University of California, Davis
Department of Chemical Engineering
ECH 267
Advanced Process Control

M. Ellis **Homework 2** **Winter 2021**

Reading Assignment: Lecture notes; Khalil Chapter 4.1-4.7
Due date: Friday, February 12 at 6:00PM PST

1. Exercise 4.1 (Khalil pg. 181). Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable:

   (1) stable node
   (2) unstable node
   (3) stable focus
   (4) unstable focus
   (5) center

   Justify your answer using phase portraits (sketches of phase portraits are acceptable).

2. Exercise 4.2 (Khalil pg. 181). Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k\,|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

3. Exercise 4.3 (Khalil pg. 181). For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

   Hint: For the g.a.s case, you may want to try non-quadratic Lyapunov functions, such as $V(x) = \frac{1}{2a}x_1^{2a} + \frac{1}{2b}x_2^{2b}$ with $a,\ b$ positive integers, and show that their derivatives are negative for all $(x_1, x_2) \neq (0, 0)$.

   1. $$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

   2. $$\begin{aligned}\dot{x}_1 &= -x_2 - x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 - x_2^2)\end{aligned}$$

   3. $$\begin{aligned}\dot{x}_1 &= x_2(1 - x_1^2) \\ \dot{x}_2 &= -(x_1 + x_2)(1 - x_1^2)\end{aligned}$$

   4. $$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= 2x_1 - x_2^3\end{aligned}$$

4. Exercise 3.4 (Khalil Second Edition pg. 155). Using $V(x) = x_1^2 + x_2^2$, study stability of the origin of the system

   $$\begin{aligned}\dot{x}_1 &= x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2) \\ \dot{x}_2 &= -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2)\end{aligned}$$

   when (a) $k = 0$ and (b) $k \neq 0$.

5. Exercise 3.8 (Khalil Second Edition pg. 155). Consider the system

   $$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \text{sat}(2x_1 + x_2)\end{aligned}$$

(a) Show that the origin is asymptotically stable.

(b) Show that all trajectories starting in the first quadrant to the right of the curve $x_1x_2 = c$ (with sufficiently large $c > 0$) cannot reach the origin.

Hint: In part (b), consider $V(x) = x_1x_2$; calculate $\dot{V}(x)$ and show that on the curve $V(x) = c$ the derivative $\dot{V}(x) > 0$ when c is large enough.

(c) Show that the origin is not globally asymptotically stable.

6. Exercise 4.13 (Khalil pg. 183). For each of the following systems, show that the origin is unstable:

(1)
$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_1^2x_2\\ \dot{x}_2 &= -x_2 + x_2^2 + x_1x_2 - x_1^3\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2\\ \dot{x}_2 &= x_1^6 - x_2^3\end{aligned}$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

7. Exercise 3.22 (Khalil First Edition pg. 158). Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2\end{aligned}$$

(a) Show that the origin is the unique equilibrium point.
(b) Show, using linearization, that the origin is asymptotically stable.
(c) Show that the origin is globally asymptotically stable.

Hint: Use a simple quadratic Lyapunov function and the inequality $|sin x_1| < 1$.

8. Exercise 4.20 (Khalil pg. 185). Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

9. Exercise 4.26 (Khalil pg. 186). Let $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

(a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

(b) Show that $x = 0$ is stable (asymptotically stable/unstable) if and only if $z = 0$ is stable (asymptotically stable/unstable).

10. Determine the three equilibrium points of the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2.\end{aligned}$$

Investigate the stability of the equilibrium point $x = 0$. Verify the conclusions about the phase portrait and region of attraction of the system. In particular, generate its phase portrait by simulations and plot the level sets of the Lyapunov function $V(x) = \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2$ (if using Matlab, use the commands `meshgrid` and `contour`). Verify that the set $\{V(x) < \frac{9}{8}\}$ is (approximately) the largest estimate of the region of attraction that can be generated from this Lyapunov function.

11. Exercise 3.15 (Khalil Second Edition pg. 157). Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2\text{sat}(x_2^2 - x_3^2) \\ \dot{x}_3 &= x_3\text{sat}(x_2^2 - x_3^2)\end{aligned}$$

where $\text{sat}(\cdot)$ is the saturation function. Show that the origin is the unique equilibrium point, and use $V(x) = x^T x$ to show that it is globally asymptotically stable.

12. Exercise 3.16 (Khalil Second Edition pg. 157). The origin $x = 0$ is an equilibrium point of the system

$$\begin{aligned}\dot{x}_1 &= -kh(x)x_1 + x_2 \\ \dot{x}_2 &= -h(x)x_2 - x_1^3\end{aligned}$$

Let $D = \{x \in \mathbb{R}^2 \mid \|x\|_2 < 1\}$. Using $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, investigate stability of the origin in each of the following cases.

(1) $k > 0,\ h(x) > 0,\ \forall x \in D$.
(2) $k > 0,\ h(x) > 0,\ \forall x \in \mathbb{R}^2$
(3) $k > 0,\ h(x) < 0,\ \forall x \in D$
(4) $k > 0,\ h(x) = 0,\ \forall x \in D$
(5) $k = 0,\ h(x) > 0,\ \forall x \in D$
(6) $k = 0,\ h(x) > 0,\ \forall x \in \mathbb{R}^2$

13. Exercise 3.19 (Khalil Second Edition pg. 158). Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - kx_1 - dx_2 - cx_3 \\ \dot{x}_3 &= -x_3 + x_2\end{aligned}$$

where all coefficients are positive and $k > a$. Using

$$V(x) = 2a\int_0^{x_1} \sin y dy + kx_1^2 + x_2^2 + px_3^2$$

with some $p > 0$, show that the origin is globally asymptotically stable.

14. Exercise 4.18 (Khalil pg. 184). The mass-spring system of Exercise 1.11 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}\,|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

15. Exercise 4.23 (Khalil pg. 185). Consider the linear system $\dot{x} = \left(A - BR^{-1}B^TP\right)x$, where (A, B) is controllable, $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0$$

$R = R^T > 0$, and $Q = Q^T \geq 0$. Using $V(x) = x^TPx$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when:

(1) $Q > 0$.
(2) $Q = C^TC$ and (A, C) is observable. *Hint:* Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0\ \forall t$ if and only if $x = 0$.

16. Exercise 4.28 (Khalil pg. 186). Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2\end{aligned}$$

(a) Show that $x = 0$ is the unique equilibrium point.
(b) Show, by using linearization, that $x = 0$ is asymptotically stable.
(c) Show that $\Gamma = \{x \in \mathbb{R}^2 | x_1 x_2 \geq 2\}$ is a positively invariant set.
(d) Is $x = 0$ globally asymptotically stable?

17. Exercise 3.30 (Khalil Second Edition pg. 161). For each of the following systems, use linearization to show that the origin is asymptotically stable. Then, show that the origin is globally asymptotically stable.

(1)
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -ax_1 - bx_2, \; a, b > 0\end{aligned}$$

18. Exercise 4.36 (Khalil pg. 188). Is the origin of the scalar system $\dot{x} = -x/(t+1)$, $t \geq 0$, uniformly asymptotically stable?

19. Exercise 4.37 (Khalil pg. 188). For each of the following linear systems, use a quadratic Lyapunov function to show that the origin is exponentially stable:

(1)
$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x, \; |\alpha(t)| \leq 1$$

(2)
$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

(3)
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x, \; \alpha(t) \geq 2$$

(4)
$$\dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

In all cases, $\alpha(t)$ is continuous and bounded for all $t \geq 0$.

20. Exercise 4.41 (Khalil pg. 189). Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2\end{aligned}$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.
(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, then $x(t)$ approaches $\begin{bmatrix} t & 1 \end{bmatrix}^T$ as $t \to \infty$.

21. Exercise 4.44 (Khalil pg. 189). Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + (x_1^2 + x_2^2)\sin t \\ \dot{x}_2 &= -x_1 - x_2 + (x_1^2 + x_2^2)\cos t\end{aligned}$$

Show that the origin is exponentially stable and estimate the region of attraction.

22. Exercise 4.48 (Khalil pg. 190). Consider two systems represented by $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of the first system is exponentially stable if and only if the origin of the second system is exponentially stable.