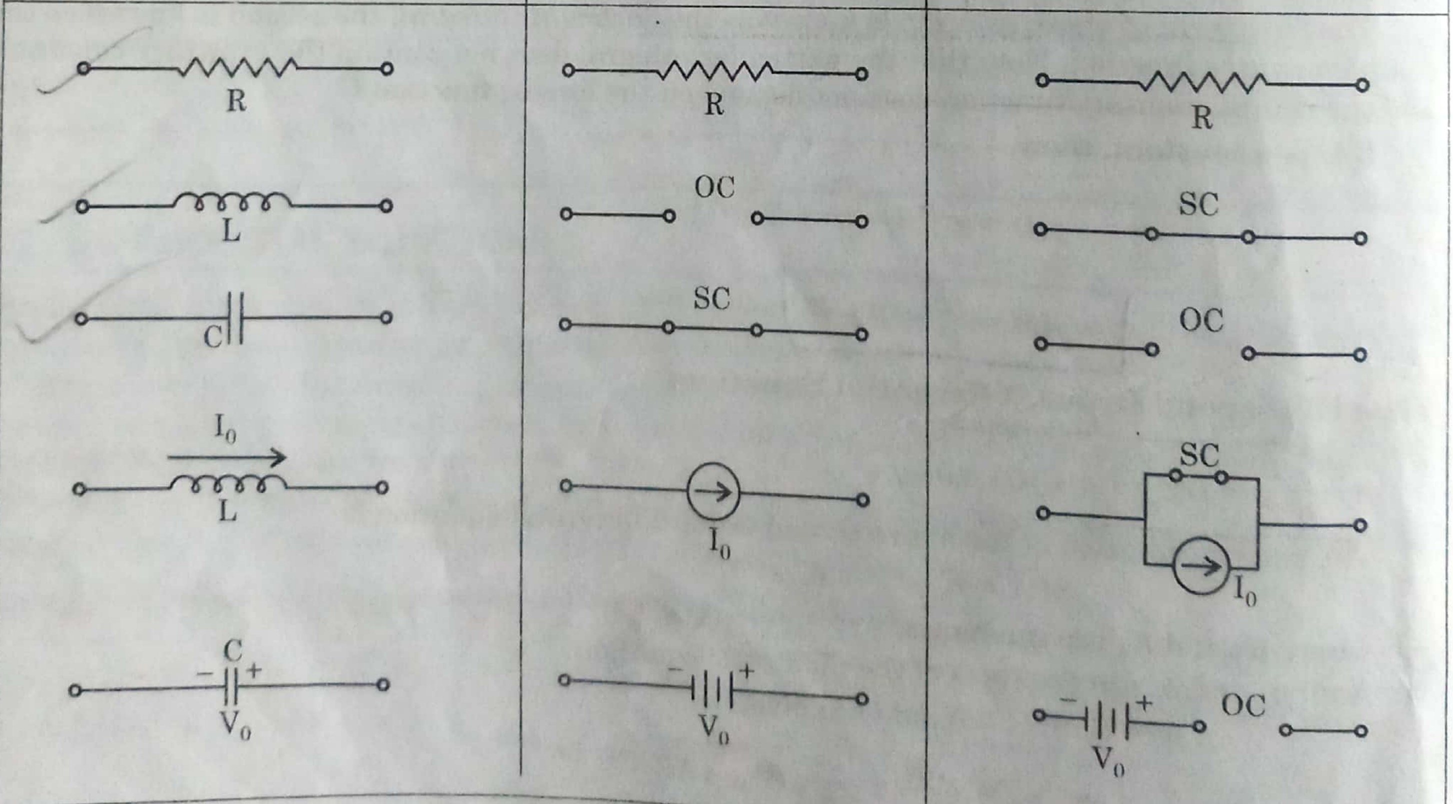
(ii) Replace all capacitors with short circuits or with voltage sources having the value $V_0 = \frac{q_0}{C}$ if there is an initial charge, q_0.

(iii) Resistors are left in the network without change.

The equivalent circuit for the three parameters (R, L and C) at $t = 0^+$ and $t = \infty$ are shown in Table 3.2.

Table 3.2: The Equivalent circuit of the parameter

Element with initial conditions	Equivalent circuit at $t = 0^+$	Equivalent circuit at $t = \infty$



OC - Open circuit

SC - Short circuit

Some of the important characteristics of the inductor/capacitor are now discussed :

1. There is no voltage/current across/through the inductor/capacitor if the current/voltage through/across it is not changing with time. Therefore, the inductor/capacitor is the short/open circuit to d.c.
2. A finite amount of energy can be stored in the inductor/capacitor even if the voltage/current across/through the inductor/capacitor is zero, such as when the current/voltage through/across it is constant.
3. The inductor/capacitor never dissipates energy, but only stores it. Although this is true for the mathematical model, it is not true for a physical inductor/capacitor.

3.4. VARIOUS RESPONSES

3.4.1. Transient Response: (*Transient means short lived*)

The values of voltage and current during the transient period are known as the transient responses. It is also defined as the part of the total time response that goes to zero as time becomes large. It depends upon the network elements alone and independent of the forcing function (source).

The complementary function is the solution of the differential equation with forcing function set to zero and hence, the complementary function represents the *source-free response* or simply *free response* or *natural response* or *transient response.*

3.4.2. Steady State Response:

The values of voltage and current after the transient has died out are known as the steady-state responses. It is also defined as the part of the total time response which remains after the transient has passed. It depends on both the network elements and forcing function.

The particular integral represents the *forced response* or *steady state response*. It satisfies the differential equation but not the initial conditions.

Note:
The complete or total response of a network is the sum of the transient response and the steady state response or the sum of the natural and forced responses.

3.4.3. Zero Input Response:

The values of voltage and current that result from initial conditions when the excitation (input) or forcing function is zero are known as zero input responses.

3.4.4. Zero State Response:

The values of voltage and current for an excitation which is applied when all initial conditions are zero are known as zero state responses. Such a network is also said to be at rest or initially relaxed.

3.5. TRANSIENT RESPONSE OF SERIES R-L-C CIRCUIT HAVING DC EXCITATION

EXAMPLE 3.1 **Consider the RLC series circuit shown in figure 3.2. V_s = 2V; R = 6Ω; L = 2H; C = 0.25F. Determine $i(0^+)$; $\frac{di}{dt}(0^+)$, $\frac{d^2i}{dt^2}(0^+)$ and $i(t)$.**

Note : The forcing function may be a direct voltage or current source, a ramp function, an exponential function or a sinusoidal source or any other function.

the inductor

Solution : The differential equation for the current in the circuit of figure 3.2 is given by Kirchhoff's law as

Int.-diff b

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)\,dt = V_s \qquad \text{...(1)}$$

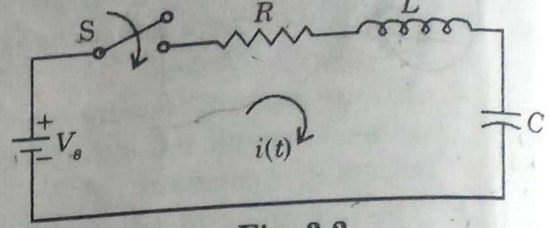


Fig. 3.2.

Differentiating and using numerical values for R, L and C gives

$$2\frac{d^2i(t)}{dt^2} + 6\frac{di(t)}{dt} + \frac{1}{0.25}i(t) = 0$$

or

$$\frac{d^2i(t)}{dt^2} + 3\frac{di(t)}{dt} + 2i(t) = 0 \qquad \text{...(2)}$$

$p^2 i(t) + 3p i(t) + 2 i(t) =$

Substituting p^2 for $\frac{d^2i(t)}{dt^2}$ and p for $\frac{di(t)}{dt}$; thus

$$p^2 + 3p + 2 = 0$$

root

This equation has the roots $p_1 = -1$ and $p_2 = -2$, so that the general solution is

$$i(t) = K_1 e^{-t} + K_2 e^{-2t} \qquad \text{...(3)}$$

The constants K_1 and K_2 can be evaluated for a specific problem by a knowledge of the initial conditions.

always find K1 & K2 by using intial condi

P_1 & P_2 are roots

If the switch S is closed at $t = 0$, then

$$\boldsymbol{i(0^+) = 0} \qquad \text{...(4)}$$

(because current cannot change instantaneously in the inductor or inductor behaves as a open circuit at $t = 0^+$)

In equation (1), the second and third voltage terms are zero at the instant of switching, $Ri(0^+)$ being zero becaue $i(0^+) = 0$ and $\frac{1}{C}\int_{-\infty}^{0^+} i\,dt$ being zero because it is the initial voltage across the capacitor.

Hence

$V_s = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$

$$\frac{di}{dt}(0^+) = \frac{V_s}{L} = \frac{2}{2}$$

or

$$\boldsymbol{\frac{di}{dt}(0^+) = 1 \text{ A/sec}} \qquad \text{...(5)}$$

$V_3 = 0$ because cap => short ckt in term of vtg

From equation (2),

$i(0) = K_1 + K_2$

$$\frac{d^2i}{dt^2}(0^+) + 3\frac{di}{dt}(0^+) + 2i(0^+) = 0$$

i.e.

$$\frac{d^2i}{dt^2}(0^+) = -3 \times 1 - 2 \times 0$$

or

$$\boldsymbol{\frac{d^2i}{dt^2}(0^+) = -3 \text{ A/sec}^2} \qquad \text{...(6)}$$

The above two initial conditions (4) and (5), substituted into the general solution, equation (3), give the equations

$$K_1 + K_2 = 0 \quad \text{and} \quad -K_1 - 2K_2 = 1$$

The solution of these equations is $K_1 = 1$, and $K_2 = -1$, hence the particular solution is

$$\boldsymbol{i(t) = e^{-t} - e^{-2t}} \text{ A}$$

A plot of the separate parts and their combination is shown in figure 3.3.

R-C = 1st order
R-L = 1st order
L-C = 2nd order
R-L-C = 2nd order

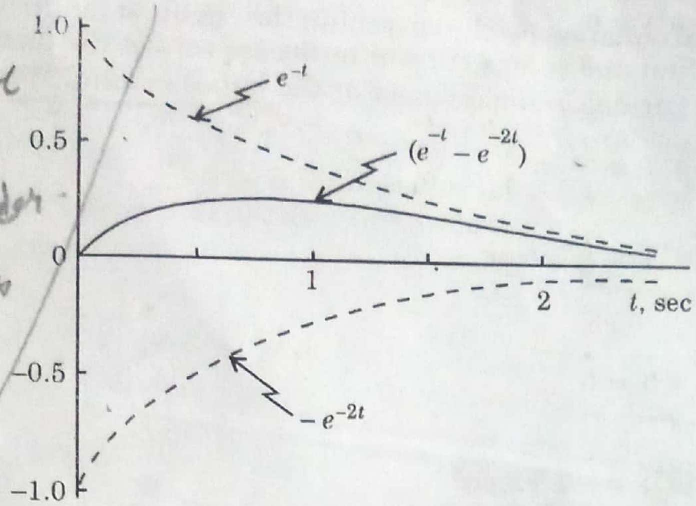


Fig. 3.3. *Response including the two component parts that sum to give the response.*

EXAMPLE 3.2 **Consider the RLC parallel circuit shown in figure 3.4. $I_s = 2A$;**

$$R = \frac{1}{16}\,\Omega;\ L = \frac{1}{16}\,H;\ C = 4F.\ \text{Determine } v(0^+),\ \frac{dv}{dt}(0^+),\ \frac{d^2v}{dt^2}(0^+) \text{ and } v(t).$$

$v = L\frac{di(t)}{dt}$

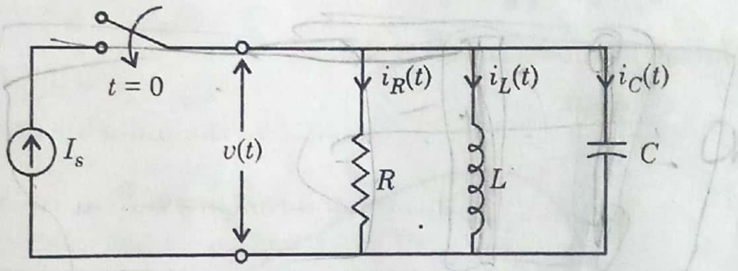


Fig. 3.4.

$v_L(0^+) = i_L(0^-)$

$V_C(0^+) = V_C(0^-)$

Solution: The circuit equation by *KCL* is

$$I_s = i_R(t) + i_L(t) + i_C(t)$$

or
$$I_s = \frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt + C\frac{dv(t)}{dt} \quad ...(1)$$

Differentiating and using numerical values for R, L and C gives

$$0 = 16\frac{dv(t)}{dt} + 16v(t) + 4\frac{d^2v(t)}{dt^2}$$

or
$$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 4v(t) = 0 \quad ...(2)$$

Substituting p^2 for $\frac{d^2v(t)}{dt^2}$ and p for $\frac{dv(t)}{dt}$; thus

$$p^2 + 4p + 4 = 0$$

This equation has the repeated roots $p_{1,2} = -2$

Thus, the general solution to our problem with repeated roots

$$v(t) = K_1\,e^{-2t} + K_2\,t\,e^{-2t} \quad ...(3)$$

To obtain a particular solution for this problem will require knowledge of two initial conditions. From the circuit of figure 3.4, $v(0^+)$ must equal zero, since the capacitor acts as a short circuit at the initial instant. *i.e.*,

$$v(0^+) = 0 \qquad \text{...(4)}$$

In equation (1), the first and second current terms are zero at the instant of switching, because $v(0^+) = 0$ and there is no current in the inductor at the initial instant. Hence

$$\frac{dv}{dt}(0^+) = \frac{I_s}{C} = \frac{2}{4} \qquad \text{...(5)}$$

or

$$\frac{dv}{dt}(0^+) = \frac{1}{2}\ \text{V/sec}$$

From equation (2),

$$\frac{d^2v}{dt^2}(0^+) + 4.\frac{1}{2} + 4.0 = 0 \qquad \text{...(6)}$$

or

$$\frac{d^2v}{dt^2}(0^+) = -2\ \text{V/sec}^2$$

The above two initial conditions (4) and (5), substituted into equation (3), give the equations

$$K_1 = 0 \quad \text{and} \quad -2K_1 + K_2 = \frac{1}{2} \quad \text{or} \quad K_2 = \frac{1}{2}$$

Therefore, the particular solution of our problem is

$$v(t) = \frac{1}{2}\, t \cdot e^{-2t}\ \text{V} \qquad \text{...(7)}$$

A plot of this solution is shown in figure 3.5.

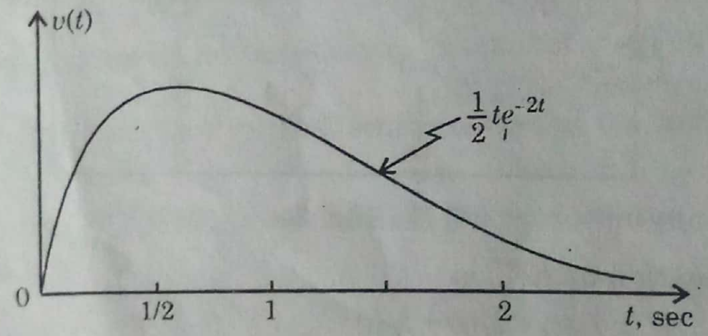


Fig. 3.5. Voltage response of network of Figure 3.4 as given by equation (7).

3.6. TRANSIENT RESPONSE OF SERIES R-L CIRCUIT HAVING DC EXCITATION

EXAMPLE 3.3 **Consider a series *R-L* circuit, as shown in figure 3.6. The switch S is closed at time $t = 0$. Find the current $i(t)$ through and voltage across the resistor and inductor.**

Solution: Applying *KVL*,

G.Q.

$$L\frac{di(t)}{dt} + Ri(t) = V$$

or

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V}{L}$$

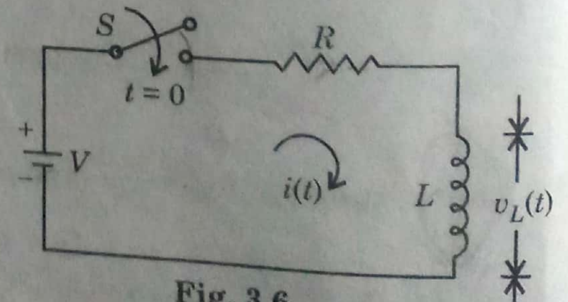


Fig. 3.6.

General solution of this differential equation is given as

$$i(t) = \frac{V}{R} + K e^{-\frac{R}{L}t}$$

Since inductor behaves as a open circuit at switching

$$i(0^+) = 0$$

$\therefore$

$$0 = \frac{V}{R} + K$$

or $$K = -\frac{V}{R}$$

Therefore, $$i(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t} = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

And voltage across the resistor and inductor are given as

$$v_R(t) = i(t)\cdot R$$

or $$\boldsymbol{v_R(t) = V\left(1 - e^{-\frac{R}{L}t}\right)}$$

$$v_L(t) = L\frac{di(t)}{dt} \qquad [\text{or } V - v_R(t)]$$

$$= L.\frac{V}{R}\left[0 - \left(\frac{-R}{L}\right)e^{-\frac{R}{L}t}\right]$$

or $$\boldsymbol{v_L(t) = Ve^{-\frac{R}{L}t}}$$

at $t = 0$; $i(t) = 0$ and $v_L(t) = V$, $v_R(t) = 0$

at $t = \infty$; $i(t) = \frac{V}{R}$ and $v_L(t) = 0$, $v_R(t) = V$

at $t = \frac{L}{R} = \tau$; $i(t) = \frac{V}{R}(1 - e^{-1}) = 0.632\frac{V}{R}$

and $v_L(t) = V.e^{-1} = 0.368\ V$, $v_R(t) = 0.632$ V

These $i(t)$ and $v_R(t)$ and $v_L(t)$ are plotted in figure 3.7(a) and (b).

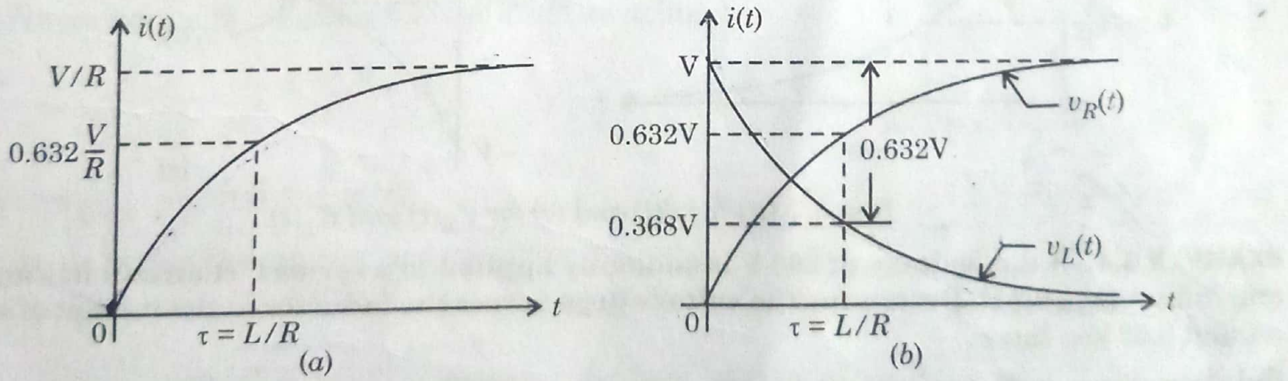


Fig. 3.7. *(a) for $i(t)$ and (b) for $v_R(t)$ and $v_L(t)$.*

> ***Note***
>
> $\tau = \frac{L}{R}$ is known as the time constant of the circuit and is defined as the interval after which current or voltage changes 63.2% of its total change.

Let us now analyse another transient condition of the R-L circuit as the circuit reaches at steady state (at $t = \infty$) and suddenly the voltage is withdrawn by opening the switch S and throwing it to S' as shown in figure 3.7 (c) at $t = 0$.

Then,

$$L\frac{di'(t)}{dt} + Ri'(t) = 0$$

General solution of this differential equation is given as

$$i'(t) = Ke^{-\frac{R}{L}t}$$

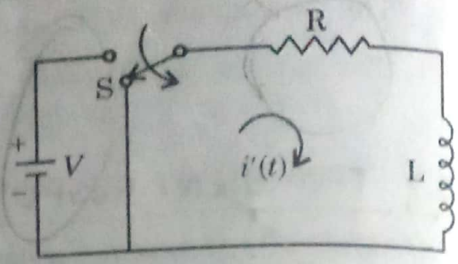


Fig. 3.7 (c).

However, at $t = 0^+$ the inductor keep the steady state current $i'(0^+) = i(\infty) = \frac{V}{R}$

Here $i(0^+)$ means after switching when the circuit is in steady state.

or $$\frac{V}{R} = K'e^0 \text{ or } K' = \frac{V}{R}$$

Therefore, $$i'(t) = \frac{V}{R}e^{-\frac{R}{L}t}$$

The corresponding voltages across the resistor and inductor are then given as

$$v'_R(t) = i'(t) \cdot R = Ve^{-\frac{R}{L}t}$$

and $$v'_L(t) = L\frac{di'(t)}{dt} = -Ve^{-\frac{R}{L}t}$$

$i_L(0^-) = i_L(0^+)$

Here $i_L(0^-) = i_L(\infty)$

[As $v'_R(t) + v'_L(t) = 0$]

These $i'(t)$ and $v'_R(t)$ and $v'_L(t)$ are plotted in figure 3.7 (*d*) and (*e*).

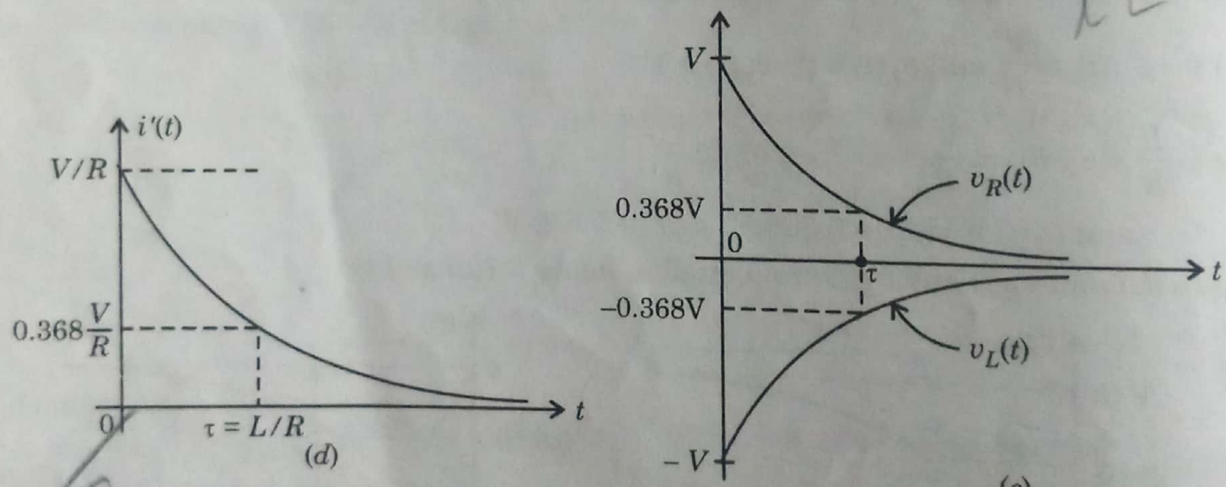


Fig. 3.7 (*d*) *for $i'(t)$ and (e) for $v'_R(t)$ and $v'_L(t)$.*

EXAMPLE 3.4 A d.c. voltage of 200 V is suddenly applied to a series L-R circuit having $R = 20\Omega$ and inductance 0.2 H. Determine the voltage drop across the inductor at the instant of switching on and 0.02 sec later.

Solution: Since inductor initially behaves as a open circuit at $t = 0^+$, then $i(0^+) = 0$

Hence, at instant of switching, the voltage drop across inductor is **200 V.**

After the instant of switching the current is given by

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) = \frac{200}{20}\left(1 - e^{-\frac{20}{0.2}t}\right) = 10\,(1 - e^{-100t}).$$

at $$t = 0.02 \text{ sec}, \; i(t) = 10(1 - e^{-100 \times 0.02}) = 8.646\text{A}$$

$$Ri(t) + L\frac{di(t)}{dt} = 200$$

or $$L\frac{di(t)}{dt} = 200 - Ri(t)$$

The voltage across the inductor after lapse of 0.02 sec from switching is given as

$$L\frac{di(0.02)}{dt} = 200 - 20 \times 8.646 = 27\text{V}$$

3.7. TRANSIENT RESPONSE OF SERIES R-C CIRCUIT HAVING DC EXCITATION

EXAMPLE 3.5 Consider a series R-C circuit, as shown in figure 3.8. The switch S is closed at time $t = 0$. Find the current $i(t)$ through and voltage across the resistor and capacitor.

Solution: By Kirchhoff's voltage law,

G.Q.

$$Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = V$$

or $$Ri(t) + \frac{1}{C}\int_{0}^{t} i(t)dt + v_C(0^+) = V$$

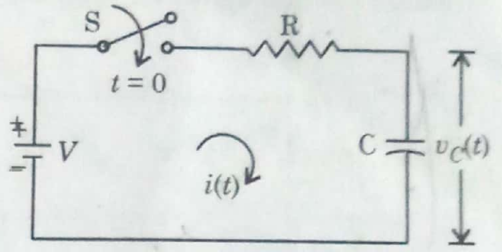


Fig. 3.8.

Assume, Initially, capacitor was uncharged, *i.e.*,

$\therefore$ $$v_C(0^+) = 0$$

Differentiating, we get

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

or $$\frac{di(t)}{dt} + \frac{1}{RC}i(t) = 0$$

General solution of this differential equation is

$$i(t) = K \cdot e^{-\frac{1}{RC}t}$$

at $$t = 0^+,\ i(0^+) = \frac{V}{R}$$

(Since capacitor behaves as a short circuit at switching)

or $$\frac{V}{R} = K{\cdot}e^0 = K$$

Therefore, $$\boldsymbol{i(t) = \frac{V}{R}e^{-\frac{1}{RC}t}}$$

And voltage across the resistor and capacitor are

$$v_R(t) = i(t)\cdot R$$

or $$\boldsymbol{v_R(t) = Ve^{-\frac{1}{RC}t}}$$

$$v_C(t) = \frac{1}{C}\int_0^t i(t)dt \qquad [\text{or } V - v_R(t)]$$

$$= \frac{1}{C}\cdot\frac{V}{R}\cdot\left(e^{-\frac{1}{RC}t}\right)\Bigg|_0^t(-RC) = -V\cdot e^{-\frac{1}{RC}t}\Bigg|_0^t = -V\left(e^{-\frac{1}{RC}t} - 1\right)$$

or $$\boldsymbol{v_C(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)}$$

at $t = 0$; $i(t) = \frac{V}{R}$ and $v_C(t) = 0$, $v_R(t) = V$

at $t = \infty$; $i(t) = 0$ and $v_C(t) = V$, $v_R(t) = 0$

at $t = RC = \tau$; $i(t) = \frac{V}{R}e^{-1} = 0.368\frac{V}{R}$

and $v_C(t) = V(1 - e^{-1}) = 0.632\ V$, $v_R(t) = 0.368\ V$

These $i(t)$ and $v_R(t)$ and $v_C(t)$ are plotted in figure 3.9 (*a*) and (*b*).

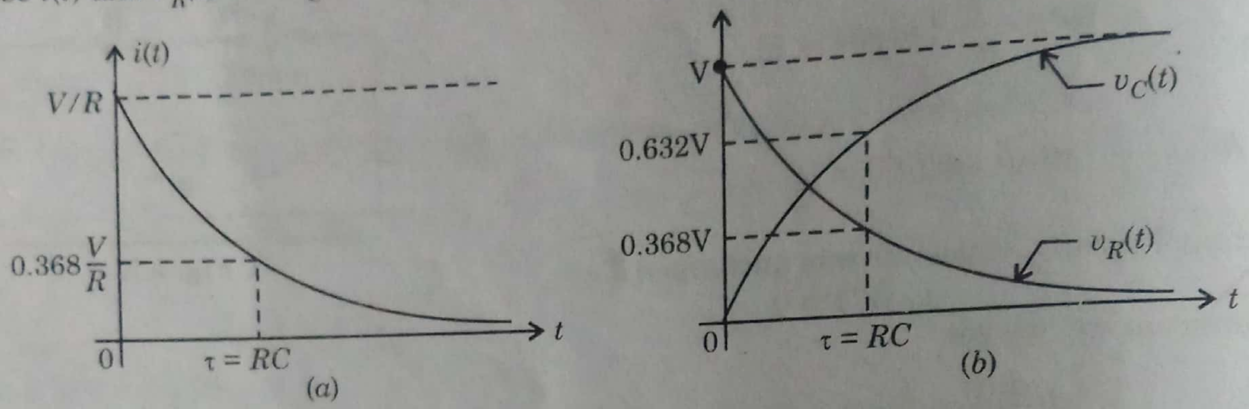


Fig. 3.9 *(a) for i(t), and (b) for $v_R(t)$ and $v_C(t)$.*

Note:
Here $\tau = RC$ is known as time constant of the circuit.

Let us now analyse another transient condition of the *R-C* circuit as the circuit reaches at steady state (at $t = \infty$) and suddenly the voltage is withdrawn by opening the switch *S* and throwing it to *S'* as shown in figure 3.9 (*c*) at $= 0$.

Then by *KVL*,

$$Ri'(t) + \frac{1}{C}\int_0^t i'(t) + v_C(0^+) = 0$$

Differentiating, we get

$$R\frac{di'(t)}{dt} + \frac{1}{C}i'(t) = 0$$

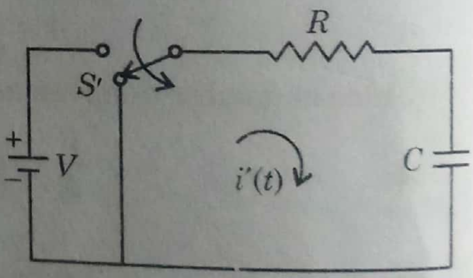


Fig. 3.9 (c).

Its solution is $i'(t) = K'e^{-\frac{1}{RC}t}$

However, at $t = 0^+$, the capacitor keeps the steady state voltage $v_C(0^+) = V$. And the direction of $i'(t)$ during discharge is negative. Thus

$$i'(0^+) = -\frac{V}{R}$$

or $-\frac{V}{R} = K'e^0$ or $K' = -\frac{V}{R}$

Therefore, $$i'(t) = -\frac{V}{R}e^{-\frac{1}{RC}t}$$

The corresponding transient voltages across the resistor and capacitor are given by

$$v'_R(t) = i'(t) \cdot R$$

or $$v'_R(t) = -Ve^{-\frac{1}{RC}t}$$

and $$v'_C(t) = \frac{1}{C}\int i'(t)\,dt$$

or $$v'_C(t) = Ve^{-\frac{1}{RC}t}$$

[Obviously $v'_R(t) + v'_C(t) = 0$]

These $i'(t)$ and $v'_R(t)$ and $v'_C(t)$ are plotted in figure 3.9 (*d*) and (*e*).

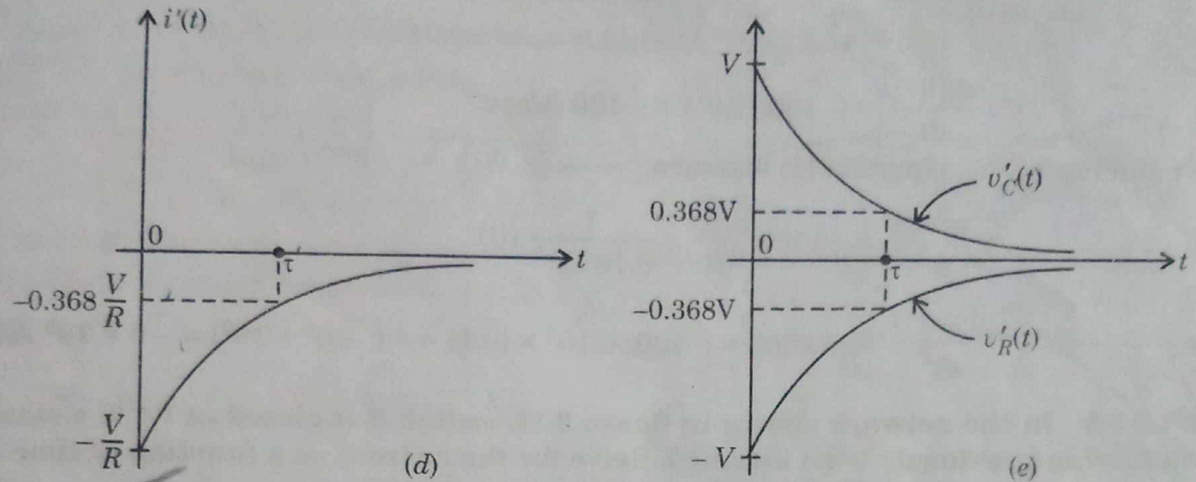


Fig. 3.9. *(d) for i' (t) and (e) for* $v'_R(t)$ *and* $v'_C(t)$.

EXAMPLE 3.6 A resistance R and 5 μF capacitor are connected in series across a 100 V d.c. supply. Calculate the value of R such that the voltage across the capacitor becomes 50 V in 5 sec after the circuit is switched on.

Solution: In case of charging, the voltage at any time across the capacitor is given as

$$v_C(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)$$

or $$50 = 100\left(1 - e^{-\frac{5}{R\times 5\times 10^{-6}}}\right)$$

or $$0.5 = \left(1 - e^{-\frac{10^6}{R}}\right)$$

or $$e^{-\frac{10^6}{R}} = 0.5$$

$\therefore$ $$R = 1.45 \times 10^6\ \Omega$$

EXAMPLE 3.7 In the given circuit as shown in figure 3.10, switch S is changed from position 'a' to 'b' at $t = 0$. Find values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

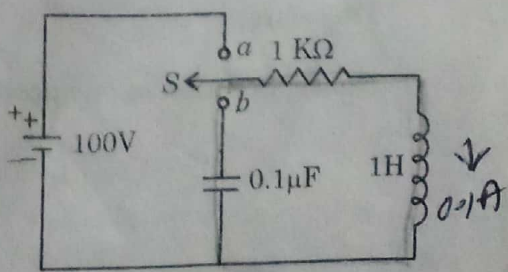


Fig. 3.10.

Solution: At position 'a':

The steady state value of current $i(0^-)$

$$= \frac{100}{1000} = 0.1 \text{ A}$$

At position 'b':

$$i(0^+) = i(0^-) = \mathbf{0.1\ A}$$

Ye voltage h

Applying *KVL*, we have

$$1000\, i(t) + 1.\frac{di(t)}{dt} + \frac{1}{0.1\times10^{-6}}\int_{-\infty}^{t} i(t)\, dt = 0 \qquad ...(1)$$

Since initially capacitor is uncharged and at switching instant capacitor behaves as a short circuit, *i.e.* last term of left hand side of above equation is zero. Then from equation (1);

$$1000\, i(t) + \frac{di(t)}{dt} = 0 \quad \text{— (2)}$$

$$\frac{di(0^+)}{dt} = -\,1000\, i(0^+) = -\,\mathbf{100\ A/sec}$$

On differentiating, equation (1) becomes; — & not (2)

$$\frac{d^2 i(t)}{dt^2} = -1000\frac{di(t)}{dt} - \frac{1}{0.1\times10^{-6}}\, i(t)$$

or

$$\frac{d^2 i(0^+)}{dt^2} = -\,1000\times(-100) - 10^7\times(0.1) = -\,(-10^5 + 10^6) = -\,\mathbf{9\times10^5\ A/sec^2}$$

EXAMPLE 3.8 In the network shown in figure 3.11, switch S is closed at $t = 0$, a steady state current having previously been attained. Solve for the current as a function of time.

Solution: Steady state current (before the switching action takes place)

$$i(0^-) = \frac{V}{R_1 + R_2}$$

(Since inductor behaves as a short circuit at $t = \infty$)

When switch is closed: R_2 is short-circuited

Applying *KVL*

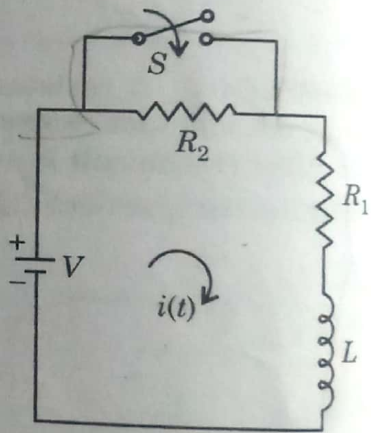


Fig. 3.11.

$$V = L\frac{di(t)}{dt} + R_1\, i(t)$$

or

$$\frac{di(t)}{dt} + \frac{R_1}{L}\, i(t) = \frac{V}{L}$$

The general solution of the above V equation is given as differential

$$i(t) = \frac{V}{R_1} + Ke^{-\frac{R_1}{L}t}$$

and

$$i(0^+) = i(0^-) = \frac{V}{R_1 + R_2}$$

or

$$\frac{V}{R_1 + R_2} = \frac{V}{R_1} + K$$

Therefore;

$$K = -\,V.\frac{R_2}{R_1(R_1 + R_2)}$$

Hence, Solution of our problem is

$$i(t) = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)}.e^{-\frac{R_1}{L}t}$$

or

$$\boldsymbol{i(t) = \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2}e^{-\frac{R_1}{L}t}\right) A}$$

$i_L(0^+) = i_L(0^-)$
$v_C(0^+) = v_C(0^-)$

EXAMPLE 3.9 **The circuit shown in figure 3.12, is in the steady state with the switch S closed. The switch is opened at $t = 0$.**

Determine voltage across the switch v_s and $\frac{dv_s}{dt}$ at $t = 0^+$.

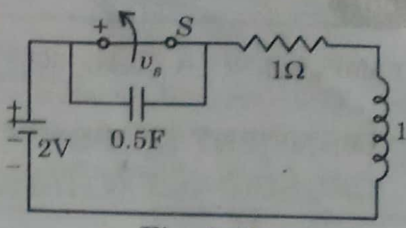


Fig. 3.12.

Solution : When, Circuit is in the steady state with the switch S closed, capacitor is short circuited *i.e.* voltage across capacitor is zero.

or $\quad v_s(0^-) = 0$, and steady state current $i(0^-) = \frac{2}{1} = 2\text{A}$.

And, when switch is opened at $t = 0$, the capacitor behaves as a short circuit, so

$$v_s \text{ at } t = 0^+ \quad \text{or} \quad v_s(0^+) = 0$$

$$i(0^+) = i(0^-) = 2\text{ A}$$

$$C\frac{dv_s}{dt} = i(t) \quad \text{or} \quad \frac{dv_s}{dt} = \frac{1}{C}i(t)$$

Hence
$$\frac{dv_s}{dt}(0^+) = \frac{1}{C}i(0^+) = \frac{1}{0.5}.2$$

or
$$\frac{dv_s}{dt}(0^+) = 4\text{ V/sec}$$

EXAMPLE 3.10 **Using classical method of solution of differential equations, find the value of $v_C(t)$ for $t > 0$ in the circuit shown in figure 3.13. Assume initial condition $v_C(0^-) = 9\text{V}$.**

Solution: Let the current in the circuit is $i(t)$. Applying KVL in the circuit; we have

$$1 = 4i(t) + v_C(t)$$

and
$$v_C(t) = 9 + \frac{1}{C}\int_0^t i(t)\,dt$$

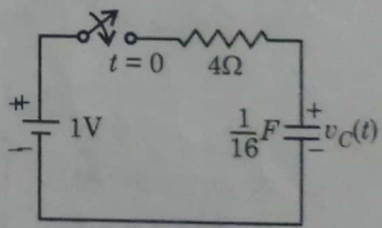


Fig. 3.13.

Therefore
$$1 = 4i(t) + 9 + 16\int_0^t i(t)\,dt$$

or
$$4i(t) + 16\int_0^t i\,dt = -8$$

or
$$i(t) + 4\int_0^t i(t)\,dt = -2$$

On differentiating,

$$\frac{di(t)}{dt} + 4i(t) = 0$$

The general solution of the above differential equation is

$$i(t) = Ke^{-4t}$$

Since initial voltage across capacitor is 9V, therefore, initial current

$$i(0^+) = \frac{1-9}{4} = -2 = K\cdot e^0$$

$\therefore$
$$K = -2$$

So, The value of the current; $i(t) = -2e^{-4t}$

Then
$$v_C(t) = 9 + \frac{1}{C}\int_0^t i(t)\,dt$$

$V_L(0^+) = -V_C(0^+)$

$$v_C(t) = 9 + 16.\left(\frac{-2e^{-4t}}{-4}\right)\Bigg|_0^t = 9 + 8e^{-4t}\Big|_0^t = 9 + (8e^{-4t} - 8)$$

or $$v_C(t) = 1 + 8\,e^{-4t}\ \mathrm{V}$$

EXAMPLE 3.11 The circuit shown in figure 3.14, is in the steady state with the switch S closed. The switch is opened at $t = 0$. Determine $i_L(t)$ in the circuit.

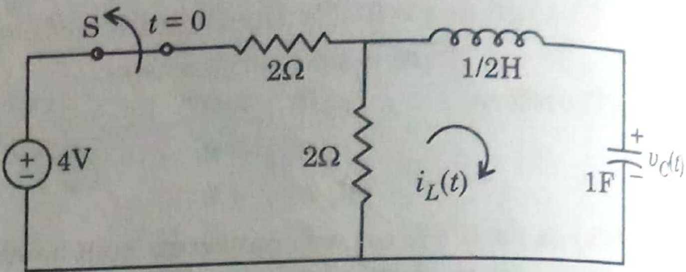


Fig. 3.14.

Solution: At steady state with the switch S closed. The capacitor behaves as a open circuit. Therefore $v_C(0^-) = 2\ V$ and $i_L(0^-) = 0$

Now, switch S is opened, then applying KVL with $v_C(0^+) = v_C(0^-)$ and

$$i_L(0^+) = i_L(0^-)$$

$$2i_L(t) + \frac{1}{2}\frac{di_L(t)}{dt} + \frac{1}{1}\int_0^t i_L(t) + v_C(0^-) = 0$$

On Differentiating, we have

$$2\frac{di_L(t)}{dt} + \frac{1}{2}\frac{d^2 i_L(t)}{dt^2} + i_L(t) = 0$$

$\frac{1}{2}\frac{d(i_L 0^+)}{dt} = -$

or $$\frac{d^2 i_L(t)}{dt^2} + 4\frac{di_L(t)}{dt} + 2i_L(t) = 0$$

Characteristic equation is

$$p^2 + 4p + 2 = 0$$

having the roots $\quad p_1 = -3.414,\ p_2 = -0.586$

Thus $\quad i_L(t) = K_1\, e^{-3.414t} + K_2\, e^{-0.586\,t}$

$i_L(0^+) = 0$ requires that $K_1 + K_2 = 0$

Now $v_L(0^+) = -v_c(0^+) = -2 = L\dfrac{di_L(0^+)}{dt}$

or $$\frac{di_L(0^+)}{dt} = \frac{-2}{L} = \frac{-2}{\frac{1}{2}} = -4$$

Another method at $t = 0^+$ C is SC ∴ in ① $\frac{1}{2}\frac{di_L(t)}{dt} + 2 = 0$, $\frac{di_L(0^+)}{dt} = -4$

$$-3.414\,K_1 - 0.586\,K_2 = -4$$

Solving for K_1 and K_2 yields

$$K_1 = 1.414 \quad \text{and} \quad K_2 = -1.414$$

finally, $$i_L(t) = 1.414(e^{-3.414t} - e^{-0.586\,t})\ \mathrm{A}$$

EXAMPLE 3.12 The 12 V battery in figure 3.15 is disconnected (opened) at $t = 0$. Find the inductor current and voltage as a function of time.

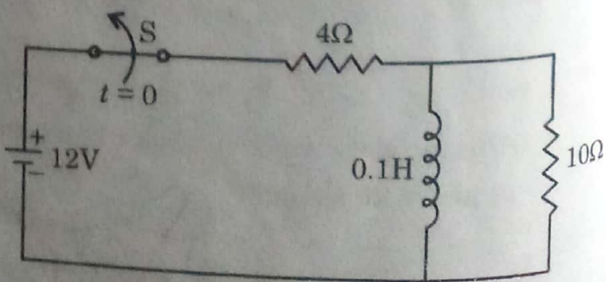


Fig. 3.15.

Solution: Assume the switch S has been closed for a long time before $t = 0$. The inductor behaves as a short circuit. *i.e.*

$$v_L(0^-) = 0$$

and $$i_L(0^-) = \frac{12}{4} = 3\mathrm{A}$$

After the battery is disconnected, at $t > 0$, Applying KVL

$$0.1\frac{di_L(t)}{dt} + 10i_L(t) = 0$$

$$\frac{di_L(t)}{dt} + 100i_L(t) = 0$$

This gives $i_L(t) = K e^{-100t}$

Since $i_L(0^+) = i_L(0^-) = 3 = K \cdot e^0.$

or $K = 3$

Therefore, $i_L(t) = \mathbf{3e^{-100\,t}}$

$$v_L(t) = L\frac{di_L(t)}{dt} = 0.1 \times 3.\ (-100)\ e^{-100\,t} = \mathbf{-30\ e^{-100\,t}\ V}$$

EXAMPLE 3.13 The switch in figure 3.16 has been in position 1 for a long time; it is moved to 2 at $t = 0$. Obtain the expression for i, for $t > 0$. *(U.P.T.U., 2002)*

Solution: With the switch on 1,

$$i\,(0^-) = \frac{50}{40} = 1.25 \text{ A}$$

When switch at 2,

$$i(0^+) = i(0^-) = 1.25 \text{ A}$$

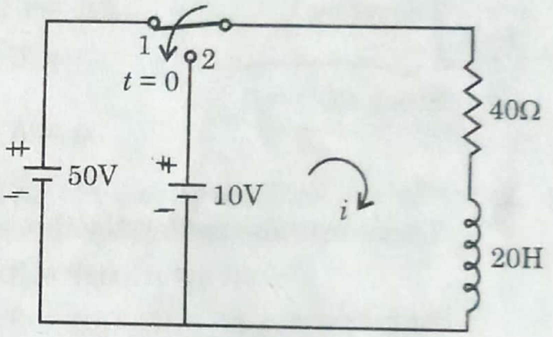


Fig. 3.16.

Applying *KVL*,

$$10 = 40\,i(t) + 20\frac{di(t)}{dt}$$

or $$\frac{di(t)}{dt} + 2i\,(t) = 0.5$$

or $$i(t) = \frac{0.5}{2} + Ke^{-2t}$$

Putting $i(0^+) = 1.25$ in above equation

$$1.25 = 0.25 + K$$

or $K = 1.00$

Therefore,

Doubt

$$\mathbf{i(t) = 0.25 + e^{-2t}\ A}$$

EXAMPLE 3.14 In figure 3.17, the switch S is closed. Find the time when the current from the battery reaches to 500 mA.

Solution: Let the current through 50 Ω be I_1 and through 70 Ω (or 100 μF) be I_2 after the switch S is closed.

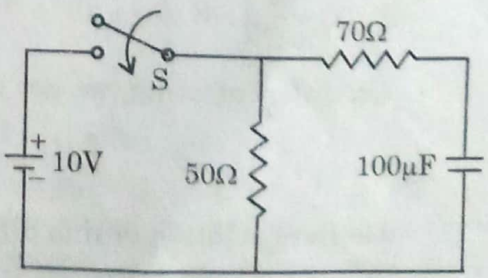


Fig. 3.17.

$$I_1 = \frac{10}{50} = 0.2 \text{ A} = 200 \text{ mA}$$

However, $I = I_1 + I_2$

[I being the current from the supply]

or $500 = 200 + I_2$

$\therefore$ $I_2 = 300$ m A

This I_2 equal to $\frac{10}{70}\ e^{\frac{t}{70\times100\times10^{-6}}}$ $\quad [\therefore\ i_C(t) = \frac{V}{R}\ e^{-t/RC}]$

This gives $\boldsymbol{t}$ **= 5.2 msec.**

EXAMPLE 3.15 The circuit of figure 3.18 was under steady state before the switch was opened. If $R_1 = 1\Omega$, $R_2 = 2\Omega$ and $C = 0.167$ F, determine $v_c(0^-)$ and $v_c(0^+)$. Also find $i(0^+)$.

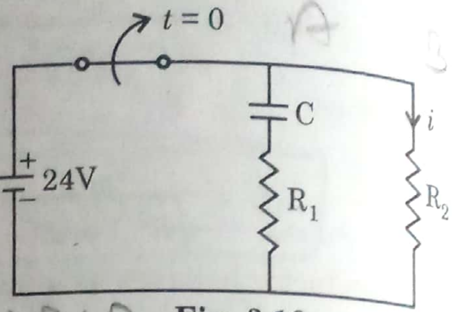


Fig. 3.18.

Solution: Since at steady state capacitor behaves as a open circuit or the voltage can not change instantaneously in the capacitor, we have

$$v_c(0^-) = v_c(0^+) = 24 \text{ V}$$

After the switch is opened at $t = 0^+$,

$$v_c = v_{R_1} + v_{R_2}$$

or $$24 = i(1+2)$$

Hence, $$i(0^+) = 8\text{A}$$

At t=0+ switch is open ∴ loop is ABCD

EXAMPLE 3.16 Consider the first order *R-L* series circuit shown in figure 3.19 with $R = 5\ \Omega$, $L = 1\text{H}$, $V_s = 48$ V.

Determine:

(*a*) the expression $i(t)$, $V_R(t)$, $V_L(t)$, and $\frac{di}{dt}$ for $t \geq 0$;

(*b*) $\frac{di}{dt}$ at $t = 0^+$;

(*c*) the time at which $V_R = V_L$;

(*d*) the resistance is decreased from 5 to 4 Ω at $t = 0.5$ sec, determine $i(t)$.

Assume $i_L(0^-) = 0$

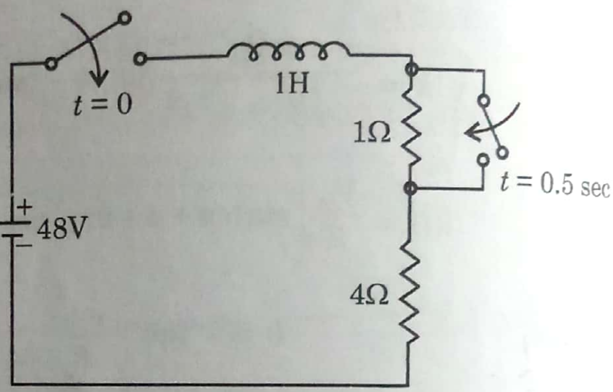


Fig. 3.19.

Solution: (*a*) Applying *KVL*,

$$48 = 1.\frac{di(t)}{dt} + 5i(t)$$

This gives, $$i(t) = 9.6\,(1 - e^{-5t}) \text{ A} \quad (\text{since } i_L(0^-) = 0)$$

or iLt coz this current passes through L also.

Then $$V_L(t) = L\frac{di}{dt} = 48\, e^{-5t} \text{ V}$$

And, $$V_R(t) = Ri(t) = 48\,(1 - e^{-5t}) \text{ V}$$

$$\frac{di}{dt} = 48e^{-5t} \text{ A/sec.}$$

(*b*) $$\frac{di}{dt}(\text{at } t = 0^+) = 48e^{-5t}\Big|_{t=0^+} = 48 \text{ A/sec.}$$

(*c*) Say at time $t = t_1$ at which $V_R = V_L$, *i.e.*

$$48(1 - e^{-5t1}) = 48e^{-5t1}$$

Solving, we get $$t_1 = 0.1386 \text{ sec}$$

(*d*) at $t = t_2 = 0.5$ sec

$$i(t_2) = 9.6\,[1 - e^{-5 \times 0.5}] = 8.812 \text{ A}$$

Now, let resistance is decreased from 5 to 4Ω at $t = 0$. Then $i(0^-) = 8.812$ A. Then applying *KVL*,

$$48 = L\frac{di(t)}{dt} + 4\,i(t)$$

$$\frac{di(t)}{dt} + 4\,i(t) = 48$$

$$i(t) = 12 - Ke^{-4t}$$

at $$t = 0^-, \quad i(t) = 8.812$$

Closed open OT
Voltage short
OT

This gives $K = 3.188$

Therefore, $i(t) = 12 - 3.188\, e^{-4t}\, A$

EXAMPLE 3.17 **For the network shown in figure 3.20. Find i_1, i_2 and V_1 at**

(*a*) $t = 0^-$ (*b*) $t = 0^+$

(*c*) $t = \infty$ (*d*) $t = 50$ m sec.

(*I.P. Univ., 2001*)

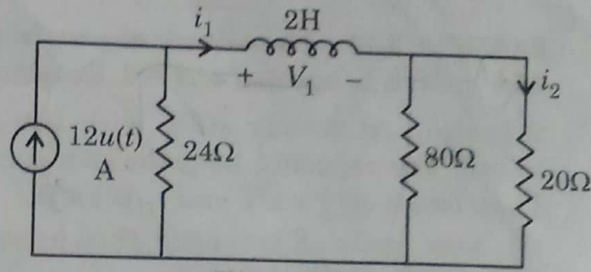


Fig. 3.20.

Solution: (*a*) **at $t = 0^-$:** By the Definition of step signal. at $t = 0^-$, There is no energy source.

Therefore,
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$V_1(0^-) = 0$$

(*b*) **at $t = 0^+$:** First we convert current source of $12u(t)$ in an equivalent voltage source $= 12\,u(t) \times 24 = 288\,u(t)$ as shown in figure 3.21.

Applying *KVL*,

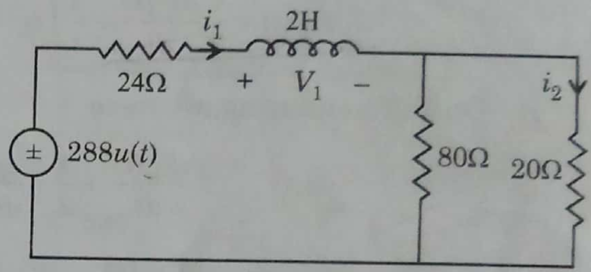


Fig. 3.21.

$$288\,u(t) = 24i_1 + 2\frac{di_1}{dt} + 80\,(i_1 - i_2) \quad ...(1)$$

and
$$i_2 = i_1 \cdot \frac{80}{20+80} \quad \text{(By current division rule)}$$
$$i_2 = 0.8\,i_1 \quad ...(2)$$

From equations (1) and (2), we have

$$288\,u(t) = 24\,i_1 + 2\frac{di_1}{dt} + 80(i_1 - .08i_1) = 40\,i_1 + 2\frac{di_1}{dt}$$

V = IR

or
$$\frac{di_1}{dt} + 20\,i_1 = 144\,u(t)$$

The general solution of above differential equation is given as

$$i_1(t) = \frac{144}{20} + Ke^{-20t}$$

at $t = 0^+$, inductor behaves as a open circuit, *i.e.*

$$i_1(0^+) = 0, \text{ This gives } K = -\frac{144}{20}$$

Therefore,
$$i_1(t) = \frac{144}{20}(1 - e^{-20t}) = 7.2\,(1 - e^{-20\,t}) \quad ...(3)$$

and
$$i_2(t) = 0.8i_1\,(t) = 5.76\,(1 - e^{-20\,t}) \quad ...(4)$$

so
$$i_1(0^+) = \mathbf{0}; \quad i_2\,(0^+) = \mathbf{0}; \quad V_1(0^+) = \mathbf{288\ V}$$

(*c*) **at $t = \infty$:** From equations (3) and (4),

$$i_1\,(\infty) = \mathbf{7.2\ A}; \quad i_2(\infty) = \mathbf{5.76\ A}$$

and
$$V_1\,(\infty) = L\frac{di_1(\infty)}{dt} = 2 \times 0 = \mathbf{0}$$

(*Alternatively:* since inductor behaves as a short circuit at $t \to \infty$ so $V_1(\infty) = 0$)

(*d*) **at $t = 50$ msec:**

$$i_1 = 7.2\,(1 - e^{-20 \times 50 \times 10^{-3}}) = 7.2(1 - e^{-1}) = \mathbf{4.55A}$$
$$i_2 = 5.76\,(1 - e^{-1}) = \mathbf{3.64A}$$

(or from equation (2) $i_2 = 0.8\,i_1 = 0.8 \times 4.55 = 3.64$ A)

$$V_1 = L\frac{di_1}{dt}\bigg|_{\text{at } 50\text{ msec}} = 2 \times (7.2 \times 20\, e^{-20\times 50\times 10^{-3}}) = 105.95 \text{ V}$$

3.8. TRANSIENT RESPONSE OF SERIES R-L CIRCUIT HAVING SINUSOIDAL EXCITATION

EXAMPLE 3.18 **Consider a series R-L circuit excited by a sinusoidal voltage source as shown in figure 3.22. The switch S is closed at time $t = 0$. Find the response (current) $i(t)$.**

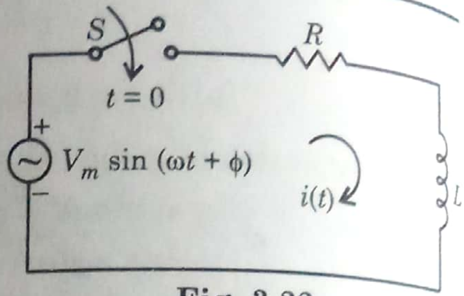


Fig. 3.22.

Solution: Applying KVL,

$$L.\frac{di(t)}{dt} + R.i(t) = V_m \sin(\omega t + \phi)$$

or
$$\frac{di(t)}{dt} + \frac{R}{L}.i(t) = \frac{V_m}{L}\sin(\omega t + \phi) \qquad \ldots(1)$$

This is a non-homogeneous equation. The current $i(t)$ consists of the sum of complementary function $i_c(t)$ and particular integral $i_p(t)$, *i.e.*

$$i(t) = i_c(t) + i_p(t)$$

The complementary function of equation (1) is

$$\boxed{i_c(t) = K.e^{-\frac{R}{L}t}}$$

cos θ = c

And the particular integral of equation (1) is

$$i_p(t) = e^{-\frac{R}{L}t}\int \frac{V_m}{L}\sin(\omega t + \phi).e^{\frac{R}{L}t}\, dt = \frac{V_m.e^{-\frac{R}{L}t}}{2jL}\int\left\{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right\}e^{\frac{R}{L}t}\, dt$$

$$= \frac{V_m.e^{-\frac{R}{L}t}}{2jL}.\left[\frac{e^{j(\omega t+\phi)+\frac{R}{L}t}}{j\omega + \frac{R}{L}} - \frac{e^{-j(\omega t+\phi)+\frac{R}{L}t}}{-j\omega + \frac{R}{L}}\right]$$

$$= \frac{V_m}{2jL}\left[\frac{e^{j(\omega t+\phi)}\left(-j\omega + \frac{R}{L}\right) - e^{-j(\omega t+\phi)}\left(j\omega + \frac{R}{L}\right)}{\left(j\omega + \frac{R}{L}\right)\left(-j\omega + \frac{R}{L}\right)}\right] \Rightarrow \frac{V_m}{2jL}\left[-j\omega\, e^{-j\omega t} \ldots\right.$$

$$= \frac{V_m}{L\left(\omega^2 + \frac{R^2}{L^2}\right)}\left[\frac{R}{L}\sin(\omega t + \phi) - \omega\cos(\omega t + \phi)\right]$$

$$= \frac{V_m}{R^2 + \omega^2 L^2}\left[R\sin(\omega t + \phi) - \omega L\cos(\omega t + \phi)\right]$$

This can be reduced to a single sinusoid in the form

$$i_p(t) = \frac{V_m}{R^2 + \omega^2 L^2}\left[C\sin(\omega t + \phi + \theta)\right]$$

where C and θ can be determined as

$$C\cos\theta = R \quad \text{and} \quad C\sin\theta = -\omega L$$

i.e.,
$$C = \sqrt{R^2 + \omega^2 L^2} \quad \text{and} \quad \theta = -\tan^{-1}\frac{\omega L}{R}$$

Substituting C and θ,

$$i_p(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t+\phi-\tan^{-1}\frac{\omega L}{R}\right)$$

Therefore,

$$\boxed{i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t+\phi-\tan^{-1}\frac{\omega L}{R}\right)+Ke^{-\frac{R}{L}t}}$$

Since inductor behaves as a open circuit at switching

$\therefore \quad i(0^+) = 0$

or $$0 = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi-\tan^{-1}\frac{\omega L}{R}\right)+K$$

or $$K = -\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi-\tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore \quad i(t) = \frac{V_m}{Z}\left(\sin(\omega t+\phi+\theta)-\sin(\phi+\theta)e^{-\frac{R}{L}t}\right)$$

where, $$\theta = -\tan^{-1}\frac{\omega L}{R}$$

and $$Z = \sqrt{R^2+\omega^2L^2}\ ;\ \text{impedance of } R\text{-}L \text{ circuit.}$$

The $i(t)$ is plotted in figure 3.23.

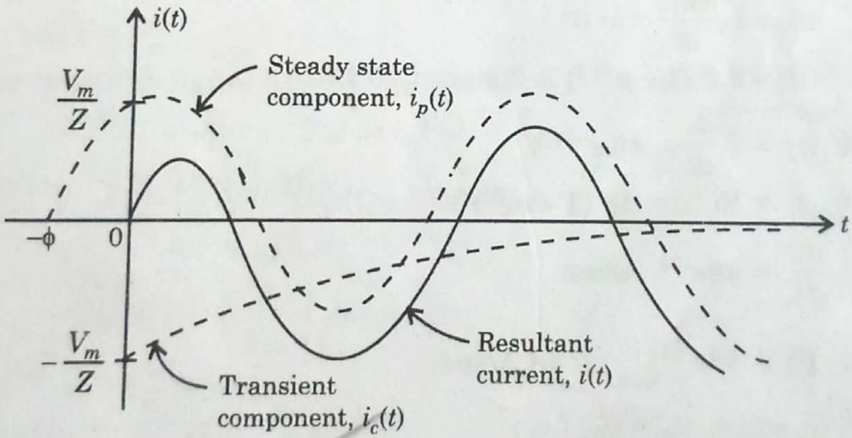


Fig. 3.23. *Variation of i(t) with time t.*

It is observed that if the angle ϕ which represents the angle of the sinusoid at the time switch S is closed, has the value

$$\phi = \theta\left(=-\tan^{-1}\frac{\omega L}{R}\right)$$ $\Rightarrow \phi = \tan^{-1}\frac{\omega L}{R}$

the constant K will have zero value, and the transient current $i_c(t)$ will vanish. In other words, if the switch is closed at the proper instant, there will be no transient.

The first term of $i(t)$ is the steady state current which lags the applied voltage by $\theta = -\tan^{-1}\left(\frac{\omega L}{R}\right)$, and the second term is the transient current with decay factor $e^{-\frac{R}{L}t}$, which dies out with time constant $\frac{L}{R}$.

EXAMPLE 3.19 **A 50 Hz 300V (peak value) sinusoidal voltage is applied at $t = 0$ to a series *R-L* circuit having resistance 2.5 Ω and inductance 0.1H. Find an expression of current at any instant *t*. Also, calculate the value of the transient current, steady state current and resultant current 0.01 sec after switching on.**

Solution: Impedance of *R-L* circuit

$$Z = \sqrt{(2.5)^2 + (2\pi \times 50 \times 0.1)^2} = 31.5\ \Omega$$

$$\text{Transient current; } i_c(t) = K e^{-\frac{R}{L}t} = K e^{-25t}$$

$$\text{Steady state current; } i_p(t) = \frac{V_m}{Z} \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) = \frac{300}{31.5} \sin(314t - 85.45°)$$

(Since voltage is applied at $t = 0$, *i.e.* $\phi = 0$ and $\tan^{-1}\dfrac{314 \times 0.1}{2.5} = 85.45°$)

$$= 9.52 \sin(314t - 1.49) \qquad \text{[Since } 85.45° \equiv 1.49 \text{ radian]}$$

Therefore, $\quad i(t) = i_c(t) + i_p(t)$

ductors

$$= K e^{-25t} + 9.52 \sin(314t - 1.49)$$

≅ Since $i(0^+) = 0$

open

$\therefore \quad 0 = K + 9.52 \sin(-1.49)$

circuit or $\quad K = 9.49$

Therefore, the required expression of current is given by

$$\mathbf{i(t) = 9.49\, e^{-25t} + 9.52 \sin(314t - 1.49)}$$

At $t = 0.01$ sec, $\quad i_c(t) = 9.49\, e^{-25 \times 0.01} = \mathbf{7.39\ A}$

$$i_p(t) = 9.52 \sin(314 \times 0.01 - 1.49) = \mathbf{9.49\ A}$$

$$i(t) = i_c(t) + i_p(t) = \mathbf{16.88\ A}$$

3.9. TRANSIENT RESPONSE OF SERIES R-C CIRCUIT HAVING SINUSOIDAL EXCITATION

EXAMPLE 3.20 **Consider a series *R-C* circuit excited by a sinusoidal voltage source as shown in figure 3.24. The switch S is closed at time $t = 0$. Find the current $i(t)$.**

Solution: Applying *KVL*,

$$R.i(t) + \frac{1}{C}\int_0^t i(t)\, dt + v_c(0^+) = V_m \sin(\omega t + \phi)$$

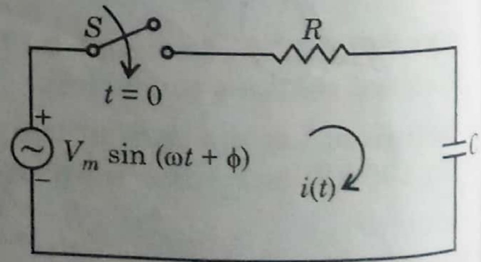


Fig. 3.24.

On differentiating, we get

$$\frac{di(t)}{dt} + \frac{1}{RC} i(t) = \frac{V_m \omega}{R} \cos(\omega t + \phi)$$

General solution of this differential equation is

$$i(t) = i_c(t) + i_p(t)$$

$$= K e^{-t/RC} + \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

[The particular integral $i_p(t)$ has been obtained in a similar way as has been done for the case of R-L circuit of Article 3.7.]

Since capacitor behaves as a short circuit at switching.

$$\therefore \quad i(0^+) = \frac{V_m \sin\phi}{R}$$

$$\text{or} \quad \frac{V_m \sin\phi}{R} = K + \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$\text{or} \quad K = \frac{V_m \sin\phi}{R} - \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$\therefore \quad i(t) = \left[\frac{V_m \sin\phi}{R} - \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)\right] e^{-\frac{t}{RC}}$$

$$+ \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$\text{or} \quad i(t) = \left[\frac{V_m \sin\phi}{R} - \frac{V_m}{Z}\sin(\phi + \theta)\right] e^{-t/RC} + \frac{V_m}{Z}\sin(\omega t + \phi + \theta)$$

where $\theta = \tan^{-1}\dfrac{1}{\omega CR}$

and $Z = \sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}$; impedance of R-C circuit.

The first term if $i(t)$ is the transient current with decay factor $e^{-t/RC}$, which dies out with time constant RC, and the second term is the steady state current which leads the applied voltage by $\theta = \tan^{-1}\dfrac{1}{\omega CR}$.

same as $\theta \geq 0$

EXAMPLE 3.21 A voltage $v = 300 \sin 314t$ is applied at $t = 2.14$ m sec to a series R-C circuit having resistance 10Ω and capacitance 200 μF. Find an expression for current. Also, find the value of current 1 msec after switching on.

Solution: It may be noted that the voltage is not applied at $t = 0$, but at ϕ where $\phi = 2.14$ msec $= 2.14 \times 10^{-3} \times 314$

or $\phi = 0.672$ rad

$2.14 \times 10^{-3} \times$

Impedance of R-C circuit

$$Z = \sqrt{(10)^2 + \left(\frac{1}{314 \times 200 \times 10^{-6}}\right)^2} = 18.8\ \Omega$$

Transient current;

$$i_c(t) = K e^{-\frac{t}{RC}} = K e^{-\frac{t}{10 \times 200 \times 10^{-6}}} = K e^{-500t}$$

Steady state current;

$$i_p(t) = \frac{V_m}{Z}\sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$= \frac{300}{18.8}\sin\left[314t + 0.672 + \tan^{-1}\left(\frac{1}{314\times 200\times 10^{-6}\times 10}\right)\right]$$

$$= 15.96 \sin(314\,t + 0.672 + 1.59)$$

Therefore, $i(t) = i_c(t) + i_p(t) = Ke^{-500t} + 15.96\sin(314t + 0.672 + 1.59)$

Since capacitor behaves as a short-circuit at switching.

$$\therefore\ i(2.14 \text{ msec}) = \frac{300\sin(314\times 2.14\times 10^{-3})}{10} = 18.67\text{ A}$$

Hence, we get

$$18.67 = K(1) + 15.96\sin(0.672 + 1.59)$$

or $$K = 18.67 - 12.29 = 6.38$$

Therefore, the required expression of current is given by

$$i(t) = 6.38\,e^{-500t} + 15.96\sin(314\,t + 2.262)$$

After 1 msec the current becomes

$$i = 6.38e^{-500\times 10^{-3}} + 15.96\sin(314\times 10^{-3} + 2.262) = 3.87 + 8.55 = 12.42\text{ A}$$

EXAMPLE 3.22 **The switch in circuit of figure 3.25 has been closed for a very long time. It opens at $t = 0$. Find $v_C(t)$ for $t > 0$ using differential equation approach.** *(U.P.T.U., 2001)*

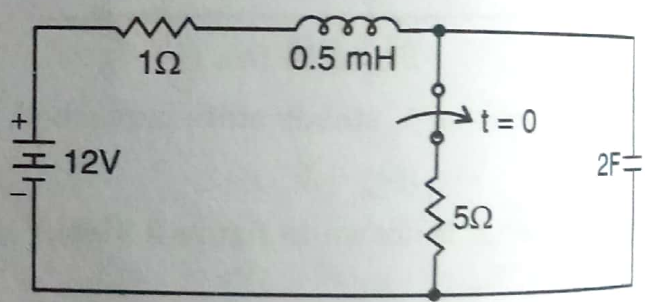


Fig. 3.25.

Solution: At steady state (with switch is closed), inductor behaves as a short circuit while capacitor bahaves as a open circuit, (circuit is shown in figure 3.26 (*a*)).

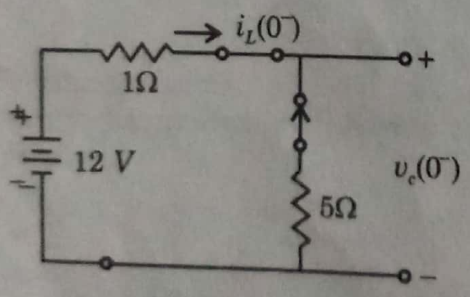


Fig. 3.26 (*a*).

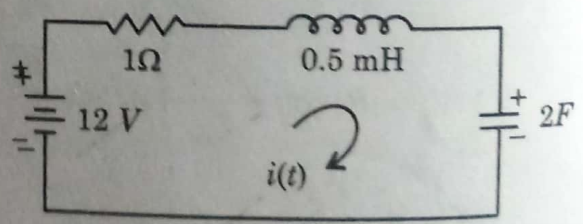


Fig. 3.26 (*b*).

Therefore,

$$i_L(0^-) = \frac{12}{1+5} = 2\text{ A}$$

$$v_c(0^-) = 2\times 5 = 10\text{ V}$$

Now at $t = 0$, switch is open, then applying KVL (circuit is shown in figure 3.26(*b*))

$$12 = 1\cdot i(t) + 0.5\frac{di(t)}{dt} + \frac{1}{2}\int_0^t i(t)\,dt + 10$$

or $$2 = i(t) + 0.5\frac{di(t)}{dt} + \frac{1}{2}\int_0^t i(t)\,dt$$

Scanned by CamScanner

On differentiating, we get

$$0 = \frac{di(t)}{dt} + 0.5\frac{d^2i(t)}{dt^2} + \frac{1}{2}i(t)$$

or $$\frac{d^2i(t)}{dt^2} + \frac{2di\,(t)}{dt} + i(t) = 0$$

Characteristic equation is

$$p^2 + 2p + 1 = 0 \quad \text{or} \quad (p+1)^2 = 0$$

having the roots, $p_{1,\,2} = -1$

Thus, the general solution

$$i(t) = (K_1 + K_2\,t)\,e^{-t}$$

At $t = 0^+$

(i) $$i(0^+) = i_L(0^-) = 2 = K_1\,e^0$$

or $$K_1 = 2$$

(ii) $$v_c(0^+) = v_c(0^-) = 10 = \frac{1}{2}\int_0^t i(t)\,dt\Big|_{t=0} + 10$$

gives, $$K_2 = 2$$

Hence, $$i(t) = (2 + 2t)\,e^{-t}$$

So $$v_c(t) = \frac{1}{2}\int_0^t i(t)\,dt + 10 = \int_0^t (1+t)\,e^{-t}dt + 10 = (1+t)\cdot -e^{-t}\Big|_0^t - \int_0^t 1.-e^{-t}\Big|_0 dt + 10$$

$$= (1+t)\,(1-e^{-t}) - \int_0^t (1-e^{-t})\,dt + 10 = 1 - t\,e^{-t} - e^{-t} + t - \left\{t + e^{-t}\right\}\Big|_0^t + 10$$

$$= 1 - t\,e^{-t} - e^{-t} + t - (t + e^{-t} - 1) + 10$$

or $$\boldsymbol{v_c(t) = 12 - 2\,e^{-t} - t\,e^{-t}}$$

EXAMPLE 3.23 **Consider the network shown in figure 3.27. The switch is initially closed for a long time. The switch is opened at $t = 0$. Find differential equation relating $i_L(t)$ with $v(t)$ and also evaluate initial conditions.** *(U.P.T.U., 2002)*

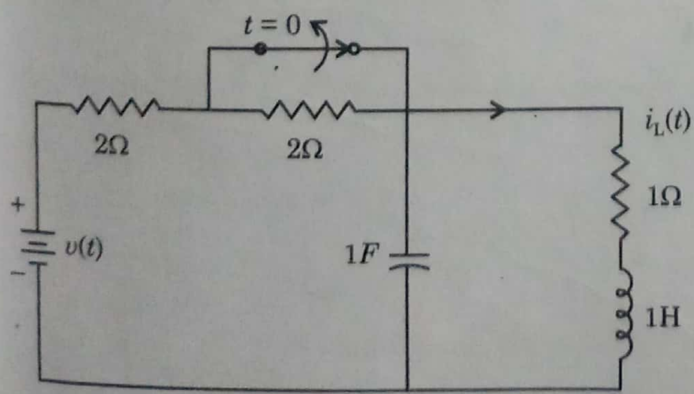


Fig. 3.27.

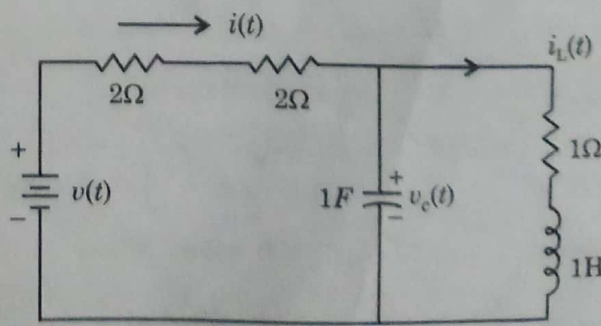


Fig. 3.28.

Solution: When switch is initially closed for a long time, the capacitor behaves as an open circuit and the inductor behaves as an open circuit and the inductor behaves as a short circuit, *i.e.*,

$v_c(0^-) = \dfrac{v(t)}{3}$ and $i_L(0^-) = \dfrac{v(t)}{3}$. After switch is opened, the circuit is redrawn as shown in figure 3.28.

Applying KVL in outer loop, we have

$$v(t) = 4i(t) + 1 \cdot i_L(t) + 1 \cdot \frac{di_L(t)}{dt}$$

or $$v(t) = 4\,i(t) + i_L(t) + \frac{di_L(t)}{dt} \qquad ...(1)$$

But $$i(t) = C\frac{dv_c(t)}{dt} + i_L(t)$$

or $$i(t) = 1 \cdot \frac{d}{dt}\left[1.i_L(t) + 1.\frac{di_L(t)}{dt}\right] + i_L(t) = \frac{d\,i_L(t)}{dt} + \frac{d^2 i_L(t)}{dt^2} + i_L(t) \qquad ...(2)$$

From equations (1) and (2), we have

$$v(t) = 4\left[\frac{d\,i_L(t)}{dt} + \frac{d^2\,i_L(t)}{dt^2} + i_L(t)\right] + i_L(t) + \frac{d\,i_L(t)}{dt}$$

or $$v(t) = 4\,\frac{d^2\,i_L(t)}{dt^2} + 5\,\frac{d\,i_L(t)}{dt} + 5\,i_L(t)$$

EXAMPLE 3.24 **In the circuit of figure 3.29, at time t_0 after the switch S was closed, it is found that $v_2 = +5$V. We are required to determine the values of $i_2(t_0)$ and $\frac{di_2(t_0)}{dt}$.**

Solution: Applying KCL,

$$i(t) = i_1(t) + i_2(t) \qquad ....(i)$$

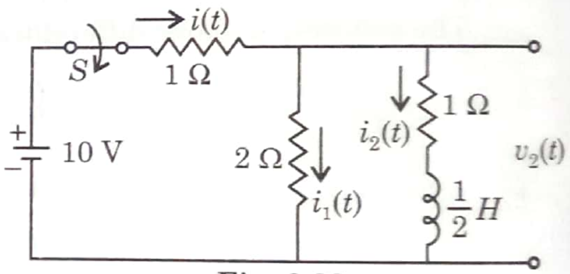


Fig. 3.29.

where $$i_1(t_0) = \frac{v_2}{2} = \frac{5}{2}\,A$$

Now, applying KVL,

$$10 = 1.\,i(t) + 2i_1(t) \qquad ....(ii)$$

From equations (i) and (ii), we have

$$10 = i_1(t) + i_2(t) + 2i_1(t)$$

or $$i_2(t) = 10 - 3i_1(t)$$

At $$t = t_0, \quad i_2(t_0) = 10 - 3\,i_1(t_0) = \mathbf{2.5A}$$

And $$v_2(t) = 1.i_2(t) + \frac{1}{2}\frac{di_2(t)}{dt}$$

At $$t = t_0,$$

$$5 = 1.(2.5) + \frac{1}{2}\frac{di_2(t_0)}{dt}$$

or $$\frac{di_2(t_0)}{dt} = \mathbf{5\ A/sec}$$

EXAMPLE 3.25 **In the circuit of figure 3.30, it is given that $v_2(t_0) = 2$ V, and $\frac{dv_2(t_0)}{dt} = -10$V/sec, where t_0 is the time after the the switch S was closed. Determine the value of C.**

Solution: From figure 3.30, using two Kirchhoff's Laws,

We have, $$i(t) = i_1(t) + i_2(t) \qquad ...(i)$$

and $$3 = 2i(t) + i_1(t) \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$$3 = 3i_1(t) + 2i_2(t) \qquad \text{...(iii)}$$

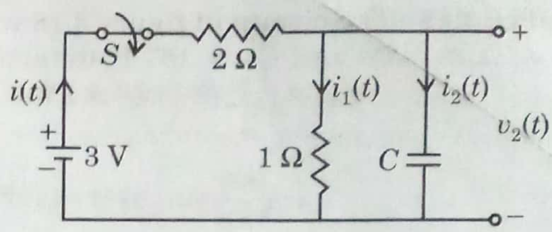


Fig. 3.30.

At $\quad t = t_0, \quad i_1(t_0) = \dfrac{v_2(t_0)}{1} = 2$

Now, from equation (*iii*),

$$3 = 3.2 + 2\, i_2(t_0)$$

$$i_2(t_0) = -\frac{3}{2} \equiv C\,\frac{dv_2(t_0)}{dt}$$

or $\quad C = -\dfrac{3}{2}\left(-\dfrac{1}{10}\right) = \mathbf{0.15\ F}$

EXAMPLE 3.26 In the circuit of figure 3.31(*a*), the switch *S* is moved from position 1 to 2 at *t* = 0, having been in position 1 for a long time before *t* = 0. Capacitor C_2 is uncharged at *t* = 0. (*a*) Find the particular solution for *i*(*t*) for *t* > 0. (*b*) Find the particular solution for $v_2(t)$ for *t* > 0.

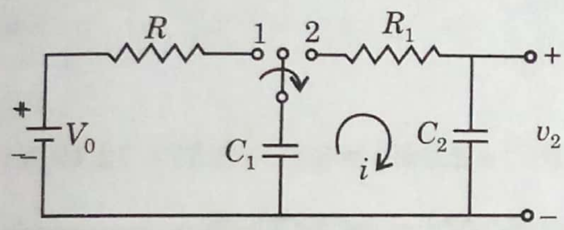


Fig. 3.31 (*a*).

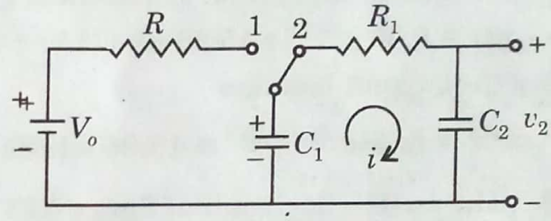


Fig. 3.31 (*b*).

Solution: At position 1, steady state is reached. Therefore,

$$v_{C_1}(0^-) = V_o$$

At *t* > 0, circuit is shown in figure 3.31(*b*).

Applying KVL,

$$V_o = \frac{1}{C_1}\int_0^t i(t)\,dt + R_1 i(t) + \frac{1}{C_2}\int_0^t i(t)\,dt$$

or $$V_o = R_1\, i(t) + \frac{1}{C_{eq}}\int_0^t i(t)\,dt$$

where $$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$$

Solution of above equation gives,

$$i(t) = \frac{V_o}{R_1}.e^{-t/R_1C_{eq}}$$

And $$v_2(t) = \frac{1}{C_2}\int_0^t i(t)\,dt = \frac{1}{C_2}\int_0^t \frac{V_o}{R_1}.e^{-t/R_1C_{eq}}\,dt$$

$$= \frac{V_o}{C_2R_1}\cdot\frac{e^{-t/R_1C_{eq}}}{-1/R_1C_{eq}}\Bigg|_0^t = \frac{V_oC_{eq}}{C_2}\left(-e^{-t/R_1C_{eq}}\right)\Big|_0^t$$

or $$v_2(t) = \frac{C_1}{C_1 + C_2}\,V_o\left(1 - e^{-t/R_1C_{eq}}\right)$$

EXAMPLE 3.27 **In the circuit of figure 3.32 the switch S is in position 'a' for a long time. At $t = 0$, the switch is moved from 'a' to 'b'. Find $v_2(t)$ using numerical values given in the circuit. Assume that the initial current in 2H inductor is zero.**

Solution: At position 'a'

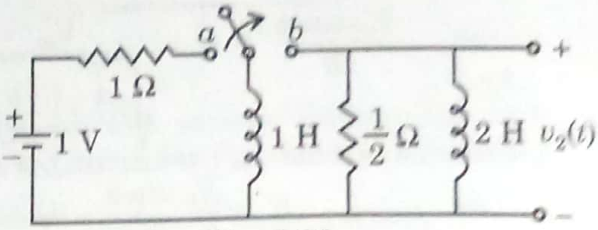


Fig. 3.32.

$$i_L(0^-) = \frac{1}{1} = 1\text{ A}$$

Now at position 'b'

Applying KCL,

$$\left[1 + \frac{1}{1}\int_0^t v_2(t)\,dt\right] + \frac{v_2(t)}{1/2} + \frac{1}{2}\int_0^t v_2(t)\,dt = 0$$

On differentiating, we get

$$\frac{dv_2(t)}{dt} + \frac{3}{4}v_2(t) = 0$$

or $\quad v_2(t) = K\,e^{-\frac{3}{4}t}$

But at $\quad t = 0,\ v_2(t) = (-1)\frac{1}{2} = -\frac{1}{2}$ (Since both the inductors behave as open circuit)

This gives $\quad K = -\frac{1}{2}$

or $\quad v_2(t) = -\frac{1}{2}\,e^{-\frac{3}{4}t}$

EXAMPLE 3.28 **The circuit of figure 3.33 reaches a steady state in position 2 and at $t = 0$ the switch S is moved to position 1. Find $i(t)$.**

Solution: At position 2, steady state is reached. Therefore,

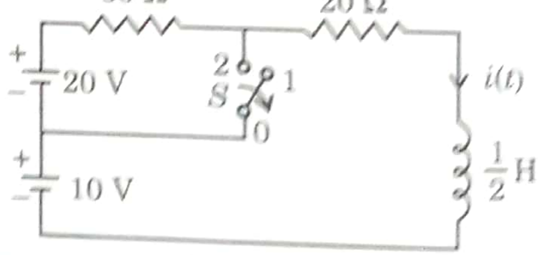


Fig. 3.33.

$$i\,(0^-) = \frac{10}{20} = 0.5\text{ A}$$

Now switch is opened. Applying KVL

$$30 = 30i(t) + 20i(t) + \frac{1}{2}\frac{di(t)}{dt}$$

or $\quad \frac{di(t)}{dt} + 100i(t) = 60$

or $\quad i(t) = \frac{60}{100} + Ke^{-100t}$

Since at $\quad t = 0^+,\ i(0^+) = i(0^-) = 0.5$

or $\quad 0.5 = 0.6 + K$ or $K = -0.1$

Hence, $\quad i(t) = \mathbf{0.6 - 0.1\,e^{-100t}}$ **A**

EXAMPLE 3.29. **In the given circuit of figure 3.34, $v_1(t) = e^{-t}$ for $t \geq 0$ and is zero for all $t < 0$. If the capacitor is initially uncharged, find $v_2(t)$. Let $R_1 = 10\Omega$, $R_2 = 20\Omega$, and $C = \frac{1}{20}$ F, and for these values sketch $v_2(t)$. Also determine values of v_2, $\frac{dv_2}{dt}$, $\frac{d^2v_2}{dt^2}$ and $\frac{d^3v_2}{dt^3}$ at $t = 0^+$.**

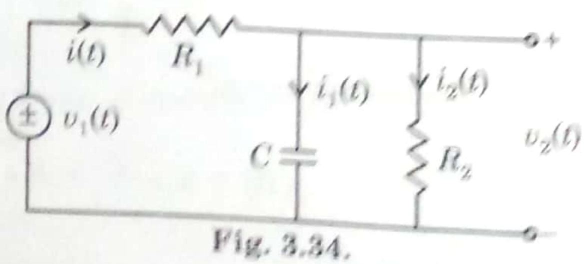


Fig. 3.34.

Solution: From figure 3.34,

$$i(t) = i_1(t) + i_2(t) \quad ...(i)$$

Applying KVL,

$$v_1(t) = R_1 i(t) + v_2(t) \quad ...(ii)$$

And $$v_2(t) = \frac{1}{C}\int i_1(t)\,dt = R_2\, i_2(t) \quad ...(iii)$$

From equations (*i*), (*ii*) and (*iii*), we have

$$v_1(t) = R_1\,[i_1(t) + i_2(t)] + R_2\, i_2(t) = R_1\, i_1(t) + (R_1 + R_2)\, i_2(t)$$

$$= R_1\, i_1(t) + (R_1 + R_2)\,.\,\frac{1}{R_2 C}\int i_1(t)\,dt$$

Putting the element values, we have

$$e^{-t} = 10 i_1(t) + \frac{30}{20.\frac{1}{20}}\int i_1(t)\,dt$$

On differentiating, we have

$$-e^{-t} = 10\frac{di_1(t)}{dt} + 30 i_1(t)$$

or $$\frac{di_1(t)}{dt} + 3i_1(t) = -\frac{1}{10}e^{-t}$$

The solution of above differential equation is

$$i_1(t) = e^{-3t}\int\left(-\frac{1}{10}e^{-t}\right).\,e^{3t}dt + Ke^{-3t} = -\frac{e^{-3t}}{10}\int e^{2t}\,dt + Ke^{-3t}$$

$$= -\frac{e^{-3t}}{10}\,.\,\frac{e^{2t}}{2} + Ke^{-3t} = -\frac{1}{20}e^{-t} + Ke^{-3t}$$

Since at $$t = 0^+,\ i(t) = \frac{v_1(0^+)}{R_1} = \frac{1}{10}$$

or $$\frac{1}{10} = -\frac{1}{20} + K$$

or $$K = \frac{3}{20}$$

Therefore,

$$i_1(t) = -\frac{1}{20}e^{-t} + \frac{3}{20}e^{-3t}$$

And hence,

$$v_2(t) = \frac{1}{C}\int i_1(t)\,dt$$

$$= 20\int\left(-\frac{1}{20}e^{-t} + \frac{3}{20}e^{-3t}\right)dt = e^{-t} - e^{-3t}$$

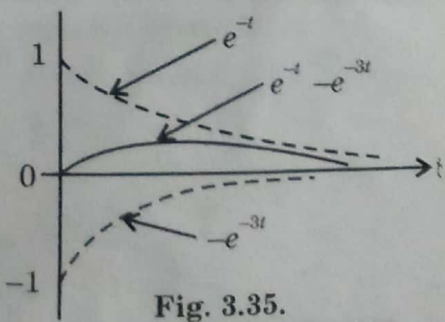


Fig. 3.35.

The plot of $v_2(t)$ is shown in figure 3.35.

$$v_2(0^+) = e^0 - e^0 = \mathbf{0}$$

$$\frac{dv_2}{dt}(0^+) = [-e^{-t} + 3e^{-3t}]\Big|_{t=0^+} = \mathbf{2\ V/sec}$$

Scanned by CamScanner

$$\frac{d^2v_2}{dt^2}(0^+) = [e^{-t} - 9e^{-3t}]\Big|_{t=0^+} = \mathbf{-8V/sec^2}$$

$$\frac{d^3v_2}{dt^3}(0^+) = [-e^{-t} + 27e^{-3t}]\Big|_{t=0^+} = \mathbf{26V/sec^3}$$

EXAMPLE 3.30 **In the circuit as shown in figure 3.36, the voltage source follows the law $v(t) = Ve^{-\alpha t}$, where α is a constant. The switch is closed at $t = 0$. (*a*) Solve for the current assuming that $\alpha \neq \frac{R}{L}$. (*b*) Solve for the current when $\alpha = \frac{R}{L}$.**

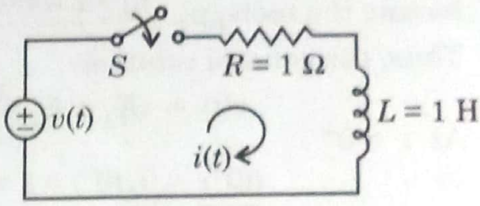


Fig. 3.36.

Solution: At $t = 0^+$, applying KVL,

$$Ve^{-\alpha t} = Ri(t) + L\frac{di(t)}{dt}$$

or
$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V}{L}e^{-\alpha t}$$

General solution of the above differential equation is

$$i(t) = e^{-\frac{R}{L}t}\int \frac{V}{L}e^{-\alpha t}\,.\,e^{\frac{R}{L}t}dt + Ke^{-\frac{R}{L}t}$$

$$= \frac{V}{L}e^{-\frac{R}{L}t}\int e^{\left(\frac{R}{L}-\alpha\right)t}dt + Ke^{-\frac{R}{L}t} \qquad \text{...}(i)$$

(*a*) When $\alpha \neq \frac{R}{L}$, from equation (*i*),

$$i(t) = \frac{V}{R - \alpha L}e^{-\alpha t} + Ke^{-\frac{R}{L}t}$$

Since $\quad i(0^+) = 0$

or
$$K = -\frac{V}{R - \alpha L}$$

Hence,
$$i(t) = \frac{V}{R - \alpha L}\left(e^{-\alpha t} - e^{-\frac{R}{L}t}\right) \mathbf{A}$$

(*b*) When $\alpha = \frac{R}{L}$, from equation (*i*),

$$i(t) = \frac{V}{L}e^{-\frac{R}{L}t}\int e^0\,dt + Ke^{-\frac{R}{L}t} = \frac{V}{L}e^{-\frac{R}{L}t}.t + Ke^{-\frac{R}{L}t}$$

Again $\quad i(0^+) = 0$ gives $K = 0$

Hence,
$$i(t) = \frac{Vt}{L}e^{-\frac{R}{L}t} \mathbf{A}$$

EXAMPLE 3.31 **In the circuit of the figure 3.37(*a*), the switch *S* is open and the circuit reaches a steady state. At $t = 0$, the *S* is closed. Find the current in the inductor for $t > 0$.**

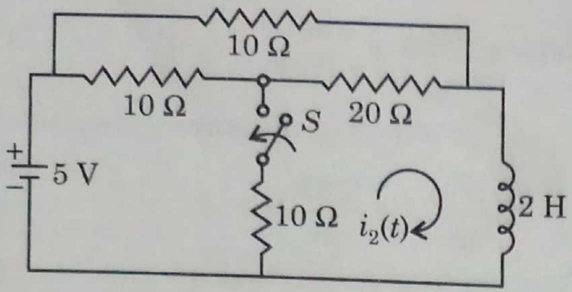


Fig. 3.37 (*a*).

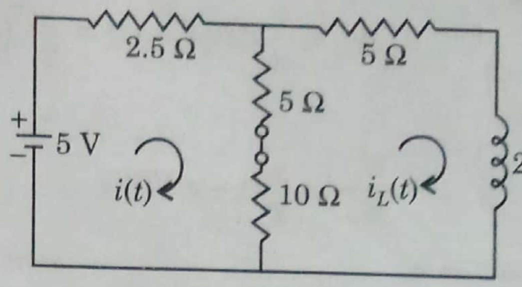


Fig. 3.37 (*b*).

Solution: At steady state with the switch S is opened

$$i_L(0^-) = \frac{5}{(10+20)||10} = 0.667\text{A}$$

Now at $t = 0^+$, the switch S is closed. And using $\Delta - Y$ transformation, the circuit becomes as shown in figure 3.37(*b*). Applying KVL in both the loops, we have

$$5 = 2.5\, i(t) + 15\,[i(t) - i_L(t)]$$

or $$i(t) = \frac{1}{3.5}[1 + 3i_L(t)] \qquad \text{...}(i)$$

And $$15[i_L(t) - i(t)] + 5i_L(t) + 2\frac{di_L(t)}{dt} = 0$$

or $$2\frac{di_L(t)}{dt} + 20i_L(t) = 15\, i(t) \qquad \text{....}(ii)$$

From equations (*i*) and (*ii*), we get

$$\frac{di_L(t)}{dt} + \frac{25}{7}i_L(t) = \frac{15}{7}$$

or $$i_L(t) = \frac{15/7}{25/7} + Ke^{-\frac{25}{7}t}$$

At $t = 0^+$, $i_L(0^+) = i_L(0^-) = 0.667$ or $K = 0.067$

Therefore, $i(t) = \mathbf{0.6 + 0.067\, e^{-3.57t}}$ A

EXAMPLE 3.32 **A switch is closed at $t = 0$, connecting a battery of voltage V with a series RC circuit. (*a*) Determine the ratio the energy delivered to the capacitor to the total energy supplied by the sourece as a function of time. (*b*) show that this ratio approaches 0.50 as $t \to \infty$.**

Solution: The charging voltage across the capacitor in a series RC circuit excited by a voltage source V is given as

$$v_c(t) = V(1 - e^{-t/RC})$$

And current in the circuit is

$$i(t) = \frac{V}{R}e^{-t/RC}$$

(*a*) Now, the total energy supplied by the source is

$$W_T = V.i(t).t = V.\frac{V}{R}e^{-t/RC}.t = \frac{V^2t}{R}e^{-t/RC}$$

And the energy delivered to the capacitor is

$$W_C = \frac{1}{2}q_c(t).v_c(t)$$

$$W_C = \frac{1}{2} i(t).t.v_c(t) = \frac{1}{2}\frac{V^2 t}{R}(1 - e^{-t/RC}).e^{-t/RC}$$

Therefore,

$$\frac{W_C}{W_T} = \frac{1}{2}(1 - e^{-t/RC})$$

(*b*) And as $t \to \infty$,

$$\frac{W_C}{W_T} = \frac{1}{2}$$

EXAMPLE 3.33 **In the circuit of the figure 3.38, the switch S is closed at $t = 0$ with the capacitor uncharged. Find values for i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$, for element values as follows: $V = 100$V, $R = 1000\Omega$ and $C = 1\mu$F.**

Solution: Applying KVL,

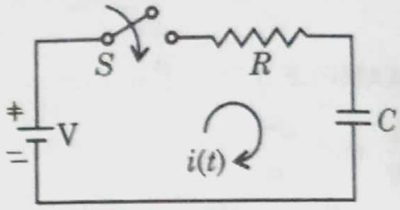


Fig. 3.38.

$$V = Ri(t) + \frac{1}{C}\int i(t)\,dt \qquad ...(i)$$

Differentiating and using element values gives

$$0 = 1000\frac{di(t)}{dt} + \frac{1}{1 \times 10^{-6}} i(t) \qquad ...(ii)$$

Again differentiating equation (*ii*) becomes,

$$0 = 1000\frac{d^2i(t)}{dt^2} + \frac{1}{1 \times 10^{-6}} . \frac{di(t)}{dt} \qquad ...(iii)$$

At $t = 0^+$, capacitor behaves as a short circuit, hence the second voltage term of equation (*i*) is zero. Therefore, from equation (*i*),

$$i(0^+) = \frac{V}{R} = \frac{100}{1000} = 0.1 \text{ A}$$

From equation (*ii*),

$$1000\frac{di}{dt}(0^+) = -\frac{1}{1 \times 10^{-6}} i(0^+)$$

or
$$\frac{di}{dt}(0^+) = -\frac{1}{1000}\left\{\frac{1}{1 \times 10^{-6}}.(0.1)\right\} = -100 \text{ A/sec}$$

And from equation (*iii*),

$$\frac{d^2i}{dt^2}(0^+) = -\frac{1}{1000}\left\{\frac{1}{1 \times 10^{-6}}.(-100)\right\} = 1 \times 10^5 \text{ A/sec}^2$$

EXAMPLE 3.34 **In the given circuit of figure 3.39, S is closed at $t = 0$ with zero current in the inductor. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$, if $R = 10\ \Omega$, $L = 1$ H and $V = 100$ V.**

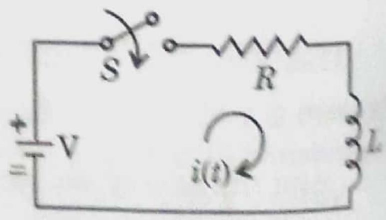


Fig. 3.39.

Solution: Applying KVL,

$$V = Ri(t) + L\frac{di(t)}{dt} \quad ...(i)$$

On differentiating equation (*i*) becomes,

$$0 = R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt^2} \quad ...(ii)$$

At $t = 0^+$, inductor behaves as an open circuit, hence

$$i(0^+) = \mathbf{0}$$

And from equation (*i*)

$$\frac{di}{dt}(0^+) = \frac{V}{L} = \frac{100}{1} = \mathbf{100\ A/sec}$$

Now, from equation (*ii*),

$$\frac{d^2i}{dt^2}(0^+) = -\frac{R}{L}\frac{di}{dt}(0^+) = -\frac{10}{1}.100 = \mathbf{-1000\ A/sec^2}$$

EXAMPLE 3.35 **The circuit shown in figure 3.40 has the switch S opened at $t = 0$. Solve for v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ at $t = 0^+$, if $I = 1$ amp, $R = 100\Omega$ and $L = 1$H. Also find the expression for $v(t)$.**

Solution: Applying KCL,

$$I = \frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt \quad ...(i)$$

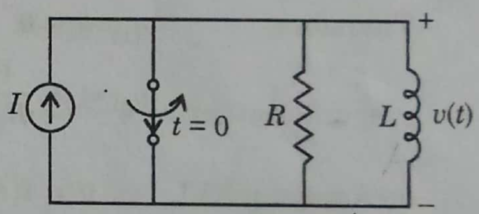


Fig. 3.40.

Differentiating and using numerical values gives

$$0 = \frac{1}{100}\frac{dv(t)}{dt} + v(t) \quad ...(ii)$$

On differentiating equation (*ii*) becomes

$$0 = \frac{1}{100}\frac{d^2v(t)}{dt^2} + \frac{dv(t)}{dt} \quad ...(iii)$$

At $t = 0^+$, inductor behaves as an open circuit, hence

$$v(0^+) = IR = \mathbf{100\ V}$$

And from equation (*ii*),

$$\frac{dv}{dt}(0^+) = -100\,v(0^+) = -10000 = \mathbf{-10^4\ V/sec}$$

Now, from equation (*iii*),

$$\frac{d^2v}{dt^2}(0^+) = -100\frac{dv}{dt}(0^+) = \mathbf{10^6\ V/sec^2}$$

Now, the general solution of the equation (*ii*) is

$$v(t) = K\,e^{-100t}$$

But at $t = 0^+$, $v(0^+) = 100$ gives $K = 100$

Therefore, $\quad v(t) = \mathbf{100\ e^{-100t}\ V}$

EXAMPLE 3.36 **In the circuit shown in figure 3.41 (*a*), the switch S is closed at $t = 0$ with zero capacitor voltage and zero inductor current. Solve for**

(*a*) v_1 and v_2 at $t = 0^+$

(*b*) v_1 and v_2 at $t = \infty$

(c) $\dfrac{dv_1}{dt}$ and $\dfrac{dv_2}{dt}$ at $t = 0^+$

(d) $\dfrac{d^2v_2}{dt^2}$ at $t = 0^+$.

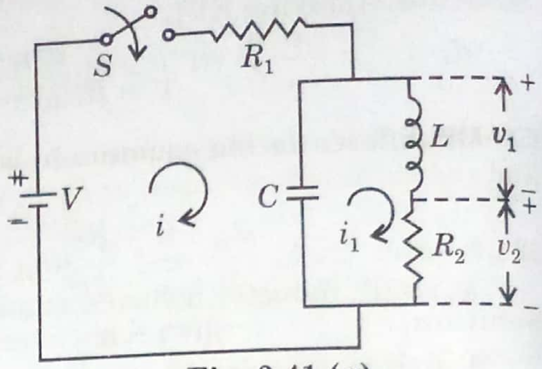


Fig. 3.41 (a).

Solution: (a) At $t = 0^+$, capacitor behaves as a short circuit and inductor behaves as an open circuit, shown in figure 3.41 (b). Therefore,

$$v_1(0^+) = 0$$

And $\quad v_2(0^+) = 0 \qquad$ (As $i_1(0^+) = 0$)

(b) At $t = \infty$, capacitor and inductor will behave as an open circuit and a short circuit respectively as shown in figure 3.41 (c).

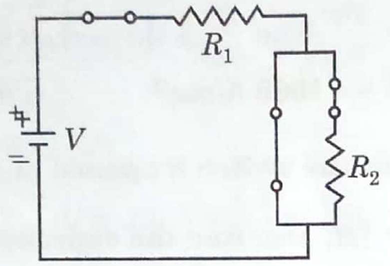


Fig. 3.41 (b). *Circuit at* $t = 0^+$.

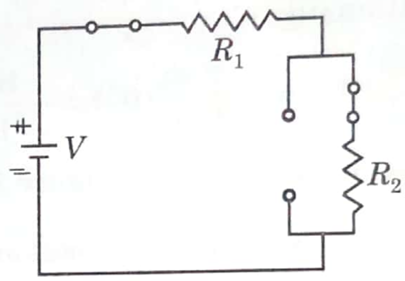


Fig. 3.41 (c). *Circuit at* $t = \infty$.

Therefore, $\quad v_1(\infty) = 0$

$$v_2(\infty) = \frac{V.R_2}{R_1 + R_2}$$

(c) Applying KVL, $\quad V = i(t).R_1 + L\dfrac{di_1(t)}{dt} + R_2 i_1(t) \qquad$ ...(i)

and $\quad L\dfrac{di_1(t)}{dt} + R_2\, i_1(t) + \dfrac{1}{C}\displaystyle\int [i_1(t) - i(t)]\, dt = 0 \qquad$ ...(ii)

Putting the value of $i(t)$ from equation (i) to equation (ii), we have

$$L\frac{di_1(t)}{dt} + R_2\, i_1(t) + \frac{1}{C}\int\left[i_1(t) - \frac{V}{R_1} + \frac{L}{R_1}\frac{di_1(t)}{dt} + \frac{R_2}{R_1}\, i_1(t)\right] dt = 0$$

On differentiating, we have

$$L\frac{d^2 i_1(t)}{dt} + \left(R_2 + \frac{L}{R_1 C}\right)\frac{di_1(t)}{dt} + \left(\frac{1}{C} + \frac{R_2}{R_1 C}\right) i_1(t) = \frac{V}{R_1 C}$$

Therefore,

$$\frac{dv_1}{dt}(0^+) = L\frac{d^2 i}{dt^2}(0^+)$$

$$= \frac{V}{R_1 C} - \left(R_2 + \frac{L}{R_1 C}\right)\frac{di_1}{dt}(0^+) - \left(\frac{1}{C} + \frac{R_2}{R_1 C}\right) i_1(0^+)$$

$$= \frac{V}{R_1 C} - \left(R_2 + \frac{L}{R_1 C}\right)\frac{v_1(0^+)}{L} - 0 = \frac{V}{R_1 C} \quad \text{(As } v_1(0^+) = 0)$$

and $\quad \dfrac{dv_2}{dt}(0^+) = R_2\dfrac{di_1}{dt}(0^+) = R_2\dfrac{v_1(0^+)}{L} = \mathbf{0}$ (As $v_1(0^+) = 0$)

(*d*) $$\frac{d^2v_2}{dt^2}(0^+) = R_2\frac{d^2i_1}{dt^2}(0^+) = \frac{R_2}{L}\frac{dv_1}{dt}(0^+) = \frac{R_2}{L}\cdot\frac{V}{R_1C} = \frac{VR_2}{R_1LC}$$

EXAMPLE 3.37 **In the circuit shown in figure 3.42, the capacitor C_1 is charged to voltage V_0 and the switch S is closed at $t = 0$. When $R_1 = 2$ MΩ, $V_0 = 1000$ V, $R_2 = 1$ MΩ, $C_1 = 10$ µF and $C_2 =$ 20 µF, solve for i_2, $\frac{di_2}{dt}$ and $\frac{d^2i_2}{dt^2}$ at $t = 0^+$**

Solution: At $t = 0^+$, both the capacitors behave as the short circuit. Hence V_0 is the voltage across R_2 as well as R_1. Therefore,

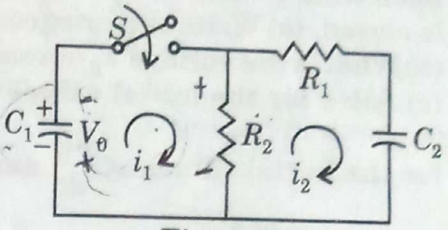


Fig. 3.42.

$$i_2(0^+) = \frac{V_0}{R_1} = \frac{1000}{2\times10^6} = \mathbf{500 \times 10^{-6}\ A}$$

Now applying KVL in both the loops, we have

$$V_0 = \frac{1}{C_1}\int i_1(t)\,dt + R_2[i_1(t) - i_2(t)] \qquad ...(i)$$

And $$R_2[i_2(t) - i_1(t)] + R_1 i_2(t) + \frac{1}{C_2}\int i_2(t)\,dt = 0$$

or $$(R_1 + R_2)\,i_2(t) + \frac{1}{C_2}\int i_2(t)\,dt = R_2\,i_1(t) \qquad ...(ii)$$

On differentiating equations (*i*) and (*ii*) become,

$$0 = \frac{1}{C_1}i_1(t) + R_2\left[\frac{di_1(t)}{dt} - \frac{di_2(t)}{dt}\right] \qquad ...(iii)$$

And $$(R_1 + R_2)\frac{di_2(t)}{dt} + \frac{1}{C_2}i_2(t) = R_2\frac{di_1(t)}{dt} \qquad ...(iv)$$

Putting the values of $i_1(t)$ and $R_2\frac{di_1(t)}{dt}$ from equations (*ii*) and (*iv*) in equation (*iii*), we have

$$0 = \frac{1}{C_1}\left[\frac{R_1+R_2}{R_2}i_2(t) + \frac{1}{R_2C_2}\int i_2(t)\,dt\right] + (R_1+R_2)\frac{di_2(t)}{dt} + \frac{1}{C_2}i_2(t) - R_2\frac{di_2(t)}{dt}$$

or $$\left[\frac{R_1+R_2}{C_1R_2} + \frac{1}{C_2}\right]i_2(t) + \frac{1}{C_1R_2C_2}\int i_2(t)\,dt = -R_1\frac{di_2(t)}{dt}$$

or $$\frac{di_2(t)}{dt} = \left(\frac{-C_2R_1 - C_2R_2 - C_1R_2}{R_1R_2C_1C_2}\right)i_2(t) - \frac{1}{R_1R_2C_1C_2}\int i_2(t)\,dt \qquad ...(v)$$

At $t = 0^+$, $\frac{1}{C_2}\int i_2(t)dt = 0$ (Since initially capacitor C_2 behaves as a short circuit).

$$\frac{di_2}{dt}(0^+) = -\left(\frac{C_2R_1 + C_2R_2 + C_1R_2}{R_1R_2C_1C_2}\right)i_2(0^+)$$

$$= -\left(\frac{40+20+10}{400}\right)(500\times10^{-6}) = \mathbf{-87.5 \times 10^{-6}\ A/sec}$$

On differentiating equation (*v*) becomes,

$$\frac{d^2 i_2(t)}{dt^2} = -\left(\frac{C_1R_1 + C_2R_2 + C_1R_2}{R_1R_2C_1C_2}\right)\frac{di_2(t)}{dt} - \frac{1}{R_1R_2C_1C_2} i_2(t)$$

At $t = 0^+$,

$$\frac{d^2 i_2(t)}{dt^2}(0^+) = -\frac{70}{400}(-87.5 \times 10^{-6}) - \frac{1}{400}(500 \times 10^{-6}) = \mathbf{14.06 \times 10^{-6}\ A/sec^2}$$

EXAMPLE 3.38 **In the circuit shown in figure 3.43, a steady state is reached with the switch S open with $V = 100$ V, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 20\ \Omega$, $L = 1$ H and $C = 1\ \mu$F. At time $t = 0$, the switch is closed. (a) Write the integro-differential equations for the circuit after the switch is closed. (b) What is the voltage V_0 across C before the switch is closed. (c) Solve for the initial values of i_1 and i_2 at $t = 0^+$. (d) Solve for the initial values of $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$ at $t = 0^+$. (e) What is the value of $\frac{di_1}{dt}$ at $t = \infty$.**

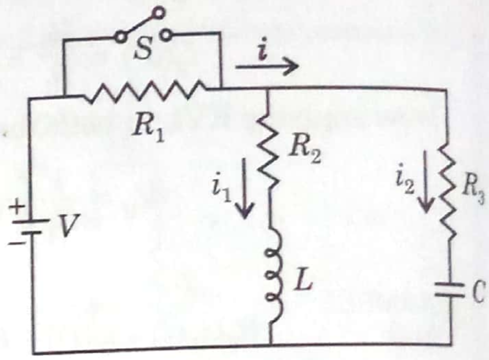


Fig. 3.43.

Solution: At steady state with switch S open, L and C behave as the short and open circuit respectively. Hence,

$$i_1(0^-) = \frac{V}{R_1 + R_2} = \frac{100}{10 + 20} = 3.33\ \text{A}$$

and
$$v_c(0^-) = R_2\, i_1(0^-) = \frac{VR_2}{R_1 + R_2} = 66.6\text{V}$$

(*a*) Now, switch is closed at $t = 0$. (Resistance R_1 is short circuited). Applying KVL, we have

$$V = R_2\, i_1(t) + L\frac{di_1(t)}{dt} \quad \text{with} \quad i_1(0^-) = i_1(0^+) = \frac{V}{R_1 + R_2} \qquad \text{...}(i)$$

and
$$V = R_3\, i_2(t) + \frac{1}{C}\int_0^t i_2(t)\,dt + v_c(0^-) \qquad \text{...}(ii)$$

(*b*) Voltage across the capacitor (before the switch is closed) is given as

$$V_0 = v_c(0^-) = v_c(0^+) = \mathbf{66.6\ V}$$

(*c*) At $t = 0^+$,

$$i_1(0^+) = i_1(0^-) = \frac{V}{R_1 + R_2} = \mathbf{3.33\ A}$$

And
$$i_2(0^+) = \frac{V - v_c(0^+)}{R_3} = \frac{100 - 66.6}{20} = \mathbf{1.66\ A}$$

(*d*) From equation (*i*),

$$L\frac{di_1}{dt}(0^+) = V - R_2 i_1(0^+)$$

or
$$\frac{di_1}{dt}(0^+) = 100 - 20(3.33) = \mathbf{33.3 A/sec}$$

On differentiating equation (*ii*) becomes,

$$0 = R_3\frac{di_2}{dt}(t) + \frac{1}{C}\, i_2(t) + 0$$

At $t = 0^+$

$$\frac{di_2}{dt}(0^+) = -\frac{1}{R_3C}\,i_2(0^+) = -\frac{1}{20 \times 1 \times 10^{-6}}\,.(1.66) = -\,\mathbf{83{,}000\ A/sec}$$

(*e*) Again, from equation (*i*) with initial condition, we have

$$i_1(t) = \frac{V}{R_2} + K\,e^{-\frac{R_2}{L}t} = 5 + Ke^{-20t}$$

At $t = 0^+$, $\quad i_1(0^+) = 3.33$

or $\quad 3.33 = 5 + K\,e^0$ or $K = -\,1.67$

Hence $\quad i_1(t) = 5 - 1.67\,e^{-20t}$

or $$\frac{di_1(t)}{dt} = -\,1.67.\,(-20)\,e^{-20t} = 33.4\,e^{-20t}$$

As $t \to \infty$ $$\frac{di_1}{dt}(\infty) = \mathbf{0}$$

EXAMPLE 3.39 **The circuit as shown in figure 3.44 has two indepedent node pairs. If the switch S is opened at $t = 0$. Find the following quantities at $t = 0^+$: (*a*) v_1, (*b*) v_2, (*c*) $\frac{dv_1}{dt}$, (*d*) $\frac{dv_2}{dt}$.**

Solution: At $t = 0^+$, inductor and capacitor behave as an open circuit and short circuit respectively. Hence,

$$v_1(0^+) = \boldsymbol{R_1\,i(0^+)}$$
$$v_2(0^+) = \mathbf{0}$$

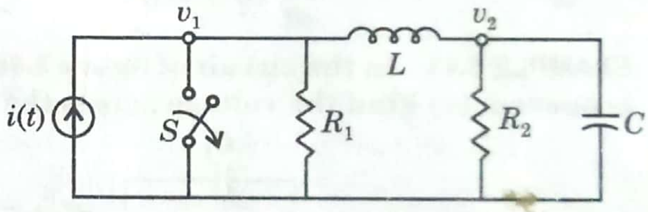


Fig. 3.44.

At any time t,

$$v_1(t) = R_1\left[i(t) - \frac{1}{L}\int\{v_1(t) - v_2(t)\}\,dt\right]$$

On differentiating, we have

$$\frac{dv_1(t)}{dt} = R_1\left[\frac{di(t)}{dt} - \frac{1}{L}\{v_1(t) - v_2(t)\}\right]$$

At $t = 0^+$,

$$\frac{dv_1}{dt}(0^+) = \boldsymbol{R_1\left[\frac{di}{dt}(0^+) - \frac{R_1}{L}\,i(0^+)\right]} \qquad (\because\ v_1(0^+) = R_1 i(0^+) \text{ and } v_2(0^+) = 0)$$

And also at $t = 0^+$, applying KCL,

$$0 = \frac{v_2(0^+)}{R_2} + C\,\frac{dv_2(0^+)}{dt}$$

or $$\frac{dv_2(0^+)}{dt} = \mathbf{0}$$

EXAMPLE 3.40 **In the circuit of figure 3.45, the switch S is closed at $t = 0$. At $t = 0^-$, all capacitor voltages and inductor current are zero. Three node-to-datum voltages are indentified as v_1, v_2, and v_3. Find: (*a*) v_1 and $\frac{dv_1}{dt}$ at $t = 0^+$ (*b*) v_2 and $\frac{dv_2}{dt}$ at $t = 0^+$ and (*c*) v_3 and $\frac{dv_3}{dt}$ at $t = 0^+$.**

Solution: At $t = 0^+$, all capacitors behave as the short circuit. Therefore,

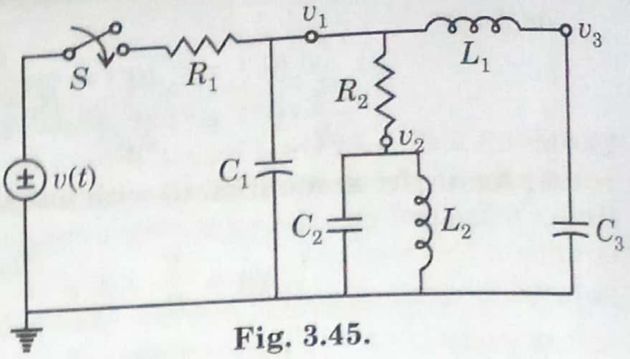


Fig. 3.45.

$$v_1(0^+) = 0$$
$$v_2(0^+) = 0$$
and
$$v_3(0^+) = 0$$

And also $$C_1 \frac{dv_1(0^+)}{dt} = \frac{v(0^+) - v_1(0^+)}{R_1}$$

(Since capacitor C_1 behaves as short circuit at $t = 0^+$)

or $$\frac{dv_1}{dt}(0^+) = \frac{v(0^+)}{R_1C_1}$$

$$C_2 \frac{dv_2}{dt}(0^+) = 0$$

(Since current flow through the least resistance path *i.e.*,through C_1)

or $$\frac{dv_2}{dt}(0^+) = 0$$

$$C_3 \frac{dv_3}{dt}(0^+) = 0$$

(Since inductor L_1 behaves as an open circuit at $t = 0^+$)

or $$\frac{dv_3}{dt}(0^+) = 0$$

EXAMPLE 3.41 In the circuit of figure 3.46(*a*), a steady state is reached, and $t = 0$, the switch S is opened. (*a*) Find the voltage across the switch, v_s at $t = 0^+$.

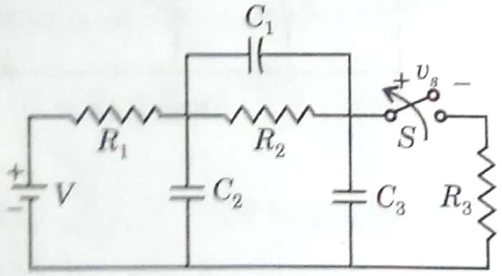


Fig. 3.46 (*a*).

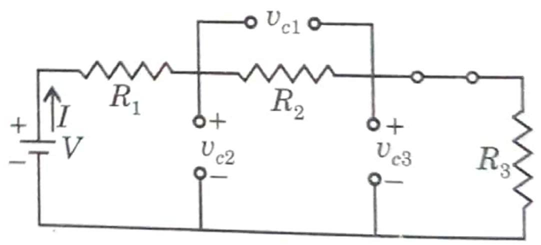


Fig. 3.46 (*b*).

Solution: At seady state with S is closed all the capacitors behaves the open circuit as shown in figure 3.46(*b*).

Applying KVL, we have

$$I = \frac{V}{R_1 + R_2 + R_3}$$

Therefore, the steady state voltages across the capacitors are given as

$$v_{c1} = IR_2 = \frac{VR_2}{R_1 + R_2 + R_3}$$

$$v_{c2} = V - IR_1 = \frac{V(R_2 + R_3)}{R_1 + R_2 + R_3}$$

and $$v_{c3} = IR_3 = \frac{VR_3}{R_1 + R_2 + R_3}$$

Now, when S is open at $t = 0$, the voltage across the S is

$$v_s(0^+) = v_{c3}(0^-) = \frac{V R_3}{R_1 + R_2 + R_3}.$$

EXAMPLE 3.42 **In the circuit shown in figure 3.47, capacitor C has an initial voltage $V_C = 10$ volts and at the same instant, current through inductor L is zero. The switch K is closed at time $t = 0$. Find out the expression for the voltage $v(t)$ across the inductor L using differential equation formulation.**

(U.P.T.U., 2003 C.O.)

Solution: Applying KVL,

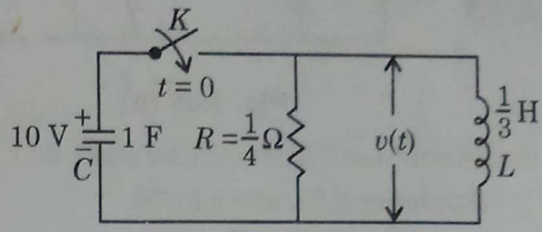


Fig. 3.47.

$$10 = \frac{1}{1}\int_0^t i(t)\,dt + v(t)$$

(where $i(t)$ be the current through the capacitor)

On differentiating,

$$0 = i(t) + \frac{dv(t)}{dt} \quad \text{...(1)}$$

Now, applying KCL,

$$i(t) = \frac{v(t)}{1/4} + \frac{1}{1/3}\int_0^t v(t)\,dt$$

or

$$i(t) = 4\,v(t) + 3\int_0^t v(t)\,dt \quad \text{...(2)}$$

From equations (1) and (2), we have

$$0 = 4\,v(t) + 3\int_0^t v(t)\,dt + \frac{dv(t)}{dt}$$

On differentiating,

$$0 = 4\frac{dv(t)}{dt} + 3v(t) + \frac{d^2v(t)}{dt^2}$$

The general solution of the above differential equation is

$$v(t) = K_1 e^{-t} + K_2\,e^{-3t} \quad \text{...(3)}$$

But at $t = 0^+$, $\quad v(0^+) = 10 = K_1 + K_2 \quad$ ...(4)

$$i_L(0^+) = 3\int v(0^+)\,dt = 0$$

(Since inductor behaves as an open circuit at $t = 0^+$)

or

$$3\left[-K_1 - \frac{K_2}{3}\right] = 0$$

or

$$K_1 = -\frac{K_2}{3} \quad \text{...(5)}$$

From equations (4) and (5),

$$K_1 = -5, \quad K_2 = 15 \quad \text{...(6)}$$

Now, from equation (3) and (6), we have

$$v(t) = \mathbf{-5\,e^{-t} + 15\,e^{-3t}\ V}$$

EXAMPLE 3.43 **Find differential equation relating $v(t)$ and $i(t)$ in the network shown in figure 3.48(a).** *(U.P.T.U., 2003)*

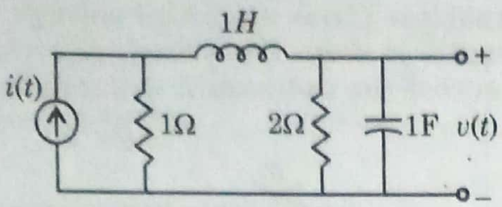


Fig. 3.48 (a).

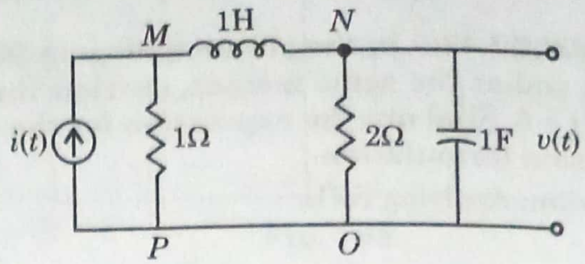


Fig. 3.48 (b).

Solution: Let $i_1(t)$ be the current in 1 H.

Applying KCL at node 'N',

$$i_1(t) = \frac{v(t)}{2} + 1.\frac{dv(t)}{dt} \qquad ...(i)$$

Applying KVL in the loop PMNO,

$$0 = -1.[i(t) - i_1(t)] + 1.\frac{di_1(t)}{dt} + v(t)$$

or

$$i_1(t) - i(t) + \frac{di_1(t)}{dt} + v(t) = 0 \qquad ...(ii)$$

From equations (i) and (ii),

$$\frac{v(t)}{2} + \frac{dv(t)}{dt} - i(t) + \frac{d}{dt}\left[\frac{v(t)}{2} + \frac{dv(t)}{dt}\right] + v(t) = 0$$

$$3\frac{v(t)}{2} + \frac{3}{2}\frac{dv(t)}{dt} + \frac{d^2v(t)}{dt^2} - i(t) = 0$$

EXAMPLE 3.44 **Consider the circuit shown in figure 3.49, Assume that $i_{L1}(0^-) = I_1$ and $i_{L2}(0^-) = I_2$. Formulate this problem in the form of differential equation with initial conditions. Find out $i_R(t)$ for $t \geq 0$ by solving the above differential equation.**

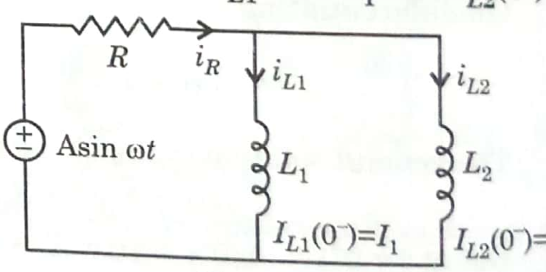


Fig. 3.49.

Solution: From the figure 3.49,

$$i_R(t) = i_{L1}(t) + i_{L2}(t) \qquad ...(1)$$

$$A \sin\omega t = R\, i_R(t) + L_1\frac{di_{L1}(t)}{dt} \qquad ...(2)$$

$$L_1\frac{di_{L1}(t)}{dt} = L_2\frac{di_{L2}(t)}{dt} \qquad ...(3)$$

On integrating equation (3), we get

$$L_1\, i_{L1}(t) = L_2\, i_{L2}(t) \qquad ...(4)$$

From equations (1) and (4),

$$i_R(t) = i_{L1}(t) + \frac{L_1}{L_2}\, i_{L1}(t)$$

$$i_{L1}(t) = \frac{i_R(t)}{1 + \frac{L_1}{L_2}} = \frac{L_2}{L_1 + L_2}\, i_R(t) \qquad ...(5)$$

Now, from equations (2) and (5),

$$A \sin\omega t = R\, i_R(t) + \frac{L_1 . L_2}{L_1 + L_2}\frac{d\, i_R(t)}{dt}$$

or $$\frac{d\, i_R(t)}{dt} + \frac{R(L_1 + L_2)}{L_1 . L_2} i_R(t) = \frac{L_1 + L_2}{L_1 . L_2} A \sin\omega t$$

with $$i_R(0^-) = i_{L1}(0^-) + i_{L2}(0^-) = I_1 + I_2$$

General solution of the above differential equation is

$$i_R(t) = e^{-\frac{R}{L_{eq}}t} \int \frac{1}{L_{eq}} A \sin\omega t\, e^{\frac{R}{L_{eq}}t}\, dt + K e^{-\frac{R}{L_{eq}}t} \qquad \left(\text{where } L_{eq} = \frac{L_1 . L_2}{L_1 + L_2}\right)$$

$$= \frac{Ae^{-\frac{R}{L_{eq}}t}}{L_{eq}} \int \sin\omega t\, e^{\frac{R}{L_{eq}}t}\, dt + K e^{-\frac{R}{L_{eq}}t}$$

$$I = \int \sin\omega t\, e^{\frac{R}{L_{eq}}t}\, dt$$

$$= e^{\frac{R}{L_{eq}}t} \frac{(-\cos\omega t)}{\omega} - \int e^{\frac{R}{L_{eq}}t} . \frac{R}{L_{eq}} \frac{(-\cos\omega t)}{\omega} dt = -\frac{e^{\frac{R}{L_{eq}}t} \cos\omega t}{\omega} + \frac{R}{L_{eq}\omega} \int e^{\frac{R}{L_{eq}}t} \cos\omega t\, dt$$

$$= -\frac{e^{\frac{R}{L_{eq}}t} \cos\omega t}{\omega} + \frac{R}{L_{eq}\omega} \left[e^{\frac{R}{L_{eq}}t} \frac{\sin\omega t}{\omega} - \int e^{\frac{R}{L_{eq}}t} . \frac{R}{L_{eq}} \left(\frac{\sin\omega t}{\omega}\right) dt \right]$$

$$= -\frac{e^{\frac{R}{L_{eq}}t} \cos\omega t}{\omega} + \frac{R}{L_{eq}\omega^2} e^{\frac{R}{L_{eq}}t} \sin\omega t - \left(\frac{R}{L_{eq}\omega}\right)^2 \int e^{\frac{R}{L_{eq}}t} \sin\omega t\, dt$$

$$= -\frac{e^{\frac{R}{L_{eq}}t} \cos\omega t}{\omega} + \frac{R}{L_{eq}\omega^2} e^{\frac{R}{L_{eq}}t} \sin\omega t - \left(\frac{R}{L_{eq}\omega}\right)^2 I$$

Hence, $$I = -\frac{e^{\frac{R}{L_{eq}}t} \cos\omega t}{\omega\left(1 + \frac{R^2}{L_{eq}^2 \omega^2}\right)} + \frac{R\, e^{\frac{R}{L_{eq}}t} \sin\omega t}{L_{eq}\omega^2 \left(1 + \frac{R^2}{L_{eq}^2 \omega^2}\right)}$$

Therefore, $$i_R(t) = -\frac{A \cos\omega t}{\omega L_{eq}\left(1 + \frac{R^2}{\omega^2 L_{eq}^2}\right)} + \frac{A\, R \sin\omega t}{\omega^2 L_{eq}^2 + R^2} + K e^{-\frac{R}{L_{eq}}t}$$

At $t = 0^+$;

$$i_R(0^+) = i_R(0^-) = I_1 + I_2$$

or $$I_1 + I_2 = -\frac{A}{\omega L_{eq}\left(1 + \frac{R^2}{\omega^2 L_{eq}^2}\right)} + K$$

or $$K = I_1 + I_2 + \frac{A}{\omega L_{eq}\left(1 + \frac{R^2}{\omega^2 L_{eq}^2}\right)}$$

$$i_R(t) = -\frac{A\cos\omega t}{\omega L_{eq}\left(1+\dfrac{R^2}{\omega^2 L_{eq}^2}\right)} + \frac{AR\sin\omega t}{\omega^2 L_{eq}^2 + R^2} + \left\{I_1 + I_2 + \frac{A}{\omega L_{eq}\left(1+\dfrac{R^2}{\omega^2 L_{eq}^2}\right)}\right\} e^{-\frac{R}{L_{eq}}t}$$

with $L_{eq} = \dfrac{L_1 . L_2}{L_1 + L_2}$

EXAMPLE 3.45 **Calculate the voltage, $V_C(t)$ and current $i_R(t)$ for $t \geq 0$ for the circuit shown in figure 3.50. Assume that switch S was closed for a long time before being opened at $t = 0$.**

Solution: At steady state with switch S closed,

$$v_c(0^-) = 1 \text{ V}$$

when switch is opened at $t = 0$, applying KVL,

$$1 = 2\, i_R(t) + \frac{1}{1}\int_0^t i_R(t)$$

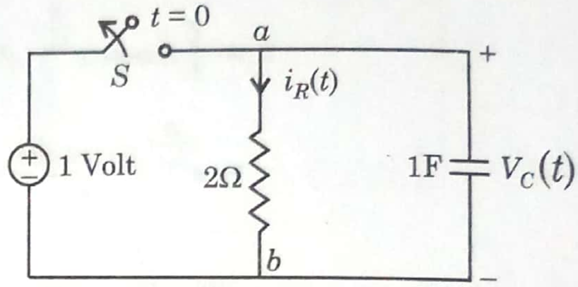


Fig. 3.50.

On differentiating, we get

$$2\frac{d\, i_R(t)}{dt} + i_R(t) = 0$$

Its solution is

$$i_R(t) = K\, e^{-\frac{1}{2}t}$$

At $t = 0^+$; $\quad i_R(0^+) = \dfrac{1}{2} = K$

Therefore, $\quad i_R(t) = \dfrac{1}{2} e^{-\frac{1}{2}t}$

and $$-v_c(t) = 1 - \frac{1}{1}\int_0^t i_R(t) = 1 - \int_0^t \frac{1}{2} e^{-\frac{1}{2}t}\, dt$$

$$= 1 - \frac{1}{2}\left.\frac{e^{-\frac{1}{2}t}}{-\frac{1}{2}}\right|_0^t = 1 - \left(1 - e^{-\frac{1}{2}t}\right) = e^{-\frac{1}{2}t}$$

or $$v_c(t) = -\, e^{-\frac{1}{2}t}.$$

EXERCISES

3.1. What are the initial conditions? Explain the procedure to evaluate the initial conditions.

3.2. Define all the types of responses.

3.3. Find the transient responses of (*a*) series *R-L* (b) series *R-C* circuits having sinusoidal excitation.

3.4. Find the transient responses of (*a*) series *R-L-C*, (*b*) parallel *R-L-C*, (c) series *R-L*, (*d*) series *R-C* circuits having DC excitation.

PROBLEMS

3.1. In the given circuit shown in figure P.3.1, the switch S is changed from position a to position b at time $t = 0$. Find out an expression for current $i(t)$, $\frac{di(t)}{dt}$ and $\frac{d^2i(t)}{dt^2}$ at $t = 0^+$.

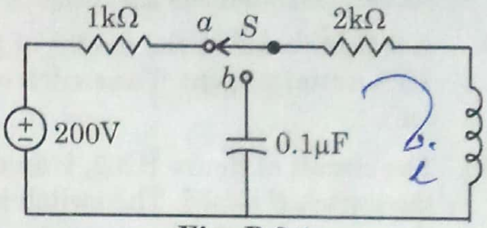


Fig. P.3.1.

3.2. A *dc* voltage of 100*V* is applied in the adjoining circuit (figure *P.3.2*) and the switch S is open. The switch S is closed at $t = 0$. Find the complete expression for the current.

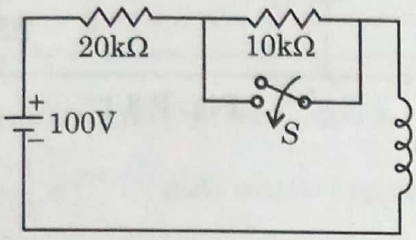


Fig. P.3.2.

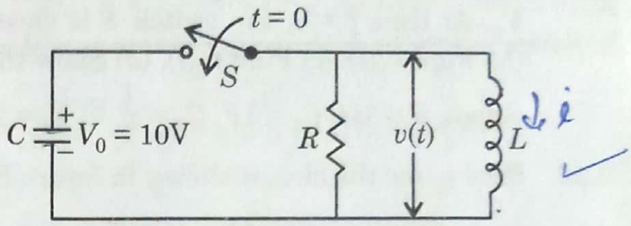


Fig. P.3.3.

3.3. Figure P.3.3 represents a parallel *RLC* circuit where $R = 0.1\Omega$, $L = 0.5$ H and C is 1 *F*. Capacitor C has an initial voltage of 10 *V* (polarity being shown in the figure). The switch S is closed at $t = 0$. Obtain $v(t)$.

3.4. The step voltage applied to a series *R-L* circuit is 36 *V* with $R = 15\Omega$. Determine the value of inductance L required to make the current of 1.0 A at 250 μ sec. assume the initial current is zero.

3.5. For the initially related circuit shown in figure P.3.5, the switch is closed at $t = 0$ and then open at $t = 1$ sec. Determine the voltage across the capacitor at $t = 2$ sec.

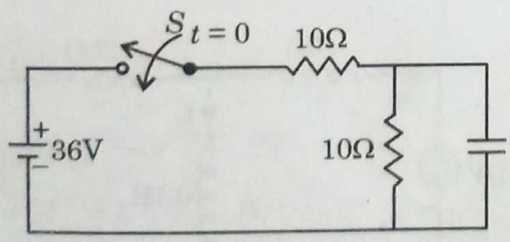


Fig. P.3.5.

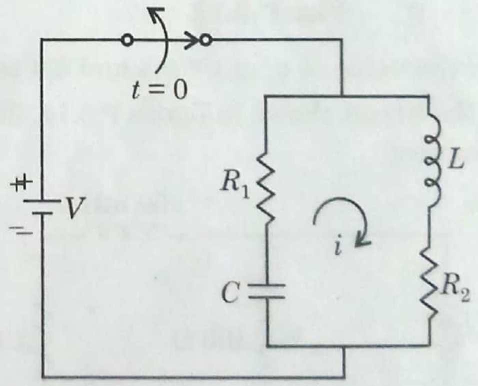


Fig. P.3.6.

3.6. The switch in the circuit of figure P.3.6, is opened at $t = 0$. Determine the current i and its derivative at $t = 0^+$.

3.7. In the circuit of figure P.3.7, the switch is moved from position 1 to 2 at $t = 0$. Determine i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

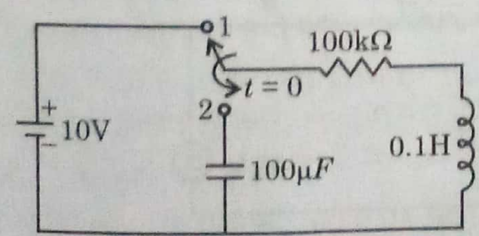


Fig. P.3.7.

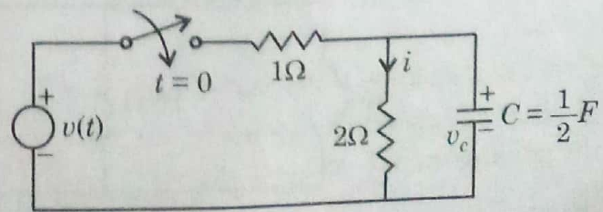


Fig. P.3.8.

3.8. Find the voltage across the uncharged capacitor of the circuit of figure P.3.8, if $v(t) = 2e^{-t}$ V. Also calculate the current through the 2Ω resistor.

3.9. A 3-H inductor in the circuit of figure P.3.9 carries a 10-A initial current. The switch is closed at $t = 0$. Solve for i.

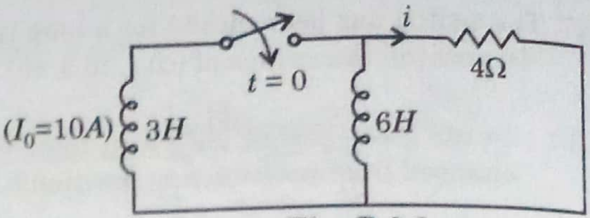


Fig. P.3.9.

3.10. The circuit of figure P.3.2, is in the steady state with the switch S closed. The switch is opened at $t = 0$. Find the expression for the current.

3.11. In the circuit of figure P.3.11, the initial voltage on C_1 is V_1 and on C_2 is V_2 such that $v_1(0) = V_1$ and $v_2(0) = V_2$. At time $t = 0$, the switch S is closed. (a) Find $i(t)$, (b) Find $v_1(t)$, (c) Find $v_2(t)$, (d) Show that $v_1(\infty) = v_2(\infty)$. Given, $R = 1\Omega$, $C_1 = 1$F, $C_2 = \frac{1}{2}$ F, $V_1 = 2$V and $V_2 = 1$V.

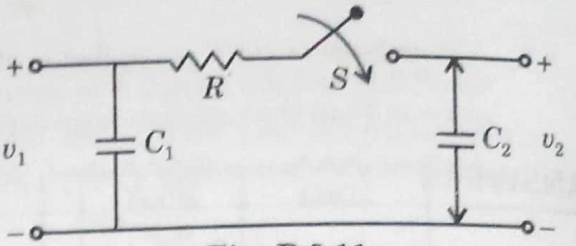


Fig. P.3.11.

3.12. Find i_L for the circuit shown in figure P.3.12.

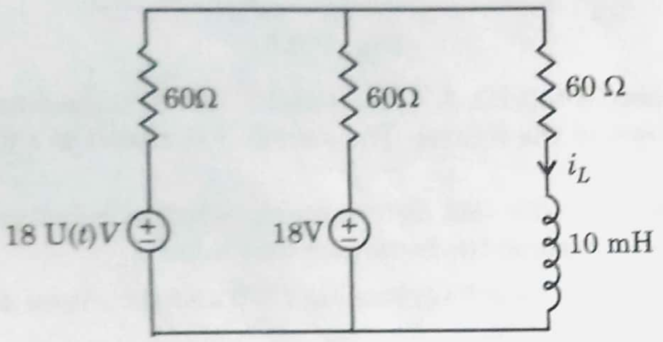


Fig. P.3.12.

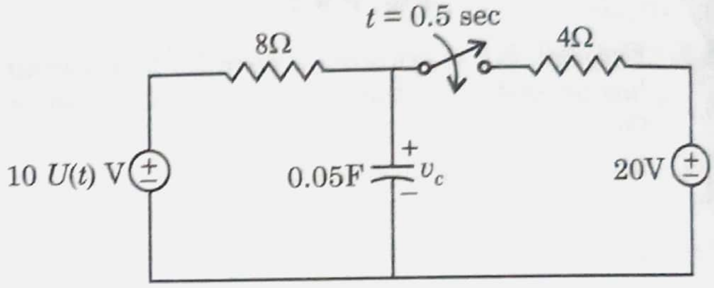


Fig. P.3.13.

3.13. Find the value of v_c at $t = 0.4$ and 0.8 sec for the circuit shown in figure P.3.13.

3.14. For the circuit shown in figure P.3.14, find the values of i_L and v_1 at t equal to (a) 0^-, (b) 0^+, (c) ∞, (d) 0.2 m sec.

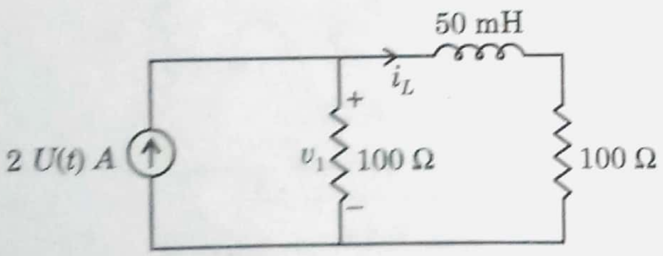


Fig. P.3.14.

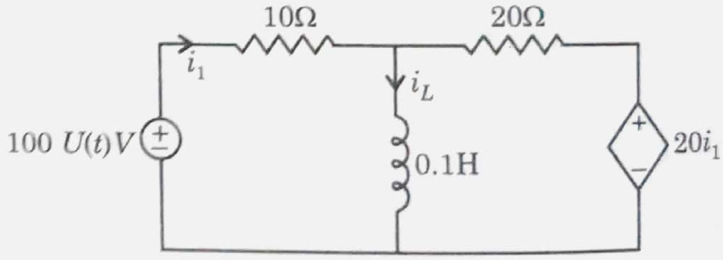


Fig. P.3.15.

3.15. For the circuit shown in figure P.3.15, calculate (a) $i_L(t)$, (b) $i_l(t)$.

3.16. Assume that the circuit shown in figure P.3.16 has been in the form shown for a very long time. Find $v_c(t)$ for all t after the switch opens.

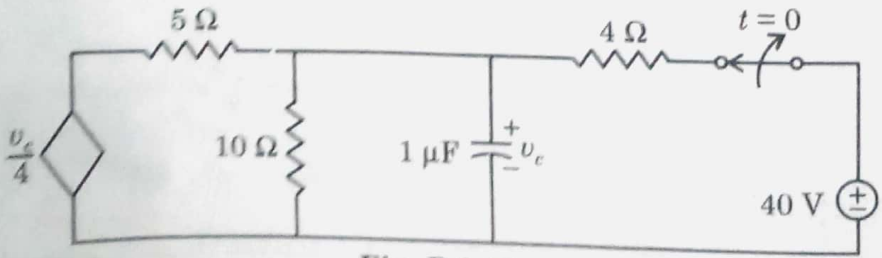


Fig. P.3.16.

3.17. The switch has been closed for a long time as shown in figure P.3.17. The switch is opened at $t = 0$. Determine the values of (a) $i_s(0^-)$, (b) $i_x(0^-)$, (c) $i_x(0^+)$, (d) $i_s(0^+)$, (e) $i_x(0.4 \text{ sec})$.

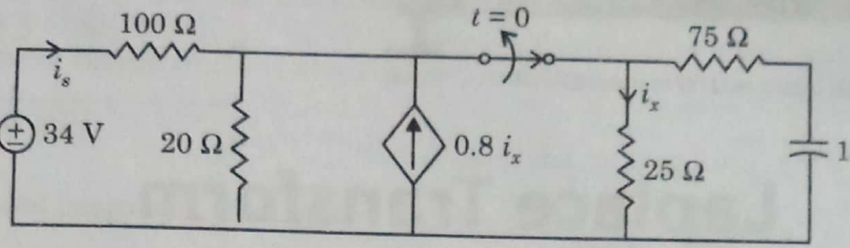


Fig. P.3.17.

ANSWERS

3.1. (66.6 mA, – 133.2 A/sec, – 400 × 10^3 A/sec^2)

3.2. $5\left(1-\frac{1}{3}e^{-200\,t}\right)$ mA; where t in msec.

3.3. $10.21e^{-9.796t} - 0.21e^{-0.204t}$

3.4. (6.957 H)

3.5. (5.726V)

3.6. $i(0^+) = \frac{V}{R_2}$ A

$\frac{di}{dt}(0^+) = -\frac{VR_1}{R_2L}$ A/sec

3.7. $i(0^+) = 10^{-4}$ A

$\frac{di}{dt}(0^+) = -100$ A/sec

$\frac{d^2i}{dt^2}(0^+) = -10^8$ A/sec^2

3.8. $v_c(t) = 2(e^{-t} - e^{-3t})$ V

$i(t) = (e^{-t} - e^{-3t})$ A

3.9. $i(t) = 10e^{-2t}$ A

3.10. $i(t) = 5\left(\frac{2}{3}+\frac{1}{3}e^{-300t}\right)$ mA; where t in msec.

3.11. $i(t) = e^{-3t}$, $v_1(t) = \frac{5}{3}+\frac{1}{3}e^{-3t}$, $v_2(t) = \frac{5}{3}-\frac{2}{3}e^{-3t}$

3.12. $i_L(t) = 0.1 + (0.1 - 0.1e^{-9000t})\, U(t)$ A

3.13. 6.32 and 15.66 V

3.14. (a) 0, 0; (b) 0, 200; (c) 1,100; (d) 0.55, 144.9

3.15. (a) $20(1 - e^{-40t})\, U(t)\, A$; (b) $(10 - 8e^{-40t})\, U(t)\, A$

3.16. $v_c(t) = 20e^{-2.5 \times 10^5 t}\, V$

3.17. (a) 0.29 A; (b) 0.2 A; (c) 0.05 A; (d) 0.277 A; (e) 0.0335 A