Solving System of equations

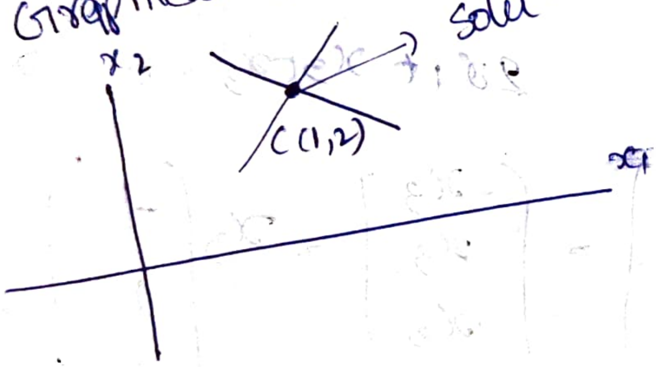
$$x_1 + 3x_2 = 5$$
$$2x_1 + 2x_2 = 6$$

① Elimination method

$$\cancel{2x_1} + 6x_2 = 10$$
$$\cancel{2x_1} + 2x_2 = 6$$

$$4x_2 = 4 \Rightarrow x_2 = 1$$
$$2x_1 = 4 \Rightarrow x_1 = 2$$

② Graphical method



③ Cramer's rule

$$x_1 + 3x_2 = 5$$
$$2x_1 + 2x_2 = 6$$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{B}$$

$$|A| = 2 - 6 = -4$$

$$A_1 = \begin{bmatrix} 5 & 3 \\ 6 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$$

$$|A_1| = -8, \quad |A_2| = 6 - 10 = -4$$

$x_1 = \frac{|A_1|}{|A|} = \frac{-8}{-4} = 2$, $x_2 = \frac{-4}{-4} = 1 = \frac{|A_2|}{|A|}$

④ Inverse Approach

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$AX = B$
is invertible

$|A| \neq 0$, A^{-1} exists

$X = A^{-1}B$

$$AX = B$$

$$X = A^{-1}B \quad , \quad |A| \neq 0$$

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

• If A is orthogonal matrix ($AA^T = I$)

$$A^T = A^{-1}$$

✗ $n \times n$ system of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}a_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$AX = B$, $B \neq 0$, Non-homogenous system

$AX = 0$, $B = 0$ Homogenous system

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

① Guass Elimination
② LU decomposition (cholekeys method)

} Relation b/w These two method

Gauss Elimination :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

step:1- $a_{11} \neq 0$, else $a_{11} = 0$ then interchange the rows (1st pivoting element)

step:2-4: $R_2 \xrightarrow{E_1} R_2 - \frac{a_{21}}{a_{11}} R_1$, $R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$

Elementary ~~ele~~ ele min

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

3tep:- $a'_{12} \neq 0$ (second pivoting) or else interchange

l_2: $R_3 \longrightarrow R_3 - \frac{a'_{32}}{a'_{22}} R_2$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a'_{33} & b''_3 \end{array}\right]$$

Transforming matrix A to an upper triangular matrix form

→ Base substitution then you will get the solution

② LU decomposition :-

Ex: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

A B

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 2 \\ 4 & 6 & 8 & 1 \end{array}\right]$$

L_1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$L_1 A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 4 & -3 \end{array}\right]$ A

L_1 = Apply These operation to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Then we will get L_1

Then

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$L_2 : R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & -3 \end{array}\right]$$

U

$$\boxed{L_2(L_1 A) = U}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 1$$
$$x_2 + 3x_3 = 0$$
$$-2x_3 = -3$$
$$x_3 = 3/2$$

$x_2 = -9/2$

$x_1 + x_2 + x_3 = 1$

$x_1 - 9/2 + 3/2 = 1$

$x_1 = 4$

$(L_2 L_1) A = U$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow A = \underbrace{(L_2 L_1)^{-1}}_{\rightarrow L} U = L_1^{-1} L_2^{-1} U \quad (\because (AB)^{-1} = B^{-1} A^{-1})$$

Inverse of a lower Δ^{le}lar matrix is again a lower Δ^{le}lar matrix

$A = LU \rightarrow$ upper Δ^{le}lar

$\quad \hookrightarrow$ Lower Δ^{le}lar

$(L_2 L_1) A = U$

$\Downarrow$

$\bar{L} A = U$

$A = (\bar{L})^{-1} U$

$A = LU$

LU decomposition :-

$AX = B$

$\hookrightarrow A = LU$

$(LU) X = B$

$\Rightarrow L(UX) = B \quad — ①$

Assume $UX = Z \quad — ②$

from ① and ② $\quad LZ = B \quad — ③$

$\lim_{(x,y)\to} f(x,y)$

LU Decomposition :-

$\cancel{\boxed{A = LU}}$

$\cancel{(LU)X = B}$

$\cancel{\boxed{UX = Y}}$

$\cancel{LY = B}$

First solve eq (3) to equaluate Z, then we will solve eq (2) to find vector X

* Ex :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{31}U_{12}+L_{32}U_{22} \\ L_{31}U_{11} & L_{21}U_{13}+U_{23} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}}$$

$$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}$$

$A = LU$

$(LU)X = B$

$\Rightarrow \cancel{UX = Y}$ $\quad Ly = b$

$\cancel{LZ = B}$ $\quad UX = y$

$$* \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Use LU decomposition to solve the given system?

Doolittle's Decomposition Method:

Decomposition phase: Doolittle's decomposition is closely related to Guass elimination. To illustrate the relationship, consider 3×3 matrix A and assume that there exists triangular matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$A = LU$$

$$A = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}L_{21} & U_{12}L_{21} + U_{22} & U_{13}L_{21} + U_{23} \\ U_{11}L_{31} & U_{12}L_{31} + U_{22}L_{32} & U_{13}L_{31} + U_{23}L_{32} + U_{33} \end{bmatrix}$$

$$R_2 \longrightarrow R_2 - L_{21} \times R_1$$

$$R_3 \longrightarrow R_3 - L_{31} \times R_1$$

$$A' = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & U_{22}U_{32} & U_{23}L_{32}+U_{33} \end{bmatrix}$$

$$R_3 \rightarrow R_3 - L_{32} \times R_2$$

$$A'' = U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Indirect Methods:

1. Gauss Jacobi iteration method
2. Gauss siedel method

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 \qquad = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$\rightarrow AX = B$

1. Gauss Jacobi Method:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad (2)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad (3)$$

first should be
diagonally dominant

~~a11 > the~~ i.e.

$$|a_{11}| \geq |a_{12}| + |a_{13}|$$
$$|a_{22}| \geq |a_{21}| + |a_{23}|$$
$$|a_{33}| \geq |a_{31}| + |a_{32}|$$

Arrange the given system of equations as diagonally dominant system from 1, 2 & 3

$$a_{11}x_1 = b_1 - (a_{12}x_2 + a_{13}x_3)$$

from 1
$$\Rightarrow x_1 = \frac{1}{a_{11}}(b_1 - (a_{12}x_2 + a_{13}x_3))$$

from 2
$$x_2 = \frac{1}{a_{22}}(b_2 - (a_{21}x_1 + a_{23}x_3))$$

from 3
$$x_3 = \frac{1}{a_{33}}(b_3 - (a_{31}x_1 + a_{32}x_2))$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix} \longrightarrow \text{To Calculate } X^1$$

(*) $X_0 \longrightarrow X^1$
To cal X^1

$\underline{K=1}$

$$x_1^1 = \frac{1}{a_{11}}(b_1 - [a_{12}x_2^0 + a_{13}x_3^0)]$$

$$x_2^1 = \frac{1}{a_{22}}[b_2 - (a_{21}x_1^0 + a_{23}x_3^0)]$$

$$x_3^1 = \frac{1}{a_{33}}[b_3 - (a_{31}x_1^0 + a_{32}x_2^0)]$$

$$X^0 \rightarrow X^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{bmatrix}$$

$X^1 \rightarrow X^2$

$k=2$

$$x_1^2 = \frac{1}{a_{11}}\left(b_1 - (a_{12}x_2^1 + a_{13}x_3^1)\right)$$

$$x_2^2 = \frac{1}{a_{22}}\left(b_2 - (a_{21}x_1^1 + a_{23}x_3^1)\right)$$

$$x_3^2 = \frac{1}{a_{33}}\left(b_3 - (a_{31}x_1^1 + a_{32}x_2^1)\right)$$

General form

$$\begin{cases} x_1^{k+1} = \frac{1}{a_{11}}\left[b_1 - (a_{12}x_2^k + a_{13}x_3^k)\right) \\ x_2^{k+1} = \frac{1}{a_{22}}\left(b_2 - (a_{21}x_1^k + a_{23}x_3^k)\right) \\ x_3^{k+1} = \frac{1}{a_{33}}\left(b_3 - (a_{31}x_1^k + a_{32}x_2^k)\right) \end{cases}$$

represent in matrix representation

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{B}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{13} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} A &= (L+D+U) \\ X^{K+1} &= \frac{1}{D}\left[B-(L+U)X^{K}\right] \end{aligned}}$$

Stoping criteria : $\|X^K - X^{K+1}\| \le \epsilon,\ \epsilon \geq 0$

$$X^K \cong X^{K+1}$$

① Gauss siedel Method:

Here imidiate updation will be

$$X^1 = (x_1^1, x_2^1, x_3^1)$$

$$x_1^2 = \frac{1}{a_{11}}\left(b_1 - (a_{12}x_2^1 + a_{13}x_3^1)\right)$$

$$x_2^2 = \frac{1}{a_{22}}\left(b_2 - (a_{21}x_1^2 + a_{23}x_3^1)\right)$$

$$x_3^2 = \frac{1}{a_{33}}\left(b_3 - (a_{31}x_1^2 + a_{32}x_2^2)\right)$$

General form

$$x_1^{K+1} = \frac{1}{a_{11}}\left(b_1 - (a_{12}x_2^K + a_{13}x_3^K)\right)$$

$$x_2^{K+1} = \frac{1}{a_{22}}\left[b_2 - (a_{21}x_1^{K+1} + a_{23}x_3^K)\right]$$

$$x_3^{k+1} = \underbrace{\frac{1}{a_{33}}}_{D} [\underbrace{b_3}_{B} - \underbrace{(a_{31} x_1^{k+1} + a_{32} x_2^{k+1})}_{L+U}]$$

$$\boxed{X^{k+1} = \frac{1}{D} [B - (L X^{k+1} + U X^k)]}$$

$$D X^{k+1} + L X^{k+1} = B - U X^k$$

$$(D+L) X^{k+1} = B - U X^k$$

$$\boxed{X^{k+1} = (D+L)^{-1} B - (D+L)^{-1} U X^k}$$

End

$$\| X^{k+1} - X^k \| \leq 0.001$$

Ex:

Q solve the following system of equations by
a) Gauss Jacobi
b) Gauss Siedel

$$6x + 2y - z = 4$$
$$2x + y + 4z = 3$$
$$x + y + 4z = 27$$

$X^1 = (0, 0, 0)$

Sol Gauss Jacobi:

Check diagonally dominance, if they are not change them

$$\begin{cases} 6x + 2y - z = 4 \\ x + 3y + z = 27 \\ 2x + y + 4z = 3 \end{cases}$$

$x = \frac{1}{6}(4 - (2y - z))$

$y = \frac{1}{3}(27 - x - z)$ $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$z = \frac{1}{4}(3 - 2x - y)$

$x = 0, y = 0, z = 0$

$x^2 = \frac{1}{6}[4 - 0] = 4/6 = 2/3$

$y^2 = 1/3 [27 - 0] = 27/3 = 9$

$z^2 = \frac{1}{4}(3) = 3/4$

$X^2 = [2/3, 9, 3/4]$, $x = 2/3, y = 9, z = 3/4$

$x^3 =$

$y^3 =$

$z^3 =$

do ——

Ex: Use the Jacobi method to approriate the soluⁿ of the following system of equⁿs

Power Method:-

(1) spectrum = $\{\lambda_1, \lambda_2 \ldots \lambda_n\}$
(2) Spectral radius : $\rho(A)$ = largest Eigen value
$\rho(A) < 1$

$A_{n \times n} = \lambda_1, \lambda_2 \ldots \lambda_n$ - Eigen values

$|\lambda_1| < |\lambda_2| < |\lambda_3| < |\lambda_4| \ldots < |\lambda_n|$

$|\lambda_1|$ → smallest Eigen value

$|\lambda_n|$ → largest Eigen value

(Reigh's)
Power method: } → Iterative method

(1) Finding largest Eigen value
(2) Finding smallest Eigen value
(3) Nearest Eigen value of λ_0

* Finding largest Eigen value:-

1. Take a initial vector (X_0)

$$Y_0 = AX_0 = \lambda_0 X_1$$

$$AX_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ → select the largest magnitude value

For ex x_2 is the largest magnitude

$$= x_2 \begin{bmatrix} x_1/x_2 \\ 1 \\ x_3/x_2 \end{bmatrix} \rightarrow \text{new eign vector } (X_1)$$

(λ_0)

new eigen value

$Y_0 = AX_0 = \lambda_0 X_1$

$Y_1 = AX_1 = \lambda_1 X_2$

$Y_2 = AX_2 = \lambda_2 X_3$

$X_{n+1} \simeq X_n \Rightarrow X_{n+1} - X_n \rightarrow 0$

$\lambda_n \simeq \lambda_{n+1}$

then you can stop

largest Eigen value

$$\boxed{AX_n = \lambda_n X_n}$$

eigen value

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find largest Eigen value

$$AX_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\lambda_0 = 3 \ \& \ X_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2.33 \\ 5 \end{bmatrix}$$

$$= 5\begin{bmatrix} 2.33/5 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2.33/5 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.466 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.466 \\ 5.399 \end{bmatrix}$$

$$= 5.399\begin{bmatrix} 2.466/5.399 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 5.399, \quad X_3 = \begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.25 \\ 5.35 \end{bmatrix}$$

$$= 5.35\begin{bmatrix} 2.25/5.35 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.245 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45 \\ 5.35 \end{bmatrix}$$

$$= 5.35\begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$X_3 = X_4 = \lambda' = 5.35$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \lambda^2 - (a_{11}+a_{22})\lambda + |A| = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25+8}}{2}$$

$$\lambda_1 = \frac{5-\sqrt{33}}{2} \quad \& \quad \lambda_2 = \frac{5+\sqrt{33}}{2} = 5.37$$

$$\lambda_1 \approx -0.3$$

largest

Same question take smallest magnitude value of the vector as common.

Ex: Find the largest Eigen value of $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with initial gauss $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(take upto 2 decimals and perform two iterations)

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 3/8 \\ 3/8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3/8 \\ 3/8 \end{bmatrix} = \begin{bmatrix} 1 + 18/8 + 3/8 \\ 1 + 6/8 \\ 9/8 \end{bmatrix}$$

$$= \begin{bmatrix} 29/8 \\ 14/8 \\ 9/8 \end{bmatrix}$$

Smallest Eigen values

$\lambda \rightarrow A$ 3 largest Eigen value of A

$1/\lambda \rightarrow A^{-1}$

$1/3 \rightarrow$ smallest eigen values of A^{-1}

$$AX = \lambda X$$

$$A^{-1}AX = A^{-1}\lambda X$$

$$\frac{1}{\lambda} X = A^{-1} X$$

$$A^{-1} X = \frac{1}{\lambda} X$$

$A^{-1} \rightarrow B$

$1/\lambda \rightarrow \beta$

$$\boxed{BX = \beta X}$$

$\rightarrow$ The largest Eigen value of B is smallest Eigen value of A.

* Find smallest Eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Find $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = B$$

$$BX_0 = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \end{pmatrix}$$

$$= -2\begin{bmatrix} 1 \\ \frac{-1.5}{2} \end{bmatrix}$$

(i) taking smallest magnitude value common (for matrix A)

(ii) Find A^{-1} and then finding largest Eigen value

(iii) Nearest Eigen value of λ_0

Eigen value of A

$(A - \lambda_0 I) = B$

smallest Eigen value

Find largest Eigen value for $(A - \lambda_0 I)^{-1}$ or B^{-1}

iii) Nearest eigen value of $\lambda_k \approx P$

$$A \longrightarrow \lambda_1, \lambda_2, \cdots \lambda_k \cdots \lambda_n$$

'p' is the nearest Eigen value of λ_k

$$(A-PI) \longrightarrow \lambda_1 - P,\ \lambda_2 - P,\ \cdots\ \lambda_k - P \cdots \lambda_n - P$$

Eigen value

Very small

Smallest eigen value of $A-PI$

is the nearest Eigen value of $\lambda_k \to A$

* Finding the nearest eigen value of λ_k is nothing but finding smallest Eigen value of $A-PI$

* Finding smallest eigen value of $(A-PI)$ is similar of Finding largest Eigen value of $(A-PI)^{-1}$

* Find the nearest Eigen value of 5 for

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \text{ with } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Sol

Step① $(A-5I) = \begin{bmatrix} -2 & -1 & 0 \\ -2 & -1 & -3 \\ 0 & -1 & -4 \end{bmatrix} = B$

Smallest Eigen value of $(A-5I)$

= largest Eigen value of $(A-5I)^{-1}$ is

~~nothing but~~ ~~the~~ that will be ur starting nearest Eigen value of 5 of A[ ]

Draw backs of power method

① choosing intial gauss

$X_K \rightarrow \lambda_K \rightarrow$ dominating eigen values

$X_0 \simeq$ eigen vector for λ_1

② Complex eigen values

③ knowing all Eigen values is tough