iv) $P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - [0.0821 + 0.2052 + 0.2565]$

$= \underline{0.4562}$

Markov Chain → Discrete
→ Continuous $= \{x(t) \mid t \in T\}$
↳ f^n of Time
↳ Random variable

↳ Example of discrete stochastic process

Q-1:-
A Stochastic process $X(t)$ is based on a outcome a tossing a dice and observing the number on the face.

1) Give a graphical representation of stochastic process.

2) Compute a) $P(X(0) = -1)$

b) $P(X(1) \leq 0)$

c) $P(X(0) = 0, X(1) = -1)$

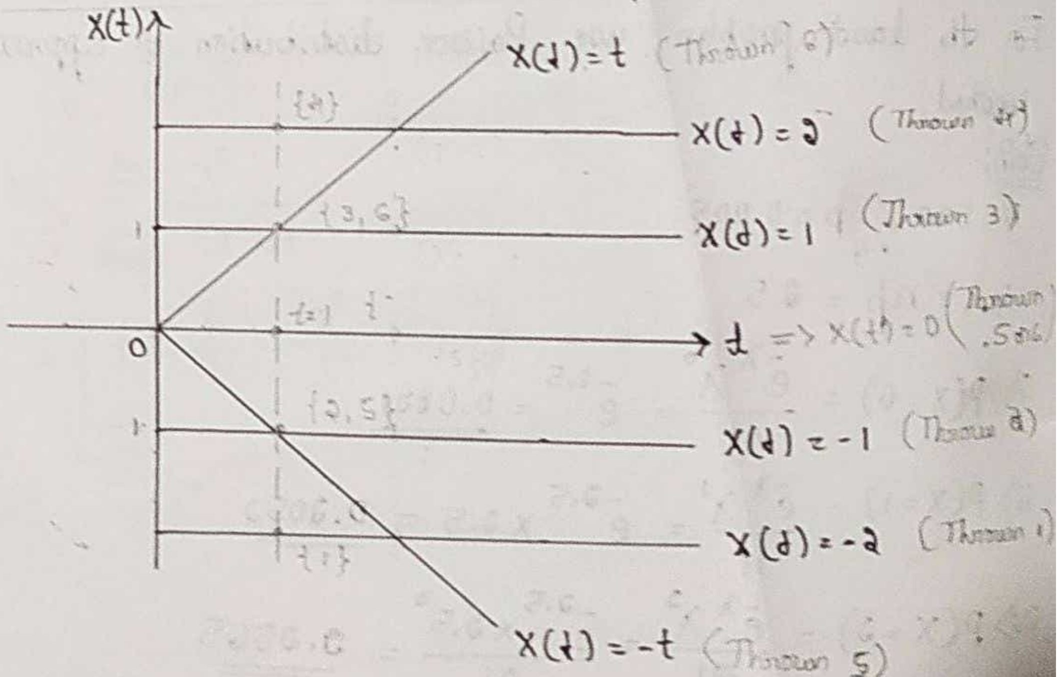
d) $P(X(1) = -1 \mid X(0) = 0)$

Given,

Outcome	1	2	3	4	5	6
$X(t)$	-2	-1	1	2	$-t$	t

Soln:-

1)



This is known as Ensemble set of all waveform

a) $P(X(0) = -1) = P(\{2\}) = \frac{1}{6}$

b) $P(X(t) \le 0) = P(\{5, 1, 2, 6\}) = \cancel{\tfrac{4}{6}}\ \frac{3}{6}$

c) $P(X(0) = 0,\ X(1) = -1) = P(X(0) = 0 \cap X(1) = -1)$

$= P(\{5, 6\} \cap \{2, 5\}) = P(\{5\})$

$= \frac{1}{6}$

d) $P(X(1) = -1 \mid X(0) = 0) = \dfrac{P(X(1) = -1,\ X(0) = 0)}{P(X(0) = 0)}$

$= \dfrac{P(X(1) = -1 \cap X(0) = 0)}{P(X(0) = 0)}$

$= \dfrac{P(\{2, 5\} \cap \{5, 6\})}{P(\{5, 6\})}$

$= \dfrac{P(\{5\})}{P(\{5, 6\})} = \dfrac{1/6}{2/6} = \frac{1}{2}$

For the same question, Given

Q-2:-

Outcome	1	2	3	4	5	6
X(t)	-2	-1	1	2	-1/2	1/2

Soln:-

For the question, Q-1:- Find expectation, $E(X(t))$ or the average

Soln:-

$$E(X) = \sum x\, P(X = x)$$

$$\therefore E(X(t)) = \sum X(t)\, P(X(t))$$

$$= (-2)\tfrac{1}{6} + (-1)\tfrac{1}{6} + 1\tfrac{1}{6} + 2\tfrac{1}{6} - t\tfrac{1}{6} + t\tfrac{1}{6}$$

$$= \underline{0}$$

$\therefore$ This is Mean of Stochastic Process.

$$\therefore E(X^2(t)) = \sum X^2(t)\, P(X(t))$$

$$= (-2)^2\tfrac{1}{6} + (-1)^2\tfrac{1}{6} + 1^2\tfrac{1}{6} + 2^2\tfrac{1}{6} + (-t)^2\tfrac{1}{6} + t^2\tfrac{1}{6} = \frac{10+2t^2}{6}$$

$$= \frac{5+t^2}{3}$$

$$\therefore V(X(t)) = E(X^2(t)) - [E(X(t))]^2$$

$$= \frac{5+t^2}{3} - 0^2$$

$$= \frac{5+t^2}{3}$$

$$\therefore Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$\therefore Cov(X(s), X(t)) = C(s,t)$$

$$= E(X(s)X(t)) - E(X(s))\underbrace{E(X(t))}_{\to 0}$$

$$= E(X(s)X(t)) - 0$$

∴

Outcome	1	2	3	4	5	6
$X(t)$	-2	-1	1	2	$-t$	t
$X(s)$	-2	-1	1	2	$-s$	s
$X(s)X(t)$	4	1	1	4	st	st

$$\therefore Cov(X(s), X(t)) = E(X(s)X(t))$$

$$= 4\times\tfrac{1}{6} + 1\cdot\tfrac{1}{6} + \tfrac{1}{6} + 4\cdot\tfrac{1}{6} + \tfrac{st}{6} + \tfrac{st}{6}$$

$$= \frac{10+2st}{6}$$

$$= \frac{5+st}{3}$$

Put $t=s$ in $E(X(t)X(s))$ then

$$C(t,t) = E(X^2(t)) - [E(X(t))]^2$$

$$= V(X(t))$$

$$= \frac{t^2+5}{3}$$

$C(s,t) = E(X(t)\,X(s)) - E(X(t)) \cdot E(X(s))$

$E(X(t)) = \sum X(t)\, P(X(t))$

$= (-2)\frac{1}{6} + (-1)\frac{1}{6} + 1\frac{1}{6} + 2\left(\frac{1}{6}\right) - \frac{t}{6} + \frac{t}{6}$

$= \underline{\underline{0}}$

Auto Correlation Coefficiend:-

$$r(s,t) = \frac{C(s,t)}{\sqrt{C(s,s) \cdot C(t,t)}}$$

$$= \frac{\frac{st+5}{3}}{\sqrt{\left(\frac{s^2+5}{3}\right)\left(\frac{t^2+5}{3}\right)}}$$

$$= \frac{st+5}{\sqrt{(s^2+5)(t^2+5)}}$$

∴ The value between -1 and 1. this is correlation coefficiend.

Eg:- $$r(10, 20) = \frac{(10 \times 20)+5}{\sqrt{(100+5)(400+5)}}$$

$$= \underline{\underline{0.9941}}$$

$$r(1000, 3000) = \frac{(1000 \times 3000)+5}{\sqrt{[(1000)^2+5][(3000)^2+5]}}$$

$$= \underline{\underline{0.9999}}$$

Definition:

Wide sense stationary;

$E(X(t)) \longrightarrow$ Independend of t

$V(X(t)) \longrightarrow$ Independend of t

$C(s,t) = f(t-s)$

Q:- Consider a stochastic process;

$$X(t) = A\cos\omega t + B\sin\omega t$$

where
$A, B \rightarrow$ ~~Unknown~~ Uncorrelated random variables with mean 0, variance 1.

$\omega \rightarrow$ +ve constant,

Find
1) $E(X(t))$ 2) $V(X(t))$
3) Auto covariance $C(s,t)$
4) Test whether the stochastic process is WSS (Wide Sence Stationary)

Soln:-
1) $E(X(t)) = E(A\cos\omega t + B\sin\omega t)$

$= E(A)\cos\omega t + E(B)\sin\omega t$

$= \underline{\underline{0}}$ (independent of t)

2) $V(X(t))$ = Put $s=t$ in $C(s,t)$

$= \cos\omega(t-s)$

$= \cos\omega(t-t)$

$= \underline{\underline{0}}$ (Independent of t)

3) Auto covariance, $C(s,t) = E(X(t)X(s)) - \underbrace{E(X(t))}_{0}\ \underbrace{E(X(s))}_{0}$

$= E(X(t)X(s))$

$= E[(A\cos\omega t + B\sin\omega t)(A\cos\omega s + B\sin\omega s)]$

$\therefore C(s,t) = E(A^2\cos\omega t\cos\omega s + AB\cos\omega t\sin\omega s + AB\sin\omega t\cos\omega s + B^2\sin\omega t\sin\omega s)$

Hence A & B are uncorrelated

which means that $E(AB) = E(A)\,E(B)$
$\underbrace{}_{0 \text{ Given}}$

$= \underline{0}$

$\therefore C(s,t) = E(A^2)[\cos\omega t \cos\omega s] + E(B^2)[\sin\omega t \sin\omega s]$

$V(A) = E(A^2) - [E(A)]^2$

$= E(A^2) - 0$

$V(A) = \underline{E(A^2)}$

But $V(A) = V(B) = 1$ (Given)

$E(A^2) = E(B^2) = 1$

$\therefore C(s,t) = \cos\omega t \cos\omega s + \sin\omega t \sin\omega t$

$= \cos(\omega t - \omega s)$

$= \cos\omega(t - s)$

$= \underline{f(|t - s|)}$

4) $C(s,t)$ is a function of $(t-s)$

$E(X(t))$ and $V(X(t))$ are independent of t.

$\therefore$ Stochastic process is wide sense stationary (WSS)

5) Auto Correlation Coefficient

$$\gamma(s,t) = \frac{C(s,t)}{\sqrt{C(s,s)\,C(t,t)}}$$

$$= \frac{\cos\omega(t-s)}{\sqrt{\cos\omega(s-s) \cdot \cos\omega(t-t)}}$$

$$= \frac{\cos\omega(t-s)}{\sqrt{\cos 0 \cdot \cos 0}}$$

$\therefore \gamma(s,t) = \underline{\cos\omega(t-s)}$

if $\omega = 1$,

$\gamma(0.5, 0.75) = \underline{\underline{0.9689}}$

$\gamma(-0.3, 0.5) = \underline{\underline{0.6967}}$

$\gamma(0.01, \pi/2) = \underline{\underline{0.0099}}$

Poisson Stochastic Process:-

stochastic process $\{X(t) \mid t \in T\}$ with

$$P(X(t) = n) = \frac{e^{-at}(at)^n}{n!}, \quad a > 0 \quad \text{when } n = 0, 1, 2, \cdots$$

is called Poisson Stochastic Process.

$$\text{Mean} = E(X(t))$$

$$= \sum_{n=0}^{\infty} X(t)\, P(X(t) = n)$$

$$= \sum_{n=1}^{\infty} n \cdot \frac{e^{-at}(at)^n}{n!}$$

$$= e^{-at} \sum_{n=1}^{\infty} \frac{(at)^n}{(n-1)!}$$

$$= e^{-at}\, at \sum_{n=1}^{\infty} \frac{(at)^{n-1}}{(n-1)}$$

$$= e^{-at}\, at\, e^{at}$$

$\underline{\underline{= at}}$ → Depends on t so it is Evolutionary

$$\text{Variance} = E(X^2(t)) - [E(X(t))]^2$$

$$E(X^2(t)) = \sum_{n=0}^{\infty} X^2(t)\, P(X(t) = n)$$

$$= \sum_{n=1}^{\infty} n^2 \frac{e^{-at}(at)^n}{(n \cancel{-1})!}$$

$$= e^{-at} \sum_{n=1}^{\infty} \frac{n^2 (at)^n}{n!} \qquad [n^2 = n(n-1) + n]$$

$$= e^{-at} \sum_{n=1}^{\infty} [n(n-1)+n] \frac{(at)^n}{n!}$$

$$= e^{-at} \sum_{n=2}^{\infty} \frac{(at)^n}{(n-2)!} + \sum_{n=1}^{\infty} n \frac{(at)^n}{n!}$$

$$= e^{-at} (at)^2 \sum_{n=2}^{\infty} \frac{(at)^{n-2}}{(n-2)!} + at$$

$$= e^{-at} (at)^2 e^{at} + at$$

$$= (at)^2 + at$$

$$\therefore V(X(t)) = (at)^2 + at - (at)^2$$

$$= at$$

Suppose that customer arrive at the bank according to the Poisson Stochastic Process with mean rate of 3 per minute. Find the prob. that during the time interval of 2 minutes

1) Exactly 4 customers arrives

2) Greater than 4 customer arrives.

Soln:-

Mean rate of 3 per minutes $= at$

$3 = a(1)$ → Per minute

$a = 3$

$$P(X(t) = n) = \frac{e^{-at} (at)^n}{n!}, \quad n = 0, 1, 2, 3, \ldots$$

1) time interval of 2 minutes, then $t=2$.

$\therefore P(X(2)=4) = \dfrac{e^{-6}(6)^4}{4!} = 0.1339 = 13.39\%$

2) $P(X(2)>4) = 1 - P(X(2) \le 4)$

$= 1 - [P(X(2)=0) + P(X(2)=1) + P(X(2)=2) + P(X(2)=3) + P(X(2)=4)]$

$= 1 - e^{-6}\left[\dfrac{6^0}{0!} + \dfrac{6^1}{1!} + \dfrac{6^2}{2!} + \dfrac{6^3}{3!} + \dfrac{6^4}{4!}\right]$

$= 0.7149 = 71.49$

Auto ~~Correla~~ Covariance :-

$t > s$

$E(X(s)X(t)) = E(X(s)(X(t) - X(s) + X(s)))$

$= \underbrace{E(X(s)(X(t) - X(s)))}_{\text{Independent}} + E(X^2(s))$

$= (as)(at - as) + ((as)^2 + as)$

$= a^2st + as$

$= as(at+1)$

$\therefore C(s,t) = E(X(s)X(t)) - E(X(t))E(X(s))$

$= as(at+1) - (at)(as)$

$= a^2st + as - a^2st$

$= as$

$\therefore C(s,t) = \begin{cases} as, & \text{if } t>s \\ at & \text{if } s>t \end{cases}$

$= a \min\{s,t\}$

$= a(s \wedge t)$

$$\therefore r(s,t) = \frac{C(s,t)}{\sqrt{C(s,s)\cdot C(t,t)}}$$

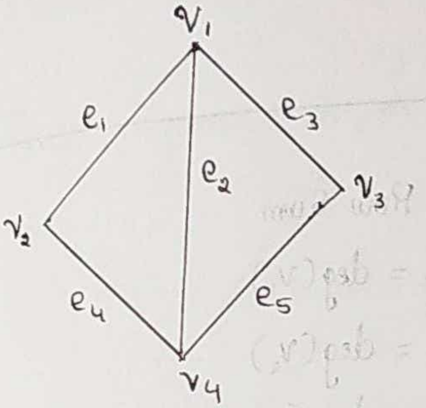
$$= \frac{a \min\{s,t\}}{\sqrt{(as)(at)}} = \frac{\alpha \min\{s,t\}}{\alpha\sqrt{st}}$$

$$= \frac{\min\{s,t\}}{\sqrt{st}}$$

$$r(s,t) = \begin{cases} \sqrt{t/s}, & s>t \\ \sqrt{s/t}, & t>s \end{cases}$$

Eg: $r(2,9) = \sqrt{2/9} = r(9,2) = \underline{0.4714}$

Graph Theory



Adjacency Matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

∴ Symmetric for undirected graph

∴ Eigen values are real.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix} \rightarrow \text{degree}(v_i) \rightarrow \text{Walk of length 2.}$$

$\Rightarrow \text{Trace}(A^2) = \sum \deg(v_i) = \underline{10}$

$$A^2(A) = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 2 & 5 \\ 5 & 2 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{bmatrix} \rightarrow \text{Walk of length 3.}$$

$\text{Trace}(A^3) = 4 + 2 + 2 + 4 = \underline{12}$

$\frac{1}{3!} \text{Trace}(A^3) = $ no. of Δ^{les} in graph

$\frac{1}{6} \times 12 = \underline{\underline{2}}$

∴ Incidence Matrix

$$\begin{array}{c|ccccc|l} & e_1 & e_2 & e_3 & e_4 & e_5 & \text{Row Sum} \\ \hline v_1 & 1 & 1 & 1 & 0 & 0 & 3 = \deg(v_1) \\ v_2 & 1 & 0 & 0 & 1 & 0 & 2 = \deg(v_2) \\ v_3 & 0 & 0 & 1 & 0 & 1 & 2 = \deg(v_3) \\ v_4 & 0 & 1 & 0 & 1 & 1 & 3 = \deg(v_4) \\ \hline \text{Column Sum} & 2 & 2 & 2 & 2 & 2 & \end{array}$$

$$\boxed{\begin{array}{l} \therefore \text{Row Sum} = \text{Column Sum} \\ \sum_{i=1}^{n} \deg(v_i) = 2(\text{Total Edges}) = 2e \end{array}}$$

∴ This is 1st Theorem in Graph Theory.

Tree:-

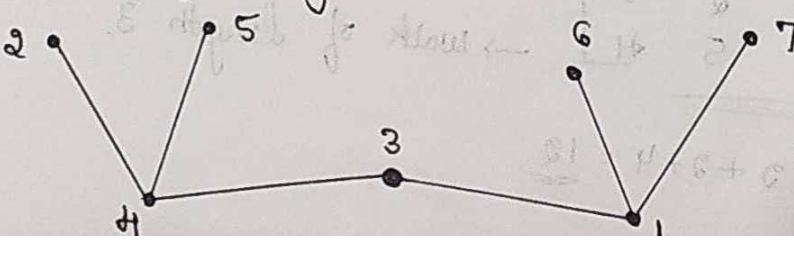
Connect graph, with any cycles or circuit

Prufer Sequence:-

Consider a tree 'T' with vertices $\{1, 2, 3, \dots, n\}$ at step i,

1) Remove the leaf (deg 1 vertex) with smallest label

2) Set the i^{th} element of the Prufer Sequence to be the label of this leaf's neighbour.

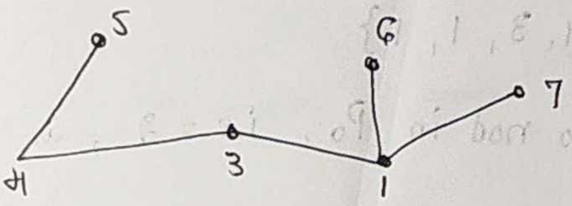


= T

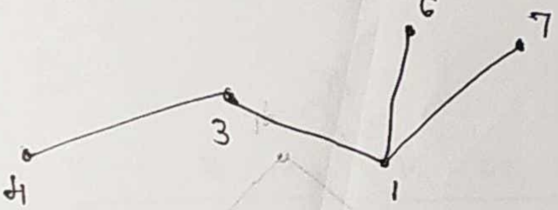
Soln:-

* Leafs : 2, 5, 6, 7 $\longrightarrow$ smallest = 2.
* Remove 2 and note down immediate neighbour.

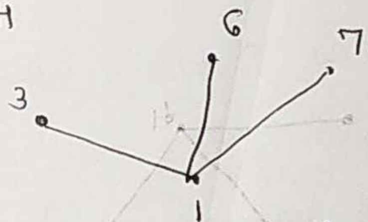
Prufer Sequence : 4, 4, 3, 1, 1



* Leaf : 5, 6, 7 $\longrightarrow$ smallest = 5
* Remove 5 & note the immediate neighbour.

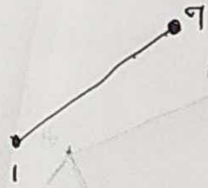


* Leaf : 4, 6, 7 $\longrightarrow$ smallest = 4
* Remove 4



* Leaf : 3, 6, 7 $\longrightarrow$ smallest = 3
* Remove 3

* Leaf : 6, 7 $\longrightarrow$ smallest = 6
* Remove 6



* Stop the recursive elemination, our sequence = 4, 4, 3, 1, 1

Drawing Tree:-

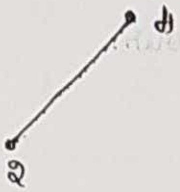
No. of vertices = (n + 2) of sequence

= 5 + 2 = 7

∴ 7 vertices, $S_0 = \{1, 2, 3, 4, 5, 6, 7\}$

$P_0 = \{4, 4, 3, 1, 1\}$

1st term in S_0 which is not in P_0, i.e = 2., is have edge with 1st term in P_0

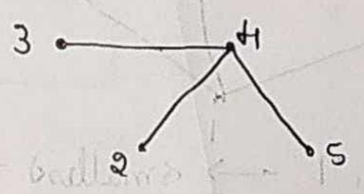


$S_0 = \{1, 3, 4, 5, 6, 7\}$

$P_0 = \{4, 3, 1, 1\}$ ⇒

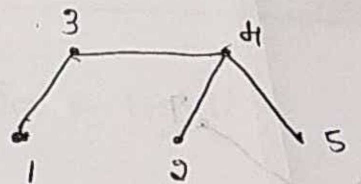
$S_0 = \{1, 3, 4, 6, 7\}$

$P_0 = \{3, 1, 1\}$ ⇒



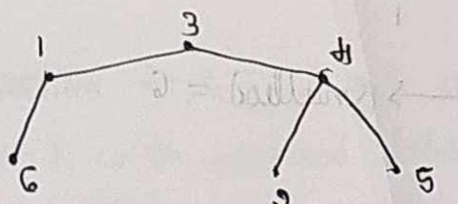
$S_0 = \{1, 3, 6, 7\}$

$P_0 = \{1, 1\}$ ⇒



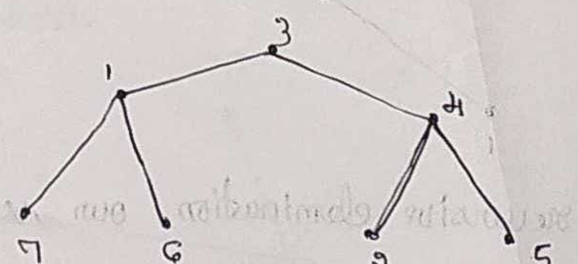
$S_0 = \{1, 6, 7\}$

$P_0 = \{1, 1\}$



$S_0 = \{1, 7\}$

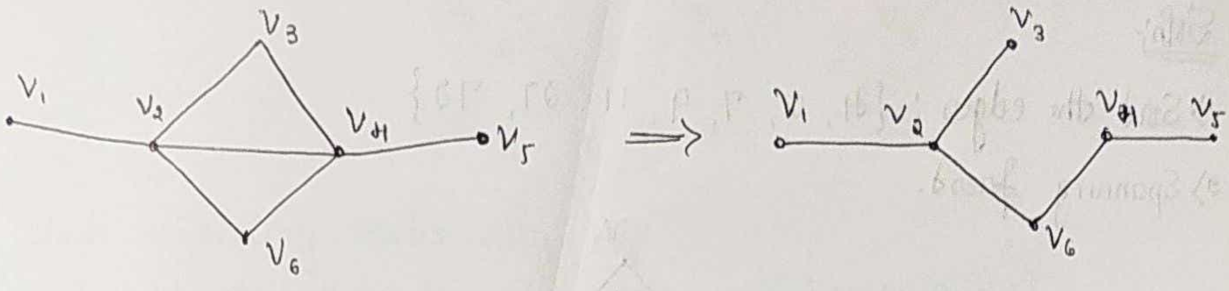
$P_0 = \{\phi\}$ ⇒



Connected Graph:-

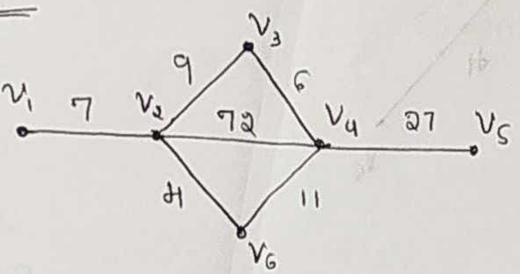
The graph which has atleast one spanning tree.

Spanning tree → a Subtree with all connected vertices and no cycle exists.



In a given graph or connected graph the number of labelled trees is $\underline{n^{n-2}}$

Proof:-



Take a labelled tree 'T' with 'n' vertices says
$$\{1, 2, 3, \ldots\ldots, n\}$$

There is a Prufer Sequence for T, $p_1, p_2, \ldots, p_{n-2}$
$\downarrow$
n-ways

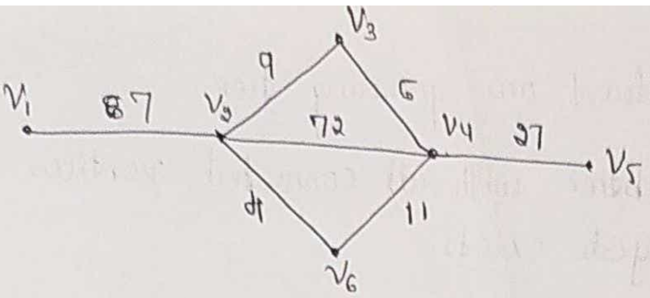
$\therefore$ Total ways is $\underline{\underline{n^{n-2}}}$

Kruskal's Algorithm:-

Steps:-

1) Sort the edges by the weights
2) Build the Spanning forest (collection of trees) by adding the edge with minimum weight which when added does not form a cycle.

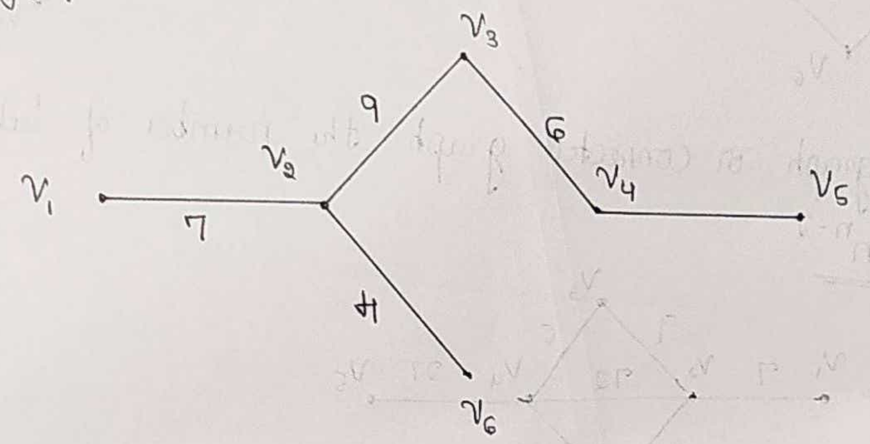
Eg:-



Soln:-

1) Sort the edges :- {4, 6, 7, 9, 11, 27, 72}

2) Spanning forest.



① $v_2 v_6 = 4$

② $v_3 v_4 = 6$

③ $v_1 v_2 = 7$

④ $v_2 v_3 = 9$

⑤ $v_4 v_5 = 27$

∴ If 'n' vertices in a tree then spanning tree has n-1 edges

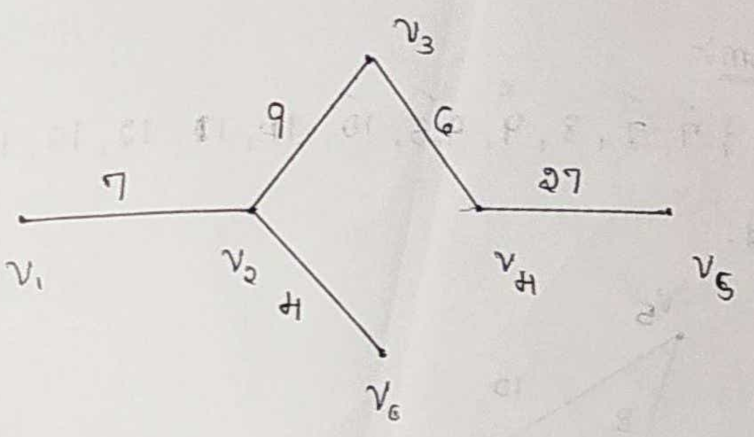
∴ Total weight is = <u>53</u>

Prim's Algorithm:-

1) Build a tree one vertex at a time.

2) Start with any point in a tree and add the edge with minimum weight among those with one point in 'T' and the other not in T.

Eg:- For the [illegible]



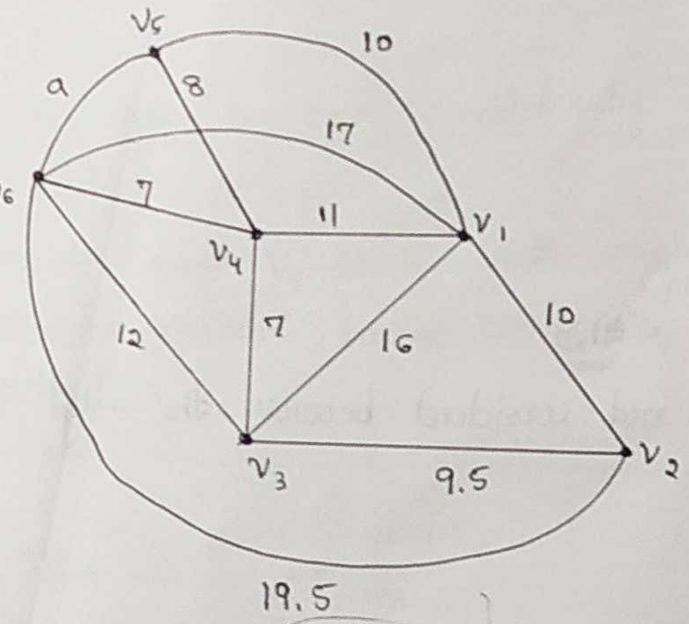
Starts with any vertex say 'v_2'
and add all the edges from v_2 are = {4, 7, 9, 72}
select with minimum weight i.e, 4

① $v_2 v_c = 4$
② $v_1 v_2 = 7$
③ $v_2 v_3 = 9$
④ $v_3 v_4 = 6$
⑤ $v_4 v_5 = 27$.

$\therefore$ Total weight = 53

Options v_2 : {7, 9, 72, 4}
v_6 = {11}
v_1 = ~~{7}~~ ϕ
v_3 = {6}
v_4 = {27}

Eg:-

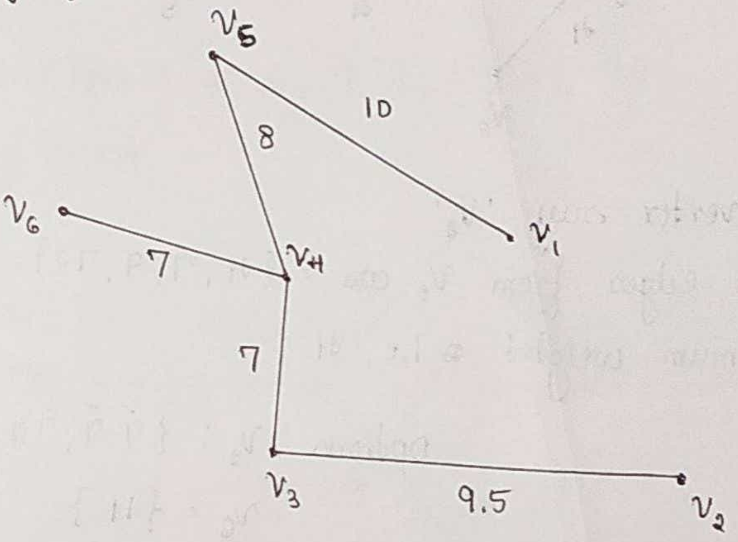
$G =$ 

Soln:-

Kruskal's Algorithm:-

1) Sort the edges = { 7, 7, 8, 9, 9.5, 10, 10, 12, 12, 16, 17, 19.5 }

2) Spanning forest:



n = 6 then n - 1 = 5 steps.

① $v_4 v_6 = 7$

② $v_4 v_3 = 7$

③ $v_4 v_5 = 8$

④ $v_3 v_2 = 9.5$

⑤ $v_5 v_1 = 10$

∴ Total weight is = 41.5

∴ In this '9' is not considered because the edge $v_6 - v_5$ form a cycle.

Prim's Algorithm:-

Steps:-

① $V_4 V_3 = 7$

② $V_4 V_6 = 7$

③ $V_4 V_5 = 8$

④ $V_3 V_2 = 9.5$

⑤ $V_2 V_1 = 10$

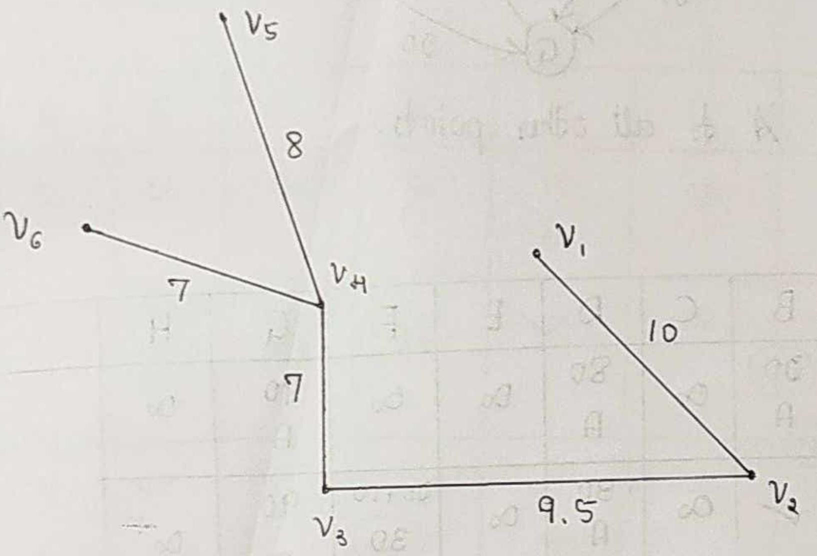
Options

$V_4 : \{7, 11, 8, 7\}$ (① v_3, ③, ② v_6)

$V_3 : \{9.5, 16, 19\}$ (v_2)

$V_6 : \{9, 19.5, 17\}$

$V_5 : \{10\}$



∴ Total weight = 41.5

∴ In this algorith cycle we not consider by if the vertex is not yet explored.

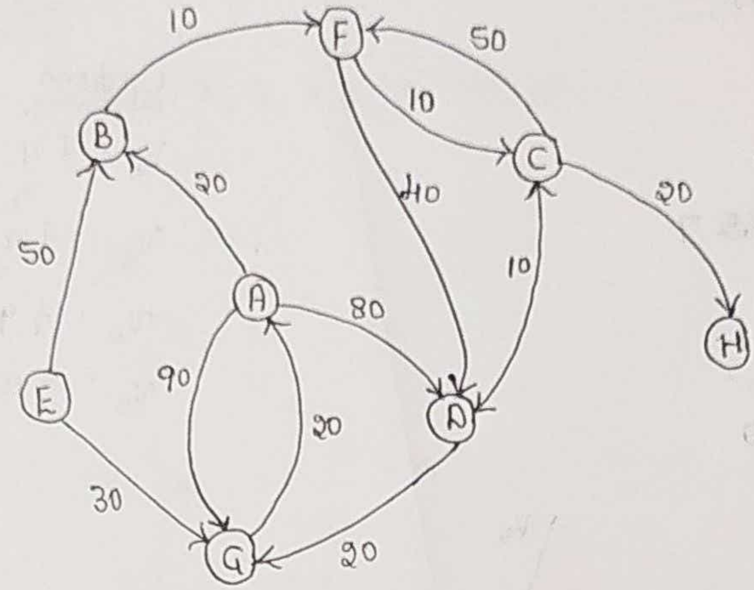
So we not consider edge V_6 to V_5 with weight '9' because V_6 & V_5 is already explored with vertex V_4 with weight 7 & 8 respectively.

Algorithms:-

1) Min - Spanning Tree → Kruskal's, → Prims

2) Shortest Path Algorithm → Dijkstra's

Dijktra's Algorithm:-



Staring Point 'A' to all other points.

Soln:-

Starting Point	B	C	D	E	F	G	H
A	20 A min	∞	80 A	∞	∞	90 A	∞
B (20)	✓	∞	80 A no change	∞	20+10 30 B min	90 A	∞
F (30)	✓	30+10 40 F min	30+40 70 F	∞	✓	90 A	∞
C (40)	✓	✓	40+10 50 C min	∞	✓	∞	40+20 60 C
D (50)	✓	✓	✓	∞	✓	50+20 70 D	60 C min
H (60)	✓	✓	✓	∞	✓	70 D min	✓
~~D~~G (70)	✓	✓	✓	∞	✓	✓	✓
E (∞)	✓	✓	✓	✓	✓	✓	✓

→ Immediate Parent

D has max but while from 'F' it is min so update

	B	C	D	E	F	G	H
A	20	40	50	∞	30	70	60
Parent	A	F	C	—	B	D	C

∴ A to H : 60

$A \xleftarrow{20} B \xleftarrow{10} F \xleftarrow{10} C \xleftarrow{20} H$

For the same graph if starting vertex is 'B'

Soln:-

Start Point	A	C	D	E	F	G	H
B	∞	∞	∞	∞	10 min B	∞	∞
F (10)	∞	10+10 20 min F	10+40 50 F	∞	✓	∞	∞
C (20)	∞	✓	20+10 30 min C	∞	✓	∞	20+20 40 C
D (30)	∞	✓	✓	∞	✓	30+20 50 D	20+20 40 min C
H (40)	∞	✓	✓	∞	✓	50 min D	✓
G (50)	50+20 70 min G	✓	✓	∞	✓	✓	✓
A (70)	✓	✓	✓	∞ min	✓	✓	✓
E (∞)	✓	✓	✓	✓	✓	✓	✓

	A	C	D	E	F	G	H
B	70	20	30	∞	10	50	40
Parend	G	F	C	–	B	D	C

B to H : 40

$$B \xleftarrow{10} F \xleftarrow{10} C \xleftarrow{20} H$$

Game Theory

Odd - Even Game:

$$\begin{array}{cc} & \text{Player - II} \\ & \begin{array}{cc} 1 & 2 \end{array} \\ \text{Player I} \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \\ & \begin{array}{cc} q & 1-q \end{array} \end{array}$$

p (Prob. that player - I select op^n 1)

$1-p$ (" " " option 2)

Expectation of player I should be idependent of whether player - 2 chooses option 1 or 2.

If Player - 2 chooses op^n - 1   If player - 2 chooses op^n - 2

Expectation of player - 1 = Expection of player - 1

$$(-2)p + 3(1-p) = 3p + (-4)(1-p)$$

$$-5p - 7p = -7$$

$$-12p = -7$$

$$p = 7/12$$

$$\therefore q = 1 - p = 1 - 7/12 = \underline{\underline{5/12}}$$

value of the game $= -2p + 3(1-p)$

$$= -2(7/12) + 3(1 - 7/12)$$

$$= \underline{\underline{1/12}}$$

if value $> 0 \rightarrow$ You will be player - 1

$= 0 \rightarrow$ fair game

$< 0 \rightarrow$ You will be player - 2

$^{|||ly}$

$$(-2)q + 3(1-q) = 3q + (-4)(1-q)$$

$$q = 7/12, \quad 1 - q = 5/12$$

values of game $= (-2)q + 3(1-q)$

$= (-2)\,{}^{7}/_{12} + 3({}^{5}/_{12})$

$= \underline{{}^{1}/_{12}}$ $\left[\begin{matrix} >0, & \text{player-1} \\ <0, & \text{player-2} \end{matrix}\right]$

Even - Odd Game:-

Player-II

$$\begin{array}{ccc} & 1 & 2 & \\ \text{Player-I} \quad \begin{matrix} 1 \\ 2 \end{matrix} & \left[\begin{matrix} 2 & -3 \\ -3 & 4 \end{matrix}\right] & \begin{matrix} p \\ 1-p \end{matrix} \\ & q \quad 1-q & \end{array}$$

$2p - 3(1-p) = -3p + 4(1-p)$

$2p - 3 + 3p = -3p + 4 - 4p$

$\cancel{1\,4p} + 2p = 7$

$p = {}^{7}/_{12}, \quad 1-p = {}^{5}/_{12}$

$\therefore$ Value of game $= 2p - 3(1-p) = 2({}^{7}/_{12}) - 3({}^{5}/_{12})$

$= \underline{-{}^{1}/_{12}}$

$\cancel{2q + 4(}\; 2q - 3(1-q) = -3q + 4(1-q)$

$q = {}^{7}/_{12}, \quad 1-q = {}^{5}/_{12}$

$\therefore$ Value of game $= 2q - 3(1-q) = \underline{-{}^{1}/_{12}}$

$\therefore$ Mixed Strategies:-

$$\begin{array}{cc} & p \quad 1-p \\ \text{Player - 1} & \left[\begin{matrix} {}^{7}/_{12} & {}^{5}/_{12} \end{matrix}\right] \end{array}$$

$$\begin{array}{cc} & q \quad 1-q \\ \text{Player - 2} & \left[\begin{matrix} {}^{7}/_{12} & {}^{5}/_{12} \end{matrix}\right] \end{array}$$

Value of game $= -{}^{1}/_{12}$ //

Given, $\left[\begin{matrix} 0 & -10 \\ 1 & 2 \end{matrix}\right]$ solve the game.

Soln:-

Player-2

$$\text{Player-1} \quad \left[\begin{matrix} 0 & -10 \\ \boxed{1} & \boxed{2} \end{matrix}\right] \quad \text{row min } \bigcirc$$

col max $\square$

$\therefore$ Saddle Point $= 1$ (value of game).

$\therefore$ Saddle point is the value which includes both row min and col max.

$\therefore$ Pure Strategies, Player I $[0, 1]$

Player II $[1, 0]$

Odd-Even Game.

$\left[\begin{matrix} -2 & 3 \\ 3 & -4 \end{matrix}\right]$

$3-(-4) = 7 \Rightarrow p = \frac{7}{5+7} = \underline{\frac{7}{12}}$

$3-(-2) = 5 \Rightarrow 1-p = \underline{{}^{5}/_{12}}$

$3-(-2) = 5 \Rightarrow q = {}^{5}/_{12}$

$3-(-4) = 7 \Rightarrow 1-q = \underline{{}^{7}/_{12}}$

$\therefore$ Value of the game $= (-2)\,{}^{7}/_{12} + 3({}^{5}/_{12}) = \underline{{}^{1}/_{12}}$

Even - Odd Game

$\left[\begin{matrix} 2 & -3 \\ -3 & 4 \end{matrix}\right]$

$4-(-3) = 7 \Rightarrow p = {}^{7}/_{12}$

$2-(-3) = 5 \Rightarrow 1-p = {}^{5}/_{12}$

$2-(-3) = 5 \Rightarrow q = {}^{5}/_{12}$

$4-(-3) = 7 \Rightarrow q = {}^{7}/_{12}$.

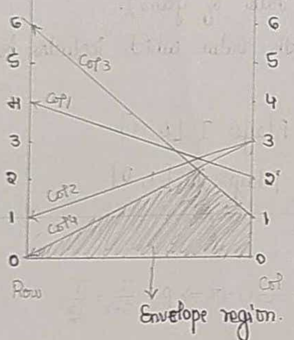
$\therefore$ Value of game $= 2({}^{7}/_{12}) + (-3)({}^{5}/_{12}) = -{}^{1}/_{12}$ //

2×m Game:- graphical method

Solve this game,

$$\begin{array}{c|cccc} & Col1 & Col2 & Col3 & Col4 \\ \hline & 2 & 3 & 1 & 3 \\ row & 4 & 1 & 6 & 0 \end{array}_{2\times 4}$$

Soln:-



~~Delete the envelop region lines, i.e.~~

Delete the other lines not included in envelope region i.e, Col1 and Col2. (they are redundant).

∴ The matrix reduces to,

$$\begin{array}{cc} Col3 & Col4 \end{array}$$
$$\begin{bmatrix} 1 & 3 \\ 6 & 0 \end{bmatrix}$$

In this there is no saddle ~~sag~~ point. so,

$$\begin{bmatrix} 1 & 3 \\ 6 & 0 \end{bmatrix}$$

$6-0 \Rightarrow p = 6. \Rightarrow 6/8$

$3-1 \Rightarrow 1-p = 2 \Rightarrow 2/8$

$6-1 = 5$ $\quad$ $3-0 = 3$

$q = 5/8$ $\quad$ $1-q = 3/8$

∴ Value of game = $1(6/8) + 6(2/8) = \underline{\underline{?/8}} > 0. \rightarrow$ Player-1

Analysis / Mixed Strategies:

Player-1 : $[\,6/8 \quad \underline{\underline{2/8}}\,]$

Player-2 : $[\,0 \quad 0 \quad \underline{\underline{3/8 \quad 5/8}}\,]$

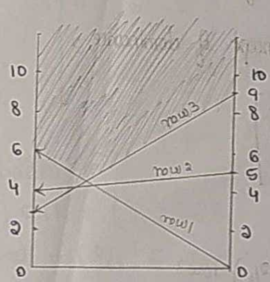
---

n×2 Game:-

Solve this game,

$$\begin{array}{c} row1 \\ row2 \\ row3 \end{array}\begin{bmatrix} 0 & 6 \\ 5 & 4 \\ 9 & 3 \end{bmatrix}_{3\times 2}$$

Soln:-



Row1 and Row3 decide the upper envelope and hence the game delete row 2.

$$\begin{bmatrix} 0 & 6 \\ 9 & 3 \end{bmatrix}$$

$9-3 = 6 \Rightarrow p = 6/12$

$6-0 \Rightarrow 6 \Rightarrow 1-p = 6/12$

$9-0 \Rightarrow q = 9/12.$

$6-3 = 3 \Rightarrow 1-q = 3/12$

∴ Value of the game $= 0(6/12) + 9(6/12) = \boxed{9/2} > 0 \Rightarrow$ player-1

Mixed Strategies :-

Player-1 : $[6/12 \quad 0 \quad 6/12]$

Player-2 : $[3/12 \quad 9/12]$

Dominance :-

If the column is greater than another column delete the ~~lesser~~ greater column.

If the row is smaller than the another row then delete the lesser row.

A row(column) is removed if ~~the~~ it is dominated by a probability distribution of other rows(columns).

Solve this game using concept of dominance,

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

$$\begin{array}{c} \text{row1} \\ \text{row2} \\ \text{row3} \end{array} \begin{bmatrix} 0 & 6 \\ 5 & 4 \\ 9 & 3 \end{bmatrix}$$

First solve column

Soln:-

Consider the weights for PDE are, $(1/3, 2/3)$ or $(1/2, 1/2)$

$$\text{row 2} \leq (1/3)\,\text{row1} + (2/3)\,\text{row3}$$

$$5 \leq (1/3)0 + (2/3)9 = 6$$

$$4 \leq (1/3)6 + (2/3)3 = 4.$$

$\therefore$ Row 2 is lower than others, so Row1 & Row3 dominate the row2.

$\therefore$ Delete the row 2.

$\therefore$ Reduced game is,

$$\begin{bmatrix} 0 & 6 \\ 9 & 3 \end{bmatrix}$$

Now solving this game,

$$\begin{bmatrix} 0 & 6 \\ 9 & 3 \end{bmatrix}$$

$\quad p = 6/12$

$9-3 = 6 \Rightarrow 1-p = 6/12$

$9-0 \Rightarrow 9 \Rightarrow q = 9/12$

$6-3 = 3 \Rightarrow 1-q \Rightarrow 3/12$

$\therefore$ Value of game $= 0(6/12) + 9(6/12) = \underline{9/2} > 0 \longrightarrow$ player-I

Mixed Strategies :-

Player-I : $[6/12 \quad 0 \quad 6/12]$

Player-II : $[9/12 \quad 3/12]$

Solve this game using concept of dominance.

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

Soln:-

1) Column elemination,

$$\text{col2} \geq (1/3)\,\text{col1} + (2/3)\,\text{col3}$$

$$4 \geq (1/3)0 + (2/3)6 = 4$$

$$7 \geq (1/3)5 + (2/3)4 = 13/3 = 4.3$$

$$6 \geq (1/3)9 + (2/3)3 = 5$$

$\therefore$ Delete col2. & reduced matrix is,

$$\begin{bmatrix} 0 & 6 \\ 5 & 4 \\ 9 & 3 \end{bmatrix}$$

2) Row elemination,

$$\text{row2} \leq (1/3)\,\text{row1} + (2/3)\,\text{row3}$$

$$5 \leq (1/3)0 + (2/3)9 = 6$$

$$4 \leq (1/3)6 + (2/3)3 = 4$$

∴ Now delete row 2 & resulted matrix game is,

$$\begin{bmatrix} 0 & 6 \\ 9 & 3 \end{bmatrix} \quad \begin{array}{l} p = 6/12 \\ 1-p = 6/12 \\ q = 9/12 \\ 1-q = 3/12 \end{array}$$

∴ Value of game $= 0(6/12) + 9(6/12) = 9/2 > 0 \longrightarrow$ Player - I

∴ Mixed Strategies,

Player-I : $[6/12 \quad 0 \quad 6/12]$

Player-II : $[9/12 \quad 3/12]$

Odd-Even Game (3-Options)

$$\begin{array}{cc|c} & & \text{Player-II} \\ & & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{Player-I} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -2 & 3 & -4 \\ 3 & -4 & 5 \\ -4 & 5 & -6 \end{bmatrix} \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \end{array}$$

Soln:-

$-2p_1 + 3p_2 - 4p_3 = v$ —— ①

$3p_1 - 4p_2 + 5p_3 = v$ —— ②

$-4p_1 + 5p_2 - 6p_3 = v$ —— ③

and $p_1 + p_2 + p_3 = 1$ —— ④

① - ②

$-5p_1 + 7p_2 - 9p_3 = 0$ —— ⑤

② - ③

$7p_1 - 9p_2 + 11p_3 = 0$ —— ⑥

Using ⑤, ⑥ and ④

∴ $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 1/4$

∴ Value of the game $= -2(p_1) + 3p_2 - 4p_3$

$= -2(1/4) + 3(1/2) - 4(1/4)$

$= \underline{0} \longrightarrow$ Fair game.

∴ This is also called row player expectation.

∴ Column player expectation,

$-2q_1 + 3q_2 - 4q_3 = v$ —— ⑦

$3q_1 - 4q_2 + 5q_3 = v$ —— ⑧

$-4q_1 + 5q_2 - 6q_3 = v$ —— ⑨

$q_1 + q_2 + q_3 = 1$ —— ⑩.

∴ This is same as previous,

∴ $q_1 = 1/4$, $q_2 = 1/2$, $q_3 = 1/4$

∴ Value of game $= -2(1/4) + 3(1/2) - 4(1/4) = 0 \longrightarrow$ Fair game.

∴ Mixed Strategies,

Player-I : $[1/4, 1/2, 1/4]$; $v = 0$

Player-II : $[1/4, 1/2, 1/4]$; $v = 0$

$\underline{\underline{v=0}}$

Triangular Game :-

Ex :-

$$\begin{bmatrix} -2 & 3 & -4 \\ 0 & -4 & 5 \\ 0 & 0 & -6 \end{bmatrix}$$

Soln :-

$-2p_1 = v$ —— ①

$3p_1 - 4p_2 = v$ —— ②

$-4p_1 + 5p_2 - 6p_3 = v$ —— ③

$p_1 + p_2 + p_3 = 1$ —— ④

$\therefore$ ① $\Rightarrow p_1 = -v/2$

② $\Rightarrow 3(-v/2) - 4p_2 = v \Rightarrow v + \frac{3v}{2} = \frac{5v}{2}$

$\therefore p_2 = \frac{-5v}{8}$

③ $\Rightarrow -4\left(\frac{-v}{2}\right) + 5\left(\frac{-5v}{8}\right) - 6p_3 = v$

$-6p_3 = v - \frac{4v}{2} + \frac{25v}{8} = \frac{8v - 16v + 25v}{8} = \frac{17v}{8}$

$\therefore p_3 = \frac{-17v}{48}$

$\therefore$ ④ $\Rightarrow \frac{-v}{2} - \frac{5v}{8} - \frac{17v}{48} = 1$

$-\frac{24v - 30v - 17v}{48} = 1$

$\frac{-71v}{48} = 1 \Rightarrow v = \frac{-48}{71}$

$\therefore p_1 = -v/2 = \frac{\frac{48}{71}}{2} = \frac{24}{71}$

$p_2 = \frac{-5v}{8} = -\frac{5}{8}\left(\frac{-48}{71}\right) = +\frac{30}{71}$

$p_3 = \frac{-17v}{48} = \frac{-17}{48}\left(\frac{-48}{71}\right) = +\frac{17}{71}$

$\therefore$ Value of the game $= -2\left(\frac{24}{71}\right) + 0\left(\frac{30}{71}\right) + 0\left(\frac{17}{71}\right)$

$= -\frac{48}{71} < 0 \longrightarrow$ Player - II

Diagonal Game:-

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Soln:-

$-2p_1 = v \Rightarrow p_1 = -v/2$

$-4p_2 = v \Rightarrow p_2 = -v/4$

$-6p_3 = v \Rightarrow p_3 = -v/6$

$(+)\quad 1 = -\frac{v}{2} - \frac{v}{4} - \frac{v}{6}$

$1 = \frac{-11v}{12}$

$\therefore v = \frac{-12}{11}$

$\therefore p_1 = \frac{-v}{2} = \frac{6}{11}$

$p_2 = -v/4 = 3/11$

$p_3 = -v/6 = 2/11$

$\therefore$ Value of the game $= -2(6/11) + 0(3/11) + 0(2/11)$

$= -12/11 < 0 \rightarrow$ Player-II

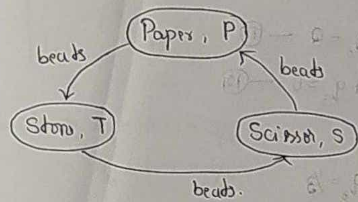
Symmetric Game:-

A matrix is called Skew Symmetric if $A = -A^T$

A game is called symmetric if

1) If the matrix is square matrix
2) If the matrix is skew symmetric.

Ex:- Stone, Paper, Scissor.



$\Rightarrow$

$$\begin{array}{c|c} & \text{Player-II} \\ & \begin{array}{ccc} P & T & S \end{array} \\ \text{Player-I} \begin{array}{c} P \\ T \\ S \end{array} & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Win = 1
Lost = -1

$$0p_1 - p_2 + p_3 = v \quad \text{—— (1)}$$
$$p_1 + 0p_2 - p_3 = v \quad \text{—— (2)}$$
$$-p_1 + p_2 + 0p_3 = v \quad \text{—— (3)}$$
$$p_1 + p_2 + p_3 = 1 \quad \text{—— (4)}$$

$$(1) - (2) \Rightarrow -p_1 - p_2 + 2p_3 = 0 \quad \text{—— (5)}$$
$$(2) - (3) \Rightarrow 2p_1 - p_2 - p_3 = 0 \quad \text{—— (6)}$$

Using (4), (5) & (6),

$\therefore p_1 = 1/3, \; p_2 = 1/3, \; p_3 = 1/3$

$\therefore$ Value of the game $= 0(1/3) - (1/3) + (1/3) = 0 \longrightarrow$ Fair game.

$$0q_1 + q_2 - q_3 = v \quad \text{—— (7)}$$
$$-q_1 + 0q_2 + q_3 = v \quad \text{—— (8)}$$
$$q_1 - q_2 + 0q_3 = v \quad \text{—— (9)}$$
$$q_1 + q_2 + q_3 = 1 \quad \text{—— (10)}$$

$$(7) - (8) \Rightarrow q_1 + q_2 - 2q_3 = 0 \quad \text{—— (11)}$$
$$(8) - (9) \Rightarrow -2q_1 + q_2 + q_3 = 0 \quad \text{—— (12)}$$

Using (10), (11) & (12)

$\therefore q_1 = 1/3, \; q_2 = 1/3, \; q_3 = 1/3$

$\therefore$ Value of the game $= 0(1/3) - (1/3) + (1/3) = 0 \longrightarrow$ Fair game.

$\therefore$ Mixed Strategies,

Player - I : $[1/3, 1/3, 1/3]$; $v = 0$

Player - II : $[1/3, 1/3, 1/3]$; $v = 0$.

Non-Zero Sum game:

Prisoner's Dilemma:-

Prisoner I, II $\longrightarrow$ Combined Crime, and they can't believe (trust) each other, then

$$\begin{array}{c|cc} & \text{Deny} & \text{Confess} \\ \hline \text{Deny} & -1, -1 & -3, \boxed{0} \\ \text{Confess} & \boxed{0}, -3 & \boxed{-2, -2} \end{array}$$

$\longrightarrow$ Saddle point

$\begin{matrix} -1, -3 \\ 0, -2 \end{matrix} \Big\} \to$ 1st comp

$\begin{matrix} -1, 0 \\ -3, -2 \end{matrix} \Big\}$ 2nd comp

$\begin{matrix} -1, 0 \\ -3, -2 \end{matrix} \Big\}$ 1st

$\begin{matrix} -1, -3 \\ 0, -2 \end{matrix} \Big\}$ 2nd

$O \to$ min, $\square \to$ max

1st component $\to$ min row; max col

2nd component $\to$ max in row; min in col

$\therefore$ Solution is saddle point (Nash Equilibrium / Pure Strategy)

$\therefore$ Confess, Confess.

Mixed Strategy Nash Equilibrium:-

$$\begin{array}{c|cc} & q_1 & q_2 \\ \hline p_1 & \boxed{2}, -3 & 1, \boxed{2} \\ p_2 & 1, \boxed{1} & \boxed{4}, -1 \end{array} \Rightarrow \underset{\text{1st comp}}{\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}}, \; \underset{\text{2nd comp}}{\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}}$$

v_{p_1}, v_{p_2}

1st component, $q_1 = 4 - 1 = 3/4$

$q_2 = 2 - 1 = 1/4$

2nd component, $p_1 = 1 - (-1) = 2 \Rightarrow 2/7$

$p_2 = 2 - (-3) = 5 \Rightarrow 5/7$

$$v_{p_1} = -3p_1 + 1p_2 = -3(2/7) + 1(5/7) = -1/7$$
$$v_{p_2} = 2q_1 + 1q_2 = 2(3/4) + 1(1/4) = 7/4$$

$\therefore P1 : [2/7, 5/7]$

$P_2 : [3/4, 1/4]$

Q: Solve,

$$\begin{bmatrix} 1,1 & 4,-1 \\ 2,-3 & 1,2 \end{bmatrix}$$

Soln:-

$$\begin{bmatrix} 1,1 & 4,-1 \\ 2,-3 & 1,2 \end{bmatrix}$$

1st Component, $q_1 = 4 - 1 = 3 \Rightarrow 3/4$

$q_2 = 2 - 1 = 1 \Rightarrow 1/4$

2nd Component, $p_1 = 2 - (-3) = 5 \Rightarrow 5/7$

$p_2 = 1 - (-1) = 2 \Rightarrow 2/7$

$\therefore V_{p_1} = 4p_1 + 1p_2 = 4(5/7) + (2/7) = \frac{22}{7}$

$\therefore V_{p_2} = 1q_1 + 2q_2 = 3/4 + 2(1/4) = 5/4$

$\therefore$ Player - I : $[5/7, 2/7]$

Player - II : $[3/4, 1/4]$

Q: Solve,

$$\begin{bmatrix} -3,2 & 2,1 \\ 1,1 & -1,4 \end{bmatrix}$$

Soln:-

1st components, $q_1 = 2 - (-1) = 3 \Rightarrow 3/7$

$q_2 = 1 - (-3) = 4 \Rightarrow 4/7$

2nd component, $p_1 = 4 - 1 = 3 \Rightarrow 3/4$

$p_2 = 2 - 1 = 1 \Rightarrow 1/4$

$\therefore V_{p_1} = 2p_1 + (+1)p_2 = 2(3/4) + (1/4) = \cancel{8/4}\ 7/4$

$V_{p_2} = -3q_1 + 2q_2 = -3(3/7) + 4/7 \times 2 = \cancel{-8/7}\ -1/7$

$\therefore$ Player - I : $[3/4, 1/4]$

Player - II : $[3/7, 4/7]$

Advanced Numerical Methods

Finite different method (Taylor Series):-

$$y(x) = y(x_0) + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

expansion about $x = x_0$

$\therefore y(x) = e^x \quad , \quad x_0 = 0$

$y'(x) = e^x \ , \ y'(x_0) = y'(0) = e^0 = 1$

$y''(x) = e^x ,$

$\vdots$

$$y(x) = e^x = 1 + (x-0)1 + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(1) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{0.01} = 1 + 0.01 + \frac{(0.01)^2}{2!} + \cdots\cdots$$

$= 1.01005 \cdots$ for $0 \le n \le 10$.

Let $h = x - x_0$ ← small

$$\therefore y(x) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$\therefore y_{i+1} = y(x_i + h)$

$$\therefore y(x_i+h) = y_i + h\, y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \cdots \quad \text{—— (1)}$$

$$y_{i-1} = y(x_i - h)$$
$$= y_i - h y_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \cdots \quad \text{—— (2)}$$

(1) + (2):

$$y_{i+1} + y_{i-1} = 2y_i + 2\frac{h^2}{2!} y_i'' + O(h^4)$$ → order of h^4

(1) − (2):

$$y_{i+1} - y_{i-1} = 2h y_i' + 2\frac{h^3}{3!} y_i''' + O(h^5)$$

OR

$$\boxed{y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \quad \text{and} \quad y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}}$$

Given. $y(1) = 1$, $y(2) = 2$, i) $h = 0.5$, ii) $h = 0.25$

solve $xy'' + y = 0$

Soln:- i) $h = 0.5$

$y(x_0) = y_0$

$y(1) = 1, \quad x_0 = 1; \quad y_0 = 1$

$x_1 = 1.5; \quad y_1 = ? = y(x_1)$

$x_2 = 2 \ , \ y_2 = y(x_2) = y(2) = 2$

$y(2) = 2$

$x_i y_i'' + y_i = 0$

$$x_i \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{(0.5)^2} \right) + y_i = 0$$

Put $i = 1$,

$$x_1 \left(\frac{y_2 - 2y_1 + y_0}{(0.5)^2} \right) + y_1 = 0$$

$$(1.5) \left(\frac{2 - 2y_1 + 1}{(0.5)^2} \right) + y_1 = 0$$

$6(2-2y_1+1)+y_1=0$

$12-12y_1+6+y_1=0$

$-11y_1=-18$

$y_1=\frac{18}{11}=\underline{1.6366}$

Partial Differential Equations:-

Given,

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+F(x,y,u,u_x,u_y)=0 \quad \text{—— (I)}$$

(I) said to be,

i> Parabolic if $AC-B^2=0$

ii> Elliptical if $AC-B^2>0$

iii> Hyperbolic if $AC-B^2<0$

Soln:

W.K.T

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \longrightarrow \text{Laplace Equ}^n \quad \text{—— (1)}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \longrightarrow \text{Poisson Equ}^n \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow \text{Head Equ}^n \quad \text{—— (3)}$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow \text{Wave Equ}^n \quad \text{—— (4)}$$

(1) let $u_{xx}=\frac{\partial^2 u}{\partial x^2}$, $u_{yy}=\frac{\partial^2 u}{\partial y^2}$ then

$A=1,\ B=0,\ C=1$ and $AC-B^2=1-0=\underline{1>0}$

$\therefore$ This is Elliptical

(2) Here also $A=1,\ B=0,\ C=1$ then $AC-B^2=1>0$

$\therefore$ This is Elliptical.

(3) Rearrange $\frac{\partial u}{\partial t}-c_1^2\frac{\partial^2 u}{\partial x^2}$ here, $\frac{\partial^2 u}{\partial x^2}=u_{yy}$, $\frac{\partial u}{\partial t}=u_{xy}$

$A=0,\ B=1,\ C=-c_1^2$

$\therefore AC-B^2=\cancel{+}0_1^2 \approx \underline{0}$

$\therefore$ This is ~~Hyperbolic~~. Parabolic.

(*) Rearrange

$$\frac{\partial^2 u}{\partial t^2} - C_1^2 \frac{\partial^2 u}{\partial x^2} \quad \text{Here, } u_{xx} = \frac{\partial^2 u}{\partial t^2}, \ \frac{\partial^2 u}{\partial x^2} = u_{yy}, \ A=1, \ B=0, \ C=-c_1^2$$

$$\therefore AC - B^2 = 1 \times (-c_1^2) - 0 = -c_1^2 < 0$$

∴ This is Hyperbolic.

$$u(x_{i+h}, y_j) = u(x_i, y_j) + h \frac{\partial u}{\partial x}(x_i, y_j) + \frac{h^2}{2!} \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \dots$$

general Equ^n → (*)

Notation ↓

$$u_{i+1,j} = u_{ij} + h \frac{\partial u}{\partial x} ij + \frac{h^2}{2!} \frac{\partial^2 u_{ij}}{\partial x} + \dots \longrightarrow (1)$$

$$u_{i-1,j} = u(x_i - h, y_j) = u_{ij} - h \frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 u_{ij}}{\partial x^2} + \dots \longrightarrow (2)$$

(1) + (2)

$$u_{i+1,j} + u_{i-1,j} = 2u_{ij} + h^2 \frac{\partial^2 u_{ij}}{\partial x^2} + O(h^4)$$

$$\boxed{\begin{aligned} \therefore \frac{\partial^2 u_{ij}}{\partial x^2} &= \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} \\ \therefore \frac{\partial^2 u_{ij}}{\partial y^2} &= \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} \end{aligned}}$$

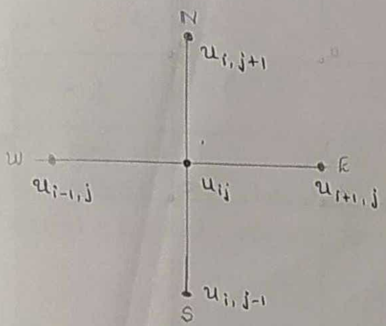
Consider Laplace Equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

then by using previous formula,

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} = 0$$

$$\therefore u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$



Consider Poisson Equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

then,

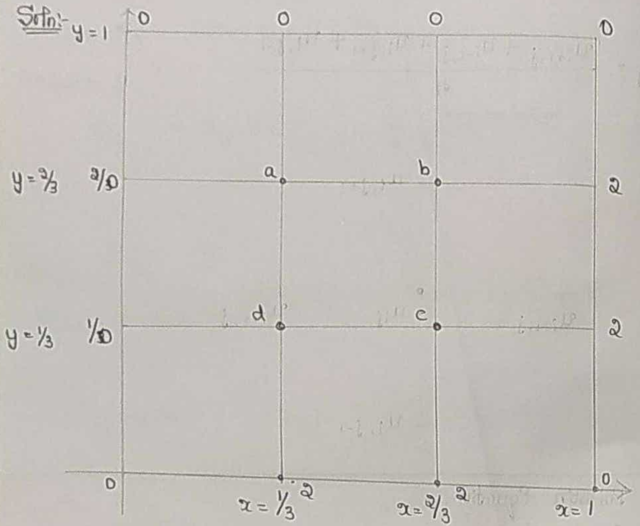
$$\frac{u_{i+1,j} + 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} = f_{ij}$$

$$\therefore u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{ij}}{4}$$

Solve this with step size $h = 1/3$, $0 \le x \le 1$ and $0 \le y \le 1$ and boundary conditions are, $u(x,1) = 0$, $u(0,y) = 0$, $u(1,y) = 9(y - y^2)$ and $u(x,0) = 9(x - x^2)$ for the Laplace Equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad u_{xx} + u_{yy} = 0$$

Soln:-



a = NESW

$$\therefore a = \frac{1}{4}[0 + b + d + 0] \implies a = \frac{b+d}{4}$$

$$b = \frac{1}{4}[0 + 2 + c + a] \implies b = \frac{a+c+2}{4}$$

$$c = \frac{1}{4}[b + 2 + 2 + d] \implies c = \frac{b+d+4}{4}$$

$$d = \frac{1}{4}[a + c + 2 + 0] \implies d = \frac{a+c+2}{4}$$

$\therefore$ $b = d$ then

$$a = \frac{1}{4}[2b] = \frac{b}{2} \implies 2a - b = 0 \quad \text{—— (1)}$$

by taking 'b',

$$b = \frac{a+c+2}{4} \implies 4b = a + c + 2$$

$$\implies a - 4b + c = -2 \quad \text{—— (2)}$$

by taking 'c'

$$c = \frac{b+d+4}{4} \implies 4c = b + d + 4$$

$$\implies 4c = b + b + 4 \quad [b = d]$$

$$4c = 2b + 4$$

$$4c - 2b = -4 \quad \text{—— (2)}$$

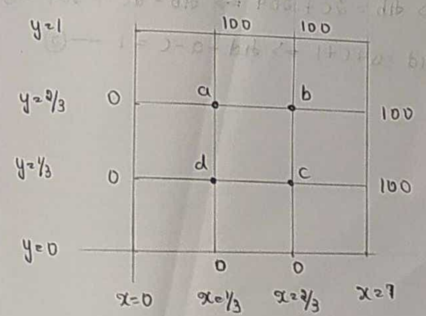
by using (1), (2) & (3)

$a = 1/2$, $b = 1$, $c = 3/2$ and

$$d = \frac{a+c+2}{4} = \underline{\underline{1}}$$

Solve this with step size $h = 1/3$, $u_{xx} + u_{yy} = -81xy$ where $0 < x < 1$ and $0 < y < 1$ and boundary conditions are $u(0,y) = u(x,0) = 0$ and $u(1,y) = u(x,1) = 100$.

Soln:-



This is Poisson Equⁿ so use 2nd formula.

$$u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 g(x_i, y_j)}{4}$$

$$a = \frac{1}{4}\left[100 + b + d + 0 - \frac{1}{9} g(1/3, 2/3)\right] = 1/4\,[b + d + 100 + 2]$$

$$= \frac{1}{4}[b + d + 102]$$

$$b = 1/4\left[100 + 100 + c + a - \frac{1}{9} g(2/3, 2/3)\right]$$

$$= 1/4\,[c + a + 204]$$

$$c = 1/4\left[b + 100 + 0 + d - 1/9\, g(2/3, 1/3)\right]$$

$$= 1/4\,[b + d + 102]$$

$$d = 1/4\left[a + c + 0 + 0 - 1/9\, g(1/3, 1/3)\right]$$

$$= 1/4\,[a + c + 1]$$

Since, $a = c$.

$b = 1/4\,[c + c + 204] \Rightarrow 4b = 2c + 204 \Rightarrow 4b - 2c = 204$ — ①

$d = 1/4\,[c + c + 1] \Rightarrow 4d = a + c + 1 \Rightarrow 4d - a - c = 1$ — ②

$\therefore$

$\therefore$