I) The boundary conditions do not involve y'

Consider (by setting $x = x_i$ in ⊛)

$$p_i y_i'' + q_i y_i' + r_i y_i = f_i \quad \text{——} \quad ⓐ$$

$p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$, $f_i = f(x_i)$

$y_i' = y'(x_i)$, $y_i'' = y''(x_i)$, $y_i = y(x_i)$

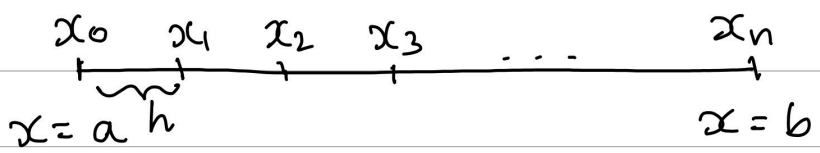
Let $y(a) = y_a$, $y(b) = y_b$ —— ⓑ

Replace y_i' and y_i'' in ⓐ by ③ & ④

$$\Rightarrow p_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + q_i \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + r_i y_i = f_i \quad \text{——} \quad ⓒ$$

$i = 1, 2, \ldots, n-1$

The eqn ⓒ gives rise to system of equations of order $n-1$.



$x_1 = x_0 + h$

$x_2 = x_1 + h$

$\vdots$

Given $y(x_0) = y_0 = y(a) = y_a$

$y(x_n) = y_n = y(b) = y_b$

Our aim is to find $x_1, x_2, x_3, \ldots$

1. Solve $xy'' + y = 0$, $y(1) = 1$, $y(2) = 2$ with $h = 0.5$ and $h = 0.25$

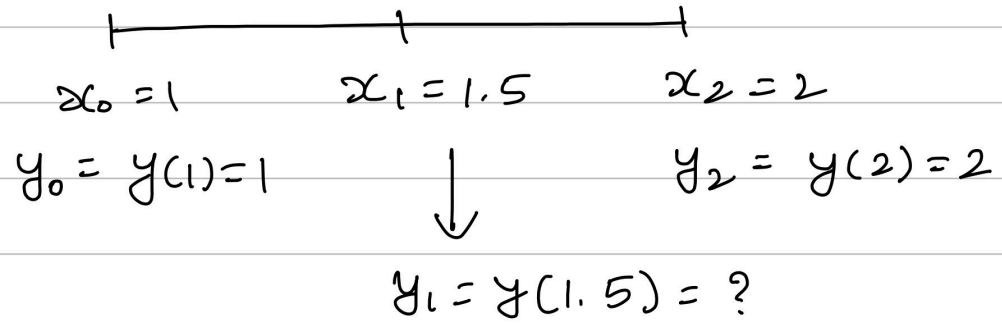
Sol^n :-

a) $h = 0.5$

Here $x_0 = 1 = a$

$x_1 = x_0 + h = 1.5$

$x_2 = x_1 + h = 2 = b$



Consider $xy'' + y = 0$

Replace x by x_i y by y_i

$$x_i y_i'' + y_i = 0 \quad \text{——— (1)}$$

$$x_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right] + y_i = 0$$

Here, $h = 0.5 = 1/2 \Rightarrow h^2 = 1/4$

$$\Rightarrow \quad x_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{1/4} \right] + y_i = 0$$

$$\Rightarrow \quad 4x_i \left(y_{i+1} - 2y_i + y_{i-1} \right) + y_i = 0$$

$$\Rightarrow \quad 4x_i y_{i+1} - 8x_i y_i + 4x_i y_{i-1} + y_i = 0 \quad \text{—— (2)}$$

Put $i = 1$

$$\Rightarrow \quad 4x_1 y_2 - 8x_1 y_1 + 4x_1 y_0 + y_1 = 0$$

$$\Rightarrow \quad 4(1.5)(2) - 8(1.5)\, y_1 + 4(1.5)(1) + y_1 = 0$$

$$\Rightarrow \quad 12 - 12y_1 + 6 + y_1 = 0$$

$$\Rightarrow \quad -11y_1 + 18 = 0$$

$$\Rightarrow \quad y_1 = \frac{18}{11} = 1.6364$$

$$\Rightarrow \quad y(1.5) = 1.6364$$

b) $h = 0.25$

$x_0 = 1, \quad x_1 = 1.25, \quad x_2 = 1.5, \quad x_3 = 1.75, \quad x_4 = 2$

$y_0 = y(1) = 1 \qquad y_1 = ? \qquad y_2 = ? \qquad y_3 = ? \qquad y_4 = y(2) = 2$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$y_0 = 1 \quad y_1 = ? \quad y_2 = ? \quad y_3 = ? \quad y_4 = 2$

$$x_i y_i'' + y_i = 0$$

$$x_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right] + y_i = 0$$

Put $h = 0.25 = 1/4 \Rightarrow h^2 = 1/16$

$$\Rightarrow 16x_i (y_{i+1} - 2y_i + y_{i-1}) + y_i = 0$$

$$\Rightarrow 16x_i y_{i+1} - 32x_i y_i + 16x_i y_{i-1} + y_i = 0$$

Put $i=1 \Rightarrow 16x_1 y_2 - 32x_1 y_1 + 16x_1 y_0 + y_1 = 0$

$$\Rightarrow 16(1.25)y_2 - 32(1.25)y_1 + 16(1.25)(1) + y_1 = 0$$

$$\Rightarrow 20y_2 - 40y_1 + 20 + y_1 = 0$$

$$\Rightarrow -39y_1 + 20y_2 = -20 \quad \text{——— (1)}$$

Put $i=2 \Rightarrow 16x_2 y_3 - 32x_2 y_2 + 16x_2 y_1 + y_2 = 0$

$$\Rightarrow 16(1.5)y_3 - 32(1.5)y_2 + 16(1.5)y_1 + y_2 = 0$$

$$\Rightarrow 24y_3 - 48y_2 + 24y_1 + y_2 = 0$$

$$\Rightarrow 24y_1 - 47y_2 + 24y_3 = 0 \quad \text{——— (2)}$$

Put $i = 3 \Rightarrow 16x_3y_4 - 32x_3y_3 + 16x_3y_2 + y_3 = 0$

$\Rightarrow 16(1.75)(2) - 32(1.75)y_3 + 16(1.75)y_2 + y_3 = 0$

$\Rightarrow 56 - 56y_3 + 28y_2 + y_3 = 0$

$\Rightarrow 28y_2 - 55y_3 = -56$ —— ③

Consider ①, ② and ③

$-39y_1 + 20y_2 = -20$ —— ①

$24y_1 - 47y_2 + 24y_3 = 0$ —— ②

$28y_2 - 55y_3 = -56$ —— ③

② /24 & ③ /55

$\Rightarrow y_1 - \frac{47}{24}y_2 + y_3 = 0$

$\frac{28}{55}y_2 - y_3 = \frac{-56}{55}$

Add the above equations

$y_1 - 1.4492y_2 = -1.0182$ —— ④

① /39 $\Rightarrow -y_1 + \frac{20}{39}y_2 = \frac{-20}{39}$ —— ⑤

Add ④ & ⑤

$$-0.9364\, y_2 = -1.531$$

$$\Rightarrow y_2 = 1.6349$$

$$-y_1 = -\left(\frac{20}{39}\right)(1.6349) - \frac{20}{39}$$

$$\Rightarrow y_1 = 1.3513$$

$$y_3 = \frac{28\,(1.6349) + 56}{55} = 1.8505$$

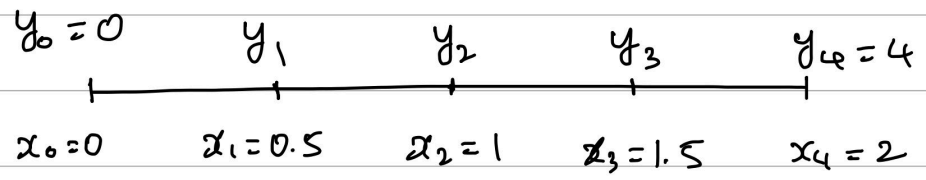
$$\Rightarrow y_3 = 1.8505$$

$$\boxed{\begin{aligned} \therefore\ y_1 &= y(1.25) = 1.3513 \\ y_2 &= y(1.5) = 1.6349 \\ y_3 &= y(1.75) = 1.8505 \end{aligned}}$$

2. Solve $(x^3+1)y'' + x^2y' - 4xy = 2$,
$y(0)=0$, $y(2)=4$ with $h=0.5$

Solⁿ:

$$(x_i^3+1)\,y_i'' + x_i^2\,y_i' - 4x_iy_i = 2$$



$$(x_i^3+1)\left[\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right] + x_i^2\left[\frac{y_{i+1}-y_{i-1}}{2h}\right] - 4x_iy_i = 2$$

Put $h = 0.5 = 1/2 \Rightarrow h^2 = 1/4$

$$\Rightarrow 4(x_i^3+1)(y_{i+1}-2y_i+y_{i-1}) + x_i^2(y_{i+1}-y_{i-1})$$

$$-4x_iy_i = 2 \quad \text{——} \quad (*) \qquad i=1,2,3$$

Put $i=1$

$$\Rightarrow 4(x_1^3+1)(y_2-2y_1+y_0) + x_1^2(y_2-y_0) - 4x_1y_1 = 2$$

$$\Rightarrow 4(0.5^3+1)(y_2-2y_1+0) + 0.5^2(y_2-0) - 4(0.5)y_1 - 2 = 0$$

$$\Rightarrow (y_2-2y_1) + 0.25y_2 - 2y_1 - 2 = 0$$

$$\Rightarrow 4.5y_2 - 9y_1 + 0.25y_2 - 2y_1 - 2 = 0$$

$$\Rightarrow -11y_1 + 4.75y_2 - 2 = 0$$

$$\Rightarrow -11y_1 + 4.75y_2 = 2 \quad \text{——} \quad (1)$$

Put $i=2$

$$4(x_2^3+1)(y_3-2y_2+y_1)+x_2^2(y_3-y_1)$$
$$-4x_2y_2=2$$

$$\Rightarrow 4(1+1)(y_3-2y_2+y_1)+(y_3-y_1)-4y_2=2$$

$$\Rightarrow 8y_3-16y_2+8y_1+y_3-y_1-4y_2=2$$

$$\Rightarrow 9y_3-20y_2+7y_1=2$$

$$\Rightarrow 7y_1-20y_2+9y_3=2 \quad \text{—— (2)}$$

Put $i=3$,

$$4(x_3^3+1)(y_4-2y_3+y_2)+x_3^2(y_4-y_2)$$
$$-4x_3y_3=2$$

$$\Rightarrow 4(1.5^3+1)(4-2y_3+y_2)+1.5^2(4-y_2)$$
$$-4\times1.5\times y_3=2$$

$$\Rightarrow 17.5(4-2y_3+y_2)+2.25(4-y_2)-6y_3=2$$

$$\Rightarrow 70-35y_3+17.5y_2+9-2.25y_2-6y_3=2$$

$$\Rightarrow 15.25y_2-41y_3=-77 \quad \text{—— (3)}$$

$$\Rightarrow \boxed{y_1 = y(x_1) = y(0.5) = 0.25}$$

$$\Rightarrow \boxed{y_2 = y(x_2) = y(1) = 1}$$

$$\Rightarrow \boxed{y_3 = y(x_3) = y(1.5) = 2.25}$$

H.W

Solve $x^2y'' + xy' + (x^2-3)y = 0$ with $y(1) = 0$, $y(2) = 2$ and $h = 0.25$

<u>Ans:-</u> 0.6044, 1.1304, 1.5973.

II) The boundary conditions involve y'.

Let the boundary conditions be

$$\alpha_0 \, y(a) + \alpha_1 \, y'(a) = \alpha_2$$

$$\beta_0 \, y(b) + \beta_1 \, y'(b) = \beta_2$$

Changing the derivatives into finite differences, we get

$$\alpha_0 y_0 + \alpha_1 y_0' = \alpha_2$$
$$\beta_0 y_n + \beta_1 y_n' = \beta_2$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \Rightarrow y_0' = \frac{y_1 - y_{-1}}{2h}$$

$$y_n' = \frac{y_{n+1} - y_{n-1}}{2h}$$

$$\therefore \quad \alpha_0 y_0 + \alpha_1 \left[\frac{y_1 - y_{-1}}{2h} \right] = \alpha_2 \quad \text{——— (a)}$$

$$\beta_0 y_n + \beta_1 \left[\frac{y_{n+1} - y_{n-1}}{2h} \right] = \beta_2 \quad \text{——— (b)}$$

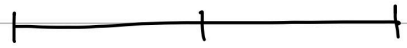
Here the problem involves finding y_{-1}, y_{n+1} (the values of y outside the interval of interest).

The eqn (a) will enable us to find y_{-1} in terms of y_0 and y_1

The eqn (b) will enable us to find y_{n+1} in terms of y_{n-1} and y_n.

1. Solve $xy'' + y = 0$, $y'(1) = 0$, $y(2) = 1$, $h = 0.5$

Soln:-



$x_0 = 1 \quad x_1 = 1.5 \quad x_2 = 2$

$y_0 = ? \quad y_1 = ? \quad y_2 = y(2) = 1$

$x_0 = 1$

$x_1 = 1 + 0.5 = 1.5$

$x_2 = 1.5 + 0.5 = 2$

Given $y'(1) = y'(x_0) = 0 \Rightarrow y_0' = 0$

Consider $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$

$$y_0' = \frac{y_1 - y_{-1}}{2 \times 1/2}$$

$$0 = y_1 - y_{-1}$$

$$\Rightarrow \boxed{y_1 = y_{-1}}$$

Given $xy'' + y = 0$

$$\Rightarrow x_i y_i'' + y_i = 0$$

$$x_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right] + y_i = 0 \quad , \; i = 0, 1 \quad \text{——— } *$$

Put $i=0$ in ⊛ and $h = 1/2$

$$x_0 \left[\frac{y_1 - 2y_0 + y_{-1}}{1/4}\right] + y_0 = 0$$

$$4x_0 (y_1 - 2y_0 + y_1) + y_0 = 0 \qquad (\because y_{-1} = y_1)$$

$$\Rightarrow 4(1)(2y_1 - 2y_0) + y_0 = 0$$
$$\Rightarrow 8y_1 - 8y_0 + y_0 = 0$$
$$\Rightarrow 8y_1 - 7y_0 = 0 \quad \text{——— (a)}$$

Put $i=1$ in ⊛ and $h = 1/2$

$$x_1 \left[\frac{y_2 - 2y_1 + y_0}{1/4}\right] + y_1 = 0$$

$$\Rightarrow 4x_1 (y_2 - 2y_1 + y_0) + y_1 = 0$$
$$\Rightarrow 4(1.5)(1 - 2y_1 + y_0) + y_1 = 0$$
$$\Rightarrow 6(1 - 2y_1 + y_0) + y_1 = 0$$
$$= -11y_1 + 6y_0 = -6 \quad \text{——— (b)}$$

$$\Rightarrow y_0 = y(1) = \frac{48}{29} = 1.6552$$

$$y_1 = y(1.5) = \frac{42}{29} = 1.4483$$

2. Solve $y'' + (1+x)y' - y = 0$, $y(0) = y'(0)$, $y(1) + y'(1) = 1$ with $h = 0.5$

Sol^n:-

$y_0 = ?$ $\quad$ $y_1 = ?$ $\quad$ $y_2 = ?$

$x_0 = 0$ $\quad$ $x_1 = 0.5$ $\quad$ $x_2 = 1$

Given $y'(0) = y(0) \Rightarrow y_0' = y_0$

Consider $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$

$$y_0' = \frac{y_1 - y_{-1}}{2 \times 1/2}$$

$$y_0 = y_1 - y_{-1} \qquad (\because y_0' = y_0)$$

$$\Rightarrow \boxed{y_{-1} = y_1 - y_0}$$

Given $y(1) + y'(1) = 1$

$$\Rightarrow y_2 + y_2' = 1$$

But $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$

$$y_2' = \frac{y_3 - y_1}{2 \times 1/2} = y_3 - y_1$$

$$\Rightarrow y_2 + (y_3 - y_1) = 1$$

$$\Rightarrow \boxed{y_3 = 1 + y_1 - y_2}$$

Consider

$$y_i'' + (1+x_i)\, y_i' - y_i = 0$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + (1+x_i)\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] - y_i = 0$$

$i = 0, 1, 2$

$\underline{i = 0}$

$$\frac{y_1 - 2y_0 + y_{-1}}{1/4} + (1+x_0)\left[\frac{y_1 - y_{-1}}{2 \times 1/2}\right] - y_0 = 0$$

$$\Rightarrow 4(y_1 - 2y_0 + y_{-1}) + (1+0)(y_1 - y_{-1}) - y_0 = 0$$

$$\Rightarrow 4(y_1 - 2y_0 + y_1 - y_0) + (y_1 - y_1 + y_0) - y_0 = 0$$

$$\Rightarrow 8y_1 - 12y_0 = 0$$

$$\Rightarrow 2y_1 - 3y_0 = 0 \quad \text{——— (a)}$$

$\underline{i = 1}$

$$4(y_2 - 2y_1 + y_0) + (1+x_1)(y_2 - y_0) - y_1 = 0$$

$$\Rightarrow 4(y_2 - 2y_1 + y_0) + (1.5)(y_2 - y_0) - y_1 = 0$$

$$\Rightarrow 5.5y_2 - 9y_1 + 2.5y_0 = 0 \quad \text{— (b)}$$

$\underline{i=2}$

$$4(y_3 - 2y_2 + y_1) + (1+1)(y_3 - y_1) - y_2 = 0$$

$$\Rightarrow 6y_3 - 9y_2 + 2y_1 = 0$$

But $y_3 = 1 + y_1 - y_2$

$$\Rightarrow 6 + 6y - 6y_2 - 9y_2 + 2y_1 = 0$$

$$\Rightarrow -15y_2 + 8y_1 = -6 \quad \text{—— (c)}$$

On solving (a), (b) & (c)

$y_0 = y(x_0) = y(0) = 1/3$

$y_1 = y(x_1) = y(0.5) = 1/2$

$y_2 = y(x_2) = y(1) = 2/3$

Solution of BVP in Partial differential equation (PDE)

Consider

$$A u_{xx} + 2B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0 \quad \text{—— (1)}$$

Where, $u_x = \dfrac{\partial u}{\partial x}$ $\quad u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$

$u_{xy} = \dfrac{\partial^2 u}{\partial y \, \partial x}$, $\quad u_y = \dfrac{\partial u}{\partial y}$, $\quad \partial_{yy} = \dfrac{\partial^2 u}{\partial y^2}$

Equation (1) is

(i) elliptic if $B^2 - AC < 0$

(ii) parabolic if $B^2 - AC = 0$

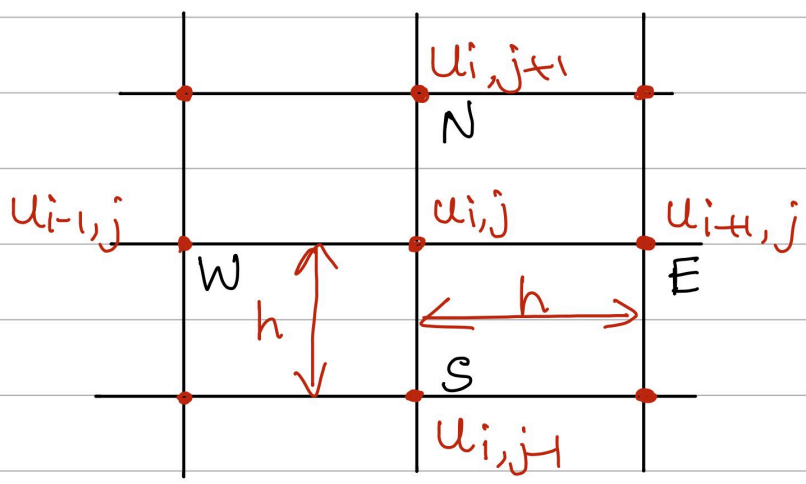
(iii) hyperbolic if $B^2 - AC > 0$

1) Elliptic equations

Laplace's equation :- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Poisson's equation :- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = F(x,y)$

To solve Laplace's and Poisson's equation in a rectangular region, divide the region into small squares by drawing lines parallel to the sides of the rectangle. The points of the intersection of these lines are called mesh points or lattice points or grid points.



Let (x_0, y_0) be an arbitrary interior corner point and $x_i = x_0 + ih$

$y_j = y_0 + jh$

Let $u_{ij} = u(x_i, y_j)$

$u_{i+1,j} = u(x_i+h, y_j)$

$u_{i-1,j} = u(x_i-h, y_j)$

By Taylor series expansion,

$$u_{i+1,j} = u_{ij} + h\frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} + \cdots$$

$$u_{i-1,j} = u_{ij} - h\frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u_{ij}}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} + \cdots$$

Adding these two eqns we get,

$$u_{i+1,j} + u_{i-1,j} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u_{ij}}{\partial x^4} + \cdots$$

As h is very small, we can neglect $h^4, h^6, \ldots$

$$\Rightarrow u_{i+1,j} + u_{i-1,j} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u_{ij}}{\partial x^2} = \frac{u_{i+1,j} - 2u_{ij} - u_{i-1,j}}{h^2}$$

(with error $O(h^2)$)

Similarly

$$\frac{\partial^2 u_{ij}}{\partial y^2} = \frac{u_{i,j+1} - 2u_{ij} - u_{i,j-1}}{h^2}$$

(with error $O(h^2)$)

To solve Laplace eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Set $x = x_i$, $y = y_j$

$$\Rightarrow \frac{u_{i+1,j} - 2u_{ij} - u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} - u_{i,j-1}}{h^2} = 0$$

$$\Rightarrow u_{ij} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$

$$u_{ij} = \frac{1}{4}\left(N + E + W + S\right)$$

(This is to remember easily. N, E, W, S stand for the values of u at the four neighbouring points in north, each, west and south directions respectively)