Chapter

5

Circuit Analysis by Laplace Transform

Inside this Chapter

5.1. INTRODUCTION

The circuit analysis in time domain were presented in the third chapter using the classical method. Having introduced the definition of Laplace transformation and learned to obtain the Laplace transform of many functions in previous chapter we shall now be exposed to the remarkable power of the Laplace transform as a mathematical tool to find the circuit responses in terms of voltages and currents, subject to any arbitrary input functions.

5.2. SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

The Laplace transformation is used to determine the solution of integro-differential equation. A differential equation of the general form

$$a_o \frac{d^n i}{dt^n} + a_1 \frac{d^{n-1} i}{dt^{n-1}} + \ldots\ldots + a_{n-1}\frac{di}{dt} + a_n i = v(t)$$

becomes, as a result of the Laplace transformation, an algebraic equation which may be solved for the unknown as

$$I(s) = \frac{£[v(t)] + \text{initial condition terms}}{a_o s^n + a_1 s^{n-1} + \ldots\ldots\ldots + a_{n-1}s + a_n}$$

where $\quad I(s) = £\,[i(t)]$

$I(s)$ is a ratio of two polynomials in s. Let the numerator and denominator polynomials be designated $P(s)$ and $Q(s)$, respectively, as

$$I(s) = \frac{P(s)}{Q(s)}$$

Note that $Q(s) = 0$ is the characteristic equation. If the transform term $P(s)/Q(s)$ can now be found in a table of transform pairs, the solution $i(t)$ can be written directly. In general, however, the transform expression for $I(s)$ must be broken into simpler terms before any practical transform table can be used.

Next, we factor the denominator polynomial, $Q(s)$,

$$Q(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$
$$\equiv a_0(s + s_1) \ldots (s + s_n)$$

or very compactly,

$$Q(s) = a_0 \prod_{j=1}^{n} (s + s_j)$$

Where Π indicates a product of factors, and $s_1, s_2, \ldots s_n$ are the n roots of the characteristic equation $Q(s) = 0$. Now the possible forms of these roots are discussed as :

(*i*) Partial Fraction Expansion When All the Roots of $Q(s)$ are Simple :

If all roots of $Q(s) = 0$, are simple, then

$$I(s) = \frac{P(s)}{(s+s_1)(s+s_2 \ldots (s+s_n)} = \frac{K_1}{s+s_1} + \frac{K_2}{s+s_2} + \ldots + \frac{K_n}{s+s_n}$$

where the K's are real constants called **residues**. Any of the residues $K_1, K_2, \ldots, K_n$ can be found by multiplying $I(s)$ by the corresponding denominator factor and setting $(s + s_j)$ equal to zero, *i.e.*

$s = -s_j$.As

$$K_j = \left[(s+s_j)\frac{P(s)}{Q(s)}\right]_{s=-s_j}$$

(*ii*) Partial Fraction Expansion When Some Roots of $Q(s)$ are of Multiple Order :

If a root of $Q(s) = 0$, is if multiplicity r, then

$$I(s) = \frac{P(s)}{(s+s_1).^r Q_1(s)} = \frac{K_{11}}{s+s_1} + \frac{K_{12}}{(s+s_1)^2} + \ldots + \frac{K_{1r}}{(s+s_1)^r} + \ldots$$

The following equations may be used for the evaluation of coefficients of repeated roots.

$$K_{1r} = (s+s_1)^r \cdot I(s)\Big|_{s=-s_1}$$

$$K_{1(r-1)} = \frac{d}{ds}\left[(s+s_1)^r \cdot I(s)\right]\Big|_{s=-s_1}$$

$$K_{1(r-2)} = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+s_1)^r \cdot I(s)\right]\Big|_{s=-s_1}$$

$$K_{11} = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left[(s+s_1)^r \cdot I(s)\right]\Big|_{s=-s_1}$$

(*iii*) Partial Fraction Expansion When two roots of $Q(s)$ are of Complex Conjugate Pair :

If two roots of $Q(s) = 0$, which form a complex conjugate pair, then

$$I(s) = \frac{P(s)}{(s+\alpha+j\omega)(s+\alpha-j\omega).Q_1(s)} = \frac{K_1}{(s+\alpha+j\omega)} + \frac{K_1^*}{(s+\alpha-j\omega)} + \ldots$$

Where $\quad K_1 = (s+\alpha+j\omega)\cdot I(s)\big|_{s=-(\alpha+j\omega)}$

and K_1^* is the complex conjugate of K_1. An expression of the type shown above is necessary for each pair of complex conjugate roots.

Several examples will illustrate the partial fraction expansion and the evaluation of K's.

EXAMPLE 5.1 **Solve the Differential Equation**

$$x'' + 3x' + 2x = 0,\ x(0^+) = 2,\ x'(0^+) = -3,$$

Solution : Taking the Laplace transform,

$$s^2X(s) - sx(0^+) - x'(0^+) + 3sX(s) - 3x(0^+) + 2X(s) = 0$$

$$(s^2 + 3s + 2)X(s) = sx(0^+) + x'(0^+) + 3x(0^+)$$

$$(s^2 + 3s + 2)X(s) = 2s + 3 \quad -3 \quad 6 \qquad \text{[By putting initial conditions]}$$

or
$$X(s) = \frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)}$$

Hence, all roots of denominator polynomial are simple. Then by partial fraction expansion,

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

Where
$$K_1 = (s+1)\cdot X(s)\Big|_{s=-1}$$

$$= \frac{2s+3}{s+2}\Bigg|_{s=-1} = \frac{-2+3}{1+2} = 1$$

and
$$K_2 = (s+2)\cdot X(s)\,|_{s=-2} = \frac{2s+3}{s+1}\Bigg|_{s=2} = 1$$

The result of the partial fraction expansion is thus,

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}$$

Therefore, the solution of given differential equation is

$$x(t) = £^{-1}[X(s)] = £^{-1}\left[\frac{1}{s+1} + \frac{1}{s+2}\right]$$

or $\quad$ $$\boldsymbol{x(t) = e^{-t} + e^{-2t}}$$

EXAMPLE 5.2 **Find $i(t)$, if $I(s) = \dfrac{1}{s(s+1)^2(s+2)}$.**

Solution :
$$I(s) = \frac{1}{s(s+1)^2(s+2)} = \frac{K_1}{s} + \frac{K_{21}}{s+1} + \frac{K_{22}}{(s+1)^2} + \frac{K_3}{s+2}$$

Where
$$K_1 = sI(s)\,|_{s=0} = \frac{1}{2},\quad K_3 = (s+2)\,I(s)\,|_{s=-2} = -\frac{1}{2}$$

$$K_{22} = (s+1)^2\,I(s)\Big|_{s=-1} = -1$$

$$K_{21} = \frac{d}{ds}[(s+1)^2\,I(s)]\,|_{s=-1}$$

$$= \frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\Bigg|_{s=-1} = \frac{-(2s+2)}{s^2(s+2)^2}\Bigg|_{s=-1} = 0$$

The complete expansion is

$$I(s) = \frac{1}{2s} - \frac{1}{(s+1)^2} - \frac{1}{2(s+2)}$$

Therefore,

$$\boldsymbol{i(t) = \frac{1}{2} - te^{-t} - \frac{1}{2}e^{-2t}\ A}$$

EXAMPLE 5.3 **If $I(s) = \dfrac{s^2+5s+9}{s^3+5s^2+12s+8}$; find $i(t)$.**

Solution : $I(s) = \dfrac{s^2+5s+9}{s^3+5s^2+12s+8} = \dfrac{s^2+5s+9}{(s+1)(s^2+4s+8)}$

$$I(s) = \frac{s^2+5s+9}{(s+1)\left[(s+2)^2-(j2)\right]} = \frac{[s^2+5s+9]}{(s+1)(s+2+j2)(s+2-j2)}$$

or $$I(s) = \frac{K_1}{s+1} + \frac{K_2}{(s+2+j2)} + \frac{K_2^*}{(s+2-j2)}$$

$$K_1 = (s+1)\, I(s)\,|_{s=1} = \left.\frac{s^2+5s+9}{s^2+4s+8}\right|_{s=-1} = 1$$

$$K_2 = (s+2+j2)\, I(s)\,|_{-(2+j2)} = -\frac{1}{j4}$$

$\therefore$ $$K_2^* = \frac{1}{j4}$$

Therefore, the complete expansion is

$$I(s) = \frac{1}{s+1} + \frac{-\frac{1}{j4}}{(s+2+j2)} + \frac{\frac{1}{j4}}{(s+2-j2)}$$

Hence, $$i(t) = e^{-t} - \left(\frac{1}{j4}\right)e^{-(2+j2)t} + \left(\frac{1}{j4}\right)e^{-(2-j2)t} = e^{-t} - \frac{1}{j4}e^{-2t}[e^{-j2t} - e^{j2t}]$$

or $$\boldsymbol{i(t) = e^{-t} + \frac{1}{2}e^{-2t} \sin 2t \ A}$$

Alternatively, $I(s) = \dfrac{s^2+5s+9}{(s+1)(s^2+4s+8)} = \dfrac{1}{s+1} + \dfrac{1}{s^2+4s+8} = \dfrac{1}{s+1} + \dfrac{1}{2}\left[\dfrac{2}{(s+2)^2+(2)^2}\right]$

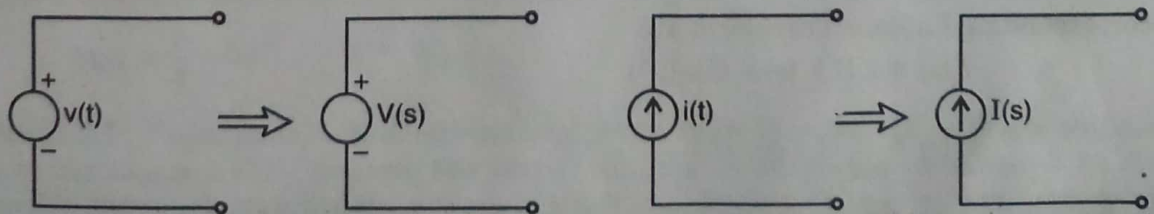
Therefore, $$i(t) = \pounds^{-1}[I(s)] = e^{-t} + \frac{1}{2}e^{-2t} \sin 2t \ A$$

as $$\pounds^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at} \sin \omega t$$

5.3. TRANSFORMED CIRCUIT COMPONENTS REPRESENTATION

5.3.1. Independent Sources

The sources $v(t)$ and $i(t)$ may be represented by their transformations, namely $V(s)$ and $I(s)$ respectively as shown in figure 5.1.



(a) (b)

Fig. 5.1. Representation of (a) voltage source (b) current source

5.3.2. Resistance Parameter

By Ohm's law, the v–i relationship for a resistor in t-domain is

$$v_R(t) = Ri_R(t)$$

In the complex-frequency domain (s-domain), above equation becomes

$$V_R(s) = RI_R(s)$$

From above two equations, we observe that the representation of a resistor in t-domain and s-domain are one and the same as shown in figure 5.2.

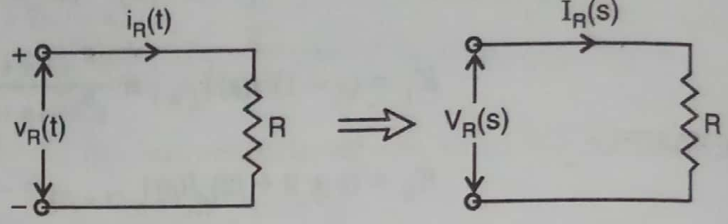


Fig. 5.2. Representation of a resistor

5.3.3. Inductance Parameter

The v-i relationship for an inductor is

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{0^+}^{t} v_L(t)\, d(t) + i_L(0^+)$$

The corresponding Laplace transforms are

$$V_L(s) = sL\, I_L(s) - Li_L(0^+)$$

or $$I_L(s) = \frac{1}{sL}V(s) + \frac{i_L(0^+)}{s}$$

From above equations, we get the transformed circuit representation for the inductor as shown in figure 5.3.

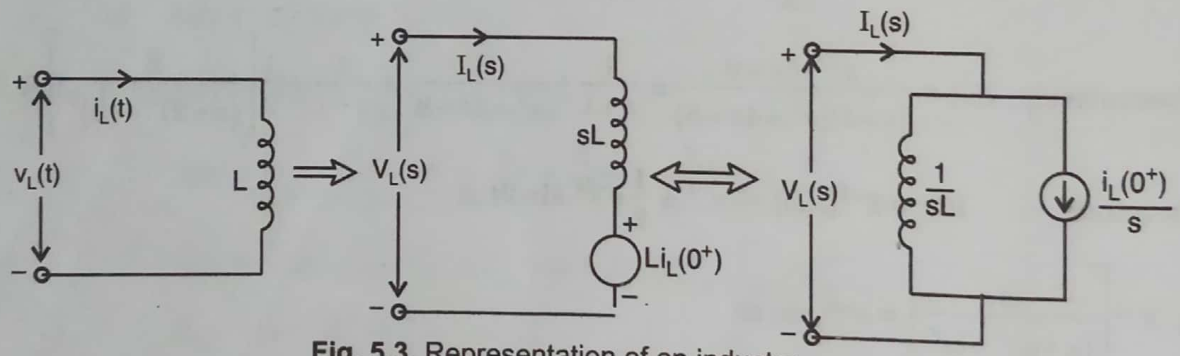


Fig. 5.3. Representation of an inductor

5.3.4. Capacitance Parameter

For a capacitor, the v–i relationship is

$$i_c(t) = C\frac{dv_c(t)}{dt} \quad \text{or} \quad v_c(t) = \frac{1}{C}\int_{0^+}^{t} i_c(t)\, dt + v_c(0^+)$$

The corresponding Laplace Transform are

$$I_c(s) = s\,C\,V_c(s) - C\,v_c(0^+)$$

or $$V_c(s) = \frac{1}{sC}I_c(s) + \frac{v_c(0^+)}{s}$$

From above equations, we get the transformed circuit representation for the capacitor as shown in figure 5.4.

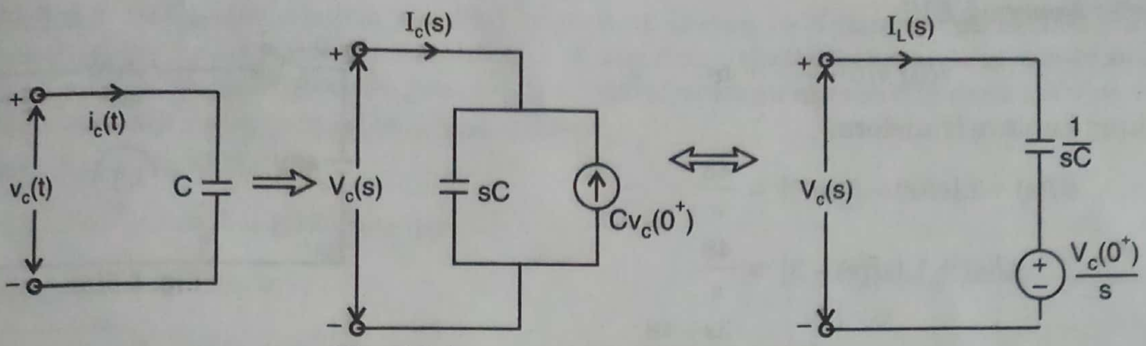


Fig. 5.4. Representation of a capacitor

EXAMPLE 5.4 Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = 5\,U(t)$$

Where $U(t)$ is unit-step function. The initial conditions are $y(0^+) = -1$ and $\frac{dy}{dt}(0^+) = 2$. Determine $y(t)$ for $t \geq 0$.

Solution : Taking Laplace transform on both sides of given differential equation :

$$s^2Y(s) - sy(0^+) - y'(0^+) + 3sY(s) - 3y(0^+) + 2Y(s) = \frac{5}{s}$$

Substituting the values of initial conditions and solving for $Y(s)$, we get

$$s^2Y(s) + s - 2 + 3sY(s) + 3 + 2Y(s) = \frac{5}{s}$$

or $$Y(s) = \frac{-s^2 - s + 5}{s(s^2 + 3s + 2)} = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

Expanded by partial fraction expansion,

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

$$K_1 = sY(s)|_{s=0} = \frac{5}{2}$$

$$K_2 = (s+1)\,Y(s)|_{s=-1} = \frac{-1+1+5}{(-1)(1)} = -5$$

$$K_3 = (s+2)\,Y(s)|_{s=-2} = \frac{-4+2+5}{(-2)(-1)} = \frac{3}{2}$$

Therefore, $$Y(s) = \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$$

Hence, taking the inverse Laplace transform, we get the complete solution as

$$y(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t} \quad ; t \geq 0$$

EXAMPLE 5.5 **Consider the R–L circuit with $R = 4\ \Omega$ and L = 1H excited by a 48V d.c. source as shown in figure 5.5(a). Assume the initial current through the inductor is 3A. Using the Laplace transform determine the current i (t); $t \geq 0$. Also draw the s-domain representation of the circuit.**

Solution : Applying *KVL*,

$$R(i) + L\frac{di(t)}{dt} = 48$$

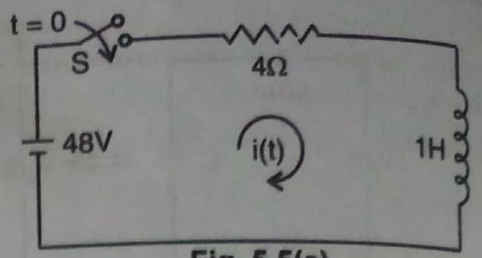


Fig. 5.5(a).

Taking Laplace transform

$$RI(s) + L[sI(s) - i_L(0^+)] = \frac{48}{s}$$

$$4I(s) + 1.[sI(s) - 3] = \frac{48}{s}$$

or $$I(s) = \frac{3s+48}{s(s+4)}$$

Applying the partial fraction expansion, we get

$$I(s) = \frac{3s+48}{s(s+4)} = \frac{K_1}{s} + \frac{K_2}{s+4}$$

where $$K_1 = s.I(s)\,|_{s=0} = \left.\frac{3s+48}{s+4}\right|_{s=0} = 12$$

and $$K_2 = (s+4).I(s)\,|_{s=-4}$$

$$= \left.\frac{(3s+48)}{s}\right|_{s=-4} = -9$$

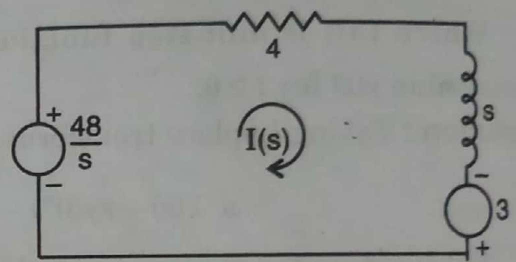


Fig. 5.5(b).

Then, $$I(s) = \frac{12}{s} - \frac{9}{s+4}$$

or $$i(t) = \mathcal{L}^{-1}[\,I(s)] = 12 - 9e^{-4t} \text{ A}$$

And the s-domain representation is shown in figure 5.5(b).

EXAMPLE 5.6 Consider a series R-L-C circuit with the capacitor initially charged to voltage of 1 V as indicated in figure 5.6(*a*). Find the expression for *i*(*t*). Also draw the s-domain representation of the circuit.

Solution : The differential equation for the current $i(t)$ is

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_c(0^+) = 0$$

and the corresponding transform equation is

$$L[sI(s) - i(0^+)] + RI(s) + \frac{1}{Cs}I(s) + \frac{v_c(0^+)}{s} = 0$$

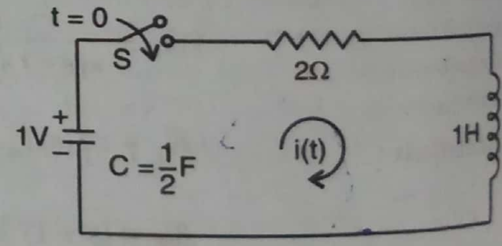


Fig. 5.6(a).

The parameters have been specified as $C = \frac{1}{2}F$, $R = 2\Omega$, $L = 1H$, and $v_c(0^+) = -1V$ (with the given polarity). The initial current $i(0^+) = 0$, because initially inductor behaves as a open circuit. The transform equation $I(s)$ then becomes

$$sI(s) + 2I(s) + \frac{2}{s}I(s) - \frac{1}{s} = 0$$

or $$I(s) = \frac{1}{s^2 + 2s + 2}$$

or, Completing the square,

$$I(s) = \frac{1}{(s+1)^2 + 1}$$

Therefore, $i(t) = \mathcal{L}^{-1}[I(s)]$

or $\boldsymbol{i(t) = e^{-t} \sin t}$ **A**

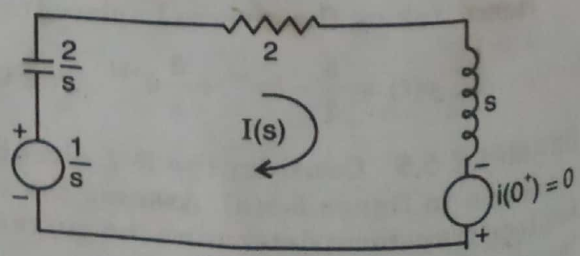


Fig. 5.6(b).

The s–domain representation is shown in figure 5.6(b).

EXAMPLE 5.7 **Consider the *R-C* parallel circuit as shown in figure 5.7(*a*) with $R = 0.5\Omega$ and $C = 4F$ excited by d.c. current source of 10 A. Determine the voltage across the capacitor by applying Laplace transformation. Assume the initial voltage across the capacitor as 2V. Also draw the s-domain representation of the circuit.**

Solution : Applying *KCL*,

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} = 10$$

Taking Laplace transform

$$\frac{V(s)}{R} + C[sV(s) - v(0^+)] = \frac{10}{s}$$

$$2V(s) + 4[sV(s) - 2] = \frac{10}{s}$$

or $$V(s) = \frac{8s+10}{s(4s+2)}$$

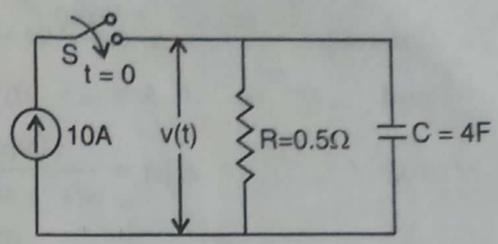


Fig. 5.7(a).

Applying the partial fraction expansion,

$$V(s) = \frac{2s+2.5}{s(s+0.5)} = \frac{K_1}{s} + \frac{K_2}{s+0.5}$$

$$K_1 = sV(s)|_{s=0} = 5$$

and $$K_2 = (s+0.5)V(s)|_{s=-0.5} = -3$$

or $$V(s) = \frac{5}{s} + \frac{-3}{s+0.5}$$

Therefore, $v(t) = \pounds^{-1}[V(s)] = 5 - 3e^{-0.5t}$ V

The *s*-domain representation of the circuit is shown in figure 5.7 (*b*).

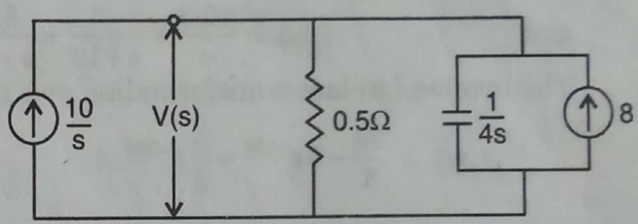


Fig. 5.7(b).

EXAMPLE 5.8 **In the network shown in figure 5.8, the switch *S* is closed at $t = 0$. With the network parameter values shown, find the expressions for $i_1(t)$ and $i_2(t)$, if the network is unenergized before the switch is closed.**

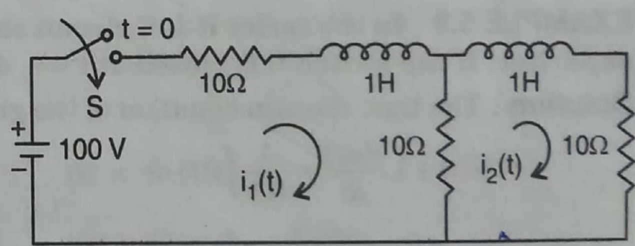


Fig. 5.8.

Solution : Applying *KVL*, Loop equations are

$$\frac{di_1(t)}{dt} + 10i_1(t) + 10[i_1(t) - i_2(t)] = 100$$

or $$\frac{di_1(t)}{dt} + 20i_1(t) - 10i_2(t) = 100 \qquad ...(i)$$

And, $$\frac{di_2(t)}{dt} + 10i_2(t) + 10[i_2(t) - i_1(t)] = 0$$

or $$\frac{di_2(t)}{dt} + 20i_2(t) - 10i_1(t) = 0 \qquad ...(ii)$$

The transform equations of (*i*) and (*ii*) may be written as (keeping initial conditions are zero, as given) :

$$(s+20)\,I_1(s) - 10I_2(s) = \frac{100}{s}$$

and $$-10I_1(s) + (s+20)\,I_2(s) = 0$$

Writing in matrix form,

$$\begin{bmatrix} s+20 & -10 \\ -10 & s+20 \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{100}{s} \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$I_1(s) = \frac{\Delta_1}{\Delta} \text{ and } I_2(s) = \frac{\Delta_2}{\Delta}$$

Where, $\Delta_1 = \begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}$; and $\Delta = \begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}$

Therefore, $\Delta_1 = \frac{100}{s}(s+20)$; $\Delta_2 = \frac{1000}{s}$

and $\Delta = (s+20)^2 - 100 = s^2 + 40s + 300$

So, $$I_1(s) = \frac{100(s+20)}{s(s^2+40s+300)}$$

and $$I_2(s) = \frac{1000}{s(s^2+40s+300)}$$

The partial fraction expansion of above expressions $I_1(s)$ and $I_2(s)$ are

$$I_1(s) = \frac{20/3}{s} + \frac{-5}{s+10} + \frac{-5/3}{s+30}$$

and $$I_2(s) = \frac{10/3}{s} + \frac{-5}{s+10} + \frac{5/3}{s+30}$$

The inverse Laplace transformation give $i_1(t)$ and $i_2(t)$ as

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t} \text{ A}$$

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t} \text{ A}$$

Which is the required solution.

EXAMPLE 5.9 **In the series R-L-C circuit shown in figure 5.9, There is no initial charge on the capacitor. If the switch S is closed at t = 0, determine the resulting current.**

Solution : The time-domain equation of the given circuit is

$$2i(t) + 1.\frac{di(t)}{dt} + \frac{1}{0.5}\int i(t)\,dt = 50$$

or $$2i(t) + \frac{di(t)}{dt} + 2\int i(t)\,dt = 50 \quad (\therefore v_c(0^-) = 0)$$

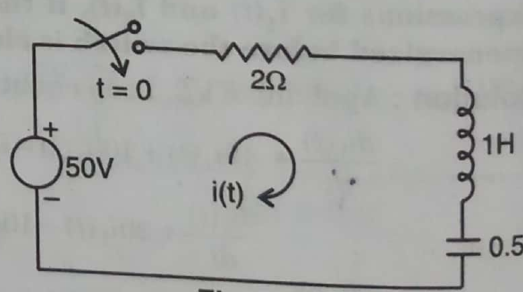


Fig. 5.9.

Taking Laplace transform,

$$2I(s) + sI(s) - i(0^+) + 2\frac{I(s)}{s} = \frac{50}{5}$$

Because $i(0^+) = 0$, therefore,

$$I(s) = \frac{50}{s^2+2s+2} = \frac{50}{(s+1)^2+1}$$

Therefore,

$$i(t) = £^{-1}[I(s)]$$

$$\boldsymbol{i(t) = 50\,e^{-t}\sin t \text{ A}}$$

EXAMPLE 5.10 **Repeat Example 3.8 as shown in figure 3.11 using Laplace transform.**

Solution : Before the switching action takes place,

$$V = i'(t)(R_1 + R_2) + L\frac{di'(t)}{dt}$$

Taking Laplace transform,

$$\frac{V}{s} = I'(s)\,(R_1 + R_2) + L[s \cdot I'(s) - i'(0^+)]$$

Since $i'(0^+) = 0$, therefore,

$$I'(s) = \frac{V}{s(R_1 + R_2 + Ls)} = \frac{\frac{V}{L}}{s\left(s + \frac{R_1 + R_2}{L}\right)} = \frac{V}{R_1 + R_2}\left[\frac{1}{s} - \frac{1}{s + \frac{R_1 + R_2}{L}}\right]$$

$$i'(t) = £^{-1}[I'(s)] = \frac{V}{R_1 + R_2}\left(1 - e^{-\frac{R_1 + R_2}{L}t}\right)$$

Therefore, $$i'(\infty) = \frac{V}{R_1 + R_2}$$

When switch is closed : R_2 is short circuited. Then

$$V = L\frac{di(t)}{dt} + R_1\, i(t)$$

Taking Laplace transform,

$$\frac{V}{s} = L[sI(s) - i(0^+)] + R_1 I(s)$$

$$i(0^+) = i'(\infty) = \frac{V}{R_1 + R_2}$$

$$\frac{V}{s} = (Ls + R_1)\, I(s) - \frac{LV}{R_1 + R_2}$$

$$(Ls + R_1)\, I(s) = \frac{V}{s} + \frac{LV}{R_1 + R_2}$$

$$I(s) = \frac{V}{s(Ls + R_1)} + \frac{LV}{(R_1 + R_2)(Ls + R_1)}$$

$$I(s) = \frac{V}{R_1}\left[\frac{1}{s} - \frac{1}{s + \frac{R_1}{L}}\right] + \frac{V}{R_1 + R_2} \cdot \frac{1}{s + \frac{R_1}{L}}$$

$$= \frac{V}{R_1} \cdot \frac{1}{s} - \frac{1}{s + \frac{R_1}{L}}\left\{\frac{V}{R_1} - \frac{V}{R_1 + R_2}\right\} = \frac{V}{R_1} \cdot \frac{1}{s} - \frac{VR_2}{R_1(R_1 + R_2)} \cdot \frac{1}{s + \frac{R_1}{L}}$$

Therefore, $$i(t) = £^{-1}[I(s)] = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)}e^{-\frac{R_1}{L}t}$$

or $$\boldsymbol{i(t) = \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2}e^{-\frac{R_1}{L}t}\right) A}$$

EXAMPLE 5.11 **Repeat Example 3.10 as shown in figure 3.13 using Laplace transform.**

Solution : Applying *KVL*,

$$1 = Ri(t) + \frac{1}{C}\int_0^t i(t)\, dt + v_c(0^+)$$

Taking Laplace transform and putting $R = 4$, $C = \frac{1}{16}$ and $v_c(0^+) = 9V$, we have

$$\frac{1}{s} = 4I(s) + \frac{16}{s}I(s) + \frac{9}{s}$$

$$-\frac{8}{s} = \left(4 + \frac{16}{s}\right)I(s)$$

$$I(s) = \frac{8}{(4s+16)} = -\frac{2}{s+4}$$

$$i(t) = \pounds^{-1}[I(s)] = -2e^{-4t}$$

Therefore, $$v_c(t) = 16\int_0^t i(t)\,dt + 9 = 16(-2)\int_0^t e^{-4t}\,dt + 9 = 8\cdot\left[e^{-4t}\right]_0^t + 9$$

or $$v_c(t) = 1 + 8e^{-4t} \text{ V}$$

EXAMPLE 5.12 **Repeat Example 3.13 as shown in figure 3.16 using Laplace transform.**

Solutions : With the switch on 1,

$$50 = 40i'(t) + 20\frac{di'(t)}{dt}$$

Taking Laplace transform,

$$\frac{50}{s} = 40\,I'(s) + 20[sI'(s) - i'(0^+)]$$

Since $i'(0^+) = 0$, therefore,

$$I'(s) = \frac{50}{s(40+20s)} = \frac{2.5}{s(s+2)}$$

Using partial fraction expansion,

$$I'(s) = \left(\frac{2.5}{2}\right)\cdot\frac{1}{s} + \left(\frac{2.5}{-2}\right)\cdot\frac{1}{s+2} = 1.25\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

Therefore, $$i'(t) = \pounds^{-1}[I'(s)] = 1.25(1 - e^{-2t})$$

as $t \to \infty$

$$i'(\infty) = 1.25 \text{ A}$$

With the switch on 2,

$$10 = 40\,i(t) + 20\frac{di(t)}{dt}$$

Taking Laplace transform,

$$\frac{10}{s} = 40\,I(s) + 20[sI(s) - i(0^+)]$$

As $i(0^+) = i'(\infty) = 1.25$

Therefore, $$\frac{10}{s} = (40 + 20s)\,I(s) - 20 \times 1.25$$

$$I(s) = \frac{\left(\frac{10}{s} + 25\right)}{(40+20s)} = \frac{10+25s}{s(40+20s)}$$

or $$I(s) = \frac{25(s+0.4)}{20s(s+2)} = 1.25\left[\frac{s+0.4}{s(s+2)}\right]$$

$$I(s) = 1.25\left[\frac{\left(\frac{0.4}{2}\right)}{s}+\frac{\left(\frac{-1.6}{-2}\right)}{s+2}\right] = \frac{0.25}{s}+\frac{1}{s+2}$$

Therefore, $\quad i(t) = £^{-1}[I(s)]$

or $\quad \boldsymbol{i(t) = 0.25 + e^{-2t}}$ **A**

EXAMPLE 5.13 **Solve for $i(t)$ in circuit as shown in figure 5.10 (a) in which 3F capacitor is initially charged to 20V, the 6-F capacitor to 10V, and the switch is closed at $t = 0$. Also draw the trans-formed circuit.**

Solution :

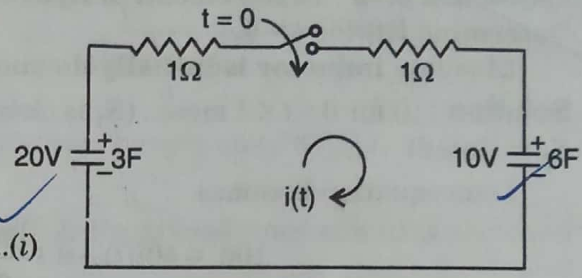


Fig. 5.10(a).

$$\frac{1}{3}\int_{-\infty}^{t} i(t)\,dt + 1i(t) + 1i(t)) + \frac{1}{6}\int_{-\infty}^{t} i(t)dt = 0$$

or $$\frac{1}{3}\int_{0}^{t} i(t)dt - 20 + 2i(t) + \frac{1}{6}\int_{0}^{t} i(t)dt + 10 = 0 \qquad ...(i)$$

Taking Laplace transform

$$\frac{1}{3s}I(s) - \frac{20}{s} + 2I(s) + \frac{1}{6s}I(s) + \frac{10}{s} = 0 \qquad ...(ii)$$

or $$\left(\frac{1}{3s}+\frac{1}{6s}\right)I(s) + 2I(s) = \frac{10}{s}$$

or $$\frac{1}{2s}I(s) + 2I(s) = \frac{10}{s}$$

or $$I(s) = \frac{20}{4s+1} = \frac{5}{s+0.25}$$

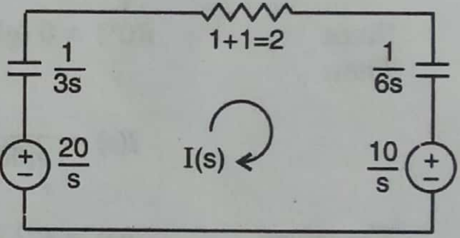


Fig. 5.10(b).

Therefore, $i(t) = £^{-1}[I(s)] = 5\,e^{-0.25\,t}$ **A**

The transformed circuit from equation (ii) is shown in figure 5.10(b).

EXAMPLE 5.14 **At $t = 0$, S is closed in the circuit of figure 5.11 find $v_c(t)$ and $i_c(t)$. All initial conditions are zero.**

Solution : Applying KCL,

$$I = \frac{v_c(t)}{R} + \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt}$$

Taking Laplace transform,

$$\frac{I}{s} = \frac{V_c(s)}{R} + \frac{V_c(s)}{R} + C[sV_c(s) - v_c(0^+)]$$

As $\quad v_c(0^+) = 0$ (given)

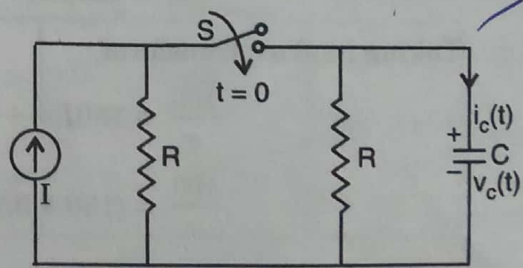


Fig. 5.11.

$$V_c(s) = \frac{I}{Cs}\left[\frac{1}{s+\frac{2}{RC}}\right] = \frac{I}{C}\left[\frac{1}{s\left(s+\frac{2}{RC}\right)}\right]$$

Therefore, $v_c(t) = £^{-1}[V_c(s)] = \dfrac{I}{C}£^{-1}\left[\dfrac{RC}{2}\cdot\dfrac{1}{s} - \dfrac{RC}{2}\cdot\dfrac{1}{s+\frac{2}{RC}}\right]$

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$$v_c(t) = \frac{IR}{2}\left(1 - e^{-\frac{2}{RC}t}\right)$$

And $$i_c(t) = C\frac{d}{dt}[v_c(t)] = C\cdot\frac{IR}{2}\left[0 - \left(\frac{-2}{RC}\right)e^{-\frac{2}{RC}t}\right]$$

or $$i_c(t) = I.e^{-\frac{2}{RC}t}\ A$$

EXAMPLE 5.15 **In the circuit of figure 5.12, S_1 is closed at $t = 0$, and S_2 is opened at $t = 4$ msec. Determine $i(t)$ for $t > 0$.**

(Assume inductor is initially de-energised)

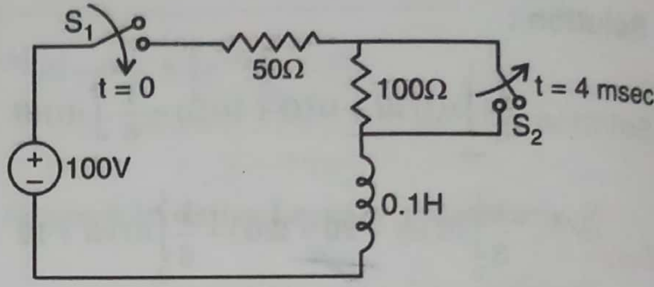


Fig. 5.12.

Solution : (*i*) for $0 \le t \le 4$ msec : (S_1 is closed and also S_2 is closed)

Loop equation becomes

$$100 = 50i(t) + 0.1\frac{di(t)}{dt}$$

Taking Laplace transform,

$$\frac{100}{s} = 50\,I(s) + 0.1\,[sI(s) - i(0^+)]$$

Since $i(0^+) = 0$ (given).

Then

$$I(s) = \frac{100}{s(50 + 0.1s)} = \frac{1000}{s(s + 500)}$$

Therefore, $$i(t) = £^{-1}[I(s)] = £^{-1}\left[\frac{2}{s} - \frac{2}{s + 500}\right] = \mathbf{2(1 - e^{-500\,t})\ A}$$

Thus $$i(4\times10^{-3}) = \mathbf{1.729\ A}$$

(*ii*) For 4 msec $\le t < \infty$: If $t' = t - 4 \times 10^{-3}$, then $0 \le t' < \infty$

Loop equation becomes

$$100 = 150\,i(t) + 0.1\frac{di(t')}{dt}$$

Taking Laplace transform,

$$\frac{100}{s} = 150I(s) + 0.1[sI(s) - i(4\times 10^{-3})]$$

$$\frac{100}{s} = (150 + 0.1s)\,I(s) - 0.1 \times 1.729$$

$$I(s) = \frac{100 + 0.1729s}{s(0.1s + 150)} = \frac{1.729\,s + 1000}{s(s + 1500)}$$

$$I(s) = \frac{0.667}{s} + \frac{1.062}{s + 1500} \quad \text{(By partial fraction expansion)}$$

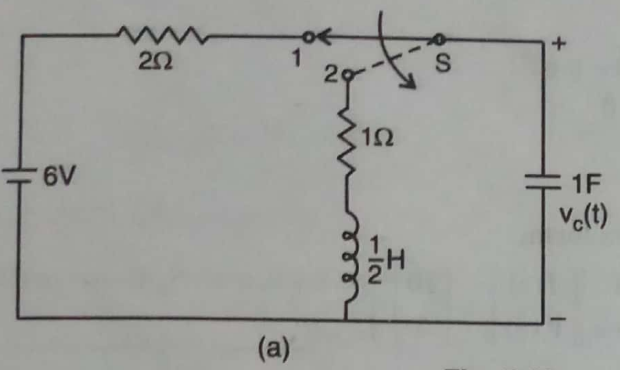
Therefore, $$i(t') = 0.667 + 1.062\cdot e^{-1500t'}$$

or $$i(t) = 0.667 + 1.062\cdot e^{-1500(t - 4\times10^{-3})}$$

$$= 0.667 + 1.062\,e^{6}\cdot e^{-1500t}$$

or $$\mathbf{i(t) = 0.667 + 428.4\,e^{-1500\,t}\ A}$$

EXAMPLE 5.16 **In the circuit shown in figure 5.13(*a*), switch S is in position 1 for a long time and moved to position 2 at $t = 0$. Find the voltage across the capacitor $v_c(t)$ for $t > 0$.**



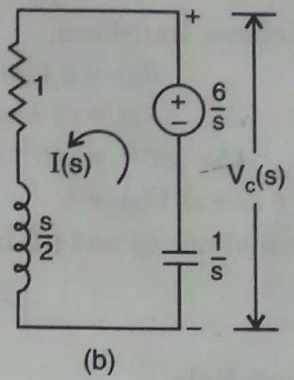


Fig. 5.13.

Solution : When the S is in position 1 for a long time, capacitor charges up to $6V$. *i.e.*

$$v_c(0^+) = 6$$

Now when the S is in position 2 for $t > 0$. Applying KVL, in the transformed circuit is shown in figure 5.13(b).

The current $I(s)$ in the loop is given by

$$I(s)\left[1+\frac{s}{2}+\frac{1}{s}\right] = \frac{6}{s}$$

$$I(s) = \frac{12}{s^2+2s+2}$$

And $$V_c(s) = I(s)\cdot\left[1+\frac{s}{2}\right] \text{ or } \frac{6}{s} - I(s).\frac{1}{s}$$

$$= \frac{6(s+2)}{s^2+2s+2} = \frac{6(s+1)}{(s+1)^2+1} + \frac{6}{(s+1)^2+1}$$

Therefore, $$v_c(t) = \pounds^{-1}[V_c(s)] = 6e^{-t}\,\text{Cos}\,t + 6\,e^{-t}\,\text{Sin}\,t = \mathbf{6e^{-t}\,(Cos\,t + Sin\,t)\ V}$$

EXAMPLE 5.17 **In the circuit shown in figure 5.14 steady state is reached with S open. S is closed at $t = 0$. Determine $i(t)$ and $v(t)$ for $t > 0$.**

Solution : At $t = 0^-$, as steady state is reached L is short circuited and C is open circuited.

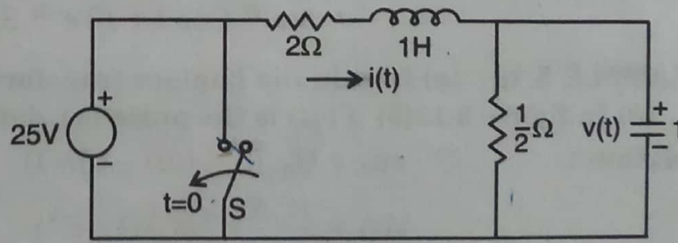


Fig. 5.14.

Hence $$i(0^-) = \frac{25}{2+\frac{1}{2}} = 10\text{ A}$$

and $$v(0^-) = 10\times\frac{1}{2} = 5\text{ V}$$

Now after S is closed. Applying KVL,

$$2\,i(t) + 1\cdot\frac{di(t)}{dt} + v(t) = 0$$

Taking Laplace transform

$$(2+s)\,I(s) - i(0^+) + V(s) = 0$$

or $$(2+s)I(s) - 10 + V(s) = 0$$

$$(\text{As } i(0^+) = i(0^-) = 10\text{A})$$

or $$(2+s)\,I(s) + V(s) = 10 \quad ...(i)$$

And, $$i(t) = \frac{v(t)}{1/2} + 1 \cdot \frac{dv(t)}{dt}$$

Taking Laplace transform,

$$I(s) = 2\,V(s) + sV(s) - v\,(0^+)$$

$$I(s) = (2 + s)\,V(s) - 5$$

$$(\text{As } v(0^+) = v(0^-) = 5\text{V})$$

$$-I(s) + (2 + s)\,V(s) = 5 \quad \text{...}(ii)$$

Writing equations (*i*) and (*ii*) in Matrix form,

$$\begin{bmatrix} 2+s & 1 \\ -1 & 2+s \end{bmatrix}\begin{bmatrix} I(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

By Cramer's Rule,

$$I(s) = \frac{\Delta_1}{\Delta} \text{ and } V(s) = \frac{\Delta_2}{\Delta}$$

Where $$\Delta = \begin{vmatrix} 2+s & 1 \\ -1 & 2+s \end{vmatrix} = (2 + s)^2 + 1$$

$$\Delta_1 = \begin{vmatrix} 10 & 1 \\ 5 & 2+s \end{vmatrix} = 20 + 10s - 5 = 10s + 15$$

and $$\Delta_2 = \begin{vmatrix} 2+s & 10 \\ -1 & 5 \end{vmatrix} = 10 + 5s + 10 = 5s + 20$$

$$I(s) = \frac{10s+15}{(s+2)^2+1} = \frac{10(s+2)}{(s+2)^2+1} - \frac{5}{(s+2)^2+1}$$

Therefore, $$i(t) = \pounds^{-1}[I(s)] = 10\,e^{-2t}\,\text{Cos}\,t - 5\,e^{-2t}\,\text{Sin}\,t \quad \text{A}$$

And $$V(s) = \frac{5s+20}{(s+2)^2+1} = \frac{5(s+2)}{(s+2)^2+1} + \frac{10}{(s+2)^2+1}$$

therefore, $$v(t) = \pounds^{-1}[V(s)]$$

$$= 5\,e^{-2t}\,\text{Cos}\,t + 10\,e^{-2t}\,\text{Sin}\,t \quad \text{A}$$

EXAMPLE 5.18 **(*a*) Obtain the Laplace transform of the pulse shown in figure 5.15(*a*). (*b*) In figure 5.15(*b*) if *v*(*t*) is the pulse (*a*), determine *i*(*t*).**

Solution : $$v(t) = G_{0,1}(t) = U(t) - U(t-1)$$

$$V(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = \frac{1}{s}(1 - e^{-s})$$

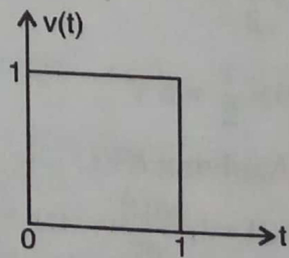


Fig. 5.15(a).

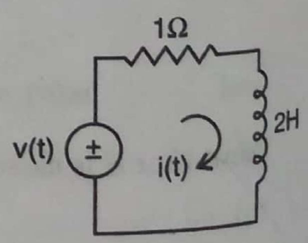


Fig. 5.15(b).

For the circuit in figure (*b*), applying *KVL*,

$$v(t) = 1.i(t) + 2\,\frac{di(t)}{dt}$$

Taking Laplace transform, (Assuming inductor is initially denergised)

$$V(s) = (1 + 2s)\,I(s)$$

or $$I(s) = \frac{V(s)}{(2s+1)}$$

If $v(t)$ is $U(t)$, then

$$I_1(s) = \frac{1}{s}\cdot\frac{1}{2\left(s+\frac{1}{2}\right)} = \frac{1}{s} - \frac{1}{s+\frac{1}{2}}$$

$$i_1(t) = \left(1 - e^{-\frac{t}{2}}\right)U(t)$$

Similarly, if $v(t)$ is $U(t-1)$, then

$$I_2(s) = \frac{1}{s}e^{-s}\cdot\left(\frac{1}{2\left(s+\frac{1}{2}\right)}\right) = e^{-s}\left[\frac{1}{s} - \frac{1}{s+\frac{1}{2}}\right]$$

$$i_2(t) = \left(1 - e^{-\left(\frac{t-1}{2}\right)}\right)U(t-1)$$

Therefore, $$i(t) = i_1(t) - i_2(t) = \left(1 - e^{-\frac{t}{2}}\right)U(t) - \left(1 - e^{-\frac{(t-1)}{2}}\right)U(t-1)\ A$$

EXAMPLE 5.19 **In the network shown in figure 5.16, find current $i_2(t)$ in thecircuit.**

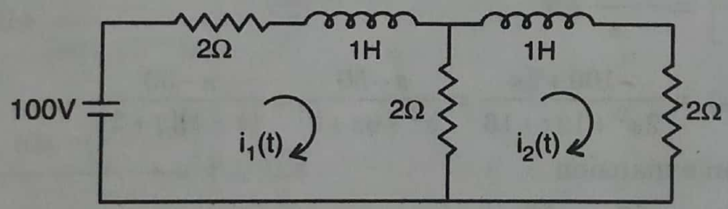


Fig. 5.16.

Solution : Applying *KVL*,

Loop 1: $$100 = 2i_1(t) + 1\cdot\frac{di_1(t)}{dt} + 2\,[i_1(t) - i_2(t)]$$

Taking Laplace transform, with $i_1(0^+) = 0$

$$\frac{100}{s} = 2I_1(s) + sI_1\,(s) + 2\,[I_1(s) - I_2\,(s)]$$

or $$\frac{100}{s} = (4+s)\,I_1(s) - 2I_2(s) \qquad \text{...(1)}$$

Loop 2 : $$2[i_2(t) - i_1(t)] + 1\cdot\frac{di_2(t)}{dt} + 2i_2(t) = 0$$

Taking Laplace transform, with $i_2(0^+) = 0$

$$2[I_2(s) - I_1(s)] + sI_2(s) + 2I_2(s) = 0$$

$$(4+s)\,I_2(s) = 2I_1(s) \qquad \text{...(2)}$$

From equation (2), putting the value of $I_1(s)$ in equation (1), we have

$$\frac{100}{s} = \left[(4+s)\cdot\frac{(4+s)}{2} - 2\right]I_2(s)$$

$$I_2(s) = \frac{200}{s(s^2+8s+12)} = \frac{200}{s(s+2)(s+6)}$$

Using partial fraction expansion

$$I_2(s) = \frac{50/3}{s} - \frac{25}{s+2} + \frac{25/3}{s+6}$$

Therefore, $$i_2(t) = £^{-1}[I_2(s)] = \left[\frac{50}{3} - 25e^{-2t} + \frac{25}{3}e^{-6t}\right] U(t)\ A$$

EXAMPLE 5.20 **In the circuit of figure 5.17, L = 2H, R = 12Ω and C = 62.5 mF. The initial conditions are $v_c(0^+)$ = 100V and $i_L(0^+)$ = 1.0 A. The switch is closed at t = 0. Find $i(t)$.**

Solution : Applying *KVL*,

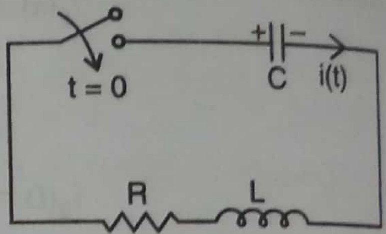


Fig. 5.17.

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_c(0^+) = 0$$

Taking Laplace transform

$$L[sI(s) - i_L(0^+)] + RI(s) + \frac{1}{Cs}I(s) + \frac{v_c(0^+)}{s} = 0$$

$$2[sI(s) - 1] + 12I(s) + \frac{1}{(62.5\times10^{-3})}I(s) + \frac{100}{s} = 0$$

$$I(s).\left[2s + 12 + \frac{16}{s}\right] = -\frac{100}{s} + 2$$

$$I(s) = \frac{-100+2s}{2s^2+12s+16} = \frac{s-50}{s^2+6s+8} = \frac{s-50}{(s+4)(s+2)}$$

Using partial fraction expansion

$$I(s) = \frac{27}{s+4} - \frac{26}{s+2}$$

Therefore, $$i(t) = £^{-1}[I(s)] = [27\,e^{-4t} - 26\,e^{-2t}]\ U(t)$$

EXAMPLE 5.21 **Find the current $i(t)$ for the network shown in figure 5.18, if the voltage source $v(t) = 2\,e^{-0.5\,t}\,U(t)$ and $v_c(0^+) = 0$.**

Solution : Applying *KVL*,

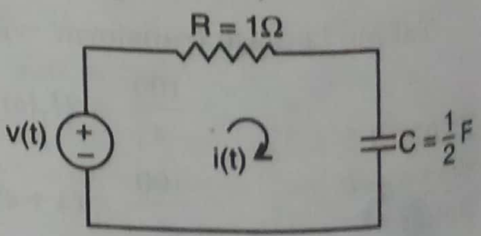


Fig. 5.18.

$$2\,e^{-0.5t}\,U(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$2\,e^{-0.5t}\,U(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_c(0^+)$$

Taking Laplace transform, with $R = 1\Omega$, $C = \frac{1}{2}$ F and $v_c(0^+) = 0$.

$$\frac{2}{s+0.5} = I(s) + \frac{2}{s}I(s) + 0$$

$$I(s) = \frac{2s}{(s+2)(s+0.5)}$$

Using partial fraction expansion

$$I(s) = \frac{8/3}{s+2} - \frac{2/3}{s+0.5}$$

Therefore, $$i(t) = £^{-1}[I(s)] = \frac{8}{3}e^{-2t} - \frac{2}{3}e^{-0.5t}$$

or $$\boldsymbol{i(t) = \frac{2}{3}(4e^{-2t} - e^{-0.5t})\,U(t)\;A}$$

EXAMPLE 5.22 In the series *R-C* circuit of figure 5.19, the capacitor has initial charge 2.5 mC. At $t = 0$, the switch is closed and a constant voltage source $V = 100$ V is applied, find $i(t)$.

Solution : Applying *KVL*,

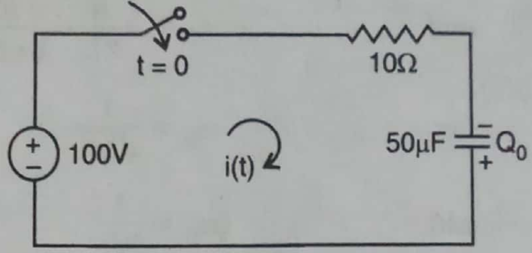


Fig. 5.19.

$$100 = 10\,i(t) + \frac{1}{50\times10^{-6}}\int_0^t i(t)dt + v_c(0^+)$$

Taking Laplace transform, with $v_c(0^+)$

$$= \frac{-Q_0}{C} = \frac{-2.5\times10^{-3}}{50\times10^{-6}}$$

$$= -50 \text{ V} \qquad \text{(with given polarity)}$$

$$\frac{100}{s} = 10I(s) + \frac{1}{50\times10^{-6}}\cdot\frac{I(s)}{s} - \frac{50}{s}$$

$$I(s)\left[10 + \frac{1}{(50\times10^{-6})s}\right] = \frac{150}{s}$$

$$I(s)\left[\frac{10s+(2\times10^4)}{s}\right] = \frac{150}{s}$$

$$I(s) = \frac{150}{10s+(2\times10^4)} = \frac{15}{s+(2\times10^3)}$$

Therefore, $$i(t) = £^{-1}[I(s)] = 15e^{-2\times10^3 t}$$

or $$\boldsymbol{i(t) = 15\,e^{-2t}}\text{. } \mathbf{A}\text{; where } t \text{ in } msec\,.$$

EXAMPLE 5.23 Series RLC circuit with a step input voltage. In the circuit of figure 5.20, let the switch be closed at time $t = 0$. Find the current $i(t)$.

Solution : The integro-differential equation for the current is

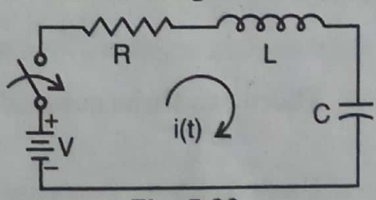


Fig. 5.20.

$$Ri + L\frac{di}{dt} + v_C(0^+) + \frac{1}{C}\int_{0^+}^{t} i\,dt = VU(t) \qquad ...(1)$$

Taking the Laplace transformation, we get

$$RI(s) + L[sI(s) - i_L(0^+)] + \frac{v_C(0^+)}{s} + \frac{I(s)}{sC} = \frac{V}{s} \qquad ...(2)$$

Assume that there is no stored energy in the circuit at $t = 0$. Hence, $i_L(0^+) = 0$ and $v_C(0^+) = 0$. Equation (2) reduces to

$$\left(R + sL + \frac{1}{sC}\right)I(s) = \frac{V}{s}$$

or
$$I(s) = \frac{V/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad ...(3)$$

The roots of the denominator on the right-hand side of Eq. (3) are

$$s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad ...(4)$$

and
$$s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad ...(5)$$

Let
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad ...(6)$$

and
$$\zeta\omega_0 = \frac{1}{2L} \quad ...(7)$$

Then
$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \quad ...(8)$$

and
$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \quad ...(9)$$

In terms of partial fraction expansion, we write

$$I(s) = \frac{V/L}{(s - s_1)(s - s_2)} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

The coefficient A is found by multiplying through by $(s - s_1)$ and putting $s = s_1$. Similarly, the coefficient B is found by multiplying through by $(s - s_2)$ and putting $s = s_2$. Hence,

$$A = \frac{V/L}{s_1 - s_2} \quad \text{and} \quad B = \frac{V/L}{s_2 - s_1}$$

or
$$A = \frac{V/L}{2\omega_0\sqrt{\zeta^2 - 1}} = -B$$

∴
$$I(s) = \frac{V}{2L\omega_0\sqrt{\zeta^2 - 1}}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right] \quad ...(10)$$

Inverse transformation gives

$$i(t) = \frac{V}{2L\omega_0\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t}\left[e^{\omega_0\sqrt{\zeta^2 - 1}\, t} - e^{-\omega_0\sqrt{\zeta^2 - 1}\, t}\right] \quad ...(11)$$

There are three cases of interest : $\zeta > 1$, $\zeta = 1$ and $\zeta < 1$. We shall consider these cases below.

Case 1: $\zeta > 1$. This case corresponds to $\frac{R^2}{4L^2} > \frac{1}{LC}$ or $Q < 1/2$, where $Q = \frac{\omega_0 L}{R}$ is the Q-factor of the circuit. The solution for the current is given by equation (11) which may be written as

$$i(t) = \frac{V}{\omega_0 L\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t} \sin h\, \omega_0\sqrt{\zeta^2 - 1}\, t \quad ...(12)$$

The circuit is said to be **overdamped** in this case. The variation of $i(t)$ with time given by equation (12) is shown in figure 5.21(a).

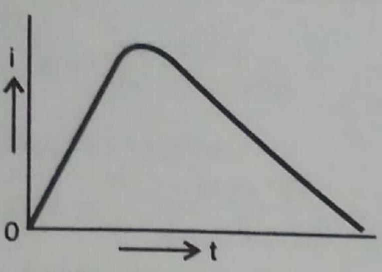


Fig. 5.21(a). Response for the overdamped case

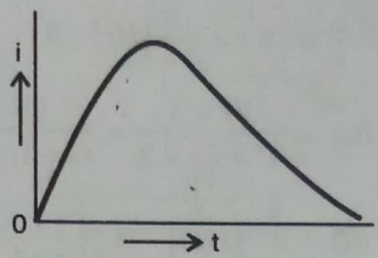


Fig. 5.21(b). Response for the critically damped case

Case 2 : $\zeta = 1$. This case corresponds to $R^2/4L^2 = 1/(LC)$, or $Q = 1/2$. The transform of the current is now given by

$$I(s) = \frac{V/L}{(s+\omega_0)^2} \quad \text{...(13)}$$

The inverse transform gives

$$i(t) = \frac{V}{L} t\, e^{-\omega_0 t} \quad \text{...(14)}$$

This is the **critically damped** case. The current curve is shown in figure 5.21(*b*) and looks similar to that of figure 5.21(*a*). If R_C is the resistance for critical damping. Equation (7) gives

$$\zeta = \frac{R}{R_C} \quad \text{...(15)}$$

i.e., ζ is the dimensionless ratio between the two resistances, and is referred to as the *damping ratio*.

Case 3 : $\zeta < 1$. This case corresponds to $R^2/4L^2 < 1/(LC)$, or $Q > 1/2$. Equation (11) gives

$$i(t) = \frac{V}{\omega_0 L\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin \omega_0\sqrt{1-\zeta^2}\, t \quad \text{...(16)}$$

The network is now said to be **underdamped** or **oscillatory**. The variation of current with time is shown in figure 5.21(*c*). The frequency of oscillations is $\omega_0\sqrt{1-\zeta^2}$; the factor $\zeta\omega_0$ determines how quickly the sinusoid decays.

When $R = 0$, *i.e.*, $\zeta = 0$, the oscillations are *underdamped* or *sustained*. The frequency of the undamped oscillations is ω_0. Therefore, ω_0 is referred to as the undamped natural frequency.

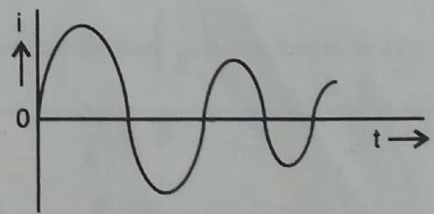


Fig. 5.21(c).

EXAMPLE 5.24 **In the circuit shown in figure 5.22 switch K is moved from position 1 to position 2 at $t = 0$. Find $i(t)$, if $i_L(0^-) = 2$A and $v_c(0^-) = 2$ V.**

Solution : When switch at position 1,

$i_L(0^-) = 2$A and $v_c(0^-) = 2$ V

Now, when switch at position 2, KVL gives

$$5 = 3i(t) + 1\cdot\frac{di(t)}{dt} + 2\int_0^t i(t)\,dt + 2$$

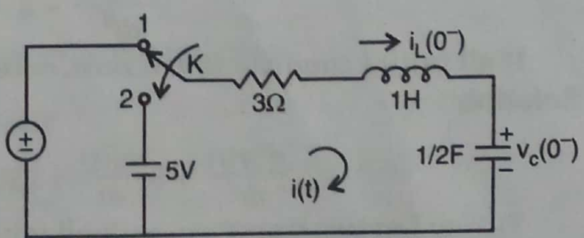


Fig. 5.22.

Taking laplace transform,

$$\frac{5}{s} = 3I(s) + sI(s) - i_L(0^+) + 2.\frac{I(s)}{s} + \frac{2}{s}$$

or $$\frac{3}{s} = \left[3 + s + \frac{2}{s}\right] I(s) - 2$$

or $$I(s) = \frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)}$$

$$i(t) = \pounds^{-1}\left[I(s)\right] = \pounds^{-1}\left[\frac{1}{s+1} + \frac{1}{s+2}\right]$$

or $$\boldsymbol{i(t) = e^{-t} + e^{-2t}}$$

EXAMPLE 5.25 **The current in a circuit is periodic. Two values of current are $i(0^+) = 1$, $i\left(\frac{\pi}{2}\right) = -1$. Find $i(t)$ if it is governed by**

$$\frac{d^2 i(t)}{dt^2} + 9i(t) = \text{Cos } 2t.$$

Solution : $$\frac{d^2 i(t)}{dt^2} + 9i(t) = \text{Cos } 2t.$$

Taking Laplace transform and substituting $i(0^+) = 1$

$$s^2 I(s) - si(0^+) - \frac{di}{dt}(0^+) + 9I(s) = \frac{s}{s^2+4}$$

$$I(s)\,(s^2+9) - s - K = \frac{s}{s^2+4}\text{ ; where } K = \frac{di}{dt}(0^+)$$

$$I(s) = \frac{s}{s^2+9} + \frac{K}{s^2+9} + s\left[\frac{1}{(s^2+4)(s^2+9)}\right] = \frac{s}{s^2+9} + \frac{K}{s^2+9} + s\left[\frac{1/5}{s^2+4} - \frac{1/5}{s^2+9}\right]$$

Hence, $$i(t) = \pounds^{-1}[I(s)] = \frac{4}{5}\text{ Cos } 3t + \frac{K}{3}\text{ Sin } 3t + \frac{1}{5}\text{ Cos } 2t$$

$$i\left(\frac{\pi}{2}\right) = -1 = 0 - \frac{K}{3} - \frac{1}{5}$$

$\therefore$ $$K = \frac{12}{5}$$

Therefore, $$\boldsymbol{i(t) = \frac{4}{5}(\text{Cos } 3t + \text{Sin } 3t) + \frac{1}{5}\text{Cos } 2t \text{ A}}$$

EXAMPLE 5.26 **The charge q in an electric circuit is given by**

$$\frac{d^2 q(t)}{dt^2} + 8\frac{dq(t)}{dt} + 25\, q(t) = 150$$

If all initial conditions are zero, determine the current through the circuit.

Solution :

$$\frac{d^2 q(t)}{dt^2} + 8\frac{dq(t)}{dt} + 25\, q(t) = 150$$

Taking Laplace transform with all initial conditions are zero.

$$s^2 Q(s) + 8s\, Q(s) + 25Q(s) = \frac{150}{s}$$

$$Q(s) = \frac{150}{s(s^2+8s+25)} = \frac{150}{s\left[(s+4)^2+9\right]}$$

Using partial fraction expansion,

$$Q(s) = \frac{6}{s} - \frac{6(s+8)}{(s+4)^2+(3)^2} = \frac{6}{s} - \frac{6(s+4)}{(s+4)^2+3^2} - \frac{24}{(s+4)^2+(3)^2}$$

Hence, $q(t) = £^{-1}[Q(s)] = 6 - 6\,e^{-4t}\,\text{Cos}\,3t - \frac{24}{3}\,e^{-4t}\,\text{Sin}\,3t$

Therefore; $i(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\,[6 - 2e^{-4t}\,(3\,\text{Cos}\,3t + 4\,\text{Sin}\,3t)]$

$= -2[e^{-4t}\cdot(-9\,\text{Sin}\,3t + 12\,\text{Cos}\,3t) + (3\,\text{Cos}\,3t + 4\,\text{Sin}\,3t)\cdot(-4\,e^{-4t})]$

or $\boldsymbol{i(t) = 50\,e^{-4t}\,\text{Sin}\,3t\ \text{A}}$

5.4. TRANSFER FUNCTION

The concept of a transfer function is of great importance in system studies. It provides a method of completely specifying the behaviour of a system subjected to arbitrary inputs. We define the transfer function of a fixed, linear system as the ratio of the Laplace transform of the system output to the Laplace transform of the system input, when all initial conditions are zero. For a perticular system, this transfer function is also called as system function $H(s)$, may be expressed as,

$$H(s) = \left.\frac{Y(s)}{X(s)}\right|_{\text{all initial Conditions are zero.}}$$

Where $Y(s)$ is the Laplace transform of the system output $y(t)$ and $X(s)$ is the Laplace transform of the system input $x(t)$.

This relation must hold true for any input. Suppose that $x(t) = \delta(t)$. Then it is obvious that $H(s)$ is the Laplace transform of the unit impulse response of the system, since $X(s) = 1$. Therefore, inverse Laplace transform of $H(s)$ is the unit impulse response of the system.

Note :

Unit impulse response of the system (also called as weighting function of the system) is the inverse Laplace transform of the system function.

Now, $Y(s) = H(s)\cdot X(s)$

Taking inverse Laplace transform, we have

$$y(t) = £^{-1}[H(s)\cdot(X(s)]$$

$$= \int_0^t x(\tau)\,.\,h(t-\tau)\,d\tau = \int_0^t x(t-\tau)\,.\,h(\tau)\,d\tau$$

EXAMPLE 5.27 **Find the response of the system whose system function**

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1} \text{ for the input}$$

(i) $x(t) = \delta(t)$ (i.e. impulse response)
(ii) $x(t) = e^{-2t}$

Solution : $H(s) = \frac{1}{s+1}$ or $h(t) = e^{-t}$

(i) $x(t) = \delta(t)$ (impulse function)

$X(s) = 1$

Therefore, $Y(s) = \dfrac{1}{s+1}$

Hence, $\boldsymbol{y(t) = e^{-t}}$

(*ii*) $x(t) = e^{-2t}$

or $X(s) = \dfrac{1}{s+2}$

$$y(t) = \int_0^t x(t-\tau).\,h(\tau)\,d\tau = \int_0^t e^{-2(t-\tau)}e^{-\tau}\,d\tau = e^{-2t}\int_0^t e^{\tau}\,d\tau = e^{-2t}\,[e^{t}-1]$$

or $\boldsymbol{y(t) = e^{-t} - e^{-2t}}$

Alternatively : $Y(s) = H(s)\cdot X(s) = \dfrac{1}{(s+2)(s+1)}$

Using partial fraction expansion, we have

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

or $\boldsymbol{y(t) = e^{-t} - e^{-2t}}$

EXAMPLE 5.28 Find the response of a network if

$$\boldsymbol{H(s) = \frac{s^2+3s+5}{(s+1)(s+2)}} \text{ **and excitation** } \boldsymbol{x(t) = e^{-3t}}.$$

Solution : $x(t) = e^{-3t}$

or $X(s) = \dfrac{1}{s+3}$

$$Y(s) = H(s)\cdot X(s)$$

or $Y(s) = \dfrac{s^2+3s+5}{(s+1)(s+2)(s+3)}$

Using partial fraction expansion, we get

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$

$$A = \left.\frac{s^2+3s+5}{(s+2)(s+3)}\right|_{s=-1} = \frac{1-3+5}{(1)(2)} = \frac{3}{2}$$

$$B = \left.\frac{s^2+3s+5}{(s+1)(s+3)}\right|_{s=-2} = \frac{4-6+5}{(-1).(1)} = -3$$

$$C = \left.\frac{s^2+3s+5}{(s+1)(s+2)}\right|_{s=-3} = \frac{9-9+5}{(-2)(-1)} = \frac{5}{2}$$

Now, $Y(s) = \dfrac{\frac{3}{2}}{s+1} - \dfrac{3}{s+2} + \dfrac{\frac{5}{2}}{s+3}$

Therefore, the response of the network is given by

$$y(t) = \pounds^{-1}[Y(s)] = \frac{3}{2}e^{-t} - 3e^{-2t} + \frac{5}{2}e^{-3t}$$

EXAMPLE 5.29 **Find the current $i(t)$ for $t > 0$ for the circuit in figure 5.23(*a*). Assume that the circuit has reached steady state at $t = 0^-$.** *(I.P. Univ., 2000)*

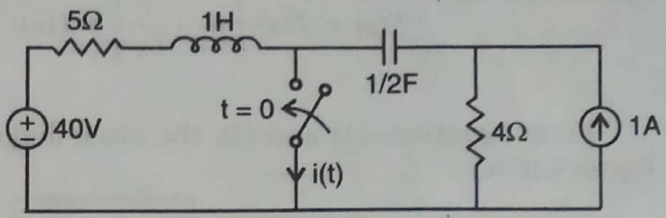


Fig. 5.23(a).

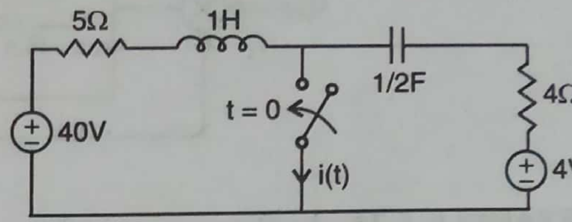


Fig. 5.23(b).

Solution : First we converted current source into equivalent voltage source as shown in figure 5.23(*b*).

At steady state, inductor behaves as a short circuit and capacitor as a open circuit. So $i_L(0^-) = 0$ and, $v_c(0^-) = 40 - 4 = 36$ V

Let $i_1(t)$ and $i_2(t)$ be the currents due to 40 V and 4 V voltage sources respectively.

Then, $$i(t) = i_1(t) + i_2(t)$$

Applying *KVL*, $$40 = 5i_1(t) + \frac{di_1(t)}{dt}$$

Taking Laplace transform,

$$\frac{40}{s} = [5 + s]\, I_1(s)$$

or $$I_1(s) = \frac{40}{s(s+5)} = 8.\left[\frac{1}{s} - \frac{1}{s+5}\right]$$

Therefore, $$i_1(t) = \pounds^{-1}[I_1(s)] = 8(1 - e^{-5t})$$

And, $$4 = 4i_2(t) + 2\int_0^t i_2(t)\, dt - 36$$

$$\frac{40}{s} = \left(4 + \frac{2}{s}\right) I_2(s)$$

or $$I_2(s) = \frac{40}{4s+2} = \frac{10}{s+0.5}$$

Therefore, $$i_2(t) = \pounds^{-1}[I_2(s)] = 10\, e^{-0.5\,t}$$

Hence, $$\boldsymbol{i(t) = 8(1 - e^{-5t}) + 10\, e^{-0.5\,t}}.$$

EXAMPLE 5.30 **Draw the block diagram representation of the circuit of figure 5.24. The variables x and y are the input and output variables respectively.** *(U.P.T.U., 2001)*

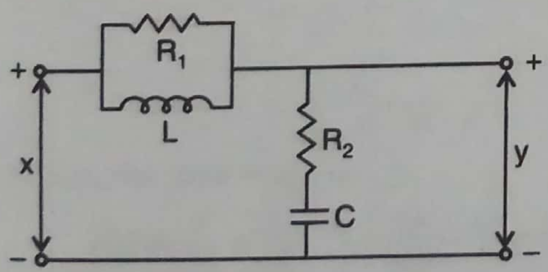


Fig. 5.24.

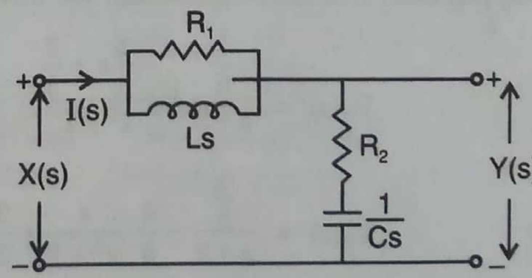


Fig. 5.25(a).

Solution : Redrawn the circuit as shown in figure 5.25(*a*).

$$X(s) - Y(s) = I(s) \cdot [R_1 \| Ls]$$

or $$X(s) - Y(s) = I(s) \cdot \frac{R_1 \cdot Ls}{R_1 + Ls}$$

or $$X(s) - Y(s) = I(s) \cdot \frac{R_1 s}{s + \frac{R_1}{L}} \qquad ...(1)$$

$$Y(s) = I(s)\cdot\left[R_2+\frac{1}{Cs}\right] = I(s)\cdot\frac{R_2Cs+1}{Cs} = I(s)\cdot\frac{s+\dfrac{1}{R_2C}}{\dfrac{1}{R_2}s} \qquad ...(2)$$

From equations (1) and (2), the block-diagram representation of the given circuit is shown in figure 5.25(*b*).

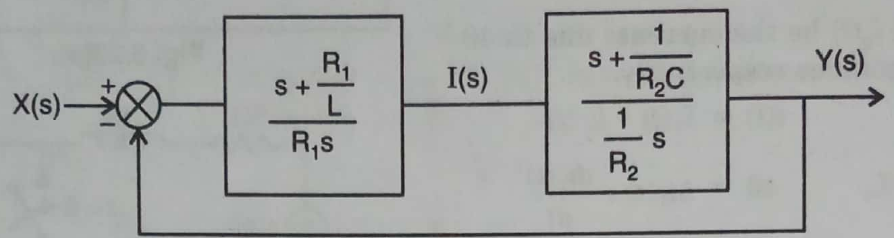


Fig. 5.25(b).

EXAMPLE 5.31 **In the network shown in figure 5.26(*a*), find $v_2(t)$ using Laplace Transform technique.**

(U.P.T.U., 2002)

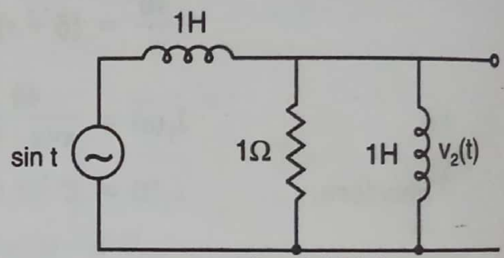


Fig. 5.26(a).

Solution : First draw the transformed circuit diagram as shown in figure 5.26(*b*), and which can reduced as figure 5.26(*c*).

From figure 5.26(*c*),

$$V_2(s) = \frac{1}{s^2+1}\cdot\frac{\dfrac{s}{s+1}}{\dfrac{s}{s+1}+s} = \frac{1}{s^2+1}\cdot\frac{s}{s^2+2s} = \frac{1}{(s^2+1)(s+2)}$$

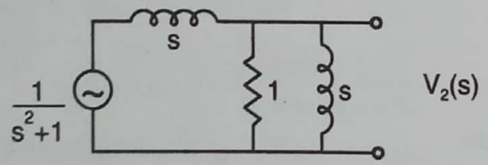


Fig. 5.26(b).

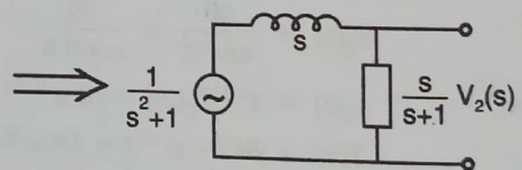


Fig. 5.26(c).

Using partical fraction expension, we have

$$V_2(s) = \frac{\frac{1}{5}}{s+2} + \frac{-\frac{1}{5}s+\frac{2}{5}}{s^2+1}$$

or $$V_2(s) = \frac{\frac{1}{5}}{s+2} - \frac{1}{5}\cdot\frac{s}{s^2+1} + \frac{2}{5}\cdot\frac{1}{s^2+1}$$

Therefore $$v_2(t) = \frac{1}{5}e^{-2t} - \frac{1}{5}\cos t + \frac{2}{5}\sin t$$

or $$v_2(t) = \frac{1}{5}(e^{-2t} - \cos t + 2\sin t)$$

EXAMPLE 5.32 **A system has the following transfer function**

$$\frac{C(s)}{R(s)} = \frac{(s+1)(s+3)}{(s+2)(s^2+8s+32)}$$

Determine time response c(t) of the system for a unit step input. *(U.P.T.U., 2002)*

Solution : $\dfrac{C(s)}{R(s)} = \dfrac{(s+1)(s+3)}{(s+2)(s^2+8s+32)}$

Given, $r(t) = U(t)$

or $R(s) = \dfrac{1}{s}$

$$C(s) = \frac{(s+1)(s+3)}{s(s+2)(s^2+8s+32)}$$

Using partial fraction expansion, we have

$$C(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+8s+32}$$

$$A = \frac{(1)\cdot(3)}{(2)\cdot(32)} = \frac{3}{64}$$

$$B = \frac{(-1)\cdot(1)}{(-2)\cdot(4-16+32)} = \frac{-1}{(-2).(20)} = \frac{1}{40}$$

$$s^2 + 4s + 3 \equiv A\,(s+2)(s^2+8s+32) + Bs\,(s^2+8s+32) + (Cs+D)\,(s)\,(s+2)$$

Coeff. of $s^3 \longrightarrow 0 = A + B + C$

or $C = -(A+B) = -\dfrac{23}{320}$

Coeff. of $s^2 \longrightarrow 1 = 8A + 2A + 8B + 2C + D$

or $10A + 8B + 2C + D = 1$

or $D = 1 - 10\cdot\dfrac{3}{64} - 8\cdot\dfrac{1}{40} - 2\cdot\left(-\dfrac{23}{320}\right)$

$$= 1 - \frac{15}{32} - \frac{1}{5} + \frac{23}{160} = \frac{19}{40}$$

Therefore, $C(s) = \dfrac{3}{64}\cdot\dfrac{1}{s} + \dfrac{1}{40}\cdot\dfrac{1}{s+2} + \dfrac{-\dfrac{23}{320}s + \dfrac{19}{40}}{(s+4)^2+4^2}$

$$= \frac{3}{64}\cdot\frac{1}{s} + \frac{1}{40}\cdot\frac{1}{s+2} - \frac{23}{320}\cdot\frac{s+4}{(s+4)^2+4^2} + \left\{\frac{23}{320} + \frac{1}{4}\cdot\frac{19}{40}\right\}\cdot\frac{4}{(s+4)^2+(4)^2}$$

$$= \frac{3}{64}\cdot\frac{1}{5} + \frac{1}{40}\cdot\frac{1}{s+2} - \frac{23}{320}\cdot\frac{(s+4)}{(s+4)^2+(4)^2} + \frac{61}{320}\cdot\frac{4}{(s+4)^2+(4)^2}$$

Hence, the time response

$$c(t) = \frac{3}{64} + \frac{1}{40}e^{-2t} - \frac{23}{320}\cdot e^{-4t}\cos 4t + \frac{61}{320}\cdot e^{-4t}\sin 4t$$

EXAMPLE 5.33 **In the circuit of figure 5.27, the switch S is closed and a steady state is reached in the circuit. At $t = 0$, the switch is opened. Find an expression for the current in the inductor, $i_L(t)$.**

Solution : At steady state, the inductor behaves as a short circuit and capacitor behaves as an open circuit, Therefore,

$$i_L(0^-) = \frac{100}{10} = 10\text{ A}$$

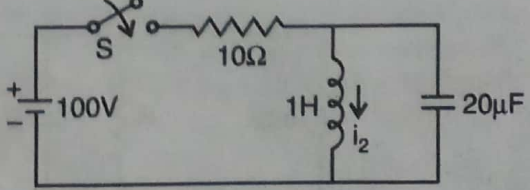


Fig. 5.27.

Now, switch is opened, applying KVL

1. $$\frac{di_L(t)}{dt} + \frac{1}{20\times10^{-6}}\int_0^t i_L(t)\,dt = 0 \text{ ; [with } i_L(0^+) = i_L(0^-) = 10 \text{ A]}$$

Taking Laplace transform, we have

$$sI_L(s) - i_L(0^+) + \frac{1}{20\times10^{-6}}\frac{I_L(s)}{s} = 0$$

$$\left[s + \frac{5\times10^4}{s}\right] I_L(s) = i_L(0^+) = 10$$

or $$I_L(s) = \frac{10\,s}{s^2 + 5\times10^4} = \frac{10\,s}{s^2 + (223.6)^2}$$

Taking inverse Laplace transform, we have

$$\mathbf{i_L(t) = 10 \cos 223.6t \ \ A}$$

EXAMPLE 5.34 **In the circuit of the figure 5.28, the switch S is closed at $t = 0$ with the capacitor initially unenergised. For the numerical values given, find $i(t)$.**

Solution : Applying KVL,

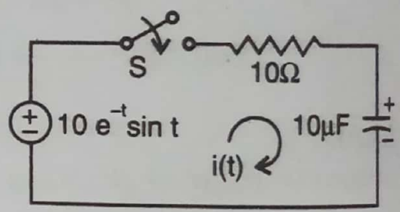


Fig. 5.28.

$$10\,e^{-t}\sin t = 10\,i(t) + \left\{\frac{1}{10\times10^{-6}}\int_0^t i(t)\,dt + 0\right\}$$

Taking laplace transform,

$$\frac{10}{(s+1)^2+1} = 10\,I(s) + 10\times10^4\,\frac{I(s)}{s}$$

$$I(s) = \frac{10\,s}{\{(s+1)^2+1\}(10\,s+10\times10^4)} = \frac{s}{\{(s+1)^2+1\}(s+10^4)}$$

Using partial fraction expansion, we have

$$I(s) = \frac{As+B}{(s+1)^2+1} + \frac{C}{s+10^4} \cong \frac{10^{-4}\,s}{(s+1)^2+1} - \frac{10^{-4}}{s+10^4}$$

$$I(s) = 10^{-4}\left[\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right] - 10^{-4}\,\frac{1}{s+10^4}$$

Taking inverse Laplace transform, we have

$$\mathbf{i(t) = 10^{-4}\,e^{-t}(\cos t - \sin t) - 10^{-4}\,e^{-10^4 t}}.$$

EXAMPLE 5.35 **Draw the transformed circuit diagram of the circuit in figure 5.29(a) and obtain the appropriate transformed nodal equation. Given $R = 1/3\ \Omega$, $L = 0.5\ H$, $C = 1F$, $e(t) = 10$ V, $i_L(0^-) = 15$ A and $v_c(0^-) = 5$V. Than solve for $v(t)$.**

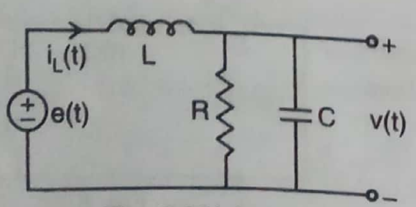


Fig. 5.29(a).

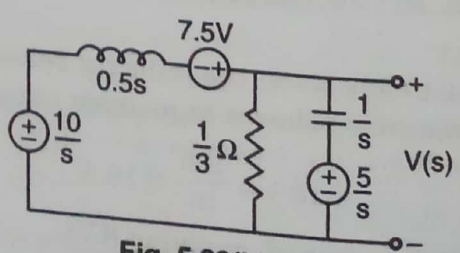


Fig. 5.29(b).

Solution : Transformed circuit diagram of the circuit of figure 5.29(*a*) is drawn in figure 5.29(*b*).

Applying KCL, Nodal equation is given as,

$$\frac{\frac{10}{s}+7.5-V(s)}{0.5s}=\frac{V(s)}{1/3}+\frac{V(s)-\frac{5}{s}}{\frac{1}{s}}$$

or $$\frac{20}{s^2}+\frac{15}{s}=\frac{2}{s}V(s)+3V(s)+sV(s)-5$$

$$(20+15s+5s^2)=(2s+3s^2+s^3)\,V(s)$$

or $$V(s)=\frac{5\left(s^2+3s+4\right)}{s(s+1)(s+2)}$$

Using partial fraction expansion, we have

$$V(s)=\frac{10}{s}-\frac{10}{s+1}+\frac{5}{s+2}$$

Therefore,

$$v(t)=\mathbf{10-10\,e^{-t}+5\,e^{-2t}\ V}$$

EXAMPLE 5.36 **For the circuit in figure 5.30(*a*) with $R = 1\Omega$, $C = 1F$ and $V_c(0^-) = 0V$, determine output (response) $v(t)$ when input $i(t)$ is (*a*) unit- impulse function and (*b*) unit-step function.**

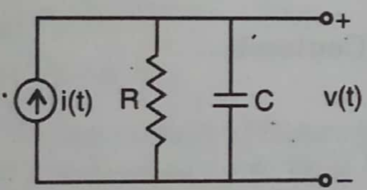


Fig. 5.30(a).

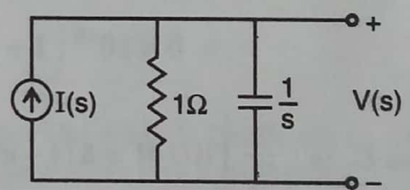


Fig. 5.30(b).

Solution : Using transformed circuit diagram as shown in figure 5.30(*b*), we have

$$V(s)=I(s).\left[1\,\Big\|\,\frac{1}{s}\right]=\frac{I(s)}{s+1}$$

(*a*) $$i(t)=\delta(t)\ ;\ i.e.,\ I(s)=1$$

Then $$V(s)=\frac{1}{s+1}\quad\text{or}\quad v(t)=\boldsymbol{e^{-t}\,U(t)}$$

(*b*) $$i(t)=U(t)\ ;\ i.e.,\quad I(s)=\frac{1}{s}$$

Then $$V(s)=\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1}$$

or $$v(t)=\boldsymbol{(1-e^{-t})\,U(t)}$$

EXAMPLE 5.37 **In the circuit shown in figure 5.31, C_1 is initially charged to a voltage V_0. At time $t = 0$, switch S is closed. Obtain the expressions for (*i*) current, (*ii*) voltage across R, (*iii*) charge across C_1, (*iv*) voltage across C_2 as a function of time, if $C_1 = C_2 = 1\mu$ F, $V_0 = 10V$ and $R = 10\ \Omega$. Also calculate the values of (*i*) to (*iv*) after $t = 10\ \mu$ sec.**

Solution : Applying KVL,

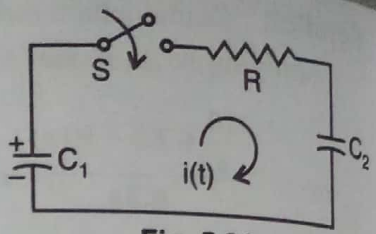


Fig. 5.31.

$$\frac{1}{C_1}\int_0^t i(t)\,dt - V_0 + Ri(t) + \frac{1}{C_2}\int_0^t i(t)\,dt = 0$$

Taking Laplace transform, we have

$$I(s)\left[\frac{1}{C_1 s} + R + \frac{1}{C_2 s}\right] = \frac{V_0}{s}$$

$$I(s) = \frac{V_0}{\frac{1}{C_1} + Rs + \frac{1}{C_2}} = \frac{10}{10s + 2\times10^6} = \frac{1}{s + 2\times10^5}$$

Hence, $i(t) = e^{-2\times10^5 t}$ **A**

And voltage across R is; $R.i(t) = 10\, e^{-2\times10^5 t}$ **V**

Charge across C_1 is ; $C_1\left[V_0 - \frac{1}{C_1}\int_0^t i(t)\,dt\right]$

$$= C_1\left[V_0 - \frac{1}{C_1}\cdot\frac{e^{-2\times10^5 t}}{-2\times10^5}\bigg|_0^t\right]$$

$$= 1\times10^{-6}\times10 + 5\times10^{-6}\left(e^{-2\times10^5 t} - 1\right)$$

$$= 5\times10^{-6}\left(1 + e^{-2\times10^5 t}\right)\text{Coulomb}$$

Voltage across C_2 is ; $\frac{1}{C_2}\int_0^t i(t)\,dt = 5\left(1 - e^{-2\times10^5 t}\right)$ V

Value after $t = 10$ μ sec,

$$i = e^{-2\times10^5\times10\times10^{-6}} = 0.135 \text{ A}$$
$$v_R = 10\times0.135 = 1.35 \text{ V}$$
$$q_{c1} = 5\times10^{-6}(1 + e^{-2}) = 5.67\ \mu\text{ Coulomb}$$
$$v_{c2} = 5(1 - e^{-2}) = 4.32 \text{ V}$$

EXAMPLE 5.38 **In the circuit of figure 5.32, switch S is closed and steady-state conditions reached. Now, at time $t = 0$, switch S is opened. Obtain the expression for the current through the inductor.**

Solution : At steady state (with switch S closed),

$$i_L(0^-) = \frac{10}{2} = 5A\,1$$

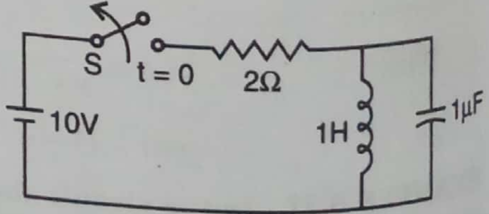


Fig. 5.32.

Find $v_c(0^-) = 0$ (as inductor behaves as a short circuit)

Now, switch is opened at $t = 0$.

$$i_L(0^+) = i_L(0^-) = 5 \text{ A}$$

And applying KVL,

$$1\cdot\frac{di_L(t)}{dt} + \frac{1}{1\times10^{-6}}\int_0^t i_L(t)\,dt = 0$$

Taking Laplace transform, we have

$$sI_L(s) - i_L(0^+) + 10^6 \frac{I_L(s)}{s} = 0$$

or
$$I_L(s) = \frac{5}{s + \frac{10^6}{s}} = 5\frac{s}{s^2 + 10^6}$$

or
$$i(t) = 5 \cos 1000\, t \text{ A}$$

EXAMPLE 5.39 **The current in a resistive-inductive ciruit is given as $i(t) = -5e^{-10t}$. Find**

(i) time constant,

(ii) initial rate of change of current,

(iii) i(t) at t = 0.4 sec, and

(iv) initial value of inductor current.

Solution :

(i) time constant ; $\tau = \frac{L}{R} = \frac{1}{10} = \mathbf{0.1}$ **sec** (on comparing with $K\, e^{-\frac{R}{L}t}$)

(ii) $\frac{di(t)}{dt} = -5\, e^{-10t}.(-10) = 50\, e^{-10t}$

or
$$\frac{di}{dt}(0^+) = \mathbf{50\ A/sec}$$

(iii) $i(t)\big|_{t=0.4\,\text{sec}} = -5e^{-10\times 0.4} = \mathbf{-0.09\ A}$

(iv) $i(0^+) = -5\, e^0 = \mathbf{-5\ A}$

EXAMPLE 5.40 **In the circuit of figure 5.33(a) after the switch has been in the open position for a long time, it is closed at $t = 0$. Find the voltage across the capacitor.**

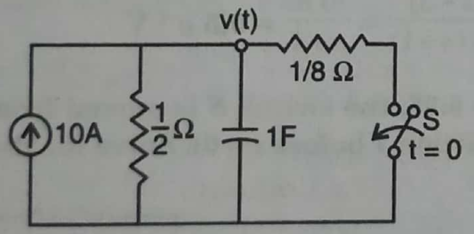


Fig. 5.33(a).

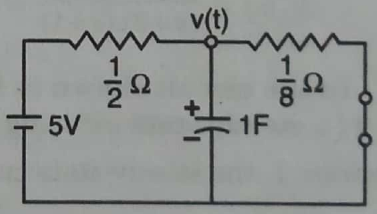


Fig. 5.33(b).

Solution: At steady state (with switch opened),

$$v(0^-) = 10 \times \frac{1}{2} = 5 \text{ V}$$

Applying source conversion; we have the circuit as shown in figure 5.33(b)

Applying KCL,

$$\frac{5 - v(t)}{1/2} = 1 \cdot \frac{dv(t)}{dt} + \frac{v(t)}{1/8}$$

$$10 = 2v(t) + \frac{dv(t)}{dt} + 8v(t) = \frac{dv(t)}{dt} + 10v(t)$$

Taking Laplace transform,

$$\frac{10}{s} = sV(s) - v(0^+) + 10\, V(s)$$

$$\frac{10}{s} + 5 = (s+10)\,V(s) \qquad \text{(As } v(0^+) = v(0^-) = 5 \text{ V)}$$

$$V(s) = \frac{10+5s}{s\,(s+10)} = \frac{1}{s} + \frac{4}{s+10}$$

or $$v(t) = 1 + 4e^{-10t} \text{ V}$$

EXAMPLE 5.41 **In the circuit shown in figure 5.34(*a*), the switch *S* is closed at $t = 0$ connecting a source e^{-t} to the RC circuit. At $t = 0$, it is observed that the capacitor voltage has the value $v_c(0) = 0.5$ V. For the element values given, determine $v_2(t)$.**

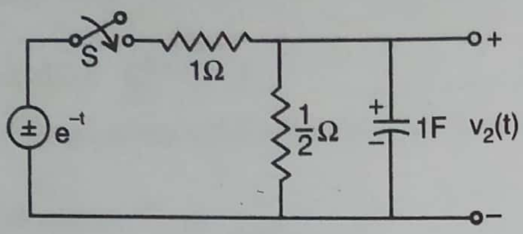


Fig. 5.34(a).

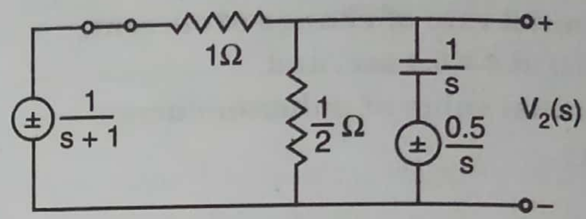


Fig. 5.34(b).

Solution : The transformed circuit diagram of the circuit of figure 5.34(*a*) at $t = 0^+$ is shown in figure 5.34(*b*).

Applying KCL,

$$\frac{\dfrac{1}{s+1} - V_2(s)}{1} = \frac{V_2(s)}{\dfrac{1}{2}} + \frac{V_2(s) - \dfrac{0.5}{s}}{\dfrac{1}{s}}$$

or $$\frac{1}{s+1} + 0.5 = (3+s)\,V_2(s)$$

or $$V_2(s) = \frac{1+0.5(s+1)}{(s+3)\,(s+1)} = \frac{0.5\,(s+3)}{(s+3)\,(s+1)} = \frac{0.5}{s+1} = 0.5\,e^{-t} \text{ V}$$

EXAMPLE 5.42 **In the circuit shown in figure 5.35, the switch *S* is moved from position 1 to position 2 at $t = 0$ (a steady state existing in position 1 before $t = 0$). Solve for the current $i_L(t)$.**

Solution : At position 1, the steady state current

$$i_L(0^-) = \frac{V}{R}$$

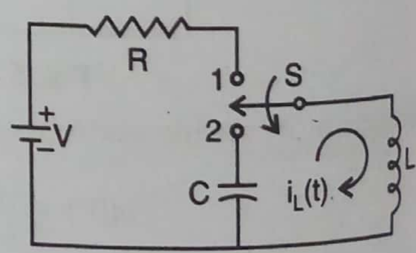


Fig. 5.35.

Now at position 2, applying KVL,

$$L\frac{di_L(t)}{dt} + \frac{1}{C}\int_0^t i_L(t)\,dt = 0$$

Taking Laplace transform, we have

$$L\left[sI_L(s) - i_L(0^+)\right] + \frac{1}{C}\frac{I_L(s)}{s} = 0$$

$$\left(Ls + \frac{1}{Cs}\right) I_L(s) = L.\frac{V}{R}$$

$$I_L(s) = \frac{LV}{R}\,\frac{Cs}{LCs^2+1} = \frac{V}{R}\cdot\frac{s}{s^2 + \dfrac{1}{LC}}$$

Hence $$i_L(t) = \frac{V}{R}\cos\left(t/\sqrt{LC}\right)\ A$$

EXAMPLE 5.43 **With switch S in a position 1, the circuit shown in figure 5.36(a) attains equilibrium. At time $t = 0$, the switch is moved to position 2. Find the voltage across 5 MΩ resistor.**

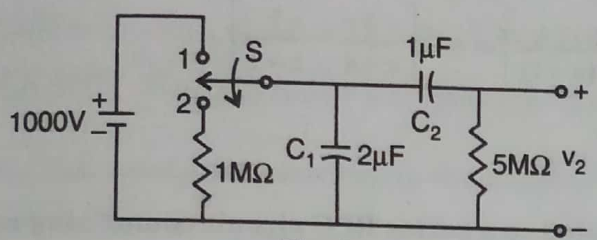


Fig. 5.36(a).

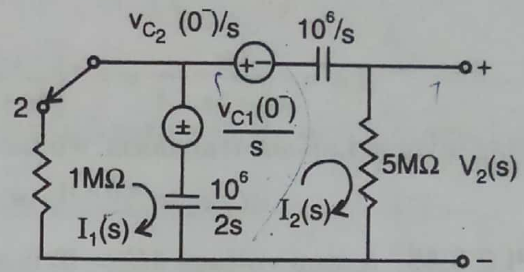


Fig. 5.36(b).

Solution : At position 1, the steady state voltages across the capacitors are

$$v_{c1}(0^-) = v_{c2}(0^-) = 1000\ V$$

Now at position 2, the transformed circuit diagram of the circuit is shown in figure 5.36(b). Applying KVL,

$$10^6\, I_1(s) + \frac{1000}{s} + \frac{10^6}{2s}\left[I_1(s) - I_2(s)\right] = 0 \qquad ...(i)$$

And $$\frac{1000}{s} = \frac{1000}{s} + \frac{10^6}{s} I_2(s) + 5\times 10^6 I_2(s) + \frac{10^6}{2s}\left[I_2(s) - I_1(s)\right]$$

$$10^6\left[\frac{1}{s} + 5 + \frac{1}{2s}\right] I_2(s) = \frac{10^6}{2s} I_1(s)$$

or $$I_1(s) = (2 + 10\,s + 1)\, I_2(s) = (10\,s + 3)\, I_2\,(s) \qquad ...(ii)$$

From equations (1) and (2), we have

$$10^6\,(10\,s+3)\, I_2(s) + \frac{1000}{s} + \frac{10^6}{2s}\left[10s + 3 - 1\right] I_2(s) = 0$$

$$10^6\left[10\,s + 3 + 5 + \frac{1}{s}\right] I_2(s) = -\frac{1000}{s}$$

$$I_2\,(s) = -\frac{1000}{10^6\left[10s^2 + 8s + 1\right]}$$

Therefore $V_2(s) = 5 \times 10^6 .\ I_2(s)$

$$= -\frac{5000}{10s^2 + 8s + 1} = -\frac{500}{(s+0.155)\,(s+0.645)}$$

$$= -\left[\frac{1020}{s+0.155} - \frac{1020}{s+0.645}\right]$$

or $$v_2(t) = \mathbf{1020\,(e^{-0.645t} - e^{-0.155t})\ V}$$

EXAMPLE 5.44 **Find the current $i(t)$ in a series R-L-C circuit comprising $R = 5$ ohm, $L = 1$ H and $C = 1/4$F when impulse voltage $3\,\delta(t-1)$ is applied.** *(U.P.T.U., 2003)*

Solution : Applying KVL, (from figure 5.37)

$$3\delta(t-1) = 5i(t) + 1.\frac{di(t)}{dt} + 4\int_0^t i(t)\,dt$$

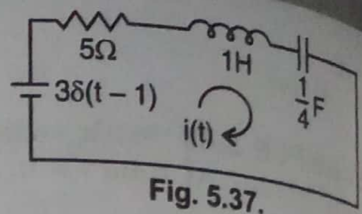


Fig. 5.37.

Taking Laplace transform,

$$3e^{-s} = \left(5 + s + \frac{4}{s}\right) I(s)$$

or
$$I(s) = \frac{3se^{-s}}{s^2+5s+4} = e^{-s}\left[\frac{3s}{(s+4)(s+1)}\right] = e^{-s}\left[\frac{4}{s+4} - \frac{1}{s+1}\right]$$

Taking inverse Laplace transform, we have

$$i(t) = 4e^{-4(t-1)} - e^{-(t-1)}\ \mathrm{A}$$

EXAMPLE 5.45 **A step voltage $3U(t-3)$ is applied to a series RLC circuit comprising resistor $R = 5\Omega$, inductor $L = 1$H and capacitor $C = \frac{1}{4}F$. Find the expression for current $i(t)$ in the circuit.** *(U.P.T.U., 2003 C.O.)*

Solution : Applying KVL, (from figure 5.84)

$$3U(t-3) = 5i(t) + 1.\frac{di(t)}{dt} + 4\int_0^t i(t)\,dt$$

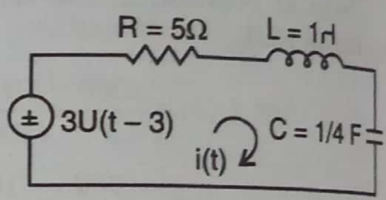


Fig. 5.38.

Taking Laplace transform,

$$\frac{3e^{-3s}}{s} = \left[5 + s + \frac{4}{s}\right] I(s) - i(0^+)$$

or
$$I(s) = \left(\frac{3e^{-3s}}{s^2+5s+4}\right)$$ (Since $i(0^+) = 0$)

Using partial fraction expansion, we have

$$I(s) = e^{-3s}\left[\frac{1}{s+1} - \frac{1}{s+4}\right]$$

or
$$i(t) = [e^{-(t-3)} - e^{-4(t-3)}]\ U(t-3)$$

EXAMPLE 5.46 **Using Laplace transformation, solve the following differential equation:**

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 8i = 8U(t)$$

Given that $i(0^+) = 3$ and $\frac{di}{dt}(0^+) = -4$. *(U.P.T.U., 2003)*

Solution : $\frac{d^2i(t)}{dt^2} + 4\frac{di(t)}{dt} + 8\,i(t) = 8U(t)$

Taking Laplace transform, we have

$$s^2I(s) - si(0^+) - i'(0^+) + 4[s\,I(s) - i(0^+)] + 8I(s) = \frac{8}{s}$$

$$(s^2 + 4s + 8)\,I(s) = \frac{8}{s} + (s+4)\,i(0^+) + i'(0^+)$$

$$= \frac{8}{s} + 3(s+4) - 4$$

(Since $i(0^+) = 3$ and $i'(0^+) = -4$)

or
$$I(s) = \frac{3s^2 + 3s + 8}{s(s^2 + 4s + 8)}$$

Using partial fraction expansion, we have

$$I(s) = \frac{1}{s} + 2\frac{(s+2)}{(s+2)^2 + (2)^2}$$

Therefore,
$$i(t) = [1 + 2\, e^{-2t} \cos 2t]\ U(t)$$

EXAMPLE 5.47 **Find $i(t)$ for $t > 0$ in the circuit shown in figure 5.39. Switch is opened at $t = 0$.**

(U.P.T.U., 2003 C.O.)

Solution : The steady state current in the inductor (with switch closed).

$$i_L(0^-) = \frac{12}{2} + \frac{16}{4} = 10 \text{ A}$$

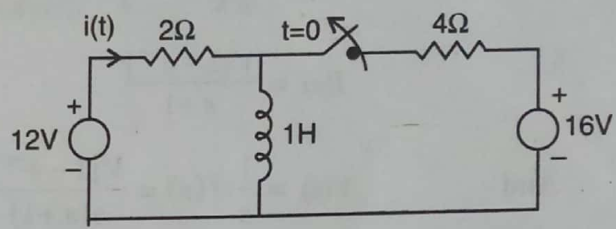


Fig. 5.39.

Now switch is opened, applying KVL,

$$12 = 2i(t) + 1\cdot\frac{di(t)}{dt}$$

Taking Laplace transform,

$$\frac{12}{s} = 2\, I(s) + sI(s) - i_L(0^-)$$

or
$$I(s) = \frac{\frac{12}{s} + 10}{s+2} = \frac{10s + 12}{s(s+2)}$$

Using partial fraction expansion, we have

$$I(s) = \frac{6}{s} + \frac{4}{s+2}$$

Therefore,
$$i(t) = 2(3 + 2e^{-2t})\ U(t)$$

EXAMPLE 5.48 **In the network shown in Figure 5.40, the switch K is in position 1 long enough to establish steady state condition. At time $t = 0$, the switch K is moved from position 1 to 2. Find expression for the current $i(t)$.**

(U.P.T.U., 2003 C.O.)

Solution : At position 1, the steady state current

$$i_L(0^-) = \frac{V_0}{R}$$

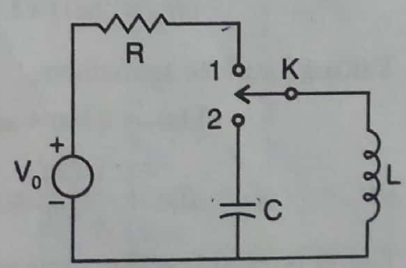


Fig. 5.40.

Now at position 2, applying KVL

$$\frac{1}{C}\int_0^t i(t) + L\frac{di(t)}{dt} = 0$$

Taking Laplace transform,

$$\frac{1}{Cs} I(s) + L[s\, I(s) - i_L(0^+)] = 0$$

$$\left(\frac{1}{Cs} + Ls\right) I(s) = L.\frac{V_0}{R} \qquad \left(\text{Since } i_L(0^+) = i_L(0^-) = \frac{V_0}{R}\right)$$

or
$$I(s) = \frac{LV_0/R}{(1 + LCs^2)/Cs} = \frac{V_0}{R}\left(\frac{s}{s^2 + \frac{1}{LC}}\right)$$

Therefore, $$i(t) = \frac{V_0}{R}\cos\left(\frac{t}{\sqrt{LC}}\right)U(t)$$

EXAMPLE 5.49. Find out impulse response of the network shown in figure 5.41. *(U.P.T.U., 2003 C.O.)*

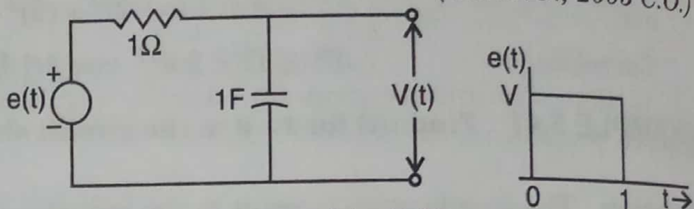


Fig. 5.41.

Solution : $$e(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt$$

where $$e(t) = V[U(t) - U(t-1)]$$

Taking Laplace transform,

[Since $R = 1\Omega$, $C = 1F$ and $v_c(0^+) = 0$]

$$E(s) = \frac{V}{s}\left[1 - e^{-s}\right] = \left(1 + \frac{1}{s}\right)I(s)$$

$$I(s) = \frac{V[1-e^{-s}]}{s+1}$$

And $$V(s) = \frac{1}{s}.I(s) = \frac{V[1-e^{-s}]}{s(s+1)}$$

So, $$\frac{V(s)}{E(s)} = H(s) = \frac{1}{s+1}$$

Therefore, impulse response,

$$h(t) = £^{-1}[H(s)] = e^{-t}\,U(t)$$

EXAMPLE 5.50 **A voltage pulse of magnitude 8 volts and duration 2 seconds extending from $t = 2$ seconds to $t = 4$ seconds is applied to a series RL circuit as shown in figure 5.42(*a*). Obtain the expression for the current *i*(*t*).** *(U.P.T.U., 2003 C.O.)*

Solution : The voltage pulse is given as [as shown in figure 5.59(*b*)]

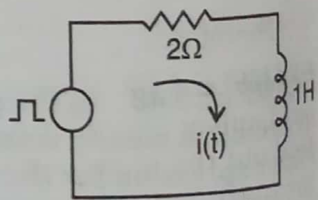


Fig. 5.42(a).

$$v(t) = 8[U(t-2) - U(t-4)]$$

or $$V(s) = \frac{8}{s}(e^{-2s} - e^{-4s})$$

Applying KVL in the circuit of figure 5.89(*a*), we have

$$v(t) = 2i(t) + 1.\frac{di(t)}{dt}$$

Taking Laplace transform,

$$V(s) = 2I(s) + sI(s)$$

or $$I(s) = \frac{V(s)}{s+2} = \frac{8(e^{-2s} - e^{-4s})}{s(s+2)}$$

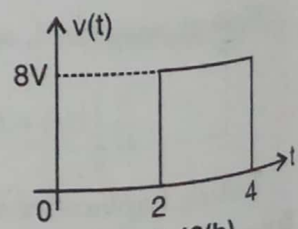


Fig. 5.42(b).

Using partial fraction expansion,

$$I(s) = e^{-2s}\left[\frac{4}{s} - \frac{4}{s+2}\right] - e^{-4s}\left[\frac{4}{s} - \frac{4}{s+2}\right]$$

or $$i(t) = 4(1 - e^{-2(t-2)})\,U(t-2) - 4\,(1 - e^{-2(t-4)})\,U(t-4)$$

EXAMPLE 5.51 **Determine the steady-state mesh currents i_1 and i_2 in the circuit of figure 5.43. There is no initial energy stored in the circuit.** *(U.P.T.U., 2004)*

Solution : After switch is closed, Applying KVL, we have

$$10 \sin t = i_1(t) + 2\left[\frac{di_1(t)}{dt} - \frac{di_2(t)}{dt}\right]$$

and
$$0 = 2\, i_2(t) + \frac{1}{1}\int_{-\infty}^{t} i_2(t)\, dt + 2\left[\frac{di_2(t)}{dt} - \frac{di_1(t)}{dt}\right]$$

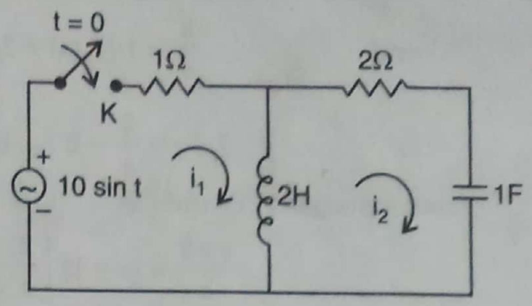


Fig. 5.43.

As there is no initial energy stored in the circuit. Taking Laplace transform, above equations become

$$\frac{10}{s^2+1} = (2s + 1)\, I_1(s) - 2s\, I_2(s)$$

$$0 = -2s\, I_1(s) + \left(2 + \frac{1}{s} + 2s\right) I_2(s)$$

Solving for $I_1(s)$ and $I_2(s)$, we have

$$I_1(s) = \frac{10\,(2s^2+2s+1)}{(s^2+1)\,(6s^2+4s+1)}$$

and
$$I_2(s) = \frac{20s^2}{(s^2+1)\,(6s^2+4s+1)}$$

Therefore, the steady state values of i_1 and i_2 is given as

$$i_1(\infty) = \lim_{s\to 0} sI_1(s) = 0$$

and
$$i_2(\infty) = \lim_{s\to 0} sI_2(s) = 0$$

Alternatively: We can easily see that the steady state value of i_1 and i_2 is zero since inductor and capacitor behave as short circuit and open circuit respectively, at steady state and hence $i_2 = 0$ and i_1 is oscillatory in nature.

EXAMPLE 5.52 Construct transformed network of the circuit shown in figure 5.44(a). Find out Laplace Transform function for $I_{R2}(s)$ and then using final value and initial value theorems find out the initial and final value of current $i_{R2}(t)$, through R_2. Verify the results by solving for $i_{R2}(t)$. *(U.P.T.U., 2004)*

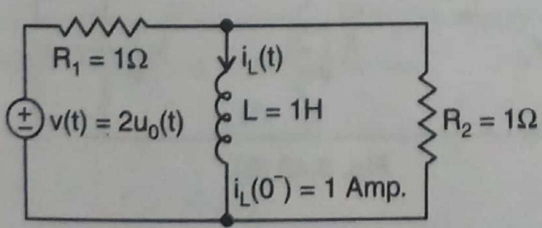


Fig. 5.44(a).

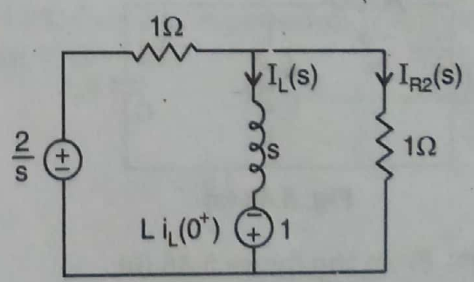


Fig. 5.44(b).

Solution : The transformed network of the circuit as shown in figure 5.44(b).

Applying KVL,

$$\frac{2}{s} + 1 = 1\cdot[I_L(s) + I_{R2}(s)] + s\cdot I_L(s)$$

$$\frac{s+2}{s} = (s + 1)\, I_L(s) + I_{R2}(s) \qquad \text{...(1)}$$

and $$\frac{2}{s} = 1\cdot[I_L(s) + I_{R2}(s)] + 1 \,.\, I_{R2}(s)$$

or $$I_L(s) = \frac{2}{s} - 2\, I_{R2}(s) \qquad \ldots(2)$$

From equations (1) and (2),

$$\frac{s+2}{s} = (s+1)\left[\frac{2}{s} - 2I_{R2}(s)\right] + I_{R2}(s)$$

$$(2s+1)\, I_{R2}(s) = 1$$

or $$I_{R2}(s) = \frac{1}{2\left(s+\frac{1}{2}\right)}$$

Initial value of $i_{R2}(t) = i_{R2}(0) = \lim\limits_{s\to\infty} s.I_{R2}(s)$

$$= \lim_{s\to\infty} \frac{s}{2\left(s+\frac{1}{2}\right)} = \frac{1}{2\left(1+\frac{1}{2s}\right)} = \frac{1}{2}$$

Final value of $i_{R2}(t) = i_{R2}(\infty) = \lim\limits_{s\to 0} s.I_{R2}(s) = \lim\limits_{s\to 0} \dfrac{s}{2\left(s+\frac{1}{2}\right)} = 0$

Taking inverse Laplace transform of $I_{R2}(s)$, we have

$$i_{R2}(t) = \frac{1}{2} e^{-\frac{1}{2}t}$$

$$i_{R2}(0) = \lim_{t\to 0} i_{R2}(t) = \lim_{t\to 0} \frac{1}{2} e^{-\frac{1}{2}t} = \frac{1}{2}$$

$$i_{R2}(\infty) = \lim_{t\to\infty} i_{R2}(t) = \lim_{t\to\infty} \frac{1}{2} e^{-\frac{1}{2}t} = 0$$

EXAMPLE 5.53 Obtain the differential equation relating i_L and V_s for the circuit shown in figure 5.45 (*a*). Also find the expression for V_C in terms of V_s. *(U.P.T.U., 2004)*

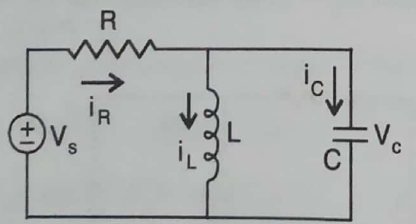


Fig. 5.45 (a).

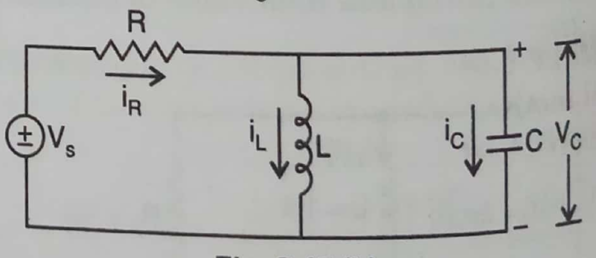


Fig. 5.45 (b).

Solution: From the figure 5.45 (*b*).

$$L\,\frac{di_L(t)}{dt} = \frac{1}{C}\int_0^t i_C(t)\, dt$$

On differentiating,

$$L\,\frac{d^2 i_L(t)}{dt^2} = \frac{1}{C}\, i_C(t)$$

or $$i_C(t) = LC \cdot \frac{d^2 i_L(t)}{dt^2}$$

Now, applying KVL

$$V_s = R\, i_R(t) + L\frac{di_L(t)}{dt}$$

$$= R\,[i_L(t) + i_C(t)] + L\frac{di_L(t)}{dt} \quad \text{[Since } i_R(t) = i_L(t) + i_C(t)]$$

$$= R\left[i_L(t) + LC\frac{d^2 i_L(t)}{dt^2}\right] + L\,\frac{di_L(t)}{dt}$$

or $$RLC\,\frac{d^2 i_L(t)}{dt^2} + L\frac{di_L(t)}{dt} + R\, i_L(t) = V_s$$

On taking Laplace transform, above equation gives,

$$I_L(s) = \frac{V_s}{s(RLCs^2 + Ls + R)}$$

Also $$V_C(t) = L\,\frac{di_L(t)}{dt}$$

or $$V_C(s) = L\, s\, I_L(s) = \frac{L\,V_s}{RLC\,s^2 + Ls + R}$$

$$= \frac{V_s}{RC\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right]} = \frac{V_s}{RC\left[\left(s + \frac{1}{2RC}\right)^2 + \frac{1}{LC} - \frac{1}{4R^2C^2}\right]}$$

$$= \frac{V_s}{RC\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}} \cdot \frac{\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}}{\left(s + \frac{1}{2RC}\right)^2 + \left[\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}\right]^2}$$

or $$v_c(t) = \frac{V_s}{\sqrt{\frac{R^2C}{L} - \frac{1}{4}}} \cdot e^{-\frac{t}{2RC}} \sin\left(\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}\right)t$$

EXAMPLE 5.54 **In this series RC circuit of figure 5.93, find an expression for $V_c(t)$ using nodal method of analysis, if $V_0(t) = u(t) - u(t-1)$ and $V_c(0) = 2$ V.** *(U.P.T.U., 2004)*

Solution : Applying nodal analysis,

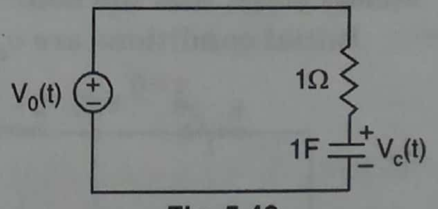


Fig. 5.46.

$$\frac{v_0(t) - v_c(t)}{1} = 1\,.\,\frac{d}{dt}\,v_c(t)$$

or $$\frac{dv_c(t)}{dt} + v_c(t) = v_0(t)$$

Taking Laplace transform,

$$sV_c(s) - v_c(0^-) + V_c(s) = V_0(s)$$

or $$(s+1)\,V_c(s) = 2 + \frac{1}{s}(1 - e^{-s})$$ [since $v_c(0^-) = 2$ V and $v_0(t) = U(t) - U(t-1)$]

or $$V_c(s) = \frac{2}{s+1} + \frac{1-e^{-s}}{s(s+1)}$$

Taking inverse Laplace transform,

$$v_c(t) = 2e^{-t}\, U(t) + (1 - e^{-t})\, U(t) - [1 - e^{-(t-1)}]\, U(t-1)$$

or
$$\boldsymbol{v_c(t) = (1 + e^{-t})\, U(t) - [1 - e^{-(t-1)}]\, U(t-1)}$$

EXAMPLE 5.55 **In the network of figure 5.47 (*a*), switch '*S*' is closed at $t = 0$. Find the driving point impedance and source current in *s*-domain. Hence find the time response of this current.**

(U.P.T.U., 2004)

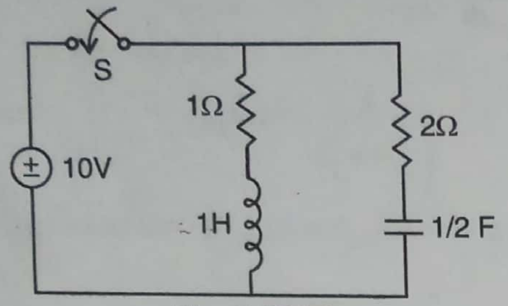


Fig. 5.47 (a).

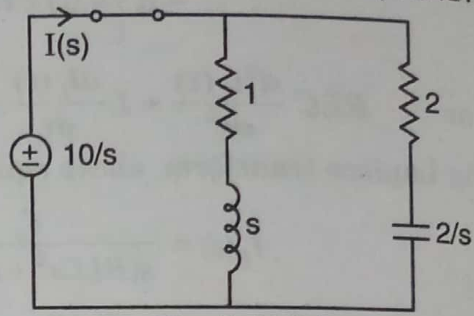


Fig. 5.47 (b).

Solution : At $t = 0^+$, redrawn the circuit in *s*-domain as shown in figure 5.94(*b*).

Driving point impedance

$$Z(s) = (s+1)\ ||\ \left(2 + \frac{2}{s}\right) = \frac{(s+1)\,.\,2\left(1+\dfrac{1}{s}\right)}{s+1+2+\dfrac{2}{s}} = \frac{2(s+1)^2}{s^2+3s+2}$$

or
$$\boldsymbol{Z(s) = \frac{2(s+1)}{s+2}}$$

$$\text{Source current, } I(s) = \frac{10/s}{\dfrac{2(s+1)}{s+2}} = \boldsymbol{\frac{5(s+2)}{s(s+1)}}$$

And time response of current,

$$i(t) = L^{-1}[I(s)] = L^{-1}\left[\frac{10}{s} - \frac{5}{s+1}\right]$$

or
$$\boldsymbol{i(t) = (10 - 5e^{-t})\, U(t)}$$

EXAMPLE 5.56 **Draw the transformed circuit of the figure 5.48 (*a*) in which switch *K* is opened at $t = 0$. Assuming the that prior to opening of switch, the circuit had been in steady state, find the node voltages $v_1(t)$ and $v_2(t)$.**

Initial conditions are $v_c(0^-) = 1$ V and $i_L(0^-) = 1$ A.

(U.P.T.U., 2005)

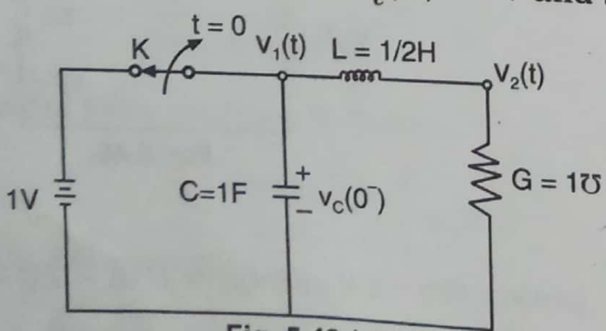


Fig. 5.48 (a).

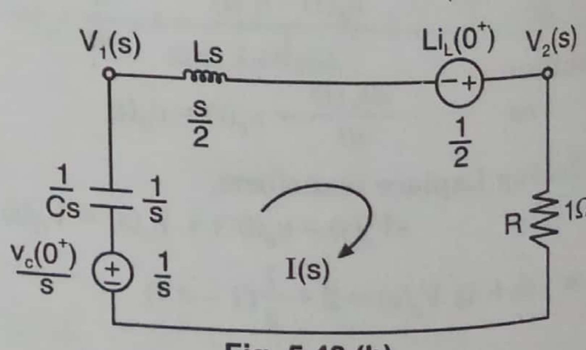


Fig. 5.48 (b).

Solution: At steady state with switch *K* closed, $v_c(0^-) = 1$V and $i_L(0^-) = 1$ A. At t = 0, the switch *K* is opened, the transformed circuit as shown in figure 5.95(*b*), with $v_c(0^+) = 1$ V and $i_L(0^+) = 1$ A. Let *I*(*s*) be the current in loop. Applying KVL,

$$\frac{1}{s}+\frac{1}{2} = I(s)\left[\frac{1}{s}+\frac{s}{2}+1\right]$$

or $$I(s) = \frac{s+2}{s^2+2s+2}$$

Now, $$V_1(s) = \frac{1}{s}-\frac{1}{s}.I(s) = \frac{1}{s}\left[1-\left(\frac{s+2}{s^2+2s+2}\right)\right]$$

$$= \frac{s^2+s}{s(s^2+2s+2)} = \frac{s+1}{(s+1)^2+(1)^2}$$

$$v_1(t) = £^{-1}[V_1(s)] = e^{-t}\cos t$$

and $$V_2(s) = 1\,.\,I(s) = \frac{s+2}{s^2+2s+2} = \frac{(s+1)+1}{(s^2+1)^2+(1)^2}$$

or $$v_2(t) = £^{-1}[V_2(s)] = e^{-t}(\cos t+\sin t)$$

EXAMPLE 5.57 **For the given Laplace transform**

$$Y(s) = \frac{17s^3+7s^2+s+6}{s^5+3s^4+5s^3+4s^2+2s}$$

Find the initial and final values of the corresponding time function $y(t)$.

(U.P.T.U., 2005)

Solution : $$y(0^+) = \lim_{s\to\infty}[s.Y(s)] = \lim_{s\to\infty}\frac{17s^3+7s^2+s+6}{s^4+3s^3+5s^2+4s+2}$$

$$= \lim_{s\to\infty}\frac{1+\frac{7}{s}+\frac{1}{s^2}+\frac{6}{s^3}}{s+3+\frac{5}{s}+\frac{4}{s^2}+\frac{2}{s^3}} = 0$$

$$y(\infty) = \lim_{s\to\infty}[s.Y(s)] = \frac{6}{2} = 3$$

EXAMPLE 5.58 **Find the inverse Laplace transform of the function $F(s) = \dfrac{s+5}{s(s^2+2s+5)}$.**

(U.P.T.U., 2006)

Solution : $$F(s) = \frac{s+5}{s(s^2+2s+5)}$$

$$F(s) = \frac{s+5}{s(s-\alpha)(s-\beta)}$$

$$s^2+2s+5 = 0$$

$$s_1 = -1+2i = \alpha$$

$$s_2 = -1-2i = \beta$$

$$\frac{A}{s}+\frac{B}{(s-\alpha)}+\frac{C}{(s-\beta)} = \frac{(s+5)}{s(s-\alpha)(s-\beta)}$$

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$$A(s-\alpha)(s-\beta) + B(s-\beta)s + C s(s-\alpha) = (s+5)$$

$$s = 0; \quad A = \frac{5}{\alpha\beta}$$

$$s = \alpha; \quad B = \frac{\alpha+5}{\alpha(\alpha-\beta)}$$

$$s = \beta; \quad C = \frac{\beta+5}{\beta(\beta-\alpha)}$$

$$F(s) = \frac{5}{\alpha\beta}\frac{1}{s} + \frac{\alpha+5}{\alpha(\alpha-\beta)}\frac{1}{(s-\alpha)} + \frac{(\beta+5)}{\beta(\beta-\alpha)}\frac{1}{(s-\beta)}$$

$$F(s) = \frac{1}{s} - \frac{1}{2}\frac{1}{[s-(-1+2i)]} - \frac{1}{2}\frac{1}{[s-(-1-2i)]}$$

Taking inverse Laplace transform,

$$f(t) = \left[1 - \frac{1}{2}e^{(-1+2i)t} - \frac{1}{2}e^{(-1-2i)t}\right]U(t)$$

EXAMPLE 5.59 **For the circuit shown in figure 5.59, determine the total current delivered by the source when the switch is closed at $t = 0$. Assume no initial charge on the capacitor.**

(U.P.T.U., 2006)

Solution :

$$Z = \frac{\frac{1}{s}\times 10}{\frac{1}{s}+10} + 5 = \frac{\frac{10}{s}}{\frac{1+10s}{s}} + 5$$

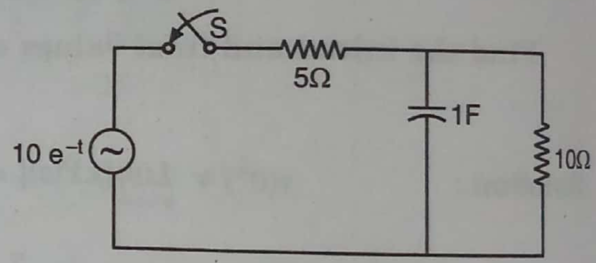

Fig. 5.49.

$$Z = \frac{15+50s}{1+10s} \text{ and } V(s) = \frac{10}{(s+1)}$$

$$I = \frac{V}{Z} = \frac{10}{(s+1)} \times \frac{1+10s}{(15+50s)}$$

$$= \frac{2(1+10s)}{(s+1)(3+10s)} = \frac{A}{(s+1)} + \frac{B}{(3+10s)}$$

Solving $\quad A = \frac{18}{7}, \quad B = -\frac{40}{7}$

$$I(s) = \frac{18}{7(s+1)} - \frac{40}{7.10\left(s+\frac{3}{10}\right)}$$

or $\quad i(t) = \left[\frac{18}{7}e^{-t} - \frac{4}{7}e^{-3/10\,t}\right]U(t)$

EXAMPLE 5.60 **Find the $v_c(t)$ and $i_L(t)$ in the circuit of figure 5.50, assuming zero initial conditions.**

(U.P.T.U., 2006)

Solution :

$$\delta(t) = i_1(t) + i_2(t) + i_L(t) \quad \text{... (1)}$$

$$\frac{1}{C}\int i_2(t)\,dt - i_1(t)R_1 = 0 \quad \text{... (2)}$$

$$i_L(t)R_2 + \frac{1}{2}\frac{di_L(t)}{dt} - \frac{1}{C}\int i_2(t)\,dt = 0 \quad \text{... (3)}$$

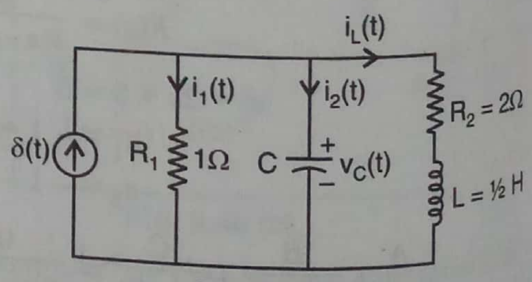

Fig. 5.50.

Taking Laplace of above three equations

$$1 = I_1(s) + I_2(s) + I_L(s) \quad \text{... (4)}$$

$$\frac{1}{sC}\; I_2(s) = I_1(s)\; R_1 \quad \text{... (5)}$$

$$I_L(s)\; R_2 + \frac{1}{2}\; sI_L(s) - \frac{1}{sC}\; I_2(s) = 0 \quad \text{... (6)}$$

Putting the values of R_1 L and R_2, we have

$$\frac{1}{Cs}\; I_2(s) = I_1(s) \quad \text{... (7)}$$

$$2I_L(s) + \frac{1}{2} s\; I_L(s) = \frac{1}{sC}\; I_2(s) \quad \text{... (8)}$$

$$I_L(s)\left[2+\frac{s}{2}\right] = \frac{1}{sC}\; I_2(s)$$

$$I_L(s) = \frac{2}{(4+s)\,sC}\, I_2(s) \quad \text{... (9)}$$

From equations (4) and (7),

$$\frac{I_2(s)}{sC} + I_2(s) + I_L(s) = 1$$

Put the value of $I_2(s)$ from equation (9),

$$I_2(s)\left[\frac{1}{sC}+1+\frac{2}{(4+s)\,sC}\right] = 1$$

$$I_2(s) = \frac{sC(4+s)}{sC(4+s)+s+6}$$

$$I_L(s) = \frac{2}{(4+s)\,sC} \times \left[\frac{(4+s)\,sC}{sC(4+s)+(s+6)}\right] = \frac{2}{sC(4+s)+(s+6)}$$

$$i_L(t) = L^{-1}\left[\frac{2}{sC(4+s)+(s+6)}\right]$$

$$v_c(t) = \frac{1}{C}\int i_2(t)\; dt$$

$$V_c(s) = \frac{1}{sC}\; I_2(s)$$

$$V_c(s) = \frac{(4+s)}{sC\,(4+s)+(s+6)}$$

$$v_c(t) = L^{-1}\left[\frac{(4+s)}{sC\,(4+s)\; +(s+6)}\right]$$

Considering $C = 1\text{F}$;

$$i_L(t) = £^{-1}\left[\frac{2}{s^2+5s+6}\right] = 2\,(e^{-2t} - e^{-3t})$$

$$v_c(t) = £^{-1}\left[\frac{s+4}{s^2+5s+6}\right] = 2\,e^{-2t} - e^{-3t}$$

EXERCISES

5.1. Draw the transformed circuits for all electrical components.

5.2. Find the current $i(t)$ in a series RLC circuit with a step input voltage (Example. 5.23).

PROBLEMS

5.1. In an R-L series circuit $R = 5\Omega$, $L = 2.5$ mH and $i(0^-) = 2$ A. If a source of $50V$ is applied at $t = 0$. Find $i(t)$ for $t > 0$.

5.2. In an R-C series circuit, the capacitor has initial charge 2.5 mC. At $t = 0$, the source of 100 V is applied. Find $i(t)$ for $t > 0$.

Hint : $V_0 = \frac{Q_0}{C} = 50$ V.

5.3. In the R-L circuit shown in figure P.5.3, the switch is in position 1 long enough to establish steady-state conditions, and at $t = 0$ it is switched to position 2. Find the resulting current.

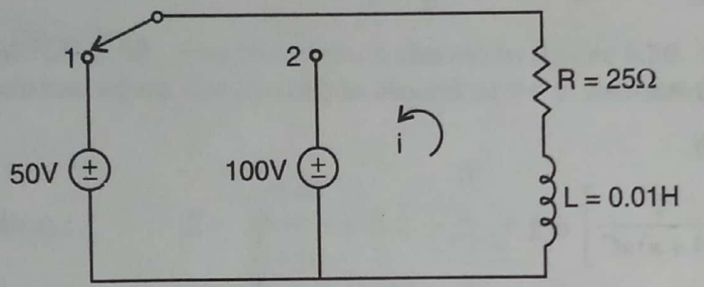

Fig. P.5.3.

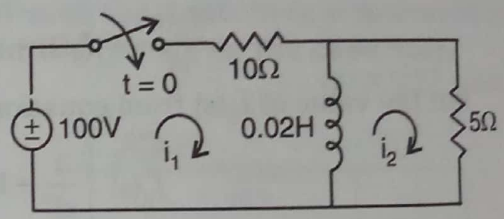

Fig. P.5.4

5.4. In the two-mesh network of figure P.5.4. Find the currents which result when the switch is closed.

5.5. Determine the initial and final values of i_1 and i_2 in above circuit of figure P.5.4.

5.6. Find $i(t)$ in the circuit of figure P.5.6, where S is opened at $t = 0$.

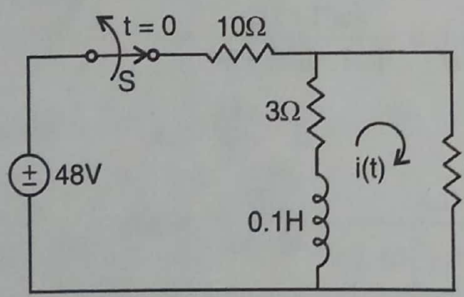

Fig. P.5.6.

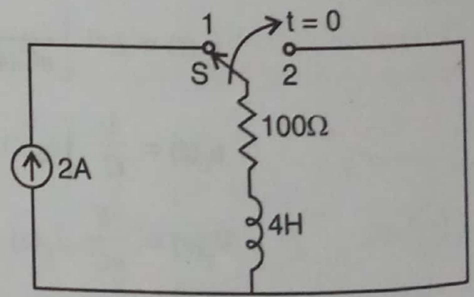

Fig. P.5.7.

5.7. In the circuit of figure P.5.7, S is moved form 1 to 2 at $t = 0$. Determine the voltage across and current through the resistor.

5.8 In the circuit of figure P.5.8, the switch is closed at $t = 0$ and there is no initial charge on either of the capacitors. Find the resulting current.

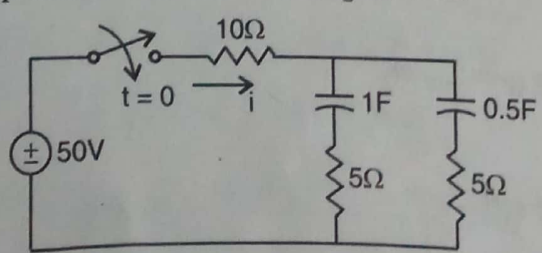

Fig. P.5.8.

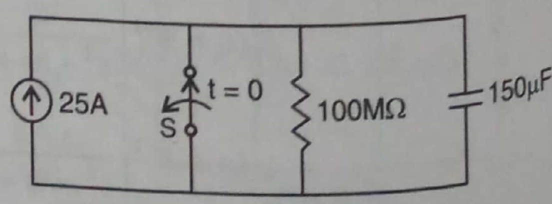

Fig. P.5.9.

5.9. Express the voltage across the capacitor for $t \geq 0$, when the switch in the circuit shown in figure P.5.9, is opened at time $t = 0$.

5.10. Use Laplace transform method to solve the differential equation

$$\frac{d^2v}{dt^2} + \frac{3dv}{dt} + 2v = 4\,U(t) \qquad \left(\text{Given } v(0^-) = 0 \text{ and } \frac{dv}{dt}(0^-) = 5\right)$$

5.11. In the circuit of figure P.5.11, the initial values are $i_L(0^-) = 5$ A, $v_c(0^-) = 10$ V, $i(t) = 10\,u(t)$. Determine $v_c(t)$ for $t > 0$..

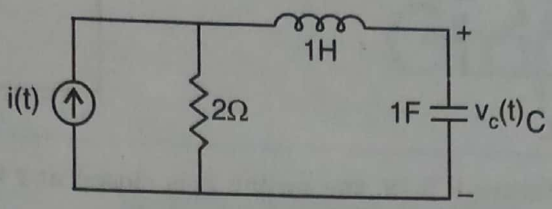

Fig. P.5.11.

5.12. In the circuit shown in figure P.5.12, obtain $v(t)$ for $t > 0$. When S is closed at $t = 0$. Also determine the current drawn from the source.

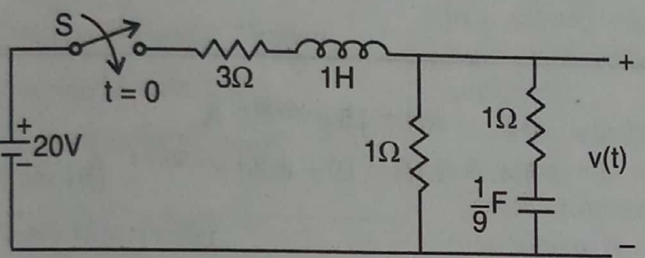

Fig. P.5.12.

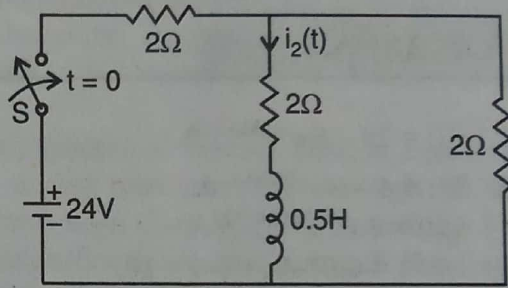

Fig. P.5.13.

5.13. Determine $i_2(t)$ in the circuit in figure P.5.13, when switch S is closed at $t = 0$. The inductor is initially deenergised.

5.14. Determine $i(t)$ in the circuit of figure P.5.6, where switch S is closed at $t = 0$. The switch is initially opened.

5.15. Determine the voltage across the capacitance C as a function of time in the figure P.5.15(a), if a voltage pulse of unit height and width T as shown in figure P.5.15(b) is applied.

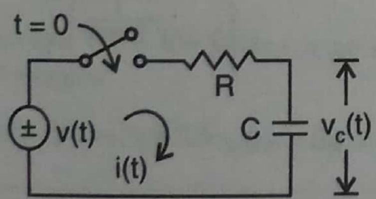

Fig. P.5.15(a).

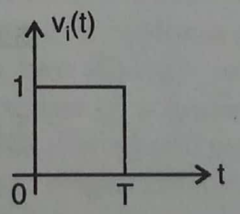

Fig. P.5.15(b).

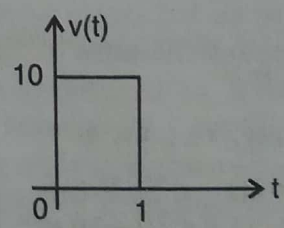

Fig. P.5.16.

5.16. Determine $i(t)$ of the circuit in figure P.5.15(a) for the pulse input voltage is as shown in figure P.5.16, with zero initial conditions. Take $R = 1\Omega$ and $C = 1$F.

5.17. A triangular wave as shown in figure P.5.17(a), is applied as input, to a series R–L circuit with $R = 2\Omega$, $L = 2$H, as shown in figure P.5.17(b). Calculate the current $i(t)$ in the circuit.

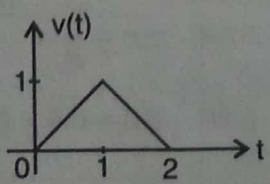

Fig. P.5.17(a).

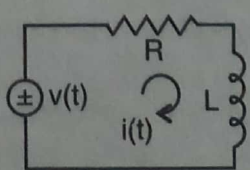

Fig. P.5.17(b).

5.18. In the circuit shown in figure P.5.18, the switch S is moved from position 1 to 2 at $t = 0$, a steady state having previously been established at position 1. Solve for the current $i(t)$.

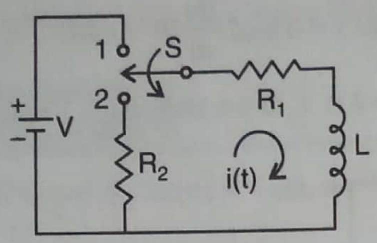

Fig. P.5.18.

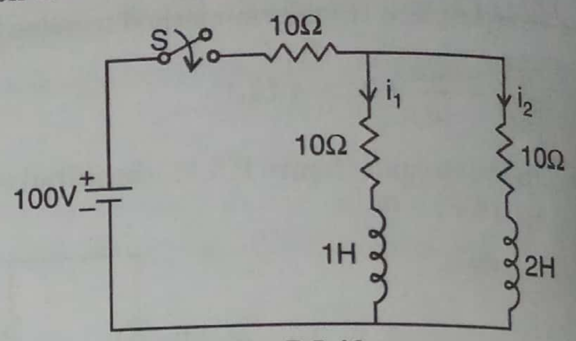

Fig. P.5.19.

5.19. In the circuit shown in figure P.5.19, the switch S is closed at $t = 0$ with the network previously unenergized. Evaluate $i_1(t)$.

5.20. Repeat Example 3.31 by Laplace transform Technique.

ANSWERS

5.1. $i(t) = 10 - 8\,e^{-2000\,t}$ A.

5.2. $i(t) = 15\,e^{-2000\,t}$ A.

5.3. $i(t) = 4 - 6\,e^{-2500\,t}$ A.
$i_2(t) = 6.67\,e^{-166.7\,t}$ A

5.4. $i_1(t) = 10 - 3.33\,e^{-166.7\,t}$ A.

5.5. $i_1(0^+) = 6.67$ A, $i_1(\infty) = 10$,
$i_2(0^+) = 6.67$ A, and $i_2(\infty) = 0$

5.6. $i(t) = -\dfrac{8}{3}\,e^{-90t}$ A

5.7. $V_R(t) = 200\,e^{-25\,t}$ V
$i_R(t) = 2\,e^{-25\,t}$ A

5.8 $i(t) = \dfrac{1}{8}\left(e^{-0.308t} + 31e^{-0.052t}\right)$ A

5.9. $v(t) = 25 \times 10^8 \left(1 - e^{-\frac{t}{15000}}\right)$ V

5.10. $v(t) = 9(e^{-t} - e^{-2\,t})$ V

5.11. $v_c(t) = 20 - 10\,e^{-t} - 5\,t\,e^{-t}$ V.

5.12. $v(t) = 5(1 - e^{-2t})$ V
$i(t) = 5(1 + e^{-2\,t}) - 10\,e^{-3t}$ A

5.13. $i_2(t) = 4(1 - e^{-6\,t})\,U(t)$

5.14. $i(t) = \dfrac{4}{3} + \dfrac{5}{3}e^{-67.5t}$

5.15. $v_c(t) = (1 - e^{-t/RC})\,U(t) - (1 - e^{-(t-T)/RC})\,U(t-T)$

5.16. $i(t) = 10[e^{-t} - e^{-(t-1)}\,U(t-1)]$

5.17. $i(t) = \dfrac{\left(t + e^{-t} - 1\right)}{2}[U(t) - 2U(t-1) + U(t-2)]$

5.18. $i(t) = \dfrac{V}{R_1}\,e^{-\frac{(R_1+R_2)t}{L}}$

5.19. $i_1(t) = 3.33 + 1.121\,e^{-6.35t} - 4.5\,e^{-23.65t}$

5.20. $i(t) = 0.6 + 0.067\,e^{-3.57t}$.