A Brief Introduction to Combinatorial Game Theory

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Introduction

There is a certain set of game attributes a game must have in order to be classified as a **combinatorial game**. We require these specifications for the sake of analysis; otherwise we would have to take a lot more information into account such as probabilities and various types of valid endgames. Thus, the following restrictions are necessary to simplify the set of games so that we may carry out our analysis:

1) Two Players

   Combinatorial Game Theory (CGT) is study of two-person games. Generally, the two players are denoted *Left* and *Right*, where *Left* is a female and *Right* is a male.

2) No Chance

   Combinatorial games do not allow dice, card shuffling, or any other devices which lead to the need for probabilities and distributions. Otherwise, the outcome from each turn would be heavily dependent upon the factors of chance rather than the abilities of the players or the nature of the game itself.

3) Perfect Information

   All combinatorial games require for all game data to be accessible to both players. That is to say that there is nothing hidden from a player's opponent. Everything pertinent to the current game being played is completely laid out on the game board for both players to see.

4) Turn-Based

   We assume that the players have agreed upon who makes the first move. Players make moves by taking turns one at a time. This ensures that speed is NOT a factor that would also need to be included in our analysis. At any point in the game, the designation *First Player* or *Second Player* depends on whose turn it is.

5) Absolute Winner

   In every combinatorial game, there must be an absolute winner: the first player to fulfill the winning condition described below. This means that there is no possibility for a Tie or a Draw. It also prevents the allowances for player resignations or any other sort of premature game termination.

6) Winning Condition

   In most combinatorial games, the winning condition is simple: the last player to make a valid move wins.

In the theory of Combinatorial Games, we assume that both players make optimal moves. In reality, though, the games are so complex that it is not uncommon for a player to make a sub-optimal move.

A game G is given by a pair of sets: $G = \{G^L \mid G^R\}$, where G^L consists of sets of options. For example, $G^L = \{A, B, C, \ldots\}$ while $G^R = \{R, S, T, \ldots\}$ These options are called **positions**.

For example, if *Left* is *First*, then she must choose one of A, B, C, …

After *Left* moves, though, the set of options for *Right* changes. Suppose *Left* chooses C. The game position becomes $C = \{$ $C=\{C^{L_1},C^{L_2}, \ldots|C^{R_1},C^{R_2}, \ldots\}$, where *Right* moves first.

In Combinatorial Game Theory, there are two types of games: *Impartial Games* and *Partisan Games*.

An **impartial game** is one in which at *any* stage in the game $G^L = G^R$. That is, both *Left* and *Right* have the same options. In this instance, $G = \{G^L \mid G^R\}$ becomes just $G = \{G_1, G_2, \ldots\}$.

A **partisan game** is one where the set of *Left* and *Right* options are *not* the same at every point.

We can consider the set of all (finite) games as the following Venn diagram:

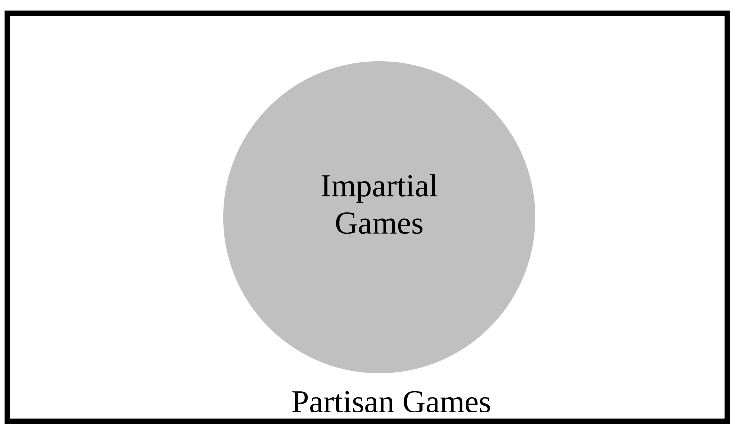


Figure 1: The Set of All Finite Games

There are many examples of Combinatorial Games, both impartial and partisan. Seven games have been included on the following page.

Examples of Combinatorial Games (Part 1)

The following are some examples of combinatorial games. While looking at them, notice how each game conforms to the requirements of a combinatorial game. Also, this list is by no means exhaustive. In fact, there are infinitely many combinatorial games. Hopefully these can inspire you to develop your own game.

Clobber

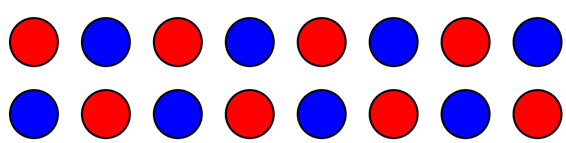
Players:

Left = bLue, usually referred to in female gender
Right = Red, usually referred to in male gender

Game Format:

Before playing, the players must decide who will play blue (Left) and who will play red (Right) as well as who is to make the first move. It should be noted that these decisions do not have any affect upon the computations that are to follow. The game board is created by arranging the red and blue pieces in alternating order. *Note*: This arrangement does not have to be rectangular; however, most game boards are made this way for the sake of simplicity.

A typical “game board” is similar to the figure below:



Clobber is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Taking a turn:

A player moves by capturing one of their opponent's pieces in an adjacent spot. Moves are made orthogonally but not diagonally. A move is only valid if a player captures an opponent's piece.

Winning the game:

A player wins the game when their opponent has no more moves remaining. In other words, their opponent cannot capture another one of their opponent's pieces.

Cutcake

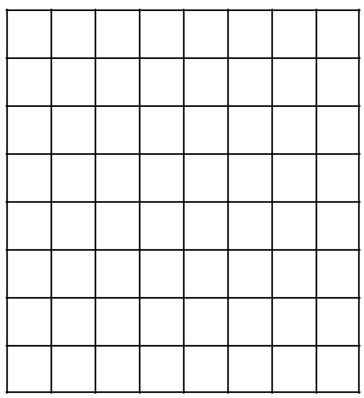
Players:

Left = verticaL (North-South) cuts, usually referred to in female gender
Right = hoRizontal (East-West) cuts, usually referred to in male gender

Game Format:

Before playing, the players must decide who will cut the grid along vertical gridlines (Left) and who will cut the grid along horizontal gridlines (Right) as well as who is to make the first move. It should be noted that these decisions do not have any affect upon the computations that are to follow. The game board is created by drawing out some sort of grid. *Note*: This does not have to be rectangular; however, most game boards are made this way for the sake of simplicity.

A typical "game board" is similar to the figure below:



Cutcake is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Taking a turn:

A player moves by making a cut (depending upon their designated "cutting" orientation) along one of the gridlines as far as the lines span on the game board. The result is a pair of subgames from the original. The breakdown continues with each turn until they are left with individual squares with no connecting grids. The single squares cannot be cut and therefore removed (erased) from the game board.

Note: One of the mistakes that people make in understanding the rules is that once you cut the cake, in the future, you only cut one piece at a time. A player may cut along an *entire* "longitude" or "latitude" whether it is a continuous segment or contains multiple segments.

Winning the game:

A player wins the game when their opponent has no more moves remaining.

Domineering

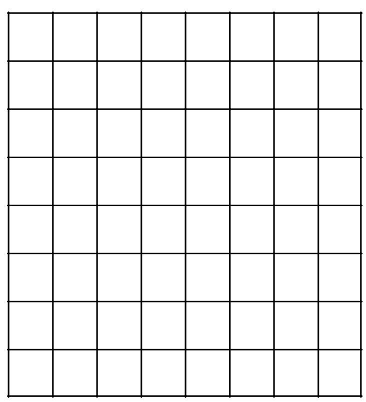
Players:

Left = verticaL dominos, usually referred to in female gender
Right = hoRizontal dominos, usually referred to in male gender

Game Format:

Before playing, the players must decide who will place dominos vertically (Left) and who will place dominos horizontally (Right) as well as who is to make the first move. It should be noted that these decisions do not have any affect upon the computations that are to follow. The game board is created by drawing out some sort of grid. *Note*: this does not have to be rectangular; however, most game boards are made this way for the sake of simplicity.

A typical "game board" is similar to the figure below:



Domineering is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Taking a turn:

A player moves by placing a domino either vertically or horizontally (depending upon their designated "placing" orientation) on the game board. General strategy follows from either player trying to reserve themselves grid areas where only they can place their dominos while, at the same time, trying to minimize the areas in which their opponents can play.

Winning the game:

A player wins the game when their opponent has no more moves remaining.

Hackenbush (Red-Blue)

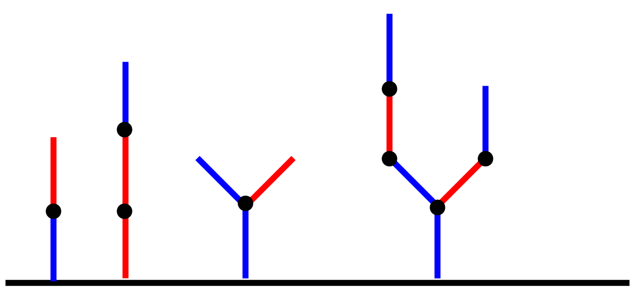
Players:

Left = bLue, usually referred to in female gender
Right = Red, usually referred to in male gender

Game Format:

The game board is created by arranging any sort of combination of red and blue sticks that extend from the ground or another stick. Before playing, the players must decide who will play blue (Left) and who will play red (Right) as well as who is to make the first move. It should be noted that these decisions do not have any affect upon the computations that are to follow.

A typical “game board” is similar to the figure below:



Hackenbush is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Taking a turn:

A player moves by “hacking” a single stick from the game board of their own designated color. Remember that if any stick becomes disconnected from the ground, it too must be removed from the game board.

Winning the game:

A player wins the game when their opponent has no more moves remaining. While at first this may seem arbitrary and only dependent upon the number of stick each player has, we reemphasize that every stick must be connected to the ground to still be playable and thus allows players to eliminate their opponent’s moves from the game board as well.

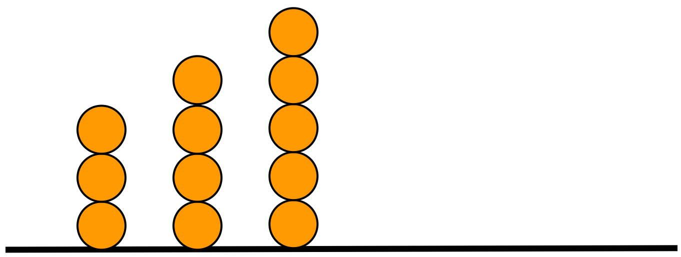
Nim

Players:

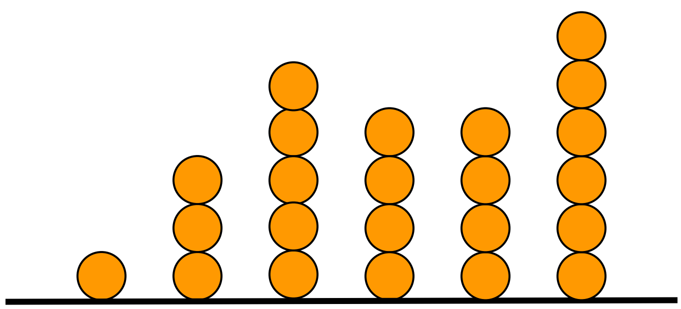
Two players. (No names need to be assigned to distinguish one from the other because both players have the same set of options each turn.)

Game Format:

Before playing, the players must decide who is to make the first move. The game board is created by grouping together piles of pennies (or whatever). A small game usually features three piles of various sizes, say three in one, four in the next, and five in the other:



Another typical “game board” is similar to the figure below:



Nim is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Taking a turn:

A player moves by removing any number of pennies from a single pile on the game board. A player *cannot* take pennies from multiple piles in the same turn. A played must take *at least one* penny every turn.

Winning the game:

A player wins the game when their opponent has no more moves remaining. In other words, there are *no more* pennies left in any piles, and whoever removed the last penny or pile of pennies is the winner.

Ski Jumps

Players:

Left = “L”-skiers (moves from Left), usually referred to in female gender
Right = “R”-skiers (moves from Right), usually referred to in male gender

Game Format:

The game board is created by drawing a grid with one row and a certain number of columns. Even before the game begins, players place their skiers on the board (most games start with all “L”-skiers lined up on the left and all “R”-skiers lined up on the right, but this is *not* a requirement).

A typical “game board” is similar to the figure below:

L							*R*
	L						
							R

Ski jumps is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Taking a turn:

A player moves by moving one of their skiers one or more slots to the right (“L”-skiers only) or to the left (“R”-skiers only) of the skier's current position provided their is not another ski in their horizontal path.

There also is a chance to “jump” an opponent's skier only if their opponent has a skier in the square directly below them. To “jump”, a player moves their skier into an empty slot down a row and to the right (“L”-skiers only) or down a row and to the left (“R”-skiers only) of the skier's its position. Skiers may *not* jump up a row.

The consequence for being “jumped” is that the skier who was “jumped” loses their confidence and can no longer jump another skier. By convention, since a “jumped” skier's pride is reduced, we keep track of this by using a lowercase letter instead of uppercase.

Winning the game:

A player wins the game when their opponent has no more moves remaining.

Toads and Frogs

Players:

Left = Toad (moves from Left), usually referred to in female gender
Right = Frog (moves from Right), usually referred to in male gender

Game Format:

The game board is created by drawing a grid with one row and a certain number of columns. Even before the game begins, players place their Toads and Frogs on the board (most games start with all Toads lined up on the left and all Frogs lined up on the right, but this is *not* a requirement). A simple game board is shown below:

T	*T*		*T*		*F*	*F*	*F*

However, Toads & Frogs becomes much more interesting by adding additional (but separate) rows to the game. Another typical "game board" is similar to the figure below:

T	*T*		*T*		*F*	*F*	*F*

T		*T*	*F*		*F*	*F*	

	T	*T*	*F*	*T*	*F*	*F*	

Toads & Frogs is a turn-based game such that each player takes one move at a time until one of the players can no longer move.

Note: You may see games where the rows are attached in one giant grid, however since Toads and Frogs are only allowed to move horizontally in the aforementioned directions it is usually best to see the rows as separate games that do not interact with each other.

Taking a turn:

A player moves by moving one of their creatures (depending upon whether they are Toads or Frogs) into the next slot to the right (Toads only) or to the left (Frogs only) of the creature's current position. There also is a chance to jump over an opponent's creature only if there is an empty slot behind its position.

Winning the game:

A player wins the game when their opponent has no more moves remaining. This means that the object is *not* for a player to move all their creatures to the end of the row and off the board because once a player has done that they have no more moves remaining.

Assigning Values to Games

Our goal is to assign numerical values to games, which will determine not only which player, *Left* or *Right* (or, in the case of impartial games, *First* and *Second*) can win, but by how many moves.

The analysis begins with the **zero game**, denoted by 0 = { | }. (Here, { | } actually means {∅|∅}.) Obviously, 0 = { | } is a second player win, since *First* has no options to choose from, and hence loses. We have four outcome classes:

Given a game $G = \{G^L \mid G^R\}$, we say that

(i) G > 0, "G is greater than zero" or "G is positive" if and only if *Left* can win
(ii) G < 0, "G is less than zero" or "G is negative" if and only if *Right* can win
(iii) G || 0, "G is confused with zero" or "G is fuzzy" if and only if *First* can win
(iv) G = 0, "G equals zero" if and only if *Second* can win.

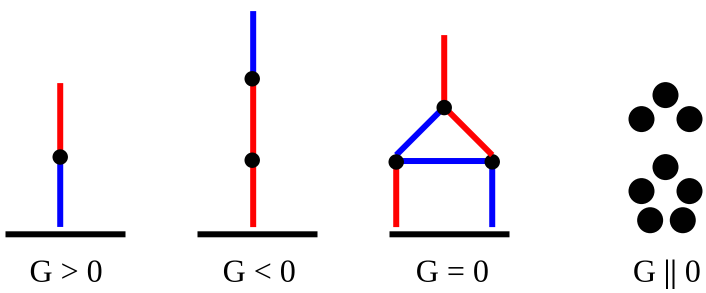


Figure 2: Examples of each outcome class

From these outcomes, it follows that

(i) G ≥ 0 if and only if *Left* or *Second* can win
(ii) G ≤ 0 if and only if *Right* or *Second* can win
(iii) G ⧐ 0 if and only if *Left* or *First* can win
(iv) G ⧏ 0 if and only if *Right* or *First* can win

These lead to the following proposition:

Proposition: Given a game $G = \{G^L \mid G^R\}$,

(i) G ≥ 0 means "If *Right* is *First*, then *Left* can win."
(ii) G ≤ 0 means "If *Left* is *First*, then *Right* can win."
(iii) G ⧐ 0 means "If *Left* is *First*, then *Left* can win."
(iv) G ⧏ 0 means "If *Right* is *First*, then *Right* can win."

Sums and Differences of Games

The basic idea is as follows: Suppose we have two games G and H. The **sum of the two games** is denoted G + H and is played as follows. One moves by choosing an option in either G or H. If G is selected, that games changes, which H remains the same.

Recall the first example Nim game. We had a heap of 3 stones, 4 stones and 5 stones.

Since it is possible to have a Nim game consisting of only one heap, we can actually think of that starting example as a sum of three games.

That is, $S = G_3 + G_4 + G_5$.

Consider the following game of Domineering in which the first nine moves have already been played.

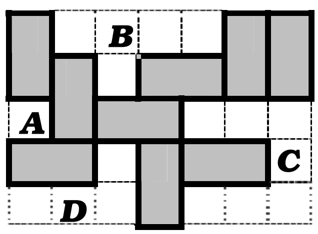


Figure 3: After nine moves of a Domineering Game

We can think of this game as a sum of four separate games. S = A + B + C + D. Notice that A = { | }, as no domino can be placed in a 1 × 1 square.

The three Hackenbush positions shown in Figure 2 can be thought of as separate games, since no edge connects each stalk. Let us denoted these games H_1, H_2 and H_3

We saw that $H_1 > 0$, $H_2 < 0$ and $H_3 = 0$. But what is the value of $S = H_1 + H_2 + H_3$? Does it happen that S > 0? Is S < 0? Or maybe S = 0?

We shall return to this question when we talk about Numbers. It turns out that all Hackenbush games have numerical values. That is not always the case with games.

For any game $G = \{G^L \mid G^R\}$, we say that the negative of G is given by $-G = \{-G^R \mid -G^L\}$.

Theorem: For any games G and H, we have

(a) G + (-G) = 0; equivalently, G = -(-G)
(b) –(G + H) = (-G) + (-H)

We use the negative of a game to prove that two games G and H are essentially equal.

To show this, we show that G + (-H) = 0. Recall, this is the same thing as showing that G + (-H) can be won by *Second*, regardless of who goes first. The game G + (-H) is referred to as the **difference game**.

If we are dealing with a partisan game, then H = -H. This follows from the fact that $H^L = H^R$ and the previous equation becomes G + H = 0.

Given the outcome classes of both G and H, what can we say about the sum G + H? The following table summarizes what we know:

	H = 0	H > 0	H < 0	H ‖ 0
G = 0	G + H = 0	G + H > 0	G + H < 0	G + H ‖ 0
G > 0	G + H > 0	G + H > 0	G + H ? 0	G + H |> 0
G < 0	G + H < 0	G + H ? 0	G + H < 0	G + H <| 0
G ‖ 0	G + H ‖ 0	G + H |> 0	G + H <| 0	G + H ? 0

Table 4: Relationship between G + H and zero

In the above table, G + H ? 0 means that nothing (in general) can be said about the comparison of G + H with zero. We also have the following relationship among games:

	H = K	H > K	H < K	H ‖ K
G = H	G = K	G > K	G < K	G ‖ K
G > H	G > K	G > K	G ? K	G |> K
G < H	G < K	G ? K	G < K	G <| K
G ‖ H	G ‖ K	G |> K	G <| K	G ? K

Table 5: Relationship between games G, H and K

Lemma: For any game $G = \{G^L \mid G^R\}$, it follows that $G^L \lhd\!\!| \; G \lhd\!\!| \; G^R$.

Proof: We claim that $G - G^R \lhd\!\!| \; 0$. This is the same thing as showing if *Right* plays first in $G - G^R$, then he can win. Suppose *Right* is *First*.

Then he plays in the first component, G, and sends the difference to 0. That is, $G - G^R = \{G^L | G^R\} - G^R \xrightarrow{\text{Right}} G^R - G^R = 0$.

Similarly, we can show that $G^L - G \lhd\!\!| \; 0$, which completes the proof.

C.L. Bouton's Solution to Nim (1902)

Next, we discuss the first paper published on the subject of Combinatorial Game Theory over 100 years ago.

To begin with, let us solve a game that consists of 5 heaps, say, in which the sizes are 30, 23, 17, 10, and 3. Bouton's solution is as follows:

1) We start by partitioning each heap size into powers of 2

30	=	16	+	8	+	4	+	2	+	0
23	=	16	+	0	+	4	+	2	+	1
17	=	16	+	0	+	0	+	0	+	1
10	=	0	+	8	+	0	+	2	+	0
3	=	0	+	0	+	0	+	2	+	1
		Odd Parity		Even Parity		Even Parity		Even Parity		Odd Parity

Figure 6: 5 Nim Heaps as Powers of 2

2) Look at the columns. The 16's and 1's columns are said to have **odd parity** since 16 occurs three times and 1 occurs three times (which are odd numbers). On the other hand, the 8's, 4's and 2's columns all have **even parity**, since each digit occurs an even number of times.

3) If there is odd parity, then *First* has a winning strategy. The strategy is to restore even parity to all of the columns. If every column has even parity to begin with, then *Second* has a winning strategy.

Since we have odd parity, *First* can win. She needs to eliminate and 16 and a 1. This can be done in three ways.

One way is to eliminate the 17-heap. Then the 16's and 1's have even parity.

		30	=	16	+	8	+	4	+	2	+	0
		23	=	16	+	0	+	4	+	2	+	1
17 - 17	=	0	=	0	+	0	+	0	+	0	+	0
		10	=	0	+	8	+	0	+	2	+	0
		3	=	0	+	0	+	0	+	2	+	1
				Even Parity		Even Parity		Even Parity		Even Parity		Even Parity

Figure 7: 5 Nim Heaps as Powers of 2

There are two other ways. Can you see them?

Second must respond to Figure 7 or the other two moves. When he does, however, the new display will have at least one column with *odd* parity. With the next move, his opponent will again restore even parity to the columns.

His opponent will persist in this strategy throughout. In this manner, starting with Figure 6, *First* is certain to win.

Numbers

A game G is a **number** if all options of G (for both *Left* and *Right*) are numbers, and no left option is greater than or equal to any right option.

This definition may seem a bit strange at first. Numbers are defined recursively. We use it repeatedly until we have a precise definition. We shall begin by showing that the game $G = \{ \mid \}$ is a number.

Theorem: The game $G = \{ \mid \} = 0$ is a number.

Proof: Suppose *G* is *not* a number. Then some option of G is not a number or some left option of G is greater than or equal to some right option. But in the first instance, we have a contradiction, since G has no options. The second one is also a contradiction for the same reason. So, we have that G is a number.

It should be noted that any number x cannot be a first player win. If x was a first player win, then both $x^L \geq 0$ and $x^R \geq 0$, but this forces $x^L \geq x^R$, which violates the definition of a number.

Theorem: For any number $x = \{x^L \mid x^R\}$, we have that $x^L < x < x^R$.

Proof: We shall show $x^L < x$. Suppose $x^L \geq x$ for some x^L. If $x^L = x$, then *Second* can win $x^L - x = x^L + \{-x^R \mid -x^L\}$. But this is a contradiction, since *Right* obviously sends the sum to zero if he plays first.

If we assume that $x^L > x$, that is $x^L + \{-x^R \mid -x^L\} > 0$, we get another contradiction for the same reason.

By building from the ground up, we can show that all Hackenbush positions are numbers. To do this, we built it up from the ground up (literally).

The zero game _____, which has no stalks, is a number. We can show that as we add stalks, the game is still a number. Furthermore, if a blue edge is erased, then the value of the stalk decreases; when a red edge is erased, the value of the stalk increases.

We end up with $G^L < G < G^R$, with both G^L and G^R numbers, thus G is a number as well.

The Integers and Their Birthdays

In the tradition of the founding book on Combinatorial Game Theory, *On Numbers and Games* by John Conway, we say that the number { | } = 0 was born on day zero. Thus, we say that its **birthday** is 0.

Furthermore, the three games {0| }, { |0} and {0|0} we all born on day one, since they are defined by objects born on day zero. We call these games 1, -1 and * (pronounced "star"), respectively. (We shall return to * later.)

On Day 2, we have numbers {0|1}, {0, 1| }, and {-1, 0|1}. It turns out that {0|1} = 1/2. There is a rich theory behind the construction of all rationals of the form $(2q + 1)/2^p$, which are known as dyadic rationals. From there, you can use limits to construct irrational numbers. These concepts, require some analytical techniques, which is beyond the scope of this text, though.

For the curious reader, a proof of why {0|1} = 1/2 is included in Appendix A.

There are also non-numbers such that {1|0}, {1|1}, and {1|-1}. However, we shall limit our discussion, for the time-being at least, to numbers.

Proceeding in this manner, we define the non-negative integers by

0 = { | }, 1 = {0| }, 2 = {1| }, 3 = {2| }, …

and the negative integers by

-1 = { |0}, -2 = { |-1}, -3 = { |-2}, …

We know that *Second* wins the game 0 = { | }. It is clear that *Left* wins all of the positive integers 1, 2, 3, … and *Right* wins all of the negative integers -1, -2, -3, …

There exists a simplicity rule for evaluating game expressions like those above. One form of it states: "If p and q are positive integers, then we have that {-p|q} = 0."

This implies that {-3|5} = {-1|17} = { | } = 0. For the sake of simplicity, we will use { | } to denote the zero game. This happens to be the **canonical form** of zero. A detailed discussion of what that means is beyond the scope of this text.

Our working definition will be the "simplest" way to write a number. By simplest, we mean using those numbers with the earliest birthdays.

Examples of Combinatorial Games (Part 2)

Earlier, we introduced the game of Cutcake. It turns out that the game-theoretic values of Cutcake are all integers, as we shall see shortly.

Cutcake

The starting position for Cutcake is an $m \times n$ rectangular grid, or "cake". See the figure below for one possible cake.

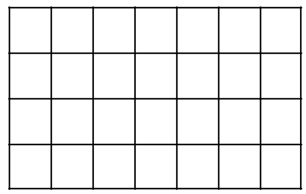


Figure 8: A 4×7 cake

Moves are made as follows:

If *Left* is *First*, she may make a vertical cut. If *Right* is *First*, he may make a horizontal cut. It should be noted that after the first cut, the other play must choose a piece to work in to make their next cut. That is, they cannot cut through multiple pieces of cake during the same move.

We say that $\square = \{ \mid \} = 0$, since no player can move from this position. That is, there is nothing left to cut.

To simplify our notation and to avoid continually drawing pictures, the above diagram will be referred to as $[4 \times 7]$.

To assign a value to this game, we need to find out all of the possible moves that can result from this game, and assign values to those. Then we can follow our rule for finding the value of a game to give the game a numerical value.

It is left to the reader to verify that

$$[4\times7] = \left\{[4\times1][4\times6],\ [4\times2][4\times5],\ [4\times3][4\times4]\ \mid\ [1\times7][3\times7],\ [2\times7][2\times7]\right\}$$

So, in order to find the value of this position, we need to know, for example, the value of the game $[4 \times 6]$.

As it turns out, though,

$$[4\times6] = \left\{[4\times1][4\times5],[4\times2][4\times4],[4\times3][4\times3] \mid [1\times6][3\times6],[2\times6][2\times6]\right\}$$

Thus, we need to know even more positions. Suffice it to say, it can be a lot of work. So, our strategy is to work from the zero game up to our current game.

Doing this, we end up with the following table of game-theoretic values for Cutcake

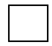
Columns

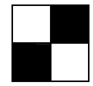
<table>
<tr><th>Rows</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>13</th><th>14</th><th>15</th><th>16</th></tr>
<tr><td>1</td><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td></tr>
<tr><td>2</td><td>-1</td><td rowspan="2" colspan="2">0</td><td rowspan="2" colspan="2">1</td><td rowspan="2" colspan="2">2</td><td rowspan="2" colspan="2">3</td><td rowspan="2" colspan="2">4</td><td rowspan="2" colspan="2">5</td><td rowspan="2" colspan="2">6</td><td rowspan="2"></td></tr>
<tr><td>3</td><td>-2</td></tr>
<tr><td>4</td><td>-3</td><td rowspan="2" colspan="2">-1</td><td rowspan="4" colspan="4">0</td><td rowspan="4" colspan="4">1</td><td rowspan="4" colspan="4">2</td><td rowspan="4"></td></tr>
<tr><td>5</td><td>-4</td></tr>
<tr><td>6</td><td>-5</td><td rowspan="2" colspan="2">-2</td></tr>
<tr><td>7</td><td>-6</td></tr>
<tr><td>8</td><td>-7</td><td rowspan="2" colspan="2">-3</td><td rowspan="4" colspan="4">-1</td><td rowspan="8" colspan="8">0</td><td rowspan="8"></td></tr>
<tr><td>9</td><td>-8</td></tr>
<tr><td>10</td><td>-9</td><td rowspan="2" colspan="2">-4</td></tr>
<tr><td>11</td><td>-10</td></tr>
<tr><td>12</td><td>-11</td><td rowspan="2" colspan="2">-5</td><td rowspan="4" colspan="4">-2</td></tr>
<tr><td>13</td><td>-12</td></tr>
<tr><td>14</td><td>-13</td><td rowspan="2" colspan="2">-6</td></tr>
<tr><td>15</td><td>-14</td></tr>
<tr><td>16</td><td>-15</td><td colspan="2"></td><td colspan="4"></td><td colspan="8"></td><td></td></tr>
</table>

Table 9: Game Theoretic Values for Cutcake

Nimbers, Not Numbers

We shall introduce a variation of Cutcake, known as **Chesscake**. This game is played on a chessboard and has one new rule not found in Cutcake: No black square may be separated from any cake.

As a result, we have three zero positions:   

This creates a new position, though. How shall we classify the position: 

Notice that no matter who goes first, the game reduces to the zero game.

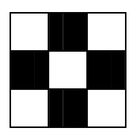
So, we have that position is given by {0 | 0}. This is not a number, since it does not conform to the definition. It is something else.

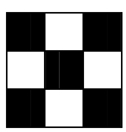
This game is equivalent to a game of Nim with only one stone. In that game, after the first move, the game is sent to zero. Thus, it is a first player win.

To denote this new game, we call it * = {0 | 0}, read "**star**". It is the first of many non-numbers. Because of their relation to the game of Nim, these non-numbers are collectively referred to as **nimbers**.

In a while, we shall prove that Sprague-Grundy Theorem, which states that *all* impartial games are equivalent to a game of Nim.

Think about some positions of games in Chesscake.

What would the value of the following game have: 

What about the game  ?

On the next page, we list some game-theoretic values for Chesscake

<table>
<tr><th>Rows \ Columns</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th></th></tr>
<tr><td>1</td><td>0</td><td>0</td><td>0,1</td><td>1</td><td>1,2</td><td>2</td><td>2,3</td><td>3</td><td>3,4</td><td>4</td><td></td></tr>
<tr><td>2</td><td>0</td><td>*</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td></td></tr>
<tr><td>3</td><td>0,-1</td><td>0</td><td>0,*</td><td>1</td><td>1,2</td><td>2</td><td>2,3</td><td>3</td><td>3,4</td><td>4</td><td></td></tr>
<tr><td>4</td><td>-1</td><td>-1</td><td>-1</td><td rowspan="4" colspan="4">0</td><td rowspan="4" colspan="3">1</td><td rowspan="4">• • •</td></tr>
<tr><td>5</td><td>-1,-2</td><td>-2</td><td>-1,-2</td></tr>
<tr><td>6</td><td>-2</td><td>-3</td><td>-2</td></tr>
<tr><td>7</td><td>-2,-3</td><td>-4</td><td>-2,-3</td></tr>
<tr><td>8</td><td>-3</td><td>-5</td><td>-3</td><td rowspan="3" colspan="4">-1</td><td rowspan="4" colspan="4">0</td></tr>
<tr><td>9</td><td>-3,-4</td><td>-6</td><td>-3,-4</td></tr>
<tr><td>10</td><td>-4</td><td>-7</td><td>-4</td></tr>
<tr><td></td><td colspan="3"></td><td colspan="4">•
•
•</td></tr>
</table>

Table 10: Game Theoretic Values for Chesscake

What would happen if we considered the impartial game of Chesscake? What stays the same? What changes? As an exercise, you might want to work through the games and come up with your own analysis.

We define the nimber $*n$ to mean $*n = \{*0, *1, *2, \ldots, *(n-1) \mid *0, *1, *2, \ldots, *(n-1)\}$. In general, though, we simply write $*0 = 0$ and $*1 = *$.

It should be noted that $*n$ refers to a Nim heap of size n.

Proposition: If $n \neq 0$, then we have that $*n \parallel 0$. That is, any non-zero nimber is a *First* player win.

Proposition: $*n + *n = 0$. That is, a Nim game specified by two heaps of equal size is a *Second* player win.

Nim Addition

In the previous section, we discussed Nimbers. But how do we add two nimbers? To proceed, we rely on the following fact (which we used earlier).

Fact: Any number can be uniquely written as the sum of powers of 2.

If we want to add compute the sum *13 + *7 +*3 + *9, we write out each number in its binary form.

	8	4	2	1	← powers of 2
13 stones	1	1	0	1	
7 stones	0	1	1	1	
3 stones	0	0	1	1	
9 stones	1	0	0	1	

Using the terminology from Bouton's solution, we look to see which columns have even parity, and which have odd parity. For columns with even parity, we write a 0 below. For columns with odd parity, we write a 1 below.

Notice that in this example, all of the columns have even parity. So, we see that the sum will be equal to $0 \cdot 8 + 0 \cdot 4 + 0 \cdot 2 + 0 \cdot 1 = 0$. So, the sum is 0.

If we consider the sum *11 + *7 + *4 + *9 + *8, again we look at the binary representation of the numbers.

	8	4	2	1	← powers of 2
11 stones	1	0	1	1	
7 stones	0	1	1	1	
4 stones	0	1	0	0	
9 stones	1	0	0	1	
8 stones	1	0	0	0	

The first and fourth columns have odd parity, so the sum is equal to $1 \cdot 8 + 0 \cdot 4 + 0 \cdot 2 + 1 \cdot 1 = 9$.

So, we have that *11 + *7 + *4 + *9 + *8 = *9.

Below is a table for adding two nimbers.

Nim Addition Table $a \oplus b$															
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	1	7	6	1	0	3	2
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 11: A Nim Addition Table

Notice that the table is symmetric. This follows our intuition, since *a + *b is the same thing as *b + *a.

The Minimum-Excluded (mex) Rule

The Nim Addition Table that appears in the previous section would take a long time to compute by hand. As it turns out, though, it can also be derived from a rule for impartial games known as the **Minimum-Excluded Rule**.

Let S be any nonempty subset of the non-negative integers {0, 1, 2, …}. The minimum excluded (mex) value of S is the smallest non-negative integer that is *not* a member of S. This value is written as m = mex(S).

For example, If S = {2, 3, 4, 6}, then mex(S) = 0. If S = {0, 1, 2, 3, 4, 6}, then mex(S) = 5. If S = {0, 1, 2, 3, 4, 5, 6}, then mex(S) = 7

Theorem: The game-theoretic value of any set of nimbers $\{n_1, n_2, \ldots, n_k\}$ is given by another nimber, namely $m = \text{mex}\{n_1, n_2, \ldots, n_k\}$.

This means that if S is a game whose options are only nimbers, then we can compute the game-theoretic value by looking at the mex value.

To compute an entry in the Nim Addition table, you need to look at the mex of what is missing in that row and column for the desired entry.

For example, if we wanted to compute the value of *6 + *4, we look at the row that *6 appears in as well as the column *4 appears in and write down those entries.

This gives us: 7, 4, 5, 5, 6, 7, 0, 1. Notice that the mex value of this set of numbers is given by 2. So, we have that *6 + *4 = *2, as desired.

The Sprague-Grundy Theorem for Impartial Games

The following theorem (and its result) have been mentioned numerous times thus far. It was developed independently by Sprague (1936) and Grundy (1939).

Theorem: Every impartial game can be regarded as a Nim-heap. That is, every impartial game has a nimber value.

What this theorem tells us is that if we know how to win Nim (which we due, thanks to Bouton's solution), all we need is a way to translate our impartial game into Nim and we can then find a winning strategy.

We can now use the Sprague-Grundy Theorem (coupled with the mex rule) to analyze some other combinatorial games.

Examples of Combinatorial Games (Part 3)

Kayles

The game of Kayles is played as follows. We have 13 bowling pins set up. (See Figure 12 below.) Imagine that *First* and *Second* are both expert bowlers who, without error, are capable of knocking down any single pin or any pair of *neighboring* pins. What is the value of this game?

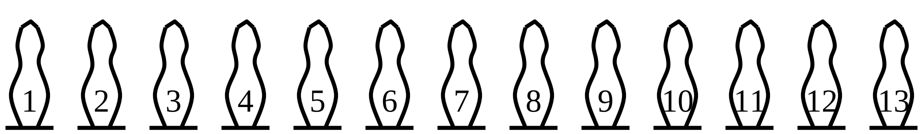


Figure 12: A game of Kayles with 13 pins

Before answering this question, we shall work our way up from the zero game. Notice that if have no pins, then it is a zero game. That is, g(0) = 0.

Here, g(n) = m says “the Sprague-Grundy value for n pins is given by m”.

Notice that g(1) = *. g(2) = {0, *} = *2. g(3) = {0, *, *2} = *3.

The table below lists the Sprague-Grundy Values for 0 to 11.

n	0	1	2	3	4	5	6	7	8	9	10	11
$g(n)$	0	*	*2	*3	*	*4	*3	*2	*	*4	*2	*6

Table 13: Sprague-Grundy Values for Kayles

We can use the mex rule to verify the values in the table.

$\odot$

$$K_1 = \{0\} = *.$$

$\odot\odot$

$$K_2 = \{0, *\} = *2.$$

$\odot\odot\odot$

$$K_3 = \{\underbrace{0}_{\text{take middle}}, \underbrace{*}_{\text{take two}}, \underbrace{*2}_{\text{take side}}\} = *3.$$

$\odot\odot\odot\odot$

$$K_4 = \{\underbrace{0}_{\text{take middle two}}, \underbrace{*+*2}_{\text{take middle one}}, \underbrace{*2}_{\text{take left two}}, \underbrace{*3}_{\text{take left one}}\} = *.$$

$\odot\odot$
$\odot\odot\odot$

$$K_5 = \{\underbrace{\underset{*\;+\;*3}{\odot\;\;\odot\odot\odot}}_{*2}, \underbrace{\underset{*2\;+\;*2}{\odot\odot\;\;\odot\odot}}_{0}, \underbrace{\underset{*\;+\;*2}{\odot\;\;\odot\odot}}_{*3}, \underbrace{\odot\odot\odot}_{*3}, \underbrace{\odot\odot\odot\odot}_{*}\} = *4.$$

$\odot\odot\odot$
$\odot\odot\odot$

$$K_6 = \{\underbrace{\underset{*\;+\;*}{\odot\;\;\odot\odot\odot\odot}}_{0}, \underbrace{\underset{*2\;+\;*3}{\odot\odot\;\;\odot\odot\odot}}_{*}, \underbrace{\odot\odot\odot\odot\odot}_{*4}, \underbrace{\odot\odot\odot\odot}_{*},$$
$$\underbrace{\underset{*\;+\;*3}{\odot\;\;\odot\odot\odot}}_{*2}, \underbrace{\underset{*2\;+\;*2}{\odot\odot\;\;\odot\odot}}_{0}\} = *3.$$

$\odot\odot\odot$
$\odot\odot\odot\odot$

$$K_7 = \{\underbrace{\underset{*\;+\;*4}{\odot\;\;\odot\odot\odot\odot\odot}}_{*5}, \underbrace{\underset{*2\;+\;*}{\odot\odot\;\;\odot\odot\odot\odot}}_{*3}, \underbrace{\underset{*3\;+\;*3}{\odot\odot\odot\;\;\odot\odot\odot}}_{0}, \underbrace{\odot\odot\odot\odot\odot\odot}_{*3},$$
$$\underbrace{\odot\odot\odot\odot\odot}_{*4}, \underbrace{\underset{*2\;+\;*3}{\odot\odot\;\;\odot\odot\odot}}_{*}, \underbrace{\underset{*\;+\;*}{\odot\;\;\odot\odot\odot\odot}}_{0}\} = *2.$$

$$\begin{array}{l}\odot\odot\odot\odot\\ \odot\odot\odot\odot\end{array}\qquad
\begin{aligned}K_8 = \{&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{*2},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*6},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot}}_{*2},\\
&\underbrace{\odot\odot\odot\odot\odot\odot\odot}_{*2},\underbrace{\odot\odot\odot\odot\odot\odot}_{*3},\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*5},\\
&\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot}}_{*3},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot}}_{0}\} = *.\end{aligned}$$

$$\begin{array}{l}\ \odot\odot\odot\odot\\ \odot\odot\odot\odot\odot\end{array}\qquad
\begin{aligned}K_9 = \{&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*2}{\odot\odot\odot\odot\odot\odot\odot}}_{*3},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{*},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*7},\\
&\underbrace{\underset{*}{\odot\odot\odot\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot}}_{0},\underbrace{\odot\odot\odot\odot\odot\odot\odot\odot}_{*},\underbrace{\odot\odot\odot\odot\odot\odot\odot}_{*2},\\
&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{*2},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*6},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot}}_{*2}\} = *4.\end{aligned}$$

$$\begin{array}{l}\odot\odot\odot\odot\odot\\ \odot\odot\odot\odot\odot\end{array}\qquad
\begin{aligned}K_{10} = \{&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot\odot\odot\odot\odot}}_{0},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*2}{\odot\odot\odot\odot\odot\odot\odot}}_{0},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{0},\\
&\underbrace{\underset{*}{\odot\odot\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*5},\underbrace{\odot\odot\odot\odot\odot\odot\odot\odot\odot}_{*4},\underbrace{\odot\odot\odot\odot\odot\odot\odot\odot}_{*},\\
&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*2}{\odot\odot\odot\odot\odot\odot\odot}}_{*3},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{*},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*7},\\
&\underbrace{\underset{*}{\odot\odot\odot\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot}}_{0}\} = *.\end{aligned}$$

$$\begin{array}{l}\ \odot\odot\odot\odot\odot\\ \odot\odot\odot\odot\odot\odot\end{array}\qquad
\begin{aligned}K_{11} = \{&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot\odot\odot\odot\odot}}_{*5},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot\odot\odot\odot\odot}}_{*3},\\
&\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*2}{\odot\odot\odot\odot\odot\odot\odot}}_{*},\ \underbrace{\underset{*}{\odot\odot\odot\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{*2},\\
&\underbrace{\underset{*4}{\odot\odot\odot\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{0},\underbrace{\odot\odot\odot\odot\odot\odot\odot\odot\odot\odot}_{*},\underbrace{\odot\odot\odot\odot\odot\odot\odot\odot\odot}_{*4},\\
&\underbrace{\underset{*}{\odot}\ \underset{+}{\ }\ \underset{*}{\odot\odot\odot\odot\odot\odot\odot\odot}}_{0},\underbrace{\underset{*2}{\odot\odot}\ \underset{+}{\ }\ \underset{*2}{\odot\odot\odot\odot\odot\odot\odot}}_{0},\underbrace{\underset{*3}{\odot\odot\odot}\ \underset{+}{\ }\ \underset{*3}{\odot\odot\odot\odot\odot\odot}}_{0},\\
&\underbrace{\underset{*}{\odot\odot\odot\odot}\ \underset{+}{\ }\ \underset{*4}{\odot\odot\odot\odot\odot}}_{*5}\} = *6.\end{aligned}$$

As an exercise, show that the original game has game theoretic value equal to *. That is, the game is a *First* player win. Can you find an optimal strategy?

Road Trip

In Road Trip, the towns A-E together with the hometown are shown below, with various roads connecting the different towns.

Initially, there are two cars, which are located in towns B and E. Game play alternates between two players. At each player's turn, the player must move a car via a road to another town.

The new town, though, must have a lower letter than the previous town. For example, a car can move from B to A, but not from B to C. At any point in the game, a car can go to the hometown (provided there is a road to Home) and more than one car is permitted at any given town.

The game ends once the two cars have returned to the hometown. The winner is the person who takes the last car to Home.

Question: Who wins this game?

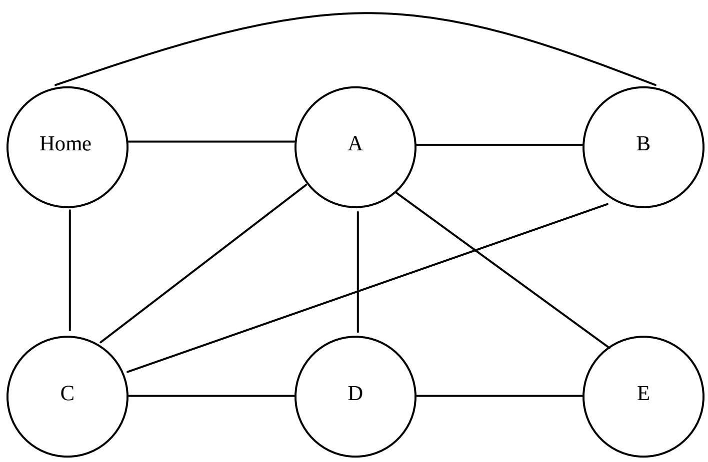


Figure 14: The Map for Road Trip

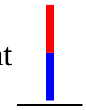
To determine who wins this game, we need to assign values to each of the towns.

We let Home = 0, since if a car is at Home, it has no available moves.

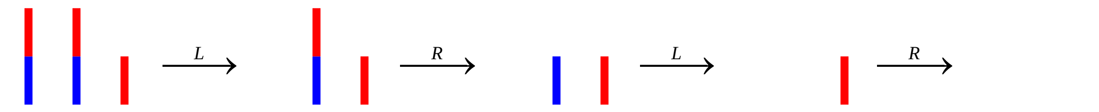
A = mex{0} = *, since the only available moves sends the game (the car) to 0 (home).
B = mex{0, *} = *2, by the mex rule.
C = mex{0, *, *2} = *3, by the mex rule.
D = mex{*, *3} = 0, by the mex rule.
E = mex{0, *} = *2, by the mex rule.

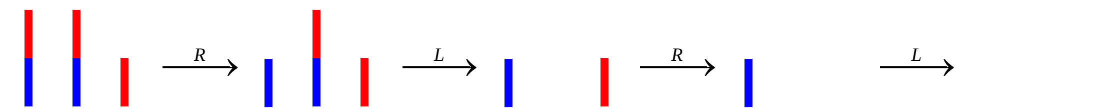
And so, we see that if we have a car at town B and town E, then the game is equal to *2 + *2 = 0, which is a *Second* player win.

Appendix A: A Sample Combinatorial Game Theory Proof

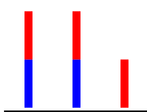
How to show that  = {0|1} = 1/2.

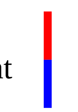
To do this, we shall show that two copies of the above plus is equal to 0.





In the above, we have shown that if *Left* is *First*, then *Right* wins. Likewise, we also saw that if *Right* is *First*, then *Left* wins. Thus, we have shown that

 = 0.

But since we know that is equal to -1, it follows that  = 1/2, completing the proof.

Note: In the above, I only showed optimal moves.

In the opening move for *Left*, either move would lead to the game to the right. However, at this point, *Right* has two options: hack the stalk above the blue stalk or hack the one to the right.

The stalk to the right is not in danger, and consequently should be reserved. The stalk above the blue stalk, however, is at risk of being eliminated if *Left* hacks her stalk.

In the opening move for *Right*, again he works from the top down, since those are in the most danger. *Left* counters with removing the her stalk that has *Right*'s stalk above it.

To be complete with the proof, I would need to show that the moves that I just glossed over were not optimal.

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