Introduction to Hidden Markov Models

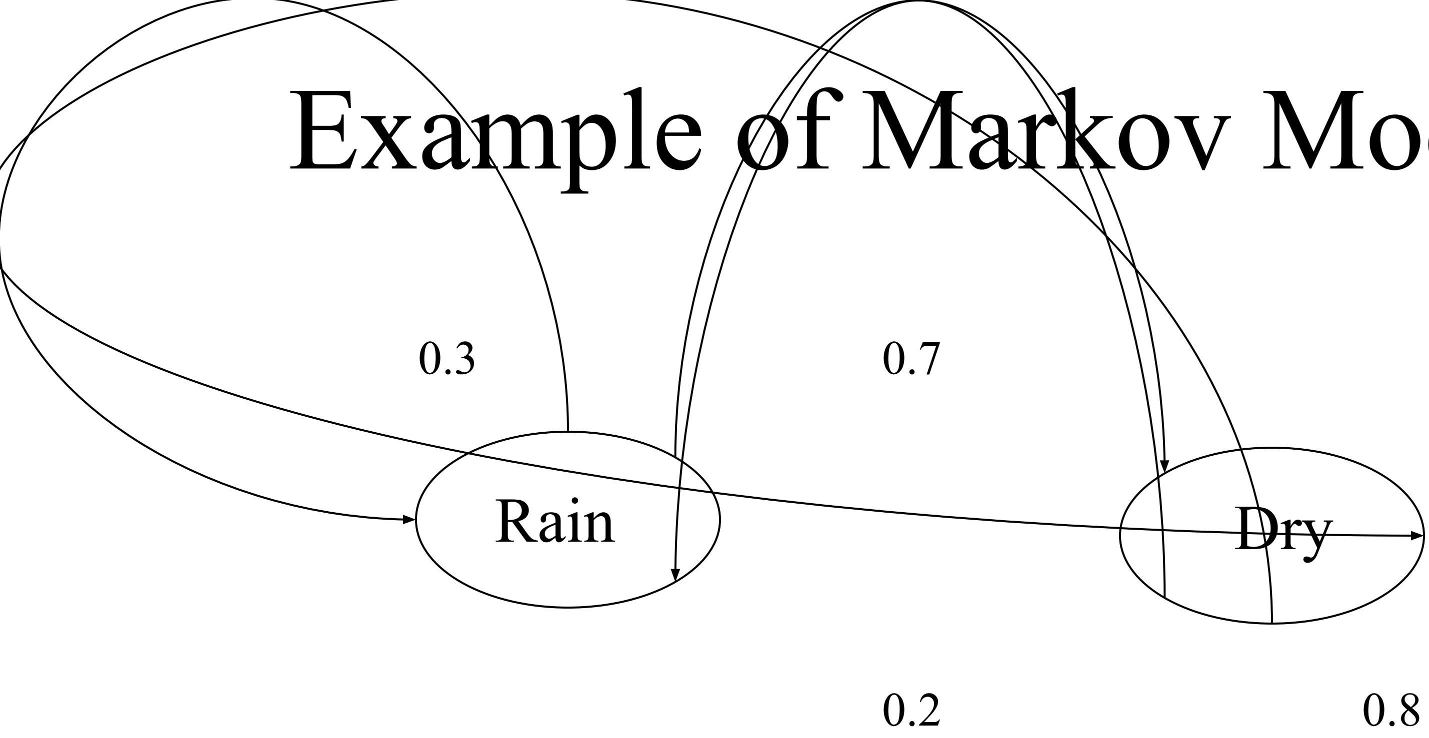
Markov Models

- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- To define Markov model, the following probabilities have to be specified: transition probabilities $a_{ij} = P(s_i \mid s_j)$ and initial probabilities $\pi_i = P(s_i)$

Example of Markov Model



- Two states : ‘Rain’ and ‘Dry’.
- Transition probabilities: $P($‘Rain’|‘Rain’$)=0.3$, $P($‘Dry’|‘Rain’$)=0.7$, $P($‘Rain’|‘Dry’$)=0.2$, $P($‘Dry’|‘Dry’$)=0.8$
- Initial probabilities: say $P($‘Rain’$)=0.4$, $P($‘Dry’$)=0.6$.

Calculation of sequence probability

- By Markov chain property, probability of state sequence can be found by the formula:

$$P(s_{i1}, s_{i2}, \ldots, s_{ik}) = P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1})$$
$$= P(s_{ik} \mid s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1}) = \ldots$$
$$= P(s_{ik} \mid s_{ik-1}) P(s_{ik-1} \mid s_{ik-2}) \ldots P(s_{i2} \mid s_{i1}) P(s_{i1})$$

- Suppose we want to calculate a probability of a sequence of states in our example, {'Dry','Dry','Rain',Rain'}.

P({'Dry','Dry','Rain',Rain'}) =

P('Rain'|'Rain') P('Rain'|'Dry') P('Dry'|'Dry') P('Dry')=

= 0.3*0.2*0.8*0.6

Hidden Markov models.

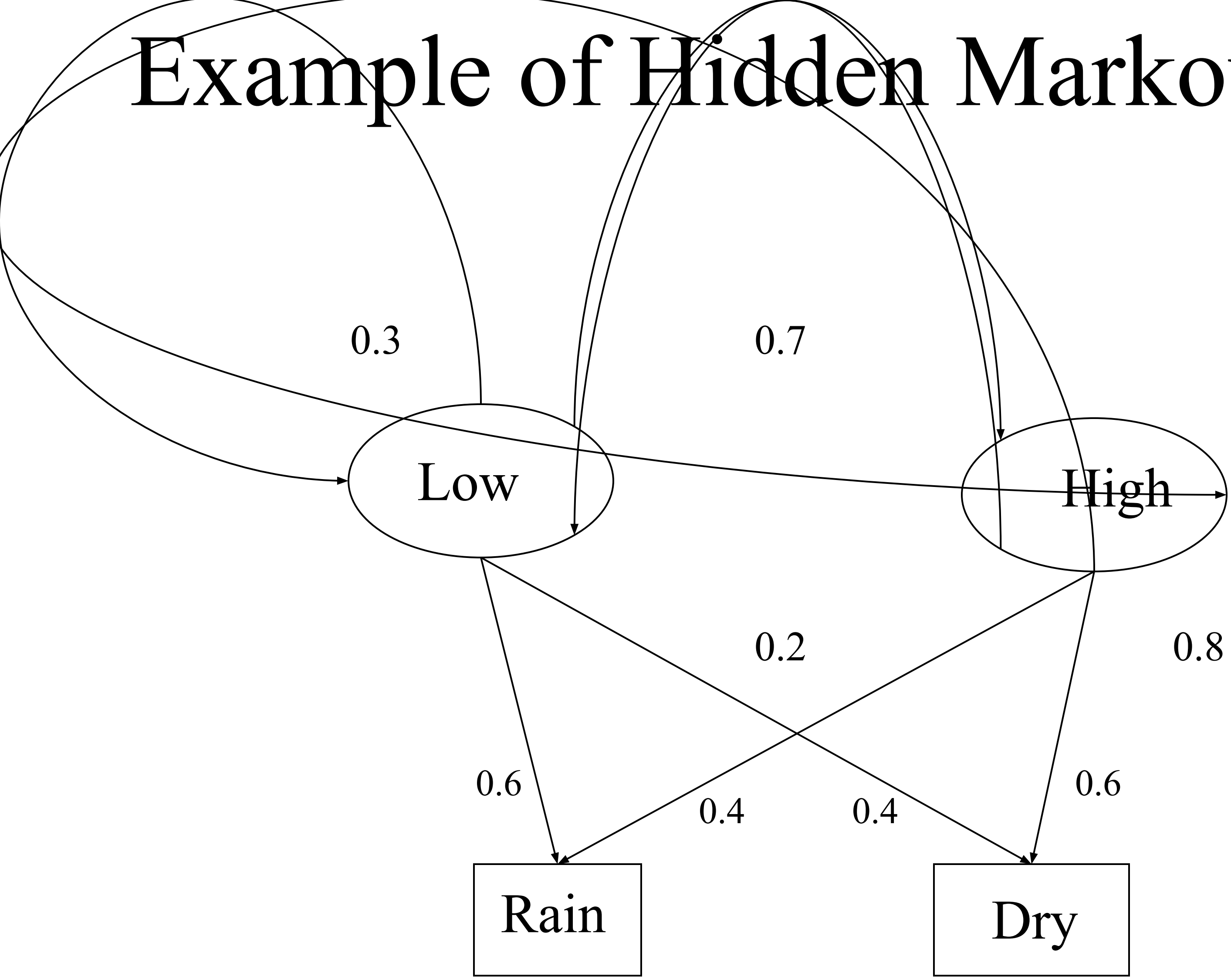
- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- States are not visible, but each state randomly generates one of M observations (or visible states) $\{v_1, v_2, \ldots, v_M\}$

- To define hidden Markov model, the following probabilities have to be specified: matrix of transition probabilities $A=(a_{ij})$, $a_{ij}= P(s_i \mid s_j)$, matrix of observation probabilities $B=(b_i(v_m))$, $b_i(v_m)= P(v_m \mid s_i)$ and a vector of initial probabilities $\pi=(\pi_i)$, $\pi_i = P(s_i)$. Model is represented by $M=(A, B, \pi)$.

Example of Hidden Markov Model



Example of Hidden Markov Model

- Two states : ‘Low’ and ‘High’ atmospheric pressure.
- Two observations : ‘Rain’ and ‘Dry’.
- Transition probabilities: $P(\text{‘Low’}|\text{‘Low’})=0.3$, $P(\text{‘High’}|\text{‘Low’})=0.7$, $P(\text{‘Low’}|\text{‘High’})=0.2$, $P(\text{‘High’}|\text{‘High’})=0.8$
- Observation probabilities : $P(\text{‘Rain’}|\text{‘Low’})=0.6$, $P(\text{‘Dry’}|\text{‘Low’})=0.4$, $P(\text{‘Rain’}|\text{‘High’})=0.4$, $P(\text{‘Dry’}|\text{‘High’})=0.3$.
- Initial probabilities: say $P(\text{‘Low’})=0.4$, $P(\text{‘High’})=0.6$.

Calculation of observation sequence probability

- Suppose we want to calculate a probability of a sequence of observations in our example, {'Dry','Rain'}.
- Consider all possible hidden state sequences:

$$P(\{\text{'Dry','Rain'}\}) = P(\{\text{'Dry','Rain'}\}, \{\text{'Low','Low'}\}) + P(\{\text{'Dry','Rain'}\}, \{\text{'Low','High'}\}) + P(\{\text{'Dry','Rain'}\}, \{\text{'High','Low'}\}) + P(\{\text{'Dry','Rain'}\}, \{\text{'High','High'}\})$$

where first term is :

$$P(\{\text{'Dry','Rain'}\}, \{\text{'Low','Low'}\}) =$$
$$P(\{\text{'Dry','Rain'}\} \mid \{\text{'Low','Low'}\})\ P(\{\text{'Low','Low'}\}) =$$
$$P(\text{'Dry'}|\text{'Low'})P(\text{'Rain'}|\text{'Low'})\ P(\text{'Low'})P(\text{'Low'}|\text{'Low})$$
$$= 0.4*0.4*0.6*0.4*0.3$$

Main issues using HMMs :

Evaluation problem. Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots o_K$, calculate the probability that model M has generated sequence O.

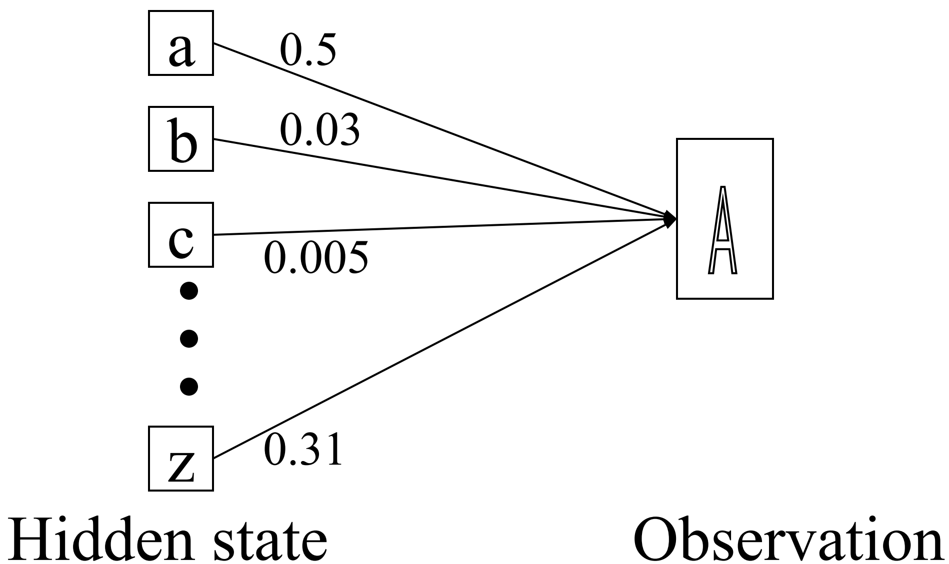
- **Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence O.
- **Learning problem.** Given some training observation sequences $O=o_1\, o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data.

$O=o_1 \ldots o_K$ *denotes a sequence of observations* $o_k \in \{v_1, \ldots, v_M\}$.

Word recognition example(1).

- Typed word recognition, assume all characters are separated.

- Character recognizer outputs probability of the image being particular character, P(image|character).



Word recognition example(2).

- Hidden states of HMM = characters.

- Observations = typed images of characters segmented from the image v_α. Note that there is an infinite number of observations

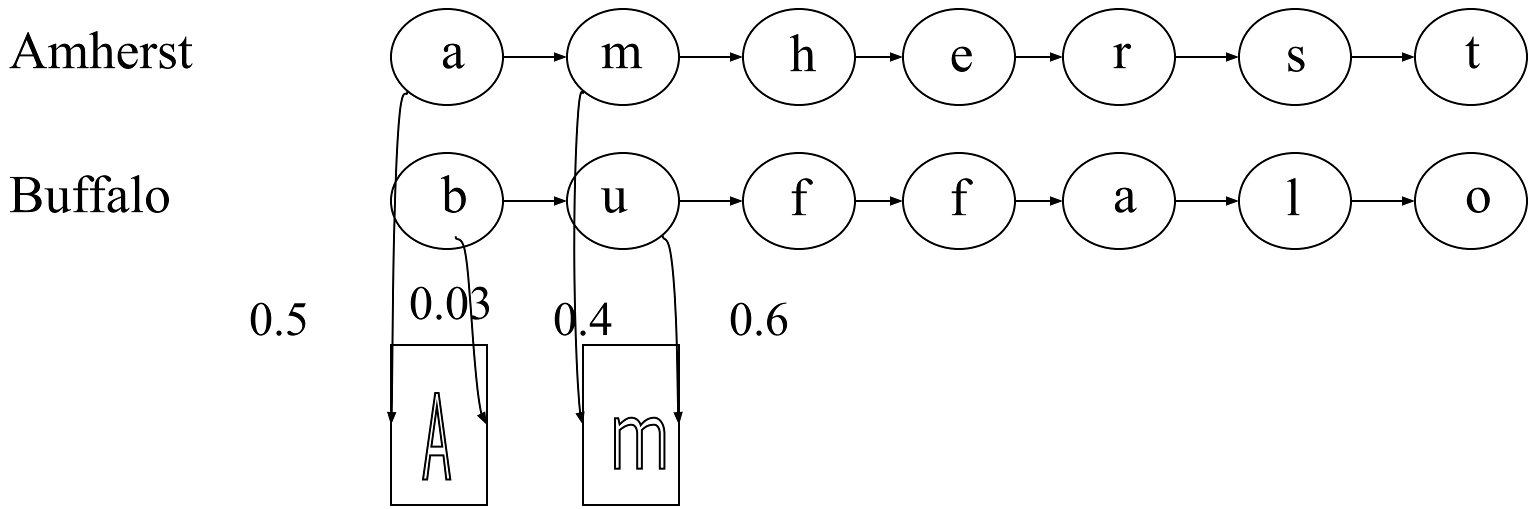
- Observation probabilities = character recognizer scores.

$$B = \left(b_i(v_\alpha)\right) = \left(P(v_\alpha \mid s_i)\right)$$

- Transition probabilities will be defined differently in two subsequent models.

Word recognition example(3).

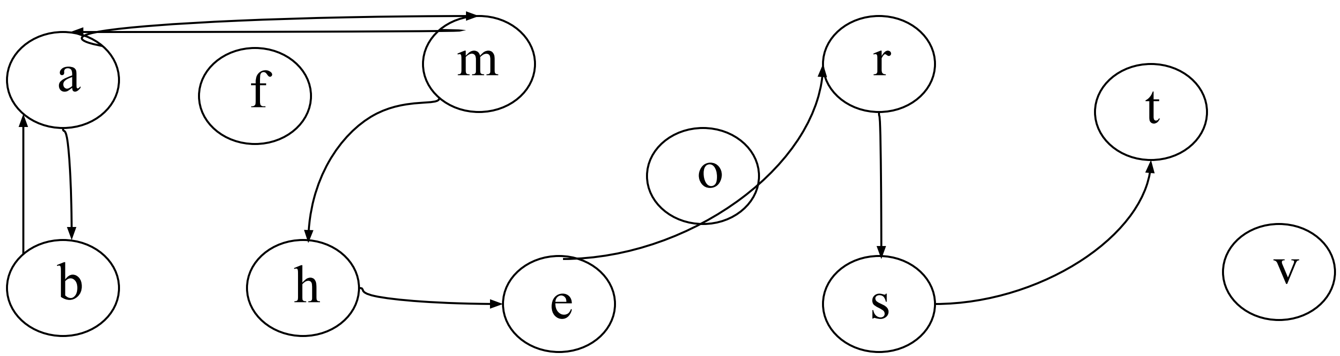
- If lexicon is given, we can construct separate HMM models for each lexicon word.



- Here recognition of word image is equivalent to the problem of evaluating few HMM models.
- This is an application of **Evaluation problem.**

Word recognition example(4).

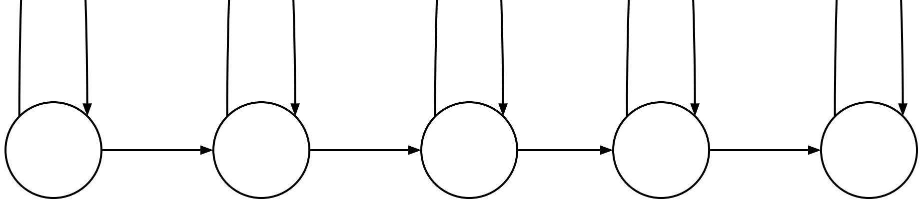
- We can construct a single HMM for all words.
- Hidden states = all characters in the alphabet.
- Transition probabilities and initial probabilities are calculated from language model.
- Observations and observation probabilities are as before.



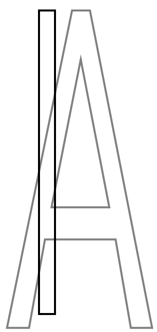
- Here we have to determine the best sequence of hidden states, the one that most likely produced word image.
- This is an application of **Decoding problem.**

Character recognition with HMM example.

- The structure of hidden states is chosen.



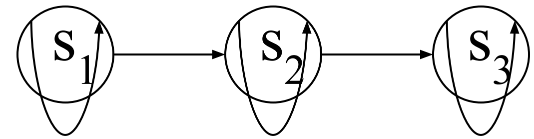
- Observations are feature vectors extracted from vertical slices.



- Probabilistic mapping from hidden state to feature vectors:
    1. use mixture of Gaussian models
    2. Quantize feature vector space.

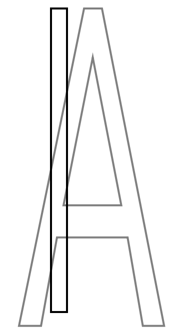
Exercise: character recognition with HMM(1)

- The structure of hidden states:



- Observation = number of islands in the vertical slice.

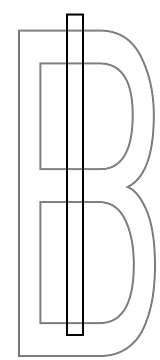
- HMM for character 'A' :



Transition probabilities: $\{a_{ij}\} = \begin{pmatrix} .8 & .2 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{pmatrix}$

Observation probabilities: $\{b_{jk}\} = \begin{pmatrix} .9 & .1 & 0 \\ .1 & .8 & .1 \\ .9 & .1 & 0 \end{pmatrix}$

- HMM for character 'B' :



Transition probabilities: $\{a_{ij}\} = \begin{pmatrix} .8 & .2 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{pmatrix}$

Observation probabilities: $\{b_{jk}\} = \begin{pmatrix} .9 & .1 & 0 \\ 0 & .2 & .8 \\ .6 & .4 & 0 \end{pmatrix}$

Exercise: character recognition with HMM(2)

- Suppose that after character image segmentation the following sequence of island numbers in 4 slices was observed:

  { 1, 3, 2, 1}

- What HMM is more likely to generate this observation sequence , HMM for ‘A’ or HMM for ‘B’ ?

Exercise: character recognition with HMM(3)

Consider likelihood of generating given observation for each possible sequence of hidden states:

- HMM for character ‘A’:

Hidden state sequence	Transition probabilities		Observation probabilities		
$s_1 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3$	.8 * .2 * .2	*	.9 * 0 * .8 * .9	=	0
$s_1 \rightarrow s_2 \rightarrow s_2 \rightarrow s_3$	.2 * .8 * .2	*	.9 * .1 * .8 * .9	=	0.0020736
$s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_3$	.2 * .2 * 1	*	.9 * .1 * .1 * .9	=	0.000324
			Total	=	0.0023976

- HMM for character ‘B’:

Hidden state sequence	Transition probabilities		Observation probabilities		
$s_1 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3$	.8 * .2 * .2	*	.9 * 0 * .2 * .6	=	0
$s_1 \rightarrow s_2 \rightarrow s_2 \rightarrow s_3$	.2 * .8 * .2	*	.9 * .8 * .2 * .6	=	0.0027648
$s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_3$	.2 * .2 * 1	*	.9 * .8 * .4 * .6	=	0.006912
			Total	=	0.0096768

Evaluation Problem.

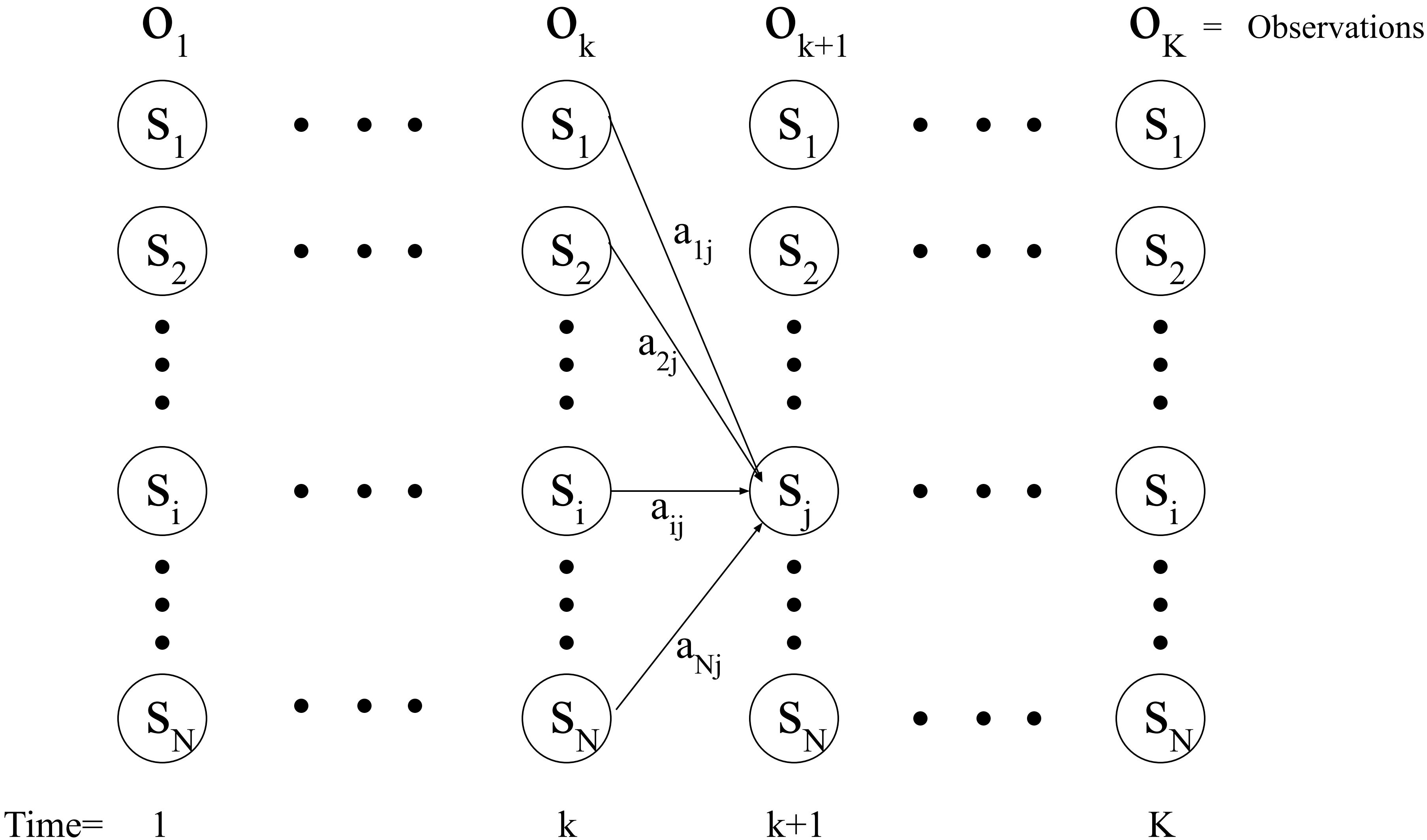
- **Evaluation problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \dots o_K$, calculate the probability that model M has generated sequence O.
- Trying to find probability of observations $O=o_1\, o_2 \dots o_K$ by means of considering all hidden state sequences (as was done in example) is impractical:

  N^K hidden state sequences - exponential complexity.

- Use **Forward-Backward HMM algorithms** for efficient calculations.

- Define the forward variable $\alpha_k(i)$ as the joint probability of the partial observation sequence $o_1\, o_2 \dots o_k$ and that the hidden state at time k is s_i : $\alpha_k(i)= P(o_1\, o_2 \dots o_k,\, q_k= s_i)$

Trellis representation of an HMM



Forward recursion for HMM

- Initialization:

$$\alpha_1(i) = P(o_1\, , q_1 = s_i) = \pi_i\, b_i(o_1)\ , \quad 1 <= i <= N.$$

- Forward recursion:

$$\alpha_{k+1}(i) = P(o_1\, o_2 \dots\, o_{k+1}\, , q_{k+1} = s_j) =$$
$$\Sigma_i\, P(o_1\, o_2 \dots\, o_{k+1}\, , q_k = s_i\, , q_{k+1} = s_j) =$$
$$\Sigma_i\, P(o_1\, o_2 \dots\, o_k\, , q_k = s_i)\, a_{ij}\, b_j(o_{k+1}) =$$
$$[\Sigma_i\, \alpha_k(i)\, a_{ij}\,]\, b_j(o_{k+1})\ , \qquad 1 <= j <= N,\ 1 <= k <= K-1.$$

- Termination:

$$P(o_1\, o_2 \dots\, o_K) = \Sigma_i\, P(o_1\, o_2 \dots\, o_K\, , q_K = s_i) = \Sigma_i\, \alpha_K(i)$$

- Complexity :

  N^2K operations.

Backward recursion for HMM

- Define the forward variable $\beta_k(i)$ as the joint probability of the partial observation sequence $o_{k+1}\, o_{k+2} \dots o_K$ given that the hidden state at time k is s_i : $\beta_k(i) = P(o_{k+1}\, o_{k+2} \dots o_K \mid q_k = s_i)$
- Initialization:

$$\beta_K(i) = 1 \quad , \ 1 <= i <= N.$$

- Backward recursion:

$$\beta_k(j) = P(o_{k+1}\, o_{k+2} \dots o_K \mid q_k = s_j) =$$

$$\Sigma_i\, P(o_{k+1}\, o_{k+2} \dots o_K, q_{k+1} = s_i \mid q_k = s_j) =$$

$$\Sigma_i\, P(o_{k+2}\, o_{k+3} \dots o_K \mid q_{k+1} = s_i)\; a_{ji}\; b_i(o_{k+1}) =$$

$$\Sigma_i\, \beta_{k+1}(i)\; a_{ji}\; b_i(o_{k+1}) , \qquad 1 <= j <= N,\ 1 <= k <= K\text{-}1.$$

- Termination:

$$P(o_1\, o_2 \dots o_K) = \Sigma_i\, P(o_1\, o_2 \dots o_K, q_1 = s_i) =$$

$$\Sigma_i\, P(o_1\, o_2 \dots o_K \mid q_1 = s_i)\; P(q_1 = s_i) = \Sigma_i\, \beta_1(i)\; b_i(o_1)\; \pi_i$$

Decoding problem

- **Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1\, o_2 \ldots\, o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence.
- We want to find the state sequence $Q= q_1 \ldots q_K$ which maximizes $P(Q \mid o_1\, o_2 \ldots\, o_K)$, or equivalently $P(Q\, ,\, o_1\, o_2 \ldots\, o_K)$.
- Brute force consideration of all paths takes exponential time. Use efficient **Viterbi algorithm** instead.
- Define variable $\delta_k(i)$ as the maximum probability of producing observation sequence $o_1\, o_2 \ldots\, o_k$ when moving along any hidden state sequence $q_1 \ldots\, q_{k-1}$ and getting into $q_k= s_i$.

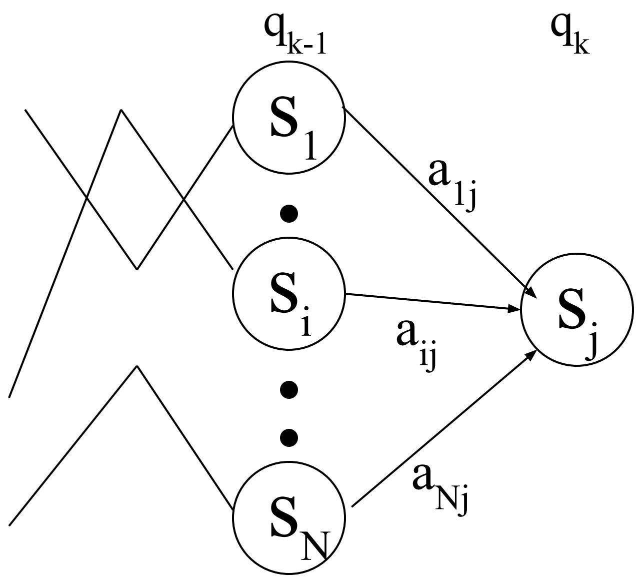
$$\delta_k(i) = \max P(q_1 \ldots\, q_{k-1}\, , q_k= s_i\, , o_1\, o_2 \ldots\, o_k)$$

where max is taken over all possible paths $q_1 \ldots\, q_{k-1}$.

Viterbi algorithm (1)

- General idea:

  if best path ending in $q_k = s_j$ goes through $q_{k-1} = s_i$ then it should coincide with best path ending in $q_{k-1} = s_i$.



- $\delta_k(i) = \max P(q_1 \ldots q_{k-1}, q_k = s_j, o_1 o_2 \ldots o_k) =$
  $\max_i [\, a_{ij}\, b_j(o_k)\ \max P(q_1 \ldots q_{k-1} = s_i, o_1 o_2 \ldots o_{k-1})\,]$
- To backtrack best path keep info that predecessor of s_j was s_i.

Viterbi algorithm (2)

- Initialization:

$$\delta_1(i) = \max P(q_1 = s_i , o_1) = \pi_i \, b_i(o_1) \, , \quad 1<=i<=N.$$

- Forward recursion:

$$\delta_k(j) = \max P(q_1 \ldots \, q_{k-1} , q_k = s_j , o_1 \, o_2 \ldots \, o_k) =$$
$$\max_i [\, a_{ij} \, b_j(o_k) \max P(q_1 \ldots \, q_{k-1} = s_i , o_1 \, o_2 \ldots \, o_{k-1}) \,] =$$
$$\max_i [\, a_{ij} \, b_j(o_k) \, \delta_{k-1}(i) \,] , \qquad 1<=j<=N, \; 2<=k<=K.$$

- Termination: choose best path ending at time K

$$\max_i [\, \delta_K(i) \,]$$

- Backtrack best path.

This algorithm is similar to the forward recursion of evaluation problem, with Σ *replaced by max and additional backtracking.*

Learning problem (1)

- **Learning problem.** Given some training observation sequences $O=o_1\, o_2 \dots\, o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data, that is maximizes $P(O \mid M)$.

- There is no algorithm producing optimal parameter values.

- Use iterative expectation-maximization algorithm to find local maximum of $P(O \mid M)$ - **Baum-Welch algorithm.**

Learning problem (2)

- If training data has information about sequence of hidden states (as in word recognition example), then use maximum likelihood estimation of parameters:

$$a_{ij} = P(s_i \mid s_j) = \frac{\text{Number of transitions from state } S_j \text{ to state } S_i}{\text{Number of transitions out of state } S_j}$$

$$b_i(v_m) = P(v_m \mid s_i) = \frac{\text{Number of times observation } V_m \text{ occurs in state } S_i}{\text{Number of times in state } S_i}$$

Baum-Welch algorithm

General idea:

$$a_{ij} = P(s_i \mid s_j) = \frac{\text{Expected number of transitions from state } S_j \text{ to state } S_i}{\text{Expected number of transitions out of state } S_j}$$

$$b_i(v_m) = P(v_m \mid s_i) = \frac{\text{Expected number of times observation } V_m \text{ occurs in state } S_i}{\text{Expected number of times in state } S_i}$$

$$\pi_i = P(s_i) = \text{Expected frequency in state } S_i \text{ at time } k=1.$$

Baum-Welch algorithm: expectation step(1)

- Define variable $\xi_k(i,j)$ as the probability of being in state s_i at time k and in state s_j at time k+1, given the observation sequence $o_1\, o_2 \ldots\, o_K$.

$$\xi_k(i,j) = P(q_k = s_i\,, q_{k+1} = s_j \mid o_1\, o_2 \ldots\, o_K)$$

$$\xi_k(i,j) = \frac{P(q_k = s_i\,, q_{k+1} = s_j\,, o_1\, o_2 \ldots\, o_k)}{P(o_1\, o_2 \ldots\, o_k)} =$$

$$\frac{P(q_k = s_i\,, o_1\, o_2 \ldots\, o_k)\, a_{ij}\, b_j(o_{k+1})\, P(o_{k+2} \ldots\, o_K \mid q_{k+1} = s_j)}{P(o_1\, o_2 \ldots\, o_k)} =$$

$$\frac{\alpha_k(i)\, a_{ij}\, b_j(o_{k+1})\, \beta_{k+1}(j)}{\Sigma_i \Sigma_j\, \alpha_k(i)\, a_{ij}\, b_j(o_{k+1})\, \beta_{k+1}(j)}$$

Baum-Welch algorithm: expectation step(2)

- Define variable $\gamma_k(i)$ as the probability of being in state s_i at time k, given the observation sequence $o_1\, o_2 \dots\, o_K$.

$$\gamma_k(i) = P(q_k = s_i \mid o_1\, o_2 \dots\, o_K)$$

$$\gamma_k(i) = \frac{P(q_k = s_i\, , o_1\, o_2 \dots\, o_k)}{P(o_1\, o_2 \dots\, o_k)} = \frac{\alpha_k(i)\, \beta_k(i)}{\sum_i \alpha_k(i)\, \beta_k(i)}$$

Baum-Welch algorithm: expectation step(3)

- We calculated $\xi_k(i,j) = P(q_k = s_i, q_{k+1} = s_j \mid o_1\, o_2 \dots o_K)$
  and $\gamma_k(i) = P(q_k = s_i \mid o_1\, o_2 \dots o_K)$

- Expected number of transitions from state s_i to state s_j =
  $= \Sigma_k\, \xi_k(i,j)$
- Expected number of transitions out of state $s_i = \Sigma_k\, \gamma_k(i)$

- Expected number of times observation v_m occurs in state s_i =
  $= \Sigma_k\, \gamma_k(i)$, k is such that $o_k = v_m$
- Expected frequency in state s_i at time k=1 : $\gamma_1(i)$.

Baum-Welch algorithm: maximization step

$$a_{ij} = \frac{\text{Expected number of transitions from state } s_j \text{ to state } s_i}{\text{Expected number of transitions out of state } s_j} = \frac{\sum_k \xi_k(i,j)}{\sum_k \gamma_k(i)}$$

$$b_i(v_m) = \frac{\text{Expected number of times observation } v_m \text{ occurs in state } s_i}{\text{Expected number of times in state } s_i} = \frac{\sum_k \xi_k(i,j)}{\sum_{k,o_k = v_m} \gamma_k(i)}$$

$$\pi_i = (\text{Expected frequency in state } s_i \text{ at time } k=1) = \gamma_1(i).$$