Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.
2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.
3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?
4. (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
   (c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.
5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.
6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.
7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology,
$\mathcal{T}_2$ = the topology of $\mathbb{R}_K$,
$\mathcal{T}_3$ = the finite complement topology,
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.
8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a,b) \mid a < b, \ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Exercises

1. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A

inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \{x \mid \tfrac{1}{2} < |x| < 1\},$$
$$B = \{x \mid \tfrac{1}{2} < |x| \le 1\},$$
$$C = \{x \mid \tfrac{1}{2} \le |x| < 1\},$$
$$D = \{x \mid \tfrac{1}{2} \le |x| \le 1\},$$
$$E = \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}.$$

4. A map $f : X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

5. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.
   (a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
   (b) Does the converse of (a) hold? Justify your answer.

6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

7. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

9. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the

Exercises

1. Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\varnothing$ and X are in
   and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Sho
   that the collection

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

   is a topology on X.

2. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

3. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times$

4. Show that if U is open in X and A is closed in X, then $U - A$ is open in X, a
   $A - U$ is closed in X.

5. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subset [a, b]$. Und
   what conditions does equality hold?

6. Let A, B, and A_α denote subsets of a space X. Prove the following:
   (a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.
   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
   (c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

7. Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

8. Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.
   (a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
   (b) $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$.
   (c) $\overline{A - B} = \bar{A} - \bar{B}$.

9. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

10. Show that every order topology is Hausdorff.

11. Show that the product of two Hausdorff spaces is Hausdorff.

12. Show that a subspace of a Hausdorff space is Hausdorff.

13. Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

14. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

15. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

16. Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.
   (a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.
   (b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

17. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

18. Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\},$$
$$B = \{(1 - 1/n) \times \tfrac{1}{2} \mid n \in \mathbb{Z}_+\},$$
$$C = \{x \times 0 \mid 0 < x < 1\},$$
$$D = \{x \times \tfrac{1}{2} \mid 0 < x < 1\},$$
$$E = \{\tfrac{1}{2} \times y \mid 0 < y < 1\}.$$

19. If $A \subset X$, we define the **boundary** of A by the equation

$$\operatorname{Bd} A = \bar{A} \cap \overline{(X - A)}.$$

(a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\bar{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.

(b) Show that $\operatorname{Bd} A = \varnothing \Leftrightarrow A$ is both open and closed.

(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \bar{U} - U$.

(d) If U is open, is it true that $U = \operatorname{Int}(\bar{U})$? Justify your answer.

20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(a) $A = \{x \times y \mid y = 0\}$

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{x \times y \mid x \text{ is rational}\}$

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

*21. (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from this collection to itself.

(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

Exercises

1. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

3. Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.
   (a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
   (b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

   are imbeddings.

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

7. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

   for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.
   (b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
   (a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

9. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$ is continuous for each α.
   (a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.
   (b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.
   (c) An indexed family of sets $\{A_\alpha\}$ is said to be *locally finite* if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

11. Let $F : X \times Y \to Z$. We say that F is *continuous in each variable separately* if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

12. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases}$$

   (a) Show that F is continuous in each variable separately.
   (b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.
   (c) Show that F is not continuous.

13. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

Exercises

1. Prove Theorem 19.2.
2. Prove Theorem 19.3.
3. Prove Theorem 19.4.
4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.
5. One of the implications stated in Theorem 19.6 holds for the box topology. Which one?
6. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?
7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.
8. Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots).$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?
9. Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq \emptyset$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.
10. Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.
    (a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.
    (b) Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$

    and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.
    (c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.
    (d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J};$$

    let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \dots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i. Given $\mathbf{x} \in U$, we find an open set V of the metric topology such that $\mathbf{x} \in V \subset U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about x_i and lying in U_i for $i = \alpha_1, \dots, \alpha_n$; choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min\{\epsilon_i/i \mid i = \alpha_1, \dots, \alpha_n\}.$$

We assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U.$$

Let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. Then for all i,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon.$$

Now if $i = \alpha_1, \dots, \alpha_n$, then $\epsilon \leq \epsilon_i/i$, so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$; it follows that $|x_i - y_i| < \epsilon_i$. Therefore, $\mathbf{y} \in \prod U_i$, as desired. ∎

Exercises

1. (a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

3. Let X be a metric space with metric d.
   (a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
   (b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.