Class- 6 and 7: Matrix Decomposition

Gram-Schmidt process and the Singular Value Decomposition

August 26, 2025

Orthogonal vectors

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1. Check whether $u = \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix}$ and $v = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$ are orthogonal vectors.

2. Determine which pair of vectors are orthogonal

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a. $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

b. $y = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

Orthogonal sets

A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in $\mathbb{R}^n$ are orthogonal if each pair of distinct vectors in the set are orthogonal, that is $u_i \cdot u_j = 0$ for $i \neq j$.

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Example

Check if $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} \frac{-1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$ are orthogonal.

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Definition

Two vectors u and v in $\mathbb{R}^n$ are orthonormal if $u.v = 0$ and $||u|| = ||v|| = 1$.

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Example

Check if $u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $v = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ are orthonormal.

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Example

Show that $\{v_1, v_2, v_3\}$ is an orthonormal set in $\mathbb{R}^3$, where $v_1 = \begin{bmatrix} \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{bmatrix}$,

$v_2 = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ and $v_3 = \begin{bmatrix} \frac{-1}{\sqrt{66}} \\ \frac{-4}{\sqrt{66}} \\ \frac{7}{\sqrt{66}} \end{bmatrix}$

Theorem

An $m \times n$ matrix U has orthonormal column if and only if $U^T U = I$. If a matrix has orthonormal columns, then the matrix is **orthogonal matrix.**

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Example

Check if U is orthogonal matrix, where $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

The Gram-Schmidt Process

The Gram-Schmidt Process is a simple algorithm for producing an orthogonal or orthonormal set in $\mathbb{R}^n$.

Gram-Schmidt orthonormalization process

Consider the vectors as columns of the matrix A. That is,

$$A = \left[a_1 \mid a_2 \mid \cdots \mid a_n \right].$$

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$$A = \left[a_1 \mid a_2 \mid \cdots \mid a_n \right].$$

Then,

$$\mathbf{u}_1 = \mathbf{a}_1, \quad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|},$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1, \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|},$$

$$\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{a}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{a}_3 \cdot \mathbf{e}_2)\mathbf{e}_2, \quad \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|},$$

$$\vdots$$

$$\mathbf{u}_{k+1} = \mathbf{a}_{k+1} - (\mathbf{a}_{k+1} \cdot \mathbf{e}_1)\mathbf{e}_1 - \cdots - (\mathbf{a}_{k+1} \cdot \mathbf{e}_k)\mathbf{e}_k, \quad \mathbf{e}_{k+1} = \frac{\mathbf{u}_{k+1}}{\|\mathbf{u}_{k+1}\|}.$$

Eigenvalues and Eigenvectors: Definition

A real number λ is an *eigenvalue* of an $n \times n$ square matrix A if there exists a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$ such that

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For example, consider $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =$

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Therefore, $\lambda = 2$ is an eigenvalue corresponding to the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

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Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$**.**

Compute the characteristic polynomial and equate it to zero:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{bmatrix} = 0.$$

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so the eigenvalues are $\boxed{\lambda \in \{0, 2, 4\}}$.

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Definition: Let A be an $m \times n$ matrix. A scalar λ is called a *singular value* of A if there exist nonzero vectors $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^m$ such that

$$A\boldsymbol{v} = \lambda\, \boldsymbol{u} \quad \text{and} \quad A^T \boldsymbol{u} = \lambda\, \boldsymbol{v}.$$

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The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are a *singular vector pair* corresponding to λ.

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The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are a *singular vector pair* corresponding to λ.

Equivalently: the singular values are the positive square roots of the eigenvalues of A^TA or AA^T.

Finding singular values: Example (i)

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then

$$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

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The eigenvalues of $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ are 9 and 1.

Therefore the singular values of A are $\boxed{3 \text{ and } 1}$.

Finding singular values: Example (ii)

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Then

$$AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}.$$

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Eigenvalues: 15 and 0.
Singular values: $\sqrt{15}$ and 0.

The Singular Value Decomposition (SVD)

Theorem (SVD)

Let A be an $m \times n$ matrix and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be the nonzero singular values. Then

$$A = U \Sigma V^T,$$

where U is $m \times m$ orthogonal, V is $n \times n$ orthogonal, and

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \mathrm{diag}(\sigma_1, \ldots, \sigma_r).$$

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Columns of U are orthonormal eigenvectors of AA^T (left singular vectors).
Columns of V are orthonormal eigenvectors of $A^T A$ (right singular vectors).

Working procedure for computing an SVD

1. Determine an orthonormal set of eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of $A^T A$ corresponding to eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \ldots, 0$ ($n - r$ times).

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3. For $1 \leq i \leq r$, set $\boldsymbol{u}_i := \dfrac{A\boldsymbol{v}_i}{\|A\boldsymbol{v}_i\|}$.
4. Choose $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ to complete $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ to an orthonormal basis of $\mathbb{R}^m$.

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5. Form $U = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$.
6. Let Σ have diagonal entries $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq r$, zeros otherwise. Then $A = U\Sigma V^T$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \leq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = A^T A$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(n - r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.

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2. Find the orthonormal set of eigenvectors of L corresponding to each eigenvalue (taken in decreasing order): $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.
3. Form $V = [\boldsymbol{v}_1 \; \cdots \; \boldsymbol{v}_n]$.

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Suppose A is $m \times n$ matrix with $m \leq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = A^T A$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(n-r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
2. Find the orthonormal set of eigenvectors of L corresponding to each eigenvalue (taken in decreasing order): $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.
3. Form $V = [\boldsymbol{v}_1 \cdots \boldsymbol{v}_n]$.
4. For $1 \leq i \leq r$, set $\boldsymbol{u}_i := \dfrac{A\boldsymbol{v}_i}{\|A\boldsymbol{v}_i\|}$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \leq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = A^T A$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(n-r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
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3. Form $V = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$.
4. For $1 \leq i \leq r$, set $\boldsymbol{u}_i := \dfrac{A\boldsymbol{v}_i}{\|A\boldsymbol{v}_i\|}$.
5. Choose $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ to complete $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ to an orthonormal vectors in $\mathbb{R}^m$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \leq n$

1. Determine the largest order matrix between A^TA and AA^T which is $L = A^TA$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(n-r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
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3. Form $V = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$.
4. For $1 \leq i \leq r$, set $\boldsymbol{u}_i := \dfrac{A\boldsymbol{v}_i}{\|A\boldsymbol{v}_i\|}$.
5. Choose $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ to complete $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ to an orthonormal vectors in $\mathbb{R}^m$.
6. Form $U = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \leq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = A^T A$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(n-r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
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3. Form $V = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$.
4. For $1 \leq i \leq r$, set $\boldsymbol{u}_i := \dfrac{A\boldsymbol{v}_i}{\|A\boldsymbol{v}_i\|}$.
5. Choose $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ to complete $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ to an orthonormal vectors in $\mathbb{R}^m$.
6. Form $U = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$.
7. Let Σ have diagonal entries $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq r$, zeros otherwise. Then $A = U\Sigma V^T$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \geq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = AA^T$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(m - r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \geq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = AA^T$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(m - r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
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2. Find the orthonormal set of eigenvectors of L corresponding to each eigenvalue (taken in decreasing order): $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.
3. Form $U = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \geq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = AA^T$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(m - r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
2. Find the orthonormal set of eigenvectors of L corresponding to each eigenvalue (taken in decreasing order): $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.
3. Form $U = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$.
4. For $1 \leq i \leq r$, set $\boldsymbol{v}_i := \dfrac{A^T \boldsymbol{u}_i}{\|A^T \boldsymbol{u}_i\|}$.

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4. For $1 \leq i \leq r$, set $\boldsymbol{v}_i := \dfrac{A^T \boldsymbol{u}_i}{\|A^T \boldsymbol{u}_i\|}$.
5. Choose $\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n$ to complete $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to an orthonormal vectors in $\mathbb{R}^n$.

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6. Form $V = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$.

Working procedure for computing an SVD

Suppose A is $m \times n$ matrix with $m \geq n$

1. Determine the largest order matrix between $A^T A$ and AA^T which is $L = AA^T$ in this case. Find the eigenvalues L and arrange in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $0, \cdots, 0$ $(m - r)$ times. Singular values are the nonnegative square root of these eigenvalues say $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and possibly 0 as well.
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3. Form $U = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$.
4. For $1 \leq i \leq r$, set $\boldsymbol{v}_i := \dfrac{A^T \boldsymbol{u}_i}{\|A^T \boldsymbol{u}_i\|}$.
5. Choose $\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n$ to complete $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to an orthonormal vectors in $\mathbb{R}^n$.
6. Form $V = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$.
7. Let Σ have diagonal entries $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq r$, zeros otherwise. Then $A = U\Sigma V^T$.