MODULE 3:
RANDOM PROCESS

Q. Define random process? Explain classification of random process with suitable example?

A random variable that is a function Of one independent variable is called random process.

Eg:X(t) is a random process, where X is the random variable & t is the independent variable.

Random process is also termed as NON-DETERMINISTIC STOCHIASTIC process.

A random process is a fn. Of one or more independent variables.

X(t)➔Fn. Of one independent variable.

X(t,u)➔Fn. Of two independent variables.

X(t,u,v)➔Fn. Of three independent variables.

CLASSIFICATION OF RANDOM PROCESS:-

1. CONTINUOUS 2.DISCRETE

3. DETERMINISTIC, 4.NON DETERMINISTIC

1. CONTINUOUS RANDOM PROCESS:-

A random process X(t ,u) is said to be continuous random process. If the random variable X is continuous & 't' can have any of the continuous values.

Eg: The maximum temperature of a particular place in the interval (0,t) is a continuous random process. The set of all possible values of X is continuous in continuous time.

2. DISCRETE RANDOM PROCESS:-

A random process X(t, u) is said to be discrete if the random variables X takes only discrete values while 't' is continuous.

Eg: X(t) represents the number of telephone calls received into a switchboard of a company in (0, t). The set of all possible values of X no. of calls is discrete while t is continuous, where u={0,1,2..}.

3. DISCRETE RANDOM SEQUENCE

A random process is said to be discrete random sequence if both random variables & time are discrete.

Eg:-If Xn represents outcome of the nth toss of a fair dice, then {Xn,n>=1} is a discrete random sequence in which T={1,2,3...} & U={1,2,3,4,5,6}.

4. CONTINUOUS RANDOM SEQUENCE

A random process X(t,s) for which the random variable is continuous but time has only discrete values is called as continuous random sequence.

eg:-If Xn represents the cool temperature at the end of the nth hour of the day, then {Xn,1≤n≤24} } is continuous random sequence as the temperature can take any value in the interval in which it is continuous.

5 .<u>DETERMINISTIC RANDOM VARIABLE</u>

A random process in which the future value of any sample function can be predicted exactly from past values is called deterministic random process. I t is also called predictable random process.

*Eg:-Consider the random process X(t)=rsin (ωt+φ),where r is the random amplitude , 'φ' is the phase angle & ω=2*3.14*t.*

From the above, we can understand that it's completely specified in terms of the random variable r & 'φ'.

6. <u>NON-DETERMINISTIC RANDOM VARIABLE</u>

If the function value of any sample function cannot be predicted exactly from the past values, it's called non deterministic random process.

Eg:-It consists of a family functions that cannot be described in terms of finite number of parameters.

<u>FIRST ORDER DISTRIBUTION OF RANDOM PROCESS</u>

Let X(t) be the distribution function with the random variable X, then, P(X(t)≤x)=F(x,t). For any real number,this function depends on x & t. The first order density of the process X(t) is denoted by f(x, t) & is given by f(x, t)=d/dx(F(x, t)).

<u>SECOND ORDER DISTRIBUTION OF RANDOM PROCESS</u>

If X(t) is the joint distribution of 2 random variables X(t1) & X(t2), then we have p(X(t)≤x1,X(t)≤x2)=F(x1,x2;t1,t2).

Second order density of the process x(t) is denoted by F(x1,x2;t1,t2) & is given by F(x1,x2;t1,t2)=δ(δ(F(x1,x2;t1,t2))) .

<u>MEAN OF THE RANDOM PROCESS</u>

Mean of the random process is the expected value of the random variable X(t) & it is denoted by μ(t) with X as subscript(also μ(t))=E(X(t))=∫(xf(x,t)),with limits negative to positive infinity(∞).

<u>AUTO-CORRELATION OF RANDOM PROCESS</u>

Let X(t1) & X(t2) be the 2 give random variables. The auto-correlation of these random variables is the product of the expected value of the random variables X(t1) & X(t2) & it is denoted as R_{xx}(t1,t2) or R(t1,t2)

R_{xx}(t1,t2).=E(X(t1).X(t2))=∫∫(f(x1,x2;t1,t2)dxdy),with limits from negative to positive infinity.

When t1=t2=t, R_{xx}(t1,t2).= E(X(t).X(t))= E(X^2(t)), which is called the mean square value or average power of the random process.

AUTO –COVARIANCE OF A RANDOM PROCESS X(t)

It is the covariance of the random variables X(t1) & X(t2) & is denoted by $C_{xx}(t1,t2)$.

$C_{xx}(t1,t2)= C(t1,t2)= R(t1,t2)-(\mu(t1)*\mu(t2))$. When t1=t2=t, $C_{xx}(t1,t2)$ is X(t).

CORRELATION COEFFICIENT

$\rho_{xx}(t1,t2)= \rho(t1,t2)= C_{xx}(t1,t2)./\sqrt{(C_{xx}(t1,t1)* C_{xx}(t2,t2).)}$

CROSS CORRELATION

The cross correlation of the random process X(t) & Y(t) is given by, $R_{xy}(t1,t2)=E(X(t1) *Y(t2))$, where X(t1) & Y(t2) are the random variables.

CROSS COVARIANCE OF 2 RANDOM PROCESS X(t) & Y(t)

$C_{xy}(t1,t2)= R_{xy}(t1,t2)-(\mu_x(t1)*\mu_y(t2))$.

CROSS CORRELATION COEFFICIENT FOR RANDOM VARIABLES X(t) & Y(t)

$R(t1,t2)= C_{xy}(t1,t2)/\sqrt{[[C_{xx}(t1,t1) *C_{yy}(t2,t2)]]}$

ORTHOGONAL PROCESS

Two random variables X(t) & Y(t) said to be orthogonal if,

R(t1,t2)=0,for every t1 & t2.

NOTE:2 Random variables X(t) & Y(t) are said uncorrelated if $C_{xy}(t1,t2)= 0$,for every t1 & t2.

NOTE:2 Random variables are said to be statistically independent if

$R_{xy}(t1,t2)=E(X(t1) * E(Y(t2))$.

STRICT SENSE STATIONARY PROCESS (SSS PROCESS)

A random process X(t) is said to be in SSS PROCESS if:

E(X(t)) is a constant & Var(X(t)) is a constant.

WIDE SENSE STATIONARY PROCESS(WSS PROCESS)

A random process X(t) is said to be WSS process if ;

E(X(t)) is a constant & auto correlation depends on time difference (ie. it is a function of t1-t2).

NOTE:If the random process is not stationary, then it is called evolutionary process.

Problems:

1. In tossing a fair coin, the RP is defined as $X(t) = \begin{cases} \sin \pi t & \text{, if head shows.} \\ 2t & \text{, if tail shows.} \end{cases}$ Find:

(a) $E(X(t))$ (b) $F(x,t)$ $\forall$ $t = 0.25$ and 1 ?

(a) $P(X(t) = \sin \pi t) = \frac{1}{2}$ &

$P(X(t) = 2t = \frac{1}{2})$.

$E(X(t)) = (\sin \pi t)(\frac{1}{2}) + (2t)(\frac{1}{2})$.

$= \frac{\sin \pi t}{2} + t$.

(b) When $t = 0.25 = \frac{1}{4}$.

$P(X(t) = \sin \pi(\frac{1}{4}) = \sin \frac{\pi}{4} = 0.707) = \frac{1}{2}$.

$P(X(t) = 2 \times \frac{1}{4} = \frac{1}{2}) = \frac{1}{2}$.

$\therefore F(x,t) = P(X(t) \leq x)$

$= \frac{1}{2}$ for $\frac{1}{2} \leq x \leq 0.707$

$F(x, 0.25) = \begin{cases} 0, & \text{if } x < \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq x < 0.707 \\ 1, & \text{if } x \geq \frac{1}{\sqrt{2}}. \end{cases}$

when $t = 1$;

$P(X(t) = \sin \pi = 0) = \frac{1}{2}$ & $P(X(t) = 2(1) = 2) = \frac{1}{2}$.

$\therefore F(x,t) = \frac{1}{2}$ if $0 \leq x < 2$

$$F(x,1) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < a \\ 1, & \text{if } x \ge a \end{cases}$$

(2) If $X(t) = A + Bt$, where A & B are idependent random variables with $E(A) = p$ & $E(B) = q$, & $Var(A) = \sigma_A^2$ & $Var(B) = \sigma_B^2$. Find:

(a) $E(X(t))$. (b) $R(t_1, t_2)$ (c) $C(t_1, t_2)$.

(a) $E(X(t)) = \mu(t) = E(A + Bt)$

$= E(A) + t \cdot E(B) = \underline{\underline{p + qt}}$.

(b) $R(t_1, t_2) = E(X(t_1) \cdot X(t_2))$

$= E((A + Bt_1)(A + Bt_2))$

$= E(A^2 + ABt_2 + ABt_1 + B^2 t_1 t_2)$.

$= E(A^2) + E(AB) \cdot (t_1 + t_2) + t_1 t_2 \cdot E(B^2)$.

given $\sigma_A^2 = E(A^2) - (E(A))^2 = E(A^2) - p^2$

$\sigma_B^2 = E(B^2) - (E(B))^2 = E(B^2) - q^2$

since $E(AB) = E(A) \cdot E(B)$, if A & B are indepen-dent variables, $E(AB) = p \cdot q$.

$\therefore R(t_1, t_2) = \sigma_A^2 + p^2 + pq(t_1 + t_2) + t_1 t_2 (\sigma_B^2 + q^2)$

(c) $C(t_1, t_2) = R(t_1, t_2) - \mu(t_1) \cdot \mu(t_2)$

$= R(t_1, t_2) - E(X(t_1)) \cdot E(X(t_2))$.

$$= \sigma_A^2 + p^2 + pq(t_1+t_2) + (\sigma_B^2 + q^2)t_1 t_2 - (p+qt_1)(p+qt_2)$$

$$= \sigma_A^2 + p^2 + t_1 pq + t_2 pq + t_1 t_2 \sigma_B^2 + t_1 t_2 q^2 - p^2 - pqt_2 - qpt_1 - q^2 t_1 t_2$$

$$= \sigma_A^2 + t_1 t_2 \sigma_B^2.$$

③. If $X(t) = \delta(t)$ is a stochastic process, find $E(X(t))$, $R(t_1,t_2)$ & $C(t_1,t_2)$?

$$E(X(t)) = \mu(t) = E(\delta(t)) = E(1)\cdot\delta(t) = \delta(t)$$

$$R(t_1,t_2) = E(X(t_1))\cdot E(X(t_2))$$
$$= E(X(t_1)\cdot X(t_2))$$
$$= E(\delta(t_1)\cdot\delta(t_2)) = \delta(t_1)\cdot\delta(t_2)$$

$$C(t_1,t_2) = R(t_1,t_2) - E(X(t_1))\cdot E(X(t_2))$$
$$= R(t_1,t_2) - \mu(t_1)\cdot\mu(t_2)$$
$$= \delta(t_1)\cdot\delta(t_2) - \delta(t_1)\cdot\delta(t_2)$$
$$= 0$$

④ Suppose $X(t)$ is a process with $\mu(t) = 3$, $R(t_1,t_2) = 9 + 4e^{-|t_1-t_2|/.5}$. Find the variance and covariance of $X(5)$ and $X(8)$.

Given: $E(X(t)) = \mu(t) = E(X(5)) = E(X(8)) = 3$

$$E(X^2(5)) = E(X(5))\cdot E(X(5))$$
$$= E(X(5)\cdot X(5)) = R(5,5)$$
$$= 9+4e^{-|5-5|/5} = 9+4e^{0}$$
$$= 9+4 = \underline{13}$$

$$E(X^2(8)) = E(X(8))\cdot E(X(8))$$
$$= E(X(8)\cdot X(8)) = R(8,8)$$
$$= 9+4e^{-|8-8|/5} = 9+4e^{0}$$
$$= \underline{13}$$

$$E(X(5)\cdot X(8)) = R(5,8)$$
$$= 9+4e^{-|5-8|/5}$$
$$= 9+4e^{-3/5} = \underline{11.195}$$

$$\sigma_X^2(5) = E(X^2(5)) - [E(X(5))]^2$$
$$= 13-(3)^2 = 13-9 = \underline{4}$$

$$\sigma_X^2(8) = E(X^2(8)) - [E(X(8))]^2$$
$$= 13-9 = \underline{4}$$

Covariance $= C(5,8)$
$$= R(5,8) - \mu(5)\cdot\mu(8)$$
$$= 11.195 - 9$$
$$= \underline{2.195}$$

(5) If $X(t) = A\cos\omega t - B\sin\omega t$, where A & B are independent normal random variables with zero mean & variance σ^2; find $E(X(t))$ & $E(X^2(t))$?

since A & B are independant,

$E(AB) = E(A) \cdot E(B)$

$E(X(t)) = E(A\cos\omega t - B\sin\omega t)$

$= E(A) \cdot \cos\omega t - E(B) \cdot \sin\omega t$

$= 0$ $\left\{ \begin{array}{l} \text{since it is given that} \\ E(A) = E(B) = 0 \end{array} \right\}$.

$E(X^2(t)) = E[(A\cos\omega t - B\sin\omega t)^2]$

$= E[A^2\cos^2\omega t + B^2\sin^2\omega t - 2AB\cos\omega t\sin\omega t]$

$= E(A^2)\cos^2\omega t + E(B^2)\sin^2\omega t -$

$E(AB) \cdot 2\cos\omega t\sin\omega t$

$= \sigma^2\cos^2\omega t + \sigma^2\sin^2\omega t - 0.$

$= \sigma^2(\cos^2\omega t + \sin^2\omega t) = \sigma^2$

$\left\{ \begin{array}{l} \text{since it is given that} \\ E(A^2) = E(B^2) = \sigma^2, \text{ie}, \\ E(A^2) = \sigma^2 - (E(A))^2 = \sigma^2 \\ E(B^2) = \sigma^2 - (E(B))^2 = \sigma^2 \end{array} \right\}$

(6) If $X(t) = r\cos(\omega t + \phi)$, where the random variables r & ϕ are independant & ϕ is uniform in $(-\pi, \pi)$, find $R(t_1, t_2)$?

Given :- $X(t) = r\cos(\omega t + \phi)$

$= r[\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)]$

$X(t_1) = r(\cos(\omega t_1)\cos\phi - \sin(\omega t_1)\sin\phi)$.

$X(t_2) = r(\cos(\omega t_2)\cos\phi - \sin(\omega t_2)\sin\phi)$

$$x(t_1)\cdot x(t_2) = r^2\left[\cos(\omega t_1)\cos\phi - \sin(\omega t_1)\sin(\phi)\right]\left[\cos(\omega t_2)\cos\phi - \sin(\omega t_2)\sin(\phi)\right].$$

$$= r^2\left[\cos(\omega t_1)\cos(\omega t_2)\cos^2\phi - \cos(\omega t_1)\sin(\omega t_2)\sin\phi\cos\phi - \sin(\omega t_1)\cos(\omega t_2)\sin\phi\cos\phi + \sin(\omega t_1)\cos(\omega t_2)\sin^2\phi.\right]$$

$$= r^2\left[\cos(\omega t_1)\cdot\cos(\omega t_2)\left(\frac{1+\cos 2\phi}{2}\right) + \sin(\omega t_1)\sin(\omega t_2)\left(\frac{1-\cos 2\phi}{2}\right) - \sin\omega(t_1+t_2)\left(\frac{\sin 2\phi}{2}\right)\right]$$

$$= \frac{r^2}{2}\left[\cos[\omega(t_1-t_2)] + \cos 2\phi\cdot\cos[\omega(t_1+t_2)] - \sin[\omega(t_1+t_2)]\sin(2\phi).\right].$$

$$= \frac{r^2}{2}\left[\cos[\omega(t_1-t_2)] + \cos(\omega(t_1+t_2)+2\phi)\right]$$

$$R(t_1,t_2) = E(x(t_1)\cdot x(t_2))$$

$$= \frac{1}{2}\cdot E(r^2)\cdot\cos[\omega(t_1-t_2)] + \frac{1}{2}\cdot E(r^2)\cdot E\left(\cos(\omega(t_1+t_2)+2\phi)\right)$$

since 'ϕ' is uniform in $(-\pi,\pi)$, its pdf is

$$\frac{1}{b-a} = \frac{1}{\pi+\pi} = \frac{1}{2\pi}$$

$$E(\cos(\omega(t_1+t_2)+2\phi)) = \int_{-\pi}^{\pi}\cos(\omega(t_1+t_2)+2\phi)\frac{1}{2\pi}\,d\phi$$

$$= \frac{1}{2\pi}\left[\frac{\sin(\omega(t_1+t_2)+2\phi)}{2}\right]_{-\pi}^{\pi} = 0.$$

$$\therefore R(t_1,t_2) = \frac{1}{2} E(r^2)\cdot \cos(\omega(t_1-t_2))$$

$$= \frac{r^2}{2}\cdot \cos[\omega(t_1-t_2)]$$

$\left\{ \text{since } E(\text{constant}) = \text{a constant} \right\}$

(7) consider the random process $x(t) = \cos(t+\phi)$, where ϕ is a random variable with density function $F(\phi) = \frac{1}{\pi}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}$. check whether the process is stationary or not?

$$x(t) = \cos(t+\phi)$$

$$F(\phi) = \frac{1}{\pi}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}.$$

$$E(x(t)) = \mu(t) = \int_{-\infty}^{\infty} x(t)\cdot F(\phi)\, d\phi.$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \cos(t+\phi)\frac{1}{\pi}\, d\phi = \frac{1}{\pi}\left[\sin(t+\phi)\right]_{-\pi/2}^{\pi/2}.$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}+t\right) - \sin\left(t-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}(\cos t + \cos t)$$

$$= \frac{2\cos t}{\pi} \neq \text{constant}.$$

$\therefore x(t)$ is a stationary process.

(8) Consider the random process $X(t) = \cos(\omega_0 t + \theta)$ where θ is uniformly distributed in the interval $(-\pi, \pi)$. Check whether $X(t)$ is stationary or not.

Since 'θ' is uniformly distributed in $(-\pi, \pi)$,

$$F(\theta) = \frac{1}{2\pi}, \quad -\pi < \theta < \pi.$$

$$E(X(t)) = \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi} \left[\sin(\omega_0 t + \theta) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[-\sin(\omega_0 t - \pi) + \sin(\omega_0 t + \pi) \right]$$

$$= \frac{1}{2\pi} \left[-\sin \omega_0 t + \sin(\pi - \omega_0 t) \right]$$

$$= \frac{1}{2\pi} \left(-\sin \omega_0 t + \sin \omega_0 t \right) = 0 = \text{a constant.}$$

$$Var(X(t)) = E(X^2(t)) - (E(X(t)))^2$$

$$E(X^2(t)) = E\left(\cos^2(\omega_0 t + \theta)\right)$$

$$= E\left(\frac{1 + \cos 2(\omega_0 t + \theta)}{2}\right)$$

$$= \frac{1}{2} \quad E(1) + E\left(\cos(2\omega_0 t + 2\theta)\right)$$

$$= \frac{1}{2} \quad \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta + \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta) \cdot \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2\pi}\left\{ (\theta)_{-\pi}^{\pi} + \left[\frac{\sin(2\omega_0 t + 2\theta)}{2}\right]_{-\pi}^{\pi} \right\}$$

$$= \frac{1}{4\pi}\cdot\left[2\pi + \frac{1}{2}\ \sin(2\omega_0 t + 2\pi) - \sin(2\omega_0 t - 2\pi)\right]$$

$$= \frac{1}{4\pi}\left\{2\pi + \frac{1}{2}\left[\sin(2\omega_0 t) - \sin(2\omega_0 t)\right]\right\}$$

$$= \frac{1}{4\pi}(2\pi) = \frac{1}{2} = \text{a constant.}$$

$$\mathrm{Var}(X(t)) = E(X^2(t)) - [E(X(t))]^2$$
$$= \frac{1}{2} - 0 = \frac{1}{2} = \text{a constant.}$$

$\therefore$ $X(t)$ is a stationary process.

(9) The process $\{X(t)\}$, where probability distribution under certain conditions is given by

$$P(X(t)=n) = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n \in N$$

$$= \frac{at}{1+at}, \quad n = 0.$$

Show that, it is not stationary?

$X(t)=n$	0	1	2	3	----
P_n	$\frac{at}{at+1}$	$\frac{1}{(at+1)^2}$	$\frac{at}{(at+1)^3}$	$\frac{(at)^2}{(at+1)^4}$	----

$$E(X(t)) = \sum_{n=0}^{\infty} n P_n$$

$$= \frac{at}{1+at} \times 0 + \frac{1}{(1+at)^2} \times 1 + \frac{(at)}{(1+at)^3} \times 2 + \frac{(at)^2}{(1+at)^4} \times 3 - - -$$

$$= 0 + \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + - -$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2\,at}{(1+at)} + 3\left(\frac{at}{1+at}\right)^2 + - - -\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2} \cdot \frac{1}{(1+at)^{-2}} = \frac{(1+at)^2}{(1+at)^2} = 1$$

$$= \text{a constant.}$$

$$\left\{ \text{using the formula, } 1 + 2x + 3x^2 + \cdots = (1-x)^{-2} \right\}$$

$$E(X^2(t)) = \sum_{n=0}^{\infty} n^2 P_n$$

$$= 1^2 \cdot \frac{1}{(1+at)^2} + 2^2 \frac{at}{(1+at)^3} + 3^2 \frac{(at)^2}{(1+at)^4}$$

$$= \frac{1}{(at+1)^2}\left[1 + \frac{2^2 at}{1+at} + 3^2\left(\frac{at}{1+at}\right)^2 + - -\right]$$

$$= \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} \frac{n^2 (at)^{n-1}}{(at+1)^{n-1}}$$

$$= \frac{1}{(1+at)^2} \cdot \sum_{n=1}^{\infty} (n(n+1)-n) \frac{(at)^{n-1}}{(at+1)^{n-1}}$$

$$= \frac{1}{(at+1)^2} \left[\sum_{n=1}^{\infty} n(n+1) \frac{(at)^{n-1}}{(at+1)^{n-1}} - \sum_{n=1}^{\infty} n \cdot \frac{(at)^{n-1}}{(at+1)^{n-1}} \right]$$

$$= \frac{1}{(at+1)^2} \left\{ \left[(1)(2) + (2)(3)\frac{at}{1+at} + (3)(4)\left[\frac{at}{1+at}\right]^2 + \cdots \right] - \left[1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \cdots \right] \right\}.$$

$$= \frac{1}{(1+at)^2} \left[2\left(1 - \frac{at}{1+at}\right)^{-3} - \left(1 - \frac{at}{1+at}\right)^{-2} \right]$$

using the formula,

$$(1-x)^{-3} = \frac{1}{2}\left[(1)(2) + (2)(3)x + (3)(4)x^2 + \cdots n(n+1)x^{n-1} \cdots\right].$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots nx^{n-1} + \cdots$$

$$= \frac{1}{(1+at)^2} \left\{ 2\left[\frac{1}{1+at}\right]^{-3} - \left[\frac{1}{1+at}\right]^{-2} \right\}.$$

$$= \frac{(1+at)^2}{(1+at)^2} \cdot \left[2(1+at) - 1\right]$$

$$= 2 + 2at - 1$$

$$= \underline{1 + 2at}$$

$$Var(x(t)) = E(x^2(t)) - (E(x(t)))^2$$

$$= 1 + 2at - 1$$

$$= 2at \neq \text{constant}.$$

$E(x(t)) =$ constant; but

$Var(x(t))$ is not a constant.

$\therefore$ the given process isn't a stationary process.

(10) Show that the random process $x(t) = A\cos(\omega t + \phi)$ is wide sense stationary if A & ω are constants & ϕ is uniformly distributed random variable in $(0, 2\pi)$?

Given $x(t) = A\cos(\omega t + \phi)$.

Since ϕ is uniformly distributed,

$$f(\phi) = \frac{1}{2\pi}, \quad 0 < \phi < 2\pi.$$

$$E(x(t)) = \int_0^{2\pi} x(t) \cdot f(\phi) \, d\phi$$

$$= \frac{1}{2\pi} A \int_0^{2\pi} \cos(\omega t + \phi) \, d\phi$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + \phi)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + 2\pi) - \sin \omega t\right]$$

$$= \frac{A}{2\pi} (\sin \omega t - \sin \omega t) = 0 = \text{a constant}.$$

The auto correlation,

$$R(t_1, t_2) = E(X(t_1) \cdot X(t_2))$$

$\because X(t_1) = A\cos(\omega t_1 + \phi),\ X(t_2) = A\cos(\omega t_2 + \phi)$

$\therefore X(t_1) \cdot X(t_2) = A\cos(\omega t_1 + \phi) \cdot A\cos(\omega t_2 + \phi)$

$$= A^2 \cdot \cos(\omega t_1 + \phi)\cos(\omega t_2 + \phi)$$

$$= \frac{A^2}{2} \cdot [\cos(\omega t_1 + \phi + \omega t_2 + \phi) + \cos(\omega t_1 + \phi - \omega t_2 - \phi)]$$

$$\boxed{2\cos A\cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{A^2}{2}[\cos(\omega(t_1 + t_2) + 2\phi) + \cos \omega(t_1 + t_2)]$$

$$E[X(t_1) \cdot X(t_2)] = \frac{A^2}{2} E(\cos(\omega[t_1 + t_2] + 2\phi)) + \frac{A^2}{2}(\cos \omega[t_1 - t_2])$$

$$\boxed{E(\text{constant}) = \text{constant}}$$

$$= \frac{A^2}{2} \int_0^{2\pi} \frac{1}{2\pi}(\cos \omega(t_1 + t_2) + 2\phi)\, d\phi + \frac{A^2}{2} \cdot \cos[\omega(t_1 + t_2)]$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin(\omega(t_1 + t_2) + 2\phi)}{2}\right]_0^{2\pi} + \frac{A^2}{2}\cos \omega(t_1 - t_2)$$

$$= \frac{A^2}{8\pi}[\sin(\omega(t_1 + t_2)) + 4\pi - \sin \omega(t_1 + t_2)] + \frac{A^2}{2} \cdot \cos \omega[t_1 - t_2]$$

$$= \frac{A^2}{8\pi}\left\{\sin[\omega(t_1+t_2)] - \sin(\omega[t_1+t_2])\right\} +$$

$$\frac{A^2}{2}\cos[\omega(t_1-t_2)].$$

$$= 0 + \frac{A^2}{2}\cos(\omega[t_1-t_2]) = \frac{A^2}{2}\cos(\omega(t_1-t_2)).$$

$R(t_1, t_2)$ is a function of $\overline{\overline{t_1 - t_2}}$.

$\therefore \{X(t)\}$ is a WSS process.

(11) Given a random variable Y with characteristic function $\phi(\omega) = E(e^{i\omega y})$ and a random process defined by $X(t) = \cos(\lambda t + y)$. Show that, $\{X(t)\}$ is stationary in the wide sense of $\phi(1) = \phi(2) = 0$?

Given that :- $\phi(1) = \phi(2) = 0$.

$$E(X(t)) = E(\cos(\lambda t + y))$$

$$= E(\cos\lambda t \cos y - \sin\lambda t \sin y)$$

$$= \cos\lambda t \cdot E(\cos y) - \sin\lambda t\, E(\sin y). \quad \text{--- (1)}$$

Given $\phi(\omega) = E(e^{i\omega y})$

$$\therefore \phi(1) = E(e^{iy}) = 0.$$

$$\boxed{e^{ix} = \cos x + i\sin x}$$

$$\phi(1) = E(e^{iy}) = E(\cos y + i\sin y) = 0.$$

$$E(\cos y) + i.\,E(\sin y) = 0$$

$$\Rightarrow E(\cos y) = 0 = E(\sin y).$$

similarly, $\phi(2) = 0 \Rightarrow E(e^{2iy}) = 0.$

$$E(\cos 2y + i\sin 2y) = 0.$$

$$E(\cos 2y) + i\,E(\sin 2y) = 0.$$

$$\Rightarrow E(\cos 2y) = 0 = E(\sin 2y).$$

(1) $\Rightarrow E(x(t)) = 0 =$ a constant.

$$R(t_1, t_2) = E(x(t_1) \cdot x(t_2))$$

$$= E(\cos(\lambda t_1 + y) \cdot \cos(\lambda t_2 + y))$$

$$= E([\cos \lambda t_1 \cos y - \sin \lambda t_1 \sin y].$$
$$[\cos \lambda t_2 \cos y - \sin \lambda t_2 \sin y])$$

$$= E[\cos \lambda t_1 \cos \lambda t_2 \cos^2 y -$$
$$\cos \lambda t_1 \sin \lambda t_2 \cos y \sin y + \sin \lambda t_1 \sin \lambda t_2 \sin^2 y +$$
$$- \sin \lambda t_1 \cos \lambda t_2 \sin y \cos y].$$

$$= \cos \lambda t_1 \cos \lambda t_2 E(\cos^2 y) -$$
$$\cos \lambda t_1 \sin \lambda t_2 E(\cos y \sin y)$$
$$- \sin \lambda t_1 \cos \lambda t_2 E(\sin y \cos y) +$$
$$\sin \lambda t_1 \sin \lambda t_2 E(\sin^2 y).$$

$$= \cos \lambda t_1 \cos \lambda t_2 \ E\left[\frac{1+\cos 2y}{2}\right]$$
$$- \cos \lambda t_1 \sin \lambda t_2 \ E\left[\frac{\sin 2y}{2}\right]$$
$$- \sin \lambda t_1 \cos \lambda t_2 \ E\left[\frac{\sin 2y}{2}\right]$$
$$+ \sin \lambda t_1 \sin \lambda t_2 \ E\left[\frac{1-\cos 2y}{2}\right].$$

$$= \cos \lambda t_1 \cos \lambda t_2 \ E\left[\frac{1}{2} + \frac{\cos 2y}{2}\right] -$$
$$\cos \lambda t_1 \sin \lambda t_2 \ E\left(\frac{\sin 2y}{2}\right) -$$
$$\sin \lambda t_1 \cos \lambda t_2 \ E\left(\frac{\sin 2y}{2}\right) +$$
$$\sin \lambda t_1 \sin \lambda t_2 \ E\left[\frac{1}{2} - \frac{\cos 2y}{2}\right].$$

$$= \frac{1}{2} \cos \lambda t_1 \cos \lambda t_2 + \frac{1}{2} \sin \lambda t_1 \sin \lambda t_2$$
$$= \frac{1}{2} \cos(\lambda(t_1 - t_2))$$

$$\boxed{E(\sin 2y) = E(\cos 2y) = 0}$$

$\because R(t_1, t_2)$ is a function of $t_1 - t_2$,

$\{x(t_1)\}$ is a WSS process.

(12) For a random process $X(t) = Y \sin \omega t$, Y is a uniform random variable in the interval $(-1, 1)$. Check whether the process is stationary or not?

$\because Y$ is uniformly distributed in the interval $(-1,1)$, $X(t) = Y\sin\omega t$.

$$f(y) = \frac{1}{2}, \quad -1 < y < 1$$

$$E(X(t)) = \int_{-1}^{1} \frac{1}{2} y \sin\omega t \, dy = \frac{1}{2} \sin\omega t \left(\frac{y^2}{2}\right)_{-1}^{1}.$$

$$= \frac{1}{2} \sin\omega t \left(\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2} \cdot \sin(\omega t(0)) = 0$$

$$R(t_1, t_2) = E(X(t_1) \cdot X(t_2))$$

$$= E[Y\sin(\omega t_1) \cdot Y\sin(\omega t_2)]$$

$$= E[Y^2 \sin(\omega t_1)\sin\omega t_2.]$$

$$= \sin(\omega t_1)\sin(\omega t_2) E(Y^2).$$

$$= \sin(\omega t_1)\sin(\omega t_2) \int_{-1}^{1} y^2 f(y) dy.$$

$$= \sin(\omega t_1)\sin(\omega t_2) \int_{-1}^{1} y^2 \frac{1}{2} \cdot dy.$$

$$= \frac{1}{2}\sin(\omega t_1)\sin(\omega t_2) \left(\frac{y^3}{3}\right)_{-1}^{1}$$

$$= \frac{1}{2} \sin(\omega t_1)\sin(\omega t_2)\left(\tfrac{1}{3} + \tfrac{1}{3}\right)$$

$$= \frac{1}{6}[2 \cdot \sin(\omega t_1)\sin(\omega t_2)].$$

$$= \frac{1}{6}[\cos(\omega t_1 - \omega t_2) - \cos(\omega t_1 + \omega t_2)]$$

$$= \frac{1}{6}[\cos(\omega(t_1 - t_2)) - \cos(\omega(t_1 + t_2))]$$

$$\boxed{2\sin A \sin B = \cos(A-B) - \cos(A+B)}$$

Here mean is constant, but the auto-correlation function is not a function of time differences of the 2 random variables.

$\therefore$ the process isn't a WSS process.

Homework : Qts:

(13) If $X(t) = \sin(\omega t + Y)$ where Y is uniformly distributed in $(0, 2\pi)$; show that $X(t)$ is a wide sense stationary process?

(14) consider the random process $X(t) = A\cos(100t + \theta)$ where A & θ are independent variables. A is a random variable with mean 0 & variance 1. θ is uniformly distributed in $(-\pi, \pi)$. Find mean & auto correlation.?

(15) Consider a random process $Y(t) = X(t)\cos(\omega_0 t + \theta)$ where $X(t)$ is widesense stationary random process. θ is a random variable independent of $X(t)$ & is distributed uniformly in $(-\pi, \pi)$ & ω_0 is a constant. Prove that $Y(t)$ is a widesense stationary process?

Answers

(13) $X(t) = \sin(\omega t + Y)$

since 'Y' is uniformly distributed within the interval $(0, 2\pi)$, $f(Y) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$,

$0 < Y < 2\pi$.

$$E(x(t)) = \int_0^{2\pi} x(t)\cdot f(Y)\,dY = \int_0^{2\pi} \sin(\omega t+Y)\frac{1}{2\pi}\,dY.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(\omega t+Y)\,dY = \frac{-1}{2\pi}\left[\cos(\omega t+Y)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi}\left(\cos(\omega t+2\pi) - \cos\omega t\right)$$

$$= \frac{-1}{2\pi}\left[\cos(\omega t) - \cos(\omega t)\right] = \underline{\underline{0}} = \text{a constant}$$

Auto correlation

$$R_{XY}(t_1, t_2) = E[x(t_1)\cdot x(t_2)]$$

$$x(t_1)x(t_2) = \sin(\omega t_1+Y)\cdot\sin(\omega t_2+Y)$$

$$= [\cos(\omega t_1+Y-\omega t_2-Y) - \cos(\omega t_1+Y+\omega t_2+Y)]\cdot\frac{1}{2}.$$

$$= \frac{1}{2}[\cos(\omega t_1\ \omega t_2) - \cos(\omega t_1+\omega t_2+2Y)]$$

$$= \frac{1}{2}[\cos[\omega(t_1\ t_2)] - \cos(\omega(t_1+t_2)+2Y)]$$

$$E(x(t_1)\cdot x(t_2)) = E\left[\frac{\cos(\omega(t_1-t_2)) - \cos(\omega(t_1+t_2)+2Y)}{2}\right]$$

$$= \frac{1}{2}\left[E(\cos\omega(t_1-t_2)) - E(\cos\omega(t_1+t_2)+2Y)\right]$$

$$= \frac{1}{2}\cdot\cos(\omega[t_1-t_2]) - \frac{1}{2}\int_0^{2\pi}\cos(\omega(t_1+t_2)+2Y)\,dY.$$

$$= \frac{1}{2}\cos(\omega(t_1-t_2)) - \frac{1}{4}\left[\sin\omega(t_1+t_2)+2Y\right]_0^{2\pi}.$$

$$= \frac{1}{2\pi} \cos[\omega(t_1-t_2)] - \frac{1}{4}\left[\sin[\omega(t_1+t_2)+2\pi\cdot 2] - \sin(\omega(t_1+t_2))\right]$$

$$= \frac{1}{2\pi} \cos(\omega(t_1-t_2)) - \frac{1}{4}\left[\sin[4\pi+\omega(t_1+t_2)] - \sin(\omega(t_1+t_2))\right]$$

$$= \frac{1}{2\pi} \cos\omega(t_1-t_2) - \frac{1}{4}\left(\sin[\omega[t_1+t_2]] - \sin(\omega(t_1+t_2))\right)$$

$$= \frac{1}{2\pi}\cdot\cos\omega(t_1-t_2) = \text{a function of } t_1-t_2.$$

$\therefore$ $X(t)$ is a wide sense stationary process.

(14) Given $X(t) = A\cos(100t+\theta)$,

$E(A)=0$ & $Var(A)=1$.

$$Var(A) = E(A^2) - (E(A))^2$$

$$1 = E(A^2) - 0^2 \Rightarrow E(A^2)=1$$

Since θ is uniformly distributed in the interval $(-\pi,\pi)$, $f(\theta) = \frac{1}{\pi+\pi} = \frac{1}{2\pi}$, $\forall -\pi<\theta<\pi$

$$\text{Mean, } E(X(t)) = \int_{-\pi}^{\pi} X(t)\cdot f(\theta)\,d\theta.$$

$$= \int_{-\pi}^{\pi}\left(A\cos(100t+\theta)\frac{1}{2\pi}d\theta\right)$$

$$= \frac{A}{2\pi}\int_{-\pi}^{\pi}\cos(100t+\theta)\,d\theta$$

$$= \frac{A}{2\pi}\left[\sin(100t+\theta)\right]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}\left[\sin(100t+\pi) - \sin(100t-\pi)\right]$$

$$= \frac{A}{2\pi}\left[2\cos\left(\frac{100t+\pi+100t-\pi}{2}\right)\sin\left(\frac{100t+\pi-100t+\pi}{2}\right)\right]$$

$$= \frac{A}{2\pi}\, 2\cos(100t)\sin\pi$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

$$= \frac{A}{\pi}\cos 100t \cdot 0 \quad = 0, \ 0 \text{ constant.}$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B).$$
$$2\sin A\cos B = \sin(A+B) + \cos(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \cos(A-B)$$

Auto correlation,

$$R(t_1, t_2) = E\left[X(t_1)\cdot X(t_2)\right]$$

$$X(t_1)\cdot X(t_2) = A\cos(100t_1+\theta)\cdot A\cos(100t_2+\theta).$$
$$= A^2\cos(100t_1+\theta)\cos(100t_2+\theta).$$
$$= \frac{A^2}{2}\left[\cos(100t_1+\theta+100t_2+\theta) + \cos(100t_1+\theta-100t_2-\theta)\right]$$

$$= \frac{A^2}{2}\left[\cos(100(t_1+t_2)+2\theta) + \cos 100(t_1-t_2)\right]$$

$$E(X(t_1)\cdot X(t_2)) = E\left[\frac{A^2}{2}\cdot\left[\cos(100(t_1+t_2)+2\theta) + \cos(100(t_1-t_2))\right]\right]$$

$$= E\left(\frac{A^2}{2}\right)\left[E(\cos 100(t_1+t_2)+2\theta) + E(\cos(100(t_1-t_2)))\right]$$

$$= \frac{1}{2}\cos[100(t_1-t_2)] + \frac{1}{2}E(\cos(100(t_1+t_2)+2\theta)).$$ since $E(A^2)=1$

$$E(\cos(100(t_1+t_2)+2\theta)) = \int x(t) f(\theta) d\theta$$

$$= \int_{-\pi}^{\pi} \cos(100(t_1+t_2)+2\theta)\cdot\frac{1}{2\pi}\, d\theta.$$

$$= \frac{1}{2\pi}\left[\frac{\sin(100(t_1+t_2)+2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\sin(100(t_1+t_2)+2\pi) - \sin(100(t_1+t_2)-2\pi)\right]$$

$$= \frac{1}{4\pi}\left\{2\cos\left[\frac{100(t_1+t_2)+100(t_1+t_2)+2\pi-2\pi}{2}\right]. \sin\left[\frac{100(t_1+t_2)+2\pi-100(t_1+t_2)+2\pi}{2}\right]\right\}.$$

$= \frac{1}{4\pi} \; 2\cos[100(t_1+t_2)] \sin 4\pi$

$\boxed{\text{Since } \sin n\pi = 0}$

$= 0$

$$E(X(t_1)\cdot X(t_2)) = \frac{1}{2}\cos[100(t_1-t_2)] + 0 \times \frac{1}{2}$$

$$= \frac{1}{2}\cos[100(t_1-t_2)]$$

ie $R(t_1,t_2) = \frac{1}{2}\cos(100(t_1-t_2))$, a function of t_1-t_2. $\therefore$

Since the mean $E(X(t))$ is a constant & auto correlation $R(t_1,t_2)$ is a function of t_1-t_2, $X(t)$ is a WSS process.

(15) $Y(t) = X(t)\cos(\omega_0 t+\theta)$.

Since θ is uniformly distributed in the interval $(-\pi,\pi)$, $f(\theta) = \frac{1}{\pi+\pi} = \frac{1}{2\pi}$, $\forall -\pi < \theta < \pi$.

$$E(Y(t)) = E(X(t)\cos(\omega_0 t+\theta))$$

$$= E(X(t))\cdot E(\cos(\omega_0 t+\theta))$$

$$E\cos(\omega_0 t+\theta) = \int_{-\pi}^{\pi} \cos(\omega_0 t+\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{1}{2\pi}\Big[\sin(\omega_0 t+\theta)\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\sin(\omega_0 t+\pi) - \sin(\omega_0 t-\pi)\right]$$

$$= \frac{1}{2}\left[-\sin(\omega_0 t) + \sin(\omega_0 t)\right] = 0$$

$E(Y(t)) = E(X(t)) \times 0 = 0$, a constant.

$$Y(t_1)\cdot Y(t_2) = X(t_1)\cos(\omega_0 t_1 + \theta) \cdot X(t_2)\cos(\omega_0 t_2 + \theta)$$

$$= X(t_1)X(t_2)\cdot \frac{\left[\cos(\omega_0 t_1 + \theta + \omega_0 t_2 + \theta) + \cos(\omega_0 t_1 + \theta - \omega_0 t_2 - \theta)\right]}{2}$$

$$= \frac{X(t_1)\cdot X(t_2)}{2}\cdot\left[\cos\left(\omega_0(t_1+t_2) + 2\theta\right) + \cos(\omega_0(t_1 - t_2))\right]$$

$$R_{YY}(t_1, t_2) = E(Y(t_1)\cdot Y(t_2))$$

$$= E\left\{\frac{X(t_1)\cdot X(t_2)}{2}\cdot\left[\cos(\omega_0(t_1+t_2)+2\theta) + \cos(\omega_0(t_1-t_2))\right]\right\}$$

$$= \frac{1}{2}E(X(t_1)\cdot X(t_2))\,E\left(\cos[\omega_0(t_1+t_2)+2\theta] + \cos\omega_0(t_1-t_2)\right)$$

$$= \frac{1}{2}E[X(t_1)\cdot X(t_2)]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(\omega_0(t_1+t_2)+2\theta)\right]d\theta + \cos(\omega_0(t_1-t_2))\right]$$

$$= \frac{1}{2}E(X(t_1)\cdot X(t_2))\left[\frac{1}{2\pi}\left[\frac{\sin(\omega_0(t_1+t_2)+2\theta)}{2}\right]_{-\pi}^{\pi} + \cos(\omega_0(t_1-t_2))\right]$$

$$= \frac{1}{4\pi} \cdot E(X(t_1)\cdot X(t_2))\left[\cdot \frac{1}{2}\left[(\sin(\omega_0(t_1+t_2)+2\pi) - \sin(\omega_0(t_1+t_2)-2\pi)\right] + \cos[\omega_0(t_1-t_2)]\right]$$

$$= \frac{1}{4\pi} \cdot E(X(t_1)X(t_2))\left[\frac{1}{2}(2\cos\omega_0(t_1+t_2)\sin 4\pi) + \cos(\omega_0(t_1-t_2))\right]$$

$$= \frac{1}{4\pi} \cdot E(X(t_1)\cdot X(t_2))\cdot \cos\cdot\omega_0(t_1-t_2)$$

Since $R(t_1, t_2)$ is a function of $t_1 - t_2$

mean $E(Y(t)) = 0$, $Y(t)$ is a WSS process.

(16) Two random process $X(t)$ & $Y(t)$ are defined by $X(t) = A\cos\omega t + B\sin\omega t$ & $Y(t) = B\cos\omega t - A\sin\omega t$. Show that $X(t)$ & $Y(t)$ are jointly wide-sense stationary, if A & B are uncorrelated random variables with zero means, same variances & ω is a constant?

$X(t)$ & $Y(t)$ are jointly wide-sense stationary if : (i) $E(X(t))$ & $E(Y(t))$ are WSS processes.
(ii) $R_{XY}(t_1, t_2)$ is a function of $t_1 - t_2$.

Given : $E(A) = E(B) = 0$.

$Var(A) = Var(B)$

ie $E(A^2) - [E(A)]^2 = E(B^2) - [E(B)]^2$

$\Rightarrow E(A^2) = E(B^2)$

Let $E(A^2) = E(B^2) = K$ (say).

Since A & B are uncorrelated, covar$(A,B) = 0$.

$\Rightarrow \quad E(A \cdot B) = E(A) \cdot E(B)$

$\quad\quad = \underline{0}$ $\quad\quad \boxed{\text{Since } E(A) = E(B) = 0}$

To prove that $X(t)$ is a WSS process, the following conditions must be verified.

(i) $E(X(t))$ must be a constant.

(ii) $R(t_1, t_2)$ is a function of $t_1 - t_2$.

Now,

$$E(X(t)) = E(A\cos\omega t + B\sin\omega t).$$

$$= \cos(\omega t) \cdot E(A) + \sin(\omega t) \cdot E(B)$$

$$= \underline{0}, \text{ a constant.} \quad \boxed{E(A) = 0 = E(B)}$$

$$R(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

$$= E[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)]$$

$$= E[A^2\cos\omega t_1\cos\omega t_2 + AB\cos\omega t_1\sin\omega t_2 + AB\sin\omega t_1\cos\omega t_2 + B^2\sin\omega t_1\sin\omega t_2].$$

$$= \cos\omega t_1\cos\omega t_2 \cdot E(A^2) + \cos\omega t_1\sin\omega t_2 \, E(AB) + \sin\omega t_1\cos\omega t_2 \cdot E(AB) + \sin\omega t_1\sin\omega t_2 \, E(B^2)$$

$= K \cos\omega t_1 \cos\omega t_2 + 0 + 0 + K \sin\omega t_1 \sin\omega t_2$

$$\boxed{E(A^2) = E(B^2) = K, \quad E(AB) = E(BA) = 0}$$

$= K(\cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2)$

$= K \cos(\omega t_1 - \omega t_2) = K \cos(\omega(t_1 - t_2))$

$$\boxed{\cos(a-b) = \cos a \cos b + \sin a \sin b}$$

Since $E(X(t))$ is '0' and $R(t_1, t_2)$ is a function of $t_1 - t_2$, $X(t)$ is a WSS process.

Similarly, we can prove that $Y(t)$ is also a WSS process.

To prove $X(t)$ & $Y(t)$ are jointly WSS process;

consider, $R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)]$

$$R_{XY}(t_1, t_2) = E[(A\cos\omega t_1 + B\sin\omega t_1)(B\cos\omega t_2 - A\sin\omega t_2)]$$

$$= E(AB\cos\omega t_1 \cos\omega t_2 - A^2 \cos\omega t_1 \sin\omega t_2 + B^2 \sin\omega t_1 \cos\omega t_2 - BA \sin\omega t_1 \sin\omega t_2)$$

$$= \cos\omega t_1 \cos\omega t_2 E(AB) - \cos\omega t_1 \sin\omega t_2 E(A^2) + \sin\omega t_1 \cos\omega t_2 E(B^2) - \sin\omega t_1 \sin\omega t_2 E(BA).$$

$= 0 - K\cos\omega t_1 \sin\omega t_2 + K\sin\omega t_1 \cos\omega t_2 - 0$

$= K(\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2)$

$= K\sin(\omega t_1 - \omega t_2)$

$= K\sin(\omega(t_1 - t_2))$

$$\boxed{\begin{array}{l} E(A^2) = E(B^2) = K \\ E(AB) = E(BA) = 0 \end{array}}$$

$$\boxed{\begin{array}{l} \sin(a+b) = \sin a \cos b + \cos a \sin b \\ \sin(a-b) = \sin a \cos b - \cos a \sin b \end{array}}$$

Since both $X(t)$ and $Y(t)$ are WSS processes & $R_{XY}(t_1, t_2)$ is a function of $t_1 - t_2$, $X(t)$ & $Y(t)$ are jointly WSS processes.

(17) If $X(t) = Y\cos t + Z\sin t$, $\forall t$, where Y & Z are independent binary random variables each of which assumes the values -1 & 2 with probabilities $\frac{2}{3}$ & $\frac{1}{3}$ respec. prove that $X(t)$ is a WSS process?

Y & Z are discrete random variables which assume the values

Y	-1	2
$P(Y)$	$2/3$	$1/3$

Z	-1	2
$P(Z)$	$2/3$	$1/3$

$E(Y) = \sum y_i P(y_i) = \sum y P(y)$

$= (-1)(\frac{2}{3}) + (2)(\frac{1}{3}) = 0$.

$E(Y^2) = \sum y_i^2 P(y_i) = \sum y^2 P(y)$

$= (-1)^2 (2/3) + (2)^2 (\frac{1}{3})$

$= 2/3 + 4/3 = \underline{\underline{2}}$

$E(Z) = \sum z_i P(z_i) = \sum z P(z) = (-1)(2/3) + (2)(1/3)$

$= \underline{\underline{0}}$.

$E(Z^2) = \sum z_i^2 P(z_i) = \sum z^2 P(z) = \underline{\underline{2}}$

Since Y & Z are independent random variables,

$E(YZ) = E(Y) \cdot E(Z) = 0$ $\boxed{E(Z) = E(Y) = 0}$

(i) $E(X(t)) = E(Y \cos t + Z \sin t)$

$= \cos t \; E(Y) + \sin t \; E(Z) = 0,$

a constant.

(ii) $R_{XX}(t_1, t_2) = E(X(t_1) \cdot X(t_2))$

$= E[(Y \cos t_1 + Z \sin t_1)(Y \cos t_2 + Z \sin t_2)]$

$= E(Y^2 \cos t_1 \cos t_2 + YZ \cos t_1 \sin t_2 + ZY \sin t_1 \cos t_2 + Z^2 \sin t_1 \sin t_2)$

$= \cos t_1 \cos t_2 \cdot E(Y^2) + \cos t_1 \sin t_2 \; E(YZ) + \sin t_1 \cos t_2 \; E(ZY) + \sin t_1 \sin t_2 \; E(Z^2)$

$= 2 \cos t_1 \cos t_2 + 0 + 0 + 2 \sin t_1 \sin t_2$.

$= 2 (\cos t_1 \cos t_2 + \sin t_1 \sin t_2)$

$= 2 \cdot \cos(t_1 - t_2)$

Since $E(X(t))$ is a constant &
$R(t_1, t_2) =$ a function of $t_1 - t_2$, $X(t)$ is a WSS process.

Another Definition for Auto correlation function & Cross correlation function.

If the process is either WSS or SSS, $E(X(t)\,X(t-\tau)) = E(X(t) \cdot X(t+\tau))$ is a function of τ, denoted by $R_{XX}(\tau)$ or $R(\tau)$.
This function $R(\tau)$ is called the auto correlation function of the process.

If the processes $X(t)$ & $Y(t)$ are jointly WSS, then the function $R_{XY}(\tau) = E(X(t)\,Y(t-\tau))$ is called the cross correlation function of the processes $X(t)$ and $Y(t)$.

Properties:

(I) $R(-\tau) = R(\tau)$ ie $R(\tau)$ is an even function.

proof:-
$$R(\tau) = E(X(t)\,X(t-\tau))$$
$$R(-\tau) = E(X(t)\,X(t+\tau))$$
$$= R(\tau)$$
Hence $R(\tau)$ is an even function of τ.

(II) $|R(\tau)| \leq R(0)$ ie $R(\tau)$ is maximum at $\tau = 0$.

proof:-
$$(R(\tau))^2 = [E(X(t) \cdot X(t-\tau))]^2$$
$$\leq E(X^2(t)) \cdot E(X^2(t-\tau))$$

$$\boxed{\text{since } (E(XY))^2 \le E(X^2)\cdot E(Y^2)}$$

$$[R(\tau)]^2 \le E(X^2(t))\cdot E(X^2(t))$$

$$\boxed{\text{put } \tau = 0}$$

$$\le [E(X^2(t))]^2$$

$$\le (R(0))^2$$

ie $[R(\tau)]^2 \le (R(0))^2$

or $\underline{\underline{R(\tau) \le R(0)}}$

(III) $R_{XY}(\tau) = R_{YX}(-\tau)$

(IV) $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\, R_{YY}(0)}$

(V) $|R_{XY}(\tau)| \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

(VI) If $E(X(t)) = 0$ and $X(t)$ has no periodic components, then $\lim_{|\tau| \to \infty} R(\tau) = \mu_X^2$.

(VII) If $X(t)$ and $Y(t)$ are orthogonal, then $R_{XY}(\tau) = 0$.

(VIII) If $X(t)$ & $Y(t)$ are independent, then $R_{XY}(\tau) = \mu_X \mu_Y$

(IX) $E[X(t+\tau) - X(t)]^2 = 2[R(0) - R(\tau)]$

Problems

(1) Suppose $X(t)$ is a WSS process with $R(\tau)= Ae^{-\alpha|\tau|}$. Find the second moment of $X(8)- X(5)$?

$$E[X(8)-X(5)]^2$$

$$= E(X^2(8))+ E(X^2(5))-2E(X(5)).E(X(8))$$

$$= R(0)+R(0)-2R(3)$$

$$\boxed{\begin{aligned} \text{since } |\tau| &= |t_1-t_2| \\ &= |8-5| \\ &= 3 \end{aligned}}$$

$$= A+A-2Ae^{-3\alpha}$$

$$= 2A(1-e^{-3\alpha})$$

(2) The process $X(t)$ is normal WSS & $E(X(t))=0$. Show that if $Z(t)= X^2(t)$, then $C_{ZZ}(\tau)=2C^2_{XX}(\tau)$?

$$E(Z(t))= E(X^2(t)) = R(0).$$

Since X is a WSS process, we use the result that X & Y are jointly normal with zero mean. Then,

$$E(X^2Y^2) = E(X^2)\cdot E(Y^2)+2E^2(XY) \quad \text{---} (1)$$

Since $X(t)$ & $X(t+\tau)$ are jointly normal

$$R_Z(\tau)= E(Z(t)\cdot Z(t+\tau))$$

$$= E(X^2(t)X^2(t+\tau))$$

$$= E(X^2(t))\cdot E(X^2(t+\tau))+$$

$$2E^2(X(t)X(t+\tau)) \quad \boxed{\text{from } (1)}$$

$$= R(0)R(0) + 2(R_X(\tau))^2$$
$$= (R(0))^2 + 2(R_X(\tau))^2 \quad \text{--- (2)}$$

$$C_{ZZ}(\tau) = R_Z(\tau) - (E(Z))^2$$
$$= R_Z(\tau) - (R(0))^2$$
$$= (R(0))^2 + 2(R_X(\tau))^2 - (R(0))^2$$
$$= 2(R_X(\tau))^2 \quad \boxed{\text{from (2)}}$$

$$C_{XX}(\tau) = R_X(\tau) - (E(X))^2$$
$$= R_X(\tau) \quad \boxed{\text{since } E(X)=0}$$

$$\therefore \; C_{ZZ}(\tau) = 2C_{XX}^2(\tau)$$

(3) The process $X(t)$ is WSS and normal with $E(X(t)) = 0$ & $R(\tau) = 4e^{-2|\tau|}$. Find $E[(X(t+1) - X(t-1))^2]$?

Since $X(t)$ is a WSS process & normal, $E(X^2) = R(0)$ & is independant of t &

$$\sigma^2 = E(X^2) - (E(X))^2 = E(X^2) \quad \boxed{E(X)=0}$$
$$= R(0)$$
$$= 4$$

$$E([X(t+1) - X(t-1)]^2)$$
$$= E(X^2(t+1)) + E(X^2(t-1)) - 2E(X(t+1)X(t-1))$$

$$= R(0) + R(0) - 2R(2)$$

$$\boxed{\text{since } \tau = t+1 - t-1 \\ = 2}$$

$$= 4 + 4 - 8e^{-4}$$

$$= 8(1 - e^{-4})$$

(4) Find the mean & variance of the stationary process with periodic components and $R(\tau) = \dfrac{125\tau^2 + 36}{5\tau^2 + 1}$;

$$R(\tau) = \frac{125\tau^2 + 36}{5\tau^2 + 1} = \frac{125\tau^2\left(1 + \frac{36}{125\tau^2}\right)}{5\tau^2\left(1 + \frac{1}{5\tau^2}\right)}$$

$$= \frac{25\left(1 + \frac{36}{125\tau^2}\right)}{\left(1 + \frac{1}{5\tau^2}\right)}$$

$$\mu^2 = \lim_{\tau \to \infty} R(\tau) = 25.$$

$$\therefore \mu = 5$$

$$E(X^2) = R(0) = 36.$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$= 36 - 25$$

$$= 9.$$

Alternate method:-

$$\begin{array}{r|l} & 25 \\ 5\tau^2+1 & 125\tau^2+36 \\ & 125\tau^2+25 \\ \hline & \quad 11 \end{array}$$

$$R(\tau) = 25 + \frac{11}{5\tau^2+1}$$

$$E(X^2) = R(0) = 36.$$

$$\mu^2 = \lim_{\tau\to\infty} R(\tau) = 25 \Rightarrow \underline{\mu = 5}$$

$$\sigma^2 = E(X^2) - (E(X))^2 = 36 - 25 = \underline{9}$$

(5) If $X(t)$ is a WSS process with auto-correlation function $R_{XX}(\tau)$ & if $Y(t) = X(t+a) - X(t-a)$, show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$?

By definition, $R_{XX}(\tau) = E(X(t)X(t+\tau))$.

Given :- $Y(t) = X(t+a) - X(t-a)$.

$$\text{LHS} = R_{YY}(\tau) = E(Y(t)\cdot Y(t+\tau))$$

ie $E(X(t+a) - X(t-a))(X(t+a+\tau) - X(t-a+\tau))$

$$\begin{aligned} = E\big(& X(t+a)X(t+a+\tau) - \\ & X(t+a)X(t-a+\tau) - \\ & X(t-a)X(t+a+\tau) + \\ & X(t-a)X(t-a+\tau)\big) \quad \text{--- (1)} \end{aligned}$$

$$\boxed{\begin{aligned} \text{since} \quad t+a+\tau &= t-a+2a+\tau \ \& \\ t-a+\tau &= t+a-2a+\tau. \end{aligned}}$$

(1) becomes

$$R_{YY}(\tau) = E\big(X(t+a)\cdot X(t+a+\tau) - X(t+a)X(t+a-2a+\tau) - X(t-a)X(t-a+2a+\tau) + X(t-a)X(t-a+\tau)\big)$$

Let $z = t+a$ & $u = t-a$.

Then

$$R_{YY}(\tau) = E\big(X(z)\,X(z+\tau) - X(z)\,X(z-2a+\tau) - X(u)\,X(u+2a+\tau) + X(u)\,X(u+\tau)\big)$$

$$= E\big(X(z)\,X(z+\tau)\big) - E\big(X(z)X(z+\tau-2a)\big) - E\big(X(u)X(u+\tau+2a)\big) + E\big(X(u)\,X(u+\tau)\big).$$

$$= R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a) + R_{XX}(\tau)$$

$$= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$$

$$= \underline{\underline{RHS}}$$

Poisson process.

Let $X(t)$ be the process with parameter λ. The probablity distribution of $X(t)$ is given by
$$P_n(t) = P(X(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \forall n \in W.$$

Mean & Variance of Poisson process.

Let $X(t)$ be the poisson process with parameter λ. Then, $P_n(t) = P(X(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \forall n \in W$

$$E(X(t)) = \sum_{n=0}^{\infty} n\, P_n(t)$$
$$= \sum_{n=0}^{\infty} n \cdot \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$
$$= e^{-\lambda t} \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n(n-1)!}.$$
$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!}$$
$$= e^{-\lambda t}\left(\lambda t + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \cdots\right)$$
$$= e^{-\lambda t} \cdot \lambda t\left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right).$$

$$= e^{-\lambda t} \cdot \lambda t e^{\lambda t}$$

$$\boxed{\text{since } e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \dots}$$

$$\therefore \boxed{\text{mean} = \lambda t}$$

$$E(X^2(t)) = \sum_{n=0}^{\infty} n^2 P_n(t)$$

$$= \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{n^2 (\lambda t)^n}{n(n-1)!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{n(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \left[\frac{n-1}{(n-1)!} + \frac{1}{(n-1)!}\right] (\lambda t)^n.$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \left(\frac{n-1}{(n-1)!} + \frac{1}{(n-1)!}\right) (\lambda t)^n.$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \left[\frac{n-1}{(n-1)(n-2)!} + \frac{1}{(n-1)!}\right] (\lambda t)^n$$

$$= e^{-\lambda t} \sum_{n=2}^{\infty} \frac{(\lambda t)^n}{(n-2)!} + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t}\left((\lambda t)^2 + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^4}{2!} + \dots\right) +$$

$$e^{-\lambda t}(\lambda t)\left[\lambda t + (\lambda t)^2 + \frac{(\lambda t)^3}{2!} + \dots\right].$$

$$= e^{-\lambda t} \cdot (\lambda t)^2 \left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right) +$$

$$e^{-\lambda t} \cdot (\lambda t) \left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right)$$

$$= e^{-\lambda t} \cdot (\lambda t)^2 (e^{\lambda t}) + e^{-\lambda t} \cdot (\lambda t)(e^{\lambda t})$$

$$= (\lambda t)^2 + (\lambda t)$$

$$\text{Variance} = E(X^2(t)) - (E(X(t)))^2$$
$$= (\lambda t)^2 + \lambda t - (\lambda t)^2 = \underline{\underline{\lambda t}}$$

ie $\boxed{\text{variance} = \lambda t}$

Autocorrelation Function (ACF)

Let $X(t)$ be a poisson process with parameter λ. We know that $E(X(t)) = \lambda t$ & $E(X^2(t)) = (\lambda t)^2 + \lambda t$

The ACF for poisson process is

$$R(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$
$$= E\left(X(t_1) \cdot [X(t_2) - X(t_1)]\right) + E(X^2(t_1))$$
$$= E(X(t_1)) \cdot E(X(t_2) - X(t_1)) + E(X^2(t_1)).$$

$$= E(X(t_1))\big(E(X(t_2)) - E(X(t_1))\big) + E(X^2(t_1))$$

$$= \lambda t_1(\lambda t_2 - \lambda t_1) + (\lambda t_1)^2 + \lambda t_1$$

$$\boxed{\text{since } E(X) = \lambda t \;\&\; E(X^2(t)) = (\lambda t)^2 + \lambda t}$$

$$= \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda^2 t_1^2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 \quad \text{if } t_2 \geq t_1 \text{ or}$$

$$R(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

<u>Covariance</u>.

$$C(t_1, t_2) = R(t_1, t_2) - E(X(t_1) \cdot X(t_2))$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda t_1 \lambda t_2.$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$$= \lambda t_1 \quad \text{if } t_2 \geq t_1 \quad \text{or}$$

$$C(t_1, t_2) = \lambda \min(t_1, t_2)$$

<u>Correlation coefficient</u>.

$$\rho(t_1, t_2) = \frac{C(t_1, t_2)}{\sigma_x t_1 \sigma_x t_2} = \frac{\lambda t_1}{\sqrt{\lambda t_1}\sqrt{\lambda t_2}}$$

$$= \frac{\lambda t_1}{\sqrt{\lambda^2 t_1 t_2}} = \frac{\lambda t_1}{\lambda \sqrt{t_1 t_2}}$$

$$= \frac{\lambda\sqrt{t_1}\sqrt{t_1}}{\lambda\sqrt{t_1}\sqrt{t_2}} = \sqrt{\frac{t_1}{t_2}} \quad \text{if } t_1 < t_2.$$

Note: To compute correlation coefficient, first compute $R_{XX}(t_1, t_2)$ & $C_{XX}(t_1, t_2)$.

Properties of poisson process.

Property 1: The sum of 2 independent poisson process is a poisson process.

proof

Let $X_1(t)$ & $X_2(t)$ be 2 independent poisson processes with parameters λ_1 & λ_2 respect.

Then, $P(X_1(t) = n) = \frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!}$ &

$$P(X_2(t) = n) = \frac{e^{-\lambda_2 t}(\lambda_2 t)^n}{n!}.$$

Let $X(t) = X_1(t) + X_2(t)$

We have to prove $X(t)$ is a poisson process.

$$P(X(t) = n) = P(X_1(t) + X_2(t) = n).$$

$$= \sum_{r=0}^{n} P(X_1(t) = r)\, P(X_2(t) = n-r)$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= e^{-\lambda_1 t} e^{-\lambda_2 t} \sum_{r=0}^{n} (\lambda_1 t)^r (\lambda_2 t)^{n-r} \frac{1}{r!(n-r)!}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} (\lambda_1 t)^r (\lambda_2 t)^{n-r} \sum_{r=0}^{n} (\lambda_1 t)^r (\lambda_2 t)^{n-r} \frac{nC_r}{n!}$$

$$\boxed{\text{since } nC_r = \frac{n!}{r!(n-r)!}}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} (\lambda_1 t + \lambda_2 t)^n.$$

$$\boxed{\text{since } nC_r (a)^r (b)^{n-r} = (a+b)^n}$$

$$P(X(t)=n) = P(X_1(t) + X_2(t) = n)$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} ((\lambda_1+\lambda_2)t)^n$$

which is the probablity law for poisson process with parameters $\lambda_1 + \lambda_2$. Hence sum of independent poisson processes is also a poisson process.

Property 2 :- Difference of two independent poisson process is not a poisson process.

proof

Let $X_1(t)$ & $X_2(t)$ be 2 independent processes with parameters λ_1 & λ_2 respect. therefore,

$$E(X_1(t)) = \lambda_1 t, \quad E(X_1^2(t)) = (\lambda_1 t)^2 + \lambda_1 t.$$

$$E(X_2(t)) = \lambda_2 t, \quad E(X_2^2(t)) = (\lambda_2 t)^2 + \lambda_2 t.$$

Let $X(t) = X_1(t) - X_2(t)$.

$$E(X(t)) = E(X_1(t) - X_2(t))$$
$$= E(X_1(t)) - E(X_2(t))$$
$$= \lambda_1 t - \lambda_2 t = (\lambda_1 - \lambda_2)t.$$

$$E(X^2(t)) = E(X_1(t) - X_2(t))^2$$
$$= E(X_1^2(t)) + E(X_2^2(t)) - 2E(X_1(t))E(X_2(t))$$
$$= (\lambda_1 t)^2 + \lambda_1 t + (\lambda_2 t)^2 + \lambda_2 t - 2\lambda_1 t \lambda_2 t.$$
$$= (\lambda_1 t - \lambda_2 t)^2 + \lambda_1 t + \lambda_2 t$$

$$E(X^2(t)) = [(\lambda_1 - \lambda_2)t]^2 + [(\lambda_1 + \lambda_2)t]$$

$$E(X^2(t)) \neq [(\lambda_1 - \lambda_2)t]^2 + [(\lambda_1 - \lambda_2)t].$$

Therefore, difference of 2 independent poisson process is not a poisson process.

Property: 3 : The interarrival time of a poisson process ie, the interval b/w 2 successive occurencens of a poisson process with parameter λ, has an exponential distribution with mean $= \frac{1}{\lambda}$.

Problems

(1) Suppose the customer arrive at a bank acc. to a poisson process with mean rate of 3/min. Find the probablity that, during a time interval of 2 minutes (a) exactly 4 customers arrive

(b) greater than 4 customers arrive.

(c) fewer than 4 customers arrive.

Let $X(t)$ denotes the no. of customers arrival during the interval $(0,t)$.

Given $X(t)$ follows poisson process &

$\lambda = 3/\text{min}, t = 2\,\text{min}.$

$$P(X(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad \forall n \in \mathbb{N}.$$

(a) P(exactly 4 customers arrive during a time interval of 2 minutes)

$$= P(X(2)=4) = \frac{e^{-6}(6)^4}{4!} = \frac{3.2124}{24}$$

$$= \underline{0.13385}$$

(b) P(greater than 4 customer arrive during a time interval of 2 min.)

$= P(X(2)>4) = 1 - P(X(2)\leq 4)$

$= 1 - [P(X(2)=0) + P(X(2)=1) + P(X(2)=1) + P(X(2)=3) + P(X(2)=4)]$

$= 1 - \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} + \frac{e^{-6}(6)^4}{4!}\right]$

$= 1 - e^{-6}\left(1+6+\frac{36}{2}+36+54\right)$

$= 1 - e^{-6}(115) \doteq 1 - 0.285$

$= \underline{\underline{0.715}}$

(c) P(less than 4 customers arrive during a time interval of 2 min.) $= P(X(2)<4)$

$= P(X(2)=0) + P(X(2)=1) + P(X(2)=2) + P(X(2)=3)$.

$= e^{-6}(61) = \underline{\underline{0.1512}}$

(2) If a customer arrive at a counter in accordance with a poisson process with mean rate of 2 per minute, find the probablity that the interval b/w 2 consecutive arrivals is (a) more than one min.

(b) b/w 1 to 2 minutes.

(c) less than or equal to 4 min.

Given $\lambda = 2/\text{min}$.

Let T be the interval b/w a consecutive arrivals. Then T follows exponential distribution with $\lambda = 2/\text{min}$.
The exponential distribution is $f(t) = 2e^{-2t}$, $\forall\, t > 0$.

(a) $P(T>1) = \int_1^{\infty} f(t)\,dt$

$$= \int_1^{\infty} 2e^{-2t}\,dt$$

$$= 2\left(\frac{e^{-2t}}{-2}\right)_1^{\infty}$$

$$= 0 + e^{-2} = e^{-2} = \underline{\underline{0.1353}}$$

(b) $P(1<T<2) = \int_1^{2} f(t)\,dt$

$$= \int_1^{2} 2e^{-2t}\,dt.$$

$$= 2\left(\frac{e^{-2t}}{-2}\right)_1^{2}$$

$$= -(e^{-4} - e^{-2}) = \underline{\underline{0.117}}$$

(c) $P(T \le 4) = \int_0^{4} f(t)\,dt$

$$= \int_0^{4} 2e^{-2t}\,dt$$

$$= 2\left(\frac{e^{-2t}}{-2}\right)_0^{4} = -[e^{-8} - e^{-16}]$$

$$= \underline{\underline{0.996}}.$$