

Homework #1

① $y^n = g_1(t, y, \dot{y}, \dots, y^{n-1}, u) + g_2(t, y, \dot{y}, \dots, y^{n-2}) \dot{u}$
 hint: $x_n = y^{n-1} - g_2(t, y, \dot{y}, \dots, y^{n-2}) u$

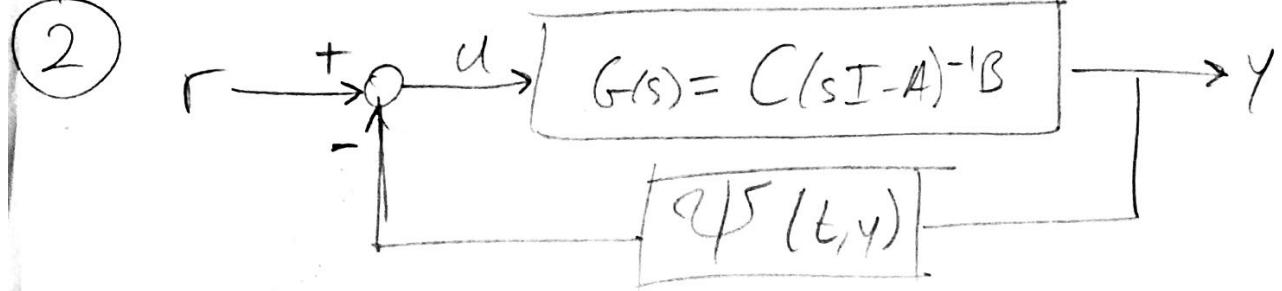
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{n-2} \\ y^{n-1} - g_2(t, y, \dot{y}, \dots, y^{n-2}) u \end{bmatrix} \quad \boxed{y = x_1}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \ddot{\dot{y}} \\ \vdots \\ \ddot{y}^{n-1} \\ y^n - g_2(t, x_1, \dots, x_{n-1}) \dot{u} - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1} \dot{x}_1 + \dots + \frac{\partial g_2}{\partial x_{n-1}} \dot{x}_{n-1} \right) u \end{bmatrix}$$

$$\dot{x}_n = \frac{d}{dt} (y^{n-1} - g_2(t, y, \dot{y}, \dots, y^{n-2}) u) = \frac{d}{dt} (y^{n-1}) - \frac{d}{dt} (g_2(t, y, \dot{y}, \dots, y^{n-2}) u)$$

$$\dot{x}_n = y^n - \frac{d}{dt} (g_2(u)) u - g_2(u) \frac{d}{dt} u = y^n - g_2(u) \dot{u} - \frac{d}{dt} (g_2(u)) u$$

$$\frac{d}{dt} (g_2(u)) = \frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1} \dot{x}_1 + \dots + \frac{\partial g_2}{\partial x_{n-1}} \dot{x}_{n-1}$$



$$y = Gu$$

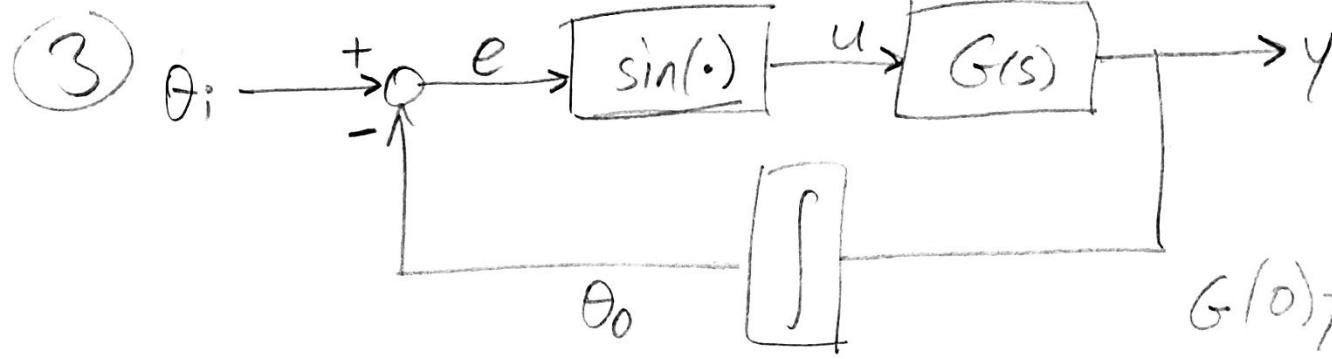
$$u = r - \psi^*(t, y) = r - \psi^*(t, Gu)$$

$$\text{let } \dot{x} = Ax + Bu$$

$$\dot{x} = Ax + B(r - \psi^*(t, y))$$

$$Cx := y$$

$$\boxed{\begin{array}{l} \dot{x} = Ax + Br - B\psi^*(t, Cx) \\ y = Cx \end{array}}$$



$$G(0) \neq 0$$

$$\theta_i = \text{const.}$$

$$\dot{z} = Az + Bs \sin e$$

$$\ddot{e} = -Cz$$

a) $y = gu = \sin(e) = g \sin(\theta_i - \theta_o)$

$$\theta_o = \int y dt$$

$$e = \theta_i - \theta_o$$

$$\dot{e} = \ddot{\theta}_i - \ddot{\theta}_o = 0 - \ddot{\theta}_o \Rightarrow \dot{e} = -\ddot{\theta}_o$$

$$Cz := \ddot{\theta}_o$$

$$\dot{z} := Az + Bu$$

$$u := \sin e$$

$$\boxed{\dot{z} = Az + Bs \sin e}$$

$$G(s) = C(sI - A)^{-1}B$$

$$G(0) = CA^{-1}B$$

b) $\dot{z} = Az + Bs \sin e$ $CA^{-1}B = G(0) \neq 0$

$$\dot{z}_{eq} = 0 = Az_{eq} + Bs \sin e_{eq} \rightarrow \sin e_{eq} = 0$$

$$z_{eq} = -A^{-1}Bs \sin e_{eq}$$

$$\dot{e}_{eq} = 0 = -Cz_{eq}$$

$$+ CA^{-1}B \sin e_{eq} = 0$$

$$\boxed{e_{eq} = \pm n\pi, n \in \mathbb{Z}}$$

$$z_{eq} = -A^{-1}Bs \sin e_{eq}$$

$$\boxed{z_{eq} = 0}$$

⑥ pendulum state space model:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml^2} T$$

$$G(s) = \frac{1}{Ts+1} = C(sI-A)^{-1}B \xrightarrow{\text{Scalar}} \frac{CB}{s-A} = \frac{C}{\frac{1}{B}s - \frac{A}{B}}$$

$$\frac{1}{B} = \tau \Rightarrow B = \frac{1}{\tau}, \underline{C=1}, -\frac{A}{B} = 1 \Rightarrow A = -\frac{1}{\tau}$$

$$\dot{z} = Az + B \sin e$$

$$\dot{z} = -\frac{1}{\tau} z + \frac{1}{\tau} \sin e$$

$$\dot{e} = -z$$

$$\begin{array}{c} \uparrow \\ \dot{x}_1 = x_2 \end{array}$$

$$\begin{array}{c} -e : x_1 \\ z : x_2 \end{array}$$

$$\dot{z} = -\frac{1}{\tau} \sin(-e) - \frac{1}{\tau} z$$

$$\begin{array}{c} \uparrow \\ \dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml^2} T \end{array}$$

$$g/l = 1/\tau, k/m = 1/\tau, T = 0$$

$$\textcircled{4} \quad F = ma = m\ddot{y} = mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

$$m\ddot{y} + c_1\dot{y} + c_2\dot{y}|\dot{y}| + ky = mg$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = [mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|]/m$$

$$\dot{x}_2 = -\frac{k}{m}y - \frac{c_1}{m}\dot{y}|\dot{y}| + g$$

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2|x_2| + g$$

$$⑤ \dot{x}(t) = [x(t)]^{1/3}, \quad x(0) = 0$$

$$\frac{dx}{dt} = x^{1/3}$$

$$\int x^{-1/3} dx = \int dt$$

$$\frac{2}{3} x^{2/3} = t + C^1$$

$$x(t) = \left[\frac{2}{3} (t + C^1) \right]^{3/2}$$

↑

C^1 is arbitrary

$\hookrightarrow x(t) = \frac{2}{3} (t + C^1)^{3/2}$ is a family of solutions

\hookrightarrow no unique solution to $\dot{x}(t) = [x(t)]^{1/3}, x(0) = 0$

(also, $x(t) = 0$ is another solution)

$$\boxed{\begin{array}{l} x=0 \\ \dot{x}=0 \end{array}}$$

Problem 7

$$(i) A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$Av = \lambda v$
 $(A - \lambda I)v = 0$

@ $\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} v = 0$

$$\det \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = (-\lambda)(-3-\lambda) - (1)(-2) = 0$$

$$3\lambda + \lambda^2 + 2 = 0 \Rightarrow \lambda = \{-2, -1\}$$

$$\lambda = -2, (\lambda+1)(\lambda+2) = 0$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{aligned} \cancel{v_1} + 1v_2 &= -2v_1 \rightarrow \\ -2v_1 - 3v_2 &= -2v_2 \\ -2v_1 + 6v_1 &= -4v_1 \end{aligned}$$

$$\lambda = -2 \rightarrow v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda = -1, \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rightarrow \cancel{v_1} + 1v_2 = -v_1$$

$$-2v_1 - 3v_2 = -v_2$$

$$\lambda_2 = -1 \rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\times \begin{array}{l} \lambda_1, \lambda_2 \in \mathbb{R} \\ \rightarrow M = [v_1 \ v_2] \Rightarrow M = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \end{array}$$

swap v_1 & v_2 and multiply by -1

$$\Rightarrow M = \boxed{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}$$

⑥ $\lambda_1, \lambda_2 \in \mathbb{R}$
 $\lambda_1, \lambda_2 > 0$

$\rightarrow (0,0)$ is an unstable node

$$\textcircled{C} \quad M = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad J_r = M^{-1}AM = \frac{1}{(1)(2) - (-1)(-1)} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$J_r = +1 \begin{bmatrix} -2 & -1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} -$$

$$\dot{z} = M^{-1}AM z = J_r z$$

$$\boxed{\begin{aligned}\dot{z}_1 &= -z_1 \\ \dot{z}_2 &= -2z_2\end{aligned}}$$

$$(ii) \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \rightarrow \lambda = \{1\} \rightarrow v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\{ \operatorname{Re}\{\lambda\} \} > 0 \rightarrow (0,0) \text{ is an unstable node}$

$$M = [v_1, v_2], \quad Av_2 = \lambda v_2 + v_1$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 1 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} + \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-1 + v_{21} + v_{22} = 0$$

$$1 - v_{21} - v_{22} = 0$$

$$-2 - 2v_{21} - 2v_{22} = 0$$

$$v_{21} + v_{22} = 1$$

choose $v_{21} = 0 \rightarrow v_{22} = 1$

$$J_r = M^{-1}AM = \frac{1}{(-1)v_{11} - (0)v_{11}} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow M = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$J_r = -1 \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = - \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \rightarrow \dot{z} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z \Rightarrow \boxed{\begin{aligned}\dot{z}_1 &= z_1 + z_2 \\ \dot{z}_2 &= z_2\end{aligned}}$$

$$(iii) A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \rightarrow \lambda = \{-1, 1\}$$

$$v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) $M = [v_1 \ v_2] = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, M^{-1}AM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\lambda \in \mathbb{R}$
 $\lambda_1 < 0, \lambda_2 > 0 \rightarrow (0,0)$ is a saddle point

(c) $\dot{z} = jz \Rightarrow \begin{cases} \dot{z}_1 = -z_1 \\ \dot{z}_2 = z_2 \end{cases}$

(iv) $A = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix} \rightarrow \lambda = \{-2i, +2i\}$

$\lambda_1 = \alpha + j\beta$
 $\lambda_2 = \alpha - j\beta$
 \downarrow
 $\alpha = 0, \beta = 2$

$v_1 = \begin{bmatrix} -1-2i \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$

$$M = [u \ v] \quad A(u+jv) = (\alpha + \beta j)(u+jv) \quad u_1 + 5u_2 = -2v_1$$

$$\begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 + jv_1 \\ u_2 + jv_2 \end{bmatrix} = (0+2j) \begin{bmatrix} u_1 + jv_1 \\ u_2 + jv_2 \end{bmatrix} \quad \begin{cases} v_1 + 5v_2 = -2u_1 \\ -u_1 - u_2 = -2v_2 \end{cases}$$

$$\begin{bmatrix} u_1 + jv_1 + 5u_2 + 5jv_2 \\ -u_1 - jv_1 - u_2 - jv_2 \end{bmatrix} = \begin{bmatrix} 2u_1 j - 2v_1 \\ 2u_2 j - 2v_2 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 2 & 0 \\ -2 & 0 & 1 & 5 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = 0$$

(b) $\operatorname{Re}(\lambda) = 0 \rightarrow$ critically stable center

$$\textcircled{1} \quad J_r = M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\dot{\vec{z}} = J_r \vec{z} \quad \boxed{\begin{aligned} \dot{z}_1 &= -2z_2 \\ \dot{z}_2 &= 2z_1 \end{aligned}}$$

(V) $A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \rightarrow \lambda_1 = 1+i, \lambda_2 = 1-i \rightarrow \lambda = \alpha \pm \beta i \Rightarrow \frac{\alpha}{\beta} = 1$

$$v_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

⑥ $\operatorname{Re}(\lambda) > 0 \rightarrow (0,0)$ is an unstable focus

$$\textcircled{2} \quad J_r = M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \boxed{\begin{aligned} \dot{z}_1 &= z_1 - z_2 \\ \dot{z}_2 &= z_1 + z_2 \end{aligned}}$$

$$M = [u \ v], \quad A(u+vj) = (\alpha+\beta j)(u+vj)$$

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 + v_1 j \\ u_2 + v_2 j \end{bmatrix} = (1+j) \begin{bmatrix} u_1 + v_1 j \\ u_2 + v_2 j \end{bmatrix} \quad \begin{aligned} v_1 &= 0, u_1 = 0 \\ \rightarrow u_2 &= v_2, u_2 = -v_2 \\ u &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 2u_1 + 2v_1 j - u_1 - v_1 j \\ 2u_1 + 2v_1 j \end{bmatrix} = \begin{bmatrix} u_1 + v_1 j + u_1 j - v_1 \\ u_2 + v_2 j + u_2 j - v_2 \end{bmatrix}$$

$$0 = \begin{bmatrix} 2u_1 + 2v_1 j - u_1 - v_1 j - u_1 - v_1 j + u_1 j + v_1 \\ 2u_1 + 2v_1 j - u_2 - v_2 j - u_2 j + v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -u_1 \\ 2u_1 - u_2 + v_2 \\ 2v_1 - v_2 - u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textcircled{B} \quad \dot{x}_1 = -x_1 - \frac{x_2}{\ln \sqrt{x_1^2 + x_2^2}}$$

$$\dot{x}_2 = -x_2 + \frac{x_1}{\ln \sqrt{x_1^2 + x_2^2}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f(x) \approx f(x_0) + \left. \frac{df}{dx} \right|_{x=x_0} (x-x_0)$$

$$\frac{\partial}{\partial x_1} \frac{1}{\ln \sqrt{x_1^2 + x_2^2}} = \frac{\partial}{\partial \ln \sqrt{x_1^2 + x_2^2}} \left(\ln \sqrt{x_1^2 + x_2^2} \right)^{-1} \cdot \frac{\partial \ln \sqrt{x_1^2 + x_2^2}}{\partial x_1} \cdot \frac{\partial \sqrt{x_1^2 + x_2^2}}{\partial x_1}$$

$$\frac{\partial}{\partial x_1} \frac{1}{\ln \sqrt{x_1^2 + x_2^2}} = -\left(\frac{-1}{(\ln \sqrt{x_1^2 + x_2^2})^2} \right) \left(\frac{1}{\sqrt{x_1^2 + x_2^2}} \right) \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right)$$

without loss of generality

$$\frac{\partial}{\partial x_2} \frac{1}{\ln \sqrt{x_1^2 + x_2^2}} = \left(\frac{-1}{\ln \sqrt{x_1^2 + x_2^2}} \right) \left(\frac{1}{\sqrt{x_1^2 + x_2^2}} \right) \left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right)$$

$$\eta := \frac{1}{(\ln \sqrt{x_1^2 + x_2^2})^2 (x_1^2 + x_2^2)}, \quad \mu := \frac{1}{\ln \sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial}{\partial x_1} \frac{1}{\ln \sqrt{x_1^2 + x_2^2}} = -\eta x_1 \frac{\partial}{\partial x_2} \frac{1}{\ln \sqrt{x_1^2 + x_2^2}} = -\eta x_2$$

$$\frac{\partial \dot{x}_1}{\partial x_1} = -1 + \eta x_1 x_2, \quad \frac{\partial \dot{x}_1}{\partial x_2} = 0 - (1)(\mu) - (x_2)(-\eta x_2)$$

$$\frac{\partial \dot{x}_2}{\partial x_1} = 0 + (1)(\mu) + (x_1)(-\eta x_1), \quad \frac{\partial \dot{x}_2}{\partial x_2} = -1 + \eta x_1 x_2$$

$$\frac{\partial \dot{x}}{\partial x} = \begin{bmatrix} -1 + \eta x_1 x_2 & -\mu + \eta x_2^2 \\ \mu - \eta x_1^2 & -1 - \eta x_1 x_2 \end{bmatrix}$$

$$f(x) \approx f(x_0) + \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x-x_0)$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad f = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\lim_{x \rightarrow 0} x_1 x_2 = \lim_{x \rightarrow 0} \frac{x_1 x_2}{(\ln \sqrt{x_1^2 + x_2^2})^2 (x_1^2 + x_2^2)} = 0 \quad \leftarrow$$

$$\lim_{x \rightarrow 0} \mu = \lim_{x \rightarrow 0} \frac{1}{\ln \sqrt{x_1^2 + x_2^2}} = 0$$

$$\Rightarrow \left. \frac{\partial f}{\partial x} \right|_{x=x_0} = \begin{bmatrix} -1+0 & -0+0 \\ 0-0 & -1-0 \end{bmatrix} \Rightarrow \boxed{\left. \frac{\partial f}{\partial x} \right|_{x=x_0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$Av - \lambda v = 0 \quad \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)^2 = 0 \Rightarrow \boxed{\lambda_1 = \lambda_2 = -1}$$

$\operatorname{Re}(\lambda_i) < 0 \Rightarrow$ origin is a stable node

$$\textcircled{8} \quad (b) \quad \textcircled{1} \quad \dot{x}_1 = -x_1 - \frac{x_2}{\ln \sqrt{x_1^2 + x_2^2}}$$

$$\textcircled{2} \quad \dot{x}_2 = -x_2 + \frac{x_1}{\ln \sqrt{x_1^2 + x_2^2}} \quad \dot{r} = -r$$

$$r := \sqrt{x_1^2 + x_2^2}, \quad \theta := \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\dot{x}_1^2 + \dot{x}_2^2 = \dot{r}^2 = x_1^2 + \frac{2x_1x_2}{\ln r} + \frac{x_2^2}{(\ln r)^2}$$

$$+ x_2^2 - \frac{2x_1x_2}{\ln r} + \frac{x_1^2}{(\ln r)^2}$$

$$\dot{r}^2 = r_0^2 + \frac{r^2}{(\ln r)^2} = r^2 \left[1 + \frac{1}{(\ln r)^2} \right]$$

$$\lim_{r \rightarrow 0} \left[1 + \frac{1}{(\ln r)^2} \right] = 1 \Rightarrow \dot{r}^2 = r^2$$

$$\dot{r} = \pm r$$

$\dot{r} = -r$ because the origin
 \downarrow is a stable node

$$r(t) = r_0 e^{-t}$$

$$\textcircled{1} \quad \dot{r} \cos \theta - r \sin(\theta) \dot{\theta} = -r \cos \theta - \frac{r \sin \theta}{\ln r}$$

$$-r \cos \theta - r \sin(\theta) \dot{\theta} = -r \cos \theta - \frac{r \sin \theta}{\ln r}$$

$$\dot{\theta} = \frac{1}{\ln r}$$

$$\dot{\theta} = \frac{1}{\ln(r_0 e^{-t})} = \frac{1}{\ln r_0 - t}$$

$$\dot{\theta} = \frac{1}{\ln r_0 - t}$$

$$u = \ln r_0 - t$$

$$du = -dt$$

$$\ln r_0 - t$$

$$\int \frac{1}{u} du = -\ln u$$

$$\left. \begin{array}{l} \ln r_0 - t \\ C \\ \ln r_0 \end{array} \right|$$

$$\theta = \int_0^t \dot{\theta} dt = \int_0^t \frac{1}{\ln r_0 - t} dt = - \int_{\ln r_0}^{\ln r_0 - t} \frac{1}{u} du = -\ln u \Big|_{\ln r_0}^{\ln r_0 - t}$$

$$\boxed{\theta = -\ln |\ln r_0 - t| - \ln |r_0| + \theta_0}$$

for $0 < r_0 < 1$, $\dot{r} < 0$ and $\dot{\theta} < 0$

$$\lim_{t \rightarrow \infty} r(t) = 0, \quad \lim_{t \rightarrow \infty} \theta(t) = -\infty$$

→ state trajectory spirals towards origin

- ③ The linearized system has a stable equilibrium at the origin, but in the non-linear system, the origin is a stable focus.

$$\textcircled{11} \quad \|x\|_2^2 = \sum_{i=1}^n |x_i|^2, \quad \|x_1\|^2 = \sum_{i=1}^n |x_i|^2 + 2 \sum_{i \neq j, i \neq j} |x_i||x_j|$$

$$\sum_{i=1}^n |x_i|^2 \leq \sum_{i=1}^n |x_i|^2 + 2 \sum_{i,j, i \neq j} |x_i||x_j|$$

$$\|x\|_2^2 \leq \|x_1\|^2$$

$$\boxed{\|x\|_2 \leq \|x_1\|}$$

2

Cauchy-Schwarz Inequality: $|x^T y| \leq \|x\|_2 \|y\|_2$

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n |x_i| \cdot 1 = z^T 1$$

$$z := \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix}$$

$$z = \|x\|_2 \|1\|_2 = \sqrt{n} \|x\|_2$$

$$\|1\|_2 = \sqrt{\sum_{i=1}^n |1|^2} = \sqrt{n}$$

by Cauchy-Schwarz Inequality,

$$|z^T 1| \leq \|z\|_2 \|1\|_2$$

$$\boxed{\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2}$$

$$\boxed{\|x_1\| \leq \sqrt{n} \|x\|_2}$$



$$\|x\|_\infty = \max(|x_i|) = (\max(|x_i|)^2)^{1/2} \leq \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \|x\|_2$$

$$\boxed{\|x\|_\infty \leq \|x\|_2}$$

$$\text{for all } i, |x_i| \leq \max(|x_i|) = \|x\|_\infty \\ \Rightarrow x_i^2 \leq \|x\|_\infty^2$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n \|x\|_\infty^2} = \sqrt{n} \|x\|_\infty$$

$$\boxed{\|x\|_2 \leq \sqrt{n} \|x\|_\infty} \rightarrow \boxed{\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty}$$

$$\|x\|_2 \leq \|x\|_1 \text{ & } \|x\|_\infty \leq \|x\|_2 \rightarrow \underline{\|x\|_\infty \leq \|x\|_1}$$

$$\|x\|_1 \leq \sqrt{n} \|x\|_2$$

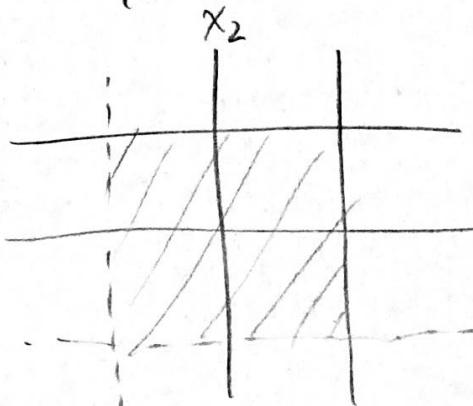
$$\times \quad \underline{\|x\|_2 \leq \sqrt{n} \|x\|_\infty}$$

$$\cancel{\|x\|_1 \|x\|_2 \leq n \|x\|_2 \|x\|_\infty}$$

$$\underline{\|x\|_1 \leq n \|x\|_\infty}$$

$$\boxed{\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty}$$

$$⑫ S = \{x \in \mathbb{R}^2 \mid -1 < x_i \leq 1, i=1,2\}$$



S is not an open set

because all the points
in the neighborhood of the
points on the closed boundary
are not members of S

$$(CB = \{(x_1=1, -1 < x_2 \leq 1) \cup (x_2=1, -1 < x_1 \leq 1)\})$$

$S' = \mathbb{R}^2 - S$ is similarly not an open set since
it has closed boundaries \rightarrow S is not a closed set

$$d(S) = \{x \in \mathbb{R}^2 \mid -1 < x_i \leq 1, i=1,2\} \quad (\text{closure})$$

$$\text{int}(S) = \{x \in \mathbb{R}^2 \mid -1 < x_i < 1, i=1,2\} \quad (\text{interior})$$

$$\begin{aligned} bd(S) = & \{x \in \mathbb{R}^2 \mid (x_1=1, -1 < x_2 \leq 1) \cup (x_1=-1, -1 \leq x_2 \leq 1) \\ & \cup (x_2=1, -1 < x_1 \leq 1) \cup (x_2=-1, -1 < x_1 \leq 1)\} \end{aligned} \quad (\text{boundary})$$

(13) $u_T(t) = \begin{cases} 0, & t < T \\ 1, & t \geq T \end{cases}$ | the interval boundary ($t=T$) has finite limits ($0, 1$) from both sides ($-$, $+$)

a) $f_1 = 0$ is continuous on $(-\infty, T)$

$f_2 = 1$ is continuous on $[T, \infty)$

$u_T(t)$ has a finite number (1) of discontinuities (at $t=T$), and is continuous otherwise,
therefore $u_T(t)$ is piecewise continuous

b) $f(t) = g(t)u_T(t) = \begin{cases} 0, & t < T \\ g(t), & t \geq T \end{cases}$

$f_1 = 0$ is continuous on $(-\infty, T)$

$f_2 = g(t)$ is continuous on $[T, \infty]$ $\leftarrow g(t)$ is cont.

$f(t)$ has a finite number (1) of discontinuities (at $t=T$), and is continuous otherwise,

therefore $f(t)$ is piecewise continuous

(B) c) $\text{sgn}(t) = \text{sgn}(\sin(\frac{2\pi t}{T})) = \begin{cases} +1, & \frac{n}{2}T < t < \frac{2n+1}{2}T \\ 0, & t = \frac{n}{2}T \\ -1, & \frac{2n+1}{2}T < t < \frac{n}{2}T \end{cases}$

$$n \in \mathbb{Z}$$

the interval boundaries
have finite limits (± 1)

$f_1 = 1$ is continuous on $\frac{n}{2}T < t < \frac{2n+1}{2}T, \forall t, n \in \mathbb{Z}$

$f_3 = -1$ is continuous on $\frac{2n+1}{2}T < t < \frac{n}{2}T, \forall t, n \in \mathbb{Z}$

$f_2 = 0, t = \frac{n}{2}T$ is the set of discontinuity points
which is finite over a finite interval

therefore the periodic square wave function
is piecewise continuous

⑭ $f(x) \in C_1$

$$\dot{x} = f(x)$$

$$f(x^*) = 0$$

$$J = \left[\frac{\partial f}{\partial x} \right]_{x=x^*} \text{ is non-singular}$$

By Implicit Function Theorem

there exists neighborhood $U \subset R^n$ of x^*

such that $f(x) = \dot{x}$ has a unique solution $x \in U$.

Since the solution, x^* , is unique within
the neighborhood, it is an isolated equilibrium point.

(15)

$$1) f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x=0 \end{cases}$$

$$① f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \left(-\frac{1}{x^2}\right)$$

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$f'(x=0)$ does not exist \rightarrow f(x) is not continuously differentiable

(b)

$$\begin{aligned} f(x)-f(y) &= x^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{y}\right) \\ &= x^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{y}\right) \\ &= (x^2 - y^2) \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{y}\right) \end{aligned}$$

$$T_1 := (x^2 - y^2) \sin\left(\frac{1}{x}\right), T_2 := y^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{y}\right),$$

$$f(x)-f(y) = T_1 + T_2$$

$$\begin{aligned} T_1 &= (x^2 - y^2) \sin\left(\frac{1}{x}\right) \leq |(x^2 - y^2) \sin\left(\frac{1}{x}\right)| = |(x+y)(x-y) \sin\left(\frac{1}{x}\right)| \\ &\leq |x-y| |x+y| |\sin\left(\frac{1}{x}\right)| \leq |x-y| |x+y| \underbrace{\left|\frac{1}{x}\right|}_{\text{by Mean Value Theorem } |\sin u| \leq |u|} \end{aligned}$$

$$\exists c \in (0,1) \mid \frac{d}{dt} \sin t = \frac{\sin x - \sin 0}{x-0} = \cos(c) = \frac{\sin x}{x}$$

$$-1 \leq \cos(c) \leq 1 \Rightarrow \left|\frac{\sin x}{x}\right| \leq 1 \Rightarrow |\sin x| \leq |x|$$

$$|x+y|\frac{1}{|x|} \leq (|x|+|y|)\frac{1}{|x|}$$

assuming without loss of generality,

$$0 < |y| \leq |x|$$

$$(|x|+|y|)\frac{1}{|x|} = 1 + \frac{|y|}{|x|} \leq 2$$

$$T_1 = (x^2 - y^2) \sin\left(\frac{1}{x}\right) \leq 2|x-y|$$

$$|T_2| = |y^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{y}\right)| \leq |y^2| \left|\frac{1}{x} - \frac{1}{y}\right| = |y^2| \left|\frac{1}{xy}(y-x)\right|$$

$$|T_2| \leq \left|\frac{y}{x}\right| |x-y| \leq |x-y|$$

$$\left|\frac{y}{x}\right| \leq 1$$

$$|f(x) - f(y)| \leq 2|x-y| + |x-y| = 3|x-y|, \text{ for } 0 < |y| \leq |x|$$

edge cases $x=0, y \neq 0$ without loss of generality

$$|f(x) - f(y)| = |f(x) - f(0)| = |x^2 \sin\left(\frac{1}{x}\right)|$$

$$\leq \left|\frac{x^2}{x}\right| = |x| = |x-0| \leq 3|x-0| + x$$

$$\Rightarrow |f(x) - f(y)| \leq 3|x-y| + x, y=0$$

$$|f(x) - f(y)| \leq 3|x-y|$$

↑

$f(x)$ is locally Lipschitz at $x=0$

$$(15) \text{ i) (c)} f(x) = 0$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, x \neq 0$$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2, x \neq 0$$

$$\lim_{x \rightarrow 0} -x^2 \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} x^2,$$

$$0 \leq \lim_{x \rightarrow 0} f(x) \leq 0$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = 0$$

↑
therefore $f(x)$ is continuous at $x=0$

(d) as previously shown

$$|f(x) - f(y)| \leq 3|x-y| \quad \forall x, y \in \mathbb{R} \quad (\text{derivative is globally bounded})$$

$f(x)$ is globally Lipschitz

(product of cont. functions is cont.)

(e) $f(x)$ is cont at $x=0$, x^2 is contin. on \mathbb{R} , $\sin\left(\frac{1}{x}\right)$ is cont on $\mathbb{R} \setminus \{0\}$, so

(f) $f(x)$ is Lipschitz at $(-1, 1)$ since it is uniformly continuous

$$2) f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x=0 \end{cases}$$

$$(a) -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x^3 \leq x^3 \sin\left(\frac{1}{x}\right) \leq x^3$$

$$\lim_{x \rightarrow 0} -x^3 \leq \lim_{x \rightarrow 0} x^3 \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0} x^3$$

$$0 \leq \lim_{x \rightarrow 0} x^3 \sin\left(\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x \rightarrow 0} x^3 \sin\left(\frac{1}{x}\right) = 0$$

squeeze theorem

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) + x^3 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right)$$

$$\lim_{x \rightarrow 0} -x \leq \lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0} x$$

$$0 \leq \lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) \leq 0$$

as seen earlier, $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ and by squeeze theorem $\lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) = 0$

therefore $\lim_{x \rightarrow 0} f'(x) = 0$, $f'(0) = \frac{d}{dx} 0 = 0$, $\lim_{x \rightarrow 0} f'(x) = f'(0)$

therefore $f(x)$ is continuously differentiable at $x=0$

(b) $f(x)$ is globally continuously differentiable

→ therefore $f(x)$ is Lipschitz continuous

(c) $f(0) = 0$, $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

→ therefore $f(x)$ is continuous at $x=0$

(d) $\lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) = \infty \rightarrow f(x) \text{ is unbounded}$
→ $f(x)$ is not globally Lipschitz

(e) $\lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) = \infty$

→ $f(x)$ is not uniformly continuous

(f) $f(x)$ is glob. cont. diff $\rightarrow f(x)$ is Lipschitz cont. on $(-1, 1)$

(15)

(3) $f(x) = \tan(\pi x/2)$

(a) $f(0) = \tan(0) = 0$

$$\lim_{x \rightarrow 0} \tan(\pi x/2) = 0$$

$$f'(x) = \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)}$$

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)} = \frac{\pi}{2} \frac{1}{\cos^2(0)} = \frac{\pi}{2}$$

\Rightarrow $\tan(\pi x/2)$ is continuously differentiable at $x=0$

(b) $f(x) = \tan(\pi x/2) \rightarrow f'(x) = \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)}$

$f(x)$ is continuously differentiable on $(-1, 1)$

\rightarrow $f(x)$ is locally Lipschitz at $x=0$

(c) $f(0) = \lim_{x \rightarrow 0} f(x) = 0 \rightarrow$ $\tan(\pi x/2)$ is continuous at $x=0$

(d) since $\tan(\pi x/2)$ is not locally Lipschitz,
 \therefore it is not globally Lipschitz

(e) $\lim_{x \rightarrow 1^-} \tan(\pi x/2) = +\infty \neq -\infty = \lim_{x \rightarrow 1^+} \tan(\pi x/2)$

\rightarrow therefore $\tan(\pi x/2)$ is not uniformly continuous on \mathbb{R}

$$(f) f'(x) = \frac{\pi/2}{\cos^2(\pi x/2)}$$

$$\lim_{x \rightarrow -1} f'(x) = +\infty, \quad \lim_{x \rightarrow -1^+} f'(x) = -\infty$$

$f'(x)$ is unbounded on $(-1, 1)$

→ therefore $f(x) = \tan(\pi x/2)$ is not

Lipschitz continuous on $(-1, 1)$

$$\textcircled{16} (1) f(x) = \begin{bmatrix} x_1 + \operatorname{sgn}(x_2) \\ x_2 \end{bmatrix}$$

$$@ \operatorname{sgn}(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases} \Rightarrow \frac{d}{dx} \operatorname{sgn}(x) = \begin{cases} 0, & x \in \mathbb{R}, x \neq 0 \\ \text{undefined}, & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & x_1, x_2 \in \mathbb{R}, x_2 \neq 0 \\ \begin{bmatrix} 1 & \text{undef.} \\ 0 & 1 \end{bmatrix}, & x_1 \in \mathbb{R}, x_2 = 0 \end{cases}$$

$f'(x)$ does not exist at $x_2 = 0$

→ $f(x)$ is not continuously differentiable

\textcircled{5}

⑯ since $\operatorname{sgn}(x_2)$ is not continuous,

(b)-(e) $f(x)$ is not continuous

→ therefore $f(x)$ is not Lipschitz or continuous

(b) $f(x)$ is locally Lipschitz on any

interval, I , that does not contain $x_2=0$

$$② f(x) = \begin{cases} x_1 + \operatorname{sat}(x_2) \\ x_1 + \sin(x_2) \end{cases}$$

a) $f(x)$ is not continuously differentiable

because $\operatorname{sat}(x)$ is not continuously differentiable

b) $f(x)$ is locally Lipschitz on interval, I ,
that does not contain $x_2=\delta$, where
 $\operatorname{sat}(\delta)$ is a critical point

c) $f(x)$ is continuous because x , $\operatorname{sat}(x)$, $\sin(x)$
are globally continuous

d) $f(x)$ is not globally Lipschitz because the
derivative of $\operatorname{sat}(x)$ is not globally defined

e) $f(x)$ is uniformly continuous on \mathbb{R}^2 b/c $x, \operatorname{sat}(x), \sin(x) \in C$,

$$(3) \quad f(x) = \begin{cases} x_3 \text{sat}(x_1 + x_2) \\ x_2^2 \\ x_1 \end{cases}$$

(a) f(x) is not continuously differentiable

because sat(x) is not continuously differentiable

(b)

(c) f(x) is continuous on \mathbb{R}^n because

$x_3 \text{sat}(x_1 + x_2)$, x_2^2 , and x_1 are all continuous

(d) f(x) is not globally Lipschitz because the slope of x^2 is unbounded as $x \rightarrow \pm\infty$

(e) f(x) is not continuously on \mathbb{R}^3 because
the slope of x^2 is unbounded as $x \rightarrow \pm\infty$

(17) $\exists c_1 > 0, c_2 > 0,$

s.t. $c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha, \forall x \in \mathbb{R}^n$

without loss of generality,

let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be Lipschitz in $\|\cdot\|_\alpha$

then $\|f(y) - f(x)\|_\alpha \leq L_\alpha \|y - x\|_\alpha$

$$c_2 \|f(y) - f(x)\|_\alpha \leq c_2 L_\alpha \|y - x\|_\alpha$$

$$\|x\|_\beta \leq c_2 \|x\|_\alpha \rightarrow \|f(y) - f(x)\|_\beta \leq c_2 \|f(y) - f(x)\|_\alpha$$

$$c_1 \|x\|_\alpha \leq \|x\|_\beta \rightarrow c_1 \|y - x\|_\alpha \leq \|y - x\|_\beta$$

$$c_2 L_\alpha \|y - x\|_\alpha \leq \frac{c_2 L_\alpha}{c_1} \|y - x\|_\beta$$

$$c_2 \|f(y) - f(x)\|_\alpha \leq \frac{c_2 L_\alpha}{c_1} \|y - x\|_\beta$$

$$\|f(y) - f(x)\|_\beta \leq \frac{c_2 L_\alpha}{c_1} \|y - x\|_\beta$$

$$\|f(y) - f(x)\|_\beta \leq L_\beta \|y - x\|_\beta$$

$\rightarrow f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz in $\|\cdot\|_\beta$

(18) $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g \in C_1$

$$f(x) = \frac{1}{1+g^T(x)g(x)} g(x)$$

Show $\dot{x} = f(x)$, $x(0) = x_0$ has a unique solution for all $t \geq 0$

$$\begin{aligned} f'(x) &= \frac{\partial}{\partial x} \left(\frac{1}{1+g^T(x)g(x)} \right) g(x) + \left(\frac{1}{1+g^T(x)g(x)} \right) \frac{\partial g}{\partial x} \\ &= \frac{1}{(1+g^Tg)^2} \frac{\partial}{\partial x} (1+g^Tg)g + \frac{1}{1+g^Tg} \frac{\partial g}{\partial x} \\ &= \frac{1}{(1+g^Tg)^2} \left(\frac{\partial g^T}{\partial x} g + g^T \frac{\partial g}{\partial x} \right) + \frac{1}{1+g^Tg} \frac{\partial g}{\partial x} \end{aligned}$$

$g \in C_1 \rightarrow f'(x)$ is continuous $\rightarrow f(x) \in C_1$

$f(x) \in C_1 \rightarrow f(x)$ is Lipschitz continuous

\rightarrow by Theorem 3.1, there exists some $\delta > 0$ such that
 the state equation $\dot{x} = f(tx)$ with $x(t_0) = x_0$
 has a unique solution over $[t_0, t_0 + \delta]$
 (local unique solution)

(18) cont

$$\|f(x)\|_2 = \left\| \frac{1}{1+g^T g} g \right\|_2 = \left\| \frac{g}{1+\|g\|_2^2} \right\|_2$$

$$\begin{aligned}\|f(x)\|_2 &= \left[\sum_{i=1}^N \left(\frac{g_i}{1+\|g\|_2^2} \right)^2 \right]^{1/2} \\ &= \left[\left(\frac{1}{1+\|g\|_2^2} \right)^2 \cdot g^T g \right]^{1/2} =\end{aligned}$$

$$\|f(x)\|_2 = \frac{\|g(x)\|_2}{1+\|g(x)\|_2^2}$$

$$u := \|g(x)\|_2 \Rightarrow \|f(x)\|_2 = \frac{u}{1+u^2} = \frac{1}{\frac{1}{u} + u}$$

$$\frac{1}{u} + u - 2 = \frac{u^2}{u} + \frac{1}{u} - \frac{2u}{u} = \frac{u^2 - 2u + 1}{u} = \frac{(u-1)^2}{u} \geq 0$$

$$\Rightarrow \frac{1}{u} + u \geq 2 \Rightarrow \frac{1}{\frac{1}{u} + u} \leq \frac{1}{2}$$

$$\Rightarrow \|f(x)\|_2 \leq \frac{1}{2}$$

$$\dot{x} \leq \frac{1}{2}$$

$$\|x(t)\|_2 \leq \|x_0\|_2 + \frac{1}{2} (t - t_0)$$

↑ therefore, the solution is defined for all $t \geq 0$,
(sol. is unique)

therefore there is a unique solution for all $t \geq 0$

$$⑯ \quad y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^t e^{-\alpha(t-\tau)} [k_2 y(\tau) + k_3] d\tau$$

$$y(t) e^{\alpha(t-t_0)} \leq k_1 e^{-\alpha(t-t_0-t+t_0)} + \int_{t_0}^t e^{-\alpha(t-\tau)} [k_2 y(\tau) e^{\alpha(t-t_0)} + k_3] d\tau$$

$$z(t) := y(t) e^{\alpha(t-t_0)} \leq k_1 + \int_{t_0}^t e^{\alpha(t-\tau)} [k_2 y(\tau) + k_3] d\tau$$

$$z(t) \leq k_1 + \int_{t_0}^t [k_2 z(\tau) + k_3 e^{\alpha(t-\tau)}] d\tau$$

$$z(t) \leq k_1 + k_3 e^{-\alpha t_0} \int_{t_0}^t e^{\alpha \tau} d\tau + \int_{t_0}^t k_2 z(\tau) d\tau$$

$$z(t) \leq k_1 + k_3 e^{-\alpha t_0} \left[\frac{1}{\alpha} e^{\alpha t} \right]_{\tau=t_0}^t + \int_{t_0}^t k_2 z(\tau) d\tau$$

$$z(t) \leq k_1 + k_3 e^{-\alpha t_0} \frac{1}{\alpha} [e^{\alpha t} - e^{\alpha t_0}] + \int_{t_0}^t k_2 z(\tau) d\tau$$

$$z(t) \leq k_1 + \frac{k_3}{\alpha} [e^{\alpha(t-t_0)} - 1] + \int_{t_0}^t k_2 z(\tau) d\tau$$

$$\lambda := k_1 + \frac{k_3}{\alpha} [e^{\alpha(t-t_0)} - 1], \quad \mu := k_2, \quad a := t_0, \quad s := \tau$$

Gronwall-Bellman Inequality

$$f(t) \leq \lambda(t) + \int_a^t \mu(s) f(s) ds$$

$$\rightarrow f(t) \leq \lambda(t) + \int_a^t \lambda(s) \mu(s) \exp \left[\int_s^t \mu(\tau) d\tau \right] ds$$

applying G-B Inequality...

$$Z(t) \leq \lambda(t) + \int_{t_0}^t \lambda(\tau) \mu(\tau) \exp \left[\int_{\tau}^t \mu(s) ds \right] d\tau$$

$$\int_{\tau}^t \mu(s) ds = \int_{\tau}^t k_2 ds = k_2(t-\tau) \quad \alpha\tau - \alpha t_0 - k_2\tau$$

$$Z(t) \leq \lambda(t) + \int_{t_0}^t \left[k_1 + \frac{k_3}{\alpha} [e^{\alpha(t-\tau)} - 1] \right] k_2 e^{k_2(t-\tau)} d\tau$$

$$Z(t) \leq \lambda(t) + \int_{t_0}^t k_1 k_2 e^{k_2 t} e^{-k_2 \tau} d\tau + \int_{t_0}^t \frac{k_2 k_3}{\alpha} [e^{\alpha(t-\tau)} - 1] e^{k_2 t} e^{-k_2 \tau} d\tau$$

$$Z(t) \leq \lambda(t) + k_1 k_2 e^{k_2 t} \left[\frac{1}{-k_2} e^{-k_2 t} \right]_{t_0}^t + \frac{k_2 k_3}{\alpha} e^{k_2 t} \left[\int_{t_0}^t e^{\alpha(t-\tau)-k_2 \tau} d\tau - \int_{t_0}^t e^{-k_2 \tau} d\tau \right]$$

$$Z(t) \leq \lambda(t) - k_1 e^{k_2 t} [e^{-k_2 t} - e^{-k_2 t_0}] + \frac{k_2 k_3}{\alpha} e^{k_2 t} \left[e^{-\alpha t_0} \left[\frac{e^{(\alpha-k_2)t}}{\alpha-k_2} \right]_{t=t_0}^t - \left[\frac{e^{-k_2 t}}{-k_2} \right]_{t=t_0}^t \right]$$

$$Z(t) \leq \lambda(t) - k_1 \left[1 - e^{k_2(t-t_0)} \right] + \frac{k_2 k_3}{\alpha} e^{k_2 t} \left[e^{-\alpha t_0} \left[\frac{e^{(\alpha-k_2)t} - e^{(\alpha-k_2)t_0}}{\alpha-k_2} \right] - \left[\frac{e^{-k_2 t} - e^{-k_2 t_0}}{-k_2} \right] \right]$$

$$\begin{aligned} Z(t) &\leq k_1 + \frac{k_3}{\alpha} [e^{\alpha(t-t_0)} - 1] - k_1 + k_1 e^{-k_2(t-t_0)} \\ &\quad + \frac{k_2 k_3}{\alpha} \frac{e^{k_2 t - \alpha t_0 + \alpha t - k_2 t} - e^{k_2 t - \alpha t_0 + \alpha t - k_2 t_0}}{\alpha - k_2} \\ &\quad - \frac{k_2 k_3}{\alpha} \frac{e^{k_2 t - k_2 t} - e^{k_2 t - k_2 t_0}}{-k_2} \end{aligned}$$

$$Z(t) \leq \frac{k_3}{\alpha} e^{\alpha(t-t_0)} - \frac{k_3}{\alpha} + k_1 e^{-k_2(t-t_0)}$$

$$+ \frac{k_2 k_3}{\alpha(\alpha-k_2)} \left[e^{\alpha(t-t_0)} - 1 \right]$$

$$+ \frac{k_3}{\alpha} \left[1 - e^{k_2(t-t_0)} \right]$$

$$Z(t) \leq \frac{k_3}{\alpha} e^{\alpha(t-t_0)} + \left(k_1 - \frac{k_3}{\alpha} \right) e^{k_2(t-t_0)} + \frac{k_2 k_3}{\alpha(\alpha-k_2)} \left[e^{\alpha(t-t_0)} - e^{k_2(t-t_0)} \right]$$

$$y(t) = Z(t) e^{-\alpha(t-t_0)} \leq \frac{k_3}{\alpha} + \frac{k_3}{\alpha} \left[1 + \frac{k_2}{\alpha-k_2} \right]$$

$$\text{II } + \left[k_1 - \frac{k_3}{\alpha} - \frac{k_2 k_3}{\alpha(\alpha-k_2)} \right] e^{(k_2-\alpha)(t-t_0)}$$

$$y(t) \leq \frac{k_3}{(\alpha-k_2)} + \left[k_1 - \frac{k_3}{(\alpha-k_2)} \right] e^{(k_2-\alpha)(t-t_0)}$$

$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha-k_2} \left[1 - e^{-(\alpha-k_2)(t-t_0)} \right]$$

(20) $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$

Show $f(x)$ is uniformly continuous on W

$$|f(x) - f(y)| \leq L|x-y|, \quad \forall x, y \in W$$

Uniform continuity: $\forall \varepsilon > 0, \exists \delta > 0 \mid \forall x, y \in I,$

$$|x-y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon$$

$$|x-y| < \delta \quad \forall x, y \in W$$

$$|f(x) - f(y)| \leq L|x-y| < L\delta \quad \forall x, y \in W$$

$$|f(x) - f(y)| < \varepsilon = L\delta \quad \forall x, y \in W$$

$\rightarrow f$ is continuous on W

②) $x: \mathbb{R} \rightarrow \mathbb{R}^n$, $x \in C$,

$$\|\dot{x}(t)\| \leq g(t), \quad \forall t \geq t_0$$

Show that $\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^t g(s)ds$

$$\int_{t_0}^t \|\dot{x}(s)\| ds \leq \int_{t_0}^t g(s)ds$$

$$\|x(t)\| - \|x(t_0)\| \leq \int_{t_0}^t g(s)ds$$

$$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^t g(s)ds$$

(22)



material balance $\frac{dC_A}{dt} = -r_1, \frac{dC_B}{dt} = r_1 - r_2, \frac{dC_C}{dt} = r_2$

C_j = concentration of j

$r_{1,2}$ = reaction rates $\left[\frac{\text{mol}}{\text{time} \cdot \text{vol}} \right]$

rate law: $r_1 = k_1 C_A, r_2 = k_2 C_B$

$$\dot{c} = \begin{bmatrix} \dot{c}_A \\ \dot{c}_B \\ \dot{c}_C \end{bmatrix} = \begin{bmatrix} -k_1 C_A \\ k_1 C_A - k_2 C_B \\ k_2 C_B \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$$

a) The model is linear

b) $\dot{x} = Ax + Bu$ $\dot{x} = Ax$
 $y = Cx + Du$ $y = Cx$

$$x = c = \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}, \dot{x} = \dot{c} = \begin{bmatrix} \dot{c}_A \\ \dot{c}_B \\ \dot{c}_C \end{bmatrix}, y = Ca$$

$$A = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix}, C = [1 \ 0 \ 0]$$

1 Problem 6

Table 1: Problem 6 Equilibrium Points Table

Part	Equilibrium Points	Eigenvalues	Type
1	(0, 0)	$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	Asym. stable focus
	($-\sqrt{6}, 0$)	-2, 1	Unstable saddle
	($\sqrt{6}, 0$)	-2, 1	Unstable saddle
2	(-7.4494, -7.4494)	$-1.5 \pm 1.1830i$	Stable focus
	(-2.5505, -2.5505)	-3.3706, 0.3706	Unstable saddle
	(0, 0)	-2.0916, -0.9083	Stable node
3	(-3, -4)	$\frac{7}{2} \pm \frac{\sqrt{73}}{2}$	Unstable node
	(0, 0)	1, 2	Unstable node
	(0, 2)	-3, -2	Stable node
	(1, 0)	-1, 2	Unstable saddle
4	(0, 0)	$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	Unstable focus
5	(0, 0)	$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	Asym. stable focus
6	($-\sqrt{1 - x_2^2}, x_2 \leq 1$)	0, 2	Critically stable arc
	($+\sqrt{1 - x_2^2}, x_2 \leq 1$)	0, 2	Critically stable arc
	(0, 0)	$-1 \pm i$	Asym. stable focus
	($-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$)	0, 2	Critically stable node
	($\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$)	0, 2	Critically stable node
7	(-1, -1)		
	(0, 0)	-4, -2	Stable node
	(1, 1)	-1, 1	Unstable saddle
	(-i, i)	-4, -2	Stable node
	(i, -i)	2, 4	Unstable node
	($-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, +\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$)	2, 4	Unstable node
	($+\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$)	$\pm\sqrt{8}i$	Critically stable center
	($-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, +\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$)	$\pm\sqrt{8}i$	Critically stable center
	($+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$)	$\pm\sqrt{8}i$	Critically stable center

2 Problem 7

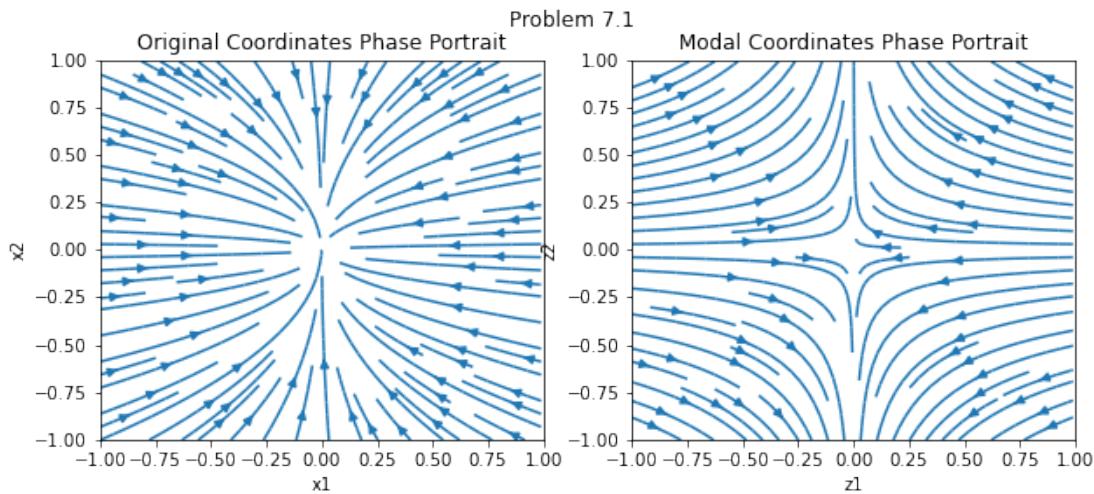


Figure 1: Problem 7.1 Phase Portraits

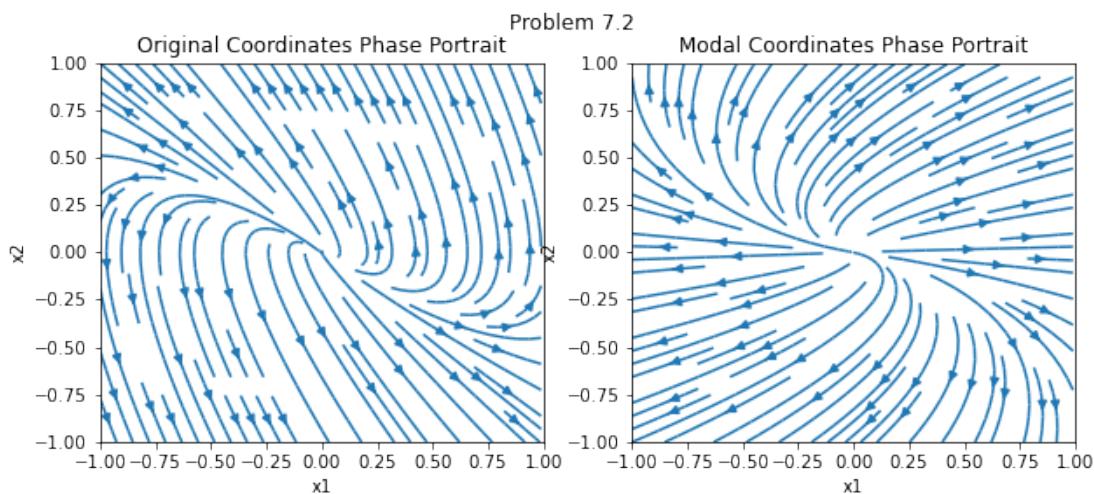


Figure 2: Problem 7.2 Phase Portraits

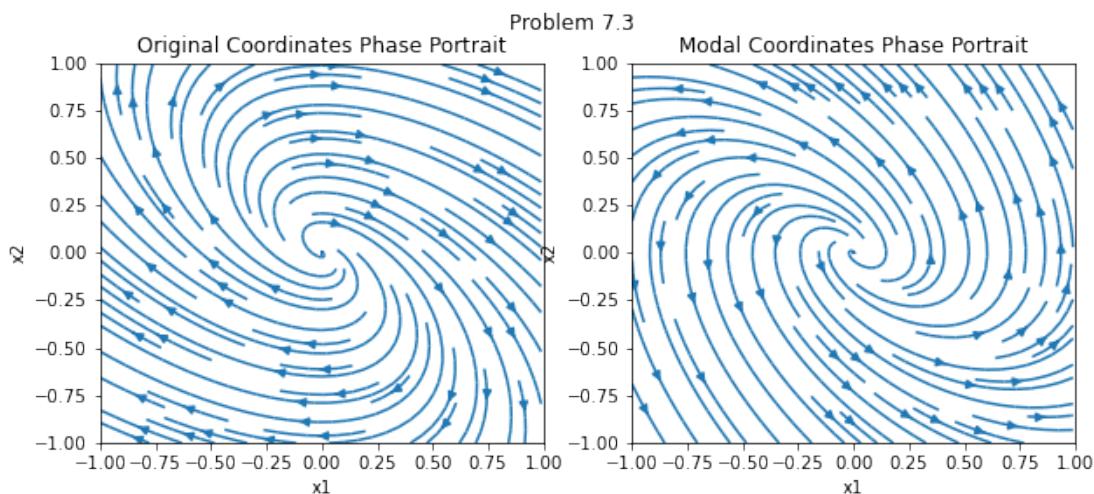


Figure 3: Problem 7.3 Phase Portraits

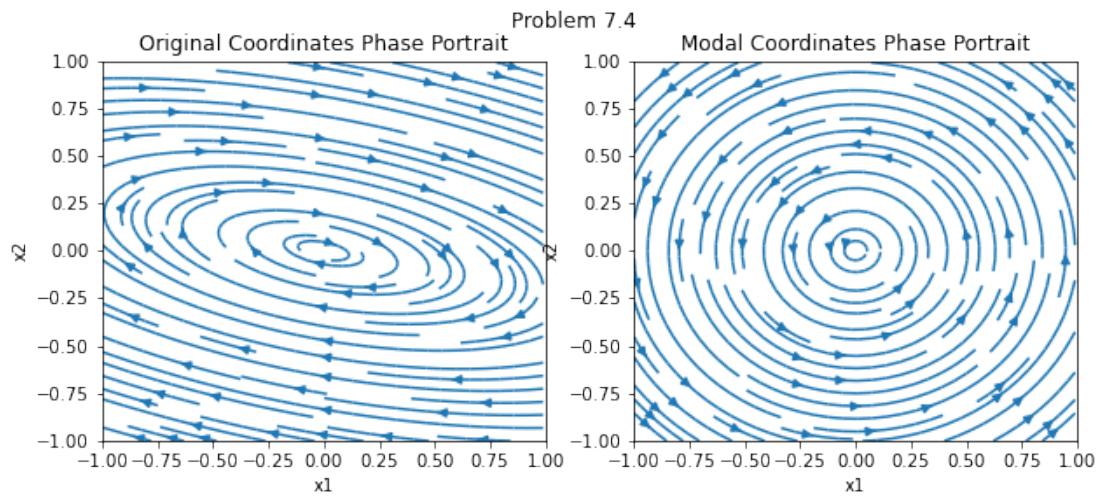


Figure 4: Problem 7.4 Phase Portraits

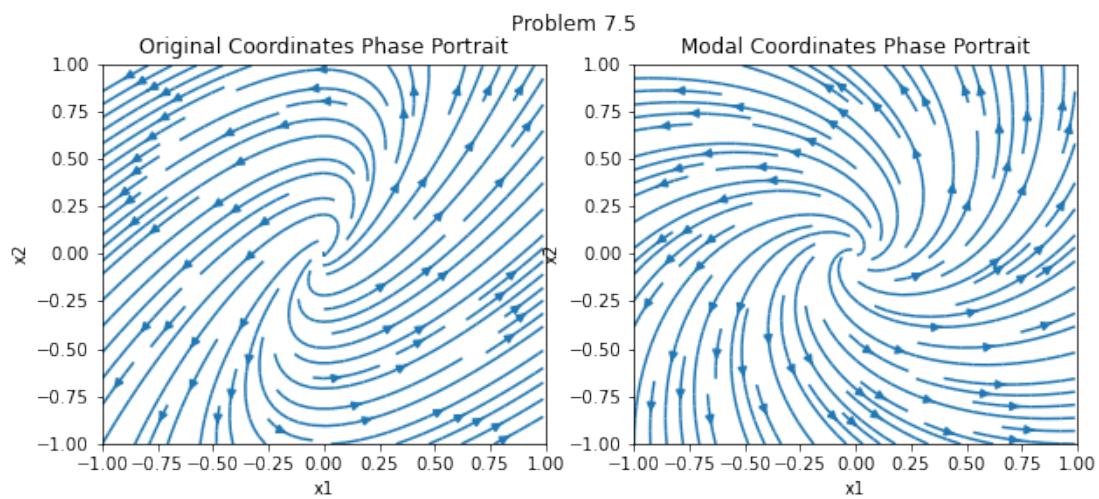


Figure 5: Problem 7.5 Phase Portraits

3 Problem 9

3.1 9.1

The system has a stable node at the origin.

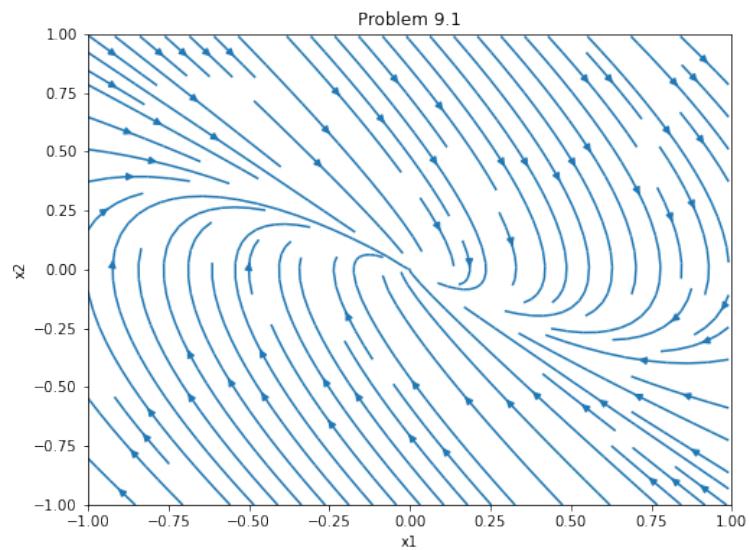


Figure 6: Problem 9.1 Phase Portraits

3.2 9.2

The system appears to converge to a single stable equilibrium trajectory centered at the origin.

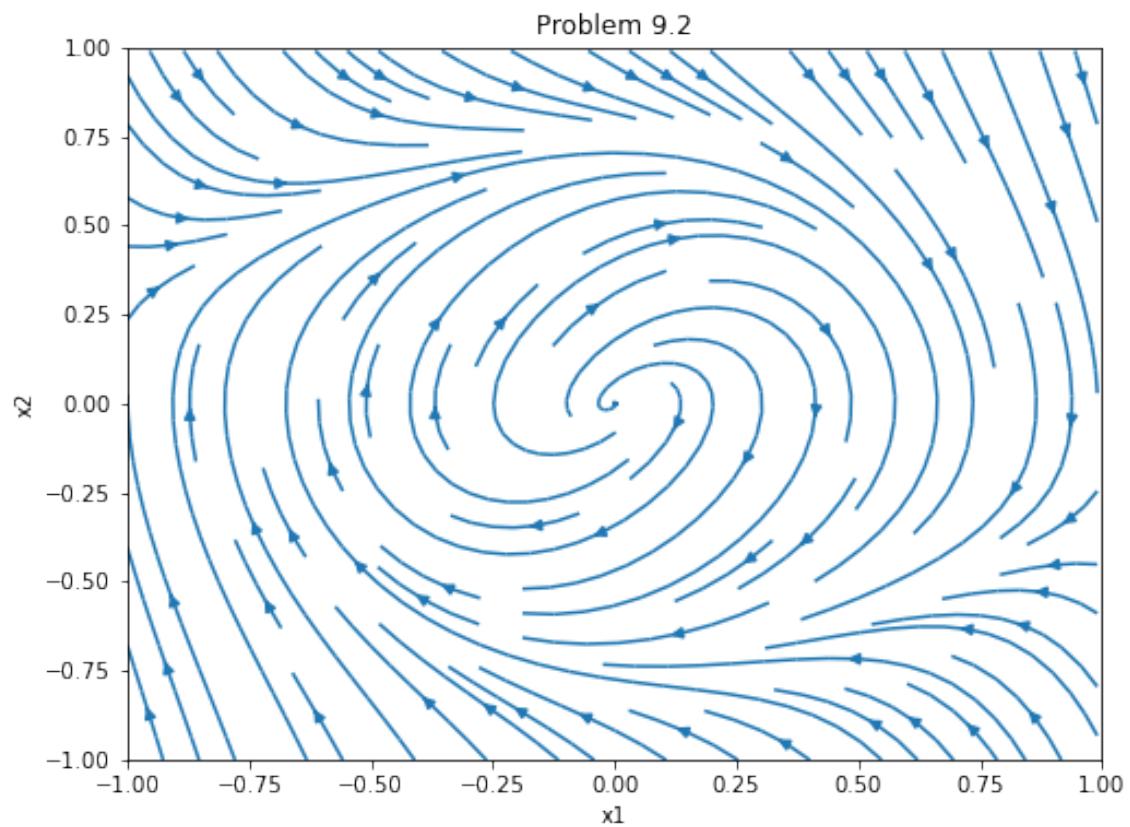


Figure 7: Problem 9.2 Phase Portraits

3.3 9.3

The system has an unstable equilibrium point at the origin and two asymptotically stable equilibrium points around $(\pm \frac{1}{2}, 1)$.

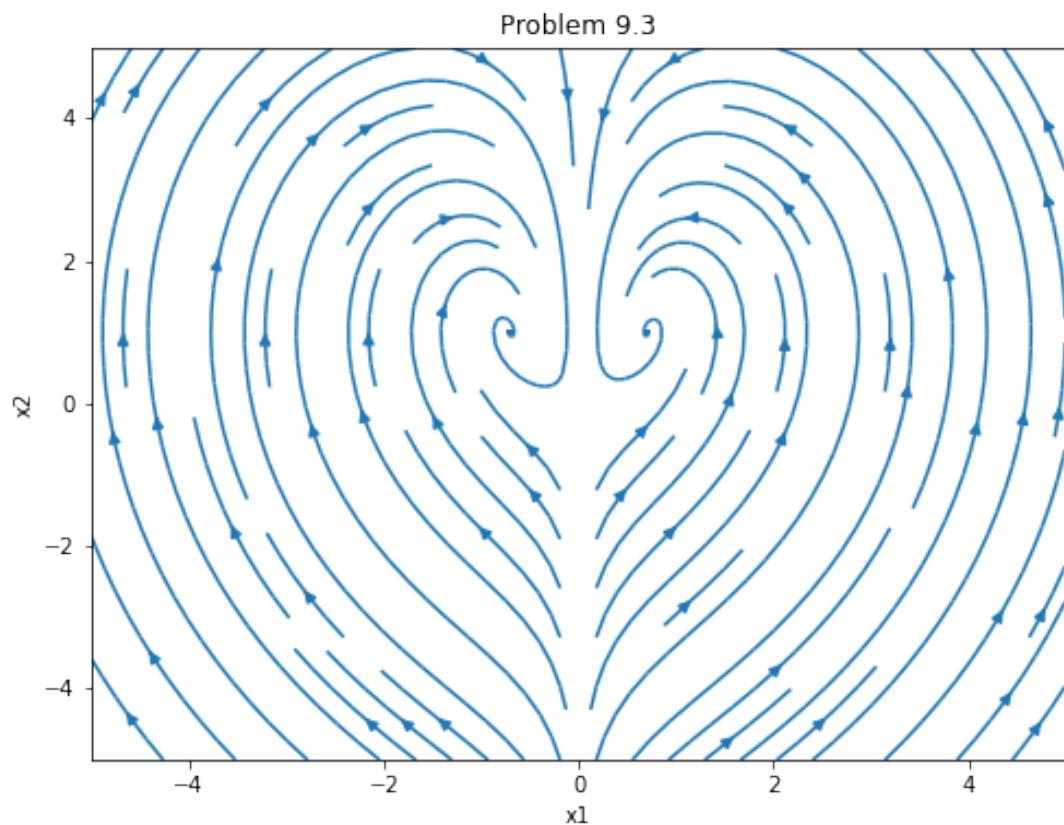


Figure 8: Problem 9.3 Phase Portraits

3.4 9.4

The system appears to converge to a single stable equilibrium trajectory centered at the origin.

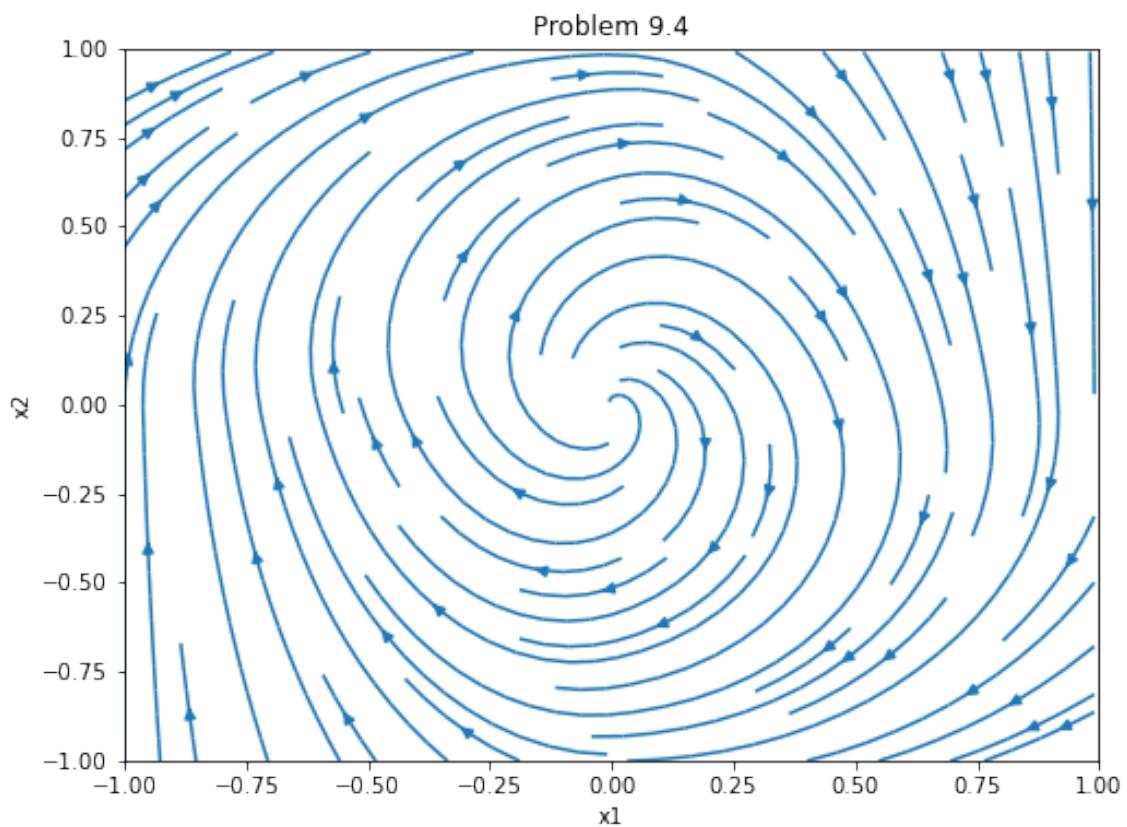


Figure 9: Problem 9.4 Phase Portraits

4 Problem 10

4.1 10.1

The arrows point in the correct direction. The system has an asymptotically stable equilibrium at the origin.

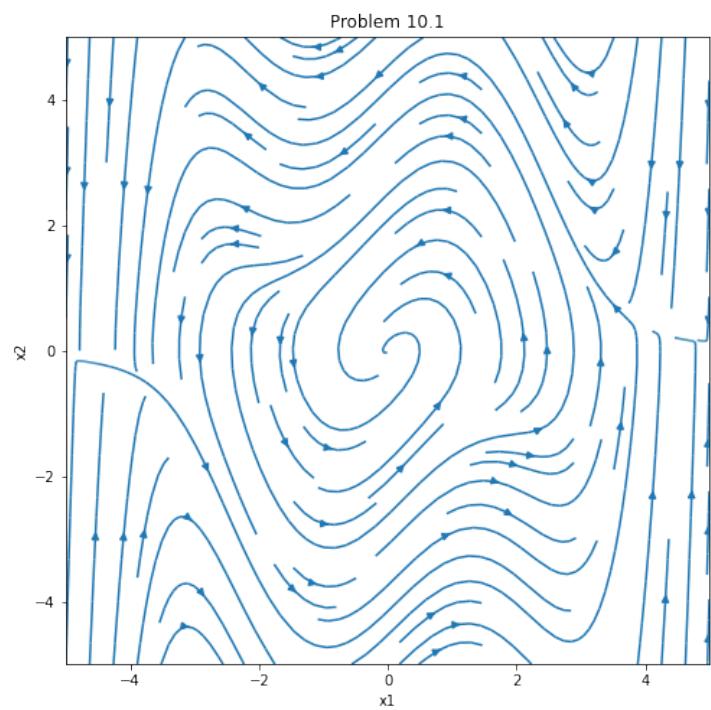


Figure 10: Problem 10.1 Phase Portrait

4.2 10.2

The arrows point in the correct direction. The system has an asymptotically stable equilibrium point at the origin with a finite region of convergence.

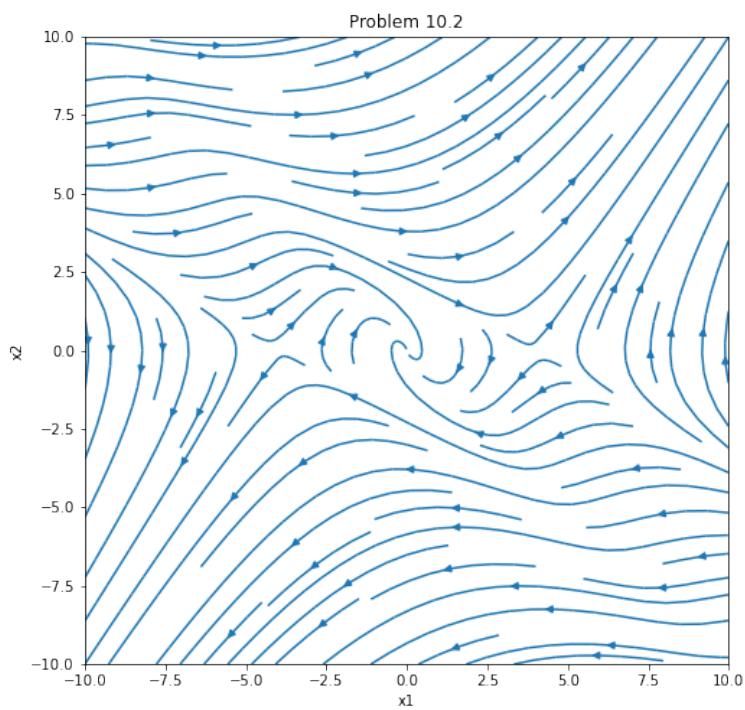


Figure 11: Problem 10.2 Phase Portrait

4.3 10.3

The arrows point in the correct direction. The system has a stable center at the origin.

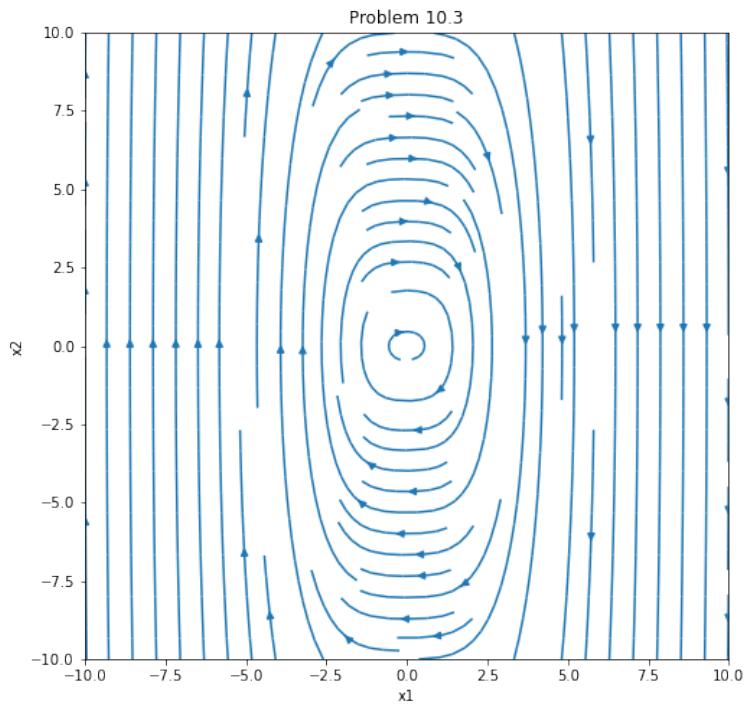


Figure 12: Problem 10.3 Phase Portrait

4.4 10.4

The arrows point in the correct direction. The system has an unstable focus and spirals outwards.

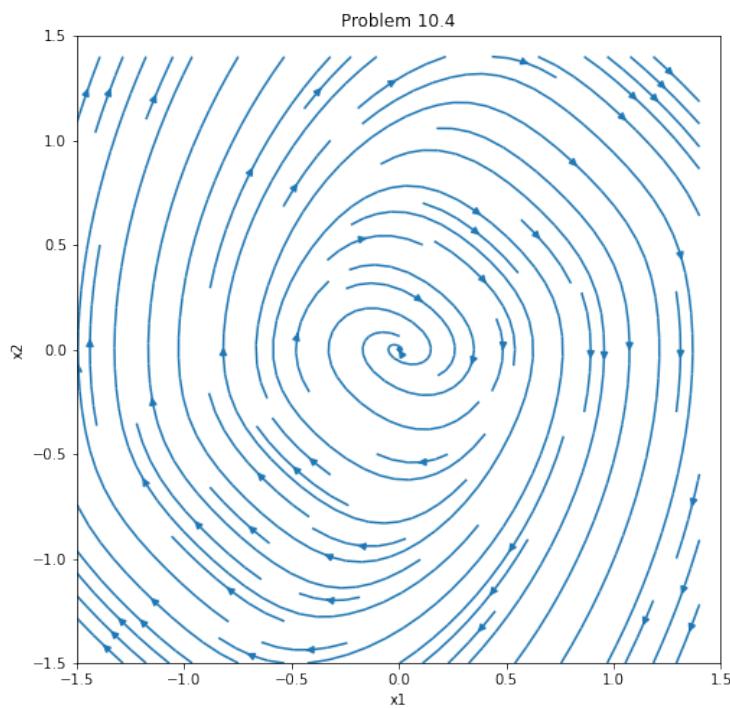


Figure 13: Problem 10.4 Phase Portrait

5 Problem 22

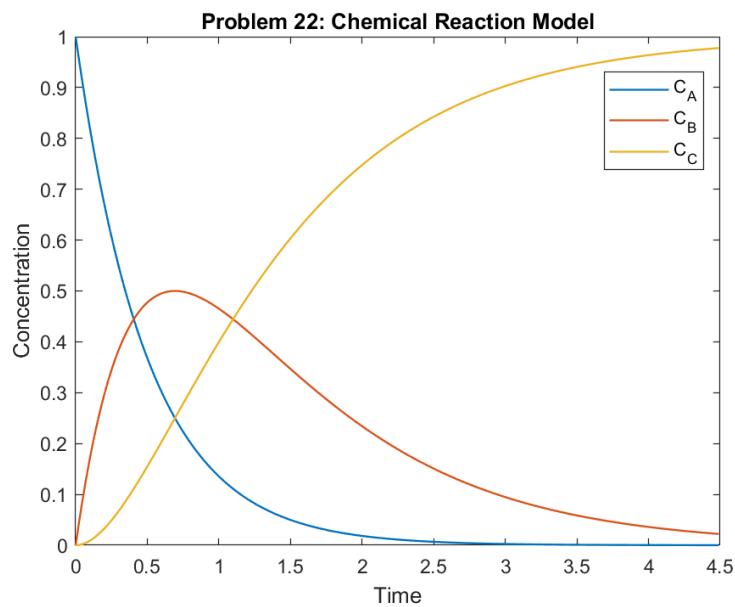


Figure 14: Chemical reaction model simulation