University of California, Davis
Department of Chemical Engineering
ECH 267
Advanced Process Control

M. Ellis **Homework 2** **Winter 2021**

Reading Assignment: Lecture notes; Khalil Chapter 4.1-4.7
Due date: Friday, February 12 at 6:00PM PST

1. Exercise 4.1 (Khalil pg. 181). Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable:

   (1) stable node

   Solution.

   A stable node is an asymptotically stable equilibrium point, since trajectories that start close to the equilibrium stay close to it and asymptotically converge to it.

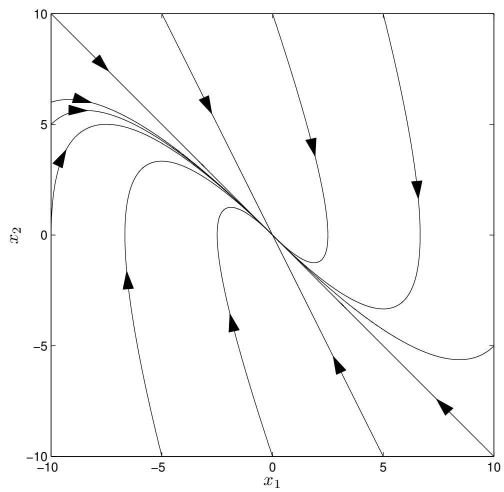

Figure 1: A stable node is an asymptotically stable equilibrium point: for a given $\epsilon > 0$, it is possible to find a $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq 0$. Moreover, trajectories starting close to the equilibrium asymptotically converge to it.

(2) unstable node

Solution.

An unstable node is an unstable equilibrium point, since no matter how close to the equilibrium a trajectory starts, it will always move out of a neighborhood of the equilibrium.

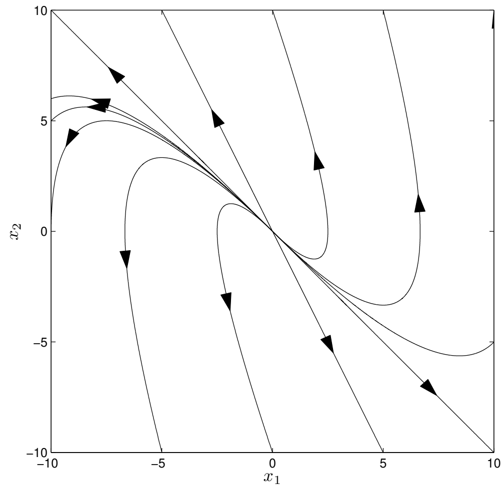

Figure 2: A unstable node is an unstable equilibrium point.

(3) stable focus

Solution.

An stable focus is an asymptotically stable equilibrium point, since trajectories that start close to the equilibrium stay close to it and asymptotically converge to it (the fact that they keep rotating as they approach the equilibrium does not matter).

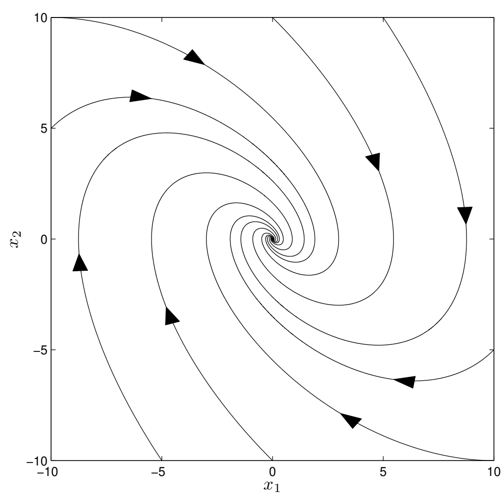

Figure 3: A stable focus is an asymptotically stable equilibrium point.

(4) unstable focus

Solution.

An unstable focus is an unstable equilibrium point, since no matter how close to the equilibrium a trajectory starts, it will always move out of a neighborhood of the equilibrium.

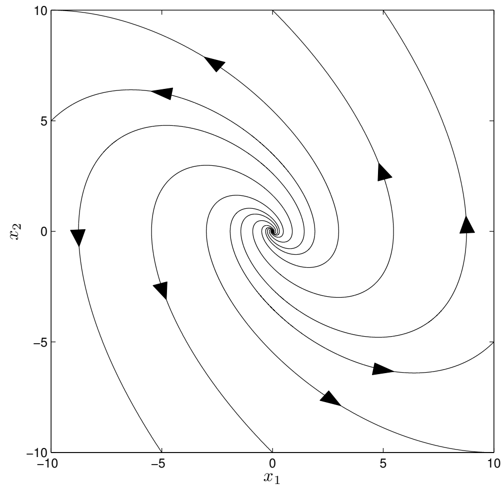

Figure 4: A unstable focus is an unstable equilibrium point.

(5) center

Solution.

A center is a stable equilibrium point (not asymptotically stable), since trajectories that start close to the equilibrium point stay close to it, but they do not converge to it.

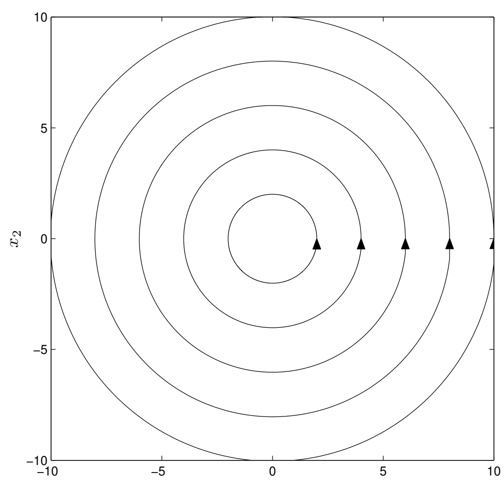

Figure 5: A center is a stable equilibrium point.

(6) saddle

Solution.

A saddle is an unstable equilibrium point, since for every neighborhood of the equilibrium (arbitrarily small), one can find trajectories that move out of the neighborhood.

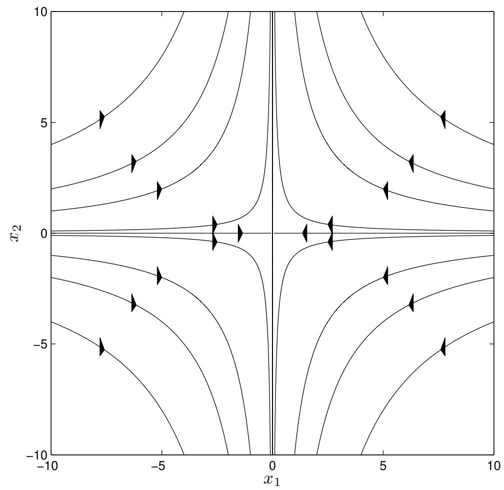

Figure 6: A saddle point is an unstable equilibrium point.

Justify your answer using phase portraits (sketches of phase portraits are acceptable).

2. Exercise 4.2 (Khalil pg. 181). Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k\,|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

   Solution. The inequality implies that $x = 0$ is an equilibrium point. Consider the Lyapunov function candidate:

$$V(x) = \frac{1}{2}x^2.$$

   Note that V is a positive definite function ($V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$). The time derivative of $V(x)$ along the trajectories of the scalar system is

$$\dot{V}(x) = ax^{p+1} + xg(x).$$

   The magnitude of the term $xg(x)$ is given by

$$\begin{aligned} |xg(x)| &\leq |x|\,|g(x)| \\ &\leq k\,|x|^{p+2}. \end{aligned}$$

   Hence,

$$\dot{V}(x) \leq ax^{p+1} + k\,|x|^{p+2}.$$

   Note that near the origin, the term ax^{p+1} is dominant. If p is odd and $a < 0$, $\dot{V}(x) < 0$ in a neighborhood of the origin and the origin is asymptotically stable. If p is odd and $a > 0$, $\dot{V}(x) > 0$ in a neighborhood of the origin, and therefore, the origin is unstable. If p is even and $a \neq 0$, consider two cases: (1) $a > 0$ and $x > 0$ and near the origin the vector field $f(x) > 0$ and (2) $a < 0$ and $x < 0$ and near the origin the vecgtor field $f(x) < 0$. In both cases, $\dot{V}(x) > 0$ and the vector field points away from the origin, so the trajectories starting close to the origin will move away from the origin making the origin unstable.

3. Exercise 4.3 (Khalil pg. 181). For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

Hint: For the g.a.s case, you may want to try non-quadratic Lyapunov functions, such as $V(x) = \frac{1}{2a}x_1^{2a} + \frac{1}{2b}x_2^{2b}$ with $a,\ b$ positive integers, and show that their derivatives are negative for all $(x_1, x_2) \neq (0, 0)$.

1.
$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2$$

Solution. The system has only one equilibrium point: $(0, 0)$. Considering the suggested Lyapunov function candidate,

$$V(x) = \frac{1}{2a}x_1^{2a} + \frac{1}{2b}x_2^{2b}$$

which is positive definite and radially unbounded. The derivative of V along the trajectories is

$$\begin{aligned}\dot{V}(x) &= \begin{bmatrix} x_1^{2a-1} & x_2^{2b-1}\end{bmatrix}\begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix} \\ &= -x_1^{2a} + x_1^{2a}x_2 - x_2^{2b}\end{aligned}$$

Picking $a = 1$ and $b = 1$ yields

$$\begin{aligned}\dot{V}(x) &= -x_1^2 + x_1^2 x_2 - x_2^2 \\ &= -(1-\theta)x_1^2 - \theta x_1^2 - x_2^2 + x_1^2 x_2 \\ &= -(1-\theta)x_1^2 - x_2^2 - x_1^2(\theta - x_2)\end{aligned}$$

for any $\theta \in (0, 1)$ and $\dot{V}(x) < 0$ for $x_2 < \theta$ and $x \neq 0$ so the origin is asymptotically stable.
The solution to the second equation is $x_2(t) = e^{-t}x_2(0)$, so $x_2(t) \to 0$ as $t \to \infty$. Substituting the solution into the first ODE yields

$$\dot{x}_1 = \left(-1 + e^{-t}x_2(0)\right)x_1$$

which is a linear time-varying system. Given that a linear system cannot have a finite escape time and after some time $x_2(t) < 1$ for any $x_2(0)$, $(-1 + e^{-t}x_2(0)) < 0$ and $\lim_{t\to\infty} x_1(t) = 0$. Thus, the origin is globally asymptotically stable.

2.
$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

Solution. The system has an equilibrium point: $(0, 0)$. Therefore, the origin cannot be globally asymptotically stable. Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^T x$$

which is a positive definite function. The derivative of V along the system trajectory is

$$\begin{aligned}\dot{V}(x) &= x^T \begin{bmatrix} -x_2 - x_1(1 - x_1^2 - x_2^2) \\ x_1 - x_2(1 - x_1^2 - x_2^2)\end{bmatrix} \\ &= -x_1x_2 - x_1^2(1 - x_1^2 - x_2^2) + x_1x_2 - x_2^2(1 - x_1^2 - x_2^2) \\ &= -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) \\ &= -\|x\|_2^2(1 - \|x\|_2^2)\end{aligned}$$

In the unit ball ($B_1 := \left\{x \in \mathbb{R}^2 |\, \|x\|_2 < 1\right\}$), $\dot{V}(x) < 0$ except for $x = 0$. Therefore, the origin is asymptotically stable.
For $\|x\|_2^2 > 1$, $\dot{V}(x)$ is positive, and trajectories starting in this region cannot approach the origin. Thus, the origin is not globally asymptotically stable.

3. $$\dot{x}_1 = x_2(1 - x_1^2)$$
$$\dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$$

Solution. The system has an equlibrium point at $(0, 0)$ as well as an equilibrium set $\{x \in \mathbb{R}^2 | |x_1| = 1\}$. Therefore, the origin cannot be globally asymptotically stable. Let $V(x) = x^T P x$ where P is a positive definite symmetric matrix. The derivative of V is

$$\dot{V}(x) = -2p_{12}x_1^2 + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2(p_{22} - p_{12})x_2^2 + \text{H.O.T.}$$

where H.O.T. stands for higher order terms. Near the origin, the quadratic term dominates the higher-order terms. Thus, $\dot{V}$ will be netative definite in the neighborhood of the origin if the quadratic term is negative definite. Choosing $p_{12} = 1$, $p_{22} = 2$ and $p_{11} = 3$ makes V positive definite and $\dot{V}$ negative definite (in a neighborhood of the origin). Hence, the origin is asymptotically stable.

4. $$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = 2x_1 - x_2^3$$

Solution. Let $V(x) = x_1^2 + (1/2)x_2^2$. The time derivative of V is

$$\dot{V}(x) = -2x_1^2 - 2x_1x_2 + 2x_2x_1 - x_4 = -2x_1^2 - x^4$$

Since V is positive definte and radially unbounded and $\dot{V}(x)$ for all $x \in R^2 \setminus \{0\}$, the origin is globally asymptotically stable.

4. Exercise 3.4 (Khalil Second Edition pg. 155). Using $V(x) = x_1^2 + x_2^2$, study stability of the origin of the system

$$\dot{x}_1 = x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2)$$
$$\dot{x}_2 = -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2)$$

when (a) $k = 0$ and (b) $k \neq 0$.

Solution.

- When $k = 0$, the system is

$$\dot{x}_1 = (-x_1 + x_2)(x_1^2 + x_2^2),$$
$$\dot{x}_2 = (-x_1 - x_2)(x_1^2 + x_2^2).$$

The system has an equilibrium point at the origin. The derivative of V is

$$\begin{aligned}\dot{V}(x) &= 2x^T f(x) \\ &= 2\left(-x_1^2 + x_1x_2\right)\left(x_1^2 + x_2^2\right) + 2(-x_1x_2 - x_2^2)(x_1^2 + x_2^2) \\ &= -2x_1^4 - 2x_1^2x_2^2 + 2x_1^3x_2 + 2x_1x_2^3 - 2x_1^3x_2 - 2x_1x_2^3 - 2x_1^2x_2^2 - 2x_2^4 \\ &= -2x_1^4 - 4x_1^2x_2^2 - 2x_2^4 \\ &= -2\left(x_1^2 + x_2^2\right)^2.\end{aligned}$$

Since the $\dot{V}$ is negative definite and V is positive definite and radially unbounded, the origin is globally asymptotically stable.

- When $k \neq 0$, the derivative of V is

$$\begin{aligned}\dot{V}(x) &= 2x^T f(x) \\ &= 2x_1^2(k^2 - x_1^2 - x_2^2) + 2x_1x_2(x_1^2 + x_2^2 + k^2) \\ &\quad - 2x_1x_2(k^2 + x_1^2 + x_2^2) + 2x_2^2(k^2 - x_1^2 - x_2^2) \\ &= -2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - k^2\right)\end{aligned}$$

which is positive definite in the ball $\{x \in \mathbb{R}^2 | \, \|x\|_2^2 < k^2\}$. By Chetaev's theorem, the origin is unstable.

5. Exercise 3.8 (Khalil Second Edition pg. 155). Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \text{sat}(2x_1 + x_2)\end{aligned}$$

(a) Show that the origin is asymptotically stable.

Solution.

For $|2x_1 + x_2| \leq 1$, the linearization about the origin is

$$\dot{x} = Ax = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} x = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

The A matrix has eigenvalues: $-0.5 \pm 0.8660i$. Hence, the origin is asymptotically stable.

(b) Show that all trajectories starting in the first quadrant to the right of the curve $x_1 x_2 = c$ (with sufficiently large $c > 0$) cannot reach the origin.

Hint: In part (b), consider $V(x) = x_1 x_2$; calculate $\dot{V}(x)$ and show that on the curve $V(x) = c$ the derivative $\dot{V}(x) > 0$ when c is large enough.

Solution.

Consider the function

$$V(x) = x_1 x_2$$

and the time derivative of V is

$$\begin{aligned}\dot{V}(x) &= x_2^2 + x_1^2 - x_1 \text{sat}(2x_1 + x_2) \\ &= x_1^2 + x_2^2 - x_1 \text{sat}(2x_1 + x_2).\end{aligned}$$

Evaluating $\dot{V}$ at $x_1 x_2 = c$ with sufficiently large $c > 0$ yields

$$\left.\dot{V}(x)\right|_{x_1 x_2 = c} = x_1^2 + \frac{c^2}{x_1^2} - x_1 \text{sat}(2x_1 + \frac{c}{x_1}).$$

If $x_1 \geq 0$, $0 \leq \text{sat}(2x_1 + \frac{c}{x_1}) \leq 1$ and

$$\begin{aligned}\left.\dot{V}(x)\right|_{x_1 x_2 = c} &= x_1^2 + \frac{c^2}{x_1^2} - x_1 \text{sat}(2x_1 + \frac{c}{x_1}) \\ &\geq x_1^2 + \frac{c^2}{x_1^2} - x_1\end{aligned}$$

$\dot{V}(x) > 0$ if $c \geq$ is sufficiently large any trajectory starting in the first quadrant to the right of curve $x_1 x_2 = c$ cannot reach the origin.

(c) Show that the origin is not globally asymptotically stable.

Solution.

The fact that the origin is not globally asymptotically stable follows from part (b).

Additionally, one could show this fact because it is not the only equilibrium point. The other equilibrium points are

$$\begin{aligned}&0 = x_2 \\ &0 = x_1 - \text{sat}(2x_1) \Rightarrow x_1 = 0 \,\text{or}\, x_1 = \pm 1.\end{aligned}$$

6. Exercise 4.13 (Khalil pg. 183). For each of the following systems, show that the origin is unstable:

(1)
$$\dot{x}_1 = x_1^3 + x_1^2 x_2$$
$$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

Solution.

Apply Chetaev's theorem with $V(x) = 1/2(x_1^2 - x_2^2)$. The function V is positive at points abitrarily close to the origin on the x_1-axis. The time derivative of V is

$$\begin{aligned}\dot{V}(x) &= x_1(x_1^3 + x_1^2 x_2) - x_2(-x_2 + x_2^2 + x_1 x_2 - x_1^3) \\ &= x_1^4 + x_1^3 x_2 - x_2^2 + x_2^3 + x_1 x_2^2 - x_1^3 x_2 \\ &= x_1^4 + 2x_1(x_1^2 x_2^2) + x_1^2 x_2^2 - x_1^2 x_2^2 + x_2^2(1 - x_2 - x_1) \\ &= (x_1^2 + x_1 x_2)^2 + x_2^2(1 - x_2 - x_1 - x_1^2)\end{aligned}$$

For any $0 < c < 1$, there is a domain around the origin where

$$1 - x_2 - x_1 - x_1^2 > c > 0$$

Hence, in this domain, we have
$$\dot{V}(x) \leq (x_1^2 + x_1 x_2)^2 + c x_2^2$$

The right-hand side of the preceding inequality is positive definite; hence all the conditions of Chetaev's theorem are satisfied and the equilibrium point at the origin is unstable.

(2)
$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

Solution.

The system has two equilibrium points at $(0,0)$ and $(1,1)$. Define the set $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ and we note that $(0,0)$ and $(1,1)$ are in Γ meaning that the set is nonempty. On the boundary $x_2 = x_1^2$, $\dot{x}_2 = x_1^6 - x_2^3 = 0$ and $\dot{x}_1 = -x_1^3 + x_2 = -x_1^3 + x_1^2 > 0$ (since $x_1 \in (0,1)$). Hence, all trajectories on this boundary move into Γ. On the boundary $x_2 = x_1^3$, $\dot{x}_1 = -x_1^3 + x_2 = 0$ and $\dot{x}_2 = x_1^6 - x_2^3 = x_1^6 - x_1^9 > 0$ ($x_1 \in (0,1)$). Hence, trajectories starting on this boundary of Γ will move to the interior of the set. Finally, for trajectories starting at each equilibrium point, they will be maintained in Γ. Thus, the set Γ is a non-empty positively invariant set.

In the interior of the set, the vector field $f(x) > 0$ for $x \in \mathrm{int}(\Gamma)$. Thus, all trajectories in the interior of Γ converge to the point $(1,1)$ proving that the origin is unstable. The phase portrait is given below in Figure 7

7. Exercise 3.22 (Khalil First Edition pg. 158). Consider the system

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = (x_1 + x_2)\sin x_1 - 3x_2$$

(a) Show that the origin is the unique equilibrium point.

Solution.

The equilibrium points are the solution to

$$\begin{aligned} x_1 &= x_2 \\ 0 &= (x_1 + x_2)\sin x_1 - 3x_2 \\ 0 &= (2\sin x_1 - 3)x_1. \end{aligned}$$

The above is only true if $x_1 = x_2 = 0$. Therefore, the origin is the unique equilibrium point.

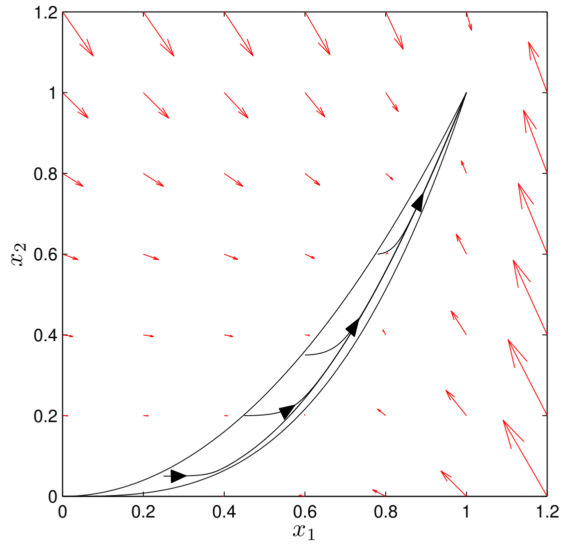

Figure 7: The set Γ and trajectories starting on the boundary of Γ.

(b) Show, using linearization, that the origin is asymptotically stable.
Solution.
The Jacobian of the vector field is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ \sin x_1 + (x_1 + x_2)\cos x_1 & \sin x_2 - 3 \end{bmatrix}$$

and evaluating at $x = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = A = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. The origin is asymptotically stable.

(c) Show that the origin is globally asymptotically stable.
Hint: Use a simple quadratic Lyapunov function and the inequality $|\sin x_1| < 1$.
Solution.
Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^T x$$

which is positive definite for all $x \neq 0$ and radially unbounded. The derivative of $V(x)$ along the trajectory is

$$\begin{aligned}
\dot{V}(x) &= x^T f(x) \\
&= -x_1^2 + x_1 x_2 + x_2 (x_1 + x_2) \sin x_1 - 3x_2^2 \\
&= -x_1^2 + x_1 x_2 + x_1 x_2 \sin x_1 + x_2^2 \sin x_1 - 3x_2^2 \\
&= -x_1^2 + x_1 x_2 (1 + \sin x_1) + x_2^2 (\sin x_1 - 3) \\
&\leq -x_1^2 + 2|x_1||x_2| - 2x_2^2 \\
&= -x_1^2 + 2\sqrt{2}|x_1||x_2| - 2x_2^2 - \left(2\sqrt{2} - 2\right)|x_1||x_2| \\
&= -\left(|x_1| - \sqrt{2}|x_2|\right)^2 - \left(2\sqrt{2} - 2\right)|x_1||x_2| < 0 \\
\dot{V}(x) &< 0.
\end{aligned}$$

This proves $\dot{V}(x)$ is negative definite for all $x \neq 0$ and therefore, the origin is globally asymptotically stable.

8. Exercise 4.20 (Khalil pg. 185). Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

Solution. Consider the class of nonlinear systems described by

$$\dot{x} = f(x), \quad x(t_0) = x_0.$$

Theorem 1. ***LaSalle's Theorem***. *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to the above nonlinear system. Let $V : \Omega \to \mathbf{R}$ be an C_1 function (continuous differentiable) such that $\dot{V}(x) \leq 0\,\forall\, x \in \Omega$. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution solution starting in Ω approaches M as $t \to \infty$.*

According to LaSalle's theorem, $x(t)$ approaches M as $t \to \infty$. Equivalently, this means given $\epsilon > 0$ there is a $T > 0$ such that

$$\inf_{y\in M} \|x(t) - y\| < \epsilon,\ \forall t > T$$

Given that M consists of a finite number of isolated points, it is possible to find a $\epsilon > 0$ small enough such that a ball of radius 2ϵ centered a given point $p \in M$ contains no other points in M.

We claim that $\lim_{t\to\infty} x(t) = p$, that is for any $\epsilon_1 > 0$ there exists a $T_1 > 0$ such that $\|x(t) - p\| < \epsilon_1$ for all $t > T_1$, for some $p \in M$.

We proceed by contradiction. At $t = t_1 > T$, let $p \in M$ be a point for which $\|x(t_1) - p\| < \epsilon$. Suppose there is time $t_2 > t_1$ such that $\|x(t_2) - p\| = \epsilon$. Let $p_1 \neq p$ be any other point of M. Then

$$\begin{aligned}\|x(t_2) - p_1\| &= \|x(t_2) - p + p - p_1\| \\ &\geq \|p - p_1\| - \|x(t_2) - p\| \geq 2\epsilon - \epsilon = \epsilon\end{aligned}$$

where the inequality follows from the reverse triangle inequality ($|\|x\| - \|y\|| \leq \|x - y\|$). The last statement implies that $\inf_{y\in M} \|x(t_2) - y\| \geq \epsilon$ because the the inequality $\|x(t_2) - p_1\| \geq \epsilon$ holds for all $p_1 \in M \setminus \{p\}$ and $\|x(t_2) - p\| = \epsilon$. This contradicts the fact that $\inf_{y\in M} \|x(t) - y\| < \epsilon$ for all $t > T$. Since it is true that $\|x(t) - p\| < \epsilon$ for all $t > T$, some $p \in M$, and for any, sufficiently small, $\epsilon > 0$, it is equivalent to $x(t) \to p$ as $t \to \infty$ and for some $p \in M$, which proves that $\lim_{t\to\infty} x(t)$ exists and equals one of the points in M.

9. Exercise 4.26 (Khalil pg. 186). Let $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

   (a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

   Solution.

   We begin by noting the inverse map $T^{-1}(z) = x$ at $z = 0$ is 0. Suppose that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x) \Rightarrow f(0) = 0$ and

$$\hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=0} = 0.$$

   Thus, $z = 0$ is an equilibrium point. We need to show that $z = 0$ is an isolated equilibrium. Proceed by contraction. Suppose it is not isolated. There, there is a $\bar{z} \neq 0$, arbitrarily close to 0, such that $\hat{f}(\bar{z}) = 0$. Define $\bar{x} = T^{-1}(\bar{z})$. Then, $f(\bar{x}) = [\partial T/\partial x]^{-1} \hat{f}(\bar{z}) = 0$; that is, $\bar{x}$ is an equilibrium point. By continuity of $T^{-1}(\cdot)$, we can make $\bar{x}$ artitrarily close to the origin, which contradicts the fact that $x = 0$ is an isolated equilibrium.

   The same argument can be repeated to show that $x = 0$ is an isolated equilibrium if $z = 0$ is an isolated equilibrium. Therefore, $x = 0$ is an isolated equilibrium point if and only if $z = 0$ is an isolated equilibrium point.

(b) Show that $x = 0$ is stable (asymptotically stable/unstable) if and only if $z = 0$ is stable (asymptotically stable/unstable).

Solution.

Suppose $x = 0$ is a stable equilibrium point. Then, given $\epsilon_1 > 0$ there is $\delta_1 > 0$ such that

$$\|x(0)\| < \delta_1 \Rightarrow \|x(t)\| < \epsilon_1, \quad \forall t \geq 0$$

Given $\epsilon_2 > 0$, there is $r > 0$ (by continuity of $T(\cdot)$) such that

$$\|x\| < r \Rightarrow \|z\| < \epsilon_2$$

Thus, there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \Rightarrow \|z(t)\| < \epsilon_2, \; \forall t \geq 0$$

By continuity of $T^{-1}(\cdot)$, there is $\delta_2 > 0$ such that

$$\|z\| < \delta_2 \Rightarrow \|x\| < \delta$$

Hence

$$\|z(0)\| < \delta_2 \Rightarrow \|x(0)\| < \delta \Rightarrow \|x(t)\| < r \Rightarrow \|z(t)\| < \epsilon_2, \quad \forall t \geq 0$$

Thus, $z = 0$ is a stable equilibrium point.

Suppose now that $x = 0$ is an asymptotically stable equililbrium point. Then

$$x(t) \to 0 \text{ as } t \to \infty$$

that is given $\epsilon_1 > 0$, there is $T_1 > 0$ such that $\|x(t)\| < \epsilon_1$ for all $t > T_1$. Now, given $\epsilon_2 > 0$, there is $r > 0$ (by continuity of $T(\cdot)$) such that

$$\|x\| < r \Rightarrow \|z\| < \epsilon_2$$

There exists $T_2 > 0$ such that

$$\|x(t)\| < r, \; \forall t > T_2 \Rightarrow \|z(t)\| < \epsilon_2, \; \forall \, t > T_2$$

Hence

$$z(t) \to 0 \text{ as } t \to \infty$$

and $z = 0$ is asymptotically stable. The opposite direction of the proof is done similarity. Now

$$x = 0 \text{ is stable} \iff z = 0 \text{ is stable}$$

is equivalent to

$$x = 0 \text{ is unstable} \iff z = 0 \text{ is unstable}$$

which completes the proof.

10. Determine the three equilibrium points of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2. \end{aligned}$$

Investigate the stability of the equilibrium point $x = 0$. Verify the conclusions about the phase portrait and region of attraction of the system. In particular, generate its phase portrait by simulations and plot the level sets of the Lyapunov function $V(x) = \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2$ (if using Matlab, use the commands `meshgrid` and `contour`). Verify that the set $\{V(x) < \frac{9}{8}\}$ is (approximately) the largest estimate of the region of attraction that can be generated from this Lyapunov function.

Solution. Consider the equilibrium points of the system:

$$0 = \frac{1}{3}x_1\left(x_1^2 - 3\right)$$

so the equilibrium points of the system are $(-\sqrt{3}, 0)$, $(0, 0)$, and $(\sqrt{3}, 0)$. A Lyapunov function candidate is

$$V(x) = \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2$$

which is positive definite in the neighborhood of the origin. The derivative of $V(x)$ along the trajectories is

$$\begin{aligned}
\dot{V}(x) &= \frac{3}{2}x_1x_2 - \frac{1}{3}x_1^3x_2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_1\left(-x_1 + \frac{1}{3}x_1^3 - x_2\right) + x_2\left(-x_1 + \frac{1}{3}x_1^3 - x_2\right) \\
&= \frac{3}{2}x_1x_2 - \frac{1}{3}x_1^3x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{6}x_1^4 - \frac{1}{2}x_1x_2 - x_1x_2 + \frac{1}{3}x_1^3x_2 - x_2^2 \\
&= -\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{6}x_1^4
\end{aligned}$$

which is negative definite in the neighborhood of the origin (quadratic terms dominate). From the phase portrait (Figure 8), we can see that the $\{V(x) < \frac{9}{8}\}$ is approximately the largest estimate of the region of attraction from this Lyapunov function (actually $V(x) < 1.17$ may be a better estimate).

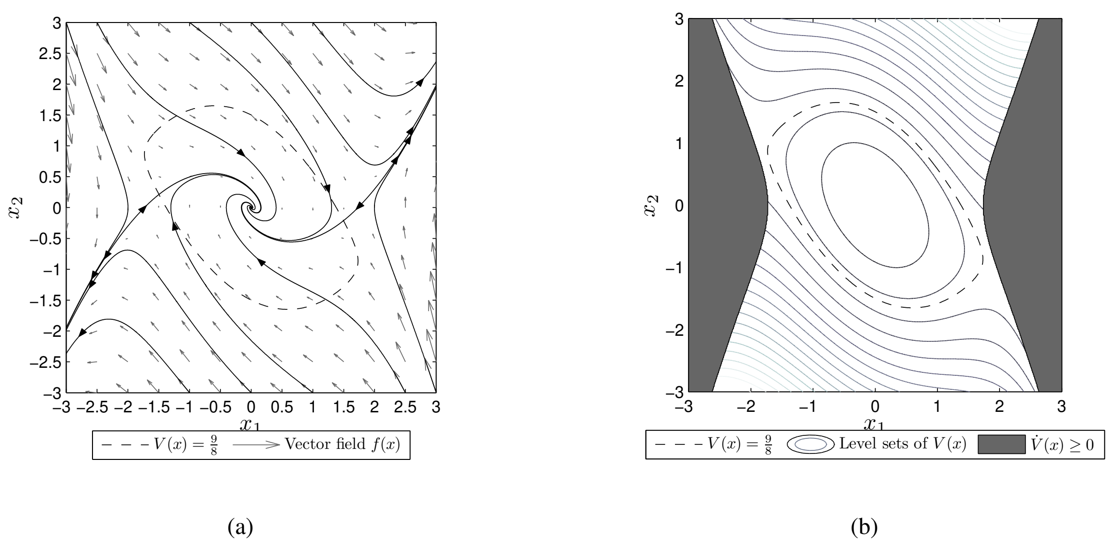

Figure 8: (a) Vector field plotted with a few trajectories. Any trajectory starting with the set $\{V(x) < \frac{9}{8}\}$ go to the origin with time. (b) Level sets of $V(x)$ plotted with the set $\{x \neq 0 \,|\, \dot{V}(x) \geq 0\}$.

11. Exercise 3.15 (Khalil Second Edition pg. 157). Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2\text{sat}(x_2^2 - x_3^2) \\
\dot{x}_3 &= x_3\text{sat}(x_2^2 - x_3^2)
\end{aligned}$$

where $\text{sat}(\cdot)$ is the saturation function. Show that the origin is the unique equilibrium point, and use $V(x) = x^Tx$ to show that it is globally asymptotically stable.

Solution. First, consider the equilibrium points of the system. An equilibrium point is a solution to the following system of algebraic equations:

$$\begin{aligned}
0 &= x_2 \\
0 &= -x_1 - x_2 \text{sat}(x_2^2 - x_3^2) \\
0 &= x_3 \text{sat}(x_2^2 - x_3^2)
\end{aligned}$$

The first and second equations imply $x_1 = x_2 = 0$. The roots of the third equation at the equilibrium point can be found by solving

$$0 = x_3 \text{sat}(-x_3^2).$$

The only root of this equation is $x_3 = 0$. Therefore, the origin is the unique equilibrium point of the system. Consider the Lyapunov function candidate

$$V(x) = x^T x$$

which is positive definite for all $x \neq 0$ and also radially unbounded. The derivative of V along the trajectory is

$$\begin{aligned}
\dot{V}(x) &= 2x^T \dot{x} \\
&= 2x^T f(x) \\
&= 2\left(x_1 x_2 - x_1 x_2 - x_2^2 \text{sat}(x_2^2 - x_3^2) + x_3^2 \text{sat}(x_2^2 - x_3^2)\right) \\
&= -2\left(x_2^2 - x_3^2\right) \text{sat}(x_2^2 - x_3^2)
\end{aligned}$$

which shows $\dot{V}(x)$ is negative semidefinite. To apply LaSalle's theorem, we must find the sets E and M:

$$\dot{V} \equiv 0 \Rightarrow x_2^2(t) \equiv x_3^2(t) \Rightarrow \dot{x}_3(t) \equiv 0$$

Hence, both $x_2(t) and $x_3(t)$ must be constants (with respect to time). This implies that $\dot{x}_2(t) \equiv 0$. From the second state equation, we conclude that $x_1(t) \equiv 0$. Then, the first state equiation implies that $x_2(t) \equiv 0$. Consequently, $x_3(t) \equiv 0$. By LaSalle's theorem (Corollary 4.2), the origin is globally asymptotically stable.

12. Exercise 3.16 (Khalil Second Edition pg. 157). The origin $x = 0$ is an equilibrium point of the system

$$\begin{aligned}
\dot{x}_1 &= -kh(x)x_1 + x_2 \\
\dot{x}_2 &= -h(x)x_2 - x_1^3
\end{aligned}$$

Let $D = \{x \in \mathbb{R}^2 \mid \|x\|_2 < 1\}$. Using $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, investigate stability of the origin in each of the following cases.

(1) $k > 0$, $h(x) > 0, \forall x \in D$.

Solution.

Consider the Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

which is positive definite and radially unbounded. The derivative of $V(x)$ along

$$\begin{aligned}
\dot{V}(x) &= -x_1^4 kh(x) + x_1^3 x_2 - h(x)x_2^2 - x_1^3 x_2 \\
&= -x_1^4 kh(x) - h(x)x_2^2.
\end{aligned}$$

For $k > 0$, $h(x) > 0, \forall x \in D$, $\dot{V}$ is negative definite for all $x \in D \setminus \{0\}$ so the origin is asymptotically stable.

(2) $k > 0$, $h(x) > 0, \forall x \in \mathbb{R}^2$.

Solution.

$\dot{V}$ is negative definite for all $x \neq 0$. Hence the origin is globally asymptotically stable.

(3) $k > 0$, $h(x) < 0, \forall x \in D$.

Solution.

In this case, $\dot{V}$ is positive definite. Hence, by Chetaev's theorem, the origin is unstable.

(4) $k > 0$, $h(x) = 0, \forall x \in D$.

Solution.

In this case, $\dot{V}(x) = 0$ for all x. Hence, the origin is stable. It is not asymptotically stable because trajectories starting on the level surface $V(x) = c$ remain on the surface for all time.

(5) $k = 0$, $h(x) > 0, \forall x \in D$.

Solution.

In this case, $\dot{V}(x) = -x_2^2 h(x) \leq 0$, for all $x \in D$. Moreover,

$$\dot{V}(x) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Hence, by LaSalle's theorem (Corollary 4.1), the origin is asymptotically stable.

(6) $k = 0$, $h(x) > 0, \forall x \in \mathbb{R}^2$.

Solution.

The same as part (5), except the conditions hold globally. Hence, the origin is globally asymptotically stable (Corollary 4.2).

13. Exercise 3.19 (Khalil Second Edition pg. 158). Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a \sin x_1 - k x_1 - d x_2 - c x_3 \\
\dot{x}_3 &= -x_3 + x_2
\end{aligned}$$

where all coefficients are positive and $k > a$. Using

$$V(x) = 2a \int_0^{x_1} \sin y dy + k x_1^2 + x_2^2 + p x_3^2$$

with some $p > 0$, show that the origin is globally asymptotically stable.

Solution. Consider the Lyapunov function candidate

$$\begin{aligned}
V(x) &= 2a \int_0^{x_1} \sin y dy + k x_1^2 + x_2^2 + p x_3^2 \\
&= 2a(1 - \cos x_1) + k x_1^2 + x_2^2 + p x_3^2 \\
&\geq k x_1^2 + x_2^2 + p x_3^2
\end{aligned}$$

which is positive definite and radially unbounded. The derivative of V is

$$\begin{aligned}
\dot{V}(x) &= 2k x_1 x_2 + 2a x_2 \sin x_1 - 2a x_2 \sin x_1 - 2k x_1 x_2 - 2d x_2^2 - 2c x_2 x_3 - 2p x_3^2 + 2c x_2 x_3 \\
&= -2d x_2^2 - 2p x_3^2 - 2k x_1 x_2 + 2c x_2 x_3
\end{aligned}$$

Taking $p = c$, we have

$$\dot{V}(x) = -d x_2^2 - p x_3^2 \leq 0$$

for all $x \in \mathbb{R}^3$. Considering solutions where $\dot{V} \equiv 0$:

$$\dot{V} \equiv 0 \Rightarrow x(t) \equiv 0, x_3(t) \equiv 0 \to a \sin(x_1(t)) + k x_1(t) \equiv 0$$

Since $k > a$,

$$a \sin(x_1(t)) + k x_1(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Using LaSalle's theorem (Corollary 4.2), we conclude that the origin is globally asymptotically stable.

14. Exercise 4.18 (Khalil pg. 184). The mass-spring system of Exercise 1.11 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}\,|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

Solution. The only equilibrium point of the system is $\dot{y} = 0$ and $y = Mg/k$. The mass-spring system in state-space form (shifted system to make the origin the equilibrium point) is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2\,|x_2|\end{aligned}$$

where $x_1 = y - Mg/k$ and $x_2 = \dot{y}$. Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^T P x$$

where P is a symmetric positive definite matrix and therefore, V is positive definite and radially unbounded. The derivative of V is

$$\begin{aligned}\dot{V}(x) &= \frac{1}{2}x^T P\dot{x} + \frac{1}{2}\dot{x}^T P x \\ &= \frac{1}{2}x^T P f(x) + \frac{1}{2}f(x)^T P x \\ &= \frac{1}{2}f_1(x)\left(x_1p_{11} + x_2p_{12}\right) + \frac{1}{2}f_2(x)\left(x_1p_{12} + x_2p_{22}\right) \\ &\quad + \frac{1}{2}x_1\left(f_1p_{11} + f_2p_{12}\right) + \frac{1}{2}x_2\left(f_1p_{12} + f_2p_{22}\right) \\ &= p_{11}x_1x_2 + p_{12}x_2^2 + \left(p_{12}x_1 + p_{22}x_2\right)\left(-\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2\,|x_2|\right).\end{aligned}$$

Let $p_{12} = 0$ and $p_{11} = \frac{kp_{22}}{M}$, we have

$$\dot{V}(x) = -\frac{c_1p_{22}}{M}x_2^2 - \frac{c_2p_{22}}{M}x_2^2\,|x_2|$$

Since $\dot{V}(x) \le 0$ for all x, consider

$$\dot{V} \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Using LaSalle's theorem (Corollary 3.2), we conclude that the origin is globally asymptotically stable.

15. Exercise 4.23 (Khalil pg. 185). Consider the linear system $\dot{x} = \left(A - BR^{-1}B^TP\right)x$, where (A, B) is controllable, $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0$$

$R = R^T > 0$, and $Q = Q^T \ge 0$. Using $V(x) = x^TPx$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when:

(1) $Q > 0$.
Solution.
Consider the Lyapunov function candidate

$$V(x) = x^TPx$$

where P is the solution to the Riccati equation. The derivative of V is

$$\begin{aligned}
\dot{V}(x) &= x^T P\dot{x} + \dot{x}^T Px \\
&= x^T \left[P\left(A - BR^{-1}B^T P\right) + \left(A - BR^{-1}B^T P\right)^T P\right] x \\
&= x^T \left[PA + A^T P - 2PBR^{-1}B^T P\right] x \\
&= -x^T \left[Q + PBR^{-1}B^T P\right] x.
\end{aligned}$$

If $Q > 0$, $\dot{V}(x)$ is negative definite on $\mathbb{R}^n$ because Q is positive definite and $PBR^{-1}B^T P$ is positive semidefinite (R and P are symmetric positive definite) so $Q + PBR^{-1}B^T P > 0$. Therefore, the origin is globally asymptotically stable.

(2) $Q = C^T C$ and (A, C) is observable. *Hint:* Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0\ \forall t$ if and only if $x = 0$.

Solution.

If $Q = C^T C$, Q, in general, is positive semidefinite, and we can only conclude that $\dot{V}(x)$ is negative semidefinite. To show globally asymptotic stability, apply LaSalle's theorem:

$$\begin{aligned}
\dot{V}(x) \equiv 0 &\Rightarrow x^T(t)(Q + PBR^{-1}B^T P)x(t) \equiv 0 \\
&\Rightarrow Cx(t) \equiv 0 \text{ and } R^{-1}B^T Px(t) \equiv 0
\end{aligned}$$

The latter observation is important because it simplifies the state equation:

$$\begin{aligned}
\dot{x} &= \left(A - BR^{-1}B^T P\right) x \\
&= Ax
\end{aligned}$$

and the solution is

$$x(t) = \exp(At)x_0.$$

Considering the first condition on $\dot{V}(x) = 0$, we have

$$C\exp(At)x_0 \equiv 0.$$

Because the pair (A, C) is observable, the above is satisfied if and only if $x_0 \equiv 0$. By LaSalle's theorem (Corollary 4.2), we conclude that the origin is globally asymptotically stable.

16. Exercise 4.28 (Khalil pg. 186). Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2
\end{aligned}$$

(a) Show that $x = 0$ is the unique equilibrium point.

Solution.

The equilibrium points are solutions to the following system of nonlinear equations:

$$\begin{aligned}
0 &= -x_1 \\
0 &= (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2
\end{aligned}$$

From the first equation, $x_2 = 0$ and from the second,

$$\begin{aligned}
0 &= -x_2^3 - x_2 \\
&= -x_2\left(x_2^2 + 1\right).
\end{aligned}$$

The only real roots of this equation is $x_1 = 0$. Therefore, $x = 0$ is the unique equilibrium point.

(b) Show, by using linearization, that $x = 0$ is asymptotically stable.

Solution.

The Jacobian of the system is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 4x_2x_1^3 - 3x_1^2 + 2x_1x_2 - 1 + x_2^2 & x_1^4 + x_1^2 + 2x_1x_2 \\ 0 & -1 \end{bmatrix}$$

and evaluating at $x = 0$,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

with eigenvalues -1 and -1 so the origin is asymptotically stable.

(c) Show that $\Gamma = \{x \in \mathbb{R}^2 | x_1x_2 \geq 2\}$ is a positively invariant set.

Solution.

To show the set Γ is positively invariant, consider

$$V(x) = x_1x_2.$$

The derivative of V is

$$\begin{aligned}
\dot{V}(x) &= \begin{bmatrix} x_2 & x_1 \end{bmatrix} f(x) \\
&= -x_1x_2 + (x_1x_2 - 1)x_1x_2^3 + (x_1x_2 - 1 + x_1^2)x_1x_2 \\
&= -x_1x_2 + x_1^2x_2^4 - x_1x_2^3 + x_1^2x_2^2 - x_1x_2 + x_1^3x_2 \\
&= x_1x_2(x_1x_2^3 + x_1^2 + x_1x_2 - x_2^2 - 2)
\end{aligned}$$

$\dot{V}(x)$ evaluated at the boundary of the set is

$$\begin{aligned}
\dot{V}(x)\Big|_{x_1x_2=2} &= 2\left(\frac{2}{x_2}x_2^3 - \frac{4}{x_2^2} + \frac{2}{x_2}x_2 - x_2^2 - 2\right) \\
&= 2\left(x_2^2 + \frac{4}{x_2^2}\right) > 0.
\end{aligned}$$

Any trajectory starting from the boundary of the set Γ will move away from the boundary and into the interior of the set Γ because $\dot{V}(x)$ is strictly positive definite on the boundary of the set Γ. Hence, the set Γ is a positively invariant set.

(d) Is $x = 0$ globally asymptotically stable?

Solution.

The origin is not globally asymptotically stable since trajectories starting in Γ do not converge to the origin.

17. Exercise 3.30 (Khalil Second Edition pg. 161). For each of the following systems, use linearization to show that the origin is asymptotically stable. Then, show that the origin is globally asymptotically stable.

(1)

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2
\end{aligned}$$

Solution.

The equilibrium points can be found by setting the derivatives to zero

$$\begin{aligned}
x_1 &= x_2 \\
0 &= (x_1 + x_2)\sin x_1 - 3x_2 \\
0 &= (2\sin x_1 - 3)x_1.
\end{aligned}$$

The above is only true if $x_1 = x_2 = 0$. Therefore, the origin is the unique equilibrium point.

The Jacobian of the vector field is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ \sin x_1 + (x_1 + x_2)\cos x_1 & \sin x_2 - 3 \end{bmatrix}$$

and evaluating at $x = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = A = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. The origin is asymptotically stable.
Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^T x$$

which is positive definite for all $x \neq 0$ and radially unbounded. The derivative of $V(x)$ along the trajectory is

$$\begin{aligned}
\dot{V}(x) &= x^T f(x) \\
&= -x_1^2 + x_1x_2 + x_2\left(x_1 + x_2\right)\sin x_1 - 3x_2^2 \\
&= -x_1^2 + x_1x_2 + x_1x_2\sin x_1 + x_2^2\sin x_1 - 3x_2^2 \\
&= -x_1^2 + x_1x_2\left(1 + \sin x_1\right) + x_2^2\left(\sin x_1 - 3\right) \\
&\leq -x_1^2 + 2\left|x_1\right|\left|x_2\right| - 2x_2^2 \\
&= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} < 0
\end{aligned}$$

for all x except $x = 0$. This proves $\dot{V}(x)$ is negative definite for all $x \neq 0$ and therefore, the origin is globally asymptotically stable.

(2)
$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + x_2 \\
\dot{x}_2 &= -ax_1 - bx_2,\ a, b > 0
\end{aligned}$$

Solution.
The equilibrium points can be found by setting the derivatives to zero

$$\begin{aligned}
ax_1 &= -bx_2 \\
0 &= -x_1^3 - \frac{b}{a}x_1 \\
0 &= -\left(x_1^2 + 1\right)x_1.
\end{aligned}$$

The only real root of the last equation is $x_1 = 0$ so the unique equilibrium point of the system is the origin.
The Jacobian of the vector field is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_1^2 & 1 \\ -a & -b \end{bmatrix}$$

and evaluating at $x = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The eigenvalues of the A matrix can be found by finding the roots of the characteristic polynomial

$$\begin{aligned}
0 &= \lambda\left(\lambda + b\right) + a \\
&= \lambda^2 + b\lambda + a.
\end{aligned}$$

The eigenvalues are clearly negative meaning the origin is asymptotically stable. To investigate if the origin is globally asymptotically stable, consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^T\text{diag}(v)x$$

where v is a vector with strictly positive real elements. $V(x)$ is positive definite for all $x \neq 0$ and radially unbounded. The derivative of $V(x)$ along the trajectory is

$$\begin{aligned}\dot{V}(x) &= x^T f(x) \\ &= -v_1 x_1^4 + v_1 x_1 x_2 - a v_2 x_1 x_2 - b v_2 x_2^2.\end{aligned}$$

Letting $v_2 = \frac{v_1}{a}$ and $v_1 = 1$, we have

$$\dot{V}(x) = -x_1^4 - \frac{b}{a} x_2^2$$

which is negative definite for all $x \neq 0$. Hence, the origin is globally asymptotically stable.

18. Exercise 4.36 (Khalil pg. 188). Is the origin of the scalar system $\dot{x} = -x/(t+1)$, $t \geq 0$, uniformly asymptotically stable?

Solution. The solution of the ODE is

$$x(t) = x(t_0)\frac{1+t_0}{1+t}$$

Since $|x(t)| \leq |x(t_0)|$ for all $t \geq t_0$, the origin is clearly stable (note that the origin is actually uniformly stable). As $t \to \infty$, $x(t) \to 0$, but the amount of time one has to wait for the solution to become smaller than ϵ depends not only on x_0, but also on t_0:

$$|x(t)| \leq \epsilon \; \forall t \geq t_0 + T \iff \left| \frac{x(t_0)(1+t_0)}{1+t} \right| \leq \epsilon \; \forall t \geq t_0 + T \iff \left| \frac{x(t_0)(1+t_0)}{t_0 + T + 1} \right| \leq \epsilon$$

$$\iff T \geq \left(\frac{|x(t_0)|}{\epsilon} - 1 \right) |t_0 - 1|$$

Hence, the origin is globally asymptotically stable, but not uniformly asymptotically stable.

19. Exercise 4.37 (Khalil pg. 188). For each of the following linear systems, use a quadratic Lyapunov function to show that the origin is exponentially stable:

(1)
$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x, \; |\alpha(t)| \leq 1$$

Solution.
Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned}\dot{V}(x) &= -x_1^2 + \alpha(t)x_1x_2 + \alpha(t)x_1x_2 - 2x_2^2 \\ &\leq -x_1^2 - 2x_2^2 + 2|x_1||x_2| \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \\ &\leq -0.382(x_1^2 + x_2^2)\end{aligned}$$

where 0.382 is the minimum eigenvalue of the 2×2 matrix. By Theorem 4.10, the origin is expontentially stable.

(2)
$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

Solution.
Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V} = -x_1^2 + \alpha(t)x_1x_2 - \alpha(t)x_1x_2 - 2x_2^2 = -x_1^2 - 2x_2^2$$

By Theorem 4.10, the origin is exponentially stable.

(3)
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x, \ \alpha(t) \geq 2$$

Solution.
Take $V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$, where $p_{11}p_{22} - p_{12}^2 > 0$, and let k be an upper bound on $|\alpha(t)|$.
$$\dot{V} = -2p_{12}x_1^2 + 2(p_{11} - p_{22} - \alpha(t)p_{12})x_1x_2 + 2(p_{12} - \alpha(t)p_{22})x_2^2$$
Take $p_{11} = p_{22} = p > 1$ and $p_{12} = 1$.
$$\begin{aligned} \dot{V} &\leq -2x_1^2 + 2k|x_1||x_2| - 2(2p-1)x_2^2 \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 2 & -k \\ -k & 2(2p-1) \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \end{aligned}$$
Choose $p > (1 + k^2/4)/2$ so that $4(2p-1) - k^2 > 0$. Then, the 2×2 matrix is positive definite and, by Theorem 4.10, the origin is exponentially stable.

(4)
$$\dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

In all cases, $\alpha(t)$ is continuous and bounded for all $t \geq 0$.
Solution.
Take $V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$, where $p_{11}p_{22} - p_{12}^2 > 0$, and let k be an upper bound on $|\alpha(t)|$.
$$\begin{aligned} \dot{V} &\leq -2p_{11}x_1^2 + 2k|x_1||x_2| - 4x_2^2 \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 2p_{11} & -k \\ -k & 4 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \end{aligned}$$
Choose $p_{11} > k^2/8$ so that $8p_{11} - k^2 > 0$. Then, the 2×2 matrix is positive definite and, by Theorem 4.10, the origin is exponentially stable.

20. Exercise 4.41 (Khalil pg. 189). Consider the system
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 \end{aligned}$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.
Solution.
$$x_2 = 1 = \dot{x}_1$$
$$2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 = 2t + 3t + 2 - 3t - 2(t+1) = 0 = \dot{x}_2$$
Thus, $x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, then $x(t)$ approaches $\begin{bmatrix} t & 1 \end{bmatrix}^T$ as $t \to \infty$.
Solution.
Let $z_1 = x_1 - t$ and $z_2 = x_2 - 1$. Then,
$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 2z_1z_2 - z_1 - 2z_2 \end{aligned}$$
We need to show that the origin $z = 0$ is uniformly asymptotically stable.
$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + 2z_2 & -2 + 2z_1 \end{bmatrix}$$
so
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$
The matrix A is Hurwitz; hence, the origin is uniformly asymptotically stable.

21. Exercise 4.44 (Khalil pg. 189). Consider the system

$$\dot{x}_1 = -x_1 + x_2 + (x_1^2 + x_2^2)\sin t$$
$$\dot{x}_2 = -x_1 - x_2 + (x_1^2 + x_2^2)\cos t$$

Show that the origin is exponentially stable and estimate the region of attraction.

Solution. Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned}\dot{V} &= -x_1^2 + x_1x_2 + x_1(x_1^2 + x_2^2)\sin t - x_1x_2 - x_2^2 + x_2(x_1^2 + x_2^2)\cos t \\ &= -(x_1^2 + x_2^2) + (x_1^2 + x_2^2)(x_1 \sin t + x_2 \cos t) \\ &\leq -\|x\|_2^2 + \|x\|_2^3\sqrt{(\sin t)^2 + (\cos t)^2} \\ &= -\|x\|_2^2 + \|x\|_2^3 \\ &\leq -(1-r)\|x\|_2^2\end{aligned}$$

for all $\|x\|_2 \leq r$, for any $r < 1$. Hence, by Theorem 4.10, the origin is exponentially stable. Since $V(x) = \frac{1}{2}\|x\|_2^2$, the region of attraction can be estimated by the set $\{\|x\|_2 \leq r\}$ for any $r < 1$.

22. Exercise 4.48 (Khalil pg. 190). Consider two systems represented by $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of the first system is exponentially stable if and only if the origin of the second system is exponentially stable.

Solution. Let $A_1 := \frac{\partial f}{\partial x}(0)$ be the linearization of $\dot{x} = f(x)$. To find the linearization of $\dot{x} = h(x)f(x)$, set $g(x) = h(x)f(x)$. Then

$$\frac{\partial g_i}{\partial x_j} = h(x)\frac{\partial f_i}{\partial x_j} + \frac{\partial h}{\partial x_j} f_i(x)$$

Hence

$$\frac{\partial g_i}{\partial x_j}(0) = h(0)\frac{\partial f_i}{\partial x_j}(0) + \frac{\partial h}{\partial x_j}(0) f_i(0) = h(0)\frac{\partial f_i}{\partial x_j}(0)$$

and

$$A_2 = \frac{\partial g}{\partial x}(0) = h(0)A_1$$

Since $h(0) > 0$, A_1 is Hurwitz if and only if A_2 is Hurwitz. By Theorem 4.15,

$$\dot{x} = f(x) \text{ is exp. stable} \iff A_1 \text{ is Hurwitz} \iff A_2 \text{ is Hurwitz} \iff \dot{x} = h(x)f(x) \text{ is exp. stable}$$