Functions of Two or More Variables

(Limit and Continuity)

9.1. FUNCTION OF TWO VARIABLES

If a quantity z has a unique, finite value for every pair of values of x and y, then z is called a function of two variables x and y. A function of two variables x and y is symbolically written as

$$f(x, y) \text{ or } F(x, y) \text{ or } \phi(x, y).$$

Domain of a function of two variables is a subset of $R^2 = R \times R = \{(x, y) : x, y \in R\}$ and range is a subset of R. Thus a function f of two variables is denoted as

$$f : S \to R \text{ where } S \subset R^2.$$

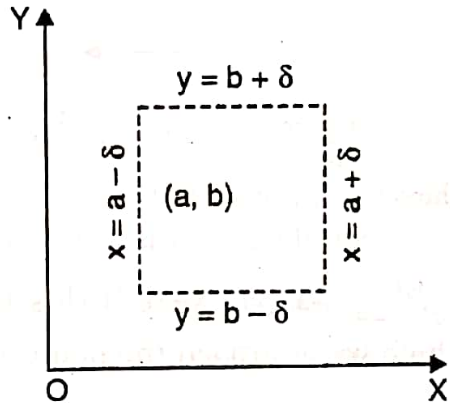
Similarly, a function f of three variables is denoted as $F : S \to R$ where $S \subset R^3$.

9.2. NEIGHBOURHOOD OF A POINT (a, b)

Every point (a, b) in R^2 has two types of neighbourhoods :

(*i*) **Square Neighbourhood**

The interior of the square with centre at (a, b), sides parallel to the coordinate axes and each side $= 2\delta$ is called a square neighbourhood of the point (a, b). For every positive value of δ, we get a square neighbourhood of (a, b).



Thus a square nbd of (a, b) is

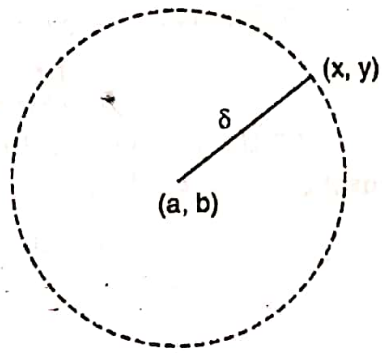
$$\{(x, y) : a - \delta < x < a + \delta, b - \delta < y < b + \delta\}$$
$$= \{(x, y) : |x - a| < \delta, |y - b| < \delta\}$$

Similarly a nbd of (a, b, c) in the form of a cube is

$$\{(x, y, z) : a - \delta < x < a + \delta, b - \delta < y < b + \delta, c - \delta < z < c + \delta\}$$
$$= \{(x, y, z) : |x - a| < \delta, |y - b| < \delta, |z - c| < \delta\}$$

(*ii*) **Circular Neighbourhood**

The interior of the circle with centre at (a, b) and radius δ is called a circular neighbourhood of the point (a, b). For every positive value of δ, we get a circular nbd of (a, b).



Thus a circular nbd of (a, b) is

$\{(x, y) : |(x, y) - (a, b)| < \delta\}$ where $|(x, y) - (a, b)|$ stands for the distance between the points (x, y) and (a, b)

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i.e., $\qquad | (x, y) - (a, b) | = \sqrt{(x-a)^2 + (y-b)^2}$

Similarly, a spherical nbd of (a, b, c) is

$\{(x, y, z) : | (x, y, z) - (a, b, c) | < \delta\}$

where $\qquad | (x, y, z) - (a, b, c) | = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

9.3. LIMIT OF A FUNCTION OF TWO VARIABLES

A function $f(x, y)$ is said to tend to a limit l as the point (x, y) tends to the point (a, b) if corresponding to any pre-assigned positive number ε, however small, we can find a positive number δ (depending on ε) such that

Def. 1. $\qquad |f(x, y) - l| < \varepsilon$

for all points (x, y) other than (a, b) for which $|x - a| < \delta$ and $|y - b| < \delta$

This definition of limit is based on square neighbourhood of a point.

Def. 2. $\qquad |f(x, y) - l| < \varepsilon$

for all points (x, y) other than (a, b) for which

$$| (x, y) - (a, b) | < \delta$$

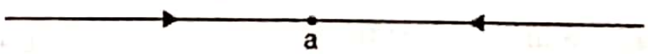
This definition of limit is based on square neighbourhood of a point.

Note 1. A function $f(x, y)$ tends to a limit l as the point (x, y) tends to point (a, b) is symbolically written as

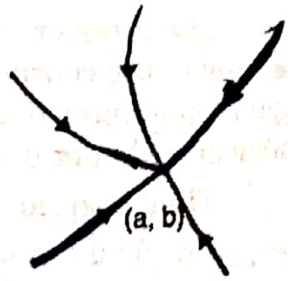
$$\underset{(x,y)\to(a,b)}{\text{Lt}} f(x, y) = l$$

Note 2. $\underset{(x,y)\to(a,b)}{\text{Lt}} f(x, y)$ if it exists, is unique

Note 3. We know that if f is a function of single variable x, then $\underset{x\to a}{\text{Lt}} f(x)$ exists iff



$\underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x)$, i.e., the limit is independent of the path along which we approach the point 'a'.



Similarly, if f is a function of two variables x and y, then $\underset{(x,y)\to(a,b)}{\text{Lt}} f(x, y)$ exists if this limit is independent of the path along which we approach the point (a, b).

9.4. CONTINUITY OF A FUNCTION OF TWO VARIABLES

A function $f(x, y)$ is said to be continuous at the point (a, b) if

$$\underset{(x,y)\to(a,b)}{\text{Lt}} f(x, y) \text{ exists and} = f(a, b)$$

Thus $f(x, y)$ is said to be continuous at the point (a, b) if given $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x, y) - f(a, b)| < \varepsilon \text{ for } | (x, y) - (a, b) | < \delta$$

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9.5. CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function $f(x, y, z)$ is said to be continuous at the point (a, b, c) if

$$\underset{(x, y, z)\to(a, b, c)}{\text{Lt}} f(x, y, z) \text{ exists and } = f(a, b, c).$$

Thus $f(x, y, z)$ is said to be continuous at the point (a, b, c) if given $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x, y, z) - f(a, b, c)| < \varepsilon$ for $|(x, y, z) - (a, b, c)| < \delta$.

ILLUSTRATIVE EXAMPLES

Example 1. *Let $f : R^2 \to R$ be defined as $f(x, y) = x^2 + y^2$.*

Show that $\underset{(x, y)\to(0, 0)}{Lt} f(x, y) = 0$

Sol. Let $\varepsilon > 0$ be given

$$|f(x, y) - 0| = |x^2 + y^2| = x^2 + y^2 < \varepsilon$$

whenever $$\sqrt{x^2 + y^2} < \sqrt{\varepsilon}$$

i.e., whenever $|(x, y) - (0, 0)| < \delta$ where $\delta = \sqrt{\varepsilon}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta(=\sqrt{\varepsilon}) > 0$ such that

$$|f(x, y) - 0| < \varepsilon \text{ whenever } |(x, y) - (0, 0)| < \delta$$

Hence by definition of limit, $\underset{(x, y)\to(0, 0)}{\text{Lt}} f(x, y) = 0$

Example 2. *Let $f : R^3 \to R$ be defined by $f(x, y, z) = x^2 + y^2 + z^2$*

Show that $\underset{(x, y, z)\to(0, 0, 0)}{Lt} f(x, y, z) = 0.$

Sol. Let $\varepsilon > 0$ be given

$$|f(x, y, z) - 0| = |x^2 + y^2 + z^2|$$

$$= x^2 + y^2 + z^2 < \varepsilon \text{ whenever } \sqrt{x^2 + y^2 + z^2} < \sqrt{\varepsilon}$$

i.e., whenever $|(x, y, z) - (0, 0, 0)| < \delta$ where $\delta = \sqrt{\varepsilon}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta(=\sqrt{\varepsilon}) > 0$ such that $|f(x, y, z) - 0| < \varepsilon$ whenever $|(x, y, z) - (0, 0, 0)| < \delta$

Hence by definition of limit, $\underset{(x, y, z)\to(0, 0, 0)}{\text{Lt}} f(x, y, z) = 0.$

Example 3. *Let $A = \{(x, y) : 0 < x < 1, 0 < y < 1, x, y \in R\}$. Let $f : A \to R$ defined by $f(x, y) = x + y$. Show that*

$$\underset{\substack{(x, y)\to\left(0, \frac{1}{2}\right) \\ (x, y)\in A}}{Lt} f(x, y) = \frac{1}{2}.$$

Sol. Let $\varepsilon > 0$ be given.

$$\left| f(x, y) - \frac{1}{2} \right| = \left| x + y - \frac{1}{2} \right|$$

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$$= \left| x + \left(y - \frac{1}{2} \right) \right| \le |x| + \left| y - \frac{1}{2} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $\quad |x| < \frac{\varepsilon}{2}$ and $\left| y - \frac{1}{2} \right| < \frac{\varepsilon}{2}$

i.e., whenever $|x - 0| < \delta$ and $\left| y - \frac{1}{2} \right| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta \left(= \frac{\varepsilon}{2} \right) > 0$ such that

$$\left| f(x, y) - \frac{1}{2} \right| < \varepsilon \text{ whenever } |x - 0| < \delta \text{ and } \left| y - \frac{1}{2} \right| < \delta$$

Hence by definition of limit, $\underset{(x,y)\to\left(0,\frac{1}{2}\right)}{\text{Lt}} f(x, y) = \frac{1}{2}$.

Example 4. *Let $f(x, y) = x + y$. Show that $f(x, y)$ is continuous at $\left(\frac{1}{2}, \frac{1}{3} \right)$.*

Sol. $\quad f\left(\frac{1}{2}, \frac{1}{3} \right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

Let $\varepsilon > 0$ be given.

$$\left| f(x, y) - f\left(\frac{1}{2}, \frac{1}{3} \right) \right| = \left| (x + y) - \left(\frac{1}{2} + \frac{1}{3} \right) \right|$$

$$= \left| \left(x - \frac{1}{2} \right) + \left(y - \frac{1}{3} \right) \right| \le \left| x - \frac{1}{2} \right| + \left| y - \frac{1}{3} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $\quad \left| x - \frac{1}{2} \right| < \frac{\varepsilon}{2}$ and $\left| y - \frac{1}{3} \right| < \frac{\varepsilon}{2}$

i.e., whenever $\left| x - \frac{1}{2} \right| < \delta$ and $\left| y - \frac{1}{3} \right| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta \left(= \frac{\varepsilon}{2} \right) > 0$ such that

$$\left| f(x, y) - f\left(\frac{1}{2}, \frac{1}{3} \right) \right| < \varepsilon \text{ whenever } \left| x - \frac{1}{2} \right| < \delta \text{ and } \left| y - \frac{1}{3} \right| < \delta$$

Hence by definition of continuity, $f(x, y)$ is continuous at $\left(\frac{1}{2}, \frac{1}{3} \right)$.

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Example 5. *Let $f : R^2 \to R$ be defined by*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Prove that $\underset{(x, y) \to (0, 0)}{\text{Lt}} f(x, y)$ *does not exist.*

Sol. We know that if $\underset{(x, y) \to (a, b)}{\text{Lt}} f(x, y)$ exists, then this limit is independent of the path along which (x, y) approaches the point (a, b).

Here, let $(x, y) \to (0, 0)$ along the path $y = mx$ where m is any real number.

As $x \to 0$, from $y = mx$, we have $y \to 0$.

Now $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = \underset{(x,y)\to(0,0)}{\text{Lt}} \dfrac{xy}{x^2 + y^2}$ (Putting $y = mx$)

$$= \underset{x\to 0}{\text{Lt}} \frac{x \,.\, mx}{x^2 + m^2x^2} = \underset{x\to 0}{\text{Lt}} \frac{mx^2}{x^2 (1+m^2)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

which is different for different values of m.

$\therefore$ $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x)$ does not exist.

Example 6. *Prove that* $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y)$ *does not exist, where*

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \ (x, y) \neq (0, 0).$$

Sol. We know that if $\underset{(x,y)\to(a,b)}{\text{Lt}} f(x, y)$ exists, then this limit is independent of the path along which (x, y) approaches the point (a, b).

Here, let $(x, y) \to (0, 0)$ along the path $y = mx$ where m is any real number.

As $x \to 0$, from $y = mx$, we have $y \to 0$

Now $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = \underset{(x,y)\to(0,0)}{\text{Lt}} \dfrac{x^2 - y^2}{x^2 + y^2}$ (Putting $y = mx$)

$$= \underset{x\to 0}{\text{Lt}} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \underset{x\to 0}{\text{Lt}} \frac{x^2 (1-m^2)}{x^2 (1+m^2)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}$$

which is different for different values of m.

$\therefore$ $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y)$ does not exist.

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Example 7. *Prove that* $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y)$ *does not exist, where*

$$f(x, y) = \frac{xy^2}{x^2 + y^4}, \ (x, y) \neq (0, 0).$$

Sol. Let $(x, y) \to (0, 0)$ along the path $y = m\sqrt{x}$.

As $x \to 0$, from $y = m\sqrt{x}$, we have $y \to 0$.

$$\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = \underset{(x,y)\to(0,0)}{\text{Lt}} \frac{xy^2}{x^2 + y^4} \qquad \text{(Putting } y = m\sqrt{x}\text{)}$$

$$= \underset{x\to0}{\text{Lt}} \frac{x \cdot m^2 x}{x^2 + m^4 x^2} = \underset{x\to0}{\text{Lt}} \frac{m^2 x^2}{x^2(1+m^4)}$$

$$= \underset{x\to0}{\text{Lt}} \frac{m^2}{1+m^4} = \frac{m^2}{1+m^4}$$

which is different for different values of m.

$\therefore \quad \underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y)$ does not exist.

Example 8. *Prove that* $\underset{(x,y)\to(0,0)}{\text{Lt}} \frac{x^2 y}{x^4 + y^2}$ *does not exist.*

Sol. Let $(x, y) \to (0, 0)$ along the path $x = m\sqrt{y}$

As $y \to 0$, from $x = m\sqrt{y}$, we have $x \to 0$.

$$\therefore \quad \underset{(x,y)\to(0,0)}{\text{Lt}} \frac{x^2 y}{x^4 + y^2} = \underset{y\to0}{\text{Lt}} \frac{m^2 y \cdot y}{m^4 y^2 + y^2} = \underset{y\to0}{\text{Lt}} \frac{m^2 y^2}{y^2(m^4 + 1)}$$

$$= \underset{y\to0}{\text{Lt}} \frac{m^2}{m^4 + 1} = \frac{m^2}{m^4 + 1}$$

which is different for different values of m.

$\therefore \quad \underset{(x,y)\to(0,0)}{\text{Lt}} \frac{x^2 y}{x^4 + y^2}$ does not exist.

Example 9. *Let* $f(x, y) = y \sin \frac{1}{x} + x \sin \frac{1}{y}$, *where* $x \neq 0, y \neq 0$. *Prove that* $f(x, y) \to 0$ *as* $(x, y) \to (0, 0)$.

Sol. Let $\varepsilon > 0$ be given.

$$|f(x, y) - 0| = \left| y \sin \frac{1}{x} + x \sin \frac{1}{y} \right| \leq \left| y \sin \frac{1}{x} \right| + \left| x \sin \frac{1}{y} \right|$$

$$= |y| \left| \sin \frac{1}{x} \right| + |x| \left| \sin \frac{1}{y} \right|$$

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$$\leq |y| + |x| \qquad \left[\because \left|\sin\frac{1}{x}\right| \leq 1 \text{ and } \left|\sin\frac{1}{y}\right| \leq 1\right]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$

i.e., whenever $|x| < \delta$ and $|y| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\left(= \frac{\varepsilon}{2}\right) > 0$ such that $|f(x, y) - 0| < \varepsilon$

whenever $|x - 0| < \delta$ and $|y - 0| < \delta$. Hence by definition of limit,

$$\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = 0.$$

Example 10. *Let $A = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ and $f : A \to R$ be defined by $f(x, y) = x + y$. Prove that f is continuous at every point of the domain A.*

Sol. Let (α, β) be any point of A.

Let us prove that $f(x, y)$ is continuous at (α, β)

i.e., $$\underset{(x,y)\to(\alpha,\beta)}{\text{Lt}} f(x, y) = f(\alpha, \beta)$$

Let $\varepsilon > 0$ be given

$$|f(x, y) - f(\alpha, \beta)| = |(x + y) - (\alpha + \beta)| = |(x - \alpha) + (y - \beta)|$$
$$\leq |x - \alpha| + |y - \beta|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|x - \alpha| < \frac{\varepsilon}{2}$ and $|y - \beta| < \frac{\varepsilon}{2}$

i.e., whenever $|x - \alpha| < \delta$ and $|y - \beta| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\left(= \frac{\varepsilon}{2}\right) > 0$ such that

$|f(x, y) - f(\alpha, \beta)| < \varepsilon$ whenever $|x - \alpha| < \delta$ and $|y - \beta| < \delta$.

Hence by definition of continuity, $f(x, y)$ is continuous at (α, β). Since (α, β) is any point of A, therefore, f is continuous at every point of A.

Example 11. *Show that the function $f : R^2 \to R$ defined by*

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right), & (x, y) \neq (0, 0) \\ 0, & \text{Otherwise} \end{cases}$$

is continuous at (0, 0).

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Sol. Let $\varepsilon > 0$ be given.

$$|f(x, y) - f(0, 0)| = \left| xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right) - 0 \right|$$

$$= |xy| \left| \frac{x^2 - y^2}{x^2 + y^2} \right|$$

$$\leq |xy| \qquad \left[\because\ |x^2 - y^2| \leq |x^2 + y^2| \ \therefore\ \left|\frac{x^2 - y^2}{x^2 + y^2}\right| \leq 1\right]$$

or $$[f(x, y) - f(0, 0)] \leq |x|\,|y|$$
$$< \sqrt{\varepsilon} \times \sqrt{\varepsilon} = \varepsilon$$

whenever $|x| < \sqrt{\varepsilon}$ and $|y| < \sqrt{\varepsilon}$

i.e., whenever $|x| < \delta$ and $|y| < \delta$ where $\delta = \sqrt{\varepsilon}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\ (= \sqrt{\varepsilon}) > 0$ such that $|f(x, y) - f(0, 0)| < \varepsilon$ whenever $|x - 0| < \delta$ and $|y - 0| < \delta$.

Hence by definition of continuity, $f(x, y)$ is continuous at $(0, 0)$.

Example 12. *Let $f(x, y) = \sqrt{|xy|}$. Show that $f(x, y)$ is continuous at the origin.*

Sol. Let $\varepsilon > 0$ be given

$$|f(x, y) - f(0, 0)| = |\sqrt{|xy|} - 0| = \sqrt{|xy|} = \sqrt{|x||y|} = \sqrt{|x|}\,.\,\sqrt{|y|} < \sqrt{\varepsilon}\,.\,\sqrt{\varepsilon} = \varepsilon$$

whenever $\sqrt{|x|} < \sqrt{\varepsilon}$ and $\sqrt{|y|} < \sqrt{\varepsilon}$

i.e., whenever $|x| < \varepsilon$ and $|y| < \varepsilon$

i.e., whenever $|x - 0| < \delta$ and $|y - 0| < \delta$ where $\delta = \varepsilon$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\ (= \varepsilon) > 0$ such that

$|f(x, y) - f(0, 0)| < \varepsilon$ whenever $|x - 0| < \delta$ and $|y - 0| < \delta$.

Hence by definition of continuity, $f(x, y)$ is continuous at $(0, 0)$ the origin.

Example 13. *Let* $\phi(y, z) = \dfrac{yz}{\sqrt{y^2 + z^2}}$, $(y, z) \neq (0, 0)$

$= 0$, *when* $(y, z) = (0, 0)$.

Show that $\phi(y, z)$ is continuous at $(0, 0)$.

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10.1. FUNCTIONS OF TWO VARIABLES

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called a function of two variables x and y, and this is denoted by $z = f(x, y)$.

z is called the dependent variable while x and y are called independent variables.

For example, the area of a triangle is determined when its base and altitude are known. Thus, area of a triangle is a function of two variables, base and altitude.

(In a similar way, a function of more than two variables can be defined).

Geometrically. Let $z = f(x, y)$ be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy-plane. Then to each point (x, y) of this area corresponds a value of z given by the relation $z = f(x, y)$. Representing all these values (x, y, z) by points in space, we get a surface.

Hence the function z = f(x, y) represents a surface.

10.2. PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z with respect to x, treating y as constant, is called partial derivative of z w.r.t. x and is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

Thus,
$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative of z with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y

Thus,
$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

[**In general,** if z is a function of two or more independent variables, then the **partial derivative of z** w.r.t. **any one** of the independent variables is the **ordinary derivative of z** w.r.t. **that variable, treating all other variables as constant.**]

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Geometrically. Let $z = f(x, y)$ be a function of two variables x and y. Then by Art. 10.1, it represents a surface S. If $y = k$, a constant, then $y = k$ represents a plane parallel to the zx-plane.

$\therefore$ $z = f(x, y)$ and $y = k$ represent a **plane curve** C which is the section of S by $y = k$.

$\frac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z).

Thus, $\frac{\partial z}{\partial x}$ gives the slope of the **tangent** drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx-plane.

Similarly, $\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to yz-plane.

10.3. PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t. x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are :

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad f_{xx} ; \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad f_{xy} ; \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{yx}$$

In general,
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad (f_{xy} = f_{yx}.)$$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$.

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots\ldots, x_n$, we get $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots\ldots, \frac{\partial z}{\partial x_n}$.

Note 2. (*i*) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} ; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

(*ii*) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}.$$

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(*iii*) If $z = \frac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(*iv*) If $z = f(u)$, where $u = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x}$; $\frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the first order partial derivatives of the following :*

(*i*) $u = \tan^{-1}\frac{x^2 + y^2}{x + y}$ (*ii*) $u = \cos^{-1}\left(\frac{x}{y}\right)$.

Sol. (*i*) $$u = \tan^{-1}\frac{x^2 + y^2}{x + y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\frac{x^2 + y^2}{x + y}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{x + y}\right)$$

$$= \frac{(x + y)^2}{(x + y^2) + (x^2 + y^2)^2}\cdot\frac{(x + y)\frac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2)\frac{\partial}{\partial x}(x + y)}{(x + y)^2}$$

$$= \frac{(x + y).2x - (x^2 + y^2).1}{(x + y)^2 + (x^2 + y^2)^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{x^2 + 2xy - y^2}{(x + y)^2 + (x^2 + y^2)^2} \quad ...(1)$$

[Since u remains the same if we interchange x and y, u is symmetrical w.r.t. x and y. Interchanging x and y in (1), we have]

Similarly, $$\frac{\partial u}{\partial y} = \frac{y^2 + 2xy - x^2}{(x + y)^2 + (x^2 + y^2)^2}$$

(*ii*) $$u = \cos^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}}\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2 - x^2}}\cdot\frac{1}{y} = \frac{-1}{\sqrt{y^2 - x^2}}.$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}}\cdot\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2 - x^2}}\left(-\frac{x}{y^2}\right) = \frac{x}{y\sqrt{y^2 - x^2}}.$$

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Example 2. *If* $z(x+y) = x^2 + y^2$, *show that* $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Sol. $$z = \frac{x^2+y^2}{x+y}$$ [z is symmetrical w.r.t. x and y]

$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 = \frac{4(x+y)^2(x-y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2}$$

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2-2xy+y^2)}{(x+y)^2} = \frac{4(x-y)^2}{(x+y)^2}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Example 3. *Prove that if* $f(x, y) = \frac{1}{\sqrt{y}} . e^{-\frac{(x-a)^2}{4y}}$, *then* $f_{xy} = f_{yx}$.

Sol. $$f(x, y) = \frac{1}{\sqrt{y}} . e^{-\frac{(x-a)^2}{4y}} = y^{-\frac{1}{2}} e^{-\frac{(x-a)^2}{4y}}$$

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}} . e^{-\frac{(x-a)^2}{4y}} . \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= y^{-\frac{1}{2}} . e^{-\frac{(x-a)^2}{4y}} . \left[-\frac{2(x-a)}{4y}\right] = -\frac{1}{2} y^{-\frac{3}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}}$$

$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2} y^{-\frac{3}{2}} . e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{1}{2}} . e^{-\frac{(x-a)^2}{4y}} . \frac{\partial}{\partial y}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-\frac{3}{2}} + y^{-\frac{1}{2}} . \frac{(x-a)^2}{4y^2}\right] = \frac{1}{4} y^{-\frac{3}{2}} e^{-\frac{(x-a)^2}{4y}} [-2 + y^{-1}(x-a)^2]$$

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$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4}y^{-\frac{3}{2}}\left\{e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right].[-2+y^{-1}(x-a)^2]+e^{-\frac{(x-a)^2}{4y}}.2y^{-1}(x-a)\right\}$$

$$= \frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}\left\{-\frac{2(x-a)}{4y}[-2+y^{-1}(x-a)^2]+2y^{-1}(x-a)\right\}$$

$$= \frac{1}{4}y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{y}}.\frac{x-a}{y}\left\{-\frac{1}{2}[-2+y^{-1}(x-a)^2]+2\right\}$$

$$= \frac{1}{4}y^{-\frac{5}{2}}(x-a)\,e^{-\frac{(x-a)^2}{4y}}\left[3-\frac{(x-a)^2}{2y}\right]$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{2}(x-a)\left[-\frac{3}{2}y^{-\frac{5}{2}}.e^{-\frac{(x-a)^2}{4y}}+y^{-\frac{3}{2}}.e^{-\frac{(x-a)^2}{4y}}.\frac{(x-a)^2}{4y^2}\right]$$

$$= -\frac{1}{4}(x-a)\,y^{-\frac{5}{2}}.e^{-\frac{(x-a)^2}{4y}}\left[-3+\frac{(x-a)^2}{2y}\right]$$

$$= \frac{1}{4}y^{-\frac{5}{2}}(x-a)\,e^{-\frac{(x-a)^2}{4y}}\left[3-\frac{(x-a)^2}{2y}\right]$$

$\therefore \quad f_{xy} = f_{yx}.$

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