University of California, Davis
Department of Chemical Engineering
ECH 267
Advanced Process Control

M. Ellis **Homework 1** **Winter 2021**

Reading Assignment: Lecture notes; Khalil Chapters 1 and 2 and Appendix A and B.
Due date: Monday, February 8 at 6:00PM PST

1. Exercise 1.2 (Khalil pg. 48). Consider a single-input-single-output system described by the nth-order differential equation

$$y^{(n)} = g_1\left(t, y, \dot{y}, \ldots, y^{(n-1)}, u\right) + g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)\dot{u}$$

where g_2 is a differentiable function of its arguments. With u as input and y as output, find a state-space model. *Hint:* Take $x_n = y^{(n-1)} - g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)u$.

2. Exercise 1.7 (Khalil pg. 29). Figure 1 shows a feedback connection of a linear time-invariant system and a nonlinear time-varying element. The variables r, u, and y are vectors of the same dimension, and $\psi(t, y)$ is a vector-valued function. With r as input and y as output, find a state-space model.

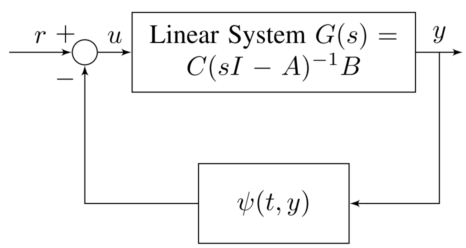

Figure 1: Exercise 1.7.

3. Exercise 1.11 (Khalil pg. 50). A phase-locked loop can be represented by the block diagram of Figure 2. Let $\{A, B, C\}$ be a minimal realization of the scalar, strictly proper transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0) \neq 0$, and $\theta_i =$ constant. Let z be the state of the realization $\{A, B, C\}$.

   (a) Show that closed-loop system can be represented by the state equations

$$\dot{z} = Az + B\sin e, \quad \dot{e} = -Cz$$

   (b) Find all the equilibrium points of the system.
   (c) Show that when $G(s) = 1/(\tau s + 1)$, the closed-loop model coincides with the model of a pendulum equation.

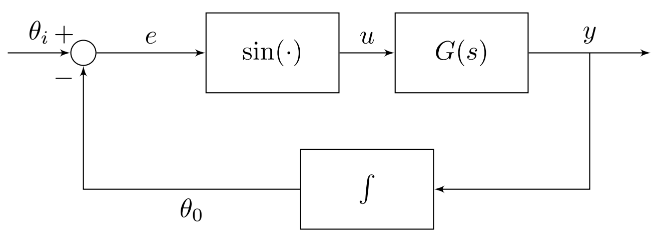

Figure 2: Exercise 1.11.

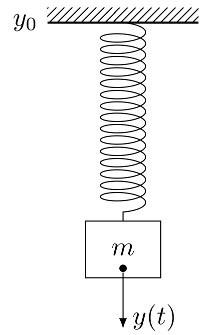

Figure 3: Mass-spring system.

4. Exercise 1.12 (Khalil pg. 51). Consider the mass-spring system shown in Figure 3. Assuming a linear spring and nonlinear viscous damping described by $c_1\dot{y} + c_2\dot{y}\,|\dot{y}|$, find a state equation that describes the motion of the system.

5. Determine whether or not the differential equation

$$\dot{x}(t) = [x(t)]^{1/3}, x(0) = 0$$

has a unique solution over $[0, \infty)$.

6. Exercise 1.13 (Khalil 2nd Edition pg. 52). For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium.

(1)
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \frac{x_1^3}{6} - x_2\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3\end{aligned}$$

(3)
$$\begin{aligned}\dot{x}_1 &= (1 - x_1)x_1 - \frac{2x_1x_2}{1 + x_1}\\ \dot{x}_2 &= \left(2 - \frac{x_2}{1 + x_1}\right)x_2\end{aligned}$$

(4)
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + x_2\left(1 - 3x_1^2 - 2x_2^2\right)\end{aligned}$$

(5)
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2(1 + x_1)\\ \dot{x}_2 &= -x_1(1 + x_1)\end{aligned}$$

(6)
$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right)\\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right)\end{aligned}$$

(7)
$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2\\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

7. For each of the A matrices below, consider the system $\dot{x} = Ax$ and:

(a) determine the matrix M that transforms A into the appropriate modal form and write the system in model coordinates ($\dot{z} = (M^{-1}AM)z$);
(b) classify the equilibrium $(0,0)$; and
(c) generate the phase portraits of the system in both the model (z) and the original (x) coordinates.

(i) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix}$

(v) $A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$

8. Exercise 2.5 (Khalil pg. 78). The system

$$\begin{aligned} \dot{x}_1 &= -x_1 - \frac{x_2}{\ln\sqrt{x_1^2 + x_2^2}} \\ \dot{x}_2 &= -x_2 + \frac{x_1}{\ln\sqrt{x_1^2 + x_2^2}} \end{aligned}$$

has an equilibrium point at the origin.

(a) Linearize the system about the origin and find the type of the origin as an equilibrium point of the linear system.

(b) Find the phase portrait of the nonlinear system near the origin, and show that the portrait resembles a stable focus. **Hint:** Transform the equations into polar coordinates.

(c) Explain the discrepancy between the results of parts (a) and (b).

9. Exercise 1.17 (Khalil Second Edition pg. 54). For each of the following systems, construct the phase portrait and discuss the qualitative behavior of the system.

(1)
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - 2\tan^{-1}(x_1 + x_2) \end{aligned}$$

(2)
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \end{aligned}$$

(3)
$$\begin{aligned} \dot{x}_1 &= x_1 - x_1 x_2 \\ \dot{x}_2 &= 2x_1^2 - x_2 \end{aligned}$$

(4)
$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(|x_1| + |x_2|) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(|x_1| + |x_2|) \end{aligned}$$

10. Exercise 1.22 (Khalil Second Edition pg. 55). The phase portraits of the following four systems are shown in Figures 4: parts (a), (b), (c), and (d) respectively. Discuss if the arrowheads are pointed in the correct direction and discuss the qualitative behavior of each system.

(1)
$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 + 0.1x_1^4) \end{aligned}$$

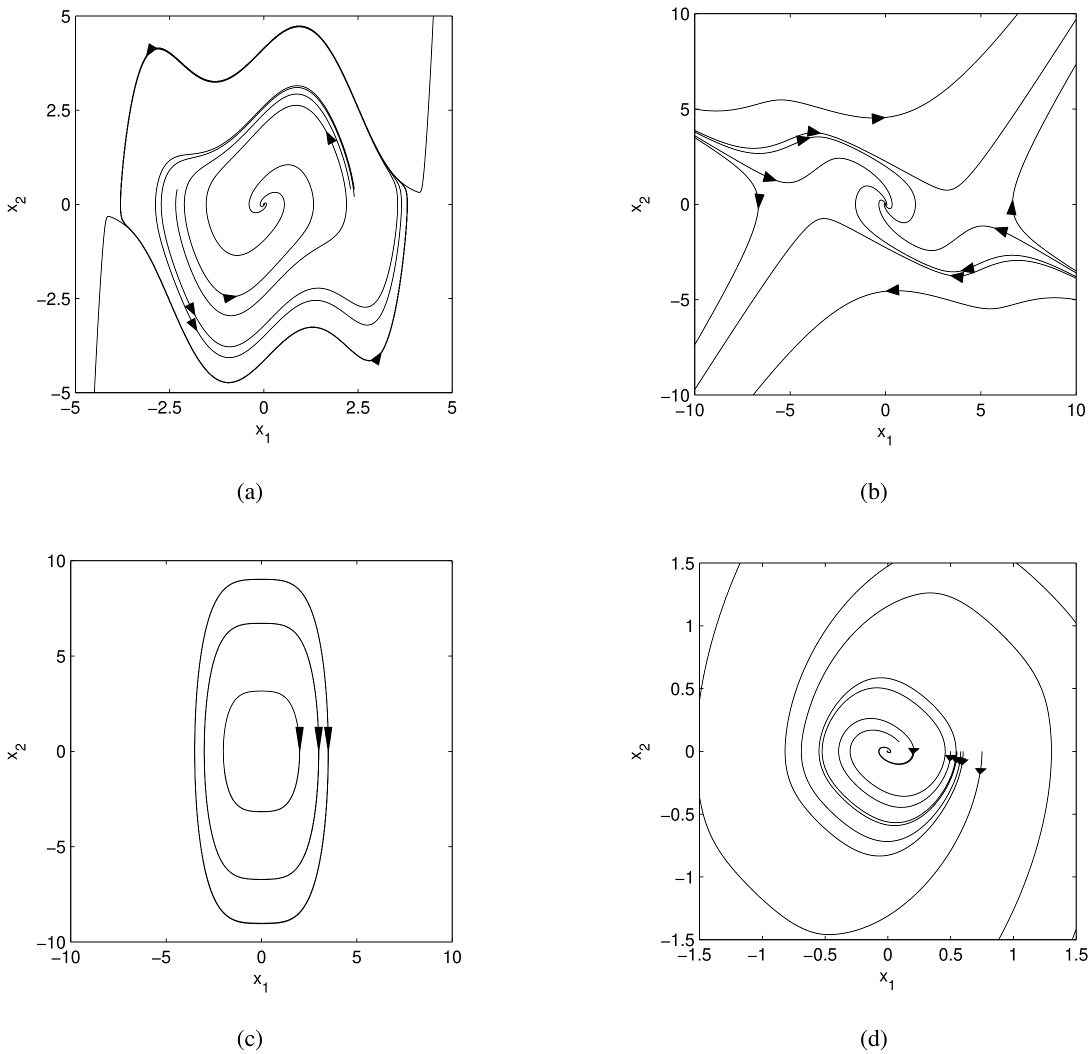

Figure 4: Exercise 1.22

(2)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 + x_2 - 3\tan^{-1}(x_1 + x_2)\end{aligned}$$

(3)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(0.5x_1 + x_1^3)\end{aligned}$$

(4)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - \psi(x_1 - x_2)\end{aligned}$$

where $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5\text{sign}(y)$ if $|y| > 1$.

11. Exercise 2.1 (Khalil Second Edition pg. 88). Show that, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\|x\|_2 &\leq \|x\|_1 \leq \sqrt{n}\|x\|_2\\ \|x\|_\infty &\leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty\\ \|x\|_\infty &\leq \|x\|_1 \leq n\|x\|_\infty\end{aligned}$$

12. Exercise 2.3 (Khalil Second Edition pg. 88). Consider the set $S = \{x \in \mathbb{R}^2 \mid -1 < x_i \leq 1,\ i = 1,2\}$. Is S open? Is it closed? Find the closure, interior, and boundary of S.

13. Exercise 2.4 (Khalil Second Edition pg. 88). Let $u_T(t)$ be the unit step function, defined by $u_T(t) = 0$ for $t < T$ and $u_T(t) = 1$ for $t \geq T$.

    (a) Show that $u_T(t)$ is piecewise continuous.

    (b) Show that $f(t) = g(t)u_T(t)$, for any continuous function $g(t)$, is piecewise continuous.

    (c) Show that the periodic square waveform is piecewise continuous.

14. Exercise 2.6 (Khalil Second Edition pg. 88). Let $f(x)$ be continuously differentiable. Show that an equilibrium point x^* of $\dot{x} = f(x)$ is isolated if the Jacobian matrix $[\partial f/\partial x](x^*)$ is nonsingular. **Hint:** Use the implicit function theorem.

15. Exercise 2.26 (Khalil Second Edition pg. 92). For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, find whether f is (a) continuously differentiable at $x = 0$; (b) locally Lipschitz at $x = 0$; (c) continuous at $x = 0$; (d) globally Lipschitz; (e) uniformly continuous on $\mathbb{R}$; (f) Lipschitz on $(-1, 1)$.

    (1) $f(x) = \begin{cases} x^2\sin(1/x), & \text{for } x \neq 0\\ 0, & \text{for } x = 0\end{cases}$

    (2) $f(x) = \begin{cases} x^3\sin(1/x), & \text{for } x \neq 0\\ 0, & \text{for } x = 0\end{cases}$

    (3) $f(x) = \tan(\pi x/2)$

16. Exercise 2.27 (Khalil Second Edition pg. 92). For each of the following functions $f : \mathbb{R}^n \to \mathbb{R}^n$, find whether f is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz; (e) uniformly continuous on $\mathbb{R}^n$.

    (1) $f(x) = \begin{bmatrix} x_1 + \text{sgn}(x_2)\\ x_2\end{bmatrix}$

    (2) $f(x) = \begin{bmatrix} x_1 + \text{sat}(x_2)\\ x_1 + \sin x_2\end{bmatrix}$

    (3) $f(x) = \begin{bmatrix} x_3\text{sat}(x_1 + x_2)\\ x_2^2\\ x_1\end{bmatrix}$

17. Exercise 3.5 (Khalil pg. 105). Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two different norms of the class of $p-$norms on $\mathbb{R}^n$. Show that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in $\|\cdot\|_\alpha$ if and only if it is Lipschitz in $\|\cdot\|_\beta$.

18. Exercise 3.7 (Khalil pg. 106). Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable for all $x \in \mathbb{R}^n$, and define $f(x)$ by
$$f(x) = \frac{1}{1 + g^T(x)g(x)} g(x).$$
Show that $\dot{x} = f(x),\ x(0) = x_0$, has a unique solution defined for all $t \geq 0$.

19. Exercise 3.18 (Khalil pg. 108). Let $y(t)$ be a nonnegative scalar function that satisfies the inequality
$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3] d\tau$$
where $k_1,\ k_2,\ k_3$ are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall-Bellman inequality, show that
$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}[1 - e^{-(\alpha-k_2)(t-t_0)}]$$
Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

20. Exercise 3.20 (Khalil pg. 108). Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$, then $f(x)$ is uniformly continuous on W.

21. Exercise 2.23 (Khalil Second Edition pg. 92). Let $x : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function that satisfies
$$\|\dot{x}(t)\| \leq g(t), \qquad \forall t \geq t_0.$$
Show that
$$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} g(s) ds.$$

22. Exercise 1.1 (Rawlings pg. 60): *State space form for chemical reaction model.* Consider the following chemical reaction kinetics for a two-step series reaction
$$A \xrightarrow{k_1} B \quad B \xrightarrow{k_2} C$$
We wish to follow the reaction in a constant volume, well-mixed, batch reactor. The material balances for the three species are
$$\frac{dc_A}{dt} = -r_1 \quad \frac{dc_B}{dt} = r_1 - r_2 \quad \frac{dc_c}{dt} = r_2$$
in which c_j is the concentration of species j, and r_1 and r_2 are the rates (mol/(time·vol)) at which the two reactions occur. Assume the rate law for the reaction kinetics are:
$$r_1 = k_1 c_A \quad r_2 = k_2 c_B$$
Substituting the rate laws into the material balances and specifying the starting concentrations, three differential equations for the three species concentrations are obtained.

(a) Is the model linear or nonlinear?
(b) Write the state space model for the deterministic series chemical reaction model. Assume the component A concentration may be measured. What are x (state vector), y (output vector), A, B, C, and D (system matrices) for this model?
(c) Simulate this model with initial conditions and parameters given by
$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$