Real Analysis

Basic Topology

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Metric Spaces

- A metric space is a set X together with a function $d : X \times X \to \mathbb{R}$ satisfying:
  - Non-negativity: $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.
  - Symmetry: $d(x, y) = d(y, x)$.
  - Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.
- Example: The Euclidean space $\mathbb{R}^n$ with the standard distance $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.
- Metric spaces provide a framework for discussing concepts like convergence, continuity, and compactness.

Metric Spaces

Let X be a metric space. All points and sets below are understood to be in X.

(a) A **neighborhood** of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$ for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a **limit point** of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, then p is called an **isolated point** of E.

(d) E is **closed** if every limit point of E belongs to E.

(e) A point p is an **interior point** of E if there exists a neighborhood N of p such that $N \subseteq E$.

(f) E is **open** if every point of E is an interior point.

Metric Spaces

(a) The **complement** of E (denoted E^c) is the set of all points $p \in X$ such that $p \notin E$.

(b) E is **perfect** if E is closed and every point of E is a limit point of E.

(c) E is **bounded** if there exists a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

(d) E is **dense** in X if every point of X is a limit point of E, or a point of E (or both).

Definition.[1] Let m be an integer. A sequence $(p_n)_{n=m}^{\infty}$ of rational numbers is any function

$$f : \{n \in \mathbb{Z} : n \geq m\} \;\rightarrow\; \mathbb{Q},$$

i.e., a mapping which assigns to each integer $n \geq m$ a rational number p_n.

More informally, a sequence $(p_n)_{n=m}^{\infty}$ of rational numbers is a collection:

$$p_m,\; p_{m+1},\; p_{m+2},\; \ldots$$

Example. The sequence defined by $p_n = \frac{1}{n}$ for $n \geq 1$ is

$$1,\; \tfrac{1}{2},\; \tfrac{1}{3},\; \tfrac{1}{4},\; \ldots$$

[1] Definition adapted from T. Tao, *Analysis I*.

Convergent Sequences

Definition.[2] A sequence p_n in a metric space X is said to **converge** if there is said a point $a \in X$ with the following property: for every $\epsilon > 0$, there is an integer N such that $d(p_n, p) < \epsilon$ whenever $n > N$ implies $d(p_n, p) < \epsilon$. In this case, we say that the sequence converges to p, or that p is the **limit** of the sequence, and we write $\lim_{n\to\infty} p_n = p$ or $p_n \to p$ as $n \to \infty$.
If p_n does not converge, we say that the sequence **diverges**.
The set of points p_n is called the **range** of the sequence. The sequence is said to be **bounded**[3] iff it is bounded by M for some rational $M \geq 0$.

[2] Definition adapted from W. Rudin, *Principles of Mathematical Analysis*.

[3] Let $M \geq 0$ be rational. A finite sequence $p_1, p_2, \ldots, p_n$ is bounded by M iff $|p_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $\{a_n\}_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$.

Convergent Sequences

Theorem 1

Let (p_n) be a sequence in a metric space X.

- **(a)** *(p_n) converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.*
- **(b)** *If $p \in X$, $p' \in X$, and if (p_n) converges to p and to p', then $p' = p$.*
- **(c)** *If (p_n) converges, then (p_n) is bounded.*
- **(d)** *If $E \subset X$ and if p is a limit point of E, then there is a sequence (p_n) in E such that $p = \lim_{n\to\infty} p_n$.*

Convergent Sequences

Theorem 2

Suppose (s_n), (t_n) *are complex sequences and* $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. *Then*

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$;

(b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c + s$ *for any number* c;

(c) $\lim_{n\to\infty} s_n t_n = st$;

(d) $\lim_{n\to\infty}(1/s_n) = 1/s$, *provided* $s_n \neq 0$ $(n = 1, 2, 3, \dots)$ *and* $s \neq 0$.

Convergent Sequences

Theorem 3

(a) Suppose $\mathbf{x}_n \in \mathbb{R}^k$ *(*$n = 1, 2, 3, \ldots$*) and* $\mathbf{x}_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$. *Then* $(\mathbf{x}_n)$ *converges to* $\mathbf{x} = (\alpha_1, \ldots, \alpha_k)$ *if and only if*

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \le j \le k). \tag{1}$$

(b) Suppose $(\mathbf{x}_n)$, $(\mathbf{y}_n)$ *are sequences in* $\mathbb{R}^k$, (β_n) *is a sequence of real numbers, and* $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. *Then*

$$\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \quad \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}, \quad \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

Sub-Sequences

Definition. Given a sequence (p_n), consider a sequence (n_k) of positive integers such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence (p_{n_k}) is called a **subsequence** of (p_n). If (p_{n_k}) converges, its limit is called a **subsequential limit** of (p_n).
A sequence (p_n) converges to p if and only if every subsequence of (p_n) converges to p.

Theorem 4

(a) *If (p_n) is a sequence in a compact metric space X, then some subsequence of (p_n) converges to a point of X.*

(b) *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.*

Theorem 5

The subsequential limits of a sequence (p_n) in a metric space X form a closed subset of X.

Definition. A sequence (p_n) in a metric space X is called a **Cauchy sequence** if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ whenever $m, n > N$.

Definition. Let E be a nonempty subset of a metric space X and let S be the set of all real numbers of the form $d(p, q)$ with $p \in E$ and $q \in E$. The $\sup S$ is called the **diameter** of E.

If (p_n) is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \ldots,$ it is clear from the two preceding definitions that (p_n) is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \text{diam } E_N = 0.$$

Cauchy Sequences

Theorem 6

(a) *If $\overline{E}$ is the closure of a set E in a metric space X, then*

$$diam\ \overline{E} = diam\ E.$$

(b) *If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \dots)$ and if*

$$\lim_{n\to\infty} diam\ K_n = 0,$$

then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Cauchy Sequences

Theorem 7

(a) *In any metric space X, every convergent sequence is a Cauchy sequence.*

(b) *If X is a compact metric space and if (p_n) is a Cauchy sequence in X, then (p_n) converges to some point of X.*

(c) *In $\mathbb{R}^k$, every Cauchy sequence converges.*

Note

The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter.

The fact that a sequence converges in $\mathbb{R}^k$ if and only if it is a Cauchy sequence is usually called the **Cauchy criterion for convergence**.

Definition. A metric space in which every Cauchy sequence converges is said to be **complete**.

Definition. A sequence (s_n) of real numbers is said to be

(a) **monotonically increasing** if $s_n \leq s_{n+1}$ $(n = 1, 2, 3, \ldots)$;

(b) **monotonically decreasing** if $s_n \geq s_{n+1}$ $(n = 1, 2, 3, \ldots)$.

Theorem 8

Suppose (s_n) is a monotonic sequence of real numbers. Then (s_n) converges if and only if it is bounded.

Upper and Lower Limits

Definition. Let (s_n) be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real M there exists an integer N such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \to -\infty.$$

Definition. Let (s_n) be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence (s_{n_k}). This set E contains all subsequential limits, plus possibly the numbers $+\infty$, $-\infty$.

Recall:

- The **supremum** (sup), or **least upper bound**, of a set is the smallest real number (or $+\infty$) that is $\geq$ every element.
- The **infimum** (inf), or **greatest lower bound**, of a set is the largest real number (or $-\infty$) that is $\leq$ every element.

Upper and Lower Limits

We now put

$$s^* = \sup E, \quad s_* = \inf E.$$

The numbers s^*, s_* are called the **upper** and **lower limits** of (s_n); we use the notation

$$\limsup_{n\to\infty} s_n = s^*, \quad \liminf_{n\to\infty} s_n = s_*.$$

Theorem 9

Let (s_n) be a sequence of real numbers. Let E and s^* have the same meaning as in Definition 3.16. Then s^* has the following two properties:

(a) $s^* \in E$.

(b) *If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.*

Moreover, s^* is the only number with the properties (a) and (b).

Upper and Lower Limits

Theorem 10

If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n, \quad \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n.$$

Some Special Series

Remark

If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

Theorem 11

(a) *If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*

(b) *If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.*

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) *If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.*

(e) *If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.*

Series

Definition. Given a sequence (a_n), we use the notation

$$\sum_{n=p}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With (a_n), we associate a sequence (s_n), where

$$s_n = \sum_{k=1}^{n} a_k.$$

For (s_n) we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n. \quad \text{(a)}$$

The symbol in (a) we call an **infinite series** or just a **series**. The numbers s_n are called the **partial sums** of the series.

Root and Ratio Tests

Theorem. The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Theorem. For any sequence (c_n) of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$