Linear Maps ≡ Matrices.

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Let $S = \{\vec{v_1}, \vec{v_2} \cdots \vec{v_n}\}$ be a basis of $\mathbb{R}^n$.

The matrix $m \times n$ is defined as

$$A_L = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ L(\vec{v_1}) & L(\vec{v_2}) & \cdots & L(v_n) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{m \times n}$$

m bit vector.

Let $\vec{v} \in \mathbb{R}^n$ with $\vec{v} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n}$

Then $L(\vec{v}) = L(\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots \alpha_n \vec{v_n})$

$= \alpha_1 L(v_1) + \alpha_2 L(\vec{v_2}) + \cdots \alpha_n L(v_n)$

Now,

$$A_L \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} L(\vec{v_1}) & L(\vec{v_2}) & \cdots & L(\vec{v_n}) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 L(\vec{v_1}) + \alpha_2 L(\vec{v_2}) + \cdots \alpha_n L(\vec{v_n}) \\ \vdots \\ \\ \end{bmatrix}_{m \times 1} \to L(\vec{v})$$

Ex:

$L: \mathbb{R}^3 \to \mathbb{R}^4$. as $\quad L((1,0,0)) = (1,0,0,0)$
$L((0,1,0)) = (0,1,0,0)$
$L((0,0,1)) = (0,0,1,0)$

$S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$

$S' = \{(1,0,0,0),(0,1,0,0),(0,0,1,0), (0,0,1,0)(0,0,0,1)\}$ is a bases of $\mathbb{R}^4$.

∘ What is

$L((3,2,1))$?

$\to 3(1,0,0) + 2(0,1,0) + 1(0,0,1)$

$\Rightarrow$

$L((3,2,1)) = 3(1,0,0,0) + 2(0,1,0,0) + 1(0,0,1,0)$

$= (3,2,1,0)$

$$A_L = \begin{bmatrix} L((1,0,0)) & L((0,1,0)) & L((0,0,1)) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}_{4\times 3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cancel{0} \\ 0 & 1 & 0 & \cancel{0} \\ 0 & 0 & 1 & \cancel{0} \\ 0 & 0 & 0 & \cancel{0} \end{bmatrix}_{4\times 3}$$

$$A_L \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}_{4\times 1}$$

For ~~the~~ reverse direction

Matrix given $\xrightarrow[\text{Linear map}]{\text{Convert into}}$

$$A = \begin{bmatrix} A^1 & A^2 & \cdots & A^n \end{bmatrix}_{m \times n}$$ where A^i is ith column.

Let $S = \{\vec{v}_1, \vec{v}_2 - - \vec{v}_n\}$ be a basis of $\mathbb{R}^n$.

Define a linear map L_A as.

$$\left.\begin{aligned} L_A(\vec{v}_1) &= A^1 \\ L_A(\vec{v}_2) &= A^2 \\ &\vdots \\ L_A(\vec{v}_n) &= A^n \end{aligned}\right\}$$

We claim that this is the linear map corresponding to given matrix A.

Consider, a vector $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + - + \alpha_n\vec{v}_n \in \mathbb{R}$

$$L_A(\vec{v}) = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha_1 A^1 + \alpha_2 A^2 + \cdot \alpha_n A^n \\ | \end{bmatrix} = \begin{bmatrix} \cdot \end{bmatrix}_{m \times 1}$$

Norm

pairing a vector real number. A norm on a vector space V over $\mathbb{R}$ is a function $\boxed{\|\cdot\| : V \to \mathbb{R}}$ $\vec{v} \to \|\vec{v}\|$, which assign each vector $\vec{v}$ its length $\|\vec{v}\|$ such that for any $\vec{v}, \vec{w} \in V$ & $\alpha \in \mathbb{R}$ should ~~follow~~ satisfy below 3 properties —

1. $\|\alpha\vec{v}\| = |\alpha|\|\vec{v}\|$
2. $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$ (triangle inequality)
3. $\|\vec{v}\| \geq 0$ for all $\vec{v} \in V$ & $\|\vec{v}\| = 0$ If and only if $\vec{v} = \vec{0}$ (positive definite property)

Ex 1:

Manhattan Norm on $\mathbb{R}^n$ is defined as

for any $\vec{v} \in \mathbb{R}^n$,

$$||v|| = \sum_{i=1}^{n} |\vec{v_i}|$$ $v_i \rightarrow$ ith element of vector $\vec{v}$

Let $n=5$,

$$||(3,2,1,0,-5)|| = |3| + |2| + |1| + |0| + |-5|$$
$$= 3+2+1+0+5$$
$$= \underline{11}$$ ✓

$$\alpha\vec{v} = \alpha(3,2,1,0,-5) = (3\alpha, 2\alpha, 1\alpha, 0, -5\alpha)$$

$$||\alpha\vec{v}|| = 3\alpha + 2\alpha + \alpha + 0 + 5\alpha$$
$$= \underline{\alpha||\vec{v}||}$$

Ex 2 Euclidean norm

For any $v \in \mathbb{R}^n$.

$$||\vec{v}||_2 = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{\vec{v}^T v}$$

Let $n=5$. $\vec{v} = (3,2,1,0,-5)$

$$||\vec{v}||_2 = \sqrt{3^2+2^2+1^2+0^2+(-5)^2}$$
$$= \sqrt{39} \approx 6.2$$

Inner product:

An inner product of a vector space V (over a field $\mathbb{F}$) is an association which to any pair of vectors $\vec{v}, \vec{w} \in V$ associates a scalar, denoted by $\langle \vec{v}, \vec{w} \rangle$ satisfies the following properties.

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \forall u, v \in V$ (symmetric property)

2. $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ for any $\vec{u}, \vec{v}, \vec{w} \in V$.

3. $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$ and
$\langle u, \alpha \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$ Where $\alpha \in \mathbb{R}$ & $u, v \in V$

these two properties are called bilinearity.

Positive definite inner product

We say $\langle \cdot, \cdot \rangle$ is a positive definite inner product if every $\vec{v} \in V$ ① $\langle \vec{v}, \vec{v} \rangle \geq 0$ &
② $\langle \vec{v}, \vec{v} \rangle = 0$ if & only if $\vec{v} = \vec{0}$.

Ex:
$\vec{v} = (1, 2, 3, 4, 5) \in \mathbb{R}^5$
$\vec{w} = (0, 1, 0, 1, 2) \in \mathbb{R}^5$

$\vec{v} \cdot \vec{w} = (0, 2, 0, 4, 10)$

$\Downarrow$

$\sum_{i=1}^{n} v_i w_i$

$$\boxed{\vec{u}\cdot(\vec{v}+\vec{w})} = (1,1,1,1,1)\cdot(1,3,3,5,7)$$
$$= (1,3,3,5,7)$$

$$\vec{u}\cdot\vec{v} = (1,2,3,4,5)$$
$$\vec{u}\cdot\vec{w} = (0,1,0,1,2)$$

$$\boxed{\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}} = (1,3,3,5,7)$$

dot product = valid inner product