Advanced Process Controls
ECH 267

Abhay Negi
914181497

Homework #1

① $y^n = g_1(t, y, \dot{y}, \dots, y^{n-1}, u) + g_2(t, y, \dot{y}, \dots, y^{n-2})\dot{u}$

hint: $x_n = y^{n-1} - g_2(t, y, \dot{y}, \dots, y^{n-2})u$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{n-2} \\ y^{n-1} - g_2(t, y, \dot{y}, \dots, y^{n-2})u \end{bmatrix}$$

$\boxed{y = x_1}$

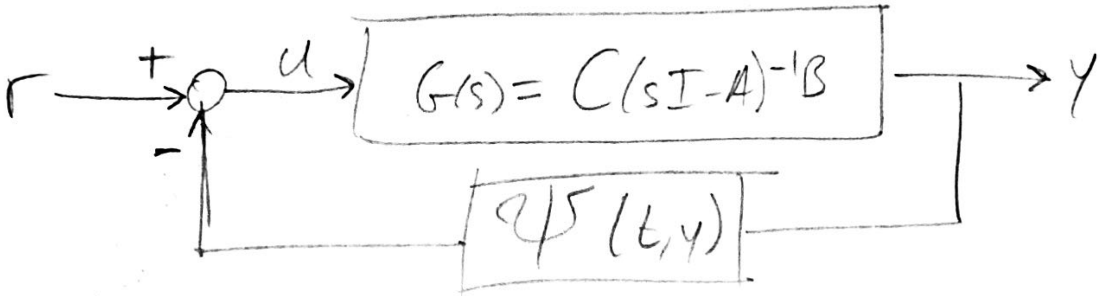
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \\ \vdots \\ y^{n-1} \\ y^n - g_2(t, x_1, \dots, x_{n-1})\dot{u} - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}\dot{x}_1 + \dots + \frac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1}\right)u \end{bmatrix}$$

$$\dot{x}_n = \frac{d}{dt}\left(y^{n-1} - g_2(t, y, \dot{y}, \dots, y^{n-2})u\right) = \frac{d}{dt}(y^{n-1}) - \frac{d}{dt}\left(g_2(t, y, \dot{y}, \dots, y^{n-2})u\right)$$

$$\dot{x}_n = y^n - \frac{d}{dt}(g_2(\sim))u - g_2(\sim)\frac{d}{dt}u = y^n - g_2(\sim)\dot{u} - \frac{d}{dt}(g_2(\sim))u$$

$$\frac{d}{dt}(g_2(\sim)) = \frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}\dot{x}_1 + \dots + \frac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1}$$

(2)



$$y = Gu$$
$$u = r - \psi(t,y) = r - \psi(t, Gu)$$

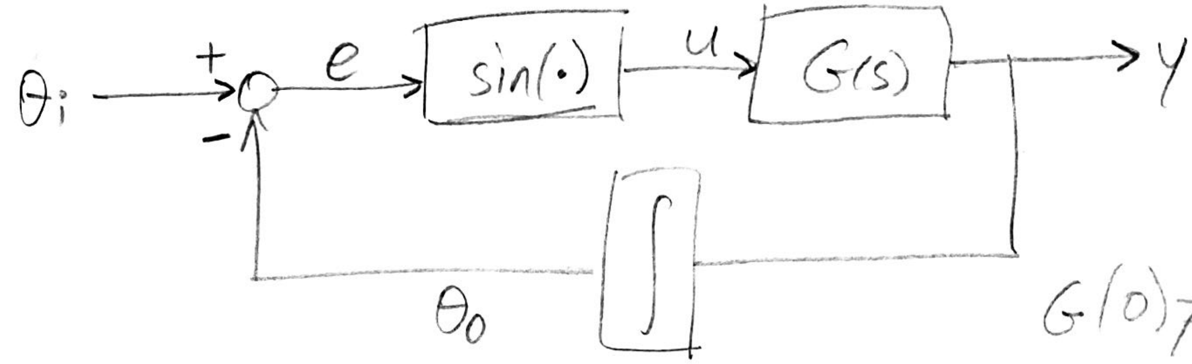
let $\dot{x} = Ax + Bu$

$$\dot{x} = Ax + B(r - \psi(t,y))$$

$$Cx := y$$

$$\boxed{\begin{aligned} \dot{x} &= Ax + Br - B\psi(t, Cx) \\ y &= Cx \end{aligned}}$$

3



$G(0) \neq 0$

$\theta_i = \text{const.}$

(a) $y = gu = \sin(e) = g\sin(\theta_i - \theta_0)$

$\dot{z} = Az + B\sin e$

$\dot{e} = -Cz$

$\theta_0 = \int y \, dt$

$e = \theta_i - \theta_0$

$\dot{e} = \dot{\theta}_i - \dot{\theta}_0 = 0 - \dot{\theta}_0 \Rightarrow \dot{e} = -\dot{\theta}_0$

$Cz := \dot{\theta}_0$

$\dot{z} := Az + Bu$

$u := \sin e$

$$\boxed{\begin{aligned} \dot{z} &= Az + B\sin e \\ \dot{e} &= -Cz \end{aligned}}$$

$G(s) = C(sI - A)^{-1}B$

$G(0) = CA^{-1}B$

(b) $\dot{z} = Az + B\sin e$

$CA^{-1}B = G(0) \neq 0$

$\dot{z}_{eq} = 0 = Az_{eq} + B\sin e_{eq} \rightarrow \sin e_{eq} = 0$

$z_{eq} = -A^{-1}B\sin e_{eq}$

$\dot{e}_{eq} = 0 = -Cz_{eq}$

$+ CA^{-1}B\sin e_{eq} = 0$

$$\boxed{e_{eq} = \pm n\pi, \ n \in \mathbb{Z}}$$

$z_{eq} = -A^{-1}B\sin e_{eq}$

$$\boxed{z_{eq} = 0}$$

(c) pendulum state space model:

$\dot{x}_1 = x_2$

$\dot{x}_2 = -g/l \sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml^2} T$

$G(s) = \frac{1}{\tau s+1} = C(sI-A)^{-1}B \underset{\text{scalar}}{=} \frac{CB}{s-A} = \frac{C}{\frac{1}{B}s - \frac{A}{B}}$

$\frac{1}{B} = \tau \Rightarrow \underline{B = \frac{1}{\tau}}, \ \underline{C = 1}, \ -\frac{A}{B} = 1 \Rightarrow \underline{A = -\frac{1}{\tau}}$

$\dot{z} = Az + B\text{sine}$

$\dot{z} = -\frac{1}{\tau} z + \frac{1}{\tau}\text{sine}$

$\dot{e} = -z$

$\Updownarrow$

$\dot{x}_1 = x_2$

$-e : x_1$
$z : x_2$

$\dot{z} = -\frac{1}{\tau}\sin(-e) - \frac{1}{\tau} z$

$\Updownarrow$

$\dot{x}_2 = -g/l \sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml^2} T$

$g/l = 1/\tau, \ k/m = 1/\tau, \ T = 0$

4 $F = ma = m\ddot{y} = mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$

$$m\ddot{y} + c_1\dot{y} + c_2\dot{y}|\dot{y}| + ky = mg$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = [mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|]/m$$

$$\dot{x}_2 = -\frac{k}{m}y - \frac{c_1}{m}\dot{y}|\dot{y}| + g$$

$$\boxed{\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2|x_2| + g \end{aligned}}$$

(5) $\dot{x}(t) = [x(t)]^{1/3}, \quad x(0) = 0$

$$\frac{dx}{dt} = x^{1/3}$$

$$\int x^{-1/3} dx = \int dt$$

$$\frac{2}{3} x^{2/3} = t + \cancel{c}\ \delta$$

$$\boxed{x(t) = \left[\frac{2}{3}(t+\delta)\right]^{3/2}}$$

$$\boxed{\begin{aligned} x &= 0 \\ \dot{x} &= 0 \end{aligned}}$$

δ is arbitrary

↳ $x(t) = \frac{2}{3}(t+\delta)^{3/2}$ is a family of solutions

↳ no unique solution to $\dot{x}(t) = [x(t)]^{1/3}, x(0) = 0$

(also, $x(t) = 0$ is another solution)

Problem 7

(i) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$Av = \lambda v$$
$$(A - \lambda I)v = 0$$

ⓐ

$$\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} v = 0$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = (-\lambda)(-3-\lambda) - (1)(-2) = 0$$

$$3\lambda + \lambda^2 + 2 = 0 \Rightarrow \lambda = \{-2, -1\}$$

$\lambda = -2,$ $\quad (\lambda+1)(\lambda+2) = 0$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\cancel{0v_1} + 1v_2 = -2v_1 \rightarrow$

$-2v_1 - 3v_2 = -2v_2$

$-2v_1 + 6v_1 = -4v_1$

$$\lambda_1 = -2 \rightarrow v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\lambda = -1,$
$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow$$

$\cancel{0v_1} + 1v_2 = -v_1$

$-2v_1 - 3v_2 = -v_2$

$$\lambda_2 = -1 \rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_1, \lambda_2 \in \mathbb{R}$

$\rightarrow M = [v_1 \ v_2] \Rightarrow \boxed{M = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}}$

swap v_1 & v_2 and multiply by -1

$$\Rightarrow \boxed{M = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}$$

ⓑ $\lambda_1, \lambda_2 \in \mathbb{R}$

$\lambda_1 \lambda_2 > 0$

$\rightarrow$ $\boxed{(0,0) \text{ is an unstable node}}$

(c) $M = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $J_r = M^{-1}AM = \frac{1}{(1)(2)-(-1)(-1)} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$J_r = +1 \begin{bmatrix} -2 & -1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

$\dot{z} = M^{-1}AMz = J_r z$

$$\boxed{\begin{aligned} \dot{z}_1 &= -z_1 \\ \dot{z}_2 &= -2z_2 \end{aligned}}$$

(ii) $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \rightarrow \lambda = \{1\} \rightarrow v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Re\{\lambda\} > 0 \rightarrow$ $\boxed{(0,0) \text{ is an unstable node}}$

$M = [v_1 \; v_2]$, $Av_2 = \lambda v_2 + v_1$

$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 1 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} + \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$

$\begin{bmatrix} -v_{22} \\ v_{21} + 2v_{22} \end{bmatrix} = \begin{bmatrix} v_{21} + v_{11} \\ v_{22} + v_{12} \end{bmatrix}$

$v_{11} + v_{21} + v_{22} = 0$

$v_{12} - v_{21} - v_{22} = 0$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$-1 + v_{21} + v_{22} = 0$

$1 - v_{21} - v_{22} = 0$

$-2 - 2v_{21} - 2v_{22} = 0$

$v_{21} + v_{22} = 1$

choose $v_{21} = 0 \rightarrow v_{22} = 1$

$J_r = M^{-1}AM = \frac{1}{(-1)(1)-(0)(1)} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ $\rightarrow \boxed{M = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}$

$J_r = -1 \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = -\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \rightarrow \underline{\dot{z} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z} \Rightarrow \boxed{\begin{aligned} \dot{z}_1 &= z_1 + z_2 \\ \dot{z}_2 &= z_2 \end{aligned}}$

(iii) $A=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix} \rightarrow \lambda=\{-1,1\}$

$V_1=\begin{bmatrix}-1\\ 2\end{bmatrix} \quad V_2=\begin{bmatrix}1\\ 0\end{bmatrix}$

(a) $M=[V_1 \ V_2]=\begin{bmatrix}-1 & 1\\ 2 & 0\end{bmatrix}, \quad M^{-1}AM=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

(b) $\lambda \in \mathbb{R}$

$\lambda_1<0, \ \lambda_2>0 \rightarrow$ $\boxed{(0,0) \text{ is a saddle point}}$

(c) $\dot{z}=J_r z \Rightarrow \boxed{\begin{aligned}\dot{z}_1&=-z_1\\ \dot{z}_2&=z_2\end{aligned}}$

(iv) $A=\begin{bmatrix}1 & 5\\ -1 & -1\end{bmatrix} \rightarrow \lambda=\{-2i,+2i\}$

$\lambda_1=\alpha+j\beta$
$\lambda_2=\alpha-j\beta$
$\alpha=0, \ \beta=2$

$V_1=\begin{bmatrix}-1-2i\\ 1\end{bmatrix} \quad V_2=\begin{bmatrix}-1+2i\\ 1\end{bmatrix}$

$M=[u \ v] \qquad A(u+jv)=(\alpha+\beta j)(u+jv)$

$\begin{bmatrix}1 & 5\\ -1 & -1\end{bmatrix}\begin{bmatrix}u_1+jv_1\\ u_2+jv_2\end{bmatrix}=(0+2j)\begin{bmatrix}u_1+jv_1\\ u_2+jv_2\end{bmatrix}$

$\begin{bmatrix}u_1+jv_1+5u_2+5jv_2\\ -u_1-jv_1-u_2-jv_2\end{bmatrix}=\begin{bmatrix}2u_1j-2v_1\\ 2u_2j-2v_2\end{bmatrix}$

$u_1+5u_2=-2v_1$
$v_1+5v_2=2u_1$
$-u_1-u_2=-2v_2$
$-v_1-v_2=2u_2$

$\begin{bmatrix}1 & 5 & 2 & 0\\ -2 & 0 & 1 & 5\\ -1 & -1 & 0 & 2\\ 0 & 2 & 1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ v_1\\ v_2\end{bmatrix}=0$

(b) $Re(\lambda)=0 \rightarrow$ $\boxed{\text{critically stable center}}$

(c) $J_r = M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$\dot{z} = J_r z$ $\boxed{\begin{array}{l} \dot{z}_1 = -2z_2 \\ \dot{z}_2 = 2z_1 \end{array}}$

(v) $A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \rightarrow \lambda_1 = 1+i,\ \lambda_2 = 1-i \rightarrow \lambda = \alpha \pm \beta i \Rightarrow \begin{array}{l} \alpha = 1 \\ \beta = 1 \end{array}$

$v_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix},\ V_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

(b) $Re(\lambda) > 0 \rightarrow$ $\boxed{(0,0) \text{ is an unstable focus}}$

(c) $J_r = M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\boxed{\begin{array}{l} \dot{z}_1 = z_1 - z_2 \\ \dot{z}_2 = z_1 + z_2 \end{array}}$

$M = [u\ v]$, $A(u+vj) = (\alpha+\beta j)(u+vj)$

$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u_1 + v_1 j \\ u_2 + v_2 j \end{bmatrix} = (1+j)\begin{bmatrix} u_1 + v_1 j \\ u_2 + v_2 j \end{bmatrix}$

$\begin{bmatrix} 2u_1 + 2v_1 j - u_1 - v_1 j \\ 2u_1 + 2v_1 j \end{bmatrix} = \begin{bmatrix} u_1 + v_1 j + u_1 j - v_1 \\ u_2 + v_2 j + u_2 j - v_2 \end{bmatrix}$

$0 = \begin{bmatrix} \cancel{2u_1 + 2v_1 j - u_1 - v_1 j - u_1 - v_1 j - u_1 j + v_1} \\ 2u_1 + 2v_1 j - u_2 - v_2 j - u_2 j + v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -u_1 \\ 2u_1 - u_2 + v_2 \\ 2v_1 - v_2 - u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$v_1 = 0,\ u_1 = 0.$

$\rightarrow u_2 = v_2,\ u_2 = -v_2$

$u = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(8) $\dot{x}_1 = -x_1 - \dfrac{x_2}{\ln\sqrt{x_1^2+x_2^2}}$

$\dot{x}_2 = -x_2 + \dfrac{x_1}{\ln\sqrt{x_1^2+x_2^2}}$ $\qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

$$\frac{\partial}{\partial x_1} \frac{1}{\ln\sqrt{x_1^2+x_2^2}} = \frac{\partial \left(\ln\sqrt{x_1^2+x_2^2}\right)^{-1}}{\partial \ln\sqrt{x_1^2+x_2^2}} \cdot \frac{\partial \ln\sqrt{x_1^2+x_2^2}}{\partial \sqrt{x_1^2+x_2^2}} \cdot \frac{\partial \sqrt{x_1^2+x_2^2}}{\partial x_1}$$

$$\frac{\partial}{\partial x_1} \frac{1}{\ln\sqrt{x_1^2+x_2^2}} = \left(\frac{-1}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2}\right)\left(\frac{1}{\sqrt{x_1^2+x_2^2}}\right)\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right)$$

without loss of generality $\dfrac{\partial}{\partial x_2} \dfrac{1}{\ln\sqrt{x_1^2+x_2^2}} = \left(\dfrac{-1}{\ln\sqrt{x_1^2+x_2^2}}\right)\left(\dfrac{1}{\sqrt{x_1^2+x_2^2}}\right)\left(\dfrac{x_2}{\sqrt{x_1^2+x_2^2}}\right)$

$$\eta := \frac{1}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2 (x_1^2+x_2^2)}, \quad \mu := \frac{1}{\ln\sqrt{x_1^2+x_2^2}}$$

$$\frac{\partial}{\partial x_1} \frac{1}{\ln\sqrt{x_1^2+x_2^2}} = -\eta x_1, \quad \frac{\partial}{\partial x_2} \frac{1}{\ln\sqrt{x_1+x_2^2}} = -\eta x_2$$

$$\frac{\partial \dot{x}_1}{\partial x_1} = -1 + \eta x_1 x_2, \quad \frac{\partial \dot{x}_1}{\partial x_2} = 0 - (1)(\mu) - (x_2)(-\eta x_2)$$

$$\frac{\partial \dot{x}_2}{\partial x_1} = 0 + (1)(\mu) + (x_1)(-\eta x_1), \quad \frac{\partial \dot{x}_2}{\partial x_2} = -1 - \eta x_1 x_2$$

$$\frac{\partial \dot{x}}{\partial x} = \begin{bmatrix} -1 + \eta x_1 x_2 & -\mu + \eta x_2^2 \\ \mu - \eta x_1^2 & -1 - \eta x_1 x_2 \end{bmatrix}$$

$$f(x) \simeq f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} (x - x_0)$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad f = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\lim_{x \to 0} x_1 x_2 \eta = \lim_{x \to 0} \frac{x_1 x_2}{\left(\ln\sqrt{x_1^2 + x_2^2}\right)^2 (x_1^2 + x_2^2)} = 0 \longleftarrow$$

$$\lim_{x \to 0} \mu = \lim_{x \to 0} \frac{1}{\ln\sqrt{x_1^2 + x_2^2}} = 0$$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \begin{bmatrix} -1+0 & -0+0 \\ 0-0 & -1-0 \end{bmatrix} \Rightarrow \boxed{\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$Av - \lambda v = 0$$
$$(A - \lambda I) \cdot v = 0 \Rightarrow \left| \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \right| = 0$$

$$(-1-\lambda)^2 = 0 \Rightarrow \boxed{\lambda_{1,2} = -1}$$

$Re(\lambda_i) < 0 \Rightarrow$ $\boxed{\text{origin is a stable node}}$

(8) (b) (1) $\dot{x}_1 = -x_1 - \dfrac{x_2}{\ln\sqrt{x_1^2+x_2^2}}$

(2) $\dot{x}_2 = -x_2 + \dfrac{x_1}{\ln\sqrt{x_1^2+x^2}}$ $\qquad \dot{r} = -r$

$r := \sqrt{x_1^2+x_2^2}\,, \quad \theta := \tan^{-1}\left(\dfrac{x_2}{x_1}\right)$

$$\dot{x}_1^2 + \dot{x}_2^2 = \dot{r}^2 = x_1^2 + \cancel{\frac{2x_1x_2}{\ln r}} + \frac{x_2^2}{(\ln r)^2}$$

$$+ x_2^2 - \cancel{\frac{2x_1x_2}{\ln r}} + \frac{x_1^2}{(\ln r)^2}$$

$$\dot{r}^2 = r^2 + \frac{r^2}{(\ln r)^2} = r^2\left[1 + \frac{1}{(\ln r)^2}\right]$$

$$\lim_{r\to 0}\left[1 + \frac{1}{(\ln r)^2}\right] = 1 \Rightarrow \dot{r}^2 = r^2$$

$$\dot{r} = \pm r$$

$\underline{\dot{r} = -r}$ because the origin is a stable node

$\downarrow$

$$\boxed{r(t) = r_0 e^{-t}}$$

(1) $\dot{r}\cos\theta - r\sin(\theta)\dot{\theta} = -r\cos\theta - \dfrac{r\sin\theta}{\ln r}$

$\cancel{-r\cos\theta} - r\sin(\theta)\dot{\theta} = \cancel{-r\cos\theta} - \dfrac{\cancel{r\sin\theta}}{\ln r}$

$$\boxed{\dot{\theta} = \frac{1}{\ln r}} \qquad \dot{\theta} = \frac{1}{\ln(r_0e^{-t})} = \frac{1}{\ln r_0 - t}$$

$$\dot{\theta} = \frac{1}{\ln r_0 - t}$$

$u = \ln r_0 - \tau$
$du = -d\tau$

$$\theta = \int_0^t \dot{\theta}\, d\tau = \int_0^t \frac{1}{\ln r_0 - \tau} d\tau = -\int_{u=\ln r_0}^{\ln r_0 - t} \frac{1}{u} du = -\ln u \Big|_{\ln r_0}^{\ln r_0 - t}$$

$$\boxed{\theta = -\ln|\ln r_0 - t| - \ln|r_0| + \theta_0}$$

for $0 < r_0 < 1$, $\dot{r} < 0$ and $\dot{\theta} < 0$

$\lim_{t \to \infty} r(t) = 0$, $\lim_{t \to \infty} \theta(t) = -\infty$

$\Rightarrow$ state trajectory spirals towards origin

(c) The linearized system has a stable equilibrium at the origin, but in the non-linear system, the origin is a stable focus.

(11) $\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2$, $\|x_1\|^2 = \sum_{i=1}^{n} |x_i|^2 + 2\sum_{i,j,\, i\neq j} |x_i||x_j|$

$$\sum_{i=1}^{N} |x_i|^2 \leq \sum_{i=1}^{n} |x_i|^2 + 2\sum_{i,j,\, i\neq j} |x_i||x_j|$$

$$\|x\|_2^2 \leq \|x_1\|^2$$

$$\boxed{\|x\|_2 \leq \|x_1\|}$$

Cauchy-Schwarz Inequality: $|x^T y| \leq \|x\|_2 \|y\|_2$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i| \cdot 1 = z^T \mathbb{1}$$

$$z := \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix}$$

$$z = \|x\|_2 \|\mathbb{1}\|_2 = \sqrt{n}\, \|x\|_2$$

$$\|\mathbb{1}\|_2 = \sqrt{\sum_{i=1}^{n} |1|^2} = \sqrt{n}$$

by Cauchy-Schwarz Inequality,

$$|z^T \mathbb{1}| \leq \|z\|_2 \|\mathbb{1}\|_2$$

$$\boxed{\|x_1\| \leq \sqrt{n}\, \|x\|_2}$$

$$\boxed{\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\, \|x\|_2}$$

$$\|x\|_\infty = \max(|x_i|) = \left(\max(|x_i|)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \|x\|_2$$

$$\boxed{\|x\|_\infty \leq \|x\|_2}$$

for all i, $|x_i| \leq \max(|x_i|) = \|x\|_\infty$

$$\Rightarrow x_i^2 \leq \|x\|_\infty^2$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sqrt{\sum_{i=1}^{n} \|x\|_\infty^2} = \sqrt{n \|x\|_\infty^2}$$

$$\boxed{\|x\|_2 \leq \sqrt{n}\, \|x\|_\infty} \rightarrow \boxed{\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\, \|x\|_\infty}$$

$\|x\|_2 \leq \|x\|_1$ & $\|x\|_\infty \leq \|x\|_2 \rightarrow \underline{\|x\|_\infty \leq \|x_1\|}$

$$\|x\|_1 \leq \sqrt{n}\, \|x\|_2$$

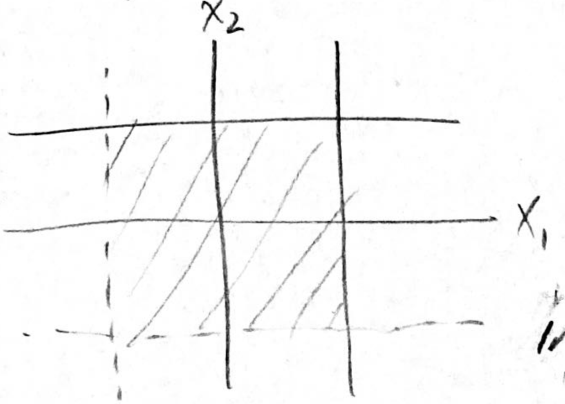
$$\times \quad \underline{\|x\|_2 \leq \sqrt{n}\, \|x\|_\infty}$$

$$\|x\|_1 \cancel{\|x\|_2} \leq n \cancel{\|x\|_2} \|x\|_\infty$$

$$\underline{\|x\|_1 \leq n \|x\|_\infty}$$

$$\boxed{\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty}$$

(12) $S = \{ x \in \mathbb{R}^2 \mid -1 < x_i \leq 1, i = 1,2 \}$



S is not an open set

because all the points in the neighborhood of the points on the closed boundary are not members of S

$(CB = \{ (x_1 = 1, -1 < x_2 \leq 1) \cup (x_2 = 1, -1 < x_1 \leq 1) \}$

$S' = \mathbb{R}^2 - S$ is similarly not an open set since it has closed boundries $\Rightarrow$ S is not a closed set

$cl(S) = \{ x \in \mathbb{R}^2 \mid -1 \leq x_i \leq 1, i = 1,2 \}$ (closure)

$int(S) = \{ x \in \mathbb{R}^2 \mid -1 < x_i < 1, i = 1,2 \}$ (interior)

$bd(S) = \{ x \in \mathbb{R}^2 \mid (x_1 = 1, -1 \leq x_2 \leq 1) \cup (x_1 = -1, -1 \leq x_2 \leq 1)$
$\cup (x_2 = 1, -1 \leq x_1 \leq 1) \cup (x_2 = -1, -1 \leq x_1 \leq 1) \}$

(boundary)

(13) $u_T(t) = \begin{cases} 0, & t<T \\ 1, & t \geq T \end{cases}$ | the interval boundary ($t=T$) has finite limits $(0,1)$ from both sides $(-,+)$

a) $f_1 = 0$ is continuous on $(-\infty, T)$

$f_2 = 1$ is continuous on $[T, \infty)$

$u_T(t)$ has a finite number (1) of discontinuities (at $t=T$), and is continuous otherwise, therefore $u_T(t)$ is piecewise continuous

b) $f(t) = g(t)u_T(t) = \begin{cases} 0, & t<T \\ g(t), & t \geq T \end{cases}$

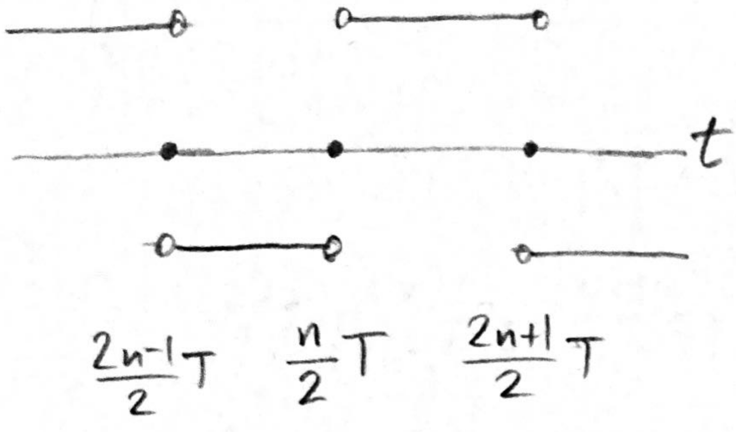
$f_1 = 0$ is continuous on $(-\infty, T)$

$f_2 = g(t)$ is continuous on $[T, \infty]$ $\leftarrow$ $g(t)$ is cont.

$f(t)$ has a finite number (1) of discontinuities (at $t=T$), and is continuous otherwise, therefore $f(t)$ is piecewise continuous

(13)

c) $sq(t) = sgn\left(\sin\left(\frac{2\pi t}{T}\right)\right) = \begin{cases} +1, & \frac{n}{2}T < t < \frac{2n+1}{2}T \\ 0, & t = \frac{n}{2}T \\ -1, & \frac{2n+1}{2}T < t < \frac{n}{2}T \end{cases}$



$n \in \mathbb{Z}$

the interval boundaries have finite limits (± 1)

$f_1 = 1$ is continuous on $\frac{n}{2}T < t < \frac{2n+1}{2}T$, $\forall t, n \in \mathbb{Z}$

$f_3 = -1$ is continuous on $\frac{2n+1}{2}T < t < \frac{n}{2}T$, $\forall t, n \in \mathbb{Z}$

$f_2 = 0$, $t = \frac{n}{2}T$ is the set of discontinuity points which is finite over a finite interval

therefore the periodic square wave function is piecewise continuous

(14) $f(x) \in C_1$

$\dot{x} = f(x)$

$f(x^*) = 0$

$J = \left[\frac{\partial f}{\partial x} \right]_{x = x^*}$ is non-singular

By Implicit Function Theorem

there exists neighborhood $U \subset \mathbb{R}^n$ of x^*

Such that $f(x) = \dot{x}$ has a unique solution $x \in U$.

Since the solution, x^*, is unique within

the neighborhood, it is an isolated equilibrium point.

(15) 1) $f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

(a) $f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

$f'(x=0)$ does not exist $\rightarrow$ f(x) is not continuously differentiable

(b) $f(x)-f(y) = x^2\sin\left(\frac{1}{x}\right) - y^2\sin\left(\frac{1}{y}\right)$

$= x^2\sin\left(\frac{1}{x}\right) - y^2\sin\left(\frac{1}{x}\right) + y^2\sin\left(\frac{1}{x}\right) - y^2\sin\left(\frac{1}{y}\right)$

$= (x^2-y^2)\sin\left(\frac{1}{x}\right) + y^2\sin\left(\frac{1}{x}\right) - y^2\sin\left(\frac{1}{y}\right)$

$T_1 := (x^2-y^2)\sin\left(\frac{1}{x}\right)$, $T_2 := y^2\sin\left(\frac{1}{x}\right) - y^2\sin\left(\frac{1}{y}\right)$,

$f(x)-f(y) = T_1+T_2$

$T_1 = (x^2-y^2)\sin\left(\frac{1}{x}\right) \leq \left|(x^2-y^2)\sin\left(\frac{1}{x}\right)\right| = \left|(x+y)(x-y)\sin\left(\frac{1}{x}\right)\right|$

$\leq |x-y||x+y|\left|\sin\left(\frac{1}{x}\right)\right| \leq |x-y||x+y|\left|\frac{1}{x}\right|$

$|\sin u| \leq |u|$

by Mean Value Theorem

$\exists c \in (0,1) \mid \frac{d}{dt}\sin t = \frac{\sin x - \sin(0)}{x-0} = \cos(c) = \frac{\sin x}{x}$

$-1 \leq \cos(c) \leq 1 \Rightarrow \left|\frac{\sin x}{x}\right| \leq 1 \rightarrow |\sin x| \leq |x|$

$|x+y||\frac{1}{x}| \leq (|x|+|y|)|\frac{1}{x}|$

assuming without loss of generality,

$0<|y|\leq|x|$

$(|x|+|y|)|\frac{1}{x}| = 1 + |\frac{y}{x}| \leq 2$

$T_1 = (x^2-y^2)\sin(\frac{1}{x}) \leq 2|x-y|$

$|T_2| = |y^2\sin(\frac{1}{x}) - y^2\sin(\frac{1}{y})| \leq |y^2||\frac{1}{x}-\frac{1}{y}| = |y^2||\frac{1}{xy}(y-x)|$

$|T_2| \leq |\frac{y}{x}||x-y| \leq |x-y|$

$|\frac{y}{x}|\leq 1$

$|f(x)-f(y)| \leq 2|x-y| + |x-y| = 3|x-y|$, for $0<|y|\leq|x|$

---

edge cases

$x=0,\ y\neq 0$ without loss of generality,

$|f(x)-f(y)| = |f(x)-f(0)| = |x^2\sin(\frac{1}{x})|$

$\leq |\frac{x^2}{x}| = |x| = |x-0| \leq 3|x-0|\ \forall x$

$\Rightarrow |f(x)-f(y)| \leq 3|x-y|\ \forall x,\ y=0$

$|f(x)-f(y)| \leq 3|x-y|$

$f(x)$ is locally Lipschitz at $x=0$

(15) 1) (c) $f(x)=0$

$-1 \le \sin(\frac{1}{x}) \le +1 \quad , x \neq 0$

$-x^2 \le x^2 \sin(\frac{1}{x}) \le x^2, \ x \neq 0$

$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} f(x) \le \lim_{x\to 0} x^2$

$0 \le \lim_{x\to 0} f(x) \le 0$

$\lim_{x\to 0} f(x) = f(0) = 0$

therefore $f(x)$ is continuous at $x=0$

(d) as previously shown

$|f(x)-f(y)| \le 3|x-y| \quad \forall x,y \in \mathbb{R}$ (derivative is globally bounded)

$\Rightarrow f(x)$ is globally lipschitz

(e) $f(x)$ is cont at $x=0$, x^2 is contin. on $\mathbb{R}$, $\sin(\frac{1}{x})$ is conti on $\mathbb{R}-\{0\}$. SO (product of cont. functions is cont.)

$f(x)$ is uniformly continuous

(f) $f(x)$ is lipschitz at $(-1,1)$ since it is globally lipschitz

2) $f(x) = \begin{cases} x^3\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x=0 \end{cases}$

(a) $-1 \le \sin(\frac{1}{x}) \le 1 \Rightarrow -x^3 \le x^3\sin(\frac{1}{x}) \le +x^3$

$\lim_{x\to 0} -x^3 \le \lim_{x\to 0} x^3\sin(\frac{1}{x}) \le \lim_{x\to 0} x^3$

squeeze theorem

$0 \le \lim_{x\to 0} x^3\sin(\frac{1}{x}) \le 0$

$\Rightarrow \lim_{x\to 0} x^3\sin(\frac{1}{x}) = 0$

$-1 \le \cos(\frac{1}{x}) \le 1$

$\lim_{x\to 0} -x \le \lim_{x\to 0} x\cos(\frac{1}{x}) \le \lim_{x\to 0} x$

$0 \le \lim_{x\to 0} x\cos(\frac{1}{x}) \le 0$

$f'(x) = 3x^2\sin(\frac{1}{x}) + x^3\cos(\frac{1}{x})(-\frac{1}{x^2})$

$= 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x})$

as seen earlier, $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$ and by squeeze theorem, $\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$

therefore $\lim_{x\to 0} f'(x) = 0$, $f'(0) = \frac{d}{dx}0 = 0$, $\underline{\lim_{x\to 0} f'(x) = f'(0)}$

therefore $f(x)$ is continuously differentiable at $x=0$

(b) $f(x)$ is globally continuously differentiable

→ therefore $f(x)$ is Lipschitz continuous

(c) $f(0)=0$, $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$

→ therefore $f(x)$ is continuous at $x=0$

(d) $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}) = \infty$ → $f'(x)$ is unbounded

→ $f(x)$ is not globally Lipschitz

(e) $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}) = \infty$

→ $f(x)$ is not uniformly continuous

(f) $f(x)$ is glob. cont. diff → $f(x)$ is Lipschitz cont. on $(-1,1)$

(15)

(3) $f(x) = \tan(\pi x/2)$

(a) $f(0) = \tan(0) = 0$

$$\lim_{x \to 0} \tan(\pi x/2) = 0$$

$$f'(x) = \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)}$$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)} = \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)} = \frac{\pi}{2}$$

$\Rightarrow$ $\tan(\pi x/2)$ is continuosly differentiable at $x=0$

(b) $f(x) = \tan(\pi x/2) \rightarrow f'(x) = \frac{\pi}{2} \frac{1}{\cos^2(\pi x/2)}$

$f(x)$ is continuosly differentiable on $(-1, 1)$

$\rightarrow$ $f(x)$ is locally Lipschitz at $x=0$

(c) $f(0) = \lim_{x \to 0} f(x) = 0 \rightarrow$ $\tan(\pi x/2)$ is continuous at $x=0$

(d) since $\tan(\pi x/2)$ is not locally Lipschitz, it is not globally Lipschitz

(e) $\lim_{x \to 1^-} \tan(\pi x/2) = +\infty \neq -\infty = \lim_{x \to 1^+} \tan(\pi x/2$

$\rightarrow$ therefore $\tan(\pi x/2)$ is not uniformly continuous on $\mathbb{R}$

(f) $f'(x) = \dfrac{\pi/2}{\cos^2(\pi x/2)}$

$\lim_{x \to 1^-} f'(x) = +\infty$, $\lim_{x \to -1^+} f'(x) = -\infty$

$f'(x)$ is unbounded on $(-1,1)$

$\rightarrow$ $\boxed{\text{therefore } f(x) = \tan(\pi x/2) \text{ is not Lipschitz continuous on } (-1,1)}$

(16) (i) $f(x) = \begin{bmatrix} x_1 + \operatorname{sgn}(x_2) \\ x_2 \end{bmatrix}$

(a) $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \Rightarrow \frac{d}{dx}\operatorname{sgn}(x) = \begin{cases} 0, & x \in \mathbb{R}, x \neq 0 \\ \text{undefined}, & x = 0 \end{cases}$

$$f'(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & x_1, x_2 \in \mathbb{R}, x_2 \neq 0 \\ \begin{bmatrix} 1 & \text{undef.} \\ 0 & 1 \end{bmatrix}, & x_1 \in \mathbb{R}, x_2 = 0 \end{cases}$$

$f'(x)$ does not exist at $x_2 = 0$

$\rightarrow$ $f(x)$ is not continuously differentiable

(b)

(1b) since $\text{sgn}(x_2)$ is not continuous,

(b)-(e) $f(x)$ is not continuous

→ therefore $f(x)$ is not Lipschitz or continuous

(b) $f(x)$ is locally lipschitz on any interval, I, that does not contain $x_2 = 0$

(2) $f(x) = \begin{bmatrix} x_1 + \text{sat}(x_2) \\ x_1 + \sin(x_2) \end{bmatrix}$

(a) $f(x)$ is not continuosly differentiable because $\text{sat}(x)$ is not continuously differentiable

(b) $f(x)$ is locally lipschitz on interval, I, that does not contain $x_2 = \delta$, where $\text{sat}(\delta)$ is a critical point

(c) $f(x)$ is continuous because x, $\text{sat}(x)$, $\sin(x)$ are globally continuous

(d) $f(x)$ is not globally Lipschitz because the derivative of $\text{sat}(x)$ is not globally defined

(e) $f(x)$ is uniformly continuous on $\mathbb{R}^2$ b/c x, $\text{sat}(x)$, $\sin(x) \in C_1$

(3) $f(x) = \begin{pmatrix} x_3 \, sat(x_1 + x_2) \\ x_2^2 \\ x_1 \end{pmatrix}$

(a) $f(x)$ isn't continuously differentiable
because $sat(x)$ is not continuosly differentiable

(b)

(c) $f(x)$ is continuous on $\mathbb{R}^n$ because
$x_3 sat(x_1 + x_2)$, x_2^2, and x_1 are all continuous

(d) $f(x)$ is not globally Lipschitz because the
slope of x^2 is unbounded as $x \to \pm\infty$

(e) $f(x)$ is not continuously on $\mathbb{R}^3$ because
the slope of x^2 is unbounded as $x \to \pm\infty$

(17) $\exists\, c_1 > 0,\ c_2 > 0,$

s.t. $\quad c_1\|x\|_\alpha \le \|x\|_\beta \le c_2\|x\|_\alpha,\ \forall x \in \mathbb{R}^n$

without loss of generality,

let $f: \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz in $\|\cdot\|_\alpha$

then $\quad \|f(y) - f(x)\|_\alpha \le L_\alpha \|y - x\|_\alpha$

$$c_2\|f(y) - f(x)\|_\alpha \le c_2 L_\alpha \|y - x\|_\alpha$$

$$\|x\|_\beta \le c_2\|x\|_\alpha \to \|f(y) - f(x)\|_\beta \le c_2\|f(y) - f(x)\|_\alpha$$

$$c_1\|x\|_\alpha \le \|x\|_\beta \to c_1\|y - x\|_\alpha \le \|y - x\|_\beta$$

$$c_2 L_\alpha \|y - x\|_\alpha \le \frac{c_2 L_\alpha}{c_1}\|y - x\|_\beta$$

$$c_2\|f(y) - f(x)\|_\alpha \le \frac{c_2 L_\alpha}{c_1}\|y - x\|_\beta$$

$$\|f(y) - f(x)\|_\beta \le \frac{c_2 L_\alpha}{c_1}\|y - x\|_\beta$$

$$\|f(y) - f(x)\|_\beta \le L_\beta \|y - x\|_\beta$$

$\to f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in $\|\cdot\|_\beta$

(18) $g: \mathbb{R}^n \to \mathbb{R}^n$, $g \in C_1$

$$f(x) = \frac{1}{1+g^T(x)g(x)} g(x)$$

show $\dot{x} = f(x)$, $x(0) = x_0$ has a unique solution for all $t \geq 0$

$$f'(x) = \frac{\partial}{\partial x}\left(\frac{1}{1+g^T(x)g(x)}\right) g(x) + \left(\frac{1}{1+g^T(x)g(x)}\right)\frac{\partial g}{\partial x}$$

$$= \frac{1}{(1+g^Tg)^2}\frac{\partial}{\partial x}(1+g^Tg)g + \frac{1}{1+g^Tg}\frac{\partial g}{\partial x}$$

$$= \frac{1}{(1+g^Tg)^2}\left(\frac{\partial g^T}{\partial x}g + g^T\frac{\partial g}{\partial x}\right) + \frac{1}{1+g^Tg}\frac{\partial g}{\partial x}$$

$g \in C_1 \to f'(x)$ is continuous $\to f(x) \in C_1$

$f(x) \in C_1 \to f(x)$ is Lipschitz continuous

$\to$ by Theorem 3.1, there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0+\delta]$
(local unique solution)

(18) cont.

$$\|f(x)\|_2 = \left\| \frac{1}{1+g^T g} g \right\|_2 = \left\| \frac{g}{1+\|g\|_2^2} \right\|_2$$

$$\|f(x)\|_2 = \left[\sum_{i=1}^{N} \left(\frac{g}{1+\|g\|_2^2} \right) \left(\frac{g}{1+\|g\|_2^2} \right) \right]^{1/2}$$

$$= \left[\left(\frac{1}{1+\|g\|_2^2} \right)^2 g^T g \right]^{1/2} =$$

$$\|f(x)\|_2 = \frac{\|g(x)\|_2}{1+\|g(x)\|_2^2}$$

$$u := \|g(x)\|_2 \Rightarrow \|f(x)\|_2 = \frac{u}{1+u^2} = \frac{1}{\frac{1}{u}+u}$$

$$\frac{1}{u}+u-2 = \frac{u^2}{u}+\frac{1}{u}-\frac{2u}{u} = \frac{u^2-2u+1}{u} = \frac{(u-1)^2}{u} \geq 0$$

$$\Rightarrow \frac{1}{u}+u \geq 2 \Rightarrow \frac{1}{\frac{1}{u}+u} \leq \frac{1}{2}$$

$$\Rightarrow \|f(x)\|_2 \leq \frac{1}{2}$$

$$\dot{x} \leq \frac{1}{2}$$

$$\|x(t)\|_2 \leq \|x_0\|_2 + \frac{1}{2}(t - \cancel{t_0}^{\,0})$$

Therefore, the solution is defined for all $t \geq 0$,

(sol. is unique)

therefore there is a unique solution for all $t \geq 0$

(19) $$y(t) \le k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}\left[k_2 y(\tau) + k_3\right] d\tau$$

$$y(t) e^{\alpha(t-t_0)} \le k_1 e^{-\alpha(t-t_0-t+t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}\left[k_2 y(\tau) e^{\alpha(\tau-t_0)} + k_3\right] d\tau$$

$$z(t) := y(t) e^{\alpha(t-t_0)} \le k_1 + \int_{t_0}^{t} e^{\alpha(\tau-t_0)}\left[k_2 y(\tau) + k_3\right] d\tau$$

$$//$$

$$z(t) \le k_1 + \int_{t_0}^{t}\left[k_2 z(\tau) + k_3 e^{\alpha(\tau-t_0)}\right] d\tau$$

$$z(t) \le k_1 + k_3 e^{-\alpha t_0} \int_{t_0}^{t} e^{\alpha\tau} d\tau + \int_{t_0}^{t} k_2 z(\tau) d\tau$$

$$z(t) \le k_1 + k_3 e^{-\alpha t_0}\left[\frac{1}{\alpha} e^{\alpha\tau}\right]_{\tau=t_0}^{t} + \int_{t_0}^{t} k_2 z(\tau) d\tau$$

$$z(t) \le k_1 + k_3 e^{-\alpha t_0} \frac{1}{\alpha}\left[e^{\alpha t} - e^{\alpha t_0}\right] + \int_{t_0}^{t} k_2 z(\tau) d\tau$$

$$z(t) \le k_1 + \frac{k_3}{\alpha}\left[e^{\alpha(t-t_0)} - 1\right] + \int_{t_0}^{t} k_2 z(\tau) d\tau$$

$$\lambda := k_1 + \frac{k_3}{\alpha}\left[e^{\alpha(t-t_0)} - 1\right], \quad \mu := k_2, \quad a := t_0, \quad s := \tau$$

Gronwall-Bellman Inequality

$$f(t) \le \lambda(t) + \int_a^t \mu(s) f(s) ds$$

$$\rightarrow f(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau) d\tau\right] ds$$

applying G-B Inequality...

$$z(t) \le \lambda(t) + \int_{t_0}^{t} \lambda(\tau)\mu(\tau)\exp\left[\int_{\tau}^{t} \mu(s)ds\right]d\tau$$

$$\alpha\tau - \alpha t_0 - k_2\tau$$

$$\int_{\tau}^{t} \mu(s)ds = \int_{\tau}^{t} k_2\, ds = k_2(t-\tau)$$

$$z(t) \le \lambda(t) + \int_{t_0}^{t} \left[k_1 + \frac{k_3}{\alpha}\left[e^{\alpha(\tau-t_0)} - 1\right]\right] k_2 e^{k_2(t-\tau)} d\tau$$

$$z(t) \le \lambda(t) + \int_{t_0}^{t} k_1 k_2 e^{k_2 t} e^{-k_2\tau} d\tau + \int_{t_0}^{t} \frac{k_2 k_3}{\alpha}\left[e^{\alpha(\tau-t_0)} - 1\right] e^{k_2 t} e^{-k_2\tau} d\tau$$

$$z(t) \le \lambda(t) + k_1 k_2 e^{k_2 t}\left[\frac{1}{-k_2} e^{-k_2\tau}\right]_{t_0}^{t} + \frac{k_2 k_3}{\alpha} e^{k_2 t}\left[\int_{t_0}^{t} e^{\alpha(\tau-t_0)-k_2\tau} d\tau - \int_{t_0}^{t} e^{-k_2\tau} d\tau\right]$$

$$z(t) \le \lambda(t) - k_1 e^{k_2 t}\left[e^{-k_2 t} - e^{-k_2 t_0}\right] + \frac{k_2 k_3}{\alpha} e^{k_2 t}\left[e^{-\alpha t_0}\left[\frac{e^{(\alpha-k_2)\tau}}{\alpha-k_2}\right]_{\tau=t_0}^{t} - \left[\frac{e^{-k_2\tau}}{-k_2}\right]_{\tau=t_0}^{t}\right]$$

$$z(t) \le \lambda(t) - k_1\left[1 - e^{k_2(t-t_0)}\right] + \frac{k_2 k_3}{\alpha} e^{k_2 t}\left[e^{-\alpha t_0}\left[\frac{e^{(\alpha-k_2)t} - e^{(\alpha-k_2)t_0}}{\alpha-k_2}\right] - \left[\frac{e^{-k_2 t} - e^{-k_2 t_0}}{-k_2}\right]\right]$$

$$z(t) \le \cancel{k_1} + \frac{k_3}{\alpha}\left[e^{\alpha(t-t_0)} - 1\right] \cancel{-k_1} + k_1 e^{-k_2(t-t_0)}$$

$$+ \frac{k_2 k_3}{\alpha} \frac{e^{k_2 t - \alpha t_0 + \alpha t - k_2 t} - e^{k_2 t - \alpha t_0 + \alpha t_0 - k_2 t_0}}{\alpha - k_2}$$

$$- \frac{k_2 k_3}{\alpha} \frac{e^{k_2 t - k_2 t} - e^{k_2 t - k_2 t_0}}{-k_2}$$

$$z(t) \le \frac{k_3}{\alpha} e^{\alpha(t-t_0)} - \frac{k_3}{\alpha} + k_1 e^{-k_2(t-t_0)}$$

$$+ \frac{k_2 k_3}{\alpha(\alpha-k_2)} \left[e^{\alpha(t-t_0)} - 1 \right]$$

$$+ \frac{k_3}{\alpha} \left[1 - e^{k_2(t-t_0)} \right]$$

$$z(t) \le \frac{k_3}{\alpha} e^{\alpha(t-t_0)} + \left(k_1 - \frac{k_3}{\alpha}\right) e^{k_2(t-t_0)} + \frac{k_2 k_3}{\alpha(\alpha-k_2)} \left[e^{\alpha(t-t_0)} - e^{k_2(t-t_0)} \right]$$

$$y(t) = z(t) e^{-\alpha(t-t_0)} \le \frac{k_3}{\alpha} + \frac{k_3}{\alpha} \left[1 + \frac{k_2}{\alpha-k_2} \right]$$

$$+ \left[k_1 - \frac{k_3}{\alpha} - \frac{k_2 k_3}{\alpha(\alpha-k_2)} \right] e^{(k_2-\alpha)(t-t_0)}$$

$$y(t) \le \frac{k_3}{(\alpha-k_2)} + \left[k_1 - \frac{k_3}{(\alpha-k_2)} \right] e^{(k_2-\alpha)(t-t_0)}$$

$$\boxed{y(t) \le k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha-k_2} \left[1 - e^{-(\alpha-k_2)(t-t_0)} \right]}$$

(20) $f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$

show $f(x)$ is uniformly continuous on W

$$|f(x) - f(y)| \leq L|x-y| \quad , \forall x,y \in W$$

---

uniform continuity: $\forall \varepsilon > 0, \exists \delta > 0 \mid \forall x,y \in I,$

$$|x-y| < \delta \to |f(x) - f(y)| < \varepsilon$$

$$|x-y| < \delta \qquad \forall x,y \in W$$

$$|f(x)-f(y)| \leq L|x-y| < L\delta \qquad \forall x,y \in W$$

$$|f(x)-f(y)| < \varepsilon = L\delta \quad \forall x,y \in W$$

$\to f$ is continuous on W

(21) $x : \mathbb{R} \to \mathbb{R}^n$, $x \in C_1$

$$\|\dot{x}(t)\| \le g(t), \quad \forall t \ge t_0$$

show that $\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^{t} g(s)\,ds$

$$\int_{t_0}^{t} \|\dot{x}(s)\|\,ds \le \int_{t_0}^{t} g(s)\,ds$$

$$\|x(t)\| - \|x(t_0)\| \le \int_{t_0}^{t} g(s)\,ds$$

$$\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^{t} g(s)\,ds$$

(22) $A \xrightarrow{k_1} B$, $B \xrightarrow{k_2} C$

material balance $\frac{dC_A}{dt} = -r_1$, $\frac{dC_B}{dt} = r_1 - r_2$, $\frac{dC_C}{dt} = r_2$

C_j = concentration of j

$r_{1,2}$ = reaction rates $\left[\frac{mol}{time \cdot vol}\right]$

rate law: $r_1 = k_1 C_A$, $r_2 = k_2 C_B$

$$\dot{C} = \begin{bmatrix} \dot{C}_A \\ \dot{C}_B \\ \dot{C}_C \end{bmatrix} = \begin{bmatrix} -k_1 C_A \\ k_1 C_A - k_2 C_B \\ k_2 C_B \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$$

(a) The model is linear

(b) $\dot{x} = Ax + \cancel{Bu}^{\,0}$ $\quad \dot{x} = Ax$

$y = Cx + \cancel{Du}^{\,0}$ $\quad y = Cx$

$$x = C = \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}, \quad \dot{x} = \dot{C} = \begin{bmatrix} \dot{C}_A \\ \dot{C}_B \\ \dot{C}_C \end{bmatrix}, \quad y = C_A$$

$$A = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix}, \quad C = [1 \ 0 \ 0]$$