Q. **Toss a coin 12 times. What is the probability of getting 7 heads?**

$$P(X=7) = {}^{12}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^5$$

$$= \frac{792}{2^{12}} = \frac{792}{2048} = .38$$

Q. **In an exam, 10 multiple choice questions are asked where only one out of four questions are correct. Find the probability of getting 5 out of 10 questions correct in an answer sheet.**

$$P(X=5) = {}^{10}C_5\, p^5 (1-p)^5$$

$$= {}^{10}C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^5$$

$$= {}^{10}C_5 \frac{3^5}{4^{10}} = 252 \times \underline{243}$$

$$= \underline{0.05}$$

Q.

If a basketball player takes 8 independent free throws, with a probability of 0.7 of getting a basket on each shot, what is the probability that she gets exactly 6 baskets?

$\mathbf{C}(8,6)(0.7)^6(0.3)^2 \approx 0.296$

What is the expected number of baskets that she gets?

$np = 8(.7) = 5.6$ (**not** 6, or 5! — expected value doesn't have to be a value that can actually occur)

Q. A student is given a multiple choice exam with 10 questions, each question with five possible answers. He guesses randomly for each question.

(a) What's P(he will get exactly 6 questions correct)?

$n = 10$, $k = 6$, $p = 0.2$, so $\mathbf{C}(10, 6)(0.2)^6(0.8)^4 \approx 0.0055$

(b) What is the probability he will get **at least** 6?

$\mathbf{C}(10, 6)(0.2)^6(0.8)^4 + \mathbf{C}(10, 7)(0.2)^7(0.8)^3 +$
$\mathbf{C}(10, 8)(0.2)^8(0.8)^2 + \mathbf{C}(10, 9)(0.2)^9(0.8)^1 +$
$\mathbf{C}(10, 10)(0.2)^{10}(0.8)^0 \approx 0.0063.$

(c) What is the expected number of correct answers, and what's the standard deviation?

$E(x) = np$
$= 10 \times 0.2$
$= 2$

$s.d = \sqrt{var(x)} = \sqrt{np(1-p)}$
$= \sqrt{10 \times 0.2 \times 0.8}$
$= \sqrt{1.6} = 1.26$ ✓

Q. Assume that the Mets and the Royals are in the world series, that the Mets have a 3/5 chance of winning any given game, and that the games are independent experiments. What is the probability of a 7 game series?

Note 1: The world series is **not** a Bernoulli experiment! (number of games is not fixed in advance)

Note 2 A seven game series will occur only when each team wins 3 of the first 6 games.

A seven game series will occur whenever the Mets win exactly 3 of the first 6 games. The probability of this is $\mathbf{C}(6, 3)(0.6)^3(0.4)^3 \approx 0.27$.

It is also true that a seven game series will occur whenever the Royals win exactly 3 of the first 6 games. The probability of this is $\mathbf{C}(6, 3)(0.4)^3(0.6)^3 \approx 0.27$.

Q. The Everlasting Lightbulb company produces light bulbs, which are packaged in boxes of 20 for shipment. Tests have shown that 4% of their light bulbs are defective.

(a)What is the probability that a box, ready for shipment, contains exactly 3 defective light bulbs?

$\mathbf{C}(20,3)(0.04)^3(0.96)^{17} \approx 0.036.$

(b) What is the probability that the box contains 3 or more defective light bulbs?

$1-(\mathbf{C}(20,2)(0.04)^2(0.96)^{18}+\mathbf{C}(20,1)(0.04)^1(0.96)^{19}+\mathbf{C}(20,0)(0.04)^0(0.96)^{20}) \approx 1-0.956=0.044.$

We can also compute the expected number of defective bulbs, $\mathbf{E}(X)=20\cdot 0.04=0.8$, and the standard deviation, $\boldsymbol{\sigma}(X)=\sqrt{20\cdot 0.04\cdot 0.96}\approx 0.876.$

Q. A random variable X is the number of successes in a Bernoulli experiment with n trials, each with a probability of success p and a probability of failure q. The probability distribution table of X is shown below:

k	$\mathbf{P}(X=k)$
0	$\frac{1}{81}$
1	$\frac{8}{81}$
2	$\frac{24}{81}$
3	$\frac{32}{81}$
4	$\frac{16}{81}$

Old Exam questions

Which of the following values of n, p, q give rise to this probability distribution?

(a) $n=4,\ p=\frac{2}{3},\ q=\frac{1}{3}$ (b) $n=4,\ p=\frac{1}{3},\ q=\frac{2}{3}$

(c) $n=4,\ p=\frac{1}{6},\ q=\frac{5}{6}$ (d) $n=5,\ p=\frac{1}{3},\ q=\frac{2}{3}$

(e) $n=5,\ p=\frac{2}{3},\ q=\frac{1}{3}$

The listed probabilities add up to 1 so they are a probability distribution and therefore $n=4$.

$\mathbf{P}(X=0)=\mathbf{C}(4,0)p^0q^4=q^4$ so $q=\frac{1}{3}$. Hence (a) is the right answer.

Normal Probabilities Practice Problems Solution

1. Most graduate schools of business require applicants for admission to take the Graduate Management Admission Council's GMAT examination. Scores on the GMAT are roughly normally distributed with a mean of 527 and a standard deviation of 112. What is the probability of an individual scoring above 500 on the GMAT?

Normal Distribution

$$Z = \frac{500 - 527}{112} = -0.24107$$

$\mu = 527$

$\sigma = 112$

$\Pr\{X > 500\} = \Pr\{Z > -0.24\} = 1 - 0.4052 = \boxed{\mathbf{0.5948}}$

2. How high must an individual score on the GMAT in order to score in the highest 5%?

Normal Distribution

$\mu = 527$

$\sigma = 112$

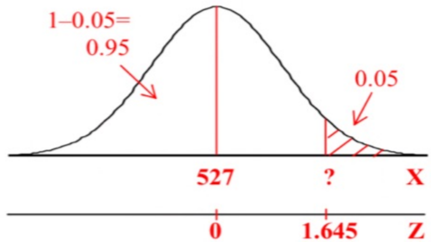
$P(X > ?) = 0.05 \Rightarrow P(Z > ?) = 0.05$

$P(Z < ?) = 1 - 0.05 = 0.95 \Rightarrow Z = 1.645$

$X = 527 + 1.645(112)$

$X = 527 + 184.24$

$X = \boxed{\mathbf{711.24}}$



3. The length of human pregnancies from conception to birth approximates a normal distribution with a mean of 266 days and a standard deviation of 16 days. What proportion of all pregnancies will last between 240 and 270 days (roughly between 8 and 9 months)?

Normal Distribution

$$Z = \frac{240 - 266}{16} = -1.625$$

$\mu = 266$

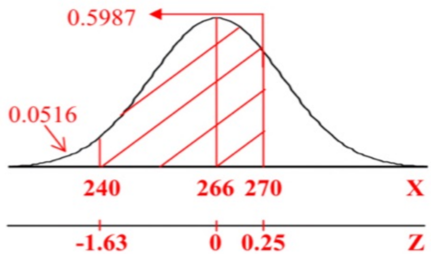
$$Z = \frac{270 - 266}{16} = 0.25$$

$\sigma = 16$

$P(240 < X < 270) = P(-1.63 < Z < 0.25)$

$P(-1.63 < Z < 0.25) = P(Z < 0.25) - P(Z < -1.63)$

$P(-1.63 < Z < 0.25) = 0.5987 - 0.0516 = \boxed{\mathbf{0.5471}}$



4. What length of time marks the shortest 70% of all pregnancies?

Normal Distribution

$\mu = 266$

$\sigma = 16$

$P(X < ?) = 0.70 \Rightarrow P(Z < ?) = 0.70 \Rightarrow Z = 0.52$

$X = 266 + 0.52(16)$

$X = 266 + 8.32$

$X = \boxed{\mathbf{274.32}}$

Normal Probabilities Practice Problems Solution

5. The average number of acres burned by forest and range fires in a large New Mexico county is 4,300 acres per year, with a standard deviation of 750 acres. The distribution of the number of acres burned is normal. What is the probability that between 2,500 and 4,200 acres will be burned in any given year?

Normal Distribution $\qquad Z = \dfrac{2500 - 4300}{750} = -2.40$

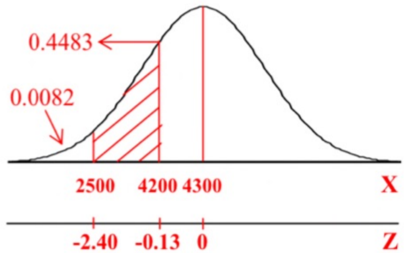
$\mu = 4300 \qquad Z = \dfrac{4200 - 4300}{750} = -0.13333$

$\sigma = 750$

$P(2500 < X < 4200) = P(-2.40 < Z < -0.13)$

$P(-2.40 < Z < -0.13) = P(Z < -0.13) - P(Z < -2.40)$

$P(-2.40 < Z < -0.13) = 0.4483 - 0.0082 = \boxed{0.4401}$



6. What number of burnt acres corresponds to the 38th percentile?

Normal Distribution

$\mu = 4300$

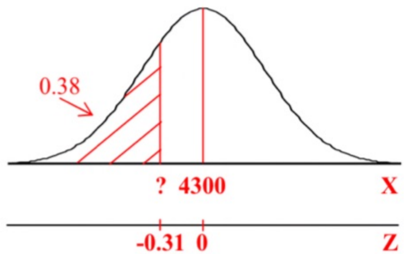
$\sigma = 750$

$P(X < ?) = 0.38 \Rightarrow P(Z < ?) = 0.38 \Rightarrow Z = -0.31$

$X = 4300 + (-0.31)(750)$

$X = 4300 - 232.5$

$X = \boxed{4067.5}$



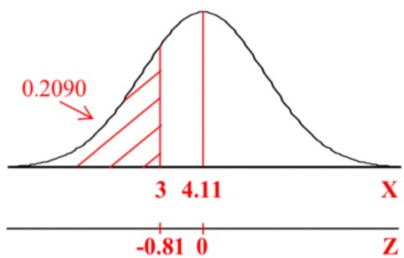
7. The Edwards's Theater chain has studied its movie customers to determine how much money they spend on concessions. The study revealed that the spending distribution is approximately normally distributed with a mean of \$4.11 and a standard deviation of \$1.37. What percentage of customers will spend less than \$3.00 on concessions?

Normal Distribution $\qquad Z = \dfrac{3.00 - 4.11}{1.37} = -0.81021$

$\mu = 4.11$

$\sigma = 1.37$

$P(X < 3.00) = P(Z < -0.81) = 0.2090 \Rightarrow \boxed{20.9\%}$



8. What spending amount corresponds to the top 87th percentile?

Normal Distribution

$\mu = 4.11$

$\sigma = 1.37$

$P(X > ?) = 0.87 \Rightarrow P(Z > ?) = 0.87$

$P(Z > ?) = 0.87 \Rightarrow P(Z < ?) = 1 - 0.87 = 0.13 \Rightarrow Z = -1.13$

$X = 4.11 + (-1.13)(1.37)$

$X = 4.11 - 1.5481$

$X = 2.5619$

$X = \boxed{\$2.56}$

Example on moment :

Example Let X be a discrete random variable having support

$$R_X = \{1,2,3\}$$

and probability mass function

$$p_X(x) = \begin{cases} 1/2 & \text{if } x = 1 \\ 1/3 & \text{if } x = 2 \\ 1/6 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

The third moment of X can be computed as follows:

$$\begin{aligned} \mu_X(3) &= \mathrm{E}[X^3] \\ &= \sum_{x \in R_X} p_X(x) x^3 \\ &= \frac{1}{2}\cdot 1^3 + \frac{1}{3}\cdot 2^3 + \frac{1}{6}\cdot 3^3 \\ &= \frac{1}{2} + \frac{8}{3} + \frac{27}{6} \\ &= \frac{3+16+27}{6} = \frac{46}{6} = \frac{23}{3} \end{aligned}$$

3^{rd} moment

$$E|X|^3 = E(x^3)$$

$$= \sum_i x_i^3 P(x_i)$$

$$= 1^3 \times \frac{1}{2} + 2^3 \times \frac{1}{3} + 3^3 \times \frac{1}{6}$$

$$= \frac{1}{2} + \frac{8}{3} + \frac{27}{6}$$

$$= \frac{3+16+27}{6} = \frac{46}{6} = \frac{23}{3}$$

Linear Map (Linear transformation)

Let V, V' be two vector space over $\mathbb{F}$. A linear mapping

$$L : V \to V'$$

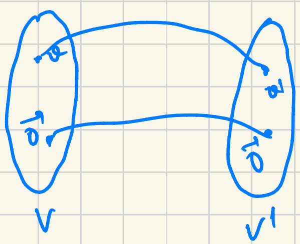
is a mapping which satisfies the following two properties.

① For every $\vec{u}, \vec{v} \in V$, we have

$$L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$$

② For all $\alpha \in \mathbb{F}$, $v \in V$

$$L(\alpha v) = \alpha L(v)$$



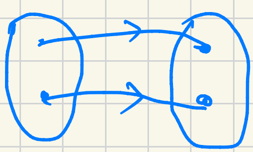
$$L(\vec{0} + \vec{v}) = L(\vec{0}) + L(\vec{v})$$

$$L(\vec{v}) = L(\vec{0}) + L(\vec{v})$$

$L(\vec{0})$ has to be $\vec{0}$ to satisfy LHS = RHS

Let $L : V \to V'$ be a mapping then,

① L is injective (or one-to-one) if all $\vec{u}, \vec{v} \in V$



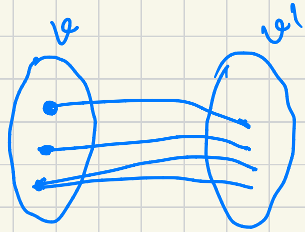
$$L(\vec{u}) = L(\vec{v}) \Rightarrow \vec{u} = \vec{v}$$

or

$$\vec{u} \neq \vec{v} \Rightarrow L(\vec{u}) \neq L(\vec{v})$$

Made with Goodnotes

(ii) L is surjective (or onto) if every $\vec{w} \in V'$ there exist some $\vec{v}$, $\vec{v} \in V$ such that $L(\vec{v}) = \vec{w}$



everybody in V' has a pre image.

(iii) L is bijective if it is injective & surjective.

Isomorphism:

Let U & V' be two V.S over some field $\mathbb{F}$

Let $L: \vec{V} \to \vec{V}'$ be a Linear map.

if L is bijective then we call L is isomorphism.

Theorem! Let V, V' be finite dimensional V.S over the same field $\mathbb{F}$. Then V is isomorphic to V', if and only if $\dim(V) = \dim(V')$

Norms:

A norm on a V.S V over $\mathbb{R}$ is a function

$$\|\cdot\|: V \to \mathbb{R}$$

$$\vec{v} \longrightarrow \|\vec{v}\|$$, which assign each vector $\vec{v}$ its length $\|\vec{v}\|$ such that for any $\vec{v}, \vec{w} \in V$ & $\alpha \in \mathbb{R}$ should satisfy below properties

① $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$

② $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (triangle inequility)

③ $\|\vec{v}\| \geq 0$ for all $\vec{u} \in V$ & $\|\vec{u}\| = 0$ if and only if $\vec{v} = \vec{0}$ (positive definite property).

Ex:

Manhatten Norm:

Manhatten Norm over $\mathbb{R}$ is defined as for any $\vec{v} \in \mathbb{R}^n$

$$\|\vec{v}\| = \sum_{i=1}^{n} |v_i|$$

v_i = ith element of vector $\vec{v}$.

Euclidean Norm:

for any $\vec{v} \in \mathbb{R}^n$

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} \vec{v}_i^2}$$

Inner product:

An inner product of a vector space V (over a field $\mathbb{F}$) is an association which to any pair of vectors $\vec{v}, \vec{w} \in V$ associated with a scalar denoted by $\langle \vec{v}, \vec{w} \rangle$ satisfies the following properties.

① $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \;\; \forall \; u, v \in V$ (symmetric property)

② $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}+\vec{w} \rangle$ for any $\vec{u}, \vec{v}, \vec{w} \in V$

③ $\langle \alpha\vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$ &

$\langle \vec{u}, \alpha\vec{v} \rangle = \alpha\langle \vec{u}, \vec{v} \rangle$ where $\alpha \in \mathbb{R}$ & $u, v \in V$

these two properties are called bilinearity.

Further, we say $\langle \cdot, \cdot \rangle$ is a positive definite inner product if for every $\vec{v} \in V$ ① $\langle \vec{v}, \vec{v} \rangle \geq 0$ &

② $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

The pair $(V, \langle \cdot, \cdot \rangle)$, where V is a v.s over $\mathbb{F}$ and $\langle \cdot, \cdot \rangle$ is an inner product defined from $V \times V \to \mathbb{F}$ is called an inner product space.

Symmetric, positive definite matrices:

Given an n-dim vector space V (over the field $\mathbb{F}$) with a positive definite inner product

$\langle \cdot, \cdot \rangle = V \times V \to \mathbb{R}$ and and ordered basis

$S = \{v_1, v_2 - - v_n\}$ For any $\vec{u}, \vec{v} \in V$ with

$\vec{u} = \alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + - - - + \alpha_n\vec{u}_n$

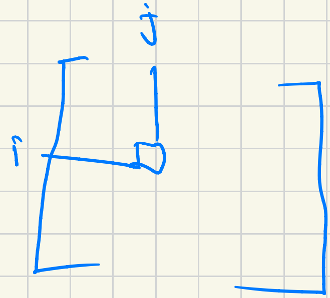
$\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdot - - \alpha_n\vec{v}_n$

$$\langle \vec{u}, \vec{v} \rangle = \left\langle \sum_{i=1}^{n} \alpha_i \vec{u}_i, \sum_{i=1}^{n} \beta_i \vec{v}_i \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \langle u_i, v_j \rangle \beta_j$$

$$= \hat{u}^T A \hat{v} \qquad \hat{u}^T = (\alpha_1, \alpha_2 - - \alpha_n)_{1 \times n}$$

$$\hat{v} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}_{n \times 1}$$

$A =$ 

$$A_{ij} = \langle \vec{u_i}, \vec{v_j} \rangle$$

The matrix A is symmetric.

(Any inner product is symmetric)

Since $\langle \cdot , \cdot \rangle$ is positive definite, we have

$\vec{v}^T A \vec{v} > 0$ for every vector $\vec{v} \in V \setminus \{\vec{0}\}$

$\downarrow$

$\langle v, v \rangle$ and $\vec{0}^T A \vec{0} = 0.$ $\longrightarrow$ eq, 1

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ that satisfies eq, 1 is called a symmetric positive definite matrix.

Length & Distance

positive definite inner products yield norms which obtained as:

$$\| \vec{v} \|_0 = \sqrt{\langle v, v \rangle}$$

check if it satisfies all norm's conditions:

① $\|\alpha\vec{v}\| = \sqrt{\langle \alpha\vec{v}, \alpha\vec{v}\rangle} = \sqrt{\alpha^2\langle v, v\rangle}$

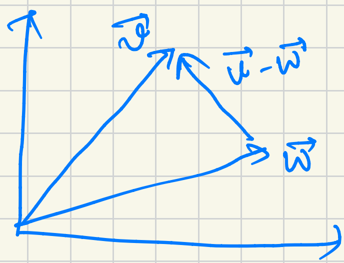
$= |\alpha| \langle v, v\rangle$

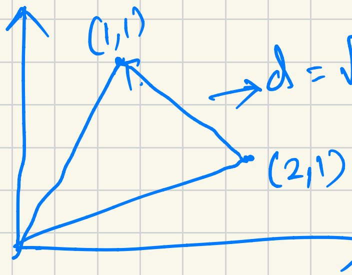
② $\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle \leq \|\vec{u}\| + \|\vec{v}\|$

③ $\|\vec{v}\| \geq 0$ & $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
(this holds since $\langle \cdot, \cdot \rangle$ is positive definite)

Consider an inner product space $(V, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is positive definite then

$$d(\vec{v}, \vec{w}) := \|\vec{v} - \vec{w}\| = \sqrt{\langle \vec{v}-\vec{w}, \vec{v}-\vec{w}\rangle}$$





$d = \sqrt{(2-1)^2 + (1-1)^2}$

$= \sqrt{1}$

$= \pm 1$

The mapping

$d : V \times V \to \mathbb{R}$ is called metric.

Like length of a vector, the distance between two vectors doesn't require an inner product, a norm is sufficient.

① d is positive definite $d(\vec{u}, \vec{v}) \geq 0$
$\forall\, \vec{u}, \vec{v} \in V$ & $d(\vec{u}, \vec{v}) = 0$ iff $\vec{u} = \vec{v}$

② d is symmetric that is

$d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

③ Triangle inequality:

$$d(\vec{u},\vec{w}) \leq d(\vec{u},\vec{v}) + d(\vec{v},\vec{w})$$

for all $\vec{u},\vec{v},\vec{w} \in V$.

□ Angles & Orthogonality:

In addition to enabling the definition of lengths of vectors as well as distance between two vectors, inner product also captures the geometry of a V.S by defining angle ω between two vectors,

Let $\vec{u} \neq \vec{0}$ & $\vec{v} \neq \vec{0}$

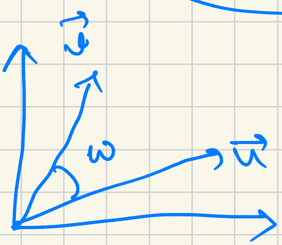
by Cauchy-Schwarz inequality

$$-1 \leq \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$$

Therefore, there exists a unique $\omega \in [0,\pi]$ with

$$\cos\omega = \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|}$$

$\to$ angle between $\vec{u}$ & $\vec{v}$



Orthogonality:

$\vec{u}$ & $\vec{v}$ are orthogonal if $\langle\vec{u},v\rangle = \vec{0}$ & we write $\vec{u} \perp \vec{v}$, In addition,

if $\|\vec{u}\| = 1 = \|\vec{v}\|$, then we say that $\vec{u}$ & $\vec{v}$ are orthonormal vector.

** $\vec{0}$ is orthogonal to every vector.

Orthogonal matrix:

A square matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal matrix if and only if its columns are orthonormal so that,

$$AA^T = I = A^TA$$

$\hookrightarrow$ Identity matrix this implies that

$$A^{-1} = A^T \text{ where } A \text{ is orthogonal.}$$

two properties of transformation by orthogonal matrix

(1) transformations by orthogonal matrices preserve the length of the vector being transformed.

suppose, the inner product under A is orthogonal matrix

$$\vec{v} \in V$$

$\vec{v}$ is transformed/mapped to $A\vec{v}$

$$\|A\vec{v}\|^2 = (A\vec{v})^T(Av) = \vec{v}^TA^T(A\vec{v}) = \vec{v}^TI\vec{v} = \vec{v}^Tv$$
$$= \|v\|^2$$

② They preserve angles too. Let dot product be the inner product under consideration. Consider two vectors $\vec{u}$ & $\vec{v}$, let ω be the angle between $A\vec{u}$ & $A\vec{v}$.

$$\cos\omega = \frac{(A\vec{u})^T(A\vec{v})}{\|A\vec{u}\| \, \|A\vec{v}\|} = \frac{\vec{u}^T A^T . A\vec{v}}{\sqrt{\vec{u}^T A^T A\vec{u} \quad \vec{v}^T A^T A\vec{v}}}$$

$$= \frac{u^T v}{\|\vec{u}\| \, \|v\|}$$

angle between $\vec{u}$ & $\vec{v}$.

Orthonormal basis:

Let V be a n-dim vectorspace over $\mathbb{R}$.

Let $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ be a basis of V.

if $\langle \vec{v_i}, \vec{v_j} \rangle = 0$ for $1 \leq i \leq j \leq n$

& $\langle \vec{v_i}, \vec{v_j} \rangle = 1 \quad \forall \ 1 \leq i \leq n.$

The basis $\{v_1, v_2 \ldots v_n\}$ is called an orthonormal basis.

Ex: standard/Canonical basis in $\mathbb{R}^n$ is orthonormal basis $\mathbb{R}^n$ over dot product.

$\vec{E_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\vec{E_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ — $\vec{E_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ then

Orthogonal Complement:

basis of $\mathbb{R}^n$ under dot product

Orthogonal complement

Let V be an n-dimensional V.S. over $\mathbb{R}$. Let $U \subseteq V$ be a k-dimensional subspace of V. Then the **orthogonal complement of U**, denoted by $U^{\perp}$, is an $(n-k)$-dimensional subspace of V and contains all vectors in V that are orthogonal to every vector in U. Further $U \cap U^{\perp} = \{\vec{0}\}$.

Suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\} \longrightarrow$ basis of U

$\{\vec{u}_1^{\perp}, \vec{u}_2^{\perp}, \dots, \vec{u}_{n-k}^{\perp}\} \longrightarrow$ basis of $U^{\perp}$

Then, every vector in $\vec{V}$ can be written as a linear combination of these vectors.

as a linear combination of these vectors.

Ex:

Example

Describe the orthogonal complement of V, $V^\perp$.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

The subspace V is a plane in $\mathbb{R}^3$, spanned by the two vectors $\vec{v_1} = (1, -3, 2)$ and $\vec{v_2} = (0, 1, 1)$. Therefore, its orthogonal complement $V^\perp$ is the set of vectors which are orthogonal to both $\vec{v_1} = (1, -3, 2)$ and $\vec{v_2} = (0, 1, 1)$.

$$V^\perp = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3)$, we get two equations from these dot products.

$$x_1 - 3x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

Put these equations into an augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

then put it into reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The rref form gives the system of equations

$$x_1 + 5x_3 = 0$$

$$x_2 + x_3 = 0$$

and we can solve the system for the pivot variables. The pivot entries we found were for x_1 and x_2, so we'll solve the system for x_1 and x_2.

$$x_1 = -5x_3$$

$$x_2 = -x_3$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^\perp = \text{Span}\left(\begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} \right)$$

orthogonal projection

$P(x, y) = \frac{}{10}(1, 3).$

Example. Let W be the plane generated by the vectors $(1,1,1)$ and $(1,0,1)$. Find the orthogonal projection $P : \mathbb{R}^3 \longrightarrow W$.

Solution. We notice first that $((1,1,1),(1,0,1)) = 2 \neq 0$, so this is not an orthogonal basis. Using Gram-Schmidt we get:
$v_1 = (1,1,1)$
$v_2 = (1,0,1) - \frac{2}{3}(1,1,1) = (\frac{1}{3}, -\frac{2}{3}), \frac{1}{3} = \frac{1}{3}(1,-2,1)$.

To avoid fractions, we can use $(1,-2,1)$ instead of $\frac{1}{3}(1,-2,1)$. Thus the orthogonal projection is:

$$\begin{aligned} P(x,y,z) &= \frac{x+y+z}{3}(1,1,1) + \frac{x-2y+z}{6}(1,-2,1) \\ &= \Bigg(\frac{2x+2y+2z}{6} + \frac{x-2y+z}{6}, \\ &\qquad \frac{2x+2y+2z}{6} - 2\frac{x-2y+z}{6}, \\ &\qquad \frac{2x+2y+2z}{6} + \frac{x-2y+z}{6}\Bigg) \\ &= \left(\frac{x+z}{2}, y, \frac{x+z}{2}\right). \end{aligned}$$

Example. Let W be the plane $\{(x,y,z) \in \mathbb{R}^3 | x + y + 2z = 0\}$. Find the orthogonal projection $P : \mathbb{R}^3 \longrightarrow W$.

Solution. We notice that our first step is to find an orthogonal basis for W. The vectors $(1,-1,0)$ and $(2,0,-1)$ are in W, but are not orthogonal.We have

$$(2,0,-1) - \frac{2}{2}(1,-1,0) = (1,1,-1) \in W$$

and orthogonal to $(1,-1,0)$. So we get:

$$\begin{aligned} P(x,y,z) &= \frac{x-y}{2}(1,-1,0) + \frac{x+y-z}{3}(1,1,-1) \\ &= \left(\frac{5x-y-2z}{6}, \frac{-x+5y-2z}{6}, \frac{-x-y+z}{3}\right). \end{aligned}$$

8.7 Exercises

1. Let $V \subset \mathbb{R}^2$ be the line $V = \mathbb{R}(1,-1)$.

Orthogonal projection:

Orthogonal Projection Examples

Example 1: Find the orthogonal projection of $\vec{y} = (2,3)$ onto the line $L = \langle (3,1) \rangle$.

Example 2: Let $V = \langle (1,0,1), (1,1,0) \rangle$. Find the vector $\vec{v} \in V$ which is closest to $\vec{y} = (1,2,3)$.