Linear Algebra

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & & & \\ \vdots & & & \\ a_m & & & \end{bmatrix}_{m \times n}$$

$\det(A) \rightarrow$ lower dimension

$A^T_{mn} \rightarrow$ Transpose matrix

$A^{-1} \rightarrow$ Inverse matrix

$\|A\| \rightarrow$ Norm of $A = \sqrt{a_{11}^2 + \cdots + a_{mn}^2} \rightarrow$ Euclidean Distance.

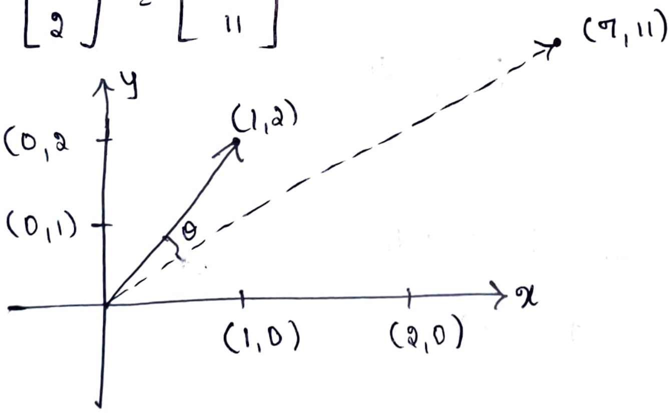
Manhattan Distance,

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

$$d(x,y) = \{|x_1 - y_1|, |x_2 - y_2|\}$$

Eigen Values, Eigen Vectors $\Rightarrow$ Eigen = Own.

Eg:- $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$



∴ The vector is rotated and expanded.

∴ If no rotation and expansion/ contract is allowed then

$$AX = \lambda X$$

where

$\lambda \rightarrow$ Eigen Value, $X \rightarrow$ Eigen Vector, $x \neq 0$

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$ → Eg-①: using Cayley-Hamilton theorem.

Soln:-

$$AX = \lambda X$$

$$AX - \lambda X = [0] \Rightarrow [A - \lambda I]X = [0]$$

if $[A - \lambda I]^{-1}$ exists then $X = [0]$

∴ We do not want $[A - \lambda I]^{-1}$ to exists.

∴ Inverse wont exists if $|A - \lambda I| = 0$. → Characteristic eqn.

∴ $|A - \lambda I| = 0$

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} = [0]$$

$$(5-\lambda)^2 - 1 = 0$$

$$(5-\lambda)^2 = 1$$

$$5 - \lambda = \pm 1$$

$$\therefore \lambda_1 = 4, \quad \lambda_2 = 6$$

∴ Sum of λ_1 & λ_2 = trace of A

$$\lambda_1 + \lambda_2 = \text{trace}(A)$$

$$4 + 6 = 5 + 5$$

∴ Product is det(A) ⇒ $\lambda_1 \times \lambda_2 = \det(A)$

$$4 \times 6 = 25 - 1$$

$$24 = 24$$

$\therefore \lambda^2 - 10\lambda + 24\lambda^0 \longrightarrow$ Characteristic / Eigen Equ^n

$A^2 - 10A + 24I$

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} - 10\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} + 24\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 26 & 10 \\ 10 & 26 \end{bmatrix} - \begin{bmatrix} 50 & 10 \\ 10 & 50 \end{bmatrix} + \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} = \underline{\underline{0}}$$

$\therefore$ Every Square matrix satisfies it's own characteristic equ^n

this is Cayley Hamilton theorem.

$\therefore \underline{\lambda = 4}$:-

$AX = \lambda X \Rightarrow \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 4\begin{bmatrix} x \\ y \end{bmatrix}$

$[A - \lambda I]X = [0]$

$$\begin{bmatrix} 5-4 & 1 \\ 1 & 5-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \left.\begin{aligned} x + y &= 0 \\ x + y &= 0 \end{aligned}\right\} \underline{\underline{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}}$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \underline{\underline{4}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore$ Here 4 is eigen value and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is eigen vectors.

$\therefore \underline{\lambda = 6}$:-

$[A - \lambda I]X = [0]$

$$\begin{bmatrix} 5-6 & 1 \\ 1 & 5-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \left.\begin{aligned} -x + y &= 0 \\ x - y &= 0 \end{aligned}\right\} \underline{\underline{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \Rightarrow 6\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ Here 6 is eigen value and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vector.

$$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1 \cdot 1) + 1(-1) = \underline{0}$$

$\underline{A^n = ?}$

$A = PDP^{-1} \Rightarrow A^2 = PD^2P^{-1} \longrightarrow$ Eigen Decomposition.

$A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$, P = Eigen Vector = $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and D = Eigen Values = $\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$

$$\therefore P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1-(-1)}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ -4 & 6 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 10 & 2 \\ 2 & 10 \end{bmatrix} = \underline{\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}} = A$$

if A^{100} then $= PD^{100}P^{-1}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4^{100} & 0 \\ 0 & 6^{100} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4^{100} & 6^{100} \\ (-4^{100}) & 6^{100} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\underline{\begin{bmatrix} 4^{100}+6^{100} & -(4^{100})+6^{100} \\ -4^{100}+6^{100} & 4^{100}+6^{100} \end{bmatrix}}$$

Eg:- $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1}$

Soln:-

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1/2-(-1/2)}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Eg: ⓐ Solve and find the inverse of it.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Soln:-

$$|A-\lambda I|X = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$\Rightarrow (2-\lambda)\left[(2-\lambda)^2-0\right] - 0 + 1\left[-(2-\lambda)\right] = 0$$

$$(2-\lambda)\left[(2-\lambda)^2-1\right] = 0$$

$$(2-\lambda)\left[4-4\lambda+\lambda^2-1\right) = 0$$

$$(2-\lambda)(\lambda^2-4\lambda+3) = 0$$

$$(2-\lambda)(\lambda-3)(\lambda-1) = 0$$

$\therefore \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ → Eigen values of A.

$$\lambda_1+\lambda_2+\lambda_3 = \text{trace}(A)$$

$$1+2+3 = 2+2+2$$

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A)$$

$$1\times2\times3 = 6$$

To find eigen vectors corresponding to

1) $\lambda = 1$

$$[A - \lambda I] X = [0]$$

$$\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + 0 + z &= 0 \\ 0 + y + 0 &= 0 \\ x + 0 + z &= 0 \end{aligned}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ Eigen vectors corresponding to $\lambda = 1$

2) $\lambda = 2$:

$$[A - \lambda I] X = [0]$$

$$\begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0 + 0 + z &= 0 \\ 0 + 0 + 0 &= 0 \\ x + 0 + 0 &= 0 \end{aligned}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Here 'y' is not set by the eqn so choose any non zero

3) $\lambda = 3$

$$[A - \lambda I] X = [0]$$

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x + 0 + z &= 0 \\ 0 - y + 0 &= 0 \\ x + 0 - z &= 0 \end{aligned}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

∴ If the ~~dot~~ product of eigen vector is not orthogonal i.e = 0 then error in calculation.

$1.1 + 0.0 + (-1 \cdot 1) = 0$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$1 \cdot 0 + 0 \cdot 1 + (-1 \cdot 0) = 0$ $\qquad$ $0.1 + 1 \cdot 0 + 0.1 = 0$

Unit Eigenvectors:-

1) $$\frac{1}{\sqrt{1^2 + 0^2 + (-1)^2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

2) $$\frac{1}{\sqrt{0^2 + 1^2 + 0^2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3) $$\frac{1}{\sqrt{1^2 + 0^2 + 1^2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$\therefore AA^{-1} = I$

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eg:- ③ :- Solve. for when matrix is large in size.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$ Solve using Power Method.

Soln:-

Consider a initial approximation as $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = X^{(0)}$

$$AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ 0 \\ 0.9 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 2.9 \\ 0 \\ 2.8 \end{bmatrix} = 2.9\begin{bmatrix} 1 \\ 0 \\ 0.965 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.965 \end{bmatrix} = \begin{bmatrix} 2.9655 \\ 0 \\ 2.9310 \end{bmatrix} = 2.9655\begin{bmatrix} 1 \\ 0 \\ 0.9884 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9884 \end{bmatrix} = \begin{bmatrix} 2.9884 \\ 0 \\ 2.9768 \end{bmatrix} = 2.9884\begin{bmatrix} 1 \\ 0 \\ 0.9961 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9961 \end{bmatrix} = \begin{bmatrix} 2.9961 \\ 0 \\ 2.9922 \end{bmatrix} = 2.9961\begin{bmatrix} 1 \\ 0 \\ 0.9987 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

∴ Largest eigenvalue after 7th iteration is 2.9961 and corresponding eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 0.9987 \end{bmatrix}$

Functions / Transformation / Mapping:-

Eg:-

$$① \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underset{\text{Original}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\text{Transformed}}{\begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}}$$

$\therefore T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x, y) = (x+2y, 3x+4y)$ ②

If ② is given then get ①

$T(1,0) = (1,3) = 1(1,0) + 3(0,1)$

$T(0,1) = (2,4) = 2(1,0) + 4(0,1)$

$$m(T) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\therefore T^2(x,y) = T(T(x,y))$

$= T(x+2y, 3x+4y)$

$= [(x+2y) + 2(3x+4y)], [3(x+2y) + 4(3x+4y)]$

$= (7x+10y, 15x+22y)$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7x+10y \\ 15x+22y \end{bmatrix}$$

$\therefore$ Multiplication of two matrices is same as composition of functions and vice versa.

Given $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$T(x,y) = (x+5y,\ 5x+y)$ the find $T^{100}(x,y)$.

Soln:-

$$T(1,0) = (1,5) = 1(1,0) + 5(0,1)$$
$$T(0,1) = (5,1) = 5(1,0) + 1(0,1)$$

$$\therefore A = m(T) = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

$\therefore A^{100}$ by finding eigen values & vectors.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-1-1}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}(-1/2)\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -4 \\ 6 & 4 \end{bmatrix}\left(1/2\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 10 \\ 10 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} = A$$

$$\therefore A^{100} = P D^{100} P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 6^{100} & 0 \\ 0 & -4^{100} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}1/2$$

$$= \frac{1}{2}\begin{bmatrix} 6^{100} & -4^{100} \\ 6^{100} & 4^{100} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6^{100} + 4^{100} & 6^{100} + 4^{100} \\ 6^{100} + 4^{100} & 6^{100} - 4^{100} \end{bmatrix}$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x,y) = (ax+by, cx+dy)$, $m(T) = ?$

Soln:-

$T(1,0) = (a,c) = a(1,0) + c(0,1)$

$T(0,1) = (b,d) = b(1,0) + d(0,1)$

$\therefore m(T) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

$\therefore T^{-1}(x,y) = A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= A^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$

$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= \frac{1}{ad-bc} \begin{bmatrix} dx-by \\ -cx+ay \end{bmatrix}$

$= \left(\frac{dx-by}{ad-bc} , \frac{-cx+ay}{ad-bc} \right)$

Given, $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x,y) = (5x+y, x+5y)$ then find $T^{-1}(x,y)$.

Soln:-

$m(T) = A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

$\therefore T^{-1}(x,y) = A^{-1} = \frac{1}{24} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5x-y}{24} \\ \frac{-x+5y}{24} \end{bmatrix}$

$= \left(\frac{5x-y}{24} , \frac{-x+5y}{24} \right)$

$\therefore$ For this characteristic eqn, $\lambda^2 - 10\lambda + 24 I = 0$

$$A^{-1}(A^2 - 10A + 24I) = A^{-1}O = O$$

$$A - 10I + 24A^{-1} = O$$

$$24A^{-1} = 10I - A$$

$$= 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{24}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

OR

$$T(x,y) = (5x+y,\ x+5y) \quad \text{—} (1)$$

$$= T \circ T^{-1}(x,y)$$

$$= T(T^{-1}(x,y))$$

$$= T\left[\underbrace{\frac{5x-y}{24}}_{u},\ \underbrace{\frac{-x+5y}{24}}_{v}\right]$$

$$= \frac{1}{24}\left[T\left[(5x, -x) + (-y, 5x)\right]\right]$$

$$= \frac{1}{24}\left(T(x(5,-1)) + T(y(-1,5))\right)$$

$$= \frac{1}{24}\left(xT(5,-1) + yT(-1,5)\right)$$

Substitute in (1)

$$= \frac{1}{24}\left[x(24,0) + y(0,24)\right]$$

$$= \frac{1}{24}(24x,\ 24y)$$

$$= (x,y)$$

Linear Transformation:-

1) $T(kx) = kT(x)$

2) $T(x+y) = T(x) + T(y)$

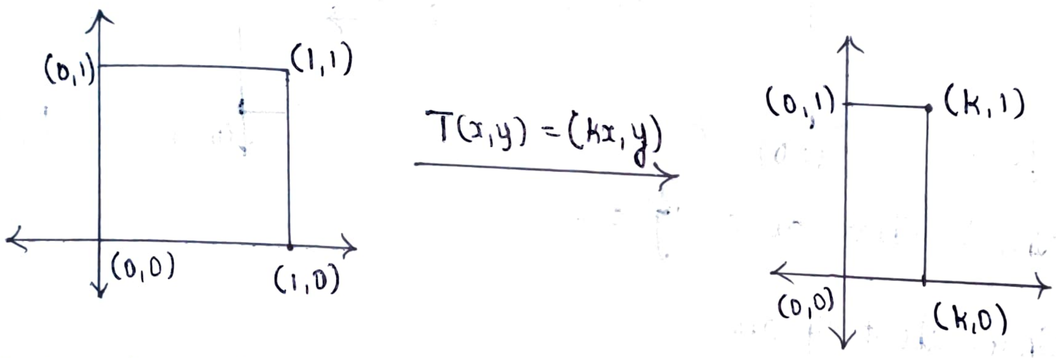
$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(x,y) = (kx, y) \Rightarrow$ Scaling along 'x'-axis. ①

$T(1,0) = (k,0) = k(1,0) + 0(0,1)$

$T(0,1) = (0,1) = 0(1,0) + 1(0,1)$

$\therefore m(T) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$



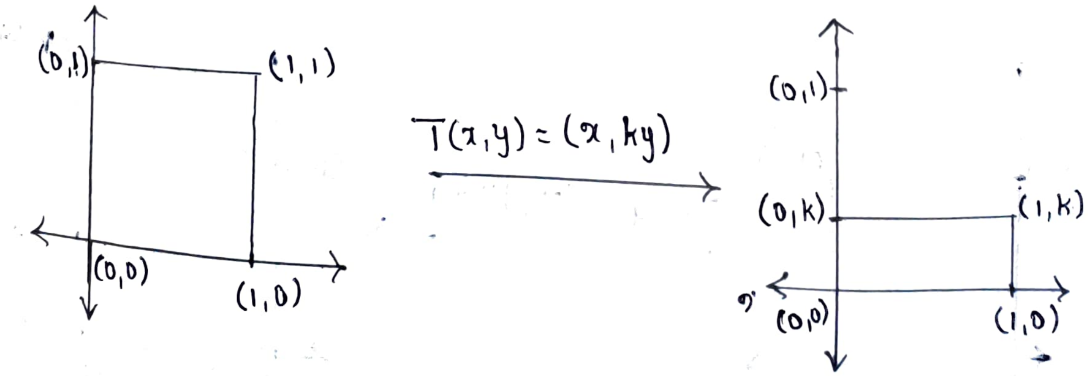
② Scaling along 'y' axis.

$T(x,y) = (x, ky)$

$T(1,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1) = (0,k) = 0(1,0) + k(0,1)$

$\therefore m(T) = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$



3) $T(x,y) = (x+ky,\ y) \quad —①$

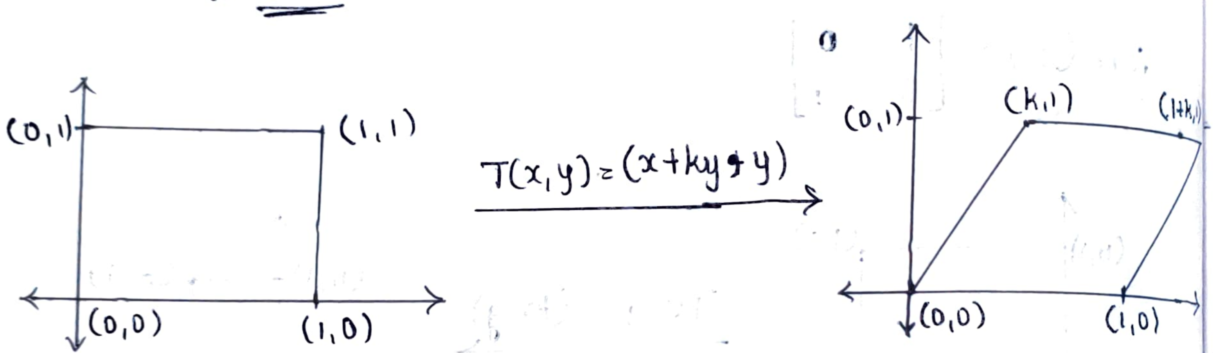
$T(1,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1) = (k,1) = k(1,0) + 1(0,1)$

$\therefore m(T) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad —②.$

From ① & ②

$\therefore ① = ②$



∴ This is Shear about 'x' axis.

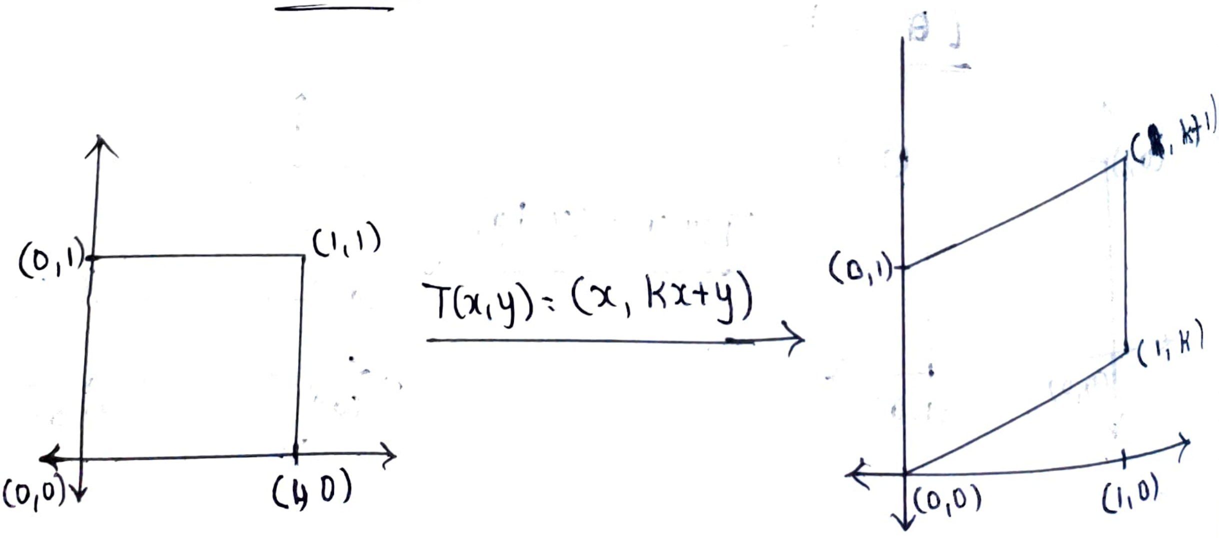
4) Shear about 'y' axis

$T(x,y) = (x,\ kx+y)$

$T(1,0) = (1,\ k)$

$T(0,1) = (0,1)$

$\therefore m(T) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$



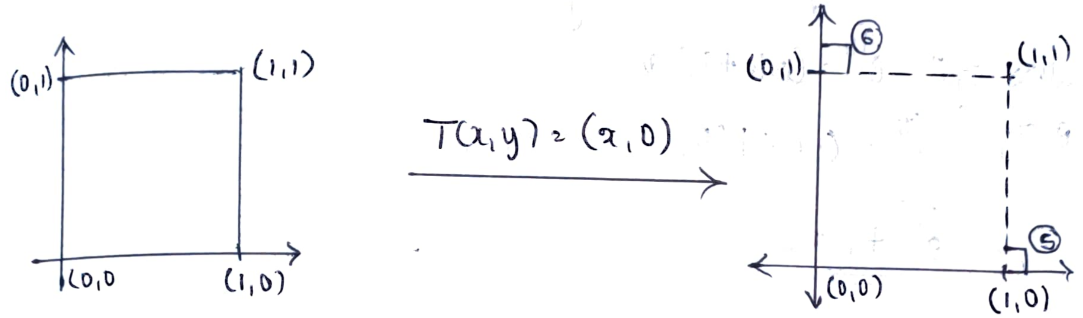
5 Projection on 'x' axis.

$T(x,y) = (x,0)$

$T(1,0) = (1,0)$

$T(0,1) = (0,0)$

$$\therefore m(T) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



6 Projection on 'y' axis,

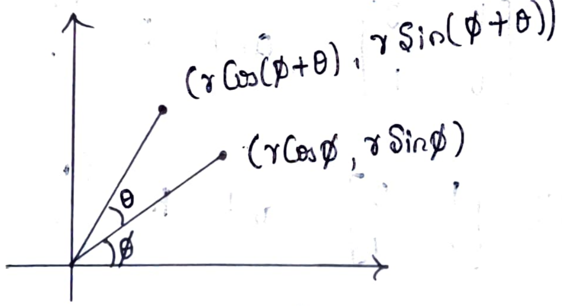
$T(x,y) = (0,y)$

$T(1,0) = (0,0)$

$T(0,1) = (0,1)$

$$\therefore m(T) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

7 Rotation



$$\bar{y} = r\sin(\phi+\theta) = r[\sin\phi\cos\theta + \cos\phi\sin\theta]$$

$$= x\sin\theta + y\cos\theta$$

$$\bar{x} = r\cos(\phi+\theta) = r[\cos\phi\cos\theta - \sin\phi\sin\theta]$$

$$= x\cos\theta - y\sin\theta$$

$$R_\theta = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{e^{\pi i} + 1 = 0}$$

→ Additive identity

→ Multiplicative identity

Eulers Equ^n, $e^{i\theta} = \cos\theta + i\sin\theta$

$\theta = \pi$, $\quad e^{i\pi} = (-1) + i(0)$

$$e^{\pi i} + 1 = 0.$$

Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and convert it to } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Soln:-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1 \quad \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$R_2 \to R_2/2 \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad E_2^{-1} = \frac{1}{(-1/2)}\begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} +1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$ → Shearing along 'y'-axis by an amound $k=3$

→ Scaling along y-axis by an amound $k=-2$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

→ Shearing along x-axis by an amound $k=2$

$$= \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= A$$

Express in terms elementary matrices of given 'A' The geometric meaning of multiplication by 'A'.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Soln:-

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ and $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$R_2 \to \dfrac{R_2}{-2}$ $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

$R_1 \to R_1 - 3R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $E^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $E_2^{-1} = -2\begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

and $E_3^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ this is shearing along y-axis by an amount $k=2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ this is scaling along y-axis by an amount $k=-2$

$E_3^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ this is shearing along x-axis by an amount $k=3$

$$\therefore E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

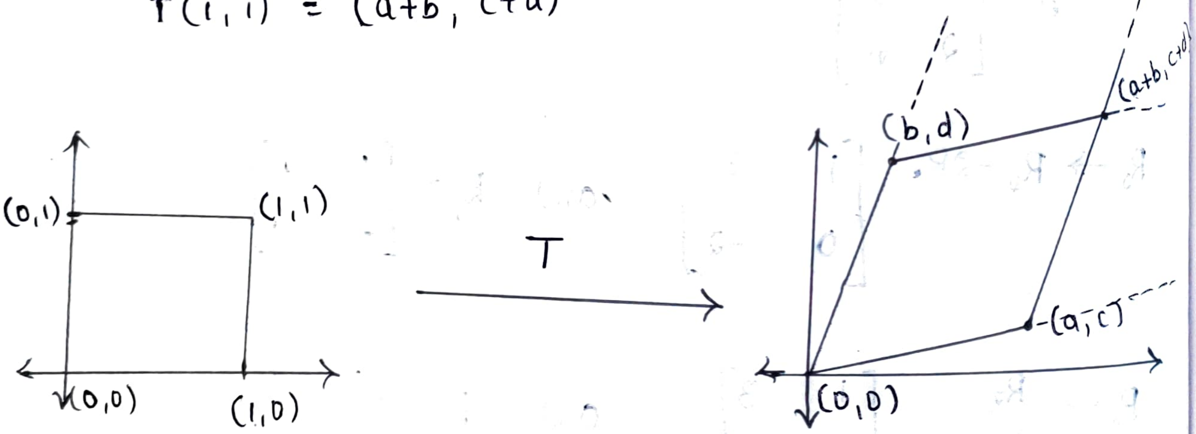
$$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = A$$

Given,

$$T(x,y) = (ax+by,\ cx+dy)$$

Soln:-

$$T(0,0) = (0,0)$$
$$T(1,0) = (a,c)$$
$$T(0,1) = (b,d)$$
$$T(1,1) = (a+b,\ c+d)$$



$\therefore$ This is Parallelogram law of vector addition.

$\therefore$ Area of this is $= ad - bc \rightarrow$ determinant.

$T : \mathbb{R}^n \to \mathbb{R}^n$

i) $T(\alpha X) = \alpha(T(x))$

ii) $T(X+Y) = T(X) + T(Y)$

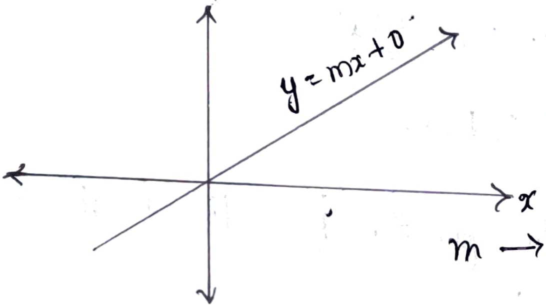
Then T is a linear transformation.

Result: The only linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is linear passing through origin.

Take a LT $\mathbb{R} \to \mathbb{R}$

$$T(x) = T(x.1)$$
$$= x\,T(1)$$
$$= xm \quad \text{where } m = T(1)$$

$$\boxed{\therefore y = T(x) = mx}$$



$m \to$ slope

Conversely:-

To show that $T(x) = mx$ is a linear transformation then

i) $T(\alpha x) = m(\alpha x)$
$$= \alpha(mx)$$
$$= \alpha T(x)$$

ii) $T(x+y) = m(x+y)$
$$= mx + my$$
$$= T(x) + T(y)$$

Both conditions are satisfied.

$\therefore T$ is a linear transformation.

$T(x) = 5x + 6 \rightarrow$ not LT.

It is not a LT since

$c = 6$ in this case

$\therefore$ from any LT, $c = 0$.

'0' is an invariant point under any linear transformation.

$T(0) = T(0+0)$

$= T(0) + T(0)$

$\therefore \underline{T(0) = 0}$

$T(x) = 5x^2$

$T(7x) = 5(7x)^2 = 49 \times 5 \times x^2$

$\therefore 7T(x) = 7(5x^2) = \underline{35x^2}$

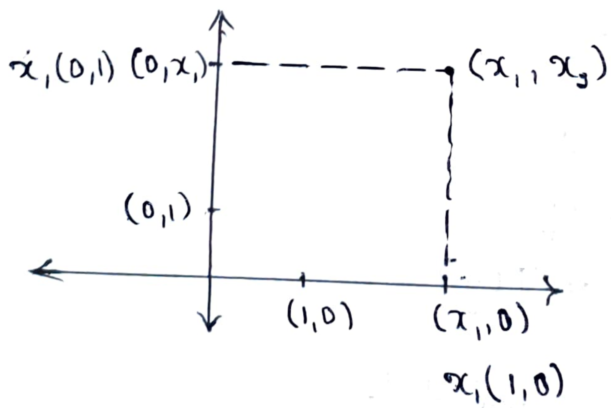
$\therefore \underline{T(7x) \neq 7T(x)}$

The only linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a plane passing through the origin.

$\therefore T(x) = T((x_1, x_2))$

$= T[x_1(1,0) + x_2(0,1)]$

$= T(\underline{x_1(1,0)}) + T(x_2(0,1))$



Here we can distinguish b/n scalar and vector

$\therefore$ Scalar : (x_1) and (x_2)

Vector : $[1,0]$ and $[0,1]$

$\therefore T(x_1, x_2) = x_1 T(1,0) + x_2 T(0,1)$

$= x_1 \alpha + x_2 \beta$ where $\alpha = T(1,0)$, $\beta = T(0,1)$

Conversely → To show that $T(x_1, x_2)$

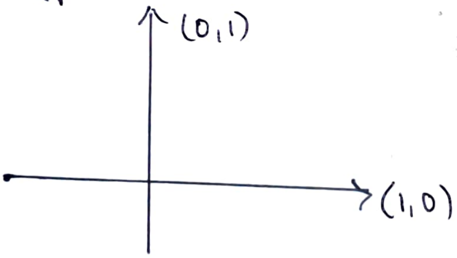
$= x_1 \alpha + x_2 \beta$ is a Linear Transformation (LT)

$$T k(x_1, x_2)) = T((kx_1, kx_2))$$
$$= (kx_1)\alpha + (kx_2)\beta$$
$$= k(x_1 \alpha + x_2 \beta)$$
$$= kT(x_1, x_2)$$

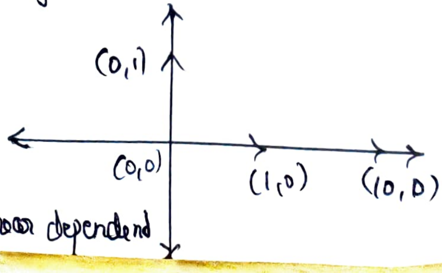
$$T(x+y) = T((x_1, x_2) + (y_1, y_2))$$
$$= T((x_1 + y_1), (x_2 + y_2))$$
$$= (x_1 + y_1)\alpha + (x_2 + y_2)\beta$$
$$= (x_1 \alpha + x_2 \beta) + (y_1 \alpha + y_2 \beta)$$
$$= T(x_1, x_2) + T(y_1, y_2)$$

Eg: $T(x_1 + x_2) = 5x + 6$ is not a LT.

$$T(x_1, x_2) = x_1 T(1,0) + x_2 T(0,1)$$
$$= x_1 \alpha + x_2 \beta$$



$B_1 = \{(1,0), (0,1)\}$ → Linear Independent Set and they only meet at origin and it has only one solution. (intersection point)
(e_1, e_2)



$\therefore B_2 \{(1,0), (100,0)\}$ Linear dependent

$|B_1| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ & $|B_3| = \begin{vmatrix} 1 & 100 \\ 0 & 0 \end{vmatrix} = 0 \rightarrow$ Linear Dependent

$\therefore B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\uparrow$

Basis of R^3 (xyz space) & $|B| = 1 \neq 0 \rightarrow$ Linear independent

$\dim \mathbb{R}^3 = n(B) = 3 \rightarrow$ no. of elements in B

$\dim \mathbb{R}^2 = n(B) = 2$

$\dim \mathbb{R} = n(B) = 1$

Find dimension of set of all 2x2 matrices

Soln:-

$$M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a,b,c,d \in R \right\}$$

$$= \{(a,b,c,d) \mid a,b,c,d \in R\}$$

$\therefore \mathbb{R}^2 = \{(a,b) \mid a,b \in R\}$

$\mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in R\}$

$\mathbb{R}^4 = \{(a,b,c,d) \mid a,b,c,d \in R\}$

Basis of $R^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$\therefore M_{n\times m} \Rightarrow \dim = n^2$

$\therefore M_{m\times n} \rightarrow \dim = mn$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

1) Projection on x-axis,

$T(x,y) = (x,0)$

$T(1,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1) = (0,0) = 0(1,0) + 0(0,1)$

$$m(T) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

$\therefore$ Characteristic equ^n,

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \underline{\underline{0}}$$

$\Rightarrow -\lambda(1-\lambda) = -\lambda + \lambda^2 = \underline{\underline{0}}$

$\Rightarrow (T^2 - T)(x,y)$

$\Rightarrow T^2(x,y) - T(x,y)$

$\Rightarrow T(T(x,y)) - T(x,y)$

$\Rightarrow T((x,0)) - (x,0)$

$\Rightarrow (x,0) - (x,0)$

$\Rightarrow \underline{\underline{(0,0)}}$.

$\boxed{\therefore f(x) = x^2 - x = 0}$ ← Annihilating polynomial of T.

2) Reflection about $y = x$,

$T(x,y) = (y,x)$

$T(1,0) = (0,1) = 0(1,0) + 1(0,1)$

$T(0,1) = (1,0) = 1(1,0) + 0(0,1)$

$$\therefore A = m(T) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Characteristic equ^n,

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \underline{\underline{\lambda^2 - 1}} = 0$

$\therefore$ Annihilating polynomial is, $f(x) = x^2 - 1$

$\therefore f(T) = T^2 - I$

$$f(T)(x,y) = (T^2 - I)(x,y)$$
$$= T^2(x,y) - I(x,y)$$
$$= T(T(x,y)) - I(x,y)$$
$$= T(y,x) - (x,y)$$
$$= (x,y) - (x,y) = \underline{(0,0)}$$

$$\left.\begin{aligned} \therefore f(x) &= x^2 - 1 \\ &= \overset{\neq 0}{(x-1)}\overset{\neq 0}{(x+1)} \\ \text{If } & (T - I) \neq 0 \\ & (T + I) \neq 0 \end{aligned}\right\} \text{Minimal Polynomial of } T.$$

Find a transformation / (matrix) which when multiplied by

1) Rotates by 90° then $\rightarrow h$

2) Reflect about $y = x$ then $\rightarrow g$

3) Project on the y-axis $\rightarrow f$ and also find the minimal polynomial matrix transformation.

Soln:-

$$(f \circ g \circ h)x = f(g(h(x)))$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \underline{\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}} = A$$

$$\therefore |A - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$
$$-\lambda(-1-\lambda) = 0$$
$$\lambda + \lambda^2 = 0$$

$\therefore$ Anhilating polynomial is,

$f(x) = x^2 + x = \underline{x(x+1)}$ ← minimal polynomial

$\therefore x \neq 0,\ T \neq 0,\ T + I = 0,\ T = -I,\ T \neq -I$

$\therefore \underline{T(x, y) = (0, -y)}$

LU Decomposition:-

$A = LU \Rightarrow$ Lower Triangular and Upper Triangular

= 

L U

Find LU Decomposition of

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

Soln:-

$R_1 \to R_1 / 6$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$ ⇒ Multiplier is $1/6$

$R_2 \to R_2 - 9R_1;$
$R_3 \to R_3 - 3R_1;$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix}$ ⇒ Multiplier is (-9) & (-3)

$R_2 \to \frac{R_2}{2}$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 8 & 5 \end{bmatrix}$ ⇒ $1/2$

$5 - \cancel{8}^{4}/\cancel{2} = 1$

$R_3 \to R_3 - 8R_2$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$ ⇒ -8

$$\therefore A = \begin{bmatrix} 6 & 0 & 0 \\ x & 2 & 0 \\ y & z & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Here 6, 2 & 1 are the reciprocal of multipliers are used to calculate the value of diagonal matrix.

$-x/3 + 2 + 0$

$-\frac{y}{3} + z$

$$A = \begin{bmatrix} 6 & -2 & 0 \\ x & -x+6/3 & 1 \\ y & \frac{-y}{3}+z & z/3+1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 0 \\ x & \frac{-x+6}{3} & 1 \\ y & \frac{-y}{3}+z & \frac{z}{2}+1 \end{bmatrix}$$

$\therefore x = 9, \quad y = 3$

$$\frac{-y}{3} + z = 7 \Rightarrow \frac{-\cancel{3}}{\cancel{3}}^{-1} + z = 7 \Rightarrow z = 8$$

$$\therefore A = \underbrace{\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Singular Value Decomposition (SVD):-

Find SVD of a matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Soln:-

Step-1:- Find eigen values and eigen vectors of AA^T

$$AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \leftarrow \text{Symmetric}$$
$= B$:

$|B - \lambda I| X = [0]$

$$\begin{vmatrix} 11 & 1 \\ 1 & 11 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(11-\lambda)^2 - 1 = 0$$
$$(11-\lambda)^2 = 1$$
$$11-\lambda = \pm 1$$
$$\lambda_1 = 10, \quad \lambda_2 = 12$$

$\lambda = 10$:-

$[B - \lambda I] X = [0]$

$$\begin{bmatrix} 11-10 & 1 \\ 1 & 11-10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x+y &= 0 \\ x+y &= 0 \end{aligned}\right\} \quad \begin{aligned} x &= 1 \\ y &= -1 \end{aligned} \quad = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{— (1)}$$

$\lambda = 12$:-

$[B - \lambda I] X = [0]$

$$\begin{bmatrix} 11-12 & 1 \\ 1 & 11-12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x + y &= 0 \\ y - x &= 0 \end{aligned}\right\} \quad \begin{aligned} x &= 1 \\ y &= 1 \end{aligned} \quad = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{— (2)}$$

∴ Unit vector for (1) $\quad u_y = \frac{1}{\sqrt{1^2 + (-1)^2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Unit vector for ② $\therefore u_y = \frac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore U_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \xRightarrow{\text{Decreasing order of } \lambda} \begin{array}{cc} \lambda=12 & \lambda=10 \\ \end{array} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

<u>Step-2</u>:- Find eigen values and vector of A^TA.

$\therefore$ Eigen values of A^TA = Eigen values of $B^T = A^TA$

$\therefore \lambda_1 = 10,\ \lambda_2 = 12,\ \lambda_3 = ?$

$$A^TA = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} = C.$$

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A^TA)$

$10 + 12 + \lambda_3 = 10 + 10 + 2 \Rightarrow \lambda_3 = 22 - 22 = \underline{\underline{0}}$

$\therefore$ <u>$\lambda_1 = 10$</u>:-

$[C - \lambda I]X = [0]$

$$\begin{bmatrix} 10-10 & 0 & 2 \\ 0 & 10-10 & 4 \\ 2 & 4 & 2-10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2z = 0 \Rightarrow z = 0$

$4z = 0$

~~2x+4y=2z~~ $2x + 4y - 2z = 0 \Rightarrow 2x = -4y$

$2 = -2 \cdot -1$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$\therefore V_1 = \frac{1}{\sqrt{1^2+(-2)^2+0}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$$

$\lambda_2 = 12$:-

$[C - \lambda I]\,[X] = [0]$

$$\begin{bmatrix} 10-12 & 0 & 2 \\ 0 & 10-12 & 4 \\ 2 & 4 & 2-12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + 0 + 2z = 0$$
$$0x - 2y + 4z = 0 \quad \Rightarrow \cancel{4y = 0}$$
$$2x + 4y - 2z = 0 \qquad \underline{y = 0}$$

$$-2x + 2z = 0$$
$$x = z \Rightarrow \underline{1 = 1}$$

$$-2y + 4z = 0 \Rightarrow z = 2.$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore V_2 = \frac{1}{\sqrt{1^2+2^2+1^2}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$\lambda_3 = 0$:-

$$\begin{bmatrix} 10-0 & 0 & 2 \\ 0 & 10-0 & 4 \\ 2 & 4 & 2-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$10x + 0y + 2z = 0 \Rightarrow x = 1,\ z = -5$$
$$0x + 10y + 4z = 0 \Rightarrow \underline{y = 2}$$
$$2x + 4y + 2z = 0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}.$$

$$\therefore V_3 = \frac{1}{\sqrt{1^2+(-5)^2+2^2}} \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{30} \\ -5/\sqrt{30} \\ 2/\sqrt{30} \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{6} & 1/\sqrt{30} \\ -1/\sqrt{5} & 2/\sqrt{6} & -5/\sqrt{30} \\ 0 & 1/\sqrt{6} & 2/\sqrt{30} \end{bmatrix} \xRightarrow{\text{Decreasing order}} \begin{array}{ccc} \lambda=12 & \lambda=10 & \lambda=0 \end{array}$$

$$\begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

Step-3:- Singular values (eigenvalues of A^TA).

$$A_{2\times3} = U_{2\times2} \underline{\begin{bmatrix} 12 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix}} V^T_{3\times3}$$ (Decreasing ord

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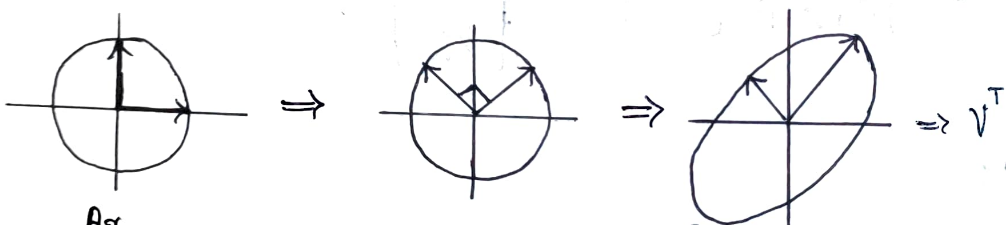
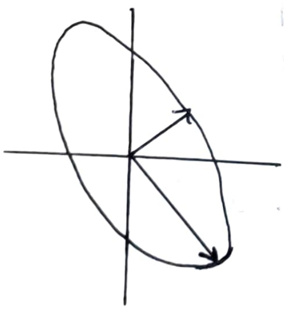
$$\therefore Ax = (U \Sigma V^T)x$$

- V^T → Orthogonal Rotation
- Σ → Diagonal Scaling
- U → Orthogonal Rotation



Ax U Σ

$V^T =$ 

Find the SVD of

$$G = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

Soln:-

if $\underset{2\times3\ \ 3\times2}{G^TG} \rightarrow \underline{\underline{2\times2}}$ and $\underset{3\times2\ \ 2\times3}{GG^T} \rightarrow 3\times3$

$$G^T G = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

$\therefore$ Eigen values, $\lambda_1 = 10$, $\lambda_2 = 12$

$\therefore$ Eigen Vectors $= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

OR

$(AB)^T = B^T A^T$ find SVD of A

$A = U \Sigma V^T$

$A^T = (U \Sigma V^T)^T$

$= U^T \Sigma^T V$

Then, $G^T = A$

$\therefore G = A^T = U^T \Sigma^T V$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

Probability

Axioms:-

1) $0 \leq P(A) \leq 1$
2) $P(S) = 1$
3) $P(A \cup B) = P(A) + P(B)$ where $A \cap B = \emptyset$

$\therefore$ These are known as, Kolmogorov's Axioms of Probability

---

Prove:-

1) $P(\emptyset) = 0$

Soln:-

$$S = S \cup \emptyset$$

$$P(S) = P(S \cup \emptyset)$$

$$P(S) = P(S) + P(\emptyset) \quad \rightarrow \text{Axiom - 3}$$

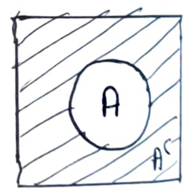
$$1 = 1 + P(\emptyset) \quad \rightarrow \text{Axiom - 2}$$

$$\therefore P(\emptyset) = 1 - 1 = 0$$

---

2) $P(A^c) = 1 - P(A)$ where, $A^c \rightarrow$ Complement of A

Soln:-



S

$$S = A \cup A^c$$

$$P(S) = P(A \cup A^c)$$

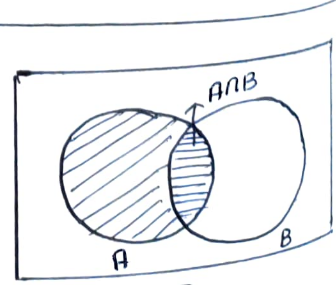
$$1 = P(A) + P(A^c) \quad \because \text{Axioms 2 \& 3.}$$

$$\therefore P(A^c) = 1 - P(A)$$

---

3) $P(A \cap B^c) = P(A) - P(A \cap B)$

Soln:-



S

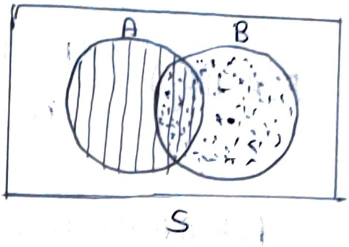
$\rightarrow$ Shadded $\qquad \rightarrow$ Horizontal line

$$A = (A \cap B^c) \cup (A \cap B)$$

$$P(A) = P(A \cap B^c) + P(A \cap B)$$

$$\therefore P(A \cap B^c) = P(A) - P(A \cap B)$$

4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ← Principle of inclusion & exclusion in combinatorics.



Soln:-

$$A \cup B = (A \cap B^c) \cup B$$

$$P(A \cup B) = P(A \cap B^c) + P(B)$$

$$= P(A) - P(A \cap B) + P(B)$$

$$\therefore \underline{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

---

5) $P(A_1 \cup A_2 \cup A_3)$

Soln:-

$$P(\underbrace{A_1}_{A} \cup \underbrace{A_2 \cup A_3}_{B}) = P(A_1) + P(A_2 \cup A_3) - P(A_1 \cap (A_2 \cap A_3))$$

$$- P(A_2 \cap A_3)$$

$$= P(A_1) + P(A_2) + P(A_3) - P((A_1 \cap A_2) \cup (A_1 \cap A_3))$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3)$$
$$+ P[(A_1 \cap A_2) \cap (A_1 \cap A_3)]$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3)$$
$$+ P(A_1 \cap A_2 \cap A_3) - P(A_2 \cap A_3)$$

$$\therefore P(A_1 \cup A_2 \cup \dots \cup A_n) = P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i \le j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i \le j < k \le n} P(A_i \cap A_j \cap A_k) + (-1)^{n-1} P(A_1 \cap A_2 \cap \dots \cap A_n)$$

Given 'n' as $\{1, 2, 3, \dots, n\}$. What is the probability that no integer takes its proper place?

Soln:-

Suppose A_i is the event where i^{th} integer takes i^{th} place

$P(A_i) = P(i^{th}$ integer taking i^{th} place$)$

$$= \frac{(n-1)!}{n!}$$

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!}, \quad P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!}$$

$$P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n = \frac{1}{n!}$$

$$\therefore P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n) = \sum_{i=1}^{n} \frac{(n-1)!}{n!} - \sum_{1 \le i \le j \le n} \frac{(n-2)!}{n!} + \sum_{1 \le i < j < k \le n} \frac{(n-3)!}{n!}$$

$$+ (-1)^{n-1} \frac{1}{n!}$$

$$= {}^nC_1 \frac{(n-1)!}{n!} - {}^nC_2 \frac{(n-2)!}{n!} + {}^nC_3 \frac{(n-3)!}{n!} - \dots$$

$$+ (-1)^{n-1} \frac{1}{n!} \, {}^nC_n$$

$$= \frac{n(n-1)!}{n!} - \frac{n(n-1)}{2!} \frac{(n-2)!}{n!} + \frac{n(n-1)(n-2)}{3!}$$

$$\frac{(n-3)!}{n!} - \dots + (-1)^{n-1} \frac{1}{n!}$$

$$= \frac{n!}{n!} - \frac{n!}{2!\,n!} + \frac{n!}{3!\,n!} - \dots + (-1)^{n-1} \frac{1}{n!}$$

P(Atleast one of the integer taking its proper place) $= 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!}$

$\therefore$ P(None of the integer is taking its proper place) $= 1 - $ P(Atleast one of the integer taking its proper place)

$$= 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!}\right)$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^n}{n!}$$

$\hookrightarrow$ Taylor Series.

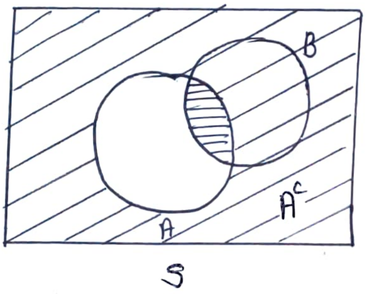
$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$\therefore e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$

$\therefore P\left[\begin{array}{l}\text{None of the integer is taking} \\ \text{its proper place}\end{array}\right] = e^{-1} = \frac{1}{e} = 0.367$

$\therefore$ Dearrangements $= 1 - e^{-1} = 0.627$

Conditional Probability:-



i.e. Probability of B given A has occured.

$\therefore P(B/A) = \frac{P(A \cap B)}{P(A)}$ if $P(A) > 0 \Rightarrow P(A \cap B) = P(B/A) \cdot P(A)$ ——— ①

Probability of 'A' given B has occured,

$\therefore P(A/B) = \frac{P(B \cap A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$

$\therefore P(A \cap B) = P(A/B) \cdot P(B)$ ——— ②

$\therefore$ ① = ②

$$\boxed{\therefore P(A/B)\, P(B) = P(B/A)\, P(A) = P(A \cap B)} \quad \text{——— ③}$$

If 'A' and 'B' are independent, then,

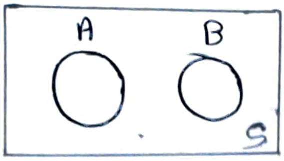
i) $P(A/B) = P(A)$

ii) $P(B/A) = P(B)$

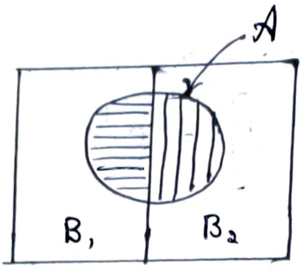
$\therefore$ If A and B are independent, ③ given,

$\boxed{\therefore P(A \cap B) = P(A)\, P(B)} = P(B)\, P(A)$

If we have independent event, we multiply the probability



$\Rightarrow A \cap B = \emptyset \rightarrow$ A & B are mutually exclusive



1> $S = B_1 \cup B_2$
2> $B_1 \cap B_2 = \emptyset$
3> $B_1 \neq \emptyset, \quad B_2 \neq \emptyset$

} B_i is partition 'S'

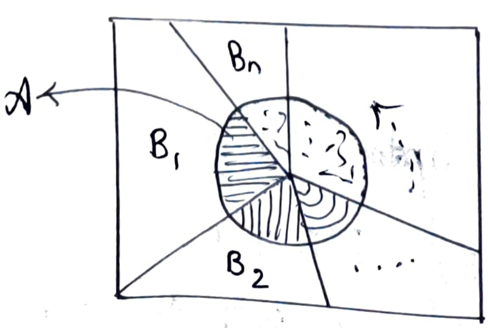
$$\therefore A = A \cap S$$
$$= A \cap (B_1 \cup B_2)$$
$$A = (B_1 \cap A) \cup (B_2 \cap A)$$
$$\therefore P(A) = P(B_1 \cap A) + P(B_2 \cap A)$$

$$\boxed{\therefore P(A) = P(A|B_1)\, P(B_1) + P(A|B_2)\, P(B_2)} \quad \text{—— (4)}$$

$$P(B_1|A) = \frac{P(A \cap B_1)}{P(A)}$$

$$\boxed{\therefore P(B_1|A) = \frac{P(A|B_1)\, P(B_1)}{P(A|B_1)\, P(B_1) + P(A|B_2)\, P(B_2)}} \quad \text{—— (5)}$$

(4) is the theorm on total probability
(5) is the Bayes theorem



$$S = B_1 \cup B_2 \cup \cdots \cup B_n$$

1) $S = B_1 \cup B_2 \cup \cdots \cup B_n$

2) $B_i \cap B_j = \phi$ for $i \neq j$

3) $B_1 \neq \phi$, $B_2 \neq \phi$, ..., $B_n \neq \phi$

$$\therefore A = A \cap S$$
$$= A \cap (B_1 \cup B_2 \cup \cdots \cup B_n)$$
$$= (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_n \cap A)$$

$$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_n \cap A)$$

$$\boxed{P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \cdots + P(A/B_n)P(B_n)}$$

$$\therefore P(B_k/A) = \frac{P(A \cap B_k)}{P(A)}$$

$$\boxed{\therefore P(B_k/A) = \frac{P(A/B_k)\,P(B_k)}{P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \cdots + P(A/B_n)P(B_n)}}$$

Experiment:-

Normal Coin	Biased Coin with both sides are Head.

If we toss the coin 4 times gives {H, H, H, H} then what is the probability that the coin is a two sided (Biased) coin?

Soln:-

A = { 4 tosses gives 4 heads }

B_1 = { coin is normal }

B_2 = { coin is biased }

$$\therefore P(B_2/A) = ?$$

$$\therefore P(B_2/A) = \frac{P(A/B_2)\,P(B_2)}{P(A)}$$

$$\therefore P(B_2|A) = \frac{P(A/B_2)\,P(B_2)}{P(A/B_1)\,P(B_1) + P(A/B_2)\,P(B_2)}$$

$\therefore P(B_1) = 1/2$, $P(B_2) = 1/2$ [$\because$ any coin we can pic out of 2]

If we select normal coin,

$P(A/B_1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^4}$ [4 times tossed].

If we selected biased coin,

$P(A/B_2) = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$

$$\therefore P(B_2/A) = \frac{1 \times \frac{1}{2}}{1/2^4 \times \frac{1}{2} + 1 \times \frac{1}{2}} = \frac{1/2}{1/2^5 + 1/2} = \frac{1/2}{1/32 + 1/2}$$

$$= \frac{\cancel{34/64}}{32} = \frac{1/2}{1/2\,(1/2^4 + 1)} = \frac{1}{1/2^4 + 1} = \frac{16}{17} = 0.9412$$

If for 'n' toss,

$P(A/B_1) = 1/2 \cdot 1/2 \cdots 1/2 = 1/2^n$

$$\therefore P(B_2/A) = \frac{2^n}{2^n + 1}$$

A bag contains 90 fair coin ($P(H) = 1/2 = P(T)$) and 10 unfair coins. A coin is picked at random and tossed 'n' times, each of the 'n' tosses gave 'n' heads.

1) What is the prob. that the picked coin is unfair coin?
2) Find the least value of 'n' that gives the prob. that the picked coin is unfair with prob. at least 90%.

Soln:-

A = {'n' tosses give n heads}

B_1	B_2
unfair	fair

$\Rightarrow$ B_1 = {Coin is unfair}

B_2 = {coin is fair}

$$\Rightarrow P(B_1/A) = \frac{P(A/B_1)\,P(B_1)}{P(A/B_1)\,P(B_1) + P(A/B_2)\,P(B_2)}$$

$\therefore P(B_1) = \frac{10}{100} = 0.1$, $P(B_2) = \frac{90}{100} = 0.9$

$$P(A/B_1) = \underset{H}{3/4} \cdot \underset{H}{3/4} \cdots \underset{H}{3/4} = (3/4)^n$$

$$P(A/B_2) = 1/2 \cdot 1/2 \cdots 1/2 = (1/2)^n$$

$$\therefore P(B_1|A) = \frac{(3/4)^n \times 0.1}{(3/4)^n \times 0.1 + (1/2)^n \times 0.9}$$

2)

n	$P(B_1 \mid A)$
1	0.1428
⋮	
5	0.457
⋮	
10	0.86
11	0.905

$\therefore$ The least value of $n = 11$ gives 90% probability that picked coin is unfair

Suppose we have screening test, the test whether a patient has a particular disease. We denote +ve & -ve results as positive & negative respectively, the 'D' denotes the person having disease in the population. Suppose that the test is not accurate & follows the following rules,

$P(\text{positive}/D) = 95\%$

$P(\text{positive}/D^c) = 8\%$

$P(D) = 0.9\%$

What is the probability that the person has disease given that the positive result?

Soln:-

A = { Test is given and it gave +ve result }

B_1 = { Person has the disease }

B_2 = { Person doesn't have disease }

$$P(B_1/A) = \frac{P(A/B_1)\,P(B_1)}{P(A/B_1)\,P(B_1) + P(A/B_2)\,P(B_2)}$$

$\therefore P(B_1) = \frac{0.9}{100}$, $P(B_2) = 1 - \frac{0.9}{100} = 0.991$

$P(A/B_1) = \frac{95}{100} = 0.95$

$P(A/B_2) = \frac{8}{100} = 0.08$

$$\therefore P(B_1/A) = \frac{0.95 \times 0.009}{0.95 \times 0.009 + 0.08 * 0.991} = 0.0973 = 9.73\%$$

For the same question, if two tests are conducted then,

$$P(B_1/A) = \frac{(0.95)^2 \times 0.009}{(0.95)^2 \times 0.009 + (0.08)^2 \times 0.991} = 0.561$$

$= 56.1\%$

Monte - Carlo Algorithm:-

1) If the algorithm returns 'Yes' then integer 'm' definitly has property 'p' is 1

$$P(\text{m has property 'p'} \,/\, \text{Algorithm return 'Yes'}) = 1$$

and

$$P(\text{Algorithm return No} \,/\, \text{Not } p) = 1$$

2) If 'm' has property 'p' then the algorithm returns 'Yes' for atleast 50% of choices for a randomly selected elements

i.e. $P(\text{Algorithm returns 'Yes'} \,/\, \text{'m' has property}) \geq \frac{1}{2}$

In a Monte-Carlo algorithm all 'N' trials returns the answer 'NO' how confident can be that our integer 'm' doesn't have the property 'p' ?

A multinational company has its branches at 3 places A, B, C where in they have 1729, 4104, 7999 employes respectively. For the purpose of dealing with transfers the company has devided employes into 2 category Type1 & Type 2. 17%, 2% & 9% of the employes are of type 2 & the place is A, B, C respectively. The company transfers the employes of Type 2 what is prob. that the employes is situated at place 'A'?

Soln:-

T = { Employee of Type 2 was transfered }

B_1 = { Employe is from A }

B_2 = { Employe is from B }

B_3 = { Employe is from C }

$$\therefore P(B_1/T) = \frac{P(T/B_1)\,P(B_1)}{P(T/B_1)\,P(B_1) + P(T/B_2)\,P(B_2) + P(T/B_3)\,P(B_3)}$$

$\therefore P(T/B_1) = 17\% = \underline{0.17}$, $P(T/B_2) = 0.02$, $P(T/B_3) = 0.09$

$$P(B_1) = \frac{1729}{1729+4104+7999} = 0.125, \quad P(B_2) = 0.2967$$

$P(B_3) = 0.5783$

$$\therefore P(B_1/T) = \frac{0.17 \times 0.125}{0.17 \times 0.125 + 0.02 \times 0.2967 + 0.09 \times 0.5783}$$

$= 0.2682$

$= \underline{\underline{26.82}}\ \%$

Collision :-

Birthday Paradox :-

P(Two student have same birthday) = 1 - P(No 2 student have same birthday)

Assume 365 days in a year. then

$$p(T) = 1 - \left(\underset{\substack{\downarrow \\ 1^{st}\text{ Student}}}{\frac{365-0}{365}}, \underset{\substack{\downarrow \\ 2^{nd}}}{\frac{365-1}{365}}, \underset{\substack{\downarrow \\ 3^{rd}}}{\frac{365-2}{365}}, \ldots \underset{\substack{\downarrow \\ n}}{\frac{365-(n-1)}{365}} \right)$$

n	P(T)
$n=2$	$1 - \frac{365}{365} \cdot \frac{364}{365} \approx 0.0027$
$n=3$	$1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = 0.0082$
$n=4$	0.016
$n=5$	0.027
$n=6$	0.040
⋮	
$n=22$	0.4757
$\underline{\underline{n=23}}$	0.5073 → min value

Functions:-

Random Variable:-

$X : S \rightarrow \mathbb{R}$

this is the property from 'Sample Space' to 'Real Numbers', or line.

Coin tossed 3 times,

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$P(H) = 1/2 = P(T)$

$X(s) \rightarrow$ Count the number of heads in the event 'S'.

$X(HHH) = 3$

$X(HHT) = 2 = X(HTH) = X(THH)$

$X(HTT) = 1 = X(THT) = X(TTH)$

$X(TTT) = 0$

Range of X = {0, 1, 2, 3}

$P(X=0) = P(TTT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$P(X=1) = P(HTT \cup TTH \cup THT) = P(HTT) + P(THT) + P(TTH)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{3}{8}$

$P(X=2) = P(HHT \cup HTH \cup THH) = P(HHT) + P(HTH) + P(THH)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{3}{8}$

$P(X=3) = P(HHH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

Rang of X	P(X = x)
0	1/8
1	3/8
2	3/8
3	1/8

→ Probability Distributive Function (PDF)

— (1)

+ = 1

For

$P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$

$P(X=0) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$

$P(X=1) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{64}$

$P(X=2) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$

$P(X=3) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$

Rang of X	P(X = x)
0	1/64
1	9/64
2	27/64
3	27/64

— (2)

+ = 1

Probability Distributive Function:-

$P(X=x)$ is known as Pdf if

i) $P(X=x) \geq 0$

ii) $\sum_{x \in S} P(X=x) = 1$

A function $X: S \rightarrow \mathbb{R}$ is called random variable if there exists a Pdf associated with this random variable.

$\therefore$ Expected Outcome,

$$E(X) = \sum_{x \in S} x\, P(X=x)$$

$\therefore$ For the table ②

$$E(X) = 0 \cdot 1/64 + 1 \times \frac{9}{64} + 2 \times \frac{27}{64} \times 3 \times \frac{27}{64} = 2.25$$

Mean / Expectation / Average : $E(X)$, μ, $\bar{X}$

$X = 5, 6, 7, 8, 9$

$$E(X) = \frac{5+6+7+8+9}{5} = 7$$

$X - E(X) = -2, -1, 0, 1, 2$

$(X - E(X))^2 = 4, 1, 0, 1, 2$

$$\therefore V(X) = E((X - E(X))^2) = \frac{4+1+0+1+2}{5} = 2$$

$\therefore \sigma^2 = V(X) = 2$

$\therefore \sigma = \sqrt{2}$ (Standard deviation)

For the table ②

$(X - E(x))^2 \Rightarrow 5.0625,\ 1.5625,\ 0.0625,\ 0.5625$

$$\therefore E((X - E(x))^2) = 5.0625 \cdot 1/64 + 1.5625 \cdot 9/64 + 0.0625 \cdot \frac{27}{64} + 0.5625 \cdot \frac{27}{64}$$

$$= 0.5625 \Rightarrow \sigma = \sqrt{0.5625} = 0.75$$

$$V(x) = E[(x-\mu)^2] \quad , \quad \mu = E(x)$$

$$= E[x^2 - 2\mu x + \mu^2]$$

$$= E(x^2) - 2\mu E(x) + E(\mu^2)$$

$$= E(x^2) - 2\mu^2 + \mu^2$$

$$= E(x^2) - \mu^2$$

$$\therefore V(x) = E(x^2) - (E(x))^2$$

$$\therefore E(x^2) = \sum_{x=0}^{3} x^2 P(X=x)$$

So, table ②'s values,

$$= 0^2 \cdot \frac{1}{64} + 1^2 \cdot \frac{9}{64} + 2^2 \cdot \frac{27}{64} + 3^2 \cdot \frac{27}{64} = 5.625$$

$$\therefore \sigma^2 = V(x) = E(x^2) - (E(x))^2$$

$$= 5.625 - (2.25)^2$$

$$= 0.5625$$

Q) A number is considered x from $\{1, 2, 3, \ldots n\}$. Find the

i) $E(x)$ and ii) $V(x)$

Soln:-

X	P(X=x)
1	1/n
2	1/n
3	1/n
⋮	⋮
n	1/n

i) $$E(x) = \sum_{x=1}^{n} x P(X=x)$$

$$= 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \cdots + n \cdot \frac{1}{n}$$

$$= \frac{1}{n}(1+2+3+\cdots+n)$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

$\therefore E(X^2) = \frac{1}{n}(1^2 + 2^2 + 3^2 + \cdots + n^2)$

$$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

$\therefore V(X) = E(X^2) - [E(X)]^2$

$$= \frac{(n+1)(2n+1)}{6} - \left[\frac{(n+1)}{2}\right]^2$$

$$= \frac{2(2n^2 + n + 2n + 1) - 3(n^2 + 2n + 1)}{12}$$

$$\sigma^2 = \frac{n^2 - 1}{12}$$

$$\therefore \sigma = \sqrt{\frac{n^2-1}{12}}$$

Q) There are 2 question and 1st question has 3 options and 2nd question has 4 options. A student selected an answer at random one for each question. If X = number of questions answered correctly.

Find i) $E(X)$ ii) $V(X)$.

Soln:-

X	$P(X=x)$
0	$6/12$
1	$5/12$
2	$1/12$

$\therefore P(X=2) \Rightarrow P(\text{I} \cap \text{II})$

$= P(\text{I})\, P(\text{II})$

$= 1/3 \cdot 1/4 = 1/12$

→ incorrect

→ correct

$\therefore P(X=1) = P(\text{I} \cap \overline{\text{II}}) + P(\overline{\text{I}} \cap \text{II})$

$$= \frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{1}{4} = \frac{5}{12}$$

$\therefore P(X=0) = P(\overline{\text{I}} \cap \overline{\text{II}})$

$$= \frac{2}{3} \cdot \frac{3}{4} = 6/12$$

i) $E(X) = \sum_{x=0}^{n} x P(X=x)$

$= 0.\frac{6}{12} + 1.\frac{5}{12} + 2.\frac{1}{12} = \frac{7}{12}$

$E(X^2) = 0^2.\frac{6}{12} + 1^2.\frac{5}{12} + 2^2\frac{1}{12} = \frac{9}{12}$

ii) $V(X) = E(X^2) - [E(X)]^2$

$= \frac{9}{12} - \left(\frac{7}{12}\right)^2$

$\sigma^2 = \frac{59}{144}$

$\therefore \sigma = \sqrt{\frac{59}{144}}$

For the same question if the question is answered correctly the 1 mark given else -0.5 mark given.

$\therefore \phi(x) = \begin{cases} -1 & , x=0 \\ 0.5 & , x=1 \\ 2 & , x=2 \end{cases}$

$\therefore E(X) = \cancel{0}\,{-1} \times \frac{6}{12} + 0.5 \times \frac{5}{12} + 2 \times \frac{1}{12} = -0.125$

$\therefore E(X^2) = \cancel{0}\,(-1)^2 \times \frac{6}{12} + (0.5)^2 \times \frac{5}{12} + 2^2 \times \frac{1}{12} = 0.9375$

$\therefore V(X) = E(X^2) - [E(X)]^2$

$= 0.9375 - (-0.125)^2$

$\sigma^2 = 0.921875$

$\therefore \sigma = \sqrt{0.921875}$

When a pair of dies are thrown and the faces of dice values are summed up. Then find the,

i) $E(X)$

ii) $V(X)$

Soln:-

X	P(X = x)	f(x)
2	1/36	2
3	2/36	-3
4	3/36	4
5	4/36	-5
6	5/36	6
7	6/36	-7
8	5/36	8
9	4/36	-9
10	3/36	10
11	2/36	-11
12	1/36	12

finding sum of dices

X	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

⇒ Total = 36 values,

$$\therefore f(x) = \begin{cases} a+b, & \text{if } a+b \text{ is even} \\ -(a+b), & \text{if } a+b \text{ is odd} \end{cases}$$

$$\therefore E(X) = \sum_{x=1}^{12} f(x)\, P(X=x)$$

$$= 2\times\frac{1}{36} + (-3)\frac{2}{36} + \ldots\ldots + 12\times {}^{1}/_{36}$$

$$= \underline{0} \text{ (fair game)}.$$

$$\therefore E(X^2) = 2^2\times\frac{1}{32} + (-3)^2\times\frac{2}{36} + \ldots\ldots + (12)^2\times {}^{1}/_{36} = \underline{54.8333}$$

i) $\therefore V(f(x)) = E(x^2) - [E(x)]^2$

$= 54.8333 - 0$

$\therefore \sigma = \sqrt{54.8333}$

Discrete:-

i) $P(X = x) \geq 0$

ii) $\sum P(X = x) = 1$

Continuous:-

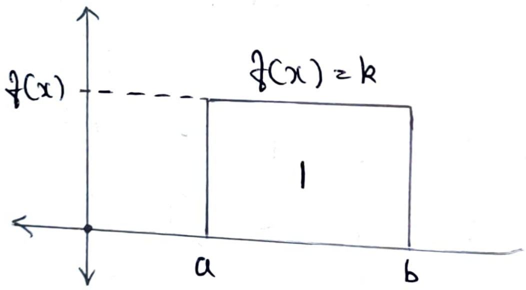
$E(X) = \int x f(x)\,dx$

$E(X^2) = \int x^2 f(x)\,dx.$

$\therefore f(x)$ is a Pdf if

i) $f(x) \geq 0$

ii) $\int f(x)\,dx = 1$



Finding k such that area is = 1 ?

$$\int_a^b f(x)\,dx = 1$$

$$\int_a^b k\,dx = 1$$

$$k(b-a) = 1$$

$$k = 1/b-a$$

$$\therefore E(x) = \int_a^b x f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_b^a$$

$$= \frac{1}{b-a}\ \frac{(b^2-a^2)}{2}$$

$$= \frac{b+a}{2}$$

$$\therefore E(x^2) = \int x^2 f(x)dx = \int_a^b \frac{1}{b-a}\left[\frac{x^2}{\cancel{2}}\right]dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_b^a$$

$$= \frac{1}{b-a}\ \frac{(b^3-a^3)}{3} = \frac{1}{3\cancel{(b-a)}}\cancel{(b-a)}(b^2+ab+a^2)$$

$$\therefore \quad = \frac{b^2+ab+a^2}{3\cancel{(b-a)}}$$

$$\therefore V(X) = E(X^2) - (E(X))^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

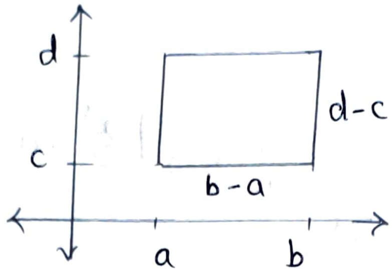
$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

Random Variable	Discrete	Continuous	Dimension
X	$\sum P(X=x) = 1$	$\int_a^b f(x) = 1$	1
(X,Y)	$\sum\sum P(X=x, Y=y) = 1$	$\iint_R f(x,y)\,dx\,dy = 1$	2

Q:- (x, y) is a uniformly distributed random variable over
$$R = [a, b] \times [c, d]$$
then, $E(X)$, $E(Y)$, $V(X)$, $V(Y)$.

Soln:-



$$\therefore f(x,y) = \begin{cases} \dfrac{1}{(b-a)(d-c)} & , (x,y) \in R \\ 0 & , (x,y) \notin R \end{cases}$$

$$\therefore \iint f(x,y)\, dx . dy = \int_a^b \int_c^d \frac{1}{(b-a)(d-c)}\, dy . dx$$

$$= \int_a^b \frac{1}{(b-a)(d-c)} [y]_c^d\, dx$$

$$= \frac{1}{(b-a)\cancel{(d-c)}} \int_a^b \cancel{(d-c)}\, dx$$

$$= \frac{1}{b-a} [x]_a^b = \frac{1}{\cancel{b-a}} \cancel{(b-a)}$$

$$= 1$$

$$\therefore E(X) = \iint_R x\, f(x,y)\, dy\, dx$$

$$= \int_{x=a}^{b} \int_{y=c}^{d} x \frac{1}{(b-a)(d-c)}\, dy . dx$$

$$= \frac{1}{(b-a)(d-c)} \int_{x=a}^{b} x [y]_c^d\, dx$$

$$= \frac{1}{(b-a)\cancel{(d-c)}} \int_{x=a}^{b} x\,\cancel{(d-c)}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{(b-a)2}(b^2-a^2) = \frac{1}{2\cancel{(b-a)}}(b+a)\cancel{(b-a)}$$

$$= \frac{b+a}{2}$$

$$E(Y) = \iint_R y\, f(x,y)\, dy.dx$$

$$= \int_{x=a}^{b}\int_{y=c}^{d} y \frac{1}{(b-a)(d-c)}\, dy.dx$$

$$= \frac{1}{(b-a)(d-c)}\int_{x=a}^{b}\left[\frac{y^2}{2}\right]_c^d dx$$

$$= \frac{1}{2(b-a)\cancel{(d-c)}}\int_{x=a}^{b}(d^2-c^2)\,dx$$

$$= \frac{(d+c)}{2(b-a)}\int_{x=a}^{b} dx = \frac{d+c}{2(b-a)}\left[x\right]_a^b = \frac{c+d}{2\cancel{(b-a)}}\cancel{(b-a)}$$

$$= \frac{c+d}{2}$$

$$E(X^2) = \iint_R x^2 f(x,y)\, dy.dx$$

$$= \int_{x=a}^{b}\int_{y=c}^{d} x^2 \frac{1}{(b-a)(d-c)}\, dy.dx$$

$$= \cancel{f}\frac{x^2}{(b-a)(d-c)}\int_{x=a}^{b} \cancel{\left[\frac{x^3}{3}\right]_c^d dy}\left[y\right]_c^d dx$$

$$= \frac{x^2}{(b-a)\cancel{(d-c)}}\int_{x=a}^{b}\cancel{(d-c)}\,dx$$

$$= \frac{1}{(b-a)} \left[\frac{x^3}{3} \right]_a^b = \frac{1}{3(b-a)} (b^3 - a^3)$$

$$= \frac{1}{3\cancel{(b-a)}} \cancel{(b-a)} (b^2 + ab + a^2)$$

$$= \frac{b^2 + ab + a^2}{3}$$

$$\therefore E(Y^2) = \iint_R y^2 f(x,y)\, dy.dx$$

$$= \int_{x=a}^{b} \int_{y=c}^{d} y^2 \frac{1}{(b-a)(d-c)}\, dy.dx$$

$$= \frac{1}{(b-a)(d-c)} \int_{x=a}^{b} \left[y^3/3 \right]_c^d dx$$

$$= \frac{1}{3(b-a)(d-c)} \int_{x=a}^{b} (d^3 - c^3)\, dx = \frac{\cancel{(d-c)}(d^2+dc+c^2)}{3(b-a)\cancel{(d-c)}} \int_{x=a}^{b} dx$$

$$= \frac{d^2 + dc + c^2}{3(b-a)} [x]_a^b$$

$$= \frac{d^2 + dc + c^2}{3\cancel{(b-a)}} \cancel{(b-a)}$$

$$= \frac{d^2 + dc + c^2}{3}$$

$$\cancel{\therefore V(X) = E(X^2) - [E(X)]^2}$$

$$\cancel{= \frac{d^2 - dc + c^2}{3} - \left[\frac{b+a}{2} \right]^2}$$

$$\cancel{= \frac{d^2 - dc + c^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{4d^2 - 4dc + 4c^2 - 3b^2 - 6ab + 3a^2}{12}}$$

$$\cancel{= \frac{3a^2 - 3b^2 + 4c^2 + 4d^2 - 4dc - 6ab}{12}}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{b^2 - ab + a^2}{3} - \left[\frac{a+b}{2}\right]^2 = \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{d^2 + dc + c^2}{3} - \left[\frac{c+d}{2}\right]^2$$

$$= \frac{d^2 + dc + c^2}{3} - \frac{c^2 + 2cd + d^2}{4}$$

$$= \frac{4d^2 + 4dc + 4c^2 - 3c^2 - 6cd - 3d^2}{12}$$

$$= \frac{d^2 - 2dc + c^2}{12} = \frac{(c-d)^2}{12}$$

Discreate

$X = \{-1, 0, 1\}$, $Y = \{1, 2, 3\}$

$x \backslash y$	1	2	3	Marginal Pdf of x = $g(x)$
-1	$1/4$	0	$1/4$	$1/2$
0	0	0	0	0
1	$1/4$	0	$1/4$	$1/2$
Marginal Pdf of y = $h(y)$	$1/2$	0	$1/2$	$\sum_x \sum_y f(x,y) = 1$

$$\boxed{\therefore f(x,y) = g(x) \cdot h(y)}$$

x, y are independent if $f(x,y) = g(x) \cdot h(y)$ for all x, y.

$$E(XY) = \sum_x \sum_y xy\, f(x,y)$$

$$= \sum_x \sum_y xy\, g(x)\, h(y)$$

$$= \left[\sum_x x\, g(x)\right]\left[\sum_y y\, h(y)\right]$$

$$E(XY) = E(X)\, E(Y)$$

This holds if and only if x, y are independent.

Covariance:-

$$\therefore \mathrm{Cov}(XY) = E(XY) - E(X)\,E(Y)$$

Correlation Coefficient:-

$$\therefore \rho_{xy} = \frac{\mathrm{Cov}(XY)}{\sqrt{V(X)\, V(Y)}}$$

Example:- Given,

x	-1	0	1
y	-4	1	6
xy	4	0	6

Soln:-

$$E(X) = \frac{-1+0+1}{3} = 0$$

$$E(Y) = \frac{-4+1+6}{3} = 1$$

$$E(XY) = \frac{4+0+6}{3} = 10/3$$

$$\therefore \mathrm{Cov}(XY) = E(XY) - E(X)\,E(Y)$$

$$= \frac{10}{3} - (0)(1)$$

$$= \frac{10}{3}$$

$$\therefore E(X^2) = \frac{(-1)^2 + 0^2 + 1^2}{3} = \frac{2}{3}, \quad E(Y^2) = \frac{(-4)^2 + 1^2 + 6^2}{3} = \frac{53}{3}$$

$\therefore V(X) = E(X^2) - [E(X)]^2 = \frac{2}{3} - 0 = 2/3$

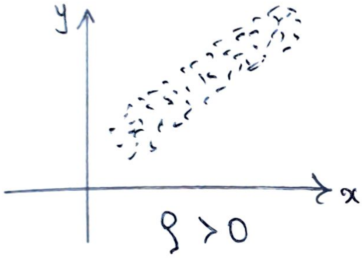
$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{53}{3} - 1 = \underline{\frac{50}{3}}$

$\therefore \cancel{\rho_{xy}}\ \cancel{=}\ Cov(XY) = E(XY) - E(X)E(Y)$

$\cancel{= \frac{10}{3} - 4}$

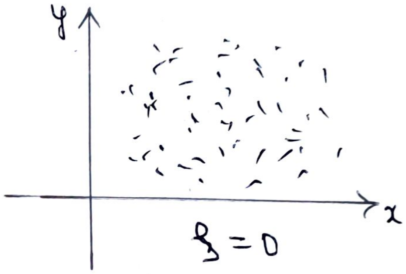
$$\therefore \rho_{xy} = \frac{Cov(XY)}{\sqrt{V(X)\, V(Y)}}$$

$$= \frac{\frac{10}{3}}{\sqrt{2/3 \cdot 50/3}} = \frac{10/3}{10/3} = \underline{\underline{1}}$$



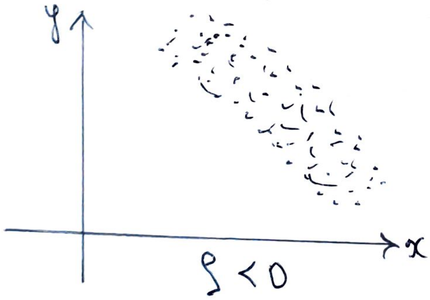
$\rho > 0$

+ve Correlation



$\rho = 0$

No Correlation



$\rho < 0$

-ve Correlation.

Example:- Given

x	-1	0	1
y	5	3	1
xy	-5	0	1

Soln:-

$E(X) = \frac{-1+0+1}{3} = \underline{0}$

$E(Y) = \frac{5+3+1}{3} = \underline{3}$

$E(XY) = \frac{-5+0+1}{3} = \underline{-4/3}$

$$Cov(XY) = E(XY) - E(X)E(Y)$$
$$= \frac{-4}{3} - (0)(3) = \underline{\underline{\frac{-4}{3}}}$$

$$E(X^2) = \frac{(-1)^2 + 0^2 + 1^2}{3} = \underline{\underline{2/3}}$$

$$E(Y^2) = \frac{5^2 + 3^2 + 1^2}{3} = \underline{\underline{\frac{35}{3}}}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2}{3} - (0)^2 = \underline{\underline{\frac{2}{3}}}$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{35}{3} - 3^2 = 8/3$$

$$\therefore \rho_{xy} = \frac{Cov(XY)}{\sqrt{V(X)\,V(Y)}}$$

$$= \frac{-4/3}{\sqrt{2/3 \cdot 8/3}} = \frac{-4/3}{4/3} = \underline{\underline{-1}}$$

$$E[[X - E(X)][Y - E(Y)]] = E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$
$$= E(XY) - E(Y)E(X) - \cancel{E(X)E(Y)} + \cancel{E(X)E(Y)}$$
$$= E(XY) - E(X)E(Y)$$
$$= \underline{Cov(XY)}$$

<u>Result</u> :- $-1 \le \rho \le 1$

<u>Proof</u> :- $E[(u \pm v)^2] \ge 0$ $\quad \because$ Expectation of non-negative random variable is non-negative.

$$\Rightarrow E[u^2 \pm 2uv + v^2] \ge 0$$

$$\Rightarrow E(u^2) \pm 2E(uv) + E(v^2) \ge 0$$

choose, $u = \frac{X - E(X)}{\sqrt{V(X)}}$, $v = \frac{Y - E(Y)}{\sqrt{V(Y)}}$

$$\Rightarrow E\left[\frac{(X - E(X))^2}{V(X)}\right] \pm 2E\left[\frac{X - E(X)}{\sqrt{V(X)}} \cdot \frac{Y - E(Y)}{\sqrt{V(Y)}}\right] + E\left[\frac{(Y - E(Y))^2}{V(Y)}\right] \ge 0$$

$$\Rightarrow \frac{V(X)}{V(X)} \pm 2\rho + \frac{V(Y)}{V(Y)} \geq 0$$

$$\Rightarrow 1 \pm 2\rho + 1 \geq 0$$

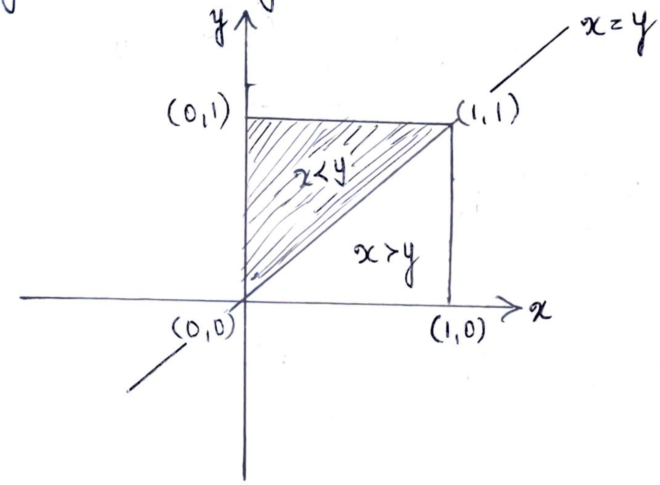
$$\Rightarrow 2 \pm 2\rho \geq 0$$

$$\Rightarrow 1 \pm \rho \geq 0$$

$$\boxed{\therefore \; -1 \leq \rho \leq 1}$$

Suppose x & y are uniformly distributed, $R = \{(x,y) \mid 0 \leq x < y \leq 1\}$ over this region. Find ρ_{xy}?

Soln:-



$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}, \quad E(X) = \iint_R x f(x,y)\, dy \cdot dx$$

$$f(x) = \begin{cases} \dfrac{1}{\text{Area of upper triangle or } R}, & (x,y) \in R \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{1/2 \times 1 \times 1} \\ 0 \end{cases}$$

$$= \begin{cases} 2 \\ 0 \end{cases}$$

$$\therefore E(X) = \int_{x=0}^{1} \int_{y=x}^{1} x\, 2\, dy \cdot dx = 2\int_{x=0}^{1} x \left[y \right]_x^1 dx$$

$$= 2\int_{x=0}^{1} x(1-x)\,dx$$

$$= 2\int_{x=0}^{1} x - x^2 .\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\tfrac{1}{2}(1-0) - \tfrac{1}{3}(1-0)\right]$$

$$= \underline{\tfrac{1}{3}}$$

$$\therefore E(X^2) = \iint_R x^2 f(x)\, dy.dx = \int_{x=0}^{1}\int_{y=x}^{1} 2x^2\, dy.dx$$

$$= 2\int_{x=0}^{1} x^2 \left[y\right]_x^1 dx$$

$$= 2x^2\int_{x=0}^{1}(1-x)\,dx = 2\int_0^1 x^2 - x^3\, dx = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\left[\tfrac{1}{3}(1-0) - \tfrac{1}{4}(1-0)\right] = \underline{\tfrac{1}{6}}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \underline{\tfrac{1}{18}}$$

$$E(Y) = \iint_R y\, f(x,y)\, dy.dx = \int_{x=0}^{1}\int_{y=x}^{1} 2y\, dy.dx$$

$$= \cancel{2}\int_{x=0}^{1}\left[\frac{y^2}{\cancel{2}}\right]_x^1 dx = \int_{x=0}^{1}\left[1 - x^2\right] dx = \int_{x=0}^{1} 1 - x^2\, dx$$

$$= -\cancel{\left[\frac{x^3}{3}\right]_0^1} \ne \cancel{\tfrac{1}{3}[1-0]} = \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \tfrac{1}{3}$$

$$= \underline{\tfrac{2}{3}}$$

$$E(Y^2) = \iint_R y^2 f(x,y)\, dy.dx = \int_0^1 \int_x^1 2y^2\, dy.dx$$

$$= 2\int_0^1 \left[\frac{y^3}{3}\right]_x^1 dx = \frac{2}{3}\int_0^1 [1 - x^3]\, dx$$

$$= \frac{2}{3}\left[x - \frac{x^4}{4}\right]_0^1 = \frac{2}{3}\left[1 - \frac{1}{4}\right] = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}_2}$$

$$= \frac{1}{2}$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$E(XY) = \iint_R xy\, f(x,y)\, dy.dx = \int_0^1 \int_x^1 2xy\, dy.dx$$

$$= \cancel{2}x\int_0^1 \left[\frac{y^2}{\cancel{2}}\right]_x^1 dx = x\int_0^1 [1 - x^2]\, dx = \int_0^1 x - x^3\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore \rho_{xy} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)\,V(Y)}}$$

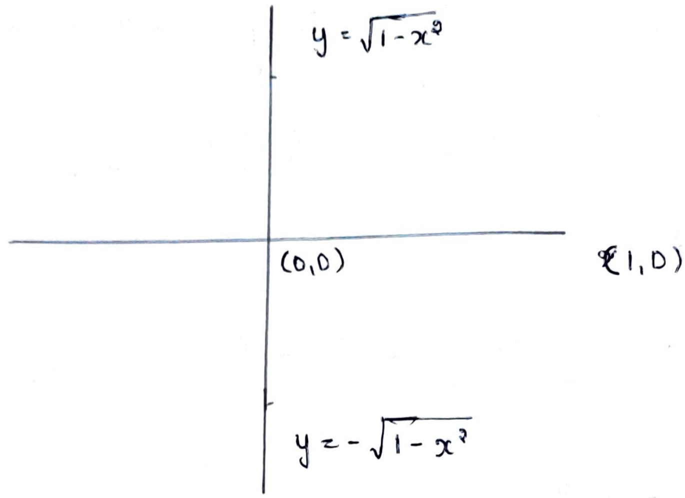
$$= \frac{1/4 - 1/3 \cdot 2/3}{\sqrt{1/18 \cdot 1/18}}$$

$$= \frac{1/36}{1/18} =$$

$$= \frac{1}{2}$$

Suppose x & y are uniformly distributed over region formed by $x^2+y^2=1$ (circle) then find ρ_{xy}.

Soln:-



$$f(x,y)=\begin{cases}\dfrac{1}{\pi\cdot(1)^2}, & x,y\in R\\ 0, & \text{otherwise.}\end{cases}$$

$$E(X)=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\cdot f(x,y)\,dy.dx. = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\cdot\frac{1}{\pi}\,dy.dx$$

Here, $x=r\cos\theta$, $y=r\sin\theta$ and $dy.dx=r\,dr\,d\theta$

$$\therefore E(X)=\frac{1}{\pi}\int_0^{2\pi}\int_0^1 r\cos\theta\; r\,dr\,d\theta.$$

$$=\frac{1}{\pi}\left[\int_0^{2\pi}\cos\theta\,d\theta\right]\left[\int_0^1 r^2\,dr\right]$$

$$=\frac{1}{\pi}\Big[\sin\theta\Big]_0^{2\pi}\left[\frac{r^3}{3}\right]_0^1$$

$$=\underline{\underline{0}}$$

$$Cov(X,Y)=E(XY)-\underbrace{E(X)}_{0}E(Y)$$

$$E(XY) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy \frac{1}{\pi}\, dy\, dx = \frac{1}{\pi}\int_{0}^{2\pi}\int_{0}^{1} r\cos\theta\,(r\sin\theta)\, r\, dr\, d\theta.$$

$$= \frac{1}{\pi}\left[\int_{0}^{2\pi} \cos\theta \sin\theta\, d\theta\right]\left[\int_{0}^{1} r^3\, dr\right]$$

$$= \frac{1}{\pi}\left[\cancel{\sin\theta}\ \frac{t^2}{2}\right]_0^0 \left[\frac{r^4}{4}\right]_0^1$$

$$= \underline{\underline{0}}$$

$\sin\theta = t$

$\cos\theta\, d\theta = dt$

Where, $\theta = 0$, $t = 0$

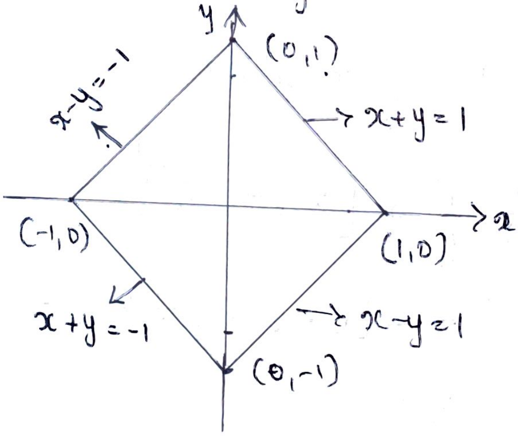
$\theta = 2\pi$, $t = 0$

$\therefore Cov(X, Y) = \underline{\underline{0}}$

$\therefore \rho_{xy} = 0$

Suppose, x, y are uniformly distributed over region formed by square with vertices, Find ρ_{xy}.

Soln:-



$$f(x,y) = \begin{cases} \dfrac{1}{\text{Area of } R}, & (x,y) \in R \\ 0, & \text{Otherwise,} \end{cases}$$

$\sqrt{(1-0)^2+(0-1)^2}$

$\Rightarrow d = \sqrt{1+1} = \sqrt{2}$

$\therefore$ Area $= d^2 = \underline{2}$

$$= \begin{cases} 1/2, & (x,y) \in R \\ 0, & \text{otherwise,} \end{cases}$$

OR

$$\text{Area} = 2\int_{0}^{1}\int_{x-1}^{1-x} dy\, dx = 2\int_0^1 [y]_{x-1}^{1-x}\, dx$$

$$= 2\int_0^1 (1-x) - (x-1)\, dx = 2\int_0^1 (2-2x)\, dx$$

$$= 2\left[2x - \cancel{2}\frac{x^2}{\cancel{2}}\right]_0^1 = 2\left[2(1-0) - (1^2-0^2)\right] = 2(2-1)$$

$$= \underline{\underline{2}}$$

$$\therefore E(x) = \iint_{(x,y)\in R} x \cdot \frac{1}{2}\, dy \cdot dx$$

$$= \frac{1}{2}\int_{-1}^{0}\int_{-1-x}^{x+1} x \cdot dy \cdot dx + \frac{1}{2}\int_{0}^{1}\int_{x-1}^{1-x} x\, dy \cdot dx$$

$$= \frac{x}{2}\int_{-1}^{0} \cdot \left[y\right]_{-1-x}^{x+1} dx + \frac{1}{2}\int_0^1 x\left[y\right]_{x-1}^{1-x} dx$$

$$= \frac{x}{2}\int_{-1}^{0} (x+1) - (-1-x)\, dx + \frac{x}{2}\int_0^1 (1-x) - (x-1)\, dx$$

$$= \frac{x}{\cancel{2}}\int_{-1}^{0} \cancel{2}x + \cancel{2}\, dx + \frac{x}{\cancel{2}}\int_0^1 2 - 2x\, dx$$

$$= \int_{-1}^{0} x^2 + x\, dx + \int_0^1 x - x^2\, dx$$

$$= \left[x^3/3 + \frac{x^2}{2}\right]_{-1}^{0} + \left[x^2/2 - x^3/3\right]_0^1$$

$$= \frac{1}{3}\left[0 - (-1)^3\right] + \frac{1}{2}\left[0 - (-1)^2\right] + 1/2(1-0) - 1/3(1-0)$$

$$= \cancel{1/3} - \cancel{1/2} + \cancel{1/2} - \cancel{1/3}$$

$$= \underline{\underline{0}}$$

$$E(XY) = \iint_{x,y\in R} xy\, 1/2\, dy \cdot dx = 1/2\int_{-1}^{0}\int_{1-x}^{x+1} xy\, dy\, dx + 1/2\int_0^1\int_{x-1}^{1-x} xy\, dy\, dx$$

$$= \frac{x}{2}\int_{-1}^{0} \left[y^2/2\right]_{1-x}^{x+1} dx + x/2\int_0^1 \left[y^2/2\right]_{x-1}^{1-x} dx$$

$$= \frac{x}{4}\int_{-1}^{0} (x+1)^2 - (1-x)^2\, dx + \frac{x}{4}\int_0^1 (1-x)^2 - (x-1)^2\, dx$$

$$= \frac{x}{4}\int_{-1}^{0} \cancel{x^2} + \cancel{2x} + \cancel{1} - \cancel{1} + \cancel{2x} - \cancel{x^2}\, dx + \frac{x}{4}\int_0^1 \cancel{1} - \cancel{2x} + \cancel{x^2} - \cancel{x^2} + \cancel{2x} - \cancel{1}\, dx$$

$= \frac{x}{\cancel{4}} \int_{-1} \cancel{4x}\, dx) + 0$

$= \int_{-1}^{0} \cancel{x^2 dx} = \cancel{\left[\frac{x^3}{3}\right]_{-1}^{0}} = \cancel{\frac{1}{3}[0-8}$

$= 0 + 0 = \underline{0}$

$\therefore Cov(X, Y) = E(XY) - E(X)E(Y) = \underline{0}$

$\therefore \rho_{xy} = \underline{0}$

Find the value of 'k' such that, $f(x,y) = k e^{-2x-3y}$ then find ρ_{xy} where $x \geq 0$, $y \geq 0$.

Soln:-

$f(x,y)$ is a valid pdf if it has satisfy,

i) $f(x,y) \geq 0$

ii) $\iint_R f(x,y)\, dy \cdot dx = 1$

$\therefore \int_0^\infty \int_0^\infty k e^{-2x-3y}\, dy\, dx = 1$

$k \int_0^\infty e^{-2x} \left[\frac{e^{-3y}}{-3}\right]_0^\infty dx = 1$

$\frac{-k}{3} \int_0^\infty e^{-2x} [0-1]\, dx = 1$

$\frac{k}{3} \left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1$

$\frac{-k}{6}[0-1] = 1$

$\therefore \underline{k = 6}$

To find ρ_{xy}.

$$E(x) = \int_0^\infty \int_0^\infty x\,(6e^{-2x-3y})\,dy\,dx$$

$$= 6\int_0^\infty x e^{-2x}\left[\frac{e^{-3y}}{-3}\right]_0^\infty dx$$

$$= -2\int_0^\infty x e^{-2x}[0-1]\,dx$$

$$= +2\int_0^\infty \underset{u}{x}\,\underset{v}{e^{-2x}}\,dx \qquad \because \int uv\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$$

$$= +2\left\{\left[x\frac{e^{-2x}}{-2}\right]_0^\infty - \int_0^\infty (1)\left[\frac{e^{-2x}}{-2}\right]dx\right\}$$

$$= +2\left\{0 + \tfrac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^\infty\right\}$$

$$= +2\left[\frac{-1}{4}[0-1]\right\}$$

$$= \tfrac{1}{2}$$

$$E(Y) = \int_0^\infty \int_0^\infty y\,(6e^{-2x-3y})\,dy\,dx$$

$$= 6e^{-2x}\int_0^\infty \int_0^\infty y e^{-3y}\,dy\,dx$$

$$= 6e^{-2x}\int_0^\infty \left\{\left[y\cdot\frac{e^{-3y}}{-3}\right]_0^\infty - \int_0^\infty (1)\left[\frac{e^{-3y}}{-3}\right]dy\right\}dx$$

$$= 6e^{-2x}\int_0^\infty \left\{0 + \frac{1}{3}\left[\frac{e^{-3y}}{-3}\right]_0^\infty\right\}dx$$

$$= 6e^{-2x}\int_0^\infty -\tfrac{1}{9}[0-1]\,dx$$

$$= \frac{6}{9}\int_0^\infty e^{-2x}\,dx = \frac{6}{9}\left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{-\cancel{6}^{\,1}}{+\cancel{18}_{\,3}}[0-1]$$

$$= \tfrac{1}{3}$$

$$E(XY) = \int_0^\infty \int_0^\infty xy \, 6 e^{-2x-3y} \, dy.dx$$

$$= 6xe^{-2x} \int_0^\infty \int_0^\infty y e^{-3y} \, dy. \, dx$$

$$= 6xe^{-2x} \int_0^\infty \left\{ \left[y . \frac{e^{-3y}}{-3} \right]_0^\infty - \int_0^\infty (1) \left[\frac{e^{-3y}}{-3} \right] dy \right\} dx$$

$$= 6xe^{-2x} \int_0^\infty \left\{ 0 + \tfrac{1}{3} \left[\frac{e^{-3y}}{-3} \right]_0^\infty \right\} dx$$

$$= 6xe^{-2x} \int_0^\infty -\tfrac{1}{9} [0 - 1] \, dx$$

$$= \frac{6}{9} \int_0^\infty x e^{-2x} \, dx$$

$$= \frac{6}{9} \left\{ \left[x . \frac{e^{-2x}}{-2} \right]_0^\infty - \int_0^\infty (1) \left[\frac{e^{-2x}}{-2} \right] dx \right\}$$

$$= \frac{6}{9} \left\{ 0 + \tfrac{1}{2} \left[\frac{e^{-2x}}{-2} \right]_0^\infty \right\}$$

$$= \frac{6}{9} \left\{ \frac{-1}{4} [0 - 1] \right\}$$

$$= \frac{\cancel{6}^{\,2}}{\cancel{9}_3} \times \frac{1}{\cancel{4}\,2}$$

$$= \underline{\underline{\frac{1}{6}}}$$

$$\therefore Cov(X, Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{6} - \frac{1}{2} \times \frac{1}{3}$$

$$= \underline{\underline{0}}$$

$$\therefore \rho_{xy} = \frac{Cov(X,Y)}{\sqrt{V(X) V(Y)}}$$

$$= \underline{\underline{0}}$$

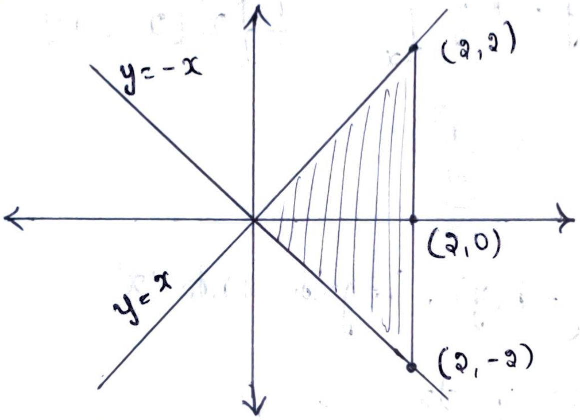
Find the value of 'k' if the joint pdf.

$$f(x,y) = \begin{cases} kx(x-y), & 0 \le x \le 2 \text{ and } -x \le y \le x \\ 0, & \text{otherwise.} \end{cases}$$

and then find

i) k ii) $g(x)$ iii) $h(y)$

Soln:-



$$\iint_R f(x)\, dy\, dx = 1$$

$$\int_0^2 \int_{-x}^{x} k\, x(x-y)\, dy\, dx = 1$$

$$k\int_0^2 \int_{-x}^{x} [x^2 - xy]\, dy\, dx = 1$$

$$k\int_0^2 \left[x^2 y - x\frac{y^2}{2}\right]_{-x}^{x} dx = 1$$

$$k\int_0^2 x^2[x-(-x)] - \frac{x}{2}[x^2-(-x)^2] = 1$$

$$k\int_0^2 2x^3\, dx = 1$$

$$2k\left[\frac{x^4}{4_2}\right]_0^2 = 1$$

$$\frac{k}{2}[\overset{8}{16} - 0] = 1$$

$$8k = 1 \Rightarrow \underline{k = 1/8}$$

Marginal pdf of $x = g(x)$, integrate w.r.to 'y'

$$g(x) = \int_{R} f(x,y)\,dy = \int_{-x}^{x} \frac{1}{8} x(x-y)\,dy$$

$$= \frac{1}{8}\int_{-x}^{x} x^2 - xy\,dy$$

$$= \frac{1}{8}\left[x^2 y - x\frac{y^2}{2}\right]_{-x}^{x} = \frac{1}{8}\left\{x^2[x-(-x)] - \frac{x}{2}\left[\cancel{x^2} - \cancel{(-x)^2}\right]\right\}$$

$$= \frac{1}{8_4}(\cancel{2}x^3) = \underline{\underline{\frac{x^3}{4}}}$$

~~Marginal pdf of $y = h(y)$~~, integrate w.r.to 'x'

$$\cancel{h(y) = \int_{R} \frac{1}{8} x(x-y)\,dx}$$

$$\cancel{= \frac{1}{8}\int_{0}^{2} x^2 - xy\,dx}$$

$$\cancel{= \frac{1}{8}\left\{\left[\frac{x^3}{3}\right.\right.}$$

$$\therefore h(y) = \begin{cases} \int \frac{1}{8} x(x-y), & -2 \le y \le 0 \\ \int \frac{1}{8} x(x-y), & 0 \le y \le 2 \end{cases}$$

$$= \int_{-2}^{0} \frac{1}{8} x(x-y)\,dx + \int_{0}^{2} \frac{1}{8} x(x-y)\,dx$$

$$= \frac{1}{8}\int_{-2}^{0} x^2 - xy\,dx + \frac{1}{8}\int_{0}^{2} x^2 - xy\,dx$$

$$= \frac{1}{8}\left\{\left[\frac{x^3}{3} - \frac{x^2}{2}y\right]_{-2}^{0}\right\} + \frac{1}{8}\left\{\left[\frac{x^3}{3} - \frac{x^2}{2}y\right]_{0}^{2}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{3}[0-(-2)^3] - \frac{y}{2}[0-(-2)^2]\right\} + \frac{1}{8}\left\{\frac{1}{3}[8-0] - \frac{y}{2}[4-0]\right\}$$

$$= \frac{1}{8}\left[\frac{-8}{3} + \frac{\cancel{4}^{2} y}{\cancel{2}}\right] + \frac{1}{8}\left[\frac{8}{3} - \frac{\cancel{4}^{2} y}{\cancel{2}}\right]$$

$$= \frac{+1}{3} + \frac{y}{4} + \frac{1}{3} - \frac{y}{4} = \underline{\underline{0}}$$

Marginal pdf of $y = h(y)$, integration w.r.t. to x:

$$h(y) = \begin{cases} \int_{-y}^{2} \frac{1}{8} x(x-y), & -2 \le y \le 0 \\ \int_{y}^{2} \frac{1}{8} x(x-y), & 0 \le y \le 2 \end{cases}$$

$$= \int_{-y}^{2} \frac{1}{8} x(x-y)\,dx + \int_{y}^{2} \frac{1}{8} x(x-y)\,dx$$

$$= \frac{1}{8}\int_{-y}^{2} x^2 - xy\,dx + \frac{1}{8}\int_{y}^{2} x^2 - xy\,dx$$

$$= \frac{1}{8}\left\{\left[\frac{x^3}{3} - y\frac{x^2}{2}\right]_{-y}^{2}\right\} + \frac{1}{8}\left\{\left[\frac{x^3}{3} - y\frac{x^2}{2}\right]_{y}^{2}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{3}[8+y^3] - \frac{y}{2}[4-y^2]\right\} + \frac{1}{8}\left\{\frac{1}{3}[8-y^3] - \frac{y}{2}[4-y^2]\right\}$$

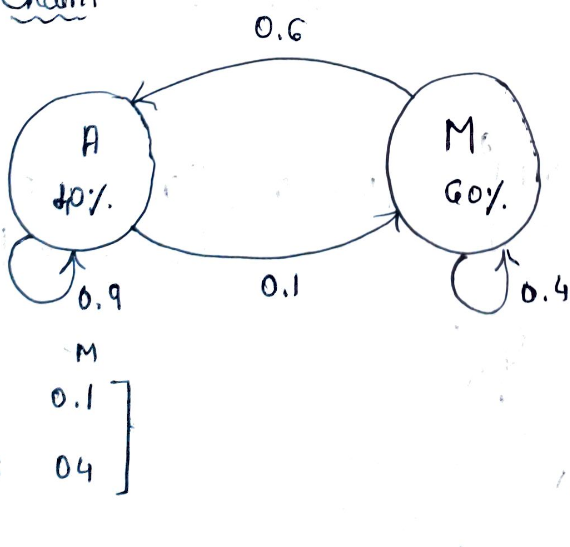
$$= \frac{1}{8}\left\{\frac{8}{3} + \frac{y^3}{3} - 2y + \frac{y^3}{2}\right\} + \frac{1}{8}\left\{\frac{8}{3} - \frac{y^3}{3} - 2y + \frac{y^3}{2}\right\}$$

$$= \frac{1}{3} + \cancel{\frac{y^3}{24}} - \frac{y}{4} + \frac{y^3}{16} + \frac{1}{3} - \cancel{\frac{y^3}{24}} - \frac{y}{4} + \frac{y^3}{16}$$

$$= \frac{2}{3} - \frac{y}{2} + \frac{y^3}{8} = \underline{\underline{\frac{y^3}{8} - \frac{y}{2} + \frac{2}{3}}}$$

Markov Chain:-

Eg:-



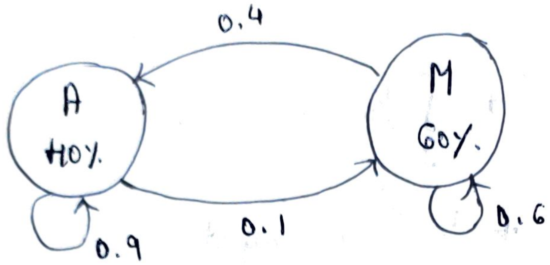
$$\therefore \begin{array}{c c} & \begin{array}{cc} A & M \end{array} \\ \begin{array}{c} A \\ M \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

$\therefore$ This is transition probability matrix.

$$\begin{bmatrix}0.4 & 0.6\end{bmatrix}\begin{bmatrix}0.9 & 0.1\\ 0.6 & 0.4\end{bmatrix} = \begin{bmatrix}0.4\times0.9+0.6\times0.6, & 0.4\times0.1+0.6\times0.4\end{bmatrix}$$

$$= \begin{bmatrix}0.72, & 0.28\end{bmatrix}$$

Eg:-



Soln:-

$$\underset{\substack{\uparrow\\ \text{Initial State}}}{\begin{bmatrix}0.4 & 0.6\end{bmatrix}}\begin{bmatrix}0.9 & 0.1\\ 0.4 & 0.6\end{bmatrix} = \begin{bmatrix}0.4\times0.9+0.6\times0.4, & 0.4\times0.1+0.6\times0.6\end{bmatrix}$$

$$= \begin{bmatrix}0.6, & 0.4\end{bmatrix} \leftarrow \text{latest state-1}$$

↳ After 1 year

After 2 year

$$\underset{\substack{\uparrow\\ \text{latest state-1}}}{\begin{bmatrix}0.6, & 0.4\end{bmatrix}}\begin{bmatrix}0.9 & 0.1\\ 0.4 & 0.6\end{bmatrix} = \begin{bmatrix}0.6\times0.9+0.4\times0.4, & 0.6\times0.1+0.4\times0.6\end{bmatrix}$$

$$= \begin{bmatrix}0.7, & 0.3\end{bmatrix}$$

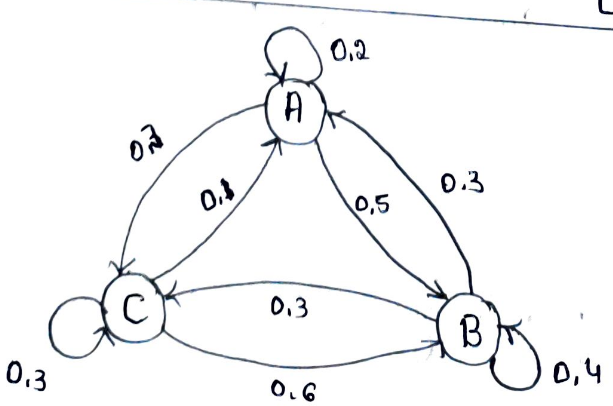
∴ This method is complex for 'n' iterations.

∴ Initial State × M^n $\qquad$ M → matrix

For 2 year.

$$\begin{bmatrix}0.4 & 0.6\end{bmatrix}\begin{bmatrix}0.9 & 0.1\\ 0.4 & 0.6\end{bmatrix}^2 = \begin{bmatrix}0.4 & 0.6\end{bmatrix}\begin{bmatrix}0.85 & 0.15\\ 0.6 & 0.4\end{bmatrix} = \begin{bmatrix}0.7 & 0.3\end{bmatrix}$$

Given,



Find TPM.

Soln:-

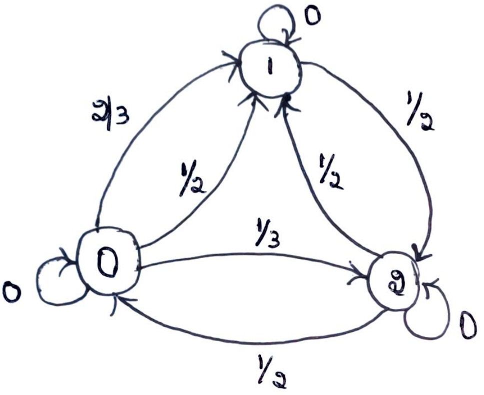
$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0.2 & 0.5 & 0.3 \\ B & 0.3 & 0.4 & 0.3 \\ C & 0.1 & 0.6 & 0.3 \end{array}$$

Given

$$TPM = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Draw the Markov chain and find stationary Distribution.

Soln:-



W.K.T

$$VT = V$$

$$V[T - I] = [0]$$

$$[v_1 \; v_2 \; v_3] \begin{bmatrix} -1 & 2/3 & 1/3 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{bmatrix} = [0 \; 0 \; 0]$$

$$-v_1 + \frac{1}{2} v_2 + \frac{1}{2} v_3 = 0 \quad \text{—— (1)}$$

$$\frac{2}{3} v_1 - v_2 + \frac{1}{2} v_3 = 0 \quad \text{—— (2)}$$

$$\frac{1}{3} v_1 + \frac{1}{2} v_2 - v_3 = 0 \quad \text{—— (3)}$$

and $\boxed{v_1 + v_2 + v_3 = 1}$ → (4)

Using (1), (2) & (4)

$$v_1 = \frac{9}{27}, \quad v_2 = \frac{10}{27}, \quad \frac{8}{27}$$

Assume that 3 values are equally distributed, then,

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 6/18, & \underline{7/18}, & 5/18 \end{bmatrix}$$

Assume that 3 values are equally distributed, then,

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 6/18, & \underline{7/18}, & 5/18 \end{bmatrix}$$

Stationary Distribution:-

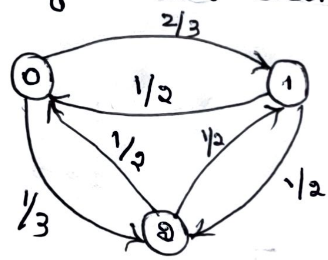
* This is can also be computed by

$$\nu_K = \frac{C_K}{\sum_j C_j}$$

$$C_j = \sum_j w_i (T_j)$$

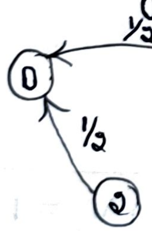
The sum is the sum of all entries to the point 'j' in the transition diagram of Markov Chain.

Eg:-

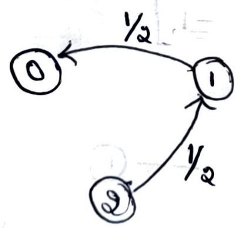


Soln:-

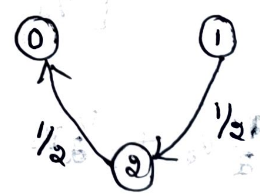
Intree at $j=0$



or



or



T_1, $W_1 = 1/2 \cdot 1/2 = 1/4$

T_2, $W_2 = 1/2 \cdot 1/2 = 1/4$

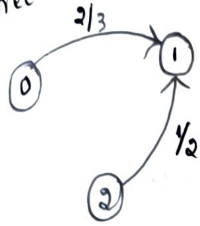
T_3, $W_3 = 1/2 \cdot 1/2 = 1/4$.

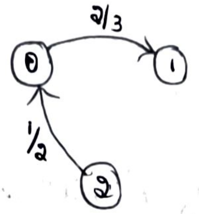
$$\therefore C_0 = W_1(T_1) + W_2(T_2) + W_3(T_3)$$

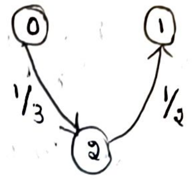
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= \underline{3/4}$$

Intree at $j=1$,



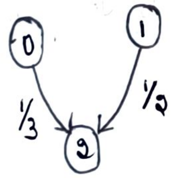


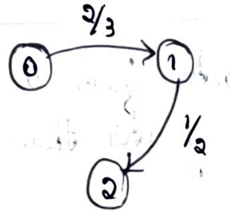


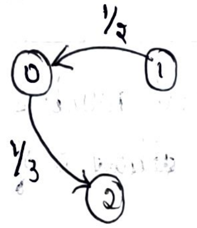
$w_1(T_1) = 2/3 \cdot 1/2 = 1/3$ $\quad$ $w_2(T_2) = 1/3$, $\quad$ $w_3(T_3) = 1/6$

$$\therefore C_1 = w_1(T_1) + w_2(T_2) + w_3(T_3)$$
$$= 1/3 + 1/3 + 1/6$$
$$= \frac{5}{6}$$

Intree at $j=2$,







$w_1(T_1) = 1/3 \cdot 1/2 = 1/6$ $\quad$ $w_2(T_2) = 1/3$ $\quad$ $w_3(T_3) = 1/6$

$$\therefore C_2 = w_1(T_1) + w_2(T_2) + w_3(T_3)$$
$$= 1/6 + 1/3 + 1/6$$
$$= 2/3$$

$$\therefore \sum_j C_j = C_0 + C_1 + C_2 = \frac{3}{4} + \frac{5}{6} + \frac{2}{3} = \frac{9}{4}$$

$$\therefore v_0 = \frac{C_0}{9/4} = \frac{3/4}{9/4} = 1/3$$

$$v_1 = \frac{C_1}{9/4} = \frac{5/6}{9/4} = \frac{10}{27}$$

$$v_2 = \frac{C_2}{9/4} = \frac{2/3}{9/4} = \frac{8}{27}$$

Binomial Experiments :-

$P(X=r) = {}^nC_r\, p^r q^{n-r} \geq 0$ → Binomial Distribution

∴ This is the probability of 'r' successes in 'n' tosses.

$$\therefore \sum_{r=0}^{n} P(X=r) = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r}$$

$$= 1 \cdot p^0 q^{n-0} + n \cdot p^1 q^{n-1} + \frac{n(n-1)}{2!} p^2 q^{n-2} + \dots + 1 \cdot p^n q^0$$

$$= (p+q)^n$$

$$= 1^n$$

$$= 1$$

Q:- 5 real numbers are selected from 0 to 2. What is the prob. that atleast 2 of them are greater than 1.

Soln:-

Binomial expt → Yes (p - prob. of success) / → No

p = prob. of success.

= prob. that a number is >1

$$= \int_1^2 \frac{1}{2-0}\, dx = \frac{1}{2}[x]_1^2 = \frac{1}{2}(2-1) = \frac{1}{2}$$

↳ Pdf of uniform dist.

$\therefore q = 1-p \Rightarrow 1 - \frac{1}{2} = \frac{1}{2}$

∴ Prob. that atleast 2 of them are greater than 1. is

$\Rightarrow P(Y \geq 2)$

$= 1 - [P(Y=0) + P(Y=1)]$

$= 1 - [{}^5C_0 (1/2)^0 (1/2)^{5-0} + {}^5C_1 (1/2)^1 (1/2)^{5-1}]$

$= 1 - [1/32 + 5/32] = 26/32$

Q:- 5 determinations of a random variable X, are done over the interval [0, 2] & pdf of X is given by then,

$$f(x) = \begin{cases} kx^2 & , 0 \le x \le 2 \\ 0 & , \text{Otherwise} \end{cases}$$

i) $k = ?$

ii) What is the probability that atleast 2 of them > 1

Soln:-

$$\int_0^2 kx^2 \, dx = 1$$

$$k\left[\frac{x^3}{3}\right]_0^2 = 1$$

$$\frac{k}{3}[8-0] = 1$$

$$\therefore k = 3/8$$

$\therefore$ Binomial Expt $\rightarrow$ Yes (p - prob. of success) / $\rightarrow$ No.

p = prob. of success.

= prob. that a no. is > 1

$$= \int_1^2 f(x)\, dx$$

$$= \int_1^2 3/8\, x^2\, dx = 3/8\left[\frac{x^3}{3}\right]_1^2 = 1/8[8-1] = 7/8$$

$$\therefore q = 1 - p \Rightarrow 1 - 7/8 = 1/8$$

$\therefore$ Prob. that atleast 2 of them are greater than 1.

$\Rightarrow 1 - P(Y \ge 2)$

$= 1 - [P(Y=0) + P(Y=1)]$

$= 1 - \left[{}^5C_0 (7/8)^0 (1/8)^{5-0} + {}^5C_1 (7/8)^1 (1/8)^{5-1}\right]$

$= 1 - \left[1/8^5 + 5(7/8^5)\right]$

$= \dfrac{8183}{8192}$

Q:- A binary communication channel is a model that consists of a transmitter sending a binary signal together with a receiver. Suppose 500 bit message was send with the prob. of 0.005 that an error will occur in each coordinate. then

i) Find the prob. that the message was send error free.

ii) Message was send exactly with 1 error.

iii) " " " " " 2 error.

iv) " " " " " 3 or more error.

Soln:-

$n = 500$ (trials)

p = prob. of success.

= prob. that error will occur = 0.005.

$\therefore q = 1 - p = 1 - 0.005 = \underline{0.995}$

i) $P(X = 0) = {}^{500}C_0 (0.005)^0 (0.995)^{500-0} = \underline{0.0816}$

ii) $P(X = 1) = {}^{500}C_1 (0.005)^1 (0.995)^{500-1} = \underline{0.2049}$

iii) $P(X = 2) = {}^{500}C_2 (0.005)^2 (0.995)^{500-2} = \underline{0.2569}$

iv) $P(X \geq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$

$= 1 - [0.0816 + 0.2049 + 0.2569]$

$= \underline{0.4566} \Rightarrow (45.66\%)$

$E(x) = np$, $V(x) = ?$

$$E(X) = \sum_{x=0}^{n} x\, p(X = x)$$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_x\, p^x q^{n-x}$$

$$= 0 + 1 \cdot {}^{n}C_1\, p^1 q^{n-1} + 2 \cdot {}^{n}C_2\, p^2 q^{n-2} + \dots + n \cdot 1 \cdot p^n q^0$$

$= n \cdot p^1 q^{n-1} + 2 \cdot \frac{n(n-1)}{1.2} p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{1.2.3} p^3 q^{n-3} + \dots + np^n$

$= np \left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2!} p^2 q^{n-3} + \dots + p^{n-1} \right]$

$= np [q+p]^{n-1}$

$E(x) = np$

$\therefore E(x^2) = \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$ let $x^2 = x(x-1) + x$

$= \sum_{x=0}^{n} [x(x-1) + x] \, {}^nC_x \, p^x q^{n-x}$

$= \sum_{x=0}^{n} x(x-1) \, {}^nC_x \, p^x q^{n-x} + \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$

$= \sum_{x=0}^{n} x(x-1) \, {}^nC_x \, p^x q^{n-x} + np$

$= \left[0 + 0 + 2(2-1) \frac{n(n-1)}{1.2} p^2 q^{n-2} + 3(3-1) \frac{n(n-1)(n-2)}{1.2.3} p^3 q^{n-3} \right.$
$\left. + \dots + n(n-1) p^n q^0 \right] + np$

$= \left[n(n-1) p^2 q^{n-2} + \frac{n(n-1)(n-2)}{1!} p^3 q^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2!} p^4 q^{n-4} \right.$
$\left. + \dots + n(n-1) p^n \right] + np$

$= n(n-1)p^2 \left[q^{n-2} + \frac{(n-2)}{1!} p q^{n-3} + \frac{(n-2)(n-3)}{2!} p^2 q^{n-4} + \dots \right.$
$\left. + p^{n-2} \right] + np$

$= n(n-1)p^2 [q+p]^{n-2} + np$

$\therefore E(x^2) = n(n-1)p^2 + np$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= n(n-1)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np - np^2$

$\boxed{q = 1-p}$

$= np(1-p)$

$= npq$

Poisson Distribution :-

$\therefore P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

if $\displaystyle\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$

$= e^{-\lambda}\left(1 + \dfrac{\lambda}{1!} + \dfrac{\lambda^2}{2!} + \ldots\right)$ $\rightarrow$ Tailor Series,

$= e^{-\lambda} e^{\lambda}$

$= 1$

$\therefore E(X) = \displaystyle\sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$ if $x=0$, $E(X)=0$.

$= \displaystyle\sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$

$= e^{-\lambda} \displaystyle\sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$

$= e^{-\lambda}\lambda \displaystyle\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda}\lambda\left[1 + \dfrac{\lambda}{1!} + \dfrac{\lambda^2}{2!} + \ldots\right] = e^{-\lambda}\lambda e^{\lambda}$

$E(X) = \lambda$

$$\therefore E(x^2) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \left[\sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}\right] + \lambda$$

if $x=1$ & 2 then $e = 0$

$$= e^{-\lambda}\lambda^2 \left[\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}\right] + \lambda$$

$$= e^{-\lambda}\lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right] + \lambda$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$= \underline{\underline{\lambda^2 + \lambda}}$$

$$\therefore V(x) = E(x^2) - [E(x)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \underline{\underline{\lambda}}$$

For the last problem use Poisson distribution or approximation method,

Soln:-

$n = 500$, $p = 0.005$.

$\therefore \lambda = np = 2.5$.

i) $P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-2.5} = \underline{\underline{0.0821}}$

ii) $P(X=1) = \frac{e^{-\lambda}\lambda^1}{1!} = e^{-2.5} \times 2.5 = \underline{\underline{0.2052}}$

iii) $P(X=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-2.5} \times 2.5^2}{2!} = \underline{\underline{0.2565}}$

iv) $P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - [0.0821 + 0.2052 + 0.2565]$

$= \underline{\underline{0.4562}}$