Mathematical Social Sciences 78 (2015) 80–88

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

Allocation rules for coalitional network games

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HIGHLIGHTS

- Players cooperate by forming networks and coalitions.
- The flexible approach assumes that players organize themselves the best way possible.
- We propose two allocation rules that distribute the value of the efficient structure.
- A first rule allocates directly the value to the players.
- A second rule allocates indirectly the value to the minimal forms of cooperation.

ARTICLE INFO

Article history:
Received 27 January 2014
Received in revised form
8 May 2015
Accepted 31 August 2015
Available online 24 October 2015

ABSTRACT

Coalitional network games are real-valued functions defined on a set of players organized into a network and a coalition structure. We adopt a flexible approach assuming that players organize themselves the best way possible by forming the efficient coalitional network structure. We propose two allocation rules that distribute the value of the efficient coalitional network structure: the atom-based flexible coalitional network allocation rule and the player-based flexible coalitional network allocation rule.

1. Introduction

There are many situations where agents are part of a network and belong to groups or coalitions. Regarding firms' strategies on R&D, firms can sign bilateral R&D agreements – that is, a network – and, at the same time firms may group themselves into R&D joint ventures — that is, coalitions. Individuals are living their social interactions in clubs or communities as well as through friendship networks. Countries can sign bilateral free trade agreements or multilateral free trade agreements and may belong to customs unions. Connections among different criminal gangs became a major feature of the organized crime during the 1990s.

Caulier et al. (2013a) have developed a theoretical framework that allows to study which bilateral links and coalition structures are going to emerge at equilibrium. They have proposed the notion of coalitional network to represent a network and a coalition structure, where the network specifies the nature of the relationship each individual has with her coalition members and with individuals outside her coalition.[1] They have shown that this new framework can provide insights that one cannot obtain if coalition formation and network formation are tackled separately and independently.[2]

The aim of this paper is to study the allocation of value among players who are part of a network and belong to coalitions and to assess the strategic position of each player in a coalitional network. The way the value is allocated matters, not only in terms of fairness and equity considerations, but also in determining the incentives players have to form links and coalitions.

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http://dx.doi.org/10.1016/j.mathsocsci.2015.08.003
0165-4896/

1 Caulier et al. (2013a) have used the concepts of strong stability and of contractual stability to predict the coalitional networks that are going to emerge at equilibrium. Contractual stability imposes that any change made to the coalitional network needs the consent of both the deviating players and their original coalition partners. Requiring the consent of coalition members under the simple majority or unanimity decision rule may help to reconcile stability and efficiency.

2 Caulier et al. (2013b) have considered situations where players are also part of a network and belong to coalitions. However, each player's payoff only depends on the network, and so, each player's coalition only constrains her ability to add or delete links in the network.

One of the central problems tackled by traditional cooperative game theory concerns the way to allocate among players the value generated collectively by the group of players in a fair way. In cooperative games, it is assumed that players cooperate by forming coalitions and the fruit of cooperation, the worth of a coalition, is achieved independently of the organization of the other players and can be freely distributed among the coalition members. In this context, the Shapley value proposes a way to share the worth of the grand coalition taking into account the marginal contribution of each player to the worth of each possible subcoalition. Myerson (1977) was first to augment a cooperative game by a network structure specifying which groups of players can communicate and achieve their worth.[3] The feasible groups are the ones whose members can communicate via the given network. Myerson (1977) has extended the Shapley value for this class of cooperative games, called communication games. Myerson (1980) has modeled the communication possibilities of the players by means of hypergraphs. Each element of a hypergraph is called a conference. Communication and negotiation between players can only take place within a conference if all players of the conference participate. Since a conference can consist of several players, a hypergraph is a generalization of a network, which has bilateral communication channels only. Myerson (1980) has generalized the Myerson value to this setting.[4] Jackson and Wolinsky (1996) have introduced a class of games – network games – where the value generated by a group of players depends directly on the network structure. They have extended the Myerson value to network games.

In this paper we extend the Shapley value to coalitional networks. The value generated by the coalitional network is captured by a real-valued function—called a coalitional network game. Notice that the coalition structure in a coalitional network game can vary and generates itself a value, whereas the coalition structure only restricts the possibilities for forming coalitions or networks in the recent related literature.[5] Following Jackson (2005), we propose an allocation rule for coalitional network games that shares the value generated by a given coalitional network taking into account the contribution of each player not only to the coalitional network that actually forms but also to every alternative coalitional network that could have been formed. We adopt Jackson's (2005) flexible approach because the efficient coalitional network is not necessarily the one where all players are linked to each other and belong to the grand coalition; i.e., the complete coalitional network. This means that we must care about how to allocate value to some coalitional networks that are not the complete coalitional network. In such cases, the allocation of value may depend on information about the roles of players that require calculations based on coalitional networks that are not subcoalitional networks of a given coalitional network.[6]

Observe that we develop a specific approach to adapt the Shapley value to our framework. The Shapley value is originally applied to the Boolean lattice of sets ordered by inclusion where each player is an element of the lattice. As singleton, each player is an atom (that covers the empty set) in the lattice of sets and the Shapley value allocates the value of the grand coalition to these lattice elements. In our setting, players do not appear as elements of the lattice of coalitional networks partially ordered. In order to circumvent this difficulty, we allocate a value to the atoms of the lattice under consideration, like the Shapley value for TU games, and then to the players.

The paper is structured as follows. Section 2 provides definitions for coalitional networks and presents the lattice structure of coalitional networks. Section 3 introduces coalitional network games and establishes some properties for this class of games. Section 4 presents two allocation rules for coalitional network games: the atom-based flexible coalitional network allocation rule and the player-based flexible coalitional network allocation rule. Section 5 provides the relationship with existing allocation rules.

2. Coalitional networks

Let $N = \{1, \ldots, n\}$ be the finite set of players who are connected in some network relationship and who belong to some coalitions. A coalitional network (g, P) is a pair that consists of a network g and a coalition structure or partition P.

A network g is a list of (unordered) pairs of players linked to each other and is represented by an undirected graph. A link between two players $i, j \in N$, $i \neq j$, is denoted ij or ji. For notational convenience, when the identities of linked players are not needed, we use the generic symbol l to designate a link. The set of all possible networks is denoted $G = \{g \mid g \subseteq g^N\}$, where g^N denotes the set of all subsets of N of size 2; i.e. the complete network. Let g^S denote the complete network among players in $S \subseteq N$. Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subset$ for strict inclusion. Thus, $g^{\emptyset}$ is the empty network where all players are isolated. For any network g, let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g. Let $n(g) \equiv |N(g)|$. As it is implicitly stated in the definition of G, a network is considered as a set of links and the set of all possible networks is partially ordered by inclusion. A network $g' \in G$ is a subnetwork of a network $g \in G$ if the set of links in g' is weakly included in g, $g' \subseteq g$. The infimum (meet) and supremum (join) of any two networks $g, g' \in G$ exist and are respectively written $g \cap g'$ and $g \cup g'$, and $(G, \subseteq)$ is a lattice with bottom element $g^{\emptyset}$ and top element g^N. A network g covers a network g' if $g' \subset g$ and there is no network g'' such that $g' \subset g'' \subset g$. The set of networks that cover the bottom element $g^{\emptyset}$, the set of atoms $\mathcal{A}(G, \subseteq)$, are the one-link networks $l \subset g^N$. A maximal decomposition of a network g in terms of atoms is the expression of g as the supremum of all atoms included in g. Formally,

$$g = \bigcup_{l \in \mathcal{A}(g)} l \quad \text{with } \mathcal{A}(g) = \{l \in \mathcal{A}(G, \subseteq) \mid l \subseteq g\}$$

where $\mathcal{A}(g)$ is the set of atoms (one link networks) included in g.

We say that a lattice L is ranked if there exist a function $r : L \to \mathbb{N}$ defined recursively by $r(\bot) = 0$ with $\bot \in L$ the bottom element of L and $r(x) = r(y) + 1$ with $x, y \in L$ such that x covers y. We can see that the lattice $(G, \subseteq)$ is ranked and each element $g \in G$ has rank $r(g) = |g|$. The rank of a network g is precisely the number of links in g and corresponds to the number of atoms included in the

[3] Other approaches with specific communication structures are Amer and Carreras (2005), Aumann and Drèze (1974), Bergantinos et al. (1993), and Carreras (1991).

[4] Algaba et al. (2001) have introduced the Myerson value for union stable structures (i.e. structures where the union of two intersecting feasible coalitions is also feasible) and have provided an axiomatization for it. Ui et al. (2011) have provided an extension of the Myerson value for complete coalition structures defined as sets of feasible coalitions.

[5] For instance, Vazquez-Brage et al. (1996) have proposed an allocation rule for a TU game endowed with independent of each other both a coalition structure and a communication graph on the set of players. See also Alonso-Meijide et al. (2009) and Kongo (2011). Recently, van den Brink et al. (2011) have introduced an Owen-type value for TU games endowed with two-level communication structures where players are partitioned into a coalition structure such that there exists restricted communication between and within the a priori unions of the coalition structure.

[6] Notice that in traditional cooperative games it is assumed that the grand coalition forms and the Shapley value decomposes the grand coalition in various ways to evaluate players' contributions. Hence, in the decomposition of the grand coalition, the value of every other coalition is taken into account in the computation of players' contributions.

network. The degree of an element x of a lattice L is the number of elements that x covers in L. Hence, we identify the number of atoms in g with the degree of g. Observe that if a network g covers a network g' then there exists a network $a \in \mathcal{A}(g)$ such that $g' \cup a = g$ and the network g has one more link than g', $r(g) = r(g') + 1$. For any two networks $g, g' \in G$, the rank function satisfies the following identity: $r(g)+r(g') = r(g \cup g')+r(g \cap g')$. A lattice $(L, \vee, \wedge)$ is distributive if for all $x, y, z \in L$ we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Hence $(G, \subseteq)$ is a distributive lattice.

A coalition is a subset $S \subseteq N$ and a coalition structure (or partition) is a collection of nonempty mutually disjoint coalitions whose union is N. We denote a coalition structure $P = \{S^1, \ldots, S^m\}$ such that $S^k \neq \emptyset$ for $k = 1, \ldots, m$, $S^k \cap S^{k'} = \emptyset$, $k \neq k'$ and $\bigcup_k S^k = N$. A k-partition is a partition P that consists of k coalitions; i.e., $|P| = k$. The set of possible coalition structures (or partitions) on N is denoted $\mathcal{P}$ and is partially ordered under the refinement ordering $\sqsubseteq$. Let P, P' be partitions of N. We say that P is a refinement of P' or is finer than P', denoted $P \sqsubseteq P'$, if any coalition of P is a subset of a coalition of P'. Strict refinement is denoted $\sqsubset$. The dual relation of the refinement is the coarsening relation. The infimum and supremum between any two partitions P and P' exist and are respectively $P \wedge P'$ and $P \vee P'$. The poset $(\mathcal{P}, \sqsubseteq)$ is thus a lattice. The bottom element of the partition lattice $(\mathcal{P}, \sqsubseteq)$ is the finest partition $P_\perp = \{\{1\}, \ldots, \{n\}\}$. The top element, i.e. the coarsest partition, is the grand coalition $P^\top = \{N\}$. The atoms $\mathcal{A}(\mathcal{P}, \sqsubseteq)$ are the elements that cover the finest partition and are partitions whose only non-trivial coalition is a two-element coalition. That is, $Q_{ij} \in \mathcal{A}(\mathcal{P}, \sqsubseteq)$ if there exist $i, j \in N$ such that $\{i, j\} \in Q_{ij}$ and all other coalitions of Q_{ij} are singletons. The lattice $(\mathcal{P}, \sqsubseteq)$ is ranked and each element P has rank $r(P) = n - |P|$. Any partition P' covered by P has the same coalitions as P except one that is divided in two coalitions in P'. Hence, for any $P, P' \in \mathcal{P}$ such that P covers P', we have that P' has one more coalition than P, $r(P') = r(P) + 1$. For any two partitions $P, P' \in \mathcal{P}$, the rank function satisfies $r(P)+r(P') \geq r(P \vee P')+r(P \wedge P')$, hence $(\mathcal{P}, \sqsubseteq)$ is a semimodular lattice. A lattice $(L, \vee, \wedge)$ is (upper) semimodular if for all $x, y \in L$ we have that $x \wedge y \prec x$ and $x \wedge y \prec y$ imply $x \prec x \vee y$ and $y \prec x \vee y$. A distributive lattice is semimodular, while the converse is not necessarily true.

A maximal decomposition of a partition P in terms of atoms is the expression of P as the supremum of all atoms finer than P. Formally,

$$P = \bigvee_{Q_{ij} \in \mathcal{A}(P)} Q_{ij} \quad \text{with } \mathcal{A}(P) = \left\{Q_{ij} \in \mathcal{A}(\mathcal{P}, \sqsubseteq) \mid Q_{ij} \sqsubseteq P\right\},$$

where $\mathcal{A}(P)$ is the set of atoms (partition with only one nontrivial two-element coalition) finer than P. The class of a partition $P \in \mathcal{P}$ is defined by the collection of integers $c^P = \left\{c_1^P, \ldots, c_n^P\right\}$ such that c_k^P is the number of coalitions of P consisting of exactly k players. Thus $\sum_{k=1}^n c_k^P k = n$ and $\sum_{k=1}^n c_k^P = n - r(P) = |P|$. The size s^P of a partition $P \in \mathcal{P}$ is the number of atoms finer than P. That is,

$$s^P = \sum_{k=1}^{n} c_k^P \binom{k}{2} = |\{\{i, j\} \in \mathcal{A}(P)\}|. \tag{1}$$

A coalitional network consists of a pair $(g, P) \in G \times \mathcal{P}$. We define the ordering relation $\preceq$ on $(G \times \mathcal{P}) \times (G \times \mathcal{P})$ such that $(g, P) \preceq (g', P')$ if and only if $g \subseteq g'$ in G and $P \sqsubseteq P'$ in $\mathcal{P}$. Since $(G \times \mathcal{P}, \preceq)$ is defined as the Cartesian product of two lattices, it has also a lattice structure. Moreover, it inherits the semimodularity property of the partition lattice. The bottom and top elements of the lattice $(G \times \mathcal{P}, \preceq)$ are $(g^\emptyset, P_\perp)$ and $(g^N, \{N\})$ respectively. Atom elements in $\mathcal{A}(G \times \mathcal{P}, \preceq)$ take one of the following two forms, $(l, P_\perp)$ or $(g^\emptyset, Q_{ij})$ with $l \in G$ being a one-link network and $Q_{ij} \in \mathcal{A}(\mathcal{P}, \sqsubseteq)$. If (g_a, P_a) is an atom, $i \in (g_a, P_a)$ means that player i is either a node of the one-link network or a member of the sole two-member coalition.

From direct calculations we have

$$|\mathcal{A}(G \times \mathcal{P}, \preceq)| = (n(n-1)/2) + \binom{n}{2} = n(n-1).$$

Each element (g, P), with $P = \{S_1, \ldots, S_k\}$ being a k-partition, is covered by $\binom{k}{2} + \left(|g^N| - |g|\right)$ elements and covers $\sum_{S \in P} 2^{|S|-1} - |P| + |g|$ elements. The number of atoms in a maximal decomposition of any (g, P) is $|\mathcal{A}(g, P)| = s^P + |g|$ with s^P defined in (1). Let $|\mathcal{A}(g, P)|$ be the degree of the coalitional network (g, P) and denote it by $d(g, P)$. For any player $i \in N$ and $(g, P) \in G \times \mathcal{P}$, we denote by $d_i(g, P)$ the degree of player i in the coalitional network (g, P). The degree $d_i(g, P)$ is the number of atoms to which i belongs, that is the number of links player i has in g and the number of two-player coalitions in atoms finer than P to which player i belongs. Finally, we denote by $n(g, P)$ the number of players that have at least one link in g or that are not singletons in P. That is, $n(g, P) = |N(g, P)|$ with $N(g, P) = N(g) \cup \{i \in N \mid \{i\} \notin P\}$.

We now present some properties fulfilled by the lattice of coalitional networks that are of interest for the sequel.

Proposition 1. *The lattice $(G \times \mathcal{P}, \preceq)$ is ranked. The rank function $r : (G \times \mathcal{P}) \to \mathbb{N}$ is such that $r(g, P) = n - |P| + |g|$ for all $(g, P) \in G \times \mathcal{P}$.*

By definition, a ranked lattice $(L, \vee, \wedge)$ is semimodular if and only if its rank function $r : L \to \mathbb{N}$ satisfies $r(x) + r(y) \geq r(x \vee y) + r(x \wedge y)$. The next proposition shows that the lattice of coalitional networks is semimodular:

Proposition 2. *The lattice $(G \times \mathcal{P}, \preceq)$ is semimodular.*

3. Coalitional network games

Knowing the lattice structure of coalitional networks ordered by $\preceq$, we can now study games on coalitional networks that are bottom-normalized real-valued lattice functions.

Definition 1. A coalitional network game is a function $v : G \times \mathcal{P} \to \mathbb{R}$ such that $v(g^\emptyset, P_\perp) = 0$.

A coalitional network game assigns a real value to each possible pair consisting of a network g and a partition P that represents the total value generated by the set of players when organized under (g, P). The set of all possible coalitional network games is denoted $\mathcal{V}$ and can be identified with the vector space $\mathbb{R}^{|G| \times |\mathcal{P}| - 1}$.

A coalitional network game is a richer object than a cooperative network game or a classical coalitional game because it allows the value generated to depend both on the network structure and on the organization of players into partitions. Coalitional network games can be seen as network games with externalities, where the value generated by a network depends on the organization of the set of players into mutually disjoint coalitions, and converge to classical network games in case of absence of externalities (i.e. when the partition organization of players does not influence the worth). To emphasize the richness of coalitional network games, we can compare the vector space associated to them to the corresponding space of classical network games. Classical network games take values only on the set of possible networks G. The number of possible networks in N is $|G| = 2^{n(n-1)/2}$. Network games considered as real-valued functions on $|G|$ can be identified with $\mathbb{R}^{|G|-1}$. The number of possible partitions on N is the Bell number B_n.[7] Thus, coalitional network games considered as real-valued functions on $G \times \mathcal{P}$ can be identified with $\mathbb{R}^{|G| \times B_n - 1}$.

[7] Bell numbers are defined recursively, using the Stirling numbers of the second kind, and no close form expression exists to express them.

Definition 2. A coalitional network $(g, P) \in G \times \mathcal{P}$ is efficient relative to a coalitional network game v if $v(g, P) \geq v(g', P')$ for all $(g', P') \in G \times \mathcal{P}$.

The efficient coalitional network may not be unique. Of course, in case of multiplicity, they all achieve the same maximum value. The efficient coalitional networks represent the best way to organize the set of players in terms of networks and groups.

Definition 3. For any coalitional network game $v \in \mathcal{V}$, its monotonic cover $\hat{v}$ is defined by

$$\hat{v}(g, P) = \max_{(g', P') \preceq (g, P)} v(g', P').$$

Two different interpretations can be offered to monotonic covers of coalitional network games. The first one corresponds to the one presented by Jackson (2005). The idea is that at the time of building a coalitional network, players consider all the available possibilities, and, if there is still some possibility to modify the coalitional network, then it is useful to consider which structure generates the maximum possible value. This approach is called flexible by Jackson in the context of network games without externalities. Another interpretation is the following. In classical coalitional games, it is usually assumed that the game is superadditive so that the grand coalition generates the maximum value and is thus formed. In the coalitional network games context, this is a too strong assumption, since it is often the case that forming or maintaining links induces costs and the grand coalition is not necessarily the one that maximizes the worth. Instead, we assume here that the complete network and the grand coalition form, but only activate or declare some links and groups in order to generate the maximum value. The complete network and the grand coalition have all links and subgroups at their disposal but only use some of them to cooperate. A set of players with communication links g^N can use any network $g \subseteq g^N$ to cooperate. A set of players forming a unique group $\{N\}$ are free to group themselves into smaller groups to achieve higher values.[8] Hence, the complete network and the grand coalition always get the maximum value under its monotonic cover.

Definition 4. A coalitional network game $v \in \mathcal{V}$ is monotonic if

$$(g, P) \preceq (g', P') \Rightarrow v(g, P) \leq v(g', P').$$

Notice that if a coalitional network game is monotonic, then $v = \hat{v}$. A monotonic coalitional network game attributes to a coalitional network a higher value than the value it attributes to its subcoalitional networks. This may not be a very natural property in coalitional network games since the top coalitional network structure is not always efficient. Nevertheless, we can draw some useful information about how allocation rules perform on monotonic coalitional network games.

A special family of monotonic coalitional network games consists of the unanimity coalitional network games. For a coalitional network $(g, P) \in G \times \mathcal{P}$, let $u_{g,P} \in \mathcal{V}$ denote the unanimity coalitional network game satisfying

$$u_{g,P}(g', P') = \begin{cases} 1 & \text{if } (g, P) \preceq (g', P') \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Each coalitional network game $u_{g,P}$ can be seen as a vector in $\mathbb{R}^{|G| \times B_n - 1}$. The $|G| \times B_n - 1$ different $u_{g,P}$'s are linearly independent, hence the set

$$\left\{u_{g,P} \mid (g, P) \in G \times \mathcal{P}, (g, P) \neq (g^{\emptyset}, P_{\perp})\right\}$$

of all unanimity coalitional network games forms a linear basis for $\mathbb{R}^{|G| \times B_n - 1} \equiv \mathcal{V}$ (see Gilboa and Lehrer, 1991). Each coalitional network game $v \in \mathcal{V}$ can thus be written as

$$v = \sum_{(g^{\emptyset}, P_{\perp}) \neq (g, P) \in G \times \mathcal{P}} \Delta^{g,P}(v) u_{g,P}. \tag{3}$$

Each coefficient $\Delta^{g,P}(v)$ is called the Harsanyi dividend (see Harsanyi, 1959). The dividend of a given element (g, P) of the lattice $(G \times \mathcal{P}, \preceq)$ represents the value that is left to (g, P) once all (g', P') included in (g, P) have received their corresponding dividends. By combining results from Grabisch (2010) and Caulier (2010), the numerical value of a coefficient is found by

$$\Delta^{g,P}(v) = \sum_{(g', P') \preceq (g, P)} v(g', P')(-1)^{|P'| - |P|}(n_1 - 1)! \dots (n_{|P|} - 1)!(-1)^{|g| - |g'|}.$$

4. Flexibility and equal treatment

In order to keep track of how the value generated by a coalitional network is allocated to players, we adopt the flexible approach of Jackson (2005).[9] Two different allocation rules are proposed. The atom-based allocation rule focuses on the role played by the minimal forms of cooperation among players in generating the value. The player-based allocation rule emphasizes the role of the players in achieving the value.

Definition 5. An allocation rule for a coalitional network game $v \in \mathcal{V}$ is a function $\psi : G \times \mathcal{P} \times \mathcal{V} \to \mathbb{R}^N$ such that $\sum_i \psi_i(g, P, v) = v(g, P)$ for all v, g and P.

It is important to note that an allocation rule depends on g, P and v. This allows an allocation rule to take full account of a player i's role in the network and in the coalition structure. This includes not only what the network configuration and coalition structure are, but also and how the value generated depends on the overall network and coalition structure. Note that efficiency $\left(\sum_i \psi_i(g, P, v) = v(g, P)\right)$ is assumed in the definition of an allocation rule.

Definition 6. An allocation rule ψ is a flexible coalitional network rule if $\psi(g, P, v) = \psi\left(g^N, \{N\}, \hat{v}\right)$, for all v and efficient coalitional network (g, P) relative to v.

The allocation rule only depends on the monotonic cover of the coalitional network game and distributes the value taken by the efficient configuration. This is consistent with the perspective that the coalitional network is being formed and that it can still be modified, or that the complete network together with the grand coalition is formed but only uses a subnetwork and a partition efficient relative to v. The idea from the flexible perspective is that inefficient coalitional network structures should not be reached.

Note in the definition that the equivalence is only required on efficient structures, as the value that accrues to other coalitional networks might not even be the same (i.e. $v(g, P) \neq \hat{v}(g, P)$ for inefficient (g, P)).

The next property states how the values in two different games are related. The property states the behavior followed by the players concerning the distribution of the value generated when confronted to different games.[10]

8 This is similar to essential superadditivity in coalitional games (see Wooders, 2008).

9 Navarro (2010) has proposed three modifications of Jackson's (2005) flexible network axiom when the structure of externalities across components is known.

10 Jackson (2005) uses the term additivity instead of linearity. Since the definition imposes both additivity and homogeneity, we prefer to name it linearity.

Definition 7. An allocation rule ψ is weakly linear if for any monotonic coalitional network games v and v', and scalars $a \geq 0$ and $b \geq 0$,

$$\psi(g^N, \{N\}, av + bv') = a\psi(g^N, \{N\}, v) + b\psi(g^N, \{N\}, v'),$$

and if $av - bv'$ is monotonic, then

$$\psi(g^N, \{N\}, av - bv') = a\psi(g^N, \{N\}, v) - b\psi(g^N, \{N\}, v').$$

Again, the weakly linearity condition only applies to monotonic coalitional network games, the only relevant information if we consider the coalitional network as flexible.

As a matter of equity, Jackson (2005) proposes to share the value in an unanimity game equally between essential players or, for link-based allocation rules, between essential links, whichever you consider as vital in generating value. In coalitional network games, basic ingredients are not the players. The mathematical structure in terms of lattice shows that the minimal aggregation form in a coalitional network is an atom, which takes the form of either a link between two players together with the trivial partition or a partition whose unique non-singleton coalition is a pair of players together with the empty network. In order to assess the contribution to cooperation of players in this context, we argue that the role played by each atom must first be assessed. In the network game setting, the contribution of a player may be computed in terms of the links she controls. In coalitional network games, the contribution of a player may be computed in terms of the atoms controlled by the player; that is, either the links controlled by the player in the existing network or the partitions with only one nontrivial two-element coalition to which the player belongs that are finer than the existing partition.

Hence we propose the following property:

Definition 8. An allocation rule ψ satisfies equal treatment of vital atoms if $u_{g,P} \in \mathcal{V}$ is an unanimity coalitional network game for some (g, P), then

$$\psi_i(g, P, u_{g,P}) = \begin{cases} \displaystyle\sum_{\substack{(g_a,P_a)\in\mathcal{A}(g,P),\\ i\in(g_a,P_a)}} \frac{1}{2|\mathcal{A}(g,P)|} & \text{if } i \text{ belongs to at least one atom,} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that unanimity coalitional network games of (g, P) are such that all atoms of (g, P) are members of the decomposition of (g, P), and then, the join of all these atoms is the (only) configuration that generates some value. Formally, for each $(g, P) \in G \times \mathcal{P}$ with $\mathcal{A}(g, P) \subseteq \mathcal{A}(G \times \mathcal{P}, \preceq)$, the set of atoms such that $(g_a, P_a) \in \mathcal{A}(g, P) \Rightarrow (g_a, P_a) \preceq (g, P)$, and $(g, P) = \bigcup_{(g_a,P_a)\in\mathcal{A}(g,P)} (g_a, P_a)$. In an unanimity coalitional network game $u_{g,P}$, all atoms of (g, P) are identical, in the sense that they are vital in the generation of worth, while the other atoms are not part of the structures generating worth. We thus propose to distribute equally the value generated among these vital atoms. The 1/2 reflects the fact that the value of a given vital atom, either a link or the unique nontrivial two-element coalition of the partition, is controlled by two players.

The properties described above are enough to characterize a unique solution, that we call the atom-based flexible coalitional network allocation rule.

Theorem 1. *An allocation rule for coalitional network games satisfies equal treatment of vital atoms, weak linearity and is a flexible coalitional network rule if and only if for all $v \in \mathcal{V}$ and $(g, P) \in G \times \mathcal{P}$ efficient relative to v, it is the atom-based flexible coalitional network allocation rule, ψ^a, defined as follows:*

$$\psi_i^a(g, P, v) = \sum_{(g',P')\in G\times\mathcal{P}} \left[\sum_{\substack{(g_a,P_a)\in\mathcal{A}(g',P'):\\ i\in(g_a,P_a)}} \frac{\Delta^{g',P'}(\hat{v})}{2|\mathcal{A}(g',P')|} \right]. \quad (4)$$

The idea is first to calculate the dividends for the monotonic cover of the game under consideration, next, to distribute them equally among the atoms of the coalitional networks corresponding to these dividends and, finally, to the players essential to these atoms. This allocation rule thus stresses the importance of minimal forms of cooperation among players that can take the form of links or coalitions, before sharing the global worth to individuals.

In order to show the independence of the properties used in Theorem 1, we provide Proposition 3 that shows that there exist allocation rules that satisfy all properties except one and, hence, that none of the properties is superfluous.

Proposition 3. *None of the properties used in the characterization of the atom-based flexible coalitional network allocation rule ψ^a given by (4) in Theorem 1 is superfluous.*

Theorem 1 only applies to an efficient coalitional network (g, P) relative to v. In order to have a complete definition of an allocation rule, we also need to specify how to allocate the value of inefficient coalitional networks. Following Jackson (2005), we propose to use the allocation of efficient coalitional networks as a benchmark and to allocate the value of an inefficient coalitional network proportionally.

Definition 9. An allocation rule ψ is proportional if for each i and $v \in \mathcal{V}$ either $\psi_i(g, P, v) = 0$ for all (g, P), or for any (g, P) and (g', P') such that $v(g', P') \neq 0$,

$$\frac{\psi_i(g, P, v)}{\psi_i(g', P', v)} = \frac{v(g, P)}{v(g', P')}. \quad (5)$$

Note first that the definition of proportional allocation rule covers the case where $\psi_i(g', P', v) = 0$ since it thus implies that $\psi_i(g, P, v) = 0$ for all $(g, P) \neq (g', P')$. When $\psi_i(g', P', v) \neq 0$ condition (5) applies. Notice also that, if the allocation rule is proportional, we cannot have $\psi_i(g, P, v) \neq 0$ for some i when $v(g, P) = 0$. By condition (5), and given an efficient coalitional network (g', P') relative to $v \in \mathcal{V}$ and such that $v(g', P') \neq 0$, we have that $\psi_i(g, P, v) = 0$ for all i.

A proportional allocation rule has first to be determined on an efficient coalitional network and afterwards rescaled for the final inefficient coalitional network. When there are several efficient coalitional networks relative to a value function, a proportional allocation rule gives the same allocation independently of the efficient coalitional network used to rescale the allocation for the inefficient coalitional networks. Indeed, proportionality for an allocation rule ψ implies that for all efficient coalitional networks (g, P) and (g', P') relative to $v \in \mathcal{V}$, $(g, P) \neq (g', P')$, with $v(g, P) = v(g', P')$, we have by Eq. (5) that $\psi_i(g, P, v) = \psi_i(g', P', v)$ for all i.

Theorem 2. *An allocation rule for coalitional network games satisfies equal treatment of vital atoms, weak linearity and is a flexible and proportional coalitional network rule if and only if for all $v \in \mathcal{V}$ and $(g, P) \in G \times \mathcal{P}$, it is the atom-based flexible and proportional coalitional network allocation rule, ψ^a, defined as follows:*

$$\psi_i^a(g, P, v) = \frac{v(g, P)}{\hat{v}(g^N, \{N\})} \times \sum_{(g',P')\in G\times\mathcal{P}} \left[\sum_{\substack{(g_a,P_a)\in\mathcal{A}(g',P'):\\ i\in(g_a,P_a)}} \frac{\Delta^{g',P'}(\hat{v})}{2|\mathcal{A}(g',P')|} \right]. \quad (6)$$

Note that Proposition 3 can be used to show that all the properties of Theorem 2 are independent.

If, on the contrary, we think that the emphasis should be set directly on the players rather than indirectly, we propose to adapt the equity condition in Definition 8 as follows.

Definition 10. An allocation rule ψ satisfies equal treatment of vital players if $u_{g,P} \in \mathcal{V}$ is an unanimity coalitional network game for some (g, P), then

$$\psi_i(g, P, u_{g,P}) = \begin{cases} 0 & \text{if } i \text{ is isolated in } g \text{ and a singleton in } P, \\ \dfrac{1}{n(g,P)} & \text{otherwise} \end{cases}$$

with $n(g, P)$ the number of players that have at least one link in g or that are not singletons in P.

In an unanimity coalitional network game, players not isolated in g or in P are all vital to the functioning of the coalitional network, in the sense that the value is generated by their cooperation and no other player contribute in any sense. It is not to say that a stand-alone player is not able to accomplish some valuable worth in a coalitional network, but our focus is on the worth generated through cooperation and how to share this value among cooperating players. In this case, players not isolated are considered as equals and isolated players contribute nothing. Hence, this equal treatment condition allocates the worth equally among the $n(g, P)$ players in $N(g, P)$.

Before presenting our player-based flexible allocation rule, we need the following definition:

Definition 11. The modular elements $\mathcal{P}^{mod}$ of the partition lattice $(\mathcal{P}, \sqsubseteq)$ over N are the partitions $P \in \mathcal{P}^{mod}$ containing a unique non-trivial coalition as well as $P_\perp$.

The finest partition $P_\perp$ and the coarsest partition $\{N\}$ are modular elements. Any other $P \in \mathcal{P}^{mod}$ consists in a coalition $\{S\} \in 2^N \backslash \emptyset$ together with the singletons $\{\{i\} | i \in N \backslash S\}$. Hence, each $P \in \mathcal{P}^{mod}$ can be uniquely characterized by its non-trivial coalition $\{S\} \in 2^N \backslash \emptyset$ and we may thus write with some abuse of notation the modular partition as $\{S\} \in \mathcal{P}^{mod}$. Note that the only one modular partition corresponding to all singleton coalitions $\{\{i\} | \{i\} \in 2^N\}$, is the trivial partition $P_\perp$. Hence the number of distinct modular partitions on N is $|\mathcal{P}^{mod}| = 2^n - n$, since $P_\perp$ has multiplicity n in $\mathcal{P}^{mod}$.

We now present the player-based flexible allocation with its characterizing properties.

Theorem 3. *An allocation rule for coalitional network games satisfies equal treatment of vital players, weak linearity and is a flexible coalitional network rule if and only if for all $v \in \mathcal{V}$ and $(g, P) \in G \times \mathcal{P}$ efficient relative to v, it is the player-based flexible coalitional network allocation rule ψ^p, defined by*

$$\psi_i^p(g, P, v) = \sum_{\substack{S \ni i, \\ \{S\} \in \mathcal{P}^{mod}, \\ S \subseteq N(g,P)}} \frac{\Delta^{g^S, \{S\}}(\hat{v})}{|S|}. \tag{7}$$

The proof of this theorem is a direct analog of the proof of Theorem 1, which appears in the Appendix. If the end coalitional network is not an efficient one, we can once again use Definition 9 and adapt formula (7) to hold for any (g, P), not necessarily efficient relative to v.

Theorem 4. *An allocation rule for coalitional network games satisfies equal treatment of vital players, weak linearity and is a flexible and proportional coalitional network rule if and only if for all $v \in \mathcal{V}$ and $(g, P) \in G \times \mathcal{P}$, it is the player-based flexible and proportional coalitional network allocation rule ψ^p, defined by*

$$\psi_i^p(g, P, v) = \frac{v(g, P)}{\hat{v}(g^N, \{N\})} \sum_{\substack{S \ni i, \\ \{S\} \in \mathcal{P}^{mod}, \\ S \subseteq N(g,P)}} \frac{\Delta^{g^S, \{S\}}(\hat{v})}{|S|}. \tag{8}$$

The proof of Theorem 4 follows directly from the proof of Theorem 2 which is given in the Appendix. The allocation rule (7) is close to the classical Shapley value (where Harsanyi dividends are shared equally among players). However, in this setting, we first deal with the monotonic cover of the value function as prescribed by our flexible approach and, second, players are involved in much more complicated structures consisting in both a network and a partition. To stress the similarities, let us express Eq. (7) in the following equivalent way, closer to the better known expression for the Shapley value (see Shapley, 1953):

$$\psi_i^p(g, P, v) = \sum_{\substack{\{S\} \in \mathcal{P}^{mod}, \\ S \subseteq N(g,P) \\ i \notin S}} \left(\hat{v}\left(g^{S \cup i}, \{S \cup i\}\right) - \hat{v}\left(g^S, \{S\}\right)\right) \times \left(\frac{|S|!(n - |S| - 1)!}{n!}\right). \tag{9}$$

This amounts to define a TU-game $c_{g,P,v} : 2^N \to \mathbb{R}$ such that, for all $S \in 2^N$, $c_{g,P,v}(S) = \hat{v}(g^S, S)$. Then, the player-based flexible allocation rule (7) can be rewritten $\psi^p(g, P, v) = Sh(c_{g,P,v})$, where Sh stands for the Shapley value.[11]

Example 1. Take $N = \{1, 2, 3\}$. Let $v(\{12\}, \{12|3\}) = 1$, $v(\{23\}, \{1|23\}) = 1$, $v(\{12, 23\}, \{123\}) = w \geq 1$ and $v(g, P) = 0$ for all other coalitional networks.[12] We also define $v'(g, P) = w$ for all (g, P) such that g has at least two links and $P = P^\top$ and $v'(g, P) = 1$ for all (g, P) such that g has one link and P contains one two-element coalition. Then, the link- and player-based flexible coalitional network allocation rules provide different allocations if the coalitional network that realizes is $\{\{12, 23\}, P^\top\}$

$$\psi^a\left(\{\{12, 23\}, P^\top\}, v\right) = \left(\frac{w}{4}, \frac{w}{2}, \frac{w}{4}\right)$$

$$\psi^p\left(\{\{12, 23\}, P^\top\}, v\right) = \left(\frac{w}{3} - \frac{1}{6}, \frac{w}{3} + \frac{1}{3}, \frac{w}{3} - \frac{1}{6}\right)$$

$$\psi^a\left(\{\{12, 23\}, P^\top\}, v'\right) = \left(\frac{w}{3}, \frac{w}{3}, \frac{w}{3}\right)$$

$$\psi^p\left(\{\{12, 23\}, P^\top\}, v'\right) = \left(\frac{w}{3}, \frac{w}{3}, \frac{w}{3}\right).$$

Under the game v, both allocation rules reflect correctly the importance of player 2 which is more "central" than players 1 and 3. Player 2 participates to both the one link networks and in both two-element coalitions in the partitions generating a value of 1. The presence of player 2 is also necessary in the structure achieving a value of w. The importance of player 2 is thus reflected under both allocation rules and the difference between the shares she receives pertains to whether we stress the role of the atoms to which player 2 belongs (player 2 participates to twice more important atoms than player 1 or 3, and receives thus twice their shares), or

[11] We thank an anonymous referee for drawing our attention to this point.

[12] A partition $P = \{\{a, b\}, \{c\}\}$ is denoted $\{ab|c\}$ for convenience.

if we stress directly the role played by player 2 under the player-based allocation rule. Under the game v', all players receive the same share which is consistent with the equity principle fulfilled by the rules.

Let us explain the two equity properties in Definitions 8 and 10. The proposed properties are not claimed to be logically independent, they are merely a convenient characterization of the object under study. The resemblance of the proposed allocation rules with the classical Shapley value stems for other possible characterizations with different desirable properties (such as fairness in Myerson, 1977 or population solidarity in Calvo and Gutierrez, 2010). The chosen perspective has a twofold motivation. First, it shows how should the allocation rule behave under the simplest case of an unanimity game. This approach presents a first trade-off: should we focus on cooperation among players or on players directly? Second, the setting we use deals with coalitions and networks. Hence, the adaptation of properties like fairness in Myerson (1977) is not straightforward as it would require to restate it both in terms of coalitions and networks. This would have lengthened the list of properties and these ones would have been less intuitive.

5. Relationship with existing allocation rules

The allocation rules presented in this paper are generalizations of the Jackson (2005) player-based and link-based flexible allocation rules for network games to coalitional network games in which players may also form coalitions. The fact that we opt for a presentation in terms of Möbius transforms is mainly to avoid cumbersome notation or lengthy expression and should not confuse the reader to remark the strict equivalence of the Jackson allocation rules and the ones presented in this paper when coalition structures play no role, i.e. if $v(g,P) = v(g,P')$ for all $(g,P) \in G \times \mathcal{P}$, $(g,P') \in G \times \mathcal{P}$, $P \neq P'$. In this case, partitions do not affect the value and the coalitional network game v is equivalent to a value function for network games and our player-based and atom-based flexible allocation rules for coalitional network games correspond to the player-based and link-based flexible allocation rules for network games introduced by Jackson (2005).

We could also relate the player-based flexible coalitional network allocation rule ψ^p with the classical Shapley value for cooperative game (with characteristic function). A cooperative game is a function $c : 2^N \to \mathbb{R}$ that assigns a worth $c(S)$ to each possible coalition $S \in 2^N$. The set of possible cooperative games is denoted $\mathcal{C}$. A cooperative game c is additive if $c(S \cup T) = c(S) + c(T)$ for all non-empty $S, T \subset N$, $S \cap T = \emptyset$. The set of additive cooperative games is denoted $\mathcal{C}^\circ$. A solution (or allocation rule) is a map $\Phi : \mathcal{C} \to \mathbb{R}^n$. The Shapley value Φ^S is a solution that satisfies Efficiency, Null Player, Symmetry and Linearity.

Efficiency $c(N) = \sum_{i \in N} \Phi_i^S(c)$.
Null Player If for all $S \in N \backslash i$ we have that $c(S \cup i) = c(S)$, then $\Phi_i^S(c) = 0$.
Symmetry $\Phi^S(\pi c) = \pi \Phi^S(c)$ with π a bijection from N to N.
Linearity $\Phi^S(\alpha c + c') = \alpha \Phi^S(c) + \Phi^S(c')$, $\alpha \in \mathbb{R}, c, c' \in \mathcal{C}$.

The Shapley value Φ^S is an additive cooperative game, i.e.

$$\sum_{i \in S \cup T} \Phi_i^S(c) = \sum_{i \in S} \Phi_i^S(c) + \sum_{i \in T} \Phi_i^S(c)$$

for all $c \in \mathcal{C}$, $S, T \in 2^N \backslash \emptyset$, $S \cap T = \emptyset$. This amounts to write that for all $c \in \mathcal{C} : \Phi^S(c) \in \mathcal{C}^\circ$ and Φ^S is thus a projection from $\mathcal{C}$ to $\mathcal{C}^\circ$. For all $c \in \mathcal{C}^\circ$, $\Phi^S(c) = c$.[13]

[13] This property is also called the inessential game property.

The set of additive cooperative games appears to be the subspace consisting of fixed points for solutions. One could apply this property (that a solution for an additive game should be this game itself) to allocation rules for coalitional network games and identify the set of fixed point games that are trivially their own solutions. We can define an allocation rule $\Psi : \mathcal{V} \to \mathcal{V}^\circ$, with $\mathcal{V}^\circ$ the set of additive coalitional network games. A coalitional network game v is additive if $v(g \cup g', P \vee P') = v(g,P) + v(g',P')$ for all $(g,P), (g',P') \in G \times \mathcal{P} \backslash \{(g^\emptyset, P_\perp)\}$ such that $g \cap g' = \emptyset$ and $P \wedge P' = P_\perp$. However, due to the semimodularity of $(G \times \mathcal{P}, \preceq)$, an additive coalitional network game would convey the same value to each and every element of the lattice.

Proposition 4. *If a coalitional network game v is additive, then*

$$v(g,P) = v(g',P') \quad \textit{for all } (g,P), (g',P') \in G \times \mathcal{P} \backslash \{(g^\emptyset, P_\perp)\}, \quad (g,P) \neq (g',P').$$

In our setting, additive games are not the set of games whose trivial allocation rule is the game itself, due to the semimodularity structure of $(G \times \mathcal{P}, \preceq)$. The relationship of our player-based flexible allocation rule to the Shapley value for coalitional games is direct, provided two different adaptations. The first one relates to our flexible approach. The complete or top structure being not necessarily the most efficient one, we have to deal with monotonic covers of games, which are identical to the original game when this one is monotonic. Hence the relationship to Shapley value bares on monotonic games. The second adaptation needed is simply the expression of the Shapley value in terms of Harsanyi dividends: $\Phi_i^S(c) = \sum_{S: i \in S} \Delta^S(c)/|S|$ with $\Delta^S(c)$ the Harsanyi dividend defined recursively by $\Delta^S(c) = 0$ if $S = \emptyset$ and $\Delta^S(c) = c(S) - \sum_{T \subset S} \Delta^T(c)$, $S \neq \emptyset$. Dividends are shared equally across the members of the coalition, which is exactly the case for our player-based flexible allocation rule. Hence the correspondence is clear.

Acknowledgments

We would like to thank two anonymous referees for helpful comments. We also thank Matt Jackson, Marco Slikker and Myrna Wooders for valuable comments and suggestions. Ana Mauleon and Vincent Vannetelbosch are Senior Research Associates of the National Fund for Scientific Research (FNRS). Financial support from Spanish Ministry of Economy and Competition under the project ECO 2012-35820 and from the Fonds de la Recherche Scientifique—FNRS research grant J.007315 is gratefully acknowledged.

Appendix

Proof of Proposition 1. We have $r(g^\emptyset, P_\perp) = n - n + 0 = 0$ and for any $(g,P) \in \mathcal{A}(G \times \mathcal{P}) : r(g,P) = 1$ since $r(l, P_\perp) = n - n + 1$ and $r(g^\emptyset, Q_{ij}) = n - (n-1) + 0$.

Assume that the proposition holds for $(g,P) = (g^N, \{N\})$, i.e. $r(g^N, \{N\}) = n - 1 + n(n-1)/2$. The elements (g,P) covered by $(g^N, \{N\})$ have one of the following two forms: $(g^N \backslash l, \{N\})$ or $(g^N, \{N \backslash \{i,j\}, \{i,j\}\})$. In the first case, $r(g^N \backslash l, \{N\}) = n - 1 + n(n-1)/2 - 1 = r(g^N, \{N\}) - 1$ and in the second case $r(g^N, \{N \backslash \{i,j\}, \{i,j\}\}) = n - 2 + n(n-1)/2 = r(g^N, \{N\}) - 1$. □

Proof of Proposition 2. From Proposition 1 we have that $(G \times \mathcal{P}, \preceq)$ is ranked by $r(g,P) = n - |P| + |g|$ for all $(g,P) \in G \times \mathcal{P}$. Take any $(g,P), (g',P') \in (G \times \mathcal{P})$. Then,

$$r(g,P) + r(g',P') \geq r(g \cap g', P \wedge P') + r(g \cup g', P \vee P')$$

since $|g| + |g'| = |g \cap g'| + |g \cup g'|$ and $2n - |P| - |P'| \geq 2n - |P \wedge P'| - |P \vee P'|$ because of the semimodularity of the partition lattice. □

Proof of Theorem 1. First we show that the atom-based flexible coalitional network allocation rule defined by (4) satisfies all the properties. We have

$$\sum_{(g,P)} \Delta^{g,P}(\hat{v}) = \hat{v}(g^N, \{N\}) = \max_{(g,P)\in G\times\mathcal{P}} v(g,P)$$

and thus

$$\sum_{i\in N} \psi_i^a(g,P,v) = \hat{v}(g^N, \{N\})$$

since each atom consists of two players.

The atom-based flexible coalitional network allocation rule satisfies weak-linearity. Consider any monotonic coalitional network games v and v' in $\mathcal{V}$, and scalars $a \geq 0$ and $b \geq 0$. Then $av + bv'$ is monotonic and coincides with its monotonic cover. Hence,

$$\begin{aligned} &\psi_i^a(g^N, \{N\}, av + bv') \\ &= \sum_{(g,P)\in G\times\mathcal{P}} \sum_{\substack{(g_a,P_a)\in\mathcal{A}(g,P)\\ i\in(g_a,P_a)}} \frac{\Delta^{g,P}(a\hat{v} + b\hat{v}')}{2\,|\mathcal{A}(g,P)|} \\ &= \sum_{(g,P)\in G\times\mathcal{P}} \sum_{\substack{(g_a,P_a)\in\mathcal{A}(g,P)\\ i\in(g_a,P_a)}} \frac{a\Delta^{g,P}(\hat{v}) + b\Delta^{g,P}(\hat{v}')}{2\,|\mathcal{A}(g,P)|} \\ &= a\psi_i^a(g^N, \{N\}, v) + b\psi_i^a(g^N, \{N\}, v'). \end{aligned}$$

By a similar argument if $av - bv'$ is monotonic, we have that $\psi_i^a(av - bv') = a\psi_i^a(v) - b\psi_i^a(v')$.

Equal treatment of vital atoms is easily checked to hold in (4).

Second, we verify that any allocation rule satisfying equal treatment of atoms, weak linearity, and flexible coalitional network must coincide with the atom-based flexible coalitional network allocation rule ψ^a on efficient coalitional networks. Let $v \in \mathcal{V}$ and $\phi : G \times \mathcal{P} \times \mathcal{V} \to \mathbb{R}^N$ an allocation rule satisfying the claimed properties. Given that ϕ is a flexible coalitional network allocation rule implies that $\phi(g,P,v) = \phi(g^N, \{N\}, \hat{v})$ on efficient (g,P) relative to v, and so it is enough to show that $\phi(g^N, \{N\}, \hat{v})$ is uniquely determined on an efficient coalitional network. By Eq. (3) we have that

$$\hat{v} = \sum_{(g,P)\in G\times\mathcal{P}} \Delta^{g,P}(\hat{v})\, u_{g,P}.$$

Let $G^- = \{(g,P) \mid \Delta^{g,P}(\hat{v}) < 0\}$ and $G^+ = (G\times\mathcal{P})\backslash G^-$. Hence,

$$\hat{v} = \sum_{(g,P)\in G^+} \Delta^{g,P}(\hat{v})\, u_{g,P} - \sum_{(g,P)\in G^-} \left|\Delta^{g,P}(\hat{v})\right| u_{g,P}.$$

By weak linearity, we have that $\phi\left(g^N, \{N\}, \hat{v}\right)$ is equal to

$$\phi\left(g^N, \{N\}, \sum_{(g,P)\in G^+} \Delta^{g,P}(\hat{v}) u_{g,P}\right) - \phi\left(g^N, \{N\}, \sum_{(g,P)\in G^-} \left|\Delta^{g,P}(\hat{v})\right| u_{g,P}\right).$$

By weak linearity again, we obtain

$$\phi\left(g^N, \{N\}, \hat{v}\right) = \sum_{(g,P)\in G\times\mathcal{P}} \Delta^{g,P}(\hat{v})\phi(g^N, \{N\}, u_{g,P}).$$

Since ϕ is a flexible coalitional network allocation rule then $(g^N, \{N\})$ and (g,P) take both the same value under the monotonic cover of $u_{g,P}$ for each $(g,P) \in G\times\mathcal{P}$. Finally, by equal treatment of vital atoms, the value is uniquely determined and thus, $\phi = \psi^a$. □

Proof of Proposition 3. We show that for each property in Theorem 1, there is an allocation rule different than ψ^a given by (4) that satisfies the remaining other properties.

- **Remove flexibility**: For all $v \in \mathcal{V}$ and $(g,P) \in G\times\mathcal{P}$ efficient relative to v, the allocation rule

$$\psi_i(g,P,v) = \sum_{(g',P')\preceq(g,P)} \left[\sum_{\substack{(g_a,P_a)\in\mathcal{A}(g',P'):\\ i\in(g_a,P_a)}} \frac{\Delta^{g',P'}(v)}{2|\mathcal{A}(g',P')|} \right] \quad (10)$$

satisfies weak linearity, equal treatment of vital atoms but is not a flexible rule because it does not depend on the monotonic cover of v and is calculated on sub-coalitional network games (g',P') of (g,P) only. If v is non-monotonic, $\psi_i(g,P,v)$ is different from $\psi_i(g^N, \{N\}, \hat{v})$.

- **Remove equal treatment of vital atoms**: For all $v \in \mathcal{V}$ and $(g,P) \in G\times\mathcal{P}$ efficient relative to v, the allocation rule

$$\psi_i(g,P,v) = \frac{\hat{v}(g^N, \{N\})}{n} \quad (11)$$

is a flexible allocation rule that satisfies weak linearity but violates equal treatment of vital atoms.

- **Remove weak linearity**: Let $v \in \mathcal{V}$ and $(g,P) \in G\times\mathcal{P}$. An atom $(g_a,P_a) \in (g,P)$ is a *null atom* for v in (g,P) if

$$\bigcup_{(g',P')\in\mathcal{A}(g,P)\backslash(g_a,P_a)} (g',P') \prec (g,P)$$

and

$$v(g,P) = v\left(\bigcup_{(g',P')\in\mathcal{A}(g,P)\backslash(g_a,P_a)} (g',P')\right).$$

For all the other cases, the atoms are non-null. Denote by $\mathcal{NA}(g,P,v)$ the set of atoms $(g',P') \in \mathcal{A}(g,P)$ such that (g',P') is non-null for v in (g,P).

For all $v \in \mathcal{V}$ and $(g,P) \in G\times\mathcal{P}$ efficient relative to v, the allocation rule

$$\psi_i(g,P,v) = \sum_{(g',P')\in G\times\mathcal{P}} \left[\sum_{\substack{(g_a,P_a)\in\mathcal{A}(g',P'):\\ i\in(g_a,P_a)}} \frac{\Delta^{g',P'}(\hat{v})}{2|\mathcal{NA}(g',P')|} \right] \quad (12)$$

is a flexible allocation rule that satisfies equal treatment of vital atoms (all atoms are non-null in unanimity coalitional network games) but violates weak linearity. Weak linearity is violated when the sum of the relative number of non-null atoms for the games av and bv' to which a player i belongs is not the same as the relative number of non-null atoms for the game $av + bv'$ (or $av - bv'$) to which player i belongs. □

Proof of Theorem 2. The uniqueness part mimics the one for Theorem 1 with a constant proportional rescaling at the end. We now check that the allocation rule satisfies all properties.

Let $(g,P) \in G\times\mathcal{P}$ and $v \in \mathcal{V}$ and $(g',P') \in G\times\mathcal{P}$ efficient relative to v such that $v(g',P') \neq 0$. The allocation rule

$$\psi_i^a(g,P,v) = \frac{v(g,P)}{\hat{v}(g^N, \{N\})} \times \sum_{(g',P')\in G\times\mathcal{P}} \left[\sum_{\substack{(g_a,P_a)\in\mathcal{A}(g',P'):\\ i\in(g_a,P_a)}} \frac{\Delta^{g',P'}(\hat{v})}{2|\mathcal{A}(g',P')|} \right] \quad (6)$$

satisfies all the properties stated in Theorem 2. First, it is a flexible rule as it depends on the monotonic cover and on the complete structure to be calculated. Equal treatment of vital atoms is satisfied as the only difference between formula (4) and (6) is the multiplicative constant $\frac{v(g,P)}{\hat{v}(g^N,\{N\})}$. Thus, equal atoms are treated equally. The multiplicative constant shows that proportionality is also satisfied.

Last, consider any monotonic coalitional network games v and v' in $\mathcal{V}$, and scalars $a \geq 0$ and $b \geq 0$. Then $av + bv'$ is monotonic and coincides with its monotonic cover. Hence, $(av + bv'^N, \{N\}) = (\widehat{av + bv'})(g^N, \{N\})$ so that the multiplicative constant in formula (6) is

$$\frac{(av + bv'^N, \{N\})}{(\widehat{av + bv'})(g^N, \{N\})} = 1$$

and weak linearity is proved using precisely the same steps as in the proof of Theorem 1. □

Proof of Proposition 4. We first show the following lemma that applies to general semimodular lattices.

Lemma 1. *Let $(L, \vee, \wedge)$ be a lattice and $v : L \to \mathbb{R}$ be an additive function such that $v(x) + v(y) = v(x \wedge y) + v(x \vee y)$ for all $x, y \in L$. If $(L, \vee, \wedge)$ is semimodular, then v is constant.*

Proof. If L is semimodular with cardinality at least 5, it must contain 5 elements $a, b, c, e, f \in L$ such that

$a \vee b = b \vee c = e$

$a \wedge b = b \wedge c = f$

$a < c$.

Then for every $t \in [a, c]$, an additive function v has to satisfy

$v(t) + v(b) = v(t \wedge b) + v(t \vee b) = v(f) + v(e)$.

Hence $v(t)$ is constant on the interval $[a, c]$. □

Since the lattice of coalitional networks is semimodular, any additive function defined on it has to be constant by the previous lemma. □

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