Graph Theory-Class 3

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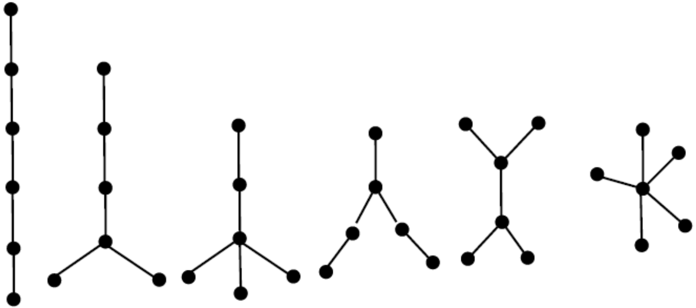
All trees with six vertices:



T_1 : T_2 : T_3 : T_4 : T_5 : T_6 :

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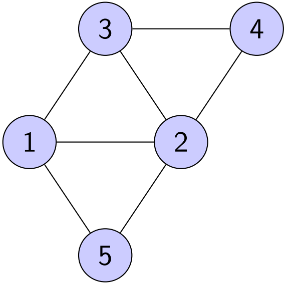

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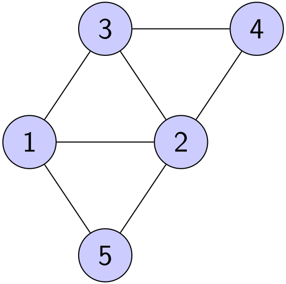

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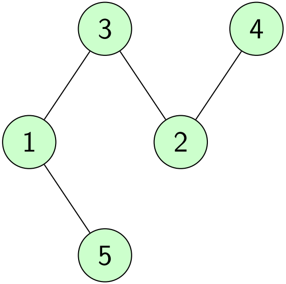

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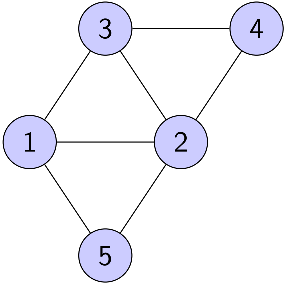

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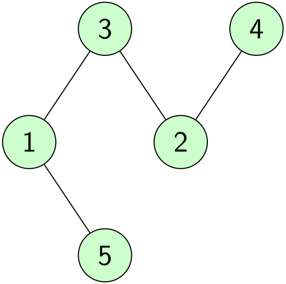

Question: Does every graph has a spanning tree?

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6. Further no vertices are removed, which gives a spanning tree.

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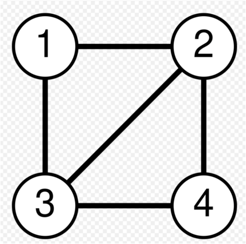

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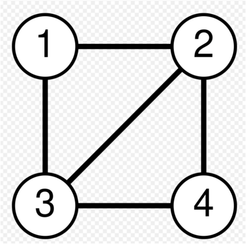

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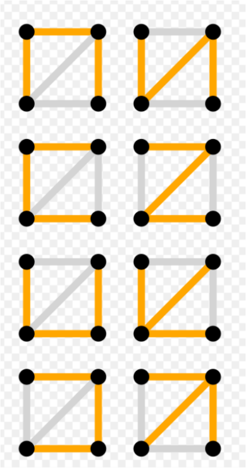

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- The value $C_{i,j}$ is independent of the choice of i and j.

Question: Determine number of spanning trees in

- previous graph G and
- K_4.