

## **Elementary Matrices and a Method for Finding $A^{-1}$ :**

### **Week 5**

- **Elementary Matrices and Row Operations**
- **Row Equivalence**
- **Using Row Operations to Find  $A^{-1}$**

# Elementary Matrices and Row Operations

## Definition

An  $n \times n$  matrix is called an *elementary matrix* if it can be obtained from the  $n \times n$  identity matrix  $I_n$  by performing a single elementary row operation.

## Example 1

Listed below are four elementary matrices and the operations that produce them.

$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
↑	↑	↑	↑
Multiply the second row of $I_2$ by $-3$ .	Interchange the second and fourth rows of $I_4$ .	Add 3 times the third row of $I_3$ to the first row.	Multiply the first row of $I_3$ by 1.



When a matrix  $A$  is multiplied on the *left* by an elementary matrix  $E$ , the effect is to perform an elementary row operation on  $A$ . This is the content of the following theorem,

## Theorem 1.5.1 (Row Operations by Matrix Multiplication)

*If the elementary matrix  $E$  results from performing a certain row operation on  $I_m$  and if  $A$  is an  $m \times n$  matrix, then the product  $EA$  is the matrix that results when this same row operation is performed on  $A$ .*

## Example 2 (Using Elementary Matrices)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of  $I_3$  to the third row. The product  $EA$  is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the same matrix that results when we add 3 times the first row of  $A$  to the third row. ✨

If an elementary row operation is applied to an identity matrix  $I$  to produce an elementary matrix  $E$ , then there is a second row operation that, when applied to  $E$ , produces  $I$  back again. For example, if  $E$  is obtained by multiplying the  $i$ th row of  $I$  by a nonzero constant  $c$ , then  $I$  can be recovered if the  $i$ th row of  $E$  is multiplied by  $\frac{1}{c}$ . The various possibilities are listed in Table 1. The operations on the right side of this table are called the **inverse operations** of the corresponding operations on the left.

**Table 1**

Row Operation on $I$ That Produces $E$	Row Operation on $E$ That Reproduces $I$
Multiply row $i$ by $c \neq 0$	Multiply row $i$ by $1/c$
Interchange rows $i$ and $j$	Interchange rows $i$ and $j$
Add $c$ times row $i$ to row $j$	Add $-c$ times row $i$ to row $j$

### Example 3 (Row Operations and Inverse Row Operations)

In each of the following, an elementary row operation is applied to the  $2 \times 2$  identity matrix to obtain an elementary matrix  $E$ , then  $E$  is restored to the identity matrix by applying the inverse row operation.

$$\begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{Multiply the} \qquad \text{Multiply the} \\ \text{second row} \qquad \text{second row} \\ \text{by } 7. \qquad \qquad \text{by } \frac{1}{7}. \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{Interchange the} \qquad \text{Interchange the} \\ \text{first and second} \qquad \text{first and second} \\ \text{rows.} \qquad \qquad \text{rows.} \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\substack{\uparrow \\ \text{Add 5 times} \\ \text{the second row} \\ \text{to the first.}}} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{\substack{\uparrow \\ \text{Add } -5 \text{ times} \\ \text{the second row} \\ \text{to the first.}}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



The next theorem gives an important property of elementary matrices.

### Theorem 1.5.2

*Every elementary matrix is invertible, and the inverse is also an elementary matrix.*

**Proof.** If  $E$  is an elementary matrix, then  $E$  results from performing some row operation on  $I$ . Let  $E_0$  be the matrix that results when the inverse of this operation is performed on  $I$ . Applying Theorem 1.5.1 and using the fact that inverse row operations cancel the effect of each other, it follows that

$$E_0 E = I \text{ and } E E_0 = I$$

Thus, the elementary matrix  $E_0$  is the inverse of  $E$ . ▀

The next theorem establishes some fundamental relationships among invertibility, homogeneous linear systems, reduced row-echelon forms, and elementary matrices. These results are extremely important and will be used many times in later sections.

### Theorem 1.5.3

*If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent, that is, all true or all false.*

- (a)  $A$  is invertible.
- (b)  $Ax = \mathbf{0}$  has only the trivial solution.
- (c) The reduced row-echelon form of  $A$  is  $I_n$ .
- (d)  $A$  is expressible as a product of elementary matrices.

**Proof.** (See text book) ▀

### Row Equivalence

If a matrix  $B$  can be obtained from a matrix  $A$  by performing a finite sequence of elementary row operations, then obviously we can get from  $B$  back to  $A$  by performing the inverses of these elementary row operations in reverse order. Matrices that can be obtained from one another by a finite sequence of elementary row operations are said to be **row equivalent**. With this terminology, it follows from parts (a) and (c) of Theorem 1.5.3 that an  $n \times n$  matrix  $A$  is invertible if and only if it is row equivalent to the  $n \times n$  identity matrix.

## A Method for Inverting Matrices

To find the inverse of an invertible matrix  $A$ , we must find a sequence of elementary row operations that reduces  $A$  to the identity and then perform this same sequence of operations on  $I_n$  to obtain  $A^{-1}$ .

### Example 4 (Using Row Operations to Find $A^{-1}$ )

Find the inverse of  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

We want to reduce  $A$  to the identity matrix by row operations and simultaneously apply these operations to  $I$  to produce  $A^{-1}$ .

The  
follows:

computations are as

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

← We added  $-2$  times the first row to the second and  $-1$  times the first row to the third.

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right]$$

← We added 2 times the second row to the third.

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We multiplied the third row by  $-1$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We added 3 times the third row to the second and  $-3$  times the third row to the first.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We added  $-2$  times the second row to the first.

Thus,  $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$  ✦

### Example 5 (Showing That a Matrix Is Not Invertible)

Consider the matrix  $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Applying the procedure of Example 4 yields

$$\begin{bmatrix} 1 & 6 & 4 & | & 1 & 0 & 0 \\ 2 & 4 & -1 & | & 0 & 1 & 0 \\ -1 & 2 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 6 & 4 & | & 1 & 0 & 0 \\ 0 & -8 & -9 & | & -2 & 1 & 0 \\ 0 & 8 & 9 & | & 1 & 0 & 1 \end{bmatrix} \quad \leftarrow \text{We added } -2 \text{ times the first row to the second and added the first row to the third.}$$
$$\begin{bmatrix} 1 & 6 & 4 & | & 1 & 0 & 0 \\ 0 & -8 & -9 & | & -2 & 1 & 0 \\ 0 & 0 & 0 & | & -1 & 1 & 1 \end{bmatrix} \quad \leftarrow \text{We added the second row to the third.}$$


Since we have obtained a row of zeros on the left side,  $A$  is not invertible.

### Example 6 (A Consequence of Invertibility)

In Example 4 we showed that  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

is an invertible matrix. From Theorem 3, it follows that the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + 8x_3 &= 0 \end{aligned}$$

has only the trivial solution 

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## Exercises and Home works

1) Which of the following are elementary matrices?

(a)  $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$

(b)  $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$

(d)  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(e)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

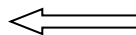
(f)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$

(g)  $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

### Solution

(a)

$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$   
elementary matrix



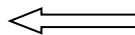
$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-5R_1 + R_2 \rightarrow R_2$$

The given  $2 \times 2$  matrix is obtained by adding -5 times the first row of  $I_2$  to the second row.

(b)

$\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$   
not elementary matrix

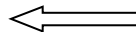


$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The given  $2 \times 2$  matrix cannot be obtained by performing a single row elementary operation on  $I_2$

(c)

$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$   
elementary matrix

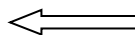


$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt{3}R_2 \rightarrow R_2$$

(d)

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$   
elementary matrix



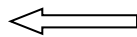
$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

(e)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

*not elementary matrix*

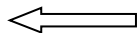


The given  $3 \times 3$  matrix cannot be obtained by performing a single row elementary operation on  $I_3$

(f)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

*Is elementary matrix*



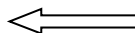
$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$9R_2 \rightarrow R_2$

(g)

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*not elementary matrix*



$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The given  $4 \times 4$  matrix cannot be obtained by performing a single row elementary operation on  $I_4$

2) Consider the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matrices  $E_1, E_2, E_3$  and  $E_4$  such that

(a)  $E_1 A = B$

(b)  $E_2 B = A$

(c)  $E_3 A = C$

(d)  $E_4 C = A$

3) Find a row operation that will restore the given elementary matrix to an identity matrix.

(a)  $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(c)  $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**Solution**



$$(a) \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad \Longrightarrow \quad 3R_1 + R_2 \rightarrow R_2$$

$$(b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \Longrightarrow \quad \frac{1}{3}R_3 \rightarrow R_3$$

$$(c) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad R_1 \leftrightarrow R_3$$

$$(d) \begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad 7R_3 + R_1 \rightarrow R_1$$

**4)** In Exercise 3 is it possible to find an elementary matrix  $E$  such  $EB = C$  ? Justify your answer.

**5)** If a  $2 \times 2$  matrix is multiplied on the left by the given matrices, what elementary row operation is performed on that matrix?.

$$(a) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(b) \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

**6)** In Exercises 6–8 use the method shown in Examples Example 4 and Example 5 to find the inverse of the given matrix if the matrix is invertible, and check your answer by multiplication.

$$(a) \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$

$$(b) \begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$$

$$(c) \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$$

7)

(a)  $\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

(b)  $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

(e)  $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

### Solution

(a)

$$\left[ \begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array} \right]$$

Interchange Rows  
1 and 2.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array} \right]$$

Add  $-3$  times Row 1  
to Row 2 and  $-2$  times  
Row 1 to 3.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{array} \right]$$

Add  $-1$  times Row 2  
to Row 3.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array} \right]$$

Add  $-4$  times Row 3  
to Row 2 and inter-  
change Rows 2 and 3.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array} \right]$$

Multiply Row 3 by  $-1/10$ . Then add  $-3$  times Row 3 to Row 1.

Thus, the desired inverse is

$$\left[ \begin{array}{ccc} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array} \right]$$

(c)

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array} \right]$$

Subtract Row 1 from Row 3.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

Subtract Row 2 from Row 3 and multiply Row 3 by  $-1/2$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

Subtract Row 3 from Rows 1 and 2.

Thus

$$\left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right]^{-1} = \left[ \begin{array}{ccc} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

8)

$$(a) \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$$

$$(b) \begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

$$(d) \begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$$

$$(e) \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$$

9) Find the inverse of each of the following matrices, where  $k_1, k_2, k_3, k_4$  and  $k$  are all nonzero.

$$(a) \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$$

$$(b) \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$$

### Solution

(b) Multiplying Row  $i$  of

$$\left[ \begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

by  $1/k_i$  for  $i = 1, 2, 3, 4$  and then reversing the order of the rows yields  $I_4$  on the left and the desired inverse

$$\begin{bmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{bmatrix}$$

on the right.

(c) To reduce

$$\left[ \begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array} \right]$$

we multiply Row  $i$  by  $1/k$  and then subtract Row  $i$  from Row  $(i + 1)$  for  $i = 1, 2, 3$ . Then multiply Row 4 by  $1/k$ . This produces  $I_4$  on the left and the inverse,

$$\begin{bmatrix} 1/k & 0 & 0 & 0 \\ -1/k^2 & 1/k & 0 & 0 \\ 1/k^3 & -1/k^2 & 1/k & 0 \\ 1/k^4 & 1/k^3 & -1/k^2 & 1/k \end{bmatrix}$$

on the right.

10). Consider the matrix  $A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

- Find elementary matrices  $E_1$  and  $E_2$  such that  $E_2 E_1 A = I$ .
- Write  $A^{-1}$  as a product of two elementary matrices.
- Write  $A$  as a product of two elementary matrices.

11). Write the matrix  $A = \begin{bmatrix} 3 & -2 \\ 3 & -1 \end{bmatrix}$

as a product of elementary matrices.

**Note** There is more than one correct solution.

12). Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

- Find elementary matrices  $E_1, E_2$  and  $E_3$  such that  $E_3 E_2 E_1 A = I$ .
- Write  $A$  as a product of elementary matrices.

### Solution

(a)  $E_3 E_2 E_1 A =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$(b) \ A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**13).** Express the matrix  $A = \begin{bmatrix} 0 & 1 & 7 & 8 \\ 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \end{bmatrix}$  in the form  $A = EFG R$  where  $E, F$  and  $G$  are elementary matrices and  $R$  is in row-echelon form.

**14)** Show that if  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$  is an elementary matrix, then at least one entry in the third row must be a zero.