University of California, Davis
Department of Chemical Engineering
ECH 267
Advanced Process Control

M. Ellis | **Homework 1** | **Winter 2021**

Reading Assignment: Lecture notes; Khalil Chapters 1-3 and Appendix A and B.
Due date: Monday, February 8 at 6:00PM PST

1. Exercise 1.2 (Khalil pg. 48). Consider a single-input-single-output system described by the nth-order differential equation

$$y^{(n)} = g_1\left(t, y, \dot{y}, \ldots, y^{(n-1)}, u\right) + g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)\dot{u}$$

where g_2 is a differentiable function of its arguments. With u as input and y as output, find a state-space model. *Hint:* Take $x_n = y^{(n-1)} - g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)u$.

Solution. Define the state vector x as

$$x := \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} - g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)u \end{bmatrix}$$

where $y^{(i)}$ denotes the ith time derivative of y. The state-space model is

$$\begin{aligned}
\dot{x}_1 &= \dot{y} = x_2 \\
&\vdots \\
\dot{x}_i &= y^{(i)} = x_{i+1} \\
&\vdots \\
\dot{x}_{n-1} &= y^{(n-1)} = x_n + g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)u \\
&= x_n + g_2\left(t, x_1, x_2, \ldots, x_{n-1}\right)u \\
\dot{x}_n &= \underbrace{y^{(n)} - g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)\dot{u}}_{=g_1\left(t, y, \dot{y}, \ldots, y^{(n-1)}, u\right)} - \frac{d}{dt}\left(g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right)\right)u \\
&= g_1\left(t, x_1, x_2, \ldots, x_n + g_2\left(t, x_1, x_2, \ldots, x_{n-1}\right)u, u\right) \\
&\quad - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1}\right)u \\
&= g_1\left(t, x_1, x_2, \ldots, x_n + g_2\left(t, x_1, x_2, \ldots, x_{n-1}\right)u, u\right) \\
&\quad - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}x_2 + \cdots + \frac{\partial g_2}{\partial x_{n-1}}\left(x_n + g_2\left(t, x_1, x_2, \ldots, x_{n-1}\right)u\right)\right)u \\
&=: g_3(t, x_1, x_2, \ldots, x_n, u) \\
y &= x_1
\end{aligned}$$

or concisely:

$$\begin{aligned}
\dot{x} &= f(t, x, u) \\
y &= h(x)
\end{aligned}$$

where $h(x) := x_1$ and

$$f(t,x,u) := \begin{bmatrix} x_2 \\ \vdots \\ x_{i+1} \\ x_n + g_2\left(t, y, \dot{y}, \ldots, y^{(n-2)}\right) u \\ g_3(t, x_1, x_2, \ldots, x_n, u) \end{bmatrix}$$

2. Exercise 1.7 (Khalil pg. 29). Figure 1 shows a feedback connection of a linear time-invariant system and a nonlinear time-varying element. The variables r, u, and y are vectors of the same dimension, and $\psi(t, y)$ is a vector-valued function. With r as input and y as output, find a state-space model.

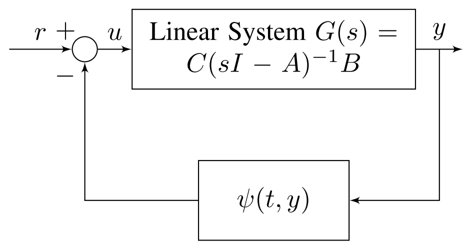


Figure 1: Exercise 1.7.

Solution. The output (in the Laplace domain) is defined as

$$\begin{aligned} \hat{y}(s) &= G(s)\hat{u}(s) \\ &= C(sI - A)^{-1}B\hat{u} \end{aligned}$$

or written as a (minimal) state-space realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where u is determined by the feedback law:

$$u = r - \psi(t, y)$$

Therefore, the closed-loop state-space model is

$$\begin{aligned} \dot{x} &= Ax - B\psi(t, Cx) + Br \\ y &= Cx \end{aligned}$$

3. Exercise 1.11 (Khalil pg. 50). A phase-locked loop can be represented by the block diagram of Figure 2. Let $\{A, B, C\}$ be a minimal realization of the scalar, strictly proper transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0) \neq 0$, and $\theta_i =$ constant. Let z be the state of the realization $\{A, B, C\}$.

   (a) Show that closed-loop system can be represented by the state equations

   $$\dot{z} = Az + B\sin e, \quad \dot{e} = -Cz$$

   Solution. The minimal realization of the scalar, strictly proper transfer function $G(s)$ is

   $$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz \end{aligned}$$

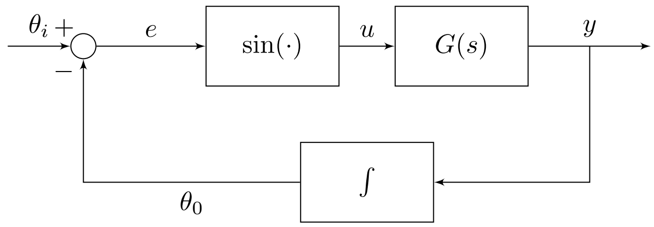


Figure 2: Exercise 1.11.

where $G(s) = C(sI - A)^{-1}B$. The input is $u = \sin(e)$ and

$$\begin{aligned} e &= \theta_i - \theta_0 \\ &= \theta_i - \int_{t_0}^{t} y(\tau)d\tau \\ &= \theta_i - \int_{t_0}^{t} Cz(\tau)d\tau \end{aligned}$$

Differentiating e with respect to time gives

$$\dot{e} = -Cz$$

Therefore, the closed-loop system is given by:

$$\dot{z} = Az + B\sin e, \qquad \dot{e} = -Cz.$$

(b) Find all the equilibrium points of the system.
Solution. The equilibrium points of the system can by found by solving the system of nonlinear equations

$$\begin{aligned} 0 &= Az + B\sin e \\ 0 &= -Cz. \end{aligned}$$

The second equation implies z must be in the nullspace of C. Rearranging the first equation (and note that A is non-singular),

$$\begin{aligned} z &= -A^{-1}B\sin e \\ Cz = 0 &= -CA^{-1}B\sin e \\ 0 &= \sin e. \end{aligned}$$

Note that the last step follows from the fact that $CA^{-1}B \neq 0$, which follows from the fact that $G(0) \neq 0$ (i.e., $G(s) = C(sI - A)^{-1}B$ and $G(0) = -CA^{-1}B \neq 0$). Thus, the equilibrium points are

$$e = \pm n\pi, \ z \in \mathcal{N}(C)$$

where $n = 0, 1, 2, 3, \dots$ and $\mathcal{N}(C)$ denotes the nullspace of C.

(c) Show that when $G(s) = 1/(\tau s + 1)$, the closed-loop model coincides with the model of a pendulum equation.
Solution. The transfer function

$$G(s) = C(sI - A)^{-1}B = 1/(\tau s + 1) = \frac{1}{\tau(s + 1/\tau)}$$

Let $C = 1$, $B = 1/\tau$, $A = -1/\tau$ (note: this choice of state-space realization is not unique and other realizations are possible). The state-space model is

$$\begin{aligned} \dot{z} &= -\frac{1}{\tau}z + \frac{1}{\tau}\sin e \\ \dot{e} &= -z \end{aligned}$$

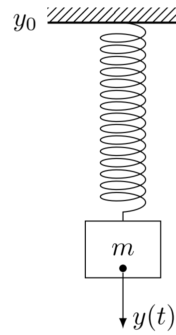


Figure 3: Mass-spring system.

which is coincides with the model of a pendulum equation ($x_1 = e$ and $x_2 = -z$):

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{\tau}\sin x_1 - \frac{1}{\tau}x_2
\end{aligned}$$

4. Exercise 1.12 (Khalil pg. 51). Consider the mass-spring system shown in Figure 3. Assuming a linear spring and nonlinear viscous damping described by $c_1\dot{y} + c_2\dot{y}\,|\dot{y}|$, find a state equation that describes the motion of the system.

   Solution. From Newton's law of motion

   $$ma + F_f + F_{sp} = F$$

   where m is the mass of the block, a is the acceleration, F_f is the dissipative force, F_{sp} is the spring force, and F is the external force. Substituting in for the forces:

   $$m\ddot{y} + c_1\dot{y} + c_2\dot{y}\,|\dot{y}| + ky = mg$$

   and solving for the acceleration

   $$\ddot{y} = -\frac{k}{m}y - \frac{c_1}{m}\dot{y} - \frac{c_2}{m}\dot{y}\,|\dot{y}| + g$$

   Let $x_1 = y$ and $x_2 = \dot{y}$. The state equation that describes the motion of the system is

   $$\begin{aligned}
   \dot{x}_1 &= x_2 \\
   \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c_1}{m}x_2 - \frac{c_2}{m}x_2\,|x_2| + g.
   \end{aligned}$$

5. Determine whether or not the differential equation

   $$\dot{x}(t) = [x(t)]^{1/3}, x(0) = 0$$

   has a unique solution over $[0, \infty)$.

   Solution. The differential equation does not have a unique solution over $[0, \infty)$. First, we note that

   $$\begin{aligned}
   \int_0^t \frac{\dot{x}(s)}{[x(s)]^{1/3}}\,ds &= \int_0^t ds \\
   \frac{3}{2}\left([x(t)]^{2/3} - x_0^{2/3}\right) &= t \\
   [x(t)]^{2/3} = \{[x(t)]^2\}^{1/3} &= \frac{2}{3}t + x_0^{2/3} \\
   |x(t)| &= \left(\frac{2}{3}t + x_0^{2/3}\right)^{3/2}
   \end{aligned}$$

So, if $x(0) = x_0 \neq 0$, the ODE has a *unique* solution on $[0, \infty)$ given by:

$$x(t) = \operatorname{sgn}(x_0)\left(\frac{2}{3}t + x_0^{2/3}\right)^{3/2}$$

where $\operatorname{sgn}(\cdot)$ denotes the signum function. For the initial condition $x(0) = 0$ this implies two solutions

$$x(t) = -\left(2/3t\right)^{3/2}$$

and

$$x(t) = (2/3t)^{3/2}.$$

Also, $x = 0$ is an equilibrium point of the ODE so another possible solution to the ODE is $x(t) = 0$. Therefore, if $x(0) = 0$, then the ODE has infinitely many solutions, which may be written as:

$$x_\delta(t) = \begin{cases} 0 & \text{if } t \leq \delta \\ \pm\left(\frac{2}{3}(t-\delta)\right)^{2/3} & \text{if } t > \delta \end{cases}$$

for any $\delta \geq 0$.

6. Exercise 1.13 (Khalil 2nd Edition pg. 52). For each of the following systems, find all equilibrium points, and determine the type of each isolated equilibrium.

(1)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{x_1^3}{6} - x_2 \end{aligned}$$

Solution. Solving the steady state equations, we have:

$$\begin{aligned} 0 &= x_2 \\ 0 &= -x_1 + \frac{x_1^3}{6} - x_2 = \frac{x_1}{6}(x_1^2 - 6) \end{aligned}$$

Thus, the equilibrium points are $(0,0)$, $(-\sqrt{6}, 0)$, and $(\sqrt{6}, 0)$. The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} 0 & 1 \\ -1 + \frac{x_1^2}{2} & -1 \end{bmatrix}$$

Linearization about $(0,0)$ gives:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -0.5 + 0.8660i$ and $\lambda_2 = -0.5 - 0.8660i$ so the equilibrium is a stable focus. Linearization about $(-\sqrt{6}, 0)$ gives:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -2$ so the equilibrium is a saddle point.
Linearization about $(\sqrt{6}, 0)$ gives:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -2$ so the equilibrium is a saddle point.

(2)

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$$

Solution. The equilibrium points are $(0,0)$, $(-2.551, -2.551)$, and $(-7.450, -7.450)$ (the work to find these roots has been omitted; solving the nonlinear system of equations may be done analytically or numerically). The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} -1 & 1 \\ 0.1 - 2x_1 - 0.3x_1^2 & -2 \end{bmatrix}$$

Linearization about $(0,0)$ gives:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -0.9084$ and $\lambda_2 = -2.0916$ so the equilibrium is a stable improper node.
Linearization about $(-2.551, -2.551)$ gives:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 3.2495 & -2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 0.3707$ and $\lambda_2 = -3.3707$ so the equilibrium is a saddle point.
Linearization about $(-7.450, -7.450)$ gives:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1.6495 & -2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -1.5 + 1.1830i$ and $\lambda_2 = -1.5 - 1.1830i$ so the equilibrium is a stable focus.

(3)

$$\dot{x}_1 = (1 - x_1)x_1 - \frac{2x_1x_2}{1 + x_1}$$
$$\dot{x}_2 = \left(2 - \frac{x_2}{1 + x_1}\right) x_2$$

Solution. The equilibrium points are $(0,0)$, $(1,0)$, $(0,2)$, and $(-3,-4)$. The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} -x_1 + (1 - x_1) - \frac{2x_2}{1+x_1} + \frac{2x_1x_2}{(1+x_1)^2} & \frac{-2x_1}{1+x_1} \\ \frac{x_2^2}{(1+x_1)^2} & 2 - \frac{2x_2}{(1+x_1)} \end{bmatrix}$$

Linearization about $(0,0)$ gives:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 2$ so the equilibrium is a unstable improper node.
Linearization about $(1,0)$ gives:

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 2$ so the equilibrium is a saddle point.
Linearization about $(0,2)$ gives:

$$\dot{x} = \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = -3$ so the equilibrium is a saddle improper node.
Linearization about $(-3,-4)$ gives:

$$\dot{x} = \begin{bmatrix} 9 & -3 \\ 4 & -2 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 7.7720$ and $\lambda_2 = -0.7720$ so the equilibrium is a saddle point.

(4)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2\left(1 - 3x_1^2 - 2x_2^2\right)$$

Solution. The equilibrium point is $(0,0)$. The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} 0 & 1 \\ -1 + 6x_1 x_2 & 1 - 3x_1^2 - 6x_2^2 \end{bmatrix}$$

Linearization about $(0,0)$ gives:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 0.5 + 0.8660i$ and $\lambda_2 = 0.5 - 0.8660i$ so the equilibrium is a unstable focus.

(5)
$$\dot{x}_1 = -x_1 + x_2(1 + x_1)$$
$$\dot{x}_2 = -x_1(1 + x_1)$$

Solution. The equilibrium point is $(0,0)$. The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} -1 + x_2 & 1 + x_1 \\ -2x_1 - 1 & 0 \end{bmatrix}$$

Linearization about $(0,0)$:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = 0.5 + 0.8660i$ and $\lambda_2 = 0.5 - 0.8660i$ so the equilibrium is a stable focus.

(6)
$$\dot{x}_1 = (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right)$$
$$\dot{x}_2 = (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right)$$

Solution. The equilibrium points are $(0,0)$ and all the points on the unit circle ($x_2^2 + x_2^2 = 1$). The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} \left(x_1^2 + x_2^2 - 1\right) + 2x_1(x_1 - x_2) & -\left(x_1^2 + x_2^2 - 1\right) + 2x_2(x_1 - x_2) \\ \left(x_1^2 + x_2^2 - 1\right) + 2x_1(x_1 + x_2) & \left(x_1^2 + x_2^2 - 1\right) + 2x_2(x_1 + x_2) \end{bmatrix}$$

Linearization about $(0,0)$:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$ so the equilibrium is a stable focus.

(7)
$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1 - x_2^3$$

Solution. The equilibrium points are $(0,0)$, $(1,1)$, and $(-1,-1)$. The Jacobian of the vector field is

$$\nabla f(x) = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

Linearization about $(0,0)$:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 1$ so the equilibrium is a saddle point.

Linearization about $(1, 1)$:

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -4$ and $\lambda_2 = -2$ so the equilibrium stable improper node.

Linearization about $(-1, -1)$:

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x$$

with eigenvalues: $\lambda_1 = -4$ and $\lambda_2 = -2$ so the equilibrium stable improper node.

7. For each of the A matrices below, consider the system $\dot{x} = Ax$ and:

   (a) determine the matrix M that transforms A into the appropriate modal form and write the system in modal coordinates ($\dot{z} = (M^{-1}AM)z$);
   (b) classify the equilibrium $(0, 0)$; and
   (c) make the phase portraits of the system in both the modal (z) and the original (x) coordinates.

   i. $$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

   Solution.

   (a) The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$ and

   $$\begin{aligned} M &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.7071 & -0.4472 \\ -0.7071 & 0.8944 \end{bmatrix} \end{aligned}$$

   where v_1 and v_2 are the eigenvectors associated with the eigevalues λ_1 and λ_2 (note: this M is not unique; other choices of M exist). The system in modal coordinates is

   $$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z.$$

   (b) The equilibrium point $(0, 0)$ is a stable improper node.
   (c) Phase portrait of the system:

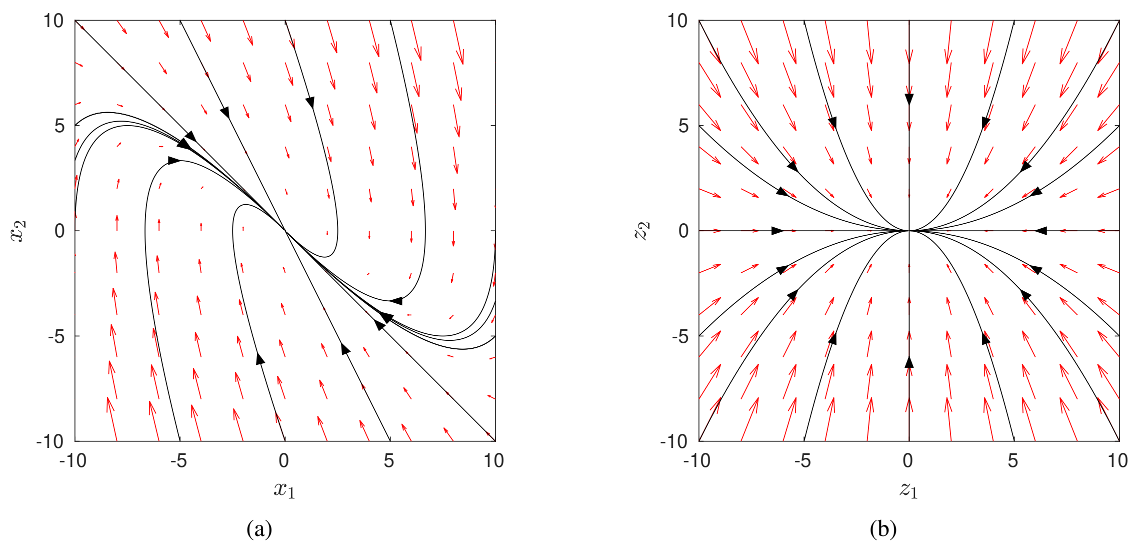


Figure 4: Phase portrait of system in (a) the original (x) coordinates and (b) the modal (z) coordinates.

ii. $$A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

Solution.

(a) The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $M = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ is an eigenvector and v_2 is the solution to

$$\begin{aligned} Av_2 &= \lambda v_2 + v_1 \\ \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} &= 1v_2 + \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix} \\ \begin{bmatrix} -v_{22} \\ v_{21} + 2v_{22} \end{bmatrix} &= \begin{bmatrix} v_{21} - 0.7071 \\ v_{22} + 0.7071 \end{bmatrix} \end{aligned}$$

The system of equations is undetermined. Picking $v_{21} = 0$ implies $v_{22} = 0.7071$. Thus,

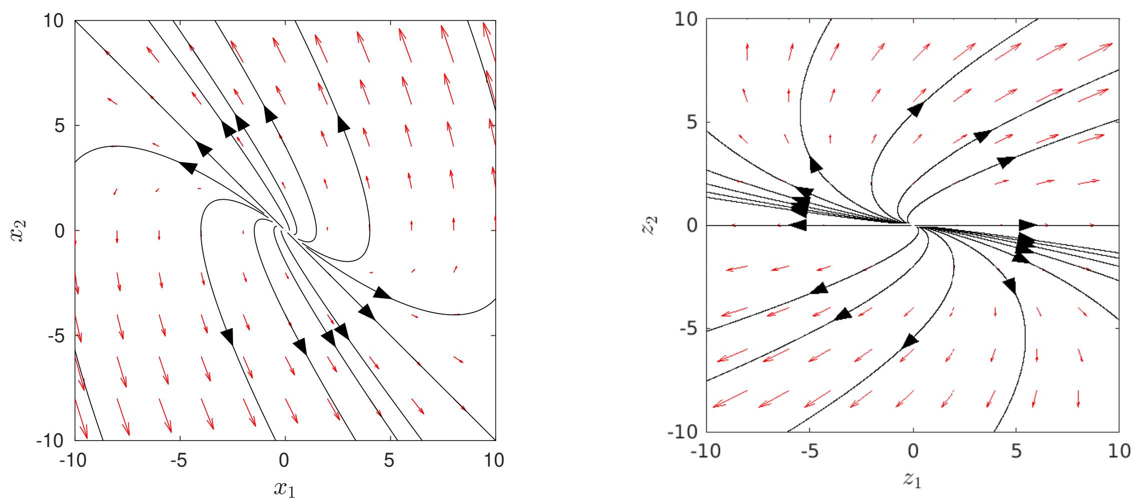
$$\begin{aligned} M &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\ &= \begin{bmatrix} -0.7071 & 0 \\ 0.7071 & 0.7071 \end{bmatrix} \end{aligned}$$

The system in modal coordinates is

$$\dot{z} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z.$$

(b) The equilibrium point $(0, 0)$ is a unstable node.

(c) Phase portrait of the system:



(a)

(b)

Figure 5: Phase portrait of system in (a) the original (x) coordinates and (b) the modal (z) coordinates.

iii. $$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Solution.

(a) The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$ and

$$\begin{aligned} M &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\ &= \begin{bmatrix} -0.4472 & 1.0000 \\ 0.8944 & 0.0000 \end{bmatrix} \end{aligned}$$

where v_1 and v_2 are the eigenvectors associated with the eigevalues λ_1 and λ_2. The system in modal coordinates is

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z.$$

(b) The equilibrium point $(0, 0)$ is a saddle point.

(c) Phase portrait of the system:

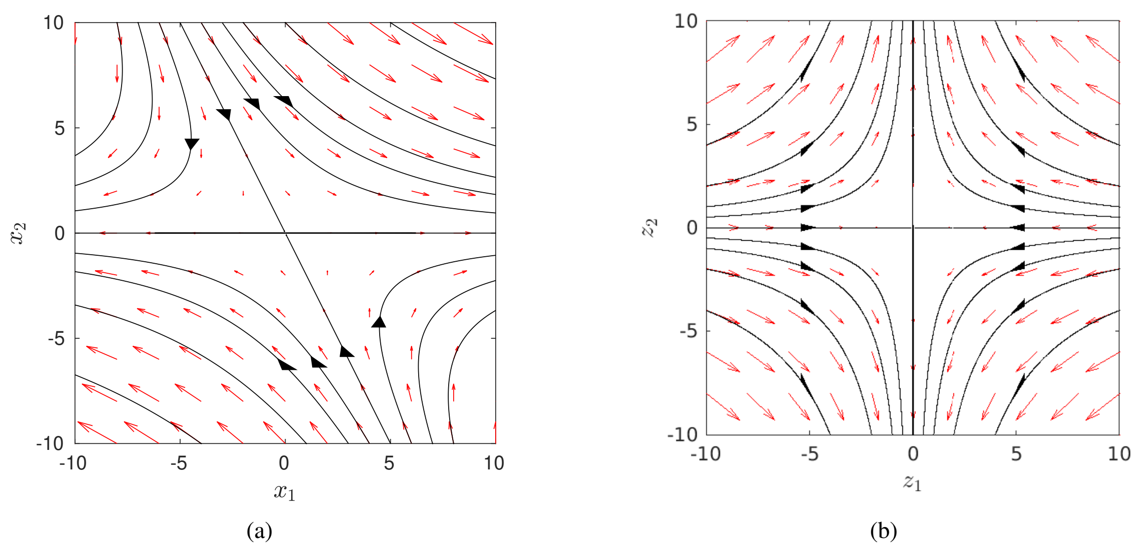


Figure 6: Phase portrait of system in (a) the original (x) coordinates and (b) the modal (z) coordinates.

iv. $$A = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix}$$

Solution.

(a) The eigenvalues of A are $\lambda_1 = 2i$ and $\lambda_2 = -2i$ and

$$M = \begin{bmatrix} 0.9129 & 0.0000 \\ -0.1826 & -0.3651 \end{bmatrix}$$

Note: this M may be found from the eigenvectors of A ($M = \begin{bmatrix} u & v \end{bmatrix}$ with $A(u+iv) = (\alpha+i\beta)(u+iv)$). The system in modal coordinates is

$$\dot{z} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} z.$$

(b) The equilibrium point $(0,0)$ is a center.

(c) Phase portrait of the system:

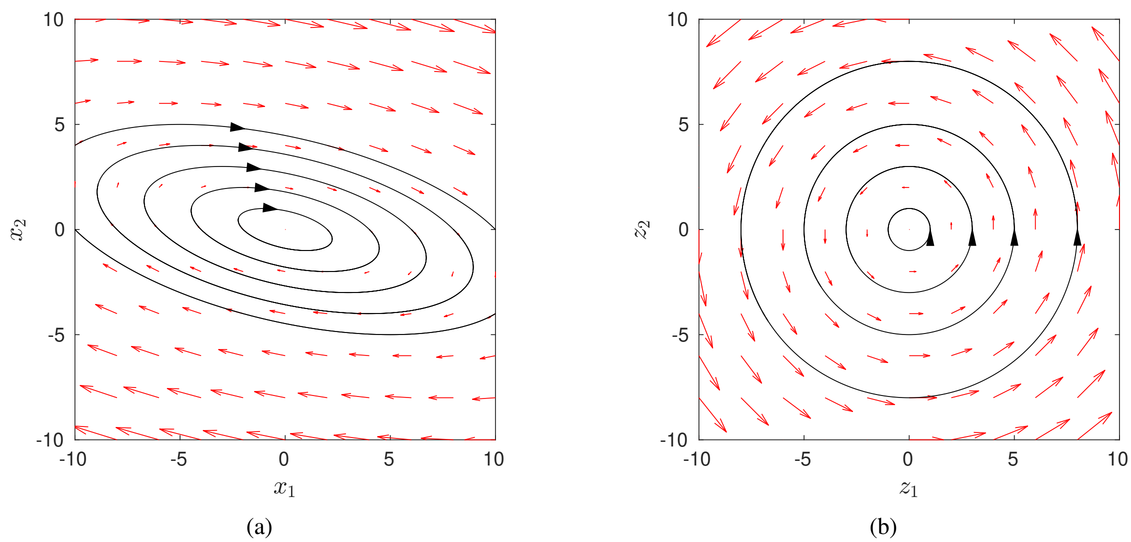


Figure 7: Phase portrait of system in (a) the original (x) coordinates and (b) the modal (z) coordinates.

v. $$A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

Solution.

(a) The eigenvalues of A are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ and

$$M = \begin{bmatrix} 0.4082 & -0.4082 \\ 0.8165 & 0.0000 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} z.$$

(b) The equilibrium point $(0, 0)$ is a unstable focus.

(c) Phase portrait of the system:

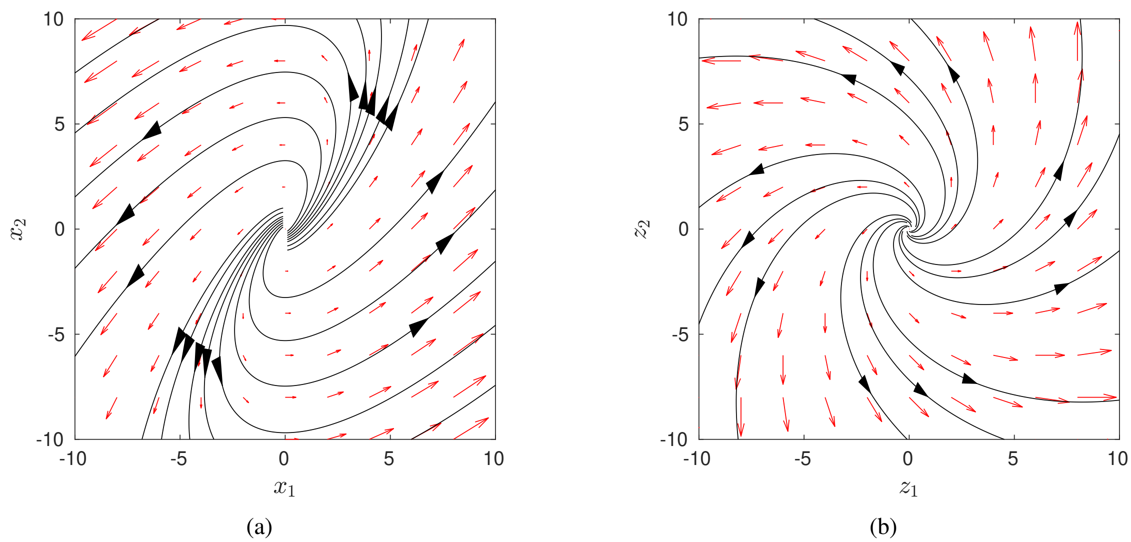


Figure 8: Phase portrait of system in (a) the original (x) coordinates and (b) the modal (z) coordinates.

8. Exercise 2.5 (Khalil pg. 78). The system

$$\dot{x}_1 = -x_1 - \frac{x_2}{\ln\sqrt{x_1^2+x_2^2}}$$
$$\dot{x}_2 = -x_2 + \frac{x_1}{\ln\sqrt{x_1^2+x_2^2}}$$

has an equilibrium point at the origin.

(a) Linearize the system about the origin and find the type of the origin as an equilibrium point of the linear system.

Solution. The Jacobian of the vector field f is

$$\nabla f = \begin{bmatrix} -1 - \frac{-x_2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2}\frac{1}{\sqrt{x_1^2+x_2^2}}\frac{2x_1}{2\sqrt{x_1^2+x_2^2}} & \frac{-1}{\ln\sqrt{x_1^2+x_2^2}} + \frac{x_2^2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2\left(x_1^2+x_2^2\right)} \\ \frac{1}{\ln\sqrt{x_1^2+x_2^2}} - x_1\frac{1}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2}\frac{1}{\sqrt{x_1^2+x_2^2}}\frac{x_1}{\sqrt{x_1^2+x_2^2}} & -1 - \frac{x_1x_2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2\left(x_1^2+x_2^2\right)} \end{bmatrix}$$
$$= \begin{bmatrix} -1 + \frac{x_1x_2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2\left(x_1^2+x_2^2\right)} & \frac{-1}{\ln\sqrt{x_1^2+x_2^2}} + \frac{x_2^2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2\left(x_1^2+x_2^2\right)} \\ \frac{1}{\ln\sqrt{x_1^2+x_2^2}} - \frac{x_1^2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2\left(x_1^2+x_2^2\right)} & -1 - \frac{x_1x_2}{\left(\ln\sqrt{x_1^2+x_2^2}\right)^2\left(x_1^2+x_2^2\right)} \end{bmatrix}$$

Linearizing about the origin (taking the limit of the Jacobian of as x_1 and x_2 approaches the origin) yields

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x$$

From inspection the eigenvalues are both -1, so the origin is a stable proper node.

(b) Find the phase portrait of the nonlinear system near the origin, and show that the portrait resembles a stable focus. **Hint:** Transform the equations into polar coordinates.

Solution. Converting x_1 and x_2 into polar coordinates:

$$x_1 = r\cos\theta,$$
$$x_2 = r\sin\theta,$$

and

$$r = \sqrt{x_1^2+x_2^2},$$
$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right).$$

The ODEs becomes

$$\begin{aligned} \dot{r} &= \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{2\sqrt{x_1^2+x_2^2}} \\ &= \frac{1}{\sqrt{x_1^2+x_2^2}}\left(-x_1^2 - \frac{x_1x_2}{\ln\sqrt{x_1^2+x_2^2}} - x_2^2 + \frac{x_1x_2}{\ln\sqrt{x_1^2+x_2^2}}\right) \\ &= \frac{-1}{\sqrt{x_1^2+x_2^2}}\left(x_1^2+x_2^2\right) \\ &= -\sqrt{x_1^2+x_2^2} \\ \dot{r} &= -r. \end{aligned}$$

and

$$\begin{aligned}
\dot{\theta} &= \frac{1}{1 + (x_2/x_1)^2} \left(-\frac{x_2}{x_1^2}\dot{x}_1 + \frac{1}{x_1}\dot{x}_2 \right) \\
&= \frac{x_1^2}{x_1^2 + x_2^2} \left(-\frac{x_2}{x_1^2}\dot{x}_1 + \frac{1}{x_1}\dot{x}_2 \right) \\
&= \frac{1}{x_1^2 + x_2^2} \left(-x_2\dot{x}_1 + x_1\dot{x}_2 \right) \\
&= \frac{1}{x_1^2 + x_2^2} \left(x_2 x_1 + \frac{x_2^2}{\ln\sqrt{x_1^2 + x_2^2}} - x_1 x_2 + \frac{x_1^2}{\ln\sqrt{x_1^2 + x_2^2}} \right) \\
&= \frac{1}{\ln\sqrt{x_1^2 + x_2^2}} \\
\dot{\theta} &= \frac{1}{\ln r}.
\end{aligned}$$

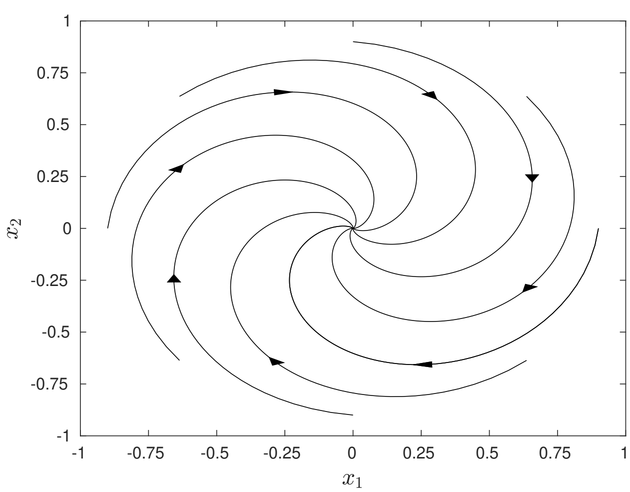
These ODE can be solved explicitly:

$$\begin{gathered}
\int_{r_0}^{r} \frac{dr}{r} = -\int_{t_0}^{t} dt \\
\ln\left(\frac{r}{r_0}\right) = -t \\
r = r_0 e^{-t}
\end{gathered}$$

and

$$\begin{gathered}
\dot{\theta} = \frac{1}{\ln r_0 - t} \\
\int_{\theta_0}^{\theta} d\theta = -\int_{t_0}^{t} \frac{dt}{t - \ln r_0} \\
\theta - \theta_0 = -\ln\left(\left| \frac{t - \ln r_0}{\ln r_0} \right| \right) \\
\theta = \theta_0 + \ln|\ln r_0| - \ln|t - (\ln r_0)|\,.
\end{gathered}$$

From these equations, phase portraits around the origin can be constructed which resembles a stable focus:



(c) Explain the discrepancy between the results of parts (a) and (b).

Solution. This is a special case described on page 54 of Khalil where the linearized state equation has a node with two equal eigenvalues and the vector field $f(x)$ is not analytic in a neighborhood near the origin.

9. Exercise 1.17 (Khalil Second Edition pg. 54). For each of the following systems, construct the phase portrait and discuss the qualitative behavior of the system.

(1)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - 2\tan^{-1}(x_1 + x_2)$$

Solution.

- The equilibrium points of the system are $(0,0)$ and $(\pm 2.331, 0)$.
- The Jacobian matrix is
$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{2}{1+(x_1+x_2)^2} & \frac{-2}{1+(x_1+x_2)^2} \end{bmatrix}$$
- Linearization about $(0,0)$:
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$
with eigenvalues: $\lambda_1 = \lambda_2 = -1$ so this is a stable node.
- Linearization about $(2.331, 0)$:
$$A = \begin{bmatrix} 0 & 1 \\ 0.6889 & -0.3111 \end{bmatrix}$$
with eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 0.6889$ so this is a saddle point.
- Linearization about $(-2.331, 0)$:
$$A = \begin{bmatrix} 0 & 1 \\ 0.6889 & -0.3111 \end{bmatrix}$$
with eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 0.6889$ so this is a saddle point.
- Phase portrait:

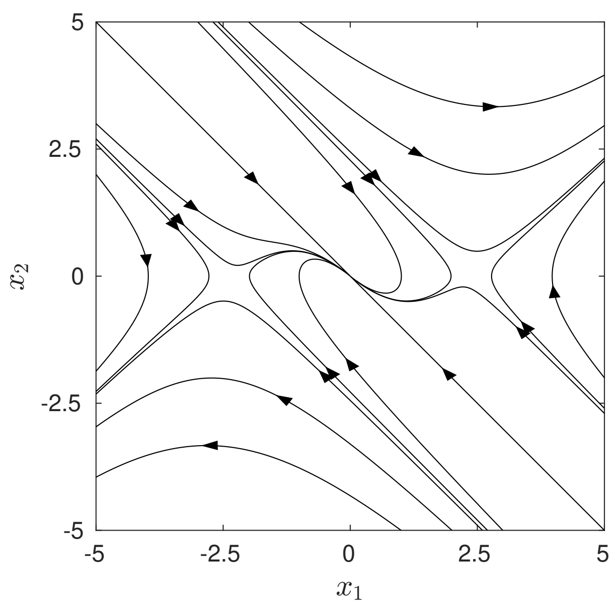


Figure 9: Exercise 1.17 (1).

The phase portrait shows that the two separatrices formed by the trajectories approach the saddle points from the $(-1, 1)$ and $(1, -1)$ directions. These two separatrices define a layer. Every trajectory starting inside this layer approaches the stable node as $t \to \infty$. Trajectories starting outside this layer approach infinity as $t \to \infty$. The line $x_2 = -x_1$ is a trajectory approaching the origin. This behavior can also be seen by defining a new variable $\sigma = x_1 + x_2$ and noting that σ satisfies the scalar equation

$$\dot{\sigma} = \sigma - 2\tan^{-1}\sigma$$

(2)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)\end{aligned}$$

Solution.

- The equilibrium point of the system is $(0,0)$.
- The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 6x_1x_2 & (1 - 3x_1^2 - 2x_2^2) - 4x_2 \end{bmatrix}$$

- Linearization about $(0,0)$:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

  with eigenvalues: $\lambda_1 = 0.5 + 0.8660i$ and $\lambda_2 = 0.5 - 0.8660i$ so this is an unstable focus.
- Phase portrait:

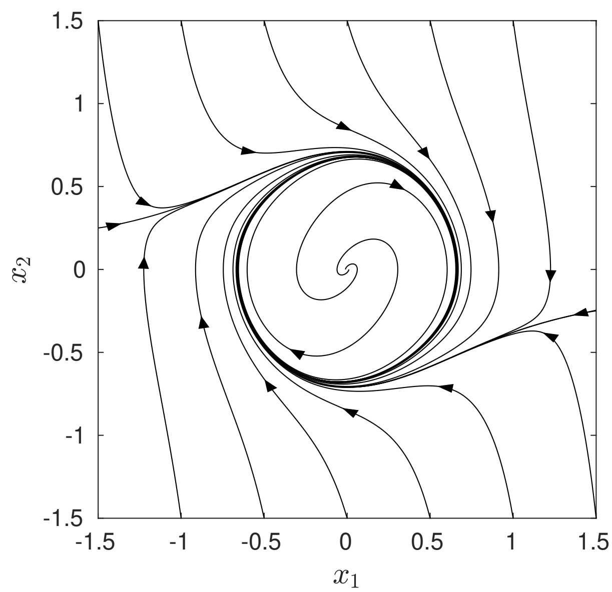


Figure 10: Exercise 1.17 (2).

- From the phase portrait, we can see an attractive limit cycle and the unstable focus. Starting from anywhere (excluding the origin), the trajectories converge to the limit cycle.

(3)
$$\dot{x}_1 = x_1 - x_1 x_2$$
$$\dot{x}_2 = 2x_1^2 - x_2$$

Solution.

- The equilibrium points of the system are $(0,0)$, $(1,2)$ and $(-1,2)$.
- The Jacobian matrix is
$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{bmatrix}$$
- Linearization about $(0,0)$:
$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$
with eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = -1$ so this is a saddle point.
- Linearization about $(1,2)$:
$$A = \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix}$$
with eigenvalues: $\lambda_1 = -0.5 + 1.9365i$ and $\lambda_2 = -0.5 - 1.9365i$ so this is a stable focus.
- Linearization about $(-1,2)$:
$$A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$$
with eigenvalues: $\lambda_1 = -0.5 + 1.9365i$ and $\lambda_2 = -0.5 - 1.9365i$ so this is a stable focus.
- Phase portrait:

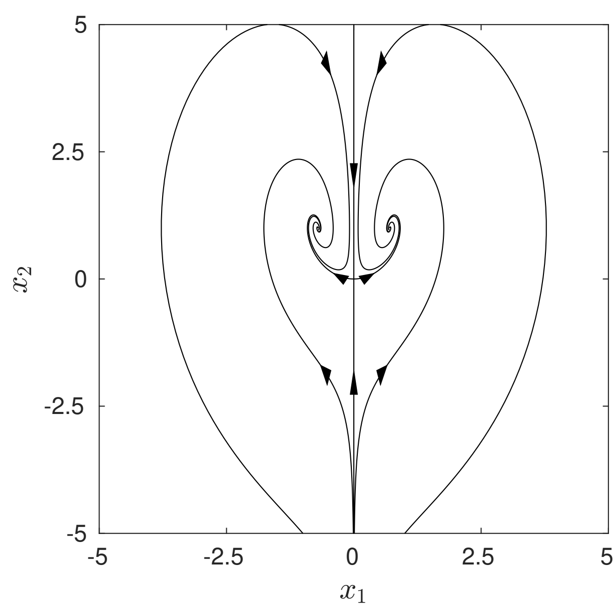


Figure 11: Exercise 1.17 (3).

- From the phase portrait, we see the two stable focus and the saddle point at the origin. Trajectories starting in the left two quadrants will converge to the stable focus at $(-1,2)$. Trajectories starting in the right two quadrants will converge to the stable focus at $(1,2)$. Trajectories that converge to the saddle point start along the x_2 axis.

(4)
$$\dot{x}_1 = x_1 + x_2 - x_1(|x_1| + |x_2|)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(|x_1| + |x_2|)$$

Solution.

- The equilibrium point of the system is $(0, 0)$.
- The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 - (|x_1| + |x_2|) - x_1 \frac{d|x_1|}{dx_1} & 1 - x_1 \frac{d|x_2|}{dx_2} \\ -2 - x_2 \frac{d|x_1|}{dx_1} & 1 - (|x_1| + |x_2|) - x_2 \frac{d|x_2|}{dx_2} \end{bmatrix}$$

- Linearization about $(0, 0)$:

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = 1 + 1.4142i$ and $\lambda_2 = 1 - 1.4142i$ so this is an unstable focus.

- Phase portrait:

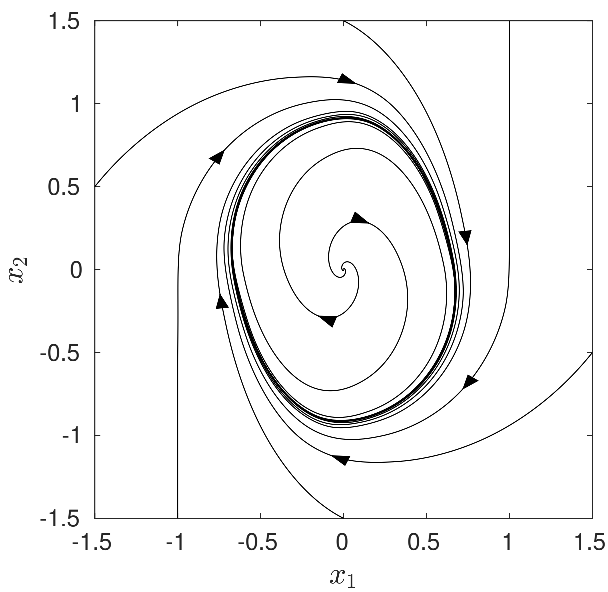
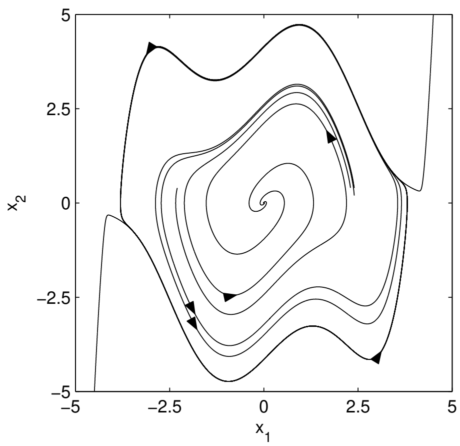
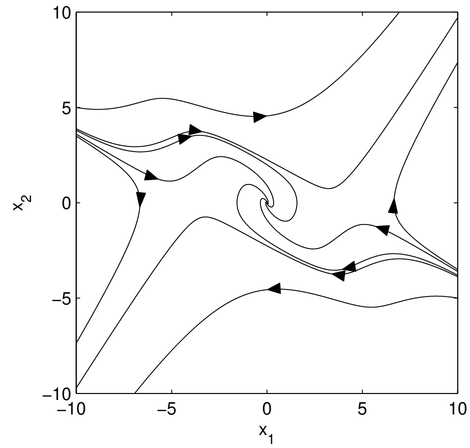


Figure 12: Exercise 1.17 (4).

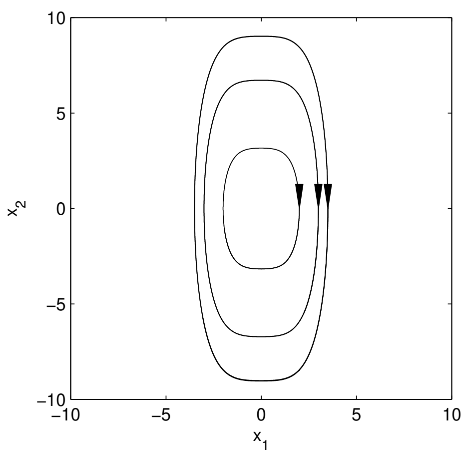
- From the phase portrait, we see an attractive limit cycle. Starting from anywhere (excluding the origin), the trajectory moves to the limit cycle.



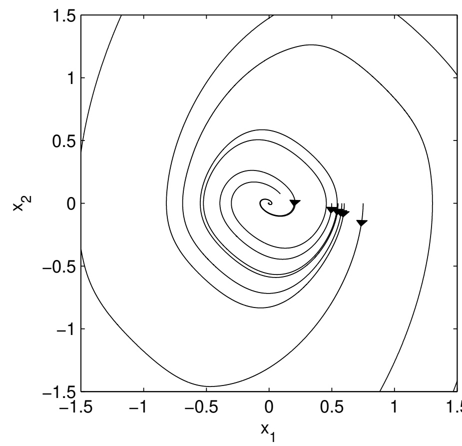
(a)



(b)



(c)



(d)

Figure 13: Exercise 1.22

10. Exercise 1.22 (Khalil Second Edition pg. 55). The phase portraits of the following four systems are shown in Figures 13: parts (a), (b), (c), and (d) respectively. Discuss if the arrowheads are pointed in the correct direction and discuss the qualitative behavior of each system.

(1)

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 + 0.1x_1^4)$$

Solution. The equilibrium point of the system is $(0,0)$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -1 \\ 1 + x_2\left(2x_1 - 0.5x_1^3\right) & -(1 - x_1^2 + 0.1x_1^4) \end{bmatrix}$$

The linearization about $(0,0)$ is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = -0.5 + 0.8660i$ and $\lambda_2 = -0.5 - 0.8660i$ so this is a stable focus. The phase portrait for this system is given by

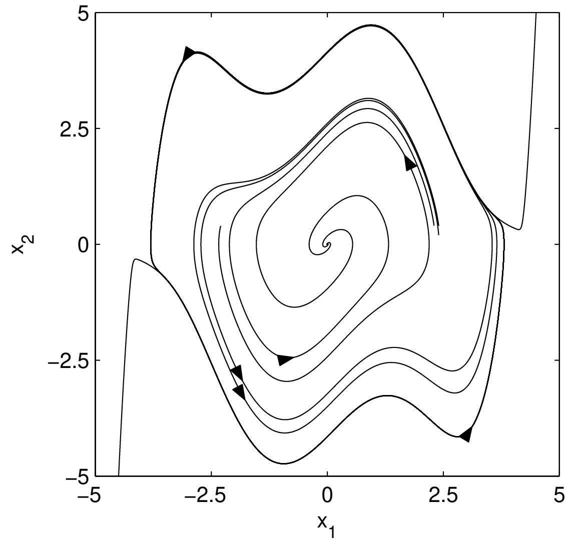


Figure 14: Exercise 1.22 (1).

The direction of the arrowheads can be determined by inspection of the vector field. In particular, since $f_1(x) = -x_2$, we see that f_1 is negative in the upper half of the plane and positive in the lower half.

From the phase portrait, we can observe two limit cycles: an attractive limit cycle on the outside and a repulsive limit cycle on the inside. Starting from inside the repulsive limit cycle, the trajectory will move to the stable focus. For any initial condition outside or on the inside limit cycle, the resulting trajectory moves towards the outside attractive limit cycle.

(2)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + x_2 - 3\tan^{-1}(x_1 + x_2)$$

Solution.

The equilibrium points of the system are $(0,0)$, $(3.9726, 0)$ and $(-3.9726, 0)$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{3}{1+(x_1+x_2)^2} & 1 - \frac{3}{1+(x_1+x_2)^2} \end{bmatrix}$$

The linearization about $(0,0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$ so this is a stable focus. The linearization about $(3.9726, 0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ 0.8212 & 0.8212 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = -0.5843$ and $\lambda_2 = 1.4055$ so this is a saddle point. The linearization about $(-3.9726, 0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ 0.8212 & 0.8212 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = -0.5843$ and $\lambda_2 = 1.4055$ so this is a saddle point. The phase portriat for this system is given by

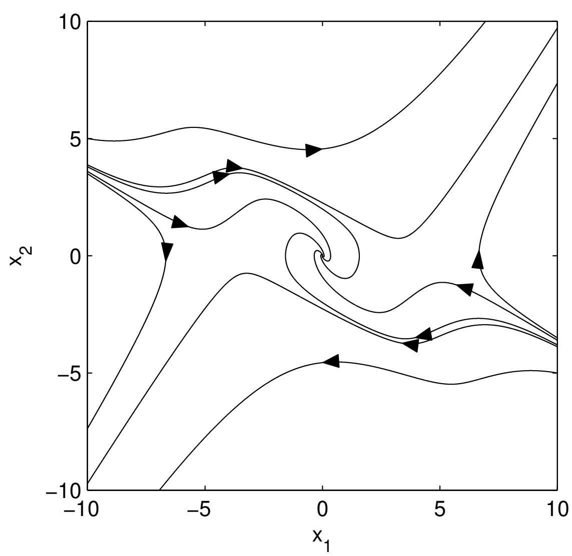


Figure 15: Exercise 1.22 (2).

The direction of the arrowheads can be determined by inspection of the vector field. Since $f_1(x) = x_2$, we see that f_1 is positive in the upper half of the plane, and negative in the lower half. Trajectories approaching the saddle points form two separatrices that divide the plane into three regions. Trajectories starting in the middle region spiral toward the origin, while those starting in the outer region approach infinity.

(3)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(0.5x_1 + x_1^3)$$

Solution.

The equilibrium point of the system is $(0, 0)$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\left(0.5 + 3x_1^2\right) & 0 \end{bmatrix}$$

The linearization about $(0, 0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = 0.7071i$ and $\lambda_2 = 0.7071i$ so this is a center in the linearization and no conclusions can be made about the nonlinear system using this linearization method. The phase portrait for this system is given by

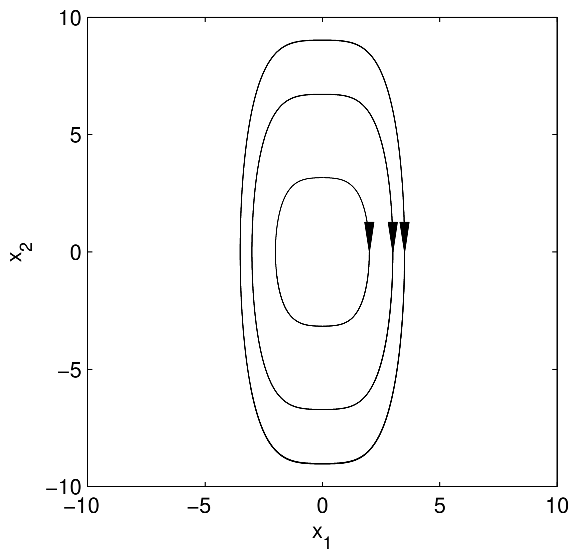


Figure 16: Exercise 1.22 (3).

The direction of the arrowheads can be determined by inspection of the vector (in a similar fashion as previous solutions). From the phase portrait, we observe that the equilibrium point at the origin is a center because the trajectories are closed cycles around the origin.

(4)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - \psi(x_1 - x_2)$$

where $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5\text{sign}(y)$ if $|y| > 1$.

Solution.

The Jacobian matrix is (assuming $|x_1 - x_2| \leq 1$):

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -3(x_1 - x_2)^2 - 0.5 & -0.5 + 3(x_1 - x_2)^2 \end{bmatrix}$$

The equilibrium point of the system is $(0, 0)$. The linearization about $(0, 0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix}$$

with eigenvalues: $\lambda_1 = -0.25 + 0.6614i$ and $\lambda_2 = -0.25 - 0.6614i$ so this is a stable focus. The phase portrait for this system is

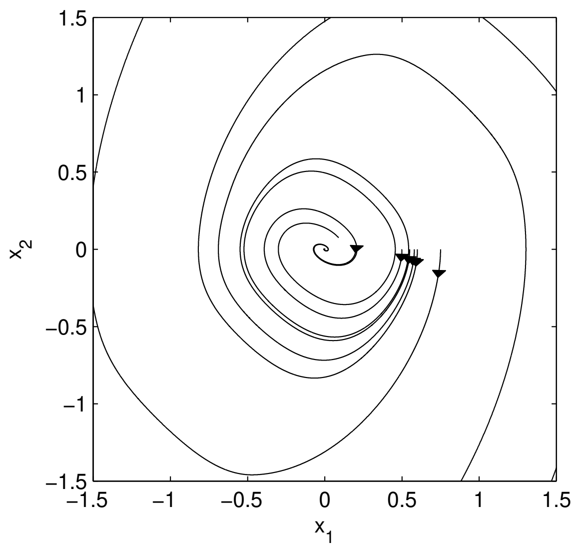


Figure 17: Exercise 1.22 (4).

From the phase portrait, we see a repulsive limit cycle. A trajectory that starts inside it will move towards the stable focus and a trajectory that starts outside it will diverge away.

11. Exercise 2.1 (Khalil Second Edition pg. 88). Show that, for any $x \in \mathbb{R}^n$, we have

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$$
$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$
$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$$

Solution. Note that for $x \in \mathbb{R}^n$:

$$\|x\|_1 := |x_1| + \cdots + |x_n|$$
$$\|x\|_2 := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$
$$\|x\|_\infty := \max_i |x_i|$$

Thus,

$$\|x\|_2^2 = |x_1|^2 + \cdots + |x_n|^2 \le (|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2$$
$$\Rightarrow \|x\|_2 \le \|x\|_1$$

Similarly,

$$\|x\|_\infty^2 = \max_i |x_i|^2 \le |x_1|^2 + \cdots + |x_n|^2 = \|x\|_2^2$$
$$\Rightarrow \|x\|_\infty \le \|x\|_2$$

and

$$\|x\|_\infty^2 = \max_i |x_i|^2 \le (|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2$$
$$\Rightarrow \|x\|_\infty \le \|x\|_1$$

Note that $\|x\|_\infty \le \|x\|_1$ also follows from the fact that the first two results ($\|x\|_\infty \le \|x\|_2 \le \|x\|_1$).
To show $\|x\|_1 \le \sqrt{n}\|x\|_2$, we will use Cauchy-Schwarz inequality:

$$\|x\|_2\|y\|_2 \ge \|x^T y\|_2$$

for all x, $y \in \mathbb{R}^n$. Thus,

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n |x_i| \cdot 1 = z^T\mathbf{1} = |z^T\mathbf{1}| \le \|z\|_2\|\mathbf{1}\|_2 = \sqrt{n}\|x\|_2$$

where $z^T := \begin{bmatrix}|x_1| & \cdots & |x_n|\end{bmatrix}$ and $\mathbf{1}^T := \begin{bmatrix}1 & \cdots & 1\end{bmatrix}$. The remaining two inequalities may be shown:

$$\|x\|_2^2 = |x_1|^2 + \cdots |x_n|^2 \le n \max_i |x_i|^2 = n\|x\|_\infty^2$$
$$\Rightarrow \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

and

$$\|x\|_1^2 = (|x_1| + \cdots |x_n|)^2 = (|x_1|^2 + |x_1||x_2| + \cdots) + (|x_2||x_1| + |x_2|^2 + |x_2||x_3| + \cdots) + \cdots \le n^2 \max_i |x_i|^2$$

The last inequality follows from the fact that there are n^2 terms which can each be bounded by $\max_i |x_i|^2$. Thus,

$$\Rightarrow \|x\|_1 \le n\|x\|_\infty$$

12. Exercise 2.3 (Khalil Second Edition pg. 88). Consider the set $S = \{x \in \mathbb{R}^2 \mid -1 < x_i \le 1,\ i = 1, 2\}$. Is S open? Is it closed? Find the closure, interior, and boundary of S.

Solution. Before addressing the problem, we first need formal definitions. We utilize definitions from *Optimization by Vector Space Methods* by D. G. Luenberger (similar definitions may be found in Khalil Appendix A).

- Let S be a subset of a normed space X. The point $s \in S$ is said to be an *interior point* of S if there is an $\epsilon > 0$ such that all vectors x satisfying $\|x - s\| < \epsilon$ are also members of S.
- The collection of all interior points of S is called the *interior* of S.
- A set S is said to be *open* if S is equal to the interior of S.
- A point $x \in X$ is said to be a *closure point* of a set S if, given $\epsilon > 0$, there is a point $s \in S$ satisfying $\|x - s\| < \epsilon$.
- The collection of all closure points of S is called the *closure* of S and is denoted by $\bar{S}$.
- A set S is said to be *closed* if S is equal to the closure of S.
- A point s is a *boundary point* of a set S if every neighborhood of s contains at least one point of S and one point not belonging to S.
- The set of all boundary points of S, denoted by ∂S, is called the *boundary* of S.
- The interior of S is denoted by $S - \partial S$ (here the minus symbol means set subtraction).

With these definitions in mind, it is clear that the set S is neither open nor closed since the closure of S is:

$$\bar{S} = \left\{x \in \mathbb{R}^2 \mid |x_i| \leq 1,\ i = 1, 2\right\}$$

and the interior of S is

$$S - \partial S = \left\{x \in \mathbb{R}^2 \mid |x_i| < 1,\ i = 1, 2\right\}$$

The boundary of S is:

$$\partial S = \left\{x \in \mathbb{R}^2 \mid |x_i| = 1,\ |x_j| \leq 1,\ i = 1, 2\ j \neq i\right\}$$

Since $S \neq \bar{S}$ and $S \neq S - \partial S$, the set is neither open nor closed.

13. Exercise 2.4 (Khalil Second Edition pg. 88). Let $u_T(t)$ be the unit step function, defined by $u_T(t) = 0$ for $t < T$ and $u_T(t) = 1$ for $t \geq T$.

   (a) Show that $u_T(t)$ is piecewise continuous.

   Solution. The unit step function is continuous on the intervals $(-\infty, T)$ and (T, ∞). The only discontinuity of $u_T(t)$ is at $t = T$. Therefore, $u_T(t)$ is piecewise continuous since it is only discontinuous at a finite number of points.

   (b) Show that $f(t) = g(t)u_T(t)$, for any continuous function $g(t)$, is piecewise continuous.

   Solution. The function f can be written as:

$$f(t) = \begin{cases} 0, & t < T \\ g(t), & t \geq T \end{cases}$$

   which, again, is continuous on the intervals $(-\infty, T)$ and (T, ∞). At $t = T$, the function $f(t)$ will be discontinuous if $g(T) \neq 0$ and continuous if $g(T) = 0$. Thus, $f(t)$ is (at least) piecewise continuous.

   (c) Show that the periodic square waveform is piecewise continuous.

   Solution. The periodic square waveform is piecewise continuous because on any bounded subinterval it is continuous expect for a finite number of discontinuities in the subinterval. In other words, the function is continuous except for a countably infinite number of discontinuity points.

14. Exercise 2.6 (Khalil Second Edition pg. 88). Let $f(x)$ be continuously differentiable. Show that an equilibrium point x^* of $\dot{x} = f(x)$ is isolated if the Jacobian matrix $[\partial f/\partial x](x^*)$ is nonsingular. **Hint:** Use the implicit function theorem.

Solution. There are many ways to show this result. Since x^* is an equilibrium point or $f(x^*) = 0$, we can define a new function $\bar{f}$ as

$$\bar{f}(x, y) := f(x) - y$$

where $f(x^*, 0) = 0$. If the Jacobian matrix $[\partial f/\partial x](x^*)$ is nonsingular, then the Jaboian matrix $[\partial \bar{f}/\partial x](x^*, 0)$ is nonsingular since:

$$\frac{\partial \bar{f}}{\partial x}(x^*, 0) = \frac{\partial f}{\partial x}(x^*)$$

By the implicit function theorem, there exist neighborhoods $U \subset \mathbb{R}^n$ of x^* and $V \subset \mathbb{R}^n$ of 0 for each $y \in V$ the equation $\bar{f}(x, y) = 0$ has a unique solution $x \in U$. This means $x = x^*$ is an isolated equilibrium point because it is a unique solution of $\bar{f}(x, y) = 0$ for $y = 0$.

15. Exercise 2.26 (Khalil Second Edition pg. 92). For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, find whether f is (a) continuously differentiable at $x = 0$; (b) locally Lipschitz at $x = 0$; (c) continuous at $x = 0$; (d) globally Lipschitz; (e) uniformly continuous on $\mathbb{R}$; (f) Lipschitz on $(-1, 1)$.

(1) $f(x) = \begin{cases} x^2 \sin(1/x), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$

Solution.

(a) Consider $x \neq 0$, the derivative of f is

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \quad \forall x \in \mathbb{R} - \{0\}.$$

so $\lim_{x\to 0} f'(x)$ does not exist. When $x = 0$, the derivative is

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} h \sin(1/h) = 0.$$

Therefore, f is not continuously differentiable at $x = 0$.

(b) Consider:

$$\begin{aligned} f(x) - f(y) &= x^2 \sin(1/x) - y^2 \sin(1/y) \\ &= x^2 \sin(1/x) - y^2 \sin(1/y) - y^2 \sin(1/x) + y^2 \sin(1/x) \\ &= (x^2 - y^2) \sin(1/x) + y^2(\sin(1/x) - \sin(1/y)) \end{aligned} \tag{1}$$

Prior to proceeding, we establish two facts that we will use in the sequel. First, we show that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$.

- Let $x = 0$:

$$|\sin(0)| = 0 = x$$

so $|\sin(x)| \leq |x|$ when $x = 0$ follows.

- Let $x \in (0, 1]$ and $f(x) = \sin(x)$. By the mean value theorem, there exist $z \in (0, 1)$ such that:

$$f'(z) = \cos(z) = \frac{\sin(x) - \sin(0)}{x - 0}$$

Since $-1 \leq \cos(z) \leq 1$ for all z, we have:

$$-1 \leq \frac{\sin(x)}{x} \leq 1$$

which implies $\sin(x) \leq x$. Repeating similar steps for $x \in [-1, 0)$, shows that $-1 \leq \sin(x)/x \leq 1$ for $x \in [-1, 0)$.

- Let $|x| > 1$. Since $0 \leq |\sin(x)| \leq 1$, it follows that $|\sin(x)| \leq |x|$.
- Thus, $|\sin(x)| \leq |x|$ for all x.

Second, we will show $|\sin(1/x) - \sin(1/y)| \leq |x-y|/|xy|$:

$$\begin{aligned}|\sin(1/x) - \sin(1/y)| &= \left|2\cos\left(\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)\right)\sin\left(\frac{1}{2}\left(\frac{1}{x}-\frac{1}{y}\right)\right)\right| \\ &\leq 2\left|\sin\left(\frac{y-x}{2xy}\right)\right| \\ &\leq 2\frac{|y-x|}{2|xy|} = \frac{|x-y|}{|xy|}\end{aligned}$$

where the first equality follows from standard trigometric relationships, the second inequality follows from the fact that $\cos(z) \leq 1$ for all z, and the last inequality follows from the fact that $|\sin(z)| \leq |z|$ Considering the first term of (1),

$$\begin{aligned}\left|(x^2-y^2)\sin(1/x)\right| &= |(x-y)(x+y)\sin(1/x)| \\ &\leq |x-y||x+y||\sin(1/x)| \\ &\leq |x-y||x+y|\left|\frac{1}{x}\right|\end{aligned}$$

where the the second to last inequality follows from the fact that $|\sin(z)| \leq |z|$ (or $|\sin(1/x)| \leq |1/x|$).

We proceed by treating two cases. In the first case, we consider neither x nor y to be zero. In the second case, we consider that one of the two variables are equal to zero. We do not need to look at the case when both x and y are zero since it is trivial to find a $L > 0$ such that the Lipschitz inequality is satisfied for this case.

Proceeding with the first case, without loss of generality, consider $0 < |y| \leq |x|$. Then,

$$|x+y|\left|\frac{1}{x}\right| \leq (|x|+|y|)\left|\frac{1}{x}\right| = 1 + \frac{|y|}{|x|} \leq 2$$

Thus,

$$\left|(x^2-y^2)\sin(1/x)\right| \leq 2|x-y| \tag{2}$$

for $0 < |y| \leq |x|$. Considering the second term of (1),

$$\begin{aligned}\left|y^2(\sin(1/x) - \sin(1/y))\right| &= |y^2|\left|\frac{1}{x}-\frac{1}{y}\right| \\ &= |y^2|\left|\frac{1}{xy}(y-x)\right| \\ &= \left|\frac{y}{x}\right||x-y| \\ &\leq |x-y| \end{aligned} \tag{3}$$

for $0 < |y| \leq |x|$. From (1), (2), and (3), we have

$$\begin{aligned}|f(x) - f(y)| &= |(x^2-y^2)\sin(1/x) + y^2(\sin(1/x) - \sin(1/y))| \\ &\leq 2|x-y| + |x-y| = 3|x-y|\end{aligned}$$

for $0 < |y| \leq |x|$.

Finally, for the second case, for $x \neq 0$ and $y = 0$,

$$|f(x) - f(y)| = |f(x) - f(0)| = \left|x^2\sin(1/x)\right| \leq \left|\frac{x^2}{x}\right| = |x| = |x-0| \leq 3|x-0|$$

Thus, the function is locally Lipschitz.

(c) Consider

$$\lim_{x\to 0} x^2 \sin(1/x) = 0$$

and $f(0) = 0$. Therefore f is continuous at $x = 0$.

(d) f is globally Lipschitz with Lipschitz constant $L = 3$.

(e) f is uniformly continuous on $\mathbb{R}$ because it is globally Lipschitz. This can also be shown by showing that the derivative is bounded on $\mathbb{R}$. First, note that on the interval $[-1,1]$ the function is continuous. Since $[-1,1]$ is a compact set and the function is continuous, the function is uniformly continuous on $[-1,1]$. Now, consider the interval $[1,\infty)$. The derivative is

$$f'(x) = 2x\sin(1/x) - \cos(1/x)$$

for $x \in [1,\infty)$. The derivative over this interval may be bounded:

$$\begin{aligned}|f'(x)| &= |2x\sin(1/x) - \cos(1/x)| \\ &\le 2\,|x\sin(1/x)| + |\cos(1/x)| \\ &\le 2|x||\sin(1/x)| + 1 \\ &\le 2|x|\left|\frac{1}{x}\right| + 1 = 3\end{aligned}$$

Thus, since the derivative may be bounded on $[1,\infty)$ it is uniformly continuous on $[1,\infty)$. Similiar steps may be shown for the interval $(-\infty,-1]$. Thus, we conclude that the function is uniformly continuous on $\mathbb{R}$.

(f) Since f is globally Lipschitz, it is Lipschitz on $(-1,1)$.

(2) $f(x) = \begin{cases} x^3\sin(1/x), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$

Solution.

(a) Consider $x \neq 0$, the derivative of f is

$$f'(x) = 3x^2\sin(1/x) - x\cos(1/x) \quad \forall x \in \mathbb{R} - \{0\}.$$

Note that as $x \to 0$, $f'(x) \to 0$. When $x = 0$, the derivative is

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} h^2 \sin(1/h) = 0.$$

Therefore, f is continuously differentiable.

(b) Since f is continuously differentiable, it implies that f is also locally Lipschitz.

(c) The Lipschitz property is stronger than continuity. Therefore, it implies continuity.

(d) f is not globally Lipschitz because f' cannot be globally bounded.

(e) f is not uniformly continuous because as $x \to \infty$, $f'(x) \to \infty$.

(f) Since f is Lipschitz on the interval $[-1,1]$, it is also Lipschitz on $(-1,1)$.

(3) $f(x) = \tan(\pi x/2)$

Solution.

(a) The derivative of f is

$$f'(x) = (\pi/2)\left(1 + \tan^2(\pi x/2)\right)$$

Thus, f is continuously differentiable at $x = 0$.

(b) Since f is continuously differentiable, it implies that f is also locally Lipschitz.

(c) The Lipschitz property is stronger than continuity. Therefore, it implies continuity.

(d) f is not globally Lipschitz because f' cannot be globally bounded.

(e) f is not uniformly continuous because as x goes to $2n-1$ where n is any integer, $f'(x) \to \infty$.

(f) f is not Lipschitz on the interval $(-1, 1)$ because as $x \to -1$ or $x \to 1$ (on the boundary of the interval), $f'(x) \to \infty$.

16. Exercise 2.27 (Khalil Second Edition pg. 92). For each of the following functions $f : \mathbb{R}^n \to \mathbb{R}^n$, find whether f is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz; (e) uniformly continuous on $\mathbb{R}^n$.

(1) $f(x) = \begin{bmatrix} x_1 + \operatorname{sgn}(x_2) \\ x_2 \end{bmatrix}$

Solution.

f is discontinuous at $x_2 = 0$. Thus, the answer to parts is no.

(2) $f(x) = \begin{bmatrix} x_1 + \operatorname{sat}(x_2) \\ x_1 + \sin x_2 \end{bmatrix}$

Solution.

The function sat is continuous, globally Lipschitz, but not continuous differential. The linear function x_1 and the function $\sin$ are continuously differentiable and globally Lipschitz.

(a) f is not continuously differentiable.

(b) f is locally Lipschitz. x_1, $\operatorname{sat}(x_2)$, and $\sin x_2$ are all globally Lipschitz and $f(x)$ is just the sum of these functions.

(c) f is continuous.

(d) f is globally Lipschitz.

(e) f is uniformly continuous on $\mathbb{R}^n$ since it is globally Lipschitz.

(3) $f(x) = \begin{bmatrix} x_3 \operatorname{sat}(x_1 + x_2) \\ x_2^2 \\ x_1 \end{bmatrix}$

Solution.

The function $x_3 \operatorname{sat}(x_1 + x_2)$ is continuous, locally Lipschitz, but not continuously differentiable nor globally Lipschitz. The function x_2^2 is continuously differentiable but not globally Lipschitz.

(a) f is not continuously differentiable

(b) f is locally Lipschitz (all functions locally Lipschitz).

(c) f is continuous (all functions are continuous).

(d) f is not globally Lipschitz (the term x_2^2 is not globally Lipschitz).

(e) f is not uniformly continuous on $\mathbb{R}^n$ since, for example, the term x_2^2 has an unbounded derivative as $x_2 \to \infty$ (this can also be shown by contradiction with the definition).

17. Exercise 3.5 (Khalil pg. 105). Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two different norms of the class of $p-$norms on $\mathbb{R}^n$. Show that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in $\|\cdot\|_\alpha$ if and only if it is Lipschitz in $\|\cdot\|_\beta$.

Solution. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in $\|\cdot\|_\alpha$ where $\|\cdot\|_\alpha$ is a norm of the class of $p-$norms on $\mathbb{R}^n$, then it satisfies the condition

$$\|f(x) - f(y)\|_\alpha \leq L \|x - y\|_\alpha$$

where $L > 0$. Owing to the equivalence of norms, then there exists positive real numbers c_1 and c_2 such that

$$c_1 \|f(x) - f(y)\|_\beta \leq \|f(x) - f(y)\|_\alpha \leq c_2 \|f(x) - f(y)\|_\beta ,$$

$$c_1 \|x - y\|_\beta \leq \|x - y\|_\alpha \leq c_2 \|x - y\|_\beta$$

where $\|\cdot\|_\beta$, a norm of the class of $p-$norms. This implies that f is Lipschitz in $\|\cdot\|_\beta$:

$$c_1 \|f(x) - f(y)\|_\beta \leq \|f(x) - f(y)\|_\alpha \leq L \|x - y\|_\alpha \leq Lc_2 \|x - y\|_\beta$$

or

$$\|f(x) - f(y)\|_\beta \leq \frac{Lc_2}{c_1} \|x - y\|_\beta .$$

A similar argument can be applied to show the converse is also true.

18. Exercise 3.7 (Khalil pg. 106). Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable for all $x \in \mathbb{R}^n$, and define $f(x)$ by

$$f(x) = \frac{1}{1 + g^T(x)g(x)} g(x).$$

Show that $\dot{x} = f(x)$, $x(0) = x_0$, has a unique solution defined for all $t \geq 0$.

Solution. The derivative of f with respect to x is

$$f'(x) = \frac{-2g(x)g^T(x)}{\left(1 + g^T(x)g(x)\right)^2} \frac{dg(x)}{dx} + \frac{1}{1 + g^T(x)g(x)} \frac{dg(x)}{dx}$$

which is differentiable for all $x \in \mathbb{R}^n$. Since it is continuously differentiable, it also satisfies a Lipschitz condition. From the local existence and uniqueness theorem (Theorem 3.1), this Lipschitz condition implies that there exists some $\delta > 0$ such that the state equation $\dot{x} = f(x)$ with $x(0) = x_0$ has a unique solution over $[0, \delta]$. If $x(t)$ is a solution of

$$\dot{x} = f(x(t)), \quad x(0) = x_0,$$

then, by integration,

$$x(t) = x_0 + \int_0^t f(x(s))ds$$

for $t \in [0, \delta]$. A bound on the norm of $x(t)$ can be obtained from above by applying the triangle inequality:

$$\begin{aligned}
\|x(t)\| &= \left\| x_0 + \int_0^t f(x(s))ds \right\| \\
&\leq \|x_0\| + \left\| \int_0^t f(x(s))ds \right\| \\
&\leq \|x_0\| + \int_0^t \|f(x(s))\| \, ds
\end{aligned}$$

We consider a bound on $\|f(x)\|$. For simplicity of analysis, we chose the Euclidean norm:

$$\begin{aligned}
\|f(x)\|_2 &= \sqrt{f(x)^T f(x)} \\
&= \frac{1}{(1 + g^T(x)g(x))} \sqrt{g^T(x)g(x)} \\
&= \frac{\|g(x)\|_2}{\left(1 + \|g(x)\|_2^2\right)}
\end{aligned}$$

The maximum function value of the function $h(y) = |y|/(1 + y^2)$ is 1/2. Thus,

$$\|f(x)\|_2 = \frac{\|g(x)\|_2}{\left(1 + \|g(x)\|_2^2\right)} \leq \frac{1}{2}$$

From this bound,

$$\begin{aligned}
\|x(t)\|_2 &\leq \|x_0\|_2 + \int_0^t \|f(x(s))\|_2 \, ds \\
\|x(t)\|_2 &\leq \|x_0\|_2 + \frac{1}{2} \int_0^t ds \\
\|x(t)\|_2 &\leq \|x_0\|_2 + \frac{1}{2} t
\end{aligned}$$

This means that for any finite t the solution is bounded in some compact set (i.e., for any $T > 0$, which could be arbitrarily large, $\|x(t)\| \leq \|x_0\|_2 + 1/2T$ for all $t \in [0, T]$). Applying Theorem 3.3, for example, the system has a unique solution defined for all $t \geq 0$.

19. Exercise 3.18 (Khalil pg. 108). Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]d\tau$$

where k_1, k_2, k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall-Bellman inequality, show that

$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}[1 - e^{-(\alpha-k_2)(t-t_0)}]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

Solution. Given

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]d\tau.$$

where k_1, k_2, and k_3 are nonnegative constants, and α is a positive constant. Let $z(t) := y(t)e^{\alpha(t-t_0)}$ so

$$\begin{aligned}
z(t) &\leq k_1 + e^{\alpha(t-t_0)} \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]d\tau \\
z(t) &\leq k_1 + \int_{t_0}^{t} e^{\alpha(\tau-t_0)}[k_2 y(\tau) + k_3]d\tau \\
z(t) &\leq k_1 + \int_{t_0}^{t} \left[k_2 z(\tau) + k_3 e^{\alpha(\tau-t_0)}\right] d\tau \\
z(t) &\leq k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right) + \int_{t_0}^{t} k_2 z(\tau)d\tau.
\end{aligned}$$

Applying the Gronwall-Bellman inequality (with $\lambda(t) = k_1 + k_3/\alpha\left(e^{\alpha(t-t_0)} - 1\right)$ and $\mu = k_2$; refer to Khalil for the statment of the Gronwall-Bellman inequality) and integrating,

$$\begin{aligned}
z(t) &\leq k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right) + \int_{t_0}^{t}\left[k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(\tau-t_0)} - 1\right)\right] k_2 e^{k_2(t-\tau)} d\tau \\
z(t) &\leq k_1 + \frac{k_3}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right) + k_1\left(e^{k_2(t-t_0)} - 1\right) - \frac{k_3}{\alpha}\left(e^{k_2(t-t_0)} - 1\right) \\
&\quad + \frac{k_3 k_2}{\alpha}\int_{t_0}^{t} e^{(\alpha-k_2)\tau - \alpha t_0 + k_2 t} d\tau \\
z(t) &\leq \frac{k_3}{\alpha}e^{\alpha(t-t_0)} + \left(\frac{k_3}{\alpha} + k_1\right)e^{k_2(t-t_0)} + \frac{k_3 k_2}{\alpha(\alpha - k_2)}\left(e^{(\alpha-k_2)t - \alpha t_0 + k_2 t} - e^{(\alpha-k_2)t_0 - \alpha t_0 + k_2 t}\right) \\
z(t) &\leq \frac{k_3}{\alpha}e^{\alpha(t-t_0)} + \left(\frac{k_3}{\alpha} + k_1\right)e^{k_2(t-t_0)} + \frac{k_3 k_2}{\alpha(\alpha - k_2)}\left(e^{\alpha(t-t_0)} - e^{k_2(t-t_0)}\right) \\
z(t) &\leq \left(\frac{k_3}{\alpha} + \frac{k_3 k_2}{\alpha(\alpha - k_2)}\right)e^{\alpha(t-t_0)} + \left(\frac{k_3}{\alpha} + k_1 - \frac{k_3 k_2}{\alpha(\alpha - k_2)}\right)e^{k_2(t-t_0)}.
\end{aligned}$$

Converting back to $y(t)$ gives the desired result:

$$\begin{aligned}
y(t) &\leq \left(\frac{k_3}{\alpha} + \frac{k_3 k_2}{\alpha(\alpha - k_2)}\right) + \left(\frac{k_3}{\alpha} + k_1 - \frac{k_3 k_2}{\alpha(\alpha - k_2)}\right)e^{-(\alpha-k_2)(t-t_0)} \\
y(t) &\leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left(\frac{\alpha - k_2}{\alpha} + \frac{k_2}{\alpha}\right) + \frac{k_3}{\alpha - k_2}\left(\frac{\alpha - k_2}{\alpha} + \frac{k_2}{\alpha}\right)e^{-(\alpha-k_2)(t-t_0)} \\
y(t) &\leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}[1 - e^{-(\alpha-k_2)(t-t_0)}].
\end{aligned}$$

20. Exercise 3.20 (Khalil pg. 108). Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$, then $f(x)$ is uniformly continuous on W.

    Solution. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$ or

    $$\|f(x) - f(y)\| \leq L \|x - y\|$$

    for all x, $y \in W$. Let $\epsilon > 0$ and $\delta = \epsilon / L$ so for all x, y satisifying $\|x - y\| < \delta$, $\|f(x) - f(y)\|$ is bounded above by $L\delta = \epsilon$ or

    $$\|x - y\| < \delta \implies \|f(x) - f(y)\| < L\delta = \epsilon.$$

    By definition, f is uniformly continuous on W.

21. Exercise 2.23 (Khalil Second Edition pg. 92). Let $x : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function that satisfies

    $$\|\dot{x}(t)\| \leq g(t), \qquad \forall t \geq t_0.$$

    Show that

    $$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} g(s) ds.$$

    Solution. Suppose $x : \mathbb{R} \to \mathbb{R}^n$ is a differentiable function that satisfies

    $$\|\dot{x}(t)\| \leq g(t)$$

    for all $t \geq t_0$. Integrating

    $$\int_{t_0}^{t} \|\dot{x}(\tau)\| d\tau \leq \int_{t_0}^{t} g(s) ds$$

    Note:

    $$\left\| \int_{t_0}^{t} \dot{x}(\tau) d\tau \right\| \leq \int_{t_0}^{t} \|\dot{x}(\tau)\| d\tau$$

    so

    $$\left\| \int_{t_0}^{t} \dot{x}(\tau) d\tau \right\| \leq \int_{t_0}^{t} g(s) ds$$
    $$\implies \|x(t) - x(t_0)\| \leq \int_{t_0}^{t} g(s) ds$$
    $$\|x(t) - x(t_0)\| + \|x(t_0)\| \leq \|x(t_0)\| + \int_{t_0}^{t} g(s) ds.$$

    By the triangle inequality:

    $$\|x(t)\| = \|x(t) - x(t_0) + x(t_0)\| \leq \|x(t) - x(t_0)\| + \|x(t_0)\| \leq \|x(t_0)\| + \int_{t_0}^{t} g(s) ds$$

    which proves:

    $$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} g(s) ds.$$

22. Exercise 1.1 (Rawlings pg. 60): *State space form for chemical reaction model.* Consider the following chemical reaction kinetics for a two-step series reaction

    $$A \overset{k_1}{\to} B \quad B \overset{k_2}{\to} C$$

    We wish to follow the reaction in a constant volume, well-mixed, batch reactor. The material balances for the three species are

    $$\frac{dc_A}{dt} = -r_1 \quad \frac{dc_B}{dt} = r_1 - r_2 \quad \frac{dc_C}{dt} = r_2$$

in which c_j is the concentration of species j, and r_1 and r_2 are the rates (mol/(time·vol)) at which the two reactions occur. Assume the rate law for the reaction kinetics are:

$$r_1 = k_1 c_A \quad r_2 = k_2 c_B$$

Substituting the rate laws into the material balances and specifying the starting concentrations, three differential equations for the three species concentrations are obtained.

(a) Is the model linear or nonlinear?

Solution.

Linear

(b) Write the state space model for the deterministic series chemical reaction model. Assume the component A concentration may be measured. What are x (state vector), y (output vector), A, B, C, and D (system matrices) for this model?

Solution.

Defining the state variables as c_A, c_B, and c_C or more specifically, the deviation of the concentrations from their corresponding steady state value:

$$x := \begin{bmatrix} c_A - c_{As} \\ c_B - c_{Bs} \\ c_C - c_{Cs} \end{bmatrix}$$

where c_{As}, c_{Bs}, and c_{Cs} denote the steady state values of the concentrations, respectively. The measured output of the reactor is c_A (or, in deviation form, x_1). From the problem statement, it is not clear that there are any inputs to the system so we assume there are no inputs. Substituting in the reaction kinetics into the material balances yields three linear ODEs:

$$\begin{aligned} \frac{dc_A}{dt} &= -k_1 c_A \\ \frac{dc_B}{dt} &= k_1 c_A - k_2 c_B \\ \frac{dc_C}{dt} &= k_2 c_B \end{aligned}$$

Converting to deviation variable form:

$$\begin{aligned} \frac{dx_1}{dt} &= -k_1 x_1 \\ \frac{dx_2}{dt} &= k_1 x_1 - k_2 x_2 \\ \frac{dx_3}{dt} &= k_2 x_2 \end{aligned}$$

Thus,

$$A = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and B and D are empty matrices.

(c) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$

Solution.

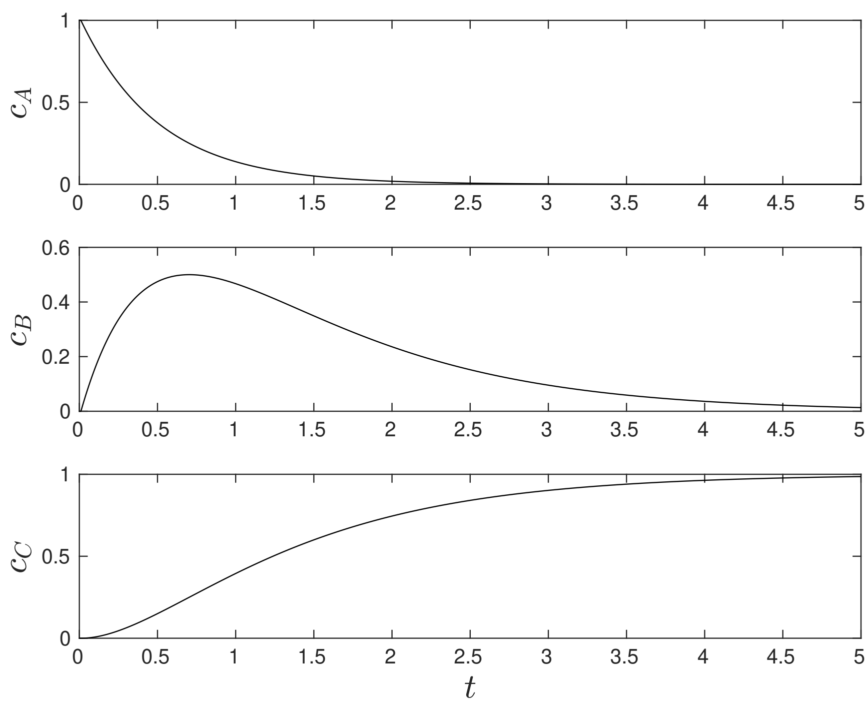


Figure 18: Resulting trajectories with $c_{A0} = 1$, $c_{B0} = c_{C0} = 0$, $k_1 = 2$, and $k_2 = 1$.