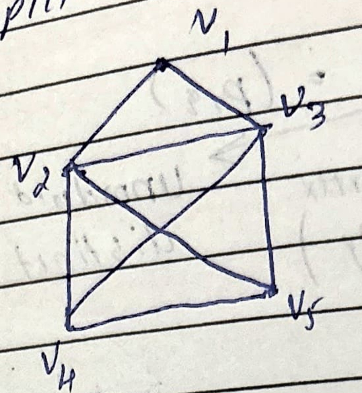
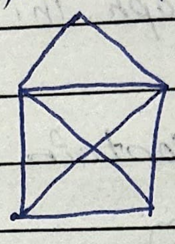
3. Complete the diagram



Sol:-

Start from V_4 then V_5 then V_2 then V_3 then V_1 then V_2 then V_4 then V_3 then (V_5).

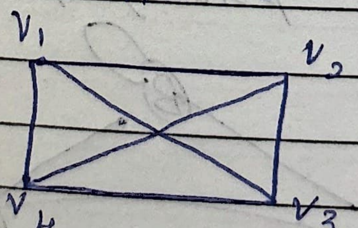


Labelled Graph: A graph in which every vertex & every edge is labeled

Degree of a vertex: The degree of vertex v is the number of edges incident on v.

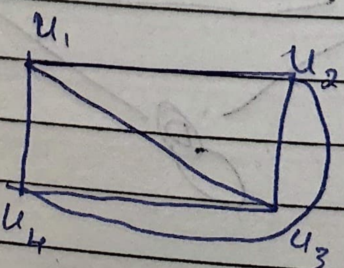
Regular graph : A graph is said to be regular if all vertices have same degree.

G_1 =



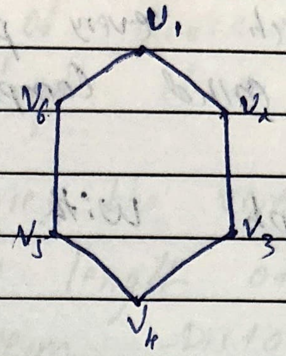
Cubic

G_2 =



G_1 & G_2 are Isomers.

Task:-



$2+2+2+2+2+2 = \underline{12}.$

Degree Sum Formula.

Theorem:

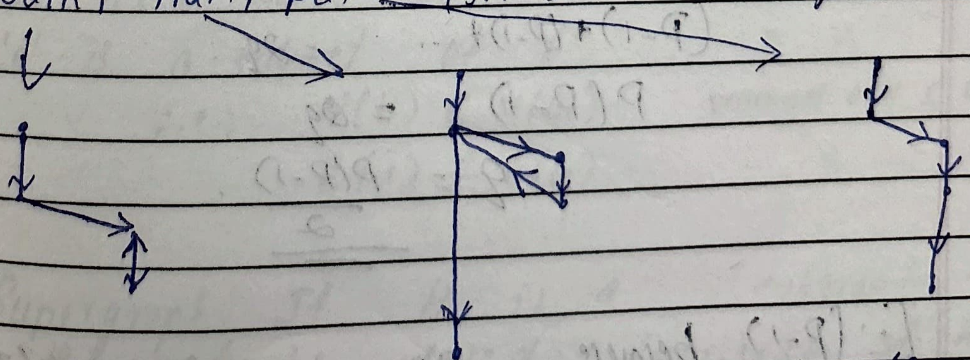
Let $G(V,E)$ be a (p,q) graph. The sum of the degrees of the vertices of G is twice the number of edges, i.e.:

$$\sum_{v\in V} deg(v) = 2q.$$

Proof:

Since every edge is incident on 2 vertices.

Walk, Trail, path. (sir showed 3 diagram of W T P)



Trial: A walk in which all edges are distinct.

Path: A walk in which all vertices are distinct.

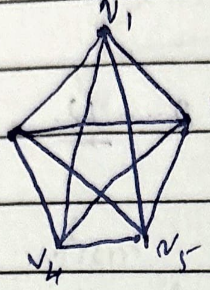
Cycle: A closed walk in which all vertices are distinct.

DATE:

PAGE:

A graph in which every pair of points are adjacent is called complete graph.

A complete graph with p vertices is denoted by K_p.



K_5

Result:

A complete graph with p vertices has $\frac{p(p-1)}{2}$.

$$\left[\because {}^{P}C_2 = \frac{P!}{(P-2)!\,2!} = \frac{P(P-1)\cancel{(P-2)!}}{\cancel{(P-2)!}\,2!} = \frac{P(P-1)}{2!} \right]$$

Proof:

$$\sum_{i=1}^{P} \text{degree}(v_i) = 2q.$$

$$(P-1)+(P-1)+\ldots = 2q.$$

$$P(P-1) = 2q$$

$$\therefore q = \frac{P(P-1)}{2}.$$

[$\because (P-1)$ because every vertex is connected to all other

$\because$ If there are P vertex then it will be $P(P-1)$.

The no. of occurances of edges in the walks - Length of a walk.

The distance b/w vertices u & v in a graph G is the length of a shortest path joining them - Distance b/w two vertices. If no such path exists, the distance is defined to be ∞.

Connectivity

Two vertices u & v are said to be connected if there existed a path b/w them in G

Connected vertices.

Connected graph: A graph is connected if every pair of vertex is connected by a path.

Subgraph: Let G(V, E) be a graph. A graph H is called a subgraph of G if ~~G'~~(v, e) where v ~~s~~ is a subset of 'V' & 'e' is a subset of 'E'

i.e: $V(H) \subseteq V(G)$: vertex set of H $\subseteq$ vertex set of G

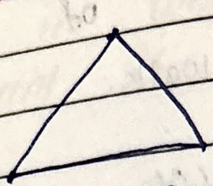
$E(H) \subseteq E(G)$

Supergraph: If H is a subgraph of G then G is called supergraph of H.

Spanning subgraph: A subgraph H of G such that V(H) ~~E~~ = V(G).

~~Sd~~

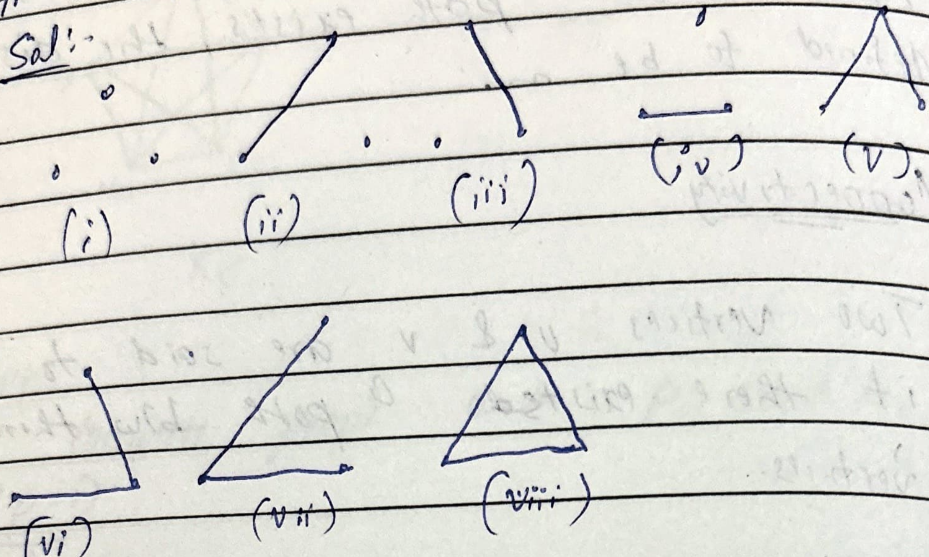
G: .



the graph C_3

Find spanning subgraph of C_3.

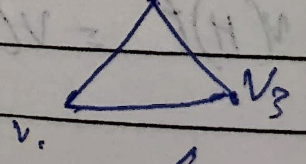
Sol:



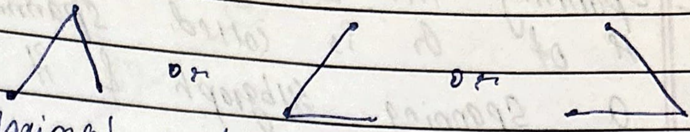
Removal of a point v: Let G be a graph & $v \in V(G)$, the subgraph $H = G - v$ consists of all points of G except v & all lines of G except those lines incident with v.

H Subgraph H' is a subgroup of G if H satisfies Property P (H) which is not satisfied by H'. Maximal Property.

Ex: P: Subgraph has no cycle.
Draw a maximal subgraph with property P.



Sol 1:



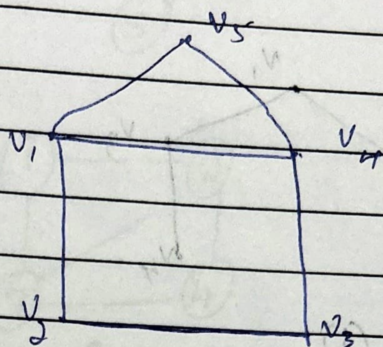
Maximal acyclic subgraph.

Induced subgraph

$V(H) = S$ &

H is not a subgraph of any supergraph H'

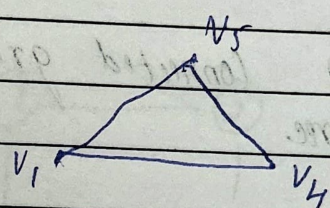
Ex:



$S = v_1, v_4, v_5$.

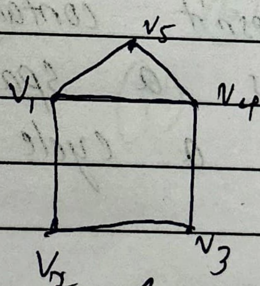
Draw induced subgraph of G with set S.

Sol:-



27/08/25

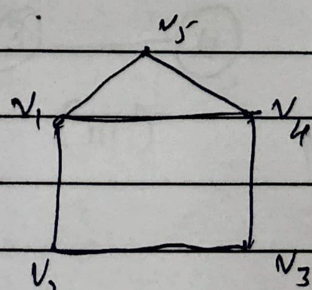
(i) Ex



G

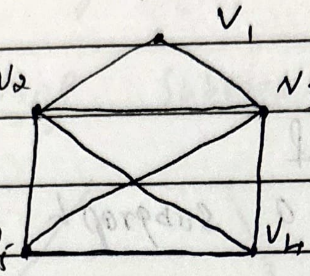
$S = \{v_1, v_2, v_3, v_4, v_5\}$. Draw induced subgraph

Sol:-

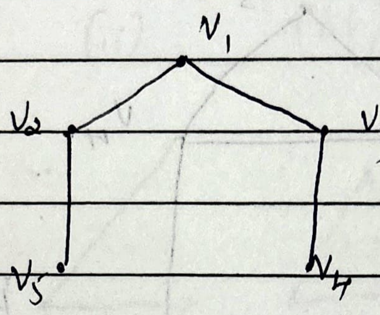


Imp: Spanning tree: Let G be a graph, a subgraph H of G is called spanning tree, if H is a spanning subgraph & it is a tree.

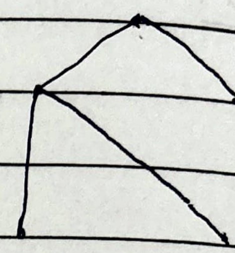
Ex:- Consider G:



Sol:-



(i)



If G is a connected graph then it contains a spanning tree

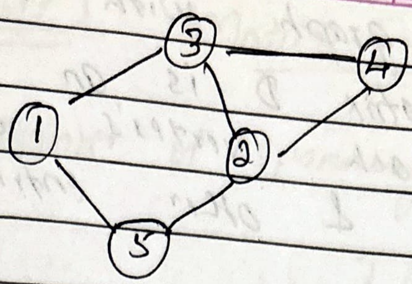
Proof:

(i) Case 1: If G doesn't contain any cycle, since it is connected, it is a spanning tree.

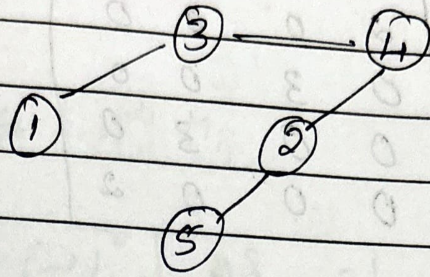
(ii) Case 2: If G has a cycle

DATE: PAGE:

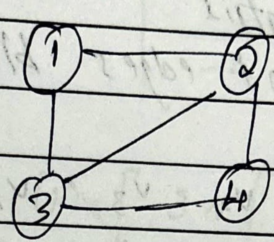
Given:



Soln:-

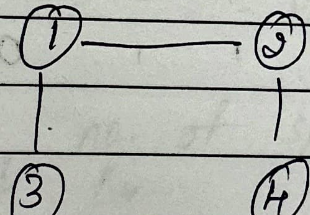


Given:

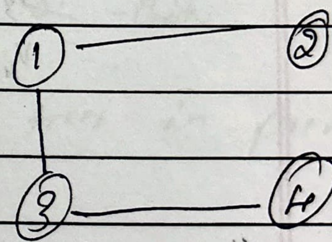


G

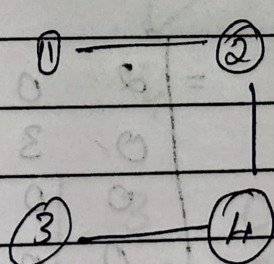
Sol:



(i)



(ii)



(iii)

Let G be a graph with n vertices. The degree matrix D is an $n \times n$ diagonal matrix with each vertex's degree on the diagonal & other entries are zero.

$$D_{4\times 4} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The adjacency matrix A is an $n \times n$ matrix where A_{ij} = no. of edges b/w vertices v_i & v_j

$$A_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

"Laplacian Matrix, for $G = (V, E)$

$$L_G := D - A$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Kirchhoff's Matrix-Tree Theorem.

The number of spanning trees in G is

$$\tau(G) = C_{i,j}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

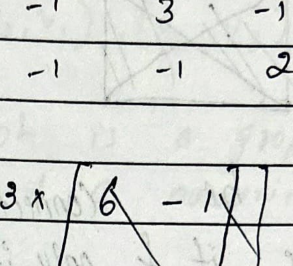
$$= (1) \times \left[3 \times (6-1) + 1(-2-1) - 1 \times (1+3) \right]$$

$$= 15 - 3 - 4 = \underline{8}$$

= ~~15~~ – Answer is 8. ~~check~~

Determine no. of spanning trees in previous graphs & K_4.

Sol: K_4 =



$$D_{4\times4} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

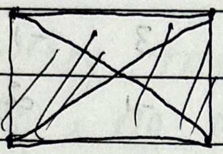
$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \quad C_{11} = (1)^2 \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

$$C_{11} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

$$= \cancel{3}\left[3(9-1) + 1(-3-1) - 1(1+3)\right]$$

$$= 3(8) - 4 - 4$$

$$= \cancel{22}\ \underline{\underline{16}}$$



Let G be Connected Graph

Note: G is tree if & only if there is unique path b/w any two vertices

Proof:

Given that G is a tree i.e it doesn't have cycle & connected.

1) There is a path because it is connected
2) If there are 2 paths b/w u & v then there is a cycle which contradicts the definition of graph.

∴ It should have unique path.

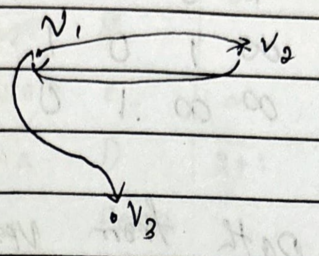
// Now Reverse

1)

A directed graph or digraph D consists of
- a finite nonempty set V of points together with
- a prescribed collection E of order pairs of distinct points.

The elements of E are directed lines or arcs.

Ex:-



Sol:-

$V = \{v_1, v_2, v_3\}$.

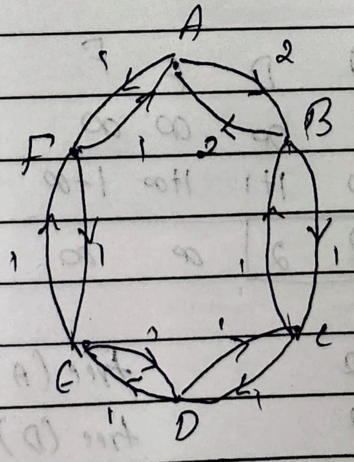
$E = \{(v_1, v_2), (v_2, v_1), (v_1, v_3)\}$

A weighted graph is a graph where each edge is assigned a numerical value, called weight.

Finding shortest path from given point to all other points.

Distance matrix $D = d_{ij}$
- $d_{ij} = 0$ if $i = j$
- $d_{ij} = \infty$ if no edge
- d_{ij} = weight of edge from v_i to v_j.

Ex:-



$$D = \begin{array}{c|ccccc} & A & B & C & D & F \\ \hline A & 0 & 2 & \infty & \infty & \\ B & 2 & 0 & 1 & \infty & \\ C & \infty & 1 & 0 & 1 & \infty \\ D & \infty & & & 0 & \\ F & & & & & \end{array}$$

DATE: PAGE:

$$D = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 2 & \infty & \infty & \infty & 1 \\ 2 & 0 & 1 & \infty & \infty & \infty \\ \infty & 1 & 0 & 1 & \infty & \infty \\ \infty & \infty & 1 & 0 & 1 & \infty \\ \infty & \infty & \infty & 1 & 0 & 1 \\ 1 & \infty & \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$

Q1: Find the shortest path from vertex B to all other vertices.

= Step 1: Find distance matrix D associated with v

Step 2: Define 2 sets K & V

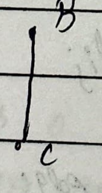
$K = \{B\}$ $V = \{A, C, D, E, F\}$

↓ Known distance   ↓ Unknown distance

	A	C	D	E	F
B	2	1	∞	∞	∞

Best distance(C) $\boxed{1}$

i.e



tree(C) = B

Now $K = \{B, C\}$, $V = \{A, D, E, F\}$.

	A	D	E	F
B	2	∞	∞	∞
Via C	1+∞	1+1	1+∞	1+∞
~~Best d(A)~~	$\boxed{2}$	$\boxed{2}$	∞	∞

— Minimum among

Best d(A) = 2   tree(A) = B

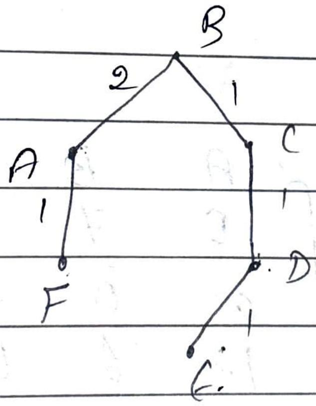
Best d(D) = 2   tree(D) = C

Now $K = \{A, B, C, D\}$ $U = \{E, F\}$

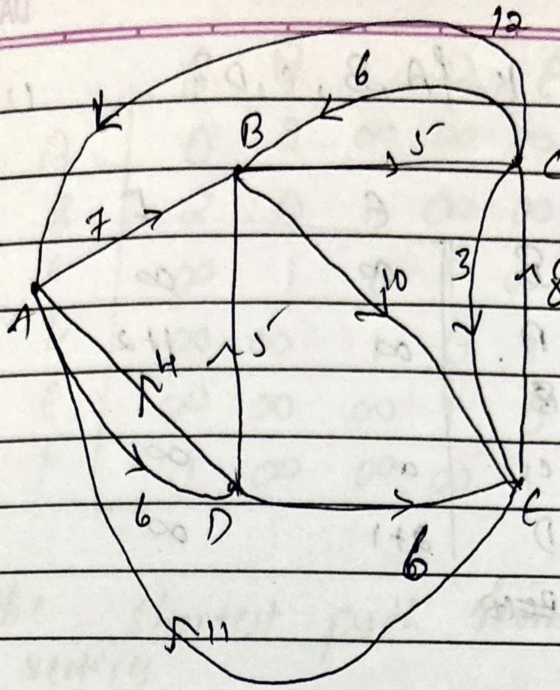
		E	F
	B	∞	∞
via	A	∞	1+2
~~via~~	~~B~~		
via	C	∞	∞
via	D	2+1	∞
~~via~~	~~BE(D)~~		
~~via~~			
min		3	3

Best $d(E) = 3$, $tree(E) = D$

Best $d(F) = 3$ $tree(F) = A$



2]



V = B.

<u>Sol:</u>

$$\cancel{D = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{cccc} A & B & C & D \\ 0 & 7 & \infty & 6 \\ \infty & 0 & 5 & \\ & & 0 & \\ & & & 0 \\ & & & \end{array}}$$

$$D = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 7 & \infty & 6 & \infty \\ B & \infty & 0 & 5 & \infty & 10 \\ C & 12 & 6 & 0 & \infty & 3 \\ D & 4 & 5 & \infty & 0 & 6 \\ E & 11 & \infty & 8 & \infty & 0 \end{array}$$

$K = \{B\}$ $\quad U = \{A, \cancel{B}, C, D, E\}$.

	A	C	D	E
B	∞	5	∞	10
min		5		

min | best $d(C) = 5$. tree $(C) = B$

$K = \{B, C\}$, $U = \{A, D, E\}$.

		A	D	E
	B	∞	∞	10
via	C	5+12	5+∞	5+3
min				8.

best_d(E) = 8 tree(E) = C.

$K = \{B, C, E\}$, $U = \{A, D\}$.

	A	D
B	∞	∞
C	5+12	5+∞
E	8+11	8+∞

min 17

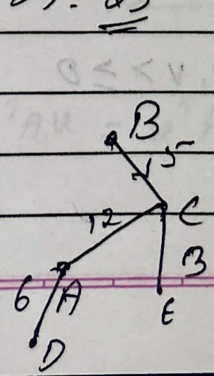
best_d(A) = 17 tree(A) = C.

$K = \{A, B, C, E\}$, $U = \{D\}$.

	D
B	∞
A	17+6
C	5+∞
E	8+∞

min = 23.

best_d(D) = 23 tree(D) = A.



	A	B	C	D	E
best d	17	0	5	23	8
tree	C	–	B	A	C

DATE: PAGE:

Chapter 2

Let u & v be any 2 vectors in R^n.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \& \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then the dot product of u & v is

$$u \cdot v = u^T v = [u_1\, u_2 \ldots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Properties:

$(u+v) \cdot w = u \cdot w + v \cdot w$

$(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

$u \cdot v = v \cdot u$

Inner Product

$$\langle \cdot , \cdot \rangle : R^n \times R^n \to R.$$

that satisfies the following..

(i) $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$

(ii) $\langle u, v \rangle = \langle v, u \rangle$

(iii) $\langle v, v \rangle \geq 0$ & $\langle v, v \rangle = 0 \iff v = 0$.

Dot product is an example of inner product.

Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Define :-

$$\langle u, v \rangle_A = u^T A v = 2u_1v_1 + 3u_2v_2$$

Sol :-

$$u^T = [u_1 \quad u_2].$$

$$u^T A v = [u_1, u_2] \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= [u_1 \quad u_2] \begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix}.$$

$$= 2u_1v_1 + 3u_2v_2.$$

Proof :-

Property 2 : $\langle u, v \rangle = \langle v, u \rangle$

$$v^T A u = [v_1, v_2] \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= [v_1, v_2] \begin{bmatrix} 2u_1 \\ 3u_2 \end{bmatrix}.$$

$$= 2u_1v_1 + 3u_2v_2.$$

$$\Rightarrow v^T A u = u^T A v$$

Property 3:

$$\langle u, u \rangle = u^T A u$$

$$= [u_1 \; u_2] \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= [u_1 \; u_2] \begin{bmatrix} 2u_1 \\ 3u_2 \end{bmatrix}$$

$$= 2u_1^2 + 3u_2^2$$

Which is +ve.

$$\langle v, v \rangle = v^T A v$$

$$= [v_1 \; v_2] \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= [v_1 \; v_2] \begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix}$$

$$= 2v_1^2 + 3v_2^2.$$

$\cancel{3u}$

Which is +ve.

Property 1:

$$\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$$

$$\cancel{= \alpha [u^T A w]}$$

$$= (\alpha u + v)^T A w$$

$$= (\alpha u^T + v^T) A w.$$

$$= \alpha u^T A w + \cancel{\alpha} v^T A w.$$

$$= \alpha (u, w) + (v, w)$$

DATE: PAGE:

(2) Compute $u \cdot v$ & $v \cdot u$ ~~from~~ for $u = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ & $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

Sol:-

$$u \cdot v = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}.$$

$$= 6 + (-10) - 3$$

$$= -7.$$

$$v \cdot u = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= 6 - 10 - 3 = -7.$$

Norm on R^n

A norm on R^n is a function.

$$\|\cdot\| : R^n \to R$$

that assigns to each vector $v \in R^n$ a nonnegative real no. & satisfies the following properties for all $u, v \in R^n$ & $\alpha \in R$.

(i) Positivity

$$\|v\| \geq 0, \quad \|v\| = 0 \Leftrightarrow v = 0.$$

(ii) $\|\alpha v\| = |\alpha| \|v\|$ – Homogeneity

(iii) $\|u + v\| \leq \|u\| + \|v\|$ – Triangle inequality

(1) $\|\cdot\| : R^2 \to R$

function: $\|a\| = \sqrt{a \cdot a}$ $\quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

Check whether the function is norm?

Sol:-

(i) $\|v\| \geq 0.$ $\quad a_1^2 + a_2^2$

$a \cdot a = a_1^2 + [a_1 \; a_2] \times [a_1 \; a_2]$

$\|a\| = \sqrt{a_1^2+a_2^2} \geq 0.$

which is +ve

(i) when $\|a\| = 0.$

$\Rightarrow \sqrt{a_1^2+a_2^2} = 0$

$\Rightarrow a_1 = 0.$ $\Rightarrow a = 0$

$a_2 = 0$

(ii) $\|\alpha a\| = \sqrt{\alpha^2 a_1^2 + \alpha^2 a_2^2}$

$\alpha a_1 \quad \alpha a_1$

$\alpha a_2 \quad \alpha a_2$

$\alpha^2 a_1^2 + \alpha^2 a_2^2$

$= \sqrt{\alpha^2 (a_1^2 + a_2^2)}$

$= \alpha \sqrt{a_1^2+a_2^2} = \alpha \|a\|$

~~||v||~~ Unit vector & norm are same.

$v = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1/\sqrt{3} \end{bmatrix}$

$||v|| = \sqrt{v \cdot v} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \underline{\underline{1}}$.

(2) $v = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

Sol:-

$||v|| = \sqrt{5}$

So normalize $= \frac{v}{||v||} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$

Now $\left|\left| \frac{v}{||v||} \right|\right| = \cancel{\frac{1}{5} + \frac{4}{5} + \frac{4}{5}} \frac{1}{\sqrt{5}} \times (\sqrt{5}$

$= \underline{\underline{1}}$.

An $m \times n$ matrix U has orthonormal column if & only if $U^T U = I$. If a matrix has orthonormal columns, then the matrix is orthogonal matrix.

Gram-Schmidt orthonormalization Process.

$$A = [a_1 | a_2 | \dots | a_n]$$

$$u_1 = a_1, \quad e_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = a_2 - (a_2 \cdot e_1)e_1, \quad e_2 = \frac{u_2}{\|u_2\|}$$

$$u_3 = a_3 - (a_3 \cdot e_1)e_1 - (a_3 \cdot e_2)e_2, \quad e_3 = \frac{u_3}{\|u_3\|}$$

$$\vdots$$

Eigenvalue & Eigenvectors

A real no. λ is an eigenvalue of an $n \times n$ square matrix A if there exists a nonzero vector $x \in R^n$ such that

$$Ax = \lambda x.$$

Ex:

$$\underbrace{\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}}_{\substack{3\times 3 \\ A}} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\substack{3\times 1 \\ v}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$Av = \lambda v.$$

Here $\lambda = 2$.

λ is an eigenvalue if determinant $(A - \lambda I) = 0$

DATE: PAGE:

Ex)- $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{bmatrix}$$

determinent of above $= (2-\lambda)\left[(2-\lambda)^2 - 0\right] + 2\left(-2(2-\lambda)\right)$

$= (2-\lambda)(2-\lambda)^2 - 4(2-\lambda)$

$= (2-\lambda)\left((2-\lambda)^2 - 4\right)$

$= (2-\lambda)(\cancel{4} + \lambda^2 - 4\lambda - \cancel{4})$

$= (2-\lambda)(\lambda^2 - 4\lambda)$.

$= (2-\lambda)(\lambda(\lambda-4))$

$= (2-\lambda)(2-\lambda)^2 - 4(2-\lambda)$.

$= (2-\lambda)\left[(2-\lambda)^2 - 4\right]$

$\Rightarrow$ $2-\lambda = 0$

or $(2-\lambda)^2 - 4 = 0$.

$\lambda = 2$ or $(2-\lambda)^2 = 4$

$2-\lambda = \pm 2$.

$\lambda = 0$ or $\lambda = 4$

DATE: PAGE:

Definition of Singular values.

$$Av = \lambda u \quad \& \quad A^T u = \lambda v.$$

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then

$$A^TA = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

A^TA eigen values:

det of $(\cancel{A^TA})$ A.

$$A = (A^TA - \lambda I)$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det(A) = \left|\begin{bmatrix} (5-\lambda) & 4 \\ 4 & (5-\lambda) \end{bmatrix}\right|$$

$$= (5-\lambda)^2 - 16.$$

$(5-\lambda)^2 - 16 = 0$

$5-\lambda^2 = \pm 4$

$5-\lambda = +4$ or $5-\lambda = -4$

$\lambda = 1$ or $\lambda = 9$..

Eigen values, $\lambda \in \{1, 9\}$.

Singular values $= \{\sqrt{1}, \sqrt{9}\}$
$= \{1, 3\}$.

(2) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

Sol:

$$AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}.$$

$$A = AA^T - \lambda I$$

$$= \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 6 \\ 6 & 12-\lambda \end{bmatrix}.$$

$$\det(A) = (3-\lambda)(12-\lambda) - 36$$

$$= \cancel{36} - 3\lambda - 12\lambda + \lambda^2 - \cancel{36}$$

$$0 = -15\lambda + \lambda^2$$

$$0 = \lambda(-15+\lambda).$$

$$\lambda = 0$$

or $\lambda = 15$

$$\therefore \lambda \in \{0, 15\}.$$

Singular values $= \{0, \sqrt{15}\}$.

DATE: PAGE:

The Singular value Decomposition.

$$A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^T_{n\times n}$$

① Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{2\times 3}$$

Sol:-

(i) Step 1: Determine the largest order matrix b/w AA^T & A^TA & find it's eigen values & singular values of A.

AA^T — 2×2

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}_{3\times 3} \rightarrow A$$

$$|A^TA - \lambda I| = 0$$

$$0 = \left| \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$0 = \begin{vmatrix} (2-\lambda) & 0 & 2 \\ 0 & (2-\lambda) & 0 \\ 2 & 0 & (2-\lambda) \end{vmatrix}$$

$$0 = (2-\lambda)\left[(2-\lambda)^2\right] + 2(-2(2-\lambda))$$

$$0 = (2-\lambda)(2-\lambda)^2 - 4(2-\lambda).$$

No. $\lambda = n = 3$.

DATE: PAGE:

$$0 = (2-\lambda)\left[(2-\lambda)^2 - 4\right]$$

$2-\lambda = 0$ or $(2-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda = 2$

$(2-\lambda)^2 = 4$

$2-\lambda = \pm 2$

$2-\lambda = 2$ or $2-\lambda = -2$

$\lambda = 0$ or $\lambda = 4$.

eigenvalues are $\{4, 2, 0\}$.

Singular values are $\{2, \sqrt{2}, 0\}$.

Step 2: Find eigenvectors of A^TA corresponding to each eigenvalue.

$\lambda = 4$:

Here $A = A^TA$

w.k.t. $AV = 4V$

$AV - 4IV = 0$

$(A-4I)V = 0$

$\boxed{AV = \lambda V}$

Let, $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A-4I)V = 0$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + 2z = 0$ — ①

$-2y = 0$ — ②

$2x - 2z = 0$ — ③

$y = 0$.

$2x = 2z$ from ①

$\Rightarrow x = z$.

$\cancel{2x = 2x \Rightarrow 2}$

DATE: PAGE:

So $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ norm $= \sqrt{1^2 + 0^2 + 1^2}$

$= \sqrt{2}$

This is not unit vector or norm $\neq 1$.

So $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$.

$\lambda = 2$:

$Av = 2v$

$Av - 2Iv = 0$

$v(A - 2I) = 0$

$$\left[\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right] v = 0.$$

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-2z = 0$

$2x = 0$

$z = 0$

$x = 0$

y we can choose. Make sure we norm it.

So we select $y = 1$.

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 0$

$\cancel{*}$ $Av = 0$.

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$2x + 2\cancel{y}^{z} = 0$

$2y = 0$

$2x + 2z = 0$.

$y = 0$.

$\cancel{2}x = -\cancel{2}z$

$x = -z$.

$v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ So normalize $v_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$$\therefore \quad V = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \end{array}$$

Step 3: To find matrix U. $u_1 = \dfrac{Av_1}{\|Av_1\|}$;

$u_2 = \dfrac{Av_2}{\|Av_2\|}$

Now A is original matrix.

$$AV_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

$$\|AV_1\| = \sqrt{4/2 + 4/2}$$

$$= \sqrt{8/2}$$

$$= \sqrt{4}$$

$$= 2.$$

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

$$AV_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|AV_2\| = \sqrt{1+1} = \sqrt{2}.$$

$$u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

DATE: PAGE:

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

2×3

$$\text{Final} = \underset{U}{\begin{array}{cc} u_1 & u_2 \\ \hline \end{array}}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}_{2\times2} \overset{\Sigma}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}^T$$

($v_1 \quad v_2 \quad v_3$ over the columns of the last matrix)

diagonal elements are singular values

(2) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

<u>Sol:-</u>

$$A^TA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(ii) $|A^TA - \lambda I| = 0$

$$0 = \left|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right|$$

$$0 = \left|\begin{bmatrix} (1-\lambda) & 0 & 0 \\ 0 & (1-\lambda) & 0 \\ 0 & 0 & (4-\lambda) \end{bmatrix}\right|$$

$$0 = (1-\lambda)\left[(1-\lambda)(4-\lambda)\right]$$

$$0 = (1-\lambda)^2(4-\lambda).$$

$\Rightarrow$ $(\lambda-1)^2 = 0$ or $4-\lambda = 0$

$\lambda - 1 = 0$ or $\underline{\underline{\lambda = 4}}$.

$\underline{\underline{\lambda = 1}}$

eigenvalues are $\{4, 1, 1\}$.

Singular values are $\{2, 1, 1\}$.

(iii) (a) $Av = \lambda v$ Here $A = A^T A$.

$Av = 4v$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$Av - 4Iv = 0.$

$(A - 4I)v = 0$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

DATE: PAGE:

$-3x = 0$

$-3y = 0$

$\cancel{z = 0}$ z we can take anything so $z = 1$

$x = 0,\ y = 0,\ z = 1.$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) $Av = 1v$

$(A - 1I)v = 0$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) v = 0.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Because no eq. of x & y so we can consider $x = 1$ and $y = 1$.

$3z = 0$

$z = 0.$

$x = t$

$y = t$

$$\cancel{v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$ so normalize $$\cancel{v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}$$

$\cancel{\|v_2\| = \sqrt{2}}$

$$\cancel{v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}$$

P.T.O.

Let $x = t$ & $y = s$.

i.e $\begin{bmatrix} t \\ s \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

This we'll consider v_2 — This we'll consider v_3,

so $V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

$$U_1 = \frac{Av_1}{\|Av_1\|}$$

$$Av_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

$$U_1 = \frac{Av_1}{\|Av_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U_2 = \frac{Av_2}{\|Av_2\|}$$

$$Av_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\|Av_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$u_3 = \frac{Av_3}{\|Av_3\|}$$

Consider

$$Av_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_3 = \frac{Av_3}{\|Av_3\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

DATE: PAGE:

3] $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times 2}$

Sol:-

$$A \cdot A^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}_{3\times 3}$$

$-3\times 6 + (-2)$

-20

$36+4$

$-18-2$

$-18-2=-20$

$|A \cdot A^T - \lambda I| = 0$

Not required.

$$\left| \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| =$$

$$\left| \begin{bmatrix} 10-\lambda & \cancel{0}\ -20 & -20 \\ -20 & \cancel{\lambda-40}\ 40-\lambda & 40 \\ -20 & 40 & 40-\lambda \end{bmatrix} \right| = 0.$$

$(10-\lambda)(40-\lambda)^2 + 20(-20\times(40-\lambda) + 80)$
$- 20(-80 + 20\times(40-\lambda)) = 0.$

$(10-\lambda)(40-\lambda)^2 - 400(40-\lambda) + 160$
$+ 160 - 400(40-\lambda) = 0$

$$0 = (10-\lambda)(40-\lambda)^2 - 400(40-\lambda) + \cancel{160}\, 320 - 400(40-\lambda)$$
not
$$0 = (40-\lambda)\left((10-\lambda)(40-\lambda) - 400 - 400\right) + 320.$$

Note:- Sum of diagonal values = Sum of eigen value.
No. of zero's as eigen values = n - rank of matrix.
To find rank we have to see how many rows are related.

Ex:- In our example we can write

$$\begin{bmatrix} R_1 \\ -2R_1 \\ -2R_1 \end{bmatrix}$$

So eigen values $= \{90, 0, 0\}$ ∵ refer above note.

Singular values $= \{\sqrt{90}, 0, 0\}$.

Step 2:

$$Av_1 = \lambda v_1$$
$$(A - 90I)v = 0.$$

$$\left[\begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} - \begin{bmatrix} 90 & 0 & 0 \\ 0 & 90 & 0 \\ 0 & 0 & 90 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$\begin{bmatrix} -80 & -20 & -20 \\ -20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$-80x - 20y - 20z = 0$. — ①
$-20x - 50y + 40z = 0$ — ②
$-20x + 40y - 50z = 0$. — ③

from ② & ③.

$-90y + 90z = 0$
$z - y = 0$
$z = y$ — ④

from ② & ③ & ①
$\cancel{-20x - 50z + 40z = 0}$
$\cancel{-20x + 40z - 50z = 0}$

DATE: PAGE:

Let $y = 1$ & $z = 1$

$-20x + 40 - 50 = 0$

$-20x - 10 = 0$

$2x + 1 = 0$

$2x = -1$

$x = -1/2$

$\sqrt{\frac{1}{4} + 1 + 1}$

$\sqrt{\frac{1}{4} + 2}$

$\sqrt{9/4} = 3/2$

$$u_1 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

So normalize.

$$u_1 = \begin{bmatrix} -1/2 \times 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = 1.$

$u_2 =$ $Au = \lambda u$

$(A - \lambda I)u = 0$

$$\left(\begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) u_2$$

$$\begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$10x - 20y - 20z = 0$ — ①
$-20x + 40y + 40z = 0$ — ②
$-20x + 40y + 40z = 0$ — ③

2 Solutions,

$$x_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 \cdot x_2 = 0$ then it is orthogonal.

$$U_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$U_3 = a_3 - (a_3 \cdot e_1) e_1 - (a_3 \cdot e_2) e_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right] \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} - (a_3 \cdot e_2) e_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \underset{3\times1}{\begin{bmatrix} -2/3 \\ 2/3 \\ 0 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}} - (a_3 \cdot e_2) e_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{2}{3} + \frac{2}{3} + 0 \end{bmatrix} e_1 - (a_3 \cdot e_2) e_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \right] \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left[\frac{4}{\sqrt{5}} \right] \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 8/5 \\ 0 \\ 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 8/5 \\ 1 \\ -4/5 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 1 \\ -4/5 \end{bmatrix}$$

$$\frac{4}{25} + 1 + \frac{16}{25} = \frac{4 + 25 + 16}{25} = \frac{\cancel{45}^{\,9}}{\cancel{25}_{\,5}}$$

$$\sqrt{9/5} = \frac{3}{\sqrt{5}} \qquad \sqrt{\frac{4}{25} + 1 + \frac{16}{25}} = \frac{\cancel{9}}{\cancel{25}}$$

$$\frac{4}{9 \times 5} + \frac{25}{9 \times 5} + \frac{16}{9 \times 5} \qquad \sqrt{\frac{4 + 25 + 16}{25}} = \frac{\sqrt{45}}{\cancel{25}}$$

$$= \frac{3\sqrt{5}}{5}$$

$$\therefore \; U_3 = \frac{1}{\frac{3\sqrt{5}}{5}} \begin{bmatrix} 2/5 \\ 1 \\ -4/5 \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}.$$

$$\frac{2/5}{\frac{3}{\sqrt5}} \quad \frac{-4}{5} \times \frac{\sqrt5}{3}$$

DATE: PAGE:

// Just trying: $3/\sqrt5$ ·

$$\frac{2\sqrt5/15}{\cancel{6\sqrt5}} + \sqrt5/3 + \frac{-4\sqrt5}{15}$$

$u_1 = a_1$

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{4\times5}{225} + \frac{5}{9} + \frac{16\times5}{225}$$

5

$$u_2 = a_2 - \langle a_2, e_1 \rangle e_1$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt5 \\ 1/\sqrt5 \\ 0 \end{bmatrix}\right] \times \begin{bmatrix} 2/\sqrt5 \\ 1/\sqrt5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/\sqrt5 \end{bmatrix} \times \begin{bmatrix} 2/\sqrt5 \\ 1/\sqrt5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/\cancel{\sqrt5}5 \\ 4/\cancel{\sqrt5}5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

$$\cancel{= \begin{bmatrix} 2-8/\sqrt5 \\ -4/\sqrt5 \\ 1 \end{bmatrix}}$$

$$\cancel{\|u\| = \sqrt{4 + \frac{64}{5} + \frac{16}{5} + 1}}$$

$$\cancel{= \sqrt{4 + \frac{80}{5} + 1}}$$

$$\cancel{= \sqrt{4 + 15 + 1} = \sqrt{20}}$$

$$\cancel{= \sqrt{5\times4}}$$

$$\cancel{= 2\sqrt5}$$

$$\cancel{\begin{bmatrix} \frac{2}{2\sqrt5} - \frac{8}{2\sqrt5 \times \sqrt5} \end{bmatrix}}$$

$$\sqrt{\frac{4}{25} + \frac{16}{25} + \frac{25}{25}} = \frac{\sqrt{45}}{\cancel{25}5}$$

$$= \frac{3\sqrt5}{5}$$

Now normalize.

$$u_2 = \frac{1}{3\sqrt5}\begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

DATE: PAGE:

$$U = \begin{bmatrix} -1/2 & 2/\sqrt{5} & \cancel{2/\sqrt{5}} \; 2/3\sqrt{5} \\ 1 & 1/\sqrt{5} & \cancel{-4/\sqrt{5}} \; -4/3\sqrt{5} \\ 1 & 0 & \cancel{1} \; 5/3\sqrt{5} \end{bmatrix}.$$

$$v_1 = \frac{A^T u_1}{\|A^T u_1\|}$$

$A^T u_1$

$$u_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} \quad A^T u_1 = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix}_{2\times 3} \cdot \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}_{3\times 1}.$$

$$= \begin{bmatrix} -1-4-4 \\ \frac{1}{3}+\frac{4}{3}+\frac{4}{3} \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} -9 \\ 3 \end{bmatrix}$$

$$\sqrt{81+9} = \sqrt{90} = \sqrt{10\times 9} = 3\sqrt{10}.$$

$$v_1 = \begin{bmatrix} -9/3\sqrt{10} \\ 3/3\sqrt{10} \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}.$$

DATE: PAGE:

w.k.T $v_1 \cdot v_2 = 0$.

$$v_1 \cdot v_2 = v_1^T v_2$$

$$0 = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$0 = -3/\sqrt{10}\, a + b/\sqrt{10}$$

$$0 = \frac{-3}{\sqrt{10}} a + \frac{b}{\sqrt{10}}$$

$$\Rightarrow \quad a = 1 \quad b = 3$$

$$\Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Normalize

$$v_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$Q.Q^T = I$ then orthogonal

QR decomposition.

Given a matrix A of size $m \times n$, we decompose A as $A = QR$ where
Q is orthogonal matrix. i.e $Q^T Q = I$
& R is upper triangular matrix

$$A = [a_1 | a_2 | \dots | a_n] = [e_1 | e_2 | \dots | e_n] \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & \dots & a_n \cdot e_1 \\ 0 & a_2 \cdot e_2 & \dots & a_n \cdot e_2 \\ \vdots & & & \\ 0 & 0 & \dots & a_n \cdot e_n \end{bmatrix}$$

$= QR.$

① Find QR decomposition of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Sol:-

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$u_2 = a_2 - (a_2 \cdot e_1) e_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right] \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \cancel{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \times \frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$+\frac{1}{2} - \frac{1}{6} = +\frac{3}{6} - \frac{1}{6} = \frac{4}{6}.$

DATE: PAGE:

$$e_2 = \begin{bmatrix} \frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{3}} \\ 1 \times \sqrt{2}/\sqrt{3} \end{bmatrix}$$

$\sqrt{\frac{1}{4} + \frac{1}{4} + 1}$

$\sqrt{\frac{1}{2} + 1}$

$\sqrt{\frac{3}{2}}$

$= \frac{\sqrt{3}}{\sqrt{2}}.$

$$= \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix} \quad \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

$-\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$

1

$$u_3 = a_3 - (a_3 \cdot e_1)e_1 - (a_3 \cdot e_2)e_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\left[\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right] \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}_{x} - \underbrace{\left[\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}\right] \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}}_{y}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right] - y.$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} - \left[\frac{-1}{\sqrt{2}\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\right] \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} - \left[\frac{1}{\sqrt{3}}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}\right]$$

$$= \begin{bmatrix} -1/2 \\ \frac{1}{2} \\ 1 \end{bmatrix} - \left[\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}\right]$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -4/6 \\ 4/6 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

DATE: PAGE:

$$e_3 = \frac{1}{2\sqrt{3}}\begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{\frac{12}{9}} = 1$$

$\frac{12}{9}$

$\frac{\sqrt{12}}{3}$

$2\sqrt{3}$

$$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{bmatrix}$$

$$a_1 \cdot e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$a_2 \cdot e_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$a_3 \cdot e_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$a_2 \cdot e_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$

$$a_3 \cdot e_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \frac{-1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}$$

$$a_3 \cdot e_3 = \begin{bmatrix}0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$= 0 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$R = \begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 3/\sqrt{6} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$$

$A = Q.R$

$$= \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 3/\sqrt{6} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$$

$$= \begin{bmatrix}1 & \frac{1}{2}+\frac{3}{2} & \frac{1\times3}{2\times3} - \frac{1}{6} - \frac{2\times2}{3\times2}\\ 1 & \frac{1}{2} - \frac{3}{6} & \frac{1}{2} + \frac{1}{6} + \frac{2}{3}\\ 0 & \frac{6}{6}+0 & -\frac{2}{6}+\frac{2}{3}\end{bmatrix}$$

$\frac{1}{2} - \frac{1}{6} + \frac{2}{3}$

$3-1-4$

$$= \begin{bmatrix}1+0+0 & \frac{1\times3}{2\times3} + \frac{3}{6} + 0 & \frac{1\times3}{2\times3} + \frac{1}{6} - \frac{2\times2}{3\times2}\\ 1+0+0 & \frac{1\times3}{2\times3} - \frac{3}{6} + 0 & \frac{1\times3}{2\times3} - \frac{1}{6} + \frac{2\times2}{3\times2}\\ 0+0+0 & 0+\frac{6}{6}+0 & 0-\frac{2}{6}+\frac{2\times2}{3\times2}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 0 - \cancel{2/6} = \cancel{-1/3}\\ 1 & 0 & \cancel{6/6} = 1\\ 0 & 1 & 1 \quad \cancel{2/6} = 1/3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$$

DATE: PAGE:

(2) Find QR decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Sol:-

$u_1 = a_1$

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ $\sqrt{1+1+0} = \sqrt{2}$

$u_2 = a_2 - (a_2 \cdot e_1) e_1$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right] \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \sqrt{2}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 1/2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$e_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \times \frac{\sqrt{2}}{\sqrt{3}}$ $\sqrt{\frac{1}{4} + \frac{1}{4} + 1}$

$\sqrt{\frac{1}{2} + 1}$ $\sqrt{3/2}$

Note: $Q^T A = R$

$\frac{1}{2} \times \frac{\sqrt{2}}{\sqrt{3}}$

$$e_2 = \cancel{\begin{bmatrix} \sqrt{2}/\sqrt{3} \\ -\sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}}$$

$$e_2 = \begin{bmatrix} 1/\sqrt{2}\sqrt{3} \\ -1/\sqrt{2}\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} a_1 e_1 & a_2 e_1 \\ 0 & a_2 e_2 \end{bmatrix}$$

$$a_1 \cdot e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$a_2 \cdot e_1^T = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$a_2 \cdot e_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \frac{3}{\sqrt{6}} = \frac{\sqrt{3} \times \cancel{\sqrt{3}}}{\sqrt{2} \times \cancel{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix}$$

DATE: PAGE:

3) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ Find eigenvalues & singular values

Sol:-

$$A \cdot A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 2 & 0+4+0 \\ 2+0+0 & 4+0+4 & 0+0+2 \\ 0+4+0 & 0+0+2 & 0+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 4 \\ 2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

$$|A \cdot A^T - \lambda I| = 0$$

$$\left| \begin{bmatrix} 5 & 2 & 4 \\ 2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (5-\lambda) & 2 & 4 \\ 2 & (8-\lambda) & 2 \\ 4 & 2 & (5-\lambda) \end{vmatrix} = 0.$$

$$0 = (5-\lambda)\left[(8-\lambda)(5-\lambda)-4\right] - 2\left[2(5-\lambda)-8\right]$$

$$+4\left[4-4(8-\lambda)\right]$$

DATE: PAGE:

$0 = (8-\lambda)(5-\lambda)^2 - 4(5-\lambda) - 4(5-\lambda) + 16$
$+ 16 - 16(8-\lambda)$

$0 = (5-\lambda)[(8-\lambda)(5-\lambda) - 4 - 4] + 32 - 16(8-\lambda)$
$0 = (5-\lambda)[(8-\lambda)(5-\lambda) - 8] + 32 - 16(8-\lambda)$

If $\lambda = 2$

$0 = (3)((6)(3) - 8) + 32 - 16(8-2)$
$= 3(10) + 32 - 16(6)$
$= 62 - 96$

If $\lambda = 1$

$0 = 4(7 \times 4 - 8) + 32 - 16(7)$
$=$

$Av = \lambda v$
$A(Av) = \lambda v$
$A^2 v = A(Av) = A(\lambda v)$
$= \lambda(Av) = \lambda(\lambda v) = \lambda^2 v.$

DATE:

PAGE:

Boundary Value Problems for Differential Equations

A differential eq is an equation involving an Unknown function & its derivatives with respect to independent variable..

Ex:-

$$\frac{dy}{dx} + y = 0. \qquad \text{— Ordinary differential eq}$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad = \quad ''$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{— partial differential eq}$$

(∵ 2 or more independent variable).

Laplace eq.: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Order & degree

Ex:- $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^{50} + y^2 = 0.$

Order = 2

degree = 1

Initial value problem

Ex:- $\frac{dy}{dx} = y$ when $y(0) = 1$

$\Rightarrow y = ce^x$

$y(0) = 1$

$1 = ce^0$

$c = 1$ $\Rightarrow y(x) = e^x.$

DATE: PAGE:

Finite difference method to solve BVP.

$$x\frac{d^2y}{dx^2} + y = 0$$

Type 1: $y(0) = 1$; $y(2) = 2$ – Dirichlet type

Type 2: $y'(0) = \alpha$; $y'(2) = \beta$ – Neumann type

Type 3: $\lambda_1 y(0) + \lambda_2 y'(0) = \alpha$, $\mu_1 y(2) + \mu_2 y'(2) = \beta$ } Mixed type

Consider the 2nd order linear ODE given by

$$p(x)\cdot\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = f(x)$$

with boundary conditions $y(a) = \alpha$ & $y(b) = \beta$

Sol:-

1. Our Domain is $[a, b]$
2. We split the domain say into n equal parts.

$$h = \frac{b-a}{n}, \quad x_i = a + ih$$

i.e $x_1 = a$

$x_2 = a + h$

$x_3 = a + 2h$

$\vdots$

$x_n = a + nh = a + n\times\left(\frac{b-a}{n}\right) = b.$

3. For each x_i we write our modelled eq.

i.e $$p(x_i)\frac{d^2y(x_i)}{dx^2} + q(x_i)\frac{dy(x_i)}{dx} + r(x_i)y(x_i) = f(x_i) \quad \text{(a)}$$

$1 \le i \le n-1.$

4. We expand the modelled eq. using central differentiation approximation

DATE: PAGE:

$$y_i'' = \frac{d^2y(x_i)}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad \text{--- (1)}$$

$$= \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \quad \text{--- (2)}$$

Substitute (1) & (2) in (a)

$$P_i\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + q_i\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + r_i y_i = f_i$$

$\forall \; 1 \leq i \leq n-1$

Multiply $2h^2$

$$2P_i\left[y_{i+1} - 2y_i + y_{i-1}\right] + q_i h\left[y_{i+1} - y_{i-1}\right] + 2h^2 r_i y_i = 2h^2 f_i$$

$$2P_i y_{i+1} - 4P_i y_i + 2P_i y_{i-1} + q_i h y_{i+1} - q_i h y_{i-1} + 2h^2 r_i y_i = 2h^2 f_i$$

$$(2P_i - q_i h)y_{i-1} + (2h^2 r_i - 4P_i)y_i + (2P_i + q_i h)y_{i+1} = 2h^2 f_i$$

$$\boxed{a\, y_{i-1} + b\, y_i + c\, y_{i+1} = d}$$

DATE: PAGE:

① Solve the following BVP using FDM

$xy'' + y = 0$; $y(1) = 1$ & $y(2) = 2$

Where $h = 0.5$

Sol:-

(i) Domain $[1, 2]$.

(ii) $x_0 = 1$

$x_1 = 1 + 1\times0.5 = 1 + 0.5 = 1.5$

$x_2 = 1 + 2\times0.5 = 2$.

(iii) $x_i\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + y_i = 0$

$i = 1$

$x_1\left[\frac{y_2 - 2y_1 + y_0}{1/4}\right] + y_1 = 0$.

$1.5\times4\left[2 - 2y_1 + 1\right] + y_1 = 0$.

$6\left[3 - 2y_1\right] + y_1 = 0$

$18 - 12y_1 + y_1 = 0$.

$18 - 11y_1 = 0$

$y_1 = 18/11 = 1.6364$.

DATE:
PAGE:

③ Solve the following BVP using FDM.

$xy'' + y = 0$; $y(1) = 1$ & $y(2) = 2$ where $h = 0.25$

Sol:-

(i) Domain $= [1, 2]$

(ii)

$x_0 = 1$

$x_1 = 1 + 0.25 \times 1 = 1.25$

$x_2 = 1 + 0.25 \times 2 = 1.5$

$x_3 = 1 + 0.25 \times 3 = 1.75$

$x_4 = 1 + 0.25 \times 4 = 2.$

(iii)

$$x_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right] + y_i = 0$$

$i = 1$

$$\frac{x_1}{(0.25)^2}[y_2 - 2y_1 + y_0] + y_1 = 0$$

$$\frac{1.25}{(0.25)^2}[y_2 - 2y_1 + 1] + y_1 = 0.$$

$$20[y_2 - 2y_1 + 1] + y_1 = 0$$

$$20y_2 - 40y_1 + 20 + y_1 = 0$$

$$20y_2 - 39y_1 + 20 = 0 . \quad - \cancel{②}$$

$$20y_2 - 39y_1 = -20 \quad - ①$$

DATE: PAGE:

$i = 2$

$$\frac{x_2}{(0.25)^2}\left[y_3 - 2y_2 + y_1\right] + y_2 = 0.$$

$$\frac{1.5}{(0.25)^2}\left[y_3 - 2y_2 + y_1\right] + y_2 = 0.$$

$$24y_3 - 48y_2 + 24y_1 + y_2 = 0$$

$$24y_3 - 47y_2 + 24y_1 = 0. \quad -(2)$$

$i = 3$

$$\frac{x_3}{(0.25)^2}\left[y_4 - 2y_3 + y_2\right] + y_3 = 0$$

$$\frac{1.75}{(0.25)^2}\left[2 - 2y_3 + y_2\right] + y_3 = 0$$

$$56 - 56y_3 + 28y_2 + y_3 = 0$$

$$\cancel{28}\,\cancel{56}\,y_2 - 55y_3 + 56 = 0. \quad -(3)$$

From eq. (2)

$$24y_3 = 47y_2 - 24y_1$$

$$y_3 = \frac{47}{24}y_2 - y_1 \quad -(4).$$

$$\Rightarrow 56y_2 - 55\times\left(\frac{47}{24}y_2 - y_1\right) + 56 = 0$$

$$56y_2 - \frac{2585}{24}y_2 + 55y_1 = -56.$$

$$-\frac{1241}{24}y_2 + 55y_1 = \cancel{-56} - 56$$

$$y_2 = \cancel{3.58827}\ 1.6347 \qquad y_1 = \cancel{2.3501}\ 1.3513$$

DATE: PAGE:

$$y_3 = \cancel{1.6660}\ 1.8505$$

$$\begin{bmatrix} -39 & 20 & 0 \\ 24 & -47 & 24 \\ 0 & 28 & -55 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ -56 \end{bmatrix}.$$

(3) Solve $(x^3+1)y'' + x^2y' - 4xy = 2$; $y(0) = 0$
$y(2) = 4$ & $h = 0.5$

Sol:-

$x_0 = 0$

$x_1 = 0 + 0.5 \times 1 = 0.5$

$x_2 = 1$

$x_3 = 1.5$

$x_4 = 2$

$$(x_i^3+1)\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + x_i^2\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] - 4x_iy_i = 2$$

$i = 1$

$$(x_1^3+1)\left[\frac{y_2 - 2y_1 + y_0}{(0.5)^2}\right] + (x_1)^2\left[\frac{y_2 - y_0}{2\times 0.5}\right] - 4x_1y_1 = 2.$$

$$((0.5)^3+1)\left[\frac{y_2 - 2y_1 - 0}{(0.5)^2}\right] + (0.5)^2\left[y_2 - 0\right] - 4\times 0.5\, y_1 = 2$$

$$\cancel{(1.125)}\ 4.5\times[y_2 - 2y_1] + (0.5)^2 y_2 - 2y_1 = 2$$

$$4.5y_2 - 9y_1 + (0.5)^2 y_2 - 2y_1 = 2.$$

$$4.75\, y_2 - 11y_1 = 2. \quad —(1)$$

DATE: PAGE:

$i=2$

$$4\times(1+1)\left[y_3-2y_2+y_1\right]+1\times(y_3-y_1)-4y_2=2$$

$$8y_3-16y_2+8y_1+y_3-y_1-4y_2=2.$$

$$9y_3-20y_2+7y_1=2 \quad -\ (2).$$

$i=3$

$$4\times((1.5)^3+1)\left[y_4-2y_3+y_2\right]+(1.5)^2(y_4-y_2)-4y_3\times1.5=2$$

$$17.5y_4-35y_3+17.5y_2+(1.5)^2y_4-1.5y_2-6y_3=2$$

$$70-35y_3+17.5y_2+6-(1.5)^2y_2-6y_3=2$$

$$-\cancel{35}\,41y_3+\cancel{16}\,15.25y_2+70=2$$

$$-41y_3+\cancel{16}\,15.25y_2=-\cancel{68}77 \quad \text{check}$$

$$\begin{bmatrix}-11 & \cancel{5}\,4.75 & 0\\ 7 & -20 & 9\\ 0 & \cancel{16}\,15.25 & -41\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}2\\ 2\\ -77\end{bmatrix}$$

$y_1=\cancel{0.2463}$ 0.25

$y_2=\cancel{0.8759}$ 1

$y_3=\cancel{2.0003}.$ 2.25

(4) Solve $x^2y''+xy'+(x^2-3)y=0$ with $y(1)=0$

$y(2)=2$ & $h=0.25$

Sol:

DATE:

PAGE:

$$x_i^2\left[\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right]+x\left[\frac{y_{i+1}-y_{i-1}}{2h}\right]+(x^2-3)y_i=0$$

$x_0 = 1$

$x_1 = 1.25$

$x_2 = 1.5$

$x_3 = 1.75$

$x_4 = 2$

$i = 1$

$$\overset{16}{\cancel{4}}\,x_1^2\left[y_2-2y_1+y_0\right]+\frac{x_1}{2}\left[y_2-y_0\right]+(x_1^2-3)y_1=0$$

$$16\times(1.25)^2\left[y_2-2y_1+1\right]+\frac{1.25}{2}\left[y_2-1\right]+(1.25^2-3)y_1=0$$

$$25y_2-\cancel{25}\,50y_1+25+0.625y_2-0.625-1.4375y_1=0-$$

$$25.625y_2-51.4375y_1=-24.375 \quad -(1).$$

DATE: PAGE:

Type 3 BVP's: Mixed condition

Let 2nd order linear DE be

$$p(x)y'' + q(x)y' + r(x)y = f(x).$$

$$\alpha_0 y(a) + \alpha_1 y'(0) = \alpha_2$$

$$\beta_0 y(b) + \beta_1 y'(b) = \beta_2$$

1. Change the boundary condition in terms of finite differences.

$$\alpha_0 y_0 + \alpha_1 y_0' = \alpha_2 \quad - \text{(I)}$$

$$\beta_0 y_n + \beta_1 y_n' = \beta_2 \quad - \text{(II)}$$

2. W.K.T by central difference formulae

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_0' = \frac{y_1 - y_{-1}}{2h} \quad - (a)$$

$$y_n' = \frac{y_{n+1} - y_{n-1}}{2h} \quad - (b).$$

Sub (a) & (b) in (I) & (II)

$$\alpha_0 y_0 + \alpha_1 \left[\frac{y_1 - y_{-1}}{2h}\right] = \alpha_2 \quad - (c)$$

$$\beta_0 y_n + \beta_1 \left[\frac{y_{n+1} - y_{n-1}}{2h}\right] = \beta_2 \quad - (d)$$

Using eq (c) & (d) we can find y_{-1} in terms of y_0 & y_1. Similarly y_{n+1} in terms of y_{n-1} & y_n

DATE: PAGE:

3. Expand the modelled eq at each x_i: $0 \le i \le n$ using central difference formulae

① Solve $y'' + (1+x)y' - y = 0$

$y(0) - y'(0) = 0$;

$y(1) + y'(1) = 1$; with $h = 0.5$.

Sol:

① Domain is $[0, 1]$.

$$y_0 - y_0' = 0 \quad - (I)$$

$\cancel{y_1 + y_1' = 1}$

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

so $n = 2$ at x_2.

$$y_2 + y_2' = 1 \quad - (II)$$

Central difference formulae

$$y_0' = \frac{y_1 - y_{-1}}{2 \times 0.5}, \quad y_2' = \frac{y_3 - y_1}{2 \times 0.5}$$

$$y_0' = y_1 - y_{-1} \quad -(a), \quad y_2' = y_3 - y_1 \quad -(b)$$

Sub (a) & (b) in (I) & (II)

$y_0 - (y_1 - y_{-1}) = 0$	$y_2 + (y_3 - y_1) = 1$
$y_0 - y_1 + y_{-1} = 0$	$y_2 + y_3 - y_1 = 1$
$\boxed{y_{-1} = y_1 - y_0}$ $-(c)$	$\boxed{y_3 = 1 + y_1 - y_2}$ $-(d)$

DATE: PAGE:

$y_i'' + (1+x_i)y_i' - y_i = 0$ from question

$i=0$

$y_0'' \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + (1+x_i)\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] - y_i = 0$

$i=0$

$$\left(\frac{y_1 - 2y_0 + y_{-1}}{(0.5)^2}\right) + (1+x_0)\left[\frac{y_1 - y_{-1}}{2\times0.5}\right] - y_0 = 0$$

$$4y_1 - 8y_0 + 4y_{-1} + (1+0)\left[y_1 - y_{-1}\right] - y_0 = 0$$

$$4y_1 - 8y_0 + 4y_{-1} + y_1 - y_{-1} - y_0 = 0$$

$$4y_1 - 8y_0 + 3y_{-1} + y_1 - y_0 = 0$$

$$5y_1 - 9y_0 + 3y_{-1} = 0$$

$$5y_1 - 9y_0 + 3(y_1 - y_0) = 0$$

$$5y_1 - 9y_0 + 3y_1 - 3y_0 = 0$$

$$\underline{8y_1 - 12y_0 = 0} \quad -(1)$$

DATE: PAGE:

$i=1$

$$\frac{y_2 - 2y_1 + y_0}{(0.5)^2} + (1+0.5)\frac{[y_2 - y_0]}{1} - y_1 = 0$$

$$4y_2 - 8y_1 + 4y_0 + 1.5y_2 - 1.5y_0 - y_1 = 0.$$

$$5.5y_2 - 9y_1 + 2.5y_0 = 0 \quad -②.$$

$i=2$ (it should be $8y_1 - 15y_2 = -6$)

$$4(y_3 - 2y_2 + y_1) + (2)(y_3 - y_1) - y_2 = 0.$$

$$4y_3 - 8y_2 + 4y_1 + 2y_3 - 2y_1 - y_2 = 0$$

$$6y_3 - 9y_2 + 2y_1 = 0$$

$$6(1 + y_1 - y_2) - 9y_2 + 2y_1 = 0$$

$$6 + \cancel{6y_1} - 6y_2 - 9y_2 + 2y_1 = 0.$$

$$6 + 8y_1 - 15y_2 = 0$$

$\cancel{6 = 15y_2}$ $\quad 8y_1 - 15y_2 = -6. - ③$

$\cancel{y_2 = 6/15}$

$- \cancel{③}$

$\cancel{15y_2 = 6}$

$$\begin{bmatrix} -12 & 8 & 0 \\ 2.5 & -9 & 5.5 \\ 0 & 8 & -15 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}$$

$\cancel{y_0 = 0.2}$ $\quad \cancel{y_2 = 0.4} \quad y_0 = 1/3$

$\cancel{y_1 = 0.3}$ $\quad y_1 = 1/2$

$y_2 = 2/3$

Solve $xy'' + y = 0$,

$y'(1) = 0$ & $y(2) = 1$ $h = 0.5$.

Sol:

$x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $\cancel{x_3 = 1.5}$, $\cancel{x_4 = 2}$

$n = \cancel{4}\ 2$, $h = 0.5$

$y_0' = 0$, $y_2' = 0$

$y_0' = \frac{y_1 - y_{-1}}{1}$, $y_2' = \cancel{\frac{y_3 - y_1}{1}}\ y_3 - y_1$

$y_{-1} = y_1 - y_0'$, $\cancel{y_3 = y_2' + 3y_1}$ $y_3 = y_2' + y_1$

$y_{-1} = y_1$, $\cancel{y_3 = 3y_1}$ $\cancel{y_3 = y_1}$

$i = 0$

$x_0 y_0'' + y_0 = 0$

$x_0 \left[\frac{y_1 - 2y_0 + y_{-1}}{(0.5)^2} \right] + y_0 = 0.$

$1 \times 4 \left[y_1 - 2y_0 + y_1 \right] + y_0 = 0$

$4y_1 - 8y_0 + 4y_1 + y_0 = 0.$

$8y_1 - 7y_0 = 0.$

$$x_i\frac{[y_{i+1} - 2y_i + y_{i-1}]}{h^2} + y_i = 0.$$

DATE: PAGE:

$i = 1$

$$x_1\frac{[y_2 - 2y_1 + y_0]}{(0.5)^2} + y_1 = 0$$

$$1.5\times 4[y_2 - 2y_1 + y_0] + y_1 = 0$$

$$6y_2 - 12y_1 + 6y_0 + y_1 = 0$$

$(\because y_2 = 1)$

$$6y_0 - 11y_1 + 6y_2 = 0. \Rightarrow 6y_0 - 11y_1 = -6$$

$i = 2$

Not required as y_2 is given

$$x_2\frac{[y_3 - 2y_2 + y_1]}{(0.5)^2} + y_2 = 0.$$

$$2\times 4[y_3 - 2y_2 + y_1] + y_2 = 0.$$

$$8y_3 - 16y_2 + 8y_1 + y_2 = 0$$

$$8\times 1 - 16y_2 + 8y_1 + y_2 = 0.$$

$$8 - 15y_2 + 8y_1 = 0$$

$$8y_1 - 15y_2 = -8$$

$$\begin{bmatrix} -7 & 8 & 0 \\ 6 & -11 & 0 \\ 0 & 8 & -15 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 8 \\ 6 & -11 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} \Rightarrow y_0 = \frac{48}{29}$$

$$y_1 = \frac{42}{29}$$

$$y_2 = 1$$

DATE: PAGE:

Solutions for BVP in BDE.

Consider the PDE given by

$$AU_{xx} + 2BU_{xy} + CU_{yy} + F(x,y,u,U_x,U_y) = 0 \quad \text{①}$$

where A, B, C are functions in x & y.

$$U_{xx} = \frac{\partial^2 [u(x,y)]}{\partial x^2} \; ; \; U_{xy} = \frac{\partial^2}{\partial y \partial x}(u(x,y))$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(u(x,y))\right]$$

$$U_{yy} = \frac{\partial^2}{\partial y^2}(u(x,y)) \; ; \; U_x = \frac{\partial}{\partial x}(u(x,y))$$

$$U_y = \frac{\partial}{\partial y}(u(x,y))$$

We say eq. ① is

1. elliptic if $B^2 - 4AC < 0$
2. Parabolic if $B^2 - 4AC = 0$
3. Hyperbolic if $B^2 - 4AC > 0$

① $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — Laplace's eq.

$A = 1$, $B = 0$, $C = 1$

$0 - 4 \times 1 \times 1 = -4 < 0$

So elliptic.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ — Poisson's eq.

DATE: PAGE:

(1) Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad 0<x<1 \,;\, 0<y<1$$

$h = 1/3$

$u(x,1) = u(0,y) = 0$

$u(1,y) = 9(y-y^2)$

$u(x,0) = 9(x-x^2)$

Sol:-

(i) To solve Laplace's & Poisson's equation in a rectangular region, divide the region into small squares by drawing lines parallel to the sides of the rectangle. The points of the intersections of these lines are called Mesh points

Let (x_0, y_0) be any arbitrary interior corner point.

$x_i = x_0 + ih$ & $y_i = y_0 + ih$

Let $u_{ij} = u(x_i, y_j)$ is the notation for convinience.

$u_{i+1,j} = u(x_{i+1}, y_j)$

$u_{i-1,j} = u(x_{i-1}, y_j)$

$$\frac{\partial^2 u_{ij}}{\partial x^2} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} \quad -(i) \quad \boxed{= \frac{\partial^2 [u(x_i, y_j)]}{\partial x^2}}$$

$$\frac{\partial^2 u_{ij}}{\partial y^2} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} \quad -(ii) \quad \boxed{\frac{\partial^2 [u(x_i, y_j)]}{\partial y^2}}$$

To solve Laplace eq. , $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ -(*)

Set $x = x_i$ & $y = y_j$ then expand (*)

DATE:

PAGE:

$$\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{\partial^2 u_{ij}}{\partial y^2} = 0$$

Sub (i) & (ii) in (2).

$$0 = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2}$$

$$\frac{u_{i+1,j} - 4u_{ij} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{h^2} = 0$$

$$u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$