Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

4. (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

   (c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

7. Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{the standard topology,}\\
\mathcal{T}_2 &= \text{the topology of } \mathbb{R}_K,\\
\mathcal{T}_3 &= \text{the finite complement topology,}\\
\mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a, b] \text{ as basis,}\\
\mathcal{T}_5 &= \text{the topology having all sets } (-\infty, a) = \{x \mid x < a\} \text{ as basis.}
\end{aligned}$$

   Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Exercises

1. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A

inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?
3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned} A &= \{x \mid \tfrac{1}{2} < |x| < 1\}, \\ B &= \{x \mid \tfrac{1}{2} < |x| \leq 1\}, \\ C &= \{x \mid \tfrac{1}{2} \leq |x| < 1\}, \\ D &= \{x \mid \tfrac{1}{2} \leq |x| \leq 1\}, \\ E &= \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}. \end{aligned}$$

4. A map $f : X \to Y$ is said to be an ***open map*** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.
5. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.
   (a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
   (b) Does the converse of (a) hold? Justify your answer.
6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

7. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?
8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.
9. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.
10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the

Exercises

1. Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\varnothing$ and X are in and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Sho that the collection

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

is a topology on X.

2. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

3. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times$

4. Show that if U is open in X and A is closed in X, then $U - A$ is open in X, ar $A - U$ is closed in X.

5. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subset [a, b]$. Und what conditions does equality hold?

6. Let A, B, and A_α denote subsets of a space X. Prove the following:
   (a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.
   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
   (c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

7. Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

8. Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.
   (a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
   (b) $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$.
   (c) $\overline{A - B} = \bar{A} - \bar{B}$.

9. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

10. Show that every order topology is Hausdorff.

11. Show that the product of two Hausdorff spaces is Hausdorff.

12. Show that a subspace of a Hausdorff space is Hausdorff.

13. Show that X is Hausdorff if and only if the ***diagonal*** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

14. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

15. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

16. Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.
   (a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.
   (b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

17. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

18. Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\},$$
$$B = \{(1 - 1/n) \times \tfrac{1}{2} \mid n \in \mathbb{Z}_+\},$$
$$C = \{x \times 0 \mid 0 < x < 1\},$$
$$D = \{x \times \tfrac{1}{2} \mid 0 < x < 1\},$$
$$E = \{\tfrac{1}{2} \times y \mid 0 < y < 1\}.$$

19. If $A \subset X$, we define the ***boundary*** of A by the equation

$$\operatorname{Bd} A = \bar{A} \cap \overline{(X - A)}.$$

(a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\bar{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.
(b) Show that $\operatorname{Bd} A = \varnothing \Leftrightarrow A$ is both open and closed.
(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \bar{U} - U$.
(d) If U is open, is it true that $U = \operatorname{Int}(\bar{U})$? Justify your answer.

20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$.
(a) $A = \{x \times y \mid y = 0\}$
(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \{x \times y \mid x \text{ is rational}\}$
(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

***21.** (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from this collection to itself.
(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.
(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained

Exercises

1. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.
2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?
3. Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.
   (a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
   (b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.
4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \qquad \text{and} \qquad g(y) = x_0 \times y$$

   are imbeddings.
5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$
6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.
7. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

   for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.
   (b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.
8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
   (a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous [*Hint:* Use the pasting lemma.]

9. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$, is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

(c) An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

11. Let $F : X \times Y \to Z$. We say that F is ***continuous in each variable separately*** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

12. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0 \end{cases}$$

(a) Show that F is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

(c) Show that F is not continuous

13. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f