

## Exercises

1. Let  $X$  be a topological space; let  $A$  be a subset of  $X$ . Suppose that for each  $x \in A$  there is an open set  $U$  containing  $x$  such that  $U \subset A$ . Show that  $A$  is open in  $X$ .
2. Consider the nine topologies on the set  $X = \{a, b, c\}$  indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.
3. Show that the collection  $\mathcal{T}_c$  given in Example 4 of §12 is a topology on the set  $X$ . Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on  $X$ ?

4. (a) If  $\{\mathcal{T}_\alpha\}$  is a family of topologies on  $X$ , show that  $\bigcap \mathcal{T}_\alpha$  is a topology on  $X$ . Is  $\bigcup \mathcal{T}_\alpha$  a topology on  $X$ ?  
 (b) Let  $\{\mathcal{T}_\alpha\}$  be a family of topologies on  $X$ . Show that there is a unique smallest topology on  $X$  containing all the collections  $\mathcal{T}_\alpha$ , and a unique largest topology contained in all  $\mathcal{T}_\alpha$ .  
 (c) If  $X = \{a, b, c\}$ , let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , and the largest topology contained in  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

5. Show that if  $\mathcal{A}$  is a basis for a topology on  $X$ , then the topology generated by  $\mathcal{A}$  equals the intersection of all topologies on  $X$  that contain  $\mathcal{A}$ . Prove the same if  $\mathcal{A}$  is a subbasis.
6. Show that the topologies of  $\mathbb{R}_\ell$  and  $\mathbb{R}_K$  are not comparable.
7. Consider the following topologies on  $\mathbb{R}$ :

$\mathcal{T}_1$  = the standard topology,

$\mathcal{T}_2$  = the topology of  $\mathbb{R}_K$ ,

$\mathcal{T}_3$  = the finite complement topology,

$\mathcal{T}_4$  = the upper limit topology, having all sets  $(a, b]$  as basis,

$\mathcal{T}_5$  = the topology having all sets  $(-\infty, a) = \{x \mid x < a\}$  as basis.

Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on  $\mathbb{R}$ .

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on  $\mathbb{R}$ .

## Exercises

1. Show that if  $Y$  is a subspace of  $X$ , and  $A$  is a subset of  $Y$ , then the topology  $A$

### 92 Topological Spaces and Continuous Functions

Ch. 2

inherits as a subspace of  $Y$  is the same as the topology it inherits as a subspace of  $X$ .

2. If  $\mathcal{T}$  and  $\mathcal{T}'$  are topologies on  $X$  and  $\mathcal{T}'$  is strictly finer than  $\mathcal{T}$ , what can you say about the corresponding subspace topologies on the subset  $Y$  of  $X$ ?
3. Consider the set  $Y = [-1, 1]$  as a subspace of  $\mathbb{R}$ . Which of the following sets are open in  $Y$ ? Which are open in  $\mathbb{R}$ ?

$$A = \{x \mid \frac{1}{2} < |x| < 1\},$$

$$B = \{x \mid \frac{1}{2} < |x| \leq 1\},$$

$$C = \{x \mid \frac{1}{2} \leq |x| < 1\},$$

$$D = \{x \mid \frac{1}{2} \leq |x| \leq 1\},$$

$$E = \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}.$$

4. A map  $f : X \rightarrow Y$  is said to be an **open map** if for every open set  $U$  of  $X$ , the set  $f(U)$  is open in  $Y$ . Show that  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are open maps.
5. Let  $X$  and  $X'$  denote a single set in the topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively; let  $Y$  and  $Y'$  denote a single set in the topologies  $\mathcal{U}$  and  $\mathcal{U}'$ , respectively. Assume these sets are nonempty.
  - (a) Show that if  $\mathcal{T}' \supset \mathcal{T}$  and  $\mathcal{U}' \supset \mathcal{U}$ , then the product topology on  $X' \times Y'$  is finer than the product topology on  $X \times Y$ .
  - (b) Does the converse of (a) hold? Justify your answer.
6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for  $\mathbb{R}^2$ .

7. Let  $X$  be an ordered set. If  $Y$  is a proper subset of  $X$  that is convex in  $X$ , does it follow that  $Y$  is an interval or a ray in  $X$ ?
8. If  $L$  is a straight line in the plane, describe the topology  $L$  inherits as a subspace of  $\mathbb{R}_\ell \times \mathbb{R}$  and as a subspace of  $\mathbb{R}_\ell \times \mathbb{R}_\ell$ . In each case it is a familiar topology.
9. Show that the dictionary order topology on the set  $\mathbb{R} \times \mathbb{R}$  is the same as the product topology  $\mathbb{R}_d \times \mathbb{R}$ , where  $\mathbb{R}_d$  denotes  $\mathbb{R}$  in the discrete topology. Compare this topology with the standard topology on  $\mathbb{R}^2$ .
10. Let  $I = [0, 1]$ . Compare the product topology on  $I \times I$ , the dictionary order topology on  $I \times I$  and the topology  $I \times I$  inherits as a subspace of  $\mathbb{R} \times \mathbb{R}$  in the

## Exercises

1. Let  $\mathcal{C}$  be a collection of subsets of the set  $X$ . Suppose that  $\emptyset$  and  $X$  are in  $\mathcal{C}$  and that finite unions and arbitrary intersections of elements of  $\mathcal{C}$  are in  $\mathcal{C}$ . Show that the collection

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

is a topology on  $X$ .

2. Show that if  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .
3. Show that if  $A$  is closed in  $X$  and  $B$  is closed in  $Y$ , then  $A \times B$  is closed in  $X \times Y$ .
4. Show that if  $U$  is open in  $X$  and  $A$  is closed in  $X$ , then  $U - A$  is open in  $X$ , and  $A - U$  is closed in  $X$ .
5. Let  $X$  be an ordered set in the order topology. Show that  $\overline{(a, b)} \subset [a, b]$ . Under what conditions does equality hold?

6. Let  $A$ ,  $B$ , and  $A_\alpha$  denote subsets of a space  $X$ . Prove the following:
  - (a) If  $A \subset B$ , then  $\bar{A} \subset \bar{B}$ .
  - (b)  $\overline{A \cup B} = \bar{A} \cup \bar{B}$ .
  - (c)  $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$ ; give an example where equality fails.
7. Criticize the following "proof" that  $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$ : if  $\{A_\alpha\}$  is a collection of sets in  $X$  and if  $x \in \overline{\bigcup A_\alpha}$ , then every neighborhood  $U$  of  $x$  intersects  $\bigcup A_\alpha$ . Thus  $U$  must intersect some  $A_\alpha$ , so that  $x$  must belong to the closure of some  $A_\alpha$ . Therefore,  $x \in \bigcup \bar{A}_\alpha$ .
8. Let  $A$ ,  $B$ , and  $A_\alpha$  denote subsets of a space  $X$ . Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions  $\supset$  or  $\subset$  holds.
  - (a)  $\overline{A \cap B} = \bar{A} \cap \bar{B}$ .
  - (b)  $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$ .
  - (c)  $\overline{A - B} = \bar{A} - \bar{B}$ .

9. Let  $A \subset X$  and  $B \subset Y$ . Show that in the space  $X \times Y$ ,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

10. Show that every order topology is Hausdorff.
11. Show that the product of two Hausdorff spaces is Hausdorff.
12. Show that a subspace of a Hausdorff space is Hausdorff.
13. Show that  $X$  is Hausdorff if and only if the **diagonal**  $\Delta = \{x \times x \mid x \in X\}$  is closed in  $X \times X$ .
14. In the finite complement topology on  $\mathbb{R}$ , to what point or points does the sequence  $x_n = 1/n$  converge?
15. Show the  $T_1$  axiom is equivalent to the condition that for each pair of points of  $X$ , each has a neighborhood not containing the other.
16. Consider the five topologies on  $\mathbb{R}$  given in Exercise 7 of §13.
  - (a) Determine the closure of the set  $K = \{1/n \mid n \in \mathbb{Z}_+\}$  under each of these topologies.
  - (b) Which of these topologies satisfy the Hausdorff axiom? the  $T_1$  axiom?
17. Consider the lower limit topology on  $\mathbb{R}$  and the topology given by the basis  $\mathcal{C}$  of Exercise 8 of §13. Determine the closures of the intervals  $A = (0, \sqrt{2})$  and  $B = (\sqrt{2}, 3)$  in these two topologies.
18. Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\},$$

$$B = \{(1 - 1/n) \times \frac{1}{2} \mid n \in \mathbb{Z}_+\},$$

$$C = \{x \times 0 \mid 0 < x < 1\},$$

$$D = \{x \times \frac{1}{2} \mid 0 < x < 1\},$$

$$E = \{\frac{1}{2} \times y \mid 0 < y < 1\}.$$

19. If  $A \subset X$ , we define the *boundary* of  $A$  by the equation

$$\text{Bd } A = \bar{A} \cap \overline{(X - A)}.$$

- (a) Show that  $\text{Int } A$  and  $\text{Bd } A$  are disjoint, and  $\bar{A} = \text{Int } A \cup \text{Bd } A$ .
  - (b) Show that  $\text{Bd } A = \emptyset \Leftrightarrow A$  is both open and closed.
  - (c) Show that  $U$  is open  $\Leftrightarrow \text{Bd } U = \bar{U} - U$ .
  - (d) If  $U$  is open, is it true that  $U = \text{Int}(\bar{U})$ ? Justify your answer.
20. Find the boundary and the interior of each of the following subsets of  $\mathbb{R}^2$ .
- (a)  $A = \{x \times y \mid y = 0\}$
  - (b)  $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
  - (c)  $C = A \cup B$
  - (d)  $D = \{x \times y \mid x \text{ is rational}\}$
  - (e)  $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
  - (f)  $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$
- \*21. (Kuratowski) Consider the collection of all subsets  $A$  of the topological space  $X$ . The operations of closure  $A \rightarrow \bar{A}$  and complementation  $A \rightarrow X - A$  are functions from this collection to itself.
- (a) Show that starting with a given set  $A$ , one can form no more than 14 distinct sets by applying these two operations successively.
  - (b) Find a subset  $A$  of  $\mathbb{R}$  (in its usual topology) for which the maximum of 14 is obtained

## Exercises

1. Prove that for functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the  $\epsilon$ - $\delta$  definition of continuity implies the open set definition.
2. Suppose that  $f : X \rightarrow Y$  is continuous. If  $x$  is a limit point of the subset  $A$  of  $X$ , is it necessarily true that  $f(x)$  is a limit point of  $f(A)$ ?
3. Let  $X$  and  $X'$  denote a single set in the two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively. Let  $i : X' \rightarrow X$  be the identity function.
  - (a) Show that  $i$  is continuous  $\Leftrightarrow \mathcal{T}'$  is finer than  $\mathcal{T}$ .
  - (b) Show that  $i$  is a homeomorphism  $\Leftrightarrow \mathcal{T}' = \mathcal{T}$ .
4. Given  $x_0 \in X$  and  $y_0 \in Y$ , show that the maps  $f : X \rightarrow X \times Y$  and  $g : Y \rightarrow X \times Y$  defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

5. Show that the subspace  $(a, b)$  of  $\mathbb{R}$  is homeomorphic with  $(0, 1)$  and the subspace  $[a, b]$  of  $\mathbb{R}$  is homeomorphic with  $[0, 1]$
6. Find a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that is continuous at precisely one point.
7. (a) Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is "continuous from the right," that is,

$$\lim_{x \rightarrow a^+} f(x) = f(a),$$

for each  $a \in \mathbb{R}$ . Show that  $f$  is continuous when considered as a function from  $\mathbb{R}_\ell$  to  $\mathbb{R}$ .

- (b) Can you conjecture what functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  are continuous when considered as maps from  $\mathbb{R}$  to  $\mathbb{R}_\ell$ ? As maps from  $\mathbb{R}_\ell$  to  $\mathbb{R}_\ell$ ? We shall return to this question in Chapter 3.
8. Let  $Y$  be an ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous.
    - (a) Show that the set  $\{x \mid f(x) \leq g(x)\}$  is closed in  $X$

(b) Let  $h : X \rightarrow Y$  be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that  $h$  is continuous [Hint: Use the pasting lemma.]

9. Let  $\{A_\alpha\}$  be a collection of subsets of  $X$ ; let  $X = \bigcup_\alpha A_\alpha$ . Let  $f : X \rightarrow Y$ ; suppose that  $f|_{A_\alpha}$  is continuous for each  $\alpha$ .

(a) Show that if the collection  $\{A_\alpha\}$  is finite and each set  $A_\alpha$  is closed, then  $f$  is continuous.

(b) Find an example where the collection  $\{A_\alpha\}$  is countable and each  $A_\alpha$  is closed, but  $f$  is not continuous.

(c) An indexed family of sets  $\{A_\alpha\}$  is said to be **locally finite** if each point  $x$  of  $X$  has a neighborhood that intersects  $A_\alpha$  for only finitely many values of  $\alpha$ . Show that if the family  $\{A_\alpha\}$  is locally finite and each  $A_\alpha$  is closed, then  $f$  is continuous.

10. Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous functions. Let us define a map  $f \times g : A \times C \rightarrow B \times D$  by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that  $f \times g$  is continuous.

11. Let  $F : X \times Y \rightarrow Z$ . We say that  $F$  is **continuous in each variable separately** if for each  $y_0$  in  $Y$ , the map  $h : X \rightarrow Z$  defined by  $h(x) = F(x \times y_0)$  is continuous, and for each  $x_0$  in  $X$ , the map  $k : Y \rightarrow Z$  defined by  $k(y) = F(x_0 \times y)$  is continuous. Show that if  $F$  is continuous, then  $F$  is continuous in each variable separately.

12. Let  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0 \end{cases}$$

(a) Show that  $F$  is continuous in each variable separately.

(b) Compute the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $g(x) = F(x \times x)$ .

(c) Show that  $F$  is not continuous

13. Let  $A \subset X$ ; let  $f : A \rightarrow Y$  be continuous; let  $Y$  be Hausdorff. Show that if  $f$  may be extended to a continuous function  $g : \bar{A} \rightarrow Y$ , then  $g$  is uniquely determined by  $f$



## Exercises

1. Prove Theorem 19.2
2. Prove Theorem 19.3.
3. Prove Theorem 19.4
4. Show that  $(X_1 \times \cdots \times X_{n-1}) \times X_n$  is homeomorphic with  $X_1 \times \cdots \times X_n$ .
5. One of the implications stated in Theorem 19.6 holds for the box topology. Which one?
6. Let  $x_1, x_2, \dots$  be a sequence of the points of the product space  $\prod X_\alpha$ . Show that this sequence converges to the point  $x$  if and only if the sequence  $\pi_\alpha(x_1), \pi_\alpha(x_2), \dots$  converges to  $\pi_\alpha(x)$  for each  $\alpha$ . Is this fact true if one uses the box topology instead of the product topology?
7. Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences that are "eventually zero," that is, all sequences  $(x_1, x_2, \dots)$  such that  $x_i \neq 0$  for only finitely many values of  $i$ . What is the closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  in the box and product topologies? Justify your answer.
8. Given sequences  $(a_1, a_2, \dots)$  and  $(b_1, b_2, \dots)$  of real numbers with  $a_i > 0$  for all  $i$ , define  $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  by the equation

$$h((x_1, x_2, \dots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots).$$

Show that if  $\mathbb{R}^\omega$  is given the product topology,  $h$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself. What happens if  $\mathbb{R}^\omega$  is given the box topology?

9. Show that the choice axiom is equivalent to the statement that for any indexed family  $\{A_\alpha\}_{\alpha \in J}$  of nonempty sets, with  $J \neq \emptyset$ , the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.

10. Let  $A$  be a set; let  $\{X_\alpha\}_{\alpha \in J}$  be an indexed family of spaces; and let  $\{f_\alpha\}_{\alpha \in J}$  be an indexed family of functions  $f_\alpha : A \rightarrow X_\alpha$ .
  - (a) Show there is a unique coarsest topology  $\mathcal{T}$  on  $A$  relative to which each of the functions  $f_\alpha$  is continuous.
  - (b) Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$

and let  $\mathcal{S} = \bigcup \mathcal{S}_\beta$ . Show that  $\mathcal{S}$  is a subbasis for  $\mathcal{T}$

- (c) Show that a map  $g : Y \rightarrow A$  is continuous relative to  $\mathcal{T}$  if and only if each map  $f_\alpha \circ g$  is continuous.
- (d) Let  $f : A \rightarrow \prod X_\alpha$  be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J};$$

let  $Z$  denote the subspace  $f(A)$  of the product space  $\prod X_\alpha$ . Show that the image under  $f$  of each element of  $\mathcal{T}$  is an open set of  $Z$ .

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where  $U_i$  is open in  $\mathbb{R}$  for  $i = \alpha_1, \dots, \alpha_n$  and  $U_i = \mathbb{R}$  for all other indices  $i$ . Given  $\mathbf{x} \in U$ , we find an open set  $V$  of the metric topology such that  $\mathbf{x} \in V \subset U$ . Choose an interval  $(x_i - \epsilon_i, x_i + \epsilon_i)$  in  $\mathbb{R}$  centered about  $x_i$  and lying in  $U_i$  for  $i = \alpha_1, \dots, \alpha_n$ ; choose each  $\epsilon_i \leq 1$ . Then define

$$\epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \dots, \alpha_n\}.$$

We assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U.$$

Let  $\mathbf{y}$  be a point of  $B_D(\mathbf{x}, \epsilon)$ . Then for all  $i$ ,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon.$$

Now if  $i = \alpha_1, \dots, \alpha_n$ , then  $\epsilon \leq \epsilon_i / i$ , so that  $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$ ; it follows that  $|x_i - y_i| < \epsilon_i$ . Therefore,  $\mathbf{y} \in \prod U_i$ , as desired. ■

## Exercises

1. (a) In  $\mathbb{R}^n$ , define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \dots + |x_n - y_n|.$$

Show that  $d'$  is a metric that induces the usual topology of  $\mathbb{R}^n$ . Sketch the basis elements under  $d'$  when  $n = 2$ .

- (b) More generally, given  $p \geq 1$ , define

$$d'(\mathbf{x}, \mathbf{y}) = \left[ \sum_{i=1}^n |x_i - y_i|^p \right]^{1/p}$$

for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Assume that  $d'$  is a metric. Show that it induces the usual topology on  $\mathbb{R}^n$ .

2. Show that  $\mathbb{R} \times \mathbb{R}$  in the dictionary order topology is metrizable.

3. Let  $X$  be a metric space with metric  $d$ .

(a) Show that  $d : X \times X \rightarrow \mathbb{R}$  is continuous.

(b) Let  $X'$  denote a space having the same underlying set as  $X$ . Show that if  $d : X' \times X' \rightarrow \mathbb{R}$  is continuous, then the topology of  $X'$  is finer than the topology of  $X$ .

One can summarize the result of this exercise as follows: If  $X$  has a metric  $d$ , then the topology induced by  $d$  is the coarsest topology relative to which the function  $d$  is continuous.

4. Consider the product, uniform, and box topologies on  $\mathbb{R}^\omega$ .

(a) In which topologies are the following functions from  $\mathbb{R}$  to  $\mathbb{R}^\omega$  continuous?

$$f(t) = (t, 2t, 3t, \dots),$$

$$g(t) = (t, t, t, \dots),$$

$$h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \dots).$$

(b) In which topologies do the following sequences converge?

$$\mathbf{w}_1 = (1, 1, 1, 1, \dots), \quad \mathbf{x}_1 = (1, 1, 1, 1, \dots),$$

$$\mathbf{w}_2 = (0, 2, 2, 2, \dots), \quad \mathbf{x}_2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots),$$

$$\mathbf{w}_3 = (0, 0, 3, 3, \dots), \quad \mathbf{x}_3 = (0, 0, \frac{1}{3}, \frac{1}{3}, \dots),$$

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$$\mathbf{y}_1 = (1, 0, 0, 0, \dots), \quad \mathbf{z}_1 = (1, 1, 0, 0, \dots),$$

$$\mathbf{y}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots), \quad \mathbf{z}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots),$$

$$\mathbf{y}_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots), \quad \mathbf{z}_3 = (\frac{1}{3}, \frac{1}{3}, 0, 0, \dots),$$

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5. Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences that are eventually zero.

What is the closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  in the uniform topology? Justify your answer.

6. Let  $\tilde{\rho}$  be the uniform metric on  $\mathbb{R}^\omega$ . Given  $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega$  and given  $0 < \epsilon < 1$ , let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots.$$

(a) Show that  $U(\mathbf{x}, \epsilon)$  is not equal to the  $\epsilon$ -ball  $B_{\tilde{\rho}}(\mathbf{x}, \epsilon)$ .

(b) Show that  $U(\mathbf{x}, \epsilon)$  is not even open in the uniform topology.

(c) Show that

$$B_{\tilde{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

7. Consider the map  $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  defined in Exercise 8 of §19; give  $\mathbb{R}^\omega$  the uniform topology. Under what conditions on the numbers  $a_i$  and  $b_i$  is  $h$  continuous? a homeomorphism?

8. Let  $X$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $\mathbf{x}$  such that  $\sum x_i^2$  converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[ \sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

defines a metric on  $X$ . (See Exercise 10.) On  $X$  we have the three topologies it inherits from the box, uniform, and product topologies on  $\mathbb{R}^\omega$ . We have also the topology given by the metric  $d$ , which we call the  $\ell^2$ -topology. (Read “little ell two.”)

(a) Show that on  $X$ , we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set  $\mathbb{R}^\infty$  of all sequences that are eventually zero is contained in  $X$ . Show that the four topologies that  $\mathbb{R}^\infty$  inherits as a subspace of  $X$  are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in  $X$ ; it is called the **Hilbert cube**. Compare the four topologies that  $H$  inherits as a subspace of  $X$ .

9. Show that the euclidean metric  $d$  on  $\mathbb{R}^n$  is a metric, as follows: If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ , define

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n), \\ c\mathbf{x} &= (cx_1, \dots, cx_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \dots + x_n y_n.\end{aligned}$$

(a) Show that  $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$ .

(b) Show that  $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ . [Hint: If  $\mathbf{x}, \mathbf{y} \neq 0$ , let  $a = 1/\|\mathbf{x}\|$  and  $b = 1/\|\mathbf{y}\|$ , and use the fact that  $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$ .]

(c) Show that  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ . [Hint: Compute  $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$  and apply (b).]

(d) Verify that  $d$  is a metric.

10. Let  $X$  denote the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $(x_1, x_2, \dots)$  such that  $\sum x_i^2$  converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if  $\mathbf{x}, \mathbf{y} \in X$ , then  $\sum |x_i y_i|$  converges. [Hint: Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let  $c \in \mathbb{R}$ . Show that if  $\mathbf{x}, \mathbf{y} \in X$ , then so are  $\mathbf{x} + \mathbf{y}$  and  $c\mathbf{x}$ .

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[ \sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well-defined metric on  $X$ .

\*11. Show that if  $d$  is a metric for  $X$ , then

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

is a bounded metric that gives the topology of  $X$ . [Hint: If  $f(x) = x/(1+x)$  for  $x > 0$ , use the mean-value theorem to show that  $f(a+b) - f(b) \leq f'(a)$ .]

## §21 The Metric Topology (continued)

In this section, we discuss the relation of the metric topology to the concepts we have previously introduced.

*Subspaces* of metric spaces behave the way one would wish them to; if  $A$  is a subspace of the topological space  $X$  and  $d$  is a metric for  $X$ , then the restriction of  $d$  to  $A \times A$  is a metric for the topology of  $A$ . This we leave to you to check.

About *order topologies* there is nothing to be said; some are metrizable (for instance,  $\mathbb{Z}_+$  and  $\mathbb{R}$ ), and others are not, as we shall see.

The *Hausdorff axiom* is satisfied by every metric topology. If  $x$  and  $y$  are distinct points of the metric space  $(X, d)$ , we let  $\epsilon = \frac{1}{2}d(x, y)$ ; then the triangle inequality implies that  $B_d(x, \epsilon)$  and  $B_d(y, \epsilon)$  are disjoint.

The *product topology* we have already considered in special cases; we have proved that the products  $\mathbb{R}^n$  and  $\mathbb{R}^\omega$  are metrizable. It is true in general that countable products of metrizable spaces are metrizable; the proof follows a pattern similar to the proof for  $\mathbb{R}^\omega$ , so we leave it to the exercises.

About *continuous functions* there is a good deal to be said. Consideration of this topic will occupy the remainder of the section.

When we study continuous functions on metric spaces, we are about as close to the study of calculus and analysis as we shall come in this book. There are two things we want to do at this point.

First, we want to show that the familiar " $\epsilon$ - $\delta$  definition" of continuity carries over to general metric spaces, and so does the "convergent sequence definition" of continuity.

Second, we want to consider two additional methods for constructing continuous functions, besides those discussed in §18. One is the process of taking sums, differences, products, and quotients of continuous real-valued functions. The other is the process of taking limits of uniformly convergent sequences of continuous functions.

**Theorem 21.1.** Let  $f : X \rightarrow Y$ ; let  $X$  and  $Y$  be metrizable with metrics  $d_X$  and  $d_Y$ , respectively. Then continuity of  $f$  is equivalent to the requirement that given  $x \in X$  and given  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

*Proof.* Suppose that  $f$  is continuous. Given  $x$  and  $\epsilon$ , consider the set

$$f^{-1}(B(f(x), \epsilon)),$$

## Exercises

1. Let  $A \subset X$ . If  $d$  is a metric for the topology of  $X$ , show that  $d|_{A \times A}$  is a metric for the subspace topology on  $A$ .
2. Let  $X$  and  $Y$  be metric spaces with metrics  $d_X$  and  $d_Y$ , respectively. Let  $f : X \rightarrow Y$  have the property that for every pair of points  $x_1, x_2$  of  $X$ ,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that  $f$  is an imbedding. It is called an *isometric imbedding* of  $X$  in  $Y$ .

3. Let  $X_n$  be a metric space with metric  $d_n$ , for  $n \in \mathbb{Z}_+$ .  
(a) Show that

$$\rho(x, y) = \max\{d_1(x_1, y_1), \dots, d_n(x_n, y_n)\}$$

is a metric for the product space  $X_1 \times \dots \times X_n$ .

- (b) Let  $\tilde{d}_i = \min\{d_i, 1\}$ . Show that

$$D(x, y) = \sup\{\tilde{d}_i(x_i, y_i)/i\}$$

is a metric for the product space  $\prod X_i$ .

4. Show that  $\mathbb{R}_\ell$  and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)
5. *Theorem.* Let  $x_n \rightarrow x$  and  $y_n \rightarrow y$  in the space  $\mathbb{R}$ . Then

$$x_n + y_n \rightarrow x + y,$$

$$x_n - y_n \rightarrow x - y,$$

$$x_n y_n \rightarrow xy,$$

and provided that each  $y_n \neq 0$  and  $y \neq 0$ ,

$$x_n/y_n \rightarrow x/y.$$

[Hint: Apply Lemma 21.4; recall from the exercises of §19 that if  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $x_n \times y_n \rightarrow x \times y$ .]

6. Define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by the equation  $f_n(x) = x^n$ . Show that the sequence  $(f_n(x))$  converges for each  $x \in [0, 1]$ , but that the sequence  $(f_n)$  does not converge uniformly.

verge uniformly.

7. Let  $X$  be a set, and let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of functions. Let  $\bar{\rho}$  be the uniform metric on the space  $\mathbb{R}^X$ . Show that the sequence  $(f_n)$  converges uniformly to the function  $f : X \rightarrow \mathbb{R}$  if and only if the sequence  $(f_n)$  converges to  $f$  as elements of the metric space  $(\mathbb{R}^X, \bar{\rho})$ .
8. Let  $X$  be a topological space and let  $Y$  be a metric space. Let  $f_n : X \rightarrow Y$  be a sequence of continuous functions. Let  $x_n$  be a sequence of points of  $X$  converging to  $x$ . Show that if the sequence  $(f_n)$  converges uniformly to  $f$ , then  $(f_n(x_n))$  converges to  $f(x)$ .
9. Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

See Figure 21.1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the zero function.

- (a) Show that  $f_n(x) \rightarrow f(x)$  for each  $x \in \mathbb{R}$ .
  - (b) Show that  $f_n$  does not converge uniformly to  $f$ . (This shows that the converse of Theorem 21.6 does not hold; the limit function  $f$  may be continuous even though the convergence is not uniform.)
10. Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of  $\mathbb{R}^2$ :

$$A = \{x \times y \mid xy = 1\},$$

$$S^1 = \{x \times y \mid x^2 + y^2 = 1\},$$

$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}.$$

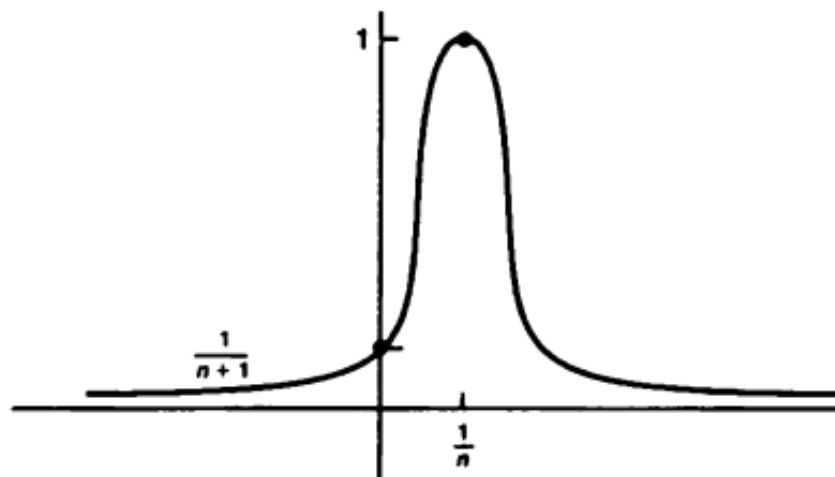


Figure 21.1

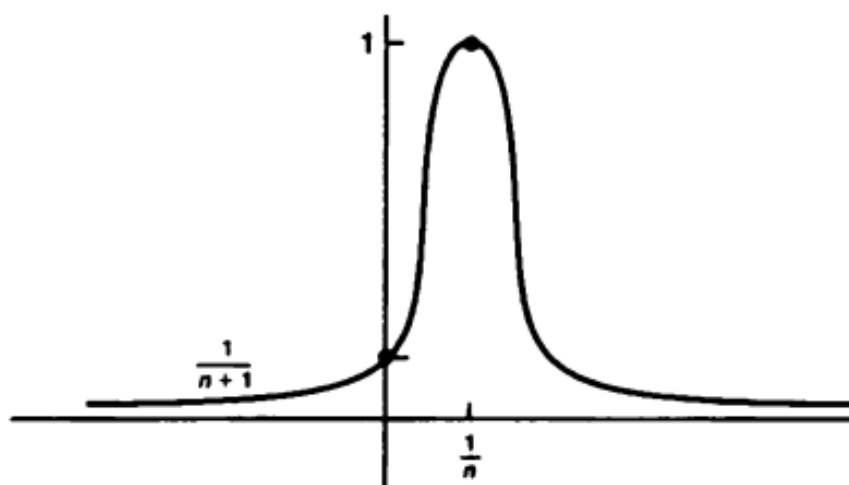


Figure 21.1

The set  $B^2$  is called the (closed) **unit ball** in  $\mathbb{R}^2$ .

11. Prove the following standard facts about infinite series:

- (a) Show that if  $(s_n)$  is a bounded sequence of real numbers and  $s_n \leq s_{n+1}$  for each  $n$ , then  $(s_n)$  converges.
- (b) Let  $(a_n)$  be a sequence of real numbers; define

$$s_n = \sum_{i=1}^n a_i.$$

If  $s_n \rightarrow s$ , we say that the **infinite series**

$$\sum_{i=1}^{\infty} a_i$$

converges to  $s$  also. Show that if  $\sum a_i$  converges to  $s$  and  $\sum b_i$  converges to  $t$ , then  $\sum (ca_i + b_i)$  converges to  $cs + t$ .

- (c) Prove the **comparison test** for infinite series: If  $|a_i| \leq b_i$  for each  $i$ , and if the series  $\sum b_i$  converges, then the series  $\sum a_i$  converges. [Hint: Show that the series  $\sum |a_i|$  and  $\sum c_i$  converge, where  $c_i = |a_i| + a_i$ .]
- (d) Given a sequence of functions  $f_n : X \rightarrow \mathbb{R}$ , let

$$s_n(x) = \sum_{i=1}^n f_i(x).$$

Prove the **Weierstrass M-test** for uniform convergence: If  $|f_i(x)| \leq M_i$  for all  $x \in X$  and all  $i$ , and if the series  $\sum M_i$  converges, then the sequence  $(s_n)$  converges uniformly to a function  $s$ . [Hint: Let  $r_n = \sum_{i=n+1}^{\infty} M_i$ . Show that if  $k > n$ , then  $|s_k(x) - s_n(x)| \leq r_n$ ; conclude that  $|s(x) - s_n(x)| \leq r_n$ .]



12. Prove continuity of the algebraic operations on  $\mathbb{R}$ , as follows: Use the metric  $d(a, b) = |a - b|$  on  $\mathbb{R}$  and the metric on  $\mathbb{R}^2$  given by the equation

$$\rho((x, y), (x_0, y_0)) = \max\{|x - x_0|, |y - y_0|\}.$$

- (a) Show that addition is continuous. [Hint: Given  $\epsilon$ , let  $\delta = \epsilon/2$  and note that

$$d(x + y, x_0 + y_0) \leq |x - x_0| + |y - y_0|.]$$

- (b) Show that multiplication is continuous. [Hint: Given  $(x_0, y_0)$  and  $0 < \epsilon < 1$ , let

$$3\delta = \epsilon/(|x_0| + |y_0| + 1)$$

and note that

$$d(xy, x_0y_0) \leq |x_0||y - y_0| + |y_0||x - x_0| + |x - x_0||y - y_0|.]$$

- (c) Show that the operation of taking reciprocals is a continuous map from  $\mathbb{R} - \{0\}$  to  $\mathbb{R}$ . [Hint: Show the inverse image of the interval  $(a, b)$  is open. Consider five cases, according as  $a$  and  $b$  are positive, negative, or zero.]
- (d) Show that the subtraction and quotient operations are continuous.

## \*§22 The Quotient Topology<sup>†</sup>

Unlike the topologies we have already considered in this chapter, the quotient topology is not a natural generalization of something you have already studied in analysis. Nevertheless, it is easy enough to motivate. One motivation comes from geometry, where one often has occasion to use “cut-and-paste” techniques to construct such geometric objects as surfaces. The *torus* (surface of a doughnut), for example, can be constructed by taking a rectangle and “pasting” its edges together appropriately, as in Figure 22.1. And the *sphere* (surface of a ball) can be constructed by taking a disc and collapsing its entire boundary to a single point; see Figure 22.2. Formalizing these constructions involves the concept of quotient topology.

## Exercises

1. Check the details of Example 3.
2. (a) Let  $p : X \rightarrow Y$  be a continuous map. Show that if there is a continuous map  $f : Y \rightarrow X$  such that  $p \circ f$  equals the identity map of  $Y$ , then  $p$  is a quotient map.
- (b) If  $A \subset X$ , a **retraction** of  $X$  onto  $A$  is a continuous map  $r : X \rightarrow A$  such that  $r(a) = a$  for each  $a \in A$ . Show that a retraction is a quotient map.

3. Let  $\pi_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be projection on the first coordinate. Let  $A$  be the subspace of  $\mathbb{R} \times \mathbb{R}$  consisting of all points  $x \times y$  for which either  $x \geq 0$  or  $y = 0$  (or both); let  $q : A \rightarrow \mathbb{R}$  be obtained by restricting  $\pi_1$ . Show that  $q$  is a quotient map that is neither open nor closed.
4. (a) Define an equivalence relation on the plane  $X = \mathbb{R}^2$  as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0 + y_0^2 = x_1 + y_1^2.$$

Let  $X^*$  be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? [Hint: Set  $g(x \times y) = x + y^2$ .]

- (b) Repeat (a) for the equivalence relation

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

5. Let  $p : X \rightarrow Y$  be an open map. Show that if  $A$  is open in  $X$ , then the map  $q : A \rightarrow p(A)$  obtained by restricting  $p$  is an open map.
6. Recall that  $\mathbb{R}_K$  denotes the real line in the  $K$ -topology. (See §13.) Let  $Y$  be the quotient space obtained from  $\mathbb{R}_K$  by collapsing the set  $K$  to a point; let  $p : \mathbb{R}_K \rightarrow Y$  be the quotient map.
  - (a) Show that  $Y$  satisfies the  $T_1$  axiom, but is not Hausdorff.
  - (b) Show that  $p \times p : \mathbb{R}_K \times \mathbb{R}_K \rightarrow Y \times Y$  is not a quotient map. [Hint: The diagonal is not closed in  $Y \times Y$ , but its inverse image is closed in  $\mathbb{R}_K \times \mathbb{R}_K$ .]