14.1 Functions of Several Variables

Real-valued functions of several independent real variables are defined analogously to functions of a single variable. Points in the domain are now ordered pairs (triples, quadruples, n-tuples) of real numbers, and values in the range are real numbers.

> **DEFINITIONS** Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number
>
> $$w = f(x_1, x_2, \ldots, x_n)$$
>
> to each element in D. The set D is the function's **domain.** The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

If f is a function of two independent variables, we usually call the independent variables x and y and the dependent variable z, and we picture the domain of f as a region in the xy-plane (Figure 14.1). If f is a function of three independent variables, we call the independent variables x, y, and z and the dependent variable w, and we picture the domain as a region in space.

In applications, we tend to use letters that remind us of what the variables stand for. To say that the volume of a right circular cylinder is a function of its radius and height, we might write $V = f(r, h)$. To be more specific, we might replace the notation $f(r, h)$ by the formula

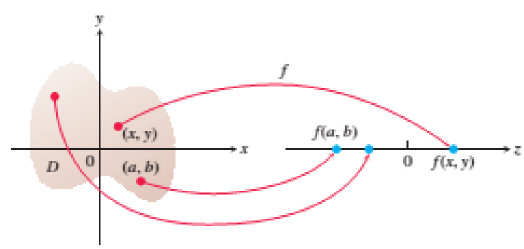


FIGURE 14.1 An arrow diagram for the function $z = f(x, y)$.

that calculates the value of V from the values of r and h, and write $V = \pi r^2 h$. In either case, r and h would be the independent variables and V the dependent variable of the function.

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable. For example, the value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

Domains and Ranges

In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If $f(x, y) = \sqrt{y - x^2}$, then y cannot be less than x^2. If $f(x, y) = 1/(xy)$, then xy cannot be zero. The domain of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly. The range consists of the set of output values for the dependent variable.

EXAMPLE 1

(a) These are functions of two variables. Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z.

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

(b) These are functions of three variables with restrictions on some of their domains.

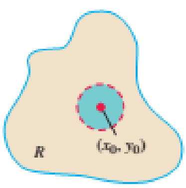
Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

Functions of Two Variables

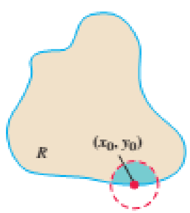
Regions in the plane can have interior points and boundary points just like intervals on the real line. Closed intervals $[a, b]$ include their boundary points, open intervals (a, b) don't include their boundary points, and intervals such as $[a, b)$ are neither open nor closed.

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

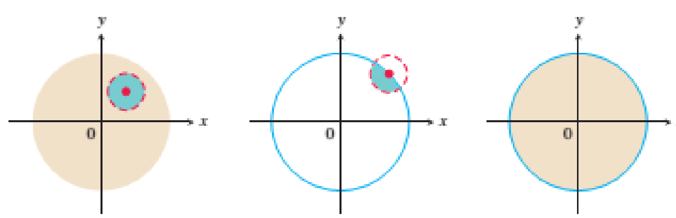


(a) Interior point



(b) Boundary point

FIGURE 14.2 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk. Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk. Contains all boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

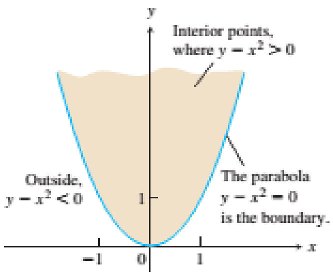


FIGURE 14.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

> **DEFINITIONS** A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

EXAMPLE 2 Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 14.4. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

14.2 Limits and Continuity in Higher Dimensions

In this section we develop limits and continuity for multivariable functions. The theory is similar to that developed for single-variable functions, but since we now have more than one independent variable, there is additional complexity that requires some new ideas.

Limits for Functions of Two Variables

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that when (x_0, y_0) lies in the interior of f's domain, (x, y) can approach (x_0, y_0) from any direction, not just from the left or the right. For the limit to exist, the same limiting value must be obtained whatever direction of approach is taken. We illustrate this issue in several examples following the definition.

> **DEFINITION** We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write
>
> $$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$
>
> if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,
>
> $$|f(x, y) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

The definition of limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is made sufficiently small (but not 0). The definition applies to interior points (x_0, y_0) as well as boundary points of the domain of f, although a boundary point need not lie within the domain. The points (x, y) that approach (x_0, y_0) are always taken to be in the domain of f. See Figure 14.12.

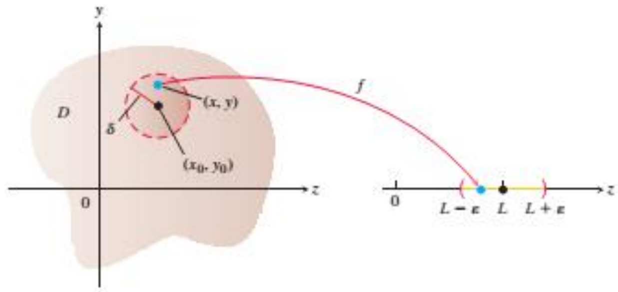


FIGURE 14.12 In the limit definition, δ is the radius of a disk centered at (x_0, y_0). For all points (x, y) within this disk, the function values $f(x, y)$ lie inside the corresponding interval $(L - \varepsilon, L + \varepsilon)$.

As for functions of a single variable, it can be shown that

$$\lim_{(x, y) \to (x_0, y_0)} x = x_0$$

$$\lim_{(x, y) \to (x_0, y_0)} y = y_0$$

$$\lim_{(x, y) \to (x_0, y_0)} k = k \qquad \text{(any number } k\text{).}$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\varepsilon > 0$ is chosen. If we let δ equal this ε, we see that if

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \varepsilon,$$

then

$$\begin{aligned} \sqrt{(x - x_0)^2} &< \varepsilon && (x - x_0)^2 \le (x - x_0)^2 + (y - y_0)^2 \\ |x - x_0| &< \varepsilon && \sqrt{a^2} = |a| \\ |f(x, y) - x_0| &< \varepsilon. && x = f(x, y) \end{aligned}$$

That is,

$$|f(x, y) - x_0| < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So a δ has been found satisfying the requirement of the definition, and therefore we have proved that

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = \lim_{(x, y) \to (x_0, y_0)} x = x_0.$$

THEOREM 1—Properties of Limits of Functions of Two Variables
The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \qquad \text{and} \qquad \lim_{(x, y) \to (x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} kf(x, y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} [f(x, y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n}$,
   n a positive integer, and if n is even, we assume that $L > 0$.

EXAMPLE 1 In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the x- and y-values of the point being approached into the functional expression to find the limiting value.

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

EXAMPLE 2 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} && \text{Multiply by a form equal to 1.} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\
&= 0(\sqrt{0} + \sqrt{0}) = 0 && \text{Known limit values}
\end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (where we would have $x - y = 0$) is *not* in the domain of the function

$$f(x, y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

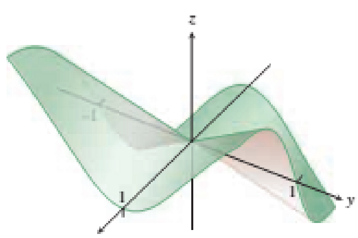


FIGURE 14.13 The surface graph shows the limit of the function in Example 3 must be 0, if it exists.

EXAMPLE 3 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}$ if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0 (see Figure 14.13). To see if this is true, we apply the definition of limit.

Let $\varepsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}. \qquad \frac{y^2}{x^2 + y^2} \leq 1$$

So if we choose $\delta = \varepsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \le 4\sqrt{x^2+y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

It follows from the definition that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

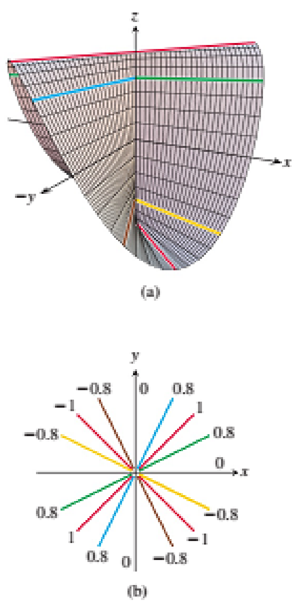


FIGURE 14.14 (a) The graph of

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

The function is continuous at every point except the origin. (b) The values of f are different constants along each line $y = mx$, $x \ne 0$ (Example 5).

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f. The only requirement is that each point (x, y) near (x_0, y_0) be in the domain of f.

A consequence of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

Solution The function f is continuous at every point (x, y) except $(0, 0)$ because its values at points other than $(0, 0)$ are given by a rational function of x and y, and therefore at those points the limiting value is simply obtained by substituting the values of x and y into that rational expression.

At $(0, 0)$, the value of f is defined, but f has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m, the function f has a constant value on the "punctured" line $y = mx$, $x \ne 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2+y^2}\Bigg|_{y=mx} = \frac{2x(mx)}{x^2+(mx)^2} = \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1+m^2}.$$

This limit changes with each value of the slope m. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous at the origin.

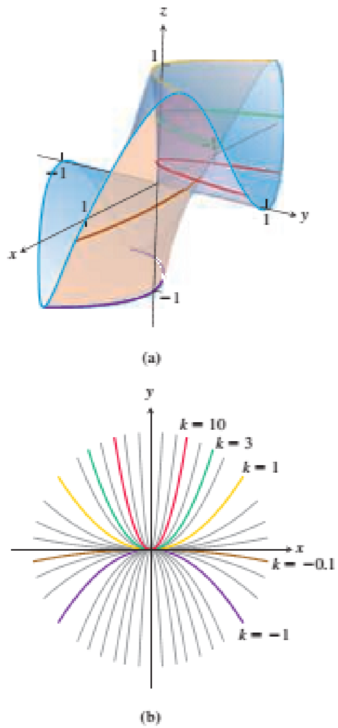


FIGURE 14.15 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. (b) Along each path $y = kx^2$ the value of f is constant, but varies with k (Example 6).

> **Two-Path Test for Nonexistence of a Limit**
> If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ does not exist.

EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as (x, y) approaches $(0, 0)$.

Solution The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$. We examine the values of f along parabolic curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\bigg|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\bigg|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y) \to (0,0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y) \to (0,0)} \left[f(x, y)\bigg|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$. ■

It can be shown that the function in Example 6 has limit 0 along every straight line path $y = mx$ (Exercise 57). This implies the following observation:

> Having the same limit along all straight lines approaching (x_0, y_0) does not imply that a limit exists at (x_0, y_0).

> **Continuity of Compositions**
> If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composition $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos \frac{xy}{x^2 + 1}, \qquad \ln (1 + x^2y^2)$$

are continuous at every point (x, y).