CSE 548: Analysis of Algorithms

Lecture 6
(Divide-and-Conquer Algorithms: Akra-Bazzi Recurrences)

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Deterministic Select

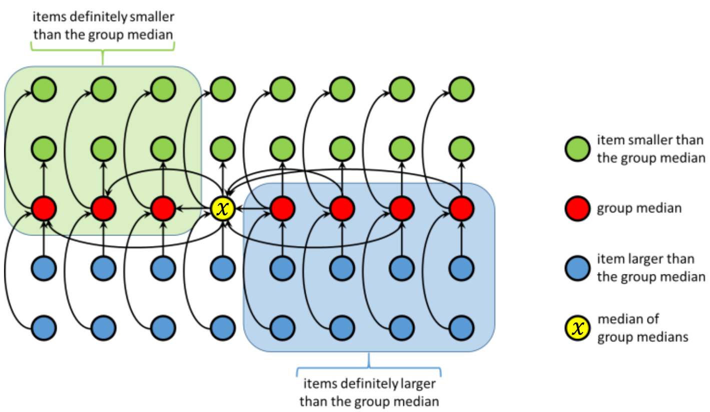
Input: An array $A[\ q : r\ ]$ of distinct elements, and integer $k \in [1, r-q+1]$.

Output: An element x of $A[\ q : r\ ]$ such that $rank(x, A[\ q : r\ ]) = k$.

```
Select ( A[ q : r ], k )
1.  n ← r − q + 1
2.  if n ≤ 140 then
3.      sort A[ q : r ] and return A[ q + k - 1 ]
4.  else
5.      divide A[ q : r ] into blocks B_i's each containing 5 consecutive elements
            ( last block may contain fewer than 5 elements )
6.      for i ← 1 to ⌈ n / 5 ⌉ do
7.          M[ i ] ← median of B_i using sorting
8.      x ← Select ( M[ 1 : ⌈ n / 5 ⌉ ], ⌊( ⌈ n / 5 ⌉ + 1 ) / 2 ⌋ )  { median of medians }
9.      t ← Partition ( A[ q : r ], x )    { partition around x which ends up at A[ t ] }
10.     if k = t − q + 1 then return A[ t ]
11.     else if k < t − q + 1 then return Select ( A[ q : t − 1 ], k )
12.         else return Select ( A[ t + 1 : r ], k − t + q − 1 )
```

Deterministic Select

SELECT(A, k): Given an unsorted set A of n $(= |A|)$ items, find the k^{th} smallest item in the set



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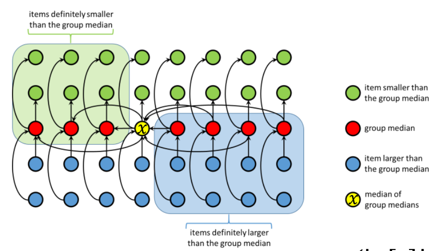

#items definitely smaller than x is $\geq 3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 1\right) \geq \frac{3n}{10} - 6$

#items definitely larger than x is $\geq 3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 1\right) \geq \frac{3n}{10} - 6$

#items in any recursive call (lines 11/12) $\leq n - \left(\frac{3n}{10} - 6\right) = \frac{7n}{10} + 6$

Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm (given in Section 9.3 of CLRS):

$$T(n) \le \begin{cases} \Theta(1), & if\ n < 140, \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

Dropping the ceiling for simplicity, and observing that $\frac{7n}{10} + 6 \le \frac{8n}{10}$ when $n \ge 60$, we obtain the following upper bound on $T(n)$.

$$T'(n) = \begin{cases} \Theta(1), & if\ n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

How do you solve for $T'(n)$?

Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm (given in Section 9.3 of CLRS):

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Dropping the ceiling for simplicity, and observing that $\frac{7n}{10} + 6 \le \frac{7.5n}{10}$ when $n \ge 120$, we obtain the following upper bound on $T(n)$.

$$T''(n) = \begin{cases} \Theta(1), & if\ n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

How do you solve for $T''(n)$?

Akra-Bazzi Recurrences

Consider the following recurrence:

$$T(x) = \begin{cases} \Theta(1), & if\ 1 \le x \le x_0, \\ \sum_{i=1}^{k} a_i T(b_i x) + g(x), & if\ x > x_0; \end{cases}$$

where,

1. $k \ge 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 \le i \le k$
3. $b_i \in (0,1)$ is a constant for $1 \le i \le k$
4. $x \ge 1$ is a real number
5. $x_0 \ge \max\left\{\frac{1}{b_i}, \frac{1}{1-b_i}\right\}$ is a constant for $1 \le i \le k$
6. $g(x)$ is a nonnegative function that satisfies a *polynomial-growth condition* (to be specified soon)

Polynomial-Growth Condition

We say that $g(x)$ satisfies the *polynomial-growth condition* if there exist positive constants c_1 and c_2 such that for all $x \ge 1$, for all $1 \le i \le k$, and for all $u \in [b_i x, x]$,

$$c_1 g(x) \le g(u) \le c_2 g(x),$$

where x, k, b_i and $g(x)$ are as defined in the previous slide.

The Akra-Bazzi Solution

Consider the recurrence given in the previous two slides under the conditions specified there:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \le x \le x_0, \\ \sum_{i=1}^{k} a_i T(b_i x) + g(x), & \text{if } x > x_0. \end{cases}$$

Let p be the unique real number for which $\sum_{i=1}^{k} a_i b_i^p = 1$. Then

$$T(x) = \Theta\left(x^p\left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\,du\right)\right)$$

Deterministic Select

$$T'(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & \text{if } n \ge 140. \end{cases}$$

From $\left(\frac{1}{5}\right)^p + \left(\frac{4}{5}\right)^p = 1$ we get $p = 1$.

Hence, $T'(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{p+1}}\,du\right)\right)$

$= \Theta\left(n\left(1 + \int_1^n \frac{du}{u}\right)\right)$

$= \Theta(n \ln n)$

Deterministic Select

$$T''(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & \text{if } n \ge 140. \end{cases}$$

From $\left(\frac{1}{5}\right)^p + \left(\frac{3}{4}\right)^p = 1$ we get $p < 1$.

Hence, $T''(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{p+1}}\,du\right)\right)$

$= \Theta\left(n^p\left(1 + \int_1^n \frac{du}{u^p}\right)\right)$

$= \Theta\left(\left(\frac{1}{1-p}\right)n - \left(\frac{p}{1-p}\right)n^p\right)$

$= \Theta(n)$

Examples of Akra-Bazzi Recurrences

Example 1: $T(x) = 2T\left(\frac{x}{4}\right) + 3T\left(\frac{x}{6}\right) + \Theta(x \log x)$

Then $p = 1$ and $T(x) = \Theta\left(x\left(1 + \int_1^x \frac{u \log u}{u^2}\,du\right)\right) = \Theta(x \log^2 x)$

Example 2: $T(x) = 2T\left(\frac{x}{2}\right) + \frac{8}{9}T\left(\frac{3x}{4}\right) + \Theta\left(\frac{x^2}{\log x}\right)$

Then $p = 2$ and $T(x) = \Theta\left(x^2\left(1 + \int_1^x \frac{u^2/\log u}{u^3}\,du\right)\right) = \Theta(x^2 \log\log x)$

Example 3: $T(x) = T\left(\frac{x}{2}\right) + \Theta(\log x)$

Then $p = 0$ and $T(x) = \Theta\left(1 + \int_1^x \frac{\log u}{u}\,du\right) = \Theta(\log^2 x)$

Example 4: $T(x) = \frac{1}{2}T\left(\frac{x}{2}\right) + \Theta\left(\frac{1}{x}\right)$

Then $p = -1$ and $T(x) = \Theta\left(\frac{1}{x}\left(1 + \int_1^x \frac{1}{u}\,du\right)\right) = \Theta\left(\frac{\log x}{x}\right)$

A Helping Lemma

Lemma: If $g(x)$ is a nonnegative function that satisfies the polynomial-growth condition, then there exist positive constants c_3 and c_4 such that for $1 \le i \le k$ and all $x \ge 1$,

$$c_3 g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le c_4 g(x).$$

Proof: $b_i x \le u \le x$

$$\Rightarrow \frac{1}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \le \frac{1}{u^{p+1}} \le \frac{1}{\min\{(b_i x)^{p+1}, x^{p+1}\}}$$

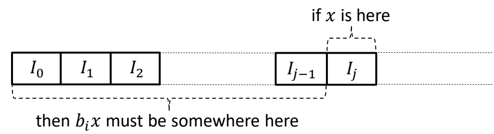
$$\Rightarrow \frac{x^p c_1 g(x)}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^{x} du \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le \frac{x^p c_2 g(x)}{\min\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^{x} du$$

$$\Rightarrow \frac{(1-b_i)c_1}{\max\left\{1, b_i^{p+1}\right\}} g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le \frac{(1-b_i)c_2}{\min\left\{1, b_i^{p+1}\right\}} g(x)$$

$$\Rightarrow c_3 g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \le c_4 g(x)$$

Partitioning the Domain of x

Let $I_0 = [1, x_0]$ and $I_j = (x_0 + j - 1, x_0 + j]$ for $j \ge 1$.

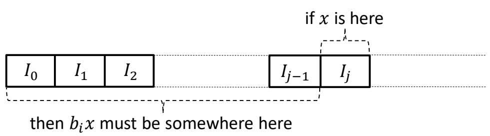


That allows us to use induction in the proof of:

$$T(x) = \begin{cases} \Theta(1), & if\ 1 \le x \le x_0, \\ \sum_{i=1}^{k} a_i T(b_i x) + g(x), & if\ x > x_0. \end{cases}$$

Partitioning the Domain of x

Let $I_0 = [1, x_0]$ and $I_j = (x_0 + j - 1, x_0 + j]$ for $j \ge 1$.



Proof:

$$x_0 + j - 1 < x \le x_0 + j$$
$$\Rightarrow b_i(x_0 + j - 1) < b_i x \le b_i(x_0 + j)$$
$$\Rightarrow b_i x_0 < b_i x \le b_i x_0 + j$$
$$\Rightarrow 1 < b_i x \le x_0 + j - (1 - b_i)x_0$$
$$\Rightarrow 1 < b_i x \le x_0 + j - 1$$

Derivation of the Akra-Bazzi Solution

Lower Bound: There exists a constant $c_5 > 0$ such that for all $x > x_0$,

$$T(x) \ge c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right).$$

Proof: By induction on the interval I_j containing x.

Base case ($j = 0$) follows since $T(x) = \Theta(1)$ when $x \in I_0 = [1, x_0]$.

Induction: $T(x) = \sum_{i=1}^{k} a_i T(b_i x) + g(x) \ge \sum_{i=1}^{k} a_i c_5 (b_i x)^p \left(1 + \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du\right) + g(x)$

$$= c_5 x^p \sum_{i=1}^{k} a_i b_i^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du\right) + g(x)$$

$$\ge c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \frac{c_4}{x^p} g(x)\right) \sum_{i=1}^{k} a_i b_i^p + g(x)$$

$$= c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right) + (1 - c_4 c_5) g(x) \ge c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)$$

(assuming $c_4 c_5 \le 1$)

Derivation of the Akra-Bazzi Solution

Upper Bound: There exists a constant $c_6 > 0$ such that for all $x > x_0$,

$$T(x) \le c_6 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right).$$

Proof: Similar to the lower bound proof.