Unit-3
Linear Algebric Equation

2140706 – Numerical & Statistical Methods

Humanities & Science Department

Matrix Equation

- ✓The matrix notation for following linear system of equation is as follow:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array}\right\}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

The above linear system is expressed in the matrix form as
$A \cdot X = B$

Elementary Transformation or Operation on a Matrix

Operation	Meaning
R_{ij} or $R_i \leftrightarrow R_j$	Interchange of i^{th} and j^{th} rows
$k \cdot R_i$	Multiplication of all the elements of i^{th} row by non zero scalar k.
$R_{ij}(k)$ or $R_j + k \cdot R_i$	Multiplication of all the elements of i^{th} row by nonzero scalar k and added into j^{th} row.

Row Echelon Form of Matrix

✓To convert the matrix into row echelon form follow the following steps:

1. Every zero row of the matrix occurs below the non zero rows.
2. Arrange all the rows in strictly decreasing order.
3. Make all the entries zero below the leading (first non zero entry of the row) element of 1st row.
4. Repeat step-3 for each row.

Reduced Row Echelon Form of Matrix

✓ To convert the matrix into reduced row echelon form follow the following steps:

1. Convert given matrix into row echelon form.
2. Make all leading elements 1(one).
3. Make all the entries zero above the leading element 1(one) of each row.

Numerical Methods For Solution Of A Linear Equation

1. **Direct Methods**
2. **Iterative Methods**

✓ **Direct Methods**

- This method produce the exact solution after a finite number of steps but are subject to errors due to round-off and other factors.
- We will discuss two direct methods :

1.Gauss Elimination method

2.Gauss-Jordan method

- **Indirect Method(Iterative Method)**
  - In this method, an approximation to the true solution is assumed initially to start method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off error are also smaller.

1.Gauss seidel method

2.Gauss jacobi method

Gauss Elimination Method

- ✓ To solve the given linear system using Gauss elimination method, follow the following steps:

1. Start with augmented matrix $[A : B]$.
2. Convert matrix A into row echelon form.(we take leading element of each row is one(1).)
3. Apply back substitution for getting equations.
4. Solve the equations and find the unknown variables (i.e. solution).

Solve by Gauss Elimination method:
$x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40.$

Solution:-

By Augmented matrix,

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}$$

We have to make zero to these three elements.

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix} \quad \boxed{R_{12}(-2)} \quad \boxed{R_{13}(-3)}$$

$(\because R_{13}(-3)$ means multiply row 1 by -3 and add it into row 3)

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 0 & \frac{12}{5} & : & 12 \end{bmatrix}$$

$$R_{23}\left(\frac{1}{5}\right)$$

Now, these three elements are zeros

Now, solving equations by back-substitution

$$\therefore \frac{12}{5}z = 12$$

$$\therefore \; z = 5$$

$$\therefore -5y + 2z = -5$$

$$\therefore \; -5y + 10 = -5$$

$$\therefore \; y = 3$$

$$\therefore \; x + y + z = 9$$

$$\therefore \; x + 3 + 5 = 9$$

$$\therefore \; x = 1$$

Answer : $(x, y, z) = (1,3,5)$

Solve by Gauss Elimination method:
$x + 2y + z = 3, 2x + 3y + 3z = 10, 3x - y + 2z = 13.$

Solution:-

By Augmented matrix,

$$\begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 3 & 3 & : & 10 \\ 3 & -1 & 2 & : & 13 \end{bmatrix}$$

We have to make zero to these three elements.

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 1 & : & 4 \\ 0 & -7 & 1 & : & 4 \end{bmatrix} \quad \boxed{R_{12}(-2)} \quad \boxed{R_{13}(-3)}$$

$(\because R_{13}(-3)$ means multiply row 1 by -3 and add it into row 3$)$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 1 & : & 4 \\ 0 & 0 & -8 & : & -24 \end{bmatrix}$$

$R_{23}(-7)$

Now, these three elements are zeros

Now, solving equations by back-substitution

$\therefore -8z = -24$

$\therefore \ z = 3$

$\therefore -y + z = 4$

$\therefore \ y = -1$

$\therefore \ x + 2y + z = 3$

$\therefore \ x = 2$

Answer : $(x, y, z) = (3, -1, 2)$

Gauss Elimination Method With Partial Pivoting

✓ To solve the given linear system using Gauss elimination method with partial pivoting, follow the following steps:

1. Find largest absolute value(pivot element) in first column.
2. Make the pivot element row to first row.
3. Eliminate x_1 below the pivot element.
4. Again find pivot element in 2^{nd} and 3^{rd} row.
5. Make the pivot element row to second row.
6. Eliminate x_2 below the pivot element.
7. Apply back substitution for getting equations.
8. Solve the equations and find the unknown variables (i.e. solution).

Solve by Gauss Elimination method with partial pivoting
$8x_2 + 2x_3 = -7, 3x_1 + 5x_2 + 2x_3 = 8, 6x_1 + 2x_2 + 8x_3 = 26$

Solution:-

By Augmented matrix,

$$\begin{bmatrix} \boxed{\begin{matrix} 0 \\ 3 \\ 6 \end{matrix}} & \begin{matrix} 8 \\ 5 \\ 2 \end{matrix} & \begin{matrix} 2 \\ 2 \\ 8 \end{matrix} & \begin{matrix} : \\ : \\ : \end{matrix} & \begin{matrix} -7 \\ 8 \\ 26 \end{matrix} \end{bmatrix}$$

Find Largest absolute value in first column and make that row into first row

$$\sim \begin{bmatrix} 6 & 2 & 8 & : & 26 \\ 3 & 5 & 2 & : & 8 \\ 0 & 8 & 2 & : & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

Make these two elements to zero

$$\sim \begin{bmatrix} 6 & 2 & 8 & : & 26 \\ 3 & 5 & 2 & : & 8 \\ 0 & 8 & 2 & : & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 6 & 2 & 8 & : & 26 \\ 0 & \boxed{4} & -2 & : & -5 \\ 0 & 8 & 2 & : & -7 \end{bmatrix} \quad R_{12}\left(-\frac{1}{2}\right)$$

Find Largest absolute value in second column in 2nd and 3rd row and interchange accordingly.

$$\sim \begin{bmatrix} 6 & 2 & 8 & : & 26 \\ 0 & 8 & 2 & : & -7 \\ 0 & 4 & -2 & : & -5 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 2 & 8 & : & 26 \\ 0 & 8 & 2 & : & -7 \\ 0 & 4 & -2 & : & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 6 & 2 & 8 & : & 26 \\ 0 & 8 & 2 & : & -7 \\ 0 & 0 & -3 & : & -\frac{3}{2} \end{bmatrix} \quad R_{23}\left(-\frac{1}{2}\right)$$

By back substitution,

$\therefore -3x_3 = -\frac{3}{2} \Rightarrow x_3 = \frac{1}{2}$

$\therefore 8x_2 + 2x_3 = -7 \Rightarrow 8x_2 + 1 = -7 \Rightarrow x_2 = -1$

$\therefore 6x_1 + 2x_2 + 8x_3 = 26 \Rightarrow 6x_1 - 2 + 4 = 26 \Rightarrow x_1 = 4$

$\therefore (x_1, x_2, x_3) = (4, -1, \frac{1}{2})$

Gauss-Jordan Method

- ✓ This method is modification of the gauss elimination method. This method solves a given system of equation by transforming the coefficient matrix into unit matrix.
- ✓ Steps to solve Gauss-Jordan method:
  1. Write the matrix form of the system of equations.
  2. Write the augmented matrix.
  3. Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
  4. Write the corresponding linear system of equations to obtain the solution.

Solve by Gauss Jordan method

$10x_1 + x_2 + x_3 = 12, x_1 + 10x_2 - x_3 = 10, x_1 - 2x_2 + 10x_3 = 9$

Solution:-

By Augmented matrix,

$$\sim \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 1 & 10 & -1 & : & 10 \\ 1 & -2 & 10 & : & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 10 & -1 & : & 10 \\ 10 & 1 & 1 & : & 12 \\ 1 & -2 & 10 & : & 9 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 10 & -1 & : & 10 \\ 0 & -99 & 11 & : & -88 \\ 0 & -12 & 11 & : & -1 \end{bmatrix} \quad \begin{matrix} R_{12}(-10) \\ R_{13}(-1) \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 10 & -1 & : & 10 \\ 0 & 1 & -\frac{1}{9} & : & \frac{8}{9} \\ 0 & -12 & 11 & : & -1 \end{bmatrix} \quad R_2\left(-\frac{1}{99}\right)$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{1}{9} & : & \frac{10}{9} \\ 0 & 1 & -\frac{1}{9} & : & \frac{8}{9} \\ 0 & 0 & \frac{87}{9} & : & \frac{87}{9} \end{bmatrix} \quad \begin{matrix} R_{21}(-10) \\ R_{23}(12) \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{1}{9} & : & \frac{10}{9} \\ 0 & 1 & -\frac{1}{9} & : & \frac{8}{9} \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \quad R_3\left(\frac{9}{87}\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \quad \begin{matrix} R_{31}\left(-\frac{1}{9}\right) \\ R_{32}\left(\frac{1}{9}\right) \end{matrix}$$

By back substitution,

$$\therefore (x, y, z) = (1,1,1)$$

Gauss Jacobi Method

- ✓ This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.
- ✓ Consider the system of equations.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

- ✓ Where co-efficient matrix $\boldsymbol{A}$ must be diagonally dominant,

$|a_1| \geq |b_1| + |c_1|$

$|b_2| \geq |a_2| + |c_2|$

$|c_3| \geq |a_3| + |b_3| \ldots\ldots (1)$

And the inequality is strictly greater than for at least one row.

Solving the system (1) for x, y, z respectively , we obtain

$$\mathrm{x} = \frac{1}{\mathrm{a}_1}(\mathrm{d}_1 - \mathrm{b}_1\mathrm{y} - \mathrm{c}_1\mathrm{z})$$

$$\mathrm{y} = \frac{1}{\mathrm{b}_2}(\mathrm{d}_2 - \mathrm{a}_2\mathrm{x} - \mathrm{c}_2\mathrm{z})$$

$$\mathrm{z} = \frac{1}{\mathrm{c}_3}(\mathrm{d}_3 - \mathrm{a}_3\mathrm{x} - \mathrm{b}_3\mathrm{y}) \ldots\ldots\ (2)$$

✓We start with $x_0 = 0, y_0 = 0$ & $z_0 = 0$ in equ. (2)

$$\therefore x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$\therefore y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$\therefore z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Again substituting these value $x_{1,}$ $y_{1,}$ z_1 in Eq. (2), the next approximation is obtained.

This process is continued till the values of x, y, z are obtained to desired degree of accuracy.

Solve by Gauss Jacobi method up to three iteration.
$20x + y - 2z = 17, 2x - 3y + 20z = 25, 3x + 20y - z = -18$

Solution:-

$|20| > |1| + |-2|$
$|-3| \ngeq |2| + |20|$

So, It is not diagonally dominant.

We need to rearrange the equations.

$20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$

$|20| > |1| + |-2|$
$|20| > |3| + |-1|$
$|20| > |2| + |-3|$

So, All Equations are diagonally dominant.

(Make subject x, y, z from diagonally dominant equations.)

Here,

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 3x + 3y)$$

Let the initial values are $x = y = z = 0.$

1^{st} iteration,

$$x^1 = \frac{1}{20}(17 - 0 + 0) = 0.85$$

$$y^1 = \frac{1}{20}(-18 - 0 + 0) = 0.9$$

$$z^1 = \frac{1}{20}(25 - 0 + 0) = 1.25$$

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 3x + 3y)$$

Iteration	x	y	z
1	0.85	1.00	1.25
2	1.02	-0.97	1.03
3	1.00	-1.00	1.00

Ans: $(x, y, z) = (1, -1, 1)$

Gauss Seidel Method

- ✓ This is a modification of Gauss-Jacobi method. In this method we replace the approximation by the corresponding new ones as soon as they are calculated.
- ✓ Consider the system of equations:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

- ✓ Where co-efficient matrix A must be diagonally dominant,

$|a_1| \geq |b_1| + |c_1|$

$|b_2| \geq |a_2| + |c_2|$

$|c_3| \geq |a_3| + |b_3| \dots \dots (1)$

And the inequality is strictly greater than for at least one row.

Solving the system (1) for x, y, z respectively, we obtain,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \dots \dots \ (2)$$

✓ We start with $x_0 = 0, y_0 = 0$ & $z_0 = 0$ in equ. (2)

$$\therefore x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Now substituting $x = x_1$ & $z = z_0$ in the second equ. Of (2)

$$\therefore y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Now substituting $x = x_1$ & $\mathrm{y} = \mathrm{y}_1$ in the second equ. Of (2)

$$\therefore z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is continued till the values of x, y, z are obtained to desired degree of accuracy.

Solve by gauss-Seidel method correct up two decimal places.
$20x + 2y + z = 30, x - 40y + 3z = -75, 2x - y + 10z = 30$

Solution:-

$|20| > |2| + |1|\,, |-40| > |1| + |3|\,, |10| > |2| + |-1|$

So, All Equations are diagonally dominant.

By Gauss Seidel method,

$$x = \frac{1}{20}(30 - 2y - z)$$

$$y = \frac{1}{-40}(-75 - x - 3z)$$

$$z = \frac{1}{10}(30 - 2x + y)$$

✓Let the initial values are $x_1 = x_2 = x_3 = 0$.

$$x = \frac{1}{20}(30 - 2y - z) = \frac{1}{20}(30 - 0 - 0) = 1.5$$

$$y = \frac{1}{-40}(-75 - x - 3z) = \frac{1}{-40}(-75 - 1.5 - 0) = 1.91$$

$$z = \frac{1}{10}(30 - 2x + y) = \frac{1}{10}(30 - 2(1.5) + 1.91) = 2.89$$

Iteration	x	y	z
1	1.5	1.91	2.89
2	1.16	2.12	2.98
3	1.14	2.13	2.99
4	1.14	2.13	2.99

In 3^{rd} & 4^{th} iteration all values are almost same.

So, $(x, y, z) = (1.14,\ 2.13,\ 2.99)$