Power Method

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11:18 AM

① Spectrum = $\{\lambda_1, \lambda_2 \cdots \lambda_n\}$

② Spectral radius = $\rho(A)$ = largest Eigen value

$\rho(A) < 1$

$A_{n\times n} \longrightarrow \lambda_1, \lambda_2 \cdots \lambda_n$

$|\lambda_1| < |\lambda_2| < |\lambda_3| \cdots < |\lambda_n|$

Smallest Eigenvalue

largest Eigen value

Power Method : Iterative Method.

① Finding largest Eigen value

② Finding Smallest Eigen value

③ Nearest Eigen value of λ_0

Finding largest Eigen value :-

Initial vector X_0

$Y_0 = AX_0 = \lambda_0 X_1$

$AX_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ select the largest magnitude value

$= x_2 \begin{bmatrix} x_1/x_2 \\ 1 \\ x_3/x_2 \end{bmatrix}$

$= \lambda_0$

$Y_0 = AX_0 = \lambda_0 X_1$

$Y_1 = AX_1 = \lambda_1 X_2$

$Y_2 = AX_2 = \lambda_2 X_3$

$\vdots \quad \vdots \quad \vdots$

$AX_n = \lambda_n X_n$

$X_{n+1} \simeq X_n \Rightarrow X_{n+1} - X_n \rightarrow 0$

$\lambda_n \simeq \lambda_{n+1}$

largest Eigen value

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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largest Eigen value

$$Ax_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\lambda_0 = 3 \quad \& \quad x_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2.33 \\ 5 \end{bmatrix}$$

$$= 5\begin{bmatrix} 2.33/5 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2.33/5 \\ 1 \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.466 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.466 \\ 5.399 \end{bmatrix}$$

$$= 5.399\begin{bmatrix} 2.466/5.399 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 5.399, \quad x_3 = \begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.45 \\ 5.35 \end{bmatrix}$$

$|\lambda_{n-1} - \lambda_n| \le \epsilon$

$|x_n - x_{n-1}| < \epsilon$

$$= \underline{5.35}\begin{bmatrix} \frac{2.45}{5.35} \\ 1 \end{bmatrix}$$

$$Ax_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45 \\ 5.35 \end{bmatrix}$$

$$= 5.35\begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$= \underline{5.35} \begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$

$$x_3 = x_4, \quad \lambda^* = 5.35$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \lambda^2 - (a_{11} + a_{22})\lambda + |A| = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 + 8}}{2}$$

$$\lambda_1 = \frac{5 - \sqrt{33}}{2} \quad \& \quad \lambda_2 = \frac{5 + \sqrt{33}}{2} = \underline{5.372}$$

λ_2 → largest

$$\lambda_1 \simeq \underline{\underline{-0.3}}$$

(x) Same Question take Smallest magnitude value of the vector as common.

Ex:- Find the largest Eigen value of $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with initial guess $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(take upto 2 decimals & perform two iterations)

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 3/8 \\ 3/8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3/8 \\ 3/8 \end{bmatrix} = \begin{bmatrix} 1 + 18/8 + 3/8 \\ 1 + 6/8 \\ 9/8 \end{bmatrix}$$

$$= \begin{bmatrix} 29/8 \\ 14/8 \\ 9/8 \end{bmatrix}$$

Smallest Eigen value

3 largest Eigen value

Smallest Eigen value

$\boxed{\lambda \longrightarrow A}$,

$1/\lambda \longrightarrow A^{-1}$

3 largest Eigen value of A

$1/3 \longrightarrow$ Smallest Eigen value of A^{-1}

λ_k is the largest Eigen value of A^{-1}

λ_k Smallest Eigen value of A.

A^{-1} $= B$

$$AX = \lambda X$$

$$A^{-1}AX = A^{-1}\lambda X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

$$A^{-1}X = \frac{1}{\lambda}X,$$

$$\boxed{BX = \beta X}$$

$A^{-1} \longrightarrow B$

$1/\lambda \longrightarrow \beta$

The largest Eigen value of B is Smallest Eigen value of A

(x) Find Smallest Eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\text{Find } A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = B$$

$$BX_0 = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \end{bmatrix}$$

$$= -2\begin{bmatrix} 1 \\ -1.5/2 \end{bmatrix}$$

(i) taking Smallest magnitude value common (for matrix A)

(ii) Find A^{-1}, and then finding largest Eigen value

$(\lambda_2) \longrightarrow A$

(ii) Find A^{-1}, and then finding largest Eigen

(iii) Nearest Eigen Value of $\lambda_0 \longrightarrow A$

$(A - \lambda_0 I) = B$

Smallest Eigen Value

Find Largest Eigen value for $(A-\lambda_0 I)^{-1}$ or B^{-1}

$\dfrac{x^T A x}{\|Ax\|}$

(iii) Nearest Eigen value of $\lambda_k \simeq p$

$A \longrightarrow \lambda_1, \lambda_2, \ldots \lambda_k \ldots \lambda_n$

'p' is the nearest Eigen value of λ_k

$(A - pI) \longrightarrow \lambda_1 - p, \lambda_2 - p, \ldots \lambda_k - p \ldots \lambda_n - p$

very small

Smallest Eigen value of $A - pI$

Smallest $\lambda \longrightarrow A$

largest $\frac{1}{\lambda} \longrightarrow A^{-1}$

Smallest Eigen value of $(A - pI)$ is the nearest Eigen value of $\lambda_k \longrightarrow A$

Finding largest Eigen value of $(A-pI)^{-1}$

(*) Find the nearest Eigen value of 5 for

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \text{ with } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A^{-1} largest

$\rightarrow \lambda_1, \lambda_2, p$ and $5 = p$

$$(A - 5I) = \begin{bmatrix} -2 & -1 & 0 \\ -2 & -1 & -3 \end{bmatrix} = B$$

A^{-1} larger

$(A-5I)$ $A_{3\times3} \to \lambda_1, \lambda_2, P$ $(5I = P)$

nearest Eigen value of $5-P$

$\lambda_1 - P$, $\lambda_2 - P$ smallest

$(A-5I) \leftarrow$ is same as finding largest eigen value of $(A-5I)^{-1}$

Step:-

$$(A-5I) = \begin{bmatrix} -2 & -1 & 0 \\ -2 & -1 & -3 \\ 0 & -1 & -4 \end{bmatrix} = B$$

Smallest Eigen value of $(A-5I)$

$=$ Largest Eigen value of $(A-5I)^{-1}$

$B^{-1} = C$

$CX_0 = Y_1 = \lambda_1 X_1$ → largest magnitude

$CX_1 = Y_2 = \lambda_2 X_2$ → largest magnitude

Drawbacks

① Choosing initial guess

$X_k \to \lambda_k$ dominating Eigen value

X_0 = Eigen vector for λ_1

= Actual Eigen value $|\lambda_n|$ approximated λ

② Complex Eigen values

③ knowing all possible Eigen values is tough

* QR decomposition to give all possible Eigen values of the matrix without using characteristic Equation

$A = [x_1\ x_2\ x_3]$

$Q = [y_1, y_2, y_3]$

$A = QR$

$Q^{-1}A = R$ upper triangular

$Q^{-1}A = R$

H.W

Using the power method obtain the dominant Eigen value and associated Eigen vector of $A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}$

value and associated Eigen vector of A $\begin{bmatrix} -2 & 4 & -5 \\ 0 & -1 & 1 \end{bmatrix}$

with $X^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (2 iterations)

14 iterations you have to perform

Define

$A = [\ ]$

$X_0 = [\ ]$

for $i = 1, 2, \ldots$

$Y_i = AX_i$

$X_i := X_{i-1}$

$\lambda_i = \max(|Y_i|)$

$X_{i+1} = Y_i / \lambda_i$

$|X_i - X_{i+1}| < 0.001$

end

X_{i+1}

i) First 2 iterations do by hand

ii) Make an algorithm and Execute through any known programming