Boolean Algebra

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Table of Contents

What is Boolean Algebra?

- A branch of algebra in which the values of the variables are the truth values *true* and *false*, usually denoted 1 and 0 respectively.
- It is used to analyze and simplify digital circuits.
- It is also called Binary Algebra or logical Algebra.
- George Boole, an English mathematician, developed this algebra in 1854.

Basic Operations

The basic operations in Boolean algebra are:

- **Conjunction** ($\wedge$): Corresponds to the AND operator.
- **Disjunction** ($\vee$): Corresponds to the OR operator.
- **Negation** ($\neg$): Corresponds to the NOT operator.

AND Operator

The AND operator is denoted by a dot $(\cdot)$ or by the absence of an operator.

- $A \cdot B = Y$ or $AB = Y$
- The expression is true (1) only if both A and B are true (1).

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

Table: Truth Table for AND

OR Operator

The OR operator is denoted by a plus sign $(+)$.

- $A + B = Y$
- The expression is true (1) if either A or B or both are true (1).

A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

Table: Truth Table for OR

NOT Operator

The NOT operator is denoted by a prime (A').

- $A' = Y$
- It inverts the input. If the input is true (1), the output is false (0).

A	NOT A
0	1
1	0

Table: Truth Table for NOT

Basic Laws

Commutative Law

- $A + B = B + A$
- $A \cdot B = B \cdot A$

Associative Law

- $(A + B) + C = A + (B + C)$
- $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

Distributive Law

- $A \cdot (B + C) = A \cdot B + A \cdot C$
- $A + (B \cdot C) = (A + B) \cdot (A + C)$

Other Important Laws

- **Identity Law**: $A + 0 = A$, $A \cdot 1 = A$
- **Annulment Law**: $A + 1 = 1$, $A \cdot 0 = 0$
- **Idempotent Law**: $A + A = A$, $A \cdot A = A$
- **Complement Law**: $A + A' = 1$, $A \cdot A' = 0$
- **Involution Law**: $(A')' = A$
- **De Morgan's Laws**:
  - $(A + B)' = A' \cdot B'$
  - $(A \cdot B)' = A' + B'$

Minterms

- A minterm is a product term (AND operation) that contains all variables of the function, either in their normal or complemented form.
- For a given row of a truth table, the minterm is 1 if and only if the input combination is the one for that row.
- A variable appears in its normal form (X) if its value is 1 in the input combination.
- A variable appears in its complemented form (X') if its value is 0 in the input combination.
- Example for two variables A and B:
  - $A'B'$ (for A=0, B=0)
  - $A'B$ (for A=0, B=1)
  - AB' (for A=1, B=0)
  - AB (for A=1, B=1)

Maxterms

- A maxterm is a sum term (OR operation) that contains all variables of the function, either in their normal or complemented form.
- For a given row of a truth table, the maxterm is 0 if and only if the input combination is the one for that row.
- A variable appears in its normal form (X) if its value is 0 in the input combination.
- A variable appears in its complemented form (X') if its value is 1 in the input combination.
- Example for two variables A and B:
  - $A + B$ (for A=0, B=0)
  - $A + B'$ (for A=0, B=1)
  - $A' + B$ (for A=1, B=0)
  - $A' + B'$ (for A=1, B=1)

Sum of Products (SOP)

- A Boolean expression can be represented as a sum of minterms.
- This form is called the Sum of Products (SOP) or Disjunctive Normal Form (DNF).
- To get the SOP form from a truth table, we sum (OR) all the minterms for which the function's output is 1.

Product of Sums (POS)

- A Boolean expression can also be represented as a product of maxterms.
- This form is called the Product of Sums (POS) or Conjunctive Normal Form (CNF).
- To get the POS form from a truth table, we multiply (AND) all the maxterms for which the function's output is 0.

Example: Generating Compound Statements

Let's consider the compound statement "If A, then B", which is denoted as $A \to B$. The truth table is as follows:

A	**B**	**A $\to$ B**
0	0	1
0	1	1
1	0	0
1	1	1

Table: Truth Table for $A \to B$

Example: SOP and POS from Truth Table

From the truth table for $A \to B$:

- **Sum of Products (SOP):** We look for rows where the output is 1.
  - Row 1 (A=0, B=0): Minterm is $A'B'$
  - Row 2 (A=0, B=1): Minterm is $A'B$
  - Row 4 (A=1, B=1): Minterm is AB

  The SOP expression is $Y = A'B' + A'B + AB$. This can be simplified to $A' + B$.
- **Product of Sums (POS):** We look for rows where the output is 0.
  - Row 3 (A=1, B=0): Maxterm is $A' + B$

  The POS expression is $Y = A' + B$.

In this case, the simplified SOP and the POS are the same, which is often the case. The expression for "If A, then B" is logically equivalent to "Not A or B".

Derivation of the Simplification

Let's derive the simplification of the expression $Y = A'B' + A'B + AB$.

$$
\begin{aligned}
Y &= A'B' + A'B + AB \\
&= A'(B' + B) + AB && \text{Distributive Law} \\
&= A'(1) + AB && \text{Complement Law: } B + B' = 1 \\
&= A' + AB && \text{Identity Law: } A \cdot 1 = A \\
&= (A' + A)(A' + B) && \text{Distributive Law} \\
&= (1)(A' + B) && \text{Complement Law: } A + A' = 1 \\
&= A' + B && \text{Identity Law: } 1 \cdot A = A
\end{aligned}
$$

This shows that the expression $A'B' + A'B + AB$ simplifies to $A' + B$. This is a common simplification that is useful to remember, and it corresponds to the logical statement for implication, $A \to B \equiv \neg A \vee B$.