CSE 548: Analysis of Algorithms

Lecture 7
(Generating Functions)

Rezaul A. Chowdhury
Department of Computer Science
SUNY Stony Brook
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An Impossible Counting Problem

Suppose you went to a grocery store to buy some fruits. There are some constraints though:

A. The store has only two **apples** left: one red and one green. So you cannot take more than 2 apples.

B. All but 3 **bananas** are rotten. You do not like rotten bananas.

F. **Figs** are sold 6 per pack. You can take as many packs as you want.

M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.

P. They sell 4 **peaches** per pack. Take as many packs as you want.

Now the question is: in how many ways can you buy n fruits from the store?

Generating Functions

Generating functions represent sequences by coding the terms of a sequence as coefficients of powers of a variable in a formal power series.

For example, one can represent a sequence $s_0, s_1, s_2, \ldots$ as:

$$S(z) = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots + s_n z^n + \ \ldots$$

So s_n is the coefficient of z^n in $S(z)$.

An Impossible Counting Problem

A. The store has only two **apples** left: one red and one green. So you cannot take more than 2 apples.

$$A(z) = 1 + 2z + z^2 = (1 + z)^2$$

B. All but 3 **bananas** are rotten. You do not like rotten bananas.

$$B(z) = 1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z}$$

F. **Figs** are sold 6 per pack. You can take as many packs as you want.

$$F(z) = 1 + z^6 + z^{12} + z^{18} + \cdots = \frac{1}{1 - z^6}$$

M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.

$$M(z) = 1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2}$$

P. They sell 4 **peaches** per pack. Take as many packs as you want.

$$P(z) = 1 + z^4 + z^8 + z^{12} + \cdots = \frac{1}{1 - z^4}$$

An Impossible Counting Problem

Suppose you can choose n fruits in s_n different ways.

Then the generating function for s_n is:

$$\begin{aligned} S(z) = A(z)B(z)F(z)M(z)P(z) &= (1+z)^2 \times \frac{1-z^4}{1-z} \times \frac{1}{1-z^6} \times \frac{1-z^6}{1-z^2} \times \frac{1}{1-z^4} \\ &= \frac{1+z}{(1-z)^2} \\ &= (1+z)\sum_{n=0}^{\infty}(n+1)z^n \\ &= \sum_{n=0}^{\infty}(2n+1)z^n \end{aligned}$$

Equating the coefficients of z^n from both sides:

$$s_n = 2n+1$$

Fibonacci Numbers

Recurrence for *Fibonacci numbers*:

$$f_n = \begin{cases} 0 & if\ n=0, \\ 1 & if\ n=1, \\ f_{n-1}+f_{n-2} & otherwise. \end{cases}$$

$$\Rightarrow f_n = f_{n-1} + f_{n-2} + [n=1]$$

Generating function: $F(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \ \ldots$

$$\begin{aligned} F(z) = \sum_n f_n z^n &= \sum_n f_{n-1} z^n + \sum_n f_{n-2} z^n + \sum_n [n=1] z^n \\ &= \sum_n f_n z^{n+1} + \sum_n f_n z^{n+2} + z \\ &= zF(z) + z^2F(z) + z \end{aligned}$$

Fibonacci Numbers

$$F(z) = zF(z) + z^2F(z) + z$$

$$\begin{aligned} \Rightarrow F(z) &= \frac{z}{1-z-z^2} \\ &= \frac{z}{(1-\varphi z)(1-\hat{\varphi} z)}, \text{ where } \varphi = \frac{1+\sqrt{5}}{2} \ \&\ \hat{\varphi} = \frac{1-\sqrt{5}}{2} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\varphi z} - \frac{1}{1-\hat{\varphi} z}\right) \\ &= \frac{1}{\sqrt{5}}\sum_n (\phi^n - \hat{\phi}^n) z^n \end{aligned}$$

Equating the coefficients of z^n from both sides:

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Average Case Analysis of Quicksort

Quicksort

Input: An array $A[1:n]$ of n distinct numbers.

Output: Numbers of $A[1:n]$ rearranged in increasing order of value.

Steps:

1. **Pivot Selection:** Select pivot $x = A[1]$.
2. **Partition:** Use a stable partitioning algorithm to rearrange the numbers of $A[1:n]$ such that $A[k] = x$ for some $k \in [1, n]$, each number in $A[1:k-1]$ is smaller than x, and each in $A[k+1:n]$ is larger than x.
3. **Recursion:** Recursively sort $A[1:k-1]$ and $A[k+1:n]$.
4. **Output:** Output $A[1:n]$.

Stable Partitioning: If two numbers p and q end up in the same partition and p appears before q in the input, then p must also appear before q in the resulting partition.

Average Number of Comparisons by Quicksort

We will average the number of comparisons performed by *Quicksort* on all possible arrangements of the numbers in the input array.

Let t_n = average #comparisons performed by *Quicksort* on n numbers.

Then

$$t_n = \begin{cases} 0 & if\ n < 1, \\ n - 1 + \frac{1}{n}\sum_{k=1}^{n}(t_{k-1} + t_{n-k}) & otherwise. \end{cases}$$

The recurrence can be rewritten as follows.

$$t_n = \begin{cases} 0 & if\ n < 1, \\ n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} t_k & otherwise. \end{cases}$$

Average Number of Comparisons by Quicksort

The recurrence: $t_n = \begin{cases} 0 & if\ n < 1, \\ n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} t_k & otherwise. \end{cases}$

Let $T(z)$ be an ordinary generating function for t_n's:

$$T(z) = t_0 + t_1 z + t_2 z^2 + \cdots + t_n z^n + \cdots$$

$$= t_0 + \sum_{n=1}^{\infty} t_n z^n$$

$$= t_0 + \sum_{n=1}^{\infty}\left(n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} t_k\right) z^n$$

Average Number of Comparisons by Quicksort

We have: $T(z) = t_0 + \sum_{n=1}^{\infty}\left(n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} t_k\right) z^n$

Differentiating:

$$T'(z) = \sum_{n=1}^{\infty}\left(n(n-1) + 2\sum_{k=0}^{n-1} t_k\right) z^{n-1}$$

$$= z\sum_{n=2}^{\infty} n(n-1) z^{n-2} + 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} t_k\right) z^n$$

$$= z\frac{d^2}{dz^2}\left(\left(\sum_{n=0}^{\infty} z^n\right) - 1 - z\right) + 2\sum_{n=0}^{\infty}\left(t_n z^n\left(\sum_{k=0}^{\infty} z^k\right)\right)$$

Average Number of Comparisons by Quicksort

$$T'(z) = z\frac{d^2}{dz^2}\left(\left(\sum_{n=0}^{\infty} z^n\right) - 1 - z\right) + 2\sum_{n=0}^{\infty}\left(t_n z^n\left(\sum_{k=0}^{\infty} z^k\right)\right)$$

$$= z\frac{d^2}{dz^2}\left((1-z)^{-1} - 1 - z\right) + 2(1-z)^{-1}\sum_{n=0}^{\infty} t_n z^n$$

$$= \frac{2z}{(1-z)^3} + \frac{2}{1-z}T(z)$$

Rearranging: $(1-z)^2 T'(z) - 2(1-z)T(z) = \frac{2z}{1-z}$

$$\Rightarrow \frac{d}{dz}\left((1-z)^2 T(z)\right) = \frac{d}{dz}\left(-2\ln(1-z) - 2z\right)$$

Integrating: $(1-z)^2 T(z) = -2\ln(1-z) - 2z + c$ (c is a constant)

Average Number of Comparisons by Quicksort

We have, $(1-z)^2 T(z) = -2\ln(1-z) - 2z + c$ (c is a constant)

Putting $z = 0$, $T(0) = c \Rightarrow t_0 = c \Rightarrow c = 0$

Hence, $(1-z)^2 T(z) = -2\ln(1-z) - 2z$

$$\Rightarrow T(z) = 2(-\ln(1-z) - z)(1-z)^{-2}$$

$$= 2\left(\sum_{j=2}^{\infty}\frac{z^j}{j}\right)\left(\sum_{k=0}^{\infty}(k+1)z^k\right)$$

Equating coefficients of z^n from both sides,

$$t_n = 2\left(\sum_{k=2}^{n}\frac{n+1-k}{k}\right) = 2(n+1)\sum_{k=1}^{n}\frac{1}{k} - 4n = 2(n+1)H_n - 4n,$$

where $H_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right)$ is the n^{th} harmonic number.

Average Number of Comparisons by Quicksort

We have, $t_n = 2(n+1)H_n - 4n$,

where $H_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right)$ is the n^{th} *harmonic number*.

But we know, $H_n = \ln n + \mathrm{O}(1)$ (prove it)

Hence, $\boxed{t_n = 2(n+1)\left(\ln n + \mathrm{O}(1)\right) - 4n = \Theta(n\log n).}$