Functions

Abhijeet

IGIDR

August 9, 2025

Relation

- When we talk about a **relation** R from a **set** A to a **set** B, we simply mean that R is a subset of ordered pairs, where the first coordinate of the pair belongs to A, and the second coordinate belongs to B.
- Let's suppose $\mathbf{A} = \{a, b, c\}$ and $\mathbf{B} = \{1, 2, 3\}$. We now take a subset of the Cartesian product $\mathbf{A} \times \mathbf{B}$: $\{(a, 1), (a, 3), (c, 2)\}$. We denote this subset by R.
- $R = \{(a, 1), (a, 3), (c, 2)\}$

Terminologies

- $dom(R) = \{a \in A : (a, b) \in R \quad \text{for some} \quad b \in B\}$
- $range(R) = \{b \in A : (a, b) \in R \quad \text{for some} \quad a \in A\}$
- $R^{-1} = \{(b, a) : (b, a) \in R\}$
- Q. Determine the inverse relation R^{-1} for the relation $R = \{(x, y) : x + 4y \quad \text{is odd}\}$ defined on N.
- Let A and B be sets with $|A| = |B| = 4$.
  1. Prove or disprove: If R is a relation from A to B where $|R| = 9$ and $R = R^{-1}$, then $A = B$.
  2. Show that by making a small change in the statement in (a), a different response to the resulting statement can be obtained.

Properties of Relations

- A relation R defined on a set A is called **reflexive** if $x \; R \; x$ for every $x \in A$.
- A relation R defined on a set A is called **symmetric** if whenever $x \; R \; y$, then $y \; R \; x$ for all $x, y \in A$.
- A relation R defined on a set A is called **transitive** if whenever $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$, for all $x, y, z \in A$.
- A relation R defined on a set A is called an **equivalence** relation if R is reflexive, symmetric and transitive.
- Prove that The relation R defined on Z by x R y if x + 3y is even is an equivalence relation.

Functions

- Let A and B be nonempty sets. By a **function** f from A to B, written $f : A \to B$, we mean a relation from A to B with the property that every element a in A is the first coordinate of exactly one ordered pair in f.
- Since f is a relation, the set A in this case is the **domain** of f, denoted by $\text{dom}(f)$. The set B is called the **codomain** of f.
- Let A and B be nonempty sets. By a **correspondence** R from A to B, written $R : A \to B$, we mean a relation from A to B with the property that every element a in A is the first coordinate of at least one ordered pair in R.
- $\text{range}(f) = \{b \in B \mid \text{b is an image under } f \text{ of some element of } A\} = \{f(x) \mid x \in A\}$

One-to-One Functions(Injective)

- A function f from a set A to a set B is called **one-to-one** or **injective** if every two distinct elements of A have distinct images in B. In symbols, a function $f : A \to B$ is one-to-one if whenever $x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$.
- Thus, if a function $f : A \to B$ is not **one-to-one**, then there exist distinct elements w and z in A such that $f(w) = f(z)$.
- Suppose that a function $f : A \to B$ is **one-to-one**, where A and B are finite sets. Since every two distinct elements of A have distinct images in B, there must be at least as many elements in B as in A, that is, $|A| \leq |B|$.
- The function $f : R \to R$ defined by $y = x + 2$ is a one to one function and $y = x^2$ is not.

Onto Functions(Surjective)

- A function $f : A \to B$ is called **onto** or **surjective** if every element of the codomain B is the image of some element of A. Equivalently, f is onto if $f(A) = B$.
- For finite sets A and B, a function $f : A \to B$ is **surjective** (or **onto**) if and only if $|B| \leq |A|$.
- The function $f : R \to R$ defined by $y = x + 2$ is a onto function.

One-to-One and Onto Functions(Bijective)

- A function $f : A \rightarrow B$ is called **onto** or **surjective** if every element of the codomain B is the image of some element of A. Equivalently, f is onto if $f(A) = B$.
- The function $f : R \rightarrow R$ defined by $y = x + 2$ is a onto function.

Composition of Functions

- The composition $g \circ f$ of f and g is the function from A to C defined by $(g \circ f)(x) = g(f(x))$ for all $a \in A$.
- **Example:** Let $f(x) = \sin x$, and $g(x) = x^2$.

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sin(x^2)$$

$$(g \circ f)(x) = g(f(x)) = g(\sin x) = (\sin x)^2 = \sin^2 x$$

- Let $f : A \to B$ and $g : B \to C$ be two functions, then prove:
  - (a) If f and g are injective, then so is $g \circ f$.
  - (b) If f and g are surjective, then so is $g \circ f$.

Inverse Functions

- For a relation R from a set A to a set B, the inverse relation R^{-1} from B to A is defined as

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

- **Example:** If $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$ and
  $R = \{(a, 1), (a, 3), (c, 2), (c, 3), (d, 1)\}$
  is a relation from A to B, then
  $R^{-1} = \{(1, a), (3, a), (2, c), (3, c), (1, d)\}$.
- Let $f : A \to B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. Furthermore, if f is bijective, then f^{-1} is also bijective.
- For all values in its Domain, $f(x) = x^2$ doesn't have an inverse