Newton Raphson/Iteration Method. 1

345

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

1. Find the +'ve root of $x^4 - x = 10$?

$F(x) = x^4 - x - 10 = 0$.

$F'(x) = 4x^3 - 1 = 0$.

$F(0) = -10$, $F(1) = -10$, $F(2) = 4$.

Take $x_0 = 2$ (+'ve valued)

Let $n=0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{31} = \underline{\underline{1.871}}$

Let $n=1$, $x_2 = x_1 - f(x_1)/f'(x_1) = 1.871 - 0.0152$

$= \underline{\underline{1.856}}$

Let $n=2$, $x_3 = x_2 - f(x_2)/f'(x_2) = 1.856 - 4.1533 \times 10^{-4}$

$= 1.856$

Let $n=3$, $x_4 = x_3 - f(x_3)/f'(x_3) = 1.856 - 4.1533 \times 10^{-4}$

$= 1.856$

$\therefore$ approximate root $= \underline{\underline{1.856}}$

2. Find the root of $x^3 = 6x - 4$ b/w 0 & 1 ?

take $x_0 = 1$? $\cancel{F(x) = x^3 - 6x + 4 = 0}$.

~~Let $n=0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - 1$~~

$F(x) = x^3 - 6x + 4 = 0$, $F'(x) = 3x^2 - 6 = 0$.

Let $x_0 = 1$

$n=0$ $x_1 = x_0 - f(x_0)/f'(x_0) = 0.667$

$n=1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.667 - (-0.0632)$

$= 0.73$

$n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73 - (-2.049 \times 10^{-3})$

$= 0.732$

$n=3$, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.732 - (-5.08 \times 10^{-5})$

$= 0.732$

$\therefore$ approximate root $= 0.732$

3. Find the root of $F(x) = 3x - \cos x - 1 = 0$?

$F'(x) = 3 + \sin x = 0$.

$F(0) = -2$, $F(1) = 1.459$

Take $x_0 = 1$.

$n=0$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1.459}{3.841} = 0.6202$

$n=1$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6202 - \frac{0.0468}{3.5812}$

$= 0.6071$

(2)

347

$\underline{n=2}$ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071 - \left(\frac{-1.648 \times 10^{-6}}{347}\right)$

$= \underline{0.607}$

$\underline{n=3}$ $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.607 - (-1.016 \times 10^{-4})$

$= \underline{0.607}$

$\therefore$ approximate root $= \underline{0.607}$

4. Find the root of $\log x = 1.2$?
5. Find the root of $\cos x = xe^{x}$?
6. Find the root of $x^3 - 2x - 5 = 0$?

4. $\log x = 1.2$. $f(x) = x - e^{1.2} = 0$.

$F(x) = \log x - 1.2 = 0$, $F'(x) = \frac{1}{x}$

$F(x) = F(1) = -1.2$, $F(2) = -0.899$

$F(3) = -0.722$

$F'(x) = 1$. $F(0) = -3.320$

$F(1) = -2.32$, $F(2) = -1.32$.

$F(3) = -0.32$, $F(4) = \underline{0.679}$

take $x_0 = 4$.

Solution of a system of linear equations.

I. Gauss Elimination Method.

Consider the 3 linear eqtns. with 3 unknowns.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{——} (1)$$

change (1) to matrix notation,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ which is of the form $AX = B$.

A = coefficient Matrix.

$$[AB] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$, augmented matrix of the given set of equations (1).

Applying row transformation, convert the matrix AB into upper triangular form. ie

$$[AB] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{bmatrix}$$

The given system of linear eqtn's is equivalent to $AX = B$.

ie $$\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & 0 & c_3'' \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2' \\ d_3'' \end{bmatrix}$$

$\Rightarrow a_1x + b_1y + c_1z = d_1$

$b_2'y + c_2'z = d_2'$

$c_3''z = d_3''$

On solving these eqtn's we get

$z = \dfrac{d_3''}{c_3''}$, $y = \dfrac{1}{b_2'c_3''}(d_2'c_3'' - c_2'd_3'')$,

$$x = \frac{1}{a_1 b'_2 c''_3}\left(d_1 b'_2 c''_3 - b_1 d'_2 c'_3 + b_1 c'_2 d''_3 - b'_2 c_1 d''_3\right).$$

<u>Note</u> This methods are only applicable for row transformation not column transformation.

1. Solve the system of equations by Gauss Elimination Method ?

$10x - 2y + 3z = 23$.

$2x + 10y - 5z = -33$.

$3x - 4y + 10z = 41$

$$[AB] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 \times 5 - R_1 \\ R_3 \to 10R_3 - 3R_1 \end{array}$$

$$= \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_3 \to 52R_3 + 34R_2$$

$$\begin{bmatrix} 10 & -2 & 3 \\ 0 & 52 & -28 \\ 0 & 0 & 3780 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -188 \\ 11340 \end{bmatrix}$$

$$\left.\begin{aligned} 10x - 2y + 3z &= 23 \\ 52y - 28z &= -188 \\ 3780z &= 11340 \end{aligned}\right\} \text{on solving,}$$

$\therefore\ x = 1,\ \underline{y = -2},\ z = 3$

II. Gauss Siedel Iteration Method.

Working Rule

Consider the following system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad -(1)$$

Let $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$ and $|c_3| > |a_3| + |b_3|$.

Solve the following equations for x, y and z;

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \quad \text{--- (1)}$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \text{--- (2)}$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \quad \text{--- (3)}$$

First approximation :-

put $y = z = 0$ in (1) & find $x = x_1$.

Put $x = x_1, z = 0$ in (2) & find $y = y_1$.

Put $x = x_1, y = y_1$ and find $z = z_1$.

Second Approximation :-

Put $y = y_1$ and $z = z_1$ in (1) and find $x = x_2$

Put $x = x_2, z = z_1$ in (2) and find $y = y_2$.

Put $x = x_2, y = y_2$ in (3) and find $z = z_2$.

Repeat the above steps until the required solution is obtained as unique.

1. Solve the following system of eqtn's by Gauss-Siedel Iteration Method;

$$8x - 3y + 2z = 20$$
$$6x + 3y + 12z = 35$$
$$4x + 11y - z = 33$$

$$8x - 3y + 2z = 20 \quad \text{--- (1)}$$
$$4x + 11y - z = 33 \quad \text{--- (2)}$$
$$6x + 3y + 12z = 35 \quad \text{--- (3)}$$

$$x = \frac{(20 - 2z + 3y)}{8} \quad \text{--- (4)}$$

$$y = \left(\frac{33 - 4x + z}{11}\right) \quad \text{--- (5)}$$

$$z = \frac{(35 - 6x - 3y)}{12} \quad \text{--- (6)}$$

Approximation I

Put $y = z = 0$ in (4), $x_1 = \frac{1}{8}(20) = \underline{\underline{2.5}}$

Put $x = 2.5, z = 0$ in (5), $y_1 = \underline{\underline{2.090}}$

Put $x = 2.5, y_1 = 2.090$ in (6), $z_1 = \underline{\underline{1.14}}$

Approximation II

$$x_2 = \frac{1}{8}\left(20 + 3(2.090) - 2(1.144)\right)$$
$$= \underline{\underline{2.997}}$$

$$y_2 = \frac{1}{11}\left(33 - 4(2.997) + 1.144\right)$$
$$= \underline{\underline{2.014}}$$

$$z_2 = \frac{1}{12}\left(35 - 6(2.997) - 3(2.014)\right)$$
$$= \underline{\underline{0.914}}$$

Approximation III

$$x_3 = \frac{1}{8}\left(20 + 3(2.014) - 2(0.914)\right)$$
$$= \underline{\underline{3.026}}.$$

$$y_3 = \frac{1}{11}\left(33 - 4(3.026) + 0.914\right)$$
$$= \underline{\underline{1.982}}$$

$$z_3 = \frac{1}{12}\left(35 - 6(3.026) - 3(1.982)\right)$$
$$= \underline{\underline{0.908}}$$

Approximation IV

$$x_4 = \frac{1}{8}\left(20 + 3(1.982) - 2(0.908)\right)$$
$$= 3.016.$$

$$y_4 = \frac{1}{11}\left(33 - 4(3.016) + 0.908\right) = 1.985.$$

$$z_4 = \frac{1}{12}(35 - 6(3.016) - 3(1.985))$$
$$= \underline{\underline{0.912}}$$

Approximation V

$$x_5 = \frac{1}{8}(20 + 3(1.985) - 2(0.912))$$
$$= 3.016.$$

$$y_5 = \frac{1}{11}(33 - 4(3.016) + 0.912)$$
$$= 1.986.$$

$$z_5 = \frac{1}{12}(35 - 6(3.016) - 3(1.986))$$
$$= 0.912.$$

Approximation VI.

$$x_6 = \frac{1}{8}(20 + 3(1.986) - 2(0.912))$$
$$= 3.016$$

$$y_6 = \frac{1}{11}(33 - 4(3.016) + 0.912)$$
$$= 1.986.$$

$$z_6 = \frac{1}{12}(35 - 6(3.016) + - 3(1.986))$$
$$= \underline{\underline{0.912}}$$

$$\Rightarrow \underline{\underline{x = 3.016,\ y = 1.986,\ z = 0.912}}$$

2. Solve using GSIM,

$$10x-2y-z-\omega=3$$
$$-2x+10y-z-\omega=15$$
$$-x-y+10z-2\omega=27$$
$$-x-y-2z+10\omega=-9 \quad ?$$

$$x=(3+2y+z+\omega)/10 \quad \text{—} \; (1)$$
$$y=(15+2x+z+\omega)/10 \quad \text{—} \; (2)$$
$$z=(27+x+y+2\omega)/10 \quad \text{—} \; (3)$$
$$\omega=(-9+x+y+2z)/10 \quad \text{—} \; (4)$$

Approximation I

$$x_1 = 0.3$$

$$y_1 = \frac{1}{10}(15+2(0.3)) \qquad z=\omega=0$$

$$= 1.56$$

$$z_1 = (27+0.3+1.56)/10 = \underline{\underline{2.886}}$$

$$\omega_1 = \frac{1}{10}(-9+0.3+1.56+2(2.886)) = \underline{\underline{\bar{0}.136}}$$

Approximation II

$$x_2 = \frac{1}{10}(3+2(1.56)+2.886-0.136)$$

$$= 0.887$$

$$y_2 = \frac{1}{10}(15 + 2(0.887) + 2.886 - 0.136)$$
$$= \underline{\underline{1.952}}$$

$$z_2 = \frac{1}{10}(27 + 0.887 + 1.952 + 2(-0.136))$$
$$= \underline{\underline{2.956}}$$

$$w_2 = \frac{1}{10}(-9 + 0.887 + 1.952 + 2(2.956))$$
$$= \underline{\underline{-0.024}}$$

Approximation III

$$x_3 = \frac{1}{10}(3 + 2(1.952) + 2.956 - 0.024)$$
$$= \underline{\underline{0.983}}$$

$$y_3 = \frac{1}{10}(15 + 2(0.983) + 2.956 - 0.024)$$
$$= 1.989.$$

$$z_3 = \frac{1}{10}(27 + 0.983 + 1.989 + 2(-0.024))$$
$$= \underline{\underline{2.992}}$$

$$w_3 = \frac{1}{10}(-9 + 0.983 + 1.989 + 2(2.992))$$
$$= \underline{\underline{-0.004}}$$

Approximation IV

$x_4 = \underline{\underline{0.996}}$, $y_4 = \underline{\underline{1.998}}$, $z_4 = \underline{\underline{2.998}}$

$w_4 = \underline{\underline{-0.001}}$

Approximation V

$$x_5 = \frac{1}{10}(3+2(1.998)+2.998-0.001)$$
$$= 0.999$$

$y_5 = 1.999$, $z_5 = 2.999$, $\omega_5 = -0.000$

Approximation VI

$x_6 = 0.999$, $y_6 = 1.999$.

$z_6 = 2.999$, $\omega_6 = -0.000$.

$\therefore x = 0.999$, $y = 1.999$, $z = 2.999$

$\omega = 0$

Interpolation

Suppose we are given the following values of $y = f(x)$ for a set of values of x.

$x : x_0 \ x_1 \ x_2 \ ------ \ x_n$.

$y : y_0 \ y_1 \ y_2 \ ------- \ y_n$.

Then the process of finding the values of y corresponding to any value of $x = x_i$, b/w x_0 and x_n is called interpolation.

Difference operators.

I. Forward Difference Operator(Δ).

Suppose that the function $y=f(x)$ is tabulated for the equality spaced values $x=x_0, x_0+h, x_0+2h, \ldots x_0+nh$, giving $y=y_0, y_1, y_2, \ldots y_n$.

The differences $y_1-y_0, y_2-y_1 \ldots y_n-y_{n-1}$ are denoted by $\Delta y_0, \Delta y_1 \ldots \Delta y_{n-1}$ respectively are called the first forward difference, where Δ is the forward difference operator.

Thus the first forward differences are $\Delta y_r = y_{r+1}-y_r$.

Similarly, the second forward difference are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r.$$

In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$ defines the p^{th} forward difference.

These differences are systematically set out as follows in what is called a forward difference table.

Forward Difference Table.

Value of x	Value of y	1st diff	2nd diff	3rd diff	4th diff	5th diff.
x_0	y_0					
		Δy_0				
x_0+h	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_0+2h	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_0+3h	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0+4h.$	y_4		$\Delta^2 y_3.$			
		Δy_4				
$x_0+5h.$	y_5					

In a difference table, x is called the argument and y, the function or the entry-y_0 the first entry is called the leading term and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc are called the leading differences.

II. Backward Difference Operator (∇)

The differences $y_1 - y_0, y_2 - y_1, \dots y_n - y_{n-1}$ are denoted by $\nabla y_1, \nabla y_2, \nabla y_3 \dots \nabla y_n$ respectively, are called the first backward differences, where ∇ is the backward difference operator. Similarly, we define higher order backward differences. Thus we have

$$\nabla y_r = y_r - y_{r-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

Backward Difference Table.

Value of x	Value of y	Ist diff.	IInd diff.	IIIrd diff.	IVth diff	Vth diff.
x_0	y_0					
x_0+h	y_1	∇y_1	$\nabla^2 y_1$			
x_0+2h	y_2	∇y_2	$\nabla^2 y_2$	$\nabla^3 y_1$	$\nabla^4 y_1$	
x_0+3h	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_2$	$\nabla^4 y_2$	$\nabla^5 y_1$
x_0+4h	y_4	∇y_4		$\nabla^3 y_3$		
x_0+5h	y_5	∇y_5	$\nabla^2 y_4$			

Newton's forward Interpolation Formula

Let the function $y=f(x)$ tak the values $y_0, y_1, y_2 \cdots y_n$ corresponding to the values $x_0, x_0+h, x_0+2h, \cdots x_0+ph$ of x. Suppose it is required to evaluate $f(x)$ for $x=x_0+ph$ where p is any real number.

ie $$y_p^{(a)} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

Newton's Backward Interpolation Formula

Let the function $y=f(x)$ take the values $y_0, y_1, y_2 \cdots y_n$ corresponding to the values $x_0, x_0+h, \cdots x_0+ph$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n+ph$, where p is any real number, then we have

$$y_p(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Qn 7. Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x=5$.

x	4	6	8	10
y	1	3	8	10

?

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4 (x_0)	1 (y_0)			
		$3-1=2$ (Δy_0)		
6 (x_1)	3 (y_1)		$5-2=3$ $(\Delta^2 y_0)$	
		$8-3=5$ (Δy_1)		$-3-3=-6$ $(\Delta^3 y_0)$
8 (x_2)	8 (y_2)		$2-5=-3$ $(\Delta^2 y_1)$	
		$10-8=2$ (Δy_3)		
10 (x_3)	10 (y_3)			

$$y_p(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{x-4}{2}$$

(Here $x_0 = 4$, $h = 6-4 = 2$ (difference)).

$$y_p(x) = 1 + \frac{\left(\frac{x-4}{2}\right)(2)}{1!} + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)(3)}{2!} + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)(-6)}{3!}$$

$$= 1 + x - 4 + \frac{(x-4)(x-6)}{8}(3) + (x-4)(x-6)(x-8)\left(\frac{-6}{48}\right)$$

$$= x - 3 + \frac{3}{8}(x^2 - 10x + 24) - \frac{1}{8}\left((x^2 - 10x + 24)(x-8)\right)$$

$$= \frac{1}{8}(-x^3 + 21x^2 - 126x + 240).$$

Put $x=5$,

$$y_p(5) = \frac{1}{8}\left(-5^3+21(5)^2-126(5)+240\right)$$
$$= \underline{\underline{1.25}}$$

2. Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find $\sin 52°$ using Newton's forward interpolation formula ?

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-7×10^{-4}
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$$x = x_0 + ph \Rightarrow p = \frac{x-x_0}{h} = \frac{x-45}{5}$$

Here $x = 52$.

$$p = \frac{52-45}{5} = \underline{\underline{1.4}}$$

$$y_p(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 0.7071 + (1.4)(0.0589) + \frac{(1.4)(0.4)(-0.0057)}{2!} + \frac{(1.4)(0.4)(-0.6)(7\times10^{-4})}{3!}$$

$$= 0.7071 + 0.08246 - 0.00160 + 0.00004 = \underline{\underline{0.7880}}$$

ie $\sin 52° = \underline{\underline{0.7880}}$.

(3) Using Newton's forward Interpolation formula, from the following table, estimate the no. of students who obtained marks b/w 40 & 45.

marks	30-40	40-50	50-60	60-70	70-80
no. of students	31	42	51	35	31

First we prepare the cumulative frequency table as follows.

marks (less than x)	40	50	60	70	80
no. of students (y)	31	73	124	159	190.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$$x = x_0 + ph. \Rightarrow p = \frac{x - x_0}{h}$$

x = no. students with marks less than 45

Take $x_0 = 40$ and $h = 10$.

$$p = \frac{45-40}{10} = \underline{\underline{0.5}}$$

$$y_p(x) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2!}(9) + \frac{(0.5)(-0.5)(-1.5)(-25)}{3!} + \frac{(0.5)(-0.5)(-1.5)(-2.5)(37)}{4!}$$

$$= 47.87$$

The number of students with marks less than 45 is 47.87 ie 48. But the no. of student with marks less than 40 is 31. Hence the no. of students getting marks b/w 40 and 45 is 48 - 31 ie $\underline{\underline{17}}$.

(4) From the following table, find the value of tan (0.28) using newton's backward interpolation formula.

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2533	0.3093

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2533		0.0014		
		0.0540			
0.30	0.3093				

$x = x_n + ph \Rightarrow p = \dfrac{x - x_n}{h}$

$\therefore p = \dfrac{0.28 - 0.30}{0.05} = \underline{\underline{-0.4}}$

$$y_p(x) = 0.3093 - \frac{0.4}{1!}(0.0540) + \frac{(-0.4)(-0.4+1)(0.0014)}{2!} + \frac{(-0.4)(-0.4+1)(-0.4+2)(0.0004)}{3!} + \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(0.0002)}{4!}$$

$= 0.309 - 0.0296 - 0.000168$

$= 0.0000256 - 0.00000832$

$= \underline{\underline{0.28720}}$

Legrange's formula for uneven / unequal intervals.

Let $y = F(x)$ be a function which takes the values of $y_0\ y_1\ y_2 - - y_n$ when x assumes the values $x_0, x_1, x_2 - - x_n$ respectively.

If the values of x are at equal intervals, we use newton's forward or backward interpolation formula. If the values of x are not at equal intervals, we use the following formula

$$F(x) = \frac{(x-x_1)(x-x_2) - - (x-x_n)}{(x_0-x_1)(x_0-x_2) - - (x_0-x_n)} y_0 +$$

$$\frac{(x-x_0)(x-x_2) - - (x-x_n)}{(x_1-x_0)(x_1-x_2) - - (x_1-x_n)} y_1 +$$

$$\frac{(x-x_1)(x-x_0) - - (x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3) - (x_2-x_n)} y_2 +$$

$$- - - - - - - - - -$$

$$\frac{(x-x_0)(x-x_1) - - (x-x_n)}{(x_n-x_0)(x_n-1) - - (x_n-x_{n-1})} y_n$$

This is called Legrange's interpolation formula for unequal intervals.

Qns.

1. Using Legrange's interpolation formula find the value of y when $x = 10$ from the following table.

x	5	6	9	11
y	12	13	14	16

$x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11$

$y_0 = 12,\ y_1 = 13,\ y_2 = 14,\ y_3 = 16$

$$F(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_0)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)(x-x_0)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$F(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) +$$

$$\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)(14)}{(9-5)(9-6)(9-11)}$$

$$+ \frac{(10-5)(10-6)(10-9)(16)}{(11-5)(11-6)(11-9)}$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \underline{\underline{14.66}}$$

2. Using Lagrange's interpolation formula, find $F(4)$ s.t. $F(0)=2$, $F(1)=3$, $F(2)=12$, $F(15)=3587$?

x_0	x_1	x_2	x_3
0	1	2	15
y_0	y_1	y_2	y_3
2	3	12	3587

$$F(4) = \frac{(4-1)(4-2)(4-15)}{(0-1)(0-2)(0-3)}(2) +$$

$$\frac{(4-0)(4-2)(4-15)}{(1-0)(1-2)(1-15)}(3) +$$

$$\frac{(4-1)(4-0)(4-15)}{(2-0)(2-1)(2-15)}(12) +$$

$$\frac{(4-1)(4-2)(4-0)}{(15-0)(15-1)(15-2)}(3587)$$

$$= \frac{132}{30} - \frac{264}{14} + \frac{1584}{26} + \frac{86088}{2730} = \underline{\underline{78}}.$$

3. Find the missing term in the following table using Legrange's interpolation formula.

x	x_0 0	x_1 1	x_2 2	x 3	x_3 4
y	y_0 1	y_1 3	y_2 9	—	y_3 81

$$F(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 +$$

(17)

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(3-1)(3-2)(3-4)}{(-1)(-2)(-4)}(1) + \frac{(3-0)(3-2)(3-4)(3)}{(1)(-1)(-3)} + \frac{(3-0)(3-1)(3-4)}{(2)(1)(-2)} 9 + \frac{(3-0)(3-1)(3-2)}{(4)(3)(2)} 81$$

$$= 2/8 + -3 + 27/2 + 81/4 = \underline{\underline{31}}$$

4. Using Legrange's formula, prove that $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} + y_{-5})$ nearly.?

y_{-5}, y_{-3}, y_3, y_5 occur in the answers. So we can have the table

x	-5	-3	3	5
y	y_{-5}	y_{-3}	y_3	y_5

Let $x_0 = -5,\ x_1 = -3,\ x_2 = 3,\ x_3 = 5$

and $y_0 = y_{-5},\ y_1 = y_{-3},\ y_2 = y_3,$

$y_3 = y_5$ ..

By Legrange's formula,

$$y(x) = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} +$$

$$\frac{(x+5)(x-3)(x-5)}{(3+5)(3-3)(3-5)} y_{-3} + \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)} y_3 +$$

$$\frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} y_5$$

$$y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} +$$

$$\frac{(6)(4)(-4)}{(8)(6)(-2)} y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$$= -0.2y_{-5} + 0.5y_{-3} + y_3 - 0.3y_5$$

$$= -0.2y_{-5} + (0.3 + 0.2)y_{-3} + y_3 - 0.3y_5$$

$$= y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

Inverse Interpolation

The process of finding a value of x for the corresponding value of y is called inverse interpolation.

On Interchanging x & y in the Lagrange's formula, we obtain

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 +$$

$$\frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y-y_n)} x_1 +$$

- - - - - - - - -

- - - - - - - - - -

$$\frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n.$$

This is called inverse interpolation.

Qns

1. Find the age corresponding to the annuity value 13.6 from the given

table;

Age(x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}$$

$$\times 30 + \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)(35)}{(14.1-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} +$$

$$(45)\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} +$$

$$(40)\frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)} +$$

$$(50)\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)}$$

$$\Rightarrow x(y=13.6) = 43$$

Numerical Integration

The process of computing the value of a definite integral $\int_a^b y\,dx$ from a set of numerical values of the integrand 'y' is called numerical integration.

Let $I = \int_a^b y\,dx$ where y takes the values $y_0, y_1, y_2 - - y_n$ for $x = x_0, x_1, x_2 - - x_n$. Let the interval of integration (a,b) be divided into n equal subintervals, each of width $h = \frac{b-a}{n}$ s.t.

Trapezoidal Rule.

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx.$$

$$= \frac{h}{2}\left\{(y_0+y_n) + 2(y_1+y_2+ - - +y_{n-1})\right\}$$

$$= \frac{h}{2}\left\{(\text{sum of the first \& last term}) + 2 \times (\text{sum of the remaining ordinates})\right\}$$

Simpson's One Third Rule.

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{3}\left\{(y_0+y_n) + 2\left\{y_2+y_4+\cdots+y_{n-2}\right\} + 4\left\{y_1+y_3+\cdots+y_{n-1}\right\}\right\}.$$

Qts

(1) Evaluate the integral $\int_1^2 \frac{dx}{1+x^2}$ using trapezoidal rule with 2 sub intervals?

Dividing the interval (1, 2) into 2 equal parts, each of width

$$h = \frac{b-a}{2n} = \frac{2-1}{2} = \frac{1}{2} = 0.5.$$

Here $f(x) = \frac{1}{1+x^2}$ and $h = 0.5$.

$x_0 = 1, f(1) = 0.5$

$x = 1.5, f(1.5) = 0.307$

$x = 2, f(2) = 0.2$

x	1	1.5	2.
y	0.5	0.3077	0.2.

2. Use trapezoidal rule to evaluate $\int_0^1 x^3 dx$ considering 5 sub intervals. (20)

Ans Here $f(x) = x^3$.

$h = \frac{b-a}{n} = \frac{1-0}{5} = \underline{0.2}$

sub intervals = $(0, 0.2), (0.2, 0.4), (0.4, 0.6)$
$(0.6, 0.8) (0.8, 1.0)$.

$f(0) = 0, f(0.2) = 8 \times 10^{-3}, f(0.4) = 0.064$.
$f(0.6) = 0.216, \quad f(0.8) = 0.512, f(1) = 1$.

x	0	0.2	0.4	0.6	0.8	1
y.	0	0.008	0.064	0.216	0.512	1

$$\int_{a=0}^{b=1} x^3 dx = \frac{0.2}{2} \{(0+1) + 2 \times (0.008 + 0.064 + 0.216 + 0.512)\}$$

$$= \cancel{\frac{0.2}{2}\{(1)+0.8\}} = \cancel{0.108}$$

$$= \underline{0.26}$$

$$\int_1^2 \frac{dx}{1+x^2} = \frac{0.5}{2}\left\{(0.5+0.2)+ 2\times 0.3077\right\}.$$

$$= \frac{0.5}{2}\,(0.7+0.6154) = \underline{0.32885}$$

3. By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x\,dx$ by trapezoidal and simpson's rule. Verify the answers with integration?

$$h = \frac{\pi - 0}{10} = \underline{0.3142}$$

~~x 0 0.3142~~

x	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$
y	0 (y_0)	0.309 (y_1)	0.5878 (y_2)	0.809 (y_3)	0.9511 (y_4)	1 (y_5)

x	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
y	0.9511 (y_6)	0.809 (y_7)	0.5878 (y_8)	0.309 (y_9)	0 (y_{10})

(1) <u>Trapezoidal Rule</u>.

$$\int_0^{\pi} \sin x\,dx = \frac{\pi}{10}\cdot\frac{1}{2}\Big\{0+2\big(0.309+0.809+0.5878+0.9511+1+0.9511+0.809+0.5878+0.309\big)\Big\}.$$

$$= \frac{\pi}{20} = \underline{\underline{1.984}}$$

Simpson's Rule

$$\int_0^{\pi} \sin x\,dx = \frac{\pi}{10}\cdot\frac{1}{3}\Big\{0+2\times\{0.5878+0.9511+0.9511+0.5878+0\}+4\times\{0.309+0.809+1+0.809+0.309\}\Big\}.$$

$$= \frac{\pi}{30}\Big\{6.1556+12.944\Big\} = \underline{\underline{2.000}}$$

By actual Integration

$$\int_0^{\pi} \sin x\,dx = -(\cos x)_0^{\pi} = -(\cos \pi - \cos 0)$$

$$= -(-2) = \underline{\underline{2}}$$

4. Evaluate $\int_0^5 \frac{dx}{4x+5}$ by Simpson's one third rule & hence find the value of $\log_e 5$, when $n=10$?.

$$h = \frac{b-a}{n} = \frac{5-0}{10} = \underline{\underline{0.5}}$$

x	0	0.5	1	1.5	2
y	0.2 y_0	0.1429 y_1	0.1112 y_2	0.0909 y_3	0.0769 y_4

x	2.5	3	3.5	4	4.5	5
y	0.0667 y_5	0.0588 y_6	0.0526 y_7	0.0476 y_8	0.0434 y_9	0.04 y_{10}

$$\int_0^5 \frac{dx}{4x+5} = \frac{0.5}{3}\Big\{(0.2+0.04)+2(0.1112+0.0769+0.0588+0.0476+\cancel{0.0004})$$

$$+4(0.1429+0.0909+0.0667+0.0526+0.0434)\Big\}$$

0.589

~~0.669~~ 1.586

$$= \frac{0.5}{3}\left(\cancel{0.669}+\cancel{2.7676}+0.24\right)$$

$$= \cancel{0.8161} \quad \cancel{0.4158} \quad 0.4025$$

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{\log(4x+5)}{4}\right]_0^5 = \frac{1}{4}(\log 25 - \log 5)$$

$$\frac{1}{4}(\log 25 - \log 5) = \cancel{0.4158} \quad 0.4025$$

$$\log 25 - \log 5 = \cancel{1.6632}$$

$$0.4025 \times 4$$

$\rightarrow \log\left(\frac{85}{5}\right) = \log 5 = \cancel{1.6632}\ \underline{1.61}$ (22)