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Core–periphery and nested networks emerging from a simple model of network formation[☆]

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ABSTRACT

This paper studies a simple model inspired by the seminal connections model of Jackson and Wolinsky (1996). Node-players can invest in links to connect with other nodes forming networks, but only direct links transmit value. Nodes may have different values and different decreasing returns abilities/technologies to form/strengthen links. The strength of a link depends on the amounts invested in it by the two nodes that it connects and is determined by a separable function of the investments of the players which form the link. Nash-stable and efficient networks are characterized, and a variety of their architectures identified for different configurations of values and technologies.

1. Introduction

The point of this paper is to study a simple model of network formation inspired by the seminal connections models of Jackson and Wolinsky (1996) and Bala and Goyal (2000). As in the seminal models, node-players can invest in links to connect with other nodes forming networks. Unlike in the seminal models: (i) Nodes may have different values and different abilities/technologies with decreasing returns to form links; (ii) the strength of a link depends on the amounts invested in it by the two nodes that it connects, and is determined by a separable function of these amounts; and (iii) only direct links transmit value.

The paper by Bloch and Dutta (2009) is the first that introduces endogenous link strength in a connections model by replacing Jackson and Wolinsky's discrete technology by a separable technology, as we do here. However, they assume that all players have the same value and the same individual increasing returns technology, while we assume that players may have different values and different individual *decreasing* returns technologies. Unlike in the models of Jackson and Wolinsky (1996), Bala and Goyal (2000), Bloch and Dutta (2009), and all the extensions considered so far by us,[1] in the model studied here only direct links transmit value, as in Ballester et al. (2006) and all papers in its wake, but their quadratic payoff function is completely different from the one considered here. Thus, our model can be seen as an intermediate model between the connections model of Jackson and Wolinsky (1996), the variation of Bloch and Dutta (2009) and the model of Ballester et al. (2006) and those in its wake, but clearly distinct from them all, as well as the results obtained.

Apart from those few papers, clearly at the root of this work, among the large piece of literature on network formation, we briefly comment some works with weighted link formation.[2] Cabrales et al. (2011) and Galeotti and Merlino (2014) feature network formation models with weighted links, although there links' investments result from socialization and cannot be directed towards specific nodes. Some more recent work with weighted link formation deserves some comment. In Bauman (2021) players' investments in links are strategic complements, instead of substitutes as in our model, and players can also invest in self-links. In Kinateder and Merlino (2022) players' investments are substitutes as here but all have the same linking technology, while players' heterogeneity derives from a local public good game that is simultaneously played on the network, which imply that linking to players who provide more public good is more profitable.

In the model studied here, briefly outlined above, we characterize Nash networks and efficient networks, and study a variety of interesting

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[1] Olaizola and Valenciano (2020, 2021, 2023).
[2] Excellent surveys on economic networks are Goyal (2007), Jackson (2008), Vega-Redondo (2007). See also Bramoullé et al. (2015).

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architectures (core–periphery, nested split graph networks and bipartite networks) that emerge as the only stable or efficient ones for different configurations of the many degrees of freedom of the model (values and technologies of the nodes). Core–periphery networks, consisting of a core of nodes which are highly interconnected and a periphery of nodes connected only with some nodes in the core but not with each other, appear in different contexts, such as sociology, international relations and economics.[3] Nested split graph networks are core–periphery networks of a particular type which also appear in different economic contexts. These are hierarchical networks where nodes are ranked by their number of neighbors. Bipartite networks appear in ecology and economics, e.g. world trade analysis. The fact that these three architectures emerge as stable or efficient networks from this simple model of networks deserves to be emphasized.

The paper is organized as follows. Section 2 introduces basic notation and terminology. Section 3 introduces the model. Section 4 establishes characterizing conditions of Nash networks and shows a variety of stable architectures that emerge for different configurations of the many degrees of freedom of the model. Section 5 addresses the question of efficiency and shows a variety of efficient architectures for different configurations of the many degrees of freedom of the model. Section 6 studies a variation of the model introducing budget constraints. Section 7 concludes.

2. Preliminaries

A *directed weighted network* is a pair (N, g) where $N = \{1, 2, \dots, n\}$ $(n \geq 3)$ is a set of *nodes* and g is a set of *links* specified by an *adjacency matrix* $g = (g_{ij})_{i,j \in N}$ of real numbers $g_{ij} \in [0, 1)$, with $g_{ii} = 0$ for all i. When no ambiguity arises we omit N and refer to g as a network. When $g_{ij} > 0$ it is said that an $\overline{ij}$*-link of strength* g_{ij} connects i to j. We distinguish two (possibly overlapping) subsets of neighbors of node i:

$$N_i^-(g) := \{j \in N : g_{ji} > 0\} \quad \text{and} \quad N_i^+(g) := \{j \in N : g_{ij} > 0\},$$

Nodes in $N_i^-(g)$ are *in-neighbors* of i, and nodes in $N_i^+(g)$ are *out-neighbors* of i, and $N_i(g) := N_i^-(g) \cup N_i^+(g)$ the set of *neighbors* of node i. The *in* (*out*)*-degree* of node i is the cardinality $\left|N_i^-(g)\right|$ $(\left|N_i^+(g)\right|)$, and $\left|N_i(g)\right|$ its *degree*. A *path* connecting nodes i and j is a sequence of distinct nodes of which the first is i, the last is j, and every two consecutive nodes are connected by a link. A *connected* network is one where any two nodes are connected by a path. An *empty network* is the one for which $g_{ij} = 0$ for all $i, j \in N$. A *complete network* is one where $g_{ij} > 0$ for all $i \neq j$. A directed network is *oriented* if for all i, j, $g_{ij} = 0$ or $g_{ji} = 0$.

Here we consider core–periphery structures in the following sense.

Definition 1. In a core–periphery directed network $g^{\mathbf{c}}$ the set of nodes is partitioned into two sets: A core $Co(g^{\mathbf{c}})$ and a periphery $Pe(g^{\mathbf{c}})$ s.t.

(i) For all $i, j \in Co(g^{\mathbf{c}})$, $g_{ij}^{\mathbf{c}} > 0$;
(ii) For all $i, j \in Pe(g^{\mathbf{c}})$, $g_{ij}^{\mathbf{c}} = 0$;
(iii) For all $i \in Pe(g^{\mathbf{c}})$, $\exists j \in Co(g^{\mathbf{c}})$ s.t. $g_{ij}^{\mathbf{c}} > 0$ or $g_{ji}^{\mathbf{c}} > 0$.

That is, every pair of nodes in the core is connected bidirectionally, no link connects any pair of peripheral nodes and every peripheral node is connected either unidirectionally or bidirectionally with at least one in the core. A *star* is a core–periphery network whose core is a singleton.

Definition 2. A *bipartite directed network* is a network where nodes are partitioned into two sets, V and W, so that links connect only nodes in different sets of the partition. A bipartite network is *complete* if every pair $(i, j) \in V \times W$ is connected by a link.

An *undirected* weighted network $\overline{g}$ is a directed network whose adjacency matrix is symmetric, i.e. $\overline{g}_{ij} = \overline{g}_{ji}$ for all $i, j \in N$.[4] Such networks can be specified as a map $\overline{g} : N_2 \to [0, 1)$, where N_2 denotes the set of all subsets of N with cardinality 2. In what follows $\overline{ij}$ stands for $\{i, j\}$ and $\overline{g}_{\overline{ij}}$ for $\overline{g}(\{i, j\})$ for any $\{i, j\} \in N_2$. In an undirected network $\overline{g}$, when $\overline{g}_{\overline{ij}} > 0$ it is said that a *link of strength* $\overline{g}_{\overline{ij}}$ connects i and j. Note also that in an undirected network $\overline{g}$:

$$N_i(\overline{g}) = N_i^-(\overline{g}) = N_i^+(\overline{g}).$$

Two nodes i, j have *nested neighborhoods* if $N_i(\overline{g}) \subseteq N_j(\overline{g}) \cup \{j\}$ or $N_j(\overline{g}) \subseteq N_i(\overline{g}) \cup \{i\}$, and *strictly nested neighborhoods* if $N_i(\overline{g}) \subseteq N_j(\overline{g})$ or $N_j(\overline{g}) \subseteq N_i(\overline{g})$.[5]

An important type of undirected network is the following.

Definition 3. A nested split graph network (NSG-network) is a undirected network $\overline{g}$ such that for all $i, j \in N$ $(i \neq j)$,

$$\left|N_i(\overline{g})\right| \leq \left|N_j(\overline{g})\right| \Rightarrow N_i(\overline{g}) \subseteq N_j(\overline{g}) \cup \{j\}. \tag{1}$$

Undirected NSG-networks have a hierarchical architecture, as nodes can be ranked according to their *degrees* or numbers of neighbors, i.e. the nodes' neighborhoods are nested according to their *degree*.

Given that undirected networks are a particular type of directed networks, the notions of connected, complete, core–periphery and bipartite network apply also to undirected networks.

3. The model

We consider a situation specified by $(N; (v_i, \beta_i)_{i \in N})$, where $N = \{1, 2, \dots, n\}$ is a set of *nodes* or *players*,[6] each of which is endowed with a *value* $v_i > 0$ and an *individual/personal* technology, which is a function $\beta_i : \mathbb{R}_+ \to [0, \frac{1}{2})$ increasing, differentiable, strictly concave, and s.t. $\beta_i(0) = 0$. The value v_i of each node can be interpreted as an endowment of valuable information that can be partially accessed by other nodes through links. Alternatively, v_i can be interpreted as the value of node i as a valuable contact. Under this interpretation, the model can be interpreted as a stylized model of a contacts network. Nodes can form links according to a separable technology s.t. the strength of a link between nodes i and j in which i invests c_{ij} and j invests c_{ji} is

$$\beta_i(c_{ij}) + \beta_j(c_{ji}).$$

If the model is interpreted in terms of flow of information, the strength of a link is the level of fidelity of the transmission of information through it. If the model is interpreted as a contacts network, the strength of a link is a measure of its quality/intensity/reliability, i.e. the "strength of a tie" (Granovetter, 1973). Thus function β_i can be interpreted as the personal/individual ability of player i to create/strengthen links with other players.

A *link-investment* is an n-tuple $\mathbf{c} = (\mathbf{c}_i)_{i \in N}$, where $\mathbf{c}_i \in \mathbb{R}_+^{n-1} = (c_{ij})_{j \in N \setminus \{i\}}$, where c_{ij} is the investment of node i in its link with j. Link-investment $\mathbf{c}$ yields an *undirected* weighted network denoted by $\overline{g}^{\mathbf{c}}$, where

$$\overline{g}_{\overline{ij}}^{\mathbf{c}} = \beta_i(c_{ij}) + \beta_j(c_{ji})$$

is the strength of the resulting link $\overline{ij}$, which is the level of fidelity of the transmission of value between i and j (or the strength of the tie

[3] Empirical evidence suggests that financial networks exhibit a core–periphery structure. Several models in the literature give theoretical support to this fact (see In't Veld et al. (2020), and Babus and Hu (2017). Core–periphery networks also emerge in models of communication network formation models (see Hojman and Szeidl (2008), and in models on the formation of trading networks (see Goyal and Vega-Redondo (2007) and Bedayo et al. (2016)).

[4] To convey the distinction between directed and undirected networks in the notation, when we deal with an undirected network we use a bar, i.e. we always write $\overline{g}$ to denote an undirected network.

[5] See Boeckner (2018).

[6] We often prefer the neutral term "nodes" to avoid bias in language.

between them), which is never perfect ($0 \leq \beta_i(c_{ij}) + \beta_j(c_{ji}) < 1$).[7] We assume that *each node receives value only from its neighbors*. Thus, node i receives from node j the fraction of node j's value, i.e. v_j, which reaches i through the link that connects i and j, i.e. $v_j \overline{g}^{\mathbf{c}}_{ij}$. Thus, i's overall revenue from $\overline{g}^{\mathbf{c}}$ is

$$I_i(\overline{g}^{\mathbf{c}}) = \sum_{j \in N_i(\overline{g}^{\mathbf{c}})} v_j \; \overline{g}^{\mathbf{c}}_{ij},$$

and i's investment

$$C_i(\overline{g}^{\mathbf{c}}) = \sum_{j \in N_i(\overline{g}^{\mathbf{c}})} c_{ij}.$$

Thus, i's *payoff* is the value received minus i's investment:

$$\pi_i(\mathbf{c}) := I_i(\overline{g}^{\mathbf{c}}) - C_i(\overline{g}^{\mathbf{c}}) = \sum_{j \in N_i(\overline{g}^{\mathbf{c}})} (v_j \overline{g}^{\mathbf{c}}_{ij} - c_{ij}), \tag{2}$$

and the *net value* of the network resulting is the aggregate payoff, i.e. the total value received by the nodes minus the total cost of the network, given by

$$v(\overline{g}^{\mathbf{c}}) := \sum_{i \in N} \pi_i(\mathbf{c}) = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} \left[(\beta_i(c_{ij}) + \beta_j(c_{ji})) v_j - c_{ij} \right]. \tag{3}$$

A network $\overline{g}^{\mathbf{c}}$ (a link-investment $\mathbf{c}$) *dominates* a network $\overline{g}^{\mathbf{c}'}$ (a link-investment $\mathbf{c}'$) if $v(\overline{g}^{\mathbf{c}}) \geq v(\overline{g}^{\mathbf{c}'})$. A network is *efficient* if it dominates any other.

4. Stability

Payoff functions (2) specify a trivial noncooperative game where a strategy of player i is a vector of investments $\mathbf{c}_i = (c_{ij})_{j \in N \setminus \{i\}} \in \mathbb{R}^{n-1}_{+}$. A *Nash link-investment* is a link-investment $\mathbf{c}^* = (\mathbf{c}^*_i)_{i \in N}$, with $\mathbf{c}^*_i = (c^*_{ij})_{j \in N \setminus \{i\}}$, such that for all $i \in N$, and all $\mathbf{c}_i = (c_{ij})_{j \in N \setminus \{i\}} \in \mathbb{R}^{n-1}_{+}$:

$$\pi_i(\mathbf{c}^*) \geq \pi_i(\mathbf{c}^*_{-i}, \mathbf{c}_i).$$

Abusing language, we often refer to the network $\overline{g}^{\mathbf{c}}$ which results from a Nash link-investment $\mathbf{c}$ as a "Nash network".

The following proposition characterizes the unique Nash equilibrium in dominant strategies of this game.

Proposition 1. *If payoffs are given by (2), there exists a unique Nash equilibrium in dominant strategies $\mathbf{c}^* = (\mathbf{c}^*_i)_{i \in N}$, where, for all i, k ($i \neq k$), node i invests in a link with node k if and only if $\beta'_i(0) > 1/v_k$, and c^*_{ik} is characterized by the condition*

$$\beta'_i(c^*_{ik}) = 1/v_k. \tag{4}$$

Proof. The payoff function (2) can be rewritten like this

$$\pi_i(\mathbf{c}) = \sum_{k \in N_i(\overline{g}^{\mathbf{c}})} (v_k(\beta_i(c_{ik}) + \beta_k(c_{ki})) - c_{ik}).$$

So that the impact of c_{ik} on i's payoff is $v_k \beta_i(c_{ik}) - c_{ik}$, which is an increasing, differentiable, strictly concave function. Therefore, a necessary condition for $\mathbf{c}^*_i$ to maximize i's payoff is

$$\frac{\partial \pi_i(\mathbf{c})}{\partial c_{ik}} = \left. \frac{d(v_k \beta_i(c_{ik}) - c_{ik})}{d c_{ik}} \right|_{c_{ik} = c^*_{ik}} = v_k \beta'_i(c^*_{ik}) - 1 = 0,$$

for all k s.t. $c^*_{ik} > 0$. Otherwise, if $v_k \beta'_i(\overline{c}_{ik}) - 1 > 0$ (< 0) an increase (decrease) in i's investment in link ik would increase i's payoff. This yields (4). Given the assumptions on β_i, such a (unique) $c^*_{ik} > 0$ is sure to exist if and only if $\beta'_i(0) > \frac{1}{v_k}$. Therefore, a *dominant strategy* of player i is to invest $c^*_{ik} > 0$ s.t. (4) in a link with node k if and only if $\beta'_i(0) > \frac{1}{v_k}$ holds. ■

[7] As we show below, the situation can also be described by incorporating more information as a *directed* network. This permits a better understanding of the structure of Nash networks and efficient networks.

Thus the Nash equilibrium network $\overline{g}^{\mathbf{c}}$ is non-empty if and only there exist i and j ($i \neq j$) s.t. $\beta'_i(0) > \frac{1}{v_j}$. The following examples graphically illustrate the characterizing conditions (Example 1) and the complexity of comparing technologies (Example 2).

Example 1. Consider two nodes: a node i with technology $\beta_i(x) = 0.2(1 - e^{-2x})$ and another node j with value $v_j = 10$. As $\beta'_i(x) = 0.4e^{-2x}$, $\beta'_i(0) = 0.4 > \frac{1}{10}$, and it will be profitable for node i to invest in a link with a node of value 10. Namely, i's dominant strategy is to invest in a link with j an amount c_{ij} s.t. $\beta'_i(c_{ij}) = 0.4e^{-2c_{ij}} = \frac{1}{10}$, which yields $c_{ij} = \ln 2 = 0.693$ (see Fig. 1). ■

Beyond the case where $\beta_i(x) > \beta_j(x)$ for all $x > 0$, and obviously i's technology is better than j's technology, comparing technologies is not obvious.

Example 2. Consider two nodes with technologies $\beta_1(0) = 0.1(1 - e^{-10x})$ and $\beta_2(0) = 0.2(1 - e^{-2x})$, and consequently $\beta'_1(0) = 1 > \beta'_2(0) = 0.4$. Thus, node 1 can invest profitably with any node i s.t. $v_i > 1$, while node 2 can only invest profitably with nodes whose value is greater than 2.5 (see Fig. 2). That is, node 1 is better than 2 at making friends and possibly has more out-neighbors. However, node 2 may possibly have fewer out-neighbors, but they will be connected by stronger out-links. ■

Note that the payoffs, given by (2), are given in terms of the undirected network $\overline{g}^{\mathbf{c}}$. Nevertheless, to better understand the architecture of Nash equilibrium networks, the situation can alternatively and more richly be described by the associated *directed* network $g^{\mathbf{c}}$ defined by:

$$g^{\mathbf{c}}_{ij} := \beta_i(c_{ij}).$$

Network $g^{\mathbf{c}}$ represents the strength of links that connect each node *due to its own investments*. Note that $g^{\mathbf{c}}$ is directed, as in general $g^{\mathbf{c}}_{ij} \neq g^{\mathbf{c}}_{ji}$. As $\overline{g}^{\mathbf{c}}_{ij} = g^{\mathbf{c}}_{ij} + g^{\mathbf{c}}_{ji}$, for all $i, j \in N$, $g^{\mathbf{c}}$ embodies all the information necessary to describe $\overline{g}^{\mathbf{c}}$, while the reciprocal is not true.

We then have the following.

Proposition 2. *In the directed network $g^{\mathbf{c}}$ associated with a Nash link-investment $\mathbf{c} = (\mathbf{c}_i)_{i \in N}$, for all $i, j \in N$:*

(i) $v_i \leq v_j \Rightarrow N^-_i(g^{\mathbf{c}}) \subseteq N^-_j(g^{\mathbf{c}}) \cup \{j\}$.
(ii) $\beta'_i(0) \leq \beta'_j(0) \Rightarrow N^+_i(g^{\mathbf{c}}) \subseteq N^+_j(g^{\mathbf{c}}) \cup \{j\}$.
(iii) $\left|N^-_i(g^{\mathbf{c}})\right| < \left|N^-_j(g^{\mathbf{c}})\right| \Rightarrow N^-_i(g^{\mathbf{c}}) \subseteq N^-_j(g^{\mathbf{c}}) \cup \{j\}$.
(iv) $\left|N^+_i(g^{\mathbf{c}})\right| < \left|N^+_j(g^{\mathbf{c}})\right| \Rightarrow N^+_i(g^{\mathbf{c}}) \subseteq N^+_j(g^{\mathbf{c}}) \cup \{j\}$.

Proof. Let $\mathbf{c} = (\mathbf{c}_i)_{i \in N}$ be a Nash link-investment.

(*i*) Assume $N^-_i(g^{\mathbf{c}}) \nsubseteq N^-_j(g^{\mathbf{c}}) \cup \{j\}$. Therefore there is a $k \neq i, j$ s.t. $g^{\mathbf{c}}_{ki} > 0$ (i.e. $\beta'_k(0) > \frac{1}{v_i}$) and $g^{\mathbf{c}}_{kj} = 0$ (i.e. $\beta'_k(0) \leq \frac{1}{v_j}$), which obviously implies that $v_i > v_j$.

(*ii*) Assume $N^+_i(g^{\mathbf{c}}) \nsubseteq N^+_j(g^{\mathbf{c}}) \cup \{j\}$. Therefore there is a $k \neq i, j$ s.t. $g^{\mathbf{c}}_{ik} > 0$ (i.e. $\beta'_i(0) > \frac{1}{v_k}$) and $g^{\mathbf{c}}_{jk} = 0$ (i.e. $\beta'_j(0) \leq \frac{1}{v_k}$), which obviously implies that $\beta'_i(0) > \beta'_j(0)$.

(*iii*) Assume $N^-_i(g^{\mathbf{c}}) \nsubseteq N^-_j(g^{\mathbf{c}}) \cup \{j\}$. Then there is a $k \neq i, j$ s.t. $c_{ki} > 0$ and $c_{kj} = 0$, which obviously implies that $v_i > v_j$. Thus, any node which invests in linking j is sure to invest in linking i. Therefore, either $\left|N^-_i(g^{\mathbf{c}})\right| > \left|N^-_j(g^{\mathbf{c}})\right|$ or $\left|N^-_i(g^{\mathbf{c}})\right| = \left|N^-_j(g^{\mathbf{c}})\right|$ (if j invests in a link with i and i does not in a link with j). In other words $\left|N^-_i(g^{\mathbf{c}})\right| \geq \left|N^-_j(g^{\mathbf{c}})\right|$. This proves (*iii*).

(*iv*) The proof is similar to that of (*iii*). ■

Note the similarity of the right-hand terms in conditions (*i*) and (*ii*) and the right-hand side of condition (1) for undirected NSG-networks. While NSG-networks rank the nested neighborhoods of nodes according to their degrees, in this model Nash networks rank the *in*-neighborhoods of nodes *according to their values* (and *out*-neighborhoods *according to their marginal strength at zero of their technologies*). Conditions (*iii*) and (*iv*) call for a comparison with condition (1) for

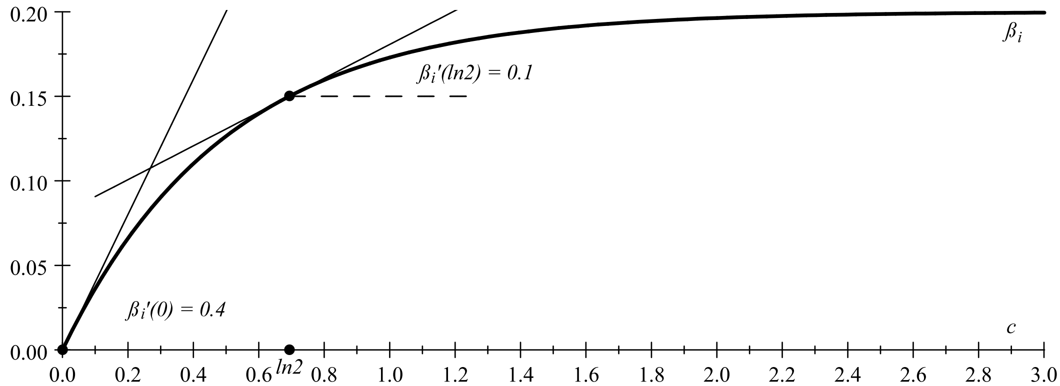

Fig. 1. Example 1.

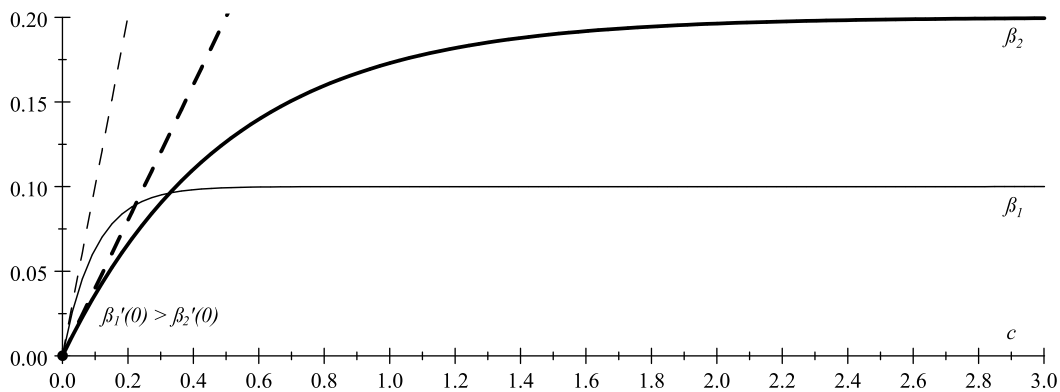

Fig. 2. Example 2.

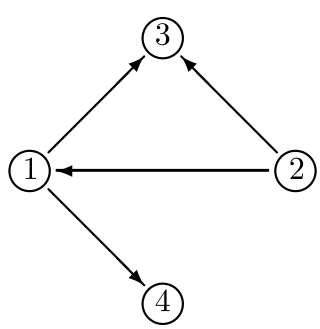

Fig. 3. Example 3.

undirected NSG-networks. The similarity of implications (*iii*) and (*iv*) with (1) suggests that Nash networks are "quasi" nested-split-graph in terms of in-neighbors and in terms of out-neighbors. The similarity is less than perfect (hence the "quasi") because implications (*iii*)/(*iv*) may not hold when the in-degree/out-degree is the same. The following counterexamples show why this is so.

Example 3. Let $N = \{1, 2, 3, 4\}$, with values and technologies such that the only positive Nash investments are c_{31}, c_{32}, c_{12}, and c_{41}. That is, 3's technology is good enough to invest in links with 1 and 2, but not with 4, while 1 only invests in a link with 2, and node 4 only in a link with 1 (see Fig. 3). This is possible if $v_1 > v_2 > v_3 = v_4$ and $\beta_1'(0) = \beta_3'(0) > \beta_4'(0) > \beta_2'(0)$. Thus, $N_1^-(g^{\mathbf{c}}) = \{3, 4\}$ and $N_2^-(g^{\mathbf{c}}) = \{1, 3\}$, that is $\left|N_1^-(g^{\mathbf{c}})\right| = \left|N_2^-(g^{\mathbf{c}})\right|$ and

$$N_1^-(g^{\mathbf{c}}) = \{3, 4\} \nsubseteq N_2^-(g^{\mathbf{c}}) \cup \{2\} = \{1, 2, 3\}.$$

A numerical example that fits these conditions is $v_1 = 50$, $v_2 = 20$ and $v_3 = v_4 = 10$, with technologies such that $\beta_1'(0) = \beta_3'(0) = 1/15$, $\beta_4'(0) = 1/30$ and $\beta_2'(0) = 1/50$.■

Example 4. Let $N = \{1, 2, 3, 4\}$, with values $v_1 = v_3 = 10$, $v_2 = 2$, $v_4 = 5$, and technologies s.t. $\beta_1'(0) = 1/3$, $\beta_2'(0) = 1/6$, $\beta_3'(0) = \beta_4'(0) = 1/10$. Then for the Nash investment $\mathbf{c}$, $\left|N_1^+(g^{\mathbf{c}})\right| = \left|N_2^+(g^{\mathbf{c}})\right| = 2$ and

$$N_1^+(g^{\mathbf{c}}) = \{3, 4\} \nsubseteq N_2^+(g^{\mathbf{c}}) \cup \{2\} = \{1, 2, 3\}.■$$

Note that in Example 3 the condition fails because two nodes have the same number of out-neighbors but different values ($v_1 > v_2$), while in Example 4 two nodes have the same number of in-neighbors but different technologies ($\beta_1'(0) > \beta_2'(0)$). However, this cannot occur when values and technologies rank nodes in the same order.

Proposition 3. *If values and technologies of nodes are equally ranked, i.e. if*

$$v_i \leq v_j \Leftrightarrow \beta_i'(0) \leq \beta_j'(0), \tag{5}$$

then the Nash undirected network $\overline{g}^{\mathbf{c}}$ is a NSG-network which ranks nodes according to that ranking.

Proof. Assume that $\overline{g}^{\mathbf{c}}$ is a NSG-network and $v_i \leq v_j \Leftrightarrow \beta_i'(0) \leq \beta_j'(0)$. If $\left|N_i(\overline{g}^{\mathbf{c}})\right| < \left|N_j(\overline{g}^{\mathbf{c}})\right|$, then there must exist some $k \in N_j(\overline{g}^{\mathbf{c}}) \setminus N_i(\overline{g}^{\mathbf{c}})$. This implies that $v_i < v_j$ or $\beta_i'(0) < \beta_j'(0)$, which by (5) implies $v_i < v_j$ *and* $\beta_i'(0) < \beta_j'(0)$. Thus, as $N_i(\overline{g}^{\mathbf{c}}) = N_i^-(g^{\mathbf{c}}) \cup N_i^+(g^{\mathbf{c}})$, it follows from Proposition 2-(*iii-iv*) that $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$. Assume now that $\left|N_i(\overline{g}^{\mathbf{c}})\right| = \left|N_j(\overline{g}^{\mathbf{c}})\right|$ and $k \in N_i(\overline{g}^{\mathbf{c}}) \setminus (N_j(\overline{g}^{\mathbf{c}}) \cup \{j\})$. Thus, $v_i > v_j$ or $\beta_i'(0) > \beta_j'(0)$ which by (5) implies that $v_i > v_j$ and $\beta_i'(0) > \beta_j'(0)$. But then $N_j(\overline{g}^{\mathbf{c}}) \subsetneq N_i(\overline{g}^{\mathbf{c}})$, which contradicts $\left|N_i(\overline{g}^{\mathbf{c}})\right| = \left|N_j(\overline{g}^{\mathbf{c}})\right|$. ■

Thus, the structure of the Nash network is clear from the two points of view provided by $g^{\mathbf{c}}$. However, these results deserve some comments. The combined effects of the heterogeneity in values and technologies may give rise to complex architectures of Nash equilibriums. In general, undirected network $\overline{g}^{\mathbf{c}}$ is *not* a NSG-network. On the other hand, the structure of an undirected NSG-network is hierarchical in terms of neighbors (nodes are ranked by their number of neighbors), while in directed networks there are two points of view. In the case of directed network $g^{\mathbf{c}}$, for a Nash equilibrium things are somewhat more complex. From Proposition 2 it follows that the greater the value of a node the more it benefits from the externalities due to the investments of the other nodes in links with it. More precisely, more nodes will invest in links with it *and they will invest more in those links*. In the case of the benefits of a node due to its own investments the situation is more subtle: Their rank by the number of out-neighbors is determined by the rank of their technologies *based on their marginal strength at zero*. Thus, from Proposition 2, it follows that the greater the marginal strength at origin of the technology of a node, the more nodes it will invest in, *but it does not necessarily follow that it will invest more in links with them* (see Example 2).

Now we concentrate in nonempty Nash equilibrium networks.

Proposition 4. *In a Nash network $\overline{g}^{\mathbf{c}}$ the maximal length of a path between two nodes is* 3.

Proof. Assume that the Nash network $\overline{g}^{\mathbf{c}}$ is not empty, and let $v_{mx} = \max_{i\in N} v_i$ and $\beta'_{mx} = \max_{i\in N} \beta'_i(0)$. If only one node, say i, has such a value and such a technology, then any node which invests in any link will surely invest in a link with node i; and any node which receives any investment from any node will surely receive some investment from node i. Therefore in this case the maximal distance possible is 2. If there are two different nodes i and j such that $v_i = v_{mx}$ and $\beta'_j(0) = \beta'_{mx}$, it is certain that $g^{\mathbf{c}}_{ji} > 0$ and any node which invests in any link will surely invest in a link with node i; and any node which receives some investment from any node will surely receive some investment from node j. In this case the maximal distance possible is 3. ■

In view of Proposition 1, the following is immediately apparent:

Proposition 5. *Networks $\overline{g}^{\mathbf{c}}$ and $g^{\mathbf{c}}$ associated with a Nash link-investment $\mathbf{c}$ are core–periphery (Definition 1) if and only if the set of nodes is partitioned into two sets, a core $Co(g^{\mathbf{c}})$ and a periphery $Pe(g^{\mathbf{c}})$ s.t.*

(i) For all $i,j \in Co(g^{\mathbf{c}})$, $\beta'_i(0) > \frac{1}{v_j}$;

(ii) For all $i,j \in Pe(g^{\mathbf{c}})$, $\beta'_i(0) \leq \frac{1}{v_j}$;

(iii) For all $i \in Pe(g^{\mathbf{c}})$, $\exists j \in Co(g^{\mathbf{c}})$ s.t. $\beta'_i(0) > \frac{1}{v_j}$ or $\beta'_j(0) > \frac{1}{v_i}$.

Thus, a core–periphery Nash network occurs when the nodes with sufficiently high values *and* sufficiently good technologies are concentrated into one group, while the rest are nodes of low value and/or with poorer technologies. If the values of the nodes of the periphery are sufficiently low their links with the core will be unidirectional, they alone sustain these links. A similar situation occurs if the technologies of the peripheral nodes are sufficiently bad. In this case their links with the core will be supported by core nodes only. However, in general, a core–periphery Nash network may have unidirectional links in both directions and also bidirectional links.

We now discuss the different architectures of the networks $\overline{g}^{\mathbf{c}}$ and $g^{\mathbf{c}}$ that emerge from a Nash investment vector $\mathbf{c} = (\mathbf{c}_i)_{i\in N}$, with $\mathbf{c}_i = (c_{ij})_{j\in N\setminus\{i\}} \in \mathbb{R}^{n-1}_+$, in different contexts.

Homogeneity in technologies and values

Assume all nodes have the same value and the same individual technology.

Proposition 6. *If all nodes have the same value and the same technology, the Nash network is either complete or the empty network.*

Proof. Assume $v_i = v$ and $\beta_i = \beta$, for all $i \in N$. From Proposition 1, in this case, if $\beta'(0) > 1/v$, the Nash network is complete and for all pair $i,j \in N$: $c_{ij} = c$ s.t. $\beta'(c) = 1/v$; while if $\beta'(0) \leq 1/v$, $c_{ij} = 0$ for all pair $i,j \in N$. ■

Unlike in other models where all players have the same value and the same technology, as Jackson and Wolinsky (1996), Bala and Goyal (2000) or Bloch and Dutta (2009), in this model the star cannot ever be a Nash stable network. The reason is clear, only direct links transmit value. Consequently indirect connections do not pay.

Homogeneity in technologies and heterogeneity in values

Assume now that a single technology is available to all players, whose values may differ.

Proposition 7. *If all nodes have the same technology β but their values may differ, the Nash network is a complete core–periphery network whose core consists of the nodes with values greater than $1/\beta'(0)$, fully connected by links in which both nodes invest, and a periphery of nodes with lower values, each connected with all nodes in the core by links supported only by the peripheral partner. In the extreme cases when one of the sets of the partition core/periphery is empty, then the Nash network is either complete or empty. If the core consists of a single node, the Nash network is a periphery sponsored star centered at that node.*

Proof. Assume $\beta_i = \beta$, for all $i \in N$. By Proposition 1, in equilibrium a player i will invest in a link with k only if $\beta'(0) > 1/v_k$. That is, the set of players is partitioned into two sets according to their values:

$$Co(g^{\mathbf{c}}) = \{i \in N : v_i > 1/\beta'(0)\} \quad \text{and} \quad Pe(g^{\mathbf{c}}) = \{j \in N : v_j \leq 1/\beta'(0)\}.$$

The Nash network is thus a core–periphery network whose core consists of the nodes with values greater than $1/\beta'(0)$, fully connected by links in which both nodes invest, namely for all pair $i,k \in Co(g^{\mathbf{c}})$, i invests in link ik, c_{ik} s.t. $\beta'(c_{ik}) = 1/v_k$; and a periphery of nodes with lower values, each connected with all nodes in the core by links, each of them supported only by the peripheral partner. That is, for all $i \in Pe(g^{\mathbf{c}})$ and all $k \neq i$, c_{ik} is s.t. $\beta'(c_{ik}) = 1/v_k$, if $k \in Co(g^{\mathbf{c}})$, while $c_{ik} = 0$ for all $k \in Pe(g^{\mathbf{c}})$. ■

It is worth noting a certain similarity of this result with Theorem 1 in Kinateder and Merlino (2022) where all players have the same technology. In spite of their setting to be completely different, in their model a sociable Nash equilibrium always exists and is a complete core–periphery network, where players who provide more public good than the linking cost (i.e., the large contributors) form a core of interconnected players.[8] In a completely different setting, Bedayo et al. (2016) study a model in which heterogeneous agents form a trading network and then a buyer and a seller randomly selected bargain through a chain of intermediaries. They show that a trading network is pairwise stable if and only if it is a core–periphery network where the core consists of all patient agents, while impatient agents, each of them connected with one in the core, form the periphery. In their model patience, i.e. discount rate (with two types in their Proposition 2) is the source of discrimination between core and periphery, while in Proposition 7 it is value heterogeneity. Namely, the core consists of those players whose value is enough to make it worth linking, i.e. above $1/\beta'(0)$. Thus, in very stylized terms, it is as if in their model patience meant value at the end of the day.

Homogeneity in values and heterogeneity in technologies

Assume now that all nodes have the same value, while their technologies may differ.

[8] A Nash equilibrium is sociable if no node can increase the weight of one link without decreasing its payoff.

Proposition 8. *If all nodes have the same value v but their technologies may differ, the Nash network is a complete core–periphery network with a core of nodes fully connected by links in which both nodes invest, and a periphery of nodes with poorer technologies, each of them connected with all nodes in the core by links, each of them supported only by the core partner. In the extreme cases when the periphery or the core is empty, the Nash network is either complete or empty. If the core consists of a single node, the Nash network is a star centered at that node and sponsored by it.*

Proof. Assume $v_i = v$, for all $i \in N$. Then, by Proposition 1, in equilibrium player i will invest in a link with k only if $\beta_i'(0) > 1/v$. In this case, the set of players is partitioned into two sets according to their technologies:

$$Co(g^{\mathbf{c}}) = \{i \in N : \beta_i'(0) > 1/v\} \quad \text{and} \quad Pe(g^{\mathbf{c}}) = \{j \in N : \beta_j'(0) \leq 1/v\}.$$

The Nash network is thus a core–periphery network whose core consists of nodes fully connected by links in which both nodes invest, and a periphery of nodes with poorer technologies, each connected with all nodes in the core by links, each of them supported only by the core partner. ■

In contrast with the case where there is heterogeneity only in values, where nodes with smaller value are pushed to the periphery, when there is heterogeneity only in technologies, the nodes with worse technologies are pushed to the periphery. Again, there is a certain similarity with Theorem 1 in Kinateder and Merlino (2022), where the large contributors form a core.

Bipartite networks

Bipartite networks appear in different social and economic contexts. Under the assumptions of this model, bipartite networks emerge in equilibrium only in very precise conditions.

Proposition 9. *A Nash network $\overline{g}^{\mathbf{c}}$ (and its associated $g^{\mathbf{c}}$) is a bipartite connected network if and only if the set of nodes is partitioned into two sets V and W s.t. nodes in W are more valuable than those in V (except perhaps the node with the best technology in V if there is only one), while V contains nodes with better technologies than those in W, and it is an oriented network where all links are entirely supported by nodes in V.*

Proof. Assume that $\mathbf{c}$ is an equilibrium investment vector s.t. $\overline{g}^{\mathbf{c}}$ (and consequently its associated $g^{\mathbf{c}}$) is a bipartite connected network. Then N must be partitioned into two sets V and W s.t. links connect only pairs of nodes which belong to different sets of this partition. By Proposition 1, for all i, j $(i \neq j)$:

$$c_{ij} > 0 \Leftrightarrow \beta_i'(c_{ij}) = \frac{1}{v_j}.$$

As $g^{\mathbf{c}}$ is not empty, there must be $c_{ij} > 0$ for some pair (i, j). Assume w.l.o.g. that $(i, j) \in V \times W$ and assume i is a node with the best technology in V, i.e. $i \in \arg\max_{k \in V} \beta_k'(0)$. Then $N_i^+(g^{\mathbf{c}}) = \cup_{j \in V} N_j^+(g^{\mathbf{c}})$ and, as links only connect nodes in different sets of the partition, for all $(k, l) \in V \times W$, s.t. $l \in N_i^+(g^{\mathbf{c}})$ and $k \in V \setminus \{i\}$, $v_k < v_l$, otherwise $c_{ik} > 0$. That is, the value of any node in $N_i^+(g^{\mathbf{c}})$ is greater than the value of any node in V, except perhaps node i if it were the only one with the best technology in V. Note that if another node j in V has a technology as good as that of i, i.e. $\beta_i'(0) = \beta_j'(0)$, then the bipartite character of $g^{\mathbf{c}}$ implies $v_i < v_l$ and $v_j < v_l$ for all $l \in N_i^+(g^{\mathbf{c}})$. Nodes in W do not invest in links with nodes in W, so for any node $l \in W$ and any node $k \in \cup_{j \in W} N_j^-(g^{\mathbf{c}})$, $\beta_k'(0) > \beta_l'(0)$. Thus the only links in $g^{\mathbf{c}}$ are those connecting nodes in $\cup_{j \in W} N_j^-(g^{\mathbf{c}})$ and nodes in $N_i^+(g^{\mathbf{c}})$, and those links are supported only by nodes in $\cup_{j \in W} N_j^-(g^{\mathbf{c}}) \subseteq V$. Finally, as $g^{\mathbf{c}}$ is connected, it must be $N_i^+(g^{\mathbf{c}}) = W$ and $\cup_{j \in W} N_j^-(g^{\mathbf{c}}) = V$. The reciprocal is obvious. ■

If there is a single node in V with the best technology, its value may be greater than that of some nodes in W. However, in any case, if i is the single node i in V with the best technology, it must hold that

$$v_i < \max\{v_j : j \in W \ \& \ \exists k \in V \setminus \{i\} \text{ s.t. } \beta_k'(0) > \frac{1}{v_j}\}$$

for the bipartite character of the network to be preserved. Thus, bipartite connected networks may arise in equilibrium when the set of nodes is partitioned into two sets V and W with values $(v_i)_{i \in N}$ and technologies $(\beta_i)_{i \in N}$ s.t.

(i) For the node in V with the best technology, i.e. $i_{best\beta} := \arg\max_{i \in V} \beta_i$,

$$\frac{1}{v_j} < \beta_{i_{best\beta}}'(0) \leq \frac{1}{v_k}$$

for all $j \in W$ and all $k \in V \setminus \{i\}$.

(ii) For all $i \in V$, $\frac{1}{v_j} < \beta_i'(0) \leq \frac{1}{v_k}$ for some $j \in W$ and all $k \in V \setminus \{i\}$.

(iii) For all $j \in W$, $\beta_j'(0) \leq \frac{1}{v_k}$ for all $k \in W \setminus \{j\}$.

This means that any node in W is more valuable than any node in V (except perhaps the node with the best technology in V if there is only one), and that technologies in V are better than in W and good enough to make it worth linking some nodes in W but not to link any node in V. The node in V with the highest technology is connected with all nodes in W and all links are supported only by nodes in V. Note also that if (i) is assumed for *all* $i \in V$ (in which case (ii) is superfluous), the bipartite network is complete.

It is worth noting that bipartite Nash stable networks are *strictly* nested oriented networks in a restricted sense.[9] Namely, for all $i, j \in V$:

$$\beta_i'(0) \leq \beta_j'(0) \Leftrightarrow \left|N_i^+(g^{\mathbf{c}})\right| \leq \left|N_j^+(g^{\mathbf{c}})\right| \Leftrightarrow N_i^+(g^{\mathbf{c}}) \subseteq N_j^+(g^{\mathbf{c}}),$$

and for all $k, l \in W$:

$$v_k \leq v_l \Leftrightarrow \left|N_i^-(g^{\mathbf{c}})\right| \leq \left|N_j^-(g^{\mathbf{c}})\right| \Leftrightarrow N_i^-(g^{\mathbf{c}}) \subseteq N_j^-(g^{\mathbf{c}}).$$

Moreover, "+" and "−" can be dropped in the right-hand side of the two implications above, given that $N_i(g^{\mathbf{c}}) = N_i^+(g^{\mathbf{c}})$ for all $i \in V$, and $N_j(g^{\mathbf{c}}) = N_j^-(g^{\mathbf{c}})$ for all $j \in V$. Note also that no bipartite network can arise in equilibrium when there is homogeneity in the values of nodes or in their technologies.

5. Efficiency

Now we turn our attention to efficiency.

Proposition 10. *If payoffs are given by (2), there exists a unique efficient investment $\mathbf{c}^* = (\mathbf{c}_i^*)_{i \in N}$, with $\mathbf{c}_i^* = (c_{ij}^*)_{j \in N \setminus \{i\}}$, where, for all pair i, k, node i invests in a link with node k if and only if $\beta_i'(0) > \frac{1}{v_i + v_k}$, and c_{ik}^* is characterized by the condition:*

$$\beta_i'(c_{ik}^*) = \frac{1}{v_i + v_k}. \tag{6}$$

Proof. Maximizing

$$v(\overline{g}^{\mathbf{c}}) = \sum_{\underline{kl} \in N_2} ((v_k + v_l)(\beta_k(c_{kl}) + \beta_l(c_{lk})) - c_{kl} - c_{lk}),$$

s.t. $c_{kl} \geq 0$, for all $k, l \in N$ $(k \neq l)$,

means maximizing the sum of the payoffs of the two nodes that each link connects, that is, maximizing

$$\max((v_k + v_l)(\beta_k(c_{kl}) + \beta_l(c_{lk})) - c_{kl} - c_{lk}),$$

s.t. $c_{kl} \geq 0$, $c_{lk} \geq 0$

[9] In a "restricted sense" because no relation holds when nodes belong to different sets, V or W, of the partition.

for all $k, l \in N$ $(k \neq l)$. Therefore, if both nodes invest in link kl, applying Kuhn–Tucker conditions, this requires

$$\frac{\partial v(\overline{g}^{\mathbf{c}})}{\partial c_{kl}} = (v_k + v_l)\beta_k'(c_{kl}) - 1 = 0,$$
$$\frac{\partial v(\overline{g}^{\mathbf{c}})}{\partial c_{lk}} = (v_k + v_l)\beta_l'(c_{lk}) - 1 = 0.$$

That is,

$$\beta_k'(c_{kl}) = \beta_l'(c_{lk}) = \frac{1}{v_k + v_l}.$$

However, this is possible only if $\beta_k'(0) > \frac{1}{v_k+v_l}$ and $\beta_l'(0) > \frac{1}{v_k+v_l}$. If only one of these conditions holds, only one node (the one with a technology with sufficient marginal strength at origin) can support the link efficiently. If neither of them holds, efficiency means that such a link will not exist. These conditions univocally characterize the efficient network. ■

Therefore, in the efficient network a link connects two nodes if and only if at least one of them has a sufficiently good technology for (6) to hold. Otherwise no link connects them. In other words, the efficient network is not empty if and only if $\beta_i'(0) > \frac{1}{v_i+v_j}$ for some pair of nodes.

We now prove that the directed network associated with an efficient investment has structural properties similar to those obtained for the directed network associated with a Nash investment in Proposition 2.

Proposition 11. *In the directed network $g^{\mathbf{c}}$, associated with an efficient investment $\mathbf{c} = (\mathbf{c}_i)_{i\in N}$, for all $i, j \in N$: $\mathbf{c} = (\mathbf{c}_i)_{i\in N}$, for all $i, j \in N$:*
(i) $v_i \le v_j \Rightarrow N_i^-(g^{\mathbf{c}}) \subseteq N_j^-(g^{\mathbf{c}}) \cup \{j\}$.
(ii) $\beta_i'(0) \le \beta_j'(0) \Rightarrow N_i^+(g^{\mathbf{c}}) \subseteq N_j^+(g^{\mathbf{c}}) \cup \{j\}$.
(iii) $\left|N_i^-(g^{\mathbf{c}})\right| < \left|N_j^-(g^{\mathbf{c}})\right| \Rightarrow N_i^-(g^{\mathbf{c}}) \subseteq N_j^-(g^{\mathbf{c}}) \cup \{j\}$.
(iv) $\left|N_i^+(g^{\mathbf{c}})\right| < \left|N_j^+(g^{\mathbf{c}})\right| \Rightarrow N_i^+(g^{\mathbf{c}}) \subseteq N_j^+(g^{\mathbf{c}}) \cup \{j\}$.

Proof. Let $\mathbf{c} = (\mathbf{c}_i)_{i\in N}$ be an efficient link-investment.

(i) By Proposition 10, for all $i \in N$:

$$N_i^-(g^{\mathbf{c}}) = \{k \in N \setminus \{i\} : \beta_k'(0) > \frac{1}{v_i + v_k}\}.$$

Thus, if $v_i = v_j$ any other node either links both i and j in $g^{\mathbf{c}}$ or neither of them. While if $v_i < v_j$ any other node $k \neq i, j$ which invests in a link with i is sure to invest in a link with j in $g^{\mathbf{c}}$. Therefore if $v_i \le v_j$ then

$$N_i^-(g^{\mathbf{c}}) \subseteq N_j^-(g^{\mathbf{c}}) \cup \{i\}.$$

(ii) Similarly, if $\mathbf{c} = (\mathbf{c}_i)_{i\in N}$ is an efficient link-investment:

$$N_i^+(g^{\mathbf{c}}) = \{j \in N \setminus \{i\} : v_i + v_j > 1/\beta_i'(0)\}, \text{ for all } i \in N.$$

Thus, if $\beta_i'(0) = \beta_j'(0)$ both i and j invest in the same links. While if $\beta_i'(0) < \beta_j'(0)$, node j invests in any link in which i does. Therefore if $\beta_i'(0) \le \beta_j'(0)$ then $N_i^+(g^{\mathbf{c}}) \subseteq N_j^+(g^{\mathbf{c}}) \cup \{j\}$.

(iii) Assume $N_i^-(g^{\mathbf{c}}) \nsubseteq N_j^-(g^{\mathbf{c}}) \cup \{j\}$. Then there is some $k \neq i, j$ s.t. $g_{ki}^{\mathbf{c}} > 0$ (i.e. $\beta_k'(0) > \frac{1}{v_i+v_k}$) and $g_{kj}^{\mathbf{c}} = 0$ (i.e. $\beta_k'(0) \le \frac{1}{v_j+v_k}$), which obviously implies that $v_i > v_j$, which by (i) implies $N_j^-(g^{\mathbf{c}}) \subseteq N_i^-(g^{\mathbf{c}}) \cup \{i\}$, which in turn implies $\left|N_j^-(g^{\mathbf{c}})\right| \le \left|N_i^-(g^{\mathbf{c}})\right|$ contradicting $\left|N_i^-(g^{\mathbf{c}})\right| < \left|N_j^-(g^{\mathbf{c}})\right|$.

(iv) The proof is similar to that of (iii). ■

The following counterexample explains why the implication of condition (iii) in Proposition 11 may not hold when $\left|N_i^-(g)\right| = \left|N_j^-(g)\right|$.

Example 5. Let $N = \{1, 2, 3, 4\}$, with values $v_1 = 50$, $v_2 = 20$ and $v_3 = v_4 = 10$, and technologies s.t. $\beta_1'(0) = 1/65$, $\beta_2'(0) = 1/70$, $\beta_3'(0) = 1/25$ and $\beta_4'(0) = 1/50$. In the efficient network $\overline{g}^{\mathbf{c}}$ (the graph of $g^{\mathbf{c}}$ is that of Fig. 3) all links are supported by only one node, i.e. the only investments are c_{31}, c_{41}, c_{12} and c_{32}. Thus, we have: $N_1^-(g^{\mathbf{c}}) = \{3, 4\}$ and $N_2^-(g^{\mathbf{c}}) = \{1, 3\}$, that is $\left|N_1^-(g^{\mathbf{c}})\right| = \left|N_2^-(g^{\mathbf{c}})\right|$, but

$N_1^-(g^{\mathbf{c}}) = \{3, 4\} \nsubseteq N_2^-(g^{\mathbf{c}}) \cup \{2\} = \{1, 2, 3\}$.■

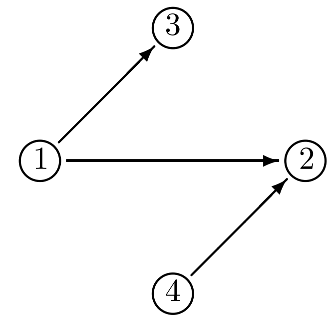

Fig. 4. Example 6.

The following counterexample shows that an efficient $\overline{g}^{\mathbf{c}}$ may not be a NSG-network.

Example 6. Let $N = \{1, 2, 3, 4\}$, with values and technologies such that the only positive efficient investments are c_{31}, c_{21} and c_{24}. That is, 2's technology is good enough to invest efficiently in links with 1 and 4, but not with 3, while 3 only invests in a link with 1, and no more investments are efficient (see Fig. 4). This is possible if $v_1 > v_4 > v_2 = v_3$ and $\beta_2'(0) > \beta_3'(0) > \beta_4'(0) = \beta_1'(0)$, with v_2, v_3, $\beta_4'(0)$ and $\beta_1'(0)$ sufficiently small. Thus, for the only efficient investment $\mathbf{c}$ we have: $N_1(\overline{g}^{\mathbf{c}}) = \{2, 3\}$ and $N_2(\overline{g}^{\mathbf{c}}) = \{1, 4\}$, that is, $\left|N_1(\overline{g}^{\mathbf{c}})\right| = \left|N_2(\overline{g}^{\mathbf{c}})\right|$ and

$N_1(\overline{g}^{\mathbf{c}}) = \{2, 3\} \nsubseteq N_2(\overline{g}^{\mathbf{c}}) \cup \{2\} = \{1, 2, 4\}$.■

Nevertheless, when the ranks of nodes according to their values and according to their technologies are the same, the efficient network is a NSG-network.

Proposition 12. *If values and technologies of nodes are equally ranked, i.e. if $v_i \le v_j \Leftrightarrow \beta_i'(0) \le \beta_j'(0)$, then the efficient network $\overline{g}^{\mathbf{c}}$ is a NSG-network which ranks nodes' neighborhoods according to that ranking.*

Proof. Let $\mathbf{c}$ be an efficient link-investment and assume $v_i \le v_j \Leftrightarrow \beta_i'(0) \le \beta_j'(0)$. Then there must exist $k \in N_j(\overline{g}^{\mathbf{c}}) \setminus N_i(\overline{g}^{\mathbf{c}})$. This implies that $v_i < v_j$ or $\beta_i'(0) < \beta_j'(0)$, which implies $v_i < v_j$ *and* $\beta_i'(0) < \beta_j'(0)$. Thus, as $N_i(\overline{g}^{\mathbf{c}}) = N_i^-(g^{\mathbf{c}}) \cup N_i^+(g^{\mathbf{c}})$, it follows from Proposition 11 that $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$. Assume now that $\left|N_i(\overline{g}^{\mathbf{c}})\right| = \left|N_j(\overline{g}^{\mathbf{c}})\right|$ and $k \in N_i(\overline{g}^{\mathbf{c}}) \setminus (N_j(\overline{g}^{\mathbf{c}}) \cup \{j\})$. This implies that $v_i > v_j$ or $\beta_i'(0) > \beta_j'(0)$, which implies $v_i > v_j$ *and* $\beta_i'(0) > \beta_j'(0)$. But then $N_j(\overline{g}^{\mathbf{c}}) \subsetneq N_i(\overline{g}^{\mathbf{c}})$, which contradicts $\left|N_i(\overline{g}^{\mathbf{c}})\right| = \left|N_j(\overline{g}^{\mathbf{c}})\right|$. ■

We now discuss the different architectures of the network $g^{\mathbf{c}}$ that emerge from an efficient investment vector in different contexts.

Homogeneity in technologies and values

Assume all nodes have the same value and the same individual technology. Then we have the following result.

Proposition 13. *If all nodes have the same value and the same technology, the only efficient network is either complete or the empty network.*

Proof. Assume $v_i = v$ and $\beta_i = \beta$, for all $i \in N$. From Proposition 10, in this case, if $\beta'(0) > 1/2v$, the efficient network is complete and for all pair $i, j \in N$: $c_{ij} = c$ s.t. $\beta'(c) = 1/2v$; while if $\beta'(0) \le 1/2v$, $c_{ij} = 0$ for all pair $i, j \in N$. ■

Homogeneity in technologies and heterogeneity in values

Assume now that a single technology β is available to all players, whose values may differ.

Proposition 14. *If all nodes have the same technology but their values may differ, the undirected efficient network $\overline{g}^{\mathbf{c}}$ is a nested split graph network where $\overline{g}_{ij}^{\mathbf{c}} > 0$ if and only if $\beta'(0) > \frac{1}{v_i+v_j}$.*

Proof. Assume $\beta_i = \beta$ for all $i \in N$. Thus, any two players i and j are connected by a link in an efficient network $g^{\mathbf{c}}$ if and only if $\beta'(0) > \frac{1}{v_i+v_j}$. That is, the set of *pairs* of players is partitioned into two sets according to their values: Those pairs i, j s.t. the sum of their values is greater than $\frac{1}{\beta'(0)}$ are connected by a link in which both nodes invest c s.t. $\beta'(c) = \frac{1}{v_i+v_j}$; while all other pairs are not connected by a link. Assume $\left|N_i(\overline{g}^{\mathbf{c}})\right| \le \left|N_j(\overline{g}^{\mathbf{c}})\right|$.

Case 1: If $\left|N_i(\overline{g}^{\mathbf{c}})\right| < \left|N_j(\overline{g}^{\mathbf{c}})\right|$, then there exists some $k \in N_j(\overline{g}^{\mathbf{c}}) \setminus N_i(\overline{g}^{\mathbf{c}})$. This can only occur if $v_i < v_j$, from which $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$ follows immediately.

Case 2: If $\left|N_i(\overline{g}^{\mathbf{c}})\right| = \left|N_j(\overline{g}^{\mathbf{c}})\right|$, two cases are possible:

Case 2.1: If $v_i = v_j$, then $N_i(\overline{g}^{\mathbf{c}}) \setminus \{j\} = N_j(\overline{g}^{\mathbf{c}}) \setminus \{i\}$. Finally, either $\overline{g}^{\mathbf{c}}_{ij} = 0$ or $\overline{g}^{\mathbf{c}}_{ij} > 0$, and in both cases $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$ follows.

Case 2.2: If $v_i < v_j$ or $v_i > v_j$, this along with their equal number of neighbors implies $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$. ■

Homogeneity in values and heterogeneity in technologies

Assume all players have the same value v, but their technologies differ.

Proposition 15. *If all nodes have the same value but technologies may differ, the undirected efficient network $\overline{g}^{\mathbf{c}}$ is a nested split graph network where $\overline{g}^{\mathbf{c}}_{ij} > 0$ if and only if $\max\{\beta_i'(0), \beta_j'(0)\} > 1/2v$.*

Proof. Assume $v_i = v$ for all $i \in N$. In an efficient network a player i should then invest in a link with j only if $\beta_i'(0) > 1/2v$. Therefore, a pair of nodes i, j is connected in an efficient network if and only if $\max\{\beta_i'(0), \beta_j'(0)\} > 1/2v$.

Assume $\left|N_i(\overline{g}^{\mathbf{c}})\right| \le \left|N_j(\overline{g}^{\mathbf{c}})\right|$.

Case 1: If $\left|N_i(\overline{g}^{\mathbf{c}})\right| < \left|N_j(\overline{g}^{\mathbf{c}})\right|$, then there exists some $k \in N_j(\overline{g}^{\mathbf{c}}) \setminus N_i(\overline{g}^{\mathbf{c}})$. This can only occur if $\beta_i'(0) < \beta_j'(0)$, from which it follows immediately that $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$.

Case 2: If $\left|N_i(\overline{g}^{\mathbf{c}})\right| = \left|N_j(\overline{g}^{\mathbf{c}})\right|$, two cases are possible:

Case 2.1: If $\beta_i'(0) = \beta_j'(0)$, then $N_i(\overline{g}^{\mathbf{c}}) \setminus \{j\} = N_j(\overline{g}^{\mathbf{c}}) \setminus \{i\}$. Finally, either $\overline{g}^{\mathbf{c}}_{ij} = 0$ or $\overline{g}^{\mathbf{c}}_{ij} > 0$, and in both cases $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$ follows.

Case 2.2: If $\beta_i'(0) \neq \beta_j'(0)$, this along with their equal number of neighbors implies $N_i(\overline{g}^{\mathbf{c}}) \subseteq N_j(\overline{g}^{\mathbf{c}}) \cup \{j\}$. ■

Bipartite networks

As we show below, the architecture of an efficient bipartite network is entirely similar to that of a Nash stable bipartite network.

Proposition 16. *The only efficient bipartite connected networks occur when the set of nodes is partitioned into two sets V and W s.t. nodes in W are more valuable than those in V (except perhaps the node in V which invests in the least valuable node in W if it is the only one), while nodes in V have better technologies than those in W, and it is an oriented network where all links are entirely supported by nodes in V.*

Proof. Assume that $\mathbf{c}$ is an efficient investment vector s.t. $\overline{g}^{\mathbf{c}}$ (and consequently $g^{\mathbf{c}}$) is a bipartite connected network. There is thus a partition of N into two sets V and W s.t. links connect only pairs of nodes which belong to different sets of that partition. By Proposition 10, for all i, j $(i \neq j)$:

$$c_{ij} > 0 \Leftrightarrow \beta_i'(c_{ij}) = \frac{1}{v_i + v_j}.$$

Now things are a bit more complicated than when dealing with stability because for efficiency, whether a node i invests or not in a link with j depends on β_i, v_j, *and* v_i. As $g^{\mathbf{c}}$ is not empty, there must be $c_{ij} > 0$ for some pair $i, j \in N$. Assume w.l.o.g. that $(i, j) \in V \times W$. For each $i \in V$, let $v_i^0 := \min\{v_l : l \in N_i^+(g^{\mathbf{c}})\}$, i.e. v_i^0 is the value of the least valuable node in W in which i invests. Obviously, out-neighborhoods of nodes in V are nested inversely according to the values of their v_i^0's, i.e. for all $i, j \in V$,

$$v_i^0 \le v_j^0 \Rightarrow N_i^+(g^{\mathbf{c}}) \supseteq N_j^+(g^{\mathbf{c}}).$$

Let i be a/the node in V which invests in the least valuable node in V, i.e. $N_i^+(g^{\mathbf{c}}) = \cup_{j\in V} N_j^+(g^{\mathbf{c}})$. Then, for all $l \in N_i^+(g^{\mathbf{c}})$, and all $k \in V \setminus \{i\}$, it must be that $v_k < v_l$, otherwise $c_{ik} > 0$, which would contradict $g^{\mathbf{c}}$ to be bipartite. That is, the value of any node in $N_i^+(g^{\mathbf{c}})$ is greater than the value of any node in V, except perhaps that of i if $v_i^0 < v_j^0$ for all $j \neq i$ (as far as for all $l \in W$, $\beta_l'(0) \le \frac{1}{v_i+v_l}$). But if there is another node $j \in V$ s.t. $v_i^0 = v_j^0 = \min\{v_l : l \in N_i^+(g^{\mathbf{c}})\}$, then it is also certain that $v_i < v_l$ and $v_j < v_l$.

The fact that nodes in W do not invest in links with nodes in W implies that for any node $l \in W$ and any node $k \in \cup_{j\in W} N_j^-(g^{\mathbf{c}})$, $\beta_k'(0) > \beta_l'(0)$. That is, technologies are better in V than in W. Thus the only links in $g^{\mathbf{c}}$ are those connecting nodes in $\cup_{j\in W} N_j^-(g^{\mathbf{c}})$ and nodes in $N_i^+(g^{\mathbf{c}})$, and those links are supported only by nodes in $\cup_{j\in W} N_j^-(g^{\mathbf{c}}) \subseteq V$. Finally, as $g^{\mathbf{c}}$ is connected, it must be that $N_i^+(g^{\mathbf{c}}) = W$ and $\cup_{j\in W} N_j^-(g^{\mathbf{c}}) = V$. ■

Notice that the *strict* out-nestedness in V of bipartite efficient networks is different from that of Nash stable bipartite networks in the sense that for a bipartite efficient network, for all $i, j \in V$:

$$\frac{1}{\beta_i'(0)} - v_i \ge \frac{1}{\beta_j'(0)} - v_j \Rightarrow N_i^+(g^{\mathbf{c}}) \subseteq N_j^+(g^{\mathbf{c}}),$$

while *strict* in-nestedness in W is similar to that for Nash stable bipartite networks, where for all $k, l \in W$:

$$v_k \le v_l \Rightarrow N_k^-(g^{\mathbf{c}}) \subseteq N_l^-(g^{\mathbf{c}}).$$

Note also that no bipartite network can be efficient when there is homogeneity in the values or technologies of nodes.

Efficiency vs. stability

The results in sections 4 and 5 show that Nash stable and efficient networks have a similar structure. Nevertheless, efficiency and Nash stability are incompatible. In equilibrium a link connecting nodes i and j receives an investment of $\beta_i(c_{ij})+\beta_j(c_{ji})$, with c_{ij} s.t. $\beta_i'(c_{ij}) = \frac{1}{v_j}$ and c_{ji} s.t. $\beta_j'(c_{ji}) = \frac{1}{v_i}$, if the link is supported by both. In the efficient network, a link connecting nodes i and j which is supported by both receives $\beta_i(c_{ij}) + \beta_j(c_{ji})$ with c_{ij} s.t. $\beta_i'(c_{ij}) = \frac{1}{v_i+v_j}$ and c_{ji} s.t. $\beta_j'(c_{ji}) = \frac{1}{v_i+v_j}$. A similar situation occurs if it is only supported by one. That is, in equilibrium players invest less than what is required for efficiency. It may even be the case that the efficient network has more links than the Nash network. Therefore, a Nash network is not efficient. This is not a surprise. On the contrary this is the rule in network formation models as players do not internalize the positive externality that a link they establish exerts on players they link to.

Example 7. Assume a set nodes of value $v = 2$ is partitioned into two sets, some nodes have a technology $\beta_1(0) = 0.1(1 - e^{-10x})$ and others $\beta_2(0) = 0.2(1 - e^{-2x})$, as in Example 2. In equilibrium a core–periphery network results where only nodes with technology β_1 invest and do so in links with all other nodes because

$$\beta_1'(0) = 1 > \frac{1}{2} > \beta_2'(0) = \frac{2}{5}.$$

In contrast with this, the efficient network is a complete network with stronger links where all nodes invest because

$$\beta_1'(0) = 1 > \frac{1}{4} \text{ and } \beta_2'(0) = \frac{2}{5} > \frac{1}{4}.$$

It may also occur that both the Nash network and the efficient network have the same architecture and only the strength of the links differ. For instance, assume the same partition of players with technologies β_1 and β_2, but with value 3 all. Then both the Nash and the efficient networks are complete, but links are weaker in the Nash network.

An extreme case occurs when the Nash network is empty, while the efficient is complete. This occurs, for instance, if all nodes have the same value $v = 0.8$ and the same technology $\beta_1 = 0.1(1 - e^{-10x})$ because $\frac{1}{1.6} < \beta_1'(0) = 1 < \frac{1}{0.8}$.

6. Model with budget constraints

In the model considered so far, node-players are assumed to have no budget constraints. This can be unrealistic in some situations, e.g. if cost means time. We now consider an enriched model $(N; (v_i, \beta_i, b_i)_{i\in N})$ where each player has a budget constraint, $b_i > 0$, which limits its total investment so that the payoff function

$$\pi_i(\mathbf{c}) = \sum_{j\in N\setminus\{i\}} (v_j(\beta_i(c_{ij}) + \beta_j(c_{ji})) - c_{ij}),$$

is constrained by

$$\sum_{k\in N\setminus\{i\}} c_{ik} \leq b_i. \tag{7}$$

Nash stability

Node i can obtain a profit from investing in a link with j only if $\beta_i'(0) > 1/v_j$ (otherwise $v_j\beta_i(c_{ij}) - c_{ij} \leq 0$ for all $c_{ij} > 0$). In these conditions, the dominant strategy of node i is to invest $\mathbf{c}_i = (c_{ij})_{j\in N\setminus\{i\}}$, where $c_{ij} = 0$ for all j s.t. $\beta_i'(0) \leq 1/v_j$, and $(c_{ij})_{j \text{ s.t. } \beta_i'(0)>1/v_j}$ is the solution of the optimization problem:

$$\max \sum_{j \text{ s.t. } \beta_i'(0)>1/v_j} (v_j\beta_i(c_{ij}) - c_{ij}),$$

$$\text{subject to} \sum_{j \text{ s.t. } \beta_i'(0)>1/v_j} c_{ij} \leq b_i.$$

Such a solution is certain to exist and to be unique because the objective function of $(c_{ij})_{j \text{ s.t. } \beta_i'(0)>1/v_j}$ is differentiable and strictly concave, and the feasible set is compact. Then Kuhn–Tucker conditions yield[10]

$$\begin{cases} (v_j\beta_i'(c_{ij}) - 1)_{j \text{ s.t. } \beta_i'(0)>1/v_j} + \lambda\mathbf{1} = \mathbf{0}, \\ \lambda(\sum_{j \text{ s.t. } \beta_i'(0)>1/v_j} c_{ij} - b_i) = 0, \\ \sum_{j \text{ s.t. } \beta_i'(0)>1/v_j} c_{ij} - b_i \leq 0, \\ \lambda \leq 0, \end{cases} \tag{8}$$

where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^{n_i}$ with $n_i = \left|\{j : \beta_i'(0) > 1/v_j\}\right|$. Therefore, in the optimal solution node i invests c_{ij} s.t. $\beta_i'(c_{ij}) = 1/v_j$ for all j s.t. $\beta_i'(0) > 1/v_j$ whenever the constraint holds with slackness (i.e. with $<$), while whenever the constraint holds without slackness (i.e. with $=$) i invests c_{ij} s.t. $\beta_i'(c_{ij}) = \frac{1-\lambda}{v_j}$ for all j s.t. $\beta_i'(0) > 1/v_j$. In other words, at the optimum, the marginal benefit of those nodes which are prevented by budget constraints from reaching the maximal benefit attainable without budget constraint (i.e. those for which $\beta_i'(c_{ij}) > 1/v_j$) are the same, i.e. $v_j\beta_i'(c_{ij}) = v_k\beta_i'(c_{ik})$ whenever $\beta_i'(c_{ij}) > 1/v_j$ and $\beta_i'(c_{ik}) > 1/v_k$. It is worth noting that the investments of some nodes in some links may decrease in the presence of budget constraints, but the set of links in which they invest remains the same as in the absence of budget constraints.

The preceding discussion yields the following result:

Proposition 17. *In the enriched model $(N; (v_i, \beta_i, b_i)_{i\in N})$, where each player has a budget constraint, $b_i > 0$, there is a unique Nash investment vector in dominant strategies characterized by Kuhn–Tucker conditions (8) and the architecture of the Nash stable network is the same as in the absence of budget constraints.*

As a corollary, the properties of the Nash networks in the model without constraints established in Proposition 2 continue to hold for the Nash networks in the model with constraints. Consequently the structures that arise in equilibrium when there is homogeneity either in values or technologies are the same as those that arise without budget. Namely, core–periphery structures as established in Propositions 7 and 8. Recently, Kinateder and Merlino (2023) introduce a local public good game in a model in which players face a budget and must decide how to allocate it between links, a local public good and a private good. A player links to others in order to free ride on others' public good provision. They obtain that Nash equilibrium networks are core–periphery graphs, a prediction that is robust to several extensions.

Efficiency

In what follows, we denote $s_{kl} := v_k + v_l$. In view of Proposition 10, in an efficient investment $\mathbf{c}$, $c_{ij} > 0$ only if $\beta_i'(0) > \frac{1}{s_{ij}}$, otherwise $c_{ij} = 0$. Therefore the efficient investment $\mathbf{c} = (\mathbf{c}_i)_{i\in N}$, with $\mathbf{c}_i = (c_{ij})_{j\in N\setminus\{i\}}$, subject to budget constraint (7), where $c_{ij} = 0$ for all j s.t. $\beta_i'(0) \leq \frac{1}{s_{ij}}$, and $(c_{ij})_{ij \text{ s.t. } \beta_i'(0)>\frac{1}{s_{ij}}}$ is the solution of the optimization problem:

$$\max \sum_{i\in N}\sum_{j\in N\setminus\{i\}} (s_{ij}\beta_i(c_{ij}) - c_{ij}),$$

$$\text{subject to} \sum_{j\in N\setminus\{i\}} c_{ij} \leq b_i \quad (\forall i \in N).$$

$$c_{ij} \geq 0 \quad (\forall i \in N,\ \forall j \in N \setminus \{i\})$$

Such a solution is certain to exist and to be unique because the objective function is differentiable and strictly concave w.r.t. the significant (i.e., possibly nonzero) variables, that is to say those c_{ij} s.t. $\beta_i'(0) > 1/s_{ij}$, and the feasible set is compact. Kuhn–Tucker conditions yield[11]

$$\begin{cases} (s_{ij}\beta_i'(c_{ij}) - 1)_{i\in N, j\in N\setminus\{i\}} + \sum_{i\in N} \lambda_i\boldsymbol{\eta}_i = \mathbf{0}, \\ \lambda_i(\sum_{j\in N\setminus\{i\}} c_{ij} - b_i) = 0 \ (\forall i \in N) \\ \sum_{j\in N\setminus\{i\}} c_{ij} \leq b_i \ (\forall i \in N) \\ \lambda_i \leq 0 \ (\forall i \in N). \end{cases} \tag{9}$$

where $\boldsymbol{\eta}, \mathbf{0} \in \mathbb{R}^{n(n-1)}$, and $\boldsymbol{\eta} := (\eta_{ij})_{i\in N, j\in N\setminus\{i\}}$ s.t.

$$\eta_{ij} = \begin{cases} 1 & \text{if } \beta_i'(0) > 1/s_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, in the efficient network under budget constraints, node i invests c_{ij} s.t. $\beta_i'(c_{ij}) = 1/s_{ij}$ for all j s.t. $\beta_i'(0) > 1/s_{ij}$ whenever i's constraint holds with slackness (i.e. with $<$) in the optimum, while whenever the constraint holds without slackness (i.e. with $=$), i invests c_{ij} s.t. $\beta_i'(c_{ij}) = \frac{1-\lambda_i-\lambda_j}{s_{ij}}$ for all j s.t. $\beta_i'(0) > 1/s_{ij}$. In other words, at the optimum, the marginal benefit of those nodes which are prevented by budget constraints from reaching the efficient investment attainable without budget constraints (i.e. those for which $\beta_i'(c_{ij}) > 1/s_{ij}$) are the same, i.e. $\beta_i'(c_{ij}) = \beta_i'(c_{ik})$ whenever $\beta_i'(c_{ij}) > 1/v_j$ and $\beta_i'(c_{ik}) > 1/v_k$. It is worth noting that the investments of some nodes in some links may decrease in the presence of a budget constraint, but the set of links in which they invest remains the same as in the absence of budget constraints.

The preceding discussion yields the following result:

Proposition 18. *In the enriched model $(N; (v_i, \beta_i, b_i)_{i\in N})$, where each player has a budget constraint, $b_i > 0$, there is a unique efficient investment vector characterized by Kuhn–Tucker conditions (9) and the architecture of the efficient network is the same as in the absence of budget constraints.*

As a corollary, the properties of the efficient networks in the model without constraints established in Proposition 11 continue to hold for the efficient networks in the model with constraints.

[10] Constraints $c_{ij} \geq 0$ play no role because every j-component of the gradient of the objective function is positive when $c_{ij} = 0$.

[11] Constraints $c_{ij} \geq 0$ play no role because every significant j -component of the gradient of the objective function is positive when $c_{ij} = 0$.

7. Concluding remarks

We introduce a very simple model of network formation and obtain characterizing conditions for both Nash networks and efficient networks. In spite of the simplicity of the model, it turns out that a variety of architectures which are significant in different contexts (core–periphery, nested split graph networks and bipartite networks) emerge for different configurations of the many degrees of freedom (values and technologies) of the model. In fact, equilibrium and efficiency lead to similar architectures, but they are incompatible because in equilibrium players do not internalize the positive externalities that the links they establish have on players they link to. A variation of the model introducing budget constraints leads to the same structures of Nash and efficient networks.

We have adopted an abstract theoretical approach, just exploring the implications of a set of assumptions that specify a simple model. In fact, in the presentation of the model we only associate with it a stylized interpretation in terms of information or contact networks. Disregarding other possible particular applications of the different variations of the model, we do not attach ourselves tightly to any particular interpretation. This is not an applied paper. However, we conclude with a few hints of possible contexts where some variations of the model, suitably adapted, could be applied and give rise to further research.[12]

An example is research collaboration, where more productive researchers hold greater value to other researchers who may want to collaborate with them (heterogeneity in values). On the other hand, different researchers may have different abilities to establish collaboration with others (heterogeneity in linking technologies) for different reasons (prestige of their institution, former collaborations). According to the model, homogeneity in only one side would lead to a core core–periphery collaboration network (Propositions 7 and 8). However, to some extent, those productive researchers may very well be better also at extracting value from their coauthors because they know interesting fields of application. In such case values and linking technologies correlate and this would lead to a hierarchical nested split collaboration network in equilibrium (Proposition 3), which intuitively seems close to what actual coauthor networks of a discipline would look like.

The structure of the only bipartite networks that can arise in equilibrium (Proposition 9) brings to mind the partition between poor countries which have poor technologies but are rich in certain valuable raw materials, and rich countries with good technologies but which are not rich in such raw materials. In this case, the need for raw materials is at the origin of value, while linking technology is not the technology to extract them, but the capital to invest in it, usually associated with the possession of that technology.

As regards to the model with budget constraints, a possible extension would be a model where players can invest residual investment budget into some generic socialization effort. An example would be job search via professional contacts or by searching vacancies online.

Finally, subsidizing by a planner who wants to implement the efficient network is another possible extension. In the general model this would require subsidizing differently each two nodes involved in each link, because this depends on each node's value and technology. Even in the case of homogeneity in technologies, subsidizing efficiently would require subsidizing differently the investments of each of the two nodes in their link if their values differ.

CRediT authorship contribution statement

Norma Olaizola: Writing – review & editing, Writing – original draft. **Federico Valenciano:** Writing – review & editing, Writing – original draft.

Data availability

No data was used for the research described in the article.

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[12] Some of them suggested by the referees.