Rice's Theorem.

Theorem. Every nontrivial property of the r.e. sets is undecidable.

Property of the r.e sets is a function.

$$P : \{\text{r.e. subsets of } \Sigma^*\} \to \{\top, \bot\}$$

Example.

Emptiness $\qquad P(A) = \begin{cases} \top & \text{if } A = \phi \\ \bot & \text{if } A \neq \phi \end{cases}$

Finiteness $\qquad P(A) = \begin{cases} \top & \text{if } A \text{ is finite} \\ \bot & \text{if } A \text{ is not finite} \end{cases}$

Regular $\qquad P(A) = \begin{cases} \top & \text{if } A \text{ is regular} \\ \bot & \text{if } A \text{ is not regular} \end{cases}$

Question. For a property P of r.e sets, is P decidable?

The set has to be given a finite presentation.

Assumption. The r.e. set is presented as a Turing machine whose language is the set.

Note. Property P is that of the set not a property of the Turing machine.

Example. Does a TM M halt on ϵ in 100 steps?
Decidable.

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Example - properties of r.e. sets.

- Is $L(M) = \phi$?
- Is $L(M)$ finite
- Is $L(M)$ regular
- Is $L(M)$ CFL
- Is $1001 \in L(M)$
- Is $\epsilon \in L(M)$

Property of the set accepted by the TM M.

Non-trivial: The property is not universally true or false.

That is, $\exists$ r.e set A & B s.t $P(A)=\top$ and $P(B)=\perp$

Proof of Rice's Theorem. Let P be non-trivial.

Wlog, assume $P(\phi)=\perp$ and $P(A)=\top$ for some A.
Let $L(K)=A$ for some TM K [Note A is r.e]

We give a reduction $HP \leq_m \underbrace{\{M \mid P(L(M))=\top\}}_{\text{Conclude: not recursive.}}$.

Given M and x, construct M' : $\sigma(M\#x)$
M' on input y works as follows:

1. Saves y on one of its tracks.

2. Write x on a separate track
3. Runs M on input x

M and x are hard coded in M'.

4. if M halts on x, M' runs K on input y ↳ $L(K) = A$
   - M' accepts if K accepts.

if M does not halt on x $\Rightarrow$ step 3 never stops.
$\Rightarrow y \notin L(M')$ for all y.

if M halts on $x \Rightarrow$ Step 4 is executed.
$\Rightarrow y \in L(M')$ iff $y \in L(K)$.

M halts on x $\Rightarrow L(M') = A \Rightarrow P(L(M')) = P(A) = T$.

M does not halt on $x \Rightarrow L(M') = \phi \Rightarrow P(L(M')) = P(\phi) = \perp$.

$HP \leq_m \{M \mid P(L(M)) = T\} \Rightarrow$ not recursive

$\therefore$ It is undecidable if $L(M)$ satisfies P.