1 Problem 1

For a nonlinear system, a:

1. Stable node is asymptotically stable,
2. Unstable node is unstable,
3. Stable focus is asymptotically stable,
4. Unstable focus is unstable,
5. Center is stable,
6. Saddle is unstable.

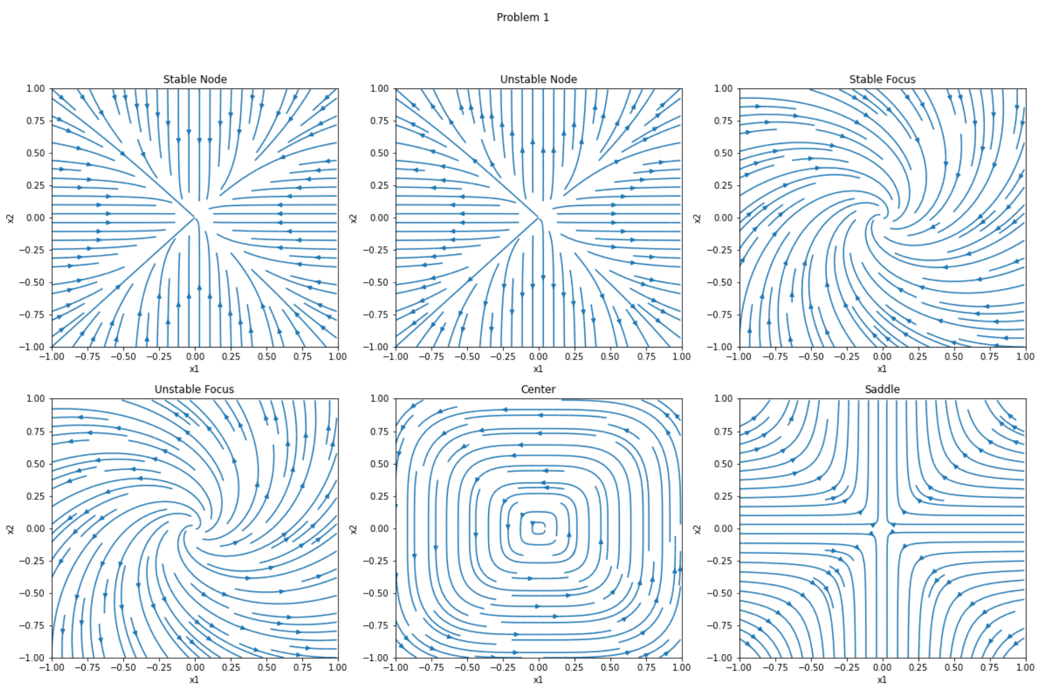

Figure 1: Problem 1 Phase Portraits

2 Problem 2

$$\dot{x} = ax^p + g(x), p \in \mathbb{N}^+ \tag{1}$$

$$|g(x)| \leq k|x|^{p+1}, \forall x : ||x|| \leq c \tag{2}$$

Note that near the origin, the ax^p term will dominate, resulting in $sign(\dot{x}) = sign(ax^p)$.

2.1 p is odd, $a < 0$

Define the following Lyapunov function,

$$V(x) = \frac{1}{2}x^2 \tag{3}$$

Then,

$$\dot{V} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = x(ax^p + g(x)) \tag{4}$$

$$x(ax^p + g(x)) = ax^{p+1} + g(x)x \tag{5}$$

Since

$$|g(x)| \leq k|x|^{p+1}, \forall x : ||x|| \leq c \tag{6}$$

In the neighborhood of the origin,

$$g(x)x \leq k|x|^{p+2} \tag{7}$$

Thus,

$$\dot{V} \leq ax^{p+1} + k|x|^{p+2} \tag{8}$$

Near the origin, since $x < 0$, the ax^{p+1} term will dominate. Since $a < 0$, $\dot{V} < 0$ near the origin. Because $\dot{V} < 0$ in the neighborhood of the origin, the origin is asymptotically stable.

2.2 p is odd, $a > 0$

If $a > 0$, $\dot{V} > 0$, thus any state near the origin will move away from the origin, making the origin unstable for this case.

2.3 p is even, $a \neq 0$

When p is even, one side of the origin will have $\dot{x}$ towards the origin, while the other side will point away from the origin, making this case unstable.

3 Problem 3

3.1 Problem 3.1

$$\dot{x_1} = -x_1 + x_1x_2 \tag{9}$$

$$\dot{x_2} = -x_2 \tag{10}$$

Define Lyapunov function,

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{11}$$

$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_1 + x_1x_2) + x_2(-x_2) = -x_1^2(1 - x_2) - x_2^2 \tag{12}$$

$$\dot{V} = -x_1^2 - x_2^2 + x_1^2x_2 \tag{13}$$

Consider the set $||x||_2 = x_1^2 + x_2^2 \leq r$. Then $|x_1| \leq r$.

$$\dot{V} = -x_1^2 - x_2^2 + x_1^2x_2 \leq -x_1^2 - x_2^2 + r|x_1||x_2| \tag{14}$$

This can be rewritten in matrix form as follows,

$$\dot{V} \leq -\begin{bmatrix}|x_1|\\|x_2|\end{bmatrix}^T \begin{bmatrix}1 & -r/2\\-r/2 & 1\end{bmatrix}\begin{bmatrix}|x_1|\\|x_2|\end{bmatrix} \tag{15}$$

$\dot{V} < 0$ for $r < 2$. Thus, the origin is asymptotically stable.
Note that the solution of the second equation is $x_2(t) = x_{2_0}e^{-t}$. Substituting this into the first equation results in the following.

$$\dot{x_1} = (x_{2_0}e^{-t} - 1)x_1 \tag{16}$$

The solution to this time-varying system does not have a finite escape time; thus, after some finite time, the coefficient of x_1 will be negative, resulting in $lim_{t\to\infty}x_1(t) = 0$. Therefore, the origin is globally asymptotically stable.

3.2 Problem 3.2

$$\dot{x_1} = -x_2 - x_1(1 - x_1^2 - x_2^2) \tag{17}$$

$$\dot{x_2} = x_1 - x_2(1 - x_1^2 - x_2^2) \tag{18}$$

Define Lyapunov function,

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{19}$$

$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_2 - x_1(1 - x_1^2 - x_2^2)) + x_2(x_1 - x_2(1 - x_1^2 - x_2^2)) \tag{20}$$

$$\dot{V} = -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2)) \tag{21}$$

$$\dot{V} = -2V(1 - 2V) \tag{22}$$

$\dot{V} < 0$ where $V < 1/2$, thus the origin is asymptotically stable. However, since $\dot{V} > 0$ for $V > 1/2$, trajectories beginning where $V > 1/2$ will never approach the origin, therefore the origin is not globally asymptotically stable.

3.3 Problem 3.3

$$\dot{x_1} = x_2(1 - x_1^2) \tag{23}$$

$$\dot{x_2} = -(x_1 + x_2)(1 - x_1^2) \tag{24}$$

Define Lyapunov function, where P is a positive definite symmetric matrix.

$$V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \tag{25}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} = (2p_{11}x_1 + 2p_{12}x_2)(x_2(1 - x_1^2)) + (2p_{12}x_1 + 2p_{22}x_2)(-(x_1 + x_2)(1 - x_1^2)) \tag{26}$$

$$\dot{V} = -2p_{12}x_1^2 + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2(p_{22} - p_{12})x_2^2 + \mathcal{O}(x) \tag{27}$$

The quadratic terms will dominate the higher order terms near the origin, resulting in $\dot{V} < 0$ in the neighborhood of the origin if the quadratic term is negative definite. The coefficients of matrix, P, can be chosen such that this is the case (ex: $p_{11} = 3, p_{12} = 1, p_{22} = 2$). Thus, the origin is asymptotically stable. The point $x = (1, 1)$ is also an equilibrium point, so the origin is not globally asymptotically stable.

3.4 Problem 3.4

$$\dot{x_1} = -x_1 - x_2 \tag{28}$$

$$\dot{x_2} = 2x_1 - x_2^3 \tag{29}$$

Define Lyapunov function,

$$V(x) = x_1^2 + \frac{1}{2}x_2^2 \tag{30}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} = 2x_1(-x_1 - x_2) + x_2(2x_1 - x_2^3) \tag{31}$$

$$\dot{V} = -2x_1^2 - x_2^4 \tag{32}$$

$\dot{V} < 0$, therefore the origin is globally asymptotically stable.

4 Problem 4

$$\dot{x}_1 = x_1\left(k^2 - x_1^2 - x_2^2\right) + x_2\left(x_1^2 + x_2^2 + k^2\right) \tag{33}$$

$$\dot{x}_2 = -x_1\left(k^2 + x_1^2 + x_2^2\right) + x_2\left(k^2 - x_1^2 - x_2^2\right) \tag{34}$$

Define Lyapunov function,

$$V(x) = x_1^2 + x_2^2 \tag{35}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = 2x_1(x_1\left(k^2 - x_1^2 - x_2^2\right) + x_2\left(x_1^2 + x_2^2 + k^2\right)) + 2x_2(-x_1\left(k^2 + x_1^2 + x_2^2\right) + x_2\left(k^2 - x_1^2 - x_2^2\right)) \tag{36}$$

$$\dot{V} = 2(x_1^2 + x_2^2)(k^2 - x_1^2 - x_2^2) \tag{37}$$

If $k = 0$,

$$\dot{V} = -2(x_1^2 + x_2^2)^2 \tag{38}$$

If $k = 0$, $\dot{V} < 0, \forall x \neq 0$, thus the origin is globally asymptotically stable for $k = 0$.
If $k \neq 0$, $\dot{V} > 0$ for $k^2 - x_1^2 - x_2^2 > 0$. In other words, if x is within a ball of radius k centered at the origin, $\dot{V} > 0$ and if x is outside the ball of radius k centered at the origin ($\|x\| < k$) , $\dot{V} < 0$. And on the circle $\|x\| = k$, $\dot{V} = 0$. Thus there exists a set of equilibrium points radius k away from the origin that is globally asymptotically stable.

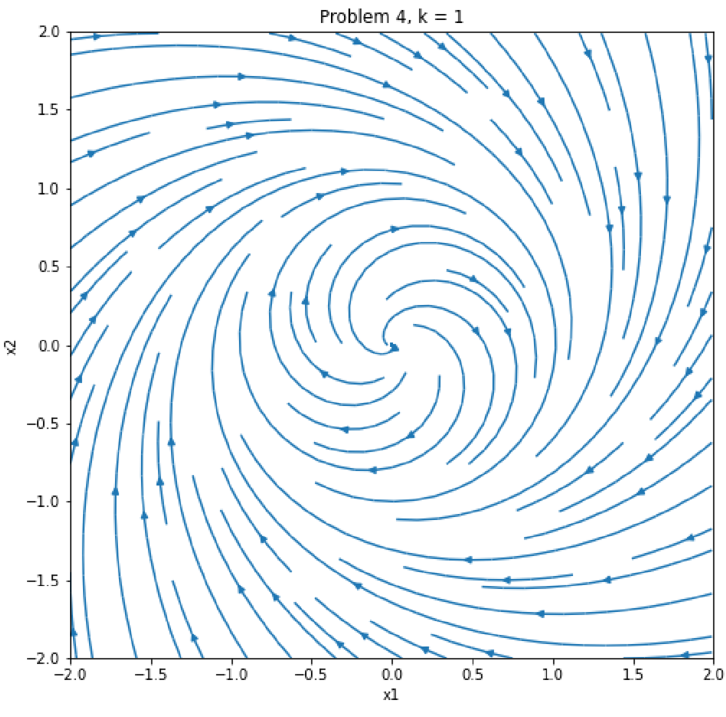

Figure 2: Problem 4 Phase Portrait, k=1

5 Problem 5

$$\dot{x_1} = x_2 \tag{39}$$

$$\dot{x_2} = x_1 - sat(2x_1 + x_2) \tag{40}$$

5.1 Problem 5a

Find the equilibrium points by setting $\dot{x} = 0$.

$$0 = x_2 \tag{41}$$

$$0 = x_1 - sat(2x_1 + x_2) = x_1 - sat(2x_1) \Rightarrow x_1 = 0 \tag{42}$$

Thus the origin is the unique equilibrium point.
Linearize the system such that it takes the form $\dot{x} = Ax$.

$$A = \frac{\partial \dot{x}}{\partial x}\bigg|_{x=0} = \begin{bmatrix} \frac{\partial \dot{x_1}}{\partial x_1} & \frac{\partial \dot{x_1}}{\partial x_2} \\ \frac{\partial \dot{x_2}}{\partial x_1} & \frac{\partial \dot{x_2}}{\partial x_2} \end{bmatrix}\Bigg|_{x=0} \tag{43}$$

$$sat(x) = \begin{cases} x, & if|x| \leq 1 \\ sign(x), & if|x| > 1 \end{cases} \tag{44}$$

Let $|2x_1 + x_2|$,

$$\frac{\partial}{\partial x} sat(2x_1 + x_2) = \frac{\partial}{\partial x}(2x_1 + x_2) \Rightarrow \frac{\partial}{\partial x_1} sat(2x_1 + x_2) = 2, \frac{\partial}{\partial x_2} sat(2x_1 + x_2) = 1 \tag{45}$$

$$A = \frac{\partial \dot{x}}{\partial x}\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{46}$$

$$det\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right) = 0 \tag{47}$$

$$0 = (-\lambda)(-1 - \lambda) - (1)(-1) = \lambda^2 + \lambda + 1 \tag{48}$$

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \tag{49}$$

For the linearization matrix, $Re(\lambda) < 0$, therefore the origin is asymptotically stable.

5.2 Problem 5b

Define Lyapunov function

$$V(x) = x_1 x_2 \tag{50}$$

$V(x) > 0$ in the first and third quadrants.

$$\dot{V}(x) = x_2 \dot{x_1} + x_1 \dot{x_2} = x_2^2 + x_1^2 - x_1 sat(2x_1 + x_2) \tag{51}$$

Suppose $2x_1 + x_2 \geq 1$, then $sat(2x_1 + x_2) = 1$.
Evaluate the Lyapunov function along the curve $x_1 x_2 = c$.

$$\dot{V} = \frac{c^2}{x_1^2} + x_1^2 - x_1 \tag{52}$$

If $c > 1$, $\dot{V} > 0 \forall x_1 \geq 1$. All trajectories in the first quadrant right of the curve $x_1 x_c$ cannot cross the curve and reach the origin. Similarly, all trajectories in the third quadrant left of the curve $x_1 x_2 = c$ cannot cross the curve and reach the origin.

5.3 Problem 5c

Since there exists a set in $\mathbb{R}$ that can not converge to the origin, the origin is not globally asymptotically stable.

6 Problem 6

6.1 Problem 6.1

$$\dot{x_1} = x_1^3 + x_1^2 x_2 \tag{53}$$

$$\dot{x_2} = -x_2 + x_2^2 + x_1 x_2 - x_1^3 \tag{54}$$

Define Lyapunov function,

$$V(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \tag{55}$$

$$\dot{V} = x_1 \dot{x_1} - x_2 \dot{x_2} \tag{56}$$

$$\dot{V} = x_1(x_1^3 + x_1^2 x_2) - x_2(-x_2 + x_2^2 + x_1 x_2 - x_1^3) \tag{57}$$

$$\dot{V} = x_1^4 + x_1^3 x_2 + x_2^2 - x_2^3 - x_1 x_2^2 + x_1^3 x_2 \tag{58}$$

$$\dot{V} = (x_1^4 + 2x_1^3 x_2 + (x_1 x_2)^2) - x_1^2 x_2^2 + x_2^2 - x_2^3 - x_1 x_2^2 \tag{59}$$

$$\dot{V} = (x_1 + x_1 x_2)^2 + x_2^2(1 - x_2 - x_1 - x_1^2) \tag{60}$$

Let $0 < c < 1$. There is a domain around the origin where,

$$0 < c < 1 - x_2 - x_1 - x_1^2 \tag{61}$$

Thus on the same domain, around the origin,

$$\dot{V} = (x_1^2 + x_1 x_2)^2 + c x_2^2 > 0 \tag{62}$$

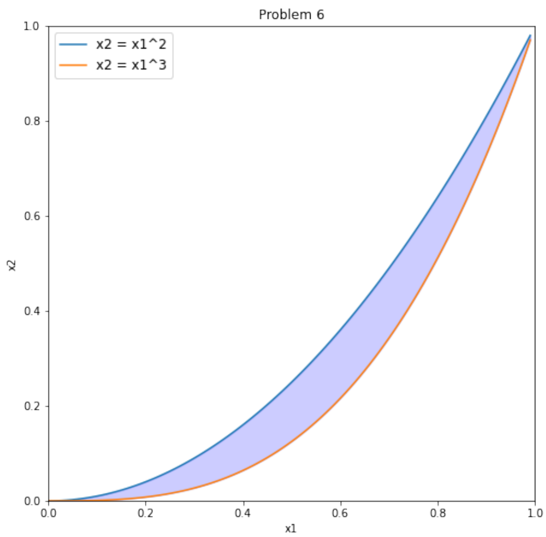

Figure 3: Problem 6

Since the origin is a boundary point of the set $G = \{x | V(x) > 0\}$ and there exists a neighborhood U of origin such that $\dot{V}(x) > 0$ for all $x \in G \cap U$, by Chetaev's Instability Theorem, the origin is an unstable equilibrium point of the system.

6.2 Problem 6.2

$$\dot{x_1} = -x_1^3 + x_2 \tag{63}$$
$$\dot{x_2} = x_1^6 - x_2^3 \tag{64}$$

Solve for the equilibrium points be solving for (x_1, x_2) such that $\dot{x_1} = 0$ and $\dot{x_2} = 0$.

$$\dot{x_1} = 0 = -x_1^3 + x_2 \Rightarrow x_2 = x_1^3 \tag{65}$$

$$\dot{x_2} = 0 = x_1^6 - x_2^3 \Rightarrow x_2^3 = x_1^6 \tag{66}$$

The equilibrium points of the system are $\{(0,0),(1,1)\}$.
Define set, Γ:

$$\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_1^3 \leq x_2 \leq x_1^2\} \tag{67}$$

On the left boundary of Γ, $x_2 = x_1^2$, $\dot{x_1} > 0$ and $\dot{x_2} = 0$ - thus all points on the left boundary will move to the right into Γ. On the right boundary of Γ, $x_2 = x_1^3$, $\dot{x_1} = 0$ and $\dot{x_2} > 0$ - thus all points on the right boundary will move upwards. Since all boundaries of Γ have trajectories going into Γ, the set Γ is positively invariant. Inside set Γ, $\dot{x_1} > 0$ and $\dot{x_2} > 0$, thus all trajectories move toward the equilibrium point $(1,1)$ and away from the origin. Since the set Γ intersects all neighborhoods of the origin, there exists no neighborhood of the origin where $\dot{x_1} < 0$ and $\dot{x_2} < 0$ for the entire neighborhood, therefore the origin can not be stable. Therefore, the origin is unstable.

7 Problem 7

$$\dot{x_1} = -x_1 + x_2 \tag{68}$$

$$\dot{x_2} = (x_1 + x_2)sinx_1 - 3x_2 \tag{69}$$

7.1 Problem 7a

Solve for the equilibrium point using $\dot{x_1} = 0$ and $\dot{x_2} = 0$.

$$\dot{x_1} = 0 = -x_1 + x_2 \Rightarrow x_1 = x_2 \tag{70}$$
$$\dot{x_2} = 0 = (x_1 + x_2)sinx_1 - 3x_2 \tag{71}$$
$$2x_1 sinx_1 = 3x_1 \tag{72}$$

$x_1 = 0$ or $2sinx_1 = 3$. But $2sinx_1 = 3$ is not possible since $sinx_1 \leq 1$, therefore $x_1 = 0$. Thus the equilibrium point is $(x_1, x_2) = (0,0)$.

7.2 Problem 7b

$$\dot{x_1} = -x_1 + x_2 \tag{73}$$

$$\dot{x_2} = (x_1 + x_2)sinx_1 - 3x_2 \tag{74}$$

The linearized system centered at the origin takes the following form:

$$\dot{x} = Ax + \dot{x}(x = 0) \tag{75}$$

Where,

$$A = \frac{\partial \dot{x}}{\partial x}\bigg|_{x=0} = \begin{bmatrix} \frac{\partial \dot{x_1}}{\partial x_1} & \frac{\partial \dot{x_1}}{\partial x_2} \\ \frac{\partial \dot{x_2}}{\partial x_1} & \frac{\partial \dot{x_2}}{\partial x_2} \end{bmatrix}\Bigg|_{x=0} \tag{76}$$

$$A = \begin{bmatrix} -1 & 1 \\ sinx_1 + (x_1 + x_2)cosx_1 & sinx_1 - 3 \end{bmatrix}\Bigg|_{x=0} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \tag{77}$$

Solve for the eigenvalues of A to assess stability.

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0 \tag{78}$$

$$det(A - \lambda I) = det\left(\begin{bmatrix} -1-\lambda & 1 \\ 0 & -3-\lambda \end{bmatrix}\right) = 0 \tag{79}$$

$$0 = (-1-\lambda)(-3-\lambda) - 1 = \lambda^2 + 4\lambda + 2 \tag{80}$$

$$\lambda = \{-2 \pm \sqrt{2}\} \tag{81}$$

$Re(\lambda) < 0$, therefore A is Hurtwitz/Stable. Because the linearization of the system about the origin is asymptotically stable, the original system is asymptotically stable.

7.3 Problem 7c

$$\dot{x_1} = -x_1 + x_2 \tag{82}$$

$$\dot{x_2} = (x_1 + x_2)sinx_1 - 3x_2 \tag{83}$$

Define Lyapunov function,

$$V(x) := \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{84}$$

$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_1 + x_2) + x_2((x_1 + x_2)sinx_1 - 3x_2) \tag{85}$$

$$\dot{V} = -x_1^2 + x_1x_2 + x_2(x_1 + x_2)sinx_1 - 3x_2^2 \tag{86}$$

$$\dot{V} = -x_1^2 + x_1x_2 + x_1x_2sinx_1 + x_2^2sinx_1 - 3x_2^2 \tag{87}$$

$$\dot{V} = -x_1^2 + x_1x_2(1 + sinx_1) - x_2^2(3 - sinx_1) \tag{88}$$

Using $sinx_1 \leq 1$,

$$\dot{V} \leq -x_1^2 + 2|x_1||x_2| - 2x_2^2 \tag{89}$$

This can be rewritten as a matrix equation,

$$\dot{V} \leq -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{90}$$

$$\dot{V} \leq -\begin{bmatrix} a|x_1| + c|x_2| & b|x_1| + d|x_2| \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} = -\begin{bmatrix} a|x_1|^2 + c|x_1||x_2| + bc|x_1||x_2| + d|x_2|^2 \end{bmatrix} \tag{91}$$

Solving for the matrix elements gives: $a = 1, d = 2, b = -1, c = -1$

$$\dot{V} \leq -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{92}$$

$$V(\dot{x}) < 0, \forall x \neq 0 \tag{93}$$

Since $V(\dot{x}) < 0, \forall x \neq 0$ and $V(\dot{x = 0}) = 0$, by Corollary 4.2, the origin is globally asymptotically stable.

8 Problem 8

By LaSalle's theorem, $\forall \epsilon > 0, \exists T > 0$ such that

$$\inf_{y \in M} \left\| x(t) - y \right\| < \epsilon, \forall t > T \tag{94}$$

Let us choose ϵ such that the neighborhood $N(p, 2\epsilon)$ of $p \in M$ contains no other points in M.
We are to prove:

$$\left\| x(t) - p \right\| < \epsilon, \forall t > T, \text{for some } p \in M \tag{95}$$

We will prove the previous statement by contradiction. Let us assume the contrary:

At $t = t_1 > T$, let $p \in M$ be a point for which $\left|x(t) - p\right| < \epsilon$. Suppose $\exists t_2 > t_1$ such that $\left|x(t_2) - p_1\right| = \epsilon$. Let $p \neq p_1$ be any other point in M.
Then, we may say the following:

$$\left|x(t_2) - p\right| = \left|x(t_2) - p_1 + p_1 - p\right| \tag{96}$$

$$\left|x(t_2) - p\right| \geq \left|x(t_2) - p_1\right| \geq 2\epsilon - \epsilon = \epsilon \tag{97}$$

This implies that

$$\inf_{y \in M} \left|x(t) - y\right| \geq \epsilon \tag{98}$$

However, the previous statement contradicts LaSalle's Theorem: $\inf_{y \in M} \left|x(t) - y\right| \leq \epsilon$. Therefore the assumption is incorrect and the following is true:

$$\left\|x(t) - p\right\| < \epsilon, \forall t > T, \text{for some } p \in M \tag{99}$$

This implies that as $t \to \infty$, $x(t) \to p$ for some $p \in M$.

9 Problem 9

9.1 Problem 9a

Claim: $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if z=0 is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

We will prove this claim by contradiction. Suppose the equilibrium point $z = 0$ is not isolated. Then there exists a point $\bar{z} \neq 0$, that is arbitrarily close to 0, such that $\hat{f}(z) = 0$.

Let us define the transformation of this equilibrium point, $\bar{x} := T^{-1}(\bar{z})$. Then, $f(\bar{x}) = \left[\partial T / \partial x\right]^{-1} \hat{f}(\bar{z}) = 0$. In other words, $\bar{x}$ is an equilibrium point. Since the inverse transformation, $T_{-1}(\cdot)$, is continuous, we can make $\bar{x}$ arbitrarily close to the origin, which contradicts the fact that the origin is an isolated equilibrium point. Thus, the assumption is incorrect and the claim is true: $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if z=0 is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

9.2 Problem 9b

9.2.1 Stable Equilibrium Point

Suppose $x = 0$ is a stable equilibrium point. Then, $\forall \epsilon_1 > 0, \exists \delta_1 > 0$ such that

$$\left|x(0)\right| < \delta_1 \Rightarrow \left|x(t)\right| < \epsilon_1, \forall t \geq 0 \tag{100}$$

Since the transformation, $T(\cdot)$ is continuous, there is $\delta_2 > 0$ such that

$$|z| < \delta_2 \Rightarrow |x| < \delta \tag{101}$$

Thus,

$$\left|z(0)\right| < \delta_2 \Rightarrow \left|x(0)\right| < \delta \Rightarrow \left|x(t)\right| < \epsilon_2, \forall t \geq 0 \tag{102}$$

Thus, $z = 0$ is a stable equilibrium point.

9.2.2 Asymptotically Stable Equilibrium Point

Suppose $x = 0$ is an asymptotically stable equilibrium point, then

$$x(t) \to 0 \text{ as } t \to \infty \tag{103}$$

$\forall \epsilon_1 > 0, \exists T_1 > 0$ such that $\left|x(t)\right| < \epsilon_1$ for all $t > T_1$. Since $T(\cdot)$ is continuous, $\forall \epsilon_2 > 0, \exists r > 0$ such that

$$|x| < r \Rightarrow |z| < \epsilon_2 \tag{104}$$

There exists $T_2 > 0$ such that

$$\left|x(t)\right| < r, \forall t > T_2 \Rightarrow \left|z(t)\right| < \epsilon_2, \forall t > T_2 \tag{105}$$

Thus,

$$z(t) \rightarrow 0 \text{ as } t \rightarrow \infty \tag{106}$$

Thus, $z = 0$ is asymptotically stable. The converse statement is also proven without loss of generality.

$$x = 0 \text{ is stable} \leftrightarrow z = 0 \text{ is stable} \tag{107}$$

9.2.3 Unstable Equilibrium Point

If an equilibrium point is not stable, it is unstable. Thus the contrapositive of the previously proven statement, must also be true:

$$x = 0 \text{ is unstable} \leftrightarrow z = 0 \text{ is unstable} \tag{108}$$

10 Problem 10

$$\dot{x_1} = x_2 \tag{109}$$

$$\dot{x_2} = -x_1 + \frac{1}{3}x_1^3 - x_2 \tag{110}$$

To solve for the equilibrium points, set $\dot{x} = 0$.

$$0 = x2 \tag{111}$$

$$0 = -x_1 + \frac{1}{3}x_1^3 - 0 \tag{112}$$

$$3x_1 = x_1^3 \tag{113}$$

Either $x_1 = 0$ or $x_1^2 = 3$.
Thus the equilibrium points are $\{(0,0), (-\sqrt{3}, 0), (+\sqrt{3}, 0)\}$.
Define the Lyapunov function

$$V(x) := \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2 \tag{114}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} \tag{115}$$

$$\dot{V} = (\frac{3}{2}x_1 - \frac{1}{3}x_1^3 + \frac{1}{2}x_2)(x_2) + (\frac{1}{2}x_1 + x_2)(-x_1 + \frac{1}{3}x_1^3 - x_2) \tag{116}$$

$$\dot{V} = -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{3}x_1^4 \tag{117}$$

Near the origin, the 2nd order terms will dominate the 4th order term, resulting in $\dot{V} < 0$ in a small neighborhood around the origin. Thus, the origin is asymptotically stable. A small neighborhood around the origin where the system converges to the origin is seen in the phase portrait and contour plot shown. The set $\{V(x) < 9/8\}$ is approximately the largest estimate of the region of attraction that can be generated from this Lyapunov function, within $\pm 1/8$.

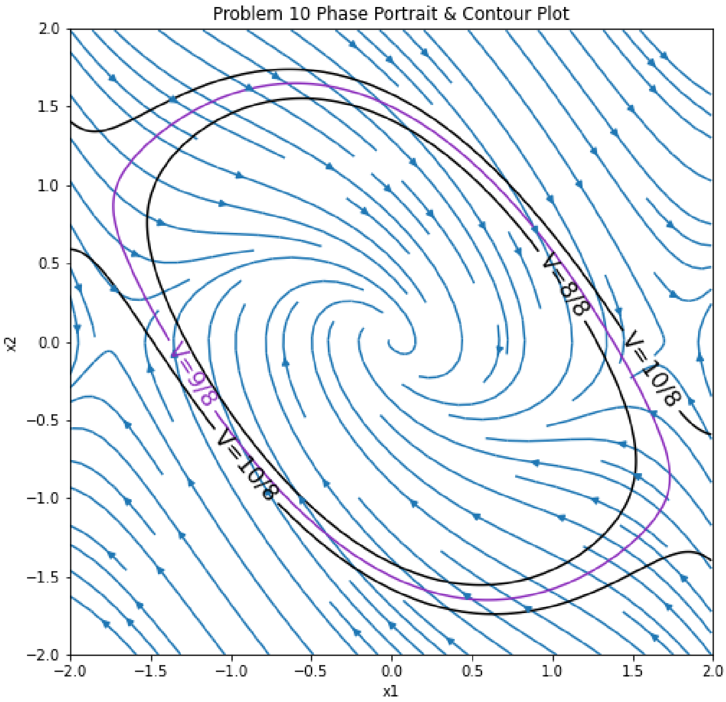

Figure 4: Problem 10: Phase Portrait and Contour Plot

11 Problem 11

$$\dot{x_1} = x_2 \tag{118}$$
$$\dot{x_2} = -x_1 - x_2 sat(x_2^2 - x_3^2) \tag{119}$$
$$\dot{x_3} = x_3 sat(x_2^2 - x_3^2) \tag{120}$$

To solve for the set of equilibrium points, set $\dot{x} = 0$:

$$\dot{x1_{eq}} = 0 = x2_{eq} \Rightarrow x2_{eq} = 0 \tag{121}$$

$$\dot{x2_{eq}} = 0 = -x2_{eq} - x2_{eq} sat(x2_{eq}^2 - x3_{eq}^2) = -x1_{eq} - 0 \Rightarrow x1_{eq} = 0 \tag{122}$$

$$\dot{x3_{eq}} = 0 = x3_{eq} sat(x2_{eq}^2 - x3_{eq}^2) = x3_{eq} sat(-x3_{eq}^2) \tag{123}$$

$$sat(-x3_{eq}) = 0 \text{ when } x_{3eq} = 0$$

Thus the origin is the unique equilibrium point.
Define the Lyapunov function,

$$V(x) = x^T x = 2x_1 + 2x_2 + 2x_3 \tag{124}$$

$$\dot{V} = 2x_1\dot{x_1} + 2x_2\dot{x_2} + 2x_3\dot{x_3} \tag{125}$$

$$\dot{V} = 2x_1x_2 + 2x_2(-x_1 - x_2 sat(x_2^2 - x_3^2)) + 2x_3 sat(x_2^2 - x_3^2) \tag{126}$$

$$\dot{V} = -2(x_2^2 - x_3^2) sat(x_2^2 - x_3^2) \tag{127}$$

$$sign(x_2^2 - x_3^2) = sign(sat(x_2^2 - x_3^2)) \Rightarrow sign((x_2^2 - x_3^2) sat(x_2^2 - x_3^2)) = +1, 0 \tag{128}$$

$$\Rightarrow \dot{V}(x) \leq 0 \tag{129}$$

$\dot{V}(x) \leq 0$, therefore the origin is asymptotically stable. For the origin to be globally asymptotically stable, $\dot{V} = 0$ must only occur at the origin.

$$\dot{V} = 0 \rightarrow -2(x_2^2 - x_3^2)sat(x_2^2 - x_3^2) = 0 \Rightarrow x_2^2 - x_3^2 = 0 \Rightarrow x_2^2 = x_3^2 \tag{130}$$

$$x_2^2 = x_3^2 \Rightarrow \dot{x_3} = x_3 sat(0) = 0 \Rightarrow x_3 = constant \tag{131}$$

$$x_3 = constant \text{ \& } x_2^2 = x_3^2 \Rightarrow x_2 = constant \tag{132}$$

$$x_2 = constant \Rightarrow \dot{x_2} = 0 \Rightarrow -x_1 - x_2 sat(x_2^2 - x_3^2) = 0 \Rightarrow x_1 = 0 \tag{133}$$

$$x_1 = 0 \Rightarrow \dot{x_1} = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 0 \tag{134}$$

The origin is the only point where $\dot{V} = 0$, therefore, by LaSalle's Theorem (Corollary 4.2), the origin is globally asymptotically stable.

12 Problem 12

$$\dot{x_1} = -kh(x)x_1 + x_2 \tag{135}$$
$$\dot{x_2} = -h(x)x_2 - x_1^3 \tag{136}$$

$$D = \{x \in \mathbb{R} \mid \|x\|_2 < 1\} \tag{137}$$

Define the Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \tag{138}$$

$$\dot{V} = x_1^3(-kh(x)x_1 + x_2) + x_2(-h(x)x_2 - x_1^3) \tag{139}$$

$$\dot{V} = -x_1^4 kh(x) - x_2^2 h(x) \tag{140}$$

Define the following terms:

$$T_1 := -x_1^4 kh(x) \tag{141}$$
$$T_2 := -x_2^2 h(x) \tag{142}$$

12.1 $k > 0, h > 0, \forall x \in D$

$$T_1 < 0, T_2 < 0, \forall x \in \{D - \{0\}\}$$

$$\dot{V}(x) < 0, \forall x \in \{D - \{0\}\} \Rightarrow \text{The origin is asymptotically stable}$$

12.2 $k > 0, h > 0, \forall x \in \mathbb{R}^2$

$$T_1 < 0, T_2 < 0, \forall x \in \mathbb{R}^2$$

$$\dot{V}(x) < 0, \forall x \in \mathbb{R}^2 \Rightarrow \text{The origin is globally asymptotically stable}$$

12.3 $k > 0, h < 0, \forall x \in D$

$$T_1 > 0, T_2 > 0, \forall x \in \{D - \{0\}\}$$

$$\dot{V}(x) > 0, \forall x \in \{D - \{0\}\} \Rightarrow \text{By Chetaev's Theorem, the origin is unstable}$$

12.4 $k > 0, h = 0, \forall x \in D$

$$T_1 = 0, T_2 = 0, \forall x \in D$$

$$\dot{V}(x) = 0, \forall x \in D \Rightarrow \text{The origin is unstable}$$

12.5 $k = 0, h > 0, \forall x \in D$

$$T_1 = 0, T_2 < 0, \forall x \in \{D - \{0\}\}$$

$$\dot{V}(x) < 0, \forall x \in \{D - \{0\}\} \Rightarrow \text{The origin is asymptotically stable}$$

12.6 $k = 0, h > 0, \forall x \in \mathbb{R}^2$

$$T_1 = 0, T_2 < 0, \forall x \in \{\mathbb{R}^2 - \{0\}\}$$

$$\dot{V}(x) < 0, \forall x \in \{\mathbb{R}^2 - \{0\}\} \Rightarrow \text{The origin is globally asymptotically stable}$$

13 Problem 13

$$\dot{x_1} = x1 \tag{143}$$

$$\dot{x_2} = -asinx_1 - kx_1 - dx_2 - cx_3 \tag{144}$$

$$\dot{x_3} = -x_3 + x_2 \tag{145}$$

$$k > a > 0, c > 0, d > 0, p > 0$$

Define the Lyapunov function

$$V(x) = 2a \int_0^{x_1} sinydy + kx_1^2 + x_2^2 + px_3^2 \tag{146}$$

$$\int_0^{x_1} sinydy = -cosy\Big|_{y=0}^{x_1} = 1 - cosx_1 \tag{147}$$

$$V(x) = 2a(1 - cosx_1) + kx_1^2 + x_2^2 + px_3^2 \tag{148}$$

$$cosx_1 \leq 1, a > 0 \Rightarrow 2a(1 - cosx_1) \geq 0$$

$$k > 0 \Rightarrow kx_1^2 \geq 0$$

$$x_2^2 \geq 0$$

$$p > 0 \Rightarrow px_3^2 \geq 0$$

All terms of $V(x)$ are positive definite, therefore the Lyapunov function is positive definite: $V(x) \geq 0$. Additionally, $\lim_{\|x\| \to \infty} V(x) = \infty$, therefore the Lyapunov function is radially unbounded.

$$\dot{V} = (2asinx_1 + 2kx_1)\dot{x_1} + 2x_2\dot{x_2} + 2px_3\dot{x_3} \tag{149}$$

$$\dot{V} = (2asinx_1 + 2kx_1)x_2 + 2x_2(-asinx_1 - kx_1 - dx_2 - cx_3) + 2px_3(-x_3 + x_2) \tag{150}$$

$$\dot{V} = 2(-dx_2^2 - cx_2x_3 - px_3^2 + px_2x_3) \tag{151}$$

Let $c = p$,

$$\dot{V} = -2dx_2^2 - 2px_3^2 \Rightarrow \dot{V}(x) \leq 0, \forall x \in \mathbb{R}^3 \tag{152}$$

Solving for the set of points where $\dot{V} = 0$,

$$\dot{V} = 0 = -2dx_2^2 - 2px_3^2 \Rightarrow x_2 = 0, x_3 = 0 \tag{153}$$

$$x_2 = 0 \Rightarrow \dot{x_2} = 0 \tag{154}$$

$$\dot{x_2} = 0 \Rightarrow -asinx_1 - kx_1 - dx_2 - cx_3 = -asinx_1 - kx_1 = 0 \tag{155}$$

$$asinx_1 + kx_1 = 0 \Rightarrow x_1 = 0 \tag{156}$$

Thus, the origin is the unique point where $\dot{V} = 0$.
$V(x)$ is positive definite ($V(x) > 0, \forall x \in \{\mathbb{R}^3 - \{0\}\}$); $V(x)$ is radially unbounded ($\lim_{\|x\|\to\infty} V(x) = \infty$); $\dot{V}(x) \leq 0, \forall x \in \mathbb{R}^3$; $x = 0$ is the unique point in $\mathbb{R}^3$ where $\dot{V}(x) = 0$, therefore, by LaSalle's Theorem (Corollary 4.2), the origin is globally asymptotically stable.

14 Problem 14

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}| \tag{157}$$

Define state variables, x_1, x_2

$$x_1 := y - Mg/k \tag{158}$$

$$x_2 := \dot{y} \tag{159}$$

$$\dot{x_1} = x_2 \tag{160}$$

$$\dot{x_2} = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| \tag{161}$$

Define the Lyapunov function,

$$V(x) = ax_1^2 + bx_2^2, a, b > 0 \tag{162}$$

The Lyapunov function defined is positive definite: $V(x) > 0, \forall x \in \{\mathbb{R}^2 - \{0\}\}$ and radially unbounded: $\lim_{\|x\|\to\infty} V(x) = \infty$.

$$\dot{V}(x) = 2ax_1\dot{x_1} + 2bx_2\dot{x_2} \tag{163}$$

$$\dot{V}(x) = 2ax_1x_2 + 2bx_2(-\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|) \tag{164}$$

To cancel terms, define $a := k/2$ and $b := M/2$.

$$\dot{V} = -c_1x_2^2 - c_2x_2^2|x_2| \tag{165}$$

$$\dot{V} \leq 0, \forall x \in \mathbb{R}^2 \tag{166}$$

Solve for the set of points where $\dot{V} = 0$

$$\dot{V} = 0 = -c_1x_2^2 - c_2x_2^2|x_2| \Rightarrow (c_1 + c_2|x_2|)x_2^2 = 0 \tag{167}$$

$$(c_1 + c_2|x_2|)x_2^2 = 0 \Rightarrow x_2 = 0 \tag{168}$$

$$x_2 = 0 \Rightarrow \dot{x_2} = 0 \Rightarrow -\frac{k}{m}x_1 - 0 = 0 \Rightarrow x_1 = 0 \tag{169}$$

Thus, $x = 0$ is the unique point where $\dot{V} = 0$.
$V(x)$ is positive definite ($V(x) > 0, \forall x \in \{\mathbb{R}^2 - \{0\}\}$); $V(x)$ is radially unbounded ($\lim_{\|x\|\to\infty} V(x) = \infty$); $\dot{V}(x) \leq 0, \forall x \in \mathbb{R}^2$; $x = 0$ is the unique point in $\mathbb{R}^2$ where $\dot{V}(x) = 0$, therefore, by LaSalle's Theorem (Corollary 4.2), the origin is globally asymptotically stable.

15 Problem 15

$$\dot{x} = (A - BR^{-1}B^TP)x \tag{170}$$

$$PA + A^TP + Q - PBR^{-1}B^TP = 0 \tag{171}$$

$$P = P^T > 0, R = R^T > 0, Q = Q^T \geq 0 \tag{172}$$

15.1 Problem 15.1

Define Lyapunov function

$$V(x) = x^TPx \tag{173}$$

From Nonlinear Systems - Khalil, 3rd ed, page 135,

$$\dot{V}(x) = x^TP\dot{x} + \dot{x}^TPx = x^T(PA + A^TP)x = -x^TMx \tag{174}$$

$$M = P(A - BR^{-1}B^TP) + (A - BR^{-1}B^TP)^TP = PA - PBR^{-1}B^TP + A^TP - P^TBR^{-1}B^TP \tag{175}$$

$$M = -Q - PBR^{-1}B^TP \tag{176}$$

$$\dot{V} = -x^T(Q + PBR^{-1}B^TP)x \tag{177}$$

Given $P > 0, R > 0$, B can be chosen such that $BR^{-1}B > 0$.

$$BR^{-1}B > 0 \Rightarrow PBR^{-1}B^TP \geq 0 \tag{178}$$

Given $Q > 0$

$$\Rightarrow Q + PBR^{-1}B^TP > 0 \Rightarrow \dot{V} = -x^T(Q + PBR^{-1}B^TP)x < 0 \tag{179}$$

Since $\dot{V} < 0$ if $Q > 0$, $x = 0$ is globally asymptotically stable if $Q > 0$.

15.2 Problem 15.2

$$Q = C^TC \geq 0 \tag{180}$$

$$C^TC + PBR^{-1}B^TP \geq 0 \tag{181}$$

$$\dot{V}(x) = 0 \Rightarrow x^T(Q + PBR^{-1}B^TP)x = 0 \Rightarrow x^TC^TCx = 0 and x^T(PBR^{-1}B^TP)x = 0 \tag{182}$$

$$x^TC^TCx = 0 \Rightarrow Cx = 0 \tag{183}$$

$$x^T(PBR^{-1}B^TP)x = 0 \Rightarrow R^{-1}B^TP)x = 0 \tag{184}$$

$$\dot{x} = (A - BR^{-1}B^TP)x \Rightarrow \dot{x} = (A - 0)x \Rightarrow \dot{x} = Ax \tag{185}$$

Solving $\dot{x} = Ax$,

$$x(t) = Cx_0exp(At) \tag{186}$$

$$Cx = 0 \Rightarrow x(t) = Cx_0exp(At) = 0 \Rightarrow x(t) = 0 \text{ if and only if } x(0) = 0 \Rightarrow \dot{V}(x) = 0 \text{ if and only if } x(0) = 0 \tag{187}$$

Since $\dot{V} < 0 \forall x \neq 0$ and $V(x = 0) = 0$, by Corollary 4.2, the origin $x = 0$ is globally asymptotically stable.

16 Problem 16

$$\dot{x_1} = -x_1 \tag{188}$$

$$\dot{x_2} = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2 \tag{189}$$

16.1 Problem 16a

To find the set of equilibrium points, set $\dot{x} = 0$.

$$\dot{x}_{1eq} = 0 = -x_{1_{eq}} \Rightarrow x_{1_{eq}} = 0 \tag{190}$$

$$\dot{x}_{2eq} = (x_{1_{eq}}x_{2_{eq}} - 1)x_{2_{eq}}^3 + (x_{1_{eq}}x_{2_{eq}} - 1 + x_{1_{eq}}^2)x_{2_{eq}} = (0-1)x_{2_{eq}}^3 + (0 - 1 + 0^2)x_{2_{eq}} \tag{191}$$

$$0 = -x_{2_{eq}}^3 - x_{2_{eq}} \Rightarrow x_{2_{eq}}^3 = -x_{2_{eq}} \tag{192}$$

$$x_{2_{eq}} = 0 \text{ or } x_{2_{eq}} = -1 \text{ but } x_2 \in \mathbb{R} \Rightarrow x_{2_{eq}} = 0$$

Thus, the origin, $x = 0$, is the unique equilibrium point.

16.2 Problem 16b

$$\dot{x} = \left[\frac{\partial \dot{x}}{\partial x}\right]_{x=0} x \tag{193}$$

$$\left[\frac{\partial \dot{x}}{\partial x}\right]_{x=0} = \begin{bmatrix} \frac{\partial \dot{x_1}}{\partial x_1} & \frac{\partial \dot{x_1}}{\partial x_2} \\ \frac{\partial \dot{x_2}}{\partial x_1} & \frac{\partial \dot{x_2}}{\partial x_2} \end{bmatrix}_{x=0} = \begin{bmatrix} -1 & 0 \\ x_2^4 + x_2^2 + 2x_1x_2 & 4x_1x_2^3 - 3x_2^2 + 2x_1x_2 - 1 + x_1^2 \end{bmatrix}_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{194}$$

Solve for the eigenvalues of the matrix

$$det\left(\left[\frac{\partial \dot{x}}{\partial x}\right]_{x=0} - \lambda I\right) = 0 \tag{195}$$

$$det\left(\begin{bmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda = -1 \tag{196}$$

$$Re(\lambda) < 0 \tag{197}$$

The matrix is Hurwitz/Stable, therefore, the origin is asymptotically stable.

16.3 Problm 16c

$$\Gamma = \{x \in \mathbb{R}^2 | x_1x_2 \geq 2\} \tag{198}$$

Define Lyapunov function,

$$V(x) = x_1x_2 \tag{199}$$

$$\dot{V}(x) = x2(-x_1) + x_1[(x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_2^2)x_2] \tag{200}$$

$$\dot{V}(x) = -2x_1x_2 + (x_1x_2)^2x_2^2 + (x_1x_2)^2 \tag{201}$$

Evaluate the Lyapunov function along the boundary of Γ

$$\dot{V}(x)|_{x_1x_2=2} = -4 + 4x_2^2 + 4 = 4x_2^2 \Rightarrow \dot{V}(x)|_{x_1x_2=2} > 0 \tag{202}$$

Since the Lyapunov function is positive definite along the boundaries of the set, Γ is a positive invariant set of the given system.

16.4 Problem 16d

Since Γ is a subset of the 2-dimensional space, $\Gamma \subset \mathbb{R}^2$, and is positive invariant, and Γ does not intersect the origin, $\Gamma \cap \{0\} = \{\}$, trajectories that begin in the set Γ will not go towards the origin, the origin $x = 0$ is not globally asymptotically stable.

17 Problem 17

17.1 Problem 17.1

$$\dot{x_1} = -x_1 + x_2 \tag{203}$$

$$\dot{x_2} = (x_1 + x_2)sinx_1 - 3x_2 \tag{204}$$

$$A = \left[\frac{\partial \dot{x}}{\partial x}\right]_{x=0} = \begin{bmatrix} -1 & 1 \\ sinx_1 - x_1cosx_1 & sinx_1 - 3 \end{bmatrix}_{x=0} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} = 0 \tag{205}$$

Solve for the eigenvalues to assess stability

$$det\left(\begin{bmatrix} -1-\lambda & 1 \\ 0 & -3-\lambda \end{bmatrix}\right) \tag{206}$$

$$0 = (-1-\lambda)(-3-\lambda) - 1 = \lambda^2 + 4\lambda + 2 \tag{207}$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2(1)} = -2 \pm \sqrt{2} \tag{208}$$

$$Re(\lambda) > 0$$

therefore A is Hurwitz, therefore the origin is asymptotically stable.

Define Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{209}$$

$$\dot{V}(x) = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_1 + x_2) + x_2((x_1 + x_2)sinx_1 - 3x_2) \tag{210}$$

$$\dot{V}(x) = -x_1^2 + x_1x_2(1 + sinx_1) - (3 - sinx_1)x_2^2 \tag{211}$$

$$sinx_1 \leq 1 \Rightarrow 1 + sinx_1 \leq 2 \Rightarrow x_1x_2(1 + sinx_1) \leq 2|x_1||x_2|$$

$$sinx_1 \leq 1 \Rightarrow -sinx_1 \geq -1 \Rightarrow 3 - sinx_1 \geq 2 \Rightarrow -(3 - sinx_1)x_2^2 \leq -2x_2^2$$

$$\dot{V} \leq -x_1^2 + 2|x_1||x_2| - 2x_2^2 = -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{212}$$

Thus, $\dot{V} < 0, \forall x \neq 0$; therefore, the origin $x = 0$ is globally asymptotically stable.

17.2 Problem 17.2

$$\dot{x_1} = -x_1^3 + x_2 \tag{213}$$

$$\dot{x_2} = -ax_1 - bx_2, a, b > 0 \tag{214}$$

$$A = \left[\frac{\partial \dot{x}}{\partial x}\right]_{x=0} = \begin{bmatrix} -3x_1^2 & 1 \\ -a & -b \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \tag{215}$$

$$det\left(\begin{bmatrix} -\lambda & 1 \\ -a & -b-\lambda \end{bmatrix} = 0\right) \tag{216}$$

$$(-\lambda)(-b-\lambda) - (1)(-a) = 0 \tag{217}$$

$$\lambda^2 + b\lambda + a = 0 \tag{218}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4(1)(a)}}{2(1)} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4a} \tag{219}$$

If $b^2 > 4a$, then $\sqrt{b^2 - 4a} < b$ since $a > 0$. If $b^2 < 4a$, then $Re(\lambda) = -b/2 < 0$ since $b > 0$. Thus, $Re(\lambda) < 0$ and A is Hurwitz. Therefore, the origin is asymptotically stable.

Define Lyapunov function

$$V = \frac{1}{2}(x_1^2 + \alpha x_2^2), \alpha > 0 \tag{220}$$

$$\dot{V}(x) = x_1\dot{x_1} + \alpha x_2 \dot{x_2} \tag{221}$$

$$\dot{V} = x_1(-x_1^3 + x_2) + \alpha x_2(-ax_1 - bx_2) \tag{222}$$

$$\dot{V} = -x_1^4 + x_1x_2 - a\alpha x_1 x_2 - b\alpha x_2^2 \tag{223}$$

Let $\alpha = 1/a$

$$\dot{V} = -x_1^4 - \frac{b}{a}x_2^2 < 0, \forall x \neq 0 \tag{224}$$

Since $\dot{V} < 0, \forall x \neq 0$, the origin $x = 0$ is globally asymptotically stable.

18 Problem 18

$$\dot{x} = -\frac{x}{1+t} \tag{225}$$

$$\frac{dx/dt}{x} = -\frac{1}{1+t} \tag{226}$$

$$\frac{1}{x}dx = -\frac{1}{1+\tau}d\tau \tag{227}$$

$$\int_{x(t_0)}^{x(t)} \frac{1}{x}dx = -\int_{t_0}^{t} \frac{1}{1+\tau}d\tau \tag{228}$$

$$\ln\left|x(t)\right| - \ln\left|x(t_0)\right| = -[\ln|t+1| - \ln|t_0+1|] \tag{229}$$

$$\ln\frac{\left|x(t)\right|}{\left|x(t_0)\right|} = -\ln\frac{t+1}{t_0+1} \tag{230}$$

$$\ln\frac{\left|x(t)\right|}{\left|x(t_0)\right|} = \ln\frac{t_0+1}{t+1} \tag{231}$$

$$\frac{x(t)}{x(t_0)} = \frac{t_0+1}{t+1} \tag{232}$$

$$x(t) = x(t_0)\frac{t_0+1}{t+1} \tag{233}$$

Given, $x(t_0) < \epsilon$,

$$x(t) < \epsilon\frac{t_0+1}{t+1} \tag{234}$$

Thus $\delta(\epsilon, t_0)$ can be defined as $\delta := \epsilon\dfrac{t_0+1}{t+1}$ such that the following statement is true:

$$\left\|x(t_0)\right\| < \delta \rightarrow \left\|x(t)\right\| < \epsilon, \forall t \geq t_0 \geq 0 \tag{235}$$

Additionally, $\lim_{t\to\infty} x(t) = 0$. By definition, the origin is a stable equilibrium point; however it is not uniformly stable since δ must be a function of t_0.

19 Problem 19

19.1 Problem 19.1

$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x, |\alpha(t)| \leq 1 \tag{236}$$

Define Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{237}$$

$$\dot{V}(x) = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_1 + \alpha x)2) + x_2(\alpha x_1 - 2x_2) \tag{238}$$

$$\dot{V} = -x_1^2 + \alpha x_1 x_2 + \alpha x_1 x_2 - 2x_2^2 \tag{239}$$

$$\dot{V} \leq -x_1^2 + 2|x_1||x_2| - 2x_2^2 \tag{240}$$

$$\dot{V} \leq -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{241}$$

$$det\left(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0 \tag{242}$$

$$(1-\lambda)(2-\lambda) - 1 = 0 \rightarrow \lambda^2 - 3\lambda + 1 = 0 \lambda = \{\frac{1}{2}(3+\sqrt{5})\} \Rightarrow \lambda = \{2.62, 0.38\} \tag{243}$$

From the properties of a positive definite matrix,

$$\dot{V} \leq -\lambda_{min}(x_1^2 + x_2^2) = -0.38(x_1^2 + x_2^2) \Rightarrow \dot{V} < 0 \forall x \neq 0 \tag{244}$$

By Theorem 4.10, the origin is exponentially stable.

19.2 Problem 19.2

$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x \tag{245}$$

Define Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 = x_2^2) \tag{246}$$

$$\dot{V}(x) = x_1\dot{x}_{1x2}\dot{x_2} = x_1(-x_1 + \alpha x_2) + x_2(-\alpha x_1 - 2x_2) \tag{247}$$

$$\dot{V}(x) = -x_1^2 - 2x_2^2 \tag{248}$$

$$\dot{V} \leq -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{249}$$

$$det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0\right) \tag{250}$$

$$(1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = \{1, 2\} \tag{251}$$

$$\dot{V} \leq -\lambda_{min} = -1 \tag{252}$$

By Theorem 4.10, the origin is exponentially stable.

19.3 Problem 19.3

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x, \alpha(t) \geq 2 \tag{253}$$

Define Lyapunov function

$$V(x) := x^T P x \tag{254}$$

Where, $P = P^T$ and $det(P) > 0$,

$$P := \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{255}$$

$$V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \tag{256}$$

$$\dot{V}(x) = 2p_{11}x_1\dot{x_1} + 2p_{12}x_2\dot{x_1} + 2p_{12}x_1\dot{x_2} + 2p_{22}x_2\dot{x_2} \tag{257}$$

$$\dot{V} = 2p_{11}x_1x_2 + 2p_{12}x_2^2 - 2p_12x_1^2 - 2\alpha p_{12}x_1x_2 - 2p_{22}x_1x_2 - 2\alpha p_{22}x_2^2 \tag{258}$$

$$\dot{V} = -2p_{12} + 2[p_{11} - p_{22} - \alpha p_{12}]x_1x_2 + 2[p_{12} - \alpha p_{22}]x_2^2 \tag{259}$$

Choose $p_{11} = p_{22} = p > 1$ and $p_{12} = 1$,

$$\dot{V} = -2px_1^2 - 2\alpha p x_1x_2 + 2(1 - \alpha p)x_2^2 \tag{260}$$

Define upper bound k, $k > |\alpha(t)|$

$$\dot{V} \leq -2x_1^2 + 2k|x_1||x_2| - 2(p-1)x_2^2 \tag{261}$$

$$\dot{V} \leq -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 2 & -k \\ -k & 2(p-1) \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{262}$$

We want the determinant of the linearization matrix to be strictly positive.

$$det\left(\begin{bmatrix} 2 & -k \\ -k & 2(p-1) \end{bmatrix}\right) > 0 \Rightarrow 2 \cdot 2(p-1) - k^2 > 0 \Rightarrow p > \frac{k^2}{4} + 1 \tag{263}$$

p can be chosen such that $p > \frac{k^2}{4} + 1$, so that the linearization matrix is positive definite. Thus, by Theorem 4.10, the origin $x = 0$ is exponentially stable.

19.4 Problem 19.4

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x \tag{264}$$

Define Lyapunov function

$$V(x) := x^T P x \tag{265}$$

Where, $P = P^T$ and $det(P) > 0$,

$$P := \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{266}$$

$$V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \tag{267}$$

$$\dot{V}(x) = 2p_{11}x_1\dot{x_1} + 2p_{12}x_2\dot{x_1} + 2p_{12}x_1\dot{x_2} + 2p_{22}x_2\dot{x_2} \tag{268}$$

$$\dot{V} = 2p_{11}x_1\dot{x_1} + 2p_{12}x_2\dot{x_1} + 2p_{12}x_1\dot{x_2} + 2p_{22}x_2\dot{x_2} \tag{269}$$

$$\dot{V} = -2p_{11}x_1^2 - 2p_{12}x_1x_2 + 2\alpha p_{12}x_1^2 - 4p_{12}x_1x_2 + 2\alpha p_{22}x_1x_2 - 4p_{22}x_2^2 \tag{270}$$

$$\dot{V} = -2[p_{11} - \alpha p_{12}]x_1^2 + 2[-p + \alpha p_{22} - 2p_{12}]x_1x_2 - 4p_{22}x_2^2 \tag{271}$$

Choose $p_{12} = 0$ and $p_{22} = 1$,

$$\dot{V} = -2p_{11}x_1^2 + 2\alpha x_1 x_2 - 4x_2^2 \tag{272}$$

$$\dot{V} \leq - \begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 2p_{11} & -k \\ -k & 4 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \tag{273}$$

We want the determinant of the linearization matrix to be strictly positive.

$$det\left(\begin{bmatrix} 2p_{11} & -k \\ -k & 4 \end{bmatrix}\right) > 0 \Rightarrow 8p_{11} - k^2 > 0 \Rightarrow p_{11} > \frac{k^2}{8} \tag{274}$$

p can be chosen such that $p_{11} > \dfrac{k^2}{8}$ so that the linearization matrix is positive definite. Thus, by Theorem 4.10, the origin $x = 0$ is exponentially stable.

20 Problem 20

$$\dot{x_1} = x_2 \tag{275}$$

$$\dot{x_2} = 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 \tag{276}$$

20.1 Problem 20a

Check that $x = [t1]^T$ is a solution.

$$\dot{x_1} = x_2 = 1\checkmark \tag{277}$$

$$\dot{x_2} = 2(t)(1) + 3t + 2 - 3t - 2(t+1)(1) = 2t + 3t - 3t - 2t + 2 - 2 = 0\checkmark \tag{278}$$

$x = [t1]^T$ is a solution.

20.2 Problem 20b

Shift the x coordinates to a new set of coordinates z, such that the solution of the system is at the origin of z.

$$z_1 = x_1 - t \tag{279}$$

$$z_2 = x_2 - 1 \tag{280}$$

$$\dot{z_1} = \dot{x_1} - 1 \tag{281}$$

$$\dot{z_1} = x_2 - 1 = z2 \tag{282}$$

$$\dot{z_2} = \dot{x_2} = 2x_1x_2 + 3t + 2 - 3x_1 - 2tx_2 - 2x_2 \tag{283}$$

$$\dot{z_2} = 2(z_1 + t)(z_2 + 1) + 3t + 2 - 3(z_1 + t) - 2t(z_2 + 1) - 2(z_2 + 1) \tag{284}$$

$$\dot{z_2} = 2z_1z_2 - z_1 - 2z_2 \tag{285}$$

$$A = \frac{\partial \dot{z}}{\partial z}\bigg|_{z=0} = \begin{bmatrix} 0 & 1 \\ 2z_2 - 1 & 2z_1 - 2 \end{bmatrix}_{z=0} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \tag{286}$$

$$det\left(\begin{bmatrix} 0 - \lambda & 1 \\ -1 & -2 - \lambda \end{bmatrix}\right) = 0 \tag{287}$$

$$(-\lambda)(-2 - \lambda) - (1)(-1) = 0 \tag{288}$$

$$\lambda^2 + 2\lambda + 1 = 0 \tag{289}$$

$$(\lambda+1)^2 = 0 \Rightarrow \lambda = \{-1\} \tag{290}$$

$Re(\lambda) < 0$, therefore A is Hurwitz. Therefore $z = 0$ is uniformly asymptotically stable. Therefore $x = [t1]^t$ is uniformly asymptotically stable.

21 Problem 21

$$\dot{x_1} = -x_1 + x_2 + (x_1^2 + x_2^2)sint \tag{291}$$
$$\dot{x_2} = -x_1 - x_2 + (x_1^2 + x_2^2)cost \tag{292}$$

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{293}$$
$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} \tag{294}$$

$$\dot{V} = x_1(-x_1 + x_2 + (x_1^2 + x_2^2)sint) + x_2(-x_1 - x_2 + (x_1^2 + x_2^2)cost) \tag{295}$$

$$\dot{V} = -x_1^2 + x_1x_2 + x_1(x_1^2 + x_2^2)sint - x_1x_2 - x_2^2 + x_2(x_1^2 + x_2)cost \tag{296}$$

$$\dot{V} = -(x_1^2 + x_2) + (x_1^2 + x_2)(x_1sint + x_2cost) = -\|x\|^2 + \|x\|^2(x_1sint + x_2cost) \tag{297}$$

$$(x_1sint + x_2cost) \leq \sqrt{x_1^2 + x_2^2} = \|x\|$$

$$\dot{V} \leq -\|x\|^2 + \|x\|^3 = -(1 - \|x\|)\|x\|^2 \tag{298}$$

$$\|x\| \leq r$$

$$\dot{V} \leq -(1-r)\|x\|^2, r < 1 \text{ (to maintain sign)} \tag{299}$$

$$\dot{V}(x) \leq -(1-r)\|x\|^2, \forall\|x\| \leq r, r < 1 \tag{300}$$

Now, we will check the conditions for Theorem 4.10.

$$\text{Condition 1: } k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a? \tag{301}$$

$$V(x) = \frac{1}{2}\|x\|$$

$$a = 1, k_1 = \frac{1}{4}, k_2 = 1$$

$$\frac{1}{4}\|x\|^1 \leq \frac{1}{2}\|x\| \leq 1 \cdot \|x\|^1 \checkmark \tag{302}$$

$$\text{Condition 2: } \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) \leq -k_3\|x\|^a? \tag{303}$$

$$\dot{V}(x) = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x}$$

$$\dot{V} + \dot{V} \leq -k_3\|x\|^a? \tag{304}$$

$$\text{let r=1/2, } \dot{V} \leq -\frac{1}{2}\|x\|$$

$$2\dot{V} \leq -\|x\|$$

$$a = 1, k_3 = 1$$

$$2\dot{V} \leq -1 \cdot \|x\|^1 \checkmark \tag{305}$$

The system satisfies the conditions for Theorem 4.10, therefore the origin $x = 0$ is exponentially stable. The region of attraction can be defined as $\|x\| \leq r, r < 1$.

22 Problem 22

$$\dot{x} = f(x), f : \mathbb{R}^n \to \mathbb{R}^n, f \in c_1 \tag{306}$$

$$\dot{x} = h(x)f(x), h : \mathbb{R}^n \to \mathbb{R}, h \in c_1 \tag{307}$$

$$f(0) = 0, h(0) > 0$$

$$g(x) := h(x)f(x) \tag{308}$$

$$A_1 := \left.\frac{\partial f}{\partial x}\right|_{x=0} \tag{309}$$

$$A_2 := \left.\frac{\partial g}{\partial x}\right|_{x=0} \tag{310}$$

$$\frac{\partial g_i}{\partial x_j} = h(x)\frac{\partial f_i}{\partial x_j} + \frac{\partial h}{\partial x_j} f_i(x) \tag{311}$$

$$\left.\frac{\partial g_i}{\partial x_j}\right|_{x=0} = h(0)\left.\frac{\partial f_i}{\partial x_j}\right|_{x=0} + \left.\frac{\partial h}{\partial x_j}\right|_{x=0} f_i(0) \tag{312}$$

$$\left.\frac{\partial g_i}{\partial x_j}\right|_{x=0} = h(0)\left.\frac{\partial f_i}{\partial x_j}\right|_{x=0} \tag{313}$$

$$A_2 = h(0)A_1 \tag{314}$$

A_1 is Hurtwitz and $h(0) > 0$, therefore A_2 is Hurwitz. Therefore, if the origin is an exponentially stable equilibrium point of the linearized system $\dot{x} = A_1 x$, then the origin is also an exponentially stable equilibrium point of the linearized system $\dot{x} = A_2 x$.

Theorem 4.15 says that given that the origin is an equilibrium point for the system $\dot{x} = f(x)$, then the origin $x = 0$ is an exponentially stable equilibrium point for the non-linear system if and only if it is the exponentially stable equilibrium point of the linearized system. Because the origin is the exponentially stable equilibrium point of the linearized systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$, it is also the exponentially stable equilibrium point of the original non-linear systems $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$.