Lecture 4:
Pixels and Filters

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What we will learn today?

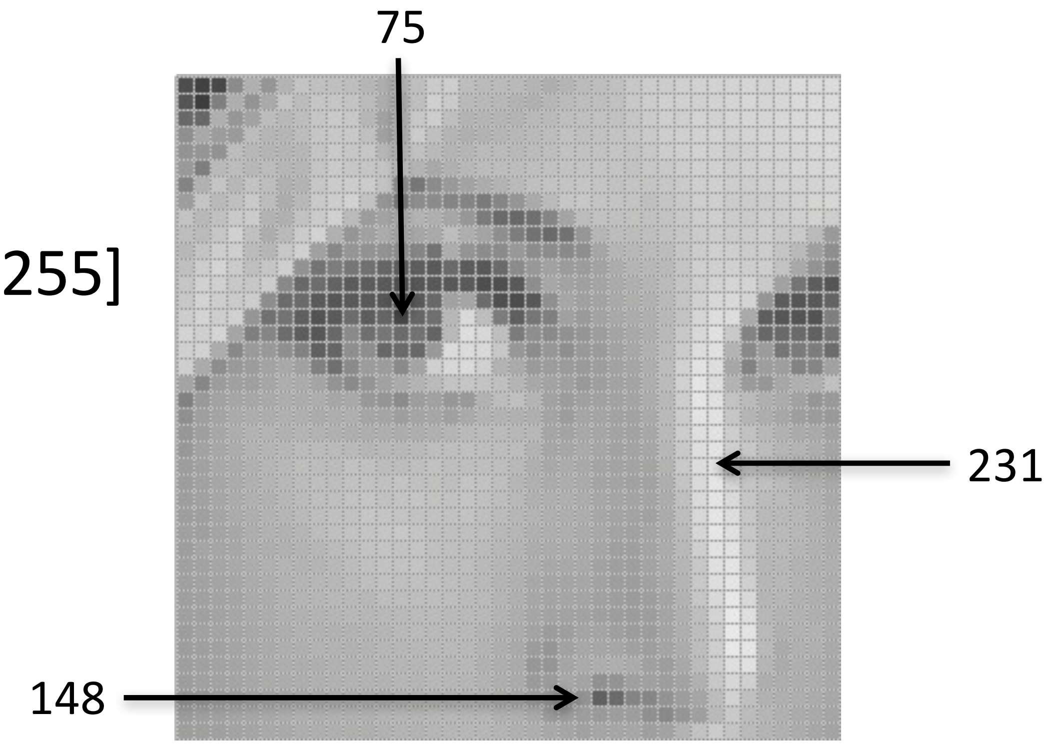
- Images as functions
- Linear systems (filters)
- Convolution and correlation

Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7

Images as functions

- An image contains discrete number of pixels
  - A simple example
  - Pixel value:
    - "grayscale"

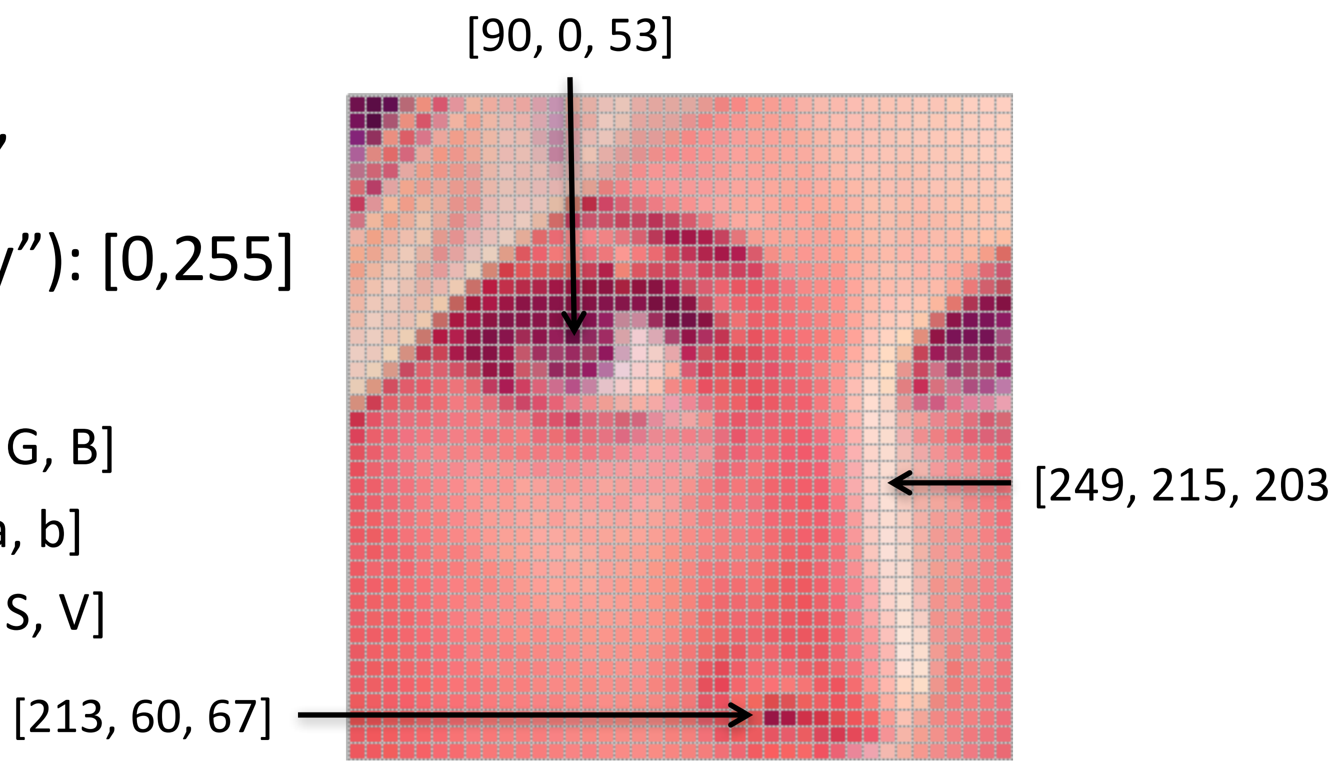
    (or "intensity"): [0,255]



Images as functions

- An image contains discrete number of pixels
  - A simple example
  - Pixel value:
    - “grayscale”

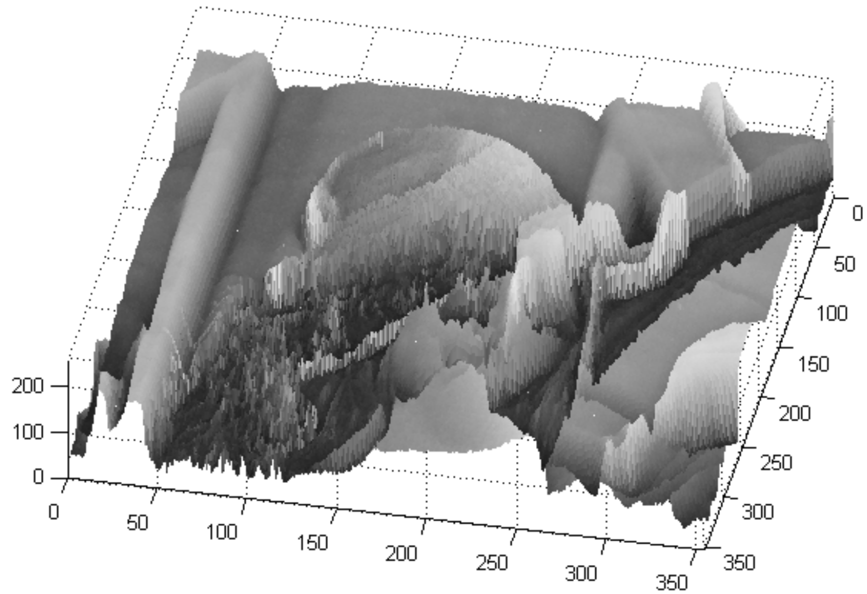
    (or “intensity”): [0,255]
    - “color”
      - RGB: [R, G, B]
      - Lab: [L, a, b]
      - HSV: [H, S, V]



Images as functions

- **An Image** as a function f from R^2 to R^M:
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Defined over a rectangle, with a finite range:

$$f: \underbrace{[a,b] \times [c,d]}_{\text{Domain support}} \rightarrow \underbrace{[0,255]}_{\text{range}}$$



Images as functions

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  - $f(x, y)$ gives the **intensity** at position (x, y)
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$$f\colon \underbrace{[a,b] \times [c,d]}_{\text{Domain support}} \rightarrow \underbrace{[0,255]}_{\text{range}}$$

- A color image: $$f(x,y) = \begin{matrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{matrix}$$

Images as discrete functions

- Images are usually **digital** (**discrete**):
  - **Sample** the 2D space on a regular grid
- Represented as a matrix of integer values

pixel

j →

i ↓

62	79	23	119	120	105	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

Images as discrete functions

Cartesian coordinates

$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,1] & f[0,1] & f[1,1] & \\ \ldots & f[-1,0] & \underline{f[0,0]} & f[1,0] & \ldots \\ & f[-1,-1] & f[0,-1] & f[1,-1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Notation for discrete functions

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- Linear systems (filters)
- Convolution and correlation

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Systems and Filters

- **Filtering:**
  - Form a new image whose pixels are a combination original pixel values

Goals:

-Extract useful information from the images

- Features (edges, corners, blobs…)

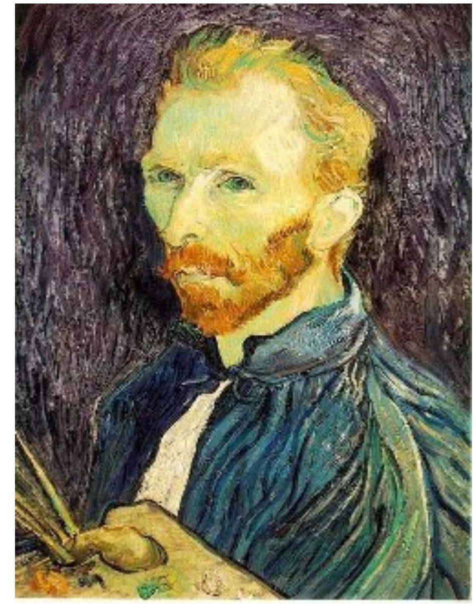
- Modify or enhance image properties:

- super-resolution; in-painting; de-noising

De-noising

Salt and pepper noise

Super-resolution



In-painting

Bertamio et al

2D discrete-space systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g = \mathcal{S}[f], \quad g[n,m] = \mathcal{S}\{f[n,m]\}$$

$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$$

Filter example #1: Moving Average

- 2D DS moving average over a 3 × 3 window of neighborhood

$$h = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$g[n,m] = \frac{1}{9}\sum_{k=n-1}^{n+1}\sum_{l=m-1}^{m+1} f[k,l]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]$$

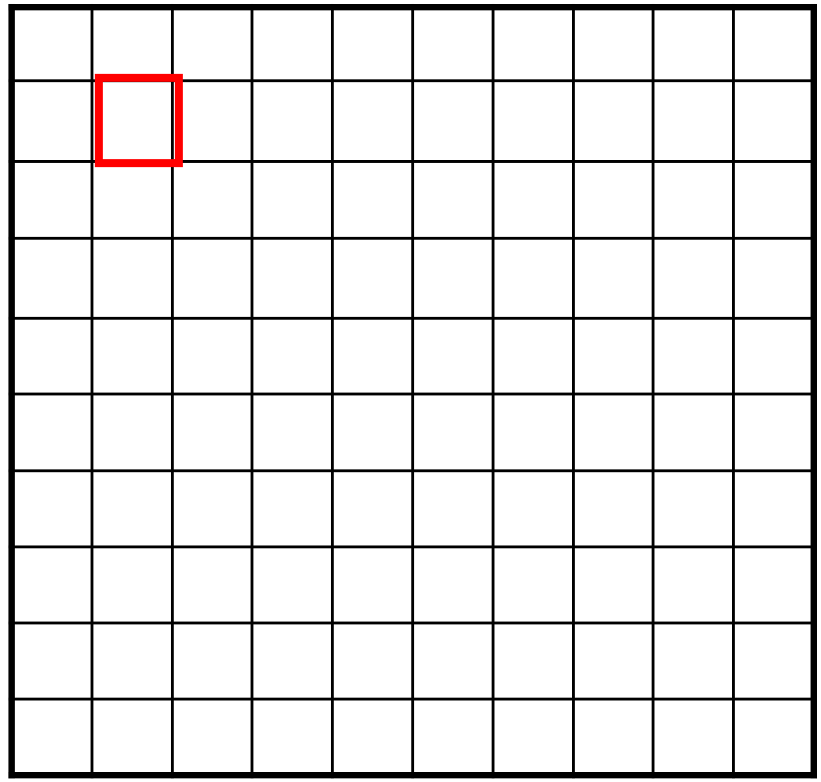
$$(f \quad h)[m,n] = \frac{1}{9} \ {}_{k,l} \ f[k,l]\, h[m \quad k, n \quad l]$$

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$



$$(f * h)[m,n] = \sum_{k,l} f[k,l]\, h[m-k, n-l]$$

Courtesy of S. Seitz

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10							

$$(f * h)[m,n] = \sum_{k,l} f[k,l]\, h[m-k, n-l]$$

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20						

$$(f * h)[m,n] = \sum_{k,l} f[k,l]\, h[m-k, n-l]$$

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30					

$$(f * h)[m,n] = \sum_{k,l} f[k,l]\, h[m-k, n-l]$$

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30				

$$(f * h)[m,n] = \sum_{k,l} f[k,l]\, h[m-k, n-l]$$

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$(f * h)[m,n] = \sum_{k,l} f[k,l]\, h[m-k, n-l]$$

Source: S. Seitz

Filter example #1: Moving Average

$$h[\cdot\,,\cdot\,]$$

$$\frac{1}{9}\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$$

In summary:

- Replaces each pixel with an average of its neighborhood.
- Achieve smoothing effect (remove sharp features)

Filter example #1: Moving Average

Filter example #2: Image Segmentation

- Image segmentation based on a simple threshold:

$$g[n,m] = \begin{cases} 255, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$


Classification of systems

- Amplitude properties
  - Linearity
  - Stability
  - Invertibility
- Spatial properties
  - Causality
  - Separability
  - Memory
  - Shift invariance
  - Rotation invariance

Shift-invariance

If $f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$ then

$$\boxed{f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]}$$

for every input image f[n,m] and shifts n_0,m_0

Is the moving average system is shift invariant?

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Is the moving average system is shift invariant?

$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]$$

$$f[n-n_0, m-m_0]$$

$$\xrightarrow{\mathcal{S}} \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[(n-n_0)-k,(m-m_0)-l]$$

$$= g[n-n_0, m-m_0]$$

Yes!

Linear Systems (filters)

$$f(x,y) \rightarrow \boxed{\mathcal{S}} \rightarrow g(x,y)$$

- Linear filtering:
  - Form a new image whose pixels are a weighted sum of original pixel values
  - Use the same set of weights at each point

- **S** is a linear system (function) iff it *S satisfies*

$$\mathcal{S}[\alpha f_1 + \beta f_2] = \alpha \mathcal{S}[f_1] + \beta \mathcal{S}[f_2]$$

superposition property

Linear Systems (filters)

$$f(x,y) \rightarrow \boxed{\mathcal{S}} \rightarrow g(x,y)$$

- Is the moving average a linear system?

- Is thresholding a linear system?
  - f1[n,m] + f2[n,m] > T
  - f1[n,m] < T
  - f2[n,m]<T

No!

LSI (linear *shift invariant*) systems

Impulse response

$$\delta_2[n,m] \rightarrow \boxed{\mathcal{S}} \rightarrow h[n,m]$$

$$\delta_2[n-k,m-l] \rightarrow \boxed{\mathcal{S}\ \text{(SI)}} \rightarrow h[n-k,m-l]$$

LSI (linear *shift invariant*) systems

Example: impulse response of the 3 by 3 moving average filter:

$$h[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$

$$= \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

h

$$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

LSI (linear *shift invariant*) systems

An LSI system is completely specified by its impulse response.

sifting property of the delta function

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\,\delta_2[n-k, m-l]$$

superposition

$$\rightarrow \boxed{\mathcal{S}\ \mathrm{LSI}} \rightarrow \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\,h[n-k, m-l]$$

$$\delta_2[n,m] \rightarrow \boxed{\mathcal{S}} \rightarrow h[n,m]$$

Discrete convolution

$$f[n,m] * h[n,m]$$

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Some background reading:
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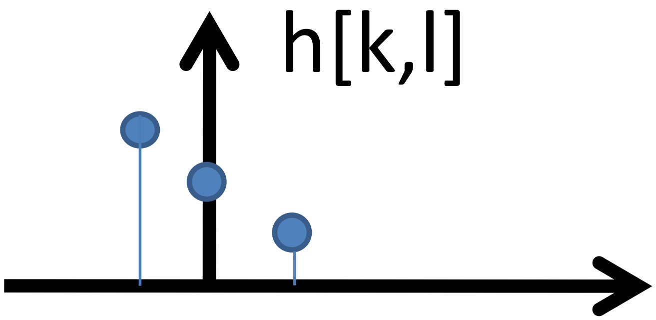
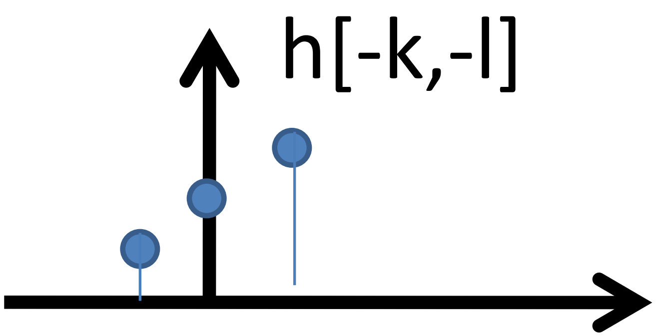
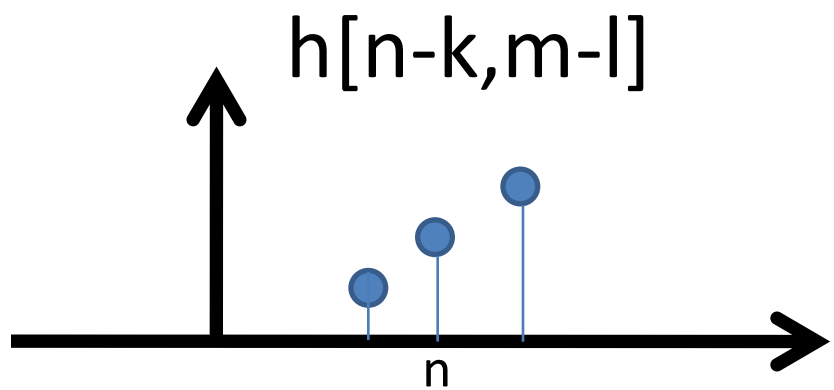
Discrete convolution (symbol: $*$)

- Fold h[k,l] about origin to form h[−k,−l]
- Shift the folded results by n,m to form h[n − k,m − l]
- Multiply h[n − k,m − l] by f[k, l]
- Sum over all k,l
- Repeat for every n,m







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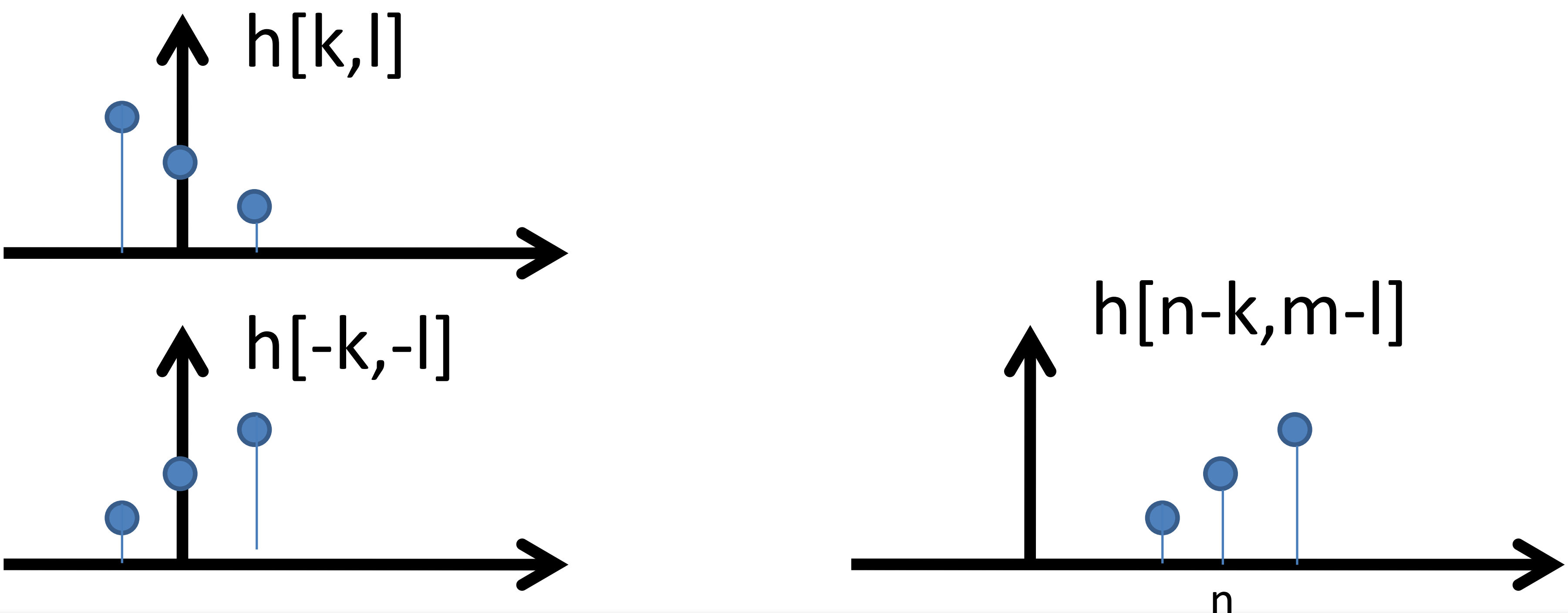

Discrete convolution (symbol: $*$)

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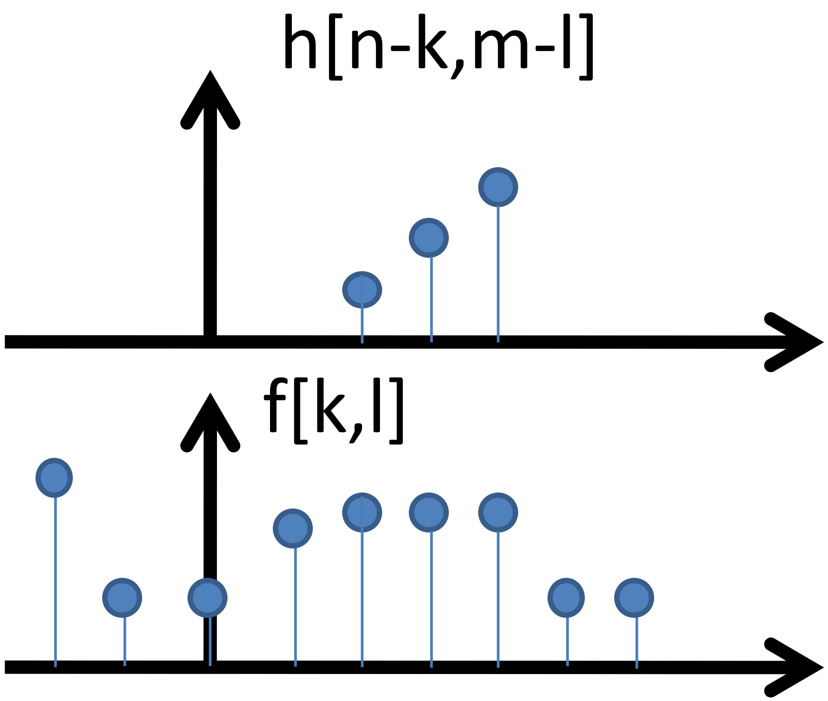
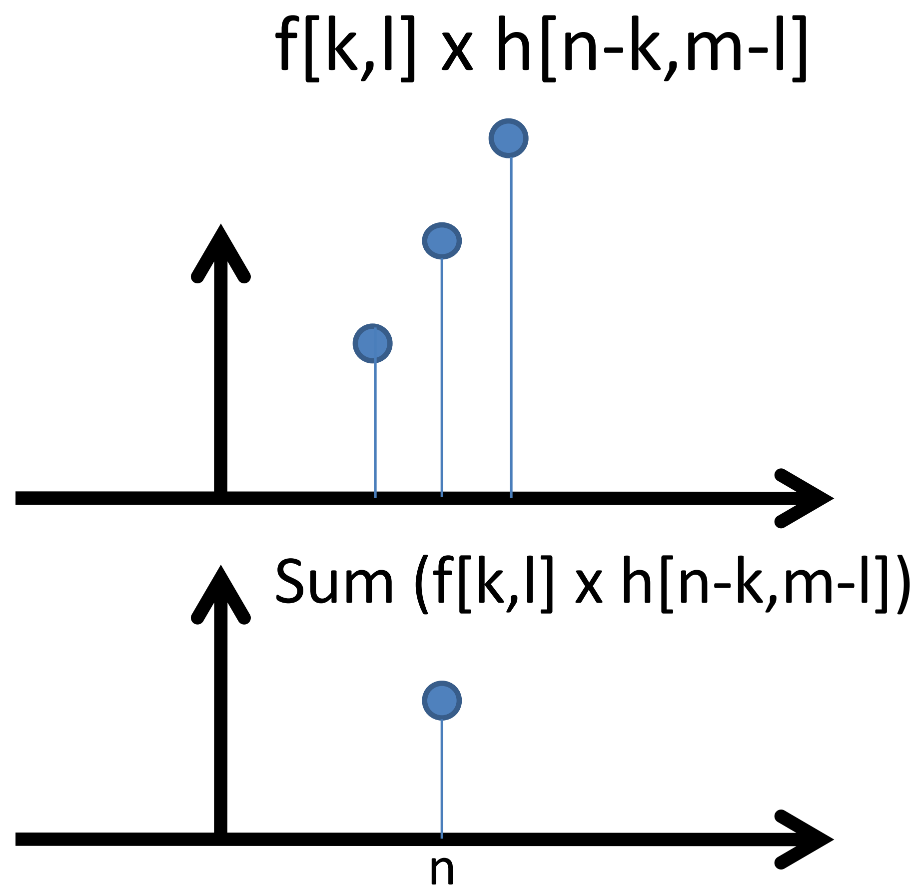




Convolution in 2D - examples

Original

*

•0	•0	•0
•0	•1	•0
•0	•0	•0

=

?

Courtesy of D Lowe

Convolution in 2D - examples

Original

$*$

•0	•0	•0
•0	•1	•0
•0	•0	•0

=

Filtered
(no change)

Courtesy of D Lowe

Convolution in 2D - examples

Original

$*$

•0	•0	•0
•0	•0	•1
•0	•0	•0

$=$?

Courtesy of D Lowe

Convolution in 2D - examples

Original

*

•0	•0	•0
•0	•0	•1
•0	•0	•0

=

Shifted right
By 1 pixel

Courtesy of D Lowe

Convolution in 2D - examples

Original

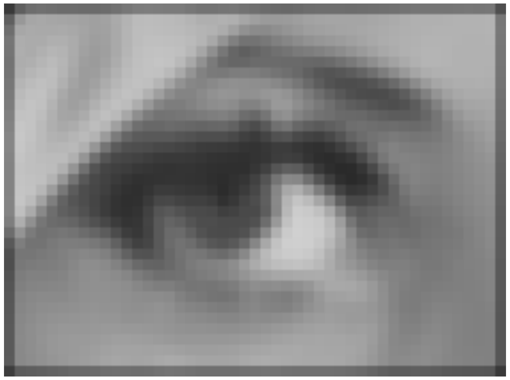
$$* \ \frac{1}{9} \begin{bmatrix} \bullet 1 & \bullet 1 & \bullet 1 \\ \bullet 1 & \bullet 1 & \bullet 1 \\ \bullet 1 & \bullet 1 & \bullet 1 \end{bmatrix} = \ ?$$

Courtesy of D Lowe

Convolution in 2D - examples

Original

$$* \ \frac{1}{9} \begin{bmatrix} \bullet 1 & \bullet 1 & \bullet 1 \\ \bullet 1 & \bullet 1 & \bullet 1 \\ \bullet 1 & \bullet 1 & \bullet 1 \end{bmatrix} =$$



Blur (with a box filter)

Courtesy of D Lowe

Convolution in 2D - examples

Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = ?$$

(Note that filter sums to 1)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}^{\text{“details of the image”}}$$

Courtesy of D Lowe

- What does blurring take away?

–

=

- Let’s add it back:

+ a

=

Convolution in 2D – Sharpening filter

Original

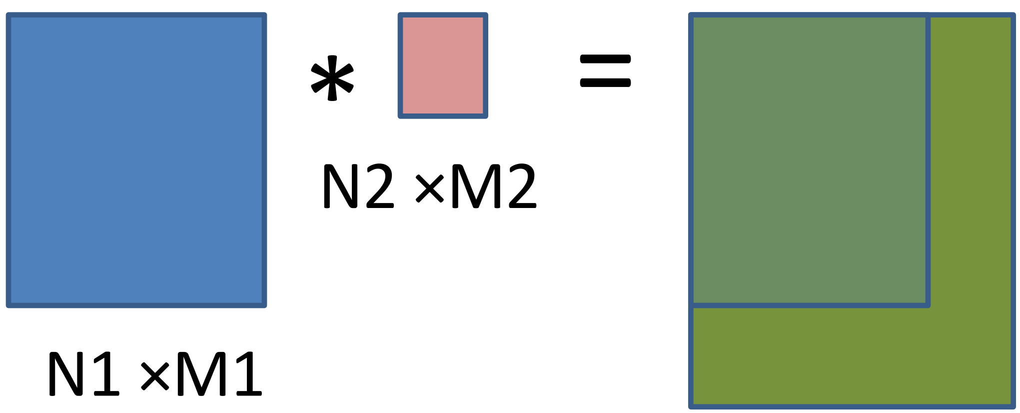
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

Sharpening filter: Accentuates differences with local average

Courtesy of D Lowe

Image support and edge effect

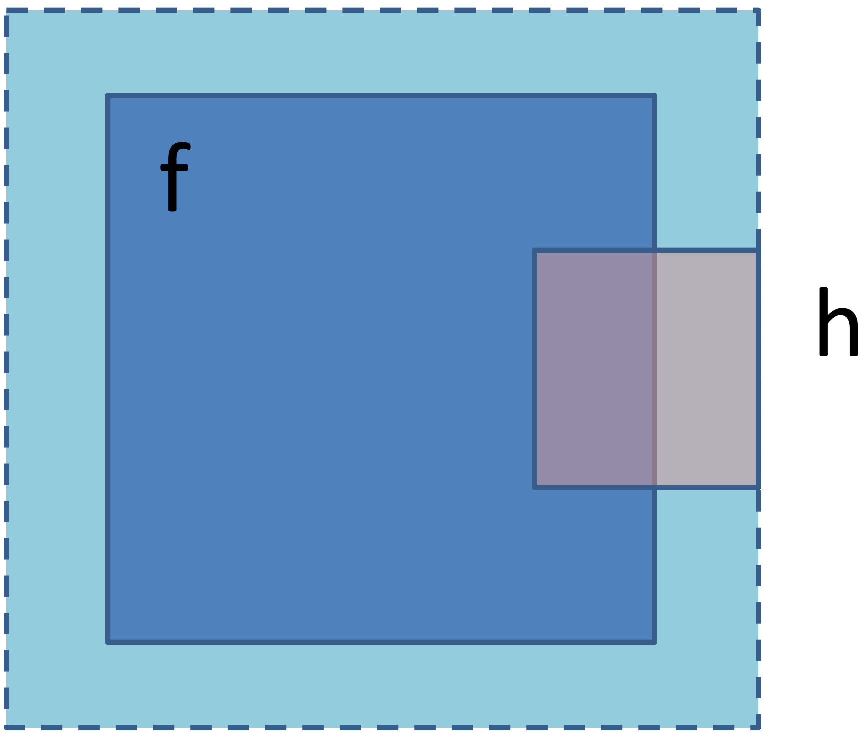
- A computer will only convolve **finite support signals.**
    - That is: images that are zero for n,m outside some rectangular region
- MATLAB's conv2 performs 2D DS convolution of finite-support signals.



$(N1 + N2 - 1) \times (M1 + M2 - 1)$

Image support and edge effect

- A computer will only convolve **finite support signals.**
- What happens at the edge?



- zero "padding"
- edge value replication
- mirror extension
- more (beyond the scope of this class)

-> Matlab conv2 uses zero-padding

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(Cross) correlation (symbol: $**$)

Cross correlation of two 2D signals f[n,m] and g[n,m]

$$r_{fg}[k,l] \triangleq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n,m]\, g^*[n-k, m-l]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n+k, m+l]\, g^*[n,m], \quad k,l \in \mathbb{Z},$$

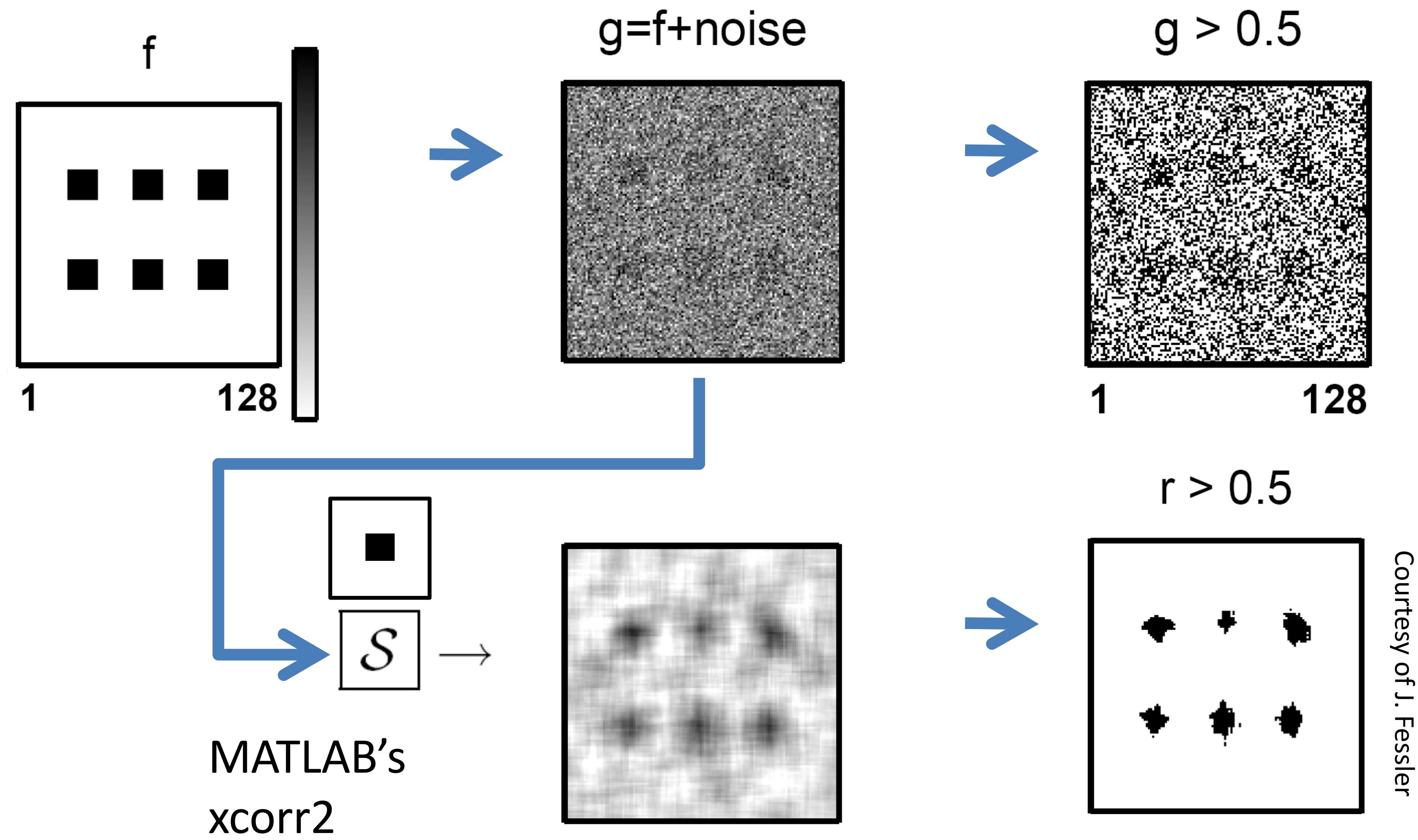
(k, l) is called the **lag**

- Equivalent to a convolution without the flip

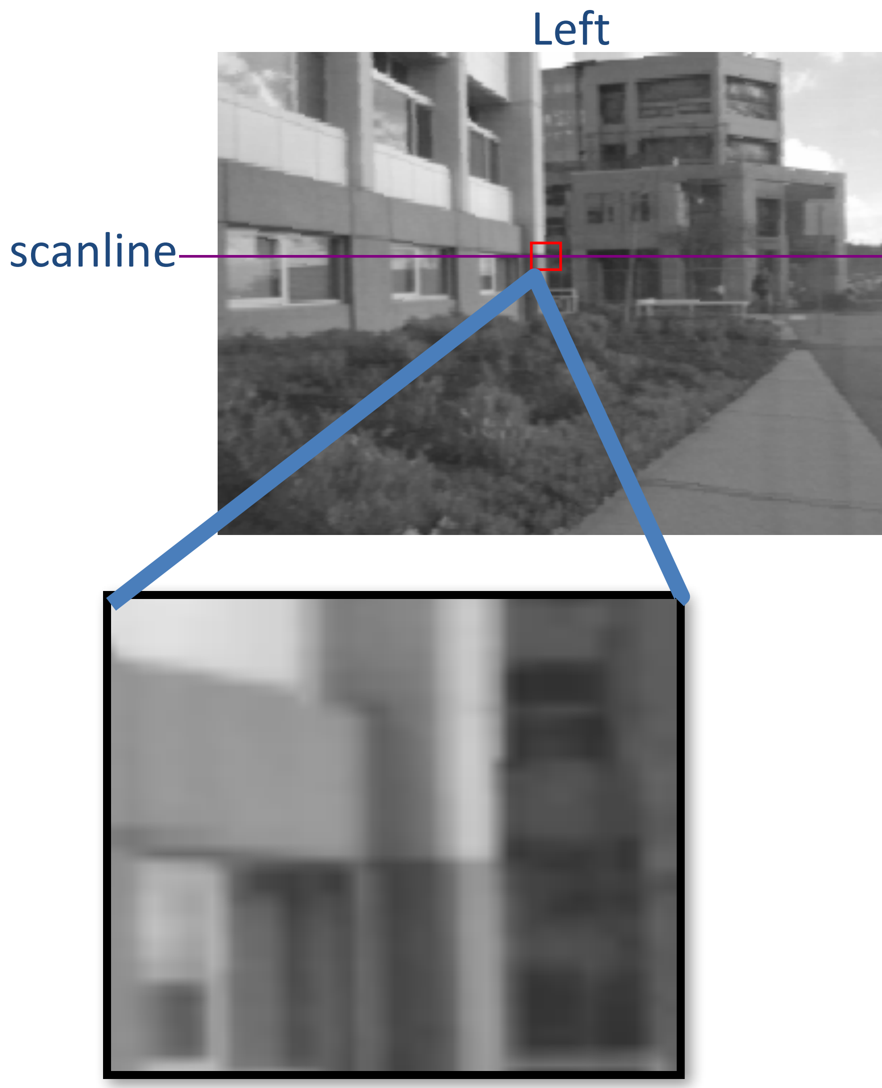
$$r_{fg}[n,m] = f[n,m] * g^*[-n,-m]$$

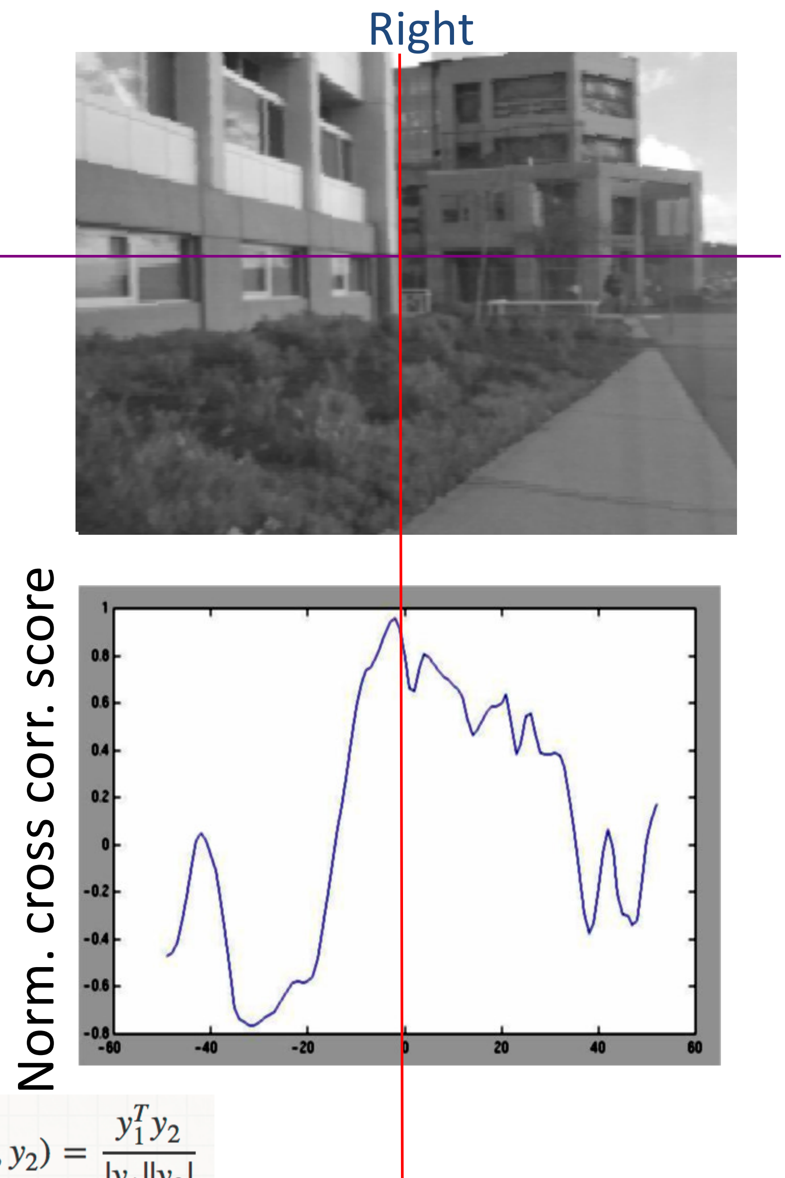
(g* is defined as the *complex conjugate* of g. In this class, g(n,m) are real numbers, hence g*=g.)

(Cross) correlation – example



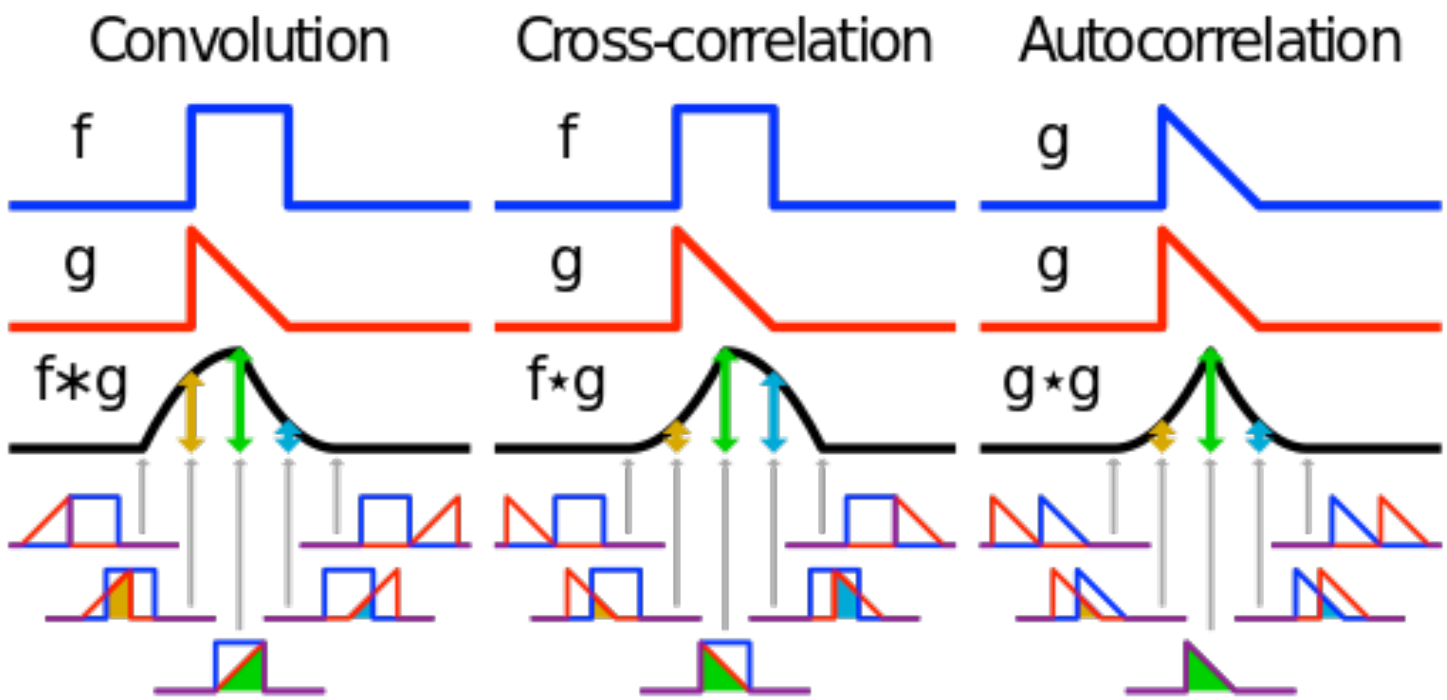
(Cross) correlation – example





$$dc(y_1, y_2) = \frac{y_1^T y_2}{|y_1||y_2|}$$

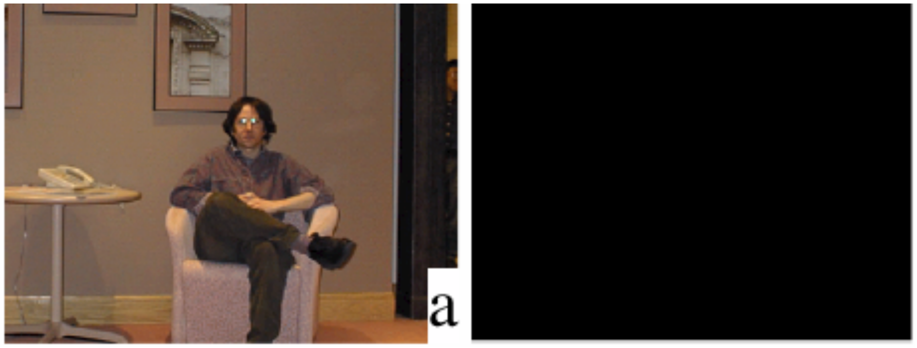
Convolution vs. (Cross) Correlation



Convolution vs. (Cross) Correlation

- A **<u>convolution</u>** is an integral that expresses the amount of overlap of one function as it is shifted over another function.
  - convolution is a filtering operation

- **<u>Correlation</u>** compares the ***similarity** of **two** sets of **data.*** Correlation computes a measure of similarity of two input signals as they are shifted by one another. The correlation result reaches a maximum at the time when the two signals match best .
  - correlation is a measure of relatedness of two signals



Cross Correlation Application: Vision system for TV remote control

- uses template matching

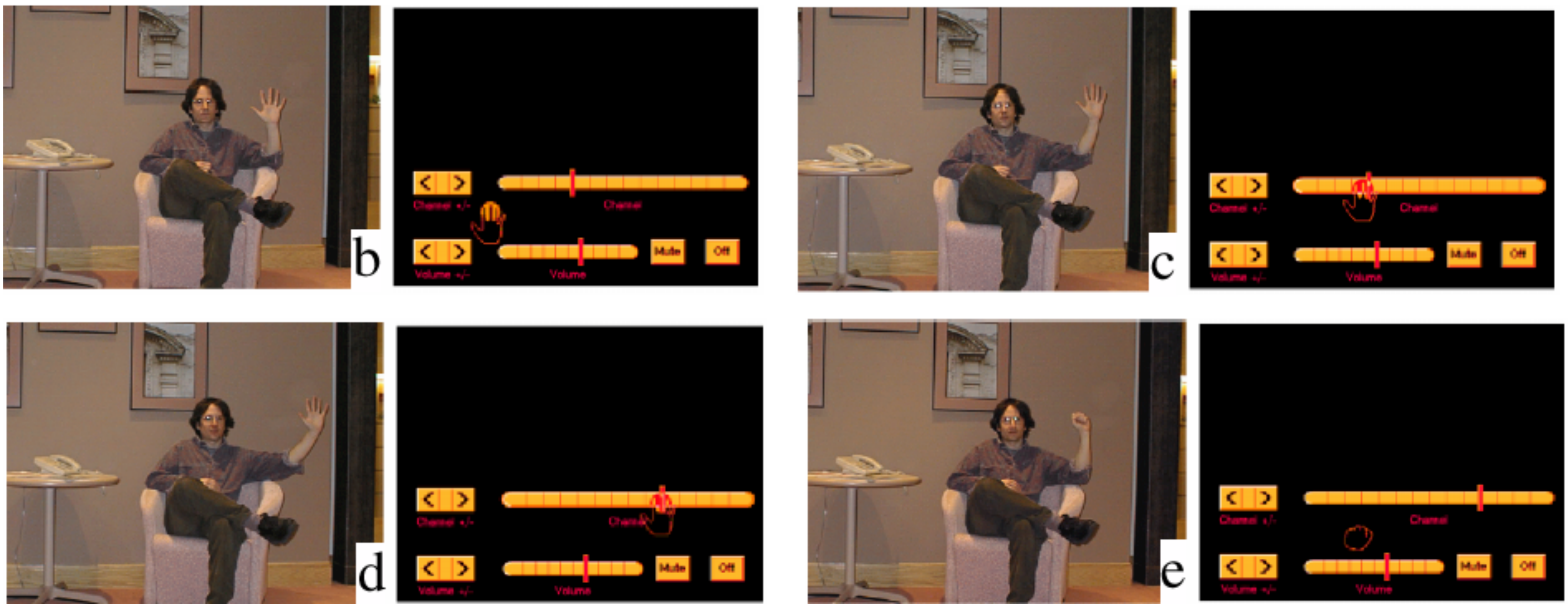


Figure from “Computer Vision for Interactive Computer Graphics,” W.Freeman et al, IEEE Computer Graphics and Applications, 1998 copyright 1998, IEEE

properties

- **Commutative property:**

$$f ** h = h ** f$$

- **Associative property:**

$$(f ** h_1) ** h_2 = f ** (h_1 ** h_2)$$

- **Distributive property:**

$$f ** (h_1 + h_2) = (f ** h_1) + (f ** h_2)$$

The order doesn't matter! $h_1 ** h_2 = h_2 ** h_1$

properties

- **Shift property:**

$$f[n,m] ** \delta_2[n-n_0, m-m_0] = f[n-n_0, m-m_0]$$

- **Shift-invariance:**

$$\begin{aligned} g[n,m] &= f[n,m] ** h[n,m] \\ &\Longrightarrow f[n-l_1, m-l_1] ** h[n-l_2, m-l_2] \\ &\qquad = g[n-l_1-l_2, m-l_1-l_2] \end{aligned}$$

What we have learned today?

- Images as functions
- Linear systems (filters)
- Convolution and correlation