QUADRATIC FORMS

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

For example, $x_1^2 + 2x_2^2 - 3x_3^2 + 5x_1x_2 - 6x_1x_3 + 4x_2x_3$ is a quadratic form in three variables.

The general form of a quadratic form, denoted by Q in n variables is

$$\begin{aligned} Q = {} & c_{11}x_1^2 + c_{12}\,x_1x_2 + \dots + c_{1n}\,x_1x_n \\ & + c_{21}\,x_2x_1 + c_{22}\,x_2^2 + \dots + c_{2n}\,x_2x_n \\ & + c_{31}\,x_3x_1 + c_{32}\,x_3x_2 + \dots + c_{3n}\,x_3\,x_n \\ & + (\text{- -}) \\ & + c_{n1}\,x_n\,x_1 + c_{n2}\,x_n\,x_2 + \dots + c_{nn}\,x_n^2 \end{aligned}$$

i.e. $$Q = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\, x_i\, x_j$$

In general, $c_{ij} \neq c_{ji}$. The coefficient of $x_i\, x_j = c_{ij} + c_{ji}$.

Now if we define $a_{ij} = \frac{1}{2}\,(c_{ij} + c_{ji})$, for all i and j, then $a_{ii} = c_{ii}$, $a_{ij} = a_{ji}$ and $a_{ij} + a_{ji} = 2a_{ij} = c_{ij} + c_{ji}$.

$\therefore Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\, x_i\, x_j$, where $a_{ij} = a_{ji}$ and hence the matrix $A = [a_{ij}]$ is a symmetric matrix. In matrix notation, the quadratic form Q can be represented as $Q = X^T AX$, where

$$A = [a_{ij}],\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad X^T = [x_1, x_2, \dots, x_n].$$

The symmetric matrix $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is called *the matrix of the quadratic form Q.*

Note ☑ To find the symmetric matrix A of a quadratic form, the coefficient of x_i^2 is placed in the a_{ii} position and $\left(\frac{1}{2} \times \text{coefficient } x_i x_j\right)$ is placed in each of the a_{ij} and a_{ji} positions.

For example, (i) if $Q = 2x_1^2 - 3x_1 x_2 + 4x_2^2$, then

$$A = \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{bmatrix}$$

(ii) if $Q = x_1^2 + 3x_2^2 + 6x_3^2 - 2x_1 x_2 + 6x_1 x_3 + 5x_2 x_3$,

$$\text{then } A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 3 & \frac{5}{2} \\ 3 & \frac{5}{2} & 6 \end{bmatrix}$$

Conversely, the quadratic form whose matrix is

$$\begin{bmatrix} 3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 6 \\ 0 & 6 & -7 \end{bmatrix} \text{ is } Q = 3x_1^2 - 7x_3^2 + x_1 x_2 + 12x_2 x_3$$

Definitions

If A is the matrix of a quadratic form Q, $|A|$ is called *the determinant* or *modulus of Q*.

The rank r of the matrix A is called *the rank of the quadratic form*.

If $r < n$ (the order of A) or $|A| = 0$ or A is singular, the quadratic form is called *singular*. Otherwise it is non-singular.

Linear Transformation of a Quadratic Form

Let $Q = X^T AX$ be a quadratic form in the n variables $x_1, x_2, \ldots, x_n$.

Consider the transformation $X = PY$, that transforms the variable set $X = [x_1, x_2, \ldots, x_n]^T$ to a new variable set $Y = [y_1, y_2, \ldots, y_n]^T$, where P is a non-singular matrix.

We can easily verify that the transformation $X = PY$ expresses each of the variables $x_1, x_2, \ldots, x_n$ as homogeneous linear expressions in $y_1, y_2, \ldots, y_n$. Hence $X = PY$ is called a non-singular linear transformation.

By this transformation, $Q = X^T AX$ is transformed to

$$\begin{aligned} Q &= (PY)^T A\,(PY) \\ &= Y^T (P^T AP)\, Y \\ &= Y^T BY, \text{ where } B = P^T AP \end{aligned}$$

Now
$$\begin{aligned} B^T = (P^T A\, P)^T &= P^T A^T P \\ &= P^T AP \qquad (\because A \text{ is symmetric}) \\ &= B \end{aligned}$$

$\therefore$ B is also a symmetric matrix.

Hence B is the matrix of the quadratic form $Y^T BY$ in the variables $y_1, y_2, \ldots, y_n$.

Thus $Y^T B Y$ is the linear transform of the quadratic form $X^T AX$ under the linear transformation $X = PY$, where $B = P^T AP$.

Canonical Form of a Quadratic Form

In the linear transformation $X = PY$, if P is chosen such that $B = P^T A P$ is a diagonal matrix of the form $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, then the quadratic form Q gets reduced as

$$Q = Y^T BY$$

$$= [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

This form of Q is called *the sum of the squares form of Q* or *the canonical form of Q*.

Orthogonal Reduction of a Quadratic Form to the Canonical Form

If, in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the quadratic form Q to the canonical form then Q is said to be reduced to the canonical form by an orthogonal transformation.

We recall that if A is a real symmetric matrix and N is the normalised modal matrix of A, then N is an orthogonal matrix such that $N^T AN = D$, where D is a diagonal matrix with the eigenvalues of A as diagonal elements.

Hence, to reduce a quadratic form $Q = X^T AX$ to the canonical form by an orthogonal transformation, we may use the linear transformation $X = NY$, where N is the normalised modal matrix of A. By this orthogonal transformation, Q gets transformed into $Y^T DY$, where D is the diagonal matrix with the eigenvalues of A as diagonal elements.

Nature of Quadratic Forms

When the quadratic form $X^T AX$ is reduced to the canonical form, it will contain only r terms, if the rank of A is r.

The terms in the canonical form may be positive, zero or negative.

The number of positive terms in the canonical form is called *the index* (p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form i.e. $p-(r-p)=2p-r$ is called the *signature(s)* of the quadratic form i.e. $s=2p-r$.

The quadratic form $Q = X^T A X$ in n variables is said to be

(i) positive definite, if $r=n$ and $p=n$ or if all the eigenvalues of A are positive.

(ii) negative definite, if $r=n$ and $p=0$ or if all the eigenvalues of A are negative.

(iii) positive semidefinite, if $r<n$ and $p=r$ or if all the eigenvalues of $A \geq 0$ and at least one eigenvalue is zero.

(iv) negative semidefinite, if $r<n$ and $p=0$ or if all the eigenvalues of $A \leq 0$ and at least one eigenvalue is zero.

(v) indefinite in all other cases or if A has positive as well as negative eigenvalues.

WORKED EXAMPLE 1(d)

Example 1.1 Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form by an orthogonal transformation. Also find the rank, index, signature and nature of the quadratic form.

$$\text{Matrix of the Q.F. is } A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

Refer to the worked example (7) in section 1(c).

The eigenvalues of A are $-1, 1, 4$.

The corresponding eigenvectors are $[0, 1, 1]^T$ $[2, -1, 1]^T$ and $[1, 1, -1]^T$ respectively.

$$\text{The modal matrix } M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{The normalised modal matrix } N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Hence $N^T AN = D(-1, 1, 4)$, where D is a diagonal matrix with $-1, 1, 4$ as the principal diagonal elements.

$\therefore$ The orthogonal transformation $X = NY$ will reduce the Q.F. to the canonical form $-y_1^2 + y_2^2 + 4y_3^2$.

Rank of the Q.F. = 3,

Index = 2

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Signature = 1

Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.2 Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction. Find also the nature of the quadratic form.

Matrix of the Q.F. is $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

Refer to worked example (8) in section 1(c).

The eigenvalues of A are $-2, 6, 6$.

The corresponding eigenvectors are $[1, 0, -1]^T$, $[1, 0, 1]^T$ and $[0, 1, 0]^T$ respectively.

Note ☑ Though two of the eigenvalues are equal, the eigenvectors have been so chosen that all the three eigenvectors are pairwise orthogonal.

The modal matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised modal matrix is given by

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Hence $N^T AN = \text{Diag}\,(-2, 6, 6)$

$\therefore$ The orthogonal transformation $X = NY$

i.e.
$$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$
$$x_2 = y_2$$
$$x_3 = -\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$

will reduce the given Q.F. to the canonical form $-2y_1^2 + 6y_2^2 + 6y_3^2$.

The Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.3 Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semidefinite. Give also a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero.

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

i.e. $$(1-\lambda)\{(2-\lambda)(1-\lambda)-1\}-(1-\lambda)=0$$

i.e. $$(1-\lambda)(\lambda^2-3\lambda)=0$$

$\therefore$ The eigenvalues of A are $\lambda = 0, 1, 3$.

When $\lambda = 0$, the elements of the eigenvector are given by $x_1 - x_2 = 0, -x_1 + 2x_2 + x_3 = 0$ and $x_2 + x_3 = 0$.

Solving these equations, $x_1 = 1, \quad x_2 = 1, \quad x_3 = -1$

$\therefore$ The eigenvector corresponding to $\lambda = 0$ is

$$[1, 1, -1]^T$$

When $\lambda = 1$, the elements of the eigenvector are given by $-x_2 = 0, -x_1 + x_2 + x_3 = 0$ and $x_2 = 0$.

Solving these equations, $x_1 = 1, \quad x_2 = 0, \quad x_3 = 1$.

$\therefore$ When $\lambda = 1$, the eigenvector is

$$[1, 0, 1]^T$$

When $\lambda = 3$, the elements of the eigenvector are given by $-2x_1 - x_2 = 0, -x_1 - x_2 + x_3 = 0$ and $x_2 - 2x_3 = 0$

Solving these equation, $x_1 = -1, \quad x_2 = 2, \quad x_3 = 1$.

$\therefore$ When $\lambda = 3$, the eigenvector is $[-1, 2, 1]^T$.

Now the modal matrix is $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

The normalised modal matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence $N^T AN = \text{Diag}\,(0, 1, 3)$

$\therefore$ The orthogonal transformation $X = NY$.

i.e. $$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = -\frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

will reduce the given Q.F. to the canonical form $0 \cdot y_1^2 + y_2^2 + 3y_3^2 = y_2^2 + 3y_3^2$.

As the canonical form contains only two terms, both of which are positive, the Q.F. is positive semi-definite.

The canonical form of the Q.F. is zero, when $y_2 = 0$, $y_3 = 0$ and y_1 is arbitrary. Taking $y_1 = \sqrt{3}$, $y_2 = 0$ and $y_3 = 0$, we get $x_1 = 1$, $x_2 = 1$ and $x_3 = -1$. These values of x_1, x_2, x_3 make the Q.F. zero.

Example 1.4 Determine the nature of the following quadratic forms without reducing them to canonical forms:

(i) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

(ii) $5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$

(iii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_3x_1$.

Note ☑ We can find the nature of a Q.F. without reducing it to canonical form. The alternative method uses the principal sub-determinants of the matrix of the Q.F., as explained below:

Let $A = (a_{ij})_{n \times n}$ be the matrix of the Q.F.

Let $$D_1 = |a_{11}| = a_{11}, \quad D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ etc.} \quad \text{and} \quad D_n = |A|$$

$D_1, D_2, D_3, \ldots D_n$ are called the principal sub-determinants or principal minors of A.

(i) The Q.F. is positive definite, if $D_1, D_2, \ldots, D_n$ are all positive i.e. $D_n > 0$ for all n.

(ii) The Q.F. is negative definite, if $D_1, D_3, D_5, \ldots$ are all negative and $D_2, D_4, D_6, \ldots$ are all positive i.e. $(-1)^n D_n > 0$ for all n.

(iii) The Q.F. is positive semidefinite, if $D_n \geq 0$ and least one $D_i = 0$.

(iv) The Q.F. is negative semidefinite, if $(-1)^n D_n \geq 0$ and at least one $D_i = 0$.

(v) The Q.F. is indefinite in all other cases.

(i) $Q = x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

Now $$D_1 = |1| = 1; \quad D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2;$$

$$D_3 = 1 \cdot (18 - 1) - 1 \cdot (6 - 2) + 2(1 - 6) = 3.$$

D_1, D_2, D_3 are all positive.

$\therefore$ The Q.F. is positive definite.

(ii) $Q = 5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$.

$$A = \begin{bmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{bmatrix}$$

Now $D_1 = 5;\quad D_2 = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24;$

$$D_3 = |A| = 5 \cdot (70 - 64) - 1 \cdot (14 - 32) - 4 \cdot (-8 + 20)$$
$$= 30 + 18 - 48 = 0$$

D_1 and D_2 are > 0, but $D_3 = 0$

$\therefore$ The Q.F. is positive semidefinite.

(iii) $Q = 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1 x_2 - 8x_2 x_3 - 4x_3 x_1$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Now $D_1 = |2| = 2;\ D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = -34;$

$$D_3 = |A| = 2 \cdot (-3 - 16) - 6 \cdot (-18 - 8) - 2\,(-24 + 2)$$
$$= -38 + 156 + 44 = 162$$

$\therefore$ The Q.F. is indefinite.

Example 1.5 Reduce the quadratic forms $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 4x_2x_3 + 18x_3x_1$ and $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$ simultaneously to canonical forms by a real non-singular transformation.

Note ☑ We can reduce two quadratic forms $X^T AX$ and $X^T BX$ to canonical forms simultaneously by the same linear transformation using the following theorem, (stated without proof):

If A and B are two symmetric matrices such that the roots of $|A - \lambda B| = 0$ are all distinct, then there exists a matrix P such that $P^T AP$ and $P^T BP$ are both diagonal matrices.

The procedure to reduce two quadratic forms simultaneously to canonical forms is given below:

(1) Let A and B be the matrices of the two given quadratic forms.

(2) Form the characteristic equation $|A - \lambda B| = 0$ and solve it. Let the eigenvalues (roots of this equation) be $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(3) Find the eigenvectors X_i $(i = 1, 2, \ldots, n)$ corresponding to the eigenvalues λ_i, using the equation $(A - \lambda_i B)\, X_i = 0$.

(4) Construct the matrix P whose column vectors are $X_1, X_2, \ldots, X_n$. Then $X = PY$ is the required linear transformation.

(5) Find $P^T AP$ and $P^T BP$, which will be diagonal matrices.

(6) The quadratic forms corresponding to these diagonal matrices are the required canonical forms.

The matrix of the first quadratic form is

$$A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

The matrix of the second quadratic form is

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is $|A - \lambda B| = 0$

i.e.
$$\begin{vmatrix} 6-2\lambda & 2-2\lambda & 9-\lambda \\ 2-2\lambda & 3-5\lambda & 2 \\ 9-\lambda & 2 & 14 \end{vmatrix} = 0$$

Simplifying,
$$5\lambda^3 - \lambda^2 - 5\lambda + 1 = 0$$

i.e.
$$(\lambda - 1)(5\lambda - 1)(\lambda + 1) = 0$$

$\therefore \quad \lambda = -1, \frac{1}{5}, 1.$

When $\lambda = -1$, $(A - \lambda B) X = 0$ gives the equations.

$8x_1 + 4x_2 + 10x_3 = 0; \quad 4x_1 + 8x_2 + 2x_3 = 0; \quad 10x_1 + 2x_2 + 14x_3 = 0.$

Solving these equations, $\dfrac{x_1}{-72} = \dfrac{x_2}{24} = \dfrac{x_3}{48}$

$\therefore$
$$X_1 = [-3, 1, 2]^T.$$

When $\lambda = \frac{1}{5}$, $(A - \lambda B) X = 0$ gives the equations $28x_1 + 8x_2 + 44x_3 = 0$; $8x_1 + 10x_2 + 10x_3 = 0$; $44x_1 + 10x_2 + 70x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-360} = \dfrac{x_2}{72} = \dfrac{x_3}{216}$

$\therefore$
$$X_2 = [-5, 1, 3]^T$$

When $\lambda = 1$, $(A - \lambda B) X = 0$ gives the equations

$$4x_1 + 8x_3 = 0; \quad -2x_2 + 2x_3 = 0; \quad 8x_1 + 2x_2 + 14x_3 = 0$$

$\therefore$
$$X_3 = [2, -1, -1]^T.$$

Now
$$P = [X_1, X_2, X_3] = \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\text{Now} \qquad P^T AP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ -1 & -1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T AX$ is reduced to the canonical form $y_1^2 + y_2^2 + y_3^2$.

$$\text{Now} \qquad P^T B P = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & -3 \\ -5 & -5 & -5 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T B X$ is reduced to the canonical form $-y_1^2 + 5y_2^2 + y_3^2$.

Thus the transformation $X = PY$ reduces both the Q.F.'s to canonical forms.

Note: $X = PY$ is not an orthogonal transformation, but only a linear non-singular transformation.