

Practice Questions

Wednesday, January 5, 2022
8:49 AM

- ① Evaluate the value of determinant of $A_{10 \times 10}$ if

$$a_{ii} = 1 \text{ if } i \\ a_{ij} = 2 \text{ if } i=j \\ \text{else } a_{ij} = 0$$

Solution

$$A_{10 \times 10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|A| = ? \\ = D_{10}$$

$2^{10} \times 2^{10}$

$n \times n$

$$A_{n \times n} \text{ then} \\ |A| = (-3)^{\frac{n(n-1)}{2}}$$

$$D_{10} = 1 \times 1 \times D_8 - 2 \times 2 \times D_8 = D_8 - 4D_8 \\ D_{10} = -3D_8, D_8 = -3D_6, D_6 = -3D_4$$

$$|A| = D_{10} = (-3)^5 \quad \text{a, b, c, d} \\ = -3D_2, D_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ = 1 - 4 = -3$$

② If $S = \left\{ A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \middle| |A| \neq 0 \right\}$ then $|S| = ?$

(a) 0 (b) 1 (c) 2 (d) 3

Skew-Symmetric $A^T = -A$ $a_{ii} = -a_{ii}$ | Every skew-symmetric matrix of odd order $|A| = 0$ ✓

③ If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 11 \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & k \end{bmatrix}$ then Row Rank, column Rank

Row Rank = 3
Column Rank = 3
max. min. Rank = 3

$\underline{\text{Amnn}}$
 $\text{Rank}(A) \leq \min(m, n)$
 $\text{Rank}(A) = 2 \Rightarrow \text{Rank}(A) = r_2 = 2$
 $\text{Rank}(AB) \leq \min(2, 3) = 2$

$$[13 + 17 = 30] \quad c_1 + c_2 = 13$$

(4) If $A = \begin{bmatrix} 13 + 17 & 30 \\ 19 + 31 & 50 \\ 42 + 61 & 103 \end{bmatrix}$ and $\det A \neq 0$ then find other Eigen values?

$\text{Rank}(A) \leq 3$

$|A| = 0$ ✓
↳ Singular
↳ at least one of the Eigen value ($\lambda_1 = 0$)

$c_1 + c_2 = 13$
 $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A) = 147$
 $\lambda_1 \lambda_2 \lambda_3 = |A| = 0, \lambda_3 = 147 - 3$

$\boxed{0, \lambda_2, 147 - \lambda_2}$

(5) If 'A' is involutory matrix of order 2022 and

$\text{Trace}(A) = 1800$ then what is $|A| = ?$

Solution

(a) 1 (b) -1 (c) 2022 (d) 1800

$|A| = 1$ or -1
Eigen Value -1 or 1

$|A| = 1$

$A_{n \times n}$

	+ve	-ve	Trace
n	0	$n-1$	$n \checkmark$
1	$n-1$	$n-2$	$2-n$
2	$n-2$	$n-3$	$4-n$
3	$n-3$	$n-4$	$6-n$
...	$n-6$	$n-7$	$...$
0	$n-6$	$n-7$	$-n$

$|A| = (-1)^{\frac{n(n-1)}{2}}$

$|A| = (-1)^{\frac{2022(2021)}{2}} = (-1)^{1011} = -1$

$\text{Trace}(A) = 1800$

$1901 + 111$
Eigen value = $2022 - 1800 = 222$
111 -ve Eigen value

$111 + 111 + 111$
 $-ve + +ve$

$$|x| = 1 \cdot 1 \cdot 1 \cdots 1 + 1, -1, -1, -1, \dots$$

↓ ↓ -ve Eigen value -ve

$$= -1 \checkmark$$

$$= x =$$

- ① Let A be a square matrix $O(A) = 2022$ and $a_{ii} = 10 \forall i$,
 $a_{ij} = 1$ otherwise, then find $|A| = ?$

Solution

$\begin{matrix} & \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \\ \leftarrow \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \end{matrix}$

$A_{n \times n} = \boxed{\begin{matrix} 10 & 1 & 1 & 1 \\ 1 & 10 & 1 & 1 \\ 1 & 1 & 10 & 1 \\ 1 & 1 & 1 & 10 \end{matrix}}$

$\boxed{A} = \boxed{\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}} \rightarrow |A - \lambda I| = \boxed{\begin{matrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{matrix}}$

$n=4$

$= (-1)^n \lambda^{n-1} (\lambda - n)$ ✓

$1-\lambda = 10$

$\Rightarrow \lambda = 1-10 = -9$

$|A| = \frac{(-1)^{2022} (-9)^{2022-1} (-9-2022)}{(-1)^3 (-9-4)} = (-1)^{2022} (-9)^{2021} (-21)$ ✓

- ② Characteristic equation of a special matrix whose all entries as one is

$\boxed{(-1)^n \lambda^{n-1} (\lambda - n)}$ ✓

$\boxed{\begin{matrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{matrix}}$

$1-\lambda = 2$

$\lambda = -1$

- ③ How many real Eigen values exist for the following matrix $\begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & -2 & -3 & 4 \end{bmatrix}$

(2)

following matrix

$$A^T = A \quad A = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 11 & 5 & 4 \\ 2 & 5 & 9 & 6 \\ 3 & 4 & 6 & 12 \end{bmatrix}$$

Ans:- All four Eigen values are real

\therefore Given A is a real symmetric matrix
 \Rightarrow It will have only real Eigen

$\lambda^2 - (\lambda - 5i)^2 \leftarrow \text{values}$

(3)

$$A = \begin{bmatrix} 0 & 7+i & i & 9 \\ -7-i & 2i & 5 & 7i \\ i & -5 & 3i & 6 \\ -9 & 7i & -6 & 5i \end{bmatrix}$$

Eigenvalues

then how many

$$|A - \lambda I| = 0$$

non/ $\neq 0$ real Eigen value will Exist?

$\lambda + 5i$
 $0+i$

(i) Skew-Hermitian matrix $(\bar{A})^T = -A$

$$A \rightarrow$$

$$|\bar{A}| = (-1)^n \cdot |A|$$

$$|A^H| = |-A|$$

$$|A| \neq 0$$

$$(\bar{\lambda})^T = -\lambda$$

Complex value

$$\boxed{\bar{\lambda} = -\lambda}$$

- * Conjugate will not effect real value
- * It effects only for sign complex value

$\Rightarrow \therefore$ possible Eigen value of a Skew-Hermitian matrix are either 0 or imaginary

(*) if $A =$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A^H = I$$

$$AA^H = I$$

Ans:-

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad 7 \times 7$$

possible Eigen values

$$AAT = I \Rightarrow A^T = I \quad \text{Eigen value}$$

$$\Rightarrow AT = A \quad \text{symmetric}$$

$$\text{also}$$

$$\lambda^7 = 1 \Rightarrow \lambda = \pm 1$$

$$\text{Trace}(A) = 1 = 1 + 1 + 1 - 1 - 1 - 1 = 1$$

$$|A| = -1$$

Ex:- Given matrix A is a nilpotent matrix
of order 3

$$\textcircled{a} A^2 = 0 \quad \textcircled{b} A^3 = 0 \quad \textcircled{c} A^{50} = 0 \quad \textcircled{d} A^6 = 0$$

Ans:- Nilpotent matrix maximum index is
it's order $A^3 = 0 \Rightarrow \lambda^6 = 0$

$$A^{50} = (\textcircled{b}) \cdot A^{44} = 0 \checkmark$$

Ex) Find characteristic polynomial and minimal
polynomial of

$$\begin{bmatrix} A^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A \neq 0 \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{ch}(A) = \lambda^4$$

$$\text{minimal}(A) = \lambda^2 \checkmark$$

$$n = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{ch}(A) = \lambda^4$$

$$\lambda \text{ is a root of } \chi \text{ if } \chi(\lambda) = 0$$

$B \neq 0$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{char}(A) = \lambda^4$$

$$\text{minimal}(A) = \lambda^3$$

λ is a minimal polynomial only if $m(\lambda) = \lambda \Rightarrow m(\lambda) = A = 0$

\otimes

$$\begin{cases} P(x, y, z, w) = x^2 + y^2 + z^2 + bw^2 \\ Q(x, y, z, w) = x^2 + y^2 + czw \end{cases}$$

$$x^2 + y^2 + z^2 + bw^2 = [x \ y \ z \ w] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$x^2 + y^2 + czw = [x \ y \ z \ w] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\text{Eigen value of } Q_1 = 1, 1, 1, b$$

$$\text{Eigen value of } Q_2 = 1, 1, \frac{c}{2}, -\frac{c}{2}$$

$\xrightarrow{\text{Diagonalisable}}$

$A \simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Symmetric matrix :- $A \simeq I$ ✓ on complex domain

$$A \simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$
 $i \cdot i = -1$