A One-Stage Model of Link Formation and Payoff Division

Slikker, M.; van den Nouweland, C.G.A.M.

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Marco Slikker[b] Anne van den Nouweland[c]

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Abstract

In this paper we introduce a strategic form model in which cooperation structures and divisions of the payoffs are determined simultaneously. We analyze the cooperation structures and payoff divisions that result according to several equilibrium concepts. We find that essentially no cycles will result and that a player need not profit from a central position in a cooperation structure.

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[a]The authors are grateful to Henk Norde and Stef Tijs for their helpful suggestions and comments.
[b]Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: M.Slikker@kub.nl

[c]Department of Economics, 435 PLC, 1285 University of Oregon, Eugene, OR 97403-1285, U.S.A.

1 Introduction

An important topic in economic research is the question which cooperation structure between economic agents will form. Another important issue deals with division of profits resulting from a specific cooperation structure. Since the cooperation structure that forms can influence the payoffs to the players, it is natural to integrate these two questions.

In the literature there is a number of models trying to link these two questions. We mention *Binmore* (1985), *Chatterjee*, *Dutta*, *Ray*, and *Sengupta* (1993), *Gul* (1989), and *Perry* and *Reny* (1994). These papers all explicitly model the negotiation proces associated with characteristic function games.

Of particular relevance for this paper is the work of *Aumann* and *Myerson* (1988) and *Dutta*, *Nouweland*, and *Tijs* (1996). Both consider a two-stage model of link formation and payoff division. Here we introduce a one-stage model which simultaneously determines a cooperation structure and a division of the payoffs.

Aumann and *Myerson* (1988) introduce a two-stage game for modelling the process of link formation and payoff division. The first stage models the negotiation process and results in a cooperation structure. According to an exogeneously given rule of order pairs of players are given the opportunity to form a link. This link is formed if and only if both players agree on forming this link. After a link is formed, it cannot be broken in a further stage of the game. The process of link formation stops when after the last link has formed all pairs of players who have not yet formed a link have had a final opportunity to form a link. In the second stage of the game the payoff to the players is determined according to the *Myerson value* (*Myerson* (1977)) of the communication situation that results from the original game and the links formed in the previous stage. *Aumann* and *Myerson* (1988) conclude that according to this game in extensive form not necessarily the grand coalition will result, which is often implicitly assumed when cooperative games are considered.

Dutta et al. (1996) consider an alternative two-stage model of link formation and payoff division, first introduced by *Myerson* (1991). As in *Aumann* and *Myerson* (1988) the links form in the first stage and in the second stage the payoff to the players is determined by an exogeneously given allocation rule. *Dutta et al.* (1996) consider a class of allocation rules, which contains the Myerson value. The first stage is modelled as a game in strategic form, where each player chooses a set of players he wants to form a link with. A link between two players is formed if and only if both players want to form this link. *Dutta et al.* (1996) find that every structure can be supported by a Nash equilibrium. They also find that the strategy vector that results in the full cooperation structure belongs to the set of undominated Nash equilibria and to the set of coalition proof Nash equilibria. Furthermore, they find that according to the undominated Nash equilibrium concept and the coalition proof Nash equilibrium concept only structures that result in the same payoff division as the full cooperation structure, are formed.

In *Slikker, Dutta, Nouweland,* and *Tijs* (1996) it is shown that the game considered by *Dutta et al.* (1996) is a weighted potential game if and only if the allocation rule is a weighted Myerson value. *Monderer* and *Shapley* (1996) point out that the argmax set of a weighted potential does not depend on the particular choice of the potential and hence can be used as an equilibrium refinement. *Slikker et al.* (1996) find that according to this equilibrium refinement the only structures that form are structures that result in the same payoff division as the complete structure.

The model in this paper is inspired by *Borm* and *Tijs* (1992) and the idea that cooperation structure formation and payoff division go hand-in-hand. *Borm* and *Tijs* (1992) introduce a non-cooperative model, where players choose a coalition they want to join. Furthermore, every player claims an amount he wants to receive for joining this coalition. It is shown that the strong core elements of a superadditive non-transferable utility game correspond to the strong Nash equilibria of this *claim game*.

This paper describes and analyzes a one-stage model of link formation and payoff division. Players claim links they want to form and certain amounts they want to receive for the formation of a link. Naturally, a player can also indicate he is not willing to cooperate with a specific player. The eventual cooperation structure and payoff division depend on the links the players are willing to form and whether the associated claims are attainable.

The goal of this paper is to analyze the structures and payoff divisions that result from the one-stage strategic form model. We will first consider the Nash equilibria of this game. Since this results in a relatively large set of solutions we will also use some refinements of the Nash equilibrium concept. Specifically, we analyze the strong Nash equilibria and the coalition proof Nash equilibria. Our main conclusion is that essentially no cycles will result in the cooperation structure. For a game with at least three players the absence of cycles implies that the full cooperation structure does not form. Furthermore, we find that strong Nash equilibria often result in at least one player receiving a payoff of zero. A coalition proof Nash equilibrium of a three-person game results, under some mild conditions on the underlying game, in a spanning tree if the core is non-empty. If the core of the underlying game is empty there exist coalition proof Nash equilibria resulting in exactly one link. If some severe restrictions on the underlying game are satisfied, the empty graph is also supported by a coalition proof Nash equilibrium.

The plan of the present paper is as follows. Section 2 deals with definitions and the formal description of our model of link formation and payoff division. In section 3 we analyze the Nash equilibria and strong Nash equilibria. Coalition proof Nash equilibria are considered in section 4. We conclude in section 5.

2 The model

In this section we will introduce a new model of link formation. We will first provide some definitions with respect to cooperative games and cooperation structures.

Let (N, v) be a cooperative game, where $N = \{1, \ldots, n\}$ is the player set and v the characteristic function, which assigns to every subset S of N a value $v(S)$, with $v(\emptyset) = 0$.[1] Throughout this paper, we will assume that $v(N) > 0$ and $v(N) \geq v(S)$ for all $S \subseteq N$. Furthermore, we will assume that v is *zero-normalized*, so $v(\{i\}) = 0$ for all $i \in N$.

A payoff vector $x \in \mathbb{R}^N$ is called an *imputation* if x is individually rational, $x_i \geq v(\{i\})$ for all $i \in N$, and x is efficient, $\sum_{i \in N} x_i = v(N)$. The set of imputations of (N, v) is denoted by $I(v)$. The *core* of the game (N, v) will be denoted by $C(v) = \{x \in I(v) \mid \forall S \subseteq N : \sum_{i \in S} x_i \geq v(S)\}$.

A cooperation structure is represented by an undirected graph (N, L), with N the set of vertices and L a set of edges, which we will mostly refer to as links. A link between two players indicates that these two players can communicate directly with each other. When two players are not connected directly they might be able to communicate with each other indirectly, via one or more other players. When two players can communicate with each other, directly or indirectly, we call these players connected. So two players are connected if and only if there exists a path in the graph (N, L) between these two players. The notion of connectedness induces a partition of the player set into *communication components*, where two players are in the same communication component if and only if they are connected. The communication component containing player i is denoted by $C_i(L)$. The communication possibilities within a subset S of N are described by $L(S) := \{\{i, j\} \in L \mid i, j \in S\}$. This set induces a partition of S into communication components and the set of these communication components will be denoted by S/L.

Myerson (1977) studied communication situations (N, v, L) where (N, v) is a cooperative game and (N, L) a (communication) graph. He introduced the *graph-restricted game* (N, v^L), defined by $v^L(S) := \sum_{C \in S/L} v(C)$, for all $S \subseteq N$. Communication only

[1] $S \subseteq N$ denotes that S is a subset of N and $S \subset N$ denotes that S is a strict subset of N.

occurs within a communication component, so the value of a coalition in the graph-restricted game is defined as the sum of the values of the communication components of this coalition in the original game.

In the current paper we are interested in the formation of cooperation structures and the simultaneous determination of payoffs. We will model the process of link formation and payoff division as a one-stage game in strategic form. In this integrated approach players demand claims they want to receive for the formation of links. Let (N, v) be a cooperative game. We define the associated *link and claim game* $\Gamma(v)$ which is described by the $2n-$tuple $(X_1, \ldots, X_n, K_1, \ldots, K_n)$, where X_i is the strategy set of player i and K_i the payoff function of player i which assigns to every strategy profile $c = (c^j)_{j \in N} \in X := \prod_{j \in N} X_j$ a payoff $K_i(c) \in \mathbb{R}$.

Define $A := \mathbb{R}_+ \cup \{P\}$ with $\mathbb{R}_+ = [0, \infty)$. The strategy space of player i is described by

$$X_i := \left\{ c^i \in A^N \mid c^i_i = P \right\}.$$

Here, $c^i_j \in \mathbb{R}_+$ indicates that player i is willing to form a link with player j, and he claims an amount c^i_j for forming this link.[2] Furthermore, $c^i_j = P$ indicates that player i is not willing to form a link with player j. Clearly, player i cannot form a link with himself, so for all strategies $c^i \in X_i$ it holds that $c^i_i = P$.

Consider a strategy profile $c \in X$. To determine the resulting payoff to the players we have to determine the cooperation structure that results. Firstly, we determine the set of links the players are willing to form according to strategy profile c. A link between two players is in this set if and only if both players are willing to form this link. We will denote this set by $l(c)$, so

$$l(c) := \left\{ \{i, j\} \mid c^j_i, c^i_j \in \mathbb{R}_+ \right\}.$$

The graph $(N, l(c))$ partitions the player set into components. Such a component can actually be formed if and only if the total payoff the players in this component claim to form the links between them, is less than or equal to the value of this component. Otherwise cooperation will break down and all players in this component will become singletons. The set of links that will actually form will be denoted by $L(c)$, so

$$L(c) := \left\{ \{i, j\} \in l(c) \mid \sum_{\{k,m\} \in l(c),\ k,m \in C_i(l(c))} \left(c^k_m + c^m_k \right) \leq v(C_i(l(c))) \right\}.$$

[2]We restrict ourselves to non-negative claims for two reasons. First, it is intuitive to exclude negative claims since players seem to be worse off by claiming a negative amount. Second, it holds that if we allow negative claims in our model, every cooperation structure and payoff vector resulting from a Nash equilibrium can also be supported by a Nash equilibrium with non-negative claims.

This construction implies that if a player is too greedy by claiming a large amount on one of the links, this player can end up being isolated, receiving zero payoff. So, greediness is punished severely.

Eventually, the players will receive the sum of the claims corresponding to the links that are actually formed. So,

$$K_i(c) := \sum_{j:\{i,j\} \in L(c)} c_j^i,$$

where the empty sum is defined to be equal to zero.

For notational convenience we write $c^S := (c^i)_{i \in S}$, $c_S^j := (c_i^j)_{i \in S}$, $X_S := \prod_{i \in S} X_i$, $K_S(c) := (K_i(c))_{i \in S}$, and $K(c) := K_N(c)$. Furthermore, we define $c^{-k} := (c^j)_{j \in N \setminus \{k\}}$, $c_{-k}^j := (c_i^j)_{i \in N \setminus \{k\}}$, $X_{-k} := \prod_{j \in N \setminus \{k\}} X_j$, and $K_{-k}(c) := (K_j(c))_{j \in N \setminus \{k\}}$. Finally, we write $c^{-S} := c^{N \setminus S}$ and $c_{-S}^j := c_{N \setminus S}^j$.

We will now describe some equilibrium concepts. Recall that a strategy profile $c \in X$ is a Nash equilibrium if and only if there is no player $i \in N$ that can improve his payoff by unilaterally deviating from c. We will denote the set of Nash equilibria of a strategic form game Γ by $\mathrm{NE}(\Gamma)$. We will also consider some Nash equilibrium refinements.

A strategy profile c is a *strong Nash equilibrium* if there is no coalition $S \subseteq N$ and strategy profile $\hat{c}^S \in X_S$ such that

$$K_S(\hat{c}^S, c^{N \setminus S}) \geq K_S(c),$$

with the inequality being strict for at least one player $i \in S$. The set of all strong Nash equilibria of the game Γ will be denoted by $\mathrm{SNE}(\Gamma)$.

A less restrictive refinement is the *coalition-proof Nash equilibrium* (CPNE). First we will introduce some notation. For every $T \subset N$ and $\hat{c}^{N \setminus T} \in X_{N \setminus T}$, let $\Gamma(\hat{c}^{N \setminus T})$ be the game induced on the players of T by the strategies $\hat{c}^{N \setminus T}$, so

$$\Gamma(\hat{c}^{N \setminus T}) = ((X_i)_{i \in T}, (K_i^*)_{i \in T})$$

where for all $i \in T$, $K_i^* : X_T \to \mathbb{R}$ is given by $K_i^*(c^T) := K_i(c^T, \hat{c}^{N \setminus T})$ for all $c^T \in X_T$.

In a one-player game with player set $N = \{i\}$, $\hat{c}^i \in X = X_i$ is a CPNE of $\Gamma = (X_i, K_i)$ if $\hat{c}^i$ maximizes K_i over X_i. Let Γ be a game with $n > 1$ players. Assume that coalition-proof Nash equilibria have been defined for games with less than n players. Then a strategy profile $\hat{c} \in X_N$ is called *self-enforcing* if for all $T \subset N$, $\hat{c}^T$ is a CPNE of $\Gamma(\hat{c}^{N \setminus T})$. Now, the stategy vector $\hat{c}$ is a CPNE of Γ if $\hat{c}$ is self-enforcing and there is no other self-enforcing strategy profile $c \in X_N$ with $K_i(c) > K_i(\hat{c})$ for all $i \in N$.

3 Analysis of the model

In this section we will analyze the model defined in the previous section. Our main question is which cooperation structures result according to some equilibrium refinements and what are the associated payoffs. For example, we want to know which payoff vectors in the imputation set of the underlying cooperative game are attainable in the corresponding link and claim game according to various equilibrium refinements.

First we will analyze the cooperation structures that result from Nash equilibria. Before moving on to the general analysis, consider the following example.

Example 3.1 Let (N, v) be the 3-person game with

$$v(S) = \begin{cases} 0 & , \text{ if } |S| = 1 \\ 30 & , \text{ if } |S| = 2 \\ 72 & , \text{ if } S = N \end{cases} .$$

Several cooperation structures are supported by Nash equilibria. The empty graph results from the strategy profile $c^i = (P, P, P)$ for all $i \in N$. Since no player can unilaterally enforce the formation of a link, this is a Nash equilibrium. The graph with only the link $\{1, 2\}$ results from the strategy profile c with $c^1 = (P, 10, P)$, $c^2 = (20, P, P)$, and $c^3 = (P, P, P)$. This is a Nash equilibrium and results in the payoff vector $(10, 20, 0)$. Clearly, there are other payoff vectors resulting from Nash equilibria where only the link $\{1, 2\}$ forms. By symmetry it follows that there exist Nash equilibria resulting in the unique link $\{1, 3\}$ or the unique link $\{2, 3\}$.

Now consider a graph with two links, $\{1, 2\}$ and $\{2, 3\}$. The strategy profile c with $c^1 = c^3 = (P, 24, P)$ and $c^2 = (12, P, 12)$ is a Nash equilibrium that results in the cooperation structure with these two links and the payoff vector $(24, 24, 24)$. The presence of two links implies that one player is a *middleman*, a player who is directly connected with at least two other players. According to c player 2 is the middleman. Player 2 can break a link with a player and still remain directly connected with another player. This deviation possibility restricts the set of payoff vectors that can be attained in a Nash equilibrium. Again note that, by symmetry, it also holds that the other graphs with exactly two links are supported by Nash equilibria.

Finally, consider a strategy profile c that results in a cooperation structure with three links. This implies $c^i_j \in \mathbb{R}_+$ for all $i, j \in N$, $i \neq j$, and $c^1_2 + c^1_3 + c^2_1 + c^2_3 + c^3_1 + c^3_2 \leq 72$. Note that if $c \in \text{NE}(\Gamma(v))$ it holds that $c^1_2 + c^1_3 + c^2_1 + c^2_3 + c^3_1 + c^3_2 = 72$, since otherwise every player could unilaterally deviate to a strategy which gives him a strictly higher payoff, by simply raising one of his claims. However, $c^1_2 + c^1_3 + c^2_1 + c^2_3 + c^3_1 + c^3_2 = 72$ implies that at least one player claims a positive amount. Assume without loss of generality that $c_2^1 > 0$. A restriction for player 1 to receive c_2^1 is that player 2 wants to form a link with player 1. So, whenever player 2 changes his strategy such that he wants to form a link with player 3 only, player 1 will not receive claim c_2^1. Since the value $v(N)$ can still be obtained, player 2 can obtain the amount c_2^1. Consider strategy $\hat{c}^2 = (P, P, c_1^2 + c_3^2 + c_2^1)$. Then $K_2(c_1, \hat{c}_2, c_3) = c_1^2 + c_3^2 + c_2^1 > c_1^2 + c_3^2 = K_2(c)$, and we conclude that c is not a Nash equilibrium.

Summarizing, we find that the full graph is the only cooperation structure that is not supported by a Nash equilibrium in the link and claim game corresponding to the 3-player game in this example. □

In example 3.1 we saw that the full cooperation structure is not supported by a Nash equilibrium. The reasoning that leads to this conclusion can be given for every strategy profile that results in a cooperation structure with a cycle and at least one positive claim corresponding to one of the links in the cycle. This implies that in a cooperation structure supported by a Nash equilibrium only cycles with corresponding claims all equal to zero can result. We formalize this in the following theorem.

Theorem 3.1 Let (N, v) be an n-person cooperative game and $\Gamma(v)$ the corresponding link and claim game. For every strategy $c \in \mathrm{NE}(\Gamma(v))$ it holds that $(N, L(c))$ is cycle-free or, if $(N, L(c))$ is not cycle-free, all claims corresponding to the links in the cycles are equal to zero.

Proof: Let (N, v) be a cooperative game and $c \in NE(\Gamma(v))$. Suppose the graph $(N, L(c))$ contains a cycle and suppose that there exists a positive claim of a player on one of the links in the cycle. Assume without loss of generality that $(N, L(c))$ contains the cycle $(1, 2, \ldots, k, 1)$ and that $c_2^1 > 0$. Let $\hat{c}^2 = (c^2_{-\{1,3\}}, \hat{c}_1^2, \hat{c}_3^2)$ with $\hat{c}_1^2 = P$ and $\hat{c}_3^2 = c_1^2 + c_3^2 + c_2^1$. Then, it follows directly that

$$K_2(c^{-2}, \hat{c}^2) = K_2(c) + c_2^1 > K_2(c),$$

and hence, $c \notin NE(\Gamma(v))$. □

The implication of this theorem is that, if $|N| \geq 3$ the full graph will never be supported by a Nash equilibrium.

In the following theorem we will describe the payoff vectors in the imputation set that, under some mild conditions, are supported by Nash equilibria. From this theorem and the subsequent example it follows that there exist 3-person cooperative games where not all payoff vectors in the imputation set are supported by Nash equilibria.

Theorem 3.2 Let (N, v) be a cooperative game with $|N| = 3$, $v(N) > v(S)$ for every S with $|S| = 2$, and let $x \in I(v)$. Then there exists a $c \in NE(\Gamma(v))$ with $K(c) = x$ if and only if at least two of the following constraints hold:

$$\begin{aligned} x_1 + x_2 &\geq v(\{1,2\}), & (1)\\ x_1 + x_3 &\geq v(\{1,3\}), & (2)\\ x_2 + x_3 &\geq v(\{2,3\}). & (3) \end{aligned}$$

Proof: First we will prove the if-part. Assume without loss of generality that the first two inequalities hold. Then c with $c^1 = (P, \frac{x_1}{2}, \frac{x_1}{2})$, $c^2 = (x_2, P, P)$, and $c^3 = (x_3, P, P)$ is a Nash equilibrium with $K(c) = x$.

To prove the only-if-part, assume that $c \in NE(\Gamma(v))$ is such that $K(c) = x$ and assume that at most one of the three inequalities holds. Since $x \in I(v)$, $v(N) > 0$, and $v(N) > v(S)$, for all $|S| = 2$, the graph $(N, L(c))$ is connected. By theorem 3.1 it follows that exactly two links will form. Let player i be the middleman. Since at most one of the inequalities (1), (2), and (3) holds there exists $j \in N \backslash \{i\}$ with $x_i + x_j < v(\{i,j\})$. Player i can improve his payoff by breaking the link with the third player, player k, and claiming an amount of $v(\{i,j\}) - x_j$ on the link with player j. Hence, c is not a Nash equilibrium. □

We apply theorem 3.2 in the following example.

Example 3.2 Consider the game (N, v) with $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & , \ |S| = 1 \\ 60 & , \ S = \{1,2\} \\ 30 & , \ S = \{1,3\} \\ 40 & , \ S = \{2,3\} \\ 90 & , \ S = N \end{cases}.$$

The set of payoff vectors in the imputation set that are obtained from Nash equilibria can easily be identified using theorem 3.2. The set of these payoff vectors is represented by the shaded area in figure 1 on page 10. Figure 1 represents the imputation set, the intersection of the hyperplane $x_1 + x_2 + x_3 = 90$ with $\mathbb{R}^3_+$. □

We see in example 3.2 that not all payoff vectors in the imputation set can be supported by a Nash equilibrium. However, every core-element satisfies inequalities (1), (2), and (3), and hence, in a 3-player game every core-element can be supported by a

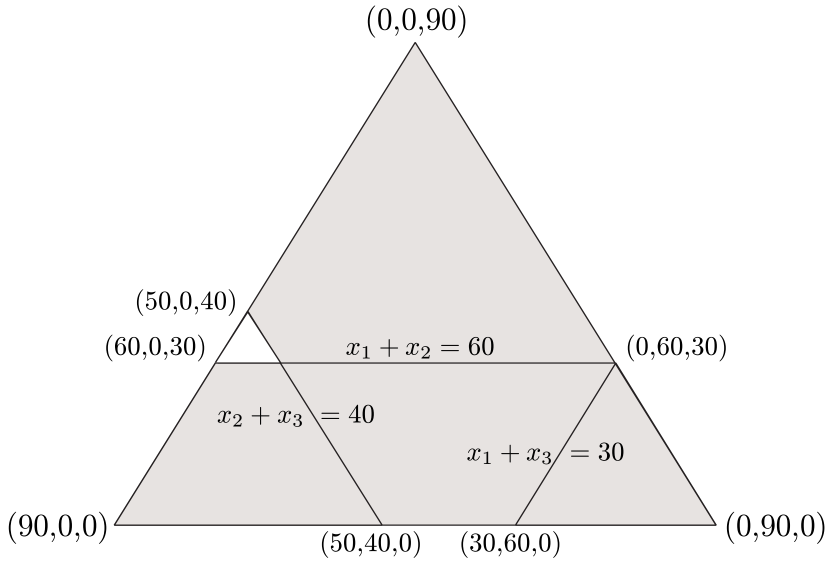

Figure 1: *Shaded area represents the payoff vectors in the imputation set resulting from Nash equilibria in example 3.2*

Nash equilibrium. In the following theorem we extend this result to general n-person cooperative games.

Theorem 3.3 Let (N, v) be an n-person cooperative game and $x \in C(v)$. Then there exists $c \in NE(\Gamma(v))$ with $K(c) = x$.

Proof: Let x be a core-allocation of (N, v). Now consider the strategy profile c defined by

$$\begin{aligned} c_k^1 &= \tfrac{x_1}{n-1} \quad , \text{ for all } k \in N\backslash\{1\} \\ c_1^1 &= P \quad , \end{aligned}$$

and, for every $j \in N\backslash\{1\}$

$$\begin{aligned} c_k^j &= P \quad , \text{ for all } k \in N\backslash\{1\} \\ c_1^j &= x_j \quad . \end{aligned}$$

This strategy results in a star graph with player 1 as the central player and payoff vector x. We will show that there is no player who can improve his payoff by unilaterally deviating. First, consider an arbitrary deviation by player 1, $\hat{c}^1 \in X_1$. By construction of c it follows that every player in $C_1(L(\hat{c}^1, c^{-1}))\backslash\{1\}$ will receive the same payoff as he received according to strategy c. Since $K(c) \in C(v)$ this implies

$$\sum_{j \in C_1(L(\hat{c}^1, c^{-1}))\backslash\{1\}} K_j(\hat{c}^1, c^{-1}) + x_1 = \sum_{j \in C_1(L(\hat{c}^1, c^{-1}))} x_j \geq v(C_1(L(\hat{c}^1, c^{-1}))).$$

Since, by definition of payoff vector K,

$$\sum_{j \in C_1(L(\hat{c}^1, c^{-1}))} K_j(\hat{c}^1, c^{-1}) \leq v(C_1(L(\hat{c}^1, c^{-1}))),$$

we conclude that $K_1(\hat{c}^1, c^{-1}) \leq x_1$. Hence, player 1 cannot profitably deviate. Now, consider player j, with $j \in N \backslash \{1\}$. The construction of c implies that player j, by unilaterally deviating, can be connected with other players via player 1 only. To get a positive payoff player j has to be connected with at least one player. Since the strategies of the other players remain the same it follows that player j cannot improve his payoff by unilaterally deviating. We conclude that $c \in NE(\Gamma(v))$ □

We have seen that Nash equilibria result in a relatively large set of possible payoff allocations. To restrict the possible payoffs we look at a subset of the set of Nash equilibria, the strong Nash equilibria. The following theorem shows that every strong Nash equilibrium results in a payoff vector in the core.

Theorem 3.4 Let (N, v) be an n-person cooperative game and $c \in \mathrm{SNE}(\Gamma(v))$. Then $K(c) \in C(v)$.

Proof: Suppose $x = K(c) \notin C(v)$. Since $x \notin C(v)$ there exists a coalition S that receives in total strictly less than its value. Let $i \in S$. Then the deviating strategy profile $\tilde{c}^S$ with

$$\begin{array}{rcll}
\tilde{c}^j_k := & P & , \text{ for all } j \in S \backslash \{i\}, k \in N \backslash \{i\}, \\
\tilde{c}^j_i := & x_j + \frac{v(S) - \sum_{k \in S} x_k}{|S|} & , \text{ for all } j \in S \backslash \{i\}, \\
\tilde{c}^i_k := & P & , \text{ for all } k \in (N \backslash S) \cup \{i\}, \\
\tilde{c}^i_k := & \frac{x_i + \frac{v(S) - \sum_{k \in S} x_k}{|S|}}{|S|-1} & , \text{ for all } k \in S \backslash \{i\},
\end{array}$$

results in a strictly higher payoff for every player in S, since $v(S) - \sum_{k \in S} x_k > 0$. This contradicts that $c \in SNE(\Gamma(v))$. □

The following example illustrates that not every payoff vector in the core is supported by a strong Nash equilibrium.

Example 3.3 Consider the game of example 3.2. First consider the payoff vector $(30, 30, 30)$, which lies in the core. From theorem 3.1 it follows that if this vector is supported by a strong Nash equilibrium, then this strategy profile results in the formation of two links and consequently of a middleman. Assume first that player 1 is the middleman. Since the payoff of player 1 is equal to 30, player 1 claims a positive amount for the formation of at least one link. Assume without loss of generality that this claim corresponds to the link with player 2. So, the strategies of the players are $c^1 = (P, c_2^1, 30 - c_2^1)$, $c^2 = (30, P, c_3^2)$, and $c^3 = (30, c_2^3, P)$, provided that players 2 and 3 do not both indicate that they want to form a link with each other, so $c_3^2 = P$ or $c_2^3 = P$. Furthermore, it holds that $c_2^1 > 0$.

We will show that this strategy profile is not a strong Nash equilibrium. Consider the deviation of players 2 and 3 where they agree on forming a link with each other, player 2 breaking the link with player 1, and players 2 and 3 dividing the claim c_2^1, so $\hat{c}^2 = (P, P, 30 + \frac{c_2^1}{2})$ and $\hat{c}^3 = (30, \frac{c_2^1}{2}, P)$. Since the total amount of claims corresponding to the two links the players will form according to strategy profile $(c^1, \hat{c}^2, \hat{c}^3)$ is equal to 90 the two links will actually be formed and it follows that

$$K_2(c) = 30 < 30 + \frac{c_2^1}{2} = K_2(c^1, \hat{c}^2, \hat{c}^3),$$
$$K_3(c) = 30 < 30 + \frac{c_2^1}{2} = K_3(c^1, \hat{c}^2, \hat{c}^3).$$

Since all players in the deviating coalition profit from the deviation, it follows that c is not a strong Nash equilibrium. A similar argument can be given when player 2 or 3 would be the middleman. We conclude that payoff vector $(30, 30, 30)$ is not supported by a strong Nash equilibrium.

The reasoning for concluding that $(30, 30, 30)$ is not supported by a strong Nash equilibrium can be repeated for every payoff vector in the imputation set with positive payoffs for all the players, since then there has to be a middleman with non-zero payoff.

This reasoning does not hold when one of the players has zero payoff. For example, the payoff vector $(35, 55, 0)$ results from the strong Nash equilibrium c with $c^1 = c^2 = (P, P, 35)$ and $c^3 = (0, 0, P)$. The payoffs corresponding to strong Nash equilibria are shown in figure 2 on page 13. □

It can be checked that in the previous example every payoff vector in the core with one of the players receiving zero payoff is supported by a strong Nash equilibrium. Furthermore, these payoff vectors are the only payoff vectors that can be supported by strong Nash equilibria. In the following theorem we generalize this result to n-person games. Note that if $|N| = 3$ then the restriction $v(N) > \sum_{i=1}^{m} v(S_i)$ for every partition $(S_1, \ldots, S_m)$ of N with $|S_i| < 3$, for all $i \in \{1, \ldots, m\}$, corresponds to $v(N) > v(S)$ for all $|S| = 2$.[3]

[3]We remind the reader that we restrict our analysis to zero-normalized games

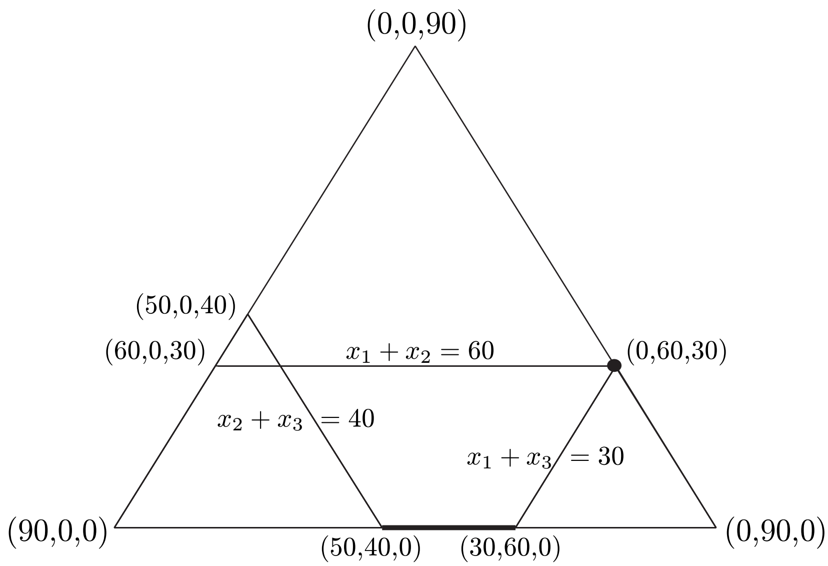

Figure 2: *Bold parts represent the payoff vectors in the imputation set resulting from strong Nash equilibria in example 3.3*

Theorem 3.5 Let (N, v) be an n-person cooperative game and $\Gamma(v)$ the corresponding link and claim game. Every core allocation of (N, v) with at least one of the players receiving zero payoff can be supported by a strong Nash equilibrium. Furthermore, when $|N| \geq 3$ and $v(N) > \sum_{i=1}^{m} v(S_i)$ for every partition $(S_1, \ldots, S_m)$ of N with $|S_i| < 3$ for all $i \in \{1, \ldots, m\}$, there are no other payoff vectors supported by strong Nash equilibria.

Proof: Let x be a core allocation of (N, v) and assume $x_i = 0$. Consider the strategy c^i of player i, with $c^i_k = 0$ for all $k \in N \backslash \{i\}$ and $c^i_i = P$. Furthermore, consider the strategy c^j for the players $j \in N \backslash \{i\}$ given by $c^j_k = P$ for all $k \in N \backslash \{i\}$ and $c^j_i = x_j$. So, the players construct a stargraph with player i as the central player and the other players directly connected only with player i.

We will show that c is a strong Nash equilibrium. Consider a deviating coalition $S \subseteq N$ and a strategy profile $\hat{c}^S \in X_S$. Suppose there exists a player $j \in S$ with $K_j(\hat{c}^S, c^{N \backslash S}) > K_j(c)$. Consider the set $C_j(L(\hat{c}^S, c^{N \backslash S})) \backslash S$, the set of players connected with player j and not in the deviating coalition S. The strategies of the players in this set were not changed by the deviation of the players in S. So, clearly if player i is in the set $C_j(L(\hat{c}^S, c^{N \backslash S})) \backslash S$ he will still receive zero payoff, while every player $k \in C_j(L(\hat{c}^S, c^{N \backslash S})) \backslash S$ with $k \neq i$ can only be connected with S via player i so, $K_k(\hat{c}^S, c^{N \backslash S}) = x_k$.

So, every player in $C_j(L(\hat{c}^S, c^{N \backslash S})) \backslash S$ receives the same payoff as he received according to strategy c. This results in the following restriction on the payoffs to the players in coalition $C_j(L(\hat{c}^S, c^{N\setminus S}))\cap S$:

$$\sum_{k\in C_j(L(\hat{c}^S,c^{N\setminus S}))\cap S} K_k(\hat{c}^S, c^{N\setminus S}) \;\;\leq\;\; v(C_j(L(\hat{c}^S, c^{N\setminus S}))) - \sum_{k\in C_j(L(\hat{c}^S,c^{N\setminus S}))\setminus S} x_k.$$

However, since x is a payoff allocation in the core of the game (N, v),

$$\sum_{k\in C_j(L(\hat{c}^S,c^{N\setminus S}))} x_k \;\;\geq\;\; v(C_j(L(\hat{c}^S, c^{N\setminus S}))).$$

Combining the last two inequalities gives us

$$\sum_{k\in C_j(L(\hat{c}^S,c^{N\setminus S}))\cap S} x_k \;\;\geq\;\; \sum_{k\in C_j(L(\hat{c}^S,c^{N\setminus S}))\cap S} K_k(\hat{c}^S, c^{N\setminus S}).$$

Since $K_j(\hat{c}^S, c^{N\setminus S}) > K_j(c) = x_j$ this implies that there exists a player $k \in S$ with $K_k(\hat{c}^S, c^{N\setminus S}) < K_k(c)$. We conclude that at least one player in S is strictly worse of by deviating to $\hat{c}^S$ and hence, c is a strong Nash equilibrium.

Now assume that $|N| \geq 3$ and $v(N) > \sum_{i=1}^m v(S_i)$ for every partition $(S_1, \ldots, S_m)$ of N with $|S_i| < 3$ for all $i \in \{1, \ldots, m\}$.

Theorem 3.4 states that $K(c) \in C(v)$ for all $c \in \mathrm{SNE}(\Gamma(v))$. Suppose $c \in \mathrm{SNE}(\Gamma(v))$ is a strategy profile that results in a payoff in the core with a positive payoff for all players. To obtain a payoff in the core, a partition of the player set into communication components $(C_1, \ldots, C_m)$ has to result with $v(N) = \sum_{r=1}^m v(C_r)$. But then there exists $r \in \{1, \ldots, m\}$ with $|C_r| \geq 3$. Consider the players in this communication component.

Every player in C_r receives a positive payoff. This implies that there is at least one middleman, player i, directly connected with at least two players, j and k. Since the payoff to this middleman is positive at least one of his claims corresponding to a link that is actually formed is positive. Without loss of generality assume player j is the other player forming this link. By theorem 3.1 we know that according to a Nash equilibrium no link between players j and k results.

Now construct the following strategies for players j and k

$$\begin{aligned}\hat{c}^j &:= (\hat{c}^j_i, \hat{c}^j_k, c^j_{N\setminus\{i,k\}}) \quad, \text{ with } \hat{c}^j_i = P \text{ and } \hat{c}^j_k = c^j_i + \frac{c^i_j}{2},\\ \hat{c}^k &:= (\hat{c}^k_j, c^k_{N\setminus\{j\}}) \quad, \text{ with } \hat{c}^k_j = \frac{c^i_j}{2}.\end{aligned}$$

This deviation implies that players j and k now divide the claim c^i_j, which can be attained since player i will not receive this claim any more, and communication component C_r will still form. Since

$$\begin{aligned}K_j(c) < K_j(c) + \frac{c^i_j}{2} = K_j(\hat{c}^{\{j,k\}}, c^{N\setminus\{j,k\}}),\\ K_k(c) < K_k(c) + \frac{c^i_j}{2} = K_k(\hat{c}^{\{j,k\}}, c^{N\setminus\{j,k\}}),\end{aligned}$$

it is clear that c is not a strong Nash equilibrium. □

Clearly, a 2-person game with zero value for the one-person coalitions and a positive value for the grand coalition has strong Nash equilibria in the corresponding link and claim game that result in positive payoffs for both players.

The following example shows that the condition $v(N) > \sum_{i=1}^{m} v(S_i)$, for every partition $(S_1, \ldots, S_m)$ of N with $|S_i| < 3$ for all $i \in \{1, \ldots, m\}$, cannot be omitted in theorem 3.5.

Example 3.4 Let (N, v) be a 4-person game with

$$v = u_{\{1,2\}} + u_{\{3,4\}}.$$

We define the following strategies, where the players are divided into two components and the players in a component divide the corresponding value. So, $c_2^1 = c_1^2 = c_4^3 = c_3^4 = \frac{1}{2}$ and $c_j^i = P$ otherwise. Clearly, no subcoalition can deviate to a strategy that results in at least the same payoff for all players in the coalition and to a higher payoff for at least one of the players in the coalition. Hence, $c \in \text{SNE}(\Gamma(v))$. □

In the following theorem we describe a class of games where the core coincides with the payoffs associated with strong Nash equilibria in the corresponding link and claim game.

Theorem 3.6 Let (N, v) be a cooperative game with n even and $\Gamma(v)$ the corresponding link and claim game. If there exists a partition $(S_1, \ldots, S_m)$ of N with $|S_i| = 2$ for all $i \in \{1, \ldots, m\}$ and $v(N) = \sum_{i=1}^{m} v(S_i)$ then $C(v) = \{x \mid x = K(c),\ c \in \text{SNE}(\Gamma(v))\}$.

Proof: From theorem 3.4 it follows that $C(v) \supseteq \{x \mid x = K(c),\ c \in \text{SNE}(\Gamma(v))\}$. Let $x \in C(v)$ and let $(S_1, \ldots, S_m)$ be a partition of N with $|S_i| = 2$ for all $i \in \{1, \ldots, m\}$ and $v(N) = \sum_{i=1}^{m} v(S_i)$. Then it holds for all $i \in \{1, \ldots, m\}$ that $\sum_{j \in S_i} x_j = v(S_i)$. We will construct a strong Nash equilibrium c that supports this core-element. Consider an arbitrary $j \in S_i$ and denote the other player in S_i by k. Let c^j be the strategy of player j with $c_k^j = x_j$ and $c_l^j = P$, for all $l \in N \backslash \{k\}$. Obiviously $K(c) = x$. Now consider a deviation of coalition $S \subseteq N$, $\hat{c}^S$. Let $k \in S$ and denote the communication component of player k that is formed according to $(\hat{c}^S, c^{N \backslash S})$ by C_k. By construction of c it holds that $K_j(c) = K_j(\hat{c}^S, c^{N \backslash S})$ for all $j \in C_k \backslash S$. Since $x = K(c)$ is a core-element, this implies

$$\sum_{j \in S \cap C_k} K_j(c) \geq \sum_{j \in S \cap C_k} K_j(\hat{c}^S, c^{N \backslash S}).$$

But then $\hat{c}^S$ cannot be a profitable deviation for all players in S, with the payoff improvement being strict for at least one player in S. Thus, $c \in$SNE$(\Gamma(v)$ and hence, $C(v) \subseteq \{x \mid x = K(c),\ c \in \text{SNE}(\Gamma(v))\}$. This completes the proof. □

A partitioning of the player set in a game with an odd number of players results in at least one isolated player or a communication component with at least three players. Using theorem 3.5 the following theorem results.

Theorem 3.7 Let (N, v) be an n-person cooperative game with $n \geq 3$ odd and $v(N) \geq \sum_{i=1}^{m} v(S_i)$ for every partition $(S_1, \ldots, S_m)$ of N. There exists a strong Nash equilibrium resulting in payoff vector x if and only if $x \in C(v)$ and there exists $i \in N$ with $x_i = 0$.

Proof: If $v(N) > \sum_{i=1}^{m} v(S_i)$ for all partitions $(S_1, \ldots, S_m)$ of N, then theorem 3.5 applies. Now assume that there exists a partition $(S_1, \ldots, S_m)$ of N with $v(N) = \sum_{i=1}^{m} v(S_i)$. The if-part follows from theorem 3.5 again. It remains to show the only-if-part. Since n is odd it follows that there exists an $i \in \{1, \ldots, m\}$ with $|S_i| = 1$ or $|S_i| \geq 3$. If $|S_i| = 1$ then all core elements attribute zero to the player in S_i. Using theorem 3.4 we conclude that this player receives zero according to every payoff vector resulting from a strong Nash equilibrium. Now assume $|S_i| \geq 3$ and consider a strategy profile resulting in communication component S_i. This implies that there is at least one middleman. However, using the proof of theorem 3.5, we find that in a strong Nash equilibrium, the middleman will always receive zero payoff. Furthermore, theorem 3.4 states that every strong Nash equilibrium results in a payoff vector in the core. This completes the proof. $\square$

4 Coalition proof Nash equilibria

In the previous section we have seen that in general a large part of the imputation set of 3-player games can be supported by Nash equilibria of the corresponding link and claim game. It appeared that the core is a subset of the set of payoff vectors that are supported by Nash equilibria. However, when we consider the strong Nash equilibria of the link and claim game we find that under mild conditions on (N, v), only a relatively small part of the imputation set results, in fact only a relatively small part of the core. Although all strong Nash equilibria correspond to payoff vectors in the core, every strong Nash equilibrium results in at least one player receiving a payoff equal to zero.

In this section we analyze coalition proof Nash equilibria. The set of coalition proof Nash equilibria is a superset of the set of strong Nash equilibria. The strong Nash equilibrium concept demands that no coalition can deviate to a profile that improves the payoff for all players in the coalition, with a strict improvement for at least one of the players. The coalition proof Nash equilibrium concept has similar requirements, but the set of allowed deviations is restricted. Every player in the deviating coalition should strictly improve his payoff and the strategy of the deviating players has to be stable with respect to deviations of subcoalitions.

We will first show an example to illustrate the coalition proof Nash equilibrium concept and to illustrate the curiosities that arise.

Example 4.1 Consider the game of example 3.2 and the associated link and claim game. Furthermore consider the payoff vector $x = (50, 15, 25)$, which is not supported by any strong Nash equilibrium.

Consider the following strategy profile that would lead to a communication graph with player 2 as middleman and x as payoff vector, defined by $c^1 = (P, 50, P)$, $c^2 = (c^2_1, P, 15 - c^2_1)$, with $0 \le c^2_1 \le 15$, and $c^3 = (P, 25, P)$.

Note that the three players divide the value of the grand coalition and that no player can unilaterally deviate to a strategy that gives him a higher payoff. This implies that we have to consider only deviations by two player coalitions.

First consider a deviation of coalition $\{1, 2\}$. The sum of the payoffs to players 1 and 2 is equal to the value of coalition $\{1, 2\}$. So, to improve their payoffs, they need to deviate to a strategy profile where they are connected with player 3. However, given the strategy of player 3 this implies that player 3 will still receive 25 after the deviation. So players 1 and 2 cannot deviate and both improve their payoffs. Since the same reasoning holds for a deviation of coalition $\{2, 3\}$, it remains to consider a deviation by coalition $\{1, 3\}$.

Players 1 and 3 will deviate only if they can both improve their payoffs, so the grand coalition has to be the unique component that forms. To improve their payoffs they have to break one of the links with player 2, with a positive claim of player 2 on this link. Assume without loss of generality that $c^2_1 > 0$ and that the link between players 1 and 2 will be broken, so

$$\begin{aligned} \hat{c}^1 = &\ (P, P, 50 + \alpha c^2_1) &&, \text{ with } 0 < \alpha < 1, \\ \hat{c}^3 = &\ \quad (\hat{c}^3_1, \hat{c}^3_2, P) &&, \text{ with } 25 < \hat{c}^3_1 + \hat{c}^3_2 = 25 + \beta c^2_1 \text{ and } 0 < \beta \le 1 - \alpha. \end{aligned}$$

However, this deviation is not stable since player 3 can deviate from strategy profile $(\hat{c}^1, c^2, \hat{c}^3)$ by playing $\tilde{c}^3 = (P, 25 + c^2_1, P)$, and thus improving his payoff from $25 + \beta c^2_1$ to $25 + c^2_1 = 40 - c^2_3 = v(\{2, 3\}) - c^2_3$. Since this deviation by player 3 is a coalition proof Nash equilibrium in the corresponding reduced game, it follows that deviation $(\hat{c}^1, \hat{c}^3)$ is not self-enforcing. Since $(\hat{c}^1, \hat{c}^3)$ was an arbitrarily chosen deviation it follows that $c \in$ CPNE(N, v). □

So, we found a coalition proof Nash equilibrium that results in a payoff vector in the core, while this payoff vector is not supported by any strong Nash equilibrium. However, this does not imply that every payoff vector in the core can be supported by a coalition proof Nash equilibrium.

Example 4.2 Consider the game of example 3.2 and the corresponding link and claim game. Furthermore, consider the payoff vector $y = (30, 40, 20)$ which lies in the interior of the core. This payoff vector is not supported by any coalition proof Nash equilibrium. The proof follows by theorem 4.2 later in this section. □

The rest of this section is dedicated to the problem of finding all payoff vectors corresponding to coalition proof Nash equilibria in the link and claim game associated with a general 3-person cooperative game. A remarkable result is that we can find cooperative games with a coalition proof Nash equilibrium resulting in a payoff vector outside the core.

Theorem 4.2 characterizes the set of payoff vectors corresponding to coalition proof Nash equilibria. The proof of this theorem is divided in several steps and every step is proven in a lemma.

First we will show that for a large class of 3-person games, all cooperation structures that are supported by coalition proof Nash equilibria resulting in payoff vectors in the imputation set, have exactly two links.

Lemma 4.1 Let (N, v) be a 3-person cooperative game with associated link and claim game $\Gamma(v)$. If $v(N) > v(S)$ for all S with $|S| = 2$, then for all $c \in \text{CPNE}(N, v)$ with $K(c) \in I(v)$ it holds that $|L(c)| = 2$.

Proof: Suppose $v(N) > v(S)$ for all S with $|S| = 2$. Let $c \in \text{CPNE}(\Gamma(v))$ with $K(c) \in I(v)$. Since $v(N) > v(S)$ for all S with $|S| = 2$, $v(N) > 0$, and $v(i) = 0$ for all $i \in N$, it follows that the grand coalition is the unique component in $(N, L(c))$. Since $\sum_{i \in N} K_i(c) > 0$ and $\text{CPNE}(\Gamma(v)) \subseteq \text{NE}(\Gamma(v))$ it follows by theorem 3.1 that the graph $(N, L(c))$ contains no cycles. We conclude that $|L(c)| = 2$. □

In the following lemma we show that all payoff vectors in the core that result from a strong Nash equilibrium can also be supported by a coalition proof Nash equilibrium. Note that this results holds for general n-person games.

Lemma 4.2 Let (N, v) be a cooperative game with associated link and claim game $\Gamma(v)$. For all $x \in C(v)$ with $x_i = 0$ for a player $i \in N$ there exists $c \in \text{CPNE}(\Gamma(N, v))$ with $K(c) = x$.

Proof: Since every strong Nash equilibrium is a coalition proof Nash equilibrium the lemma follows from theorem 3.5. □

If we concentrate on 3-person games again, we find that, besides the payoffs that are also supported by strong Nash equilibria in the corresponding link and claim game, there is another set of payoff vectors that result from coalition proof Nash equilibria in the corresponding link and claim game.

Lemma 4.3 Let (N, v) be a 3-person cooperative game and $x \in I(v)$. If at least one player j receives exactly his marginal contribution, $x_j = v(N) - v(N\backslash\{j\})$ and at least one other player k receives at most his marginal contribution, $x_k \leq v(N) - v(N\backslash\{k\})$, then there exists $c \in \text{CPNE}(N, v)$ with $K(c) = x$.

Proof: Let $x \in I(v)$ and assume without loss of generality that $x_2 = v(N) - v(\{1, 3\})$ and $x_3 \leq v(N) - v(\{1, 2\})$. Since $x \in I(v)$, so $x_1 + x_2 + x_3 = v(N)$, it follows for the payoff of the remaining player that $x_1 \geq v(\{1, 2\}) + v(\{1, 3\}) - v(N)$. Consider the strategy profile defined by

$$\begin{aligned} c^1 = &\quad (P, x_1, 0), \\ c^2 = &\quad (x_2, P, P), \\ c^3 = &\quad (x_3, P, P). \end{aligned}$$

Since $x_1 + x_2 \geq v(\{1, 2\})$ and $x_1 + x_3 = v(\{1, 3\})$, coalition $\{2, 3\}$ is the only coalition that can deviate to a strategy profile that results in a higher payoff for all members of the coalition. Given the strategy of player 1, coalition $\{2, 3\}$ has at most two possibilities to deviate and obtain a higher payoff for both players in the coalition. First, they might break exactly one of the links with player 1 and form a link within the coalition. Since $c^1_3 = 0$ and $c^1_2 > 0$, player 2 will break the link with player 1. This is represented by the strategies

$$\begin{aligned} \hat{c}^2 = &\quad (P, P, x_2 + \alpha x_1) &&, \text{ with } 0 < \alpha < 1, \\ \hat{c}^3 = &\quad (\hat{c}^3_1, \hat{c}^3_2, P) &&, \text{ with } \hat{c}^3_1 + \hat{c}^3_2 = x_3 + \beta x_1,\ 0 < \beta \leq 1 - \alpha. \end{aligned}$$

Provided that the value of the coalition $\{2, 3\}$ is large enough, i.e. $v(\{2, 3\}) > x_2 + x_3$, they could also form a coalition on their own. This results from the following strategies:

$$\begin{aligned} \dot{c}^2 = &\quad (P, P, \dot{c}^2_3), \\ \dot{c}^3 = &\quad (P, \dot{c}^3_2, P), \end{aligned}$$

with $\dot{c}_3^2 > x_2$, $\dot{c}_2^3 > x_3$, and $\dot{c}_3^2 + \dot{c}_2^3 \leq v(\{2,3\})$. Since $\dot{c}_3^2 > x_2$ and $x_1 + x_2 + x_3 = v(N) \geq v(\{2,3\})$ it holds that $x_3 < \dot{c}_2^3 < x_3 + x_1$. Now let $\beta \in (0,1)$ be such that $\dot{c}_2^3 = x_3 + \beta x_1$.

In both cases player 3 can achieve a further improvement in his payoff by breaking the link with player 2 and forming a link with player 1 only,

$$\tilde{c}^3 = (x_1 + x_3, P, P).$$

Hence, player 3 improves his payoff from $x_3 + \beta x_1$ to $x_3 + x_1 = v(N) - x_2 = v(\{1,3\})$. Since player 1 claims zero on the link with player 3, the claims on the link $\{1,3\}$ are indeed attainable. We conclude that both $(\hat{c}^2, \hat{c}^3)$ and $(\dot{c}^2, \dot{c}^3)$ are not self-enforcing. Since these were the only possible deviations that would result in higher payoffs for all members of the deviating coalition, we conclude that c is a coalition proof Nash equilibrium. $\square$

In lemmas 4.2 and 4.3 we have seen two sets of payoff vectors that result from coalition proof Nash equilibria in the corresponding link and claim game.

We will now show, for every coalition proof Nash equilibrium that results in a communication graph with exactly two links and a payoff vector in the imputation set, that this payoff vector belongs to the union of the two sets described by lemmas 4.2 and 4.3.

For convenience we will denote the middleman by i and the two other players by j and k. Then the strategies we have to consider are of the following type, where $c^m = (c_i^m, c_j^m, c_k^m)$ for all players $m \in N$:

$$\begin{aligned} c^i = &\quad (P, c_j^i, c_k^i) \quad , \ c_j^i, c_k^i \in \mathbb{R}^+, \\ c^j = &\quad (c_i^j, P, c_k^j) \quad , \ c_i^j \in \mathbb{R}^+, \\ c^k = &\quad (c_i^k, c_j^k, P) \quad , \ c_i^k \in \mathbb{R}^+, \end{aligned}$$

restricted that $c_k^j = P$ or $c_j^k = P$ and $c_j^i + c_k^i + c_i^j + c_i^k = v(N)$.

The following two lemmas deal with the games and strategies described above.

Lemma 4.4 Let (N,v) be a 3-person cooperative game with associated link and claim game $\Gamma(v)$ and strategy profile c resulting in a communication graph with exactly two links and middleman player i. If $c_j^i + c_k^i = 0$ and $K(c) \in I(v) \backslash C(v)$, then $c \notin CPNE(\Gamma(v))$.

Proof: Suppose $K(c) \in I(v) \backslash C(v)$ and $c_j^i + c_k^i = 0$. Then, using the fact that $v(\{i\}) = v(\{j\}) = v(\{k\}) = 0$, we find that at least one of the following three inequalities holds:

$$c_i^j < v(\{i,j\}), \tag{4}$$

$$c_i^k < v(\{i,k\}), \tag{5}$$

$$c_i^j + c_i^k < v(\{j,k\}). \tag{6}$$

Inequality (6) will not hold since $c_i^j + c_i^k = v(N)$ and $v(S) \leq v(N)$ for all $S \subseteq N$. Inequalities (4) and (5) are of the same type, so assume without loss of generality that inequality (4) holds. Then, player i can unilaterally improve his payoff by playing strategy

$$\hat{c}^i = (P, v(\{i,j\}) - c_i^j, P).$$

We conclude that $c \notin \text{CPNE}(\Gamma(v))$. □

The following lemma gives another sufficient condition for a strategy profile not to be a coalition proof Nash equilibrium.

Lemma 4.5 Let (N, v) be a 3-person cooperative game with associated link and claim game $\Gamma(v)$ and a strategy profile c resulting in a communication graph with exactly two links and middleman player i. If $c_j^i + c_k^i > 0$, $c_i^j \neq v(N) - v(\{i,k\})$, and $c_i^k \neq v(N) - v(\{i,j\})$ then $c \notin \text{CPNE}(\Gamma(v))$.

Proof: When $K_i(c) + K_j(c) = c_j^i + c_k^i + c_i^j < v(\{i,j\})$ or $K_i(c) + K_k(c) = c_j^i + c_k^i + c_i^k < v(\{i,k\})$ we find that player i can deviate to a strategy that strictly improves his payoff by breaking exactly one link and claiming the highest possible payoff on the other link (e.g. in the first case break the link with player k). So $c \notin \text{CPNE}(\Gamma(v))$.

From now on assume

$$c_j^i + c_k^i + c_i^j \geq v(\{i,j\}) \tag{7}$$

and

$$c_j^i + c_k^i + c_i^k \geq v(\{i,k\}). \tag{8}$$

Using the efficiency, $c_j^i + c_k^i + c_i^j + c_i^k = v(N)$, $c_i^j \neq v(N) - v(\{i,k\})$, and $c_i^k \neq v(N) - v(\{i,j\})$ it follows that both (7) and (8) cannot hold with equality, so

$$c_j^i + c_k^i + c_i^j > v(\{i,j\}) \tag{9}$$

and

$$c_j^i + c_k^i + c_i^k > v(\{i,k\}). \tag{10}$$

Suppose without loss of generality that $c_k^i > 0$. We will construct a self-enforcing deviation for coalition $\{j,k\}$. Player k will break the link with player i, and players j and k will form a new link. Furthermore, we have to make sure that both players improve their payoffs. Denote $x = K(c)$ and let

$$\begin{aligned} \hat{c}^j &= (x_j, P, \hat{c}_k^j) \quad , \text{ with } \hat{c}_k^j = \max\left\{\frac{c_k^i}{2}, c_k^i + v(\{i,j\}) - x_i - x_j\right\}, \\ \hat{c}^k &= (P, \hat{c}_j^k, P) \quad , \text{ with } \hat{c}_j^k = x_k + \min\left\{\frac{c_k^i}{2}, x_i + x_j - v(\{i,j\})\right\}. \end{aligned}$$

By construction of $\hat{c}^j$ and $\hat{c}^k$ it follows that $\hat{c}^j_k + \hat{c}^k_j = x_k + c^i_k$ and hence, the claims are attainable. Since $c^i_k > 0$ and $x_i + x_j - v(\{i,j\}) = c^i_j + c^i_k + c^j_i - v(\{i,j\}) > 0$ it follows that $\hat{c}^j_k > 0$ and $\hat{c}^k_j > x_k$, so both players improve their payoff.

The deviation is self-enforcing since player k clearly cannot improve his payoff any further, and player j cannot improve his payoff any further since $c^i_j + x_j + \hat{c}^j_k \geq c^i_j + x_j + c^i_k + v(\{i,j\}) - x_i - x_j = v(\{i,j\})$. We conclude that $c \notin \text{CPNE}(\Gamma(v))$. This completes the proof. □

We can now characterize payoff vectors corresponding to coalition proof Nash equilibria of 3-person cooperative games resulting in a communication graph with exactly two links.

Theorem 4.1 Let (N, v) be a 3-person cooperative game and $x \in \mathbb{R}^N$. There exists $c \in \text{CPNE}(\Gamma(v))$ with $|L(c)| = 2$ and $K(c) = x$ if and only if

$$x \in C(v) \text{ and } \exists j \in N: \; x_j = 0 \tag{11}$$

or

$$x \in I(v), \; \exists j \in N: \; x_j = v(N) - v(N \backslash \{j\}) \text{ and } \exists k \in N \backslash \{j\}: \; x_k \leq v(N) - v(N \backslash \{k\}). \tag{12}$$

Proof: The if-part of the theorem follows by lemmas 4.2 and 4.3. It remains to show the only-if-part. If $x \notin I(v)$, then some player could improve his payoff by unilaterally deviating. So, assume that $x \in I(v)$. Also, assume that $c \in \text{CPNE}(\Gamma(v))$ with $|L(c)| = 2$ and $K(c) = x$. We will show that if (11) is not satisfied, then (12) is satisfied. So, assume that (11) is not satisfied.

Since two links result, there will be a middleman, say player i. Since condition (11) is not satisfied, $x_j \neq 0$, for all $j \in N$ or there exist $j \in N$ with $x_j = 0$ and $x \notin C(v)$. First suppose $x \notin C(v)$ and there exist $j \in N$ with $x_j = 0$. By lemma 4.4 it follows that $x_i > 0$ and consequently $i \neq j$. Denote the third player by player k. Since c is a Nash equilibrium it follows that $x_i + x_j = x_i \geq v(\{i,j\})$ and $x_i + x_k \geq v(\{i,k\})$, since otherwise player i could deviate to a more profitable strategy. By lemma 4.5 it follows that $x_j = v(N) - v(\{i,k\})$ or $x_k = v(N) - v(\{i,j\})$. If $v(N) > v(\{i,k\})$ it follows that $x_k = v(N) - v(\{i,j\})$ and $x_j = 0 < v(N) - v(\{i,k\})$ and hence condition (12) is satisfied. Otherwise, if $v(N) = v(\{i,k\})$ then $x_j = v(N) - v(\{i,k\})$. Furthermore, since $x_i \geq v(\{i,j\})$ it follows that $x_k \leq v(N) - v(\{i,j\})$ and again we conclude that equation (12) is satisfied.

Now suppose $x_j \neq 0$ for all $j \in N$. By considering the deviation possibilities of the middleman, player i, it follows that

$$x_i + x_j \geq v(\{i,j\}) \text{ and } x_i + x_k \geq v(\{i,k\}), \tag{13}$$

where players j and k denote the remaining players in the game. By lemma 4.5 it follows that $x_j = v(N) - v(\{i,k\})$ or $x_k = v(N) - v(\{i,j\})$. Assume w.l.o.g. that $x_j = v(N) - v(\{i,k\})$.

By (13) and $x_i + x_j + x_k = v(N)$ it follows that $x_i \geq v(\{i,j\}) + v(\{i,k\}) - v(N)$ and $x_k \leq v(N) - v(\{i,j\})$ so condition (12) is satisfied. This completes the proof. □

The following example illustrates the theorem above.

Example 4.3 Consider the game of example 3.2 and the corresponding link and claim game. With lemma 4.1 it follows that exactly two links result according to a coalition proof Nash equilibrium with the associated payoff vector in the imputation set. Now, using Theorem 4.1, we can exactly determine the set of payoff vectors in the imputation set, supported by coalition proof Nash equilibria. The payoff vectors in the imputation set that result from coalition proof Nash equilibria are represented in figure 3. □

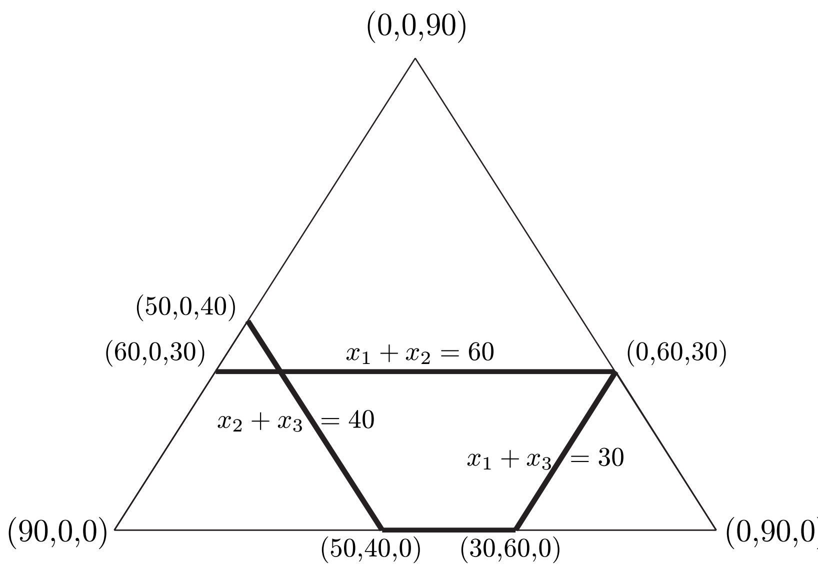

Figure 3: *Bold parts represent the payoff vectors in the imputation set resulting from coalition proof Nash equilibria in example 4.3*

Besides the coalition proof Nash equilibria that result in the formation of two links there are also coalition proof Nash equilibria that result in the formation of exactly one link.

The following lemma describes this second set of coalition proof Nash equilibria. However, since only one link will be formed, a strategy profile in this set does not necessarily result in a payoff vector in the imputation set. For a specific link between two players the attainable payoff vectors can be represented by a line segment in $\mathbb{R}^N$, with zero payoff for the remaining player.

Lemma 4.6 Let (N, v) be a 3-person cooperative game and x a payoff vector. There exists $c \in \text{CPNE}(\Gamma(v))$ with $K(c) = x$ if

$$\begin{aligned} \exists i,j,k \in N: \quad & x_i > v(N) - v(\{j,k\}),\ x_i \geq v(N) - v(\{j,k\}) + v(\{i,k\}) - v(\{i,j\}), \\ & x_j > v(N) - v(\{i,k\}),\ x_j \geq v(N) - v(\{i,k\}) + v(\{j,k\}) - v(\{i,j\}), \\ & x_i + x_j = v(\{i,j\}),\ \text{and } x_k = 0. \end{aligned} \tag{14}$$

Also, if $v(N) > v(\{i,j\})$, then all coalition proof Nash equilibria that result in the unique link $\{i,j\}$ belong to the set of payoff vectors described by (14).

Proof: Assume without loss of generality that $x_1 > v(N) - v(\{2,3\})$, $x_1 \geq v(N) - v(\{2,3\}) + v(\{1,3\}) - v(\{1,2\})$, $x_2 > v(N) - v(\{1,3\})$, $x_2 \geq v(N) - v(\{1,3\}) + v(\{2,3\}) - v(\{1,2\})$, $x_1 + x_2 = v(\{1,2\})$, and $x_3 = 0$. Now consider the strategy profile c defined by

$$\begin{aligned} c^1 &= (P, x_1, v(N) - v(\{1,2\})), \\ c^2 &= (x_2, P, v(N) - v(\{1,2\})), \\ c^3 &= (P, P, P), \end{aligned}$$

so that $K(c) = x$. No player can unilaterally deviate to a strategy that gives him a higher payoff. Consider an arbitrary self-enforcing deviation of the grand coalition, which improves the payoffs of all players. Obviously, all players will receive a positive payoff after this deviation. Then it follows that exactly two links will form, since all players have to be connected since they all receive a positive payoff and by theorem 3.1 no cycles will result. Since the strategy profile is self-enforcing no player can achieve a further improvement in his payoff by unilaterally deviating from this self-enforcing strategy profile. This implies that the players divide the value of the grand coalition. Thus, the self-enforcing strategy profile appears to be a coalition proof Nash equilibrium. Then, by theorem 4.1 it follows that at least two players will receive at most their marginal contribution to the grand coalition. Hence, at least one of the players 1 and 2 will receive a lower payoff according to the deviation. We conclude that there is no self-enforcing deviation of the grand coalition that gives all players a higher payoff.

Since coalition $\{1,2\}$ has no possibility to deviate to a strategy profile that results in a higher payoff for both players it remains to consider deviations by coalitions $\{1,3\}$ and $\{2,3\}$.

Consider a deviation of coalition $\{2,3\}$ where they form a communication component on their own,

$$\begin{aligned} \hat{c}^2 &= (P, P, x_2 + \alpha(v(\{2,3\}) - x_2)) \quad , \text{ with } 0 < \alpha < 1, \\ \hat{c}^3 &= (P, \beta(v(\{2,3\}) - x_2), P) \quad , \text{ with } 0 < \beta \leq 1 - \alpha. \end{aligned}$$

Note that $x_2 = v(\{1,2\}) - x_1 \leq v(N) - x_1 < v(\{2,3\})$. Hence, deviation $(\hat{c}^2, \hat{c}^3)$ results in a strictly higher payoff for both players. However, player 3 can achieve a further payoff improvement by playing

$$\tilde{c}^3 = (v(\{1,2\}) + v(\{1,3\}) - v(N), P, P),$$

since $c^1_3 + \tilde{c}^3_1 = v(\{1,3\})$ and

$$\beta(v(\{2,3\}) - x_2) < v(\{2,3\}) - x_2 \leq v(\{1,2\}) + v(\{1,3\}) - v(N).$$

The last equation follows since $x_2 \geq v(N) - v(\{1,3\}) + v(\{2,3\}) - v(\{1,2\})$. We conclude that the deviation $(\hat{c}^2, \hat{c}^3)$ is not self-enforcing.

Coalition $\{2,3\}$ can also deviate to a strategy profile where the communication component $\{1,2,3\}$ is formed. First, consider the deviation that results in the formation of links $\{1,3\}$ and $\{2,3\}$. This is described by a strategy profile,

$$\begin{aligned} \dot{c}^2 &= \quad (P, P, x_2 + \alpha x_1) \quad , \text{ with } 0 < \alpha < 1, \\ \dot{c}^3 &= \quad (\dot{c}^3_1, \dot{c}^3_2, P) \quad , \text{ with } \dot{c}^3_1 + \dot{c}^3_2 = \beta x_1,\ 0 < \beta \leq 1 - \alpha. \end{aligned}$$

Here, players 2 and 3 divide $v(\{1,2\}) = x_1 + x_2$ since $c^1_3 = v(N) - v(\{1,2\})$. This deviation is not self-enforcing since

$$\beta x_1 < x_1 = v(\{1,2\}) - x_2 < v(\{1,2\}) + v(\{1,3\}) - v(N),$$

so player 3 improves his payoff by playing $\tilde{c}^3$.

Secondly, if and only if $v(N) > v(\{1,2\})$, coalition $\{2,3\}$ can deviate to a strategy profile that results in the formation of the links $\{1,2\}$ and $\{2,3\}$ and both players receiving a strictly higher payoff. This is described by

$$\begin{aligned} \bar{c}^2 &= \quad (\bar{c}^2_1, P, \bar{c}^2_3) \quad , \text{ with } \bar{c}^2_1 + \bar{c}^2_3 = x_2 + \alpha(v(N) - v(\{1,2\})), 0 < \alpha < 1, \\ \bar{c}^3 &= \quad (P, \bar{c}^3_2, P) \quad , \text{ with } \bar{c}^3_2 = \beta(v(N) - v(1,2)), 0 < \beta \leq 1 - \alpha. \end{aligned}$$

Since $x_1 > v(N) - v(\{2,3\})$, the sum of the payoffs to players 2 and 3 will be strictly less than $v(\{2,3\})$, so player 2 can break the link with player 1 and improve his payoff. Finally, note that it is not possible for coalition $\{2,3\}$ to deviate to a strategy profile that results in the formation of the links $\{1,2\}$ and $\{1,3\}$ and both players improving their payoff.

Since the deviation possibilities of coalition $\{1,3\}$ are similar to the deviation possibilities of coalition $\{2,3\}$ it follows that deviations by coalition $\{1,3\}$ are also not stable. This proves the first part of the theorem.

Now, suppose there exist two players $i, j \in N$: $v(N) > v(\{i,j\})$. Assume without loss of generality that $v(N) > v(\{1,2\})$. We will show by contradiction that every coalition proof Nash equilibrium that results in the unique link $\{1,2\}$ belongs to the set described by (14) with $i=1$, $j=2$, and $k=3$. Let c be a coalition proof Nash equilibrium that results in the formation of exactly one link, $\{1,2\}$ and assume that $K(c)$ does not belong to the set described by (14) with $i=1$, $j=2$, and $k=3$. Without loss of generality we can distinguish two cases:

$$\begin{array}{lc} (i) & K_1(c) = c_2^1 \leq v(N) - v(\{2,3\}), \\ (ii) & K_1(c) = c_2^1 < v(N) - v(\{2,3\}) + v(\{1,3\}) - v(\{1,2\}), \\ & K_1(c) > v(N) - v(\{2,3\}), \text{ and } K_2(c) > v(N) - v(\{1,3\}). \end{array}$$

Since c is a coalition proof Nash equilibrium by assumption, it holds that $K_1(c)+K_2(c) = v(\{1,2\})$. Furthermore, since player 3 cannot unilaterally deviate to a strategy that gives him a higher payoff it holds that $c_3^i \geq v(N) - v(\{1,2\})$ or $c_3^i = P$ for $i = 1, 2$.

Firstly, consider (i). Consider the following deviation of players 2 and 3:

$$\begin{aligned} \check{c}^2 &= \quad (c_1^2 + \alpha(v(N) - v(\{1,2\})), P, 0) \quad , \text{ with } 0 < \alpha < 1, \\ \check{c}^3 &= \quad (P, (1-\alpha)(v(N) - v(\{1,2\})), P). \end{aligned}$$

Since $K_2(c^1, \check{c}^2, \check{c}^3) + K_3(c^1, \check{c}^2, \check{c}^3) = v(N) - c_2^1 \geq v(N) - v(N) + v(\{2,3\}) = v(\{2,3\})$ and $c_3^1 \geq v(N) - v(\{1,2\})$ or $c_3^1 = P$ players 2 and 3 have no opportunity to achieve a further improvement in their payoff and hence, this deviation is self-enforcing. Since both players improved their payoff by playing $(\check{c}^2, \check{c}^3)$, c is not a coalition proof Nash equilibrium.

Secondly, consider (ii). Now, consider the following deviation of players 1 and 3.

$$\begin{aligned} \hat{c}^1 &= (P, P, v(N) - v(\{2,3\}) + v(\{1,3\}) - v(\{1,2\})), \\ \hat{c}^3 &= (v(\{2,3\}) + v(\{1,2\}) - v(N), P, P). \end{aligned}$$

Player 1 cannot unilaterally improve his payoff any further since $c_1^2 > v(N) - v(\{1,3\})$. Since

$$v(N) - v(\{2,3\}) < K_1(c) < v(N) - v(\{2,3\}) + v(\{1,3\}) - v(\{1,2\})$$

it follows that $v(\{1,3\}) > v(\{1,2\})$. This implies that $c_3^2 > v(N) - v(\{1,2\}) > v(N) - v(\{1,3\})$, so player 3 cannot improve his payoff by deviating to a strategy that results in the links $\{1,3\}$ and $\{2,3\}$. Furthermore, player 3 cannot improve his payoff by deviating to a strategy that results in the unique link $\{2,3\}$, since

$$\begin{aligned} c_3^2 + K_3(\hat{c}^1, c^2, \hat{c}^3) &= c_3^2 + \hat{c}_1^3 \\ &> v(N) - v(\{1,2\}) + v(\{2,3\}) + v(\{1,2\}) - v(N) = v(\{2,3\}). \end{aligned}$$

Hence, $(\hat{c}^1, \hat{c}^3)$ is self-enforcing. This implies that c is not a coalition proof Nash equilibrium. This completes the proof. $\square$

Note that there exists a payoff vector x satisfying condition (14) only if the core of the underlying game is empty, since if (14) holds then

$$v(\{i,j\}) = x_i + x_j > v(N) - v(\{i,k\}) + v(N) - v(\{j,k\}),$$

and hence, $v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\}) > 2v(N)$ so the balancedness condition for a zero-normalized game is not satisfied. Further it can be shown that the reverse statement also holds, if the core is empty there exists a payoff vector x satisfying condition (14).[4]

Lemma 4.7 describes the necessary and sufficient conditions on a 3-person cooperative game for the existence of a coalition proof Nash equilibrium in the corrresponding link and claim game resulting in the empty graph and the payoff vector $(0,0,0)$.

Lemma 4.7 Let (N, v) be a 3-person cooperative game and x a payoff vector. The empty graph and payoff vector x result from a strategy profile $c \in \text{CPNE}(\Gamma(v))$ if and only if

$$\exists S, T \subseteq N,\ S \neq T,\ |S| = |T| = 2,\ v(S) = v(T) = v(N),\ \text{and } x = (0,0,0). \tag{15}$$

Proof: First we will prove the only-if-part. Let $N = \{1,2,3\}$. Assume w.l.o.g. that $v(\{1,2\}) \geq v(\{2,3\}) \geq v(\{1,3\})$. If $v(N) > v(\{2,3\}) > 0$ then x with $x_1 = v(N) - v(\{2,3\})$, $x_2 = \min\left\{\frac{1}{2}v(\{2,3\}), v(N) - v(\{1,3\})\right\}$, and $x_3 = v(\{2,3\}) - x_2$ is supported by a coalition proof Nash equilibrium (see theorem 4.1). If $v(N) = v(\{1,2\})$ and $v(\{2,3\}) = v(\{1,3\}) = 0$, then there is no coalition proof Nash equilibrium c resulting in payoff vector $(0,0,0)$, since then $(\hat{c}^1, \hat{c}^2)$ with $\hat{c}^1 = (P, \frac{1}{2}v(N), P)$ and $\hat{c}^2 = (\frac{1}{2}v(N), P, P)$ is a self-enforcing deviation that strictly improves the payoffs of both players 1 and 2. Finally, consider the case $v(N) > v(\{1,2\})$ and $v(\{2,3\} = v(\{1,3\}) = 0$. Then x, with $x_1 = x_2 = \frac{1}{2}v(\{1,2\})$ and $x_3 = v(N) - v(\{1,2\})$, is supported by a coalition proof Nash equilibrium (see theorem 4.1). Hence, we can always find a self-enforcing deviation from a strategy profile resulting in payoff vector $(0,0,0)$ if condition (15) is not satisfied. We conclude that there is no coalition proof Nash equilibrium that results in payoff vector $(0,0,0)$ if condition (15) is not satisfied.

To prove the if-part, first consider the case with exactly one two-person coalition receiving a smaller value than the grand coalition. Assume without loss of generality

[4] Assume that the core is empty and assume w.l.o.g. that $v(\{i,j\}) \geq v(\{i,k\}) \geq v(\{j,k\})$. Then x with $x_i = v(N) - v(\{j,k\}) + \frac{1}{2}[v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\}) - 2v(N)]$, $x_j = v(N) - v(\{i,k\}) + \frac{1}{2}[v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\}) - 2v(N)]$, and $x_k = 0$ belongs to the set described by (14).

that $v(\{1,2\}) < v(N)$. Consider the following strategy profile

$$\begin{aligned} c^1 &= (P, v(N), 0), \\ c^2 &= (v(N), P, 0), \\ c^3 &= (\frac{v(N)}{2}, \frac{v(N)}{2}, P). \end{aligned}$$

Clearly, c results in the empty graph and payoff vector $(0,0,0)$. We will show that $c \in \text{CPNE}(\Gamma(v))$. It is obvious that an individual player cannot improve his payoff by unilaterally deviating. Since every self-enforcing strategy profile of the grand coalition in which all players improve their payoffs, results in a connected graph with the sum of the payoffs equal to the value of the grand coalition, it follows that such a self-enforcing strategy profile is a coalition proof Nash equilibrium. Since there is only one two-player coalition with a smaller value than the value of the grand coalition it follows that there is no coalition proof Nash equilibrium with all players receiving a (strictly) positive payoff, since, according to theorem 4.1, such a coalition proof Nash equilibrium should result in a player receiving his marginal contribution and another player receiving at most his marginal contribution, so at least one player should receive zero payoff. It remains to consider deviations by two-player coalitions.

Consider an arbitrary deviation of coalition $\{1,3\}$, in which both players improve their payoffs. Firstly, they can form communication component $\{1,3\}$. Then, player 3 can achieve a further improvement in his payoff by playing $(P, v(N), P)$ which leaves player 1 isolated. Secondly, if players 1 and 3 form communication component $\{1,2,3\}$, player 1 will not be directly linked with player 2 since $c_1^2 = v(N)$. Again player 3 can improve his payoff further by playing $(P, v(N), P)$, leaving player 1 isolated. We have now considered all deviations of coalition $\{1,3\}$ that improve the payoff of both players in the coalition.

Since the deviation possibilities of coalition $\{2,3\}$ are similar to those of coalition $\{1,3\}$ it remains to consider deviations by coalition $\{1,2\}$. Firstly, if players 1 and 2 form communication component $\{1,2\}$, at least one player in the coalition $\{1,2\}$ receives strictly less than $\frac{v(N)}{2}$, since $v(\{1,2\}) < v(N)$, and hence, this player can improve his payoff by playing $(P, P, \frac{v(N)}{2})$. If they form communication component $\{1,2,3\}$ the middleman will receive less than $\frac{v(N)}{2}$ since player 3 receives $\frac{v(N)}{2}$ and the remaining player receives a positive payoff. Hence, the middleman can improve his payoff by playing $(P, P, \frac{v(N)}{2})$. We have now considered all profitable deviations by coalition $\{1,2\}$ and we conclude that $c \in \text{CPNE}(\Gamma(v))$.

Finally, let $v(S) = v(N)$, for all $|S| = 2$. Consider $\hat{c}$ with $\hat{c}^1 = (P, v(N), 0)$, $\hat{c}^2 = (0, P, v(N))$, and $\hat{c}^3 = (v(N), 0, P)$. By theorem 4.1 it follows that there is no coalition proof Nash equilibrium resulting in a positive payoff for all players. So there is no self-enforcing deviation of the grand coalition resulting in a positive payoff for all players. A deviation of a two-person coalition is not stable since the third player claims zero on one of the links. Obviously, a player cannot unilaterally improve his payoff. We conclude that c is a coalition proof Nash equilibium. This completes the proof. □

The following theorem describes all payoff vectors supported by coalition proof Nash equilibria in a link and claim game corresponding to a 3-person cooperative game.

Theorem 4.2 Let (N, v) be a 3-person cooperative game and x a payoff vector. There exists $c \in \text{CPNE}(\Gamma(v))$ with $K(c) = x$ if and only if at least one of the conditions (11), (12), (14) and (15) is satisfied.

Proof: By theorem 3.1 it follows that no cycles will result according to a Nash equilibrium[5] and hence, no cycles will result according to a coalition proof Nash equilibrium. Theorem 4.1, lemma 4.6 and lemma 4.7 then characterize all payoff vectors attainable according to all communication structures that can result. Note that if $v(N) = v(\{i,j\})$ then the payoff vector corresponding to a coalition proof Nash equilibrium resulting in the unique link $\{i,j\}$ might not belong to the set described by (14). Therefore, assume $v(N) = v(\{i,j\})$ and consider $x = K(c)$, where x does not belong to the set described by (14). Then, assume without loss of generality that

$$x_i \leq v(N) - v(\{j,k\})$$

or

$$\begin{aligned} x_i &< v(N) - v(\{j,k\}) + v(\{i,k\}) - v(\{i,j\}) \\ &= v(\{i,k\}) - v(\{j,k\}) \\ &\leq v(N) - v(\{j,k\}). \end{aligned}$$

Then, since $x_k = 0 = v(N) - v(\{i,j\})$ this payoff vector belongs to the set described by (12). □

5 Conclusions

The one-stage game in this paper describes the simultaneous determination of a cooperation structure and payoff distribution for a given cooperative game. It appears, according to the Nash equilibrium concept, that the resulting cooperation structures

[5] We remind the reader that $v(N) > 0$.

will essentially be cycle-free. The notion that the full cooperation structure will not necessarily result, conforms with the structures *Aumann* and *Myerson* (1988) find for their two-stage model.

Since the Nash equilibrium concept usually results in a large set of possible payoff vectors and several possible cooperation structures, we also looked at strong Nash equilibria and coalition proof Nash equilbria. The strong Nash equilibrium concept seems to be too restrictive for this model. For a large set of cooperative games we find that a strong Nash equilibrium strategy results in at least one player receiving zero payoff. Actually, we find that one of the players receiving zero payoff will be connected with several other players, possibly even with all other players. The fact that a player with a central position in the cooperation structure receives zero payoff contradicts for example the allocation the *Myerson value* attributes to the players. According to the Myerson value, a player usually benefits from a central position.

Finally, we looked at coalition proof Nash equilibria, concentrating on 3-person games. It appears that several cooperation strucures are supported by coalition proof Nash equilibria. The empty graph forms only under severe restrictions on the underlying game. The graph with exactly one link is supported by a coalition proof Nash equilibrium if the core of the underlying game is empty. If there is no coalition with a larger value than the value of the grand coalition, then we can always find a coalition proof Nash equilibrium resulting in a cooperation structure with exactly two links. These structures do not conform to the structures *Dutta et al.* (1996) find for their model. They consider a two-stage model and conclude that, according to the coalition proof Nash equilibrium concept, only strategy profiles that result in the same payoff vector as the strategy profile resulting in the full graph will be played.

The model in this paper predicts, according to the coalition proof Nash equilibrium concept, that the middleman will never receive a payoff larger than his marginal contribution to the grand coalition. However, we find that one of the other players may receive more than his marginal contribution to the grand coalition. Again this contradicts the Myerson value, where a player usually benefits from his central position. Note that all payoff vectors on the boundary of the core are supported by coalition proof Nash equilibria. However, there exists no payoff vector in the interior of the core supported by a coalition proof Nash equilibrium. Finally, it holds that every combination of an extreme point of the core with all players receiving a positive payoff and a cooperation structure with exactly two links is supported by a coalition proof Nash equilibrium.

Summarizing, we have analyzed a simple one-stage model of link formation and payoff distribution. Opposed to a number of models in the existing literature, the division of the profit is left to the players. We find that players do not always profit from a central position in a cooperation structure, or even stronger, we find that players in central positions are exploited.

References

Aumann, R. and R. Myerson (1988). *Endogenous formation of links between players and coalitions: an application of the Shapley value.* In: The Shapley value (Ed. A. Roth), Cambridge University Press, Cambridge, United Kingdom, 175-191.

Binmore, K. (1985). *Bargaining and coalitions.* In: Game Theoretic Models of Bargaining, Cambridge University Press, Cambridge, United Kingdom, 269-304.

Borm, P. and S. Tijs (1992). *Strategic claim games corresponding to an NTU-game.* Games and Economic Behavior 4, 58-71.

Chatterjee, K., B. Dutta, D. Ray, and K. Sengupta (1993). *A Non-Cooperative Theory of Coalitional Bargaining.* Review of Economic Studies 60, 463-477.

Dutta, B., A. van den Nouweland, and S. Tijs (1996). *Link formation in cooperative situations.* Working paper.

Gul, F. (1989). *Bargaining Foundations of the Shapley value.* Econometrica 57, 615-636.

Monderer, D. and L. Shapley (1996). *Potential games.* Games and Economic Behavior 14, 124-143.

Myerson, R. (1977). *Graphs and cooperation in games.* Mathematics of Operations Research 2, 225-229.

Myerson, R. (1991). *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts.

Perry, M. and P. Reny (1994). *A Non-Cooperative view of Coalition Formation and The Core.* Econometrica 62, 795-817.

Slikker, M., B. Dutta, A. van den Nouweland, S. Tijs (1996). *Potential maximizers and network formation.* Working paper, Department of Econometrics and CentER, Tilburg University, Tilburg, The Netherlands.