Example 4. *If $u = x^y$, show that* $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.

$$u = x^y$$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = yx^{y-1} \log x + x^y \cdot \frac{1}{x} = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad ...(1)$$

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1} + yx^{y-1} \log x = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad ...(2)$$

From (1) and (2), $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Example 5. *If* $\theta = t^n e^{-\frac{r^2}{4t}}$, *find the value of n which will make* $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial\theta}{\partial r}\right) = \dfrac{\partial\theta}{\partial t}$.

(U.P.T.U. 2006 ; K.U. 2006)

Sol.
$$\theta = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial\theta}{\partial r} = t^n . e^{-\frac{r^2}{4t}} . \left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-\frac{r^2}{4t}}$$

$\therefore$
$$r^2 \frac{\partial\theta}{\partial r} = -\frac{1}{2} r^3 . t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[3r^2 e^{-\frac{r^2}{4t}} + r^3 e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right] = -\frac{1}{2} t^{n-1} r^2 e^{-\frac{r^2}{4t}}\left[3 - \frac{r^2}{2t}\right]$$

$\therefore$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right)$$

Also
$$\frac{\partial\theta}{\partial t} = n t^{n-1} e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}} . \left(\frac{r^2}{4t^2}\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

Since
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t} \qquad \text{[Given]}$$

$\therefore$
$$\frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

$\Rightarrow$
$$\frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t} \quad \therefore \quad n = -\frac{3}{2}.$$

Example 6. *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that*

$$\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$$ (M.D.U. May 2006, Dec. 2006 ; K.U.K. 2005)

Sol. $u = (1 - 2xy + y^2)^{-1/2} = V^{-1/2}$, where $V = 1 - 2xy + y^2$

$$\frac{\partial u}{\partial x} = -\frac{1}{2} V^{-3/2} . \frac{\partial V}{\partial x} = -\frac{1}{2} V^{-3/2} (-2y) = y V^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = y . \frac{\partial}{\partial x}(V^{-3/2}) = y . \left(-\frac{3}{2}\right) V^{-5/2} . \frac{\partial V}{\partial x} = -\frac{3}{2} y V^{-5/2} (-2y) = 3y^2 V^{-5/2}$$

$\therefore$
$$\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} = (1 - x^2) . \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} . \frac{\partial}{\partial x}(1 - x^2)$$
$$= (1 - x^2) . 3y^2 V^{-5/2} + y V^{-3/2} (-2x) = y V^{-3/2} [3y V^{-1} (1 - x^2) - 2x] \qquad (1)$$

Also
$$\frac{\partial u}{\partial y} = -\frac{1}{2} V^{-3/2} \frac{\partial V}{\partial y} = -\frac{1}{2} V^{-3/2}. (-2x + 2y) = V^{-3/2} . (x - y)$$

$$\frac{\partial^2 u}{\partial y^2} = V^{-3/2} . \frac{\partial}{\partial y} (x - y) + (x - y) . \frac{\partial}{\partial y} (V^{-3/2})$$

$$= V^{-3/2} . (-1) + (x - y). \left(-\frac{3}{2} V^{-5/3}\right) . \frac{\partial V}{\partial y}$$

$$= -V^{-3/2} - \frac{3}{2} (x - y) V^{-5/2} . (-2x + 2y) = -V^{-3/2} + 3(x - y)^2 V^{-5/2}$$

$\therefore$
$$\frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = y^2 . \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} . \frac{\partial}{\partial y} (y^2)$$

$$= y^2[-V^{-3/2} + 3(x - y)^2 V^{-5/2}] + V^{-3/2} (x - y) . 2y$$

$$= y V^{-3/2} [-y + 3y(x - y)^2 V^{-1} + 2(x - y)]$$

$$= y V^{-3/2} [3y (x - y)^2 V^{-1} + (2x - 3y)] \quad ...(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1 - x^2) \frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = yV^{-3/2} [3yV^{-1}(1 - x^2) - 2x + 3y (x - y)^2 V^{-1} + 2x - 3y]$$

$$= y V^{-3/2}[3yV^{-1} (1 - x^2 + x^2 - 2xy + y^2) - 3y]$$

$$= yV^{-2/3}[3yV^{-1}(1 - 2xy + y^2) - 3y] \qquad | \because \quad V = 1 - 2xy + y^2$$

$$= yV^{-3/2} [3y - 3y]$$

$$= 0.$$

Example 7. *If $u = \log (x^3 + y^3 + z^3 - 3xyz)$, show that*

(K.U.K. 2006 ; U.P.T.U. 2006)

(*i*) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x + y + z)^2}$$

(*ii*) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} = \frac{-9}{(x + y + z)^2}.$$

Sol. (*i*)
$$u = \log (x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}; \quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

Adding,
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{x + y + z}$$

$$[\because \quad x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)]$$

Now
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= -\frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} = -\frac{9}{(x + y + z)^2} \quad ...(1)$$

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 u}{\partial z\partial y}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}$$

$$\left[\because \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 u}{\partial z\partial y}=\frac{\partial^2 u}{\partial y\partial z}, \frac{\partial^2 u}{\partial x\partial z}=\frac{\partial^2 u}{\partial z\partial x}\right]$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}$$

[from (1)]

Example 8. *If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x\partial y} = -(x \log ex)^{-1}$.*

Sol. $x^x y^y z^z = c$ defines z as a function of x and y.

Taking logs, $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y, we have

$$y\cdot\frac{1}{y}+1\cdot\log y+z\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial y}+1\cdot\log z\cdot\frac{\partial z}{\partial y}=0$$

or
$$1+\log y+(1+\log z)\frac{\partial z}{\partial y}=0 \qquad \ldots(1)$$

or
$$\frac{\partial z}{\partial y}=-\frac{1+\log y}{1+\log z}$$

Similarly,
$$\frac{\partial z}{\partial x}=-\frac{1+\log x}{1+\log z} \qquad \ldots(2)$$

Differentiating (1) partially w.r.t. x, we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y}+(1+\log z)\frac{\partial^2 z}{\partial x\partial y}=0 \quad \text{or} \quad \frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{z(1+\log z)}\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \qquad \ldots(3)$$

When $x = y = z$

From (2), $\frac{\partial z}{\partial y}=-1, \frac{\partial z}{\partial x}=-1$

From (3), $\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x(1+\log x)}(-1)(-1)=-\frac{1}{x(\log e+\log x)}=-\frac{1}{x(\log ex)}=-(x\log ex)^{-1}.$

10.7. COMPOSITE FUNCTIONS

(*i*) If $u = f(x, y)$ where $x = \phi(t), y = \psi(t)$

then u is called a composite function of (the **single variable**) t and we can find $\frac{du}{dt}$.

(*ii*) If $z = f(x, y)$ where $x = \phi(u, v), y = \psi(u, v)$

then z is called a composite function of (**two variables**) u and v so that we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

10.8. DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t), y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Proof. Here $\quad u = f(x, y) \quad$ ...(1)

Let δt be an increment in t and δx, δy, δu the corresponding increments in x, y and u respectively. Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y) \quad \text{...(2)}$$

Subtracting (1) from (2), we get

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$\frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta x}{\delta t} \quad \text{...(3)}$$

As $\delta t \to 0$, δx and δy both $\to 0$, so that

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt}, \quad \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \quad \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

$$\lim_{\delta x \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$$

and

$$\lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$$

$\therefore$ From (1), $\quad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$\frac{du}{dt}$ is called the **total derivative** of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Cor. 1. If $u = f(x, y, z)$ and x, y, z are function of t, then y is a composite function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

Cor. 2. If $z = f(x, y)$ and x, y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Cor. 3. If $u = f(x, y)$ where $y = \phi(x)$ then since $x = \psi(x)$, u is a composite function of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

Cor. 4. If we are given an **implicit function f(x, y) = c,** then $u = f(x, y)$ where $u = c$

Using Cor. 3, we have $\quad \dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$

But $\dfrac{du}{dx} = 0 \quad \therefore \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx} = 0$ or $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Hence the differential coefficient of $f(x, y)$ w.r.t. x is $\dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx}$.

Cor. 5. If $f(x, y) = c$, then by Cor. 4, we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

Differentiating again w.r.t. x, we get

$$\frac{d^2u}{dx^2} = -\frac{f_y \dfrac{d}{dx}(f_x) - f_x \dfrac{d}{dx}(f_y)}{f_y^{\,2}} = -\frac{f_y \left[\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_x}{\partial y} \cdot \dfrac{dy}{dx}\right] - f_x \left[\dfrac{\partial f_y}{\partial x} + \dfrac{\partial f_y}{\partial y} \cdot \dfrac{dy}{dx}\right]}{f_y^{\,2}}$$

$$= -\frac{f_y \left[f_{xx} - f_{yx} \cdot \dfrac{f_x}{f_y}\right] - f_x \left[f_{xy} - f_{yy} \cdot \dfrac{f_x}{f_y}\right]}{f_y^{\,2}} = -\frac{f_{xx} f_y^{\,2} - f_x f_y f_{xy} - f_x f_y f_{xy} - f_{yy} f_x^{\,2}}{f_y^{\,3}}$$

Hence $\quad \mathbf{\dfrac{d^2y}{dx^2} = -\dfrac{f_{xx} f_y^{\,2} - 2 f_x f_y f_{xy} + f_{yy} f_x^{\,2}}{f_y^{\,3}}}.$

ILLUSTRATIVE EXAMPLES

Example 1. *If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that* $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$.

Sol. The given equations define u as a composite function of t.

$$\therefore \quad \frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$=\frac{1}{\sqrt{1-(x-y)^2}}.3+\frac{1}{\sqrt{1-(x-y)^2}}(-1).12t^2$$

$$=\frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}=\frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}=\frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$=\frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}=\frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}=\frac{3}{\sqrt{1-t^2}}.$$

Example 2. *If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm per second when it passes through the value $x = 3$ cm, show that if y is passing through the value $y = 1$ cm, y must be decreasing at the rate of $2\frac{2}{15}$ cm per second, in order that z shall remain constant.*

Sol. Given : $z = 2xy^2 - 3x^2y$ and $\frac{dx}{dt} = 2$ cm/sec when $x = 3$ cm, we have to find $\frac{dy}{dt}$ when $y = 1$ cm.

Now $$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=(2y^2-6xy)\frac{dx}{dt}+(4xy-3x^2)\frac{dy}{dt}$$

Since z remains constant, $\frac{dz}{dt}=0$

$$\therefore \quad 0=(2y^2-6\times 3\times y)\times 2+(4\times 3\times y-3\times 3^2)\frac{dy}{dt}$$

$$\Rightarrow \quad 0=(4y^2-36y)+(12y-27)\frac{dy}{dt}$$

When $y = 1$ cm, we have

$$0=(4-36)+(12-27)\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt}=-\frac{32}{15} \text{ which is negative}$$

$\therefore$ When $y = 1$ cm, y is decreasing at the rate of $2\frac{2}{15}$ cm/sec.

Example 3. *If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that*

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$ (V.T.U. 2006)

Sol. Here z is a composite function of u and v.

$$\therefore \quad \frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}.e^u+\frac{\partial z}{\partial y}(-e^{-u})$$

and $$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})$$

Subtracting, $$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=(e^u+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^v)\frac{\partial z}{\partial y}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

Example 4. *If $u = f(y - z, z - x, x - y)$, prove that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

Sol. Here $\qquad u = f(X, Y, Z)$ where $X = y - z, Y = z - x, Z = x - y$

$\therefore\ u$ is a composite function of x, y and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding, $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

Example 5. *If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

(Madras, 2006 ; V.T.U. 2005)

Sol. The given equations define w as a composite function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\cdot\sin\theta$$

or $$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad \text{...(1)} \quad [\because\ w = f(x, y)]$$

Also $$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

or $$\frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Example 6. *If u is a homogeneous function of nth degree in x, y, z, prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Sol. Since u is a homogeneous function of degree n in x, y, z, let

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$$

or $$u = x^n f(t, s) \quad \text{where} \quad t = \frac{y}{x}, \quad s = \frac{z}{x}$$

Here f is a composite function of x, y, z.

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f(t, s) + x^n\left(\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}\right)$$

$$= nx^{n-1} f(t, s) + x^n\left[\frac{\partial f}{\partial t}\cdot\left(-\frac{y}{x^2}\right) + \frac{\partial f}{\partial s}\left(-\frac{z}{x^2}\right)\right]$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = nx^n f(t,s) - yx^{n-1}\frac{\partial f}{\partial t} - zx^{n-1}\frac{\partial f}{\partial s} \quad ...(1)$$

$$\frac{\partial u}{\partial y} = x^n\left(\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y}\right) = x^n\left[\frac{\partial f}{\partial t}\cdot\frac{1}{x} + \frac{\partial f}{\partial s}\cdot 0\right]$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y} = yx^{n-1}\frac{\partial f}{\partial t} \quad ...(2)$$

$$\frac{\partial u}{\partial z} = x^n\left[\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z}\right] = x^n\left[\frac{\partial f}{\partial t}\cdot 0 + \frac{\partial f}{\partial s}\cdot\frac{1}{x}\right]$$

$$\Rightarrow \quad z\frac{\partial u}{\partial z} = zx^{n-1}\frac{\partial f}{\partial s} \quad ...(3)$$

Adding (1), (2) and (3), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nx^n f(t, s) = nu$$

Example 7. *If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, show that*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right).$$

Sol. Here $\quad f(x, y) = \theta(u, v)$ and $u = x^2 - y^2$, $v = 2xy$

$\Rightarrow$ f is a function of u, v and u, v are functions of x, y

$\Rightarrow$ f is a composite function of x, y.

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial\theta}{\partial u}.2x + \frac{\partial\theta}{\partial v}.2y \quad [\because f = \theta \text{ (given)}]$$

$$= 2\left(x\frac{\partial\theta}{\partial u} + y\frac{\partial\theta}{\partial v}\right) = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\theta$$

$$\Rightarrow \quad \frac{\partial}{\partial x} = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right) \quad ...(1)$$

Also
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial\theta}{\partial u}.(-2y) + \frac{\partial\theta}{\partial v}.2x \quad [\because f = \theta \text{ (given)}]$$

$$= 2\left(-y\frac{\partial\theta}{\partial u} + x\frac{\partial\theta}{\partial v}\right) = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right)\theta$$

$$\Rightarrow \quad \frac{\partial}{\partial y} = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right) \quad ...(2)$$

Now
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial\theta}{\partial u} + 2y\frac{\partial\theta}{\partial v}\right) \quad \text{[Using (1)]}$$

$$= 4\left(x^2\frac{\partial^2\theta}{\partial u^2} + xy\frac{\partial^2\theta}{\partial u\partial v} + yx\frac{\partial^2\theta}{\partial v\partial u} + y^2\frac{\partial^2\theta}{\partial v^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 4\left(x^2\frac{\partial^2\theta}{\partial u^2} + 2xy\frac{\partial^2\theta}{\partial u\partial v} + y^2\frac{\partial^2\theta}{\partial v^2}\right) \quad ...(3) \quad \left[\because \frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial^2\theta}{\partial v\partial u}\right]$$

and $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial \theta}{\partial u} + 2x\frac{\partial \theta}{\partial v}\right) \quad \text{[Using (2)]}$$

$$= 4\left(y^2\frac{\partial^2 \theta}{\partial u^2} - yx\frac{\partial^2 \theta}{\partial u \partial v} - xy\frac{\partial^2 \theta}{\partial v \partial u} + x^2\frac{\partial^2 \theta}{\partial v^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = 4\left(y^2\frac{\partial^2 \theta}{\partial u^2} - 2xy\frac{\partial^2 \theta}{\partial u \partial v} + x^2\frac{\partial^2 \theta}{\partial v^2}\right) \quad \ldots(4) \quad \left[\because \frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial^2 \theta}{\partial v \partial u}\right]$$

Adding (3) and (4), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\left[(x^2 + y^2)\frac{\partial^2 \theta}{\partial u^2} + (y^2 + x^2)\frac{\partial^2 \theta}{\partial v^2}\right]$$

$$= 4\,(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right).$$

Example 8. *If* $x + y = 2e^{\theta}\cos\phi$ *and* $x - y = 2ie^{\theta}\sin\phi$, *show that* $\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}$, $(i = \sqrt{-1})$.

Sol. Here $\quad x + y = 2e^{\theta}\cos\phi \quad$ and $\quad x - y = 2ie^{\theta}\sin\phi$

Adding $\quad 2x = 2e^{\theta}(\cos\phi + i\sin\phi)$

$\Rightarrow \quad x = e^{\theta}\cdot e^{i\phi} = e^{\theta + i\phi}$ (By Euler's Theorem)

Subtracting $\quad 2y = 2e^{\theta}(\cos\phi - i\sin\phi)$

$\Rightarrow \quad y = e^{\theta}\cdot e^{-i\phi} = e^{\theta - i\phi}$

Now u is a function of x, y and x, y are functions of θ, ϕ

$\Rightarrow \quad u$ is a composite function of θ, ϕ.

$$\therefore \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\cdot e^{\theta + i\phi} + \frac{\partial u}{\partial y}\cdot e^{\theta - i\phi}$$

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u$$

$$\Rightarrow \quad \frac{\partial}{\partial \theta} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \ldots(1)$$

Also $$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x}\cdot ie^{\theta + i\phi} + \frac{\partial u}{\partial y}\cdot(-ie^{\theta - i\phi})$$

$$= ix\frac{\partial u}{\partial x} - iy\frac{\partial u}{\partial y} = \left(ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}\right)u$$

$$\Rightarrow \quad \frac{\partial}{\partial \phi} = ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y} \quad \ldots(2)$$

$$\therefore \quad \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \quad \text{[Using (1)]}$$

$$= x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} + yx \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial \theta^2} = x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \quad \ldots(3) \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$

and

$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right)$$

$$= \left(ix \frac{\partial}{\partial x} - iy \frac{\partial}{\partial y}\right)\left(ix \frac{\partial u}{\partial x} - iy \frac{\partial u}{\partial y}\right) \quad \text{[Using (2)]}$$

$$= i^2 \left(x^2 \frac{\partial^2 u}{\partial x^2} - xy \frac{\partial^2 u}{\partial x \partial y} - yx \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2}\right)$$

$$= -\left(x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right) \quad \ldots(4) \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}$$