7.3 The Jacobi and Gauss-Siedel Iterative Techniques

In this section we describe the Jacobi and the Gauss-Seidel iterative methods, classic methods that date to the late eighteenth century. Iterative techniques are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques such as Gaussian elimination. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type arise frequently in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations.

An iterative technique to solve the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ starts with an initial approximation $\mathbf{x}^{(0)}$ to the solution $\mathbf{x}$ and generates a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converges to $\mathbf{x}$.

Jacobi's Method

The **Jacobi iterative method** is obtained by solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n.$$

For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n. \tag{7.5}$$

Example 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1 &: \; 10x_1 - x_2 + 2x_3 = 6, \\
E_2 &: \; -x_1 + 11x_2 - x_3 + 3x_4 = 25, \\
E_3 &: \; 2x_1 - x_2 + 10x_3 - x_4 = -11, \\
E_4 &: \; 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Carl Gustav Jacob Jacobi (1804–1851) was initially recognized for his work in the area of number theory and elliptic functions, but his mathematical interests and abilities were very broad. He had a strong personality that was influential in establishing a research-oriented attitude that became the nucleus of a revival of mathematics at German universities in the 19th century.

Solution We first solve equation E_i for x_i, for each $i = 1, 2, 3, 4$, to obtain

$$\begin{aligned}
x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}, \\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}, \\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}, \\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}.
\end{aligned}$$

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$\begin{aligned}
x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} = 0.6000, \\
x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727, \\
x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000, \\
x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} = 1.8750.
\end{aligned}$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 7.1.

Table 7.1

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after ten iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} < 10^{-3}.$$

In fact, $\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 0.0002$. ∎

In general, iterative techniques for solving linear systems involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for some fixed matrix T and vector $\mathbf{c}$. After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c},$$

for each $k = 1, 2, 3, \ldots$. This should be reminiscent of the fixed-point iteration studied in Chapter 2.

The Jacobi method can be written in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ by splitting A into its diagonal and off-diagonal parts. To see this, let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A. With this notation,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is split into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$= D - L - U.$$

The equation $A\mathbf{x} = \mathbf{b}$, or $(D - L - U)\mathbf{x} = \mathbf{b}$, is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists, that is, if $a_{ii} \neq 0$ for each i, then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, \ldots. \tag{7.6}$$

Introducing the notation $T_j = D^{-1}(L + U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$ gives the Jacobi technique the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j. \tag{7.7}$$

In practice, Eq. (7.5) is used in computation and Eq. (7.7) for theoretical purposes.

Example 2 Express the Jacobi iteration method for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6,\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25,\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11,\\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Solution We saw in Example 1 that the Jacobi method for this system has the form

$$\begin{aligned}
x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5},\\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11},\\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10},\\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}.
\end{aligned}$$

Hence we have

$$T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}.$$ ■

Algorithm 7.1 implements the Jacobi iterative technique.

ALGORITHM 7.1

Jacobi Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i \le n$ of $\mathbf{b}$; the entries XO_i, $1 \le i \le n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \le N$) do Steps 3–6.

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1\\ j\ne i}}^{n} (a_{ij}XO_j) + b_i\right].$$

Step 4 If $\|\mathbf{x} - \mathbf{XO}\| < TOL$ then OUTPUT ($x_1, \ldots, x_n$);
(*The procedure was successful.*)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful.*)
STOP. ∎

Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$. To speed convergence, the equations should be arranged so that a_{ii} is as large as possible. This subject is discussed in more detail later in this chapter.

Another possible stopping criterion in Step 4 is to iterate until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$$

is smaller than some prescribed tolerance. For this purpose, any convenient norm can be used, the usual being the l_∞ norm.

The *NumericalAnalysis* subpackage of the Maple *Student* package implements the Jacobi iterative method. To illustrate this with our example we first enter both *NumericalAnalysis* and *LinearAlgebra*.

with(*Student*[*NumericalAnalysis*]): *with*(*LinearAlgebra*):

Colons are used at the end of the commands to suppress output for both packages. Enter the matrix with

$A := Matrix([[10, -1, 2, 0, 6], [-1, 11, -1, 3, 25], [2, -1, 10, -1, -11], [0, 3, -1, 8, 15]])$

The following command gives a collection of output that is in agreement with the results in Table 7.1.

IterativeApproximate(*A*, *initialapprox* = *Vector*([0., 0., 0., 0.]), *tolerance* = 10^{-3}, *maxiterations* = 20, *stoppingcriterion* = *relative*(*infinity*), *method* = *jacobi*, *output* = *approximates*)

If the option *output* = *approximates* is omitted, then only the final approximation result is output. Notice that the initial approximations was specified by [0., 0., 0., 0.], with decimal points placed after the entries. This was done so that Maple will give the results as 10-digit decimals. If the specification had simply been [0, 0, 0, 0], the output would have been given in fractional form.

Phillip Ludwig Seidel (1821–1896) worked as an assistant to Jacobi solving problems on systems of linear equations that resulted from Gauss's work on least squares. These equations generally had off-diagonal elements that were much smaller than those on the diagonal, so the iterative methods were particularly effective. The iterative techniques now known as Jacobi and Gauss-Seidel were both known to Gauss before being applied in this situation, but Gauss's results were not often widely communicated.

The Gauss-Seidel Method

A possible improvement in Algorithm 7.1 can be seen by reconsidering Eq. (7.5). The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$. But, for $i > 1$, the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than are $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$. It seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values. That is, to use

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right], \tag{7.8}$$

for each $i = 1, 2, \ldots, n$, instead of Eq. (7.5). This modification is called the **Gauss-Seidel iterative technique** and is illustrated in the following example.

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \quad &= 6, \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$\begin{aligned} x_1^{(k)} &= \qquad\qquad\quad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \qquad\qquad + \frac{3}{5}, \\ x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \qquad\qquad + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11}, \\ x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \qquad\qquad + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10}, \\ x_4^{(k)} &= \qquad\qquad - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \qquad\qquad + \frac{15}{8}. \end{aligned}$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution. Note that Jacobi's method in Example 1 required twice as many iterations for the same accuracy. ∎

To write the Gauss-Seidel method in matrix form, multiply both sides of Eq. (7.8) by a_{ii} and collect all kth iterate terms, to give

$$a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \cdots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} - \cdots - a_{in}x_n^{(k-1)} + b_i,$$

for each $i = 1, 2, \ldots, n$. Writing all n equations gives

$$\begin{aligned} a_{11}x_1^{(k)} \qquad\qquad &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1, \\ a_{21}x_1^{(k)} + a_{22}x_2^{(k)} \qquad &= \qquad\qquad -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2, \\ &\ \ \vdots \\ a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= \qquad\qquad\qquad\qquad\qquad\qquad b_n; \end{aligned}$$

with the definitions of D, L, and U given previously, we have the Gauss-Seidel method represented by

$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

and

$$\mathbf{x}^{(k)} = (D-L)^{-1}U\mathbf{x}^{(k-1)} + (D-L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots. \tag{7.9}$$

Letting $T_g = (D-L)^{-1}U$ and $\mathbf{c}_g = (D-L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g. \tag{7.10}$$

For the lower-triangular matrix $D-L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.

Algorithm 7.2 implements the Gauss-Seidel method.

ALGORITHM 7.2

Gauss-Seidel Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i \le n$ of $\mathbf{b}$; the entries XO_i, $1 \le i \le n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \le N$) do Steps 3–6.

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}XO_j + b_i\right].$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
(*The procedure was successful.*)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful.*)
STOP. ■

The comments following Algorithm 7.1 regarding reordering and stopping criteria also apply to the Gauss-Seidel Algorithm 7.2.

The results of Examples 1 and 2 appear to imply that the Gauss-Seidel method is superior to the Jacobi method. This is almost always true, but there are linear systems for which the Jacobi method converges and the Gauss-Seidel method does not (see Exercises 9 and 10).

The *NumericalAnalysis* subpackage of the Maple *Student* package implements the Gauss-Siedel method in a manner similar to that of the Jacobi iterative method. The results in Table 7.2 are obtained by loading both *NumericalAnalysis* and *LinearAlgebra*, the matrix A, and then using the command

IterativeApproximate(A, *initialapprox* = *Vector*([0., 0., 0., 0.]), *tolerance* = 10^{-3}, *maxiterations* = 20, *stoppingcriterion* = *relative*(*infinity*), *method* = *gaussseidel*, *output* = *approximates*)

If we change the final option to *output* = [*approximates*, *distances*], the output also includes the l_∞ distances between the approximations and the actual solution.

General Iteration Methods

To study the convergence of general iteration techniques, we need to analyze the formula

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k = 1, 2, \ldots,$$

where $\mathbf{x}^{(0)}$ is arbitrary. The next lemma and Theorem 7.17 on page 449 provide the key for this study.

Lemma 7.18 If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j. \qquad \blacksquare$$

Proof Because $T\mathbf{x} = \lambda\mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of $I - T$. But $|\lambda| \le \rho(T) < 1$, so $\lambda = 1$ is not an eigenvalue of T, and 0 cannot be an eigenvalue of $I - T$. Hence, $(I - T)^{-1}$ exists.

Let $S_m = I + T + T^2 + \cdots + T^m$. Then

$$(I - T)S_m = (1 + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1},$$

and, since T is convergent, Theorem 7.17 implies that

$$\lim_{m\to\infty} (I - T)S_m = \lim_{m\to\infty} (I - T^{m+1}) = I.$$

Thus, $(I - T)^{-1} = \lim_{m\to\infty} S_m = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$. $\blacksquare\ \blacksquare\ \blacksquare$

Theorem 7.19 For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k \ge 1, \tag{7.11}$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if $\rho(T) < 1$. $\blacksquare$

Proof First assume that $\rho(T) < 1$. Then,

$$\begin{aligned}
\mathbf{x}^{(k)} &= T\mathbf{x}^{(k-1)} + \mathbf{c} \\
&= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\
&= T^2\mathbf{x}^{(k-2)} + (T + I)\mathbf{c} \\
&\ \ \vdots \\
&= T^k\mathbf{x}^{(0)} + (T^{k-1} + \cdots + T + I)\mathbf{c}.
\end{aligned}$$

Because $\rho(T) < 1$, Theorem 7.17 implies that T is convergent, and

$$\lim_{k\to\infty} T^k \mathbf{x}^{(0)} = \mathbf{0}.$$

Lemma 7.18 implies that

$$\lim_{k\to\infty} \mathbf{x}^{(k)} = \lim_{k\to\infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j\right)\mathbf{c} = \mathbf{0} + (I-T)^{-1}\mathbf{c} = (I-T)^{-1}\mathbf{c}.$$

Hence, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the vector $\mathbf{x} \equiv (I-T)^{-1}\mathbf{c}$ and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k\mathbf{z} = \mathbf{0}$. By Theorem 7.17, this is equivalent to $\rho(T) < 1$.

Let $\mathbf{z}$ be an arbitrary vector, and $\mathbf{x}$ be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$. Define $\mathbf{x}^{(0)} = \mathbf{x} - \mathbf{z}$, and, for $k \ge 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$. Then $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$. Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right),$$

so

$$\mathbf{x} - \mathbf{x}^{(k)} = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right) = T^2\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right) = \cdots = T^k\left(\mathbf{x} - \mathbf{x}^{(0)}\right) = T^k\mathbf{z}.$$

Hence $\lim_{k\to\infty} T^k\mathbf{z} = \lim_{k\to\infty} T^k\left(\mathbf{x} - \mathbf{x}^{(0)}\right) = \lim_{k\to\infty}\left(\mathbf{x} - \mathbf{x}^{(k)}\right) = \mathbf{0}$.

But $\mathbf{z} \in \mathbb{R}^n$ was arbitrary, so by Theorem 7.17, T is convergent and $\rho(T) < 1$. ▪ ▪ ▪

The proof of the following corollary is similar to the proofs in Corollary 2.5 on page 62. It is considered in Exercise 13.

Corollary 7.20 If $\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

(i) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$; (ii) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \frac{\|T\|^k}{1-\|T\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$. ▪

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \quad \mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g,$$

using the matrices

$$T_j = D^{-1}(L+U) \quad \text{and} \quad T_g = (D-L)^{-1}U.$$

If $\rho(T_j)$ or $\rho(T_g)$ is less than 1, then the corresponding sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ will converge to the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. For example, the Jacobi scheme has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}$, then

$$\mathbf{x} = D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This implies that

$$D\mathbf{x} = (L+U)\mathbf{x} + \mathbf{b} \quad \text{and} \quad (D-L-U)\mathbf{x} = \mathbf{b}.$$

Since $D - L - U = A$, the solution $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$.

We can now give easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods. (To prove convergence for the Jacobi scheme see Exercise 14, and for the Gauss-Seidel scheme see [Or2], p. 120.)

Theorem 7.21 If A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$. ∎

The relationship of the rapidity of convergence to the spectral radius of the iteration matrix T can be seen from Corollary 7.20. The inequalities hold for any natural matrix norm, so it follows from the statement after Theorem 7.15 on page 446 that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \approx \rho(T)^k \|\mathbf{x}^{(0)} - \mathbf{x}\|. \tag{7.12}$$

Thus we would like to select the iterative technique with minimal $\rho(T) < 1$ for a particular system $A\mathbf{x} = \mathbf{b}$. No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system. In special cases, however, the answer is known, as is demonstrated in the following theorem. The proof of this result can be found in [Y], pp. 120–127.

Theorem 7.22 **(Stein-Rosenberg)**

If $a_{ij} \le 0$, for each $i \ne j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$, then one and only one of the following statements holds:

(i) $0 \le \rho(T_g) < \rho(T_j) < 1$; **(ii)** $1 < \rho(T_j) < \rho(T_g)$;

(iii) $\rho(T_j) = \rho(T_g) = 0$; **(iv)** $\rho(T_j) = \rho(T_g) = 1$. ∎

For the special case described in Theorem 7.22, we see from part (i) that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method. Part (ii) indicates that when one method diverges then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b.
$$\begin{aligned} 10x_1 - x_2 \phantom{{}- 2x_3} &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ - 2x_2 + 10x_3 &= 6. \end{aligned}$$

c.
$$\begin{aligned} 10x_1 + 5x_2 &= 6, \\ 5x_1 + 10x_2 - 4x_3 &= 25, \\ - 4x_2 + 8x_3 - x_4 &= -11, \\ - x_3 + 5x_4 &= -11. \end{aligned}$$

d.
$$\begin{aligned} 4x_1 + x_2 + x_3 + x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$$

2. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5, \\ -x_1 + 3x_2 + x_3 &= -4, \\ 2x_1 + 2x_2 + 5x_3 &= 1. \end{aligned}$$

b.
$$\begin{aligned} -2x_1 + x_2 + \tfrac{1}{2}x_3 &= 4, \\ x_1 - 2x_2 - \tfrac{1}{2}x_3 &= -4, \\ x_2 + 2x_3 &= 0. \end{aligned}$$

c.
$$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2, \\ x_1 + 4x_2 - x_3 - x_4 &= -1, \\ -x_1 - x_2 + 5x_3 + x_4 &= 0, \\ x_1 - x_2 + x_3 + 3x_4 &= 1. \end{aligned}$$

d.
$$\begin{aligned} 4x_1 - x_2 - x_4 &= 0, \\ -x_1 + 4x_2 - x_3 - x_5 &= 5, \\ - x_2 + 4x_3 - x_6 &= 0, \\ -x_1 + 4x_4 - x_5 &= 6, \\ - x_2 - x_4 + 4x_5 - x_6 &= -2, \\ - x_3 - x_5 + 4x_6 &= 6. \end{aligned}$$

EXAMPLE 3

Solve

$$54x + y + z = 110$$
$$2x + 15y + 6z = 72$$
$$-x + 6y + 27z = 85$$

by Gauss–Seidel method.

Solution.

From the given equations, we have

$$x = \frac{110 - y - z}{54},\quad y = \frac{72 - 2x - 6z}{15},\ \text{and}\ z = \frac{85 + x - 6y}{27}.$$

We take the initial approximation as $x_0 = y_0 = z_0 = 0$. Then the first approximation is given by

$$x_1 = \frac{110}{54} = 2.0370$$

$$y_1 = \frac{72 - 2x_1 - 6z_0}{15} = 4.5284$$

$$z_1 = \frac{85 + x_1 - 6y_1}{27} = 2.2173.$$

The second approximation is given by

$$x_2 = \frac{110 - y_1 - z_1}{54} = 1.9122$$

$$y_2 = \frac{72 - 2x_2 - 6z_1}{15} = 3.6581$$

$$z_2 = \frac{85 + x_2 - 6y_2}{27} = 2.4061.$$

The third approximation is

$$x_3 = \frac{110 - y_2 - z_2}{54} = 1.9247$$

$$y_3 = \frac{72 - 2x_3 - 6z_2}{15} = 3.5809$$

$$z_3 = \frac{85 + x_3 - 6y_3}{27} = 2.4237.$$

The fourth approximation is

$$x_4 = \frac{110 - y_3 - z_3}{54} = 1.9258$$

$$y_4 = \frac{72 - 2x_4 - 6z_3}{15} = 3.5738$$

$$z_4 = \frac{85 + x_4 - 6y_4}{27} = 2.4253.$$

The fifth approximation is

$$x_5 = \frac{110 - y_4 - z_4}{54} = 1.9259$$

$$y_5 = \frac{72 - 2x_5 - 6z_4}{15} = 3.5732$$

$$z_5 = \frac{85 + x_5 - 6y_5}{27} = 2.4254.$$

Thus, the required solution, correct to three decimal places, is

$$x = 1.926, y = 3.573, z = 2.425.$$

EXAMPLE 4

Solve

$$\begin{aligned} 28x + 4y - z &= 32 \\ 2x + 17y + 4z &= 35 \\ x + 3y + 10z &= 24 \end{aligned}$$

by Gauss–Seidel method.

Solution.

From the given equations, we have

$$x = \frac{32-4y+z}{28},\ y = \frac{35-2x-4z}{17},\ \text{and } z = \frac{24-x-3y}{10}.$$

Taking first approximation as $x_0 = y_0 = z_0 = 0$, we have

$x_1 = 1.1428571,\quad y_1 = 1.9243697,\quad z_1 = 1.7084034$
$x_2 = 0.9289615,\quad y_2 = 1.5475567,\quad z_2 = 1.8428368$
$x_3 = 0.9875932,\quad y_3 = 1.5090274,\quad z_3 = 1.8485325$
$x_4 = 0.9933008,\quad y_4 = 1.5070158,\quad z_4 = 1.8485652$
$x_5 = 0.9935893,\quad y_5 = 1.5069741,\quad z_5 = 1.8485488$
$x_6 = 0.9935947,\quad y_6 = 1.5069774,\quad z_6 = 1.8485473.$

Hence the solution, correct to four decimal places, is

$$x = 0.9935,\ y \approx 1.5069,\ z = 1.8485.$$

EXAMPLE 5

Solve the equation by Gauss–Seidel method:

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25. \end{aligned}$$

Solution.

The given equation can be written as

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 3x + 3y].$$

Taking the initial rotation as $(x_0, y_0, z_0) = (0,0,0)$, we have by Gauss–Seidal method,

$$x_1 = \frac{1}{20}[17 - 0 + 0] = 0.85$$

$$y_1 = \frac{1}{20}[-18 - 3(0.85) + 1] = -1.0275$$

$$z_1 = \frac{1}{20}[25 - 2(0.85) - 3(1.0275)] = 1.0108$$

$$x_2 = \frac{1}{20}[17 + 1.0275 + 2(1.0108)] = 1.0024$$

$$y_2 = \frac{1}{20}[-18 - 3(1.0024) + 1.0108] = -0.9998$$

$$z_2 = \frac{1}{20}[25 - 2(1.0024) + 3(-0.9998)] = 0.9998$$

$$x_3 = \frac{1}{20}[17 + 0.9998 + 2(0.9998)] = 0.99997$$

$$y_3 = \frac{1}{20}[-18 - 3(0.99997) + 0.9998] = -1.00000$$

$$z_3 = \frac{1}{20}[25 - 2(0.99997) + 3(-1.0000)] = 1.00000.$$

The second and third iterations show that the solution of the given system of equations is $x = 1$, $y = -1$, $z = 1$.