Beta and Gamma functions and their properties

Beta Function: If $m>0$ and $n>0$, then beta function of m and n is denoted by $\beta(m,n)$ and it is defined as

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Properties of Beta Function:

(i) $\beta(m,n)=\beta(n,m)$

(ii) $\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

(iii) $\beta(m,n)=2\int_0^{\pi/2} \sin^{m-1}\theta\cos^{2n-1}\theta d\theta$

Gamma Function

Gamma Function: If $n>0$, then gamma function of n is denoted by Γn and it is defined as

$$\Gamma n=\int e^{-x}x^{n-1}dx$$

Such that $\Gamma 0=\infty$ and $\Gamma(-n)=\infty$

Properties of Gamma function

1. $\Gamma 1=1$
2. $\Gamma n+1=n\Gamma n$ and if n is positive integer then $\Gamma n+1=\lfloor n$
3. $\frac{\Gamma n}{2^n}=\int_0^\infty e^{-zx}x^{n-1}dx$
4. $\Gamma n=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy=\int_0^1(\log y)^{n-1}dy$

5. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

6. $\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{a}$

7. $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

8. $\Gamma n\Gamma(n-1)=\frac{\pi}{\sin n\pi}$, when 0<n<1

Practice Sheet

1. Express $\int_0^1 x^m\left(1-x^n\right)^{\rho}\,dx$ in terms of beta function and hence prove that
   $\int_0^1 x^5\left(1-x^3\right)^{10}\,dx=\frac{1}{396}$
2. Prove that the following
   (i) $\beta(m,n)=\beta(m+1,n)+\beta(m,n+1)$
   (ii) $\frac{\beta(m,n+1)}{\beta(m,n)}=\frac{n}{m+n}$
   (iii) $\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n+1)}{n}=\frac{\beta(m,n)}{m+n}$
3. Prove that $\int_0^{\infty}\frac{x^c}{c^x}\,dx=\frac{\Gamma c+1}{(\log c)^{c+1}}, c>1$
4. Prove that $\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1}dx=\frac{\Gamma m}{n^m}$
5. Prove that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx=(b-a)^{m+n-1}\beta(m,n)$

6. Prove that $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \lfloor n}{(m+1)^{n+1}}$

7. Prove that

(i) $\int_0^\infty \sqrt{x} e^{-3\sqrt{x}} dx = \frac{4}{27}$ (ii) $\int_0^\infty x^{5/2} e^{-4x} dx = \frac{15\sqrt{\pi}}{1024}$

(iii) $\int_0^2 x\left(8 - x^3\right)^{1/3} dx = \frac{16\pi}{9\sqrt{3}}$

8. Prove that $\int_0^\infty x^n e^{-k^2 x^2} dx = \frac{1}{2k^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$

9. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}} dx = \frac{\beta(m,n)}{(b+c)^m b^n}$

10. Prove that $\int_0^\infty 4\sqrt{x} e^{\sqrt{x}} dx = \frac{3}{2}\sqrt{\pi}$

1. Evaluate: $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$

Solution: Suppose $I = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$ … (1)

Putting, $x = \log \frac{1}{y} = -\log y$

$\Rightarrow$ $y = e^{-x} \quad dy = -e^{-x} dx$

Upper Limit: $x = \log\left(\frac{1}{1}\right) = 0$ Lower Limit: $x = \log\left(\frac{1}{0}\right) = \infty$

From equation (1), we get

$$I = \int_\infty^0 (x)^{n-1}\left(-e^{-x} dx\right) = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$\Rightarrow$ $I = \Gamma n$

Thus, $\boxed{\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy = \Gamma n}$ **Answer**

2. Prove that: $\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$ **, $m>0$ and $n>0$**

Solution: We know that if z is independent of x, then

$$\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1}\,dx \qquad \ldots (1)$$

where x is independent of z and vice versa.

Equation (1), both sides multiplying by $z^{m-1}\,e^{-z}$ and integrating w. r. t. z from 0 to ∞, then we get

$$\int_0^\infty \Gamma n\left(z^{m-1}\,e^{-z}\right)dz = \int_0^\infty \left[\left(z^{m-1}\,e^{-z}\right) z^n \int_0^\infty e^{-zx} x^{n-1}\,dx\right]dz$$

$$\Rightarrow \quad \Gamma n \int_0^\infty z^{m-1}\,e^{-z}\,dz = \int_0^\infty x^{n-1}\left[\int_0^\infty e^{-z-zx} z^{m+n-1}\,dz\right]dx$$

$$\left[\because\ \Gamma n = \int_0^\infty e^{-x}\,x^{n-1}\,dx\right]$$

$$\Rightarrow \quad \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left[\int_0^\infty e^{-z(1+x)} z^{m+n-1}\,dz\right]dx$$

Putting, $y = z(1+x) \quad \Rightarrow \quad z = \frac{y}{1+x} \quad i.e.\ dz = \frac{dy}{1+x}$

$$\therefore \quad \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left[\int_0^\infty e^{-y}\left(\frac{y}{1+x}\right)^{m+n-1}\frac{dy}{1+x}\right]dx$$

$$\Rightarrow \quad \Gamma n\,\Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left[\int_0^\infty e^{-y} y^{m+n-1}\,dy\right]dx$$

$$\Rightarrow \quad \Gamma n\,\Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left[\Gamma(m+n)\right]dx$$

$$\left[\because\ \Gamma n = \int_0^\infty e^{-x}\,x^{n-1}\,dx\right]$$

$$\Rightarrow \quad = \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \Gamma(m+n)\,\beta(m,n)$$

Hence, $\boxed{\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$

Proved

Practice Sheet

1. Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

2. Prove that $\int_0^\infty \sqrt{x}e^{-3\sqrt{x}}dx=\frac{4}{27}$

3. Prove that $\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n+1)}{n}=\frac{\beta(m,n)}{m+n}$

4. Prove that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx=(b-a)^{m+n-1}\beta(m,n)$

5. Prove that relation between Beta and Gamma function

   (i) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ *(ii)* $B(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$

6. Prove that the duplication formula

   $$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

7. Prove that (i) $B(m,n)=B(m+1,n)+B(m,n+1)$

   (ii) $\frac{1}{m+n}B(m,n)=\frac{1}{m}B(m+1,n)+\frac{1}{n}B(m,n+1)$

8. Prove that $\int_0^\infty \frac{x^c}{c^x}dx=\frac{\Gamma(c+1)}{(\log c)^{c+1}}$

9. Prove that (i) $\int_0^1 x^m(\log x)^n dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$ (ii) $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1}dx=\frac{\Gamma(p)}{q^p}$

10. Evaluate (i)) $\int_0^\infty e^{-4x}x^{5/2}dx$ (ii) $\int_0^\infty \sqrt{x}\,e^{-3\sqrt{x}}dx$

11. Express the integral $\int_0^1 x^m(1-x^n)^p dx$ in term of gamma function and hence evaluate

   (i) $\int_0^1 x^3(1-x^2)^4 dx$ (ii) $\int_0^1 x^5(1-x^3)^{10}dx$ RGPV June-2012, Dec. 2012