Exercises

1. Let $A \subset X$. If d is a metric for the topology of X, show that $d|A \times A$ is a metric for the subspace topology on A.
2. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X,
$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$
Show that f is an imbedding. It is called an ***isometric imbedding*** of X in Y.
3. Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.
(a) Show that
$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$
is a metric for the product space $X_1 \times \cdots \times X_n$.

(b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that
$$D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$
is a metric for the product space $\prod X_i$.
4. Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)
5. *Theorem. Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then*
$$x_n + y_n \to x + y,$$
$$x_n - y_n \to x - y,$$
$$x_n y_n \to x y,$$
and provided that each $y_n \neq 0$ and $y \neq 0$,
$$x_n / y_n \to x / y.$$
[*Hint:* Apply Lemma 21.4; recall from the exercises of §19 that if $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to x \times y$.]
6. Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$, but that the sequence (f_n) does not converge uniformly.

verge uniformly.

7. Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.
8. Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.
9. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$
See Figure 21.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.
   (a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.
   (b) Show that f_n does not converge uniformly to f. (This shows that the converse of Theorem 21.6 does not hold; the limit function f may be continuous even though the convergence is not uniform.)
10. Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of $\mathbb{R}^2$:
$$A = \{x \times y \mid xy = 1\},$$
$$S^1 = \{x \times y \mid x^2 + y^2 = 1\},$$
$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}.$$

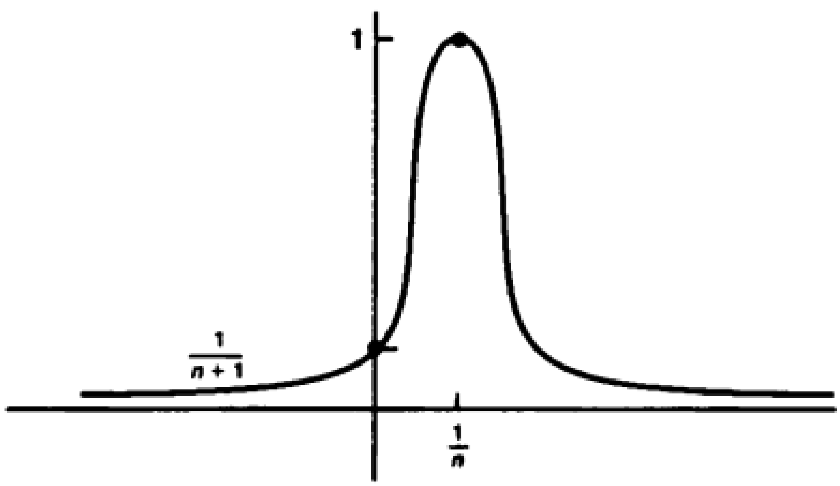

Figure 21.1

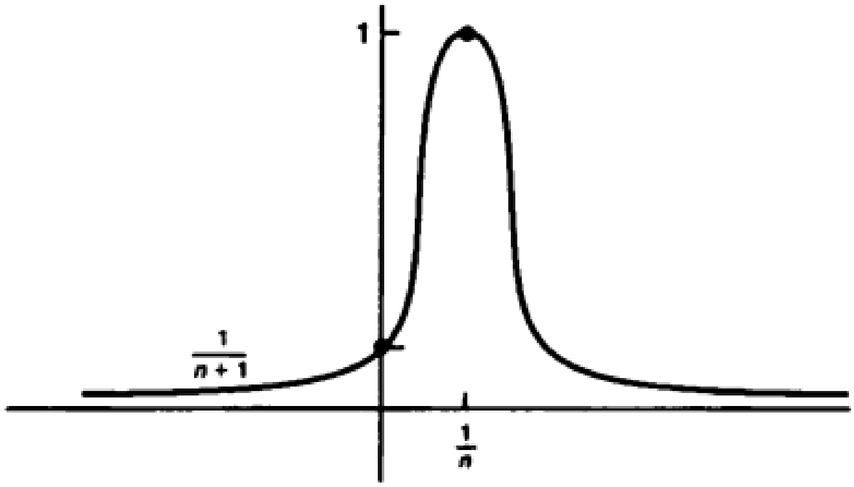

Figure 21.1

The set B^2 is called the (closed) ***unit ball*** in $\mathbb{R}^2$.

11. Prove the following standard facts about infinite series:

(a) Show that if (s_n) is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for each n, then (s_n) converges.

(b) Let (a_n) be a sequence of real numbers; define

$$s_n = \sum_{i=1}^{n} a_i .$$

If $s_n \to s$, we say that the ***infinite series***

$$\sum_{i=1}^{\infty} a_i$$

converges to s also. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum(ca_i + b_i)$ converges to $cs + t$.

(c) Prove the ***comparison test*** for infinite series: If $|a_i| \leq b_i$ for each i, and if the series $\sum b_i$ converges, then the series $\sum a_i$ converges. [*Hint:* Show that the series $\sum |a_i|$ and $\sum c_i$ converge, where $c_i = |a_i| + a_i$.]

(d) Given a sequence of functions $f_n : X \to \mathbb{R}$, let

$$s_n(x) = \sum_{i=1}^{n} f_i(x).$$

Prove the ***Weierstrass M-test*** for uniform convergence: *If $|f_i(x)| \leq M_i$ for all $x \in X$ and all i, and if the series $\sum M_i$ converges, then the sequence (s_n) converges uniformly to a function s.* [*Hint:* Let $r_n = \sum_{i=n+1}^{\infty} M_i$. Show that if $k > n$, then $|s_k(x) - s_n(x)| \leq r_n$; conclude that $|s(x) - s_n(x)| \leq r_n$.]

12. Prove continuity of the algebraic operations on $\mathbb{R}$, as follows: Use the metric $d(a, b) = |a - b|$ on $\mathbb{R}$ and the metric on $\mathbb{R}^2$ given by the equation

$$\rho((x, y), (x_0, y_0)) = \max\{|x - x_0|, |y - y_0|\}.$$

(a) Show that addition is continuous. [*Hint:* Given ϵ, let $\delta = \epsilon/2$ and note that

$$d(x + y, x_0 + y_0) \leq |x - x_0| + |y - y_0|.]$$

(b) Show that multiplication is continuous. [*Hint:* Given (x_0, y_0) and $0 < \epsilon < 1$, let

$$3\delta = \epsilon/(|x_0| + |y_0| + 1)$$

and note that

$$d(xy, x_0y_0) \leq |x_0||y - y_0| + |y_0||x - x_0| + |x - x_0||y - y_0|.]$$

(c) Show that the operation of taking reciprocals is a continuous map from $\mathbb{R} - \{0\}$ to $\mathbb{R}$. [*Hint:* Show the inverse image of the interval (a, b) is open. Consider five cases, according as a and b are positive, negative, or zero.]

(d) Show that the subtraction and quotient operations are continuous.

*§22 The Quotient Topology†

Unlike the topologies we have already considered in this chapter, the quotient topology is not a natural generalization of something you have already studied in analysis. Nevertheless, it is easy enough to motivate. One motivation comes from geometry, where one often has occasion to use "cut-and-paste" techniques to construct such geometric objects as surfaces. The *torus* (surface of a doughnut), for example, can be constructed by taking a rectangle and "pasting" its edges together appropriately, as in Figure 22.1. And the *sphere* (surface of a ball) can be constructed by taking a disc and collapsing its entire boundary to a single point; see Figure 22.2. Formalizing these constructions involves the concept of quotient topology.

Exercises

1. Check the details of Example 3.
2. (a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.
   (b) If $A \subset X$, a ***retraction*** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

3. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.
4. (a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0 + y_0^2 = x_1 + y_1^2.$$

   Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? [*Hint:* Set $g(x \times y) = x + y^2$.]
   (b) Repeat (a) for the equivalence relation

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

5. Let $p : X \to Y$ be an open map. Show that if A is open in X, then the map $q : A \to p(A)$ obtained by restricting p is an open map.
6. Recall that $\mathbb{R}_K$ denotes the real line in the K-topology. (See §13.) Let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point; let $p : \mathbb{R}_K \to Y$ be the quotient map.
   (a) Show that Y satisfies the T_1 axiom, but is not Hausdorff.
   (b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map. [*Hint:* The diagonal is not closed in $Y \times Y$, but its inverse image is closed in $\mathbb{R}_K \times \mathbb{R}_K$.]