Example 3. *If* $u = \dfrac{x}{y-z}, v = \dfrac{y}{z-x}, w = \dfrac{z}{x-y}$, *show that* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$

Sol. Here $$u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}$$

$\Rightarrow$
$$\log u = \log x - \log (y - z) \quad ...(1)$$
$$\log v = \log y - \log (z - x) \quad ...(2)$$
$$\log w = \log z - \log (x - y) \quad ...(3)$$

Differentiating (1) partially w.r.t. x

$$\frac{1}{u}\cdot\frac{\partial u}{\partial x} = \frac{1}{x} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{u}{x}$$

Differentiating (1) partially w.r.t. y

$$\frac{1}{u}\cdot\frac{\partial u}{\partial y} = -\frac{1}{y-z} \quad \Rightarrow \quad \frac{\partial u}{\partial y} = \frac{-u}{y-z}$$

Differentiating (1) partially w.r.t. z

$$\frac{1}{u}\cdot\frac{\partial u}{\partial z} = -\frac{1}{y-z}(-1) \quad \Rightarrow \quad \frac{\partial u}{\partial z} = \frac{u}{y-z}$$

Similarly from (2) and (3), we have

$$\frac{\partial v}{\partial x} = \frac{v}{z-x}, \quad \frac{\partial v}{\partial y} = \frac{v}{y}, \quad \frac{\partial v}{\partial z} = \frac{-v}{z-x}$$
$$\frac{\partial w}{\partial x} = \frac{-w}{x-y}, \quad \frac{\partial w}{\partial y} = \frac{w}{x-y}, \quad \frac{\partial w}{\partial z} = \frac{w}{z}$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{u}{x} & \frac{-u}{y-z} & \frac{u}{y-z} \\ \frac{v}{z-x} & \frac{v}{y} & \frac{-v}{z-x} \\ \frac{-w}{x-y} & \frac{w}{x-y} & \frac{w}{z} \end{vmatrix}$$ Taking out u, v, w from R_1, R_2, R_3 respectively

$$= uvw \begin{vmatrix} \frac{1}{x} & \frac{-1}{y-z} & \frac{1}{y-z} \\ \frac{1}{z-x} & \frac{1}{y} & \frac{-1}{z-x} \\ \frac{-1}{x-y} & \frac{1}{x-y} & \frac{1}{z} \end{vmatrix}$$ Multiplying R_1, R_2, R_3 by $y-z, z-x, x-y$ respectively

$$= \frac{uvw}{(y-z)(z-x)(x-y)} \begin{vmatrix} \frac{y-z}{x} & -1 & 1 \\ 1 & \frac{z-x}{y} & -1 \\ -1 & 1 & \frac{x-y}{z} \end{vmatrix}$$ Multiplying C_1, C_2, C_3 by x, y, z respectively

$$= \frac{uvw}{xyz(y-z)(z-x)(x-y)} \begin{vmatrix} y-z & -y & z \\ x & z-x & -z \\ -x & y & x-y \end{vmatrix}$$ Operating $C_1 \to C_1 + C_2 + C_3$

$$= \frac{1}{(y-z)^2 (z-x)^2 (x-y)^2} \begin{vmatrix} 0 & -y & z \\ 0 & z-x & -z \\ 0 & y & x-y \end{vmatrix} = 0$$

$$\left[\because \frac{u}{x} = \frac{1}{y-z}, \frac{v}{y} = \frac{1}{z-x}, \frac{w}{z} = \frac{1}{x-y}\right.$$

EXERCISE 10.4

1. If $u = x^2 - 2y$, $v = x + y$, prove that $\frac{\partial(u, v)}{\partial(x, y)} = 2x + 2$.

2. (*a*) If $u = x(1 - y)$, $v = xy$; prove that $JJ' = 1$.

(*b*) Prove that $JJ' = 1$ if $x = uv, y = \dfrac{u}{v}$

[**Hint.** $u = vy \Rightarrow x = v^2 y \quad \therefore \quad v = \sqrt{\dfrac{x}{y}}$ and $u = \sqrt{xy}$]

3. (*a*) In polar coordinates, $x = r\cos\theta, y = r\sin\theta$, show that $\dfrac{\partial(x, y)}{\partial(r, \theta)} = r$.

(*b*) If $x = r\cos\theta, y = r\sin\theta$, verify that $\dfrac{\partial(x, y)}{\partial(r, \theta)} \cdot \dfrac{\partial(r, \theta)}{\partial(x, y)} = 1$.

$\left[\textbf{Hint. } r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\dfrac{y}{x}.\right]$

4. If $x = r\cos\theta, y = r\sin\theta, z = z$, evaluate $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$.

Note. Here (x, y, z) and (r, θ, z) are respectively the Cartesian and cylindrical coordinates of a r t.

5. (*a*) If $y_1 = \dfrac{x_2 x_3}{x_1}, \quad y_2 = \dfrac{x_3 x_1}{x_2}, \quad y_3 = \dfrac{x_1 x_2}{x_3}$, show that $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$. (*U.P.T.U. 2006*)

(*b*) If $u = \dfrac{yz}{x}, \quad v = \dfrac{zx}{y}, \quad w = \dfrac{xy}{z}$, show that $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \dfrac{1}{4}$.

6. If $u = \dfrac{y^2}{2x}, v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

10.11. TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

We know that by Taylor's theorem for a function $f(x)$ of single variable x,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

Now let $f(x, y)$ be a function of two independent variables x and y. **If y is kept constant,** then by Taylor's theorem for a function of a single variable x, we have

$$f(x+h, \mathbf{y}+\mathbf{k}) = f(x, \mathbf{y}+\mathbf{k}) + h\frac{\partial}{\partial x} f(x, \mathbf{y}+\mathbf{k}) + \frac{h^2}{2!}\cdot\frac{\partial^2}{\partial x^2} f(x, \mathbf{y}+\mathbf{k}) + \frac{h^3}{3!}\cdot\frac{\partial^3}{\partial x^3} f(x, \mathbf{y}+\mathbf{k}) + \ldots\ldots \quad \ldots(1)$$

Now **keeping x constant** and applying Taylor's theorem for a function of a single variable y, we have

$$f(\mathbf{x}, y+k) = f(\mathbf{x}, y) + k\frac{\partial}{\partial y} f(\mathbf{x}, y) + \frac{k^2}{2!}\cdot\frac{\partial^2}{\partial y^2} f(\mathbf{x}, \mathbf{y}) + \frac{k^3}{3!}\cdot\frac{\partial^3}{\partial y^3} f(\mathbf{x}, \mathbf{y}) + \ldots\ldots \quad \ldots(2)$$

Using (2), we can write (1) as

$$\begin{aligned} f(x+h, y+k) = &\left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\cdot\frac{\partial^2}{\partial y^2} f(x,y) + \frac{k^3}{3!}\cdot\frac{\partial^3}{\partial y^3} f(x,y) + \ldots\ldots\right] \\ &+ h\frac{\partial}{\partial x}\left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\cdot\frac{\partial^2}{\partial y^2} f(x,y) + \ldots\ldots\right] \\ &+ \frac{h^2}{2!}\cdot\frac{\partial^2}{\partial x^2}\left[f(x,y) + k\frac{\partial}{\partial y} f(x,y) + \ldots\ldots\right] + \frac{h^3}{3!}\cdot\frac{\partial^3}{\partial x^3}\,[f(x, y) + \ldots.] + \ldots\ldots \end{aligned}$$

$$= \left[f(x,y) + k\frac{\partial f}{\partial y} + \frac{k^2}{2!}\cdot\frac{\partial^2 f}{\partial y^2} + \frac{k^3}{3!}\cdot\frac{\partial^3 f}{\partial y^3} + \ldots\ldots\right]$$

$$+ \left[h\frac{\partial f}{\partial x} + hk\frac{\partial^2 f}{\partial x\partial y} + \frac{hk^2}{2!}\cdot\frac{\partial^3 f}{\partial x\partial y^2} + \ldots\ldots\right]$$

$$+ \left[\frac{h^2}{2!}\cdot\frac{\partial^2 f}{\partial x^2} + \frac{h^2 k}{2!}\cdot\frac{\partial^3 f}{\partial x^2\partial y} + \ldots\ldots\right]$$

$$+ \frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \ldots\ldots$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \left(\frac{h^2}{2!}\cdot\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x\partial y} + \frac{k^2}{2!}\cdot\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \left(\frac{h^3}{3!}\cdot\frac{\partial^3 f}{\partial x^3} + \frac{h^2 k}{2!}\cdot\frac{\partial^3 f}{\partial x^2\partial y} + \frac{hk^2}{2!}\cdot\frac{\partial^3 f}{\partial x\partial y^2} + \frac{k^3}{3!}\cdot\frac{\partial^3 f}{\partial y^3}\right) + \ldots\ldots$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2 k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \ldots\ldots$$

$$= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots\ldots$$

Cor. 1. Putting $x = a$ and $y = b$, we have

$$f(a + h, b + k) + f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)]$$

$$+ \frac{1}{3!}[h^3 f_{xxx}(a, b) + 3h^2 kf_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)] + \ldots$$

Cor. 2. In Cor. 1, putting $a + h = x$ and $b + k = y$ so that $h = x - a$ and $k = y - b$, we have

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)] + \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \ldots$$

Cor. 3. Putting $a = 0, b = 0$ in Cor. 2, we have

$$f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots$$

This is called Maclaurin's theorem for two variables.

Note. Cor. 3 is used to expand $f(x, y)$ in powers of x and y [or to expand $f(x, y)$ in the neighbourhood of origin (0, 0)].

Cor. 2 is used to expand $f(x, y)$ in the neighbourhood of (a, b).

ILLUSTRATIVE EXAMPLES

Example 1. *Expand $e^x \sin y$ in powers of x and y as far as terms of the third degree.*

Sol. Here

$$
\begin{aligned}
f(x, y) &= e^x \sin y\,; & f(0, 0) &= 0\\
f_x(x, y) &= e^x \sin y, & f_x(0, 0) &= 0\\
f_y(x, y) &= e^x \cos y, & f_y(0, 0) &= 1\\
f_{xx}(x, y) &= e^x \sin y, & f_{xx}(0, 0) &= 0\\
f_{xy}(x, y) &= e^x \cos y, & f_{xy}(0, 0) &= 1\\
f_{yy}(x, y) &= -e^x \sin y, & f_{yy}(0, 0) &= 0\\
f_{xxx}(x, y) &= e^x \sin y, & f_{xxx}(0, 0) &= 0\\
f_{xxy}(x, y) &= e^x \cos y, & f_{xxy}(0, 0) &= 1\\
f_{xyy}(x, y) &= -e^x \sin y, & f_{xyy}(0, 0) &= 0\\
f_{yyy}(x, y) &= -e^x \cos y, & f_{yyy}(0, 0) &= -1
\end{aligned}
$$

....

$$
\begin{aligned}
\therefore\quad e^x \sin y &= f(x, y)\\
&= f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]\\
&\quad + \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots\\
&= 0 + [x\,.\,0 + y\,.\,1] + \frac{1}{2!}[x^2\,.\,0 + 2xy\,.\,1 + y^2\,.\,0] + \frac{1}{3!}[x^3.\,0 + 3x^2y\,.\,1 + 3xy^2\\
&\quad + 0 + y^3(-1)] + \ldots\\
&= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \ldots
\end{aligned}
$$

Example 2. *Expand $f(x, y) = \tan^{-1}\dfrac{y}{x}$ in powers of $(x - 1)$ and $(y - 1)$ upto third degree terms. Hence compute $f(1.1, 0.9)$ approximately.*

(M.D.U. Dec. 2007 ; U.P.T.U. 2006 ; J.N.T.U. 2006)

Sol. Here $f(x, y) = \tan^{-1}\dfrac{y}{x}$, $f(1, 1) = \tan^{-1} 1 = \dfrac{\pi}{4}$

$$f_x(x, y) = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}, \quad f_x(1, 1) = -\frac{1}{2}$$

$$f_y(x, y) = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}, \quad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx}(x, y) = -y(-1)(x^2+y^2)^{-2}.\,2x = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(x^2+y^2).1 - x\,.\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1, 1) = 0$$

$$f_{yy}(x, y) = x(-1)(x^2+y^2)^{-2}\,.\,2y = -\frac{2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1, 1) = -\frac{1}{2}$$

$$f_{xxx}(x, y) = 2y\,.\,\frac{(x^2+y^2)^2\,.\,1 - x\,.\,2(x^2+y^2)\,.\,2x}{(x^2+y^2)^4} = 2y\,.\,\frac{(x^2+y^2)-4x^2}{(x^2+y^2)^3}$$

$$= \frac{2y(y^2 - 3x^2)}{(x^2 + y^2)^3}, \quad f_{xxx}(1, 1) = -\frac{1}{2}$$

$$f_{xxy}(x, y) = \frac{(x^2 + y^2)^2(-2x) - (y^2 - x^2).2(x^2 + y^2).2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)(-2x) - 4x(y^2 - x^2)}{(x^2 + y^2)^3} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3}, \quad f_{xxy}(1, 1) = -\frac{1}{2}$$

$$f_{xyy}(x, y) = -2y \cdot \frac{(x^2 + y^2)^2.1 - x.2(x^2 + y^2).2x}{(x^2 + y^2)^4} = \frac{-2y(x^2 + y^2 - 4x^2)}{(x^2 + y^2)^3}$$

$$= \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}, \quad f_{xyy}(1, 1) = \frac{1}{2}$$

$$f_{yyy}(x, y) = -2x \cdot \frac{(x^2 + y^2)^2.1 - y.2(x^2 + y^2).2y}{(x^2 + y^2)^4} = \frac{-2x(x^2 + y^2 - 4y^2)}{(x^2 + y^2)^3}$$

$$= \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3}, \quad f_{yyy}(1, 1) = \frac{1}{2}$$

$$\therefore \tan^{-1}\frac{y}{x} = f(x, y) = f(1, 1) + [(x-1)f_x(1, 1) + (y-1)f_y(1, 1)]$$

$$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1)f_{xy}(1, 1) + (y-1)^2 f_{yy}(1, 1)]$$

$$+ \frac{1}{3!}[(x-1)^3 f_{xxx}(1, 1) + 3(x-1)^2(y-1)f_{xxy}(1, 1)$$

$$+ 3(x-1)(y-1)^2 f_{xyy}(1, 1) + (y-1)^3 f_{yyy}(1, 1)] + \ldots\ldots$$

$$= \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

$$+ \frac{1}{6}\left[(x-1)^3\left(-\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2\left(\frac{1}{2}\right) + (y-1)^3\left(\frac{1}{2}\right)\right] + \ldots\ldots$$

$$= \frac{\pi}{4} - \frac{1}{2}[(x-1) - (y-1)] + \frac{1}{4}[(x-1)^2 - (y-1)^2] - \frac{1}{12}[(x-1)^3$$

$$+ 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3] + \ldots\ldots$$

Putting $x = 1.1$ and $y = 0.9$, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{4}(0) - \frac{1}{12}[(0.1)^3 + 3(0.1)^2(-0.1) - 3(0.1)(-0.1)^2 - (-0.1)^3]$$

$$= 0.7854 - 0.1 + 0.0003 = 0.6857.$$

Example 3. *Expand $e^x \log(1 + y)$ in powers of x and y upto terms of third degree.*

(J.N.T.U. 2006 ; M.D.U. May 2008 ; P.T.U. 2006)

Sol. Here

$f(x, y) = e^x \log(1 + y)$, $\quad f(0, 0) = 0$

$f_x(x, y) = e^x \log(1 + y)$. $\quad f_x(0, 0) = 0$

$f_y(x, y) = \dfrac{e^x}{1 + y}$, $\quad f_y(0, 0) = 1$

$f_{xx}(x, y) = e^x \log(1 + y)$, $\quad f_{xx}(0, 0) = 0$

$f_{xy}(x, y) = \dfrac{e^x}{1 + y}$, $\quad f_{xy}(0, 0) = 1$

$$f_{yy}(x, y) = -\frac{e^x}{(1+y)^2}, \qquad f_{yy}(0, 0) = -1$$

$$f_{xxx}(x, y) = e^x \log(1+y), \qquad f_{xxx}(0, 0) = 0$$

$$f_{xxy}(x, y) = \frac{e^x}{1+y}, \qquad f_{xxy}(0, 0) = 1$$

$$f_{xyy}(x, y) = -\frac{e^x}{(1+y)^2}, \qquad f_{xyy}(0, 0) = -1$$

$$f_{yyy}(x, y) = \frac{2e^x}{(1+y)^3}, \qquad f_{yyy}(0, 0) = 2$$

....

$$\therefore \quad e^x \log(1+y) = f(x, y)$$

$$= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2y\, f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots$$

$$= 0 + [x.0 + y.1] + \tfrac{1}{2}[x^2.0 + 2xy.1 + y^2(-1)] + \frac{1}{6}[x^3.0 + 3x^2y.1 + 3xy^2.(-1) + y^3.2] + \ldots$$

$$= y + xy - \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2y - \tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3 \ldots\ldots$$

Example 4. *Expand* $\dfrac{(x+h)(y+k)}{x+h+y+k}$ *in powers of h, k upto and inclusive of the second degree terms.*

Sol. Here $$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

Putting $h = k = 0$, we have $f(x, y) = \dfrac{xy}{x+y}$

$$f_x = \frac{(x+y).y - xy.1}{(x+y)^2} = \frac{y^2}{(x+y)^2}; \qquad f_y = \frac{x^2}{(x+y)^2}, \text{ by symmetry}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3}, \; f_{yy} = -\frac{2x^2}{(x+y)^3}$$

$$f_{xy} = \frac{(x+y)^2.2x - x^2.2(x+y)}{(x+y)^4} = \frac{2x(x+y) - 2x^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$\therefore \quad \frac{(x+h)(y+k)}{x+h+y+k} = f(x+h, y+k) = f(x, y) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}] + \ldots$$

$$= \frac{xy}{x+y} + \left[h.\frac{y^2}{(x+y)^2} + k.\frac{x^2}{(x+y)^2}\right]$$

$$+ \frac{1}{2}\left[h^2.\frac{-2y^2}{(x+y)^3} + 2hk\frac{2xy}{(x+y)^3} + k^2.\frac{-2x^2}{(x+y)^3}\right] + \ldots$$

$$= \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}.h + \frac{x^2}{(x+y)^2}.k - \frac{y^2}{(x+y)^3}.h^2$$

$$+ \frac{2xy}{(x+y)^3}.hk - \frac{x^2}{(x+y)^3}.k^2 + \ldots$$

Example 5. *Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's Theorem.*
(M.D.U. May 2006, Dec. 2006 ; U.P.T.U. 2006 ; Anna 2005)

Sol. Expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] + \frac{1}{2!}[(x - a)^2 f_{xx}(a, b)$$
$$+ 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x - a)^3 f_{xxx}(a, b)$$
$$+ 3(x - a)^2(y - b)f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b)]$$
$$+ \ldots \qquad \ldots(1)$$

Here $f(x, y) = x^2y + 3y - 2$, $a = 1$, $b = -2$

$f(1, -2) = 1^2 \times (-2) + 3(-2) - 2 = -10$

$f_x = 2xy$, $\quad f_x(1, -2) = 2(1)(-2) = -4$; $\quad f_y = x^2 + 3$, $\quad f_y(1, -2) = 1^2 + 3 = 4$

$f_{xx} = 2y$, $\quad f_{xx}(1, -2) = 2(-2) = -4$; $\quad f_{xy} = 2x$, $\quad f_{xy}(1, -2) = 2(1) = 2$

$f_{yy} = 0$, $\quad f_{yy}(1, -2) = 0$; $\quad f_{xxx} = 0$, $\quad f_{xxx}(1, -2) = 0$

$f_{xxy} = 2$, $\quad f_{xxy}(1, -2) = 2$; $\quad f_{xyy} = 0$, $\quad f_{xyy}(1, -2) = 0$

$f_{yyy} = 0$, $\quad f_{yyy}(1, -2) = 0$

All higher order partial derivatives vanish.

$\therefore$ From (1), we have

$x^2y + 3y - 1 = f(x, y)$

$$= -10 + [(x - 1)(-4) + (y + 2)(4)] + \tfrac{1}{2}[(x - 1)^2(-4) + 2(x - 1)(y + 2)(2) + (y + 2)^2(0)]$$
$$+ \tfrac{1}{6}[(x - 1)^3(0) + 3(x - 1)^2(y + 2)(2) + 3(x - 1)(y + 2)^2(0) + (y + 2)^3(0)]$$
$$= -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2).$$

EXERCISE 10.5

1. Show that $e^y \log(1 + x) = x + xy - \dfrac{x^2}{2}$ approximately.
   [**Hint.** Find the expansion at (0, 0)]
2. Expand $e^x \cos y$ in powers of x and y as far as the terms of third degree.
3. Expand $e^{ax} \sin by$ in powers of x and y as far as the terms of third degree. *(M.D.U. May 2005)*
4. Expand e^{xy} at (1, 1) in powers of $(x - 1)$ and $(y - 1)$.
5. Expand $e^x \cos y$ about $\left(1, \dfrac{\pi}{4}\right)$.
6. E...