A VECTOR is a quantity having both magnitude and direction, such as displacement, velocity, force, and acceleration.

Graphically a vector is represented by an arrow OP (Fig.1) defining the direction, the magnitude of the vector being indicated by the length of the arrow. The tail end O of the arrow is called the *origin* or *initial point* of the vector, and the head P is called the *terminal point* or *terminus*.

Analytically a vector is represented by a letter with an arrow over it, as $\vec{A}$ in Fig.1, and its magnitude is denoted by $|\vec{A}|$ or A. In printed works, bold faced type, such as **A**, is used to indicate the vector $\vec{A}$ while $|\mathbf{A}|$ or A indicates its magnitude. We shall use this bold faced notation in this book. The vector $\overline{\mathrm{OP}}$ is also indicated as $\overrightarrow{\mathrm{OP}}$ or **OP**; in such case we shall denote its magnitude by $\overline{\mathrm{OP}}$, $|\overrightarrow{\mathrm{OP}}|$, or $|\mathbf{OP}|$.

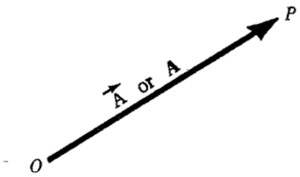

Fig.1

A SCALAR is a quantity having magnitude but no direction, e.g. mass, length, time, temperature, and any real number. Scalars are indicated by letters in ordinary type as in elementary algebra. Operations with scalars follow the same rules as in elementary algebra.

LAWS OF VECTOR ALGEBRA. If **A**, **B** and **C** are vectors and m and n are scalars, then

1. $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$	Commutative Law for Addition
2. $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$	Associative Law for Addition
3. $m\mathbf{A} = \mathbf{A}m$	Commutative Law for Multiplication
4. $m(n\mathbf{A}) = (mn)\mathbf{A}$	Associative Law for Multiplication
5. $(m+n)\mathbf{A} = m\mathbf{A}+n\mathbf{A}$	Distributive Law
6. $m(\mathbf{A}+\mathbf{B}) = m\mathbf{A}+m\mathbf{B}$	Distributive Law

A UNIT VECTOR is a vector having unit magnitude, If **A** is a vector with magnitude $A \neq 0$, then $\mathbf{A}/A$ is a unit vector having the same direction as **A**.

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

The magnitude of **A** is $\qquad A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$

In particular, the *position vector* or *radius vector* **r** from O to the point (x,y,z) is written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and has magnitude $\quad r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$.

SCALAR FIELD. If to each point (x,y,z) of a region R in space there corresponds a number or scalar $\phi(x,y,z)$, then ϕ is called a *scalar function of position* or *scalar point function* and we say that a *scalar field* ϕ has been defined in R.

Examples. (*1*) The temperature at any point within or on the earth's surface at a certain time defines a scalar field.

(*2*) $\phi(x,y,z) = x^3y - z^2$ defines a scalar field.

A scalar field which is independent of time is called a *stationary* or *steady-state scalar field.*

VECTOR FIELD. If to each point (x,y,z) of a region R in space there corresponds a vector $\mathbf{V}(x,y,z)$, then $\mathbf{V}$ is called a *vector function of position* or *vector point function* and we say that a *vector field* $\mathbf{V}$ has been defined in R.

Examples. (1) If the velocity at any point (x,y,z) within a moving fluid is known at a certain time, then a vector field is defined.

(2) $\mathbf{V}(x,y,z) = xy^2\mathbf{i} - 2yz^3\mathbf{j} + x^2z\mathbf{k}$ defines a vector field.

THE DOT OR SCALAR PRODUCT of two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\cdot\mathbf{B}$ (read $\mathbf{A}$ dot $\mathbf{B}$), is defined as the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ and the cosine of the angle θ between them. In symbols,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \qquad 0 \leqq \theta \leqq \pi$$

Note that $\mathbf{A}\cdot\mathbf{B}$ is a scalar and not a vector.

The following laws are valid:

1. $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ — Commutative Law for Dot Products
2. $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ — Distributive Law
3. $m(\mathbf{A}\cdot\mathbf{B}) = (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then
$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$
$$\mathbf{A}\cdot\mathbf{A} = A^2 = A_1^2 + A_2^2 + A_3^2$$
$$\mathbf{B}\cdot\mathbf{B} = B^2 = B_1^2 + B_2^2 + B_3^2$$
6. If $\mathbf{A}\cdot\mathbf{B} = 0$ and $\mathbf{A}$ and $\mathbf{B}$ are not null vectors, then $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

THE CROSS OR VECTOR PRODUCT of $\mathbf{A}$ and $\mathbf{B}$ is a vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ (read $\mathbf{A}$ cross $\mathbf{B}$). The magnitude of $\mathbf{A}\times\mathbf{B}$ is defined as the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ and the sine of the angle θ between them. The direction of the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and such that $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a right-handed system. In symbols,

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u}, \qquad 0 \leqq \theta \leqq \pi$$

where $\mathbf{u}$ is a unit vector indicating the direction of $\mathbf{A}\times\mathbf{B}$. If $\mathbf{A} = \mathbf{B}$, or if $\mathbf{A}$ is parallel to $\mathbf{B}$, then $\sin\theta = 0$ and we define $\mathbf{A}\times\mathbf{B} = \mathbf{0}$.

The following laws are valid:

1. $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$ — (Commutative Law for Cross Products Fails.)
2. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$ — Distributive Law
3. $m(\mathbf{A}\times\mathbf{B}) = (m\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(m\mathbf{B}) = (\mathbf{A}\times\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}, \quad \mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

6. The magnitude of $\mathbf{A}\times\mathbf{B}$ is the same as the area of a parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$.

7. If $\mathbf{A}\times\mathbf{B} = \mathbf{0}$, and $\mathbf{A}$ and $\mathbf{B}$ are not null vectors, then $\mathbf{A}$ and $\mathbf{B}$ are parallel.

TRIPLE PRODUCTS. Dot and cross multiplication of three vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ may produce meaningful products of the form $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The following laws are valid:

1. $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$

2. $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ = volume of a parallelepiped having $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ as edges, or the negative of this volume, according as $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ do or do not form a right-handed system. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$, then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ (Associative Law for Cross Products Fails.)

4. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$
   $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$

The product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is sometimes called the *scalar triple product* or *box product* and may be denoted by $[\mathbf{ABC}]$. The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is called the *vector triple product*.

In $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ parentheses are sometimes omitted and we write $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ (see Problem 41). However, parentheses must be used in $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ (see Problems 29 and 47).

RECIPROCAL SETS OF VECTORS. The sets of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are called *reciprocal sets or systems of vectors* if

$$\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1$$

$$\mathbf{a}'\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{c} = \mathbf{b}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{c} = \mathbf{c}'\cdot\mathbf{a} = \mathbf{c}'\cdot\mathbf{b} = 0$$

The sets $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are reciprocal sets of vectors if and only if

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$$