1

CSE 548: Analysis of Algorithms

Lecture 12
(Analyzing Parallel Algorithms)

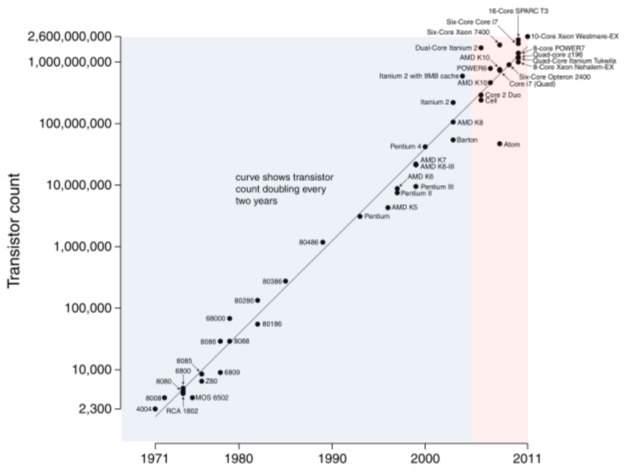
Rezaul A. Chowdhury
Department of Computer Science
SUNY Stony Brook
Fall 2019

2

Why Parallelism?

3

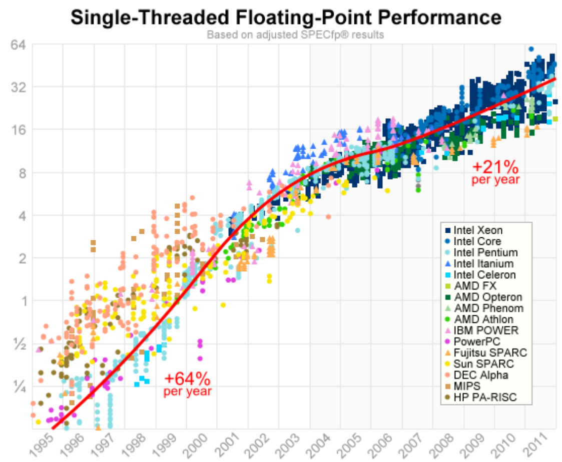
Moore's Law



Source: Wikipedia

4

Unicore Performance



Source: Jeff Preshing, 2012, http://preshing.com/20120208/a-look-back-at-single-threaded-cpu-performance/

Unicore Performance Has Hit a Wall!

5

Some Reasons

- Lack of additional ILP
  (Instruction Level Hidden Parallelism)
- High power density
- Manufacturing issues
- Physical limits
- Memory speed

Unicore Performance: No Additional ILP

6

"Everything that can be invented has been invented."

— Charles H. Duell
Commissioner, U.S. patent office, 1899

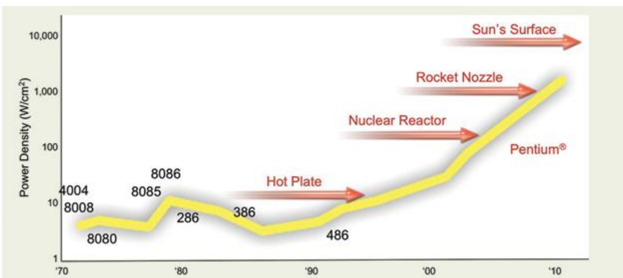
Exhausted all ideas to exploit hidden parallelism?

- Multiple simultaneous instructions
- Instruction Pipelining
- Out-of-order instructions
- Speculative execution
- Branch prediction
- Register renaming, etc.

Unicore Performance: High Power Density

7

- Dynamic power, $P_d \propto V^2 f C$
  - *V = supply voltage*
  - *f = clock frequency*
  - *C = capacitance*
- But $V \propto f$
- Thus $P_d \propto f^3$



Source: Patrick Gelsinger, Intel Developer Forum, Spring 2004 (Simon Floyd)

Unicore Performance: Manufacturing Issues

8

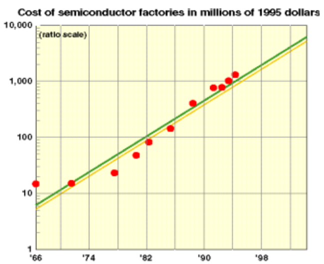
- Frequency, $f \propto 1 / s$
  - *s = feature size (transistor dimension)*
- Transistors / unit area $\propto 1 / s^2$
- Typically, die size $\propto 1 / s$
- So, what happens if feature size goes down by a factor of x?
  - Raw computing power goes up by a factor of x^4 !
  - Typically most programs run faster by a factor of x^3 without any change!

Source: Kathy Yelick and Jim Demmel, UC Berkeley

Unicore Performance: Manufacturing Issues

9

- Manufacturing cost goes up as feature size decreases
  - Cost of a semiconductor fabrication plant doubles every 4 years (Rock's Law)
- CMOS feature size is limited to 5 nm (at least 10 atoms)



Source: Kathy Yelick and Jim Demmel, UC Berkeley

Unicore Performance: Physical Limits

10

Execute the following loop on a serial machine in 1 second:

for ($i = 0$; $i < 10^{12}$; $++i$)

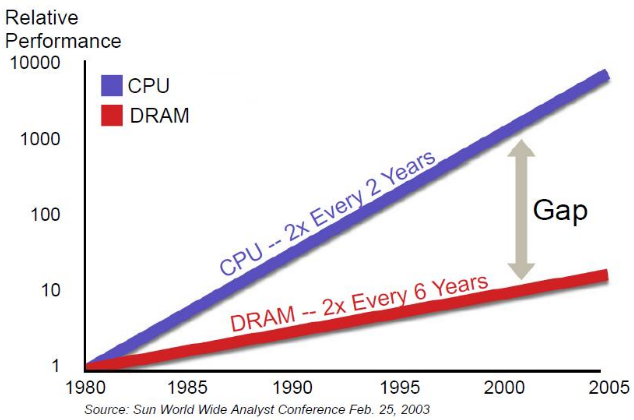
$z[\,i\,] = x[\,i\,] + y[\,i\,]$;

- We will have to access $3{\times}10^{12}$ data items in one second
- Speed of light is, $c \approx 3{\times}10^{8}$ m/s
- So each data item must be within $c \,/\, 3{\times}10^{12} \approx 0.1$ mm from the CPU on the average
- All data must be put inside a 0.2 mm × 0.2 mm square
- Each data item (≥ 8 bytes) can occupy only 1 Å^2 space! (size of a small atom!)

Source: Kathy Yelick and Jim Demmel, UC Berkeley

Unicore Performance: Memory Wall

11



Source: Rick Hetherington, Chief Technology Officer, Microelectronics, Sun Microsystems

Unicore Performance Has Hit a Wall!

12

Some Reasons

- Lack of additional ILP (Instruction Level Hidden Parallelism)
- High power density
- Manufacturing issues
- Physical limits
- Memory speed

"Oh Sinnerman, where you gonna run to?"

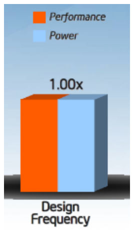
— *Sinnerman (recorded by Nina Simone)*

13

Where You Gonna Run To?

– Changing *f* by 20% changes performance by 13%

– So what happens if we overclock by 20%?



Source: Andrew A. Chien, Vice President of Research, Intel Corporation

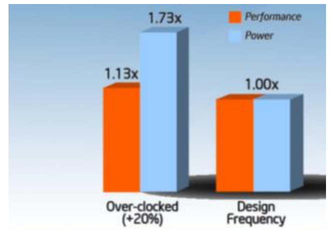
14

Where You Gonna Run To?

– Changing *f* by 20% changes performance by 13%

– So what happens if we overclock by 20%?

– And underclock by 20%?



Source: Andrew A. Chien, Vice President of Research, Intel Corporation

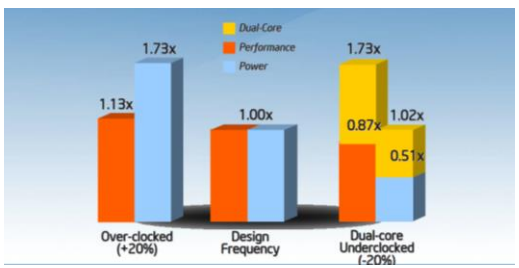
15

Where You Gonna Run To?

– Changing *f* by 20% changes performance by 13%

– So what happens if we overclock by 20%?

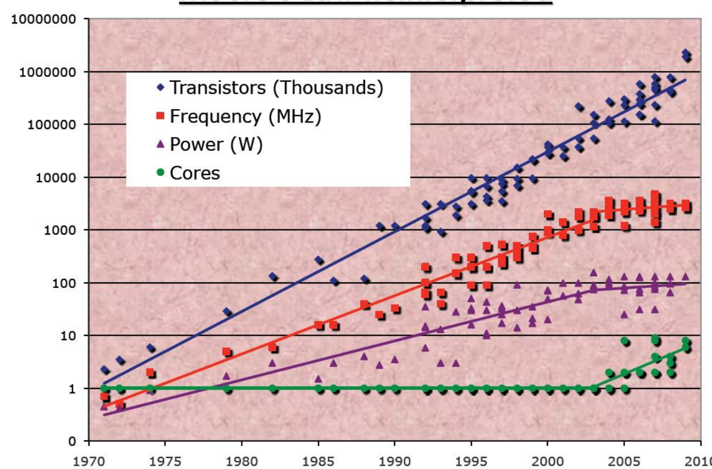
– And underclock by 20%?



Source: Andrew A. Chien, Vice President of Research, Intel Corporation

16

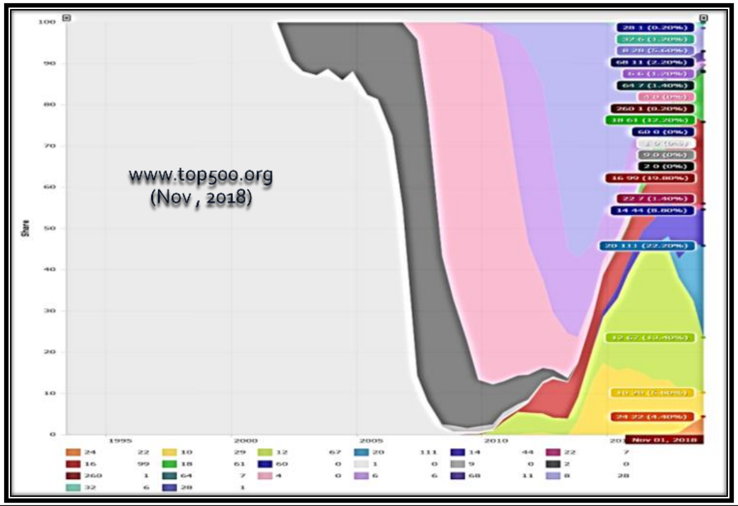
Moore's Law Reinterpreted



Source: Report of the 2011 Workshop on Exascale Programming Challenges

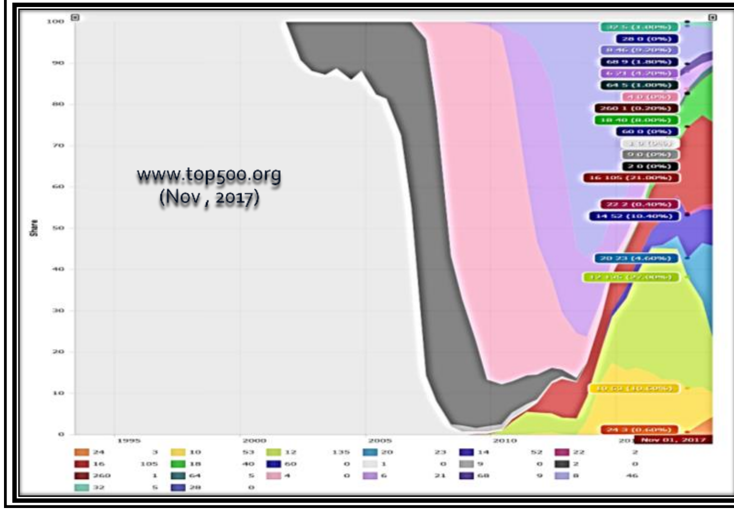
Top 500 Supercomputing Sites (Cores / Socket)

17



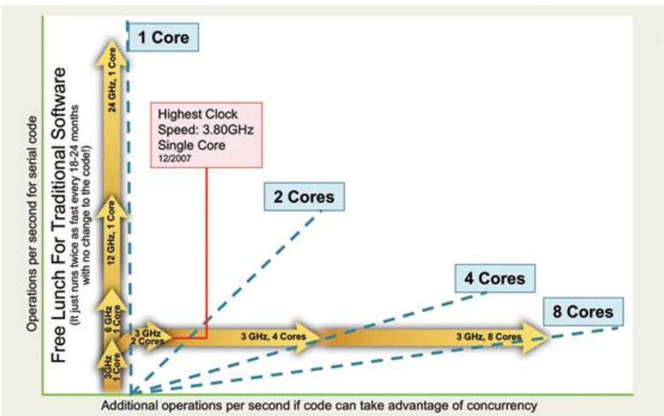
Top 500 Supercomputing Sites (Cores / Socket)

18



No Free Lunch for Traditional Software

19



19

Source: Simon Floyd, Workstation Performance: Tomorrow's Possibilities (Viewpoint Column)

Insatiable Demand for Performance

20

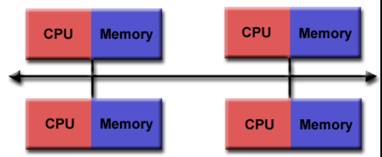
Source: Patrick Gelsinger, Intel Developer Forum, 2008

Some Useful Classifications of Parallel Computers

Parallel Computer Memory Architecture (Distributed Memory)

– Each processor has its own local memory — no global address space

– Changes in local memory by one processor have no effect on memory of other processors



Source: Blaise Barney, LLNL

– Communication network to connect inter-processor memory

– Programming
  – Message Passing Interface (MPI)
  – Many once available: PVM, Chameleon, MPL, NX, etc.

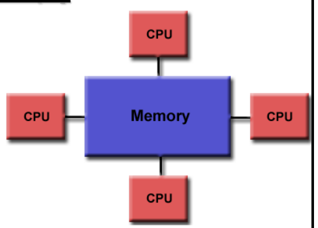
Parallel Computer Memory Architecture (Shared Memory)

– All processors access all memory as global address space

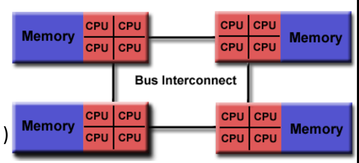
– Changes in memory by one processor are visible to all others

– Two types
  – Uniform Memory Access (UMA)
  – Non-Uniform Memory Access (NUMA)

– Programming
  – Open Multi-Processing (OpenMP)
  – Cilk/Cilk++ and Intel Cilk Plus
  – Intel Thread Building Block (TBB), etc.



UMA



NUMA

Source: Blaise Barney, LLNL

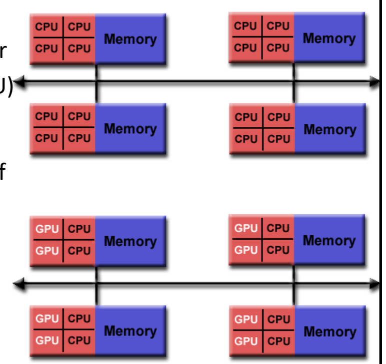
Parallel Computer Memory Architecture (Hybrid Distributed-Shared Memory)

– The share-memory component can be a cache-coherent SMP or a Graphics Processing Unit (GPU)

– The distributed-memory component is the networking of multiple SMP/GPU machines

– Most common architecture for the largest and fastest computers in the world today

– Programming
  – OpenMP / Cilk + CUDA / OpenCL + MPI, etc.



Source: Blaise Barney, LLNL

Types of Parallelism

Nested Parallelism

$^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$

```
int comb ( int n, int r )
{
  if ( r > n ) return 0;
  if ( r == 0 || r == n ) return 1;

  int x, y;

  x = comb( n - 1, r - 1 );
  y = comb( n - 1, r );

  return ( x + y );
}
```

Serial Code

Control cannot pass this point until all spawned children have returned.

Grant permission to execute the called (spawned) function in parallel with the caller.

```
int comb ( int n, int r )
{
  if ( r > n ) return 0;
  if ( r == 0 || r == n ) return 1;

  int x, y;

  x = spawn comb( n - 1, r - 1 );
  y = comb( n - 1, r );

  sync;

  return ( x + y );
}
```

Parallel Code

Loop Parallelism

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xrightarrow[\text{transpose}]{\text{in-place}} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

```
for ( int i = 1; i < n; ++i )
   for ( int j = 0; j < i; ++j )
      {
        double t = A[ i ][ j ];
        A[ i ][ j ] = A[ j ][ i ];
        A[ j ][ i ] = t;
      }
```

Serial Code

Allows all iterations of the loop to be executed in parallel.

Can be converted to spawns and syncs using recursive divide-and-conquer.

```
parallel for ( int i = 1; i < n; ++i )
   for ( int j = 0; j < i; ++j )
      {
        double t = A[ i ][ j ];
        A[ i ][ j ] = A[ j ][ i ];
        A[ j ][ i ] = t;
      }
```

Parallel Code

Recursive D&C Implementation of Parallel Loops

```
parallel for ( int i = s; i < t; ++i )
      BODY( i );
```

divide-and-conquer implementation

```
void recur( int lo, int hi )
{
  if ( hi - lo > GRAINSIZE )
    {
      int mid = lo + ( hi - lo ) / 2;
      spawn recur( lo, mid );
      recur( mid, hi );
      sync;
    }
  else
    {
      for ( int i = lo; i < hi; ++i )
         BODY( i );
    }
}

recur( s, t );
```

29

Analyzing Parallel Algorithms

30

Speedup

Let T_p = running time using p identical processing elements

Speedup, $S_p = \frac{T_1}{T_p}$

Theoretically, $S_p \leq p$

Perfect or *linear* or *ideal* speedup if $S_p = p$

31

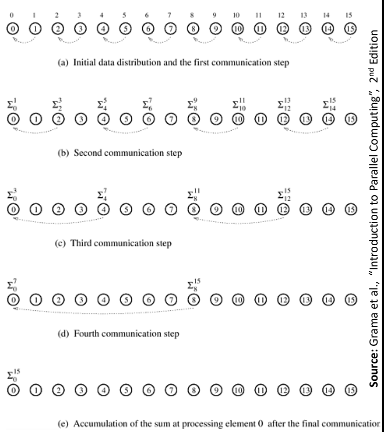
Speedup

Consider adding n numbers using n identical processing elements.

Serial runtime, $T = \Theta(n)$

Parallel runtime, $T_n = \Theta(\log n)$

Speedup, $S_n = \frac{T_1}{T_n} = \Theta\left(\frac{n}{\log n}\right)$



Source: Grama et al., "Introduction to Parallel Computing", 2nd Edition

32

Superlinear Speedup

Theoretically, $S_p \leq p$

But in practice *superlinear speedup* is sometimes observed, that is, $S_p > p$ (why?)

Reasons for superlinear speedup

- Cache effects
- Exploratory decomposition

Parallelism & Span Law

We defined, T_p = runtime on p identical processing elements

Then span, T_∞ = runtime on an infinite number of identical processing elements

Parallelism, $P = \frac{T_1}{T_\infty}$

Parallelism is an upper bound on speedup, i.e., $S_p \le P$

Span Law

$$\boxed{T_p \ge T_\infty}$$

Work Law

The cost of solving (or work performed for solving) a problem:

On a Serial Computer: is given by T_1

On a Parallel Computer: is given by pT_p

Work Law

$$\boxed{T_p \ge \frac{T_1}{p}}$$

Bounding Parallel Running Time (T_p)

A *runtime/online scheduler* maps tasks to processing elements dynamically at runtime.

A *greedy scheduler* never leaves a processing element idle if it can map a task to it.

Theorem [Graham'68, Brent'74]: For any greedy scheduler,

$$\boxed{T_p \le \frac{T_1}{p} + T_\infty}$$

Corollary: For any greedy scheduler,

$$\boxed{T_p \le 2T_p^*,}$$

where T_p^* is the running time due to optimal scheduling on p processing elements.

Work Optimality

Let T_s = runtime of the optimal or the fastest known serial algorithm

A parallel algorithm is *cost-optimal* or *work-optimal* provided

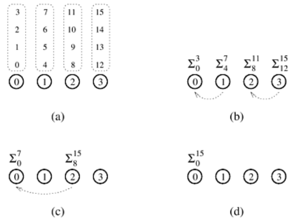
$$pT_p = \Theta(T_s)$$

Our algorithm for adding n numbers using n identical processing elements is clearly not work optimal.

37

Adding n Numbers Work-Optimality

We reduce the number of processing elements which in turn increases the granularity of the subproblem assigned to each processing element.



Source: Grama et al., "Introduction to Parallel Computing", 2nd Edition

Suppose we use p processing elements.

First each processing element locally adds its $\frac{n}{p}$ numbers in time $\Theta\left(\frac{n}{p}\right)$.

Then p processing elements adds these p partial sums in time $\Theta(\log p)$.

Thus $T_p = \Theta\left(\frac{n}{p} + \log p\right)$, and $T_s = \Theta(n)$.

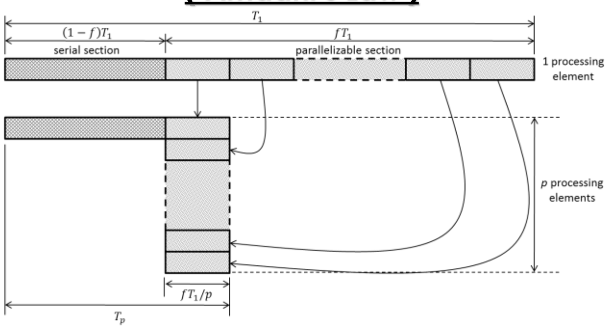
So the algorithm is work-optimal provided $n = \Omega(p \log p)$.

38

Scaling Law

39

Scaling of Parallel Algorithms (Amdahl's Law)



Suppose only a fraction f of a computation can be parallelized.

Then parallel running time, $T_p \geq (1-f)T_1 + f\frac{T_1}{p}$

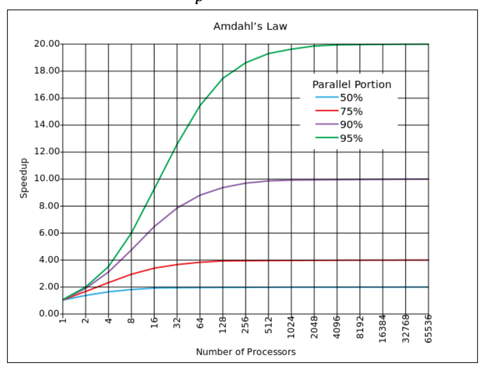
Speedup, $S_p = \frac{T_1}{T_p} \leq \frac{p}{f+(1-f)p} = \frac{1}{(1-f)+\frac{f}{p}} \leq \frac{1}{1-f}$

40

Scaling of Parallel Algorithms (Amdahl's Law)

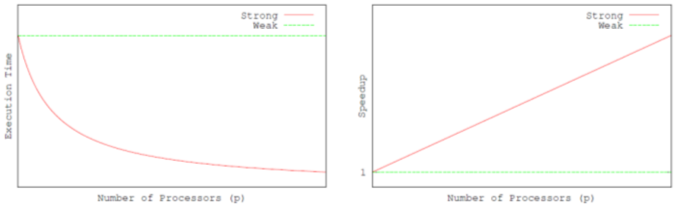
Suppose only a fraction f of a computation can be parallelized.

Speedup, $S_p = \frac{T_1}{T_p} \leq \frac{1}{(1-f)+\frac{f}{p}} \leq \frac{1}{1-f}$



Source: Wikipedia

Strong Scaling vs. Weak Scaling



Source: Martha Kim, Columbia University

Strong Scaling

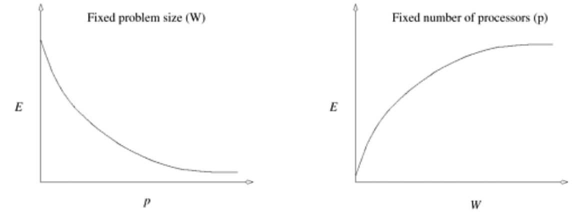
How T_p (or S_p) varies with p when the problem size is fixed.

Weak Scaling

How T_p (or S_p) varies with p when the problem size per processing element is fixed.

Scalable Parallel Algorithms

Efficiency, $E_p = \frac{S_p}{p} = \frac{T_1}{pT_p}$



Source: Grama et al., "Introduction to Parallel Computing", 2nd Edition

A parallel algorithm is called *scalable* if its efficiency can be maintained at a fixed value by simultaneously increasing the number of processing elements and the problem size.

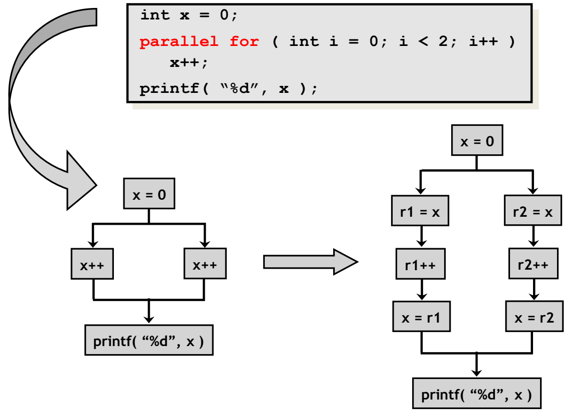
Scalability reflects a parallel algorithm's ability to utilize increasing processing elements effectively.

Races

Race Bugs

A *determinacy race* occurs if two logically parallel instructions access the same memory location and at least one of them performs a write.

```
int x = 0;
parallel for ( int i = 0; i < 2; i++ )
   x++;
printf( "%d", x );
```



Critical Sections and Mutexes


```
int r = 0;

parallel for ( int i = 0; i < n; i++ )
    r += eval( x[ i ] );
```

race

```
mutex mtx;

parallel for ( int i = 0; i < n; i++ )
    mtx.lock( );
    r += eval( x[ i ] );
    mtx.unlock( );
```

critical section
two or more strands must not access at the same time

mutex (mutual exclusion)
an attempt by a strand to lock an already locked mutex causes that strand to block (i.e., wait) until the mutex is unlocked

Problems
- lock overhead
- lock contention

Critical Sections and Mutexes


```
int r = 0;

parallel for ( int i = 0; i < n; i++ )
    r += eval( x[ i ] );
```

race

```
mutex mtx;

parallel for ( int i = 0; i < n; i++ )
    mtx.lock( );
    r += eval( x[ i ] );
    mtx.unlock( );
```

```
mutex mtx;

parallel for ( int i = 0; i < n; i++ )
    int y = eval( x[ i ] );
    mtx.lock( );
    r += y;
    mtx.unlock( );
```

- slightly better solution
- but lock contention can still destroy parallelism

Recursive D&C Implementation of Loops

Recursive D&C Implementation of Parallel Loops


```
parallel for ( int i = s; i < t; ++i )
    BODY( i );
```

divide-and-conquer implementation

```
void recur( int lo, int hi )
{
  if ( hi – lo > GRAINSIZE )
    {
      int mid = lo + ( hi – lo ) / 2;
      spawn recur( lo, mid );
      recur( mid, hi );
      sync;
    }
  else
    {
      for ( int i = lo; i < hi; ++i )
         BODY( i );
    }
}

recur( s, t );
```

Let $n = t - s$
m = running time of a single call to BODY

Span: $T_\infty(n) = \Theta(\log n + \text{GRAINSIZE} \times m)$

Parallel Iterative MM


```
Iter-MM ( Z, X, Y )          { X, Y, Z are n × n matrices,
                               where n is a positive integer }
  for i ← 1 to n do
     for j ← 1 to n do
        Z[ i ][ j ] ← 0
        for k ← 1 to n do
           Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

⇩

```
Par-Iter-MM ( Z, X, Y )      { X, Y, Z are n × n matrices,
                               where n is a positive integer }
  parallel for i ← 1 to n do
     parallel for j ← 1 to n do
        Z[ i ][ j ] ← 0
        for k ← 1 to n do
           Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

Parallel Iterative MM


```
Par-Iter-MM ( Z, X, Y )      { X, Y, Z are n × n matrices,
                               where n is a positive integer }
  parallel for i ← 1 to n do
     parallel for j ← 1 to n do
        Z[ i ][ j ] ← 0
        for k ← 1 to n do
           Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```

Work: $T_1(n) = \Theta(n^3)$

Span: $T_\infty(n) = \Theta(n)$

Parallel Running Time: $T_p(n) = O\left(\frac{T_1(n)}{p} + T_\infty(n)\right) = O\left(\frac{n^3}{p} + n\right)$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta(n^2)$

Parallel Recursive MM

$$\overset{Z}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \overset{X}{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}} \times \overset{Y}{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}$$

$$= \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

(each matrix is $n \times n$; each block is $n/2 \times n/2$)

Parallel Recursive MM


```
Par-Rec-MM ( Z, X, Y )   { X, Y, Z are n × n matrices,
                           where n = 2^k for integer k ≥ 0 }
  if n = 1 then
     Z ← Z + X · Y
  else
     spawn Par-Rec-MM ( Z11, X11, Y11 )
     spawn Par-Rec-MM ( Z12, X11, Y12 )
     spawn Par-Rec-MM ( Z21, X21, Y11 )
           Par-Rec-MM ( Z21, X21, Y12 )
     sync
     spawn Par-Rec-MM ( Z11, X12, Y21 )
     spawn Par-Rec-MM ( Z12, X12, Y22 )
     spawn Par-Rec-MM ( Z21, X22, Y21 )
           Par-Rec-MM ( Z22, X22, Y22 )
     sync
  endif
```

(Lecture 12) Analyzing Parallel Algorithms

Parallel Recursive MM


```
Par-Rec-MM ( Z, X, Y )    { X, Y, Z are n × n matrices,
                            where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else
    spawn Par-Rec-MM ( Z11, X11, Y11 )
    spawn Par-Rec-MM ( Z12, X11, Y12 )
    spawn Par-Rec-MM ( Z21, X21, Y11 )
          Par-Rec-MM ( Z21, X21, Y12 )
    sync
    spawn Par-Rec-MM ( Z11, X12, Y21 )
    spawn Par-Rec-MM ( Z12, X12, Y22 )
    spawn Par-Rec-MM ( Z21, X22, Y21 )
          Par-Rec-MM ( Z22, X22, Y22 )
    sync
endif
```

Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T_1\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(n^3) \qquad \text{[MT Case 1]}$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_\infty\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(n) \qquad \text{[MT Case 1]}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta(n^2)$

Additional Space:

$s_\infty(n) = \Theta(1)$

Recursive MM with More Parallelism

Z ($n \times n$, blocks $n/2 \times n/2$)

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} X_{11}Y_{11} & X_{11}Y_{12} \\ X_{21}Y_{11} & X_{21}Y_{12} \end{bmatrix} + \begin{bmatrix} X_{12}Y_{21} & X_{12}Y_{22} \\ X_{22}Y_{21} & X_{22}Y_{22} \end{bmatrix}$$

Recursive MM with More Parallelism


```
Par-Rec-MM2 ( Z, X, Y )    { X, Y, Z are n × n matrices,
                             where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else          { T is a temporary n × n matrix }
    spawn Par-Rec-MM2 ( Z11, X11, Y11 )
    spawn Par-Rec-MM2 ( Z12, X11, Y12 )
    spawn Par-Rec-MM2 ( Z21, X21, Y11 )
    spawn Par-Rec-MM2 ( Z21, X21, Y12 )
    spawn Par-Rec-MM2 ( T11, X12, Y21 )
    spawn Par-Rec-MM2 ( T12, X12, Y22 )
    spawn Par-Rec-MM2 ( T21, X22, Y21 )
          Par-Rec-MM2 ( T22, X22, Y22 )
    sync
    parallel for i ← 1 to n do
        parallel for j ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + T[ i ][ j ]
endif
```

Recursive MM with More Parallelism


```
Par-Rec-MM2 ( Z, X, Y )    { X, Y, Z are n × n matrices,
                             where n = 2^k for integer k ≥ 0 }
if n = 1 then
    Z ← Z + X · Y
else          { T is a temporary n × n matrix }
    spawn Par-Rec-MM2 ( Z11, X11, Y11 )
    spawn Par-Rec-MM2 ( Z12, X11, Y12 )
    spawn Par-Rec-MM2 ( Z21, X21, Y11 )
    spawn Par-Rec-MM2 ( Z21, X21, Y12 )
    spawn Par-Rec-MM2 ( T11, X12, Y21 )
    spawn Par-Rec-MM2 ( T12, X12, Y22 )
    spawn Par-Rec-MM2 ( T21, X22, Y21 )
          Par-Rec-MM2 ( T22, X22, Y22 )
    sync
    parallel for i ← 1 to n do
        parallel for j ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + T[ i ][ j ]
endif
```


Work:

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8T_1\left(\frac{n}{2}\right) + \Theta(n^2), & otherwise. \end{cases}$$

$$= \Theta(n^3) \qquad \text{[MT Case 1]}$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(\log n), & otherwise. \end{cases}$$

$$= \Theta(\log^2 n) \qquad \text{[MT Case 2]}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n^3}{\log^2 n}\right)$

Additional Space:

$$s_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 8s_\infty\left(\frac{n}{2}\right) + \Theta(n^2), & otherwise. \end{cases}$$

$$= \Theta(n^3) \qquad \text{[MT Case 1]}$$

57

Parallel Merge Sort

58

Parallel Merge Sort

```
Merge-Sort ( A, p, r )        { sort the elements in A[ p … r ] }

1.  if p < r then
2.      q ← ⌊ ( p + r ) / 2 ⌋
3.      Merge-Sort ( A, p, q )
4.      Merge-Sort ( A, q + 1, r )
5.      Merge ( A, p, q, r )
```

⇩

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p … r ] }

1.  if p < r then
2.      q ← ⌊ ( p + r ) / 2 ⌋
3.      spawn Merge-Sort ( A, p, q )
4.            Merge-Sort ( A, q + 1, r )
5.      sync
6.      Merge ( A, p, q, r )
```

59

Parallel Merge Sort

```
Par-Merge-Sort ( A, p, r )    { sort the elements in A[ p … r ] }

1.  if p < r then
2.      q ← ⌊ ( p + r ) / 2 ⌋
3.      spawn Merge-Sort ( A, p, q )
4.            Merge-Sort ( A, q + 1, r )
5.      sync
6.      Merge ( A, p, q, r )
```

Work: $T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$

$= \Theta(n \log n)$ [MT Case 2]

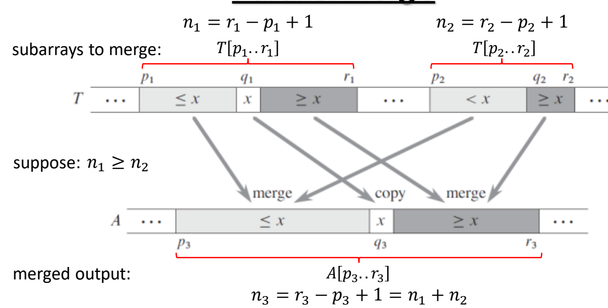
Span: $T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$

$= \Theta(n)$ [MT Case 3]

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta(\log n)$

60

Parallel Merge

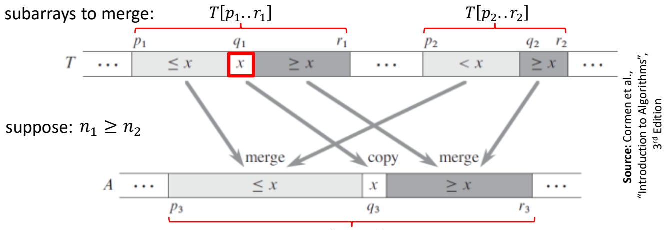


Source: Cormen et al., "Introduction to Algorithms", 3rd Edition

Parallel Merge

61

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$



merged output: $A[p_3..r_3]$

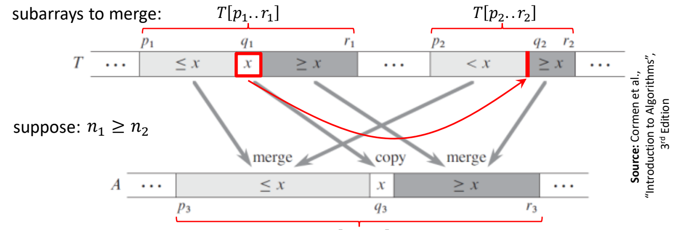
$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Step 1: Find $x = T[q_1]$, where q_1 is the midpoint of $T[p_1..r_1]$

Parallel Merge

62

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$



merged output: $A[p_3..r_3]$

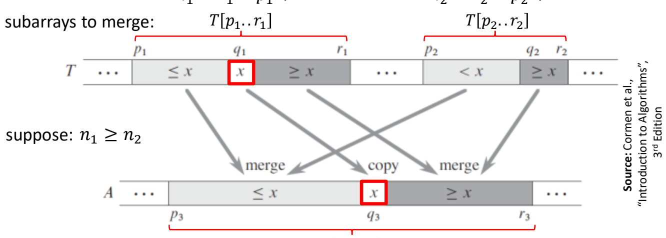
$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Step 2: Use binary search to find the index q_2 in subarray $T[p_2..r_2]$ so that the subarray would still be sorted if we insert x between $T[q_2 - 1]$ and $T[q_2]$

Parallel Merge

63

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$



merged output: $A[p_3..r_3]$

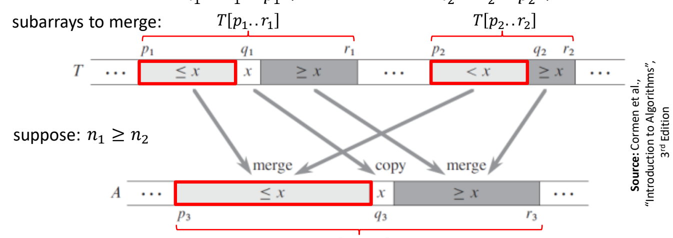
$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Step 3: Copy x to $A[q_3]$, where $q_3 = p_3 + (q_1 - p_1) + (q_2 - p_2)$

Parallel Merge

64

$n_1 = r_1 - p_1 + 1$ $\quad$ $n_2 = r_2 - p_2 + 1$



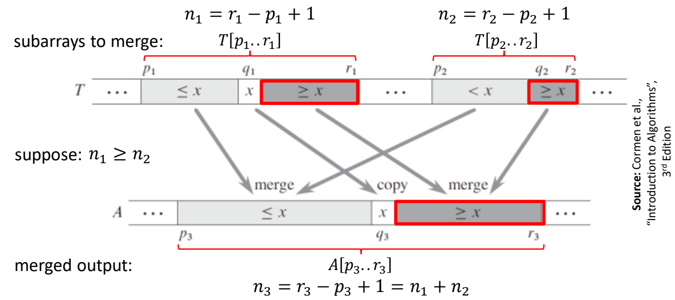
merged output: $A[p_3..r_3]$

$n_3 = r_3 - p_3 + 1 = n_1 + n_2$

Perform the following two steps in parallel.

Step 4(a): Recursively merge $T[p_1..q_1 - 1]$ with $T[p_2..q_2 - 1]$, and place the result into $A[p_3..q_3 - 1]$

Parallel Merge



Perform the following two steps in parallel.

Step 4(a): Recursively merge $T[p_1..q_1 - 1]$ with $T[p_2..q_2 - 1]$, and place the result into $A[p_3..q_3 - 1]$

Step 4(b): Recursively merge $T[q_1 + 1..r_1]$ with $T[q_2 + 1..r_2]$, and place the result into $A[q_3 + 1..r_3]$

Parallel Merge


```
Par-Merge ( T, p1, r1, p2, r2, A, p3 )
1.  n1 ← r1 - p1 + 1,   n2 ← r2 - p2 + 1
2.  if n1 < n2 then
3.      p1 ↔ p2,  r1 ↔ r2,  n1 ↔ n2
4.  if n1 = 0 then return
5.  else
6.      q1 ← ⌊ ( p1 + r1 ) / 2 ⌋
7.      q2 ← Binary-Search (  T[ q1 ],  T,  p2, r2 )
8.      q3 ← p3 + ( q1 - p1 ) + ( q2 - p2 )
9.      A[ q3 ] ← T[ q1 ]
10.     spawn Par-Merge ( T, p1, q1-1, p2, q2-1, A, p3 )
11.           Par-Merge ( T, q1+1, r1, q2+1, r2, A, q3+1 )
12.     sync
```

Parallel Merge


```
Par-Merge ( T, p1, r1, p2, r2, A, p3 )
1.  n1 ← r1 - p1 + 1,   n2 ← r2 - p2 + 1
2.  if n1 < n2 then
3.      p1 ↔ p2,  r1 ↔ r2,  n1 ↔ n2
4.  if n1 = 0 then return
5.  else
6.      q1 ← ⌊ ( p1 + r1 ) / 2 ⌋
7.      q2 ← Binary-Search (  T[ q1 ],  T,  p2, r2 )
8.      q3 ← p3 + ( q1 - p1 ) + ( q2 - p2 )
9.      A[ q3 ] ← T[ q1 ]
10.     spawn Par-Merge ( T, p1, q1-1, p2, q2-1, A, p3 )
11.           Par-Merge ( T, q1+1, r1, q2+1, r2, A, q3+1 )
12.     sync
```

We have,

$n_2 \le n_1 \Rightarrow 2n_2 \le n_1 + n_2 = n$

In the worst case, a recursive call in lines 9-10 merges half the elements of $T[p_1..r_1]$ with all elements of $T[p_2..r_2]$.

Hence, #elements involved in such a call:

$$\left\lfloor \frac{n_1}{2} \right\rfloor + n_2 \le \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_2}{2} = \frac{n_1 + n_2}{2} + \frac{2n_2}{4} \le \frac{n}{2} + \frac{n}{4} = \frac{3n}{4}$$

Parallel Merge


```
Par-Merge ( T, p1, r1, p2, r2, A, p3 )
1.  n1 ← r1 - p1 + 1,   n2 ← r2 - p2 + 1
2.  if n1 < n2 then
3.      p1 ↔ p2,  r1 ↔ r2,  n1 ↔ n2
4.  if n1 = 0 then return
5.  else
6.      q1 ← ⌊ ( p1 + r1 ) / 2 ⌋
7.      q2 ← Binary-Search (  T[ q1 ],  T,  p2, r2 )
8.      q3 ← p3 + ( q1 - p1 ) + ( q2 - p2 )
9.      A[ q3 ] ← T[ q1 ]
10.     spawn Par-Merge ( T, p1, q1-1, p2, q2-1, A, p3 )
11.           Par-Merge ( T, q1+1, r1, q2+1, r2, A, q3+1 )
12.     sync
```

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{3n}{4}\right) + \Theta(\log n), & otherwise. \end{cases}$$

$$= \Theta(\log^2 n) \qquad \text{[MT Case 2]}$$

Work:

Clearly, $T_1(n) = \Omega(n)$

We show below that, $T_1(n) = O(n)$

For some $\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have the following recurrence,

$$T_1(n) = T_1(\alpha n) + T_1\big((1 - \alpha)n\big) + O(\log n)$$

Assuming $T_1(n) \le c_1 n - c_2 \log n$ for positive constants c_1 and c_2, and substituting on the right hand side of the above recurrence gives us: $T_1(n) \le c_1 n - c_2 \log n \ = O(n)$.

Hence, $T_1(n) = \Theta(n)$.

Parallel Merge Sort with Parallel Merge


```
Par-Merge-Sort ( A, p, r )   { sort the elements in A[ p … r ] }

if p < r then
    q ← ⌊ ( p + r ) / 2 ⌋
    spawn Merge-Sort ( A, p, q )
          Merge-Sort ( A, q + 1, r )
    sync
    Par-Merge ( A, p, q, r )
```

Work: $T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$

$= \Theta(n \log n)$ [MT Case 2]

Span: $T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(\log^2 n), & otherwise. \end{cases}$

$= \Theta(\log^3 n)$ [MT Case 2]

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

Parallel Prefix Sums

Parallel Prefix Sums

Input: A sequence of n elements $\{x_1, x_2, \dots, x_n\}$ drawn from a set S with a binary associative operation, denoted by $\oplus$.

Output: A sequence of n partial sums $\{s_1, s_2, \dots, s_n\}$, where $s_i = x_1 \oplus x_2 \oplus \dots \oplus x_i$ for $1 \le i \le n$.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
5	3	7	1	3	6	2	4

$\oplus$ = binary addition

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
5	8	15	16	19	25	27	31

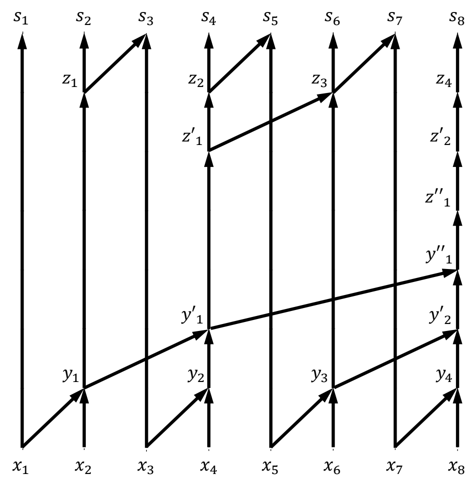
Parallel Prefix Sums


```
Prefix-Sum ( ⟨x_1, x_2, …, x_n⟩, ⊕ )   { n = 2^k for some k ≥ 0.
                                          Return prefix sums
                                          ⟨s_1, s_2, …, s_n⟩ }

if n = 1 then
    s_1 ← x_1
else
    parallel for i ← 1 to n/2 do
        y_i ← x_{2i−1} ⊕ x_{2i}
    ⟨z_1, z_2, …, z_{n/2}⟩ ← Prefix-Sum( ⟨y_1, y_2, …, y_{n/2}⟩, ⊕ )
    parallel for i ← 1 to n do
        if i = 1 then s_1 ← x_1
        else if i = even then s_i ← z_{i/2}
            else s_i ← z_{(i−1)/2} ⊕ x_i
return ⟨s_1, s_2, …, s_n⟩
```

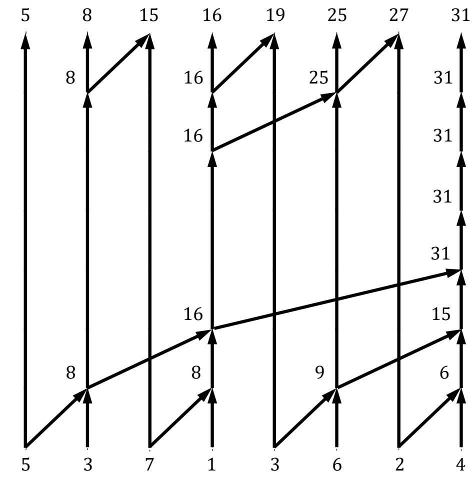
Parallel Prefix Sums

73



Parallel Prefix Sums

74



Parallel Prefix Sums

75

Prefix-Sum ($\langle x_1, x_2, \ldots, x_n \rangle, \oplus$) { $n = 2^k$ *for some* $k \geq 0$. *Return prefix sums* $\langle s_1, s_2, \ldots, s_n \rangle$ }

```
if n = 1 then
    s_1 ← x_1
else
    parallel for i ← 1 to n/2 do
        y_i ← x_{2i−1} ⊕ x_{2i}
    ⟨z_1, z_2, …, z_{n/2}⟩ ← Prefix-Sum( ⟨y_1, y_2, …, y_{n/2}⟩, ⊕ )
    parallel for i ← 1 to n do
        if i = 1 then s_1 ← x_1
        else if i = even then s_i ← z_{i/2}
            else s_i ← z_{(i−1)/2} ⊕ x_i
return ⟨s_1, s_2, …, s_n⟩
```

Work:

$$T_1(n) = \begin{cases} \Theta(1), & \textit{if } n = 1, \\ T_1\left(\frac{n}{2}\right) + \Theta(n), & \textit{otherwise.} \end{cases}$$

$$= \Theta(n)$$

Span:

$$T_\infty(n) = \begin{cases} \Theta(1), & \textit{if } n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(1), & \textit{otherwise.} \end{cases}$$

$$= \Theta(\log n)$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log n}\right)$

Observe that we have assumed here that a *parallel for loop* can be executed in $\Theta(1)$ time. But recall that *cilk_for* is implemented using divide-and-conquer, and so in practice, it will take $\Theta(\log n)$ time. In that case, we will have $T_\infty(n) = \Theta(\log^2 n)$, and parallelism $= \Theta(n/\log^2 n)$.

76

Parallel Partition

Parallel Partition

Input: An array $A[\ q : r\ ]$ of distinct elements, and an element x from $A[\ q : r\ ]$.

Output: Rearrange the elements of $A[\ q : r\ ]$, and return an index $k \in [\ q, r\ ]$, such that all elements in $A[\ q : k - 1\ ]$ are smaller than x, all elements in $A[\ k + 1 : r\ ]$ are larger than x, and $A[\ k\ ] = x$.

```
Par-Partition ( A[ q : r ], x )
n ← r − q + 1
if n = 1 then return q
array B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
parallel for i ← 0 to n − 1 do
    B[ i ] ← A[ q + i ]
    if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
    if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
lt [ 0: n − 1 ] ← Par-Prefix-Sum ( lt[ 0: n − 1 ], + )
gt[ 0: n − 1 ] ← Par-Prefix-Sum ( gt[ 0: n − 1 ], + )
k ← q + lt [ n − 1 ], A[ k ] ← x
parallel for i ← 0 to n − 1 do
    if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
    else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
return k
```

Parallel Partition

A:	9	5	7	11	1	3	8	14	4	21

***x* = 8**

Parallel Partition

A:	9	5	7	11	1	3	8	14	4	21

***x* = 8**

	0	1	2	3	4	5	6	7	8	9
B:	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
***lt*:**	0	1	1	0	1	1	0	0	1	0

	0	1	2	3	4	5	6	7	8	9
***gt*:**	1	0	0	1	0	0	0	1	0	1

Parallel Partition

A:	9	5	7	11	1	3	8	14	4	21

***x* = 8**

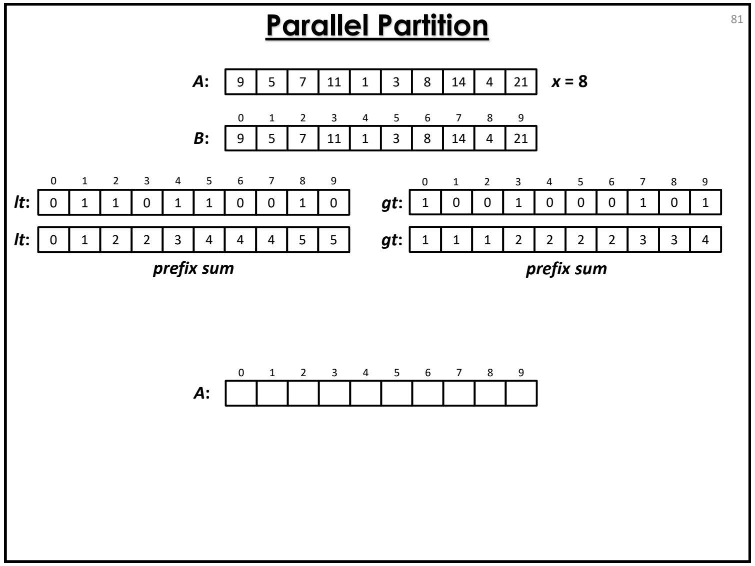
	0	1	2	3	4	5	6	7	8	9
B:	9	5	7	11	1	3	8	14	4	21

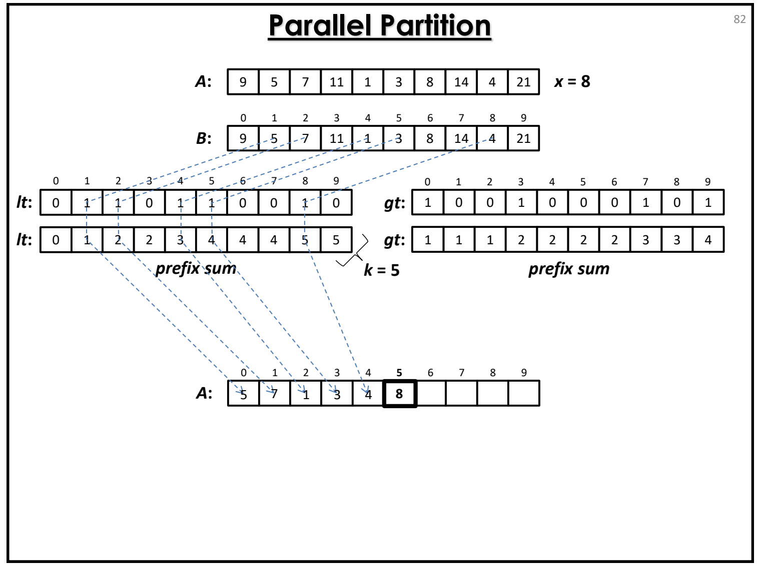
	0	1	2	3	4	5	6	7	8	9
***lt*:**	0	1	1	0	1	1	0	0	1	0
***lt*:**	0	1	2	2	3	4	4	4	5	5

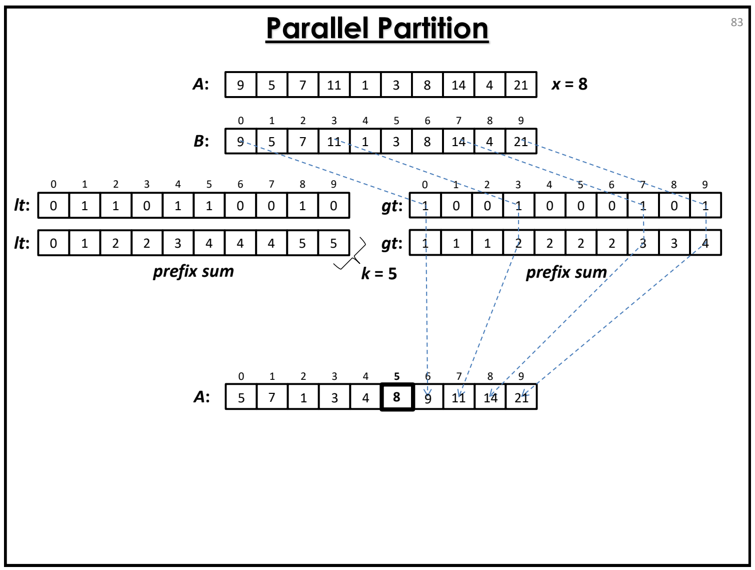
prefix sum

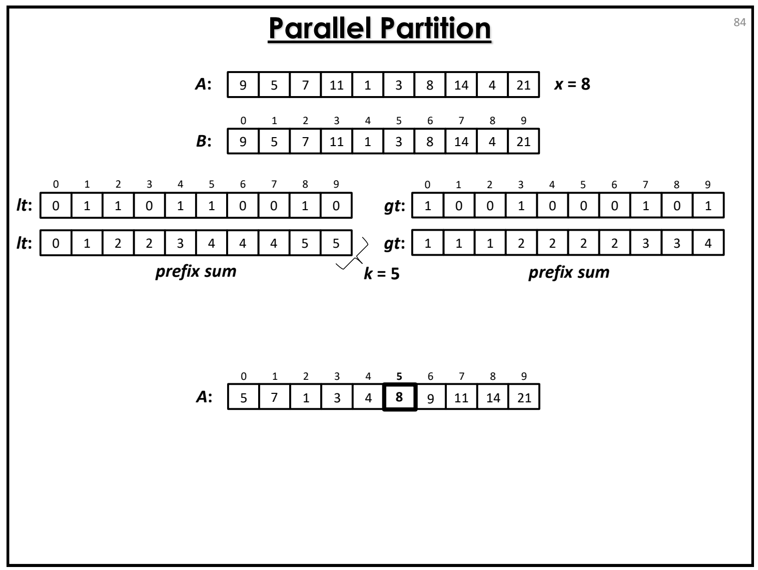
	0	1	2	3	4	5	6	7	8	9
***gt*:**	1	0	0	1	0	0	0	1	0	1
***gt*:**	1	1	1	2	2	2	2	3	3	4

prefix sum


Parallel Partition
81
A: 9 5 7 11 1 3 8 14 4 21
x = 8
B: 9 5 7 11 1 3 8 14 4 21
lt: 0 1 1 0 1 1 0 0 1 0
gt: 1 0 0 1 0 0 0 1 0 1
lt: 0 1 2 2 3 4 4 4 5 5
gt: 1 1 1 2 2 2 2 3 3 4
prefix sum
prefix sum
A:


Parallel Partition
82
A: 9 5 7 11 1 3 8 14 4 21
x = 8
B: 9 5 7 11 1 3 8 14 4 21
lt: 0 1 1 0 1 1 0 0 1 0
gt: 1 0 0 1 0 0 0 1 0 1
lt: 0 1 2 2 3 4 4 4 5 5
gt: 1 1 1 2 2 2 2 3 3 4
k = 5
prefix sum
prefix sum
A: 5 7 1 3 4 8


Parallel Partition
83
A: 9 5 7 11 1 3 8 14 4 21
x = 8
B: 9 5 7 11 1 3 8 14 4 21
lt: 0 1 1 0 1 1 0 0 1 0
gt: 1 0 0 1 0 0 0 1 0 1
lt: 0 1 2 2 3 4 4 4 5 5
gt: 1 1 1 2 2 2 2 3 3 4
k = 5
prefix sum
prefix sum
A: 5 7 1 3 4 8 9 11 14 21


Parallel Partition
84
A: 9 5 7 11 1 3 8 14 4 21
x = 8
B: 9 5 7 11 1 3 8 14 4 21
lt: 0 1 1 0 1 1 0 0 1 0
gt: 1 0 0 1 0 0 0 1 0 1
lt: 0 1 2 2 3 4 4 4 5 5
gt: 1 1 1 2 2 2 2 3 3 4
k = 5
prefix sum
prefix sum
A: 5 7 1 3 4 8 9 11 14 21

(Lecture 12) Analyzing Parallel Algorithms

Parallel Partition: Analysis

85

```
Par-Partition ( A[ q : r ], x )
1.  n ← r − q + 1
2.  if n = 1 then return q
3.  array B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
4.  parallel for i ← 0 to n − 1 do
5.      B[ i ] ← A[ q + i ]
6.      if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
7.      if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
8.  lt [ 0: n − 1 ] ← Par-Prefix-Sum ( lt[ 0: n − 1 ], + )
9.  gt[ 0: n − 1 ] ← Par-Prefix-Sum ( gt[ 0: n − 1 ], + )
10. k ← q + lt [ n − 1 ], A[ k ] ← x
11. parallel for i ← 0 to n − 1 do
12.     if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
13.     else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
14. return k
```

Work:

$$\begin{aligned} T_1(n) &= \Theta(n) && \text{[lines 1 – 7]} \\ &+ \Theta(n) && \text{[lines 8 – 9]} \\ &+ \Theta(n) && \text{[lines 10 – 14]} \\ &= \Theta(n) \end{aligned}$$

Span:

Assuming $\log n$ depth for *parallel for* loops:

$$\begin{aligned} T_\infty(n) &= \Theta(\log n) && \text{[lines 1 – 7]} \\ &+ \Theta(\log^2 n) && \text{[lines 8 – 9]} \\ &+ \Theta(\log n) && \text{[lines 10 – 14]} \\ &= \Theta(\log^2 n) \end{aligned}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

86

Parallel Quicksort

Randomized Parallel QuickSort

87

Input: An array $A[\ q : r\ ]$ of distinct elements.

Output: Elements of $A[\ q : r\ ]$ sorted in increasing order of value.

```
Par-Randomized-QuickSort ( A[ q : r ] )
1. n ← r − q + 1
2. if n ≤ 30 then
3.     sort A[ q : r ] using any sorting algorithm
4. else
5.     select a random element x from A[ q : r ]
6.     k ← Par-Partition ( A[ q : r ], x )
7.     spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
8.     Par-Randomized-QuickSort ( A[ k + 1 : r ] )
9.     sync
```

Randomized Parallel QuickSort: Analysis

88

```
Par-Randomized-QuickSort ( A[ q : r ] )
1. n ← r − q + 1
2. if n ≤ 30 then
3.     sort A[ q : r ] using any sorting algorithm
4. else
5.     select a random element x from A[ q : r ]
6.     k ← Par-Partition ( A[ q : r ], x )
7.     spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
8.     Par-Randomized-QuickSort ( A[ k + 1 : r ] )
9.     sync
```

Lines 1—6 take $\Theta(\log^2 n)$ parallel time and perform $\Theta(n)$ work.

Also the recursive spawns in lines 7—8 work on disjoint parts of $A[\ q : r\ ]$. So the upper bounds on the parallel time and the total work in each level of recursion are $\Theta(\log^2 n)$ and $\Theta(n)$, respectively.

Hence, if D is the *recursion depth* of the algorithm, then

$$T_1(n) = O(nD) \text{ and } T_\infty(n) = O(D \log^2 n)$$

89

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
n ← r − q + 1
if n ≤ 30 then
    sort A[ q : r ] using any sorting algorithm
else
    select a random element x from A[ q : r ]
    k ← Par-Partition ( A[ q : r ], x )
    spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
    Par-Randomized-QuickSort ( A[ k + 1 : r ] )
    sync
```

We already proved that w.h.p. recursion depth, $D = O(\log n)$.

Hence, with high probability,

$$T_1(n) = O(n \log n) \text{ and } T_\infty(n) = O(\log^3 n)$$

90

91

92