Mathematics for Economists

Proof Techniques [1]

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[1] Slides Derived from, Mathematical Proofs: A Transition to Advanced Mathematics

Outline

Some Definition

- An **axiom** is a true mathematical statement whose truth is accepted without proof.
- A **theorem** is a true mathematical statement whose truth can be verified.
- A **corollary** is a mathematical result that can be deduced from, and is thereby a consequence of, some earlier result.
- A **lemma** is a mathematical result that is useful in establishing the truth of some other result.

Direct Proof

- A **direct proof** is a method of proving a mathematical statement by a straightforward chain of logical deductions from known facts, definitions, and previously established results, without assuming the contrary of what is to be proved.
- A **direct proof** is a structured way of proving a mathematical statement of the form:

$$P \implies Q$$

  where:
  - P is the **hypothesis** (assumption), and
  - Q is the **conclusion** (what we want to prove).

  We assume $P(x)$ to be true and apply reasoning, definitions, and other results to conclude that $Q(x)$ is true.

Direct Proof: Example

(1) Using Direct Proof method, prove that, if n is even integer then n^2 is also even integer.

Proof: We need to prove the claim is true given the assumption is true. What do we know about the assumption? We know that n is even. So we can write $n = 2k, k \in Z$. Now square it. We get, $n^2 = 4k^2 = 2 \times 2k^2$. Now $2k^2$ is an integer and that multiplied by 2 yield an even integer. QED

(2) Using Direct Proof Method, prove that, if n is an odd integer, then $3n + 7$ is an even integer.

Proof: We know n is an odd integer. So we can write, $n = 2k + 1$, where $k \in Z$. Substituting the value of $n = 2k + 1$ in $3n + 7$, we get, $6k + 10 = (2 \times 3k + 5)$, $3k + 5$ is an ineteger, that multiplied by 2 yield an even integer. $6k + 8 = 3n + 7$, given $n = 2k + 1, k \in Z$ is even. QED

Proof by Contrapositive

- For statements P and Q, the contrapositive of the implication $P \implies Q$ is the implication $\neg Q \implies \neg P$
- For statements P and Q, the contrapositive of the implication $P \implies Q$ is the implication $\neg Q \implies \neg P$
- If Q is false, then P is also false.

Proof by Contrapositive: Examples

(1) Using proof by contrapositive, Prove that if $5x - 7$ is even then x is odd.
Proof: We assume, x to be not odd, that is x is even, and we will try to prove that if x is even then, $5x - 7$ is odd. We can write, $x = 2k, k \in Z$. So, $5x - 7 = 10k - 7 = (2 \times 5k) + 7$. Now even+odd will produce us odd. Thus, $(10k - 7) = 5x - 7$ is odd. QED
(2)Let $x \in \mathbb{Z}$. Then x^2 is even if and only if x is even:
x^2 is even $\iff$ x is even
Proof: In biconditional statements, we need to prove both ways, that is $A \implies B$ and($\wedge$) $B \implies A$
So, (i) If x^2 is even then x is even and (ii) If x is even, then x^2 is even.
(3) Let $x \in Z$. If $5x - 7$ is odd, then $9x + 2$ is even. Sometimes, its

not possible to prove a result by assuming values for all x. So, we divide into cases, like, Case 1: If x is even then B, **Case 2:** If x is odd then B.

Counter-examples

If the statement $\forall x \in S, R(x)$ is false, then there exists some element $x \in S$ for which $R(x)$ is false:
$\neg(\forall x \in S, R(x)) \equiv \exists x \in S$ such that $\neg R(x)$.
Such an element x is called a **counter-example** of the (false) statement $\forall x \in S, R(x)$.
Finding a counter-example verifies that $\forall x \in S, R(x)$ is false.

Counter-examples: Examples

Disprove the statement: Let $n \in \mathbb{Z}$. If $n^2 + 3n$ is even, then n is odd. Take $n = 2$. Now, $n^2 + 3n = 10$, which is even but n is odd. Hence, we disproved.

Proof by Contradiction

To prove a statement P, we assume the negation $\neg P$ and show that this leads to a contradiction. That is, we show:

$$\neg P \implies \text{False}$$

Therefore, P must be true.
If R is the quantified statement

$$R : \forall x \in S,\ P(x) \implies Q(x),$$

then a proof by contradiction might begin with the assumption:
Assume, to the contrary, that there exists some element $x \in S$ for which $P(x)$ is true and $Q(x)$ is false, i.e.,

$$\exists x \in S \text{ such that } P(x) \text{ is true and } Q(x) \text{ is false.}$$

The remainder of the proof then consists of showing that this assumption leads to a contradiction.

Proof by Contradiction: Examples

(1) By using the method of 'Proof by contradiction', prove that: There is no smallest positive real number.
(2) No odd integer can be expressed as the sum of three even integer.
(3) Prove that if x and y are positive real numbers, then $\sqrt{x} + \sqrt{y} \neq \sqrt{2(x+y)}$
(4) Prove that there do not exist three distinct real numbers a, b, and c such that all of the numbers $a + b + c, \quad ab, \quad ac, \quad bc, \quad abc$ are equal.

How to prove (and How not to): $\forall \in S, P(x) \implies Q(x)$

No.	First Step of "Proof"	Remarks/Goal
1	Assume for an arbitrary element $x \in S$ that $P(x)$ is true.	A direct proof is being used. Show that $Q(x)$ is true for the element x.
2	Assume for an arbitrary element $x \in S$ that $P(x)$ is false.	A mistake has been made.
3	Assume for an arbitrary element $x \in S$ that $Q(x)$ is true.	A mistake has been made.
4	Assume for an arbitrary element $x \in S$ that $Q(x)$ is false.	A proof by contrapositive is being used. Show that $P(x)$ is false for the element x.
5	Assume for an arbitrary element $x \in S$ that $P(x)$ and $Q(x)$ are true.	A mistake has been made.
6	Assume that there exists $x \in S$ such that $P(x)$ is true and $Q(x)$ is false.	A proof by contradiction is being used. Produce a contradiction.
7	Assume that there exists $x \in S$ such that $P(x)$ is false and $Q(x)$ is true.	A mistake has been made.
8	Assume that there exists $x \in S$ such that $P(x)$ and $Q(x)$ are false.	A proof by contradiction is being used. Produce a contradiction.
9	Assume that there exists $x \in S$ such that $P(x) \implies Q(x)$ is true.	A mistake has been made.
10	Assume that there exists $x \in S$ such that $P(x) \implies Q(x)$ is false.	A mistake has been made.

Proof by Mathematical Induction

For each positive integer n, let $P(n)$ be a statement. If

1. $P(1)$ is true, and
2. $\forall k \in \mathbb{N},\ P(k) \implies P(k+1)$ is true,

then $\forall n \in \mathbb{N},\ P(n)$ is true.

As a consequence of the Principle of Mathematical Induction, the quantified statement $\forall n \in \mathbb{N},\ P(n)$ can be proved to be true if

1. we can show that the statement $P(1)$ is true, and
2. we can establish the truth of the implication

$$P(k) \implies P(k+1)$$

for every positive integer k.

Proof by Mathematical Induction: Examples

Prove the following using Mathematical Induction:

- Find a formula for $1 + 4 + 7 + \cdots + (3n - 2)$ for positive integers n, and verify your formula by mathematical induction.
- Prove that $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$ for every positive integer n.
- Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for every positive integer n.
- Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n.