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The stability and efficiency of direct and star networks in a loan game

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HIGHLIGHTS

- Network formation game is applied to analyze loans in a small group.
- Players first form a network and then trade according to the Myerson value.
- Star networks and direct networks are both efficient and internally stable.
- But they may be externally unstable.

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ABSTRACT

We analyze a loan game where identical lenders and identical borrowers first form a loan network by creating bilateral links and then trade in this network. To predict the outcome of the loan game, we make two important assumptions: (i) players trade and split the surplus according to the Myerson value in any given loan network, and (ii) certain networks – networks that are pairwise stable and/or efficient under the Myerson value – are more likely to be formed. Two basic network structures, direct networks and star networks, are common in reality and sometimes coexist in a loan market. We explain these phenomena by showing that, if the link cost of each trade is small enough, these two networks are both efficient and internally stable, but they are not necessarily externally stable.

1. Introduction

A loan is a common economic and financial trade where a borrower accepts a certain amount of money from a lender at an interest rate. In reality, bilateral loans among a group of lenders and borrowers exhibit a certain network structure. Among all possible network structures, two are fundamental and can often be observed. The first one is the star network, as shown in Fig. 1(a), where all lenders and borrowers trade through an intermediary (the center of the star), such as a bank. The second one is the direct network, as in Fig. 1(b), where lenders trade directly to borrowers. Of course, some combined network structures may appear when the above two networks coexist; for example, a person who deposits some money in a bank (star network) could at the same time lend some money to a friend (direct network).

These observations involving loan behavior raise a question. Which network structures are more likely to form among a group of borrowers and lenders? In this paper, we will attempt to address this question by studying the formation and properties of loan networks—in particular, star networks and direct networks. We hope that the analysis in this paper will help us understand the monetary loan market from a new perspective, and that it will explain some phenomena—for example, why financial intermediaries such as banks exist.

In the mainstream literature, the loan of money is typically understood as taking place in a perfectly competitive market among a large population of lenders and borrowers. Each lender or borrower takes the price of money (the interest rate) as given and decides how much money he/she will lend or borrow. In equilibrium, the market clears so that the amount of money supplied is equal to the amount of money demanded. However, when the population is small, the perfect competition assumption is no longer valid; rather, more details about individual motivations and choices should be analyzed. In this situation, game theory, which studies interactions between rational decision-makers, might be a more appropriate tool for analyzing the loan market. In particular, the network formation game – a new branch of game theory – has rapidly developed over the last few decades. Roughly speaking, we assume in a network formation game that several rational agents first form a network by choosing bilateral links. They then bargain over their trades in this network. Intuitively, the network formation game might be able to analyze loans in a small group of people.

Myerson (1977) extends the Shapley value – an important solution concept in cooperative game theory – to network games.

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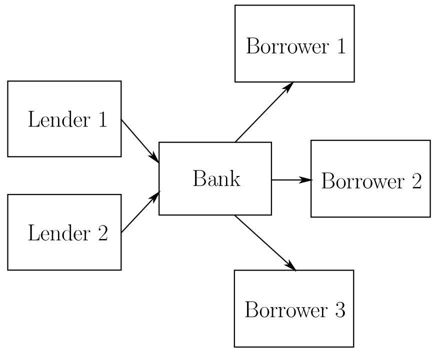
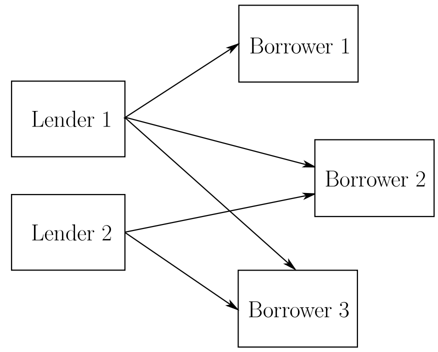


(a) A star network.



(b) A direct network.

Fig. 1. Two common loan network structures.

This solution is later named the Myerson value by Aumann and Myerson (1988). The Myerson value suggests a reasonable method to distribute cooperation surplus in a given network, but it does not recommend which network should form in the first place. Aumann and Myerson (1988) introduce a noncooperative model to explain the network that is formed as an equilibrium under the Myerson value. Jackson and Wolinsky (1996) generalize the Myerson value to more general surplus distribution manners and introduces pairwise stability as well as strong efficiency to predict which networks are likely to form; they also show the incompatibility of efficiency and stability. A large literature on network formation games has emerged since then, studying the formation and properties of networks either from the noncooperative equilibrium perspective or from the stability perspective.[1]

Network formation games have been applied in a wide range of economic situations. For instance, Johnson and Gilles (1999) discuss a spatial network model, where the distance between two areas affects the benefit and cost of their linkage. Furusawa and Konishi (2007) analyze a free trade network model, which treats the bilateral trade agreement between two countries as a link in the free trade network. Belleflamme and Bloch (2004) study a model where companies may sign bilateral market sharing agreements to form a collusive network. Corominas-Bosch (2004) and Kranton and Minehart (2001) examine the market of an indivisible good where the buyers and the sellers trade on some network structures.

To the best of my knowledge, there is little literature that applies the network formation game to analyze monetary loan (trade of a perfectly divisible good). This is exactly the goal of this paper. To this end, we build a model based on the network formation game framework. The solution to such a loan game is a prediction of which network will form and how the players will trade in this network. Following the tradition of the network formation game literature, we plan to solve a loan game in two steps. First, we assume that players trade and split the surplus according to the Myerson value in any given loan network. Roughly speaking, this suggests that players trade according to a rule that is both efficient and fair. Second, we assume that networks that are pairwise (internally and externally) stable and/or efficient under the Myerson value are more likely to be formed. By this solution method, we concentrate on the properties of the two elementary network structures mentioned above—the star network and the direct network. We find that if the link cost is small enough, these two networks are both efficient and internally stable, but they may not be externally stable. These conclusions help explain some of the phenomena described earlier in this section.

[1] We refer the reader to Jackson (2008) as a review.

Note that in reality a lender may face a risk when participating in a loan, since the borrower might not be able, or willing, to pay back the principal and/or the interest. Nevertheless, in this paper we only consider secured loans in which the borrowers pledge some asset as collateral so that no risk exists at all.

The paper is organized as follows. In Section 2, we present the basic setup of loan games and introduce the concept of trade structure to characterize the outcome of a loan game. A solution method involving the Myerson value, pairwise stability, and efficiency is introduced in Section 3, and it is applied to study the star network and the direct network in Section 4. Finally, Section 5 concludes.

2. Setup of the loan game

2.1. The market

There are n identical lenders and m identical borrowers in a loan market. The set of lenders is denoted by $L = \{l_1, l_2, \dots, l_n\}$, and the set of borrowers is denoted by $B = \{b_1, b_2, \dots, b_m\}$. Without loss of generality, we suppose $m \geq n \geq 1$. Let $N = L \cup B$ be the set of all players. Each nonempty subset of N is called a coalition. The cardinality of a coalition S is denoted by $|S|$.

Each lender is endowed with θ dollars, which is money from which she cannot benefit directly. Borrowers do not own any money initially, but they are willing to borrow ρ dollars each to invest in a project with rate of return r. Therefore, a borrower's benefit is $r\rho$ if she eventually possesses at least ρ dollars; otherwise, her benefit is zero. Suppose that

$$n\theta = m\rho. \tag{1}$$

Under this assumption, it is possible to reach the market-clearing status, since the money supplied equals the money demanded. From $n \leq m$ and (1), it follows that $\theta \geq \rho$.

Given two players $x, y \in N$, if x lends some money to y and demands some money (interest) in return, then we say there is a trade between x and y. Both players incur a fixed cost, $c \geq 0$, if there is a trade between them. This cost, called the link cost, includes expenses during negotiation and expenses from contracting and trading. In this paper, we only consider the case where c is very small.

A trade is said to be direct if it is between a lender and a borrower; a trade is indirect if it is between two lenders or two borrowers. We allow for indirect trades mainly to discuss the endogenous formation of financial intermediary. As an example, consider the loan market in Fig. 2 where $n = m = 3$. There are four trades in this market, where l_1b_1, l_1b_2, and l_3b_3 are direct trades, while l_2l_3 is an indirect trade. Note that l_3 can be regarded as an intermediary connecting l_2 and b_3.

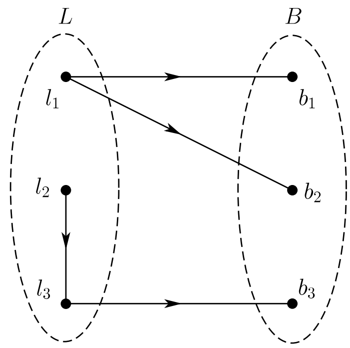


Fig. 2. A network among three lenders and three borrowers.

2.2. The network

Before formally describing the trades in the loan market, we first review some useful concepts and notations of the network formation game field. A trade between two players $x, y \in N$ can simply be denoted by an unordered pair xy, which is called a link. Let $g^N \equiv \{xy \mid x, y \in N,\ x \neq y\}$ be the set of all links. We call g^N the complete network. A nonempty set $g \subseteq g^N$ is called a network (or a graph) on N. In particular, $g = \emptyset$ is called the empty network. Let $G = \{g \mid g \subseteq g^N\}$ denote the set of all networks on N. Furthermore, given any coalition $M \subseteq N$, we define a network g on M to be a set of links that are all between players in M, and let G^M be the set of all networks on M.

Let $N(g) = \{y \in N \mid \exists x : xy \in g\}$ be the set of players that trade with some other players in g, and $N(x, g) = \{y \in N \mid xy \in g\}$ be the set of players that trade with x in g. The cardinality of these two sets is denoted by $n(g)$ and $n(x, g)$, respectively. Let $h(g) = |g|$ denote the number of links in g.

Given g, each $g' \subseteq g$ is called a subnetwork of g. If $g' \neq g$ is a subnetwork of g, then $g - g'$ is the network which results from eliminating all links in g' from g. If $g \cap g' = \emptyset$, then $g + g'$ is the network combining all links in g and g'. In particular, $g + xy$ (or $g - xy$) is the network resulting from adding a new link, xy, to g (or deleting an existing link, xy, from g).

Given a coalition M, two players $i, j \in M$, and a network $g \in G^M$, if there exists a list of links $ik_1, k_1k_2, \ldots, k_sj$ in g, then we say that i and j are connected in g. Let M/g denote the partition of M where each $S \in M/g$ is connected in g, while any two players from two different elements of M/g are not connected in g. Furthermore, given $g \in G^M$ and $S \subseteq M$, let $g(S) = \{ij \in g \mid i \in S, j \in S\}$ be the subnetwork of g that contains all links between players in S. Given $g' \subseteq g$, g' is called a component of g if there exists $S \in M/g$ such that $g' = g(S)$.

As an example, the network in Fig. 2 is $g = \{l_1b_1, l_1b_2, l_2l_3, l_3b_3\}$, where $n(g) = 6$ and $h(g) = 4$. This network contains two components: $g^1 = \{l_1b_1, l_1b_2\}$ and $g^2 = \{l_2l_3, l_3b_3\}$.

2.3. Loan game

A network alone cannot formulate the trades in the market. In fact, a network $g \in G^N$ only describes which trades take place, but it does not say anything about the principal and interest of each trade $xy \in g$. For a complete description of the trades, we introduce the following two functions defined on the network.

Given a network g, let $s_g : g \to \mathbb{R}$ and $t_g : g \to \mathbb{R}$ denote the trade function and the interest function, respectively. For all $xy \in g$, $s_g(x, y)$ is the amount of money that player x lends to player y, and $t_g(x, y)$ is the amount of interest y pays to x. In particular, let $s_g(x, y) = t_g(x, y) = 0$ if there is no trade between x and y. Note that the signs of $s_g(x, y)$ and $t_g(x, y)$ represent the direction of the trade. That is, $s_g(x, y) < 0$, $t_g(x, y) < 0$ imply that x pays $-t_g(x, y)$ dollars interest to y for borrowing $-s_g(x, y)$ dollars from y. If $s_g(x, y) \neq 0$, let $\tau_g(x, y) = \left|\frac{t_g(x,y)}{s_g(x,y)}\right|$ denote the interest rate of this trade. It follows that, for each lender $l_i \in L$:

$$\sum_{k \in N(l_i, g)} s_g(l_i, k) \leq \theta. \tag{2}$$

Given any coalition M, we call $E_g = (g, s_g, t_g)$ a trade structure on M, which is a combination of a network $g \in G^M$, as well as a trade function s_g and an interest function t_g defined on g. Note that a trade structure on N is a complete description about the trade outcomes of a loan market, and thus is what we want to determine.

If there exists a trade structure $E_g = (g, s_g, t_g)$ on N such that for each $l_i \in L$, $\sum_{k \in N(l_i, g)} s_g(l_i, k) = \theta$, while for each $b_j \in B$, $\sum_{k \in N(b_j, g)} s_g(k, b_j) = \rho$, then we say that E_g is a complete trade (on N), and g is called a sufficient network (on N). In other words, if g is a sufficient network, then we can find a trade structure defined on g such that the market clears.

Let $\overline{G}$ be the set of all sufficient networks on N. Note that $\overline{G} \neq \emptyset$, since it is obvious that $g^N \in \overline{G}$. Furthermore, the next lemma establishes that the minimal number of links of all sufficient networks on N is $n + m - \langle n, m \rangle$, where $\langle n, m \rangle$ is the greatest common divisor of n and m.

Lemma 1. $\min_{g \in \overline{G}} h(g) = n + m - \langle n, m \rangle$.

Proof. See Appendix. □

Given a trade structure E_g on N, the payoff of each lender $l_i \in L$ is her net income minus link cost:

$$u_{l_i}(E_g) = \sum_{k \in N(l_i, g)} t_g(l_i, k) - cn(l_i, g). \tag{3}$$

The payoff of each borrower $b_j \in B$ is his benefit minus total cost (including net payment and link cost):

$$u_{b_j}(E_g) = R_j(E_g) - \sum_{k \in N(b_j, g)} t_g(k, b_j) - cn(b_j, g), \tag{4}$$

where $R_j(E_g)$ is b_j's benefit from borrowing $\sum_{k \in N(b_j, g)} s_g(k, b_j)$ dollars:

$$R_j(E_g) = \begin{cases} r\rho, & \text{if } \displaystyle\sum_{k \in N(b_j, g)} s_g(k, b_j) \geq \rho \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Finally, let $u(E_g) \equiv \big(u_{l_1}(E_g), \ldots, u_{l_n}(E_g), u_{b_1}(E_g), \ldots, u_{b_m}(E_g)\big)$ denote the vector of payoffs under trade structure E_g.

A loan game defined above can be denoted by (N, u), where u is defined by (3), (4), and (5). Assume that each player $x \in N$ is rational so that his goal is to maximize his payoff u_x, subject to (2) if $x \in L$. To solve a loan game (N, u) is to predict the trade structure E_g on N, and hence the payoff vector $u(E_g)$.

3. A solution of loan games

In this section, we plan to solve a loan game in two steps. First, we determine the trades on a given network according to the Myerson value. Then, we consider which network(s) will form under the Myerson value.

3.1. The Myerson value and the normal trade rule

Given a loan game (N, u) and a network $g \in G$, we can derive an induced TU coalitional form game (N, v^g), where v^g is the

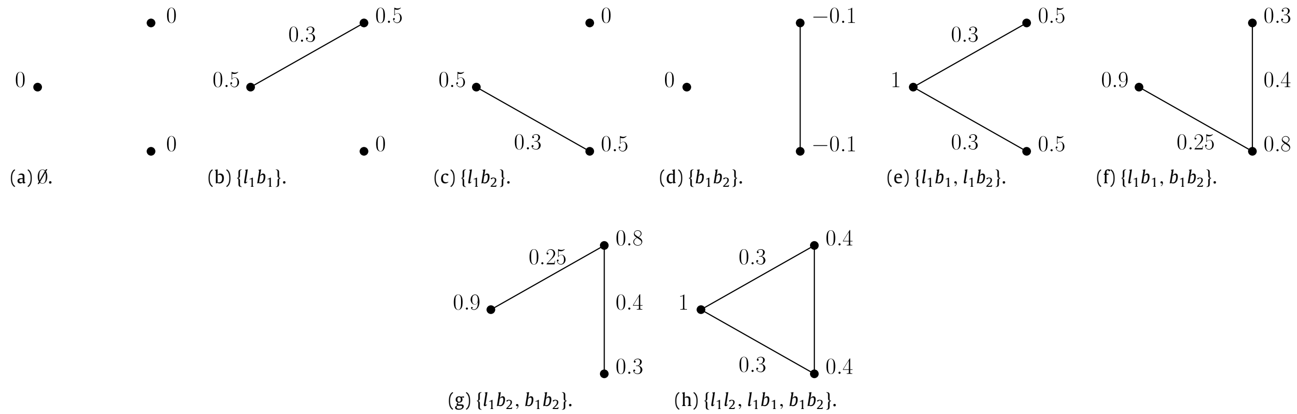


Fig. 3. The Myerson value of network g in a three-player loan game.

characteristic function that assigns to each coalition M a value $v^g(M)$. $v^g(M)$ is the maximum total payoffs of M for all possible trade structures defined on g, given that all players in M only trade within M. That is, $v^g(M) = \sum_{S\in M/g}\left[\max_{E_{g(S)}}\sum_{i\in S}u_i\left(E_{g(S)}\right)\right]$. In particular, $v^g(\emptyset)=0$.

The Shapley value of (N, v^g), denoted by $w^g = \left(w^g_{l_1},\ldots,w^g_{l_n},\right.$ $\left.w^g_{b_1},\ldots,w^g_{b_m}\right)$, is usually called the Myerson value of g, where

$$w^g_x = \sum_{S\subseteq N\setminus\{x\}}\frac{|S|!(n+m-|S|-1)!}{(n+m)!}\left[v^g(S\cup\{x\})-v^g(S)\right],\quad \forall x\in N. \tag{6}$$

As an example, consider a loan game where $N = \{l_1, b_1, b_2\}$, $\theta = 4$, $\rho = 2$, $r = 0.6$, and $c = 0.1$. Fig. 3 illustrates the Myerson value of all possible networks in this game. The points on the left, upper-right, and lower-right in each graph represent players l_1, b_1, and b_2, respectively. The number beside each point x is the Myerson value w^g_x of player x. If $s_g(x,y)\neq 0$, then the number beside the link xy is the interest rate $\tau_g(x,y)$ of the trade.[2]

In this paper, we use the Myerson value to predict the payoff allocations under a given network. Therefore, we will explain in the remainder of this subsection why the Myerson value is a feasible and reasonable payoff vector.

Define a trade rule, denoted by E, to be a mapping that assigns to each network g a trade function s_g and an interest function t_g on g. Hence, given any trade rule E, a unique trade structure E_g and a corresponding payoff vector $u(E_g)$ are associated with each network g. We can completely describe the trades in any given network by the trade rule.

A trade rule E^* is said to be normal if it satisfies the following two properties.

(A1) Component efficiency: for any $g\in G$ and any trade rule E', there does not exist $S\in N/g$ such that $\sum_{x\in S}u_x(E'_g) > \sum_{x\in S}u_x(E^*_g)$.
(A2) Balanced contribution: for any $g\in G$ and any $xy\in g$:

$$u_x(E^*_g)-u_x(E^*_{g-xy}) = u_y(E^*_g)-u_y(E^*_{g-xy}). \tag{7}$$

Intuitively, the component efficiency property implies that the players in each component coordinate their trade so as to maximize their total surplus, and the balanced contribution property suggests that this total surplus is distributed in a fair way,[3] since the marginal contributions of each link to the two players forming the link are equal. Therefore, the normal trade rule can be regarded as a reasonable trading pattern, since it is both efficient and fair.

Following Myerson (1977) and Jackson and Wolinsky (1996), we show in the theorem below that the players allocate their payoffs according to the Myerson value if they follow the normal trade rule. In addition, this theorem also shows that the normal trade rule is well defined.

Theorem 1. *Given any network $g\in G$, the Myerson value of g is the unique payoff vector that can be derived from the normal trade rule. That is, $u(E^*_g)=w^g$.*

Proof. We can derive a function $Y : G \to \mathbb{R}^{n+m}$ so that for any $g\in G$, $Y(g) = u(E^*_g)$. Hence, $Y(g)$ is the payoff allocation associated with E^*. Note that Y also satisfies both the component efficiency property and the balanced contribution property. That is, for any $g\in G$ and any other function Y', there does not exist $S\in N/g$ such that $\sum_{x\in S}Y'_x(g) > \sum_{x\in S}Y_x(g)$; for any $g\in G$ and any $xy\in g$, $Y_x(g)-Y_x(g-xy) = Y_y(g)-Y_y(g-xy)$. Jackson and Wolinsky (1996, Theorem 4)[4] state that the unique allocation rule Y that satisfies both the component efficiency property and the balanced contribution property is the Myerson value. It follows that $u(E^*_g)=Y(g)=w^g$. □

Besides the Myerson value, there are some other important single-valued solutions in network game literature. For instance, if we replace the fairness assumption (A2) by an alternative fairness assumption introduced by Herings et al. (2008, 2010), namely component fairness, then we might get an analogue of Theorem 1 that the players would distribute the surplus according to the average tree solution. In this paper, we will focus on the Myerson value, and leave the discussion of other solutions for future works.

3.2. The stability and efficiency of networks

The second step to solve a loan game is to predict which network(s) would prevail under the normal trade rule E^*. Here we apply two criteria that are commonly used in the literature of network formation games: pairwise stability and efficiency.[5]

2 The interest rate can be derived according to the normal trade rule defined below. In this example, $\tau_g(x,y)$ is not far away from, but not necessarily equal to, $r/2$. The determination of interest rate in any loan game is an interesting topic, but is beyond the scope of this paper.

3 For example, when the players have equal bargaining power.

4 This is an extension of the main theorem of Myerson (1977). In Jackson and Wolinsky (1996), the allocation rule Y is actually defined on a network g and a value function v, so that $Y_i(g,v)$ is "the payoff to player i from graph g under the value function v". However, in this paper we only consider the value function v^g defined earlier in this subsection. For notational simplicity, we write $Y(g)$ rather than $Y(g,v^g)$. In addition, Jackson and Wolinsky (1996, Theorem 4) require that the value function is component additive (the value of a network is equal to the sum of all its components' value), and so is v^g.

5 For example, see Jackson and Wolinsky (1996).

A network $g \in G$ is said to be internally stable under a trade rule E, if, for all $xy \in g$, $u_x(E_g) \geq u_x(E_{g-xy})$ and $u_y(E_g) \geq u_y(E_{g-xy})$; it is said to be externally stable under E, if, for all $xy \notin g$, $u_y(E_{g+xy}) < u_y(E_g)$ whenever $u_x(E_{g+xy}) > u_x(E_g)$. Furthermore, we say that g is pairwise stable under E, if it is both internally stable and externally stable under E.

Intuitively, a link can only be maintained under the consent of both of the players who form it, while it can unilaterally be deleted by only one of these two players. Hence, an internally stable network is one in which no individual player would like to unilaterally delete any link involving herself; an externally stable network is one in which no pair of unlinked players are willing to form a link between themselves. A pairwise stable network is stable in that there exist no individual players, or pair of players, that are willing and able to replace this network with a new one, under a certain rule of trade.

Jackson (2003) shows that in each network game, if the payoffs are allocated according to the Myerson value, there exists at least one pairwise stable network. Therefore, for each loan game (N, u), there exists a network $g \in G$ that is pairwise stable under the normal trade rule E^*. For example, consider the loan game in Fig. 3. In this game, it is easy to verify that $g = \{l_1b_1, l_1b_2\}$ is pairwise stable under E^*.

Besides pairwise stability, another possible criterion is whether a network is efficient[6]—that is, whether the total payoff of all players has been maximized. Formally, a network $g \in G$ is said to be efficient under a trade rule E if there does not exist another network $h \in G$ such that $\sum_{x\in N} u_x(E_g) < \sum_{x\in N} u_x(E_h)$. In other words, if g is efficient under E, then the players in N will not unanimously agree upon a plan to replace g with a new network g'. Therefore, an efficient network can be regarded as being "globally" stable, whereas a pairwise stable network is "locally" stable. Again, consider the game in Fig. 3 as an example. In this game, $\{l_1b_1, l_1b_2\}$, $\{l_1b_1, b_1b_2\}$, and $\{l_1b_2, b_1b_2\}$ are all efficient under E^*.

It is natural to investigate the compatibility of pairwise stability and efficiency under E^*. That is, we are interested in whether, for any loan game, there always exists a network that is both pairwise stable and efficient. Unfortunately, Jackson and Wolinsky (1996) show that under a given allocation rule, there need not exist a network that is both pairwise stable and efficient. In addition, in the following section, we present some examples of loan games that illustrate the incompatibility of pairwise stability and efficiency under the normal trade rule.

4. Star networks and direct networks

In this section, we apply the solution method described above to analyze the loan games. In particular, we are interested in star networks and direct networks. In a star network, there are some central players acting as intermediaries between other lenders and borrowers; in a direct network, there are only direct trades. The formal definitions of these network structures are given in the subsection below.

4.1. Definition

We start with a simple case. A loan game is called a coprime game if n and m are coprime; that is, $\langle n, m\rangle = 1$. The following lemma is a useful property concerning coprime games.

Lemma 2. *If $\langle n, m\rangle = 1$ and g is a sufficient network on N, then g contains only one component.*

Proof. It can be shown that there does not exist positive integers $p < n$ and $q < m$, such that $pm = qn$. Assume on the contrary that such p, q do exist. Then $p = p\langle n, m\rangle = \langle pn, pm\rangle = \langle pn, qn\rangle = n\langle p, q\rangle \geq n$, which contradicts the assumption $p < n$.

It follows from (1) that if $pm \neq qn$, then $p\theta \neq q\rho$. Hence, there does not exist any complete trade on any coalition $M \subsetneq N$. In other words, the only player set on which there is a sufficient network is N. $\square$

Now we formally define star networks and direct networks of a coprime game. Note that they are both sufficient networks, and hence have only one component. In a star network, one player $d \in N$ is chosen as the central player, to whom each lender $l_i \in L \setminus \{d\}$ lends all his endowed θ dollars, and from whom each borrower $b_j \in B\setminus\{d\}$ borrows ρ dollars. Different star networks on a coprime game differ only in the choice of the central player.

The construction of a direct network on a coprime game is more complicated and follows the algorithm below. First, let $m = k_1n + p_1$, where $0 \leq p_1 < n$. Suppose each of n lenders chooses k_1 borrowers (no borrower is chosen by more than one lender) and lends ρ dollars to each of the k_1 borrowers. Note that $p_1 > 0$, since otherwise we have $m = k_1n$ and thus $\langle n, m\rangle = n \geq 2$, which contradicts $\langle n, m\rangle = 1$. There still remain p_1 borrowers who have not borrowed enough money. Next, let $n = k_2p_1 + p_2, 0 \leq p_2 < p_1$. Suppose each of the remaining p_1 borrowers chooses k_2 lenders and borrows $\theta - k_1\rho$ dollars from each of the k_2 lenders. If $p_2 = 0$, then we have reached a complete trade and the construction ends; otherwise, $p_2 > 0$ and there remains p_2 lenders who have not lent all endowed money. The algorithm proceeds like this until $p_{q-1} = k_{q+1}p_q$, and a sufficient network is constructed. A direct network is formally defined as a network that can be constructed by the above algorithm.

As an example, consider a coprime game where $n = 3, m = 5$, $\theta = 10$, and $\rho = 6$. A direct network and a star network are both illustrated in Fig. 4. The number beside each link xy is the amount of loan $s_g(x, y)$ for the trade.

Finally, we consider more general loan games. Suppose $\langle n, m\rangle = d$. Then N can be divided into d groups, each containing $n' = n/d$ lenders and $m' = m/d$ borrowers, where $\langle n', m'\rangle = 1$. A network $g \in G$ is called a direct network (or star network) on N if it consists of d components, each of which is a direct network (or star network) on some $S \in N/g$. According to this definition, each component in a direct network (star network) of a general game is a direct network (star network) of a coprime game. In particular, if $m = kn$, then each component of a direct network is a "one-to-k" direct network.

4.2. Properties

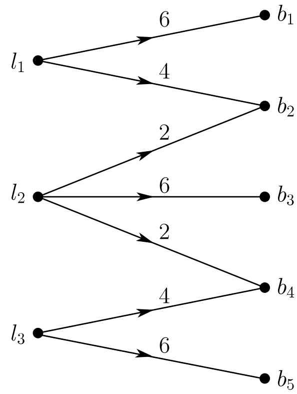
It is obvious that direct networks and star networks are all sufficient networks on N. Furthermore, combining Lemma 3 with Lemma 1, we shall see that among all sufficient networks, direct networks and star networks have the minimal number of links.

Lemma 3. *If g is a direct network or a star network, then $h(g) = n + m - \langle n, m\rangle$.*

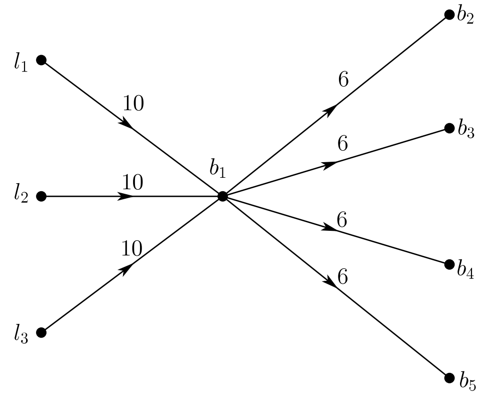
Proof. Suppose g' is a component of g. If g is a star network, then it is straightforward that $h(g') = n' + m' - 1$ since each of the $n' + m' - 1$ non-center players forms a single link with the central player, where $n' = n/\langle n, m\rangle$, $m' = m/\langle n, m\rangle$. Now suppose g is a direct network, and

$$\left.\begin{array}{ll} m' = k_1n' + p_1, & 0 < p_1 < n' \\ n' = k_2p_1 + p_2, & 0 < p_2 < p_1 \\ p_1 = k_3p_2 + p_3, & 0 < p_3 < p_2 \\ \vdots & \\ p_{q-2} = k_qp_{q-1} + p_q, & 0 < p_q < p_{q-1} \\ p_{q-1} = k_{q+1}p_q & \end{array}\right\}. \tag{8}$$

[6] Note that the players have transferable utilities. Therefore, the efficient concept used in this paper is equivalent to some other concepts in the literature, such as Pareto efficient or strong efficient.



(a) A direct network.



(b) A star network.

Fig. 4. A coprime game ($n = 3, m = 5$).

By the algorithm constructing g', we have $h(g') = k_1 n' + k_2 p_1 + \cdots + k_{q+1} p_q$. Summing up all equations in (8) leads to $h(g') = n' + m' - p_q$. On the other hand, (8) also implies $1 = \langle m', n' \rangle = \langle n', p_1 \rangle = \cdots = \langle p_{q-1}, p_q \rangle = p_q$. Hence, $h(g) = n' + m' - 1$.

Hence, the number of links of each component of g is $n'+m'-1$. Therefore, $h(g) = \langle n, m \rangle (n' + m' - 1) = n + m - \langle n, m \rangle$. $\square$

The next two propositions are the main conclusions of this subsection. They suggest that if link cost is small enough, each star network or direct network is both efficient and internally stable under the normal trade rule.

Proposition 1. *If c is sufficiently small, then each star network or direct network is efficient under E^*.*

Proof. Given any direct network or star network $g \in G$, suppose on the contrary that g is not efficient under E^*. Then there exists $g' \in G$ such that $\sum_{x \in N} w_x^g < \sum_{x \in N} w_x^{g'}$. Since g is a sufficient network, we have $\sum_{x \in N} w_x^g = mr\rho - 2ch(g)$. If g' is also a sufficient network, then $\sum_{x \in N} w_x^{g'} = mr\rho - 2ch(g')$, where $h(g') \geq h(g)$ due to Lemmas 1 and 3. However, this contradicts $\sum_{x \in N} w_x^g < \sum_{x \in N} w_x^{g'}$. Therefore, g' is not a sufficient network, and $\sum_{x \in N} w_x^{g'} = \sum_{x \in B} R_x(E_{g'}^*) - 2ch(g')$, where $\sum_{x \in B} R_x(E_{g'}^*) < mr\rho$. Since c is sufficiently small, we have $\sum_{x \in N} w_x^g > \sum_{x \in N} w_x^{g'}$, which again contradicts $\sum_{x \in N} w_x^g < \sum_{x \in N} w_x^{g'}$. Thus, g is efficient under E^*. $\square$

Proposition 2. *If c is sufficiently small, then each star network or direct network is internally stable under E^*.*

Proof. Suppose $g \in G$ is a star network and x is a central player. Since c is sufficiently small, it is obvious that for each non-center player y such that $xy \in g$, $w_y^g > 0 = w_y^{g-xy}$. It follows from the balanced contribution property that $w_x^g > w_x^{g-xy}$. Thus, g is internally stable under E^*.

Suppose $g \in G$ is a direct network, and $xy \in g$. According to (6),

$$w_x^g - w_x^{g-xy} = \sum_{S \subseteq N \setminus \{x\}} \frac{|S|!(n+m-|S|-1)!}{(n+m)!} \cdot \Big[V^g(S \cup \{x\}) - V^g(S) - V^{g-xy}(S \cup \{x\}) + V^{g-xy}(S) \Big]. \quad (9)$$

For all $S \subseteq N \setminus \{x\}$, $V^g(S) = V^{g-xy}(S)$. Also note that g is a sufficient network on N, but $g - xy$ is not. Thus, we have $V^g(S \cup \{x\}) \geq V^{g-xy}(S \cup \{x\}) - c$ for all $S \subseteq N \setminus \{x\}$, while $V^g(N) > V^{g-xy}(N)$. Since c is sufficiently small, due to (9) and the balanced contribution property, we have $w_y^g - w_y^{g-xy} = w_x^g - w_x^{g-xy} \geq 0$. Thus, g is internally stable under E^*. $\square$

There may be other network structures that also satisfy internal stability and efficiency under normal trade rule. As an example, suppose $n = 1, m = 3, \theta = 6, \rho = 2, r = 0.6, c = 0.1$, then it is easy to verify that the network $g' = \{l_1 b_1, l_1 b_2, b_2 b_3\}$ is efficient and internally stable under E^*. Note that g' can be regarded as a combined network of star and direct networks: l_1 lends some money directly to b_1, while also lends some money to b_3 through intermediary b_2.

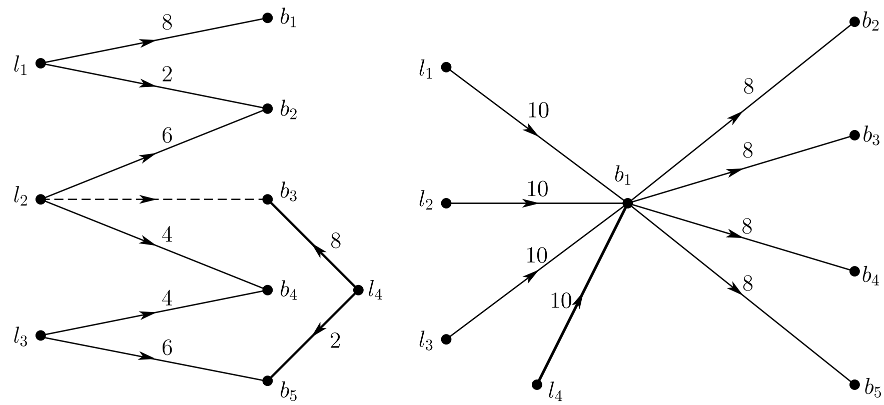
Also, notice that neither a star network nor a direct network is necessarily externally stable under E^*. For example, consider the following coprime game: $n = 2, m = 3, \theta = 6, \rho = 4$, and $r = 1$. If $c = 0$, then it is easy to verify that the direct network $g = \{l_1 b_1, l_1 b_2, l_2 b_2, l_2 b_3\}$ is not externally stable under E^*, since $w_{l_1}^{g+l_1 b_3} > w_{l_1}^g$, $w_{b_3}^{g+l_1 b_3} > w_{b_3}^g$. If $c = 0.5$, then the star network $g' = \{l_1 b_1, l_2 b_1, b_1 b_2, b_1 b_3\}$ is not externally stable under E^*, since $w_{l_1}^{g'+l_1 b_2} > w_{l_1}^{g'}$, $w_{b_2}^{g'+l_1 b_2} > w_{b_2}^{g'}$.

4.3. Discussion

Ultimately, direct networks and star networks represent two different types of loan patterns. A direct network usually emerges in a social network among closely related people such as relatives or friends, while a star network built around a financial intermediary could gather a group of strangers. Propositions 1 and 2 show us some advantages of direct networks and star networks: they are both internally stable and efficient. These might explain why direct networks and star networks are commonly observed in reality. In particular, they provide an explanation of the origin of financial intermediaries such as banks.

However, besides star networks and direct networks, there may exist other network structures that are internally stable and efficient (with small link cost). In addition, both direct networks and star networks may be externally unstable. Intuitively, this suggests that a pure star network or a pure direct network might not meet all the needs of some lenders or borrowers, since some of them may desire to form additional links. This could explain why these two trade patterns sometimes coexist in a loan market.

Further comparison between direct networks and star networks suggests that they both have their merits and drawbacks. If there is a relatively large group of persons, then loans must take place between many pairs of strangers. The star network is more appropriate in this situation, since it can reduce the cost of searching and matching and can reduce the risk of trades by building a



(a) A direct network.

(b) A star network.

Fig. 5. A coprime game ($n = 4, m = 5$).

formal supervision system through the intermediary. In contrast, direct networks are more appropriate if the trades are among a small group of people who know each other, since there is no searching and matching cost and the extra cost incurred from the supervision system can be saved.

It is worth noting that there is another advantage of star networks over direct networks. If a sudden change occurs in the game, such as adding or deleting one player, then a star network is typically more robust against this change than a direct network. This is because a star network needs less adjustment against the change than a direct network does. As an example, consider again the loan game in Fig. 4 and add to it a new lender l_4. In the new loan game, we suppose $n = 4$, $m = 5$, $\theta = 10$, and $\rho = 8$. The corresponding direct network and star network of this game are illustrated in Fig. 5. Compared to the original game in Fig. 4, only one new link is added to the star network, while three links have been changed (two links added and one link deleted) in the direct network. This phenomenon, that a direct network would suffer a larger scale of structural adjustment in response to an additional player, can be generally observed if the game remains coprime after the change.[7]

5. Conclusion

In this paper, we apply the framework of network formation games to analyze monetary loan networks. We assume that in a given loan network, players trade and split the surplus according to the Myerson value. We also assume that networks that are pairwise stable and/or efficient are more likely to arise, although a tension may exist between pairwise stability and efficiency. We focus especially on two elementary network structures: direct networks and star networks. If the link cost is sufficiently small, these two networks are both efficient and internally stable under the Myerson value, but they are not necessarily externally stable. These conclusions help us to understand some phenomena found in reality. For example, financial intermediaries such as banks are common in the loan market, since star networks have the advantages of being internally stable, efficient, and robust against small perturbations in the number of players. In addition, if direct networks and star networks are both externally unstable, there is a trend towards the combination of these two networks and hence mixed networks appear.

Some extensions may be worth exploring in future research. First, the model setup can be generalized. We may consider loan games with asymmetric lenders or borrowers, and the properties of more complex networks than direct network and star network. Second, some theoretic problems still remain unanswered. For example, we may provide some conditions to characterize which networks are externally stable, and how the interest rate varies in different network structures. Third, the network formation game approach in this paper is only applicable in a loan market among a small group of players. The analysis of loans among a large group of players involves more complicated issues that cannot be simply modeled as rational play, requiring other analysis tools such as complex networks. Finally, the model in this paper ignores the problem of asymmetric information that usually plays an important role in loan markets. By introducing asymmetric information, we would be able to investigate the impact of the risk of default on the loan networks and examine the fragility of a loan network when facing a risk such as a financial crisis.

Acknowledgments

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Appendix

Proof of Lemma 1. Within a finite number of steps, each sufficient network can be transformed into a network whose number of links is $n + m - \langle n, m\rangle$, and each step of the transformation will not increase the number of links of the network.

Given a complete trade $E_g = (g, s_g, t_g)$ on N, let A_g be a square matrix of order $n + m$ whose (i, j) entry $A_g(i, j) = s_g(x_i, y_j)$, where $i, j \in \{1, \dots, n + m\}$; $x_i = l_i$ if $i \in \{1, \dots, n\}$ and $x_i = b_i$ if $i \in \{n + 1, \dots, n + m\}$; and $y_j = l_j$ if $j \in \{1, \dots, n\}$ and $y_j = b_j$ if $j \in \{n+1, \dots, n+m\}$. Note that (a) A_g is a skew-symmetric matrix, since $s_g(x, y) = -s_g(y, x)$, and therefore $s_g(x, x) = 0$, $\forall x, y \in N$, and (b) the sum of all entries in row i is equal to θ if $i = 1, \dots, n$, or $-\rho$ if $i = n + 1, \dots, n + m$; the sum of all entries in column j is equal to $-\theta$ if $j = 1, \dots, n$, or ρ if $j = n + 1, \dots, n + m$. Let $\mathcal{A}_{n,m}$ denote the set of all square matrices of order $n + m$ that satisfy (a) and (b).

For each complete trade E_g, there exists a corresponding $A_g \in \mathcal{A}_{n,m}$. Conversely, for any $A \in \mathcal{A}_{n,m}$, there exists a sufficient network g, where $x_i y_j \in g$ if and only if $A(i, j) \neq 0$. Therefore, we can translate a transformation on $\overline{G}$ into a transformation on $\mathcal{A}_{n,m}$. Let $I(A)$ denote the number of non-zero entries in matrix A; it is double of the number of links of the corresponding network.

Now we introduce three types of transformations on $\mathcal{A}_{n,m}$. For any $A \in \mathcal{A}_{n,m}$:

[7] The case that the new game is no longer a coprime game is very complex and is not discussed in this paper.

(I) Fix i, j, where $i \neq j$, $i, j \in \{1, \ldots, n\}$ or $i, j \in \{n+1, \ldots, n+m\}$. Let A' be the matrix transformed from A by swapping row i with row j and, at the same time, swapping column i with column j.

(II) Fix $a \neq 0$ and $i_1, j_1, k_1 \in \{1, \ldots, n+m\}$ where $i_1 \neq j_1$, $j_1 \neq k_1$, and $k_1 \neq i_1$. Let A' be a square matrix of order $n+m$ transformed from A such that

$$A'(i_1, j_1) = A(i_1, j_1) - a, \qquad A'(j_1, i_1) = A(j_1, i_1) + a,$$
$$A'(i_1, k_1) = A(i_1, k_1) + a, \qquad A'(k_1, i_1) = A(k_1, i_1) - a,$$
$$A'(j_1, k_1) = A(j_1, k_1) - a, \qquad A'(k_1, j_1) = A(j_2, i_2) + a,$$

while for all other (i, j), $A'(i, j) = A(i, j)$.

(III) Fix $a \neq 0$ and $i_1, i_2 \in \{1, \ldots, n\}$, $j_1, j_2 \in \{n+1, \ldots, n+m\}$, where $i_1 \neq i_2$ and $j_1 \neq j_2$. Let A' be a square matrix of order $n+m$ transformed from A such that

$$A'(i_1, j_1) = A(i_1, j_1) + a, \qquad A'(j_1, i_1) = A(j_1, i_1) - a,$$
$$A'(i_1, j_2) = A(i_1, j_2) - a, \qquad A'(j_2, i_1) = A(j_2, i_1) + a,$$
$$A'(i_2, j_1) = A(i_1, j_2) - a, \qquad A'(j_1, i_2) = A(j_2, i_1) + a,$$
$$A'(i_2, j_2) = A(i_2, j_2) + a, \qquad A'(j_2, i_2) = A(j_2, i_2) - a,$$

while for all other (i, j), $A'(i, j) = A(i, j)$.

Let $T^1(i, j)$, $T^2(i_1, j_1, k_1; a)$, and $T^3(i_1, i_2, j_1, j_2; a)$ denote these three transformations, respectively. It is easy to see that all these transformations are mappings of $\mathcal{A}_{n,m}$ onto itself.

Given any $A_1 \in \mathcal{A}_{n,m}$, we can use finite steps of T^1 and T^2 to transform A_1 into A_2, where $I(A_2) \leq I(A_1)$ and $A_2(i, j) = 0$ if $1 \leq i, j \leq n$, or $n+1 \leq i, j \leq n+m$. For instance, if $A_1(1, 2) \neq 0$ and there exists $A(1, j) = 0$, then we can apply $T^1(2, j)$ to A_1 to get A' so that $A'(1, 2) = 0$. Suppose $A_1(1, j) \neq 0$, $2 \leq j \leq n+m$, and $A_1(1, k_1) = \min_{2 \leq j \leq n+m} A_1(1, j)$ and $A_1(1, k_2) = \max_{2 \leq j \leq n+m} A_1(1, j)$. We can first apply $T^1(2, k_1)$ to A_1 to get A' so that $A'(1, 2) = a^1$, then apply $T^2(1, 2, k_2; A_1(1, k_1))$ to A' to get A'' so that $A''(1, 2) = 0$. Repeating this algorithm, we can transform $A_1(1, 2), \ldots, A_1(1, n), A_1(2, 3), \ldots, A_1(2, n), \ldots, A_1(n, n)$ into zero in turn. Similarly, $A_1(i, j)$, $n+1 \leq i, j \leq n+m$ can also be transformed into zero. Note that each step of the transformation will not affect the entries of the matrix that have been transformed into zero in previous steps, nor will it increase the number of non-zero entries. This sequence of transformations will finally lead us to a matrix A_2 required above. Intuitively, A_2 corresponds to a network that only contains direct trades.

Next, we can use finite steps of T^1 and T^3 to transform A_2 into A_3, where $I(A_3) = n + m - \langle n, m \rangle \leq I(A_2)$. We use a T^1 (if necessary) on A_2 to get A' so that $A'(1, n+1) \geq A'(k, n+1)$, $2 \leq k \leq n$, then apply $T^3(1, k, n+1, j_2, A'_2(k, n+1))$ to A' to get A'' so that $A''(1, n+1) = \rho$, $A''(k, n+1) = 0$, $2 \leq k \leq n$, where $j_2 > n+1$. Note that this transformation is feasible since $\theta \geq \rho$ and will not increase the number of non-zero entries. Repeating this process, we transform the submatrix $\{1 \leq i \leq n, n+1 \leq j \leq n+k_1 n\}$ into k_1 diagonal matrices of order n whose main diagonal entries are ρ, where $m = k_1 n + p_1$, $0 \leq p_1 < n$; we also transform the submatrix $\{1 \leq i \leq k_2 p, n+k_1 n+1 \leq j \leq n+m\}$ into k_2 diagonal matrices of order p_1 whose main diagonal entries are $\theta - k_1\rho$, where $n = k_2 p_1 + p_2$, $0 \leq p_2 < p_1$, and so on. This algorithm will finally end up with k_{q-1} diagonal matrices of order p_q and reach A_3, where $p_{q-1} = k_{q+1} p_q$. It is easy to see that $I(A_2) \geq I(A_3) = 2(k_1 n + k_2 p_1 + \cdots + k_{q+1} p_q) = 2(n + m - p_q) = 2(n + m - \langle n, m \rangle)$.

Thus, we can transform A_1 into A_3, or equivalently transform any sufficient network into another sufficient network whose number of links is $n + m - \langle n, m \rangle$. The number of links does not increase during the transformation. This establishes that there does not exist a sufficient network whose number of links is smaller than $n + m - \langle n, m \rangle$. $\square$

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