CSE 548: Analysis of Algorithms

Lecture 4
(Divide-and-Conquer Algorithms: Polynomial Multiplication)

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Coefficient Representation of Polynomials

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$
$$= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$A(x)$ is a polynomial of degree bound n represented as a vector $a = (a_0, a_1, \cdots, a_{n-1})$ of coefficients.

The *degree* of $A(x)$ is k provided it is the largest integer such that a_k is nonzero. Clearly, $0 \leq k \leq n-1$.

Evaluating $A(x)$ at a given point:

Takes $\Theta(n)$ time using Horner's rule:

$$A(x_0) = a_0 + a_1 x_0 + a_2 (x_0)^2 + \cdots + a_{n-1}(x_0)^{n-1}$$
$$= a_0 + x_0\left(a_1 + x_0\left(a_2 + \cdots + x_0\left(a_{n-2} + x_0\left(a_{n-1}\right)\right)\cdots\right)\right)$$

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Coefficient Representation of Polynomials

Adding Two Polynomials:

Adding two polynomials of degree bound n takes $\Theta(n)$ time.

$$C(x) = A(x) + B(x)$$

where, $A(x) = \sum_{j=0}^{n-1} a_j x^j$ and $B(x) = \sum_{j=0}^{n-1} b_j x^j$.

Then $C(x) = \sum_{j=0}^{n-1} c_j x^j$, where, $c_j = a_j + b_j$ for $0 \leq j \leq n-1$.

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Coefficient Representation of Polynomials

Multiplying Two Polynomials:

The product of two polynomials of degree bound n is another polynomial of degree bound $2n-1$.

$$C(x) = A(x)B(x)$$

where, $A(x) = \sum_{j=0}^{n-1} a_j x^j$ and $B(x) = \sum_{j=0}^{n-1} b_j x^j$.

Then $C(x) = \sum_{j=0}^{2n-2} c_j x^j$ where, $c_j = \sum_{k=0}^{j} a_k b_{j-k}$ for $0 \leq j \leq 2n-2$.

The coefficient vector $c = (c_0, c_1, \cdots, c_{2n-2})$, denoted by $c = a \otimes b$, is also called the *convolution* of vectors $a = (a_0, a_1, \cdots, a_{n-1})$ and $b = (b_0, b_1, \cdots, b_{n-1})$.

Clearly, straightforward evaluation of c takes $\Theta(n^2)$ time.

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Convolution

$$\boxed{a_0} + \boxed{a_1x} + \boxed{a_2x^2} + \boxed{a_3x^3}$$

$$\boxed{b_3x^3} + \boxed{b_2x^2} + \boxed{b_1x} + \boxed{b_0}$$

$$\boxed{a_0b_0}$$

5

Convolution

$$\boxed{a_0} + \boxed{a_1x} + \boxed{a_2x^2} + \boxed{a_3x^3}$$

$$\boxed{b_3x^3} + \boxed{b_2x^2} + \boxed{b_1x} + \boxed{b_0}$$

$$\boxed{a_0b_1x} + \boxed{a_1b_0x}$$

6

Convolution

$$\boxed{a_0} + \boxed{a_1x} + \boxed{a_2x^2} + \boxed{a_3x^3}$$

$$\boxed{b_3x^3} + \boxed{b_2x^2} + \boxed{b_1x} + \boxed{b_0}$$

$$\boxed{a_0b_2x^2} + \boxed{a_1b_1x^2} + \boxed{a_2b_0x^2}$$

7

Convolution

$$\boxed{a_0} + \boxed{a_1x} + \boxed{a_2x^2} + \boxed{a_3x^3}$$

$$\boxed{b_3x^3} + \boxed{b_2x^2} + \boxed{b_1x} + \boxed{b_0}$$

$$\boxed{a_0b_3x^3} + \boxed{a_1b_2x^3} + \boxed{a_2b_1x^3} + \boxed{a_3b_0x^3}$$

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Convolution

$a_0 + a_1x + a_2x^2 + a_3x^3$

$b_3x^3 + b_2x^2 + b_1x + b_0$

$a_1b_3x^4 + a_2b_2x^4 + a_3b_1x^4$

Convolution

$a_0 + a_1x + a_2x^2 + a_3x^3$

$b_3x^3 + b_2x^2 + b_1x + b_0$

$a_2b_3x^5 + a_3b_2x^5$

Convolution

$a_0 + a_1x + a_2x^2 + a_3x^3$

$b_3x^3 + b_2x^2 + b_1x + b_0$

$a_3b_3x^6$

Coefficient Representation of Polynomials

Multiplying Two Polynomials:

We can use Karatsuba's algorithm (assume n to be a power of 2):

$$A(x) = \sum_{j=0}^{n-1} a_j x^j = \sum_{j=0}^{\frac{n}{2}-1} a_j x^j + x^{\frac{n}{2}} \sum_{j=0}^{\frac{n}{2}-1} a_{\frac{n}{2}+j} x^j = A_1(x) + x^{\frac{n}{2}} A_2(x)$$

$$B(x) = \sum_{j=0}^{n-1} b_j x^j = \sum_{j=0}^{\frac{n}{2}-1} b_j x^j + x^{\frac{n}{2}} \sum_{j=0}^{\frac{n}{2}-1} b_{\frac{n}{2}+j} x^j = B_1(x) + x^{\frac{n}{2}} B_2(x)$$

Then $C(x) = A(x)B(x)$

$$= A_1(x)B_1(x) + x^{\frac{n}{2}}[A_1(x)B_2(x) + A_2(x)B_1(x)] + x^n\, A_2(x)B_2(x)$$

But $A_1(x)B_2(x) + A_2(x)B_1(x)$

$$= [A_1(x) + A_2(x)][B_1(x) + B_2(x)] - A_1(x)B_1(x) - A_2(x)B_2(x)$$

3 recursive multiplications of polynomials of degree bound $\frac{n}{2}$.

Similar recurrence as in Karatsuba's integer multiplication algorithm leading to a complexity of $O\left(n^{\log_2 3}\right) = O\left(n^{1.59}\right)$.

Point-Value Representation of Polynomials

A point-value representation of a polynomial $A(x)$ is a set of n point-value pairs $\{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$ such that all x_k are distinct and $y_k = A(x_k)$ for $0 \le k \le n-1$.

A polynomial has many point-value representations.

Adding Two Polynomials:

Suppose we have point-value representations of two polynomials of degree bound n using the same set of n points.

$$A: \{(x_0, y_0^a), (x_1, y_1^a), \dots, (x_{n-1}, y_{n-1}^a)\}$$

$$B: \{(x_0, y_0^b), (x_1, y_1^b), \dots, (x_{n-1}, y_{n-1}^b)\}$$

If $C(x) = A(x) + B(x)$ then

$$C: \{(x_0, y_0^a + y_0^b), (x_1, y_1^a + y_1^b), \dots, (x_{n-1}, y_{n-1}^a + y_{n-1}^b)\}$$

Thus polynomial addition takes $\Theta(n)$ time.

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Point-Value Representation of Polynomials

Multiplying Two Polynomials:

Suppose we have *extended* (why?) point-value representations of two polynomials of degree bound n using the same set of $2n$ points.

$$A: \{(x_0, y_0^a), (x_1, y_1^a), \dots, (x_{2n-1}, y_{2n-1}^a)\}$$

$$B: \{(x_0, y_0^b), (x_1, y_1^b), \dots, (x_{2n-1}, y_{2n-1}^b)\}$$

If $C(x) = A(x)B(x)$ then

$$C: \{(x_0, y_0^a y_0^b), (x_1, y_1^a y_1^b), \dots, (x_{2n-1}, y_{2n-1}^a y_{2n-1}^b)\}$$

Thus polynomial multiplication also takes only $\Theta(n)$ time!

(compare this with the $\Theta(n^2)$ time needed in the coefficient form)

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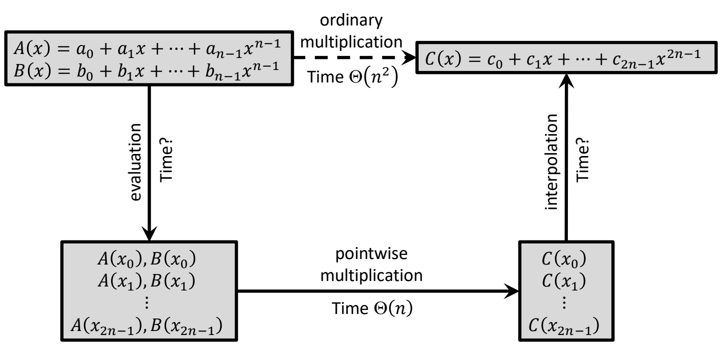
Faster Polynomial Multiplication? (in Coefficient Form)



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Faster Polynomial Multiplication? (in Coefficient Form)

Coefficient Representation ⇒ Point-Value Representation:

We select any set of n distinct points $\{x_0, x_1, \dots, x_{n-1}\}$, and evaluate $A(x_k)$ for $0 \le k \le n-1$.

Using Horner's rule this approach takes $\Theta(n^2)$ time.

Point-Value Representation ⇒ Coefficient Representation:

We can interpolate using Lagrange's formula:

$$A(x) = \sum_{k=0}^{n-1} \frac{\prod_{j \ne k}(x - x_j)}{\prod_{j \ne k}(x_k - x_j)} y_k$$

This again takes $\Theta(n^2)$ time.

In both cases we need to do much better!

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Coefficient Form ⇒ Point-Value Form

A polynomial of degree bound n: $A(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$

A set of n distinct points: $\{x_0, x_1, \ldots, x_{n-1}\}$

Compute point-value form: $\{(x_0, A(x_0)), (x_1, A(x_1)), \ldots, (x_{n-1}, A(x_{n-1}))\}$

Using matrix notation:
$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \cdot \\ \cdot \\ \cdot \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & (x_0)^2 & \cdots & (x_0)^{n-1} \\ 1 & x_1 & (x_1)^2 & \cdots & (x_1)^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_{n-1} & (x_{n-1})^2 & \cdots & (x_{n-1})^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{bmatrix}$$

We want to choose the set of points in a way that simplifies the multiplication.

In the rest of the lecture on this topic we will assume:

n is a power of 2.

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Given a Polynomial of Degree Bound 8 Find 8 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$\begin{array}{c|cclllllllllllllll}
 & A(x_0) & = & a_0 & + & a_1x_0 & + & a_2(x_0)^2 & + & a_3(x_0)^3 & + & a_4(x_0)^4 & + & a_5(x_0)^5 & + & a_6(x_0)^6 & + & a_7(x_0)^7 \\
 & A(x_1) & = & a_0 & + & a_1x_1 & + & a_2(x_1)^2 & + & a_3(x_1)^3 & + & a_4(x_1)^4 & + & a_5(x_1)^5 & + & a_6(x_1)^6 & + & a_7(x_1)^7 \\
 & A(x_2) & = & a_0 & + & a_1x_2 & + & a_2(x_2)^2 & + & a_3(x_2)^3 & + & a_4(x_2)^4 & + & a_5(x_2)^5 & + & a_6(x_2)^6 & + & a_7(x_2)^7 \\
 & A(x_3) & = & a_0 & + & a_1x_3 & + & a_2(x_3)^2 & + & a_3(x_3)^3 & + & a_4(x_3)^4 & + & a_5(x_3)^5 & + & a_6(x_3)^6 & + & a_7(x_3)^7 \\
\hline
x_4 = -x_0 & A(x_4) & = & a_0 & + & a_1x_4 & + & a_2(x_4)^2 & + & a_3(x_4)^3 & + & a_4(x_4)^4 & + & a_5(x_4)^5 & + & a_6(x_4)^6 & + & a_7(x_4)^7 \\
x_5 = -x_1 & A(x_5) & = & a_0 & + & a_1x_5 & + & a_2(x_5)^2 & + & a_3(x_5)^3 & + & a_4(x_5)^4 & + & a_5(x_5)^5 & + & a_6(x_5)^6 & + & a_7(x_5)^7 \\
x_6 = -x_2 & A(x_6) & = & a_0 & + & a_1x_6 & + & a_2(x_6)^2 & + & a_3(x_6)^3 & + & a_4(x_6)^4 & + & a_5(x_6)^5 & + & a_6(x_6)^6 & + & a_7(x_6)^7 \\
x_7 = -x_3 & A(x_7) & = & a_0 & + & a_1x_7 & + & a_2(x_7)^2 & + & a_3(x_7)^3 & + & a_4(x_7)^4 & + & a_5(x_7)^5 & + & a_6(x_7)^6 & + & a_7(x_7)^7
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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Given a Polynomial of Degree Bound 8 Find 8 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$\begin{array}{c|cclllllllllllllll}
 & A(x_0) & = & a_0 & + & a_1x_0 & + & a_2(x_0)^2 & + & a_3(x_0)^3 & + & a_4(x_0)^4 & + & a_5(x_0)^5 & + & a_6(x_0)^6 & + & a_7(x_0)^7 \\
 & A(x_1) & = & a_0 & + & a_1x_1 & + & a_2(x_1)^2 & + & a_3(x_1)^3 & + & a_4(x_1)^4 & + & a_5(x_1)^5 & + & a_6(x_1)^6 & + & a_7(x_1)^7 \\
 & A(x_2) & = & a_0 & + & a_1x_2 & + & a_2(x_2)^2 & + & a_3(x_2)^3 & + & a_4(x_2)^4 & + & a_5(x_2)^5 & + & a_6(x_2)^6 & + & a_7(x_2)^7 \\
 & A(x_3) & = & a_0 & + & a_1x_3 & + & a_2(x_3)^2 & + & a_3(x_3)^3 & + & a_4(x_3)^4 & + & a_5(x_3)^5 & + & a_6(x_3)^6 & + & a_7(x_3)^7 \\
\hline
x_4 = -x_0 & A(-x_0) & = & a_0 & - & a_1x_0 & + & a_2(x_0)^2 & - & a_3(x_0)^3 & + & a_4(x_0)^4 & - & a_5(x_0)^5 & + & a_6(x_0)^6 & - & a_7(x_0)^7 \\
x_5 = -x_1 & A(-x_1) & = & a_0 & - & a_1x_1 & + & a_2(x_1)^2 & - & a_3(x_1)^3 & + & a_4(x_1)^4 & - & a_5(x_1)^5 & + & a_6(x_1)^6 & - & a_7(x_1)^7 \\
x_6 = -x_2 & A(-x_2) & = & a_0 & - & a_1x_2 & + & a_2(x_2)^2 & - & a_3(x_2)^3 & + & a_4(x_2)^4 & - & a_5(x_2)^5 & + & a_6(x_2)^6 & - & a_7(x_2)^7 \\
x_7 = -x_3 & A(-x_3) & = & a_0 & - & a_1x_3 & + & a_2(x_3)^2 & - & a_3(x_3)^3 & + & a_4(x_3)^4 & - & a_5(x_3)^5 & + & a_6(x_3)^6 & - & a_7(x_3)^7
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$\begin{array}{c|cclllllllllllllll}
 & A(x_0) & = & a_0 & + & a_2(x_0)^2 & + & a_4(x_0)^4 & + & a_6(x_0)^6 & + & a_1x_0 & + & a_3(x_0)^3 & + & a_5(x_0)^5 & + & a_7(x_0)^7 \\
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 & A(x_3) & = & a_0 & + & a_2(x_3)^2 & + & a_4(x_3)^4 & + & a_6(x_3)^6 & + & a_1x_3 & + & a_3(x_3)^3 & + & a_5(x_3)^5 & + & a_7(x_3)^7 \\
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x_4 = -x_0 & A(-x_0) & = & a_0 & + & a_2(x_0)^2 & + & a_4(x_0)^4 & + & a_6(x_0)^6 & - & a_1x_0 & - & a_3(x_0)^3 & - & a_5(x_0)^5 & - & a_7(x_0)^7 \\
x_5 = -x_1 & A(-x_1) & = & a_0 & + & a_2(x_1)^2 & + & a_4(x_1)^4 & + & a_6(x_1)^6 & - & a_1x_1 & - & a_3(x_1)^3 & - & a_5(x_1)^5 & - & a_7(x_1)^7 \\
x_6 = -x_2 & A(-x_2) & = & a_0 & + & a_2(x_2)^2 & + & a_4(x_2)^4 & + & a_6(x_2)^6 & - & a_1x_2 & - & a_3(x_2)^3 & - & a_5(x_2)^5 & - & a_7(x_2)^7 \\
x_7 = -x_3 & A(-x_3) & = & a_0 & + & a_2(x_3)^2 & + & a_4(x_3)^4 & + & a_6(x_3)^6 & - & a_1x_3 & - & a_3(x_3)^3 & - & a_5(x_3)^5 & - & a_7(x_3)^7
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$\begin{array}{llcl}
 & A(x_0) & = & (\, a_0 + a_2(x_0)^2 + a_4(x_0)^4 + a_6(x_0)^6 \,) + x_0(\, a_1 + a_3(x_0)^2 + a_5(x_0)^4 + a_7(x_0)^6 \,) \\
 & A(x_1) & = & (\, a_0 + a_2(x_1)^2 + a_4(x_1)^4 + a_6(x_1)^6 \,) + x_1(\, a_1 + a_3(x_1)^2 + a_5(x_1)^4 + a_7(x_1)^6 \,) \\
 & A(x_2) & = & (\, a_0 + a_2(x_2)^2 + a_4(x_2)^4 + a_6(x_2)^6 \,) + x_2(\, a_1 + a_3(x_2)^2 + a_5(x_2)^4 + a_7(x_2)^6 \,) \\
 & A(x_3) & = & (\, a_0 + a_2(x_3)^2 + a_4(x_3)^4 + a_6(x_3)^6 \,) + x_3(\, a_1 + a_3(x_3)^2 + a_5(x_3)^4 + a_7(x_3)^6 \,) \\
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x_4 = -x_0 & A(-x_0) & = & (\, a_0 + a_2(x_0)^2 + a_4(x_0)^4 + a_6(x_0)^6 \,) - x_0(\, a_1 + a_3(x_0)^2 + a_5(x_0)^4 + a_7(x_0)^6 \,) \\
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x_6 = -x_2 & A(-x_2) & = & (\, a_0 + a_2(x_2)^2 + a_4(x_2)^4 + a_6(x_2)^6 \,) - x_2(\, a_1 + a_3(x_2)^2 + a_5(x_2)^4 + a_7(x_2)^6 \,) \\
x_7 = -x_3 & A(-x_3) & = & (\, a_0 + a_2(x_3)^2 + a_4(x_3)^4 + a_6(x_3)^6 \,) - x_3(\, a_1 + a_3(x_3)^2 + a_5(x_3)^4 + a_7(x_3)^6 \,)
\end{array}$$

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Find 8 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$$

$$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$$

$$\begin{array}{llcl}
 & A(x_0) & = & (\, a_0 + a_2(x_0^2) + a_4(x_0^2)^2 + a_6(x_0^2)^3 \,) + x_0(\, a_1 + a_3(x_0^2) + a_5(x_0^2)^2 + a_7(x_0^2)^3 \,) \\
 & A(x_1) & = & (\, a_0 + a_2(x_1^2) + a_4(x_1^2)^2 + a_6(x_1^2)^3 \,) + x_1(\, a_1 + a_3(x_1^2) + a_5(x_1^2)^2 + a_7(x_1^2)^3 \,) \\
 & A(x_2) & = & (\, a_0 + a_2(x_2^2) + a_4(x_2^2)^2 + a_6(x_2^2)^3 \,) + x_2(\, a_1 + a_3(x_2^2) + a_5(x_2^2)^2 + a_7(x_2^2)^3 \,) \\
 & A(x_3) & = & (\, a_0 + a_2(x_3^2) + a_4(x_3^2)^2 + a_6(x_3^2)^3 \,) + x_3(\, a_1 + a_3(x_3^2) + a_5(x_3^2)^2 + a_7(x_3^2)^3 \,) \\
\hline
x_4 = -x_0 & A(-x_0) & = & (\, a_0 + a_2(x_0^2) + a_4(x_0^2)^2 + a_6(x_0^2)^3 \,) - x_0(\, a_1 + a_3(x_0^2) + a_5(x_0^2)^2 + a_7(x_0^2)^3 \,) \\
x_5 = -x_1 & A(-x_1) & = & (\, a_0 + a_2(x_1^2) + a_4(x_1^2)^2 + a_6(x_1^2)^3 \,) - x_1(\, a_1 + a_3(x_1^2) + a_5(x_1^2)^2 + a_7(x_1^2)^3 \,) \\
x_6 = -x_2 & A(-x_2) & = & (\, a_0 + a_2(x_2^2) + a_4(x_2^2)^2 + a_6(x_2^2)^3 \,) - x_2(\, a_1 + a_3(x_2^2) + a_5(x_2^2)^2 + a_7(x_2^2)^3 \,) \\
x_7 = -x_3 & A(-x_3) & = & (\, a_0 + a_2(x_3^2) + a_4(x_3^2)^2 + a_6(x_3^2)^3 \,) - x_3(\, a_1 + a_3(x_3^2) + a_5(x_3^2)^2 + a_7(x_3^2)^3 \,)
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$$

$$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$$

$$\begin{array}{llclcl}
 & A(x_0) & = & A_{even}(x_0^2) & + & x_0\, A_{odd}(x_0^2) \\
 & A(x_1) & = & A_{even}(x_1^2) & + & x_1\, A_{odd}(x_1^2) \\
 & A(x_2) & = & A_{even}(x_2^2) & + & x_2\, A_{odd}(x_2^2) \\
 & A(x_3) & = & A_{even}(x_3^2) & + & x_3\, A_{odd}(x_3^2) \\
\hline
x_4 = -x_0 & A(-x_0) & = & A_{even}(x_0^2) & - & x_0\, A_{odd}(x_0^2) \\
x_5 = -x_1 & A(-x_1) & = & A_{even}(x_1^2) & - & x_1\, A_{odd}(x_1^2) \\
x_6 = -x_2 & A(-x_2) & = & A_{even}(x_2^2) & - & x_2\, A_{odd}(x_2^2) \\
x_7 = -x_3 & A(-x_3) & = & A_{even}(x_3^2) & - & x_3\, A_{odd}(x_3^2)
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$$

$$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$$

$$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$$

$$\begin{array}{llclcl}
 & A(x_0) & = & A_{even}(x_0^2) & + & x_0\, A_{odd}(x_0^2) \\
 & A(x_1) & = & A_{even}(x_1^2) & + & x_1\, A_{odd}(x_1^2) \\
 & A(x_2) & = & A_{even}(x_2^2) & + & x_2\, A_{odd}(x_2^2) \\
 & A(x_3) & = & A_{even}(x_3^2) & + & x_3\, A_{odd}(x_3^2) \\
\hline
x_4 = -x_0 & A(-x_0) & = & A_{even}(x_0^2) & - & x_0\, A_{odd}(x_0^2) \\
x_5 = -x_1 & A(-x_1) & = & A_{even}(x_1^2) & - & x_1\, A_{odd}(x_1^2) \\
x_6 = -x_2 & A(-x_2) & = & A_{even}(x_2^2) & - & x_2\, A_{odd}(x_2^2) \\
x_7 = -x_3 & A(-x_3) & = & A_{even}(x_3^2) & - & x_3\, A_{odd}(x_3^2)
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

We save roughly half the work.

Given a Polynomial of Degree Bound 2
Find 2 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x}$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$
$x_1 = -x_0$	$A(x_1)$	$=$	a_0	$+$	$a_1 x_1$

STRATEGY: Set $x_{1+i} = -x_i$ for $0 \le i < 1$

Given a Polynomial of Degree Bound 2
Find 2 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x}$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$
$x_1 = -x_0$	$A(-x_0)$	$=$	a_0	$-$	$a_1 x_0$

STRATEGY: Set $x_{1+i} = -x_i$ for $0 \le i < 1$

Given a Polynomial of Degree Bound 2
Find 2 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x}$

$x_0 = 1$	$A(x_0)$	$=$	a_0	$+$	a_1
$x_1 = -1$	$A(x_1)$	$=$	a_0	$-$	a_1

STRATEGY: We will evaluate any polynomial of degree bound 2 at
$x_0 = 1$
$x_1 = -1$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$	$+$	$a_2 (x_0)^2$	$+$	$a_3 (x_0)^3$
	$A(x_1)$	$=$	a_0	$+$	$a_1 x_1$	$+$	$a_2 (x_1)^2$	$+$	$a_3 (x_1)^3$
$x_2 = -x_0$	$A(x_2)$	$=$	a_0	$+$	$a_1 x_2$	$+$	$a_2 (x_2)^2$	$+$	$a_3 (x_2)^3$
$x_3 = -x_1$	$A(x_3)$	$=$	a_0	$+$	$a_1 x_3$	$+$	$a_2 (x_3)^2$	$+$	$a_3 (x_3)^3$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

$$
\begin{array}{c|ccccccccc}
 & A(x_0) & = & a_0 & + & a_1 x_0 & + & a_2(x_0)^2 & + & a_3(x_0)^3 \\
 & A(x_1) & = & a_0 & + & a_1 x_1 & + & a_2(x_1)^2 & + & a_3(x_1)^3 \\
\hline
x_2 = -x_0 & A(-x_0) & = & a_0 & - & a_1 x_0 & + & a_2(x_0)^2 & - & a_3(x_0)^3 \\
x_3 = -x_1 & A(-x_1) & = & a_0 & - & a_1 x_1 & + & a_2(x_1)^2 & - & a_3(x_1)^3
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

$$
\begin{array}{c|ccccccccc}
 & A(x_0) & = & a_0 & + & a_2(x_0)^2 & + & a_1 x_0 & + & a_3(x_0)^3 \\
 & A(x_1) & = & a_0 & + & a_2(x_1)^2 & + & a_1 x_1 & + & a_3(x_1)^3 \\
\hline
x_2 = -x_0 & A(-x_0) & = & a_0 & + & a_2(x_0)^2 & - & a_1 x_0 & - & a_3(x_0)^3 \\
x_3 = -x_1 & A(-x_1) & = & a_0 & + & a_2(x_1)^2 & - & a_1 x_1 & - & a_3(x_1)^3
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

$$
\begin{array}{c|cccccccccccc}
 & A(x_0) & = & (& a_0 & + & a_2(x_0)^2 &) & + & x_0 \, (& a_1 & + & a_3(x_0)^2 &) \\
 & A(x_1) & = & (& a_0 & + & a_2(x_1)^2 &) & + & x_1 \, (& a_1 & + & a_3(x_1)^2 &) \\
\hline
x_2 = -x_0 & A(-x_0) & = & (& a_0 & + & a_2(x_0)^2 &) & - & x_0 \, (& a_1 & + & a_3(x_0)^2 &) \\
x_3 = -x_1 & A(-x_1) & = & (& a_0 & + & a_2(x_1)^2 &) & - & x_1 \, (& a_1 & + & a_3(x_1)^2 &)
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

$$A_{even}(x) = a_0 + a_2 x$$

$$A_{odd}(x) = a_1 + a_3 x$$

$$
\begin{array}{c|cccccccccccc}
 & A(x_0) & = & (& a_0 & + & a_2(x_0^2) &) & + & x_0 \, (& a_1 & + & a_3(x_0^2) &) \\
 & A(x_1) & = & (& a_0 & + & a_2(x_1^2) &) & + & x_1 \, (& a_1 & + & a_3(x_1^2) &) \\
\hline
x_2 = -x_0 & A(-x_0) & = & (& a_0 & + & a_2(x_0^2) &) & - & x_0 \, (& a_1 & + & a_3(x_0^2) &) \\
x_3 = -x_1 & A(-x_1) & = & (& a_0 & + & a_2(x_1^2) &) & - & x_1 \, (& a_1 & + & a_3(x_1^2) &)
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$A_{even}(x) = a_0 + a_2 x$

$A_{odd}(x) = a_1 + a_3 x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\, A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\, A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\, A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\, A_{odd}(x_1^2)$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

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Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$A_{even}(x) = a_0 + a_2 x$

$A_{odd}(x) = a_1 + a_3 x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\, A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\, A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\, A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\, A_{odd}(x_1^2)$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

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Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$A_{even}(x) = a_0 + a_2 x$

$A_{odd}(x) = a_1 + a_3 x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\, A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\, A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\, A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\, A_{odd}(x_1^2)$

Observe that we evaluate both $A_{even}(x)$ and $A_{odd}(x)$ at $x = x_0^2$ and $x = x_1^2$.

But we decided to always evaluate polynomials of degree bound 2 at $x = 1$ and $x = -1$.

So, $x_0^2 = 1 \Rightarrow x_0 = 1$ and $x_1^2 = -1 \Rightarrow x_1 = \sqrt{-1} = i$.

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Given a Polynomial of Degree Bound 4 Find 4 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$A_{even}(x) = a_0 + a_2 x$

$A_{odd}(x) = a_1 + a_3 x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\, A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\, A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\, A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\, A_{odd}(x_1^2)$

So, we evaluate any polynomial of degree bound 4 at

$x_0 = 1,\ x_1 = i$

and

$x_2 = -x_0 = -1,\ x_3 = -x_1 = -i$

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Given a Polynomial of Degree Bound 8 Find 8 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$

$A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$
$A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
	$A(x_2)$	$=$	$A_{even}(x_2^2)$	$+$	$x_2\,A_{odd}(x_2^2)$
	$A(x_3)$	$=$	$A_{even}(x_3^2)$	$+$	$x_3\,A_{odd}(x_3^2)$
$x_4 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_5 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$
$x_6 = -x_2$	$A(-x_2)$	$=$	$A_{even}(x_2^2)$	$-$	$x_2\,A_{odd}(x_2^2)$
$x_7 = -x_3$	$A(-x_3)$	$=$	$A_{even}(x_3^2)$	$-$	$x_3\,A_{odd}(x_3^2)$

Observe that we evaluate both $A_{even}(x)$ and $A_{odd}(x)$ at $x = x_0^2, x = x_1^2, x = x_2^2$ and $x = x_3^2$.

But we decided to always evaluate polynomials of degree bound 4 at $x = 1, x = i, x = -1$ and $x = -i$.

So, $x_0^2 = 1 \Rightarrow x_0 = 1, x_1^2 = i \Rightarrow x_1 = \frac{1+i}{\sqrt{2}}, x_2^2 = -1 \Rightarrow x_2 = i$, and $x_3^2 = -i \Rightarrow x_3 = \frac{-1+i}{\sqrt{2}}$.

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Given a Polynomial of Degree Bound 8 Find 8 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$

$A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$
$A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
	$A(x_2)$	$=$	$A_{even}(x_2^2)$	$+$	$x_2\,A_{odd}(x_2^2)$
	$A(x_3)$	$=$	$A_{even}(x_3^2)$	$+$	$x_3\,A_{odd}(x_3^2)$
$x_4 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_5 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$
$x_6 = -x_2$	$A(-x_2)$	$=$	$A_{even}(x_2^2)$	$-$	$x_2\,A_{odd}(x_2^2)$
$x_7 = -x_3$	$A(-x_3)$	$=$	$A_{even}(x_3^2)$	$-$	$x_3\,A_{odd}(x_3^2)$

So, we evaluate any polynomial of degree bound 8 at

$x_0 = 1,\ x_1 = \frac{1+i}{\sqrt{2}},\ x_2 = i,\ x_3 = \frac{-1+i}{\sqrt{2}}$

and

$x_4 = -x_0 = -1,\ x_5 = -x_1 = -\frac{1+i}{\sqrt{2}},\ x_6 = -x_2 = -i,\ x_7 = -x_3 = -\frac{-1+i}{\sqrt{2}}$

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Given a Polynomial of Degree Bound $n = 2^k$ Find $n = 2^k$ Distinct Points to Efficiently Evaluate it at

degree bound	how did we find the points to evaluate the polynomial at?	the points	point property
2^1	… … …	$1,$ -1	all 2nd roots of unity
2^2	take positive and negative square roots of points used for degree bound 2^1 which are already the 2nd roots of unity	$1,\ i,$ $-1,\ -i$	all 4th roots of unity
2^3	take positive and negative square roots of points used for degree bound 2^2 which are already the 4th roots of unity	$1,\ \frac{1+i}{\sqrt{2}},\ i,\ \frac{-1+i}{\sqrt{2}},$ $-1,\ -\frac{1+i}{\sqrt{2}},\ -i,\ -\frac{-1+i}{\sqrt{2}}$	all 8th roots of unity
2^4	take positive and negative square roots of points used for degree bound 2^3 which are already the 8th roots of unity	$1,\ \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2},\ \ldots,\ \ldots,$ $-1,\ -\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}\right),\ \ldots,\ \ldots$	all 16th roots of unity
… … …	… … …	… … …	… … …
2^{k-1}	take positive and negative square roots of points used for degree bound 2^{k-2} which are already the 2^{k-2}th roots of unity	… … …	all 2^{k-1}th roots of unity
$n = 2^k$	take positive and negative square roots of points used for degree bound 2^{k-1} which are already the 2^{k-1}th roots of unity	… … …	all 2^kth roots of unity (i.e., n^{th} roots of unity)

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How to Find all n^{th} Roots of Unity

The n^{th} roots of unity are: $1,\ \omega_n, (\omega_n)^2, (\omega_n)^3, \ldots\ldots\ldots, (\omega_n)^{n-1}$,

where $\omega_n = \cos\frac{2\Pi}{n} + i\sin\frac{2\Pi}{n} = e^{\frac{2\Pi i}{n}}$ is known as the primitive n^{th} roots of unity.

The result above can be derived using Euler's Formula.

Euler's Formula: For any real number α, $\cos\alpha + i\sin\alpha = e^{i\alpha}$

Euler's formula follows very easily from the following three power series each of which holds for $-\infty < \alpha < +\infty$:

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} - \cdots$$

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \frac{\alpha^9}{9!} - \cdots$$

$$e^\alpha = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \frac{\alpha^5}{5!} + \frac{\alpha^6}{6!} + \frac{\alpha^7}{7!} + \frac{\alpha^8}{8!} + \cdots$$

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How to Find all n^{th} Roots of Unity

Observe that for (any) real numbers α and p,

$$(\cos\alpha + i\sin\alpha)^p = \left(e^{i\alpha}\right)^p = e^{i(p\alpha)} = \cos(p\alpha) + i\sin(p\alpha)$$

Also observe that for any integer k, $\cos(k\times 2\Pi) + i\sin(k\times 2\Pi) = 1 + i\times 0 = 1$

Then the n^{th} root of 1 (unity) is:

$$1^{\frac{1}{n}} = (\cos(k\times 2\Pi) + i\sin(k\times 2\Pi))^{\frac{1}{n}} = \cos\left(k\times\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right)$$

Observe that $\cos\left(k\times\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right)$ takes n distinct values for $0\le k<n$, and then simply repeats those values for $k<0$ and $k\ge n$.

When $k=1$, we have: $\cos\left(k\times\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right) = \cos\left(\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(\frac{2\Pi}{n}\right)$
$= \omega_n =$ primitive n^{th} root of 1.

Clearly, for any k, $\cos\left(k\times\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right) = \left(\cos\left(\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(\frac{2\Pi}{n}\right)\right)^k = (\omega_n)^k$

Hence, $1^{\frac{1}{n}} = \cos\left(k\times\frac{2\Pi}{\mathrm{n}}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right) = (\omega_n)^k$, for $k = 0, 1, 2, \ldots, n-1$.

In other words, the n^{th} roots of 1 (unity) are: $1,\ \omega_n, (\omega_n)^2, (\omega_n)^3, \ldots\ldots\ldots, (\omega_n)^{n-1}$ 41

Coefficient Form ⇒ Point-Value Form

For a polynomial of degree bound $n = 2^k$, we need to apply the trick recursively at most $\log n = k$ times.

We choose $x_0 = 1 = \omega_n^0$ and set $x_j = \omega_n^j$ for $1\le j\le n-1$.

Then we compute the following product:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_{n-1}\end{bmatrix} = \begin{bmatrix} A(1) \\ A(\omega_n) \\ A(\omega_n^2) \\ \cdot \\ \cdot \\ A(\omega_n^{n-1})\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & (\omega_n)^2 & \cdots & (\omega_n)^{n-1} \\ 1 & \omega_n^2 & (\omega_n^2)^2 & \cdots & (\omega_n^2)^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \omega_n^{n-1} & (\omega_n^{n-1})^2 & \cdots & (\omega_n^{n-1})^{n-1}\end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{n-1}\end{bmatrix}$$

The vector $y = (y_0, y_1, \cdots, y_{n-1})$ is called the *discrete Fourier transform* (DFT) of $(a_0, a_1, \cdots, a_{n-1})$.

This method of computing DFT is called the *fast Fourier transform* (FFT) method.

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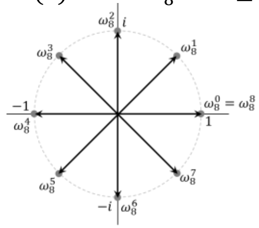
Coefficient Form ⇒ Point-Value Form

Example: For $n = 2^3 = 8$:

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$

We need to evaluate $A(x)$ at $x = \omega_8^i$ for $0\le i<8$.



complex 8^{th} roots of unity

Now $A(x) = A_{even}(x^2) + x\cdot A_{odd}(x^2)$,

where $A_{even}(y) = a_0 + a_2y + a_4y^2 + a_6y^3$
and $A_{odd}(y) = a_1 + a_3y + a_5y^2 + a_7y^3$

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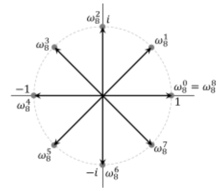
Coefficient Form ⇒ Point-Value Form

Observe that:

$$\begin{aligned}\omega_8^0 &= \omega_8^8 &&= \omega_4^0\\ \omega_8^2 &= \omega_8^{10} &&= \omega_4^1\\ \omega_8^4 &= \omega_8^{12} &&= \omega_4^2\\ \omega_8^6 &= \omega_8^{14} &&= \omega_4^3\end{aligned}$$



Also:

$$\begin{aligned}\omega_8^4 &= -\omega_8^0\\ \omega_8^5 &= -\omega_8^1\\ \omega_8^6 &= -\omega_8^2\\ \omega_8^7 &= -\omega_8^3\end{aligned}$$

$$\begin{aligned}
A(\omega_8^0) &= A_{even}(\omega_8^0) + \omega_8^0\cdot A_{odd}(\omega_8^0) &&= A_{even}(\omega_4^0) + \omega_8^0\cdot A_{odd}(\omega_4^0),\\
A(\omega_8^1) &= A_{even}(\omega_8^2) + \omega_8^1\cdot A_{odd}(\omega_8^2) &&= A_{even}(\omega_4^1) + \omega_8^1\cdot A_{odd}(\omega_4^1),\\
A(\omega_8^2) &= A_{even}(\omega_8^4) + \omega_8^2\cdot A_{odd}(\omega_8^4) &&= A_{even}(\omega_4^2) + \omega_8^2\cdot A_{odd}(\omega_4^2),\\
A(\omega_8^3) &= A_{even}(\omega_8^6) + \omega_8^3\cdot A_{odd}(\omega_8^6) &&= A_{even}(\omega_4^3) + \omega_8^3\cdot A_{odd}(\omega_4^3),
\end{aligned}$$

$$\begin{aligned}
A(\omega_8^4) &= A_{even}(\omega_8^8) + \omega_8^4\cdot A_{odd}(\omega_8^8) &&= A_{even}(\omega_4^0) - \omega_8^0\cdot A_{odd}(\omega_4^0),\\
A(\omega_8^5) &= A_{even}(\omega_8^{10}) + \omega_8^5\cdot A_{odd}(\omega_8^{10}) &&= A_{even}(\omega_4^1) - \omega_8^1\cdot A_{odd}(\omega_4^1),\\
A(\omega_8^6) &= A_{even}(\omega_8^{12}) + \omega_8^6\cdot A_{odd}(\omega_8^{12}) &&= A_{even}(\omega_4^2) - \omega_8^2\cdot A_{odd}(\omega_4^2),\\
A(\omega_8^7) &= A_{even}(\omega_8^{14}) + \omega_8^7\cdot A_{odd}(\omega_8^{14}) &&= A_{even}(\omega_4^3) - \omega_8^3\cdot A_{odd}(\omega_4^3),
\end{aligned}$$

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Coefficient Form ⇒ Point-Value Form

```
Rec-FFT ( ( a_0, a_1, ..., a_{n-1} ) )    { n = 2^k for integer k ≥ 0 }
    if n = 1 then
        return ( a_0 )
    ω_n ← e^{2πi/n}
    ω ← 1
    y^even ← Rec-FFT ( ( a_0, a_2, ..., a_{n-2} ) )
    y^odd ← Rec-FFT ( ( a_1, a_3, ..., a_{n-1} ) )
    for j ← 0 to n/2 − 1 do
        y_j ← y_j^even + ω y_j^odd
        y_{n/2+j} ← y_j^even − ω y_j^odd
        ω ← ω ω_n
    return y
```

Running time:

$$T(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n \log n)$$

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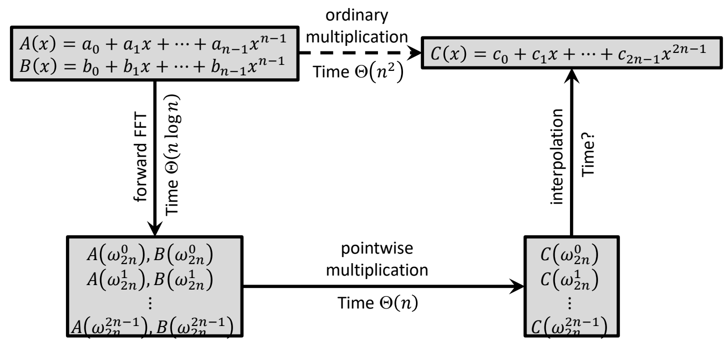
Faster Polynomial Multiplication? (in Coefficient Form)



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Point-Value Form ⇒ Coefficient Form

Given:
$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & (\omega_n)^2 & \cdots & (\omega_n)^{n-1} \\ 1 & \omega_n^2 & (\omega_n^2)^2 & \cdots & (\omega_n^2)^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \omega_n^{n-1} & (\omega_n^{n-1})^2 & \cdots & (\omega_n^{n-1})^{n-1} \end{bmatrix}}_{\substack{V(\omega_n) \\ Vandermonde\ Matrix}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{n-1} \end{bmatrix}}_{\bar{a}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_{n-1} \end{bmatrix}}_{\bar{y}}$$

$$\Rightarrow V(\omega_n) \cdot \bar{a} = \bar{y}$$

We want to solve: $\bar{a} = [V(\omega_n)]^{-1} \cdot \bar{y}$

It turns out that: $[V(\omega_n)]^{-1} = \frac{1}{n} V\left(\frac{1}{\omega_n}\right)$

That means $[V(\omega_n)]^{-1}$ looks almost similar to $V(\omega_n)$!

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Point-Value Form ⇒ Coefficient Form

Show that: $[V(\omega_n)]^{-1} = \frac{1}{n} V\left(\frac{1}{\omega_n}\right)$

Let $U(\omega_n) = \frac{1}{n} V\left(\frac{1}{\omega_n}\right)$

We want to show that $U(\omega_n)V(\omega_n) = I_n$,
where I_n is the $n \times n$ identity matrix.

Observe that for $0 \le j, k \le n-1$, the $(j,k)^{th}$ entries are:

$$[V(\omega_n)]_{jk} = \omega_n^{jk} \quad \text{and} \quad [U(\omega_n)]_{jk} = \frac{1}{n}\omega_n^{-jk}$$

Then entry (p,q) of $U(\omega_n)V(\omega_n)$,

$$[U(\omega_n)V(\omega_n)]_{pq} = \sum_{k=0}^{n-1} [U(\omega_n)]_{pk} [V(\omega_n)]_{kq} = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{k(q-p)}$$

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Point-Value Form ⇒ Coefficient Form

$$[U(\omega_n)V(\omega_n)]_{pq} = \frac{1}{n}\sum_{k=0}^{n-1}\omega_n^{k(q-p)}$$

CASE $p = q$:

$$[U(\omega_n)V(\omega_n)]_{pq} = \frac{1}{n}\sum_{k=0}^{n-1}\omega_n^{0} = \frac{1}{n}\sum_{k=0}^{n-1}1 = \frac{1}{n}\times n = 1$$

CASE $p \neq q$:

$$[U(\omega_n)V(\omega_n)]_{pq} = \frac{1}{n}\sum_{k=0}^{n-1}\left(\omega_n^{q-p}\right)^k = \frac{1}{n}\times\frac{\left(\omega_n^{q-p}\right)^n - 1}{\omega_n^{q-p} - 1}$$
$$= \frac{1}{n}\times\frac{(\omega_n^{n})^{q-p} - 1}{\omega_n^{q-p} - 1} = \frac{1}{n}\times\frac{(1)^{q-p} - 1}{\omega_n^{q-p} - 1} = 0$$

Hence $U(\omega_n)V(\omega_n) = I_n$

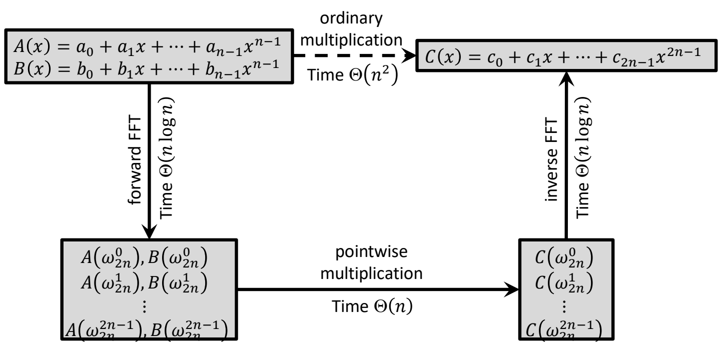
Point-Value Form ⇒ Coefficient Form

We need to compute the following matrix-vector product:

$$\underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{n-1} \end{bmatrix}}_{\bar{a}} = \frac{1}{n}\times\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{\omega_n} & \left(\frac{1}{\omega_n}\right)^2 & \cdots & \left(\frac{1}{\omega_n}\right)^{n-1} \\ 1 & \frac{1}{\omega_n^2} & \left(\frac{1}{\omega_n^2}\right)^2 & \cdots & \left(\frac{1}{\omega_n^2}\right)^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \frac{1}{\omega_n^{n-1}} & \left(\frac{1}{\omega_n^{n-1}}\right)^2 & \cdots & \left(\frac{1}{\omega_n^{n-1}}\right)^{n-1} \end{bmatrix}}_{[V(\omega_n)]^{-1}}\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_{n-1} \end{bmatrix}}_{\bar{y}}$$

This inverse problem is almost similar to the forward problem, and can be solved in $\Theta(n\log n)$ time using the same algorithm as the forward FFT with only minor modifications!

Faster Polynomial Multiplication? (in Coefficient Form)



Two polynomials of degree bound n given in the coefficient form can be multiplied in $\Theta(n\log n)$ time!

Some Applications of Fourier Transform and FFT

- Signal processing
- Image processing
- Noise reduction
- Data compression
- Solving partial differential equation
- Multiplication of large integers
- Polynomial multiplication
- Molecular docking

Some Applications of Fourier Transform and FFT

Jean Baptiste Joseph Fourier

Any periodic signal can be represented as a sum of a series of sinusoidal (sine & cosine) waves. [1807]

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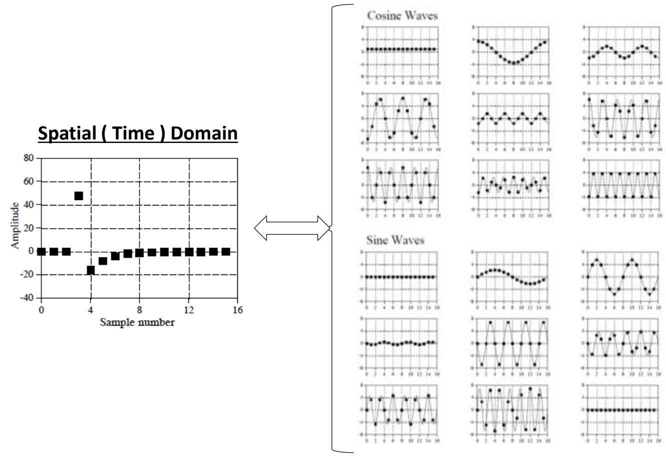
Spatial (Time) Domain ⇔ Frequency Domain

Frequency Domain



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Source: The Scientist and Engineer's Guide to Digital Signal Processing by Steven W. Smith

Spatial (Time) Domain ⇔ Frequency Domain

$s_6(x)$

Function $s(x)$ (in red) is a sum of six sine functions of different amplitudes and harmonically related frequencies. The Fourier transform, $S(f)$ (in blue), which depicts amplitude vs frequency, reveals the 6 frequencies and their amplitudes.

Source: http://en.wikipedia.org/wiki/Fourier_series#mediaviewer/File:Fourier_series_and_transform.gif (uploaded by Bob K.) 57

Spatial (Time) Domain ⇔ Frequency Domain

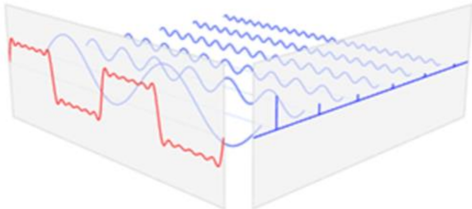
$s_6(x)$

$$a_n \cos(nx) + b_n \sin(nx)$$

Function $s(x)$ (in red) is a sum of six sine functions of different amplitudes and harmonically related frequencies. The Fourier transform, $S(f)$ (in blue), which depicts amplitude vs frequency, reveals the 6 frequencies and their amplitudes.

Source: http://en.wikipedia.org/wiki/Fourier_series#mediaviewer/File:Fourier_series_and_transform.gif (uploaded by Bob K.) 59

Spatial (Time) Domain ⇔ Frequency Domain



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Spatial (Time) Domain ⇔ Frequency Domain

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Function *s(x)* (in red) is a sum of six sine functions of different amplitudes and harmonically related frequencies. The Fourier transform, *S(f)* (in blue), which depicts amplitude vs frequency, reveals the 6 frequencies and their amplitudes.

Source: http://en.wikipedia.org/wiki/Fourier_series#mediaviewer/File:Fourier_series_and_transform.gif (uploaded by Bob K.)

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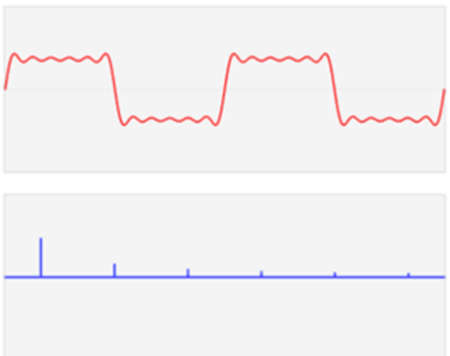
Spatial (Time) Domain ⇔ Frequency Domain



Function *s(x)* (in red) is a sum of six sine functions of different amplitudes and harmonically related frequencies. The Fourier transform, *S(f)* (in blue), which depicts amplitude vs frequency, reveals the 6 frequencies and their amplitudes.

Source: http://en.wikipedia.org/wiki/Fourier_series#mediaviewer/File:Fourier_series_and_transform.gif (uploaded by Bob K.)

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Spatial (Time) Domain ⇔ Frequency Domain (Fourier Transforms)

Let $s(t)$ be a signal specified in the time domain.

The strength of $s(t)$ at frequency f is given by:

$$S(f) = \int_{-\infty}^{\infty} s(t) \cdot e^{-2\pi i f t}\, dt$$

Evaluating this integral for all values of f gives the frequency domain function.

Now $s(t)$ can be retrieved by summing up the signal strengths at all possible frequencies:

$$s(t) = \int_{-\infty}^{\infty} S(f) \cdot e^{2\pi i f t}\, df$$

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Why do the Transforms Work?

Let's try to get a little intuition behind why the transforms work.
We will look at a very simple example.

Suppose: $s(t) = \cos(2\pi h \cdot t)$

$$\frac{1}{T}\int_{-T}^{T} s(t) \cdot e^{-2\pi i f t}\,dt = \begin{cases} 1 + \frac{\sin(4\pi f T)}{4\pi f T}, & \text{if } f = h, \\ \frac{\sin(2\pi(h-f)T)}{2\pi(h-f)T} + \frac{\sin(2\pi(h+f)T)}{2\pi(h+f)T}, & \text{otherwise.} \end{cases}$$

$$\Rightarrow \lim_{T\to\infty}\left(\frac{1}{T}\int_{-T}^{T} s(t) \cdot e^{-2\pi i f t}\,dt\right) = \begin{cases} 1, & \text{if } f = h, \\ 0, & \text{otherwise.} \end{cases}$$

So, the transform can detect if $f = h$!

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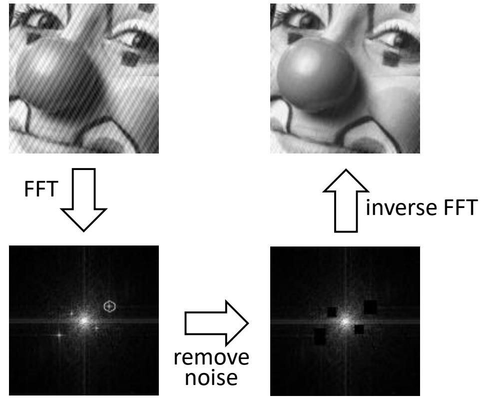
Noise Reduction



Source: http://www.mediacy.com/index.aspx?page=AH_FFTExample

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Data Compression

– Discrete Cosine Transforms (DCT) are used for lossy data compression (e.g., MP3, JPEG, MPEG)

– DCT is a Fourier-related transform similar to DFT (Discrete Fourier Transform) but uses only real data (uses cosine waves only instead of both cosine and sine waves)

– Forward DCT transforms data from spatial to frequency domain

– Each frequency component is represented using a fewer number of bits (i.e., truncated / quantized)

– Low amplitude high frequency components are also removed

– Inverse DCT then transforms the data back to spatial domain

– The resulting image compresses better

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Data Compression

Transformation to frequency domain using cosine transforms work in the same way as the Fourier transform.

Suppose: $s(t) = \cos(2\pi h \cdot t)$

$$\frac{1}{T}\int_{-T}^{T} s(t) \cdot \cos(2\pi f t)\,dt = \begin{cases} 1 + \frac{\sin(4\pi f T)}{4\pi f T}, & \text{if } f = h, \\ \frac{\sin(2\pi(h-f)T)}{2\pi(h-f)T} + \frac{\sin(2\pi(h+f)T)}{2\pi(h+f)T}, & \text{otherwise.} \end{cases}$$

$$\Rightarrow \lim_{T\to\infty}\left(\frac{1}{T}\int_{-T}^{T} s(t) \cdot \cos(2\pi f t)\,dt\right) = \begin{cases} 1, & \text{if } f = h, \\ 0, & \text{otherwise.} \end{cases}$$

So, this transform can also detect if $f = h$.

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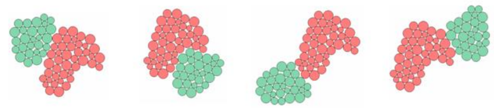
Protein-Protein Docking

- Knowledge of complexes is used in
  - Drug design
  - Structure function analysis
  - Studying molecular assemblies
  - Protein interactions
- Protein-Protein Docking: Given two proteins, find the best relative transformation and conformations to obtain a stable complex.



- Docking is a hard problem
  - Search space is huge (6D for rigid proteins)
  - Protein flexibility adds to the difficulty

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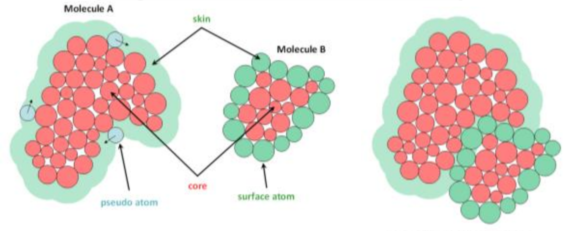
Shape Complementarity

[Wang'91, Katchalski-Katzir et al.'92, Chen et al.'03]



a possible docking solution

To maximize skin-skin overlaps and minimize core-core overlaps

- assign positive real weights to skin atoms
- assign positive imaginary weights to core atoms

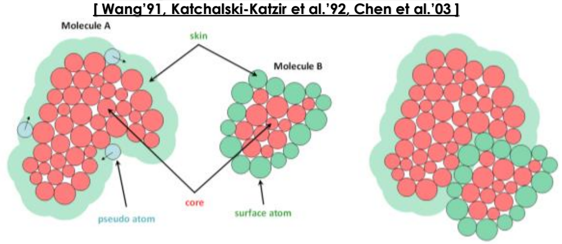
Let A' denote molecule A with the pseudo skin atoms.

For $P \in \{A', B\}$ with M_P atoms, affinity function: $f_P(x) = \sum_{k=1}^{M_P} w_k \cdot g_k(x)$

Here $g_k(x)$ is a Gaussian representation of atom k, and w_k its weight.

Shape Complementarity

[Wang'91, Katchalski-Katzir et al.'92, Chen et al.'03]



a possible docking solution

Let A' denote molecule A with the pseudo skin atoms.

For $P \in \{A', B\}$ with M_P atoms, affinity function:

$$f_P(x) = \sum_{k=1}^{M_P} w_k \cdot g_k(x)$$

For rotation r and translation t of molecule B (i.e., $B_{t,r}$),

the interaction score, $F_{A,B}(t,r) = \int_x f_{A'}(x) f_{B_{t,r}}(x)\, dx$

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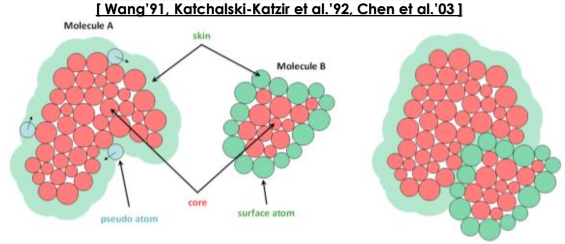
Shape Complementarity

[Wang'91, Katchalski-Katzir et al.'92, Chen et al.'03]



a possible docking solution

For rotation r and translation t of molecule B (i.e., $B_{t,r}$),

the interaction score, $F_{A,B}(t,r) = \int_x f_{A'}(x) f_{B_{t,r}}(x)\, dx$

$Re\left(F_{A,B}(t,r)\right)$ = skin-skin overlap score − core-core overlap score

$Im\left(F_{A,B}(t,r)\right)$ = skin-core overlap score

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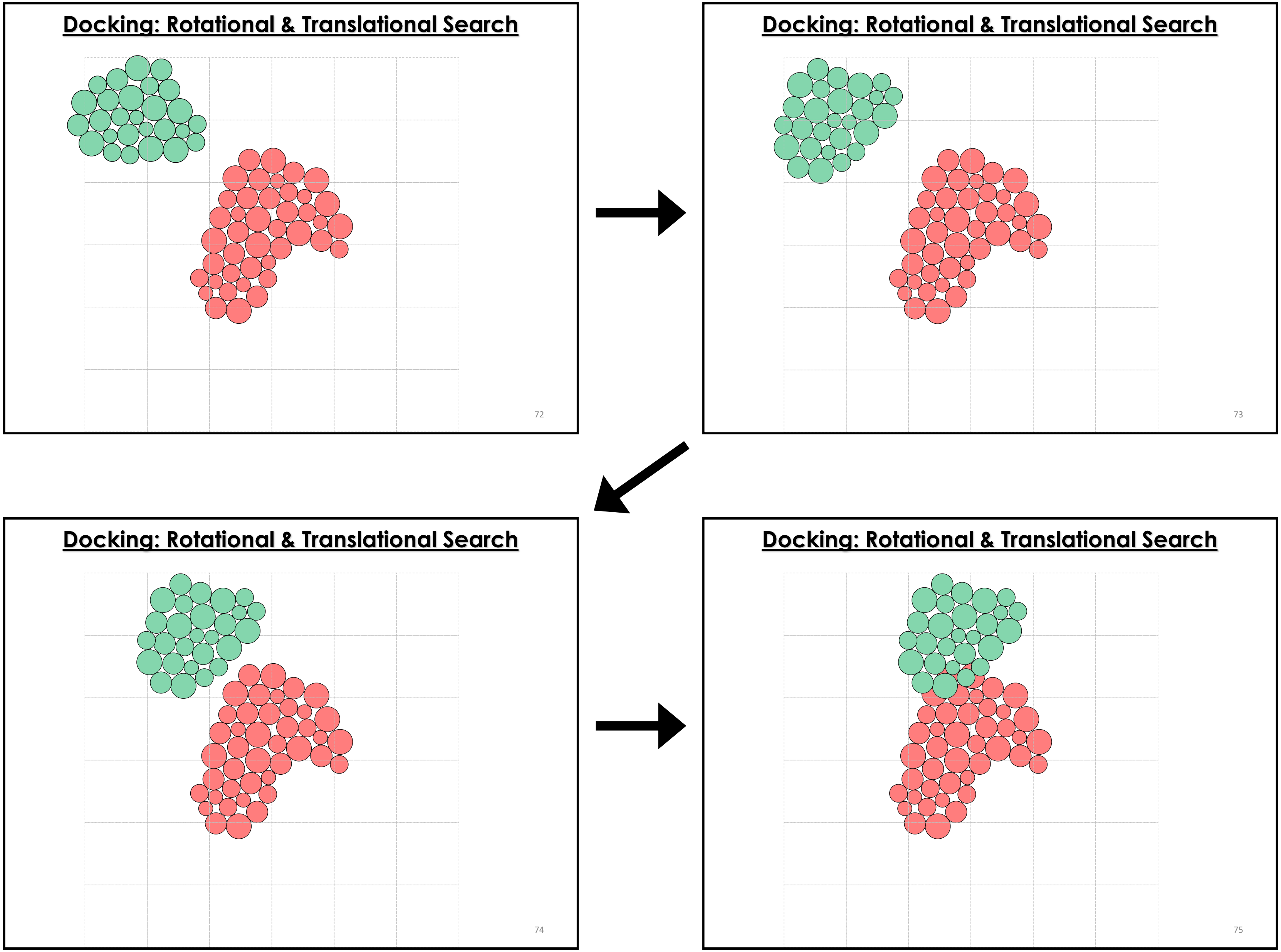
Docking: Rotational & Translational Search
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Docking: Rotational & Translational Search
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Docking: Rotational & Translational Search
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Docking: Rotational & Translational Search
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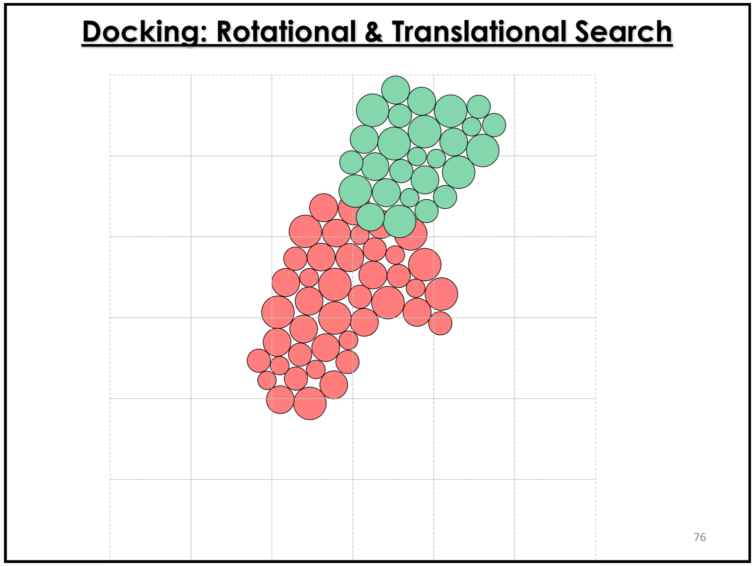
Docking: Rotational & Translational Search
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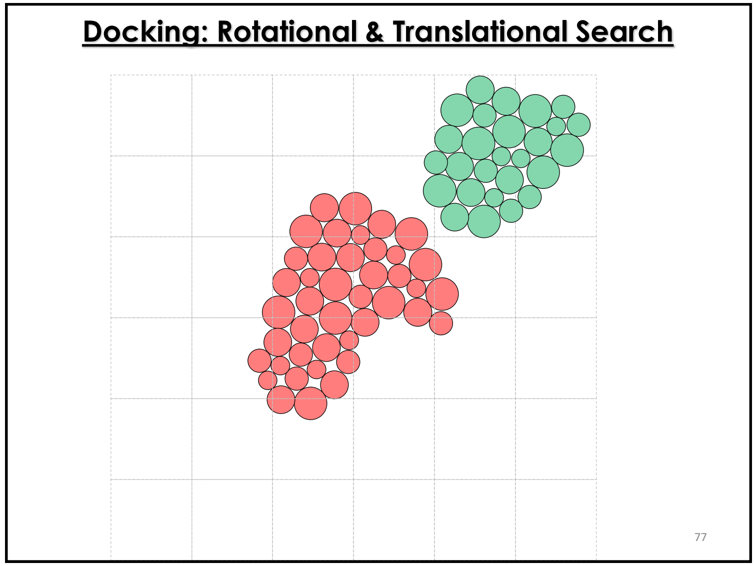
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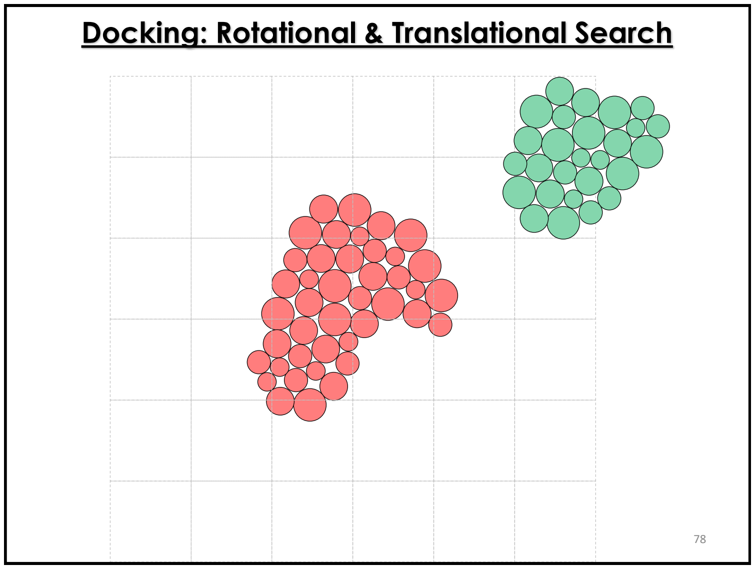
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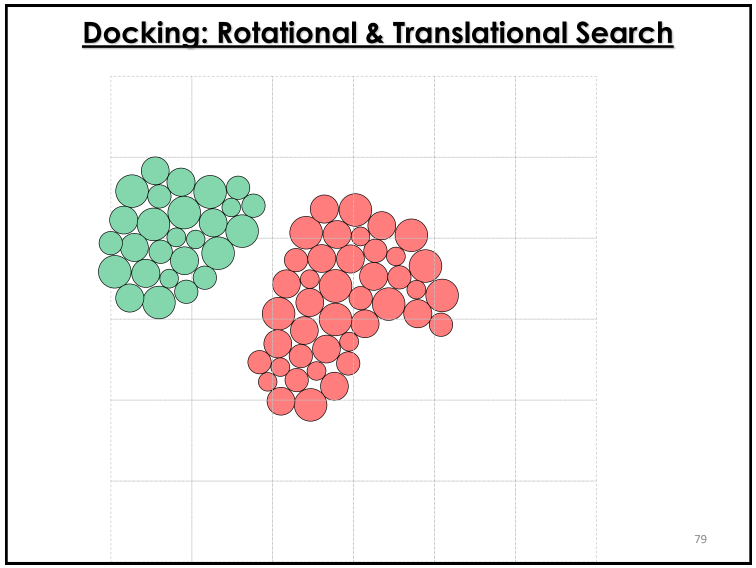
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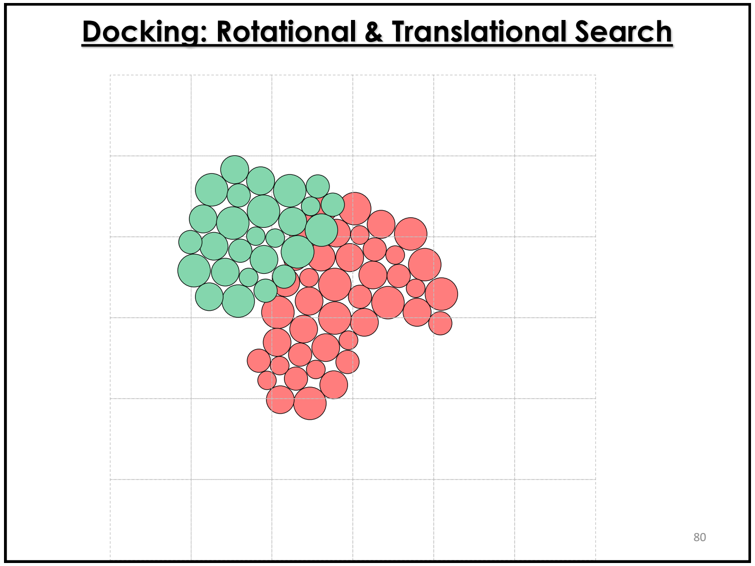
Docking: Rotational & Translational Search
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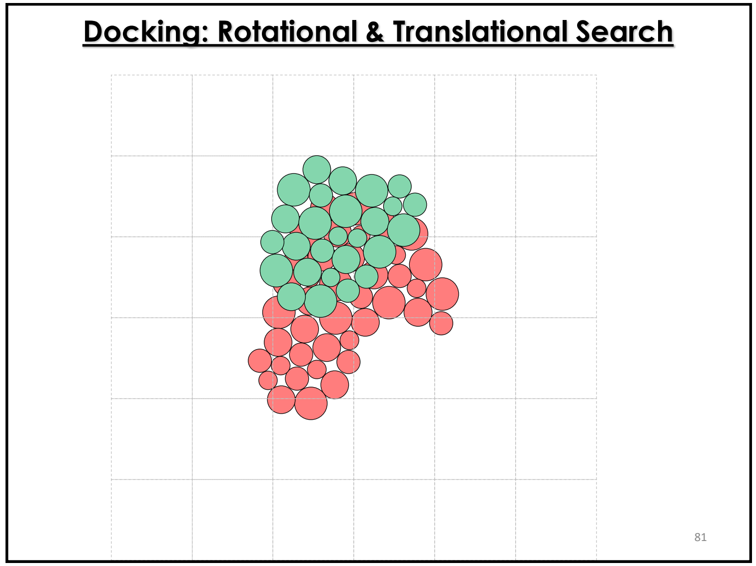
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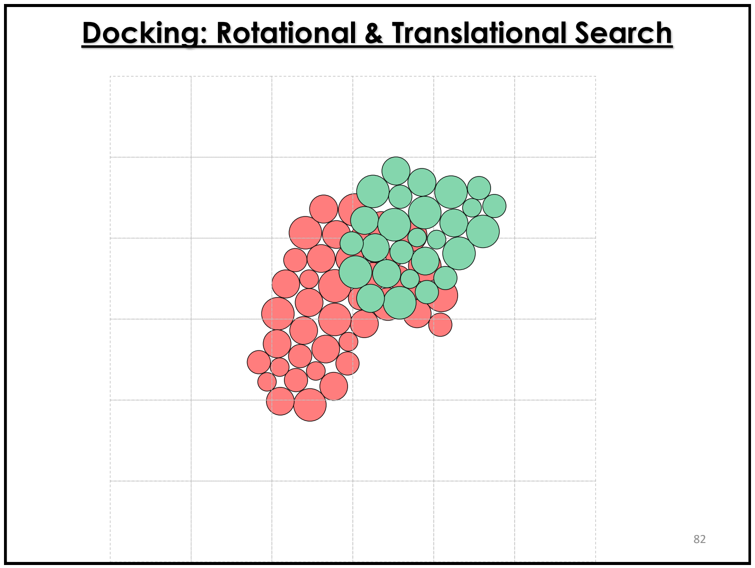
Docking: Rotational & Translational Search
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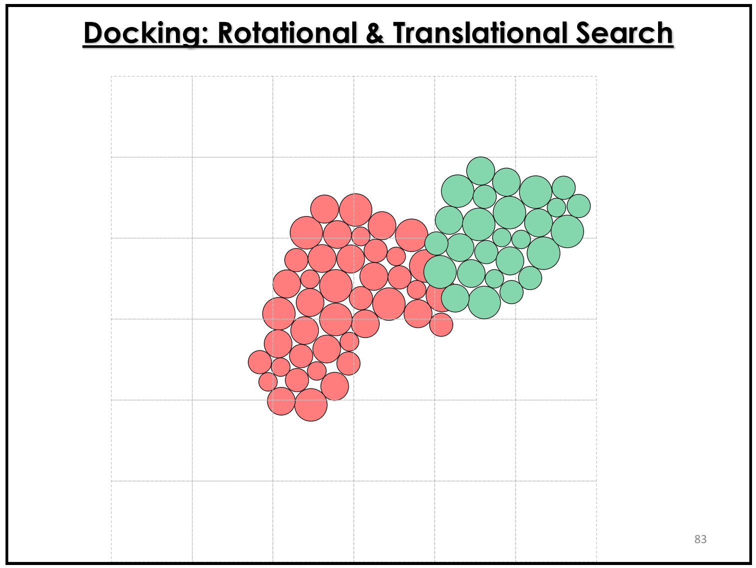
Docking: Rotational & Translational Search
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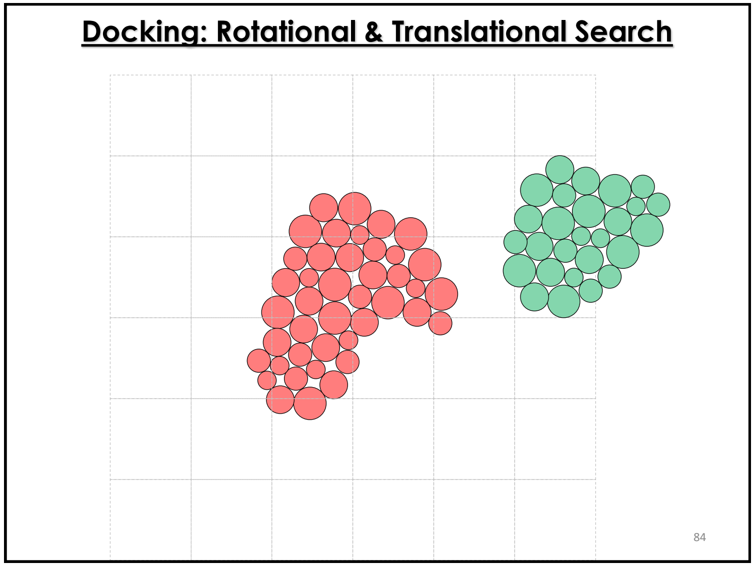
Docking: Rotational & Translational Search
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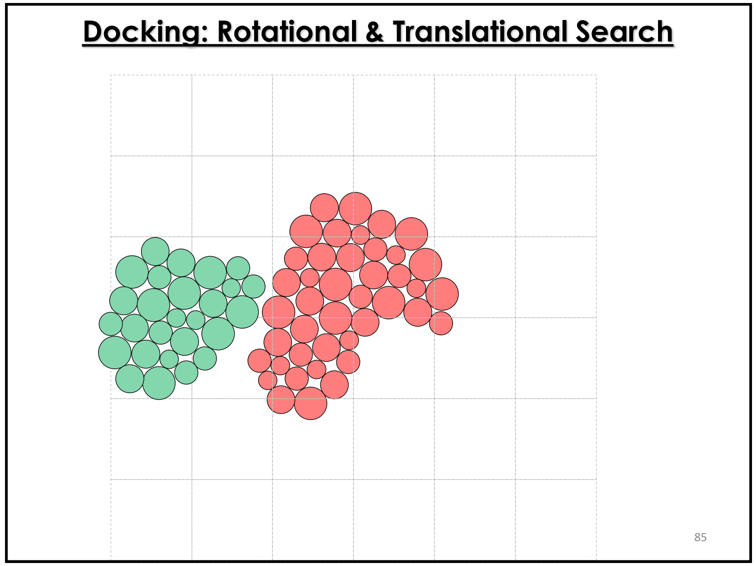
Docking: Rotational & Translational Search
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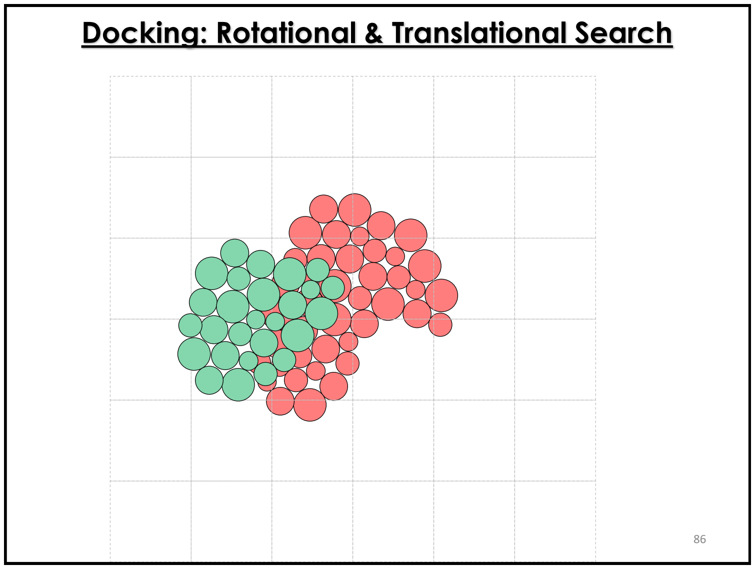
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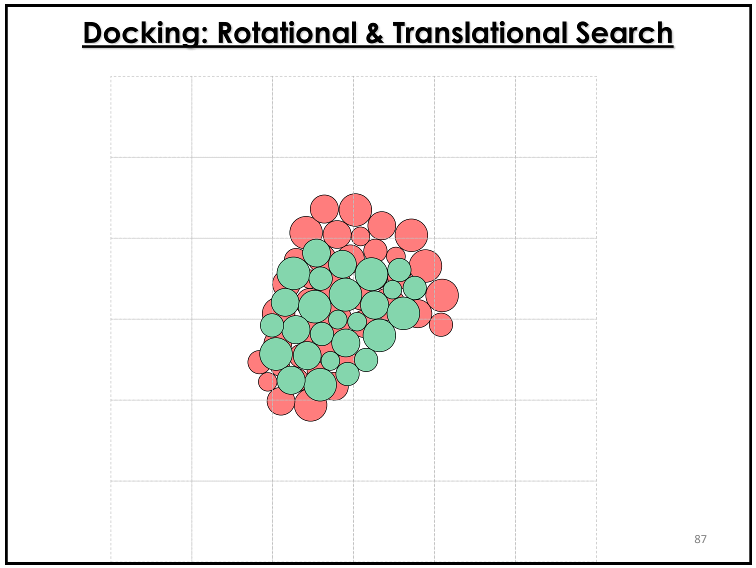
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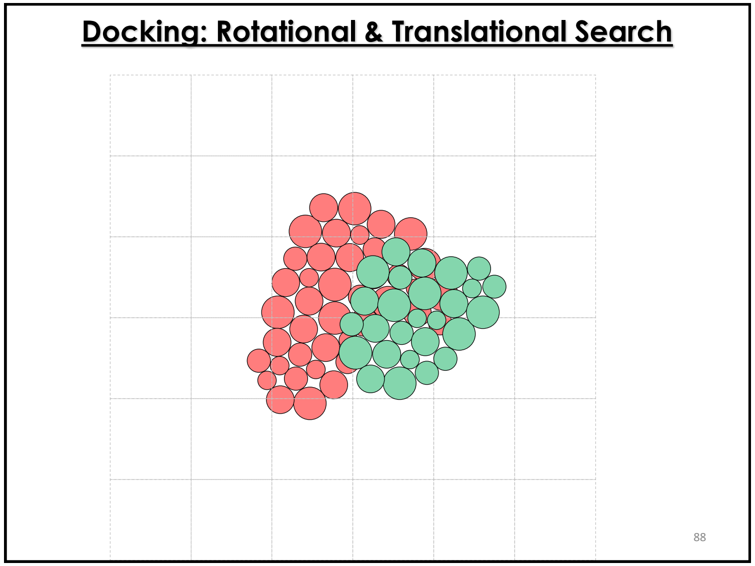
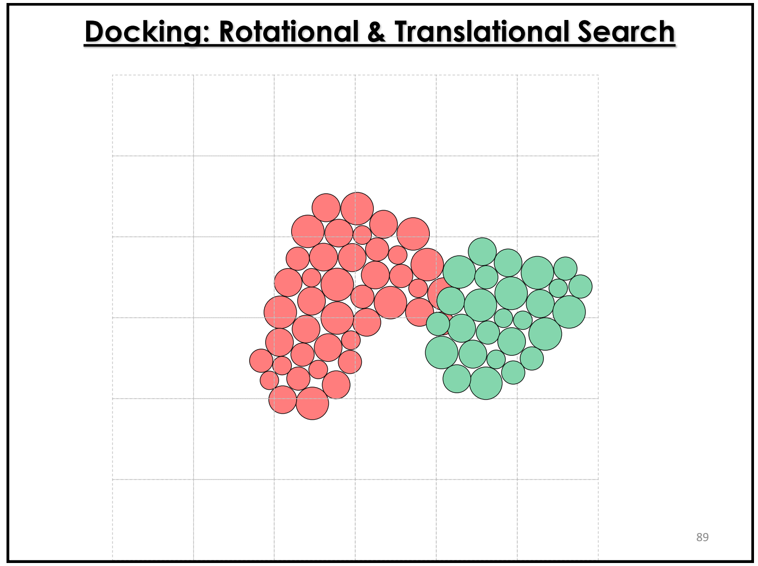
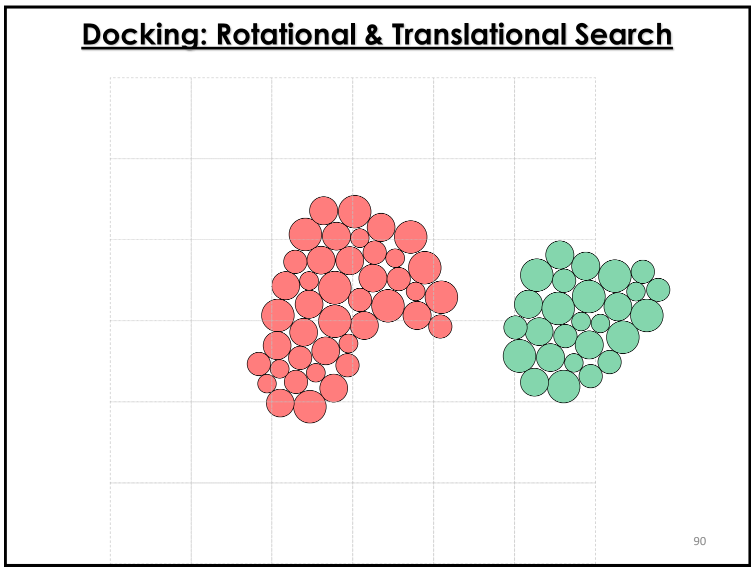
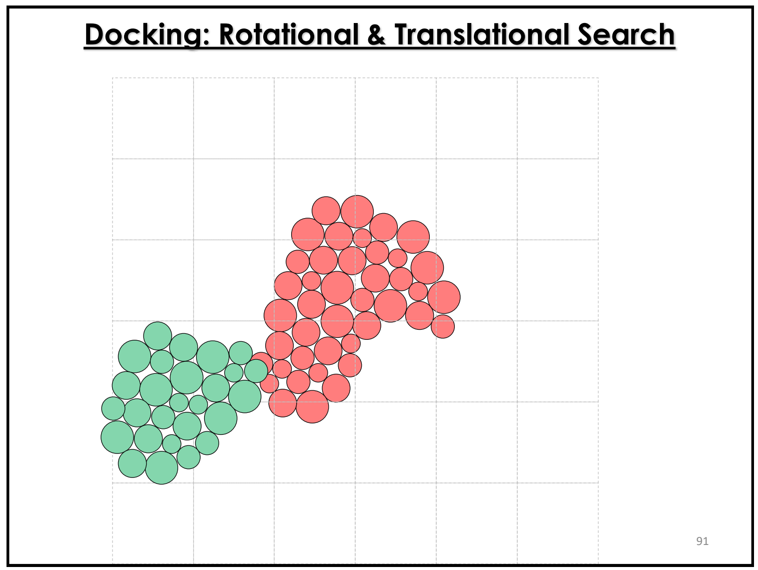
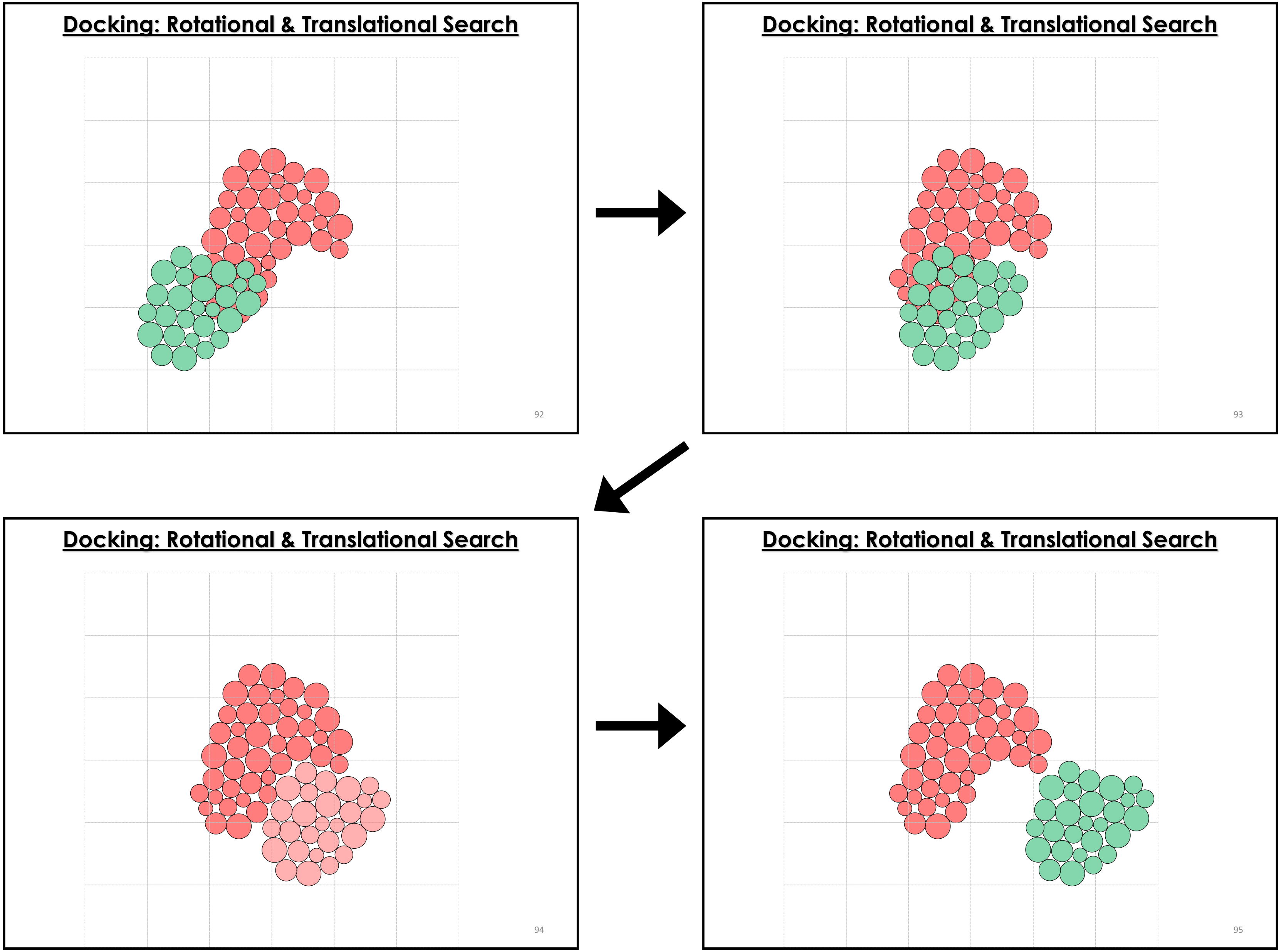
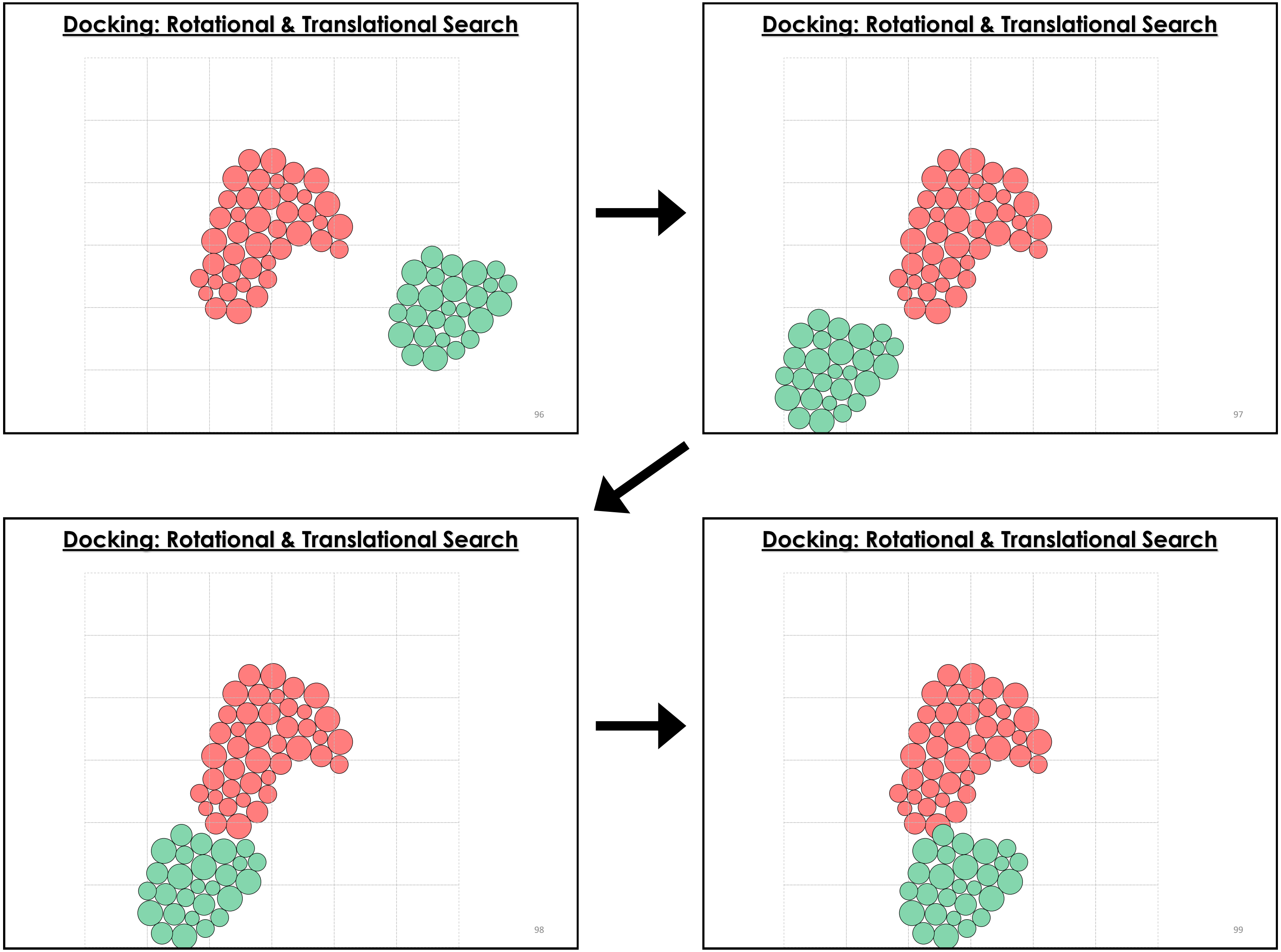

Docking: Rotational & Translational Search
88

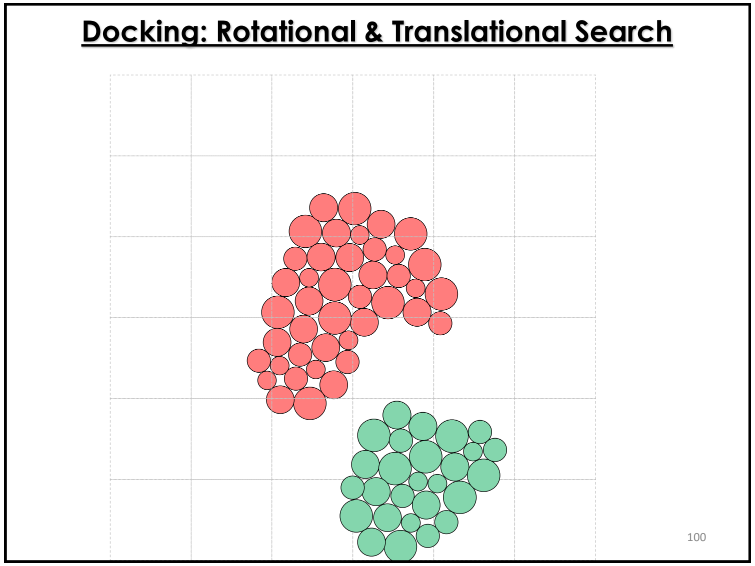

Docking: Rotational & Translational Search
89

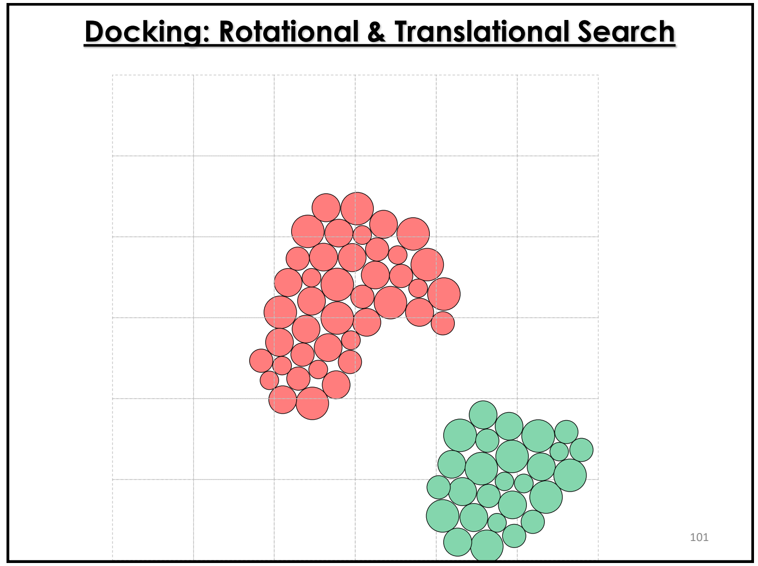

Docking: Rotational & Translational Search
90

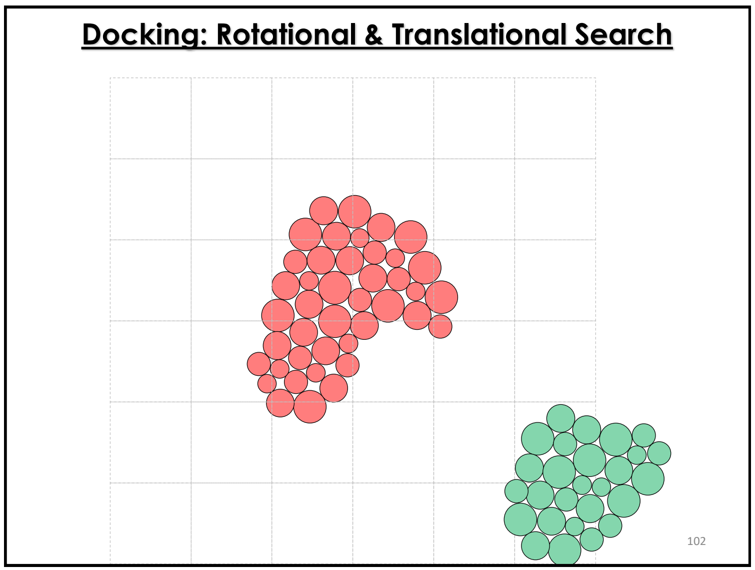

Docking: Rotational & Translational Search
91

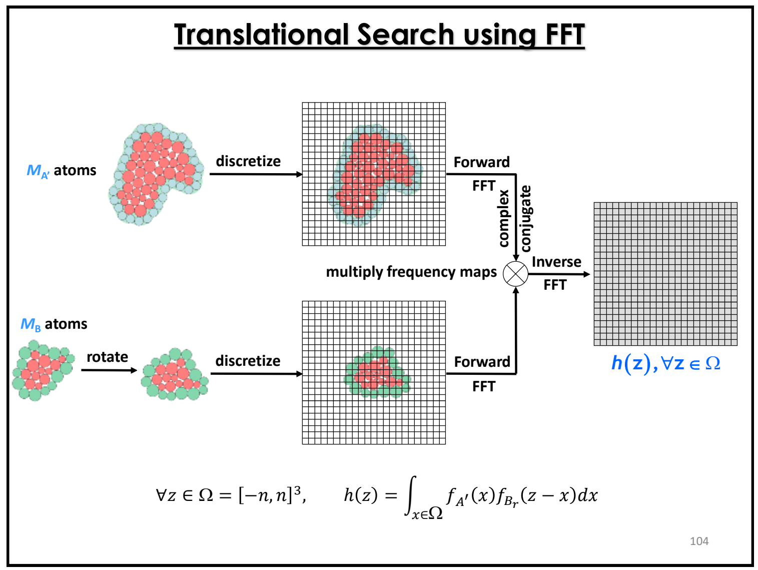

Docking: Rotational & Translational Search
92
Docking: Rotational & Translational Search
93
Docking: Rotational & Translational Search
94
Docking: Rotational & Translational Search
95

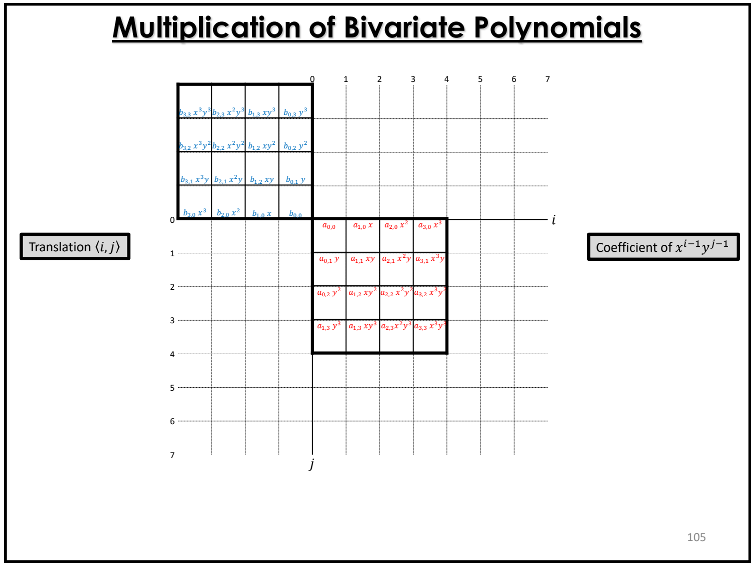

Docking: Rotational & Translational Search
96
Docking: Rotational & Translational Search
97
Docking: Rotational & Translational Search
98
Docking: Rotational & Translational Search
99

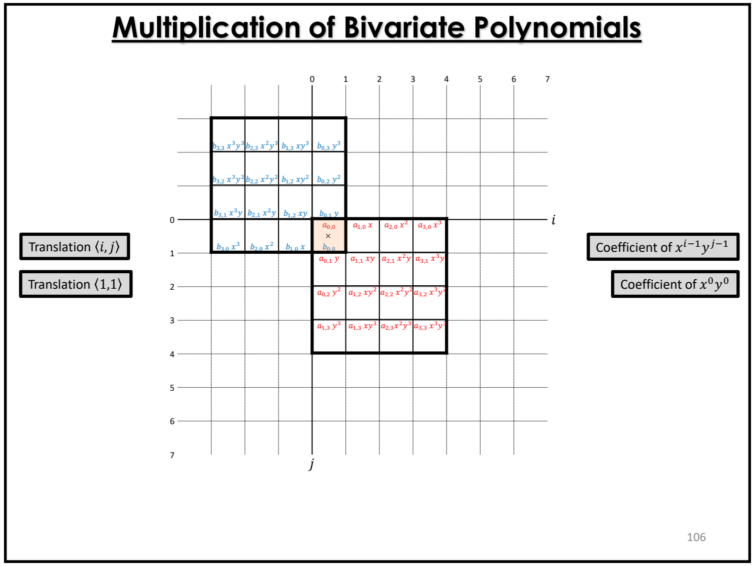

Docking: Rotational & Translational Search
100

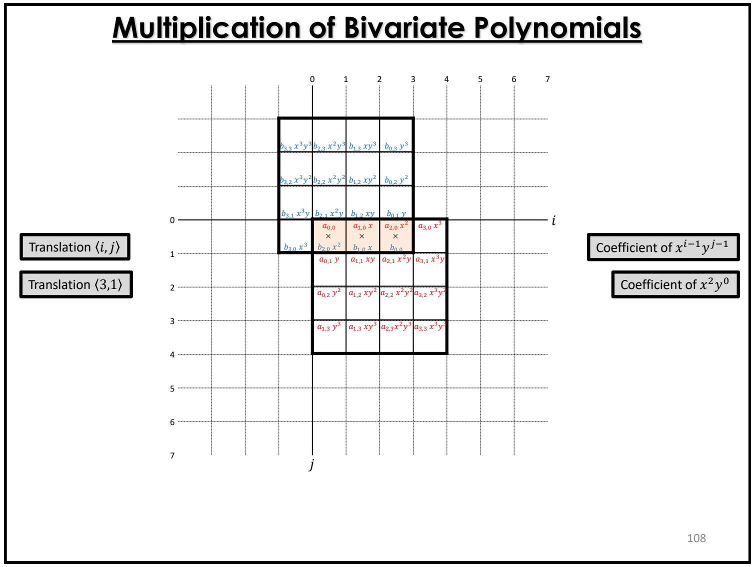

Docking: Rotational & Translational Search
101

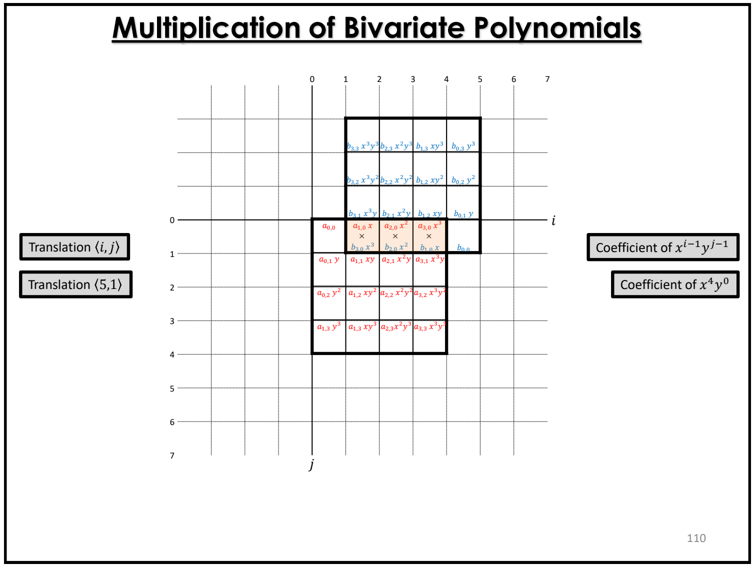

Docking: Rotational & Translational Search
102


Translational Search using FFT
$M_{A'}$ atoms
discretize
Forward FFT
complex conjugate
multiply frequency maps
Inverse FFT
M_B atoms
rotate
discretize
Forward FFT
$h(z), \forall z \in \Omega$
$\forall z \in \Omega = [-n, n]^3, \quad h(z) = \int_{x \in \Omega} f_{A'}(x) f_{B_r}(z - x) dx$
104


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Coefficient of $x^{i-1}y^{j-1}$
105


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 1,1 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^0y^0
106

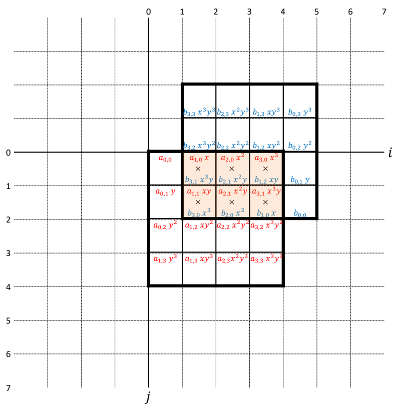

Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 3,1 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^2y^0
108


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 5,1 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^4y^0
110

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 5,2 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

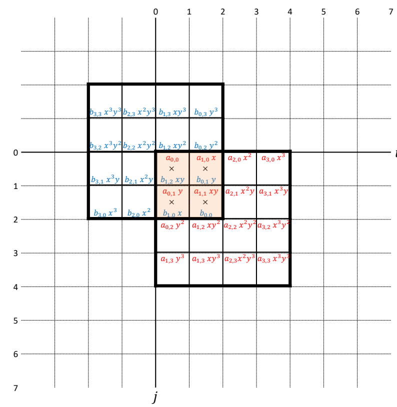
Coefficient of x^4y^1

115

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 2,2 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

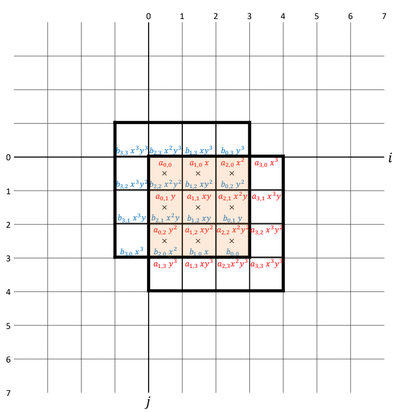
Coefficient of x^1y^1

118

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 3,3 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

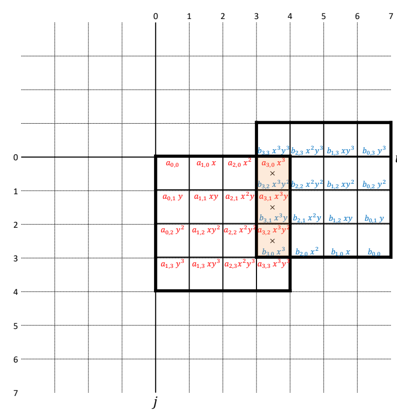
Coefficient of x^2y^2

122

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 7,3 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

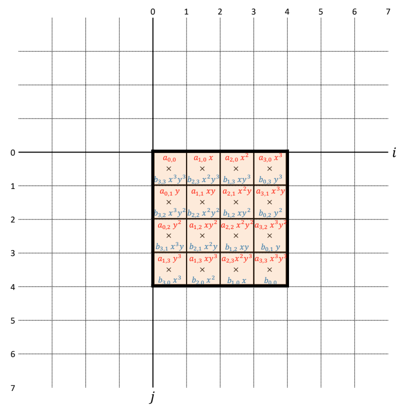
Coefficient of x^6y^2

126

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 4,4 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

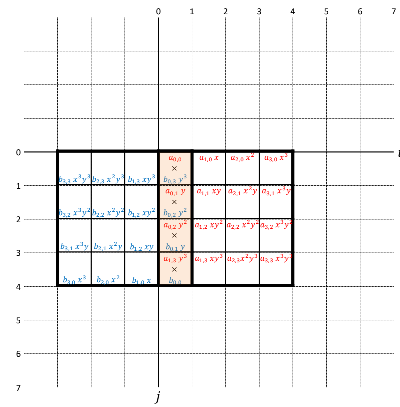
Coefficient of x^3y^3

130

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 1,4 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

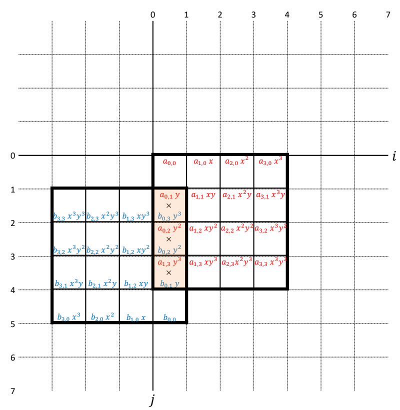
Coefficient of x^0y^3

133

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

Translation $\langle 1,5 \rangle$



Coefficient of $x^{i-1}y^{j-1}$

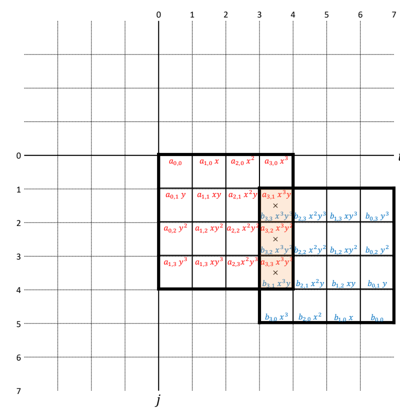
Coefficient of x^0y^4

134

Multiplication of Bivariate Polynomials

Translation $\langle i, j \rangle$

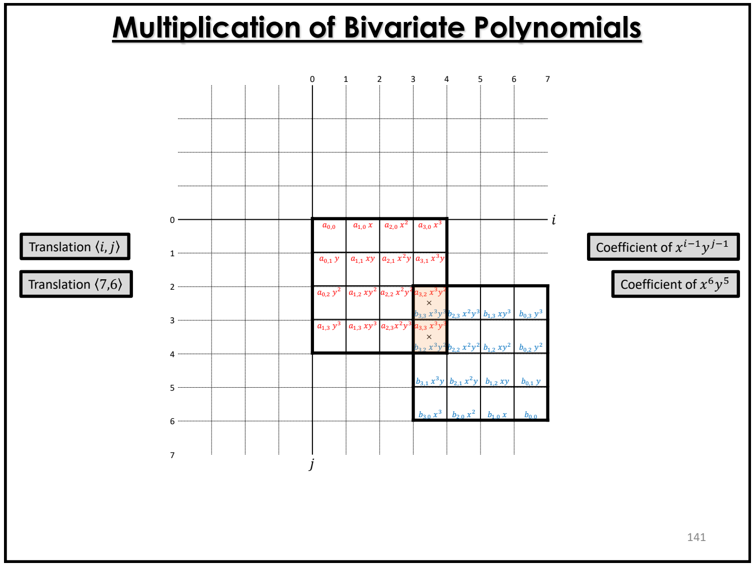
Translation $\langle 7,5 \rangle$

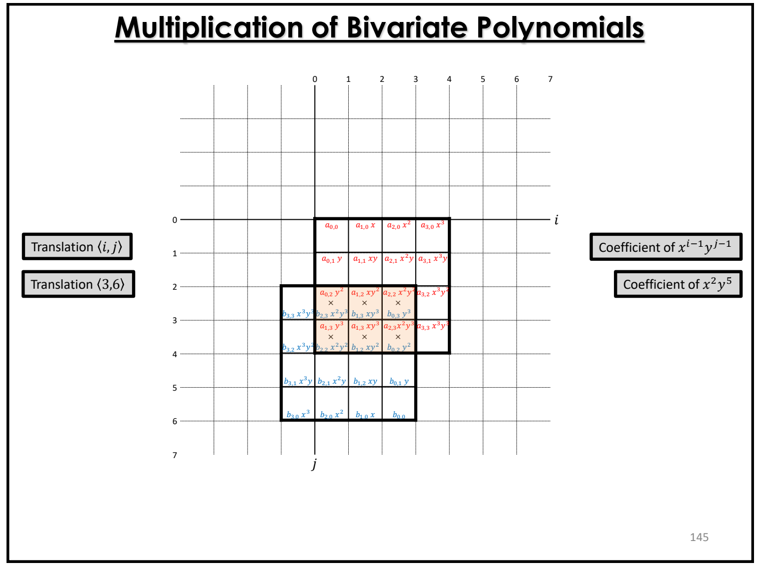


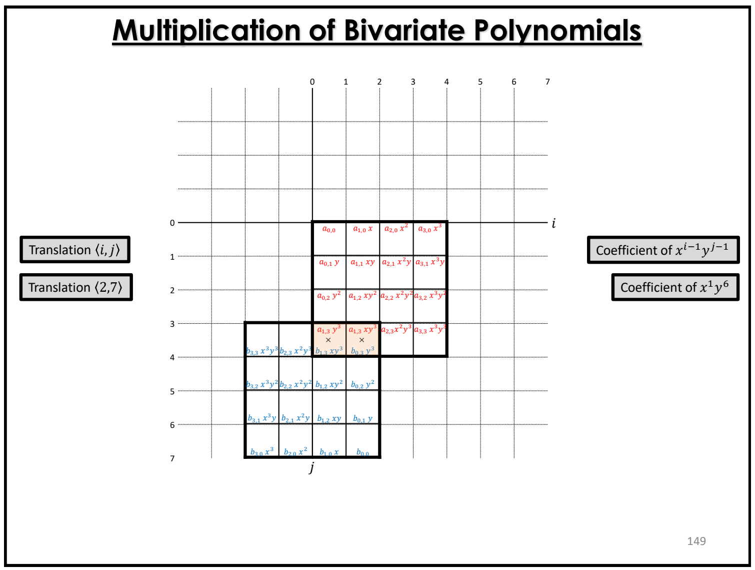
Coefficient of $x^{i-1}y^{j-1}$

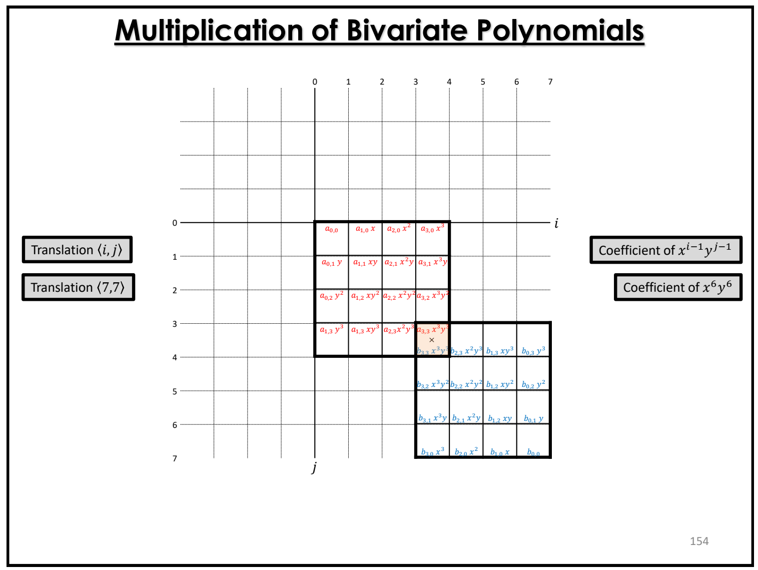
Coefficient of x^6y^4

140


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 7,6 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^6y^5
141


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 3,6 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^2y^5
145


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 2,7 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^1y^6
149


Multiplication of Bivariate Polynomials
Translation $\langle i, j \rangle$
Translation $\langle 7,7 \rangle$
Coefficient of $x^{i-1}y^{j-1}$
Coefficient of x^6y^6
154