Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i. Given $\mathbf{x} \in U$, we find an open set V of the metric topology such that $\mathbf{x} \in V \subset U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about x_i and lying in U_i for $i = \alpha_1, \ldots, \alpha_n$; choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \ldots, \alpha_n\}.$$

We assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U.$$

Let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. Then for all i,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon.$$

Now if $i = \alpha_1, \ldots, \alpha_n$, then $\epsilon \leq \epsilon_i / i$, so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$; it follows that $|x_i - y_i| < \epsilon_i$. Therefore, $\mathbf{y} \in \prod U_i$, as desired. ∎

Exercises

1. (a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

3. Let X be a metric space with metric d.
   (a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
   (b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

4. Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.
   (a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = (t, 2t, 3t, \ldots),$$
$$g(t) = (t, t, t, \ldots),$$
$$h(t) = (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots).$$

   (b) In which topologies do the following sequences converge?

$$\begin{array}{ll} \mathbf{w}_1 = (1, 1, 1, 1, \ldots), & \mathbf{x}_1 = (1, 1, 1, 1, \ldots), \\ \mathbf{w}_2 = (0, 2, 2, 2, \ldots), & \mathbf{x}_2 = (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots), \\ \mathbf{w}_3 = (0, 0, 3, 3, \ldots), & \mathbf{x}_3 = (0, 0, \tfrac{1}{3}, \tfrac{1}{3} \ldots), \\ \ldots & \ldots \\ \mathbf{y}_1 = (1, 0, 0, 0, \ldots), & \mathbf{z}_1 = (1, 1, 0, 0, \ldots), \\ \mathbf{y}_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots), & \mathbf{z}_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots), \\ \mathbf{y}_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots), & \mathbf{z}_3 = (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \ldots), \\ \ldots & \ldots \end{array}$$

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

6. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots .$$

   (a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.
   (b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.
   (c) Show that

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

7. Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

8. Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

defines a metric on X. (See Exercise 10.) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the *ℓ^2-topology*. (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in X; it is called the ***Hilbert cube***. Compare the four topologies that H inherits as a subspace of X.

9. Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n), \\ c\mathbf{x} &= (cx_1, \dots, cx_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n. \end{aligned}$$

(a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \|\mathbf{y}\|$. [*Hint:* If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \ge 0$.]

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$. [*Hint:* Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

10. Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \dots)$ such that $\sum x_i^2$ converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges. [*Hint:* Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x}, \mathbf{y} \in X$, then so are $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}$$

is a well-defined metric on X.

***11.** Show that if d is a metric for X, then

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

is a bounded metric that gives the topology of X. [*Hint:* If $f(x) = x/(1+x)$ for $x > 0$, use the mean-value theorem to show that $f(a+b) - f(b) \leq f'(a)$.]

§21 The Metric Topology (continued)

In this section, we discuss the relation of the metric topology to the concepts we have previously introduced.

Subspaces of metric spaces behave the way one would wish them to; if A is a subspace of the topological space X and d is a metric for X, then the restriction of d to $A \times A$ is a metric for the topology of A. This we leave to you to check.

About *order topologies* there is nothing to be said; some are metrizable (for instance, $\mathbb{Z}_+$ and $\mathbb{R}$), and others are not, as we shall see.

The *Hausdorff axiom* is satisfied by every metric topology. If x and y are distinct points of the metric space (X, d), we let $\epsilon = \frac{1}{2}d(x, y)$; then the triangle inequality implies that $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$ are disjoint.

The *product topology* we have already considered in special cases; we have proved that the products $\mathbb{R}^n$ and $\mathbb{R}^\omega$ are metrizable. It is true in general that countable products of metrizable spaces are metrizable; the proof follows a pattern similar to the proof for $\mathbb{R}^\omega$, so we leave it to the exercises.

About *continuous functions* there is a good deal to be said. Consideration of this topic will occupy the remainder of the section.

When we study continuous functions on metric spaces, we are about as close to the study of calculus and analysis as we shall come in this book. There are two things we want to do at this point.

First, we want to show that the familiar "ϵ-δ definition" of continuity carries over to general metric spaces, and so does the "convergent sequence definition" of continuity.

Second, we want to consider two additional methods for constructing continuous functions, besides those discussed in §18. One is the process of taking sums, differences, products, and quotients of continuous real-valued functions. The other is the process of taking limits of uniformly convergent sequences of continuous functions.

Theorem 21.1. *Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x, y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

Proof. Suppose that f is continuous. Given x and ϵ, consider the set

$$f^{-1}(B(f(x), \epsilon)),$$