1.3.2 Power Method

The method for finding the largest eigen value in magnitude and the corresponding eigen vector of the eigen value problem $\mathbf{Ax} = \lambda\, \mathbf{x}$, is called the power method.

What is the importance of this method? Let us re-look at the Remarks 20 and 23. The necessary and sufficient condition for convergence of the Gauss-Jacobi and Gauss-Seidel iteration methods is that the spectral radius of the iteration matrix **H** is less than one unit, that is, $\rho(\mathbf{H}) < 1$, where $\rho(\mathbf{H})$ is the largest eigen value in magnitude of **H**. If we write the matrix formulations of the methods, then we know **H**. We can now find the largest eigen value in magnitude of **H**, which determines whether the methods converge or not.

We assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigen values such that

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|. \tag{1.52}$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. The method is applicable if a complete system of n linearly independent eigen vectors exist, even though some of the eigen values $\lambda_2, \lambda_3, \ldots, \lambda_n$, may not be distinct. The n linearly independent eigen vectors form an n-dimensional vector space. Any vector $\mathbf{v}$ in this space of eigen vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ can be written as a linear combination of these vectors. That is,

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n. \tag{1.53}$$

Premultiplying by **A** and substituting $\mathbf{Av}_1 = \lambda_1\mathbf{v}_1$, $\mathbf{Av}_2 = \lambda_2\mathbf{v}_2, \ldots, \mathbf{Av}_n = \lambda_n\mathbf{v}_n$, we get

$$\mathbf{Av} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots + c_n\lambda_n\mathbf{v}_n$$

$$= \lambda_1\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)\mathbf{v}_n\right].$$

Premultiplying repeatedly by **A** and simplifying, we get

$$\mathbf{A}^2\mathbf{v} = \lambda_1^2\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^2\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^2\mathbf{v}_n\right]$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\mathbf{A}^k\mathbf{v} = \lambda_1^k\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n\right]. \tag{1.54}$$

$$A^{k+1}\mathbf{v} = \lambda_1^{k+1}\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1}\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1}\mathbf{v}_n\right]. \quad (1.55)$$

As $k \to \infty$, the right hand sides of (1.54) and (1.55) tend to $\lambda_1^k c_1\mathbf{v}_1$ and $\lambda_1^{k+1}c_1\mathbf{v}_1$, since $|\lambda_i/\lambda_1| < 1$, $i = 2, 3, \ldots, n$. Both the right hand side vectors in (1.54), (1.55)

$$[c_1\mathbf{v}_1 + c_2(\lambda_2/\lambda_1)^k\,\mathbf{v}_2 + \ldots + c_n\,(\lambda_n/\lambda_1)^k\,\mathbf{v}_n],$$

and

$$[c_1\mathbf{v}_1 + c_2(\lambda_2/\lambda_1)^{k+1}\,\mathbf{v}_2 + \ldots + c_n\,(\lambda_n/\lambda_1)^{k+1}\,\mathbf{v}_n]$$

tend to $c_1\mathbf{v}_1$, which is the eigen vector corresponding to λ_1. The eigen value λ_1 is obtained as the ratio of the corresponding components of $A^{k+1}\mathbf{v}$ and $A^k\mathbf{v}$. That is,

$$\lambda_1 = \lim_{k\to\infty}\frac{(A^{k+1}\mathbf{v})_r}{(A^k\mathbf{v})_r}, \quad r = 1, 2, 3, \ldots, n \quad (1.56)$$

where the suffix r denotes the rth component of the vector. Therefore, we obtain n ratios, all of them tending to the same value, which is the largest eigen value in magnitude, $|\lambda_1|$.

When do we stop the iteration The iterations are stopped when all the magnitudes of the differences of the ratios are less than the given error tolerance.

Remark 24 The choice of the initial approximation vector $\mathbf{v}_0$ is important. If no suitable approximation is available, we can choose $\mathbf{v}_0$ with all its components as one unit, that is, $\mathbf{v}_0 = [1, 1, 1, \ldots, 1]^T$. However, this initial approximation to the vector should be non-orthogonal to $\mathbf{v}_1$.

Remark 25 Faster convergence is obtained when $|\lambda_2| << |\lambda_1|$.

As $k \to \infty$, premultiplication each time by A, may introduce round-off errors. In order to keep the round-off errors under control, we normalize the vector before premultiplying by A. The normalization that we use is to make the largest element in magnitude as unity. If we use this normalization, a simple algorithm for the power method can be written as follows.

$$\mathbf{y}_{k+1} = A\mathbf{v}_k, \quad (1.57)$$

$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1} \quad (1.58)$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. Now, the largest element in magnitude of $\mathbf{v}_{k+1}$ is one unit. Then (1.56) can be written as

$$\lambda_1 = \lim_{k\to\infty}\frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3, \ldots, n \quad (1.59)$$

and $\mathbf{v}_{k+1}$ is the required eigen vector.

Remark 26 It may be noted that as $k \to \infty$, m_{k+1} also gives $|\lambda_1|$.

Remark 27 Power method gives the largest eigen value in magnitude. If the sign of the eigen value is required, then we substitute this value in the determinant $|A - \lambda_1 I|$ and find its value. If this value is approximately zero, then the eigen value is of positive sign. Otherwise, it is of negative sign.

Example 1.24 *Determine the dominant eigen value of* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *by power method.*

(A.U. Nov/Dec 2004)

Solution Let the initial approximation to the eigen vector be $\mathbf{v}_0$. Then, the power method is given by

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k,$$
$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. The dominant eigen value in magnitude is given by

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3, ..., n$$

and $\mathbf{v}_{k+1}$ is the required eigen vector.

Let $\mathbf{v}_0 = [1 \quad 1]^T$. We have the following results.

$$\mathbf{y}_1 = \mathbf{A}\mathbf{v}_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \quad m_1 = 7, \quad \mathbf{v}_1 = \frac{\mathbf{y}_1}{m_1} = \frac{1}{7}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 0.42857 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_2 = \mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.42857 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.42857 \\ 5.28571 \end{bmatrix}, \quad m_2 = 5.28571,$$

$$\mathbf{v}_2 = \frac{\mathbf{y}_2}{m_2} = \frac{1}{5.28571}\begin{bmatrix} 2.42857 \\ 5.28571 \end{bmatrix} = \begin{bmatrix} 0.45946 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_3 = \mathbf{A}\mathbf{v}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45946 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45946 \\ 5.37838 \end{bmatrix}, \quad m_3 = 5.37838,$$

$$\mathbf{v}_3 = \frac{\mathbf{y}_3}{m_3} = \frac{1}{5.37838}\begin{bmatrix} 2.45946 \\ 5.37838 \end{bmatrix} = \begin{bmatrix} 0.45729 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_4 = \mathbf{A}\mathbf{v}_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45729 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45729 \\ 5.37187 \end{bmatrix}, \quad m_4 = 5.37187,$$

$$\mathbf{v}_4 = \frac{\mathbf{y}_4}{m_4} = \frac{1}{5.37187}\begin{bmatrix} 2.45729 \\ 5.37187 \end{bmatrix} = \begin{bmatrix} 0.45744 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_5 = \mathbf{A}\mathbf{v}_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45744 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45744 \\ 5.37232 \end{bmatrix}, \quad m_4 = 5.37232,$$

$$\mathbf{v}_5 = \frac{\mathbf{y}_5}{m_5} = \frac{1}{5.37232}\begin{bmatrix} 2.45744 \\ 5.37232 \end{bmatrix} = \begin{bmatrix} 0.45743 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_6 = \mathbf{A}\mathbf{v}_5 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45743 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45743 \\ 5.37229 \end{bmatrix}.$$

Now, we find the ratios

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r} \quad r = 1, 2.$$

We obtain the ratios as

$$\frac{2.45743}{0.45743} = 5.37225, \quad 5.37229.$$

The magnitude of the error between the ratios is $| 5.37225 - 5.37229 | = 0.00004 < 0.00005$. Hence, the dominant eigen value, correct to four decimal places is 5.3722.

Example 1.25 *Determine the numerically largest eigen value and the corresponding eigen vector of the following matrix, using the power method.*

$$\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$ *(A.U. May/June 2006)*

Solution Let the initial approximation to the eigen vector be $\mathbf{v}_0$. Then, the power method is given by

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k,$$
$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. The dominant eigen value in magnitude is given by

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \qquad r = 1, 2, 3, \ldots, n$$

and $\mathbf{v}_{k+1}$ is the required eigen vector.

Let the initial approximation to the eigen vector be $\mathbf{v}_0 = [1, 1, 1]^T$. We have the following results.

$$\mathbf{y}_1 = \mathbf{A}\mathbf{v}_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 28 \\ 4 \\ -2 \end{bmatrix}, \; m_1 = 28,$$

$$\mathbf{v}_1 = \frac{1}{m_1}\mathbf{y}_1 = \frac{1}{28}\begin{bmatrix} 28 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.14286 \\ -0.07143 \end{bmatrix}.$$

$$\mathbf{y}_2 = \mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.14286 \\ -0.07143 \end{bmatrix} = \begin{bmatrix} 25.0000 \\ 1.42858 \\ 2.28572 \end{bmatrix}, \; m_2 = 25.0.$$

$$\mathbf{v}_2 = \frac{1}{m_2}\mathbf{y}_2 = \frac{1}{25.0}\begin{bmatrix} 25.00000 \\ 1.14286 \\ 2.28572 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.05714 \\ 0.09143 \end{bmatrix},$$

$$\mathbf{y}_3 = \mathbf{A}\mathbf{v}_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.05714 \\ 0.09143 \end{bmatrix} = \begin{bmatrix} 25.24000 \\ 1.17142 \\ 1.63428 \end{bmatrix}, \; m_3 = 25.24,$$

$$\mathbf{v}_3 = \frac{1}{m_3}\,\mathbf{y}_3 = \frac{1}{25.24}\begin{bmatrix} 25.24000 \\ 1.17142 \\ 1.63428 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04641 \\ 0.06475 \end{bmatrix},$$

$$\mathbf{y}_4 = \mathbf{A}\mathbf{v}_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04641 \\ 0.06475 \end{bmatrix} = \begin{bmatrix} 25.17591 \\ 1.13923 \\ 1.74100 \end{bmatrix},\ m_4 = 25.17591,$$

$$\mathbf{v}_4 = \frac{1}{m_4}\,\mathbf{y}_4 = \frac{1}{25.17591}\begin{bmatrix} 25.17591 \\ 1.13923 \\ 1.74100 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04525 \\ 0.06915 \end{bmatrix},$$

$$\mathbf{y}_5 = \mathbf{A}\mathbf{v}_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04525 \\ 0.06915 \end{bmatrix} = \begin{bmatrix} 25.18355 \\ 1.13575 \\ 1.72340 \end{bmatrix},\ m_5 = 25.18355,$$

$$\mathbf{v}_5 = \frac{1}{m_5}\,\mathbf{y}_5 = \frac{1}{25.18355}\begin{bmatrix} 25.18355 \\ 1.13575 \\ 1.72340 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04510 \\ 0.06843 \end{bmatrix},$$

$$\mathbf{y}_6 = \mathbf{A}\mathbf{v}_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04510 \\ 0.06843 \end{bmatrix} = \begin{bmatrix} 25.18196 \\ 1.13530 \\ 1.72628 \end{bmatrix},\ m_6 = 25.18196,$$

$$\mathbf{v}_6 = \frac{1}{m_6}\,\mathbf{y}_6 = \frac{1}{25.18196}\begin{bmatrix} 25.18196 \\ 1.13530 \\ 1.72628 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04508 \\ 0.06855 \end{bmatrix},$$

$$\mathbf{y}_7 = \mathbf{A}\mathbf{v}_6 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04508 \\ 0.06855 \end{bmatrix} = \begin{bmatrix} 25.18218 \\ 1.13524 \\ 1.72580 \end{bmatrix},\ m_7 = 25.18218,$$

$$\mathbf{v}_7 = \frac{1}{m_7}\,\mathbf{y}_7 = \frac{1}{25.18218}\begin{bmatrix} 25.18218 \\ 1.13524 \\ 1.72580 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04508 \\ 0.06853 \end{bmatrix},$$

$$\mathbf{y}_8 = \mathbf{A}\mathbf{v}_7 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04508 \\ 0.06853 \end{bmatrix} = \begin{bmatrix} 25.18214 \\ 1.13524 \\ 1.72588 \end{bmatrix},\ m_8 = 25.18214.$$

Now, we find the ratios

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3.$$

We obtain the ratios as

$$25.18214,\ \frac{1.13524}{0.04508} = 25.18279,\ \frac{1.72588}{0.06853} = 25.18430.$$

The magnitudes of the errors of the differences of these ratios are 0.00065. 0.00216. 0.00151, which are less than 0.005. Hence, the results are correct to two decimal places. Therefore. the largest eigen value in magnitude is $|\lambda_1| = 25.18$.

The corresponding eigen vector is $\mathbf{v}_8$,

$$\mathbf{v}_8 = \frac{1}{m_8}\mathbf{y}_8 = \frac{1}{25.18214}\begin{bmatrix} 25.18214 \\ 1.13524 \\ 1.72588 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04508 \\ 0.06854 \end{bmatrix}.$$

In Remark 26, we have noted that as $k \to \infty$, m_{k+1} also gives $|\lambda_1|$. We find that this statement is true since $|m_8 - m_7| = |25.18214 - 25.18220| = 0.00006$.

If we require the sign of the eigen value, we substitute λ_1 in the characteristic equation. In the present problem, we find that $|\mathbf{A} - 25.18\,\mathbf{I}| = 1.4018$, while $|\mathbf{A} + 25.18\,\mathbf{I}|$ is very large. Therefore, the required eigen value is 25.18.