$HP = \{ M \# x \mid M \text{ halts on } x \}.$

$MP = \{ M \# x \mid x \in L(M) \}.$

Q 1. Is HP recursive ?

Does there exists a total TM M s.t $L(M) = HP$

Q2. Is MP recursive ?

Does there exists a total TM M s.t $L(M) = MP$

Working of the universal TM U.

U takes as input an encoding of a TM M and a string x and simulates M on x.

- halts and accepts if M halts and accepts x
- halts and rejects if M halts and rejects x.
- loops if M loops on x.

U simulates M step by step.

Question. Can we do better than blind simulation?

Eg. If M halts on x then simulate M on x
if not skip the simulation.

Build a U' that takes as input $M \# x$ and

- halts and accepts if M halts and accepts x.
- halts and rejects if M halts and rejects x.
- halts and rejects if M loops on x.

Thus $L(U') = L(U) = MP$ is a recursive set.

Theorem. HP is not recursive

Proof technique. Cantor's Diagonalization

There does not exist a one-to-one correspondence between natural number N and its powerset 2^N.

Theorem. HP is not recursive

Proof technique. Cantor's Diagonalization

For $x \in \{0,1\}^*$, let M_x denote the TM with input alphabet $\{0,1\}$ whose encoding is x.

	ϵ	0	1	00	01	11	000	001 ...
M_ϵ	H	L	H	H	L	L	L	H ...
M_0	L	H	L	L	H	L	H	H
M_1	H	L	H	H	L	H	L	H
M_{00}	L	L	L	H	L	H	H	H
M_{01}	H	H	L	H	H	L	L	L
M_{11}	L	L	H	H	H	L	H	L
M_{000}	L	H	L	H	L	H	H	L
M_{001}	H	L	H	H	L	H	H	H
⋮								

x^{th} row describes for input y if M_x halts ony.

	ϵ	0	1	00	01	11	000	001	...
M_ϵ	H	L	H	H	L	L	L	H	...
M_0	L	H	L	L	H	L	H	H	
M_1	H	L	H	H	L	H	L	H	
M_{00}	L	L	L	H	L	H	H	H	
M_{01}	H	H	L	H	H	L	L	L	
M_{11}	L	L	H	H	H	L	H	L	
M_{000}	L	H	L	H	L	H	H	L	
M_{001}	H	L	H	H	L	H	H	H	

Suppose $\exists$ a total TM K s.t $L(K) = HP$.

For any x and y, K can determine the enty in the x,yth entry in the table.

On input $M\#x$, - K halts and accepts if M halts on x
- K halts and rejects if M loops on x

	ϵ	0	1	00	01	11	000	001 ---
M_ϵ	H	L	H	H	L	L	L	H --
M_0	L	H	L	L	H	L	H	H
M_1	H	L	H	H	L	H	L	H
M_{00}	L	L	L	H	L	H	H	H
M_{01}	H	H	L	H	H	L	L	L
M_{11}	L	L	H	H	H	L	H	L
M_{000}	L	H	L	H	L	H	H	L
M_{001}	H	L	H	H	L	H	H	H

Suppose $\exists$ a total TM K s.t. $L(K) = HP$.

For any x and y, K can determine the enty in the x,yth entry in the table.

On input $M\#x$, - K halts and accepts if M halts on x
- K halts and rejects if M loops on x

Consider a machine N that on input $x \in \{0,1\}^*$

1) Constructs M_x from x and writes $M_x\#x$ on the tape

2) Runs K on input $M_x\#x$, accepting if K rejects and going into a trivial loop if K accepts.

For any $x \in \{0,1\}^*$, N halts on x iff K rejects $M\#x$
iff M_x loops on x.

That is, N is different from every M_x on at least one string - the string x.
This gives a contradiction.