MASARYK UNIVERSITY
FACULTY OF INFORMATICS

Combinatorial games

BACHELOR'S THESIS

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Declaration

Hereby I declare, that this paper is my original authorial work, which I have worked out by my own. All sources, references and literature used or excerpted during elaboration of this work are properly cited and listed in complete reference to the due source.

Matej Antol

Advisor: Mgr. Michal Bulant, Ph.D.

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Abstract

This thesis is concerned with the description and analysis of the impartial and partisan combinatorial games. It is divided into three chapters.

The first chapter consists of the general formulation of combinatorial games. Also, a sample of both impartial and partisan games is presented there.

The second chapter presents ways of analysis of impartial games introduced in chapter one. In addition, it defines a basic method of analysis of games which are composed of a greater number of subgames.

The third section is concerned with a basic analysis of partisan games. A method for creating simple numbers is also indicated here. These play an important part in analysis and interpretiation of mathematical values of games.

Keywords

Game theory, impartial game, partisan game, Sprague-Grundy function, Nim, Hackenbush.

Contents

1 Introduction to mathematical games

1.1 What is a combinatorial game

The definition of a combinatorial game is usually presented using the following six conditions[5]:

- Two players: Every combinatorial game is played by two players. They are usually distinguished as blue and red, or equally as left and right. Players may also be referred to as first and second while analysing impartial games, as there is no importance in distinguishing their identities, and the only relevant difference is the order in which they make their moves. We will see a combination of these two views on the roles of both players, too.

- No chance: There are no factors of chance in combinatorial games. This means that all games are played with certain results of every move with no effect of coincidence. This condition rules out all games played with dice, shuffled cards, roulette or any element of coincidence of any kind, and also ensures accuracy of every outcome that can be deduced by analysing these games.

- Perfect information: Both players must possess the same information concerning the current game in sense that neither of the players has secret moves, and every state of the game is fully visible for both players.

- Turn based: The players take turns in all combinatorial games, with the starting player chosen in advance. Choosing a starting player may be considered as the only allowed element of chance in combinatorial games.

- Absolute winner: The winner of the game is chosen exactly, allowing no ties or draws. Chess and all similar games with the possibility of a draw are ruled out by this condition.

- Winning condition: Winning (and consequently losing) the game is defined by reaching the terminal position by one of the players. There are many varieties concerning the number of terminal positions and their meaning in specific games and their modifications.

1.2 Sorts of games

So far we have clarified the meaning of combinatorial game with no closer specification or division. This thesis concerns two types of games, which are partisan and impartial games.

A partisan game is described as a game with separately specified moves for both players. These moves are commonly distinguished by different colours of counters or direction of moves. Analysis of this type of games then concerns possibilities of both players from given positions and their advantage.

On the other hand, impartial games are always played with the same set of moves for both players. The only way to distinguish the players of impartial game is by the order of their moves.

There are two basic kinds of winning conditions in both partisan and impartial games. We normally call the game form with victory of the last player able to move the normal form of the game. The one with the victory of the first player unable to make any moves is mostly referred to as the misère form of the game.

1.3 Examples of impartial games

1.3.1 One pile game

Perhaps the simplest possible example of a meaningful impartial game is one named the One pile game. The game is played by two players with a random number of counters (commonly 21) ordered in one row. The legal move of each player consists of reducing this row by one, two or three counters from the pile. The first player unable to move is considered the loser, therefore the one who leaves the table with no counters left is the winner. There are many possible alterations of this game, for instance playing it with greater pile, or with different legal moves (for example more counters can be allowed to be taken in one move). Further in this thesis there will be shown that all these changes make no difference in difficulty of the game at all.

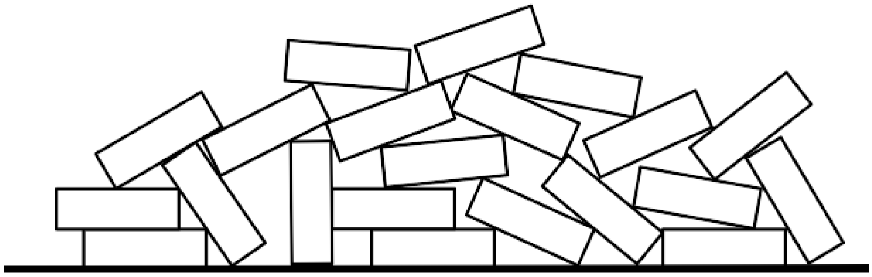

Figure 1.1: One pile game

1.3.2 The game of Nim

This game is the best one to explain the theory concerning solving impartial games. The game is, similarly to the One pile game, played with the counters in rows with difference of any number of piles of counters instead of only one. There is also a change in the definition of a legal move, which in this case consists of taking any number (greater than zero) of counters from one pile and one pile only.

Same as in the One pile game losing player is the one, who is the first unable to take any counters, because there are none left by the opponent, and the other player is, naturally, a winner. After a longer time of playing this game, a bright mind can easily find basic patterns of moves and states from which it is easy to guess the winner before the game ends, and so there are many commonly known disguises of this game with the purpose of confusing and entertaining both players at the same time.

1.3.3 Nimble

An alternation of Nim played with coins on a semi-infinite board of squares (for our purposes the border of this board will always be on the left side, which leaves the right one endless). The starting position of this game consists of a finite amount of coins randomly spread on the squares of this board. In each turn of the game, player

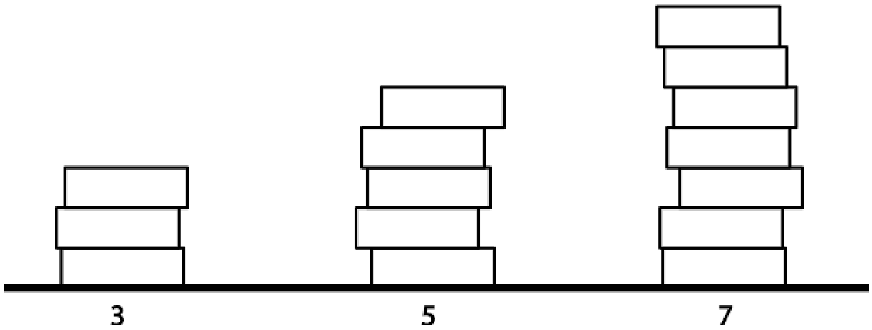

Figure 1.2: Game of Nim

moves one of the coins leftwards. The goal of the game is to reach a position from which no other move is possible. In other words, the winning position of this game is having all coins positioned on the first square, so the other player has no move to play and loses the game.

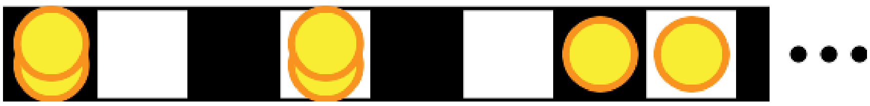

Figure 1.3: Nimble

1.3.4 The silver dollar game

Both forms of The silver dollar game (with and without the silver dollar) are significantly similar to the game of Nimble.

The version of the game without the silver dollar has only two added restrictions – no coin can be moved over any other coin, and no two or more coins can be placed at the same square of the board. The final position is then altered as a position in which all of the coins are lined-up at the beginning of the board, with no empty spaces between them, leaving no more moves for the opponent.

The alteration with the silver dollar also has the restriction of a maximum of one coin per square. However, this rule is not valid for the first square of the board, and so any number of coins can

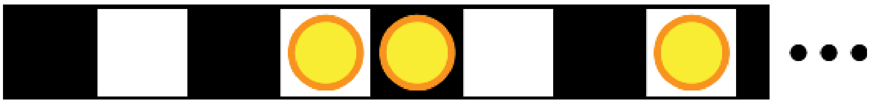

Figure 1.4: Silver dollar game without the silver dollar

be placed there. Again, jumping over other coins is not allowed. The main difference is, as the name of the game implies, in one separated coin, labelled as a silver dollar. This coin is the last one on the board (to the right of all other coins), and the goal is to be the one who moves the silver dollar on the first square and takes it off the board. This can be done if and only if are all the other coins are already placed there.

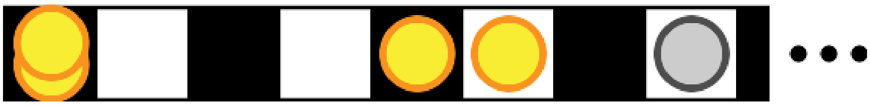

Figure 1.5: Silver dollar game with the silver dollar

The advantage of the variation with the dollar is greater visual suitability for playing the misère type of the game, by dividing the last step of moving the silver dollar and taking it to the possession into two separate moves. In this case, the player forced to move the dollar to the first square is the loser, and the other one, by taking the dollar off the board, is the winner of the game.

1.3.5 Two dimensional Nim

Two dimensional Nim is played on a quarter-infinite chess-like board with a random finite number of counters on its squares (from now on, all quarter infinite boards will be considered as a board equal to the first quadrant of Cartesian coordinate system, and so it will be infinite in the up and right directions, with solid borders at lower and left side). In one move of the game, a player must move any counter from the board either leftwards or downwards. In the normal type of game, the first player unable to make a move is the loser.

Despite the fact that the game may seem more sophisticated because of one additional dimension, this is only basic nim in disguise. The position of every counter can be specified using two numbers as coordinates. These numbers also indicate the hypothetical size of piles in nim, because of inability to change both coordinates of any counter in one move. For instance, moving one counter to the edge of the board is represented by taking all of the counters from one pile in the game of Nim. Position in Figure 1.6 is then equal to Nim position (1,2,2,3,4,5,7,7).

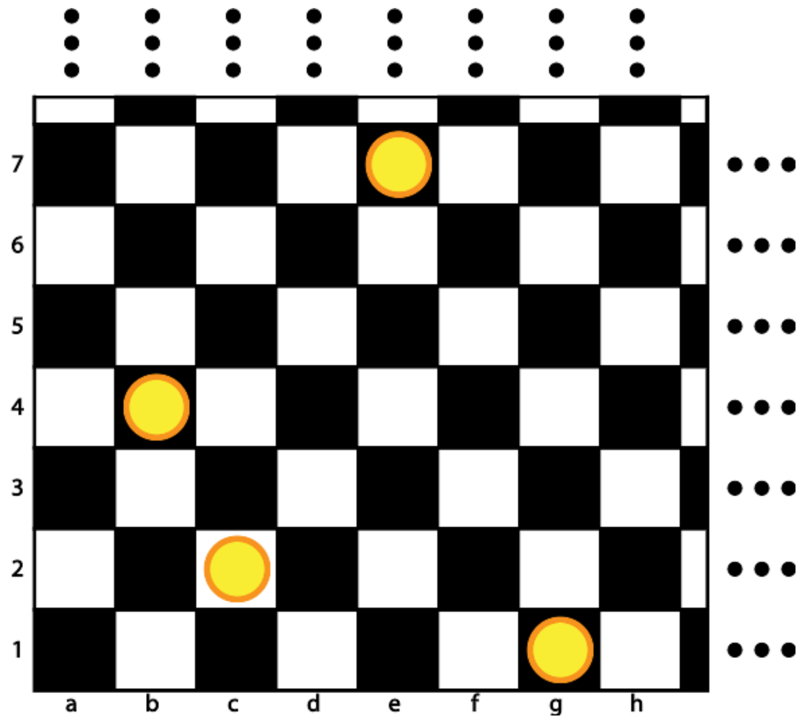

Figure 1.6: 2D Nim game

1.3.6 Wythoff's game

The origin of the game is Chinese, where it is known under the name tsyan-shizi, but it was reinvented by Willem Abraham Wythoff at the beginning of 20^{th} century. The game is played with two piles of counters, and the goal of the game is to remove the last of them, leaving no moves for the opponent. There are two types of moves in this game – each player can either remove the same number of counters from both piles, or any number of counters from one of the piles during their move.

Approximately 50 years later, Rufus Philip Isaacs, independently of Wythoff's game, introduced a game called 'Cornering the queen'. It is played on a quarter-infinite chess-like board with one chess queen, which is obliged to move strictly downwards, leftwards, or diagonally in down-left direction in one move. The winner is the player who moves the queen to position A1 (bottom-left square).

Both of these games, presented by Wythoff and Isaacs, are the same in the mathematical sense, so it is possible to say that one of them is a disguise of the other. An example of this fact can be seen in Figure 1.7, representing the same state of the game in both its forms. Two piles of Wythoff's game can be interpreted as two coordinates of the queen in Isaacs form. Diagonal move of the queen is then equal to reducing the same number of counters from both piles, and horizontal or vertical move may be similarly represented by reducing the number of counters of one pile (specifically from the pile representing the direction of the move).

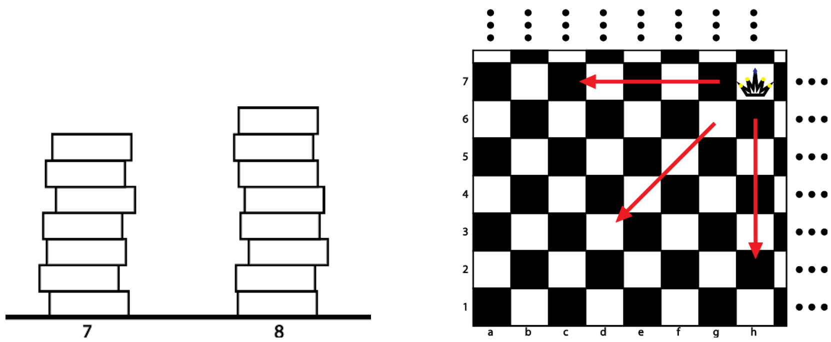

Figure 1.7: Wythoff's game

1.3.7 Green Hackenbush

This impartial form of Hackenbush is played by two players by deleting branches of graphs. The starting position of the game consists of a finite number of graphs. Every edge of this graph must be either directly or transitively connected to the ground. The game can also be played with borders around the whole playground. The rules for deleting these branches are:

- player must delete one branch of his choice in his move
- after each move, all branches that are no longer transitively connected to a border are deleted as well
- the first player who has no branches left to delete is the loser

It will be shown further in the text that some simple Green Hackenbush games may be easily represented by Nim piles, and any state of Nim can be trivially transformed into Green Hackenbush position.

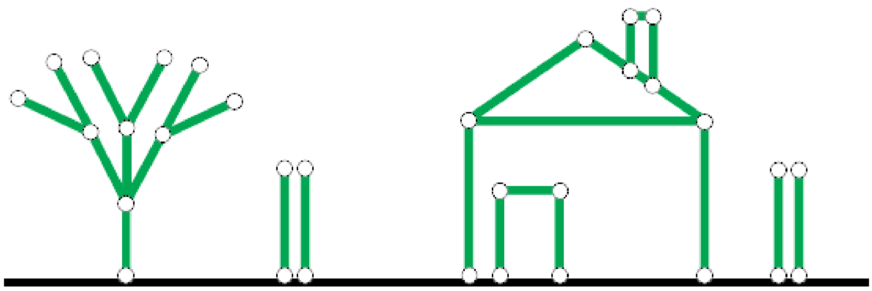

Figure 1.8: Example of Green Hackenbush game

1.4 Examples of partisan games

1.4.1 Blue-Red Hackenbush

Another branch-deleting game similar to Green Hackenbush with the difference that graphs of the game are coloured with two colours (red and blue), each of which represents moves available to one of the players. Rules of the game are as follows:

- player must delete one branch of his colour during his move
- after each move, all the branches that are no longer transitively connected to a border are deleted as well (every branch can be connected using branches of both colours)
- the first player unable to move is the loser

1.4.2 Blue-Red-Green Hackenbush

This game is a combination of Blue-Red and Green Hackenbush, using all three colours of branches and adapting rules from these two games. Both Blue-Red and Green Hackenbush may be just considered as special cases of Blue-Red-Green Hackenbush, one with exclusively common moves, and another with strictly defined moves for each player. Rules of the game are then simply modified from the previous two Hackenbush games:

- player must delete any branch of his colour or any green branch during his move
- after each move, all branches that are no longer transitively connected to the border are deleted (every branch can be connected using branches of any colour)
- the first player with no branches of his or green colour left is the loser

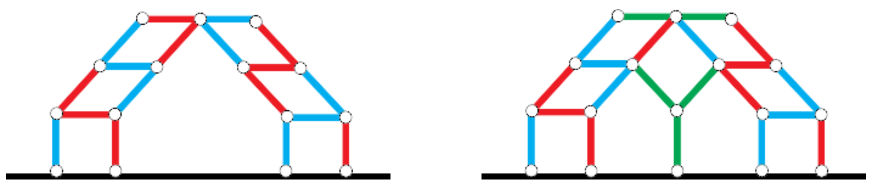

Figure 1.9: Examples of Hackenbush games with and without green branches

1.4.3 COL

The first of the two colouring games that are mentioned in this work has been presented by Colin Vout. It is played by two players, left (blue) and right (red) on a uncoloured map (map is understood as picture of white areas boarded with black lines). The task of each player is to colour the map in such a way that no adjacent areas will be shadowed with the same colour. The winner of the game is the last player able to colour any area of the map.

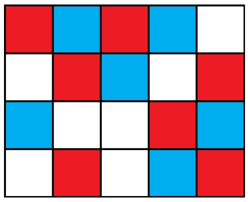

Figure 1.10: example of COL

1.4.4 SNORT

The second colouring game is attributed to Simon Norton. Players in these games represent imaginary franklins, where the left player cherishes bulls and the right one cherishes cows. In one turn, player selects one area to keep his kind of cattle in with a condition that no cows may be kept next to bulls and vice-versa.

This is represented by a map coloured by two players with different colours, and a condition that no adjacent areas can be coloured with a different colour.

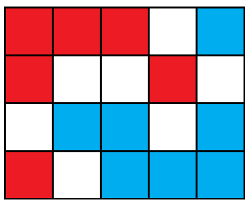

Figure 1.11: example of SNORT

1.4.5 Domineering

This game is played on a finite squared board of any shape with standard pieces of domino, or in other words with rectangular counters with a size of two squares of the board. Two players, left and right, exchange their moves. During the moves they must put one counter on any spare place on the board.

However, there is a restriction about putting dominoes on the board – the left player can only put them horizontally and the right one can only put them in vertical direction (otherwise it would be an impartial game). The goal is to force the game into a position in which there is no room for dominoes in the opponent's direction, which means defeating him in the normal form of the game.

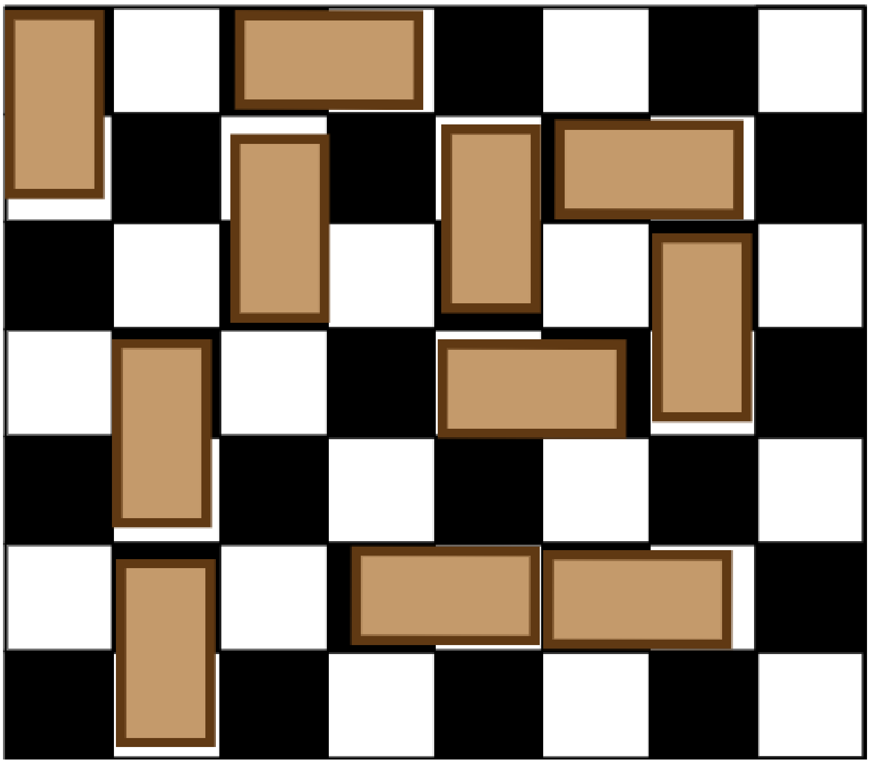

Figure 1.12: Domineering

2 Analysis of impartial games

2.1 Analysis of One pile game

Let's have a closer look at One pile game. At the beginning of the game, it is hard to tell what is the best choice from all the possible moves to go for a win. It is harder to find any formula if we consider the option of any number of counters, and so it would be better to analyse the game from a more stable point of the game. The only one that exists in every game is the terminal position.

At this point, we know that the player who is taking his move while there are no counters left is the loser. On the other hand, positions with one, two or three counters left give him the opportunity to take all of them at once, leaving no counters for the opponent. What about four counters? From a position with four counters (we will call it position 4 from now on) the possible moves are to positions 1, 2 or 3 by taking 3, 2 or 1 counter. But we have just shown that all of these options mean victory for the next player, so it seems that from position 4 there are no moves that would lead the player to a move to win the game (if it is played with a wise opponent). Position five leaves us three possibilities to play, and so positions 4, 3 and 2.

Now it starts to get a little confusing, so we will mark positions according to their meaning for the player taking the move. These will be N- and P-positions[4], meaning N for a position where the Next (actual) player to go can win by playing completely rationally, and P for a state of the game where the previous player had a move to win, leaving us with a move with no winning option. We have so far analysed positions 0 and 4 as P-positions, and positions 1, 2 and 3 as N-positions. We can also notice that there are positions from which the player can move to a P-position, which means a position when he wins and the other one loses. This fact makes them N-positions. So generally:

- Position from which it is possible to move to P-position is an N-position.
- Position from which there is no move to a P-position (therefore all possible moves are to N-positions), or there is no possible

move left, is a P-position.

Those two rules gives us a better view on position 5, which is clearly an N-position, because there is a move to a P-position, namely position 4. It will be the same for positions 6 and 7 (both of them can reach position 4, so they are all N-positions). From position 8 there are three possible moves: to positions 7, 6 and 5. But we have just stated that all of those are N-positions, and so position 8 is P-position.

We can easily see that every position of n counters where $n = 4k$, is a P-position, and all the others ($n = 4k + 1$, $n = 4k + 2$, $n = 4k + 3$) are N-positions. The logical proof of this is the fact that from each move indivisible by 4 there is a move to one that is divisible by 4 (taking one counter in position $n = 4k + 1$, two counters in position $n = 4k + 2$, and three of them in position $n = 4k + 3$). The game becomes boring when both players notice the fact that it is clear from the beginning who is going to win the game, and that after the first move they just alternate moves where one of them takes any number of counters, and the other responses by taking the residue to 4.

At this moment, it is clear that the normal form of the game with 21 counters is won by the first player, whose move will be taking one counter, leaving 20 for the opponent.

Analysis of misère form of this game is unusually simple. It is won by forcing the opponent to leave no counters left to play, which means leaving him just one counter in the last move instead of none. Therefore all P-positions will be $n = 4k + 1$, N-positions $n = 4k$, $n = 4k + 2$ and $n = 4k + 3$, and misère form of game with 21 counters will always mean loss for the starting player, since 21 is a P-position.

2.2 Analysis of Nim game

2.2.1 N,P-positions in the game of Nim

Similarly to the One pile game, finding the best moves for any game must be done by analysis from the terminal position. That is, in normal form, a position with no counters left. This position can be and is achieved only from a position with only one pile with any number of counters, and so a position with one pile is always an N-position. Game with two piles is not much more complicated. As long as they are of the same size, the first player loses the game due to the second player repeating every move, but a game with two piles with different sizes is won by the first player by reducing the greater one to the size of the smaller one. The question is how the game evolves while having more than just two piles.

Let us consider a game with three piles. The simplest one is game (1,1,1), and it is naturally won by the first player (so it is an N-position). There are three possible moves from position (1,1,2). These are positions (1,1), (1,2) or (1,1,1). Position (1,2) is an N-position, as well as position (1,1,1). On the other hand, position (1,1) is a P-position, because it is a game of two piles of the same size. Because of the existence of a move to the P-position we can proclaim that position (1,1,2) is an N-position. All possible moves from Nim position (1,2,2) are (1,1,2) (N), (1,2) (N), (2,2) (P), so it is an N-position as well. We will end the analysis with position (1,2,3). Possible moves from this state are (1,2) (N), (1,1,2) (N), (1,2,2) (N), (1,1,3) (N), (1,3) (N) and (2,3) (N), and therefore position (1,2,3) is a P-position.

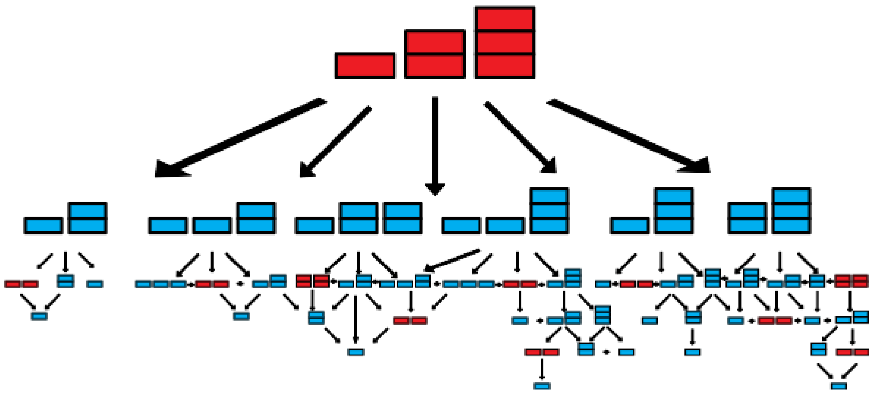

Figure 2.1: Tree of all the consequent positions after Nim position (1, 2, 3)

Transitions between all possible moves from Nim position (1,2,3) are described in Figure 2.1. Colours on this graph are representing positions as they were described before, so that red colour describes P-positions and N-positions are blue. We can also see validity of both rules there. All the red positions have only blue children, although blue parents always have every time at least one red child, or it is possible to reduce them to position 0 (which is also a P-position).

2.2.2 Nim value

All the P-positions of the Nim game can be derived from their already evaluated predecessors. However, it would be impractical while analysing games with more piles of greater sizes. Therefore a more sophisticated way of evaluating any position of the Nim game has been discovered.

One of these ways has been presented by Roland P. Sprague and Patrick M. Grundy. It uses binary code to describe every state of nim, transferring the amount of counters of every pile from decimal number system to binary. For example, pile of $(13)_{10}$ counters in decimal system would be represented as $(1101)_2$ in binary, because $1 * 10^1 + 3 * 10^0 = 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$. Consequently, every position is described by a set of binary numbers, with size of this set equal to the number of piles of the position. For instance, position $(1, 3, 9)_{10}$ would be evaluated as this set: $((1)_2, (11)_2, (1001)_2)$.

However, the value of a position is not the set itself, but sum of the piles (components of the set). Definition of addition of two piles is as follows:

$(x_n\, x_{n-1} ... x_2\, x_1\, x_0) \oplus (y_n\, y_{n-1} ... y_2\, y_1\, y_0) = (z_n\, z_{n-1} ... z_2\, z_1\, z_0)$, where every $z = (x + y)\ mod\ 2$.

Formula for calculating the value of a game of two piles can then easily be written as

$$\sum_{i=0}^{n}(((x_i + y_i)\ mod\ 2\ ) * 2^i).$$

The rule is valid for any number of piles, and the definition of the value of any position consisting of m piles with none of piles greater than 2^n is finally altered to the most general formula. Value of any position of the game is calculated in binary code as follows:

$$\sum_{i=0}^{n}(\ (\ (\ \sum_{j=0}^{m} a_{ij})\ mod\ 2\ ) * 2^i)$$

As the last step, this value can be converted back to the decimal number system for better interpretation of the result. Every position with Nim sum of its positions equal to zero is a P-position, and all positions with values other than zero are N-positions.

Let us consider our game of three piles $((1)_{10}, (2)_{10}, (3)_{10})$. It can easily be transformed to binary notation as $((1)_2, (10)_2, (11)_2)$. Sum of these three piles is consequently evaluated as $*1 + *2 + *3 = *0$:

$$\begin{array}{r} 1 \\ +\ 10 \\ +\ 11 \\ \hline 000 \end{array}$$

We can see that the result of Nim-sum is zero, which means it is a P-position. That is exactly the same result that we have achieved using the three described rules to identify N- and P-positions.

Functionality of these two methods may so far seem similar. However, analysis by Nim values has great impact while playing the game itself, because it provides a simple method of finding the best move to make from any position. While N- and P-positions just show the character of the current position with no method of specifying how to select the best possible move, using Nim values can easily specify the best option to move.

For example, let us imagine a Nim game with state (25,21,10), which is an example of a game with Nim value different to zero ($*25 + *21 + *10 = *6$). We can write the addition of these three positions in binary number system for better figuration as follows:

$$\begin{array}{r} 11001 \\ +10101 \\ +\ 1010 \\ \hline 00110 \end{array}$$

Nim value of this position truly is $(110)_2 = (6)_{10}$, which means this position is an N-position. The best choice for the player on the move should be any move with a Nim-sum of zero, which can easily be done while looking at our notation. We can decrease the pile from 21 to 19 counters, which consequently alters the whole formula:

$$\begin{array}{r} 11001 \\ +10011 \\ +\ 1010 \\ \hline 00000 \end{array}$$

This move leaves the opponent with a position with Nim value $*25 + *19 + *10 = *0$, which ensures his loss of the game.

2.3 Sprague-Grundy function

A more general way to analyse an impartial game is by using Sprague-Grundy function (S-G function)[8][6]. We must first use some new notation describing impartial games in general to be later able to define S-G function itself. This new notation will be represented by a pair (X, F), where X stands for all positions of the game, and F is a function defining all possible moves from every position $x{\in}X$. Because the fact that all moves are always done from one position of the game to a set of positions, function F gives us subset $F(x)$ of X to every position $x{\in}X$. In other words, this function assigns all reachable positions to any selected position of the game.

For example, Wythoff's game played on a field with a size of n^2 would look like

$$(\{(1,1), ...(n,n)\},\ F(x,y) = \{(x-k, y-k) \cup (x-l, y) \cup (x, y-m);\\ k, l, m \in \mathbb{Z},\ x-k, x-l, y-k, y-m \geq 0\}).$$

For instance, it would assign a subset $F(x) = \{(0,0), (1,1), (2,2), (0,1), (0,2), (1,0), (2,0)\}$ to the position $x = (2,2)$, which is a set consisting of exactly all achievable states from this position in one move.

S-G function of a game described by this pair (X, F) can be consequently defined as follows:

$$g(x) = min\{n \geq 0 : n \neq g(y),\ y \in F(x)\}$$

Function itself may be a little confusing because of recursion used in the definition. However, the algorithm becomes very easy to understand, and even easier to apply on particular games, when it is built up from terminal positions. All terminal positions (tp) have a value of 0, as far as they have no achievable positions, and set $F(x)$ is empty. Value of $g(tp)$ is then assigned as a number equal to the minimum value of set $n = \{0, 1, 2...\}$, which is 0. Considering a position whose only follower is a terminal position, the value of the function becomes $g(x) = 1$, as far as $F(x) = \{tp\}$, $y = tp$, $g(y) = 0$, and $\{n \geq 0 : n \neq g(y), y \in F(x)\}$ alters to set $\{1, 2, 3...\}$. Consequently, positions whose only followers have S-G values 0 and 1are evaluated, continuing this procedure until all positions of the game have evaluated their S-G value. Generally, the procedure of evaluating the whole game by values of Sprague-Grundy function is succeeding:

- mark all terminal positions as 0
- position that already has all achievable positions evaluated carries the smallest integer value that is not used among these positions
- all positions with a value of 0 are P-positions, and all others are N-positions

These three steps imply a big similarity in functionality between analysis by S-G function and analysis by P- and N-positions on this level. Evaluating by N- P-positions marks all states that can achieve P-position as N, while S-G function assigns number different to 0 to all this positions. On the other hand, P-position is defined as one whose only followers are N-positions, which is expressed by S-G value 0 of position with followers of S-G values different to 0.

2.4 Analysis of Wythoff's game

Figure 2.2 describes steps of evaluating individual squares of Wythoff's game from a terminal position, which is placed in lower-left corner of the field. Every position with S-G value 0 (or equally every P-position) is coloured with red colour. Every other value (N-positions) and all yet unanalysed squares are represented by blue square. In every step of analysis one (white) square with already defined S-G values of all its followers (green squares) is chosen .

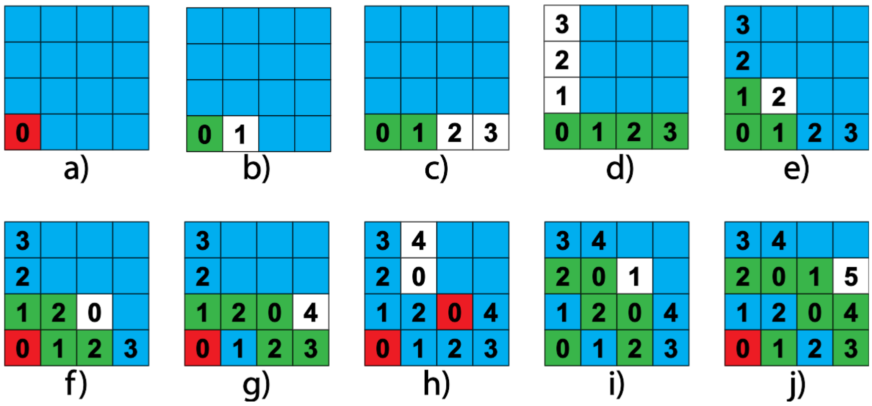

Figure 2.2: Procedure of evaluating Wythoff's game positions by values of Sprague-Grundy function

There is our field with one terminal position given by rules in picture Figure 2.2a. To evaluate white square from Figure 2.2b, we only need to take value of our one terminal position into consideration, as it is the only one achievable position from that position of the game. The smallest non-zero integer is 1, which is our S-G value of that square. Each square of the lower edge only has achievable positions on its left side, so we can evaluate the whole row by increasing values by 1, as it is depicted on Figure 2.2c. The whole field is axis-symmetric from lower left to upper right corner, so we can automatically fill squares on the left edge of the field with same values (Figure 2.2d). The white square from Figure 2.2e has one reachable square in all three directions: vertically, horizontally and diagonally. Values of these three squares are 0, 1 and 1, and so the smallest unused integer is equal to 2, which is equal to S-G function of that position.

Figure 2.2f is the first case when none of the succeeding positions has a value of 0 (values are 1, 1, 2, 2). We can now better explain relation between using analysis by S-G function and evaluating with N- P-positions. All positions with different values than zero are considered N-positions, and so the selected position must be a P-position, as there are only N-positions achievable from that position.

The other three pictures are consequent steps of the same analysis, using symmetry and S-G function. There is a fully filled table with S-G values for Wythoff's game played on field of size 100 squares in picture Figure 2.3.

9	10	11	12	8	7	13	14	15	16
8	6	7	10	1	2	5	3	4	15
7	8	6	9	0	1	4	5	3	14
6	7	8	1	9	10	3	4	5	13
5	3	4	0	6	8	10	1	2	7
4	5	3	2	7	6	9	0	1	8
3	4	5	6	2	0	1	9	10	12
2	0	1	5	3	4	8	6	7	11
1	2	0	4	5	3	7	8	6	10
0	1	2	3	4	5	6	7	8	9

Figure 2.3: Fully filled table of Sprague-Grundy function values for Wythoff's game positions

2.5 Addition of impartial games

Let us consider an impartial game consisting of two simultaneously played impartial sub-games. Example of such a game can be one consisting of standard One pile game with a pile of 21 counters, and Nim game with starting position (1,2,3), as it is displayed in Figure 2.4. The game is played with the standard rule of exchanging turns, but a player is only allowed to make his move in only one of these two sub-games. The first player who is unable to move in any of the two sub-games is considered the loser. Game with rules as they are described above is considered as sum of its consisting sub-games, and its values are described by addition of S-G functions of its sub-games.

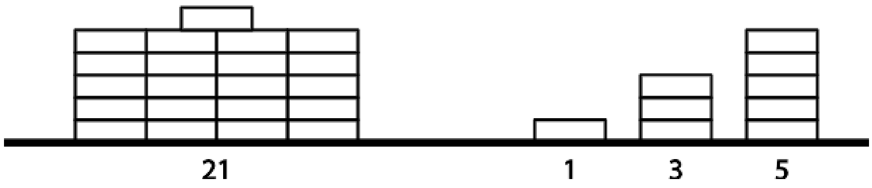

Figure 2.4: Game consisting of One pile game and Nim game

We can mark our game as $G = (X, F)$, and two sub-games as $G_1 = (X_1, F_1)$ and $G_2 = (X_2, F_2)$, where $G = G_1 + G_2$. Set of all possible states of this game is Cartesian product $X = X_1 \times X_2$, because every state $x \in G$ consists of couple (x_1, x_2), where $(x_1 \in G_1, x_2 \in G_2)$. In other words, every position of the game consists of any two positions, one from each sub-game. Set of possible moves after any position x is $F(x) = F_1(x_1) \times \{x_2\} \cup \{x_1\} \times F_2(x_2)$. In layman's terms, this means that in one move a player is able to change the state of only one game, leaving the second one untouched.

In Figure 2.5, there is a table filled with S-G values of this game. Now it can be easily seen that the number of states of this game is a Cartesian product of states of both sub-games, as we have created a game with exactly 388 states from two sub-games with a relatively little number of options to move ($X_{nim} = 14$, $X_{opg} = 22$, $X_{nim} \times X_{opg} = 388$). It is also easy to see that each player can move in only one direction, either vertical or horizontal. These two directions represents altering the state of one game, and leaving the other one ontouched.

As for the S-G values themselves, every position is evaluated as the smallest number that is not the value of any of the succeeding states, although we need to consider both directions (both games). The first row of the table represents S-G values of positions of Nim itself, because there is no move available in One pile sub-game. This is similar for the first column of the table (we can notice that those S-G values are complying with our earlier analysis of One pile game).

For instance, the state of the game (green square in the table) with 13 counters left in the One pile sub-game, and Nim position (1,1,3) has 7 followers. These are, while making move in One pile game, positions ((12 – 1,1,3), (11 – 1,1,3), (10 – 1,1,3)), and ((13 – 1,3), (13 – 1,1,2), (13 – 1,1,1), (13 – 1,1)) while altering Nim part of the game. These options are displayed in the table by white arrows, and we can see that the S-G values of these positions are: 3,0,1,3,3,0,1, so our green square carries an S-G value of 2.

This rule can be intuitively extended to a game consisting of any number of sub-games[4] $G = (X, F) = G_1 + ... + G_n = (X_1, F_1) + ... + (X_n, F_n)$, where $X = X_1 \times ... \times X_n$ and

$$\begin{aligned} F(x) = F_1(x_1) &\times \{x_2\} \times ... \times \{x_n\} \\ &\cup \{x_1\} \times F_2(x_2) \times ... \times \{x_n\} \\ &\cup ... \\ &\cup \{x_1\} \times \{x_2\} \times ... \times F_n(x_n). \end{aligned}$$

We may also notice that the position of 13 counters in the One pile game has a value of 1, and position (1,1,3) carries a value of 3. It is not a coincidence that the Nim addition of these two positions is equal to the S-G value of its addition. As a matter of fact, for every addition of impartial games, a rule applies that the value of a game created as a sum of its sub-games is equal to the Nim-sum of values of these sub-games. This is written as

$$sgf(G_1 + ... + G_n) = sgf(G_1) \oplus ... \oplus sgf(G_n),$$

where sgf stands for S-G function and $\oplus$ represents Nim addition.

We can see that this rule has already been used and partially explained in section 2.2.2. We may think of the Game of Nim as a game consisting of a number of sub-games represented by single piles. Each pile carries a Nim value of its own size, and despite the simplicity of these sub-games, the analysis of the resulting Game of Nim requires a more sofisticated approach. This approach is the Nim addition of its sub-games. This also implies a general characteristic of assembling games into one, which is the fact that the addition of trivial games can create a much more complicated game.

Nim (1,2,3)

One pile game (21)

0	0	1	0	2	1	3	3	2	0	2	3	1	1	0
1	1	0	1	3	0	2	2	3	1	3	2	0	0	1
2	2	3	2	0	3	1	1	0	2	0	1	3	3	2
3	3	2	3	1	2	0	0	1	3	1	0	2	2	3
4	0	1	0	2	1	3	3	2	0	2	3	1	1	0
5	1	0	1	3	0	2	2	3	1	3	2	0	0	1
6	2	3	2	0	3	1	1	0	2	0	1	3	3	2
7	3	2	3	1	2	0	0	1	3	1	0	2	2	3
8	0	1	0	2	1	3	3	2	0	2	3	1	1	0
9	1	0	1	3	0	2	2	3	1	3	2	0	0	1
10	2	3	2	0	3	1	1	0	2	0	1	3	3	2
11	3	2	3	1	2	0	0	1	3	1	0	2	2	3
12	0	1	0	2	1	3	3	2	0	2	3	1	1	0
13	1	0	1	3	0	2	2	3	1	3	2	0	0	1
14	2	3	2	0	3	1	1	0	2	0	1	3	3	2
15	3	2	3	1	2	0	0	1	3	1	0	2	2	3
16	0	1	0	2	1	3	3	2	0	2	3	1	1	0
17	1	0	1	3	0	2	2	3	1	3	2	0	0	1
18	2	3	2	0	3	1	1	0	2	0	1	3	3	2
19	3	2	3	1	2	0	0	1	3	1	0	2	2	3
20	0	1	0	2	1	3	3	2	0	2	3	1	1	0
21	1	0	1	3	0	2	2	3	1	3	2	0	0	1

Figure 2.5: Table of S-G values for a game consisting of two sub-games

2.6 Analysis of Green Hackenbush

2.6.1 Bamboo stalks

Three basic levels of analysis of Green Hackenbush games may be recognised. The first of these levels describes games consisting of stalks with no vertices with multiple branches and no cycles. We can generally call these games Bamboo stalks[1], and they can easily be converted to a typical Nim game by replacing every edge with a counter. In other words, every simple stalk carries a Nim-value equal to the amount of its edges as it is displayed in Figure 2.6. Any position may be evaluated as a typical game of Nim, and consequently the best movement to achieve victory can be found by altering one of the stalks and leaving game zero with Nim value.

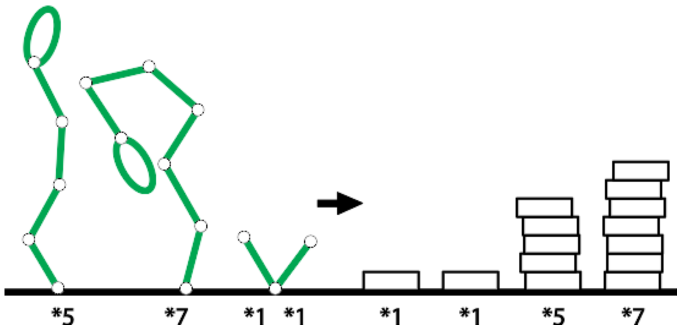

Figure 2.6: Converting an easy Green Hackenbush game to the Nim game

2.6.2 Colon principle

The second level of analysis involves a look at trees instead of simple stalks, but still with no cycles allowed. To be precise, we can claim that any game that has at least one vertex which is connecting more than two edges, is considered a forest of trees instead of bamboo stalks. We must firstly be able to simplify and evaluate the value of S-G function of one simple tree to analyse any forest later. This evaluation is done by a rule named the Colon principle[1]. The rule says that every vertex that is connecting only a certain amount of simple stalks to the rest of the tree can be considered a separate sub-game of Bamboo stalks. This whole vertex with all its stalks may consequently be replaced by another stalk with a size equal to the Nim value of its sub-game of Bamboo stalks. Figure 2.7 describes an example of such simplification. According to the Colon principle, S-G values of these four trees (or three trees and one Bamboo stalk) are equal, and it can be seen that they are also equal to the tree from the last step, as it is a simple Bamboo stalk. In the picture, we can also see partial Nim-sums of red vertices, as these are the vertices that only have Bamboo stalks above them. In the next step, these vertices are coloured blue, and new vertices with this property are coloured red.

Every single tree can be simplified into one Bamboo stalk, and we have already shown that a game consisting of nothing but Bamboo stalks is just a Nim game in disguise. This means that a forest of trees (and Bamboo stalks) is, after evaluating each tree using the

Colon principle, equal to a game of Nim at some point. The winning position can consequently be found exactly the same way it is done in Nim – by Nim addition equal to zero. Achieving this position from any N-position (position with S-G value not equal to zero) is done by deleting adequate branches of one tree, and by doing so, changing its Nim value. It is also important to notice that the Nim value of a tree can generally be both reduced and enlarged in one valid move.

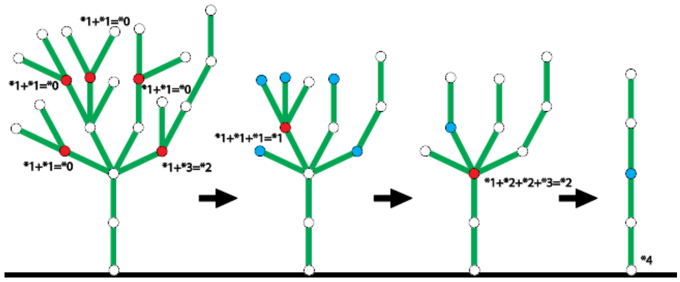

Figure 2.7: Simplifying a Green Hackenbush tree using the Colon principle

2.6.3 Fusion principle

The last level of analysis concerns Green hackenbush games with cycles. These are solved by using a method called the Fusion principle[1]. This principle says that all vertices from any cycle can be fused into one vertex without changing the original value of the game. All edges of the game must be preserved after this alteration, so if there is an edge connecting two fusing vertices, it changes to a simple cycle with only this one vertex. All the edges that are connected to a fusing vertex on one end and to an unaltered vertex on the other will also remain, and they will all connect our new fused vertex to the rest of the graph. These simple rules are described in an example in Figure 2.8. The first step of fusing this house consists of fusing the vertices of "chimney" and "door". New cycles can be seen, as we preserved all edges connecting fused vertices. Second step collapses the "roof" into one vertex (including our new vertex that represents the chimney). The last alteration is more visual than functional, because the only difference is the change of all simple cycles with only one vertex to simple stalks.

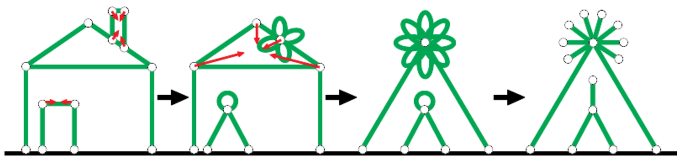

Figure 2.8: Sizing down Green Hackenbush house using Fusion principle

It is still not trivial to evaluate this game by S-G value from the last alteration we have so far. The last obstacle in our way are multiple edges connecting the graph to the ground. How would fusing all vertices connected to the ground into one vertex change the value of the game? A valid move consists of deleting one edge, and nodes are deleted only if there is no edge connected to them anymore. This means that fusing all ground nodes does not change the value of the game at all, and fusing them into one node is just another visual alteration. In Figure 2.9, this is done as the first step to the state with newly created cycles. These are again eliminated using the Fusion principle. The last step shows the simplification of four cycles with only one vertex into four stalks. The resulting position is one vertex with 13 simple nodes attached to it, so the Nim value of the whole house is equal to $13 \times *1 = *1$.

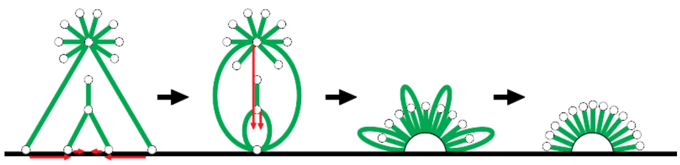

Figure 2.9: Fusing ground vertices and completing evaluating of house

3 Analysis of partisan games

So far, we have analysed games according to all possible moves for the current player from a given position, and we did not care whose turn it wos. Roles of both players are strictly specific in partisan games, and so we have to add this point of view into consideration during analysis. From now on every player will carry two roles, and so whether he is first or second, and if he is bLue or Red (respectively Left or Right).

These two roles of both players give us 4 basic possible states of every game (instead of 2 positions from impartial games, N and P), which are expressed by a number (X). If this number X is positive, the game is won by the Red (Right) player, and if it is negative, the game is won by the bLue (Left) player. There are two kinds of draw games. The game with a winning outcome for the second player is a position with a value of zero (X = 0), and the game which means victory for the player taking the turn is called a fuzzy game. We will mark fuzzy position as " * ", or use notation $X \parallel 0$.

We also need to define two sets of numbers to evaluate the exact value of the game, as we want to find out the size of advantage of the winning player, and not just whether is the game is positive or negative (impact of this is shown further in this section). These sets are R for all possible state values of the game after right's move, and L for all values after left's move. This means that the value of each position is specified recursively, using values of all states that are achievable by both players from that exact position of the game. Value of any game X is then always greater than all values of L, but lower than the whole set R, described by formula $X = \{L|R\}$. This does not, however, define the exact value of the game, so for now we will call X the simplest number in this bounds, explaining our peculiar meaning of simplicity later.

3.1 Analysis of Blue-Red Hackenbush

The main condition of every recursion is a defined starting point, so we need to specify some basic games in Hackenbush and their values to be able to analyse more complex games. The simplest position we could obtain is a game with only one edge. Considering our only edge being blue, there is only one move for left player. This move is deleting this edge, and leaving game with no edges left, which is naturally a zero game ($L = \{0\}$). However, there are no moves there in case that the right player is starting the game ($R = \{\}$), so we need to evaluate number $\{0|\}$ to find out the value of the game. This means finding the simplest number greater than 0, which is 1. Value of the same game with one red edge instead of blue is then equal to $\{0\} = -1$.

We can see that position Figure 3.1a has one move for the left player and one for the right payer, so it is obvious that this game is a draw in some sense, and we only need to decide whether it is fuzzy or zero, or in other words, whether it is won by the first or the second player to go. Player on the move must delete his only edge, leaving only one move to the opponent as well, and ending up with no move to play. If this player is the right one, the game after his one obligatory move consists of only one blue edge. We have already deduced the value of this position as 1, which means that $R = \{1\}$. It is the same for the left player with one red edge of total value $L = \{-1\}$, so we can say that the value of the game is $X = \{-1|1\} = 0$. We can as well proclaim that this game will always be won by the second player, no matter who starts the game.

Considering a game with two piles of n edges described by Figure 3.1b, one of them red and the other one blue, the value of the game will still be $\{-1|1\} = 0$. This is also valid for any two identical graphs with opposite colours of related edges (Figure 3.1c) (both games can always be played with the second player copying the moves of the first one). The most important thing about this example is the fact that one edge connected to the ground directly or only using edges of the same colour is always considered as an advantage of one move (specifically one blue edge carries a value of 1 and one red edge carries a value of-1). These three examples imply a rule that the value of a game can be calculated as a sum of its partial games. Proof of this declaration is the simple fact that every player is able to change only one of all separated graphs of any game, and so it can be considered as a simultaneous game of more games. The outcome of this game is then the sum of outcomes of all these games. This rationale is similar to the one that described an addition of S-G functions of two impartial games.

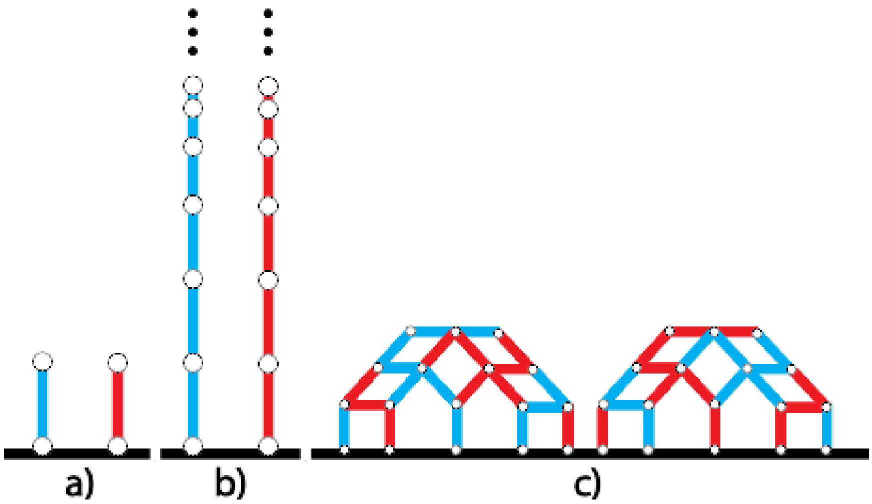

Figure 3.1: Basic Blue-Red Hackenbush games

3.1.1 Fractions and additions

Let us have a look at some of the more irregular graphs, such as these in Figure 3.2. Graph in Figure 3.2a can be clearly be won only by the blue player, because he has the only edge connected to the ground. The value of the game can quickly be calculated as $\{0|1\} = \frac{1}{2}$, which means an advantage of half a move for the blue player.

In Figure 3.2b, there is a position with a value of $-\frac{1}{2}$. We can see this either by searching through all the possible moves of both players as $\{-1|0, \frac{1}{2}\}$, or using the addition of these two stalks with values of -1 and $\frac{1}{2}$. The meaning of advantages other than a whole move may be firstly hard to imagine, although it should be clear in mathematical language, so we will prove and explain this using addition of games described in Figure 3.2c.

Using the theory we have just developed, the value of the game should be equal to $\frac{1}{2} + \frac{1}{2} - 1 = 0$. If the blue player started the game, he would certainly leave the position in Figure 3.2b by choosing any of his two possible moves, which is game with value $-\frac{1}{2}$. The red player would either change the value of one of two stalks of value

$\frac{1}{2}$, or state consisting of zero game and $\frac{1}{2}$ move, which is equal to $\frac{1}{2}$. We can clearly state that this position is a zero position, as the $\{-\frac{1}{2}|\frac{1}{2}, 1\} = 0$.

Using fractions to describe various positions enlarges our skill in analysing games from games with just one-coloured stalks (or games consisting of two opposite-coloured graphs) to stalks with any colour at each position. We can also use the addition of sub-games to analyse any number of these stalks in one game. Position in Figure 3.2d is clearly $\{0|\frac{1}{2}, 1\} = \frac{1}{4}$. Every other red edge on top of this graph decreases left's advantage of value $\frac{1}{2^{n-1}}$, where n resents the level of the edge, respectively n-1 is the number of edges under that edge. Therefore, the value of Figure 3.2e is $\frac{1}{8}$, and we can still check this by evaluating $\{0|\frac{1}{4}, \frac{1}{2}, 1\}$. This rule also works with colours of the graph changed, although with exactly opposite values, and so Figure 3.2f is a game with value -$\frac{1}{4}$.

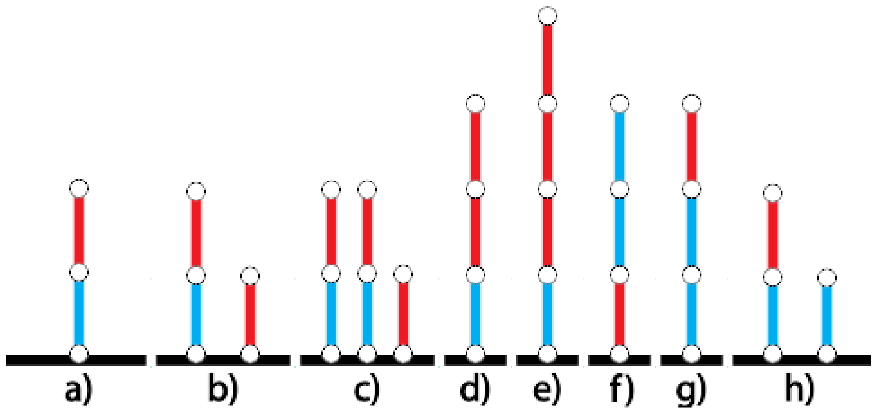

Figure 3.2: Different positions of Hackenbush with values containing fractions

3.1.2 Thea van Roode's rule

Position Figure 3.2g is a little different to previous stalks, but we will show that the idea of solving this position faster is similar to the previous ones. Firstly, we can evaluate this position with result $0, 1|2 = \frac{3}{2}$. What happens if we try to evaluate each edge as we did with all edges of one colour in the previous example? Two blue edges on the bottom of the graph would carry values $\frac{1}{2^{1-1}}$ = 1 and $\frac{1}{2^{2-1}} = \frac{1}{2}$, and the red one on the top of the stalk would have value $-\frac{1}{2^{3-1}} = -\frac{1}{4}$. The sum of these numbers would give us a result of $\frac{5}{4}$, which would obviously be different to the correct value $\frac{3}{2}$. Apparently we have done some mistake in evaluating. We evaluated the second edge with a value smaller than the lowest one, but if we had one stalk of two blue edges, it would have had a value of 2, not $\frac{3}{2}$! As long as the left player is the only one who is able to remove it, our position should be the same as position Figure 3.2h, which is equal to $\frac{3}{2}$.

All of this is described by rule presented by Thea van Roode[1], defying the value of any Blue-Red Hackenbush stalk. The rule says that every stalk can be evaluated as follows:

- all blue edges will have positive values and all red edges will have negative values
- all edges from the floor to the first colour change have an absolute value of 1
- all edges after the first one-coloured row have half the value of the previous edge with the sign changing according to the colour
- the value of the whole stalk is the sum of the values of all edges

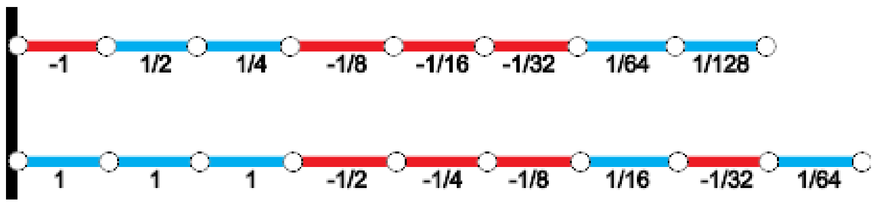

Figure 3.3: The Thea van Roode's rule

The resultant value after summation of all values from the upper stalk in Figure 3.3 is -$\frac{57}{128}$, and the lower one is equal to $\frac{139}{64}$. While considering these two stalks as one game, the value of the position would simply be the sum of these two fractions, which is $\frac{221}{128}$.

3.1.3 Hammerian trees

The methods of evaluating stalks of Blue-Red Hackenbush have already been presented. We have also seen the Colon principle for simplifying Green Hackenbush trees (see 2.6.2 Colon principle). The rule that describes the reduction and evaluation of Blue-Red Hackenbush Hammerian trees[7] is not too different from the one presented by Colon and Green Hackenbush. However, as the name of the section implies, the rule presented in this section can only be used to decompose Hammerian trees, not all Blue-Red Hackenbush trees. A Hammerian tree is described as a tree with a positive or zero value of its branches while it has a blue trunk, and a negative or zero value in case of a red trunk. The rule itself says that every Hammerian tree can be decomposed into a trunk and its branches. Each of these branches becomes a new tree that can repeatedly be decomposed until a game consisting only of simple stalks is achieved. The value of this tree is the sum of the values of the trunk and its branches. Figure 3.4 depicts the evaluation of one tree. We can see that the condition for a Hammerian tree is satisfied as the total value of the tree without the trunk is $\frac{7}{8} \geq 0$.Restricition is also valid for the cirled sub-tree, and its value without the trunk is zero. The resulting value of the whole tree is finally calculated as $\frac{15}{8}$.

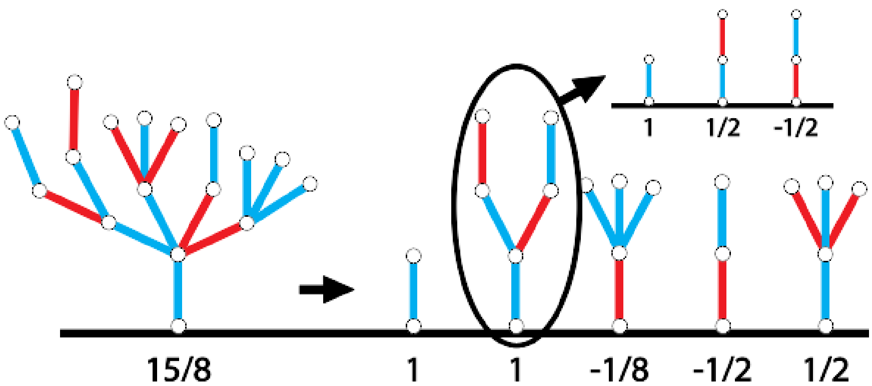

Figure 3.4: Evaluating of the Hammerian tree

3.2 Analysis of Blue-Red-Green Hackenbush

Since we have shown that Blue-Red Hackenbush is just a special case of Blue-Red-Green Hackenbush, using also green edges should not have any greater impact on the comprehensibility of this section. On the other hand, it should provide us with more flexibility by allowing moves possible for both players. For example, we are not able to create a fuzzy game in Hackenbush without using green edges. This is different while playing Blue-Red-Green Hackenbush, and we can easily create a fuzzy game as a game with only one green edge (Figure 3.5d).

It is not much complicated to deduce a very basics of analysis of the Blue-Red-Green Hackenbush after describing analysis of both Blue-Red and Green Hackenbushes. However, full analysis of this game is beyond the scope of this thesis.

3.3 Analysis of Domineering

Forasmuch as we have not taken a closer look at the Domineering game yet, we will do so in this section[1]. As we should be used to by now, the easiest way to analyse most of the games is to explore them from their basic states and extend the achieved knowledge to the more complex ones. Four basic states of partial games have been described at the beginning of this chapter. These were positive, negative, zero and fuzzy games. In Figure 3.5 there are seven examples of basic states in domineering. Six of them are displayed with equal Blue-Red-Green Hackenbush positions.

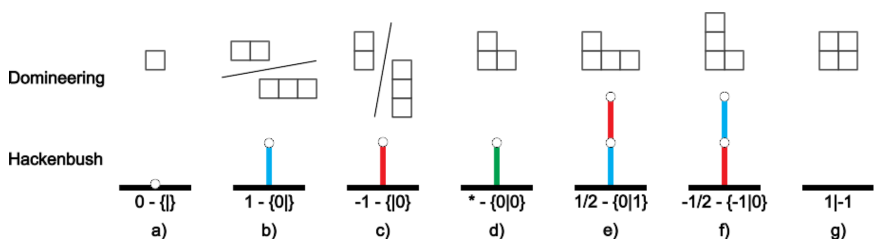

Figure 3.5: States of Domineering and Hackenbush with the same values

At the beginning of this chapter, there was an intuitive explanation of a simple number as a number greater than all values of L and smaller than all values of R. We can see that some positions of Domineering can be evaluated by $\{L|R\}$ according to this description. However, position in Figure 3.5g is different. The left player has two equal options to move, leaving a position with a total value of -1. This is the same for the right player, as he leaves a position with two squares in vertical direction with a value of 1. This position does not fulfill our description of a simple number anymore. Positions $\{x|y\}$ where $x \geq y$ will be called switches from now on.

It is not a problem to determine who is going to win a game that consists of only one switch. In our example, any starting player gains an advantage of one move, and wins the game. We can say that this state is in some way fuzzy because of the winning outcome for the starting player. On the other hand, evaluation of any game with more sub-games (or in other words a game with more separated sub-states) is done as an addition of all the sub-games, as it is described in Figure 3.6. This identifies the winning player without stating the exact value of a state. However, this is insufficient for further analysis and addition of games.

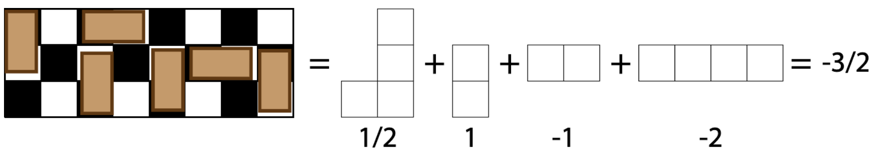

Figure 3.6: Solving a simple Domineering position

Let us consider a game consisting of more already evaluated sub-games and one switch, as it is depicted in Figure 3.7. The sum of all the sub-games is easily calculated as their natural addition. We will mark this value z. The value of the whole game is then the sum of the values of the subgames z and switch $\{x|y\}$. It has already been described that it is an advantage to make the first move in a switch sub-game, because it is fuzzy in a way. Therefore the value of the game after the left player's move is always $x+z$, and after the right player's move it is $y + z$. The addition of two sub-games, one switch and one standard, can consequently be altered as $z + \{x|y\} = \{z + x|z + y\}$.

This equation also suggests that every switch $\{x|y\}$ can be transformed into $\{a+b|a-b\} = a + \{b|-b\} = a \pm b$, where $a = \frac{1}{2}(x+y)$ and $b = \frac{1}{2}(x-y)$.

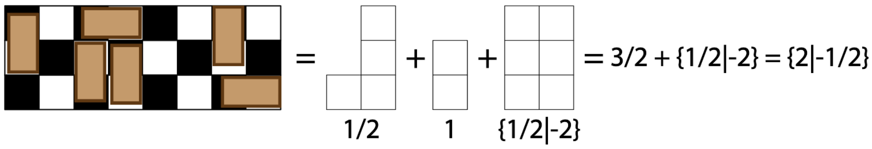

Figure 3.7: Solving a simple Domineering position with value $\frac{3}{4} \pm \frac{5}{4}$

Every game consisting of more switches has a value of $z_1 + ... + z_n + \{x_1|y_1\} + ... + \{x_m|y_m\}$. All sub-games with given values can be calculated into one: $z = \sum_{i=1}^{n} z_i$. The value of the whole game can consequently be written as $z + \{x_1|y_1\} + ... + \{x_m|y_m\} = a \pm b_1 \pm b_2 \pm ... \pm b_m$, where $a = z + \frac{1}{2}\sum_{i=1}^{m}(x_i + y_i)$ and $b_i = \frac{1}{2}(x_i - y_i)$. The resulting value is then expressed as $a + b_1 - b_2 + ...$ in case that the left player starts the game, and $a - b_1 + b_2 - ...$ if the right player is taking the turn. This value can also be written as $a + \{b_1 - b_2 + ...| - b_1 + b_2 - ...\}$.

3.4 Simple numbers

Until now there has been no exact specification of the term simple number[3]. All that we know so far is that the meaning of simplicity, while working with simple numbers, is similar to the instinctive understanding of simplicity as it is known in common language. However, we are not able to tell whether number $\{\frac{7}{9}, \frac{9}{11}\}$ is equal to $\frac{2}{3}$, $\frac{3}{4}$, or a completely different value. A closer look on what the term simple number stands for and how simple numbers are created will be presented in this chapter.

The simplest possible state to be in is $\{|\}$, as it describes a game with no possible moves for any of the two players. We already know that the value of this position is 0, because there is no winning strategy for the first player, and so we have 'created a number out of nothing'. Thereafter we can evaluate the position with one move for one of two players. There are two such positions: $\{0|\} = 1$, because the simplest number bigger than 0 is 1, and similarly $\{|0\} = -1$. These two numbers are the only ancestors of position $\{|\}$, and so we can call them the first generation. At this point, we have created two new numbers -1 and 1, which can be used again to create a new couple of numbers. The two numbers of the second generation are $\{|-1\} = -2$ and $\{1|\} = 2$. The whole set of integers $\mathbb{Z}$ can consequently be described by using this method, where every n^{th} generation consists of two numbers: $-n$ and n. But these are not all numbers that are born in this generation, end we will see further in this chapter that also both rational and real numbers are described using the similar method.

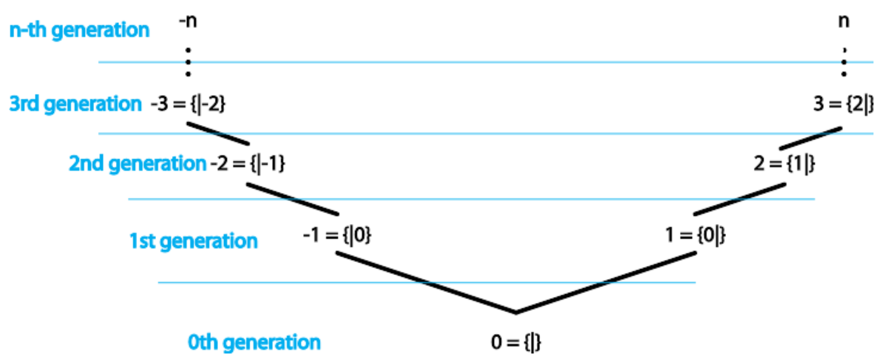

Figure 3.8: Generation tree of integer numbers $\mathbb{Z}$

We have just deduced the whole set of integers using the induction method, but we have also used a fraction while describing the games, so it would no be surprising if we were able to describe rational numbers, too. The first generation of created numbers after $\{|\}$ are two numbers: -1 and 1. We have used these to define new numbers, so every generation has been created by just one previous generation. But if we considered the option of new numbers being created by all generations so far, it would enlarge our set of numbers generation by generation with fractions. For example, the third generation would be created by combinations of -1,0 and 1 with the condition of no re-creating already created numbers. For instance, numbers created in the third generation would then be two integers $\{|-1\} = -2$ and $\{1|\} = 2$, and two fractions $\{-1|0\} = -\frac{1}{2}$, $\{0|1\} = \frac{1}{2}$.

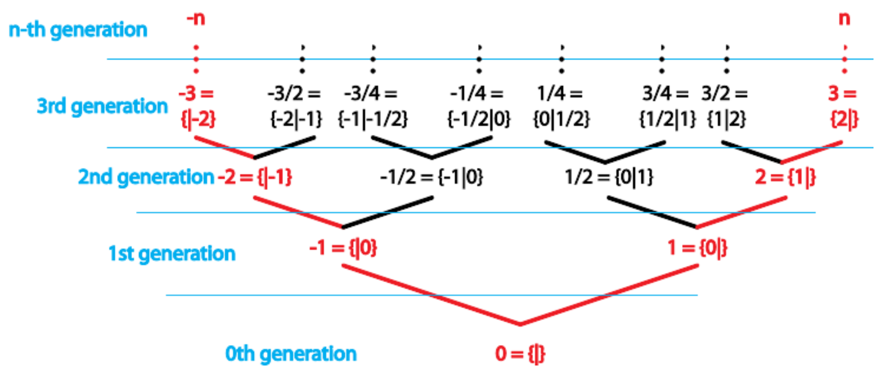

Figure 3.9: Generation tree of simple numbers

Simple number $s = \{L|R\}$ is defined as number greater or equal than every number from set L and greater or equal than every number from set R. In addition, s must always have form $\frac{a}{2^n}$, where $a \epsilon \mathbb{Z}$, variable n has the smallest possible value and variable a is as near to the value of zero as possible. Simple number $\{L|R\}$ can be also understood as first common ancestor of the greatest value of L and the least value from R. This can be easily seen and evaluated by using generation tree described on Figure 3.9.

3.5 Analysis of COL and SNORT

Both games have already been presented as being played on maps with players colouring them with unique colours. To analyse both COL and SNORT[2], we need to get used to another notation, equivalent to colouring maps. Every map, in our understanding, consists of fields divided by borders. This can be understood as a two-dimensional graph, where every field is represented by a node, and every border (which is, in a way, a relation between two fields) by an edge between two fields. An example of conversion from a map to a graph is displayed in Figure 3.10. This notation will provide us with better flexibility and clarity while analysing both of these colouring games.

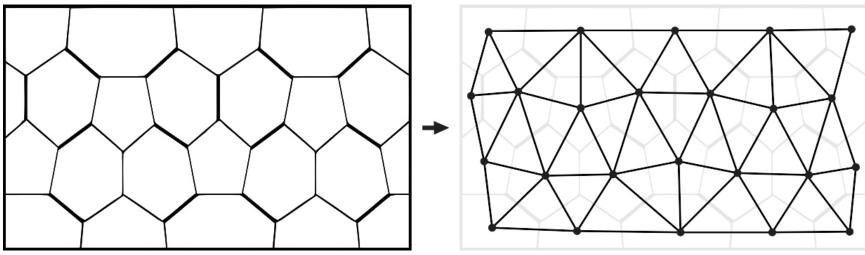

Figure 3.10: Conversion map of COL or SNORT to plain graph

3.5.1 Analysis of COL

At the beginning of the chapter, the principle of a basic transformation from a map to a plain graph has been shown. However, this gives us no advantage until we establish some additional changes. Starting point of COL game consists of a map with uncoloured fields, which means that all these fields are available for both players. This attribute is described by a small black node. We will distinguish three additional types of nodes from now on: a blue node, which can only be coloured by the blue player, a red node which is only available to the red player, and a big black node representing a field that cannot be taken by either player any more.

The nodes have a life cycle during the game: every little black node can change to a red or blue vertex, or disappear immediately when the area that it represents is coloured. Red and blue vertices can either change to a big black vertex, or disappear after being coloured by the respective player. The big black vertices, as they represent the untouchable regions, can vanish immediately after their appearance. We can see that there are two possible deaths for vertices: becoming the big black spot, or vanishing due to the respective area being coloured.

Edges of the graph are destroyed more easily than vertices. An edge must die when one of its two nodes dies, or when it connects two blue or two red vertices. All these rules concerning life cycle of vertices and deleting segments of graphs will be explained in detail in Figure 3.11. Figure 3.11a shows a simple graph of the game after three moves of both players. It is similar to the graph presented at the beginning of this section. However, we can see that we have already deleted all vertices on coloured fields, and also all edges connecting these vertices to the rest of the graph. The first step describes dividing all nodes into our four possible states. All nodes which are exclusively next to blue fields are blue and all exclusively next to red fields are red. The regions adjactent to both red and blue fields become big black nodes. All other vertices remain untouched. Figure 3.11c is the final look of the graph representing our game after deleting all redundant segments. Now we can see the significant simplification that has been achieved simply by using a more suitable notation with a better visualization of all the attributes provided by the game, as we have simplified a game played on a map with 18 uncoloured and 6 coloured fields into two games of 7 vertices.

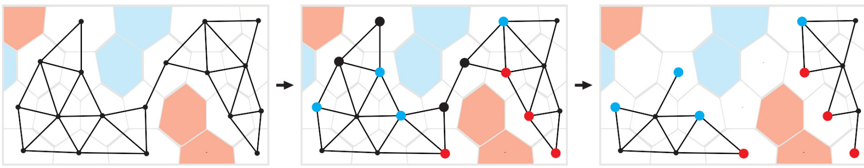

Figure 3.11: Notation of one position of COL by plain graph

3.5.2 Analysis of SNORT

All rules concerning the transformation of a map into a plain graph that have been introduced while analysing COL are also valid for SNORT. However, there is a slight difference in evaluating vertices. All nodes have the same meaning as in COL, therefore all regions neighbouring exclusively a blue field are coloured red and vice versa. Rules concerning deletion of edges are also modified: an edge is deleted if it loses one of its vertices or when it connects a red node with a blue one. This is so because there is no restriction between two adjacent fields coloured by a different colour. In Figure 3.12, we see similar transformation of notation to the one introduced in the previous subsection. This time, the game (played on the same map) is simplified into five separate games, and all that needs to be done is an evaluation of these positions. The value of the game will consequently be, as we are already used to, the sum of all its sub-games.

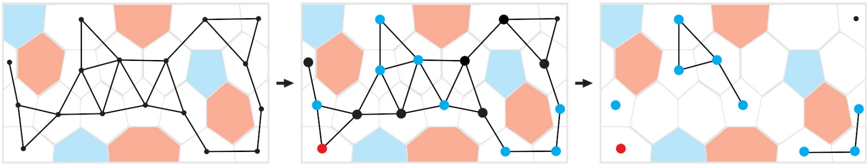

Figure 3.12: Notation of one position of SNORT by plain graph

Let us consider two games of one node in the lower left corner. Each one of them clearly means an advantage of one move for each player, as there is no legal move that can corrupt any of these two fields. This is, naturally, just a consequence of rules for deleting edges given before, and the whole model of transforming these games to graphs would be pointless. Now we can use the notation we are already familiar with. A game that consists of one field that has a blue vertex means an advantage of one move for the left player, and we can write it as $\{0|\} = 1$ (we are, however, describing a state that is impossible to create in the map model, as there is no constraining rule while having a game consisting of only one field. On the other hand, this can be imagined as a game with two fields, one of them already coloured red. This field would not be counted into the final value of the game. Since the graph description is more general, we will ignore the map model, and analyse both games using only this representation). It is the same for a game with one red vertex: {|0}=-1. Together, these two states create a game with a value of $0 = \{-1|1\}$, and we can see that the first player to make a move would really lose such game. The sub-game in the lower right corner could seem like an advantage of three moves for the blue player, but it is not hard to see that there is no way the blue player could colour all three fields. The value of this game is actually $\{1|\} = 2$. The sub-game in the upper right corner is $*$, and the one in the middle of the field is equal to 2. This means that the current state of the game has a total value of the sum of all its sub-games: $1 - 1 + 2 + 2 + * = 4 + *$. Any more complicated states of both games COL and SNORT can be evaluated by evaluating all states which are available to both players.

4 Conclusion

People commonly consider mathematics an unreasonably complicated tool for analysis of natural things. It may seem like it is not easy to describe events from the world around us using this tool without spending a long time studying this subject. The whole area of mathematics is, however, divided into many mutually interconnected branches, which can be at some level studied separately. One of such brnaches is the Combinatorial game theory.

This thesis consists of a description of combinatorial games and examples of both impartial and partisan games. It should provide the reader with good basics to understanding this area of mathematics, which is not very widespread. It is also a good way of getting familiar with the analysis of impartial games and their additions into more complex games. The very basics of analysis of partisan games are provided as well.

The whole thesis should be considered as an introduction to the theory of combinatorial games. On the other hand, it can be understood as a guide to finding similarities between various combinatorial games. The reader should consequently be able to perform a basic analysis of a variety of these games, even those which are not mentioned in this thesis.

5 Literature

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