Example 9. *If* $z = \sqrt{x^2 + y^2}$ *and* $x^3 + y^3 + 3axy = 5a^2$*, find the value of* $\frac{dz}{dx}$*, when* $x = y = a$.

Sol. The given equations are of the form $z = f(x, y)$ and $\phi(x, y) = c$

$\therefore$ z is composite function of x.

$$\Rightarrow \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad ...(1)$$

Now $$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

Also, differentiating $x^3 + y^3 + 3axy = 5a^2$ w.r.t. x, we have

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} + 3ay + 3ax \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad (y^2 + ax)\frac{dy}{dx} = -(x^2 + ay)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$$

$\therefore$ From (1), $$\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{x^2 + ay}{y^2 + ax}\right)$$

$$\left[\frac{dz}{dx}\right]_{\substack{x = a \\ y = a}} = \frac{a}{\sqrt{a^2 + a^2}} + \frac{a}{\sqrt{a^2 + a^2}} \cdot \frac{a^2 + a^2}{a^2 + a^2} = 0.$$

Example 10. *If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, find* $\frac{du}{dx}$.

Sol. Here u is a function of x, y and z while y and z are functions of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= e^y z \,.\, 1 + xe^y z \,.\, \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + xe^y \,.\, 2\sin x\cos x$$

$$= e^y\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x\sin 2x\right].$$

Example 11. *Find* $\frac{du}{dx}$ *if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$.*

Sol. The given equations are the form $u = f(x, y)$ and $\phi(x, y) = k$

$\therefore$ u is a composite function of x.

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \text{...(1)}$$

Now $$\frac{\partial u}{\partial x} = 2x\cos(x^2 + y^2), \quad \frac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$$

Also, differentiating $a^2x^2 + b^2y^2 = c$ w.r.t. x, we have

$$2a^2x + 2b^2y\,.\,\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{a^2x}{b^2y}$$

$\therefore$ From (1), $$\frac{du}{dx} = 2x\cos(x^2 + y^2) + 2y\cos(x^2 + y^2)\,.\left[-\frac{a^2x}{b^2y}\right]$$

$$= 2\left[x - \frac{a^2x}{b^2}\right]\cos(x^2 + y^2) = \frac{2(b^2 - a^2)x}{b^2}\,.\,\cos(x^2 + y^2).$$

Example 12. *Find* $\frac{dy}{dx}$, *when*

(i) $x^y + y^x = c$ *(ii)* $(\cos x)^y = (\sin y)^x$.

Sol. *(i)* Let $f(x, y) = x^y + y^x$, then $f(x, y) = c$

[Using Cor. 4] $\therefore$ $$\frac{dy}{dx} = -\frac{f_x}{f_y} \Rightarrow \frac{dy}{dx} = -\frac{yx^{y-1} + y^x\log y}{x^y\log x + xy^{x-1}}.$$

(ii) Let $f(x, y) = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y(\cos x)^{y-1}\,.\,(-\sin x) - (\sin y)^x\log(\sin y)}{(\cos x)^y\log(\cos x) - x(\sin y)^{x-1}\,.\cos y}$$

$$= \frac{y(\cos x)^{y-1}\sin x + (\cos x)^y\log(\sin y)}{(\cos x)^y\log(\cos x) - x(\cos x)^y(\sin y)^{-1}\cos y}$$

$[\because \quad (\sin y)^x = (\cos x)^y]$

$$= \frac{(\cos x)^y\left[y\,.\,\dfrac{\sin x}{\cos x} + \log\sin y\right]}{(\cos x)^y\,[\log\cos x - x\cot y]} = \frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}.$$

Example 13. *If $f(x, y) = 0$, $\phi(y, z) = 0$, show that* $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$.

Sol. $$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \qquad \text{...(1)}$$

$f(x, y) = 0$ gives $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$; $\phi(y, z) = 0$ gives $\frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$

$\therefore$ From (1), $$\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}}{\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}} \Rightarrow \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

Example 14. *If $\phi(x, y, z) = 0$, show that* $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$.

Sol. The given relation defines y as a function of x and z. Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial y}}$$

The given relation defines z as a function of x and y. Treating y as constant $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$

Similarly, $$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$$

Multiplying, we get the desired result.

10.9. JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v with respect to x, y and is denoted by the symbol $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$.

Similarly, if u, v, w be functions of x, y, z, then the Jacobian of u, v, w with respect to x, y, z is

$$J\left(\frac{u, v, w}{x, y, z}\right) \quad \text{or} \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

10.10. PROPERTIES OF JACOBIANS

I. If u, v are functions of r, s where r, s are functions of x, y, then

$$\frac{\partial(\mathbf{u}, \mathbf{v})}{\partial(\mathbf{x}, \mathbf{y})} = \frac{\partial(\mathbf{u}, \mathbf{v})}{\partial(\mathbf{r}, \mathbf{s})} \times \frac{\partial(\mathbf{r}, \mathbf{s})}{\partial(\mathbf{x}, \mathbf{y})}. \qquad \text{[Chain Rule for Jacobians]}$$

Proof. Since u, v are composite functions of x, y

$$\therefore \quad \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = u_r r_x + u_s s_x \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} = v_r r_x + v_s s_x \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} = v_r r_y + v_s s_y \end{aligned}\right] \quad \ldots(A)$$

Now $$\frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix} = \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad \text{[Using (A)]}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}.$$

II. If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y, with respect to u, v, then $J_1 J_2 = 1$ i.e., $\frac{\partial(\mathbf{u}, \mathbf{v})}{\partial(\mathbf{x}, \mathbf{y})} \cdot \frac{\partial(\mathbf{x}, \mathbf{y})}{\partial(\mathbf{u}, \mathbf{v})} = 1$.

Proof. Let $u = u(x, y)$ and $v = v(x, y)$, so that u and v are functions of x, y.
Differentiating partially w.r.t. u and v, we get

$$\left.\begin{aligned}
1 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} = u_x x_u + u_y y_u \\
0 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} = u_x x_v + u_y y_v \\
0 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} = v_x x_u + v_y y_u \\
1 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} = v_x x_v + v_y y_v
\end{aligned}\right] \quad \text{...(A)}$$

Now $$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \qquad \text{[Using (A)]}$$

III. Jacobian of Implicit Functions

If $f_1(x, y, u, v) = 0$, $f_2(x, y, u, v) = 0$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

In general,

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)} = (-1)^n \frac{\partial(f_1,f_2,\ldots,f_n)/\partial(x_1,x_2,\ldots,x_n)}{\partial(f_1,f_2,\ldots,f_n)/\partial(u_1,u_2,\ldots,u_n)}$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$, *evaluate* $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

Sol. $$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}, \qquad \frac{\partial\theta}{\partial y} = \frac{1}{1+\dfrac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}}.$$

Example 2. *If* $x = r \sin\theta \cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, *show that* $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta.$

(K.U.K., 2005)

Sol.
$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking out common factors (r from second column and $r \sin\theta$ from third column)

$$= r^2 \sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by third row

$$= r^2\sin\theta \left\{ \cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} \right\}$$

$$= r^2\sin\theta\,[\cos\theta\,(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta\,(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)]$$

$$= r^2\sin\theta\,(\cos^2\theta + \sin^2\theta) = r^2\sin\theta.$$

Note 1. Here (x, y, z) and (r, θ, ϕ) are respectively the Cartesian and spherical polar coordinates of a point.

Note 2. $\dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}} = \dfrac{1}{r^2\sin\theta}$

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