Solving system of equations

Wednesday, December 15, 2021
8:53 AM

Solving System of Equations

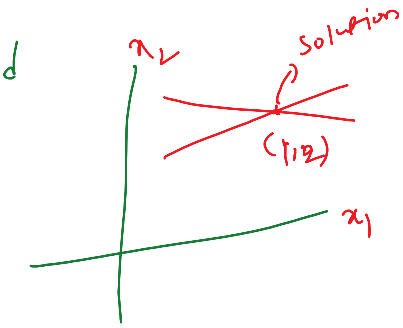
$$x_1 + 3x_2 = 5$$
$$2x_1 + 2x_2 = 6$$

(1) Elimination method

$$2x_1 + 6x_2 = 10$$
$$-\cancel{2x_1} \pm 2x_2 = \underset{-}{6}$$

$$4x_2 = 4 \Rightarrow x_2 = 1$$
$$2x_1 = 4 \Rightarrow x_1 = 2$$

(2) Graphical Method



(3) Cramer's rule

$$x_1 + 3x_2 = 5$$
$$2x_1 + 2x_2 = 6$$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{B}$$

$$|A| = 2-6 = -4, \quad A_1 = \begin{bmatrix} 5 & 3 \\ 6 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$$

$$|A_1| = -8, \quad |A_2| = 6-10 = -4$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-8}{-4} = 2, \quad x_2 = \frac{-4}{-4} = 1 = \frac{|A_2|}{|A|}$$

(4) Inverse Approach

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

(4) Inverse

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B, \quad |A| \neq 0$

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$n \times n$ system of Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$AX = B$, $B \neq 0$ Non-homogenous System

$AX = 0$, $B = 0$ Homogenous system

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

(1) Gauss Elimination
(2) LU decomposition

→ Relation b/w These two methods.

Gauss Elimination :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Step 1 :- $a_{11} \neq 0$, else $a_{11} = 0$ then interchange the rows (1st pivoting element)

Step 2 :- L_1: $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$, $R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & \boxed{a'_{32}} & a'_{33} & b'_3 \end{array}\right]$$

Step 3:- $a'_{22} \neq 0$ (second pivoting) or else interchange

$$L_2: \; R_3 \to R_3 - \frac{a'_{32}}{a'_{22}} R_2$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a'_{33} & b''_3 \end{array}\right]$$

Transforming matrix A to an upper traiangular matrix form

Back Substitution then you will get the solution.

Ex:-

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 2 \\ 4 & 6 & 8 & 1 \end{array}\right]$$

$L_1: \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 4 & -3 \end{array}\right], \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_1: \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \ldots \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ \ldots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ -4 & 0 & 1 \end{bmatrix}$$

$$\underline{L_1 A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{L_2}$: $R_3 \to R_3 - 2R_2$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$L_2 (L_1 A) = U \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & -3 \end{array}\right]$$

U

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 1$$

$$x_2 + 3x_3 = 0$$

$$-2x_3 = -3 \Rightarrow x_3 = 3/2$$

$$x_2 = -9/2$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1 - 9/2 + 3/2 = 1$$

$$x_1 = 4$$

$$(\underline{L_2 L_1}) A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$A = \boxed{(L_2 L_1)^{-1}} U = L_1^{-1} L_2^{-1} A \qquad \to L$$

Inverse of a lower traingular matrix is again a lower traingular matrix

$A = LU \to$ upper traingular matrix

$L \to$ lower traingular matrix

LU decomposition

$A = LU \rightarrow$ upper matrix

$\hookrightarrow$ lower traingular matrix

$AX = B$

$\hookrightarrow A = LU$

$(LU)X = B$

$\Rightarrow L(UX) = B \rightarrow (1)$

Assume $UX = Z \rightarrow (2)$

from (1) & (2) $LZ = B \rightarrow (3)$

First solve Eq (3) to evaluate Z

Then we will solve Eq (2) to find

vector X.