1 Problem 1

For a nonlinear system, a:

1. Stable node is asymptotically stable,
2. Unstable node is unstable,
3. Stable focus is asymptotically stable,
4. Unstable focus is unstable,
5. Center is stable,
6. Saddle is unstable.

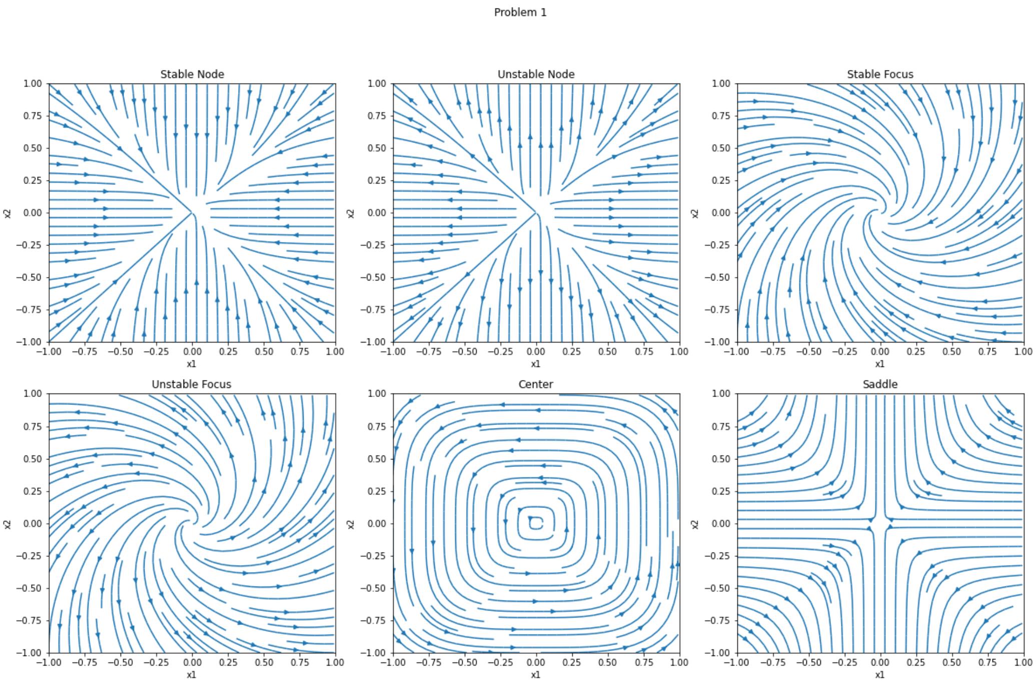

Figure 1: Problem 1 Phase Portraits

2 Problem 2

$$\dot{x} = ax^p + g(x), p \in \mathbb{N}^+ \tag{1}$$

$$|g(x)| \leq k|x|^{p+1}, \forall x : ||x|| \leq c \tag{2}$$

Note that near the origin, the ax^p term will dominate, resulting in $sign(\dot{x}) = sign(ax^p)$.

2.1 p is odd, $a < 0$

Define the following Lyapunov function,

$$V(x) = \frac{1}{2}x^2 \tag{3}$$

Then,

$$\dot{V} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = x(ax^p + g(x)) \tag{4}$$

$$x(ax^p + g(x)) = ax^{p+1} + g(x)x \tag{5}$$

Since

$$|g(x)| \leq k|x|^{p+1}, \forall x : ||x|| \leq c \tag{6}$$

In the neighborhood of the origin,

$$g(x)x \leq k|x|^{p+2} \tag{7}$$

Thus,

$$\dot{V} \leq ax^{p+1} + k|x|^{p+2} \tag{8}$$

Near the origin, since $x < 0$, the ax^{p+1} term will dominate. Since $a < 0$, $\dot{V} < 0$ near the origin. Because $\dot{V} < 0$ in the neighborhood of the origin, the origin is asymptotically stable.

2.2 p is odd, $a > 0$

If $a > 0$, $\dot{V} > 0$, thus any state near the origin will move away from the origin, making the origin unstable for this case.

2.3 p is even, $a \neq 0$

When p is even, one side of the origin will have $\dot{x}$ towards the origin, while the other side will point away from the origin, making this case unstable.

3 Problem 3

3.1 Problem 3.1

$$\dot{x_1} = -x_1 + x_1x_2 \tag{9}$$

$$\dot{x_2} = -x_2 \tag{10}$$

Define Lyapunov function,

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{11}$$

$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_1 + x_1x_2) + x_2(-x_2) = -x_1^2(1 - x_2) - x_2^2 \tag{12}$$

$$\dot{V} = -x_1^2 - x_2^2 + x_1^2x_2 \tag{13}$$

Consider the set $||x||_2 = x_1^2 + x_2^2 \leq r$. Then $|x_1| \leq r$.

$$\dot{V} = -x_1^2 - x_2^2 + x_1^2x_2 \leq -x_1^2 - x_2^2 + r|x_1||x_2| \tag{14}$$

This can be rewritten in matrix form as follows,

$$\dot{V} \leq -\begin{bmatrix}|x_1|\\|x_2|\end{bmatrix}^T \begin{bmatrix}1 & -r/2\\-r/2 & 1\end{bmatrix}\begin{bmatrix}|x_1|\\|x_2|\end{bmatrix} \tag{15}$$

$\dot{V} \leq 0$ for $r < 2$. Thus, the origin is asymptotically stable.
Note that the solution of the second equation is $x_2(t) = x_{2_0}e^{-t}$. Substituting this into the first equation results in the following.

$$\dot{x_1} = (x_{2_0}e^{-t} - 1)x_1 \tag{16}$$

The solution to this time-varying system does not have a finite escape time; thus, after some finite time, the coefficient of x_1 will be less than a negative number, resulting in $lim_{t\to\infty}x_1(t) = 0$. Therefore, the origin is globally asymptotically stable.

3.2 Problem 3.2

$$\dot{x_1} = -x_2 - x_1(1 - x_1^2 - x_2^2) \tag{17}$$

$$\dot{x_2} = x_1 - x_2(1 - x_1^2 - x_2^2) \tag{18}$$

Define Lyapunov function,

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{19}$$

$$\dot{V} = x_1\dot{x_1} + x_2\dot{x_2} = x_1(-x_2 - x_1(1 - x_1^2 - x_2^2)) + x_2(x_1 - x_2(1 - x_1^2 - x_2^2)) \tag{20}$$

$$\dot{V} = -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2)) \tag{21}$$

$$\dot{V} = -2V(1 - 2V) \tag{22}$$

$\dot{V} \leq 0$ where $V < 1/2$, thus the origin is asymptotically stable. However, since $\dot{V} \geq 0$ for $V > 1/2$, trajectories beginning where $V > 1/2$ will never approach the origin, therefore the origin is not globally asymptotically stable.

3.3 Problem 3.3

$$\dot{x_1} = x_2(1 - x_1^2) \tag{23}$$

$$\dot{x_2} = -(x_1 + x_2)(1 - x_1^2) \tag{24}$$

Define Lyapunov function, where P is a positive definite symmetric matrix.

$$V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \tag{25}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} = (2p_{11}x_1 + 2p_{12}x_2)(x_2(1 - x_1^2)) + (2p_{12}x_1 + 2p_{22}x_2)(-(x_1 + x_2)(1 - x_1^2)) \tag{26}$$

$$\dot{V} = -2p_{12}x_1^2 + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2(p_{22} - p_{12})x_2^2 + \mathcal{O}(x) \tag{27}$$

The quadratic terms will dominate the higher order terms near the origin, resulting in $\dot{V} < 0$ in the neighborhood of the origin if the quadratic term is negative definite. The coefficients of matrix, P, can be chosen such that this is the case (ex: $p_{11} = 3, p_{12} = 1, p_{22} = 2$). Thus, the origin is asymptotically stable.
The point $x = (1, 1)$ is also an equilibrium point, so the origin is not globally asymptotically stable.

3.4 Problem 3.4

$$\dot{x_1} = -x_1 - x_2 \tag{28}$$

$$\dot{x_2} = 2x_1 - x_2^3 \tag{29}$$

Define Lyapunov function,

$$V(x) = x_1^2 + \frac{1}{2}x_2^2 \tag{30}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} = 2x_1(-x_1 - x_2) + x_2(2x_1 - x_2^3) \tag{31}$$

$$\dot{V} = -2x_1^2 - x_2^4 \tag{32}$$

$\dot{V} \leq 0$, therefore the origin is globally asymptotically stable.

4 Problem 3

5 Problem 4

$$\dot{x}_1 = x_1\left(k^2 - x_1^2 - x_2^2\right) + x_2\left(x_1^2 + x_2^2 + k^2\right) \tag{33}$$

$$\dot{x}_2 = -x_1\left(k^2 + x_1^2 + x_2^2\right) + x_2\left(k^2 - x_1^2 - x_2^2\right) \tag{34}$$

Define Lyapunov function,

$$V(x) = x_1^2 + x_2^2 \tag{35}$$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} = 2x_1(x_1\left(k^2 - x_1^2 - x_2^2\right) + x_2\left(x_1^2 + x_2^2 + k^2\right)) + 2x_2(-x_1\left(k^2 + x_1^2 + x_2^2\right) + x_2\left(k^2 - x_1^2 - x_2^2\right)) \tag{36}$$

$$\dot{V} = 2(x_1^2 + x_2^2)(k^2 - x_1^2 - x_2^2) \tag{37}$$

If $k = 0$,

$$\dot{V} = -2(x_1^2 + x_2^2)^2 \tag{38}$$

If $k = 0$, $\dot{V} \leq 0$, thus the origin is globally asymptotically stable for $k = 0$.
If $k \neq 0$, $\dot{V} > 0$ for $k^2 - x_1^2 - x_2^2$. In other words, if x is within a ball of radius k centered at the origin, $\dot{V} > 0$ and if x is outside the ball of radius k centered at the origin, $\dot{V} < 0$. Thus there exists a set of equilibrium points radius k away from the origin that is globally asymptotically stable.

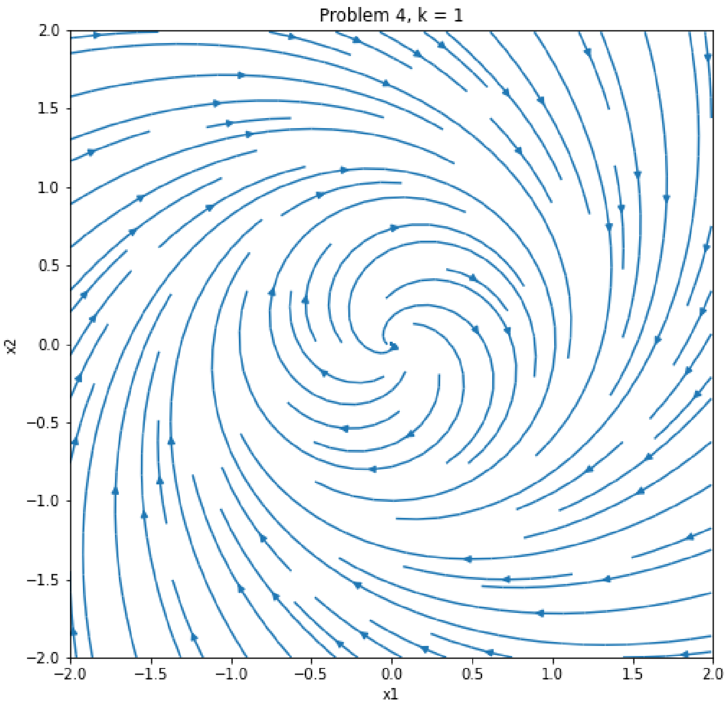

Figure 2: Problem 4 Phase Portrait, k=1