The stability and efficiency of directed communication networks

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Abstract. This paper analyzes the formation of directed networks where self-interested individuals choose with whom they communicate. The focus of the paper is on whether the incentives of individuals to add or sever links will lead them to form networks that are efficient from a societal viewpoint. It is shown that for some contexts, to reconcile efficiency with individual incentives, benefits must either be redistributed in ways depending on "outsiders" who do not contribute to the productive value of the network, or in ways that violate equity (i.e., anonymity). It is also shown that there are interesting contexts for which it is possible to ensure that efficient networks are individually stable via (re)distributions that are balanced across components of the network, anonymous, and independent of the connections of non-contributing outsiders.

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1 Introduction

Much of the communication that is important in economic and social contexts does not take place via centralized institutions, but rather through networks of decentralized bilateral relationships. Examples that have been studied range from the production and transmission of gossip and jokes, to information about job opportunities, securities, consumer products, and even information regarding the returns to crime. As these networks arise in a decentralized manner, it is important

to understand how they form and to what degree the resulting communication is efficient.

This paper analyzes the formation of such directed networks when self-interested individuals choose with whom they communicate. The focus of the paper is on whether the incentives of individuals will lead them to form networks that are efficient from a societal viewpoint. Most importantly, are there ways of allocating (or redistributing) the benefits from a network among individuals in order to ensure that efficient networks are stable in the face of individual incentives to add or sever links?

To be more precise, networks are modeled as directed graphs among a finite set of individual players. Each network generates some total productive value or utility. We allow for situations where the productive value or utility may depend on the network structure in general ways, allowing for indirect communication and externalities.

The productive value or utility is allocated to the players. The allocation may simply be the value that players themselves realize from the network relationships. It may instead represent some redistribution of that value, which might take place via side contracts, bargaining, or outside intervention by a government or some other player. We consider three main constraints on the allocation of productive value or utility. First, the allocation must be anonymous so that the allocation depends only on a player's position in a network and how his or her position in the network affects overall productive value, but the allocation may not depend on a player's label or name. Second, the allocation must respect component balance: in situations where there are no externalities in the network, the network's value should be (re)distributed inside the components (separate sub-networks) that generate the value. Third, if an outsider unilaterally connects to a network, but is not connected to by any individual in that network, then that outsider obtains at most her marginal contribution to the network. We will refer to this property as *outsider independence*.

The formation of networks is analyzed via a notion of individual stability based on a simple game of network formation in such a context: each player simultaneously selects a list of the other players with whom she wishes to be linked. Individual stability then corresponds to a (pure strategy) Nash equilibrium of this game.

We show that there is an open set of value functions for which *no* allocation rule satisfies anonymity, component balance, and outsider independence, and still has at least one efficient (value maximizing) network being individually stable. However, this result is not true if the outsider independence condition is removed. We show that there exists an allocation rule which is anonymous, component balanced and guarantees that some efficient network is individually stable. This shows a contrast with the results for non-directed networks. We go on to show that for certain classes of value functions an anonymous allocation rule satisfying component balance and outsider independence can be constructed such that an efficient network is individually stable. Finally, we show that when value accumulates from connected communication, then the value function is in

this class and so there is an allocation rule that satisfies anonymity, component balance, and outsider independence, and still ensures that at least one (in fact all) efficient networks are individually stable.

Relationship to the literature

There are three papers that are most closely related to the analysis conducted here: Jackson and Wolinsky (1996), Dutta and Mutuswami (1997), Bala and Goyal (1999).[1]

The relationship between efficiency and stability was analyzed by Jackson and Wolinsky (1996) in the context of non-directed networks. They noted a tension between efficiency and stability of networks under anonymity and component balance, and also identified some conditions under which the tension disappeared or could be overcome via an appropriate method of redistribution.

There are two main reasons for revisiting these questions in the context of directed networks. The most obvious reason is that the set of applications for the directed and non-directed models is quite different. While a trading relationship, marriage, or employment relationship necessarily requires the consent of two individuals, an individual can mail (or email) a paper to another individual without the second individual's consent. The other reason for revisiting these questions is that incentive properties turn out to be different in the context of directed networks. Thus, the theory from non-directed networks cannot simply be cut and pasted to cover directed networks. There turn out to be some substantive similarities between the contexts, but also some significant differences. In particular, the notion of an outsider to a network is unique to the directed network setting. The differences between the directed and non-directed settings are made evident through the theorems and propositions, below.

Dutta and Mutuswami (1997) showed that if one weakens anonymity to only hold on stable networks, then it is possible to carefully construct a component balanced allocation rule for which an efficient network is pairwise stable. Here the extent to which anonymity can be weakened in the directed network setting is explored. It is shown that when there is a tension between efficiency and stability, then anonymity must be weakened to hold only on stable networks. Moreover, only some (and not all) permutations of a given network can be supported even when all permutations are efficient. So, certain efficient networks can be supported as being individually stable by weakening anonymity, but not efficient network architectures.

This paper is also related to a recent paper by Bala and Goyal (1999), who also examine the formation of directed communication networks. The papers are, however, quite complementary. Bala and Goyal focus on the formation of networks in the context of two specific models (the directed connections and

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[1] Papers by Watts (1997), Jackson and Watts (1998), and Currarini and Morelli (1998) are not directly related, but also analyze network formation in very similar contexts and explore efficiency of emerging networks.

hybrid connections models discussed below) without the possibility of reallocat-ing of any of the productive value.[2] Here, the focus is instead on whether there exist equitable and (component) balanced methods of allocating (or possibly re-allocating) resources to provide efficient incentives in the context of a broad set of directed network models. Results at the end of this paper relate back to the directed connections and hybrid connections models studied by Bala and Goyal, and show that the individual stability of efficient networks in those models can be ensured (only) if reallocation of the productive value of the network is possible.

2 Definitions and examples

Players

$\{1, \ldots, N\}$ is a finite set of players. The network relations among these players are formally represented by graphs whose *nodes* are identified with the players.

Networks

We model directed networks as digraphs.

A *directed network* is an $N \times N$ matrix g where each entry is in $\{0, 1\}$. The interpretation of $g_{ij} = 1$ is that i is *linked* to j, and the interpretation of $g_{ij} = 0$ is that i is *not linked* to j. Note that $g_{ij} = 1$ does not necessarily imply that $g_{ji} = 1$. It can be that i is linked to j, but that j is not linked to i. Adopt the convention that $g_{ii} = 0$ for each i, and let G denote the set of all such directed networks. Let g_i denote the vector $(g_{i1}, \ldots, g_{iN})$.

For $g \in G$ let $N(g) = \{i \mid \exists j \text{ s.t. } g_{ij} = 1 \text{ or } g_{ji} = 1\}$. So $N(g)$ are the active players in the network g, in that either they are linked to someone or someone is linked to them.

For any given g and ij let $g+ij$ denote the network obtained by setting $g_{ij} = 1$ and keeping other entries of g unchanged. Similarly, let $g-ij$ denote the directed network obtained by setting $g_{ij} = 0$ and keeping other entries of g unchanged.

Paths

A *directed path in g connecting i_1 to i_n* is a set of distinct nodes $\{i_1, i_2, \ldots, i_n\} \subset N(g)$ such that $g_{i_k i_{k+1}} = 1$ for each k, $1 \le k \le n - 1$.

A *non-directed path in g connecting i_1 to i_n* is a set of distinct nodes $\{i_1, i_2, \ldots, i_n\} \subset N(g)$ such that either $g_{i_k i_{k+1}} = 1$ or $g_{i_{k+1} i_k} = 1$ for each k, $1 \le k \le n - 1$.[3]

Components

A network g' is a *sub-network* of g if for any i and j $g'_{ij} = 1$ implies $g_{ij} = 1$.

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[2] Also, much of Bala and Goyal's analysis is focussed on a dynamic model of formation that selects strict Nash equilibria in the link formation game in certain contexts where there also exist Nash equilibria that are not strict.

[3] Non-directed paths are sometimes referred to as semipaths in the literature.

A non-empty sub-network of g, g', is a *component of g* if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a non-directed path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$ if there is a non-directed path in g between i and j, then $j \in N(g')$. The set of components of a network g is denoted $C(g)$.

A network g is *completely connected* (or the complete network) if $g_{ij} = 1$ for all ij.

A network g is *connected* if for each distinct i and j in N there is a non-directed path between i and j in g.

A network g' is a *copy* of g if there exists a permutation π of N such that $g' = g^\pi$.

Specific network structures

A network g is a *star* if there is i such that $g_{kl} = 1$ only if $i \in \{k, l\}$. That is, a star is a network in which all connections involve a *central* node i.

A network g is a *k-person wheel* if there is a sequence of players $\{i_1, \ldots, i_k\}$ such that $g_{i_k i_1} = g_{i_j i_{j+1}} = 1$ for all $j = 1, \ldots, k-1$, and $g_{ij} = 0$ otherwise.

Value functions

A *value function* $v : G \to \mathbb{R}$, assigns a value $v(g)$ to each network g. The set of all value functions is denoted V.

In some applications the value of a network is an aggregate of individual utilities or productions, so that $v(g) = \sum_i u_i(g)$ for some profile of $u_i : G \to \mathbb{R}$.

The concepts above are illustrated in the context of the following examples.

Example 1. The Directed Connections Model.[4]   The value function $v^d(\cdot)$ is the sum of utility functions ($u_i(\cdot)$'s) that describe the benefit (net of link costs) that players obtain from direct and indirect communication with others. Each player has some information that has a value 1 to other players.[5] The factor $\delta \in [0, 1]$ captures decay of information as it is transmitted. If a player i has $g_{ij} = 1$, then i obtains δ in value from communication with j. There are different interpretations of this communication: sending or receiving. Player i could be getting value from receiving information that i has accessed from j (e.g., contacting j's web site), or it could be that i is getting value from sending j information (e.g., mailing research papers or advertising). In either case, it is i who incurs the cost of communication and is benefiting from the interaction. If the shortest directed path between i and j contains 2 links (e.g., $g_{ik} = 1$ and $g_{kj} = 1$), then i gets a value of δ^2 from the indirect communication with j. Similarly, if the shortest directed path between i and j contains m links, then i gets a value of δ^m from

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[4] This model is considered by Bala and Goyal (1999), and is also related to a model considered by Goyal (1993). The name reflects the relationship to the non-directed "connections model" discussed in Jackson and Wolinsky (1996).

[5] Bala and Goyal consider a value V. Without loss of generality this can be normalized to 1 since it is the ratio of this V to the cost c that matters in determining properties of networks, such as identifying the efficient network or considering the incentives of players to form links.

the indirect communication with j. If there is no directed path from i to j, then i gets no value from communication with j.

Note that information only flows one way on each link. Thus, j gets no value from the link $g_{ij} = 1$. This also means that i gets no value from j if there is exists a non-directed path between i and j, but no directed path from i to j.

Player i incurs a cost $c > 0$ of maintaining each direct link. Player i can benefit from indirect communication without incurring any cost beyond i's direct links.

Let $N(i, g)$ denote the set of players j for which there is a directed path from i to j. For i and any $j \in N(i, g)$, let $d(ij, g)$ denote the number of links in the minimum-length directed path from i to j. Let $n_d(i, g) = \#\{j \mid g_{ij} = 1\}$ represent the number of direct links that i maintains. The function u_i can be expressed as[6]

$$u_i(g) = \sum_{j \in N(i,g)} \delta^{d(ij,g)} - n_d(i, g)c.$$

Then $v^d(g) = \sum_i u_i(g)$.

Example 2. The Hybrid Connections Model. Consider a variation on the directed connections model where players still form directed links, but where information flows both ways along any link. This model is studied in Bala and Goyal (1999), who mention telephone calls as an example of such communication. One player initiates the link and incurs the cost, but both share the communication benefits (or losses). Another example that would fit into this hybrid model would be physical connections on a computer network like the internet. A player (who may be an individual, a university, company, or some other collection of users) incurs the cost for connecting to a network, and then others already interconnected can communicate with the player.

Let $\overline{N}(i, g)$ denote the set of players j for which there is a non-directed path between i and j. For i and any $j \in \overline{N}(i, g)$, let $\overline{d}(ij, g)$ denote the number of links in the minimum-length non-directed path from i to j. The function u_i can be expressed as

$$u_i(g) = \sum_{j \in \overline{N}(i,g)} \delta^{\overline{d}(ij,g)} - n_d(i, g)c,$$

and $v^h(g) = \sum_i u_i(g)$.

Strong efficiency

A network $g \subset g^N$ is *strongly efficient* if $v(g) \geq v(g')$ for all $g' \subset g^N$.

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[6] Player i gets no value from his or her own information. This is simply a normalization so that the value of the empty network is 0.

The term strong efficiency indicates maximal total value, rather than a Paretian notion.[7] Of course, these are equivalent if value is transferable across players. In situations where Y represents a redistribution, and not a primitive utility, then implicitly value is transferable and strong efficiency is an appropriate notion.

Allocation functions

An *allocation rule* $Y : G \times V \to \mathbb{R}^N$ describes how the value associated with each network is distributed to the individual players.

$Y_i(g, v)$ is the payoff to player i from graph g under the value function v.

In the directed connections model (without any redistribution) $Y_i(g, v) = u_i(g)$, so that players obtain exactly the utility of their communication. The definition of an allocation rule, however, also allows for situations where $Y_i(g, v) \neq u_i(g)$, so that transfers or some redistribution is considered.

Anonymity of a value function

A value function v is *anonymous* if $v(g^\pi) = v(g)$ for all g and π.

Anonymity of a value function states that the value of a network depends only on the pattern of links in the network, and not on the labels of the players who are in given positions in the network.

Anonymity of an Allocation Function

For any value function v and permutation of players π, let the value function v^π be defined by $v^\pi(g^\pi) = v(g)$ for each g.

An allocation rule Y is *anonymous* relative to a network g and value function v if, for any permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$. Y is anonymous, if it is anonymous relative to each network g and value function v.

Anonymity of an allocation rule states that if all that has changed is the names of the agents (and not anything concerning their relative positions or production values in some network), then the allocations they receive should not change. In other words, the anonymity of Y requires that the information used to decide on allocations be obtained from the value function v and the particular network g, and not from the label of a player.

Note that anonymity of an allocation rule implies that individuals who are in symmetric positions in a network are assigned the same allocation, *if* the underlying v is anonymous, but not necessarily otherwise.[8] For instance if g is such that $g_{12} = g_{21} = 1$ and $g_{ij} = 0$ for all other ij, then provided v is anonymous[9] it follows that $Y_1(g, v) = Y_2(g, v)$.

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[7] The term strong efficiency is used by Jackson and Wolinsky (1996), Dutta and Mutuswami (1997), and Jackson and Watts (1998). This is referred to as efficiency by Bala and Goyal (1999). We stick with the term strong efficiency to distinguish the notion from Pareto efficiency.

[8] This is the only implication of anonymity that is needed to establish the negative results in what follows.

[9] More explicitly, for this network the conclusion follows if $v^{\pi(12)} = v$, where $\pi(12)$ is the permutation such that 1 and 2 are switched and all other players are mapped to themselves.

Balance and component balance

An allocation rule Y is *balanced* if $\sum_i Y_i(g,v) = v(g)$ for all value functions v and networks g.

A stronger notion of balance, component balance, requires Y to allocate resources generated by any component to that component.

A value function v is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for each network g.[10]

An allocation rule Y is *component balanced* if $\sum_{i \in N(h)} Y_i(g,v) = v(h)$ for every $g \in G$ and $h \in C(g)$ and component additive $v \in V$.

Component balance requires that the value generated by a given component be redistributed only among the players in that component. It is important that the definition of component balance *only* applies when v is component additive. Thus, it is only required to hold when there are no externalities across components.

Outsiders

A stronger version of component balance turns out to be important in the context of directed networks. The following definition of outsider is important in that definition and outsider independence.

A player i is an *outsider* of a network g if

(i) $g_{ij} = 1$ for some $j \in N(g)$,
(ii) $g_{ki} = 0$ for all $k \in N(g)$, and
(iii) for every $j \neq i$, $j \in N(g)$, there exists $k \neq i$ with $k \in N(g)$ such that $g_{kj} = 1$.

Thus, an outsider is a player who has linked to some other players in a network, but to whom no other player has linked. Furthermore, a player is considered an outsider only when all other players in the network have someone (other than the outsider) linked to them, so the outsider is not important in connecting anyone else to the network. This last condition avoids the trivial case of calling player 1 an outsider in the network g where $g_{12} = 1$ and $g_{ij} = 0$ for all other ij. It also implies that there is at most one outsider to a network.

Directed component balance

Let $g - i$ denote the network obtained from network g by deleting each of player i's links, but not the links from any player $j \neq i$ to player i. That is, $(g-i)_{ij} = 0$ for all j, and $(g-i)_k = g_k$ whenever $k \neq i$.

The allocation rule Y satisfies *directed component balance* if it is component balanced, and for any component additive value function v, network g, and outsider i to g, if $v(g) = v(g-i)$, then $Y(g) = Y(g-i)$.

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[10] This definition implicitly requires that the value of disconnected players is 0. This is not necessary. One can redefine components to allow a disconnected player to be a component. One has also to extend the definition of v so that it assigns values to such components.

The situation covered by directed component balance but not by component balance is one where a single player i is initially completely unconnected under $g - i$, then connects to some other players resulting in g, but does not change the value of the network. The directed component balance condition requires that the allocation rule not change due to the addition of such an outsider. This directed version of component balance is in the same spirit as component balance. The reasoning is that a player who unilaterally links up to a component whose members are already interconnected, and who does not change the productive value of the network *in any way*, effectively should not be considered to be part of that component for the purposes of allocating productive value.

Network formation and individual stability

Let $D_i(g) = \{g' | g'_j = g_j \forall j \neq i\}$. These are the networks that i can reach from g by a unilateral change in strategy.

A network g is *individually stable* relative to Y and v if $Y_i(g', v) \leq Y_i(g, v)$ for all $g' \in D_i(g)$. [11]

The idea of individual stability is quite straightforward. A network is individually stable if no player would benefit from changing his or her directed links. The set of individually stable networks corresponds to the networks that are pure strategy Nash equilibrium outcomes of a link formation game where each player simultaneously writes down the list of players who he or she will link to, and those links are then formed. [12]

3 Individual stability and strong efficiency

Theorem 1. *If $N \geq 3$, then there is no Y which satisfies anonymity and directed component balance and is such that for each v at least one strongly efficient graph is individually stable.*

Proof. Let $N = 3$ and consider any Y which satisfies anonymity and directed component balance. The theorem is verified by showing that there exists a v such that no strongly efficient graph is individually stable.

Let g be such that $g_{12} = g_{23} = g_{31} = 1$ and all other $g_{ij} = 0$, and g' be such that $g'_{13} = g'_{32} = g'_{21} = 1$ and all other $g'_{ij} = 0$. Thus, g and g' are the 3-person wheels.

Let v be such that $v(g) = v(g') = 1 + \epsilon$ and $v(g'') = 1$ for any other graph g''. Therefore, the strongly efficient networks are the wheels, g and g'.

Consider g'' such that $g''_{12} = g''_{21} = 1$ and all other $g''_{ij} = 0$.

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[11] This notion is called 'sustainability' by Bala and Goyal (1999). The term stability is used to be consistent with a series of definitions from Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997) for similar concepts with non-directed graphs.

[12] This link formation process is a variation of the game defined by Myerson (1991, page 448). Similar games are used to model link formation by Qin (1996), Dutta, et al. (1998), Dutta and Mutuswami (1996), and Bala and Goyal (1999).

It follows from anonymity and component balance that $Y_1(v, g'') = Y_2(v, g'') = 1/2$.

It follows from directed component balance that $Y_1(v, g'' + 31) = Y_2(v, g'' + 31) = 1/2$.

It follows from anonymity and balance that $Y_1(g, v) = Y_2(g, v) = Y_3(g, v) = \frac{1+\epsilon}{3}$.

Consider the strategy profile leading to g in the link formation game. If $\epsilon < 1/6$, then this strategy profile is not a Nash equilibrium, since player 2 will benefit by deviating and adding 21 and deleting 23. (Notice that $g'' + 31$ is obtained from g by adding 21 and deleting 23.) A similar argument shows that the strategy profile leading to g' in the link formation game does not form a Nash equilibrium. The case of $N > 3$ is easily handled by extending the above v so that components with more than three players have no value.   □

The proof of Theorem 1 necessarily follows a different line of reasoning from the proof of the analogous theorem for the non-directed case in Jackson and Wolinsky (1996). This reflects the difference between individual stability in the directed setting and pairwise stability in the non-directed setting that naturally arises due to the possibility of unilateral link formation in the directed network context. In the proof here, the problematic efficient network is an anonymous one and the contradiction is reached via a comparison to the network g'' which makes use of directed component balance. In the non-directed case, the proof examines a situation where the efficient network is not anonymous, and reaches a contradiction via comparisons to anonymous super- and sub-networks. The difference between the directed and non-directed settings is further explored below.

For the case of non-directed networks, one of the main points of Dutta and Mutuswami's (1997) analysis is that one can weaken anonymity to require that it only hold on stable networks and thereby overcome the incompatibility between efficiency and stability noted by Jackson and Wolinsky (1996). This is based on an argument that one is normatively less concerned with what occurs on unstable networks (out of equilibrium), provided one expects to see stable networks form. So Dutta and Mutuswami use non-anonymous rewards and punishments out of equilibrium to support an anonymous stable allocation. It can be shown, however, that in the non-directed case there is no Y that is component balanced and for which a strongly efficient network is pairwise stable,[13] as are all anonymous permutations of that network when v is anonymous. (This follows from Theorem 1' and its proof in the appendix of Jackson and Wolinsky (1996).) The implication of this is that in order to have at least one strongly efficient network be pairwise stable and satisfy component balance, it can be that only one of the strongly efficient networks is pairwise stable even though anonymous permuta-

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[13] In the context of non-directed networks it takes the consent of two individuals to form a link. Pairwise stability requires that no individual benefit from severing one link, and no two individuals benefit (one weakly and one strictly) from adding a link. A precise definition is given in Jackson and Wolinsky (1996).

tions of it are also strongly efficient. Thus, pairwise stability may apply just to a specific efficient network with players in a fixed relationship (and not to a network structure). For example, in certain contexts one can construct a component balanced allocation rule for which a star with player 1 at the center is strongly efficient and pairwise stable, but one cannot at the same time ensure that a star with player 2 at the center is also pairwise stable even though it generates exactly the same total productive value as the star with player 1 at the center, and thus is also strongly efficient. [14] This may not be objectionable, as long as one can at least ensure an anonymous set of payoffs to players, as Dutta and Mutuswami do. But the fact that only specific efficient networks can be supported, and not a given efficient network structure, gives a very precise idea of the extent to which anonymity must be weakened in order to reconcile efficiency and stability in the face of component balance. This is stated in the context of directed networks as follows.

Theorem 2. *If $N \geq 3$, then there is no Y that satisfies anonymity relative to individually stable networks, directed component balance, has an anonymous set of individually stable networks when v is anonymous,[15] and is such that for each v at least one strongly efficient network is individually stable.*

Proof. Let $N = 3$ and consider any Y which satisfies anonymity on individually stable networks, directed component balance, has an anonymous set of stable networks when v is anonymous. The theorem is proven by showing that there exists a v such that no strongly efficient network is individually stable.

Consider g, g', g'', and v from the proof of Theorem 1. Suppose the contrary, so that either g or g' is individually stable. Since v is anonymous and g and g' are anonymous permutations of each other, it follows that both g and g' are individually stable.

Thus, anonymity on individually stable networks and balance imply that $Y_1(g, v) = Y_2(g, v) = Y_3(g, v) = \frac{1+\epsilon}{3}$ and likewise that $Y_1(v, g') = Y_2(v, g') = Y_3(v, g') = \frac{1+\epsilon}{3}$.

Also, it follows from directed component balance that $Y(v, g''+31) = Y(v, g'')$ and that $Y(v, g'' + 32) = Y(v, g'')$.

Case 1: $Y_1(v, g'') \geq 1/2$. Consider the strategy profile leading to g' in the link formation game. If $\epsilon < 1/6$, then this strategy profile is not a Nash equilibrium, since player 1 will benefit by deviating and adding 12 and deleting 13 (which results in $g'' + 32$). This is a contradiction.

Case 2: $Y_2(v, g'') > 1/2$. Consider the strategy profile leading to g in the link formation game. If $\epsilon \leq 1/6$, then this strategy profile is not a Nash equilibrium, since player 2 will benefit by deviating and adding 21 and deleting 23 (which results in $g'' + 31$). This is a contradiction.

By component balance, these two cases are exhaustive. □

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[14] Again, see the proof of Theorem 1' in the appendix of Jackson and Wolinsky (1996).
[15] g^π is individually stable whenever g, for any permutation π.

4 Outsiders

We consider next, a condition that states one cannot shift too much value to an outsider: no more than their marginal contribution to the network. A reason for exploring the role of outsiders in detail is that the value function used in the proof of Theorems 1 and 2 is special. In particular, several networks all have the same value even though their architectures are different. Moreover, that fact is important to the application of directed component balance in the proof of Theorems 1 and 2. This reliance on specific value functions is really only due to the weak way in which outsiders are addressed in directed component balance. If directed component balance is replaced by the following outsider independence condition which is more explicit about the treatment of outsiders, then the results of Theorems 1 and 2 hold for open sets of value functions.

Outsider independence

An allocation rule Y satisfies *outsider independence* if for all $g \in G$ and $v \in V$ and $i \in N(g)$ who is an outsider of g such that $v(g) \geq v(g-i)$, then $Y_j(g,v) \geq Y_j(g-i,v)$ for each $j \neq i$.

Outsider independence states that an outsider obtains at most her marginal contribution to the value of a network. The idea is that if a set of players has formed a network, and cannot prevent an outsider from linking to it, then the players should not suffer because of the outsider's actions. Such a condition is clearly satisfied in the directed connections model, and in any setting where the outsider's actions have no externalities.

Outsider independence is only required to hold in situations where the outsider's presence does not decrease the value of the network. Normatively, one might argue for it more generally.

Theorem 3. *If $N \geq 3$, there is an open subset[16] of the anonymous value functions for which any Y that satisfies anonymity on individually stable networks, component balance, and outsider independence, and has an anonymous set of individually stable networks when v is anonymous, cannot have any strongly efficient network be individually stable.*

The proof of Theorem 3 is a straightforward extension of the proofs of Theorems 1 and 2 and therefore is omitted.

It is easily seen that Theorems 1, 2, and 3 are tight in that dropping anonymity invalidates the results. For example, let $\overline{Y}$ be the equal split of value within components rule as defined below. Define Y by picking a strongly efficient g^*, and let $Y(g^*,v) = \overline{Y}(g^*,v)$. For any g such that $g_j = g_j^*$ for all $j \neq i$ for some i, set $Y_i(g,v) \leq \overline{Y}_i(g^*,v)$, set $Y_j(g,v) = \overline{Y}_j(g,v)$ for $j \notin N(h_i)$ where h_i is the

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[16] Given that the set of networks G is a finite set, a value function can be represented as a finite vector. Here, open is relative to the subspace of anonymous value functions.

component of g containing i, and let $Y_j(g, v) = \frac{v(h_i) - Y_i(g,v)}{\#N(h_i) - 1}$ for $j \in N(h_i)$, $j \neq i$. For any other g set $Y(g, v) = \overline{Y}(g, v)$.

Next, we show that weakening directed component balance or ignoring outsider independence invalidates Theorems 1, 2, and 3. If value can be allocated to outsiders without regards to their contribution to the value of a network, then it is possible to sustain efficient networks as being individually stable.

Theorem 4. *There exists an allocation rule Y that is anonymous, component balanced and such that for each v there is some strongly efficient network that is individually stable.*

Theorem 4 shows that there are important differences between the directed and non-directed network contexts. In the directed case it is always possible for any player unilaterally to become part of a network. If the allocation rule can shift value to outsiders, even when they contribute nothing to the value of a network, then one can overcome the difficulties imposed by component balance.

The proof of Theorem 4 is constructive and appears in the appendix. Here, we provide some intuition underlying the constructive proof.

Let Y be an allocation rule that we are designing to support a given strongly efficient network g^* as individually stable. So, it must be that for all i, $Y_i(g^*, v) \geq \max_{g \in D_i(g)} Y_i(g, v)$. At the same time we need to make sure that Y is anonymous and component balanced. To get a feeling for the impact of those restrictions, consider the following example.

Example 3. There are 5 players. The value function v is anonymous. A strongly efficient network g^* is such that $g^*_{12} = g^*_{23} = g^*_{34} = g^*_{45} = 1$ and $g^*_{ij} = 0$ for other ij. So, g^* is a directed line. Suppose that $v(g^*) = 5$ and that $v(g) = 5$ if g is a copy of g^*.

Let us consider the restrictions on Y imposed by anonymity, component balance, and guaranteeing that g^* is individually stable.

First, player 5 can deviate from g^* by adding the link 51, to result in the network $g^* + 51$. Let us denote that network as g^5. So, g^5 is a wheel. Since a wheel is symmetric, it must be that $Y_5(g^5, v) = v(g^5)/5$. Then, to ensure that g^* is individually stable, we need to have $Y_5(g^*, v) \geq v(g^5)/5$.

Next, player 4 can deviate from g^* by deleting link 45 and adding link 41. The resulting network, denoted g^4 is a four person wheel. Here, to ensure that g^* is individually stable and Y is anonymous and component balanced, we need to have $Y_4(g^*, v) \geq v(g^4)/4$.

Also, player 3 can deviate from g^* by deleting link 34 and adding link 31. The resulting network, denoted g^3 is a three person wheel among 1,2, and 3, together with the extra link 45. Here, to ensure that g^* is individually stable and Y is anonymous and component balanced, we need to have $Y_3(g^*, v) \geq v(h^3)/3$, where h^3 is the three person wheel among 1, 2, and 3.

There is a similar requirement for player 2. These requirements are different for different players, and so an allocation rule that simply equally splits value does not work. The proof involves showing that these requirements can all be

satisfied simultaneously, and that the type of requirements arising in this example are those arising more generally and can always be handled.

5 Efficiency and the connections models

The above results indicate that in order to find an allocation rule that reconciles individual stability and strong efficiency in general, in some cases one needs to allocate some value to non-productive outsiders. However, there are still interesting settings where strong efficiency and individual stability can be reconciled, while preserving anonymity, directed component balance, and outsider independence. We explore some such settings here.

Given a value function v and a set $K \subset N$, let $g_v^*(K)$ be a selection of a strongly efficient network restricted to the set of players K (so $N(g^*(K)) \subset K$). If there is more than one such strongly efficient network among the players K, then select one which minimizes the number of players in $N(g)$.

A value function v has *non-decreasing returns to scale* if for any $K' \subset K \subset N$

$$\frac{v(g_v^*(K))}{\#N(g_v^*(K))} \geq \frac{v(g_v^*(K'))}{\#N(g_v^*(K'))}.$$

If a value function has non-decreasing returns to scale, then *per-capita* value of the efficient network is non-decreasing in the number of individuals available. This does not imply anything about the structure of the efficient network, except that larger groups can be at least as productive per capita in an efficient configuration as smaller groups. As we shall see shortly, it is satisfied by some natural value functions.

Theorem 5. *If a component additive value function v has nondecreasing returns to scale, then there exists an allocation rule Y satisfying anonymity, directed component balance and outsider independence for which at least one strongly efficient networks is individually stable relative to v.*

The proof of Theorem 5 is given in the appendix.

The proof of Theorem 5 relies on the following allocation rule $\widehat{Y}$, which is a variant on a component-wise egalitarian rule $\overline{Y}$. Such a rule is attractive because of its strong equity properties. To be specific, define $\overline{Y}$ as follows. Consider any g and a component additive v. If i is in a component h of g (which is by definition necessarily non-empty), then $\overline{Y}_i(g,v) = \frac{v(h)}{\#N(h)}$, and if i is not in any component then $\overline{Y}_i(g,v) = 0$. For any v that is not component additive, let $Y_i(g,v) = \frac{v(g)}{N}$ for all i. $\overline{Y}$ is a component-wise egalitarian rule, and is component balanced and anonymous. It divides the value generated by a given component equally among all the members of that component, provided v is component additive (and divides resources equally among all players otherwise). It is shown in the appendix that under non-decreasing returns to scale, all strongly efficient networks are individually stable relative to $\overline{Y}$.

Unfortunately, $\overline{Y}$ does not always satisfy outsider independence. For instance, in the directed connections model it fails outsider independence for ranges of values of δ and c.[17] However, a modification of $\overline{Y}$ results in the allocation rule $\widehat{Y}$ that satisfies anonymity, directed component balance, outsider independence, and for which all strongly efficient networks are individually stable for v's that have non-decreasing returns to scale. The modified allocation rule $\widehat{Y}$ is defined as follows. For any v and strongly efficient network g^*, set $\widehat{Y}(g^*, v) = \overline{Y}(g^*, v)$. For any other g: if g has an outsider i then set $\widehat{Y}_j(g, v) = \max[\overline{Y}_j(g-i, v), \overline{Y}_j(g, v)]$ for $j \neq i$ and $\widehat{Y}_i(g, v) = v(g) - \sum_{j \neq i} \widehat{Y}_j(g, v)$; and otherwise set $\widehat{Y}(g, v) = \overline{Y}(g, v)$. As there is at most one outsider to a network, $\widehat{Y}$ is well-defined.

Both the directed connections and hybrid connections models have non-decreasing returns to scale:

Proposition 1. *The directed and hybrid connections models (v^d and v^h) have non-decreasing returns to scale. Thus, all strongly efficient networks are individually stable in the connections models, relative to the anonymous, directed component balanced and outsider independent allocation rule $\widehat{Y}$.*

The re-allocation of value under $\widehat{Y}_i$ compared with u_i^d and u_i^h is important to Proposition 1. Without any re-allocation of value, in both the directed and hybrid connections models the set of individually stable and strongly efficient networks do not intersect for some ranges of parameter values. For instance, Bala and Goyal (1999) show in the context of the directed connections model that if $N = 4$ and $\delta < c < \delta + \delta^2 - 2\delta^3$, then stars and "diamonds"[18] are the strongly efficient network structures, but are not individually stable. Similarly, in the context of the hybrid connections model if $N = 4$ and $\delta + 2\delta^2 < c < 2\delta + 2\delta^2$ then a star[19] is the strongly efficient network structure but is not individually stable. As Proposition 1 shows, reallocation of value under $\widehat{Y}$ overcomes this problem.

Let us make a couple of additional remarks about the result above. First, anonymity of $\widehat{Y}$ implies that the set of individually stable networks will be an anonymous set, so that all anonymous permutations of a given individually stable network are also individually stable. Second, in situations where $c > \delta$ (in any of the connections model) the empty network is individually stable relative to $\widehat{Y}$, even though it is not strongly efficient. The difficulty is that a single link generates negative value and so no player will want to add a link (or set of links) given that none exist. It is not clear whether an anonymous, component balanced, and outsider independent Y exists for which the set of individually stable networks exactly coincides with the set of strongly efficient networks (when $c > \delta$) in these

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[17] For example, let $N = 4$, $\delta < 1/4$ and c be close to 0 in the directed connections model. Consider the network where $g_{12} = g_{13} = g_{21} = g_{31} = 1$. Adding the link 41 results in $\overline{Y}_1(g+41, v^d) < \overline{Y}_1(g, v^d)$ even though 4 is an outsider to g.

[18] For instance a star with 1 at the center has $g_{12} = g_{13} = g_{14} = g_{21} = g_{31} = g_{41} = 1$, while a diamond has $g_{12} = g_{13} = g_{21} = g_{23} = g_{32} = g_{41} = 1$.

[19] Here, given the two-way communication on a directed link, $g_{31} = g_{21} = g_{41}$ would constitute a star, as would $g_{13} = g_{12} = g_{14}$, etc.

connections models. Such a Y would necessarily involve careful subsidization of links, in some cases violating individual rationality constraints.

Appendix

For each i, let $H^i(g) = \{h^i | h^i \in C(g^i), i \in N(h^i), g^i \in D_i(g)\}$.

Let $n_d^{-1}(i, g) = \#\{j \,|g_{ji} = 1\}$ represent the number of individuals who maintain a link with i.

We begin by stating some Lemmas that will be useful in some of the proofs that follow.

We are most grateful to Anna Bogomolnaia who provided the proof of Lemma 1.

Lemma 1. *Let $\{a_1, \ldots, a_n\}$ be any sequence of nonnegative numbers such that $\sum_{k \in S} a_k \leq a_n$ for any $S \subset \{1, \ldots, n\}$ such that $\sum_{k \in S} k \leq n$. Then,*

$$\sum_{i=1}^{n} \frac{a_i}{i} \leq a_n. \tag{1}$$

Proof. : We construct a set of n inequalities whose sum will be the left hand side of (1). We label the i-th inequality in this set as (i').

First, for each i, let (r_i, j_i) be the unique pair such that: $n = r_i i + j_i$, r_i is some integer, and $0 \leq j_i < i$.

For each $i > \frac{n}{2}$, write inequality (i') as

$$\frac{a_i}{i} + \frac{a_{n-i}}{i} \leq \frac{a_n}{i} \tag{2}$$

(Here, we adopt the convention that $a_0 = 0$.)

Now, consider any $i \leq \frac{n}{2}$, and suppose all inequalities from (n') to $(i + 1')$ have been defined. Let H_i be the sum of the coefficients of a_i in inequalities (n') to $(i + 1')$. Let us now show that $h_i = \frac{1}{i} - H_i \geq 0$.

Claim. For each $i \leq \frac{n}{2}, h_i \geq 0$.

Proof of Claim. First, we prove that

$$\#\{q \text{ is an integer } |qj + i = n \text{ for } j \text{ being an integer}, i < q\} \leq \frac{n - i}{i} \tag{3}$$

Let $P = n - i$, and note that for j being an integer, $\#\{q | q > i, P = jq\} = \#\{\frac{P}{j} | \frac{P}{j} > i, P = jq\} = \#\{\frac{P}{j} | \frac{P}{j} > j, P = jq\} = \#\{j | \frac{P}{j} > j, P = jq\} = \#\{j : \frac{P}{j} > j\} \leq \frac{P}{i}$.

So, each i appears in at most $\frac{(n-i)}{i}$ inequalities. Choose $q > i$ such that $qr_q + i = n$. Then, from (3), the coefficient of a_i in (q') is $\frac{h_q}{r_q}$. Note that since $H_q = \frac{1}{q} - h_q \geq 0$, we must have $\frac{1}{q} \geq h_q$. Hence, $\frac{h_q}{r_q} \leq \frac{1}{qr_q} = \frac{1}{n-i}$. Using (4), we get $H_i \leq (\frac{n-i}{i})(\frac{1}{n-i}) = \frac{1}{i}$.

This completes the proof of the claim. $\square$

By (1) it follows that that $r_i a_i + a_{j_i} \leq a_n$. Thus, write (i') as

$$h_i a_i + \frac{h_i}{r_i} \leq \frac{h_i}{r_i} a_n \qquad (4)$$

Note that by construction, the sum of the coefficients of a_i in inequalities (n') to (i') equals $H_i + h_i = \frac{1}{i}$, and that a_i does not figure in any inequality (k') for $k < i$. So, we have proved that the sum of the left hand side of the set of inequalities (i') equals the left hand side of (1).

To complete the proof of the lemma, we show that the sum of the right hand side of the inequalities (i') is an expression that must be less than or equal to a_n. The right hand side of the sum of the inequalities (i') is of the form Ca_n, where C is independent of the values $\{a_1, \ldots, a_n\}$. Let $a_i = \frac{i}{n}$ for all i. Then the inequalities (i') hold with equality. But, this establishes that $C = 1$ and completes the proof of the lemma. □

For any g, let $D(g) = \cup_i D_i(g)$.

Let
$$X(g, g') = \{i \,|\, \exists g'' \in D_i(g) \text{ s.t. } g'' \text{ is a copy of } g'\}.$$

So, $X(g, g')$ is the set of players who via a unilateral deviation can change g into a copy of g'.

Say that $S \subset N$ is a *dead end under* $g \in G$ if for any i and j in S, $i \neq j$, there exists a directed path from i to j, and for each $k \notin S$ $g_{ik} = 0$ for each $i \in S$.

For any g and $i \in N(g)$, either there is a directed path from i to a dead end S under g, or i is a member of a dead end of g. (Note that a completely disconnected player forms a dead end.)

Observation. Suppose that $\{S_1, \ldots, S_\ell\}$ are the dead ends of $g \in G$. Consider i and g' such that $g' \in D_i(g)$. If $i \notin S_k$ for any k, then S_k is still a dead end in g'. If $i \in S_k$ for some k, and i has a link to some $j \notin S_k$ under g', then $\{S_1, \ldots, S_\ell\} \setminus \{S_k\}$ are the dead ends of g'.

To see the second statement, note that there exists a path from every $l \in S_k$, $l \neq i$ to i, and so under g' all of the players in S_k have a directed path to j. If j is in a dead end, then the statement follows. Otherwise, there is a directed path from j to a dead end, and the statement follows.

Lemma 2. *Consider a player l, $g' \in G$, $g \in D_l(g')$ and corresponding $h^l \in C(g)$ such that $N(h^l) \subset X(g, g')$. If $C(g) \neq C(g')$, then there exists a directed path from any $i \in N(h^l)$ to any $j \in N(h^l)$.*

Proof of Lemma 2. Let $Z = N \setminus N(h)$. Consider $i \in N(h)$ and suppose that $g' \in D_i(g)$. Let $S_1, \ldots, S_\ell$ be the dead ends of g. If i is in a dead end S_k under g, then since $C(g) \neq C(g')$, i must be linked to some player in Z under g'. (Note that since i is in a dead end, there is a directed path from every player in S_k to i, and so i can only change the component structure of g by adding a link to a player outside of S_k.) From the observation above, it then follows that $\{S_1, \ldots, S_\ell\} \setminus \{S_k\}$ are the dead ends of g'.

Suppose the contrary of the lemma. This implies that there is a dead end of g, $S_k \subset N(h)$, and $\{i,j\} \subset N(h)$ such that $i \notin S_k$ and $j \in S_k$. From the Observation it follows that if $g^i \in D_i(g)$ is a copy of g', then g' has at least ℓ dead ends. However, if $g^j \in D_j(g)$ is a copy of g', then from the arguments above it follows that g^j has at most $\ell - 1$ dead ends. This implies that g^i and g^j could not both be copies of g'. This is a contradiction of the fact that $N(h) \subset X(g, g')$. $\square$

Lemma 3. *Suppose g' is connected. Choose any $i,j \in N(g')$ with $i \neq j$, and take $g^i \in D_i(g')$, $g^j \in D_j(g')$, and corresponding $h^i \in C(g^i)$ and $h^j \in C(g^j)$. If $N(h^i)$ and $N(h^j)$ are intersecting but neither is a subset of the other, then $N(h^i) \not\subset X(g^i, g')$ and $N(h^j) \not\subset X(g^j, g')$.*

Proof of Lemma 3. Suppose to the contrary of the Lemma that, say, $N(h^i) \subset X(g^i, g')$.

Consider the case where $j \notin N(h^i)$. By Lemma 2, for any $k \in N(h^i)$ with $k \neq i$, there is a directed path from k to i in h^i. Since $g'_l = h^i_l = h^j_l$ for all $l \neq i, j$, this must be a directed path in h^j as well. Hence, $i \in N(h^j)$. By this reasoning, there is a directed path from every $l \in N(h^i) \setminus \{i\}$ to i in h^i, and hence in h^j. So, $N(h^i)$ is then a subset of $N(h^j)$, which contradicts the supposition that $N(h^i)$ and $N(h^j)$ are intersecting but neither is a subset of the other.

So, consider the case where $j \in N(h^i)$. We first show that $i \in N(h^j)$. Since $N(h^j)$ is not a subset of $N(h^i)$, there exists $k \in N(h^j)$ with $k \notin N(h^i)$. Since $k \notin N(h^i)$, the only paths (possibly non-directed) connecting j and k in g' must pass through i. Thus, under g' there is a path connecting i to k that does not include j. So, since $k \in N(h^j)$, it follows that $i \in N(h^j)$. Next, for any $l \in N(h^i) \setminus \{i\}$, by Lemma 2 there is a directed path from l to i in h^i. If this path passes through j, then there is a directed path from l to j in g' (not passing through i) and so $l \in N(h^j)$. If this path does not involve j, then it is also a path in h^j. Thus, $l \in N(h^j)$ for every $l \in N(h^i) \setminus \{i\}$. Since $i \in N(h^j)$, we have contradicted the fact that $N(h^i)$ is not a subset of $N(h^j)$ and so our supposition was incorrect. $\square$

Lemma 4. *Consider i, g and g', with $g \in D_i(g')$, and $h^i \in C(g)$ such that $i \in N(h^i)$.[20] If $N(h^i) \subset X(g, g')$, then $N(h^i) \subset N(h')$ for some $h' \in C(g')$.*

Proof of Lemma 4. Suppose the contrary, so that there exists $j \in N(h^i)$ with $j \notin N(h')$, where $i \in N(h')$ and $h' \in C(g')$. Note, this implies that $C(g) \neq C(g')$. Either j is a dead end under g, or there is a path leading from j to a dead end under g. So, there exists a dead end S in h^i with $i \notin S$. This contradicts lemma 2. $\square$

Proof of Theorem 4. If $v \in V$ is not component additive, then the allocation rule defined by $Y_i(g, v) = v(g)/N$ for each player i and $g \in G$ satisfies the desired properties. So, let us consider the case where v is component additive.

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[20] Adopt the convention that a disconnected player is considered their own component.

Fix a v and pick some network g^* that is strongly efficient. Define Y^* relative to v as follows.[21]

Consider $g \in D(g^*)$. For any i, let $h^i \in C(g)$ be such that $i \in N(h^i)$.

If $i \in X(g, g^*)$, let $Y_i^*(g, v) = \overline{Y}_i(g, v)$ if $N(h^i) \subset X(g, g^*)$ and $Y_i^*(g, v) = 0$ otherwise.

If $i \notin X(g, g^*)$, let $Y_i^*(g, v) = \frac{v(h^i)}{\#\{j \,|\, j \in N(h^i), j \notin X(g, g^*)\}}$.

Let $r_i = \max_{g \in D_i(g^*)} Y_i^*(g, v)$.

Claim. $\displaystyle\sum_{j \in N(h^*)} r_j \leq v(h^*)$ for each $h^* \in C(g^*)$.

We return to prove the claim below.

Set $Y_i^*(g^*, v) = r_i + \frac{v(h^{*i}) - \sum_{j \in N(h^{*i})} r_j}{\#N(h^{*i})}$. Set Y^* on g^π which is a copy of $g \in D(g^*) \cup \{g^*\}$ according to anonymity, whenever $v(g^\pi) = v(g)$. For all remaining g, set $Y^*(g, v) = \overline{Y}(g, v)$.

By the definition of Y^*, $Y_i^*(g^*, v) \geq \max_{g \in D_i(g^*)} Y_i^*(g, v)$ for all $i \in N(g^*)$. Hence, g^* is individually stable. Also Y^* is component balanced and anonymous.

To complete the proof, we need only verify the claim.

Proof of Claim. By the definition of r_i it follows that $r_i > 0$ only if $N(h^i) \subset X(g, g^*)$. By Lemma 4, this implies that $N(h^i) \subset N(h^*)$ for some $h^* \in C(g^*)$.

For each $h^* \in C(g^*)$, let $I(h^*) = \{i \in N(h^*) | r_i > 0\}$. For each $i \in I(h^*)$, let $\bar{h}^i$ be such that $r_i = \frac{v(\bar{h}^i)}{\#N(\bar{h}^i)}$. Then, the argument in the previous paragraph establishes that each $\bar{h}^i$ is such that $N(\bar{h}^i) \subset N(h^*)$. Hence, applying lemma 3 to h^*, the set $\{\bar{h}^i \mid i \in I(h^*)\}$ can be partitioned into $\{H_1, \ldots, H_k\}$ such that
(i) Each $\bar{h}^i$ in H_1 is disjoint from every other $\bar{h}^j, j \neq i$.
(ii) For all $k = 2, \ldots K$, $H_k = \{h_1, \ldots, h_\ell\}$ is such that $N(\bar{h}_1) \subset N(\bar{h}_2) \subset \cdots N(\bar{h}_\ell)$, and elements in H_k are disjoint from elements in $H_{k'}$ if $k \neq k'$.

Define $\Delta = v(g) - \displaystyle\sum_{\bar{h}^i \in H^1} v(\bar{h}^i)$.

Since g^* is strongly efficient, $v(h^*) \geq v(h)$ for all h such that $N(h) \subset N(h^*)$. Now, one can use Lemma 1 and the fact that v is component additive, to deduce that there are numbers $\{\Delta_2, \ldots, \Delta_K\}$ such that

(i) $\displaystyle\sum_{k=2}^{K} \Delta_k \leq \Delta$.

(ii) For each $k = 2, \ldots, K$, $\Delta_k \geq \displaystyle\sum_{\bar{h}^j \in H_k} \frac{v(\bar{h}^j)}{\#N(\bar{h}^j)}$.

These inequalities prove the claim. $\square$

Proof of Theorem 5.

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[21] To ensure anonymity, work with equivalence classes of v with v^π for each π defined via the anonymity property.

$\widehat{Y}$ satisfies anonymity by definition. Since an outsider is necessarily unique, $\widehat{Y}$ satisfies directed component balance, and outsider independence relative to any $g \neq g^*$. These conditions relative to g^*, follow from the claim below.

Fix a component additive v that has non-decreasing returns to scale. We now show that $g_v^*(N)$ is individually stable relative to $\widehat{Y}$.

The following claim is useful.

Claim. Consider any component additive v that has non-decreasing returns to scale. If g is a component of $g_v^*(N)$, then for any g'

$$\frac{v(g)}{\#N(g)} \geq \frac{v(g')}{\#N(g')}.$$

Proof of Claim. Note that

$$\frac{v(g_v^*(N))}{\#N(g_v^*(N))} = \frac{\sum_{h \in C(g_v^*(N))} v(h)}{\sum_{h \in C(g_v^*(N))} \#N(h)}.$$

By non-decreasing returns,

$$\frac{\sum_{h \in C(g_v^*(N))} v(h)}{\sum_{h \in C(g_v^*(N))} \#N(h)} \geq \frac{v(h')}{\#N(h')}$$

for each $h' \in C(g_v^*(N))$. Thus,

$$\sum_{h \in C(g_v^*(N)), h \neq h'} v(h) \geq \frac{\sum_{h \in C(g_v^*(N)), h \neq h'} \#N(h)}{\#N(h')} v(h'). \tag{5}$$

Also, by non-decreasing returns,

$$\frac{\sum_{h \in C(g_v^*(N))} v(h)}{\sum_{h \in C(g)} \#N(h)} \geq \frac{\sum_{h \in C(g_v^*(N)), h \neq h'} v(h)}{\sum_{h \in C(g_v^*(N)), h \neq h'} \#N(h)},$$

for each $h' \in C(g_v^*(N))$. Thus,

$$v(h') \geq \frac{\#N(h')}{\sum_{h \in C(g_v^*(N)), h \neq h'} \#N(h)} \sum_{h \in C(g_v^*(N)), h \neq h'} v(h), \tag{6}$$

for each $h' \in C(g_v^*(N))$. Inequalities (5) and (6) then imply that

$$\frac{v(h')}{\#N(h')} = \frac{v(g_v^*(N))}{\#N(g_v^*(N))}.$$

for every $h' \in C(g_v^*(N))$. The desired conclusion then follows from non-decreasing returns. □

Consider $g^*(N)$ and some deviation by a player i, resulting in the network $g_{-i}^*(N), \overline{g}_i$. It then follows from the claim that $\overline{Y}_i(g^*(N)) \geq \overline{Y}_i(g_{-i}^*(N), \overline{g}_i)$ and $\overline{Y}_i(g^*(N)) \geq \overline{Y}_i((g_{-i}^*(N), \overline{g}_i) - j)$ for any j. Thus, if i is not an outsider at $g_{-i}^*(N), \overline{g}_i$, then from the definition of $\widehat{Y}$ it follows that $\widehat{Y}_i(g^*(N)) \geq$

$\widehat{Y}_i(g^*_{-i}(N), \overline{g}_i)$. If i is an outsider at $g^*_{-i}(N), \overline{g}_i$, then from the definition of $\widehat{Y}$, $\overline{Y}_i(g^*_{-i}(N), \overline{g}_i) \geq \widehat{Y}_i(g^*_{-i}(N), \overline{g}_i)$. So, $\widehat{Y}_i(g^*(N)) = \overline{Y}_i(g^*(N)) \geq \overline{Y}_i(g^*_{-i}(N), \overline{g}_i) \geq \widehat{Y}_i(g^*_{-i}(N), \overline{g}_i)$. Thus, $g^*_v(N)$ is individually stable. □

Proof of Proposition 1. The following claim is stronger than the stated property.

Claim. *Fix δ and c. If $g^*(K)$ is any strongly efficient network with a number[22] K players relative to the directed connections model, and g is any network with $K \geq \#N(g) > 0$, then $\frac{v^d(g^*(K))}{K} \geq \frac{v^d(g)}{\#N(g)}$. The same is true of the hybrid connections model, substituting v^h for v^d.*

Proof of the Claim. It is clear that $\frac{v^d(g^*(K))}{K} \geq 0$ ($\frac{v^h(g^*(K))}{K} \geq 0$), since the empty network is always feasible. The claim is established by showing that for each $K > 2$, $\frac{v^d(g^*(K))}{K} \geq \frac{v^d(g^*(K-1))}{K-1}$, (and $\frac{v^h(g^*(K))}{K} \geq \frac{v^h(g^*(K-1))}{K-1}$), where $g^*(K)$ denotes any selection of a strongly efficient network with K players. This implies the claim.

First, consider the directed connections model. Consider K players, with players $1, \ldots, K-1$ arranged as in $g^*(K-1)$. If $g^*(K-1)$ is empty, then the claim is clear. So suppose that $g^*(K-1)$ is not empty and consider $i \in N(g^*(K-1))$ such that $u_i(g^*(K-1)) \geq u_j(g^*(K-1))$ for all $j \in N(g^*(K-1))$, where u_i is as defined in Example 1. Thus, $u_i(g^*(K-1)) \geq \frac{v^d(g^*(K-1))}{K-1}$ Consider the network g, where $g_j = g_j^*(K-1)$ for all $j < K$, and where $g_K = g_i^*(K-1)$. It follows that $u_j(g) = u_j(g^*(K-1))$ for all $j < K$, and that $u_K(g) = u_i(g^*(K-1)) \geq \frac{v^d(g^*(K-1))}{K-1}$. Since $v^d(g) = \sum_k u_k(g)$, it follows that $v^d(g) \geq v^d(g^*(K-1)) + \frac{v^d(g^*(K-1))}{K-1}$. This implies that $v^d(g) \geq v^d(g^*(K-1)) + \frac{v^d(g^*(K-1))}{K-1}$. So $v^d(g) \geq \frac{K v^d(g^*(K-1))}{K-1}$, and thus $\frac{v^d(g)}{K} \geq \frac{v^d(g^*(K-1))}{K-1}$.

Next, consider the hybrid connections model. Again, suppose that $K > 2$.

If $2\delta + (K - 3)\delta^2 \leq c$, then a strongly efficient network for $K - 1$ players, $g^*(K-1)$ is an empty network, (or when $2\delta + (K - 3)\delta^2 = c$ then it is possible that $g^*(K-1)$ is nonempty, but still $v^h(g^*(K-1)) = 0$).[23] The result follows directly.

If $c \leq \delta - \delta^2$ then the efficient networks are those that have either $g_{ij}^* = 1$ or $g_{ji}^* = 1$ (but not both) for each ij (or when $c = \delta - \delta^2$ has a value equivalent to such a network). Then $v^h(g^*(K-1)) = (K-1)(K-2)(\delta - \frac{c}{2})$ and $v^h(g^*(K)) = (K)(K-1)(\delta - \frac{c}{2})$. This establishes the claim, since it implies that $\frac{v^h(g^*(K))}{K} = (K-1)(\delta - \frac{c}{2}) \geq \frac{v^h(g^*(K-1))}{K-1} = (K-2)(\delta - \frac{c}{2})$, and $c < 2\delta$ (or else $c = \delta = 0$ in which case $v^h(g) = 0$ for all g).

If $\delta - \delta^2 < c < 2\delta + (K - 3)\delta^2$, a star is the strongly efficient network structure for $K - 1$ players. Here, $v^h(g^*(K-1)) = (K-2)(2\delta + (K-3)\delta^2 - c)$. The value of

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[23] See Jackson and Wolinsky (1996) Proposition 1 for a proof of the characterization of efficient networks in the connections model. This translates into the hybrid connections model as noted by Bala and Goyal (1999) Proposition 5.2.

$g^*(K)$ is at least the value of a star, so that $v^h(g^*(K)) \geq (K-1)(2\delta+(K-2)\delta^2-c)$, which establishes the claim. $\square$

References

1. Bala, V., Goyal, S. (1999) Self-organization in communication networks. *Econometrica* (forth-coming)
2. Currarini, S., Morelli, M. (1998) Network formation with sequential demands. *Review of Economic Design*. (forthcoming)
3. Dutta, B., Mutuswami, S. (1997) Stable networks. *Journal of Economic Theory* 76: 322–344
4. Dutta, B., van den Nouweland, A. Tijs, S. (1998) Link formation in cooperative situations. *International Journal of Game Theory* 27: 245–256
5. Goyal, S. (1993) Sustainable communication networks. Discussion Paper TI 93-250, Tinbergen Institute, Amsterdam- Rotterdam.
6. Jackson, M., Wolinsky, A. (1996) A strategic model of social and economic networks. *Journal of Economic Theory* 71: 44–74
7. Jackson, M., Watts, A. (1998) *The evolution of social and economic networks.* Caltech WP # 1044
8. Myerson, R. (1991) *Game theory: analysis of conflict.* Harvard University Press, Cambridge, MA
9. Qin, C-Z. (1996) Endogenous formation of cooperation structures. *Journal of Economic Theory* 69: 218-226
10. Watts, A. (1997) *A dynamic model of network formation.* mimeo, Vanderbilt University