Class 5-Linear Algebra

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$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example

Compute $u.v$ and $v.u$ for $u = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

Properties of dot product

Let u, v and w be vectors in $\mathbb{R}^n$, and c be a scalar. Then

1. $(u+v).w = u.w + v.w$
2. $(cu).v = c(u.v) = u.(cv)$
3. $u.v = v.u$
4. $u.u \geq 0$ and $u.u = 0$ if and only if $u = 0$.

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3. **Positive-definiteness:**

$$\langle v, v \rangle \geq 0, \quad \text{and } \langle v, v \rangle = 0 \iff v = 0$$

Example: Dot Product

Dot Product as an Inner Product

For $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ in $\mathbb{R}^n$, define

$$\langle u, v \rangle = u \cdot v = \sum_{i=1}^{n} u_i v_i$$

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Dot product is an example of inner product on $\mathbb{R}^n$.

Example: Inner Product via Positive Definite Matrix

Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

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Note: We will mostly consider dot product from now on.

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3. **Triangle inequality:**

$$\|u + v\| \leq \|u\| + \|v\|$$

The length(or norm) of a vector

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1. Length in $\mathbb{R}^2$
2. Length in $\mathbb{R}^3$

Unit vector

A vector whose length is 1 is called the unit vector.

Example

$$v = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Normalizing a vector

If we divide a non-zero v vector by its norm $||v||$, then the resulting vector is unit vector in the direction of v.

Example

$$v = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$