CSE 548: Analysis of Algorithms

Lecture 3
(Divide-and-Conquer Algorithms: Matrix Multiplication)

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Iterative Matrix Multiplication

$$\boxed{z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}}$$

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \times \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}$$

```
Iter-MM ( Z, X, Y )          { X, Y, Z are n × n matrices,
                               where n is a positive integer }
   for i ← 1 to n do
      for j ← 1 to n do
         Z[ i ][ j ] ← 0
         for k ← 1 to n do
            Z[ i ][ j ] ← Z[ i ][ j ] + X[ i ][ k ] · Y[ k ][ j ]
```


Recursive (Divide & Conquer) Matrix Multiplication

$$\underbrace{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}}_{X} \times \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{Y}$$

$$= \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

(each block is $n/2 \times n/2$; each matrix is $n \times n$)

Recursive (Divide & Conquer) Matrix Multiplication

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

```
Rec-MM ( X, Y )     { X and Y are n × n matrices,
                      where n = 2^k for integer k ≥ 0 }
   Let Z be a new n × n matrix
   if n = 1 then
      Z ← X · Y
   else
      Z11 ← Rec-MM ( X11, Y11 ) + Rec-MM ( X12, Y21 )
      Z12 ← Rec-MM ( X11, Y12 ) + Rec-MM ( X12, Y22 )
      Z21 ← Rec-MM ( X21, Y11 ) + Rec-MM ( X22, Y21 )
      Z22 ← Rec-MM ( X21, Y12 ) + Rec-MM ( X22, Y22 )
   endif
   return Z
```

\# recursive matrix products: 8
\# matrix sums: 4

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ 8T\left(\frac{n}{2}\right) + \Theta(n^2), & \text{otherwise.} \end{cases}$$

$$= \Theta(n^3)$$

Strassen's Algorithms for Matrix Multiplication (MM)

In 1968 Volker Strassen came up with a recursive MM algorithm that runs asymptotically faster than the classical $\Theta(n^3)$ algorithm.

In each level of recursion the algorithm uses:

7 recursive matrix multiplications (instead of 8), and

18 matrix additions (instead of 4).

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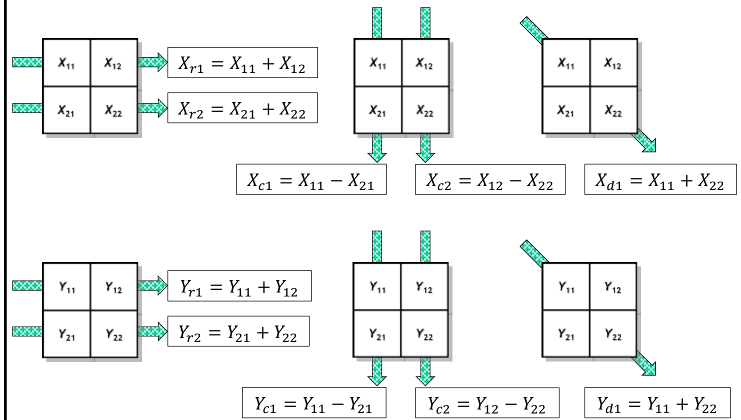
Strassen's MM: 10 Matrix Additions/Subtractions



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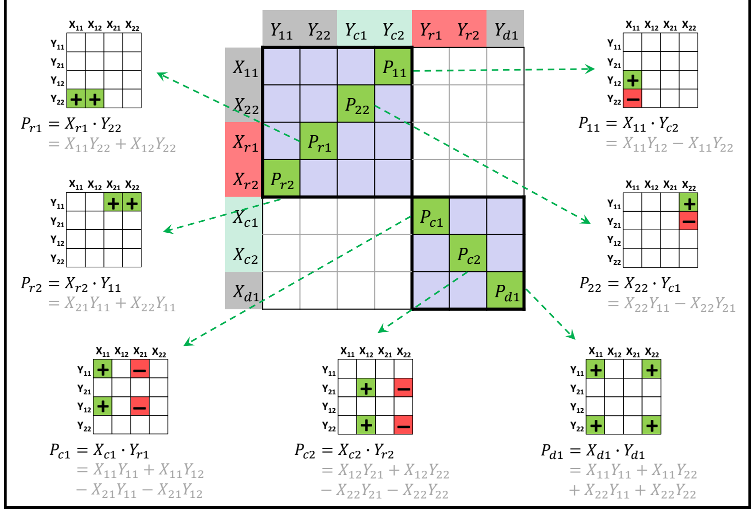
Strassen's MM: 7 Matrix Products

	Y_{11}	Y_{22}	Y_{c1}	Y_{c2}	Y_{r1}	Y_{r2}	Y_{d1}
X_{11}				P_{11}			
X_{22}			P_{22}				
X_{r1}		P_{r1}					
X_{r2}	P_{r2}						
X_{c1}					P_{c1}		
X_{c2}						P_{c2}	
X_{d1}							P_{d1}

Strassen's MM: 7 Matrix Products




Strassen's Matrix Multiplication
P11 P22 Pr1 Pr2 Pc1 Pc2 Pd1
Z11 Z12 Z21 Z22
X11 X12 X21 X22
Y11 Y21 Y12 Y22
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Strassen's Matrix Multiplication
17
Strassen's Matrix Multiplication
18
Strassen's Matrix Multiplication
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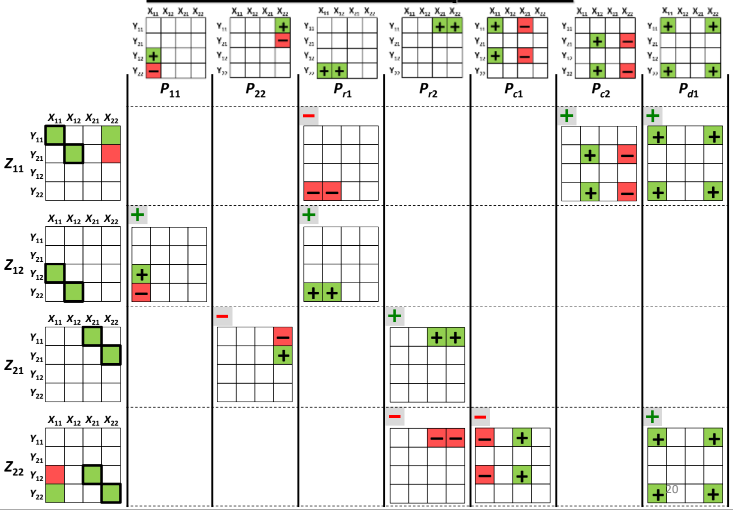
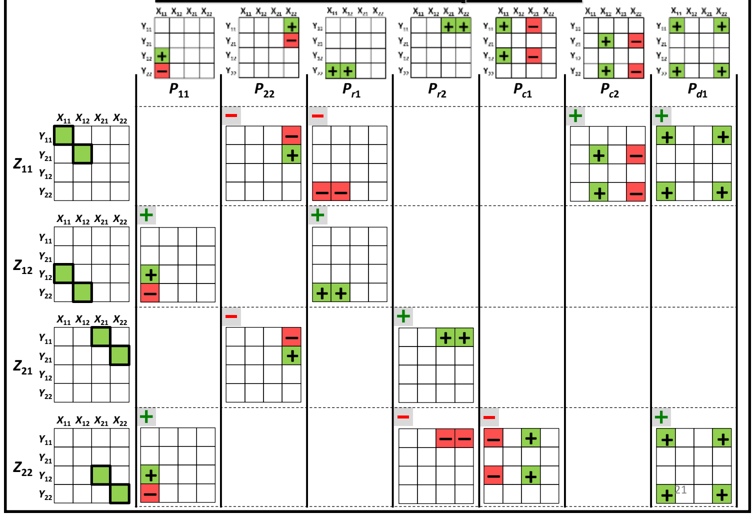
Strassen's Matrix Multiplication



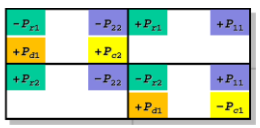
Strassen's Matrix Multiplication



Strassen's Matrix Multiplication

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \times \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} X_{11}Y_{11} + X_{12}Y_{21} & X_{11}Y_{12} + X_{12}Y_{22} \\ X_{21}Y_{11} + X_{22}Y_{21} & X_{21}Y_{12} + X_{22}Y_{22} \end{bmatrix}$$

$=$



Sums:

$X_{r1} = X_{11} + X_{12}$ $\quad$ $Y_{r1} = Y_{11} + Y_{12}$
$X_{r2} = X_{21} + X_{22}$ $\quad$ $Y_{r2} = Y_{21} + Y_{22}$
$X_{c1} = X_{11} - X_{21}$ $\quad$ $Y_{c1} = Y_{11} - Y_{21}$
$X_{c2} = X_{12} - X_{22}$ $\quad$ $Y_{c2} = Y_{12} - Y_{22}$
$X_{d1} = X_{11} + X_{22}$ $\quad$ $Y_{d1} = Y_{11} + Y_{22}$

Products:

$P_{11} = X_{11} \cdot Y_{c2}$ $\quad$ $P_{c1} = X_{c1} \cdot Y_{r1}$
$P_{22} = X_{22} \cdot Y_{c1}$ $\quad$ $P_{c2} = X_{c2} \cdot Y_{r2}$
$P_{r1} = X_{r1} \cdot Y_{22}$ $\quad$ $P_{d1} = X_{d1} \cdot Y_{d1}$
$P_{r2} = X_{r2} \cdot Y_{11}$

Running Time:

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ 7T\left(\frac{n}{2}\right) + \Theta(n^2), & \text{otherwise.} \end{cases}$$

$$= \Theta\left(n^{\log_2 7}\right) = O\left(n^{2.81}\right)$$

Deriving Strassen's Algorithm

Use the *Feynman Algorithm*:

Step 1: *write down the problem*

Step 2: *think real hard*

Step 3: *write down the solution*

Deriving Strassen's Algorithm

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}}_{X}\underbrace{\begin{bmatrix} e \\ g \\ f \\ h \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} p \\ r \\ q \\ s \end{bmatrix}}_{Z}$$

We will try to minimize the number of multiplications needed to evaluate Z using special matrix products that are easy to compute.

Type	Product	#Mults
$(\cdot)$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$	4
(A)	$\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e+g) \\ a(e+g) \end{bmatrix}$	1
(B)	$\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e+g) \\ -a(e+g) \end{bmatrix}$	1
(C)	$\begin{bmatrix} a & 0 \\ a-b & b \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} ae \\ ae + b(g-e) \end{bmatrix}$	2
(D)	$\begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} a(e-g) + bg \\ bg \end{bmatrix}$	2

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Deriving Strassen's Algorithm

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} = \underbrace{\begin{bmatrix} b & b & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ A\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ c-b & d-b & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}}_{\Delta_1}$$

$$\Delta_1 = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & c & c \end{bmatrix}}_{Type\ A\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ c-b & d-b & 0 & 0 \\ 0 & 0 & a-c & b-c \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{\Delta_2}$$

$$\Delta_2 = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ c-b & 0 & 0 & c-b \\ -(c-b) & 0 & 0 & -(c-b) \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ B\ (\ 1\ Mult\ )} + \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & d-b & 0 & b-c \\ c-b & 0 & a-c & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{\Delta_3}$$

$$\Delta_3 = \underbrace{\begin{bmatrix} a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (a-b)-(a-c) & 0 & a-c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Type\ C\ (\ 2\ Mult\ )} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & d-b & 0 & (d-c)-(d-b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{Type\ D\ (\ 2\ Mult\ )}$$

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Algorithms for Multiplying Two *n*×*n* Matrices

A recursive algorithm based on multiplying two $m \times m$ matrices using k multiplications will yield an $\mathrm{O}\left(n^{\log_m k}\right)$ algorithm.

To beat Strassen's algorithm: $\log_m k < \log_2 7 \Rightarrow k < m^{\log_2 7}$.

So, for a 3×3 matrix, we must have: $k < 3^{\log_2 7} < 22$.

But the best known algorithm uses 23 multiplications!

Inventor	Year	Complexity
Classical	—	$\Theta(n^3)$
Volker Strassen	1968	$\Theta(n^{2.807})$
Victor Pan (multiply two 70×70 matrices using 143,640 multiplications)	1978	$\Theta(n^{2.795})$
Don Coppersmith & Shmuel Winograd (arithmetic progressions)	1990	$\Theta(n^{2.3737})$
Andrew Stothers	2010	$\Theta(n^{2.3736})$
Virginia Williams	2011	$\Theta(n^{2.3727})$

Lower bound: $\Omega(n^2)$ (why?)

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