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Earth-to-Moon Transfer with a Limited Power Engine

M. Guelman*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

This paper presents a power-limited low-thrust optimal guidance law for transfer in a central gravitational field. This guidance law is obtained analytically as a particular solution to the case of transfer with free final time. The optimal thrust acceleration is shown to be colinear with the vehicle velocity vector multiplied by a time-varying gain that depends on a constant-guidance parameter. An analysis is performed of the system trajectories as obtained when this guidance law is applied. This guidance law is applied to the case of Earth-to-moon transfer. The initial condition is a low circular Earth orbit. For the final condition two cases are considered: 1) impact on the moon and 2) moon orbit injection. The solution for both cases is obtained by considering the gravity field of one body at a time. The initial Earth-centered trajectory consists of a number of revolutions during which the trajectory spirals away gradually from Earth. In the impact case, by adequately choosing the guidance parameter, it is possible to achieve, at a prescribed range, a moon-relative velocity such that the moon gravitational field will capture the spacecraft. For moon orbit injection, the trajectory is obtained by matching the outward Earth-spiral trajectory with a moon inward spiral. Both spirals are generated by the guidance law with the guidance parameter positive for the Earth outward spiral and negative for the moon inward one.

I. Introduction

ELECTRIC engines have long been known to be an efficient means of propulsion for space missions. Their principal attraction lies in their high exhaust velocity, which is normally one order of magnitude or more greater than that of chemical thrusters. The limiting factor in electrical engines is the required power.

Power-limited systems are characterized by performance indices for minimum fuel that are proportional to the integral to the acceleration squared, whereas conventional chemical propulsion systems have performance indices that are proportional to the integral of the acceleration.

Optimum-thrust numerical programs for power-limited propulsion systems were developed by Melbourne and Sauer[1] to generate rendezvous trajectories for Earth-to-Mars transfers for various flight times and launch dates, both for constant and variable thrust. Only the gravitational field of the sun was included in the calculations.

Under the assumption that the perturbations due to thrust are small and that the thrust acceleration is small compared to the acceleration of gravity, linearized equations of motion about a reference orbit can be employed to obtain analytic solutions for low-thrust rendezvous trajectories. Gobetz[2] determined the required thrust for transfer between neighboring circular orbits. Edelbaum[3] determined analytic solutions for the optimum correction of all six elements of elliptic satellite orbits. The solutions, developed in terms of orbital elements or functions of orbital elements, determined the optimal performance and thrust programs for various rendezvous and stationkeeping problems.

Minimum-fuel trajectories from a low Earth parking orbit to Lagrange points and the moon, were numerically obtained for a low-thrust limited-power spacecraft by Golan and Breakwell.[4] The numerical procedure started from an analytical description of a slightly elliptical spiral,[5] and the Earth–moon trajectory was found by matching an Earth spiral, to a moon spiral on the sphere of influence.

In order to obtain the required optimal acceleration vector, the maximum principle of Pontryagin is in general applied,[6] which in turn leads to solve a boundary value problem for a system of differential equations for the state and adjoint variables. Numerical solutions require the computation of the unknown initial values of the adjoint components that will comply with the required final conditions for the state. As is well known, the flight path depends very sensitively on the initial values of the adjoint components. Small errors in these values lead to very different trajectories. The task of obtaining the required initial values is thus extremely difficult. Schlingloff[7] developed a set of formulas in terms of thrust angles and their derivatives in order to numerically solve the two point boundary value problem more easily and accurately than with schemes using the more direct application of the adjoint variables.

The adjoint system of equation is strongly nonlinear, and analytical solutions are extremely difficult to obtain. In Ref. 8 a particular analytical solution was found to the problem of transfer in two dimensions with a limited power engine for free final time. This particular solution was applied to the case of rendezvous and landing on a small celestial object without atmosphere. The present work will further develop and extend the results obtained in Ref. 8 Application will be made of this development for the case of Earth-to-moon transfer.

II. Problem Statement

A spacecraft is moving in the space that surrounds a celestial body with graviational field $g(r)$, as depicted in Fig. 1. The system planar equations in polar cordinates r, θ, u, v for position and radial and transverse velocity components, respectively, are given by

$$\dot{r} = u \tag{1}$$

$$\dot{\theta} = v/r \tag{2}$$

$$\dot{u} = a \sin\beta + (v^2/r) - g(r) \tag{3}$$

$$\dot{v} = a \cos\beta - (vu/r) \tag{4}$$

where a and β are the control acceleration amplitude and direction, respectively.

We shall interest ourselves in the use of an electric engine to generate thrust to control the vehicle trajectory. In an electric engine it is possible to modify both the ejection speed v_c and the mass flow rate $\dot{m}$ and therefore to control independently the thrust T and power P, given by

$$T = -\dot{m} v_c \tag{5}$$

$$P = \tfrac{1}{2}|\dot{m}| v_c \tag{6}$$

From these relations it follows that

$$\frac{1}{m_f} - \frac{1}{m_0} = \int_{t_0}^{t_f} \frac{a^2}{2P}\, \mathrm{d}t \tag{7}$$

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*Visiting Professor, Faculty of Aerospace Engineering. Member AIAA.

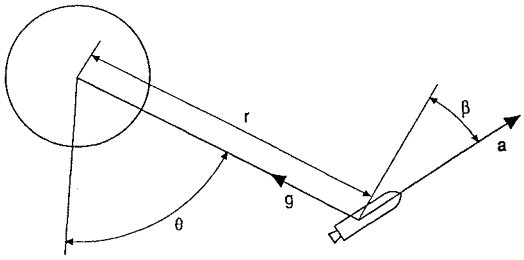

Fig. 1 Spacecraft in gravitational field.

where m is mass and $a = T/m$ is the applied thrust acceleration. For a vehicle with electric propulsion we have a limited-power system. The optimal operating point of a limited-power system uses maximum power, and the optimization criterion to minimize the fuel is to minimize the quadratic performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} a^2 \, dt \tag{8}$$

The problem to be solved is to find the required thrust acceleration amplitude and direction a and β to transfer a vehicle moving about a celestial body with gravitational field $g(r)$ from a given initial position (r_0, θ_0) and velocity (u_0, v_0) to a final position (r_f, θ_f) and velocity (u_f, v_f) at time t_f in such a way that the index J (equivalent to fuel expenditure) is a minimum. The final time of transfer t_f is free.

III. Optimal Acceleration Control

Employing the maximum principle,[6] the following Hamiltonian is formed:

$$H = \lambda_r u + \lambda_\theta \frac{v}{r} + \lambda_u\left[a\sin\beta + \frac{v^2}{r} - g(r)\right] + \lambda_v\left(a\cos\beta - \frac{vu}{r}\right) - \frac{1}{2}a^2 \tag{9}$$

The adjoint components are defined by

$$\dot\lambda_r = -\frac{\partial H}{\partial r} = \lambda_u\left(\frac{v^2}{r^2} + \frac{\mathrm{d}g(r)}{\mathrm{d}r}\right) - \lambda_v\frac{vu}{r^2} + \lambda_\theta\frac{v}{r^2} \qquad \lambda_{rf} \text{ free} \tag{10}$$

$$\dot\lambda_\theta = -\frac{\partial H}{\partial \theta} = 0 \qquad \lambda_{\theta f} \text{ free} \tag{11}$$

$$\dot\lambda_u = -\frac{\partial H}{\partial u} = \lambda_v\frac{v}{r} - \lambda_r \qquad \lambda_{uf} \text{ free} \tag{12}$$

$$\dot\lambda_v = -\frac{\partial H}{\partial v} = -\frac{\lambda_\theta}{r} - 2\lambda_u\frac{v}{r} + \lambda_v\frac{u}{r} \qquad \lambda_{vf} \text{ free} \tag{13}$$

The optimal acceleration amplitude and direction should maximize the Hamiltonian, from which it follows that

$$a\sin\beta = \lambda_u \tag{14}$$

$$a\cos\beta = \lambda_v \tag{15}$$

that is, the acceleration vector $(a\sin\beta, a\cos\beta)$ is colinear with the primer vector (λ_u, λ_v).[9]

In order to obtain the required optimal acceleration vector, the adjoint equations must be solved.

Partly motivated by the solution of the linear quadratic problem,[6] where the optimal acceleration is a linear function of position and velocity, both multiplied by a time-varying gain, let us assume for the adjoint components a solution of the form[8]

$$\lambda_u = k(t)u \tag{16}$$

$$\lambda_v = k(t)v \tag{17}$$

where $k(t)$ is a function of time yet to be determined.

Introducing Eqs. (14–17) into Eqs. (3) and (4), one obtains

$$\dot u = ku - g(r) + (v^2/r) \tag{18}$$

$$\dot v = kv - (uv/r) \tag{19}$$

Substituting now Eqs. (16–19) into Eqs. (12) and (13) and rearranging result in

$$\lambda_r = kg(r) - (\dot k + k^2)u \tag{20}$$

$$\lambda_\theta = -r(\dot k + k^2)v \tag{21}$$

For a free final time, the Hamiltonian H vanishes. Introducing (14–17) into Eq. (9), equating to zero, and rearranging, it follows that

$$H = \lambda_r u + (\lambda_\theta v/r) + \tfrac{1}{2}k^2(u^2+v^2) - kug(r) = 0 \tag{22}$$

Finally, introducing λ_r and λ_θ from Eqs. (20) and (21), respectively, into Eq. (22) and rearranging yield

$$\left(\dot k + \tfrac{1}{2}k^2\right)(u^2+v^2) = 0 \tag{23}$$

Since the total vehicle velocity is different from zero, it necessarily follows that

$$\dot k + \tfrac{1}{2}k^2 = 0 \tag{24}$$

Solving this first-order differential equation, one obtains

$$k(t) = 2/(c+t) \tag{25}$$

where c is a constant of integration with units of time.

In order to complete the verification of this particular solution, it is necessary to show that λ_r and λ_θ from Eqs. (20) and (21) are the same functions as those defined by Eqs. (10) and (11). This can be readily verified by differentiating Eqs. (20) and (21) with respect to time and comparing them with Eqs. (10) and (11), respectively, after replacing λ_u, λ_v, and $k(t)$ by their values defined in Eqs. (16), (17), and (25).

The extremal acceleration control is thus defined by

$$a\sin\beta = 2[u/(c+t)] \tag{26}$$

$$a\cos\beta = 2[v/(c+t)] \tag{27}$$

As can be seen from its derivation, this acceleration control is valid for any time-independent central gravitational field.

IV. Extremal Trajectories

The system trajectories when this extremal acceleration is applied will now be considered.

From Eq. (11) it follows that λ_θ is constant. Substituting $k(t)$ from Eq. (25) into Eq. (21) and rearranging result in

$$h/h_0 = (c+t)^2/c^2 \tag{28}$$

where $h = rv$ is the trajectory angular momentum and $h_0 = r_0 v_0 = -c^2\lambda_\theta/2$.

Substituting now Eqs. (1), (25), and (28) into Eq. (18) yields

$$\ddot r = \frac{2}{c+t}\dot r - g(r) + h_0^2\frac{(c+t)^4}{c^4 r^3} \tag{29}$$

The solutions of this equation will provide the trajectories of the vehicle when the extremal acceleration is applied.

This is a strongly nonlinear time-varying differential equation, which has no known analytical solution. In Ref. 8 a qualitative analysis of this equation was performed for the case $c = -t_f$. This analysis determined the conditions under which a spacecraft can achieve a soft landing on a small celestial body.

General, qualitative information on the system trajectory behavior can be obtained as follows: Multiplying Eq. (18) by u, introducing uv/r from Eq. (19), and rearranging yield

$$\frac{1}{2}\frac{\mathrm{d}(u^2+v^2)}{\mathrm{d}t} + g(r)u = k(t)(u^2+v^2) \tag{30}$$

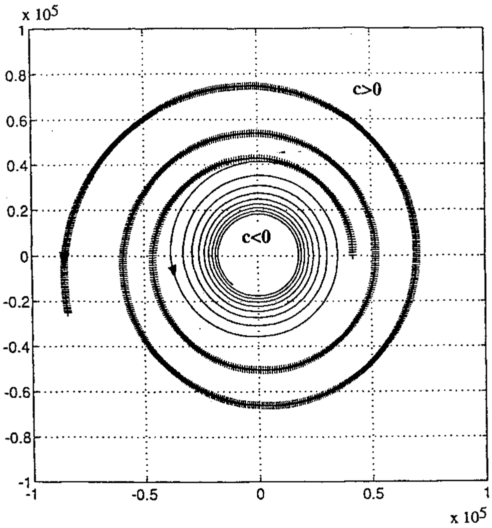

Fig. 2 Spacecraft-trajectories about central body for positive and negative values of guidance parameter *c*.

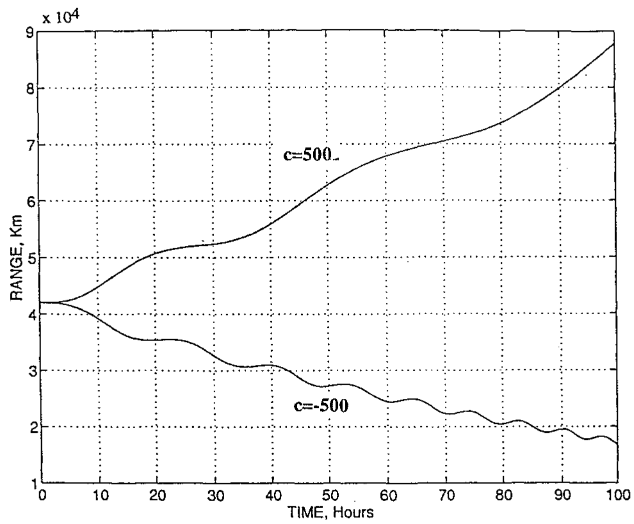

Fig. 3 Spacecraft to central body range vs time for positive and negative values of guidance constant parameter *c*.

Let us define K, U, and E such that

$$K = \tfrac{1}{2}(u^2 + v^2) \tag{31}$$

is the kinetic energy per unit mass,

$$\frac{\mathrm{d}U(r)}{\mathrm{d}r} = -g(r) \tag{32}$$

with $U(r)$ the potential energy per unit mass, and

$$E = K - U \tag{33}$$

is the total orbital energy per unit mass.

Substituting Eqs. (31–33) into Eq. (30), it follows that

$$\dot{E} = 2k(t)K \tag{34}$$

As can be readily seen from the expression of $k(t)$ given by Eq. (25) and the orbital energy rate given by Eq. (34), the orbital energy E will be either a decreasing or increasing function of time, depending on the *sign* of the constant parameter c. From expression (28) for the normalized angular momentum, it follows that h/h_0 is also either a decreasing or increasing function of time, depending on the sign of c.

For $c > 0$, E and h/h_0 are increasing functions of time.

For $c = -t_f < 0$, E and h/h_0 are decreasing functions of time for $0 < t < t_f$.

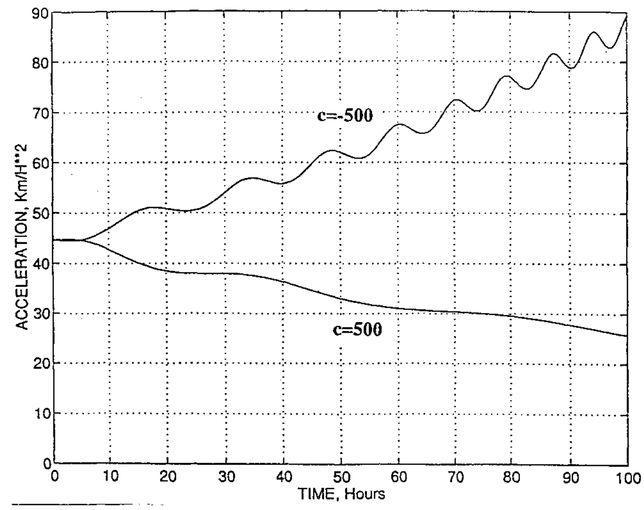

Fig. 4 Spacecraft extremal control acceleration vs time for positive and negative values of guidance constant parameter *c*.

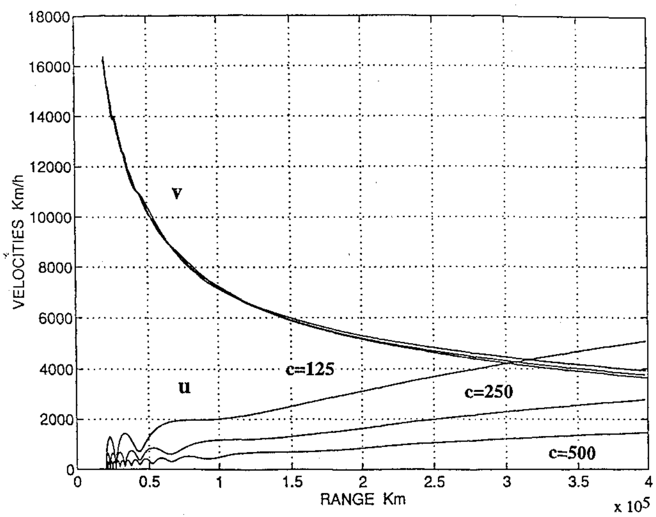

Fig. 5 Spacecraft velocity components vs range to central body for various positive values of guidance parameter *c*.

Two different classes of trajectories will be generated depending on the sign of the guidance parameter c.

Figure 2 depicts, for positive and negative values of c the spacecraft trajectories about a central body initiating from a circular orbit for the case of a spherical gravitational field, i.e.,

$$g(r) = \mu/r^2 \tag{35}$$

For positive c the spacecraft trajectory is an outward spiral with increasing angular momentum and energy. For negative c an inward spiral trajectory is generated with decreasing angular momentum and orbital energy.

Figure 3 depicts the spacecraft-to-central-body range as a function of time for positive and negative values of c. Figure 4 depicts the corresponding values of total control acceleration also as a function of time. Figure 5 depicts the transverse and radial velocities u and v for different positive values of c given in hours as functions of range. As can be seen, the transverse velocity is a decreasing function of range whereas the radial velocity is an increasing function of range. The implication is that the vehicle will finally escape the central body along a line parallel to a radius vector.

The negative c trajectories, velocities, and accelerations are identical, in the backward time direction, to the positive c trajectories generated during the time interval $0 < t < t_f$, initiating from the same final position with opposite sign velocities and $c^* = -(c + t_f)$. This can be directly verified by introducing the backward time $t^* = t_f - t$ and c^* into the equations of motion (28) and (29).

V. Earth-to-Moon Transfer

The extremal acceleration control previously obtained will now be employed to generate Earth-to-moon transfer trajectories for a spacecraft with a power-limited propulsion system.

For the Earth–moon transfer depicted in Fig. 6, the system equations describing the spacecraft trajectories correspond to those of the restricted three-body problem.[10] Assuming for both Earth and moon spherical gravitational fields, the system vector equations are given by

$$\ddot{\boldsymbol{R}}_1 = -\mu_e \frac{\boldsymbol{R}_1}{R_1^3} - \mu_m \left(\frac{\boldsymbol{R}_2}{R_2^3} + \frac{\boldsymbol{R}_m}{R_m^3} \right) + \boldsymbol{a} \tag{36}$$

$$\ddot{\boldsymbol{R}}_m = -(\mu_e + \mu_m) \frac{\boldsymbol{R}_m}{R_m^3} \tag{37}$$

where $\boldsymbol{R}_1$ and $\boldsymbol{R}_m$ are the spacecraft and moon position vectors with respect to Earth, respectively; $\boldsymbol{R}_2$ is the spacecraft-to-moon position vector, given by

$$\boldsymbol{R}_2 = \boldsymbol{R}_1 - \boldsymbol{R}_m \tag{38}$$

$\boldsymbol{a}$ is the control aceleration vector; $\mu_e = Gm_e$ and $\mu_m = Gm_m$ are the Earth and moon gravitational constants; G is the universal constant of gravitation; and m_e, m_m are the Earth and moon masses.

The Earth and moon will be assumed to move in circles around their common center of mass. The terms X_e and X_m are the Earth and moon distances to their common center of mass. The spacecraft S is assumed to be initially located in the Earth–moon orbit plane. Now, $\boldsymbol{R} = (X, Y)^T$, $\boldsymbol{R}_1 = (X + X_e, Y)^T$, $\boldsymbol{R}_2 = (X - X_m, Y)^T$, where

$$\frac{X_e}{R_m} = \frac{m_m}{m_e + m_m} = \frac{\mu_m}{\mu_e + \mu_m} \tag{39}$$

$$\frac{X_m}{R_m} = 1 - \frac{X_e}{R_m} \tag{40}$$

with R_m the constant Earth–moon range.

The equations of motion of the spacecraft S in the Earth–moon rotating coordinate system, centered at their barycenter, are given by

$$\ddot{X} - 2\omega \dot{Y} - \omega^2 X = -\left(\frac{\mu_e}{R_1^3} + \frac{\mu_m}{R_2^3} \right) X - \frac{\mu_e X_e}{R_1^3} + \frac{\mu_m X_m}{R_2^3} + a_x \tag{41}$$

$$\ddot{Y} + 2\omega \dot{X} - \omega^2 Y = -\left(\frac{\mu_e}{R_1^3} + \frac{\mu_m}{R_2^3} \right) Y + a_y \tag{42}$$

where

$$\omega \left[(\mu_e + \mu_m) / R_m^3 \right]^{\frac{1}{2}} \tag{43}$$

is the moon–Earth orbital rate.

Defining normalized coordinates $x = X/R_m$, $y = Y/R_m$, $x_e = X_e/R_m$, $x_m = X_m/R_m$, $r_1 = R_1/R_m$, and $r_2 = R_2/R_m$ and

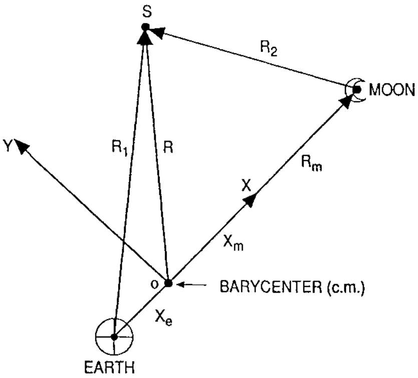

Fig. 6 Spacecraft in Earth–moon rotating frame.

normalized time $\tau = \omega t$, the normalized spacecraft equations in the Earth–moon rotating system are obtained:

$$\ddot{x} = 2\dot{y} + x - \left(\frac{x_m}{r_1^3} + \frac{x_e}{r_2^3} \right) + x_e x_m \left(\frac{1}{r_2^3} - \frac{1}{r_1^3} \right) + a_{xn} \tag{44}$$

$$\ddot{y} = -2\dot{x} + y - \left(\frac{x_m}{r_1^3} + \frac{x_e}{r_2^3} \right) y + a_{yn} \tag{45}$$

where $a_{xn} = a_x/\omega^2 R_m$ and $a_{yn} = a_y/\omega^2 R_m$.

In order to generate Earth-to-moon transfer trajectories for the spacecraft, the extermal acceleration as defined in Eqs. (26) and (27) will be employed with a single celestial body considered at a time.

The extremal control acceleration is defined by the relative velocity to the central body multiplied by a time-varying gain. When trajectories relative to Earth are considered, the spacecraft normalized acceleration $\boldsymbol{a}_n$ will be defined as

$$\boldsymbol{a}_n = (a_{xn}, a_{yn})^T = k_e \boldsymbol{v}_e \tag{46}$$

where $\boldsymbol{v}_e$ is the spacecraft-to-Earth relative velocity and is obtained by differentiating with respect to time the spacecraft-to-Earth relative position in the rotating frame, with the rotating frame normalized angular rate equal to unity, i.e.,

$$\boldsymbol{v}_e = (\dot{x} - y, \dot{y} + x + x_e)^T \tag{47}$$

When trajectories relative to the moon are considered,

$$\boldsymbol{a}_n = k_m \boldsymbol{v}_m \tag{48}$$

with

$$\boldsymbol{v}_m = (\dot{x} - y, \dot{y} + x - x_m)^T \tag{49}$$

the spacecraft-to-moon relative velocity.

Gains k_e and k_m as defined in Eq. (25) will be employed, with the constant-guidance parameters c_e and c_m for Earth and moon, respectively.

The actual Earth-to-moon trajectories will be generated numerically. The restricted three-body problem normalized equations are employed in the numerical computations. A fourth-order Runge–Kutta numerical integration with a fixed step size of 0.036 s was employed.

Two different cases will be considered: 1) moon impact and 2) moon orbit injection. For both cases the initial condition is a low circular Earth orbit. The Earth–moon constant range is taken as $R_m = 384,710$ km and the orbital period as $\omega = 2.683 \times 10^{-6}$ rad/s.

For moon impact a single spiral trajectory is generated, by considering in the guidance law only the relative velocity components to Earth. A positive value of the guidance constant c is employed to generate this outward Earth spiral.

A spacecraft trajectory achieving impact with the moon can be realized for a wide range of values of the guidance constant c. Moon impact was achieved here by adequately selecting the initial orbit location for guidance initiation. For a given value of the constant c there exists an initial value for the angular spacecraft orbit location such that impact is achieved.

Since maximum acceleration for this outward Earth trajectory from low Earth orbit occurs at guidance initiation and is inversely proportional to the guidance constant c, the value of c will be defined according to the maximum available acceleration.

Two examples of impact are presented with the constant-guidance parameter $c = 500$ h and $c = 1000$ h, respectively. The initial spacecraft circular orbit was taken with a radius $R = 6800$ km. The corresponding maximum acceleration is 0.0086 m/s^2 for $c = 500$ h and 0.0043 m/s^2 for $c = 1000$ h. The required initial angular orbit location to achieve impact was determined by simple interpolation. The initial angular orbit location as measured from the Earth–moon line is $\theta_0 = 1.51$ rad for $c = 500$ h and $\theta_0 = 3.25$ rad for $c = 1000$ h. Figure 7 depicts the spacecraft trajectory in the Earth–moon

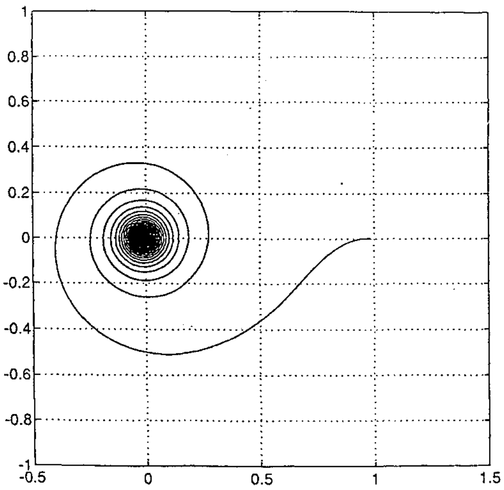

Fig. 7 Spacecraft trajectory in Earth–moon rotating frame for moon impact with $c = 500$.

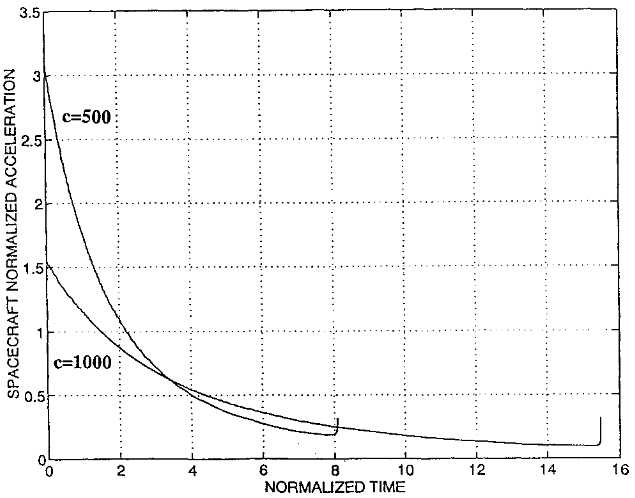

Fig. 8 Spacecraft normalized control acceleration vs normalized time for moon impact.

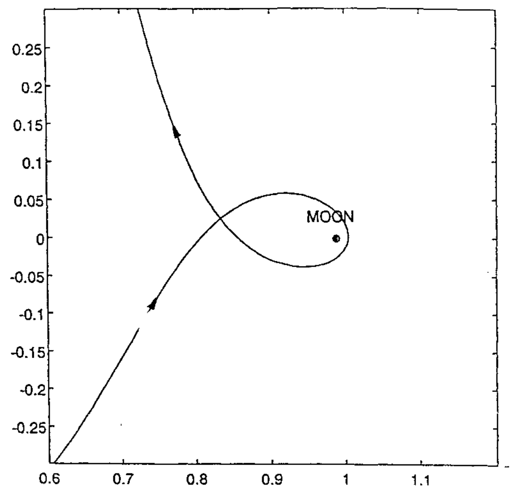

Fig. 9 Spacecraft Earth-to-moon "missing" trajectory.

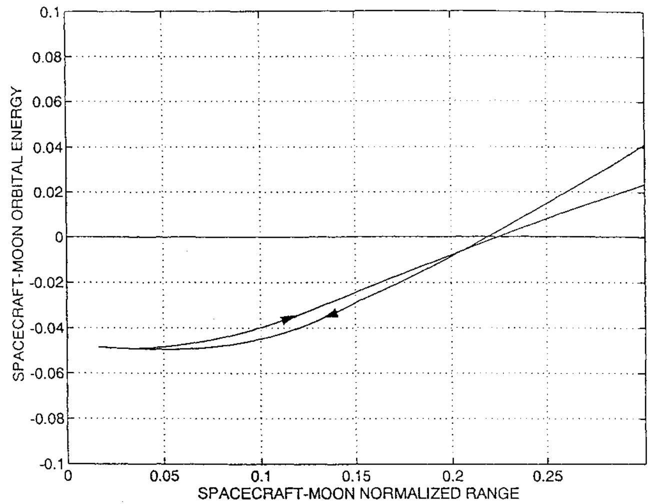

Fig. 10 Spacecraft-to-moon normalized orbital energy vs spacecraft-to-moon normalized range.

rotating system for $c = 500$ h. The spacecraft trajectory completes 59 revolutions about Earth and the total cost is $J = 13.05$ m^2/s^3. For $c = 1000$ h, the spacecraft completes 126 revolutions about Earth and the total cost is $J = 6.48$ m^2/s^3. Figure 8 depicts the spacecraft normalized control acceleration amplitude as a function of normalized time for both $c = 500$ h and $c = 1000$ h.

For the case of moon orbit injection, the trajectory is generated, as previously stated, using the restricted three-body problem normalized equations and considering for thrust acceleration definition one celestial body at a time. The outward Earth-spiral trajectory with a positive guidance parameter c is such that it will match an inward moon spiral generated with a negative c guidance parameter. In fact there is an entire region in the space defined by the spacecraft-to-moon relative range, relative velocity, from where moon orbit injection can be achieved.

The numerical procedure was implemented as follows: Initiating from the required initial angular position to achieve moon impact, this value was modified to actually miss the moon. An example of a "missing" trajectory when near the moon is shown in Fig. 9. When using low thrust, as is here the case, the actual spacecraft-to-moon relative velocity is such that, from a certain spacecraft-to-moon range, the normalized orbital energy with respect to the moon,

$$E_m = \frac{1}{2}v_m^2 - \frac{\mu_m^*}{r_2}, \qquad \mu_m^* = \frac{\mu_m}{\mu_e + \mu_m} \tag{50}$$

becomes negative, as depicted in Fig. 10. If the gravitational effects of Earth would not be present, moon orbit insertion would be achieved without any additional maneuver. Under the effects of the Earth gravitational field an additional maneuver is indeed required. Moon orbit insertion is achieved by using the extremal thrust acceleration guidance with a negative c constant.

Switch time to initiate application of the extremal thrust acceleration with respect to the moon was obtained by a simple search along the Earth outward spiral, initiating from the time where moon orbital energy becomes negative. At the switch time, the guidance law is generated by using the spacecraft–moon relative velocity and running time for thrust acceleration application is reinitialized.

The initial spacecraft circular orbit has a radius $R = 6800$ km, the constant-guidance parameter for the Earth outward spiral is $c = 500$ h, the switch to lunar injection is performed at $t = 817.8$ h, and the constant-guidance parameter for the moon inward spiral is $c = -550$ h. Figure 11 depicts the spacecraft trajectory in the Earth–moon rotating system. The spacecraft completes 59 revolutions about Earth and six revolutions about the moon before switching off thrust acceleration at $t = 1035$ h. The total cost is $J = 13.265$ m^2/s^3. Figure 12 depicts the applied normalized thrust acceleration as a function of normalized time.

Once control acceleration is brought to zero at $t = 1035$ h, the spacecraft enters into a stable elliptic orbit about the moon with semimajor axis $a = 6875$ km and eccentricity $e = 0.785$.

Final orbital conditions depend on the choice of the guidance parameter c, the time of application of the thrust acceleration with respect to the moon, and the time of thrust switch off. For instance, for a switch time $t = 807.5$ h and the same time for switch-off as previously ($t = 1035$ h), the spacecraft enters into an elliptic orbit about the moon with semimajor axis $a = 5097$ km and eccentricity $e = 0.758$.

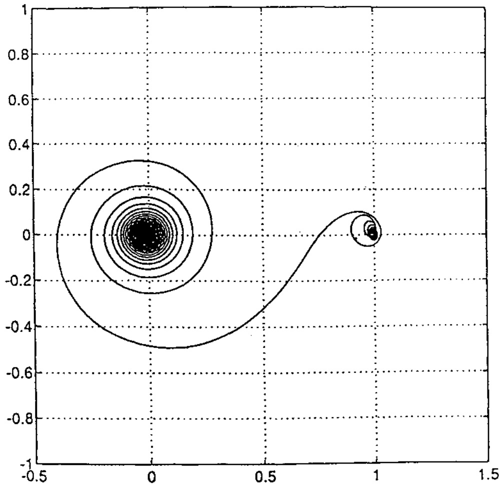

Fig. 11 Spacecraft trajectory in Earth–moon rotating frame for moon orbit injection.

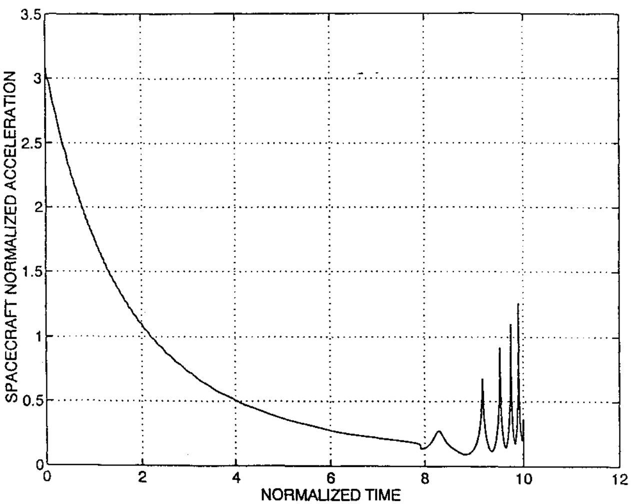

Fig. 12 Spacecraft normalized acceleration vs normalized time for moon orbit injection.

VI. Conclusions

In this work an extremal control acceleration for a power-limited spacecraft under the influence of a time-independent central gravitational field was derived. The extremal control acceleration is defined by the relative velocity to the central body multiplied by a time-varying gain, itself depending on a constant-guidance parameter. A general analysis of the spacecraft trajectories was performed. Depending on the sign of the constant-guidance parameter, the spacecraft trajectory is either an outward or inward spiral with respect to the central body. The orbital energy and trajectory angular momentum absolute value either increase or decrease with time.

This extremal control acceleration was applied for the case of Earth-to-moon transfer. Both moon impact and moon orbit injection initiating from a low Earth orbit were considered. The trajectories were obtained by considering a single celestial body at a time. These trajectories were obtained in an open-loop fashion; that is, the orbital angle at departure, as well as the switching time for moon orbit injection were employed as design parameters.

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