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Centrality and power in social networks: a game theoretic approach

Daniel Gómez[b], Enrique González-Arangüena[b], Conrado Manuel[b], Guillermo Owen[a], Mónica del Pozo[b], Juan Tejada[c,*]

[a]*Naval Postgraduate School. Department of Mathematics, Monterey, CA USA*
[b]*Dpto de Estadística e I.O. III, Escuela Universitaria de Estadística, Universidad Complutense de Madrid, Madrid, Spain*
[c]*Dpto de Estadística e I.O. I, Facultad de CC Matemáticas, Universidad Complutense de Madrid, Madrid, Spain*

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Abstract

A new family of centrality measures, based on game theoretical concepts, is proposed for social networks. To reflect the interests that motivate the interactions among individuals in a network, a cooperative game in characteristic function form is considered. From the graph and the game, the graph-restricted game is obtained. Shapley value in a game is considered as actor's power. The difference between actor's power in the new game and his/her power in the original one is proposed as a centrality measure. Conditions are given to reach some desirable properties. Finally, a decomposition is proposed.

Keywords: Social networks; Game theory; Centrality; Shapley value

JEL classification: C71

1. Introduction

In this paper, it is assumed that a *social network* is given by a graph (N, Γ), where $N = \{1, 2, \ldots, n\}$ is a finite set of individuals (*nodes*) and Γ is a collection of (unordered) pairs $\{i, j\}$ of elements of N (*edges*), which shows the possible communications; i.e.,

*Corresponding author. Tel.: +34-913-9444-24; fax: +34-913-9446-07.
E-mail address: jtejada@mat.ucm.es (J. Tejada).

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individuals i and j can communicate directly if and only if $\{i, j\} \in \Gamma$. Then, our approach will apply to networks in which relations are binary valued: they exist or they do not. If i cannot communicate directly with j, it may still be possible for them to communicate indirectly if there is some k (an *intermediary*) with whom both can communicate, or more generally, a sequence of intermediaries.

Centrality is a sociological notion which is not, however, clearly defined; it is frequently defined only in an indirect manner. For example, we are told an individual, i, has centrality in a graph if:

(i) can communicate directly with many other nodes, or
(ii) is close to many other nodes, or
(iii) there are many pairs of nodes which need i (or, can use i) as intermediary in their communications.

Previous ideas are reflected in some well-known centrality measures as:

(a) Degree centrality (Shaw, 1954; Nieminen, 1974). This approach identifies centrality with the *degree* of a node, i.e., the number of edges incident on that node. The degree centrality focuses on the level of communication activity. The more ability to communicate directly with others, the more centrality.

    For instance, the hub of the star in Fig. 1 has the highest level of direct communication as long as it has a higher degree than the other nodes.

(b) Closeness centrality (Beauchamp, 1965; Sabidussi, 1966). Another reason that makes the hub in the star more central is that it is *closer* to many more nodes than the rest of the nodes. This approach considers the sum of the *geodesic* distances between a given node and the remaining as a decentrality measure in the sense that, the lower this sum, the greater the centrality.

    Closeness centrality represents independence: the possibility to communicate with many others depending on a minimum number of intermediaries.

    We might consider degree centrality as a special case of closeness centrality in

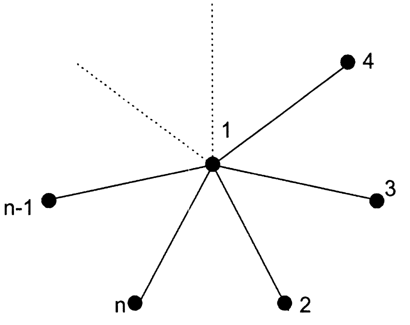


Fig. 1. Star (N, Γ^S).

which independence is essential in the sense that only communications without intermediaries are considered.

(c) Betweenness centrality (Bavelas, 1948 and Freeman, 1977). A third reason to consider the hub of a star in advantage is that it lies *between* all pair of nodes and no other node has this property. Betweenness centrality focuses on the communication control: the possibility to intermediate in the communications of others.

In this approach all possible geodesic paths between pairs of points are considered. The centrality measure of a given node is then obtained by counting the number of such paths on which it lies.

All these measures have in common the same structural element: the *geodesic pathway*. A different approach to the concept of centrality, which does not make use of geodesic paths, is given in Stephenson and Zelen (1989). They propose a centrality measure for networks based on the concept of *information* as used in the theory of statistical estimation. Their measure employs a weighting combination of all paths between pairs of nodes, each weight depending on the information contained in the corresponding path.

Another concept of centrality is suggested in the work of Bonacich (1972), that proposes as a centrality measure the eigenvector associated with the largest characteristic eigenvalue of the adjacency matrix. This approach neglects multiple paths joining nodes of the graph, but more recently Bonacich (1987) has introduced a modification, which uses a parameter to weight indirect paths. Moreover, this generalization accounts for the link between power and centrality.

This relation is another notion on which social networks researchers have failed to agree. Some of them think that centrality and power should be definitionally equivalent, while others have argued that the former determines the latter. Different papers reflect this controversy:

> *All sociologists agree that power is a fundamental property of social structure but, there is much less agreement about what power is*. (*Hanneman,* 1999).

> *Despite the once wide acceptance of the link between centrality and power, the extent to which both concepts are related is now an issue of intense controversy*. (*Mizruchi and Potts,* 1998).

In this paper our proposal is to measure centrality as variation in the power due to the social structure, using a game theoretical approach to the concept of power.

Some concepts of game theory will be useful in our development.

An n-personal game in characteristic function form is given by a pair (N, v), being $N=\{1, 2, \ldots, n\}$ a finite set of individuals (the *players* set) and v a real valued function defined on 2^N, satisfying $v(\emptyset)=0$. We can think of v as representing the economic possibilities of the several *coalitions* S (subsets of N). To measure the power of individuals or players in such a game, several power indices have been defined in the game theoretic literature.[1] Shapley value is a power index widely used. As this value indicates marginal contribution of players, it is indeed a measure for the importance and possibilities of blocking and or maintaining connections on the individual level. In that sense, it is a power index (partly) based on the social structure.

Let us consider now a social network (N, Γ) and a game (N, v), the players (of the game) being the nodes of the graph. The graph Γ indicates direct communication possibilities between pairs of players. Implicitly, it shows us which coalitions are feasible, i.e., which coalitions have all their members directly (or indirectly) communicated.

From Myerson (1977), the existence of an structure of feasible coalitions generated by the graph forces us to define a new game (N, w_Γ): *the graph-restricted game*, where the value of a coalition S equals the sum of the v-values of the connected components of the network restricted to this coalition S. Using the Shapley value we can assess the power of a node in the game v and its new power in the graph-restricted game, where the structure of communications is taken into account.

Then, to define a centrality measure based on power's variation due to the restrictions in the communication, we can take, for each node i, the difference between its Shapley value in the graph-restricted game $(\varphi_i(w_\Gamma))$ and in the original one $(\varphi_i(v))$ as such a measure.

In order to clarify and motivate our centrality measure, let us consider the following example: in a given parliament, there are members of three parties: 1, 2 and 3, the respective percentages being 40, 20 and 40%. For a bill to be passed it requires at least 2/3 of votes in favor. Let us suppose the members of each party have agreed to vote in block and nobody defects this agreement. Then, a coalition between parties 1 and 3 is essential, whereas party 2's opinion is irrelevant (2 is called a *dummy* player in game theory).

Intuitively any power index will assign null power to party 2 and, by symmetry, the same value to parties 1 and 3. Assuming that total power is one, we have:

$$\varphi_1(v) = \varphi_3(v) = 1/2, \quad \varphi_2(v) = 0.$$

This allocation of power seems reasonable when all coalitions are equally possible, but in many cases some of the coalitions are seemingly out of the question due to the players' previous relations with each other. In the previous example, if parties 1, 2 and 3 are aligned in an ideal political spectrum left–right or liberal–conservative, as in Fig. 2, it would be reasonable to suppose that the non-contiguous parties coalitions are not

[1] The term power index was firstly introduced for simple games, an special case of n-personal game in characteristic function form. Intuitively speaking, a simple game (*voting game*) is a cooperative/competitive enterprise where the only goal is to win and the only rule is a specification of which coalitions are able to do so (*winning coalitions*). In this special case of *voting games* or *weighted voting games*, it is useful a measure of the power of individual voters in the game. Shapley and Shubik (1954) proposed to use the Shapley value as such a measure. A second game theoretic power index was proposed by Banzhaf (1965).

For a general cooperative n-personal game in characteristic function form, Shapley value or any other *semi-value* (Dubey et al., 1981) or, may be, any other *allocation rule* (Myerson, 1977) could play the role that power indices do in voting games.

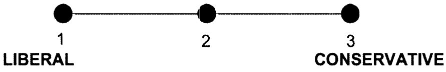


Fig. 2. Political network.

likely to be formed. In other words, a global agreement is more likely to occur than an agreement between opposite parties excluding the one in the middle. Thus, the coalition {1, 3} is not feasible but {1, 2, 3} is.

With this modification the game has changed (*graph-restricted game*). The only winning coalition is now {1, 2, 3}, where the three parties are equally indispensable. Any reasonable allocation will assign the same power to all three parties, and then, we have:

$$\varphi_1(w_\Gamma) = \varphi_2(w_\Gamma) = \varphi_3(w_\Gamma) = 1/3.$$

While restrictions have supposed an increase of power for party 2, 1 and 3 are in a worse position. This power variation (decrease or increase) could be used as a centrality measure in the sense that, the higher this difference, the more effective the connections of a player are in the graph.

In this case:

$$\begin{aligned} \gamma_1 &= \varphi_1(w_\Gamma) - \varphi_1(v) = -\frac{1}{6} = \gamma_3, \\ \gamma_2 &= \varphi_2(w_\Gamma) - \varphi_2(v) = \frac{1}{3}, \end{aligned}$$

so party 2 improves its power in $\frac{1}{3}$ of total power due to its position in the political spectrum, whereas parties 1 and 3 each reduces its power in $\frac{1}{6}$ of total power.

It could be thought that previous result depends heavily on the fact that party 2 is the hub of a star and thus, it is necessarily the most central one. Nevertheless, in our approach, game v plays a crucial role. To illustrate this fact, let us consider a slight modification of the weighted voting game from the former example, being the percentages now 30, 40, and 30% for parties 1, 2 and 3, and keeping the rest unchanged. In this game, the only winning coalitions are {1, 2}, {2, 3} and {1, 2, 3}. All these coalitions are feasible in Fig. 2 star. This leads to equality in games v and w_Γ, being for all three parties $\varphi_i(v) = \varphi_i(w_\Gamma)$; therefore, $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and no party modifies its power due to its position in the political spectrum.

The remainder of the paper is organized as follows: Section 2 is devoted to a game theoretic approach to the concept of centrality, defining new measures that satisfy, for a wide class of games, several properties that seem natural for a centrality measure. In Section 3, it is proved that the defined measures satisfy these properties. In Section 4, some particular cases are analyzed and the corresponding discussion will give rise to an interesting decomposition of the measures introduced. Section 5 is devoted to this decomposition. In Section 6, we discuss that the properties are more or less met by the well-known centrality measures, and we highlight several differences and similarities between our measures and the classical ones. Section 7 contains a final remark about the sensitivity of our measures to the initial game v.

2. A game theoretic approach

We shall say that a social network (N, Γ) is connected if it is possible to join any two nodes i and j of N by a sequence of edges from Γ. We shall say that a subset S of N is *connected in* (N, Γ) if (S, Γ_S) is connected, where Γ_S is the set of those pairs $\{h, k\} \in \Gamma$ where both h and k are elements of S.

Let (N, v) be an n-personal game in characteristic function form and (N, Γ) a social network. No particular relation is assumed between the game (N, v) and the graph (N, Γ), other than the players of the game being the nodes of the graph. When there is no ambiguity with respect to N we will refer to the graph (N, Γ) and the game (N, v) as Γ and v, respectively.

Following Myerson (1977), we define the new game w_Γ, the *graph-restricted game*, by:

$$w_\Gamma(S) = \sum_{T_k \in C_\Gamma(S)} v(T_k), \tag{1}$$

where $C_\Gamma(S)$ is the set of connected components of S in Γ. Note that, if S is connected in Γ, then $w_\Gamma(S) = v(S)$. The game w_Γ represents the economic possibilities taking into account the available communications.

If we consider some 'reasonable outcome' for these two games, the differences between the corresponding outcomes can be considered as a result of the different positions which the players have in graph Γ. Clearly, the result will depend on the particular 'reasonable outcome' which we use.

From now on, we will use the Shapley value, φ, as an index of players' power in a given game, though we could just as easily use the Banzhaf–Coleman index of power, as in Grofman and Owen (1982).[2] Then we can think of the difference between $\varphi_i(w_\Gamma)$ (the Shapley value of player i in the projected game w_Γ) and $\varphi_i(v)$ as a measure of the centrality of player i in the graph Γ, i.e.:

$$\gamma_i(v, \Gamma) = \varphi_i(w_\Gamma) - \varphi_i(v). \tag{2}$$

It represents the increase (or decrease) in i's power due to its position in the graph. Note, however, that this depends also on the game v.

From Myerson (1977) we raise the following propositions:

Proposition 2.1. *If* (N, v) *is a game and* (N, Γ) *is a social network then, removing an edge of* Γ, *the centrality of both incident nodes on that edge will change by an equal amount* (*Fairness*).

Proposition 2.2. *If* (N, v) *is a super-additive game and* (N, Γ) *is a social network then,*

[2] Choosing Shapley value as power index is not an arbitrary election: Shapley value is the only semi-value that is efficient, in the sense that allocates $v(N)$ among the players, and Myerson proves that it is also the only allocation rule that is *fair*, in the sense that, removing an edge of Γ, changes the power of both incident nodes (players) on that edge by an equal amount. Both properties are suitable for our centrality measures.

removing an edge of Γ*, the centrality of both incident nodes on that edge will decrease or, at least, not increase* (*Stability*).

In this paper, we will analyze the case in which game v deals with all players symmetrically, so that the centrality measure depends, mainly, on the graph Γ rather than on the particular role played by i in game v. Let us suppose, then, that v is symmetric, i.e.:

$$v(S) = f(s), \quad S \subset N,$$

where s is the cardinality of S, and the function f satisfies $f(0) = 0$.

Assuming v is symmetric, $\varphi_i(v) = v(N)/n$ for all players. In this case, it is possible to avoid the last term in (2). Since we are interested in comparing centralities within a network, rather than in obtaining some absolute measure of centrality, we propose:

$$\kappa_i[v, \Gamma] = \varphi_i(w_\Gamma), \tag{3}$$

as a measure for the centrality of i in Γ.

A critical aspect of network centrality analysis is the graph comparability. The centrality measure we have defined is *efficient* in the following sense: for a symmetric game (N, v), the total centrality of all nodes is invariant under graph transformations that preserve the number of connected components and their cardinalities. The following proposition proves this result.

Proposition 2.3. *If* (N, v) *is a symmetric game and* (N, Γ) *is a social network then,*

$$\sum_{i=1}^{n} \kappa_i[v, \Gamma] = \sum_{T_k \in C_\Gamma(N)} f(t_k).$$

Proof. By the Shapley value efficiency and the graph-restricted game definition:

$$\sum_{i=1}^{n} \kappa_i[v, \Gamma] = \sum_{i=1}^{n} \varphi_i(w_\Gamma) = w_\Gamma(N) = \sum_{T_k \in C_\Gamma(N)} v(T_k) = \sum_{T_k \in C_\Gamma(N)} f(t_k). \quad \square$$

In particular, for any connected graph with N nodes, the total centrality is always $v(N)$. As a direct consequence, we can avoid any normalization of centralities when we wish to compare them in different connected graphs with the same number of nodes.

In order to calculate the proposed centrality measure, it will be useful to analyze the mapping P_Γ of G_N (the vector space of all games with the set of players N) into itself, defined by $P_\Gamma(v) = w_\Gamma$, where w_Γ is given by Eq. (1). This is a linear mapping and it is not difficult to verify that it is also a projection, in the sense that $P_\Gamma \circ P_\Gamma = P_\Gamma$.

To carry out this computation, let us express the space G_N in terms of the *unanimity basis*, which consists of the $2^n - 1$ unanimity games. For each (non-empty) $S \subset N$, the unanimity game u_S is defined by:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subset T, \\ 0, & \text{if } S \not\subset T. \end{cases}$$

and it is not difficult to prove that game v can be expressed as:

$$v = \sum_{S \subset N} \Delta(S) u_S, \tag{4}$$

where $\Delta(S)$, the *Harsanyi dividend* (of S in v), is given by (Shapley, 1953):

$$\Delta(S) = \sum_{T \subset S} (-1)^{s-t} v(T), \tag{5}$$

(s and t being the cardinalities of S and T, respectively). Since G_N has dimension $2^n - 1$, it is clear that the games u_S form a basis.

We note that, if S is connected in Γ, then $P_\Gamma(u_S) = u_S$. If, on the other hand, S is not connected in Γ, we find that $P_\Gamma(u_S) = w_{S,\Gamma}$, where:

$$w_{S,\Gamma}(T) = \begin{cases} 1, & \text{if there is some connected } K \text{ with } S \subset K \subset T, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Thus, this last game, $w_{S,\Gamma}$, can be considered as the *connecting S in Γ game*. There exists an special case in which this game is easily described. If Γ is a *tree* (a connected graph with no cycles), then there is only one smallest connected K which contains S. We call it $H(S)$: the *connected hull* of S. In this case, it satisfies:

$$w_{S,\Gamma} = u_{H(S)}.$$

A general expression for $w_{S,\Gamma}$ is given in Lemma 2.1, which uses the following definition:

Definition 2.1. Given the social network (N, Γ), S, $S' \subset N$, and S' connected in Γ, we will state that S' is a minimal connection set of S in Γ if there is not $S'' \subset N$ ($S'' \neq S'$) connected in Γ and such that $S \subset S'' \subset S'$.

Let us observe that for a given $S \subset N$, several minimal connection sets of S, one, or none could exist. $\mathcal{M}_\Gamma(S)$ will denote the collection of these sets.

From (6), $\mathcal{M}_\Gamma(S) = \emptyset$ implies $w_{S,\Gamma} = \mathbf{0}$ (the null vector of G_N). Therefore in the next lemma we will only deal with the case $\mathcal{M}_\Gamma(S) \neq \emptyset$.

Lemma 2.1. *Given the social network (N,Γ) and $S \subset N$, if $\mathcal{M}_\Gamma(S)$ is non-empty and $\mathcal{M}_\Gamma(S) = \{S_i\}_{i=1}^r$, then*:

$$P_\Gamma(u_S) = \mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i}) \tag{7}$$

where $\mathbf{1}$ is the game defined by $\mathbf{1}(S) = 1$, for all $S \neq \emptyset$, i.e., the unit element of the standard inner product in G_N.

Proof. Let $T \subset N$, then:

$$(1 - u_{S_i})(T) = \begin{cases} 0, & \text{if } S_i \subset T, \\ 1, & \text{if } S_i \not\subset T, \end{cases}$$

and therefore:

$$\left(\mathbf{1}-\prod_{i=1}^{r}(\mathbf{1}-u_{S_i})\right)(T)=\begin{cases}1, & \text{if there is } S_i\subset T,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

Then, (8) coincides with (6). □

The next proposition characterizes the image of the mapping P_Γ. This result appears in Owen (1986), but we include a different and constructive proof, useful to the calculation of the centrality.

Proposition 2.4. *The image of the mapping P_Γ is the subspace of G_N spanned by the games u_T where T is connected in Γ.*

Proof. From (7), given $S\subset N$ if $\mathcal{M}_\Gamma(S)\neq\emptyset$ and $\mathcal{M}_\Gamma(S)=\{S_i\}_{i=1}^r$, we obtain:

$$P_\Gamma(u_S)=\sum_{i=1}^{r}u_{S_i}-\sum_{i<j}^{r}u_{S_i}\cdot u_{S_j}+\sum_{i<j<k}^{r}u_{S_i}\cdot u_{S_j}\cdot u_{S_k}+\cdots+(-1)^{r+1}u_{S_1}\cdot u_{S_2}\cdots u_{S_r}.$$

Note that, if $\{i_1,\ldots,i_k\}\subset\{1,\ldots,r\}$,

$$u_{S_{i_1}}\cdot u_{S_{i_2}}\cdots u_{S_{i_k}}=u_{\cup_{j=1}^{k}S_{i_j}},$$

and thus,

$$P_\Gamma(u_S)=\sum_{i=1}^{r}u_{S_i}-\sum_{i<j}^{r}u_{S_i\cup S_j}+\sum_{i<j<k}^{r}u_{S_i\cup S_j\cup S_k}+\cdots+(-1)^{r+1}u_{\cup_{i=1}^{r}S_i}. \tag{9}$$

Since $S_{i_1},\ldots,S_{i_k}$ are connected and not disjoint sets for all $\{i_1,\ldots,i_k\}\subset\{1,\ldots,r\}$, $\cup_{j=1}^{k}S_{i_j}$ is connected. Therefore, $P_\Gamma(u_S)$ is a linear combination of unanimity games u_T where T is connected. □

Using (3), (4) and the Shapley value linearity, we obtain:

$$\kappa_i[v,\Gamma]=\sum_{S\subset N}\Delta(S)\varphi_i(w_{S,\Gamma}).$$

Therefore, we only need to know the Harsanyi dividends of v and the Shapley value for the projection of every unanimity game ($w_{S,\Gamma}$).

If $\mathcal{M}_\Gamma(S)=\{\hat{S}\}$, then $w_{S,\Gamma}=u_{\hat{S}}$ and the calculation of the value $\varphi(w_{S,\Gamma})$ is done straightforward (essentially the dividend of S is evenly apportioned among all nodes of $\hat{S}$). In case of several options, calculation complexity increases. This, implicitly, means that if Γ is a tree (or a forest, defined as a union of disjoint trees), then calculation of centrality will be easy. A method, based on generating functions, is described in detail in Owen (1986).

When Γ has cycles and there exists a $S\subset N$ such that $\mathcal{M}_\Gamma(S)=\{S_l\}_{l=1}^r$, with $r>1$, using (9) and Shapley value linearity, we obtain:

$$\varphi_i(w_{S,\Gamma}) = \sum_{l=1}^{r} \varphi_i(u_{S_l}) - \sum_{l<j}^{r} \varphi_j(u_{S_l \cup S_j}) + \sum_{l<j<k}^{r} \varphi_i(u_{S_l \cup S_j \cup S_k}) - \cdots + (-1)^{r+1} \varphi_i(u_{\cup_{l=1}^{r} S_l}), \tag{10}$$

and thus,

$$\varphi_i(w_{S,\Gamma}) = \sum_{l=1}^{r} \frac{1}{s_l} \delta_{S_l}(i) - \sum_{l<j}^{r} \frac{1}{s_{l,j}} \delta_{S_{l,j}}(i) + \sum_{l<j<k}^{r} \frac{1}{s_{l,j,k}} \delta_{S_{l,j,k}}(i) - \cdots + (-1)^{r+1} \frac{1}{s_{1,\ldots,r}} \delta_{S_{1,\ldots,r}}(i), \tag{11}$$

where $s_{j_1,\ldots,j_k}$ is the cardinality of $\cup_{l=1}^{k} S_{j_l}$ and

$$\delta_{S_{j_1,\ldots,j_k}}(i) = \begin{cases} 1, & \text{if } i \in \cup_{l=1}^{k} S_{j_l}, \\ 0, & \text{otherwise.} \end{cases}$$

3. General results

In this section we will lay out general conditions for the game v under which $\kappa_i[v, \Gamma]$ satisfies several properties that seem outstanding for any centrality measure in social networks.

To this end, let us consider two classes of characteristic functions. If v is symmetric, i.e., $v(S) = f(s)$, for all $S \subset N$:

(i) v is super-additive if $f(m+n) \geq f(m) + f(n)$, for all $m, n \in \mathbb{N}$. v is strictly super-additive if the above inequality holds strictly.
(ii) v is convex if f is convex in $\mathbb{N}$, i.e., $f(s+1) - f(s) \geq f(s) - f(s-1)$, for all $s \geq 1$. v is strictly convex if the above inequality holds strictly.

From now on, we will use:

$$\mathcal{S}_N = \{v \in G_N : v \text{ symmetric}\}$$
$$\mathcal{A}_N = \{v \in G_N : v \text{ super-additive}\}$$
$$\mathcal{C}_N = \{v \in G_N : v \text{ convex}\}$$

Let us bear in mind that $\mathcal{C}_N \subset \mathcal{A}_N$.

Moreover, $\mathcal{G}_N$ will denote the set of all graphs with nodes set N.

Hereafter, if $v \in \mathcal{S}_N$, i.e.: $v(S) = f(s)$, we will use v or f equivalently.

The following Proposition 3.1 shows that, if v is symmetric, the centrality measure induced is also symmetric. Let us first introduce a definition.

Definition 3.1. We will say that a permutation $\pi: N \to N$ preserves the graph (N, Γ) when $\{i, j\} \in \Gamma$ if and only if $\{\pi(i), \pi(j)\} \in \Gamma$.

If we note $\pi\Gamma = \{\{\pi(i), \pi(j)\}: \{i, j\} \in \Gamma\}$, the previous definition tells us that π preserves the graph (N, Γ) when $\pi\Gamma = \Gamma$.

Proposition 3.1. *Let $v \in \mathscr{S}_N$ and $\Gamma \in \mathscr{G}_N$. If π is a permutation on N that preserves Γ, $\kappa_i[v, \Gamma] = \kappa_{\pi(i)}[v, \Gamma]$, for all $i \in N$.*

Proof. As $v \in \mathscr{S}_N$, $\pi v = v$.

By the symmetry of the Shapley value:

$$\varphi_i(w) = \varphi_{\pi(i)}(\pi w) = \varphi_{\pi(i)}(w),$$

and thus,

$$\kappa_i[v, \Gamma] = \kappa_{\pi(i)}[v, \Gamma]. \quad \square$$

The next proposition shows that the centrality of a node in a disconnected graph coincides with its centrality when considered as a node in the connected subgraph to which it belongs.

Proposition 3.2. *Let N_1 and N_2 be disjoint subsets of $\mathbb{N}$. Suppose $\Gamma^j \in \mathscr{G}_{N_j}$ is connected, $j = 1, 2$, and $\Gamma^1 \cup \Gamma^2 \in \mathscr{G}_{N_1 \cup N_2}$.*

If $v \in \mathscr{S}_{N_1 \cup N_2}$ and v_j is the restriction of v to N_j, $j = 1,2$, then for $i \in N_j$, $\kappa_i[v, \Gamma^1 \cup \Gamma^2] = \kappa_i[v_j, \Gamma^j]$, $j = 1, 2$.

Proof. Writing v in terms of the unanimity basis:

$$v = \sum_{S \subset N_1 \cup N_2} \Delta(S) u_S.$$

Then, by the linearity of projection $P_{\Gamma^1 \cup \Gamma^2}$:

$$\begin{aligned} w = \quad & P_{\Gamma^1 \cup \Gamma^2}(v) = \sum_{S \subset N_1 \cup N_2} \Delta(S)\, w_{S,\, \Gamma^1 \cup \Gamma^2} \\ & = \sum_{S \subset N_1} \Delta(S)\, w_{S,\, \Gamma^1 \cup \Gamma^2} + \sum_{S \subset N_2} \Delta(S)\, w_{S,\, \Gamma^1 \cup \Gamma^2} + \sum_{\substack{S \subset N \\ S \cap N_1 \neq \emptyset \\ S \cap N_2 \neq \emptyset}} \Delta(S) w_{S,\, \Gamma^1 \cup \Gamma^2}. \end{aligned}$$

As $N_1 \cap N_2 = \emptyset$, $\Gamma^1 \cup \Gamma^2$ is disconnected, by Proposition 2.4, u_S belongs to $\ker(P_{\Gamma^1 \cup \Gamma^2})$ when $S \cap N_j \neq \emptyset$, for $j = 1, 2$.

Moreover, if $S \subset N_j$, $w_{S, \Gamma^1 \cup \Gamma^2} = w_{S, \Gamma^j}$, $j = 1, 2$, due to the fact that the elements of $\mathscr{M}_{\Gamma^1 \cup \Gamma^2}(S)$ are contained in N_j. Then,

$$w = \sum_{S \subset N_1} \Delta(S) w_{S, \Gamma^1} + \sum_{S \subset N_2} \Delta(S) w_{S, \Gamma^2}.$$

Let us observe that if $i \in N_j$ and $S \subset N_k$, $k \neq j$, (11) shows that $\varphi_i(w_{S, \Gamma^k}) = 0$, as there are no sets in $\mathscr{M}_{\Gamma^k}(S)$ containing i. And then,

$$\kappa_i[v, \Gamma^1 \cup \Gamma^2] = \kappa_i[v_j, \Gamma^j]. \quad \square$$

The next proposition shows that, if v is symmetric and super-additive, isolated nodes have minimal centrality.

Proposition 3.3. *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$ and $\Gamma^0 \in \mathcal{G}_N$. If $i \in N$ is an isolated node in Γ^0, then, for all $\Gamma \in \mathcal{G}_N$, and for all $j \in N$:*

$$\kappa_i[v, \Gamma^0] \leq \kappa_j[v, \Gamma].$$

Proof. By definition of P_Γ, the centrality of an isolated node of Γ is $f(1)$.

Let $j \in N$ with degree k in Γ. If j is not isolated then $1 \leq k \leq n-1$. Let us assume $\kappa_j[v, \Gamma] < f(1)$. The node j will become an isolated node by a stepwise elimination of the k edges incident on it. From Proposition 2.1, the sequence of centralities of node j is not increasing. Then, j would be an isolated node with centrality strictly lesser than $f(1)$. This contradiction proves the result. $\square$

Lemma 3.1, whose proof can be obtained straightforward, and Proposition 3.4 guarantee that, if v is symmetric and super-additive, among all graphs with n nodes the maximal centrality is attained by the hub of a star.

Lemma 3.1. *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$, $\Gamma \in \mathcal{G}_N$ and $w = P_\Gamma(v)$. If $i \in N$, then for all $S \subset N \backslash \{i\}$:*

(i) $w(S \cup \{i\}) \leq f(s+1)$,
(ii) $w(S) \geq sf(1)$,
(iii) $w(S \cup \{i\}) - w(S) \leq f(s+1) - sf(1)$.

Proposition 3.4. *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$. Let us suppose that $\Gamma^S \in \mathcal{G}_N$ is the star with n nodes where node 1 is the hub. Then, for all $\Gamma \in \mathcal{G}_N$ and for all $i \in N$:*

$$\kappa_i[v, \Gamma] \leq \kappa_1[v, \Gamma^S].$$

Proof. Without loss of generality we can relabel node i in Γ as node 1.

Using the usual Shapley value expression for $1 \in N$ we have:

$$\kappa_1[v, \Gamma] = \varphi_1(w) = \sum_{S: 1 \notin S} \frac{s!(n-1-s)!}{n!} (w(S \cup \{1\}) - w(S)),$$

and for the hub of the star:

$$\kappa_1[v, \Gamma^S] = \sum_{S:\, 1 \notin S} \frac{s!(n-1-s)!}{n!} (f(s+1) - sf(1)),$$

and, by Lemma 3.1:

$$\kappa_1[v, \Gamma] \leq \kappa_1[v, \Gamma^S]. \quad \square$$

Proposition 3.5, which needs Lemma 3.2 (whose proof is trivial) and Lemma 3.3, proves that, if v is symmetric and convex, among all connected graphs with n nodes, the minimal centrality is attained by the end nodes in a chain.

Lemma 3.2. *Let $\Gamma \in \mathscr{G}_N$, $v \in G_N$, $S \subset N$ and Q a separated part of S in Γ, i.e., $Q \subset S$ and there are no edges joining a node in Q with a node in $S \backslash Q$. If $w = P_\Gamma(v)$, then*:

$$w(S) = w(Q) + w(S \backslash Q).$$

Lemma 3.3. *Let $\Gamma \in \mathscr{G}_N$, $v \in G_N$ and $S \subset N$. Let us suppose $i \in N \backslash S$ and $w = P_\Gamma(v)$. Then, we have: $w(S \cup \{i\}) - w(S) = v(Q) - w(Q \backslash \{i\})$, where Q is the i-component of $S \cup \{i\}$, i.e., the component of $S \cup \{i\}$ in Γ which contains i.*

Proof. $Q \backslash \{i\}$ may not be connected, but it is a separated part of S. Note that $S \cup \{i\} \backslash Q$ and $S \backslash (Q \backslash \{i\})$ are both equal to $S \backslash Q$.

As,

$$w(S \cup \{i\}) = w(Q) + w(S \backslash Q),$$

and

$$w(S) = w(Q \backslash \{i\}) + w(S \backslash Q),$$

then,

$$w(S \cup \{i\}) - w(S) = w(Q) - w(Q \backslash \{i\}).$$

As Q is connected, $w(Q) = v(Q)$. This proves the Lemma. □

Proposition 3.5. *Let $v \in \mathscr{S}_N \cap \mathscr{C}_N$. If $\Gamma^C \in \mathscr{G}_N$ is the chain with n nodes, where node 1 is an end node, then, for all connected $\Gamma \in \mathscr{G}_N$ and for all $i \in N$*:

$$\kappa_1[v, \Gamma^C] \leq \kappa_i[v, \Gamma].$$

Proof. Let us suppose that the nodes of Γ^C are labeled in the natural way. Then we relabel the nodes of Γ as follows. Let node i be relabeled as 1. Then, let 2 be any node which is adjacent to 1. Let 3 be any node which is adjacent to either 1 or 2 (or possibly both). Continuing in this manner let node k be any node which is adjacent to at least one of the nodes $1, \ldots, k-1$. Since Γ is connected, this process can be continued until all nodes have been numbered. We see then that every segment $\{1, \ldots, k\}$ is connected in Γ.

Let $w^C = P_{\Gamma^C}(v)$ and $w = P_\Gamma(v)$, then:

$$\kappa_1[v, \Gamma^C] = \varphi_1(w^C) = \sum_{S:1 \notin S} \frac{s!(n-1-s)!}{n!} (w^C(S \cup \{1\}) - w^C(S)), \tag{12}$$

whereas:

$$\kappa_1[v, \Gamma] = \varphi_1(w) = \sum_{S:1 \notin S} \frac{s!(n-1-s)!}{n!} (w(S \cup \{1\}) - w(S)). \tag{13}$$

Now, we shall show that each summand in (13) is, at least, as large as the corresponding summand in (12).

In fact, for a given S, let K and Q be the 1-components of $S \cup \{1\}$ in Γ^C and in Γ, respectively. K must be of the form $\{1, \ldots, k\}$. By Lemma 3.3, we have:

$$w^C(S \cup \{1\}) - w^C(S) = v(K) - w^C(K \backslash \{1\}).$$

Since K has k elements, $v(K) = f(k)$. Now, as $K\backslash\{1\}$ is connected in Γ^C, $w^C(K\backslash\{1\}) = v(K\backslash\{1\}) = f(k-1)$, and therefore:

$$w^C(S \cup \{1\}) - w^C(S) = f(k) - f(k-1).$$

Next, by the relabeling method used for Γ, we see that K is connected in Γ. Thus we must have $K \subset Q$, which implies that q, the number of elements in Q, cannot be smaller than k. Then:

$$w(S \cup \{1\}) - w(S) = v(Q) - w(Q\backslash\{1\}).$$

Since $v(Q) = f(q)$ and $w(Q\backslash\{1\}) \le v(Q\backslash\{1\}) = f(q-1)$, we realize that:

$$w(S \cup \{1\}) - w(S) \ge f(q) - f(q-1).$$

As $q \ge k$, and v is convex, this leads to $f(q) - f(q-1) \ge f(k) - f(k-1)$. This proves that:

$$w(S \cup \{1\}) - w(S) \ge w^C(S \cup \{1\}) - w^C(S).$$

Due to the fact that this holds for every S, it proves the proposition. □

Finally, if Γ^C is a chain and v is symmetric and convex, centrality increases from the end nodes to the median node as it is shown by Lemma 3.4 and Proposition 3.6.

Lemma 3.4. *Let $n_1, n_2 \in \mathbb{N}$, $n_1 < n_2$ and let $v \in \mathcal{S}_{N_2} \cap \mathcal{C}_{N_2}$. Let us suppose that Γ_i^C is a chain with n_i nodes ordered in the natural way. Then*:

$$\kappa_1[v, \Gamma_1^C] \le \kappa_1[v, \Gamma_2^C].$$

Proof. We will show that the result is true for $n_2 = n_1 + 1$. We have:

$$\kappa_1[v, \Gamma_2^C] - \kappa_1[v, \Gamma_1^C] = \sum_{S \subset N_2} \Delta(S)\varphi_1(w_{S,\Gamma_2^C}) - \sum_{S \subset N_1} \Delta(S)\varphi_1(w_{S,\Gamma_1^C}).$$

As 1 is an end node, $\varphi_1(w_{S,\Gamma_j^C}) = 0$, $j = 1, 2$ if $1 \notin S$, and then:

$$\begin{aligned}\kappa_1[v, \Gamma_2^C] - \kappa_1[v, \Gamma_1^C] &= \sum_{1 \in S \subset N_2} \Delta(S)\varphi_1(w_{S,\Gamma_2^C}) - \sum_{1 \in S \subset N_1} \Delta(S)\varphi_1(w_{S,\Gamma_1^C}) \\ &= \sum_{1, n_1+1 \in S \subset N_2} \Delta(S)\varphi_1(w_{S,\Gamma_2^C}).\end{aligned}$$

There are $\binom{n_1-1}{s-2}$ coalitions $S \subset N_2$ with cardinality s and such that $\{1, n_1+1\} \subset S$. For each one, $\varphi_1(w_{S,\Gamma_2^C}) = 1/n_1 + 1$. Then, the value of the previous expression is:

$$\frac{1}{n_1+1} \sum_{s=2}^{n_1+1} \binom{n_1-1}{s-2} \Delta(S).$$

From (5) and the symmetry of v:

$$\sum_{s=2}^{n_1+1} \binom{n_1-1}{s-2} \Delta(S) = \sum_{s=2}^{n_1+1} \binom{n_1-1}{s-2} \sum_{t=1}^{s} \binom{s}{t} (-1)^{s-t} f(t),$$

and, after some algebraic manipulations, we obtain:

$$f(n_1+1) - 2f(n_1) + f(n_1-1),$$

which is nonnegative when v is convex.

Hence, the centrality of the end node of a chain with n nodes is a not decreasing function on n, and thus the lemma is proved. □

Proposition 3.6. *If $v \in \mathcal{S}_N \cap \mathcal{C}_N$ and Γ^C is a chain with n nodes numbered in the natural way, then for $1 \leq i \leq n/2$:*

$$\kappa_i[v, \Gamma^C] \leq \kappa_{i+1}[v, \Gamma^C].$$

Proof. For each i, $1 \leq i \leq n/2$, removing the edge $\{i, i+1\}$, the nodes i, $i+1$ will become end nodes of chains that have i and $n-i$ nodes, respectively, $(n-i \geq i)$. Let Γ_i^C and Γ_{n-i}^C be these chains. By Lemma 3.4:

$$\kappa_i[v, \Gamma_i^C] \leq \kappa_{i+1}[v, \Gamma_{n-i}^C],$$

and from Proposition 2.2:

$$\kappa_i[v, \Gamma^C] \leq \kappa_{i+1}[v, \Gamma^C]. \quad \square$$

Let us note that, replacing the super-additiveness (convexity) condition by strict super-additiveness (strict convexity), all above inequalities become strict.

4. Some specific game functions

In the defining process of the family of centrality measures proposed in this paper, the role of the game v is essential, spreading its influence all over the process. The characteristic function, v, measures the actor's economic possibilities. In this way, $v(S)$ represents the profit of s players when there are no restricted relations. Projecting v to obtain w, we modify the economic position of coalition S so as to reflect the environmental restrictions introduced by the social network.

Granted that v is to be symmetric, defined by a function f which is to be super-additive and even convex, what function f should be chosen? Several come to mind:

- *The messages game* focuses on the level of communication activity between pairs of individuals, where the benefit of a binary communication is independent of the actors performing and it is the same for each relation. Then, if S is a subset of N, the payoff of S will be proportional to the number of pairs in S, and thus, $v_1(S) = k\binom{s}{2}$. The proportionality constant, k, is chosen as 2 in order to represent the communication in both ways, and so, $v_1(S) = f_1(s) = s^2 - s$.

  We note that the Harsanyi dividends are given by:

$$\Delta_1(S) = \begin{cases} 2, & \text{if } s = 2, \\ 0, & \text{if } s \neq 2. \end{cases}$$

  In other words, for this function, just the two-player coalitions render a profit. In essence, this corresponds to the idea of sending messages: the coalition $\{i, j\}$ (where $i \neq j$) gives rise to two possible messages as both i and j can send a message to each other.
- *The overhead game* focuses on the general cost that any set of players should pay to perform an action, considering that all players are completely related. For this purpose, we define $v_2(S) = f_2(s) = -1$ for all $s \neq 0$, where the negative sign shows that it is a cost (not a benefit). The Harsanyi dividends are given by:

$$\Delta_2(S) = (-1)^s, \quad s \geq 1.$$

  The projected game, in this case, represents the opposite of the number of connected components in each coalition needing the nodes in each to pay the general cost to perform the action.
- *The conferences game* reflects the level of communication among groups of two or more individuals in a coalition. Then, if S is a subset of N, the payoff of S will be the number of subsets in S with a cardinality greater than 1, and thus $v_3(S) = f_3(s) = 2^s - s - 1$. In this case, the Harsanyi dividends are given by:

$$\Delta_3(S) = \begin{cases} 0, & s = 0, 1, \\ 1, & s \geq 2. \end{cases}$$

  In this game, each coalition receives a unit for every possible meeting or conference among two or more of its members.

To illustrate the ideas we outlined above, let us calculate the centrality of different nodes in some particular connected graphs:

- First, let us analyze the case of a star with n nodes (Γ^S, Fig. 1). If 1 is the hub of the star then, for a symmetric game v:

$$\kappa_1[v, \Gamma^S] = \Delta(1) + \frac{1}{n}\sum_{k=2}^{n}\binom{n+1}{k+1}\Delta(k), \tag{14}$$

and for the remaining nodes ($i \neq 1$):

$$\kappa_i[v, \Gamma^S] = \Delta(1) + \sum_{k=2}^{n} \frac{1}{k+1} \left[\binom{n-1}{k-1} + \frac{1}{k}\binom{n-2}{k-2} \right] \Delta(k). \tag{15}$$

If we particularize (14) and (15) to games v_i, $i = 1, 2, 3$, we obtain:

$$\kappa_1[v_1, \Gamma^S] = \frac{n^2-1}{3}; \quad \kappa_i[v_1, \Gamma^S] = \frac{2n-1}{3}, \quad \text{for } i \neq 1,$$
$$\kappa_1[v_2, \Gamma^S] = \frac{n-3}{2}; \quad \kappa_i[v_2, \Gamma^S] = -\frac{1}{2}, \quad \text{for } i \neq 1;$$
$$\kappa_1[v_3, \Gamma^S] = \frac{1}{n}\left[2^{n+1} - \frac{n^2+3n+4}{2} \right],$$
$$\kappa_i[v_3, \Gamma^S] = \frac{2}{n(n-1)} \left[(n-2)2^{n-1} + 1 \right] - \frac{1}{2}, \quad \text{for } i \neq 1.$$

For v_1, if we standardize the centrality measure dividing by $f_1(n) = n^2 - n$, (i.e., total centrality of the n nodes), then we can study its asymptotic behaviour for the hub and for the other nodes. Noting:

$$\kappa_i^*[v_1, \Gamma^S] = \frac{\kappa_i[v_1, \Gamma^S]}{n^2-n},$$

then we have:

$$\lim_{n\to\infty} \kappa_1^*[v_1, \Gamma^S] = \frac{1}{3},$$
$$\lim_{n\to\infty} \kappa_i^*[v_1, \Gamma^S] = 0, \quad i \neq 1,$$

as in Grofman and Owen (1982).

- Let us proceed now with a six-node chain, Γ^C (Fig. 3).

  For a general game v, we get:

$$\kappa_1[v, \Gamma^C] = \kappa_6[v, \Gamma^C] = \Delta(1) + \frac{29}{20}\Delta(2) + \frac{21}{10}\Delta(3) + \frac{37}{20}\Delta(4) + \frac{13}{15}\Delta(5) + \frac{1}{6}\Delta(6).$$
$$\kappa_2[v, \Gamma^C] = \kappa_5[v, \Gamma^C] = \Delta(1) + \frac{82}{30}\Delta(2) + \frac{53}{15}\Delta(3) + \frac{27}{10}\Delta(4) + \frac{16}{15}\Delta(5) + \frac{1}{6}\Delta(6).$$
$$\kappa_3[v, \Gamma^C] = \kappa_4[v, \Gamma^C]$$
$$= \Delta(1) + \frac{199}{60}\Delta(2) + \frac{131}{30}\Delta(3) + \frac{59}{20}\Delta(4) + \frac{16}{15}\Delta(5) + \frac{1}{6}\Delta(6).$$

In the particular games v_i, $i = 1, 2, 3$ we obtain:

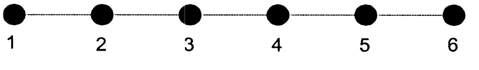


Fig. 3. Chain (Γ^C).

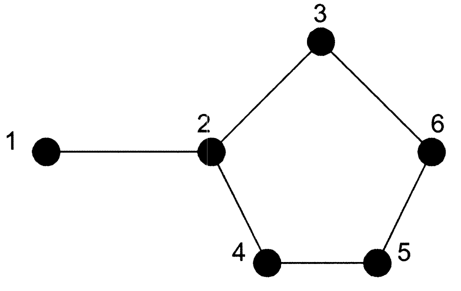


Fig. 4. Kite (Γ^K).

$$\kappa_1[v_1, \Gamma^C] = 2.9, \quad \kappa_2[v_1, \Gamma^C] = 5.47, \quad \kappa_3[v_1, \Gamma^C] = 6.63.$$
$$\kappa_1[v_2, \Gamma^C] = -0.5, \quad \kappa_2[v_2, \Gamma^C] = 0, \quad \kappa_3[v_2, \Gamma^C] = 0.$$
$$\kappa_1[v_3, \Gamma^C] = 6.43, \quad \kappa_2[v_3, \Gamma^C] = 10.2, \quad \kappa_3[v_3, \Gamma^C] = 11.87.$$

- To illustrate the calculation of centrality on graphs with cycles, we consider the *kite* (Γ^K, Fig. 4) and the messages game.

  Given that only $\Delta_1(2)$ is different from zero, we must only consider coalitions of cardinality two. Table 1 below shows how the Harsanyi dividends are allocated among the six players.

  As mentioned above, when cardinality of $\mathcal{M}_{\Gamma^K}(S)$ is greater than 1, the associated calculus becomes complex. To illustrate this situation let us consider case $S = \{1, 6\}$ in detail. For this coalition, $S_1 = \{1, 2, 3, 6\}$ and $S_2 = \{1, 2, 4, 5, 6\}$ are the two minimal connected sets that contain S. Then, using (10):

$$\varphi_i(w_{S,\Gamma^K}) = \varphi_i(u_{S_1}) + \varphi_i(u_{S_2}) - \varphi_i(u_{S_1 \cup S_2}).$$

Table 1

Coalition S	Elements of $\mathcal{M}_{\Gamma^K}(S)$	$\Delta(S)\varphi_i(w_{S,\Gamma^K})$					
		$i = 1$	2	3	4	5	6
{1,2}	{1,2}	1	1	0	0	0	0
{1,3}	{1,2,3}	0.666	0.666	0.666	0	0	0
{1,4}	{1,2,4}	0.666	0.666	0	0.666	0	0
{1,5}	{1,2,4,5}, {1,2,3,5,6}	0.566	0.566	0.066	0.166	0.566	0.066
{1,6}	{1,2,3,6}, {1,2,4,5,6}	0.566	0.566	0.166	0.066	0.066	0.566
{2,3}	{2,3}	0	1	1	0	0	0
{2,4}	{2,4}	0	1	0	1	0	0
{2,5}	{2,4,5}, {2,3,5,6}	0	0.766	0.1	0.266	0.766	0.1
{2,6}	{2,3,6}, {2,4,5,6}	0	0.766	0.266	0.1	0.1	0.766
{3,4}	{2,3,4}, {3,4,5,6}	0	0.266	0.766	0.766	0.1	0.1
{3,5}	{3,5,6}, {2,3,4,5}	0	0.1	0.766	0.1	0.766	0.266
{3,6}	{3,6}	0	0	1	0	0	1
{4,5}	{4,5}	0	0	0	1	1	0
{4,6}	{4,5,6}, {2,3,4,6}	0	0.1	0.1	0.766	0.266	0.766
{5,6}	{5,6}	0	0	0	0	1	1

Taking (11) into account:

$$\begin{aligned}
\varphi_1(w_{S,\Gamma^K}) &= 0.5 + 0.4 - 0.333 = 0.566,\\
\varphi_2(w_{S,\Gamma^K}) &= 0.5 + 0.4 - 0.333 = 0.566,\\
\varphi_3(w_{S,\Gamma^K}) &= 0.5 + 0 - 0.333 = 0.166,\\
\varphi_4(w_{S,\Gamma^K}) &= 0 + 0.4 - 0.333 = 0.066,\\
\varphi_5(w_{S,\Gamma^K}) &= 0 + 0.4 - 0.333 = 0.066,\\
\varphi_6(w_{S,\Gamma^K}) &= 0.5 + 0.4 - 0.333 = 0.566.
\end{aligned}$$

It may be seen that, when there is no alternative, i.e., if $\mathcal{M}_{\Gamma^K}(S)$ has cardinality one, an intermediary (e.g., 2 in path $1-2-3$) shares evenly with the sender and the receiver of the message. When there are alternatives (e.g., 3 or 4–5 in the paths from 2 to 6), these ‘intermediate’ players will receive much less due to the different available paths (competition).

If we now add all $\Delta(S)\varphi_i(w_{S,\Gamma^K})$ from Table 1, we obtain the vector:

$$\kappa[v_1, \Gamma^K] = (3.466, 7.466, 4.900, 4.900, 4.633, 4.633), \tag{16}$$

as centrality of the several nodes. As expected, node 2 has by far the greatest centrality, while the *nearly isolated* node 1 has the lowest one.

- Finally, Table 2 shows the centralities induced by v_i, $i = 1, 2, 3$ in each connected graph with four nodes.

5. A decomposition of the measure of centrality

In Table 2 in previous section, we showed that, for each v_i, $i = 1, 2, 3$, each node in the network Γ^1 has equal centrality to the node with the same label in the network Γ^3. This might be seen as a not reasonable result, being the degree of connection in Γ^1 higher than in Γ^3. Thus it is interesting trying to obtain more information by decomposing the current measure into two pieces.

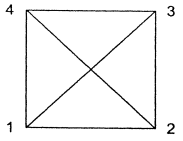
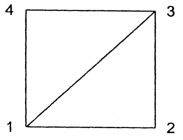
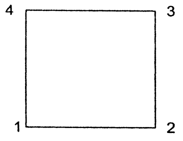
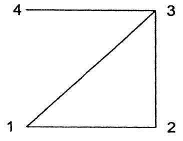
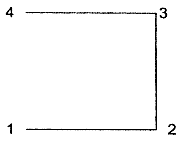
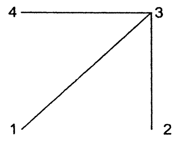
Let us consider the messages game. This game can be viewed as having two components: the ability to send and receive messages, and the ability to relay (or interrupt) messages between other individuals. So, from (16) and Table 1, we can see that, out of the 7.466 units of node 2, 4.533 come from the pairs {1, 2}, {2, 3}, {2, 4}, {2, 5} and {2, 6} (messages individual 2 sends or receives), whereas the remaining 2.933 units come from the pairs {1, 3}, {1, 4}, {1, 5}, {1, 6}, {3, 4}, {3, 5} and {4, 6} (where 2 serves as intermediary). Doing the same with all players, we realize that the centrality vector $\kappa[v_1, \Gamma]$ can be expressed as the sum of two vectors:

$$\kappa^C[v_1, \Gamma] = (3.466, 4.533, 4.200, 4.200, 4.100, 4.100),$$

and

$$\kappa^B[v_1, \Gamma] = (0, 2.933, 0.700, 0.700, 0.533, 0.533),$$

Table 2
Connected four-node graphs centralities

GRAPHS	NODES	v_1	v_2	v_3
Γ^1 	1	3	-1/4	11/4
	2	3	-1/4	11/4
	3	3	-1/4	11/4
	4	3	-1/4	11/4
Γ^2 	1	19/6	-1/6	17/6
	2	17/6	-1/3	8/3
	3	19/6	-1/6	17/6
	4	17/6	-1/3	8/3
Γ^3 	1	3	-1/4	11/4
	2	3	-1/4	11/4
	3	3	-1/4	11/4
	4	3	-1/4	11/4
Γ^4 	1	8/3	-1/3	5/2
	2	8/3	-1/3	5/2
	3	13/3	1/6	11/3
	4	7/3	-1/2	7/3
Γ^5 	1	13/6	-1/2	13/6
	2	23/6	0	20/6
	3	23/6	0	20/6
	4	13/6	-1/2	13/6
Γ^6 	1	7/3	-1/2	7/3
	2	7/3	-1/2	7/3
	3	5	1/2	4
	4	7/3	-1/2	7/3

where $\kappa^C[v_1, \Gamma]$ corresponds to sending or receiving messages (source/sink), while $\kappa^B[v_1, \Gamma]$ corresponds to relaying messages (intermediary).

More generally, for each node i in a network (N, Γ) and for $v \in \mathcal{G}_N$, we define the *communication centrality* of node i, $\kappa_i^C[v, \Gamma]$, as the portion of total centrality of node i corresponding to a payoff received as a member of different coalitions S, and the *betweenness centrality* of node i, $\kappa_i^B[v, \Gamma]$, as the payoff for i from coalitions in which i is not a member but may be needed for the coalition to be connected.

Then, we suggest the following decomposition:

$$\kappa[v, \Gamma] = \kappa^C[v, \Gamma] + \kappa^B[v, \Gamma]. \tag{17}$$

If $\delta_S \in \mathbb{R}^n$ is the characteristic vector of $S \subset N$ and for $x, y \in \mathbb{R}^n$, the coordinates of $x \circ y \in \mathbb{R}^n$ are $x_i \cdot y_i$, $i = 1, \ldots, n$, the terms of the expression (17) above are given by:

$$\kappa^C[v, \Gamma] = \sum_{S \subset N} \Delta(S)\varphi\left(\mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i})\right) \circ \delta_S$$

and:

$$\kappa^B[v, \Gamma] = \sum_{S \subset N} \Delta(S)\varphi\left(\mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i})\right) \circ \delta_{N-S}.$$

- Turning then to Table 2, for v_1 and Γ^1, the centrality in each node can be decomposed in:

  $$\kappa_i^C[v_1, \Gamma^1] = 3, \ \ \kappa_i^B[v_1, \Gamma^1] = 0, \ \ 1 \le i \le 4,$$

  whereas in Γ^3 we have:

  $$\kappa_i^C[v_1, \Gamma^3] = \frac{17}{6}, \ \ \kappa_i^B[v_1, \Gamma^3] = \frac{1}{6}, \ \ 1 \le i \le 4.$$

  The resulting decomposition for the centrality induced by v_i, $i = 1, 2, 3$ in every connected graph with four nodes appears in Table 3:
- The results corresponding to the introduced decomposition for the centrality induced by v_1 in the star with n nodes Γ^S (Fig. 1) are:

Table 3
Centrality decomposition for graphs is Table 2

Graphs	Nodes	v_1		v_2		v_3	
		$\kappa_i^C[v_1, \Gamma]$	$\kappa_i^B[v_1, \Gamma]$	$\kappa_i^C[v_2, \Gamma]$	$\kappa_i^B[v_2, \Gamma]$	$\kappa_i^C[v_3, \Gamma]$	$\kappa_i^B[v_3, \Gamma]$
Γ^1	1,2,3,4	3	0	−1/4	0	11/4	0
Γ^2	1,3	3	1/6	−1/4	1/12	11/4	1/12
	2,4	17/6	0	−1/3	0	8/3	0
Γ^3	1,2,3,4	17/6	1/6	−1/3	1/12	8/3	1/12
Γ^4	1,2	8/3	0	−1/3	0	5/2	0
	3	3	4/3	−1/4	5/12	11/4	11/12
	4	7/3	0	−1/2	0	7/3	0
Γ^5	1,4	13/6	0	−1/2	0	13/6	0
	2,3	8/3	7/6	−1/3	1/3	5/2	5/6
Γ^6	1,2,4	7/3	0	−1/2	0	7/3	0
	3	3	2	−1/4	3/4	4	5/4

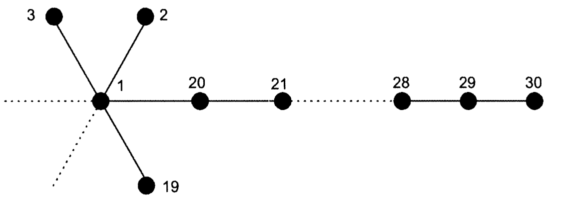


Fig. 5. Comet (Γ^{CT}).

$$\kappa_1^C[v_1, \Gamma^S] = n-1, \qquad \kappa_1^B[v_1, \Gamma^S] = \frac{(n-1)(n-2)}{3},$$
$$\kappa_i^C[v_1, \Gamma^S] = \frac{2n-1}{3}, \quad \kappa_i^B[v_1, \Gamma^S] = 0, \quad i \neq 1.$$

- In previous examples the two components of the centrality tend to go together in the sense that they order the nodes in a graph in the same way. This is not true in general. In the comet (Fig. 5) we have:

  $$\kappa_2^C[v_1, \Gamma^{CT}] = \kappa_2[v_1, \Gamma^{CT}] = 15.69, \quad \kappa_2^B[v_1, \Gamma^{CT}] = 0,$$

  whereas:

  $$\kappa_{29}^C[v_1, \Gamma^{CT}] = 8.04, \quad \kappa_{29}^B[v_1, \Gamma^{CT}] = 5.98,$$

  reflecting these figures that node 2 is in a better position to communicate but in a worse position to intermediate than node 29 is.

As expected, given the definition of $\kappa_i^B[v, \Gamma]$, for every v and every Γ, all nodes of degree 1 are not able to act as intermediaries and thus, their betweenness centrality is zero. Note that this property is not characteristic of one-degree nodes, as node 1 in graph Γ^4 shows (Tables 2 and 3).

6. The defined measures versus the classical ones

6.1. Comparing properties

In previous sections of this paper we have proved that, if v is a symmetric, strictly convex (and then strictly superadditive) game, the centrality measure induced by v satisfies the following properties:

(a) Fairness (Proposition 2.1)
(b) Stability (Proposition 2.2)
(c) Efficiency in connected graphs (Proposition 2.3)

(d) Symmetry (Proposition 3.1)
(e) Independence of the remaining connected components (Proposition 3.2)
(f) Isolated nodes have minimal centrality (Proposition 3.3)
(g) Of all graphs with n nodes, the maximal centrality is attained by the hub of a star (Proposition 3.4)
(h) Of all connected graphs with n nodes, the minimal centrality is attained by the end nodes in a chain (Proposition 3.5)
(i) In a chain, centrality increases from the end nodes to the median node (Proposition 3.6)

Let us go now to analyze the extent to which some classical centrality measures satisfy these properties:

Properties (a), (b) and (c) are direct consequences of choosing Shapley value as power index, and they are not (jointly) satisfied by classical measures. Moreover, the efficiency allows us to compare centralities of nodes in different connected graphs (with the same number of nodes) without any normalization.

Properties (e) and (f), that take into account the possibility of several connected components in a graph, are not satisfied by classical measures (except by the degree) because these are defined only for connected graphs.

On the other hand, properties (d), (g) and (i) will be satisfied for most of the classical centrality measures, that also verify a weaker version of property (h):

(h′) In a chain, end nodes have minimal centrality (which is a particular case of (h) and even a direct consequence of (i)).

But property (h) is stronger than (h′) is. Degree and betweenness centrality (Freeman, 1979) satisfy trivially property (h): degree of any terminal node is one, and its betweenness centrality must be zero, as the end nodes of a chain are never able to intermediate in the communications.

However, closeness (Freeman, 1979) or information centrality (Stephenson and Zelen, 1989) do not satisfy property (h), even if we normalize them in order to obtain an unitary total centrality in a graph.

If we consider the graphs Γ^a and Γ^b (Fig. 6) we have, for closeness centrality:

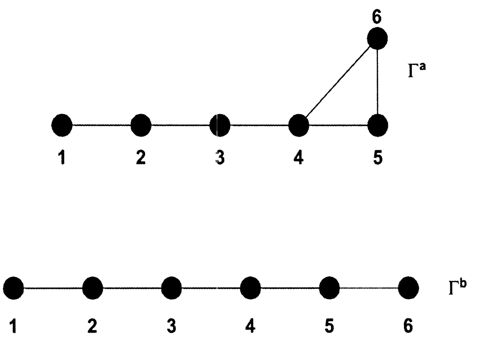


Fig. 6. Six-node graphs.

$$C^a = (0.118, 0.165, 0.207, 0.207, 0.151, 0.151),$$
$$C^b = (0.124, 0.169, 0.207, 0.207, 0.169, 0.124),$$

and for the information centrality measure:[3]

$$I^a = (0.112, 0.160, 0.204, 0.204, 0.160, 0.160),$$
$$I^b = (0.124, 0.169, 0.207, 0.207, 0.169, 0.124),$$

being, for both measures, centrality of end nodes in the chain Γ^b greater than centrality of node 1 in graph Γ^a.

As it was proved in Section 3 (Proposition 3.5), if we take a strictly convex game v (as v_1 and v_3 are, but not v_2), we must have:

$$\kappa_1[v, \Gamma^a] > \kappa_1[v, \Gamma^b].$$

For instance, if we consider the game v_1, normalized centralities for graphs Γ^a and Γ^b are:

$$\kappa^*[v_1, \Gamma^a] = (0.099, 0.188, 0.232, 0.243, 0.119, 0.119),$$
$$\kappa^*[v_1, \Gamma^b] = (0.097, 0.182, 0.221, 0.221, 0.182, 0.097).$$

Notice, also, that nodes 3 and 4 in the graph Γ^a have equal closeness or information centralities, but:

$$\kappa_4^*[v_1, \Gamma^a] > \kappa_3^*[v_1, \Gamma^a],$$

that can be viewed as a more plausible order, at least in the context of communication networks.

6.2. About the considered paths

In our approach, to calculate the centrality of a node, not only paired relations are considered, but also relations intracoalitions of any number of nodes.

Even in the special case of the game $v_1(S) = s^2 - s$, in which only coalitions of cardinality two are considered ($\Delta_1(S) = 0, \ \ s \neq 2$), our approach does not use only geodesic paths, nor all paths, but minimal paths (the minimal connected sets for coalitions with cardinality 2).

In some networks, considering minimal paths seems to be more appropriate, as illustrate the following examples:

Let us first consider the idea that centrality should only take geodesic paths between two given nodes into account. In Fig. 7, there are two possible paths between nodes 1 and 6. One path (1–2–3–4–5–6) has length 5, the other (1–11–10–9–8–7–6) has length 6. Yet it is not natural to discard the possibility of using a longer path, simply

[3]Notice that (normalized) closeness and information centralities coincide in chain Γ^b. This is not casual. If the graph is a tree, i.e., if there is an unique path joining any pair of nodes, closeness and information centralities, if normalized, must coincide.

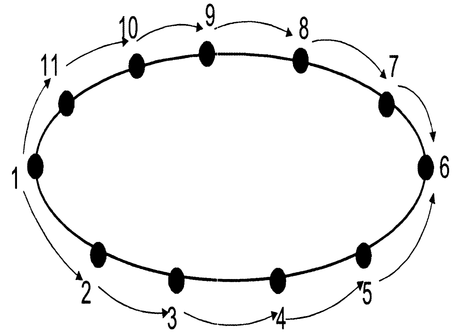


Fig. 7. Wheel.

because a (slightly) shorter one exists. However, we would tend to discard the long (10-edge) path from node 1 to 2, in favor of the one-edge direct path.

Let us assume now that all paths between two given nodes are taken into account. In Fig. 8a consider all paths between nodes 3 and 5; the geodesic path (3–2–5) has length 2, while the others (3–2–1–5, 3–4–1–5 and 3–4–1–2–5) are longer paths. In our approach, we will neglect the path 3–2–1–5 in favor of the shorter 3–2–5 path (and 3–4–1–2–5 in favor of 3–4–1–5), because it imposes an additional intermediary on the communication, which could be viewed as problematic in certain contexts.

Obviously, if edges are weighted, as in Fig. 8b, we cannot discard the path 3–2–1–4, between nodes 3 and 4, in favor of the shorter, but also more expensive, path 3–4.

6.3. Special cases related with the classical measures

In spite of the differences mentioned above, for a particular type of connected graphs, the trees (and even the forests), some of the centrality measures we have introduced are very related to certain standard measures that are used in the study of social networks.

- The centrality induced by the game:

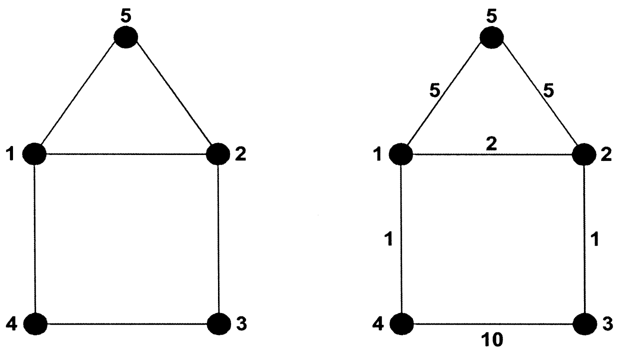


Fig. 8. Little house without (a) and with weighted edges (b).

$$v_2(S) = f_2(s) = -1, \quad s \geq 1$$

on a tree Γ is:

$$\kappa_i[v_2, \Gamma] = \frac{1}{2}\,\theta_i(\Gamma) - 1,$$

where $\theta_i(\Gamma)$ is the degree of node i in the graph Γ.

- The two components of the centrality measure induced by the game $v_1(S) = s^2 - s$ in a tree Γ are:

$$\kappa_i^C[v_1, \Gamma] = \sum_{j \neq i} \frac{2}{d(i, j) + 1}, \tag{18}$$

where $d(i, j)$ is the distance between nodes i and j, measured as the number of edges in the (unique) geodesic path that join the two nodes i and j in the graph Γ, and:

$$\kappa_i^B[v_1, \Gamma] = \sum_{\substack{j<k \\ j,k \neq i}}^{n} \delta_{jk}(i)\,\frac{2}{d(j, k) + 1}, \tag{19}$$

where

$$\delta_{jk}(i) = \begin{cases} 1, & \text{if } i \text{ is in the geodesic path that join } j \text{ and } k, \\ 0, & \text{otherwise,} \end{cases}$$

being κ_i^C a closeness measure of centrality and κ_i^B a betweenness or intermediation measure.

7. Final remark

The multiplicity and diversity of formal definitions proposed for centrality measures indicate that there is not a unique type of centrality and that different problems must give rise to different measures.

For instance, in the graph Γ^{BT} (Fig. 9), if reaching all agents in a network as quickly as possible is the issue, we will tend to choose a more local centrality measure, as degree is, that would assign to node 3 greater centrality than to node 4, whereas if we are

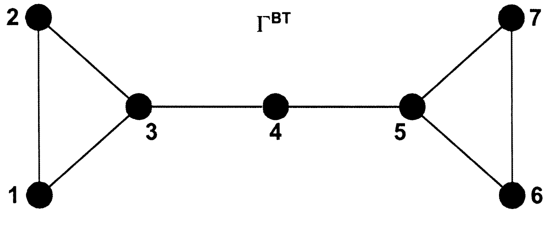


Fig. 9. Bow tie.

interested in a more global concept of centrality, we should choose a measure that assigns to node 4 the greatest centrality.

As was already announced, we have defined a *family* of centrality measures. Given a social network, each specific choice of game v determines a centrality measure, and the final centrality figures depend on this specific choice. This is not only in a cardinal sense: order may swap.

Then, if our knowledge of a priori economic (or social, or political) individual interests in the network is high enough, we will be able to choose a game that represents properly these interests.

If, on the other hand, we desire to highlight certain factors in the centrality, the game v must be selected accordingly.

In the bow-tie graph (Fig. 9), if we desire a local measure, we can consider the overhead game ($v_2(S) = -1$, $s \neq 0$), and then we would obtain centralities:

$$\kappa[v_2, \Gamma^{BT}] = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}, -\frac{1}{3}, -\frac{1}{3}\right),$$

whereas if we consider the graph as a communication network, and we are interested in a more global measure, we can consider the messages game ($v_1(S) = s^2 - s$), and we would obtain centralities:

$$\kappa[v_1, \Gamma^{BT}] = (3.966, 3.966, 8.600, 8.933, 8.600, 3.966, 3.966),$$

nodes 3 and 5 being the most central with the former, but node 4 with the latter.

To sum up, the right selection of the game is always crucial, as it spreads its influence to the centralities.

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