Mathematical Social Sciences 115 (2022) 64–77

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

Contagion in networks: Stability and efficiency

Spiros Bougheas

School of Economics, University of Nottingham, University Park, Nottingham NG7 2RD, UK

ARTICLE INFO

Article history:
Received 9 November 2020
Received in revised form 19 August 2021
Accepted 24 October 2021
Available online 3 November 2021

Keywords:
Network formation
Stability
Efficiency
Contagion

ABSTRACT

We study the formation of networks in environments where agents derive benefits from their neighbours (immediate links) but suffer losses through contagion when any agent on a path that connects them is hit by a shock. We first consider networks with undirected links (e.g. epidemics, underground resistance organizations, trade networks). We find that the only networks that satisfy strong notions of stability are comprised of disjoint subgraphs that are complete. Then, we consider networks with directed links and we find that stable networks can be asymmetric, connected but not completely connected, thus capturing the main features of production and financial networks. We also identify a trade-off between efficiency and stability.

1. Introduction

A significant part of the literature on strategic network formation has focused on variants of the 'connections' model studied by Jackson and Wolinsky (1996). The common idea of this literature is that being part of a network allows agents to benefit not only from their direct links but also from indirect connections to other agents in the network. In contrast, the costs of network participation in these models are only associated with the creation of new links. However, as Blume et al. (2011) observe in many networks studied in economics and other disciplines the structure of costs and benefits is inverted. As an example from economics they refer to the extensive body of work on issues related to systemic risk in financial networks. In those networks two institutions form a link by signing a loan agreement from which each party derives a benefit. A failure by one institution to meet its obligations inflicts costs not only on the two parties that have signed the agreement but also on other institutions connected to them, directly or indirectly, by other financial agreements.

The above observation motivated Blume et al. (2011) to study the formation of networks with this alternative general payoff structure. They have restricted their analysis to undirected graphs where shocks can travel in either direction along a link. Using stability as a solution concept that allows them to make predictions about which network structures are more likely to form they find stable networks that consist of fully connected disjoint subgraphs (cliques). Such arrangements might be offering a good description of some examples of social networks they mentioned in their paper (e.g. formation of groups that minimize the risk of disease epidemics and the organization of clandestine operations) but definitely less so for financial systems and production networks that have network structures which are connected but incomplete.[1] In such networks there is always a path connecting any of the nodes (financial institutions or firms) with every other node, however, not all nodes are directly linked with every other node.

In this paper, we argue that by distinguishing between directed and undirected graphs we can explain such variations in network structures. We begin by analysing the formation of undirected networks in a variant of the Blume et al. (2011) model. As in their model, (a) agents derive a benefit by forming a link, and (b) each agent fails independently with some fixed probability. The difference in the two models is related to the way the costs associated with such failures spread through the network. In their model after a failure each link with some probability becomes alive and shocks can only be transmitted through alive links (the network structure is determined after the shock). In our model all nodes are alive and thus potentially affected by the shock, however, the magnitude of the losses for each node depends on its distance from the one that initially failed. As Blume et al. (2011) we find that when links are undirected there exist stable networks that consist of fully connected disjoint subgraphs. We also go one step further by showing that these are the only types of networks that are Pareto-efficient. The implication of this result is that they also satisfy the notion of 'strong' stability (Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005).

Then, we turn our attention to directed networks where shocks can be only transmitted along directed paths. The Blume et al. (2011) model by having only a subset of nodes being live it mimics the transmission of shocks in directed networks. However, there is a crucial difference between their model and ours. In

[1] For a variety of examples of financial networks see the review article by Bougheas and Kirman (2015).

E-mail address: spiros.bougheas@nottingham.ac.uk.

https://doi.org/10.1016/j.mathsocsci.2021.10.006
0165-4896/

their model the paths are exogenously determined but in our model are equilibrium outcomes. The direction of the links are determined by the strategic decisions of agents. We argue that our version is better suited to capture the types of contagion observed in real financial and production networks. In financial networks links stand for loan agreements between financial institutions which are represented by the nodes of the network. There are two types of contagion that have been studied on networks formed by financial links. The first type of contagion is caused by insolvencies and flows from borrowers to lenders. When a financial institution becomes insolvent it is unable to meet its financial obligations to its creditors. The latter lose not only the benefit of their financial relationship with their creditor but they might also not be able to meet their obligations to their own creditors. The second type of contagion, which flows from lenders to borrowers, is caused when a financial institution, out of fear that its borrowers might not be able to repay their loans in the future, interrupts their funding. The crisis unfolds when the affected institutions might not be able to provide funding to their own debtors and eventually as these funding interruptions cascade through the network the market eventually freezes. In production networks the links stand for supply-chain relationships between firms and the direction of the links captures the flow of goods.[2] Contagion in such networks can be caused either because of financial factors (lack of trade credit) or because of shocks on production lines.

We establish very mild conditions such that complete directed networks (tournaments) are stable.[3] We also identify a trade-off between stability and *ex ante* efficiency. The intuition behind this result is that stability depends on the weakest link while efficiency depends on the average expected payoff. We then derive our main result by demonstrating the existence of stable networks that are not fully connected and where nodes do not belong to a closed path. In such networks some nodes are more connected than others resembling the hierarchical structures of financial and production networks. In the following section, we describe a few examples of undirected and directed networks from economics and other disciplines whose general structure is captured by our model.[4]

1.1. Contagion in social and economic networks

The first three are examples of undirected networks and the following two are examples of directed networks.

Contagious diseases and group size. As Blume et al. (2011) observe there is a trade-off related to the formation of social groups. Larger clusters increase the benefits of participants, however, they also increase the risk of contagion. This trade-off is clearly illustrated in the study by Hamilton et al. (2007) of hunter–gatherer societies where they make the distinction between cohesive and disruptive forces in the process of group formation and among the latter they identify the spread of diseases.

Underground resistance networks. This is another of the examples offered by Blume et al. (2011). Participants in such organizations benefit by working in groups but also there is a risk that the group might be infiltrated. Chai (1993) explores this trade-off in the context of groups that resist national governments while Morselli et al. (2007) do the same for criminal networks.

Globalization and the international transmission of shocks. The free movement of goods and services, intermediate inputs in production (labor and physical capital) and financial capital can be welfare enhancing during times of prosperity but they also facilitate the transmission of regional shocks around the globe. This trade-off has been studied by Imbs (2004), Kose et al. (2003) and since then it has been an active topic of research. The economic impact of the ongoing COVID 19 pandemic is a good example of the transmission of shocks through international trade linkages (see, Baldwin and Tomiura, 2020).

What all three examples mentioned above have in common is that links are symmetric and thus undirected.[5] In all three examples the size of the group is a source of tension between opposing forces. Larger groups confer benefits to participants as there are more opportunities for collaboration. However, larger groups also expose a greater number of participants to shocks in their network (a new virus, an agent caught by the authorities, a macroeconomic downturn). The size of the network will depend on balancing the costs and benefits of participation. The risk of losing a member in clandestine operations might be unacceptable and thus would keep the size of the network small. In contrast, the international trade network has been expanding both by enlisting more trading partners and reducing barriers to trade.

Financial networks and systemic risk. As we mentioned above, there are two alternative mechanisms through which shocks are transmitted in financial networks. Shocks are transmitted from borrowers to lenders when the inability of a debtor institution to meet its obligations with its lenders causes a cascade of failures through the system. The lenders of the initially failing institution might be unable to meet their own obligations and this process can keep going till the system is cleared (e.g., Eisenberg and Noe, 2001).[6] Shocks can also be transmitted from lenders to borrowers when the lenders suddenly interrupt established credit lines that they have earlier provided to the other financial institutions. Episodes of market freezes usually take place before the onset of a crisis as lenders anticipate that, in the near future, borrowers will have a hard time repaying their debts (e.g. Acharya et al., 2011; Diamond and Rajan, 2011).

Firm linkages and macroeconomic fat tails. Directed networks are also useful for understanding the causes of fat tails in the distribution of macroeconomic shocks. Recent work by Acemoglu et al. (2012, 2017b) has shown that the interaction between the distribution of idiosyncratic shocks and the structure of the network can explain 'abnormal' shocks at the aggregate level.[7] They analyse networks where nodes represent firms that buy from and sell goods to each other thus creating a web of complex supply relationships. In particular, Acemoglu et al. (2017b) show that light-tailed risks (small deviations from the normal case) in conjunction with some lack of balance in terms of economic importance across the sectors of the economy can give rise to macroeconomic fat tails.

1.2. Related literature

Our paper is related to a quickly expanding literature on the endogenous formation of economic and social networks. Early

[2] In production networks there are additional restrictions on the links that firms can form.

[3] In the case of directed networks 'complete' implies that all pairs are linked but there is no restriction on the direction of the links. While there is only one complete undirected network for any size n the number of complete undirected networks, in the absence of bi-directional links, is equal to 2^n.

[4] Many more examples of social and economic networks can be found in Jackson (2008) and Newman (2010).

[5] There are exceptions but to address them we need a more specialized model. For example, as Chai (1993) suggests while it is true that exposure to the risk of infiltration keeps the size of resistance groups small, these groups can be further protected by having a hierarchical structure where every person is in contact with no more that three other members (one above and two below).

[6] For excellent literature reviews see Acemoglu et al. (2017a), Babus and Allen (2009) and Glasserman and Young (2016).

[7] See also Carvalho (2014) for a less technical exposition of this topic.

work focused on variants of the Jackson and Wolinsky (1996) connectionist model.[8] We focus our literature review on works that consider network formation in environments with systemic risk.

As we mentioned above, the most closely related paper to ours is Blume et al. (2011). They restrict their attention to undirected graphs. However, the transmission of a shock is restricted to spread only through live links. In contrast, in the undirected graph version of our model all nodes are live but we allow for losses to be discounted as the distance of nodes from the one hit by the shock goes up. The two models become identical by setting equal to one (a) the probability of a node being live in Blume et al. (2011), and (b) the discount factor in our model. In addition, to finding, as they do, that there exist stable networks that consist of disjoint fully connected subgraphs we also show that any network satisfying well known strong notions of stability must have that structure.

Erol and Vohra (2014) consider the formation of undirected links and derive a similar result, however, they do so from a network formation game that has a quite different structure. In their model any pair of agents linked together play a coordination game, each agent deciding whether to default or not and their expected payoffs also depend on their beliefs about the default strategies of all other agents. At an earlier stage agents form undirected links anticipating the later stage possibilities.

Billand et al. (2016) explore the implications of node failure for network connectivity when links are directed. In their model when a node fails it disrupts the flow of resources in the network, however, there is no contagion to other nodes, which is the main subject of the present paper. The formation of networks with directed links was first considered by Bala and Goyal (2000). Galeotti (2006) allowed for heterogeneity among agents and Billand et al. (2012) extended the analysis by introducing benefit spillovers. In our model there are negative spillovers due to contagion caused by exogenous shocks on nodes.

Our work is also related to a number of papers in the finance literature that explore the links between network structure and systemic risk. In particular, there has been a lot of work that aims to identify the types of network structures that are more vulnerable to the two types of contagion identified in the introduction of the paper. One group of papers examines the cascade of insolvencies through the financial network caused by the inability of some debtors to meet their obligations to their creditors (e.g., Acemoglu et al., 2015; Elliott et al., 2014; Gai and Kapadia, 2010). A second group focuses on freezes in financial networks caused by creditors who, fearing that their debtors will be unable in the future to meet their financial obligations, interrupt their credit lines (e.g., Acemoglu et al., 2020; Anand et al., 2012; Gabrieli and Georg, 2014).[9]

Recently, there has also been some work on the formation of such networks. Cohen-Cole et al. (2010) study competition in the financial market where participants form undirected links. In Babus (2016) financial institutions form links to insure themselves against the probability of system wide default. In contrast, in the financial interpretation of our model financial institutions are participating in an interbank market. Acemoglu et al. (2014) and Cabrales et al. (2017) also study network formation but they exogenously restrict the links that are allowed to be formed. In our work we do not impose any restrictions on the structures of networks that can be formed.

Our work is also relevant for the growing literature in economics that studies cascades in production networks. Studying the formation of such networks requires the inclusion of additional restrictions related to the links that firms can form. In Battiston et al. (2007) cascades follow paths determined by credit chain relationships. In contrast, in Acemoglu et al. (2012, 2017b) and Acemoglu and Tahbaz-Salehi (2020) the paths are established by supply-chain relationships.

Our result on hierarchical networks is related to many studies of social, financial and trading networks that have a core–periphery structure. The interbank market has a tiered structure where money centre banks act as intermediaries (Craig and Von Peter, 2014; In 't Veld et al., 2020). A similar structure exists in the OTC dealer markets (Babus and Hu, 2017). Many types of social networks also have a core–periphery structure (Hojman and Szeidl, 2008). Such networks provide individuals access to various sources of information, including emotional and financial. Lastly, a core–periphery structure is characteristic of many trading networks. Goyal and Vega-Redondo (2007) find that the introduction of intermediaries in such networks, and, thus, the structure of the networks, depends on the size of capacity constraints. In Bedayo et al. (2016) a core–periphery structure arises when agents differ in their discount rates which affect their payoffs from bargaining.

In the following section we study undirected networks and then we turn our attention to directed ones. In the final section, we consider the implications of relaxing some of the assumptions of our model.

2. Undirected networks

There are N agents represented as nodes on a graph (network). Let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of nodes (agents). A link between two nodes indicates that the corresponding agents have a symmetric relationship. Let $\mathcal{L}$ denote the set of links and denote the network by the pair $(\mathcal{N}, \mathcal{L})$. For two agents i and j who are linked we write $ij \in \mathcal{L}$. Let $(\mathcal{N}, \mathcal{L})^C$ denote the complete network where all agents are linked with each other and let $\mathcal{G} = \{(\mathcal{N}, \mathcal{L}) \mid (\mathcal{N}, \mathcal{L}) \subseteq (\mathcal{N}, \mathcal{L})^C\}$ denote the set of all possible graphs.

The following notations will be useful. A *path* between agents i and j is defined as a sequence of agents beginning with i and ending with j such that for every pair of adjacent agents there exists a link and each agent appears only once in the sequence. We let $t(ij)$ denote the number of links in the shortest path between agents i and j. A *cycle* is formed by adding the link ij to a path between agents i and j. An *empty cycle* is a cycle where there are no links between any two agents that belong to the cycle and are not adjacent.

A *subgraph* $(\mathcal{N}', \mathcal{L}')$ of the network $(\mathcal{N}, \mathcal{L})$ is a new network where its set of nodes $\mathcal{N}'$ is a subset of $\mathcal{N}$ and its set of links $\mathcal{L}'$ is such that if and only if $i \in \mathcal{N}'$ and $j \in \mathcal{N}'$ and $ij \in \mathcal{N}$ then $ij \in \mathcal{N}'$. We say that a subgraph is *connected* if all agents that belong to the subgraph are connected by a path that itself belongs entirely to the subgraph. We say that a subgraph is *completely connected* if all agents that belong to the subgraph are connected to each other. We say that a subgraph is a *component* if for every pair of agents i and j such that $i \in \mathcal{N}'$ and $j \notin \mathcal{N}'$, $ij \notin (\mathcal{N}, \mathcal{L})$. Thus, the agents that belong to a component are not connected with any of the other agents of the original network. We let $(\mathcal{N}', \mathcal{L}')^C$ denote that the subgraph $(\mathcal{N}', \mathcal{L}')$ is complete. A complete component is referred to as a *clique*. For any network $(\mathcal{N}, \mathcal{L})$ we let $\mathcal{S}$ denote the set of all connected components with at least two nodes and $\mathcal{D}$ the set of all isolated nodes. Then $(\mathcal{N}, \mathcal{L}) = (\cup_{(\mathcal{N}', \mathcal{L}') \in \mathcal{S}} (\mathcal{N}', \mathcal{L}')) \cup \mathcal{D}$.

For any player i we define $\mathcal{T}_k^i = \{j : t(ij) = k\}$, as the set of agents with a shortest distance from agent i equal to k. Let $\left|\mathcal{T}_k^i\right|$ denote the cardinality of the set. Notice that $\left|\mathcal{T}_1^i\right|$ is equal to the *degree* of node i.

[8] See, for example, Bala and Goyal (2000), Dutta and Mutuswami (1997), Jackson and Watts (2002) and Watts (2002).

[9] See Glasserman and Young (2016) for a comprehensive review of the literature.

Next, we define the benefits and costs from network participation. With probability θ one of the nodes of the network is hit by a shock. Given that all nodes are hit by a shock with the same probability, the unconditional probability that a node is hit by a shock is equal to $\frac{\theta}{N}$.[10] Agents derive benefit b from each direct link as long as one of the following two conditions holds: Either the network is not hit by a shock or if it is hit by a shock then there is no path from them to the agent hit by the shock. There is no benefit from indirect links. The cost to an agent of being hit by a shock is equal to $c < 1$. Other agents of the network will suffer losses only if they are (directly or indirectly) connected to the agent who is hit by the shock. Suppose that agent j is hit by a shock. For any agent i connected to agent j this indirect cost is given by $\delta^{t(ij)}c$, where $0 < \delta \leqslant 1$; thus, the cost is declining with the number of links in the shortest path between the two agents.[11]

Let $\left|\mathcal{N}'\right|$ denote the cardinality of $\mathcal{N}'$. Suppose $i \in \mathcal{N}'$. Then, the expected net payoff $v_i(\mathcal{N}', \mathcal{L}')$ of agent i is given by[12]:

$$
\begin{aligned}
v_i(\mathcal{N}', \mathcal{L}') &= \left(1 - \frac{\left|\mathcal{N}'\right|\theta}{N}\right)\left|\mathcal{T}_1^i\right| b \\
&\quad - \frac{\theta}{N} c \left(1 + \delta\left|\mathcal{T}_1^i\right| + \cdots + \delta^{N-1}\left|\mathcal{T}_{N-1}^i\right|\right) \\
&= \left(1 - \frac{\left|\mathcal{N}'\right|\theta}{N}\right)\left|\mathcal{T}_1^i\right| b - \frac{\theta}{N} c \left(1 + \sum_{k=1}^{N-1}\delta^k\left|\mathcal{T}_k^i\right|\right)
\end{aligned}
$$

The first term is equal to the expected benefit derived from belonging to a connected component $(\mathcal{N}', \mathcal{L}')$. The subgraph has $\left|\mathcal{N}'\right|$ nodes and thus the probability that one of the corresponding agents is hit by a shock is equal to $\frac{\left|\mathcal{N}'\right|\theta}{N}$. As long as no agent who belongs to the subgraph is hit by the shock, agent i will derive a benefit b from each of the $\left|\mathcal{T}_1^i\right|$ direct links. The second term is equal to the corresponding expected costs. Each agent fails with probability $\frac{\theta}{N}$. When an agent is hit by the shock all agents who belong to the same connected component suffer a loss that is equal to c times a discount factor that depends on their shortest distance from the agent hit by the shock. Thus, as long as $\delta < 1$, costs due to contagion rise as the distance from the agent hit by the shock declines.

In the above set-up, agents affected by a shock face two types of costs. The first is the loss of benefits from direct links. For example, in a global trade network such a loss is related to the destruction of trade flows. The second loss captures indirect costs that can be related to the network distance from the agent who was inflicted by the initial shock. For the global trade network countries might suffer additional macroeconomic shocks as the reduction in world trade impacts on national incomes.

2.1. Stability

We will begin our analysis of network formation by applying the Jackson and Wolinsky (1996) notion of pairwise stability. While this is a very weak criterion that allows for a plethora of stable networks, as Jackson (2003) argues, it might be appropriate for networks with a large number of nodes where agents might find it difficult to either forming deviating coalitions or anticipating future changes in the network that might affect their payoffs. According to pairwise stability the formation of a new link requires the approval of both agents forming the link but any player can sever a link unilaterally. The limitations of this notion of stability are well understood. It is a local criterion that considers each link in isolation and therefore there is usually a large number of networks that satisfy its restrictions. For example, it allows for stable networks where every participant would prefer to sever all links simultaneously. However, given that our intention is to demonstrate that stable directed networks do not have the characteristics of some real-world networks our choice is not consequential. For example, we will be able to show that any network that is connected but not complete and is pairwise stable cannot survive stronger notions of stability. The derivation of this important result is greatly simplified by beginning our analysis with pairwise stable networks.

In what follows by stability we mean pairwise stability.

Definition 1. A network, $(\mathcal{N}, \mathcal{L})$, is *stable* if no agent i strictly prefers to sever a link and no pair of agents i and j strictly prefers to form link ij.

The following result addresses the two benchmark cases.[13]

Lemma 1. *(a) The empty network is stable if and only if* $\left(1 - \frac{2\theta}{N}\right) b \leqslant \frac{\theta}{N}\delta c$, *and*

(b) the complete network, $(\mathcal{N}, \mathcal{L})^C$, *is stable if and only if* $(1-\theta)\, b \geqslant \frac{\theta}{N}\delta\,(1-\delta)\, c$.

Corollary 1. *If*

$$
(1-\theta)\, b < \frac{\theta}{N}\delta\,(1-\delta)\, c < \frac{\theta}{N}\delta c < \left(1 - \frac{2\theta}{N}\right) b \tag{1}
$$

then neither the complete network nor the empty network are stable.

Notice that (1) implies that if there is no discounting the complete network is stable. The benefit of breaking a link in a complete network is very small given that the distance to all other agents remains the same. This will be still the case if the payoff of each agent from participating in the network is negative. The network is still pairwise stable as agents are not allowed to break all links simultaneously.

Our next result shows that stable networks always exist. In particular, as in Blume et al. (2011), we will show that we can always construct stable networks that consist of cliques (disjoint complete subgraphs).

Proposition 1. *Stable networks always exist.*

Example 1. Let $N = 9$, $\delta = 0.9$, $\theta = 1/3$, $b = 5$, and $c = 1$. Then the network comprised of two disjoint complete subgraphs of sizes 5 and 4, respectively, is stable (see Fig. 2.1).[14]

Proposition 1 has established that stable networks comprised of complete subgraphs always exist. The proof is a direct consequence of the observation that a subgraph that might not be

10 In Section 4, we discuss alternative specifications of the distribution of shocks.

11 In many applications discounting might not be appropriate. For example, as long as a financial institution fails during a cascade, its network distance from the institution that first defaulted might not have any effect on costs related to insolvencies. What we will see is that the crucial parameter is the size of c and not the size of δ. With that in mind we allow for no discounting.

12 Notice that through $\left|\mathcal{T}_k^i\right|$ the payoff depends on $\mathcal{L}'$.

13 All proofs are included in Appendix.

14 Notice that (A.2) in the proof given in the Appendix is satisfied for these parameter values. In addition, we need to ensure that no agent belonging to the size 4 subgraph would like to link with an agent belonging to the size 5 subgraph. The expected payoff from creating the new link is given by

$$
(1-\theta)4b - \frac{\theta}{9} c \left(1 + 4\delta + 4\delta^2\right)
$$

which is less than the expected payoff from not creating the link given by (A.1) for $N = 9$ and $\left|\mathcal{N}'\right| = 4$.

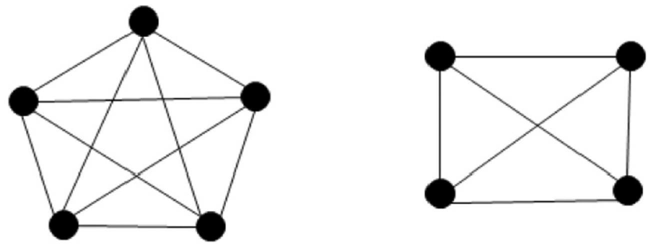

Fig. 2.1. A Stable Network ($n = 9$, $\delta = 0.9$, $\theta = 1/3$, $b = 5$, $c = 1$).

stable in the presence of isolated agents might be so in their absence. In the above example, the expected payoff of an additional link is low. While the only benefit comes from the new link there are substantial new costs as (a) the size of the network increases and therefore the probability of being affected by a shock has gone up, and (b) there are more indirect paths to new agents.

For the purpose of this study, we would also like to know if there are other stable structures. In particular, we would like to know if there are stable networks that are connected but not complete. The answer is positive as the example below demonstrates. However, we will argue that all such networks have very restricted form and later we will also prove that they do not satisfy many well known stronger notions of stability.

Example 2. Consider a symmetric network with 12 nodes such that $\left|\mathcal{T}_1^i\right| = 5$, $\left|\mathcal{T}_2^i\right| = 5$ and $\left|\mathcal{T}_3^i\right| = 1$ for all nodes $i = 1, \ldots, 12$.[15] The network corresponds to the icosahedron one of the five platonic solids. Given that the network is symmetric without any loss of generality consider node i. Suppose that node i creates a new link with an agent at distance 2. Now we have $\left|\mathcal{T}_1^i\right| = 6$, $\left|\mathcal{T}_2^i\right| = 5$ and $\left|\mathcal{T}_3^i\right| = 0$. Then, the expected payoff of adding the additional link is given by $(1-\theta)b - \frac{\theta}{N}c\delta\left(1-\delta^2\right)$. As the whole network is connected with probability $(1-\theta)$ there is an expected benefit $(1-\theta)b$ derived from the additional link. The costs are due to the following changes: The link that was 3 steps away now it is only 2 and the new link reduced the distance to the newly linked node from 2 to 1. Suppose now that node i breaks one of the existing links. In this case we have $\left|\mathcal{T}_1^i\right| = 4$, $\left|\mathcal{T}_2^i\right| = 6$ and $\left|\mathcal{T}_3^i\right| = 1$. Now the expected payoff from breaking the link is given by $-(1-\theta)b + \frac{\theta}{N}c\delta(1-\delta)$. The first term corresponds to loss of expected benefit because of the break of the link. The only other effect is that the distance from the previously linked node has increased from 1 to 2. Then, stability requires that $(1-\theta)b - \frac{\theta}{N}c\delta(1-\delta) > 0 > (1-\theta)b - \frac{\theta}{N}c\delta\left(1-\delta^2\right)$ and given that $\delta < 1$ there exist parameter values so that the two inequalities are satisfied.[16]

In the above example, all nodes and, more importantly, all links are located on the surface of a three dimensional object. Thus, if agents preferred to form any additional links, those links would have to be located in its interior. It is also clear, all agents belong to cycles and the missing links imply that these cycles are not complete. The following result states that any connected network that is stable but is not complete must have similar structure.

Proposition 2. *In any stable network that is connected all nodes are on cycles.*

Proposition 3. *Empty cycles are not stable.*

The above two results suggest that open cycles must be relative dense.

In this section, we have shown that as long as we restrict our analysis to pairwise stability we can construct networks that are connected but not complete. In the following section we will consider the efficiency of undirected networks and by showing that stable connected networks that are incomplete are also Pareto-dominated by other stable networks we will be able to conclude that such networks cannot satisfy some well known stronger notions of stability.

2.2. Ex Ante Efficiency vs Ex Post Systemic Losses

Up to this point we have been focusing on the types of network structures that are more likely to form. Now we turn our attention to relative performance. For the types of networks that we are interested in there are two alternative ways of measuring performance. The first one is having a measure of *ex ante* performance as in Jackson and Wolinsky (1996) who used the sum of expected utilities of network participants. The second way provides a measure of *ex post* performance by focusing on the magnitude of potential systemic losses after a shock on the network. Clearly, these losses are minimized by not having any links at all, that is all agents are isolated, $|\mathcal{D}| = N$. Such a measure ignores the benefits of creating networks in the first place. Nevertheless, such a measure can be beneficial to policy makers who are interested in minimizing the damage from contagion. With that in mind, we are going to identify among all stable networks the one that minimizes *ex post* losses.

Definition 2. A network $(\mathcal{N}, \mathcal{L})^*$ is *ex ante* efficient if it maximizes the sum of expected payoffs of all agents:[17]

$$\begin{aligned}(\mathcal{N}, \mathcal{L})^* &= \arg\max_{(\mathcal{N},\mathcal{L})} \textstyle\sum_{i=1}^{N} v_i(\mathcal{N}, \mathcal{L}) \\ &= \arg\max_{(\mathcal{N},\mathcal{L})} \textstyle\sum_{(\mathcal{N}',\mathcal{L}')\in\mathcal{S}} \sum_{i\in\mathcal{N}'} v_i(\mathcal{N}', \mathcal{L}') - |\mathcal{D}| \frac{\theta}{N} c\end{aligned}$$

Proposition 4. *Efficient networks are such that every $(\mathcal{N}', \mathcal{L}') \in \mathcal{S}$ is complete.*

Next, we show how to find the network that maximizes ex ante efficiency. The expected payoff of an agent i belonging to one of the complete disjoint subgraphs with $\left|\mathcal{N}'\right| = M$ is given by $\left(1 - \frac{M\theta}{N}\right)(M-1)b - \frac{\theta}{N}c\left(1 + (M-1)\delta\right)$. By maximizing this payoff with respect to M we can find the size of a complete disjoint subgraph $\hat{M}$ that offers the maximum expected payoff to its members.[18] Setting the f.o.c. equal to 0 and solving for M we get:

$$\hat{M} = \left(1 + \frac{N}{\theta} - \frac{c\delta}{b}\right)/2$$

[15] It can be shown that the four networks that correspond to the other four platonic solids (tetrahedron, cube, octahedron and icosahedron) are unstable for all parameter values.

[16] In the above example, the discount factor is crucial. If $\delta = 1$ then incomplete subgraphs can never be stable. This is because the creation of any missing link will increase the benefits of each of the two newly linked agents by $\left(1 - \frac{M\theta}{N}\right)b$ while all costs remain the same given that the original graph is connected and the costs are not affected by distance; where M is equal to the number of agents in the subgraph.

[17] Notice that the expected payoff of each $i \in D$ is equal to $-\frac{\theta}{N}c$. This expression cancels out in all our derivations with the only exception in the last section were we introduce aggregate externalities. For this reason we opted to keep it rather making an *ad hoc* introduction later in the paper.

[18] Given that $\hat{M}$ is probably not an integer we need to compare the payoffs of subgraphs of sizes equal to the first integer higher than $\hat{M}$ with the payoffs of subgraphs of sizes equal to the first integer lower than $\hat{M}$. We will ignore this complication.

If $N \bmod \hat{M} = 0$ then the network consisting of complete disjoint subgraphs of size $\hat{M}$ is the one that maximizes ex ante efficiency.[19]

The next result follows directly from the one above, however, we state it as a separate proposition because it is the one that has motivated the study of undirected networks.

Proposition 5. *The only connected network that can satisfy strong stability (Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005) is the complete network.*

Notice that the above result does not assert that the complete network always satisfies the above stronger notions of stability.[20]

What it asserts is that any connected network that is not complete even if it satisfies pairwise stability, as the one described in Example 2, it will not satisfy any of the two stronger notions of stability.[21] The crucial argument in the above result is that networks that are connected but not complete are not only *ex ante* inefficient but also Pareto-dominated which is a stronger notion of efficiency. As a consequence, while there is usually only one network that satisfies *ex ante* efficiency there might be several networks comprised of complete disjoint graphs that satisfy the stronger notions of stability. However, all such networks must be comprised of complete disjoint subgraphs.

One significant drawback of pairwise stability is that it allows for the formation of networks where the expected payoffs of all agents are negative. This is because it does not allow for the simultaneous breaking of all links and no agent can benefit by breaking a single link. All these inefficient networks can be eliminated by introducing the Goyal and Joshi (2003) notion of 'global' stability where agents can simultaneously break any number of links. In all such cases the only network that would satisfy this stronger notion of stability would be the empty network.

Lastly, we consider the relationship between stability and systemic losses. Clearly, the magnitude of systemic losses following a shock declines as the size of subgraphs gets smaller. Therefore, we are looking for the smallest complete disjoint subgraph that is stable when all other subgraphs have the same size.[22]

Example 3. Let $N = 6$, $\delta = 1$, $b = 1$, $c = 7$, and $\theta = \frac{1}{2}$. Then $\hat{M} = 3$. The above inequality is satisfied for $M = 2$ but not for $M = 1$ and thus the smallest complete disjoint subgraph that is stable has size 2. Notice that this example also trivially satisfies the stability condition for the complete network.

The above example identifies a tension between stability, efficiency and the size of systemic losses. The observation that stability can be satisfied by networks that have more connections than those that maximize efficiency is not necessarily due to the notion of stability that we use. When agents make decisions about forming or breaking a link they ignore the negative impact that these decisions have on the payoffs of other agents. This negative externality implies that agents would tend to form too many links relative to the number of links of the efficient network.[23]

In this section, we have found that stable undirected networks respond to a trade-off between large size connected networks that bring more benefits to participants and small size ones that protect them from shocks. Given that indirect connections do not confer any benefits but are still potentially harmful they are very rare in stable networks. The small size of hunter–gatherer societies might have indeed protected them against epidemics as the small size of resistance groups protects them against infiltration. In contrast, when the expected costs of contagion are low relative to the benefits of new connections the formation of large complete networks becomes possible. For example, despite the losses in welfare resulting from the transmission of macroeconomic shocks, globalization, i.e. the creation of new bilateral and multilateral links by opening international borders to allow for the movement of goods and services, inputs in production and financial capital, has been the dominant economic force over the last 30 years. However, many of the corresponding benefits and costs are being reassessed since the 2009 global financial crisis.

Many types of networks that macroeconomists and financial economists study are connected but incomplete and also hierarchical where some agents are more connected than others. In addition, although these networks are connected, many agents do not belong to a cycle. Acemoglu et al. (2012, 2017a) and Acemoglu and Azar (2020) study production networks where links represent input/output relations either between firms or industries. These networks are connected but not complete (not all firms transact with each other) and many firms are located at the end of some path in the network (dominated by vertical integration structures). Financial and banking networks have a very similar structure. There are many examples in the literature of hierarchical financial networks (Boss et al., 2004 - Austria; Inaoka et al., 2004 - Japan; Adams et al., 2010 - United Kingdom). More recently, Anderson et al. (2019) in their study of the impact of the National Banking Act in US on the interbank network do not only emphasize the connectedness and the hierarchical structure of bank networks but also show how banking regulations can encourage the formation of such networks. With the above in mind we turn our attention to directed networks.

3. Directed networks

In our model, after an initial shock contagion cascades through the network by following the paths formed by the network's links. In the previous section, we treated symmetrically two agents forming a link. That is, we have allowed cascades to flow in either of the two directions defined by a link. When any one of two linked agents is hit by a shock then the other agent also suffers losses. However, in many applications contagion flows only in one direction which depends on the nature of the relationship between the two agents. For example, for any two linked banks in a banking network there is a lender bank and a borrower bank. When the borrower bank is hit by a shock and is unable to meet its obligations to the lender bank the latter also suffers a loss. The lender bank might also play the role of a borrower bank in another link in which case the shock can be further transmitted.[24]

We will use directed links to capture these one way flows. In what follows ij captures not only the fact that agents i and j are linked but also that shocks are transmitted from agent i to agent j. Graphically, there will be an arrow between nodes i and j pointing

[19] If $N \bmod \hat{M} > 0$ then not all agents will be receiving the same payoff and the most efficient network might not feature subgraphs of size $\hat{M}$. However, for high values θ, c and δ or low values of b, $\hat{M}$ will be small relatively to N in which case it is more likely that the efficient network consists mainly of subgraphs of size $\hat{M}$.

[20] The application of strong stability requires that the decision of agents of whether or not to form a link is represented as a game in strategic form. Then a network is strong stable if it corresponds to a strong Nash equilibrium of the link formation game.

[21] It will still satisfy the intermediate strength notion of stability introduced by Belleflamme and Bloch (2004) where agents are allowed to form and break any number of links but for stability considerations the size of deviating coalitions is restricted to 2.

[22] Once more, we need to do a bit more work when N is not divisible by that particular size.

[23] As Acemoglu et al. (2014) show in financial markets ignoring this external effect leads to overlending.

[24] In many real world networks some links can be bi-directional. In order to keep the analysis simple we ignore such links.

at node j. Using the new interpretation of ij we can define, *directed paths* and *directed cycles* using the definitions for paths and cycles offered in the last section. $t(ij)$ now denotes the shortest directed path from i to j.[25] We let $\mathcal{T}(in)^i_k = \{j : t(ji) = k\}$ denote the set of agents with a shortest distance to agent i equal to k and $\mathcal{T}(out)^i_k = \{j : t(ij) = k\}$ denote the set of agents with a shortest distance from agent i equal to k. Notice that the cardinality of these sets for $k = 1$ are the *in-degree* and the *out-degree* of node i, respectively. When there is no path leading from agent i to agent j we set $t(ij) \approx \infty$. Lastly, a *Hamiltonian path* is a directed path that visits each node of the graph exactly once while a *Hamiltonian cycle* is obtained from a Hamiltonian path by adding the link from the last node to the first node.

We define the benefits and costs from participation in directed networks in a similar way as we have done for undirected networks. The unconditional probability that an agent is hit by a shock is again equal to $\frac{\theta}{N}$. As long as there is no agent hit by a shock each agent obtains a benefit b from each direct link irrespective of the link's direction.[26] As above, an agent hit by a shock suffers cost c. The only difference is that in directed networks other agents of the network will suffer losses only if they are connected to the agent who is hit by the shock by a directed path. Say agent j is hit by a shock. For any agent i connected to agent j this indirect cost is given by $\delta^{t(ji)}c$, where $0 < \delta \leqslant 1$; thus, as above we allow the cost to decline with the number of links in the shortest path between the two agents. The loss of benefits that each agent suffers following any shock will depend on the connectedness of the network. When agent j is hit by a shock all links located on directed paths beginning with agent j are affected and the corresponding benefits are lost for both agents at the ends of such links.

Next, consider the expected payoff function of agent i who belongs to a connected subgraph $(\mathcal{N}', \mathcal{L}')$ of network $(N, \mathcal{L})$.[27] Suppose that agent j who is hit by a shock belongs to the same subgraph. Then, let $(\mathcal{N}', \mathcal{L}')^j$ obtained from the $(\mathcal{N}', \mathcal{L}')$ after we have eliminated all affected links. Agent i will keep the benefits from all their remaining direct links in either direction. We can write now the expected payoff function of agent i as:

$$v_i(\mathcal{N}', \mathcal{L}') = \left(1 - \frac{|\mathcal{N}'|\,\theta}{N}\right) \left|\mathcal{T}^i_1\right| b$$
$$- \frac{\theta}{N}\left(c - \left|\mathcal{T}^i_1\right|_{(\mathcal{N}',\mathcal{L}')^i} b + \textstyle\sum_{j(\neq i)} \left(\delta^{t(ji)}c - \left|\mathcal{T}^i_1\right|_{(\mathcal{N}',\mathcal{L}')^j} b\right)\right)$$

where $\left|\mathcal{T}^i_1\right| = \left|\mathcal{T}(in)^i_1\right| + \left|\mathcal{T}(out)^i_1\right|$ and $\left|\mathcal{T}^i_1\right|_{(\mathcal{N}',\mathcal{L}')^j}$ is defined in a similar way for the network obtained after agent j is hit by a shock. The costs are calculated as in the case for undirected networks but now only those paths leading to i are included. The first term to the right of the equality sign captures the expected benefit given that no agent in $(\mathcal{N}', \mathcal{L}')$ is hit by a shock. Next, consider the expression inside the brackets to the right of $\frac{\theta}{N}$. The first two terms show the effect on the expected payoff when agent i is hit by a shock. Notice, that in contrast to the case of undirected networks agent i will keep the benefits from the incoming links (remember $(\mathcal{N}', \mathcal{L}')^j$ stands from the subgraph obtained after we have eliminated all links affected by the shock). Lastly, consider the summation. When an agent j belonging to $(\mathcal{N}', \mathcal{L}')$ is hit by a shock agent i is affected as long as there is a directed path leading from j to i. Once more, agent i will keep the benefits of unaffected incoming links. Notice that for all agents such that there is no directed path leading from them to agent i we set $\delta^{t(ji)} \approx 0$.

Comparing the above set-up to real financial and production networks we make the following observations. Firstly, the supposition that a node loses the benefits only from incoming links when is hit by a shock captures the fact that outgoing links might not be affected. For example, when a bank becomes insolvent those banks that have borrowed funds from it are still required to meet their obligations. Similarly, when an input supplier is unable to deliver goods links related to other input suppliers might still be unaffected.[28] Secondly, in financial networks contagion results when a threshold is violated. In particular, an institution becomes insolvent as its value of its liabilities exceeds the value of its assets. Given that an insolvent institution is unable to meet in whole its obligations some of its debtors might become insolvent. The likelihood that any other institution in the network will be affected will depend on its distance from the institution that was initially hit by the shock. Therefore, under risk-neutrality, the discount factor in our model can be interpreted as the probability that an institution will become insolvent after another institution is hit by a shock and c in that case captures costs associated with bankruptcy. Lastly, we have also assumed that all benefits derived by institutions affected by a shock are lost including those derived from links with institutions that are not insolvent. The benefit b in our model, in the context of financial networks, is interpreted as the profit that an institution makes from its financial link with another institution. The loss of these benefits captures losses due to the collapse of the interbank market following a systemic event like the recent global financial crisis. While this collapse can be explained by banks becoming more cautious right after the crisis (Acharya and Merrouche, 2012; Brunetti et al., 2019) there is strong evidence that the crisis had lasting consequences for the US interbank market.[29]

3.1. Weakest link, stability and efficiency

Once more, we begin with pair-wise stability.

Definition 3. A network $(\mathcal{N}, \mathcal{L})$ is *stable* if no agent i prefers to sever a link, and no pair of agents i and j prefer to form either link ij or link ji.

The definition of stability is similar as that used in the case of undirected networks. The only difference is that now stability requires that any pair of agents not linked do not want to form a link in any of the two directions.

Lemma 2. *The empty network is not stable if and only if* $(1 - \frac{\theta}{N})b > \frac{\theta}{N}\delta c$.

When the empty network is not stable a link ij is always beneficial to agent i. In contrast, whether the link is beneficial to agent j it will depend on the distribution of shortest paths that include link ij. If any agent along these paths is hit by a shock agent j will also suffer a loss. In contrast, the only benefit that agent j obtains from such paths is from the link to agent i.

We will begin the analysis by focusing on complete directed networks also known as *tournaments*.

25 Keep in mind that now it is possible that $t(ji) \neq t(ij)$.

26 The idea here is that both agents benefit from forming the link. For example, in banking networks both the lending and the borrowing banks benefit from the loan.

27 It is convenient to keep using the same notion of connectedness for directed networks (subgraphs) as the one we used for undirected networks (subgraphs). Thus, a network is connected as long as there paths (not necessarily directed paths) between all agents belonging to the network (subgraph).

28 For example, under CES production, because of input substitutability, when a supplier collapses other suppliers might benefit. Clearly, this will not be the case under Cobb–Douglas production, where the collapse of one supplier implies the collapse of production.

29 At its peak in March 2008 the total volume of loans in the interbank market was $458 billion. In December 2017 the total volume of loans was $63 billion. Source: https://fred.stlouisfed.org/series/IBLACBM027NBOG.

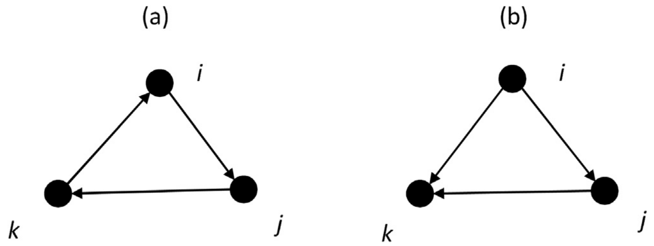

Fig. 3.1. (a) $n = 3$ Cycle; (b) $n = 3$ Complete Order.

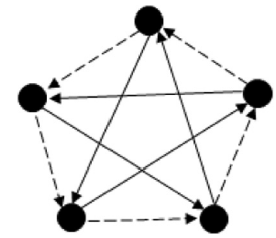

Fig. 3.2. Complete Symmetric Directed Network for $n = 5$ (— Hamiltonian Cycle)

Definition 4. A complete directed graph of size N is a *complete order* if we can label the nodes $\upsilon_1, \ldots, \upsilon_N$ such that there is a link from υ_i to υ_j, link ij, if and only if $j < i$.

Remark 1. In a complete order there are links from υ_N to all other nodes and there are links to υ_1 from all other nodes.

The following example for the case for $n = 3$ is useful for understanding networks with a large number of nodes.

Example 4. Suppose that $N = 3$ (i, j, k). There are two possible types of complete directed networks (see, Fig. 3.1)

(a) *Directed Cycle: The links are ij, jk, ki.* The network is symmetric as all agents have exactly the same expected payoff. In this case when any agent is hit by a shock all benefits are lost and all agents will suffer losses.

(b) *Complete Order: The links are ij, ik, jk.* When agent i is hit by a shock all links are affected, when agent j is hit by a shock only link jk is affected and when agent k is hit by a shock none of the links are affected. Remember that agents keep receiving benefits from links that are not affected.[30]

The example illustrates how small changes in connectivity can have large aggregate and distributional effects.

Definition 5. An agent is defined as *critical* if there exists a directed path from that agent to all other agents.

Thus, when a critical agent is hit by a shock all agents on the network will suffer losses. In the above example, in the case of a directed cycle all three agents are critical while in the case of a complete order only agent i is critical. Below we are going to show that there is a trade-off between *ex ante* efficiency and stability. Using the same definition of *ex ante* efficiency as in the case of undirected networks we can prove the following result:

Proposition 6. *Suppose that the parametric condition of Lemma 2 holds. Then, the complete order is the only ex ante efficient directed network.*

Notice that if c and θ are sufficiently high such that the average expected return of any network other than the empty one is less than $-\frac{\theta}{N}c$ (the expected return of an isolated agent) then the empty network would be *ex ante* efficient.

Fig. 3.2 shows the complete order for $N = 5$ and the corresponding Hamiltonian path. Notice that the shortest distance of any agent from the agent on the top of the order is equal to 1. The intuition behind Proposition 6 is very simple. Leighton and Dijk (2010) have shown that any complete directed network (tournament) has a Hamiltonian path. In our context this implies that every complete directed network has at least one critical agent. In the case of a complete order there is only one critical agent, the one at the top of the order. This implies that when any of the other agents is hit by a shock some of the links, and thus benefits, remain intact. From the work of Harary and Moser (1966) we also know that any complete directed network that does not have a three-agent cycle is a complete order. When there is a directed cycle and any of the agents on that cycle is hit by a shock then all other agents on that cycle are also affected. In the extreme case, we have networks with Hamiltonian Cycles where when any agent is hit by a shock all other agents are affected and, therefore, all links are broken.[31]

Although networks with Hamiltonian cycles are ex ante inefficient they are more likely to be stable than the complete order.[32] To see this point consider the following network that we will refer to it as *Complete Symmetric Directed Network*. This is a directed complete network where the in-degrees and the out-degrees of all nodes are equal to $\frac{N-1}{2}$.[33] Thus, in Fig. 3.2, we have a network with $N = 5$, where all agents are on a circle and moving counter-clockwise there are links from each agent to the next 2 agents.

Lemma 3. *Among all the complete directed networks, the complete symmetric directed network is the least ex ante efficient.*

Next, we compare the relative stability of the complete order and the complete symmetric directed network. We have the following result:

Proposition 7. *If $b < c\delta^3$ then stability of the complete order implies stability of the complete symmetric directed network but the opposite is not true.*

The above inequality is likely to hold in financial networks where the benefits of each link are relatively low, but the costs following a shock are likely to be high. The intuition behind the above result is simple. *Ex ante* efficiency depends only on the average expected payoff. In contrast, stability depends on the weakest link and in the case of the complete order this is the link between the two agents at the bottom of the order.

The conflict between stability and efficiency just described can at least be partly explained by the weak concept of stability that we are using in this paper. Had we not limit the number of links that agents are allowed to sever then networks with Hamiltonian cycles might not survive. Given that any shock on the network affects every link it is possible that even if such networks satisfy pairwise stability every agent's expected payoff is negative. Thus, it would be difficult to conceive a dynamic network formation

30 In a banking network after a bank is hit by a shock, its debtors (other banks that have borrowed from it) still need to meet their obligations.

31 There is quite a lot of work trying to establish the maximum number of hamiltonian paths and hamiltonian cycles in tournaments (e.g. Adler et al., 2001. It is well known that the number can be very large.

32 At least, according to the definition of stability used thus far. We will revisit this point below.

33 Strictly speaking the structure only applies when n is odd. But for large n the arguments about efficiency and stability are still valid when n is even by slightly modifying the network structure.

process that would converge to such networks. We can eliminate such networks by using a slightly stronger stability concept where agents are allowed to break any number of links.

Next, we turn our attention to the stability and structure of incomplete networks.

3.2. Incomplete networks

So far we have limited our analysis to the stability properties of complete networks. Our study of directed networks is motivated by the observation that many real-world networks (macroeconomics, financial) are very highly connected but are not complete. We have also shown that stable undirected networks consist of a set of disconnected subgraphs. In contrast, directed networks are very likely to be connected. The reason is that isolated agents are very unlikely to be present in stable directed networks. An isolated agent would always choose to link with any other agent as long as the link is outgoing.[34] The following result also suggests that as networks increase in size other agents would prefer to be linked with isolated agents.

Proposition 8. *Let $\hat{\theta}(N)$ denote a threshold value such that if $\theta < \hat{\theta}(N)$ networks with isolated agents are unstable. Then $\lim_{N\to\infty} \hat{\theta}(N) = 0.5$.*

That is as long as the probability that the network is hit by a shock is less than half we should not expect to see any isolated agents. Moreover, the above result was derived under the worst case scenario implying that, in general, isolated agents must be very rare. There is a further observation. Although it is still possible to have stable directed networks that consist of separated completely connected subgraphs, as it was the case with undirected networks, such networks are very unlikely to be formed. Given the earlier observation such subgraphs would be unstable in the presence of isolated agents. This implies that the subgraphs would have to be formed in isolation. For example, consider the dynamic formation process in Watts (2002) where at each point of time two agents are randomly chosen and then they decide on whether or not to form a link. Then it must be the case that the draws have to be such that for sufficient long time both agents of each drawn pair belong to the same subgroup of agents that eventually will comprise one of the subgraphs. Only when two or more such disjoint subgraphs are each sufficiently connected we can have agents that would not find it beneficial to form a link with agents belonging to other subgraphs.

The above argument only implies that stable directed networks are very likely to be connected as many real-world networks studied by economists. However, we have also argued that such networks are incomplete. Below we present the main result of this section.

Proposition 9. *Stable networks always exist. Furthermore, if $\left(\frac{(1-\theta)}{\theta}N + (N-1)\right)\frac{b}{\delta} \geqslant c > \left(\frac{(1-\theta)}{\theta}N + 1\right)\frac{b}{\delta}$ there exist stable networks that are connected, strictly hierarchical and not complete. The number of missing links increases with c.*

All incomplete networks defined in the above result are subgraphs of the complete order of size N. That is all can be obtained by eliminating links from a complete order of size N. They are hierarchical because as c increases the links that get eliminated are those connecting agents at the bottom of the order. If c does not satisfy the first inequality then the complete order will be stable and if it does not satisfy the second inequality then the empty network will be stable. The above result demonstrates that when links are directed it is straightforward to find stable networks that exhibit many of the properties of real-world networks as long as the network parameters are not sufficiently extreme (neither the empty network nor the complete network are stable).

Notice once more the conflict between stability and efficiency. Even when the complete order is not stable, completeness implies that it is *ex ante* efficient. The weakest link in the complete order is that between the agent at the bottom of the order (all links are incoming) and the one just above (the only outgoing link is the one with the agent at the bottom of the order). Thus, if the agent at the bottom of the order would like to break this link then the complete order would be unstable. The reason that the complete order is still *ex ante* efficient is because the marginal gain derived from that link from the agent just above the bottom of the order is higher in absolute terms than the marginal loss to the agent at the bottom of the order.

Lastly, notice that introducing stronger notions of stability will not eliminate any of the incomplete networks without introducing any side payments. The reason is that their hierarchical structure implies that as long as they are pairwise stable they cannot be Pareto-dominated by other networks. Altering the direction of a link cannot be Pareto-improving as one of the agents is always worse off.

4. Final remarks

In this paper, we have investigated the stability and efficiency properties of networks where agents derive benefits only from other agents linked directly with them but suffer losses when any other agent who is either directly or indirectly connected to them is hit by a shock. We first looked at networks with undirected links and we have confirmed the existence of stable networks that are comprised of complete disjoint subgraphs and then we went one step further by showing that these are the only networks that satisfy some strong notions of stability. Next, we focused on networks with directed links and we have shown that there exist networks that are connected but not complete and also their structure is hierarchical.

In the rest of this final section, we consider the implications for our main results of changing some of the main assumptions of our model.

Discounting. We have introduced discounting (decay) in our model to capture the possibility that costs related to shocks are decreasing as the shortest distance from the agent hit by the shock goes up. Further, to keep the exposition simple we have followed other examples in the literature (see, Jackson and Wolinsky, 1996; Watts, 2002) and have assumed geometric discounting. Our qualitative results do not depend neither on the exact form and they will still go through if we set the discount factor equal to 1 (no discounting).

Distribution of shocks. We have only allowed shocks that directly affect only one agent. Allowing for multiple shocks, as in Blume et al. (2011), would increase the parameter space within which the empty network is stable. For the case of undirected graphs it would also decrease the size of stable disjoint fully connected subgraphs. For the case of directed graphs it could make more likely the formation of networks that are not fully connected.

Nodes and links. In many interesting applications of directed graphs links can be bidirectional. For example, in financial networks two institutions can hold claims against each other. Generally, bankruptcy procedures do not allow the bilateral clearance of such claims following the failure of an institution as it would

[34] Notice that this is not the case with undirected networks where an isolated agent linking with any subgraph would suffer high losses each time an agent belonging to the subgraph is hit by a shock.

violate priority rules (Eisenberg and Noe, 2001). Thus, allowing for bidirectional links would not fundamentally affect our analysis. We only observe that as the number of such links increases the network would behave more as an undirected one.

Moreover, in many applications of directed graphs (financial and macroeconomic networks) links and nodes can be weighted. Weights on links would capture the size of the transaction while weights on nodes would capture the size of the institution and thus potentially the probability of being hit by a shock. Such modifications are necessary to capture the core–periphery structure of some financial, social and trading networks.

Dynamics. In the present work, we have concentrated on the properties of networks that in principle could be formed but we have ignored the dynamics of network formation and thus potentially the likelihood of these networks being formed. Our main objective in this paper has been to extend the analysis of Blume et al. (2011) to networks with directed links. The main message of the paper does not depend on any particular dynamics. However, such dynamics can be important when we consider particular applications.

Acknowledgements

I would like to thank Vincent Anesi, Yann Bramoullé, Matthias Dahm, Anne van den Nouweland, Daniel Seidmann, two referees, seminar participants at the University of Leicester and participants at the 22nd Coalition Theory Network Workshop, University of Glasgow (UK, 2017) and the Conference on Economic Design, University of York (UK, 2017) for helpful comments and suggestions. The usual disclaimer applies.

Appendix. Proofs

A.1. Proof of Lemma 1

(a) Consider the empty network and any pair of agents i and j. The probability that one of the two agents is hit by a shock is equal to $\frac{\theta}{N}$ in which case, if the link has been formed, the agent not hit by the shock will bear an indirect loss δc. With probability $1-\frac{2\theta}{N}$, none of the two agents is hit by a shock in which case each agent obtains benefit b. Thus, if $\left(1-\frac{2\theta}{N}\right)b \leqslant \frac{\theta}{N}\delta c$ the two agents will decide not to form the link.

(b) The payoff to an agent i who is part of the complete network is given by:

$$v_i(\mathcal{N},\mathcal{L})^C = (1-\theta)(N-1)\,b - \frac{\theta}{N}c\,(1+(N-1)\delta)$$

Agent i by severing a link, say with agent j, loses benefit b when there is no shock on the network and this happens with probability $1-\theta$. Given that $t(ij) = 2$ by severing the link the expected cost of participating in the network for agent i has been reduced by $\frac{\theta}{N}\delta\,(1-\delta)\,c$. The proof follows from comparing benefits and losses. □

A.2. Proof of Proposition 1

We will prove the proposition in two steps. We will first show that if we can construct a stable complete disjoint subgraph with $|\mathcal{N}'| < N$ then we can also construct a stable network with N nodes. Then we will show that a stable complete disjoint subgraph with less than N nodes always exists.

Lemma 4. *If we can construct a stable complete subgraph* $(\mathcal{N}',\mathcal{L}')^C$ *then we can also construct a stable network with all N agents.*

Proof Let $|\mathcal{N}'^C| = M$. Suppose that $N \bmod M = V$ and consider the network with $\frac{N-V}{M}$ complete subgraphs each with M nodes and 1 complete subgraph of size V. To prove the lemma we need to show that the complete subgraph of size V is stable, given that all other subgraphs are of size M and thus, by supposition, stable. The expected payoff of an agent i belonging to one of the complete subgraphs of size M is given by:

$$v_i(\mathcal{N}',\mathcal{L}')^C = \left(1-\frac{M\theta}{N}\right)(M-1)b - \frac{\theta}{N}c\,(1+(M-1)\delta) \tag{A.1}$$

One of the necessary conditions for the stability of the subgraph is that agent i does not want to sever a link. The new payoff of an agent who severs a link is given by $\left(1-\frac{M\theta}{N}\right)(M-2)b-\frac{\theta}{N}c\left(1+(M-2)\delta+\delta^2\right)$, and thus the stability condition, which by supposition holds, is given by:

$$\left(1-\frac{M\theta}{N}\right)b - \frac{\theta}{N}\delta\,(1-\delta)\,c \geqslant 0 \tag{A.2}$$

Next, we consider the stability of a complete subgraph of size V. The stability of the rest of the subgraphs implies that none of the agents belonging to the other subgraphs are willing to link with any agent belonging to another subgraph. In order to complete the proof we need to show that none of the agents in the complete subgraph of size V prefers to sever a link, which follows from the fact that the left-hand side of (A.2) is decreasing in M and the inequality $M > V$. □

Suppose that (1) holds; that is neither the empty network nor the complete network are stable. Clearly, if this is not the case existence of stable networks is trivially satisfied. Then the lemma implies that it is sufficient to show that a stable complete subgraph exists. The existence of a stable complete subgraph of cardinality M requires that two conditions are satisfied: (a) no agent prefers to break a link, that is $\left(1-\frac{M\theta}{N}\right)b > \frac{\theta}{N}\delta\,(1-\delta)\,c$, and (b) that no isolated agent would like to join the graph, that is $\left(1-\frac{(M+1)\theta}{N}\right)b < \frac{\theta}{N}c\left(\delta+(M-1)\delta^2\right)$. (Stability also requires that none of the agents belonging to the subgraph would like to be linked with agents outside the graph but this constraint does not bind. Moreover, if no isolated agent would like to join the complete subgraph then this will also be the case for any other agent belonging to any subgraph as joining an even larger network always decreases expected payoff.) Then it suffices to show that if (1) holds then there exists $M \in [2, N-1]$ such that the following inequalities are satisfied:

$$\left(1-\frac{(M+1)\,\theta}{N}\right)b < \frac{\theta}{N}\delta c < \left(1-\frac{M\theta}{N}\right)b \tag{A.3}$$

The proof of Proposition 1 follows from the observations that for $M = 2$ the second inequality in (A.3) is satisfied by (1) and for $M = N-1$ the first inequality in (A.3) is satisfied by (1). □

A.3. Proof of Proposition 2

Fig. A.1 Offers an illustration of our main argument.

By definition, in every connected network that is incomplete there must exist three agents, i, j and k, such that links ik and kj belong to the network but link ij does not. Without any loss of generality, suppose that agent i does not want to form the missing link. We will show that if the network is stable the link kj must belong to an open cycle (cycle that is not complete). We need to compare the agent k's marginal payoff from link kj with the marginal payoff that agent i would obtain by forming link ij. Given that the total payoff of each agent depends on the shortest paths to all other agents, both of these marginal payoffs would depend on all those shortest paths to other agents that because of these two links have become shorter. If agent i's marginal payoff from forming link ij is less than agent k's marginal payoff from link kj it must be because there exists at least one agent whose shortest distance from agent i has decreased by the formation of link ij while the shortest distance of the same agent from agent k is not affected by the presence of link kj. For the moment ignore Path 2 in Fig. 2.1 and suppose that the shortest path between agents k and h is through link kj and then along Path 1. Then, by forming link ij the shortest distance of agent i from agent h cannot be lower than the shortest distance of agent k from agent h. Then, it follows that the reason that agent i does not want to form link ij cannot be because the formation of the link would shortened the distance of agent i from agent h.

Now we introduce Path 2 which is an alternative path from agent k to agent h. Suppose that the number of links on Path 2 is equal to the number of links in Path 1 plus 1 (for kj). Then, by breaking link kj the shortest distance from agent k to agent h is not affected but the formation of link ij would still decrease the shortest distance from agent i to agent h by 1. Notice that for this to be the case there must be an open cycle. The reason is that if the cycle was complete then it would include the links kh and jh in which case agent i would either prefer to be linked with both j and k or none of them.

The fact that the cycle is open implies that there are other missing links which, in turn, implies that are other open cycles. However, given that the network has finite size this can only be the case if there is a finite number of cycles that support each other. This is, exactly what happens in the network described in Example 2. □

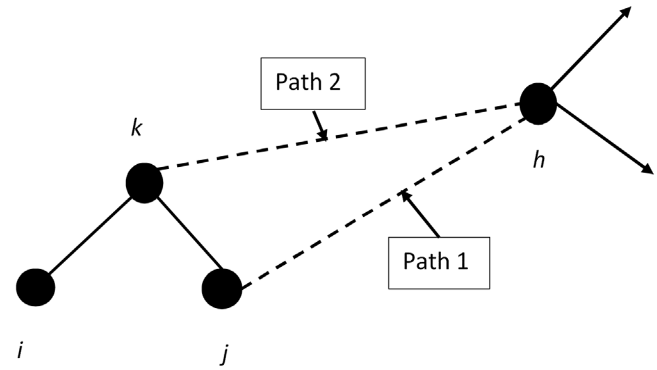

Fig. A.1. Missing Links in Stable Networks.

A.4. Proof of Proposition 3

Suppose that no agent would like to sever a link. If that is not the case the proposition trivially holds. The expected payoff of an agent who belongs to an empty cycle network with $N > 3$ nodes is given by:

$$(1-\theta)2b - \frac{\theta}{N}c\left(1 + 2\delta + 2\delta^2 + \cdots + 2\delta^{\frac{N-1}{2}}\right) \quad \text{if} \quad N \quad \text{is odd,}$$

and

$$(1-\theta)2b - \frac{\theta}{N}c\left(1 + 2\delta + 2\delta^2 + \cdots + 2\delta^{\frac{N-2}{2}} + \delta^{\frac{N}{2}}\right) \quad \text{if} \quad N \quad \text{is even.}$$

Suppose that the agent severs one of the two links. The new expected payoffs are now given by:

$$(1-\theta)b - \frac{\theta}{N}c\left(1 + \delta + \delta^2 + \cdots + \delta^{N-1}\right).$$

Thus, the expected gain from keeping the link is given by:

$$(1-\theta)b - \frac{\theta}{N}c\left(\delta + \delta^2 + \cdots + \delta^{\frac{N-1}{2}} - \delta^{\frac{N-1}{2}+1} - \cdots - \delta^{N-1}\right) \quad \text{if} \quad N \quad \text{is odd,}$$

and

$$(1-\theta)b - \frac{\theta}{N}c\left(\delta + \delta^2 + \cdots + \delta^{\frac{N-2}{2}} - \delta^{\frac{N}{2}+1} - \cdots - \delta^{N-1}\right) \quad \text{if} \quad N \quad \text{is even.}$$

Next, suppose that the agent forms a link with one of the two agents with a distance between them equal to 2. Then, the new payoffs are given by:

$$(1-\theta)3b - \frac{\theta}{N}c\left(1 + 3\delta + 2\delta^2 + \cdots + 2\delta^{\frac{N-3}{2}} + \delta^{\frac{N-1}{2}}\right) \quad \text{if} \quad N \quad \text{is odd,}$$

and

$$(1-\theta)3b - \frac{\theta}{N}c\left(1 + 3\delta + 2\delta^2 + \cdots + 2\delta^{\frac{N-2}{2}}\right) \quad \text{if} \quad N \quad \text{is even.}$$

Thus, the expected gain from forming the link is given by:

$$(1-\theta)b - \frac{\theta}{N}c\left(\delta - \delta^{\frac{N-1}{2}}\right) \quad \text{if} \quad N \quad \text{is odd,}$$

and

$$(1-\theta)b - \frac{\theta}{N}c\left(\delta - \delta^{\frac{N}{2}}\right) \quad \text{if} \quad N \quad \text{is even.}$$

When N is odd we need to show that

$$\delta - \delta^{\frac{N-1}{2}} < \delta + \delta^2 + \cdots + \delta^{\frac{N-1}{2}} - \delta^{\frac{N-1}{2}+1} - \cdots - \delta^{N-1}.$$

Rearranging we get

$$0 < \delta + \delta^2 + \cdots + 2\delta^{\frac{N-1}{2}} - \delta^{\frac{N-1}{2}+1} - \cdots - \delta^{N-1},$$

which holds given that the number of terms with positive sign is equal to the number of terms with a negative sign and the former group of terms is larger than the latter.

When N is even we need to show that

$$\delta - \delta^{\frac{N}{2}} < \delta + \delta^2 + \cdots + \delta^{\frac{N-2}{2}} - \delta^{\frac{N}{2}+1} - \cdots - \delta^{N-1}.$$

Rearranging we get

$$0 < \delta + \delta^2 + \cdots + \delta^{\frac{N-2}{2}} + \delta^{\frac{N}{2}} - \delta^{\frac{N}{2}+1} - \cdots - \delta^{N-1},$$

which holds given that there is an extra term with positive sign and those with positive sign are larger than those with a negative sign. □

A.5. Proof of Proposition 4

The expected payoff of an agent i belonging to one of the complete subgraphs of size M is given by (A.1). Subtracting $-\frac{\theta}{N}c$ and dividing by $M-1$ we get $\left(1 - \frac{M\theta}{N}\right)b - \frac{\theta}{N}\delta c$ which is equal to each agent's net expected payoff from each node. We need to consider two cases:

(a) $\left(1 - \frac{M\theta}{N}\right)b - \frac{\theta}{N}\delta c \geq 0$: Consider any other connected subgraph of cardinality M that is not complete. Then, any node is either directly or indirectly linked to other nodes. From each directly linked node an agent gets exactly the same expected payoff as the one derived from any node when being a member of a complete subgraph. However, the expected payoff derived from nodes not directly linked is negative due (a) to the absence of benefit, and (b) to connectivity. Then, given that the subgraph is incomplete there are agents whose expected payoff from belonging to the incomplete subgraph is less than the above expression. Thus, the total payoff is maximized when the subgraph is complete.

(b) $\left(1-\frac{M\theta}{N}\right)b-\frac{\theta}{N}\delta c<0$: In this case the net expected payoff received from each node of a complete subgraph is negative and clearly the payoffs of all agents would have been higher had they been isolated. □

A.6. Proof of Proposition 5

According to Jackson and van den Nouweland (2005) strongly stable networks are Pareto-efficient. From Proposition 3 we know that networks that are not complete are not efficient and to complete the proof we need to show that are also not Pareto-efficient. But from the proof of Proposition 3 we know that if the network is not complete we can make some agents better off without making any agent worse of either by competing the network (case (a) above) or by breaking all the links (case (b) above). □

A.7. Proof of Lemma 2

Consider two isolated agents i and j. It suffices to consider the expected payoff of agent j from creating the link ij who suffers losses when any of the two agents is hit by a shock. The first term of the left-hand side of the inequality shows agent j's expected payoff from creating the link in the absence of any shock. Agent j will lose the benefit of the link only when agent i is hit by a shock and, therefore, even when there is a shock on the network, with probability $\frac{\theta(N-1)}{N}$ the link remains intact and agent j obtains benefit b. The right hand side shows the net expected cost from creating the link conditional on one of the two agents is hit by a shock. Keep in mind that an isolated agent is also hit by a shock with probability $\frac{\theta}{N}$ and suffers a loss c. After the creation of the link this cost is still there, however, with probability $\frac{\theta}{N}$ agent i is hit by the shock and the expected loss to agent j equal to $\frac{\theta}{N}\delta c$. Thus, when the inequality $(1-\theta)\,b+\frac{\theta(N-1)}{N}b>\frac{\theta}{N}\delta c$ holds both agents prefer to create the link. We complete the proof by rearranging terms. □

A.8. Proof of Proposition 6

We are going to prove the Proposition using induction and we will first consider complete networks. Consider the case $N=3$. In this case there are only two possible complete networks: *Directed Cycle* (links ij, jk, ki) and *Complete Order* (links ij, ik, jk). Given that the expected payoffs of all agents in the directed cycle are equal the sum of these payoffs V^C is given by

$$V^C=3\left[(1-\theta)2b-\frac{\theta}{3}c\left(1+\delta+\delta^2\right)\right]$$

When an agent is hit by a shock all three agents suffer a loss depending on their distance from that agent. Next, consider the expected payoffs of each agent in a complete order. The expected payoff of agent i is given by

$$v_i^O=(1-\theta)2b+4\frac{\theta}{3}b-\frac{\theta}{3}c$$

Agent i's expected payoff is not affected when either agent j or agent k are hit by a shock. The expected payoff of agent j is given by

$$v_j^O=(1-\theta)2b+3\frac{\theta}{3}b-\frac{\theta}{3}c\,(1+\delta)$$

Agent j's expected payoff is not affected when agent k is hit by a shock and when agent j is hit by a shock only link jk is broken. The expected payoff of agent k is given by

$$v_k^O=(1-\theta)2b+3\frac{\theta}{3}b-\frac{\theta}{3}c\,(1+2\delta)$$

When agent k is hit by a shock no links are affected, when agent j is hit by a shock only link jk is affected and when agent i is hit by a shock both incoming links are broken. As long as any of the other two agents is hit by a shock agent k suffers a loss equal to $c\delta$. The sum of expected payoffs of the three agents V^O is given by

$$V^O=3\left[(1-\theta)2b-\frac{\theta}{3}c\,(1+\delta)\right]+10\frac{\theta}{3}b$$

Subtracting V^C from V^O we find that

$$V^O-V^C=\frac{\theta}{3}\left(10b+c\delta^2\right)>0$$

Thus, for $N=3$ the complete order is *ex ante* efficient.

Next, suppose that the complete order for a network with $N=N^*>3$ agents is *ex ante* efficient and consider complete directed networks of size N^*+1. Suppose that you begin with any complete directed network of size N^* and increase the size of the network to N^*+1 by introducing one isolated node. The addition of N^* links, independently of their direction, will increase benefits by $2(N^*)b$ as long as there is no shock on the network. Given that there are N^* new links, expected losses in benefits from shocks hitting the network must be at least equal to $N^*\frac{\theta}{N^*+1}b$; that is the only benefits lost are those related to the new links. Moreover, the expected costs must be at least equal to $N^*\frac{\theta}{N^*+1}2c(1+\delta)$. Any of the new links can be either outgoing (from the new node) or incoming. When the new link is outgoing only shocks on the new node impose additional costs (at least $c(1+\delta)$); when the new link is incoming then if the node hit by a shock is the one belonging to the old smaller network there will be a cost $c\delta$ and if the new node is hit by a shock there will be a cost c. The above minima in benefit losses and costs can be achieved by having all the new links outgoing from the new node. If the size N^* network was a complete order then the new network is also a complete order and if the old network was *ex ante* efficient the new network is, by construction, also *ex ante* efficient.

Up to this point we have shown that among all the complete directed networks the complete order is the *ex ante* efficient network. It remains to be shown that the complete order also dominates any incomplete directed network. This can be shown by using exactly the same method. This is clearly the case for $N=3$. For any $N>3$ begin with a connected subgraph of size 3. Clearly, the complete order dominates any other subgraph. Next, increase the size of the subgraph by introducing outgoing links from a previously isolated node to all the pre-existing nodes of the subgraph. For the same reason as above this will maximize the sum of expected payoffs on the new subgraph. By repeating the procedure we conclude that the complete order maximizes *ex ante* efficiency. □

A.9. Proof of Lemma 3

There exists a Hamiltonian cycle (there are many) from each agent to the next one (moving either clockwise or anti-clockwise). This implies that all benefits are lost when any of the agents is hit by a shock. The expected costs of being part of such a network are also very high. The shortest paths of any agent from half of the other agents (links incoming) is equal to 1 and the shortest paths from the other half of agents (links outgoing) is equal to 2. The proof is completed by observing that the average shortest path cannot be less than 3/2. □

A.10. Proof of Proposition 7

In the case of the complete symmetric directed network all agents are symmetric. It is clear that the expected benefit of breaking a link is higher when the link is an incoming and therefore the stability condition is given by:

$$-(1-\theta)b+\frac{\theta}{N}c\delta\left(1-\delta^2\right)\leqslant 0$$

By breaking the link there is a loss of b as long as the network is not hit by a shock. The shortest path to the corresponding agent has now increased from 1 to 3 while the shortest paths from all other agents have remained the same.

Next, consider the stability of the complete order. We focus on the agent at the bottom of the order (all links are incoming) who has the lowest expected payoff. The agent's expected payoff from the complete order is equal to

$$(1-\theta)(N-1)b+\frac{\theta}{N}(N-1)b+\frac{\theta}{N}(N-2)b+\cdots+\frac{\theta}{N}b \\ -\frac{\theta}{N}c\left(1+(N-1)\delta\right) \tag{A.4}$$

As long as the network is not hit by a shock the agent benefits from $N-1$ links. When any of the agents is hit by a shock the agent at the bottom of the order still keeps the benefits from any incoming links from agents higher up the order than the agent hit by the shock. The last expression reflects the fact that the shortest path to any agent is equal to 1. Next, consider the expected payoff after breaking the link with the agent immediately higher up the order which is given by

$$(1-\theta)(N-2)b+2\frac{\theta}{N}(N-2)b+\frac{\theta}{N}(N-3)b+\cdots+\frac{\theta}{N}b \\ -\frac{\theta}{N}c\left(1+(N-2)\delta\right) \tag{A.5}$$

Now, as long as the network is not hit by a shock the agent benefits by only $N-2$ links. Those links remain intact even when one of the two agents at the bottom of the order is hit by a shock. Expected costs are reduced as now there are only $N-2$ incoming links. Comparing the two expressions we find that the stability condition is given by

$$(1-\theta)b+\frac{\theta}{N}b-\frac{\theta}{N}c\delta\geqslant 0 \tag{A.6}$$

By comparing the two stability conditions ((A.4) and (A.5)) we arrive at the desired result.

We also need to verify that no other agent can benefit more by breaking a link; put differently, the agent at the bottom of the order is the weakest link. When any other agent breaks a link they would break the link with the agent just above the order. (This is because (a) they would only break an incoming link and (b) breaking a link with an agent higher up the order would not isolate them from shocks hit by that agent.) But whenever an agent who is lower in the order is hit by a shock they would still lose the benefit of the broken link which is not the case for the agent at the bottom of the order. □

A.11. Proof of Proposition 8

Consider any agent who is potentially linked with an isolated agent through an incoming link. The worst possible scenario is that this agent belongs to a subgraph that it is complete and symmetric (this is because such structure maximizes the losses following a shock on the isolated agent). The net expected payoff from linking with the isolated agent is given by:

$$\left(1-\frac{\left(\left|\mathcal{N}'\right|+1\right)\theta}{N}\right)b-\frac{\theta}{N}\left(\left|\mathcal{N}'\right|-1\right)b-\frac{\theta}{N}c\delta$$

where $\left|\mathcal{N}'\right|$ is equal to the size of the original subgraph. The first expression is equal to the expected benefit of an extra link conditional on the enlarged subgraph not being hit by a shock; the second expression is equal to the expected benefit losses when the agent who was previously isolated is hit by a shock (keeping in mind that before the new connection there were $\left|\mathcal{N}'\right|-1$ links; the last expression is the additional cost suffered when the previously isolated agent is hit by a shock. Notice that this expression is decreasing in $\left|\mathcal{N}'\right|$. Then setting $\left|\mathcal{N}'\right|=N$ we find that as long as $\theta<\frac{b}{2b+\frac{c\delta}{N}}$ even in this worst case scenario the link would be formed. Notice that for large N it would suffice for the formation of the link to have $\theta<0.5$.

A.12. Proof of Proposition 9

Consider a complete order of size N. We label the agents (nodes) from 0 to $N-1$ so that the labels denote the number of outgoing links of the corresponding nodes. From Proposition 7 we know that as long as agent 0 does not want to break the link 10 (that is the link between 1 to 0) the complete order is stable, and the corresponding stability condition is given by (A.6).

Next, suppose that the above weak inequality is not satisfied in which case we eliminate link 10 and we check for the stability of the new incomplete network. Notice, that in the new network nodes 0 and 1 are symmetric with each one having $N-2$ incoming links. The corresponding payoffs are given by (A.5). If these two agents were going to break any link it would be their corresponding link with agent 2 (Again, by breaking any other link they would still remain indirectly linked with the corresponding node.) Their new payoff would be equal to:

$$(1-\theta)(N-3)b+3\frac{\theta}{N}(N-3)b+\frac{\theta}{N}(N-4)b+\cdots+\frac{\theta}{N}b \\ -\frac{\theta}{N}c\left(1+(N-3)\delta\right) \tag{A.7}$$

The first term is equal to the expected benefit as long as the network is not hit by a shock. The second term captures the fact that even if nodes 0, 1 and 2 are hit by shock $N-3$ links would remain intact. The next term shows the effect on benefits when node 3 is hit by a shock in which case $N-4$ links remain intact. The penultimate term captures the fact that when node $N-2$ is hit by a shock there is only one link left intact (that with node $N-1$). The last term is equal to the costs associated with shocks on the network (nodes 0 and 1 have incoming links from $N-3$ nodes). By comparing (A.5) and (A.7) we find that agents 0 and 1 will keep their link with agent 2 as long as

$$(1-\theta)b+2\frac{\theta}{N}b-\frac{\theta}{N}c\delta\geqslant 0 \tag{A.8}$$

From (A.7) and (A.8) we find that if the following inequalities are satisfied, the network obtained by eliminating link 10 is stable:

$$(1-\theta)b+2\frac{\theta}{N}b\geqslant\frac{\theta}{N}c\delta>(1-\theta)b+\frac{\theta}{N}b. \tag{A.9}$$

If (A.8) is not satisfied we can ask if the new network obtained by eliminating links 20 and 21 (in addition to the already eliminated 10) is stable. Notice that now we have three symmetric nodes, 0, 1 and 2 all of them with $N-3$ incoming links. We can keep repeating the process to find that if

$$(1-\theta)b+k\frac{\theta}{N}b\geqslant\frac{\theta}{N}c\delta>(1-\theta)b+(k-1)\frac{\theta}{N}b \tag{A.10}$$

the network obtained from the complete order by eliminating all outgoing links from nodes 1 to $k-1$ (node 0 has no outgoing links) is stable. If the above inequalities are satisfied for $k=N-1$ then the star network, where there are $N-1$ outgoing links from node $N-1$, is stable.

Lastly, if $\frac{\theta}{N}c\delta > (1-\theta)b + (N-1)\frac{\theta}{N}b$ (which is the condition derived in Lemma 2) the empty network is stable.

Thus, we have established the existence of stable networks for every possible value of c. Moreover, if

$$(1-\theta)b + (N-1)\frac{\theta}{N}b \geqslant \frac{\theta}{N}c\delta > (1-\theta)b + \frac{\theta}{N}b$$

there exist stable networks that are connected but not complete. Rearranging we obtain the inequalities that appear in the statement of the proposition. □

References

Acemoglu, D., Azar, P., 2020. Endogenous production networks. Econometrica 88, 33–82.

Acemoglu, D., Carvalho, V., Ozdaglar, A., Tahbaz-Salehi, A., 2012. The network origins of aggregate fluctuations. Econometrica 80, 1977–2016.

Acemoglu, D., Carvalho, V., Ozdaglar, A., Tahbaz-Salehi, A., 2014. Systemic risk in endogenous financial networks, Columbia Business School research paper no. 15-7. Available at SSRN: https://ssrn.com/abstract=2553900.

Acemoglu, D., Ozdaglar, A., Siderius, J., Tahbaz-Salehi, A., 2020. Systemic credit freezes in financial lending networks. NBER Working Paper 27149. Available at https://www.nber.org/system/files/working_papers/w27149/w27149.pdf.

Acemoglu, D., Ozdaglar, A., Tahbaz-Salehi, A., 2015. Systemic risk and stability in financial networks. Amer. Econ. Rev. 105, 564–608.

Acemoglu, D., Ozdaglar, A., Tahbaz-Salehi, A., 2017a. Networks, shocks, and systemic risk. In: Bramoulle, Y., Galeotti, A., Rogers, B. (Eds.), The Oxford Handbook on the Economics of Networks. Oxford University Press, NY.

Acemoglu, D., Ozdaglar, A., Tahbaz-Salehi, A., 2017b. Microeconomic origins of macroeconomic fat tail risks. Amer. Econ. Rev. 107, 54–108.

Acemoglu, D., Tahbaz-Salehi, A., 2020. Firms, failures, and fluctuations: The macroeconomics of supply chain disruptions. MIT Mimeo.

Acharya, V., Gale, D., Yorulmazer, T., 2011. Rollover risk and market freezes. J. Finance 66, 1177–1209.

Acharya, V., Merrouche, O., 2012. Precautionary hoarding of liquidity and interbank markets: Evidence from the subprime crisis. Rev. Finance 17, 107–160.

Adams, M., Galbiati, M., Giasante, S., 2010. Liquidity costs and tiering in large-value payment systems. Bank of England Working Paper 399.

Adler, I., Alon, N., Ross, S., 2001. On the maximum number of Hamiltonian paths in tournaments. Random Struct. Algorithms 18, 291–296.

Anand, K., Gai, P., Marsili, M., 2012. Rollover risk network structure and systemic financial crises. J. Econom. Dynam. Control 36, 1088–1100.

Anderson, H., Paddrik, M., Wang, J., 2019. Bank networks and systemic risk: Evidence from the National Bank Acts. Amer. Econ. Rev. 108, 3125–3161.

Babus, A., 2016. The formation of financial networks. Rand J. Econ. 47, 239–272.

Babus, A., Allen, F., 2009. Networks in finance. In: Kleindorfer, P., Wind, J. (Eds.), Network-Based Strategies and Competencies. Wharton School Publishing, Upper Saddle River, pp. 367–382.

Babus, A., Hu, T.-W., 2017. Endogenous intermediation in over-the-counter markets. J. Financ. Econ. 125, 200–215.

Bala, V., Goyal, S., 2000. A noncooperative model of network formation. Econometrica 68, 1181–1229.

Baldwin, R., Tomiura, E., 2020. Thinking ahead about the trade impact of COVID-19. In: Baldwin, R., Weder di Mauro, B. (Eds.), Economics in the Time of COVID-19, a VoxEU.Org Book. CERP Press, London.

Battiston, S., Gatti, D. Delli, Gallegati, M., Greenwald, B., Stiglitz, S., 2007. Credit chains and bankruptcy propagation in production networks. J. Econom. Dynam. Control 31, 2061–2084.

Bedayo, M., Mauleon, A., Vannetelbosch, V., 2016. Bargaining in endogenous trading networks. Math. Social Sci. 80, 70–82.

Belleflamme, P., Bloch, F., 2004. Market sharing agreements and collusive networks. Internat. Econom. Rev. 45, 387–411.

Billand, P., Bravard, C., Iyengar, S., Kumar, R., Sarangi, S., 2016. Network connectivity under node failure. Econom. Lett. 149, 164–167.

Billand, P., Bravard, C., Sarangi, S., 2012. Spillovers with directed networks. J. Public Econ. Theory 14, 849–878.

Blume, L., Easly, D., Kleinberg, J., Kleinberg, R., Tardos, E., 2011. Network formation in the presence of contagious risk. ACM Trans. Econ. Comput. 1 (2), Article 6.

Boss, M., Elsinger, H., Summer, M., Thurner, S., 2004. The network topology of the interbank market. Quant. Finance 4, 677–684.

Bougheas, S., Kirman, A., 2015. Complex financial networks and systemic risk: A review. In: Commendatore, P., Kayam, S., Kubin, I. (Eds.), Complexity and Geographical Economics: Topics and Tools. Springer, Heidelberg, pp. 115–139.

Brunetti, C., Harris, J., Mankadc, S., Michailidis, G., 2019. Interconnectedness in the interbank market. J. Financ. Econ. 133, 520–538.

Cabrales, A., Gottardi, P., Vega-Redondo, F., 2017. Risk sharing and contagion in networks. Rev. Financ. Stud. 30, 3086–3127.

Carvalho, V., 2014. From micro to macro via production networks. J. Econ. Perspect. 28, 23–48.

Chai, S.-K., 1993. An organizational economics theory of antigovernment violence. Comp. Politics 26, 99–110.

Cohen-Cole, E., Patacchini, E., Zenou, Y., 2010. Systemic risk and network formation in the interbank market. CAREFIN research paper no. 25/2010. Available at SSRN: https://ssrn.com/abstract=1799925.

Craig, B., Von Peter, G., 2014. Interbank tiering and money center banks. J. Financ. Intermediation 23, 322–347.

Diamond, D., Rajan, R., 2011. Fear of fire sales illiquidity seeking, and credit freezes. Q. J. Econ. 126, 557–591.

Dutta, B., Mutuswami, S., 1997. Stable networks. J. Econom. Theory 76, 322–344.

Eisenberg, L., Noe, T., 2001. Systemic risk in financial systems. Manage. Sci. 47, 236–249.

Elliott, M., Golub, B., Jackson, M., 2014. Financial networks and contagion. Amer. Econ. Rev. 104, 3115–3153.

Erol, S., Vohra, R., 2014. Network formation and systemic risk. PIER Working Paper No. 15-001, Available at SSRN: https://ssrn.com/abstract=2546310 or http://dx.doi.org/10.2139/ssrn.2546310.

Gabrieli, S., Georg, C., 2014. A network view on interbank market freezes. Available at SSRN: https://ssrn.com/abstract=2542162 or http://dx.doi.org/10.2139/ssrn.2542162.

Gai, P., Kapadia, S., 2010. Contagion in financial networks. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 466, 2401–2423.

Galeotti, A., 2006. One-way flow networks: The role of heterogeneity. Econom. Theory 29, 163–179.

Glasserman, P., Young, P., 2016. Contagion in financial networks. J. Econ. Lit. 54, 779–831.

Goyal, S., Joshi, S., 2003. Networks of collaboration in oligopoly. Games Econom. Behav. 43, 57–85.

Goyal, S., Vega-Redondo, F., 2007. Structural holes in social networks. J. Econom. Theory 137, 460–492.

Hamilton, M., Milne, B., Walker, R., Burger, O., Brown, J., 2007. The complex structure of hunter–gatherer social networks. Proc. R. Soc. B 274, 2195–2222.

Harary, F., Moser, L., 1966. The theory of round robin tournaments. Amer. Math. Monthly 73, 231–246.

Hojman, D., Szeidl, A., 2008. Core and periphery in networks. J. Econom. Theory 139, 295–309.

Imbs, J., 2004. Trade, finance, specialization, and synchronization. Rev. Econ. Stat. 86, 723–734.

Inaoka, H., Ninomiya, T., Taniguchi, K., Shimizu, T., Takayasu, H., 2004. Fractal network derived from banking transactions: An analysis of network structures formed by financial institutions. Bank of Japan Working Paper 04-E-04.

Jackson, M., 2003. The stability and efficiency of economic and social networks. In: Sertel, M., Koray, S. (Eds.), Advances in Economic Design. Springer, Heidelberg, pp. 319–361.

Jackson, M., 2008. Social and Economic Networks. Princeton University Press, Princeton.

Jackson, M., van den Nouweland, A., 2005. Strongly stable networks. Games Econom. Behav. 51, 420–444.

Jackson, M., Watts, A., 2002. The evolution of social and economic networks. J. Econom. Theory 106, 265–295.

Jackson, M., Wolinsky, A., 1996. A strategic model of social and economic networks. J. Econom. Theory 71, 44–74.

Kose, A., Prasad, E., Terrones, M., 2003. How does globalization affect the synchronization of business cycles? Am. Econ. Rev. P P 93, 57–62.

Leighton, T., Dijk, D., 2010. 6.042J Mathematics for Computer Science. Massachusetts Institute of Technology, MIT OpenCourseWare, https://ocw.mit.edu, License: Creative Commons BY-NC-SA.

Morselli, C., Giguěre, C., Petit, K., 2007. The efficiency/security trade-off in criminal networks. Social Networks 29, 143–153.

Newman, M., 2010. Networks: An Introduction. Oxford University Press, NY.

In 't Veld, D., van der Leij, M., Hommes, C, 2020. The formation of a core–periphery structure in heterogeneous financial networks. J. Econom. Dynam. Control 119, http://dx.doi.org/10.1016/j.jedc.2020.103972.

Watts, A., 2002. A dynamic model of network formation. Games Econom. Behav. 34, 331–341.