In[259]:=

Kinect Energy

Kinect Energy link 1

In[260]:=
```
p1 = { l1 Cos[θ1[t]],
       l1 Sin[θ1[t]],
       0}
```
Out[260]=
$$\{\text{Cos}[\theta_1[t]]\, l_1,\ \text{Sin}[\theta_1[t]]\, l_1,\ 0\}$$

In[261]:=
```
v1 = D[ p1, {t, 1}]
```
Out[261]=
$$\{-\text{Sin}[\theta_1[t]]\, l_1\, \theta_1'[t],\ \text{Cos}[\theta_1[t]]\, l_1\, \theta_1'[t],\ 0\}$$

In[262]:=
```
Ie1 = {{Ixx1, 0, 0},
       {0, Iyy1, 0},
       {0, 0, Izz1}}
```
Out[262]=
$$\{\{\text{Ixx}_1, 0, 0\}, \{0, \text{Iyy}_1, 0\}, \{0, 0, \text{Izz}_1\}\}$$

In[263]:=
```
ω1 = { 0,
       0,
       D[θ1[t], {t, 1}]}
```
Out[263]=
$$\{0, 0, \theta_1'[t]\}$$

In[264]:=
```
KE1 = Simplify[1 / 2 m1 Dot [v1, v1]] + Simplify[ 1 / 2 Dot [ω1, Ie1 .ω1]]
```
Out[264]=
$$\frac{1}{2}\,\text{Izz}_1\,\theta_1'[t]^2 + \frac{1}{2}\,l_1^2\,m_1\,\theta_1'[t]^2$$

Kinect Energy link 2

In[265]:=
```
p2 = { l1 Cos[θ1[t]] + l2 Cos[θ1[t] + θ2[t]],
       l1 Sin[θ1[t]] + l2 Sin[θ1[t] + θ2[t]],
       0}
```
Out[265]=
$$\{\text{Cos}[\theta_1[t]]\, l_1 + \text{Cos}[\theta_1[t] + \theta_2[t]]\, l_2,\ \text{Sin}[\theta_1[t]]\, l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2,\ 0\}$$

In[266]:=
```
v2 = D[ p2, {t, 1}]
```
Out[266]=
$$\{-\text{Sin}[\theta_1[t]]\, l_1\, \theta_1'[t] - \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]),\ \text{Cos}[\theta_1[t]]\, l_1\, \theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]),\ 0\}$$

In[267]:=
```
Ie2 = {{Ixx2, 0, 0}, {0, Iyy2, 0}, {0, 0, Izz2}}
```
Out[267]=
$$\{\{\text{Ixx}_2, 0, 0\}, \{0, \text{Iyy}_2, 0\}, \{0, 0, \text{Izz}_2\}\}$$

In[268]:=
```
ω2 = { 0, 0, D[θ1[t] + θ2[t], {t, 1}]}
```
Out[268]=
$$\{0, 0, \theta_1'[t] + \theta_2'[t]\}$$

In[269]:=
```
KE2 = Simplify[1 / 2 m2 Dot [v2, v2]] + Simplify[ 1 / 2 Dot [ω2, Ie2 .ω2]]
```

Out[269]=
$$\frac{1}{2}\,Izz_2\,(\theta_1'[t]+\theta_2'[t])^2+\frac{1}{2}\,m_2\,\Big((\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t]+\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]))^2+(\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t]+\text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]))^2\Big)$$

Kinect Energy link 3

In[270]:=
```
p3 = { l1 Cos[θ1[t]] + l2 Cos[θ1[t] + θ2[t]] + l3 Cos[θ1[t] + θ2[t] + θ3[t]],
       l1 Sin[θ1[t]] + l2 Sin[θ1[t] + θ2[t]] + l3 Sin[θ1[t] + θ2[t] + θ3[t]],
       0}
```

Out[270]=
$$\{\text{Cos}[\theta_1[t]]\,l_1+\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2+\text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3,\ \text{Sin}[\theta_1[t]]\,l_1+\text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2+\text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3,\ 0\}$$

In[271]:=
```
v3 = D[ p3, {t, 1}]
```

Out[271]=
$$\{-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t]-\text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])-\text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]),\ \text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t]+\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]),\ 0\}$$

In[272]:=
```
Ie3 = {{Ixx3, 0, 0},
       {0, Iyy3, 0},
       {0, 0, Izz3}}
```

Out[272]=
$$\{\{Ixx_3,0,0\},\{0,Iyy_3,0\},\{0,0,Izz_3\}\}$$

In[273]:=
```
ω3 = { 0,
       0,
       D[θ1[t] + θ2[t] + θ3[t], {t, 1}]}
```

Out[273]=
$$\{0,\ 0,\ \theta_1'[t]+\theta_2'[t]+\theta_3'[t]\}$$

In[274]:=
```
KE3 = Simplify[1 / 2 m3 Dot [v3, v3]] + Simplify[ 1 / 2 Dot [ω3, Ie3 .ω3]]
```

Out[274]=
$$\frac{1}{2}\,Izz_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2+\frac{1}{2}\,m_3\,\Big((\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t]+\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))^2+(\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t]+\text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))^2\Big)$$

Total Kinect Energy

In[275]:=
```
KE = KE1 + KE2 + KE3
```

Out[275]=
$$\frac{1}{2} Izz_1 \theta_1'[t]^2 + \frac{1}{2} l_1^2 m_1 \theta_1'[t]^2 + \frac{1}{2} Izz_2 (\theta_1'[t] + \theta_2'[t])^2 +$$
$$\frac{1}{2} m_2 \left((\text{Cos}[\theta_1[t]]\, l_1\, \theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]))^2 + (\text{Sin}[\theta_1[t]]\, l_1\, \theta_1'[t] + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]))^2\right) +$$
$$\frac{1}{2} Izz_3 (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 +$$
$$\frac{1}{2} m_3 \left((\text{Cos}[\theta_1[t]]\, l_1\, \theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]) + \text{Cos}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))^2 + (\text{Sin}[\theta_1[t]]\, l_1\, \theta_1'[t] + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]) + \text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))^2\right)$$

Potential Energy

Potential Energy link 1

In[276]:=
```
PE1 = m1 g l1 Sin[θ1[t]]
```

Out[276]=
$$g\, \text{Sin}[\theta_1[t]]\, l_1\, m_1$$

Potential Energy link 2

In[277]:=
```
PE2 = m2 g (l1 Sin[θ1[t]] + l2 Sin[θ1[t] + θ2[t]])
```

Out[277]=
$$g\, (\text{Sin}[\theta_1[t]]\, l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2)\, m_2$$

Potential Energy link 3

In[278]:=
```
PE3 = m3 g (l1 Sin[θ1[t]] + l2 Sin[θ1[t] + θ2[t]] + l3 Sin[θ1[t] + θ2[t] + θ3[t]])
```

Out[278]=
$$g\, (\text{Sin}[\theta_1[t]]\, l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2 + \text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3)\, m_3$$

Total Potential Energy

In[279]:=
```
PE = PE1 + PE2 + PE3
```

Out[279]=
$$g\, \text{Sin}[\theta_1[t]]\, l_1\, m_1 + g\, (\text{Sin}[\theta_1[t]]\, l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2)\, m_2 + g\, (\text{Sin}[\theta_1[t]]\, l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\, l_2 + \text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3)\, m_3$$

Lagrangian Partial Derivatives

In[280]:=
```
(*Partial first order derivative of l with respect to θi'*)
```

In[281]:=
```
Dl_θ1' = D[KE - PE, {θ1'[t], 1}]
```

Out[281]=
$$\begin{aligned}
&\text{Izz}_1\,\theta_1'[t] + l_1^2\,m_1\,\theta_1'[t] + \text{Izz}_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\frac{1}{2}\,m_2\,(2\,(\cos[\theta_1[t]]\,l_1 + \cos[\theta_1[t] + \theta_2[t]]\,l_2) \\
&\qquad (\cos[\theta_1[t]]\,l_1\,\theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])) + \\
&\qquad 2\,(\sin[\theta_1[t]]\,l_1 + \sin[\theta_1[t] + \theta_2[t]]\,l_2) \\
&\qquad (\sin[\theta_1[t]]\,l_1\,\theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]))) + \\
&\text{Izz}_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]) + \frac{1}{2}\,m_3 \\
&\qquad (2\,(\cos[\theta_1[t]]\,l_1 + \cos[\theta_1[t] + \theta_2[t]]\,l_2 + \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3) \\
&\qquad\quad (\cos[\theta_1[t]]\,l_1\,\theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\qquad\qquad \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + \\
&\qquad\quad 2\,(\sin[\theta_1[t]]\,l_1 + \sin[\theta_1[t] + \theta_2[t]]\,l_2 + \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3) \\
&\qquad\quad (\sin[\theta_1[t]]\,l_1\,\theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\qquad\qquad \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])))
\end{aligned}$$

In[282]:=
```
Dl_θ2' = D[KE - PE, {θ2'[t], 1}]
```

Out[282]=
$$\begin{aligned}
&\text{Izz}_2\,(\theta_1'[t] + \theta_2'[t]) + \frac{1}{2}\,m_2\,(2\,\cos[\theta_1[t] + \theta_2[t]]\,l_2 \\
&\qquad (\cos[\theta_1[t]]\,l_1\,\theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])) + 2\,\sin[\\
&\qquad \theta_1[t] + \theta_2[t]]\,l_2\,(\sin[\theta_1[t]]\,l_1\,\theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]))) + \\
&\text{Izz}_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]) + \frac{1}{2}\,m_3 \\
&\qquad (2\,(\cos[\theta_1[t] + \theta_2[t]]\,l_2 + \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3) \\
&\qquad\quad (\cos[\theta_1[t]]\,l_1\,\theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\qquad\qquad \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + \\
&\qquad\quad 2\,(\sin[\theta_1[t] + \theta_2[t]]\,l_2 + \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3) \\
&\qquad\quad (\sin[\theta_1[t]]\,l_1\,\theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\qquad\qquad \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])))
\end{aligned}$$

In[283]:=
```
Dl_θ3' = D[KE - PE, {θ3'[t], 1}]
```

Out[283]=
$$\begin{aligned}
&\text{Izz}_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]) + \\
&\frac{1}{2}\,m_3\,(2\,\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\cos[\theta_1[t]]\,l_1\,\theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\,l_2 \\
&\qquad (\theta_1'[t] + \theta_2'[t]) + \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + \\
&\qquad 2\,\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\sin[\theta_1[t]]\,l_1\,\theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\,l_2 \\
&\qquad (\theta_1'[t] + \theta_2'[t]) + \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])))
\end{aligned}$$

In[284]:=
```
(*Partial first order derivative of Dl_θi' with respect to t, which is time*)
DlT1 = D[Dl_θi', {t, 1}]
```

Out[284]=
$$
\begin{aligned}
&\text{Izz}_1\,\theta_1''[t] + l_1^2\,m_1\,\theta_1''[t] + \text{Izz}_2\,(\theta_1''[t] + \theta_2''[t]) + \\
&\frac{1}{2}\,m_2\,\Big(2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])) \\
&\quad(-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] - \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])) + \\
&\quad 2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])) \\
&\quad(\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])) + \\
&\quad 2\,(\text{Cos}[\theta_1[t]]\,l_1 + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2) \\
&\quad\Big(-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t]^2 - \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])^2 + \\
&\quad\quad\text{Cos}[\theta_1[t]]\,l_1\,\theta_1''[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1''[t] + \theta_2''[t])\Big) + \\
&\quad 2\,(\text{Sin}[\theta_1[t]]\,l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2)\,\Big(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t]^2 + \\
&\quad\quad\text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])^2 + \text{Sin}[\theta_1[t]]\,l_1\,\theta_1''[t] + \\
&\quad\quad\text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1''[t] + \theta_2''[t])\Big)\Big) + \text{Izz}_3\,(\theta_1''[t] + \theta_2''[t] + \theta_3''[t]) + \\
&\frac{1}{2}\,m_3\,\Big(2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\quad\quad\text{Cos}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) \\
&\quad(-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] - \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) - \\
&\quad\quad\text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + \\
&\quad 2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\quad\quad\text{Cos}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) \\
&\quad(\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t]) + \\
&\quad\quad\text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + \\
&\quad 2\,(\text{Cos}[\theta_1[t]]\,l_1 + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2 + \text{Cos}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3) \\
&\quad\Big(-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t]^2 - \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])^2 - \\
&\quad\quad\text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 + \\
&\quad\quad\text{Cos}[\theta_1[t]]\,l_1\,\theta_1''[t] + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1''[t] + \theta_2''[t]) + \\
&\quad\quad\text{Cos}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1''[t] + \theta_2''[t] + \theta_3''[t])\Big) + \\
&\quad 2\,(\text{Sin}[\theta_1[t]]\,l_1 + \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2 + \text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3) \\
&\quad\Big(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t]^2 + \text{Cos}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1'[t] + \theta_2'[t])^2 + \\
&\quad\quad\text{Cos}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 + \\
&\quad\quad\text{Sin}[\theta_1[t]]\,l_1\,\theta_1''[t] + \text{Sin}[\theta_1[t] + \theta_2[t]]\,l_2\,(\theta_1''[t] + \theta_2''[t]) + \\
&\quad\quad\text{Sin}[\theta_1[t] + \theta_2[t] + \theta_3[t]]\,l_3\,(\theta_1''[t] + \theta_2''[t] + \theta_3''[t])\Big)\Big)
\end{aligned}
$$

In[285]:=
$$\mathbf{DlT_2 = D[Dl_{\theta_2'}, \{t, 1\}]}$$

Out[285]=
$$\begin{aligned}
&Izz_2\,(\theta_1''[t]+\theta_2''[t])+\\
&\frac{1}{2}m_2\Big(-2\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])\,(\cos[\theta_1[t]]\,l_1\,\theta_1'[t]+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]))+2\cos[\theta_1[t]+\theta_2[t]]\,l_2\\
&\quad(\theta_1'[t]+\theta_2'[t])\,(\sin[\theta_1[t]]\,l_1\,\theta_1'[t]+\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]))+\\
&\quad 2\cos[\theta_1[t]+\theta_2[t]]\,l_2\Big(-\sin[\theta_1[t]]\,l_1\,\theta_1'[t]^2-\\
&\qquad\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])^2+\cos[\theta_1[t]]\,l_1\,\theta_1''[t]+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1''[t]+\theta_2''[t])\Big)+2\sin[\theta_1[t]+\theta_2[t]]\\
&\quad l_2\Big(\cos[\theta_1[t]]\,l_1\,\theta_1'[t]^2+\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])^2+\\
&\qquad\sin[\theta_1[t]]\,l_1\,\theta_1''[t]+\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1''[t]+\theta_2''[t])\Big)\Big)+\\
&Izz_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])+\\
&\frac{1}{2}\\
&\ m_3\\
&\Big(2\,(\cos[\theta_1[t]]\,l_1\,\theta_1'[t]+\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))\\
&\quad(-\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])-\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\\
&\qquad(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))+2\,(\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))\\
&\quad(\sin[\theta_1[t]]\,l_1\,\theta_1'[t]+\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\\
&\qquad\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))+\\
&\quad 2\,(\cos[\theta_1[t]+\theta_2[t]]\,l_2+\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3)\\
&\quad\Big(-\sin[\theta_1[t]]\,l_1\,\theta_1'[t]^2-\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])^2-\\
&\qquad\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2+\\
&\qquad\cos[\theta_1[t]]\,l_1\,\theta_1''[t]+\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1''[t]+\theta_2''[t])+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])\Big)+\\
&\quad 2\,(\sin[\theta_1[t]+\theta_2[t]]\,l_2+\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3)\\
&\quad\Big(\cos[\theta_1[t]]\,l_1\,\theta_1'[t]^2+\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])^2+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2+\\
&\qquad\sin[\theta_1[t]]\,l_1\,\theta_1''[t]+\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1''[t]+\theta_2''[t])+\\
&\qquad\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])\Big)\Big)
\end{aligned}$$

In[286]:=
$$\mathbf{DlT_3 = D[Dl_{\theta_3'}, \{t, 1\}]}$$

Out[286]=
$$\begin{aligned}
&Izz_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])+\\
&\frac{1}{2}m_3\Big(-2\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])\\
&\quad(\cos[\theta_1[t]]\,l_1\,\theta_1'[t]+\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))+\\
&\quad 2\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])\\
&\quad(\sin[\theta_1[t]]\,l_1\,\theta_1'[t]+\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])+\\
&\qquad\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))+\\
&\quad 2\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\Big(-\sin[\theta_1[t]]\,l_1\,\theta_1'[t]^2-\sin[\theta_1[t]+\theta_2[t]]\\
&\qquad l_2\,(\theta_1'[t]+\theta_2'[t])^2-\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2+\\
&\qquad\cos[\theta_1[t]]\,l_1\,\theta_1''[t]+\cos[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1''[t]+\theta_2''[t])+\\
&\qquad\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])\Big)+\\
&\quad 2\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\Big(\cos[\theta_1[t]]\,l_1\,\theta_1'[t]^2+\cos[\theta_1[t]+\theta_2[t]]\,l_2\\
&\qquad(\theta_1'[t]+\theta_2'[t])^2+\cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2+\\
&\qquad\sin[\theta_1[t]]\,l_1\,\theta_1''[t]+\sin[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1''[t]+\theta_2''[t])+\\
&\qquad\sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])\Big)\Big)
\end{aligned}$$

In[287]:=

```
(*Partial first order derivative of l with respect to θi*)
```

In[288]:=

```
Dlθ1 = D[KE - PE, {θ1[t], 1}]
```

Out[288]=

$$
\begin{aligned}
&-g\,\text{Cos}[\theta_1[t]]\,l_1\,m_1 - g\,(\text{Cos}[\theta_1[t]]\,l_1 + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2)\,m_2 - \\
&g\,(\text{Cos}[\theta_1[t]]\,l_1 + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2 + \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3)\,m_3 + \\
&\frac{1}{2}\,m_2\,(2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])) \\
&\qquad (-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] - \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])) + \\
&\qquad 2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])) \\
&\qquad (\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]))) + \\
&\frac{1}{2}\,m_3\,(2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\qquad (-\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] - \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) - \\
&\qquad \text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) + \\
&\qquad 2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\qquad (\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])))
\end{aligned}
$$

In[289]:=

```
Dlθ2 = D[KE - PE, {θ2[t], 1}]
```

Out[289]=

$$
\begin{aligned}
&-g\,\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,m_2 - g\,(\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2 + \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3)\,m_3 + \\
&\frac{1}{2}\,m_2\,(-2\,\text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) \\
&\qquad (\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t])) + \\
&\qquad 2\,\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) \\
&\qquad (\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]))) + \\
&\frac{1}{2}\,m_3\,(2\,(\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\qquad (-\text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) - \text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3 \\
&\qquad\quad (\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) + 2\,(\text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\qquad (\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])))
\end{aligned}
$$

In[290]:=

```
Dlθ3 = D[KE - PE, {θ3[t], 1}]
```

Out[290]=

$$
\begin{aligned}
&-g\,\text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,m_3 + \\
&\frac{1}{2}\,m_3\,(-2\,\text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]) \\
&\qquad (\text{Cos}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Cos}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) + \\
&\qquad 2\,\text{Cos}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]) \\
&\qquad (\text{Sin}[\theta_1[t]]\,l_1\,\theta_1'[t] + \text{Sin}[\theta_1[t]+\theta_2[t]]\,l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \text{Sin}[\theta_1[t]+\theta_2[t]+\theta_3[t]]\,l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])))
\end{aligned}
$$

In[291]:=
```
(*τᵢ as defined by the required derivatives of the lagrangian*)
τ₁ = DlT₁ - Dl_θ₁
```

Out[291]=
$$
\begin{aligned}
&g \cos[\theta_1[t]]\, l_1\, m_1 + g\,(\cos[\theta_1[t]]\, l_1 + \cos[\theta_1[t]+\theta_2[t]]\, l_2)\, m_2 + \\
&g\,(\cos[\theta_1[t]]\, l_1 + \cos[\theta_1[t]+\theta_2[t]]\, l_2 + \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3)\, m_3 - \\
&\frac{1}{2} m_2\,(2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) \\
&\quad (-\sin[\theta_1[t]]\, l_1\, \theta_1'[t] - \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) + \\
&\quad 2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) \\
&\quad (\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]))) - \\
&\frac{1}{2} m_3\,(2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\quad (-\sin[\theta_1[t]]\, l_1\, \theta_1'[t] - \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) - \\
&\qquad \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) + \\
&\quad 2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\quad (\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t]))) + \\
&Izz_1\, \theta_1''[t] + l_1^2\, m_1\, \theta_1''[t] + Izz_2\,(\theta_1''[t]+\theta_2''[t]) + \\
&\frac{1}{2} m_2 \\
&\Big(2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) \\
&\quad (-\sin[\theta_1[t]]\, l_1\, \theta_1'[t] - \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) + \\
&\quad 2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) \\
&\quad (\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])) + \\
&\quad 2\,(\cos[\theta_1[t]]\, l_1 + \cos[\theta_1[t]+\theta_2[t]]\, l_2) \\
&\quad \Big(-\sin[\theta_1[t]]\, l_1\, \theta_1'[t]^2 - \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])^2 + \\
&\qquad \cos[\theta_1[t]]\, l_1\, \theta_1''[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1''[t]+\theta_2''[t])\Big) + \\
&\quad 2\,(\sin[\theta_1[t]]\, l_1 + \sin[\theta_1[t]+\theta_2[t]]\, l_2)\,\Big(\cos[\theta_1[t]]\, l_1\, \theta_1'[t]^2 + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])^2 + \sin[\theta_1[t]]\, l_1\, \theta_1''[t] + \\
&\qquad \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1''[t]+\theta_2''[t])\Big)\Big) + Izz_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t]) + \\
&\frac{1}{2} m_3\,\Big(2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\quad (-\sin[\theta_1[t]]\, l_1\, \theta_1'[t] - \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) - \\
&\qquad \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) + \\
&\quad 2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) \\
&\quad (\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t]) + \\
&\qquad \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])) + \\
&\quad 2\,(\cos[\theta_1[t]]\, l_1 + \cos[\theta_1[t]+\theta_2[t]]\, l_2 + \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3) \\
&\quad \Big(-\sin[\theta_1[t]]\, l_1\, \theta_1'[t]^2 - \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])^2 - \\
&\qquad \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2 + \\
&\qquad \cos[\theta_1[t]]\, l_1\, \theta_1''[t] + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1''[t]+\theta_2''[t]) + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])\Big) + \\
&\quad 2\,(\sin[\theta_1[t]]\, l_1 + \sin[\theta_1[t]+\theta_2[t]]\, l_2 + \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3) \\
&\quad \Big(\cos[\theta_1[t]]\, l_1\, \theta_1'[t]^2 + \cos[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1'[t]+\theta_2'[t])^2 + \\
&\qquad \cos[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1'[t]+\theta_2'[t]+\theta_3'[t])^2 + \\
&\qquad \sin[\theta_1[t]]\, l_1\, \theta_1''[t] + \sin[\theta_1[t]+\theta_2[t]]\, l_2\,(\theta_1''[t]+\theta_2''[t]) + \\
&\qquad \sin[\theta_1[t]+\theta_2[t]+\theta_3[t]]\, l_3\,(\theta_1''[t]+\theta_2''[t]+\theta_3''[t])\Big)\Big)
\end{aligned}
$$

In[292]:=
$$\tau_2 = \text{DlT}_2 - \text{Dl}_{\theta_2}$$

Out[292]=
$$g \cos[\theta_1[t] + \theta_2[t]]\, l_2\, m_2 + g\,(\cos[\theta_1[t] + \theta_2[t]]\, l_2 + \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3)\, m_3 -$$
$$\frac{1}{2} m_2\,(-2 \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])$$
$$(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])) +$$
$$2 \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])$$
$$(\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]))) -$$
$$\frac{1}{2} m_3\,(2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))$$
$$(-\sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) - \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3$$
$$(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + 2\,(\cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))$$
$$(\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) +$$
$$\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))) +$$
$$\text{Izz}_2\,(\theta_1''[t] + \theta_2''[t]) + \frac{1}{2} m_2\,\big(-2 \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])$$
$$(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])) +$$
$$2 \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])$$
$$(\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])) +$$
$$2 \cos[\theta_1[t] + \theta_2[t]]\, l_2\,\big(-\sin[\theta_1[t]]\, l_1\, \theta_1'[t]^2 -$$
$$\sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])^2 + \cos[\theta_1[t]]\, l_1\, \theta_1''[t] +$$
$$\cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1''[t] + \theta_2''[t])\big) + 2 \sin[\theta_1[t] + \theta_2[t]]$$
$$l_2\,\big(\cos[\theta_1[t]]\, l_1\, \theta_1'[t]^2 + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])^2 +$$
$$\sin[\theta_1[t]]\, l_1\, \theta_1''[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1''[t] + \theta_2''[t])\big)\big) +$$
$$\text{Izz}_3\,(\theta_1''[t] + \theta_2''[t] + \theta_3''[t]) +$$
$$\frac{1}{2}$$
$$m_3$$
$$\big(2\,(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))$$
$$(-\sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) - \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3$$
$$(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) + 2\,(\cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))$$
$$(\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t]) +$$
$$\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) +$$
$$2\,(\cos[\theta_1[t] + \theta_2[t]]\, l_2 + \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3)$$
$$\big(-\sin[\theta_1[t]]\, l_1\, \theta_1'[t]^2 - \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])^2 -$$
$$\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 +$$
$$\cos[\theta_1[t]]\, l_1\, \theta_1''[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1''[t] + \theta_2''[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1''[t] + \theta_2''[t] + \theta_3''[t])\big) +$$
$$2\,(\sin[\theta_1[t] + \theta_2[t]]\, l_2 + \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3)$$
$$\big(\cos[\theta_1[t]]\, l_1\, \theta_1'[t]^2 + \cos[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1'[t] + \theta_2'[t])^2 +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 +$$
$$\sin[\theta_1[t]]\, l_1\, \theta_1''[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\,(\theta_1''[t] + \theta_2''[t]) +$$
$$\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\,(\theta_1''[t] + \theta_2''[t] + \theta_3''[t])\big)\big)$$

In[293]:=
$$\tau_3 = \text{DlT}_3 - \text{Dl}_{\theta_3}$$

Out[293]=
$$g \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, m_3 -$$
$$\frac{1}{2} m_3 \left(-2 \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])\right.$$
$$(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) +$$
$$2 \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])$$
$$(\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]) +$$
$$\left.\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t]))\right) +$$
$$\text{Izz}_3\, (\theta_1''[t] + \theta_2''[t] + \theta_3''[t]) +$$
$$\frac{1}{2} m_3 \left(-2 \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])\right.$$
$$(\cos[\theta_1[t]]\, l_1\, \theta_1'[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]) +$$
$$\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) +$$
$$2 \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])$$
$$(\sin[\theta_1[t]]\, l_1\, \theta_1'[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1'[t] + \theta_2'[t]) +$$
$$\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])) +$$
$$2 \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3 \left(-\sin[\theta_1[t]]\, l_1\, \theta_1'[t]^2 - \sin[\theta_1[t] + \theta_2[t]]\right.$$
$$l_2\, (\theta_1'[t] + \theta_2'[t])^2 - \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 +$$
$$\cos[\theta_1[t]]\, l_1\, \theta_1''[t] + \cos[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1''[t] + \theta_2''[t]) +$$
$$\left.\cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1''[t] + \theta_2''[t] + \theta_3''[t])\right) +$$
$$2 \sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3 \left(\cos[\theta_1[t]]\, l_1\, \theta_1'[t]^2 + \cos[\theta_1[t] + \theta_2[t]]\, l_2\right.$$
$$(\theta_1'[t] + \theta_2'[t])^2 + \cos[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1'[t] + \theta_2'[t] + \theta_3'[t])^2 +$$
$$\sin[\theta_1[t]]\, l_1\, \theta_1''[t] + \sin[\theta_1[t] + \theta_2[t]]\, l_2\, (\theta_1''[t] + \theta_2''[t]) +$$
$$\left.\left.\sin[\theta_1[t] + \theta_2[t] + \theta_3[t]]\, l_3\, (\theta_1''[t] + \theta_2''[t] + \theta_3''[t])\right)\right)$$