Singular Value Decomposition: An Example

Problem Statement

Find the Singular Value Decomposition of

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}.$$

Solution:
Since A is of size 3×2, the SVD of A is

$$A = U\Sigma V^T,$$

where

$$U \in \mathbb{R}^{3\times 3}, \quad \Sigma \in \mathbb{R}^{3\times 2}, \quad V \in \mathbb{R}^{2\times 2}.$$

Step 1:

- Find the largest order matrix among $A^T A$ and AA^T. Here it is AA^T of size 3×3
- Find its eigenvalues and write in decresing order.
- Find the singular values of A, which are nonnegative square roots of eigenvalues of AA^T.

Step 1 continued

Compute AA^T

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}.$$

Eigenvalues of AA^T

$$\det(AA^T - \lambda I) = 0$$

gives eigenvalues

$$\lambda = 90,\ 0,\ 0.$$

Thus, singular values of A are

$$\sigma = 3\sqrt{10},\ 0,\ 0.$$

Step 2

We find eigenvectors of AA^T. Note that among U and V the largest order matrix is U. We will determine columns of U which are eigenvectors of AA^T

Eigenvectors for $\lambda = 90$: Solve

$$(AA^T - 90I)x = 0.$$

One eigenvector is

$$x = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \quad \|x\| = 3.$$

So

$$u_1 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

Step 2 continued

Eigenvectors for $\lambda = 0$: Solve

$$AA^T x = 0.$$

Two linearly independent solutions:

$$x_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

After Gram–Schmidt orthogonalization and normalization:

$$u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad u_3 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}.$$

Step 2 continued

Construct U:

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \end{bmatrix}.$$

Step 3: Construct V

From

$$v_1 = \frac{A^T u_1}{\|A^T u_1\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

But note that

$$v_2 = \frac{A^T u_2}{\|A^T u_2\|} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which cannot be column of orthogonal matrix U, as columns of U are of norm 1.

However we can find a vector which is orthogonal to v_1 as follows:

Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be any vector orthogonal to u. Orthogonality means $u^T v = 0$. Thus

$$\left(-\tfrac{3}{\sqrt{10}}\right) x + \left(\tfrac{1}{\sqrt{10}}\right) y = 0 \quad \Longrightarrow \quad -3x + y = 0.$$

Therefore $y = 3x$. Any nonzero multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is orthogonal to u.

A convenient choice is

$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Since we want a *unit* vector orthogonal to u, normalize v:

$$\|v\| = \sqrt{1^2 + 3^2} = \sqrt{10}, \qquad v_2 = \frac{v}{||v||} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Step 3 continued

Construct V.

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

Step 4: Construct Σ

$$\Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 5: Final SVD

$$A = U\Sigma V^T,$$

where

$$U = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$