Projections and Rotations

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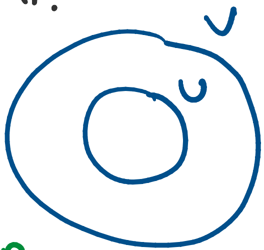
Orthogonal projections

A type of linear transformation (like rotation & reflection)

high dim data $\longrightarrow$ project it to low dim space.

Defn. Let V be a v.s. over $\mathbb{R}$ and let U be a subspace of V. A linear mapping $\pi : V \to U$ is called a projection if $\pi^2 = \pi \circ \pi = \pi$.

$$\pi(\vec{v}) = \vec{u}$$



$$\pi(\pi(\vec{v})) = \pi(\vec{u}) = \vec{u}$$

Consider the projection matrix P_π

$$P_\pi^2 = P_\pi$$

Projection onto one-dim subspaces (lines)

of the 1-D subspace

Assume we are given a line through the origin with basis vector $\vec{b} \in \mathbb{R}^n$.

$\vec{x}$: vector

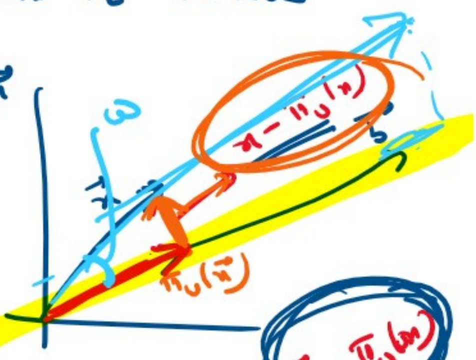
$\vec{b}$: vector

The line is a one-dim subspace $U \subseteq \mathbb{R}^n$ spanned by $\vec{b}$. When we project $\vec{x} \in \mathbb{R}^n$

spanned by $\vec{b}$. When we project $\vec{x} \in \mathbb{R}^n$ onto U, we seek the vector $\pi_U(\vec{x})$ that is closest to $\vec{x}$.

Two properties of a projection

(i) $\pi_U(\vec{x})$ is closest to $\vec{x}$. That is, $\|\vec{x} - \pi_U(\vec{x})\|$ is minimal. It follows that the segment $\pi_U(\vec{x}) - \vec{x}$ is orthogonal to U and and therefore orthogonal to the basis vector $\vec{b}$ of U.



$\pi_U(\vec{x}) + \vec{x} - \pi_U(\vec{x}) = \vec{x}$

That is,

$$\langle \pi_U(\vec{x}) - \vec{x}, \vec{b} \rangle = 0$$

$\lambda \vec{b}$

(ii) the projection $\pi_U(\vec{x})$ must be an element of U. Therefore $\pi_U(\vec{x}) = \lambda \vec{b}$, for some $\lambda \in \mathbb{R}$.

Three steps to determine λ, the projection $\pi_U(\vec{x}) \in U$, and the projection matrix P_π that maps x onto U.

(i) Finding λ.

$$\langle \vec{x} - \pi_U(x), \vec{b} \rangle = 0$$

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Since $\pi_U(\vec{x}) = \lambda \vec{b}$,

$$\langle \vec{x} - \lambda \vec{b}, \vec{b} \rangle = 0$$

$$\langle \vec{x}, \vec{b} \rangle - \lambda \langle \vec{b}, \vec{b} \rangle = 0$$

$$\lambda = \frac{\langle \vec{x}, \vec{b} \rangle}{\|b\|^2} = \frac{\langle \vec{b}, \vec{x} \rangle}{\|b\|^2}$$

If we use dot product as the inner product, then

$$\boxed{\lambda = \frac{\underset{1\times n}{\vec{b}^T} \underset{n\times 1}{\vec{x}}}{\|\vec{b}\|^2}}$$

$$\vec{b} = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix} \qquad \vec{x} \; [3 \; 5 \; 6]$$

If $\|\vec{b}\| = 1$, then $\underline{\underline{\lambda = \vec{b}^T \vec{x}}}$ $\qquad \vec{b}^T = [\qquad]$

(ii) Finding the projection point $\pi_U(\vec{x})$

$$\pi_U(\vec{x}) = \lambda \vec{b} = \frac{\langle \vec{x}, \vec{b} \rangle}{\|\vec{b}\|^2} \vec{b} = \frac{\vec{b}^T \vec{x} \, \vec{b}}{\|\vec{b}\|^2} \rightarrow \text{scalar}$$

(for dot products)

$$\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$$

From the definition of norm/length,

$$\|\pi_U(\vec{x})\| = \|\lambda \vec{b}\| = |\lambda| \, \|\vec{b}\|$$

When dot product is the inner product under

When dot product is the inner product under consideration,

$$\|\pi_U(\vec{x})\| = \frac{|\vec{b}^T \vec{x}|}{\|\vec{b}\|^2} \|\vec{b}\| = \frac{|\cos\omega| \, \|\vec{x}\| \, \|\vec{b}\| \, \|\vec{b}\|}{\|\vec{b}\|^2}$$

$$= \|\vec{x}\| \, |\cos\omega|$$

(iii) Finding the projection matrix P_π

$$\pi_U(\vec{x}) = \lambda \vec{b} = \vec{b}\lambda = \vec{b} \frac{\vec{b}^T \vec{x}}{\|\vec{b}\|^2} = \frac{\vec{b}\vec{b}^T}{\|\vec{b}\|^2} \vec{x}$$

$n \times n$

(assuming dot product as the inner product)

Thus,

$$P_\pi = \frac{\vec{b}\vec{b}^T}{\|\vec{b}\|^2}$$

$$\vec{b} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{n \times 1} \qquad \vec{b}^T = \begin{bmatrix} & & \end{bmatrix}_{1 \times n}$$

It is a symmetric matrix.

Projection onto general subspace

Projection of $\vec{x} \in \mathbb{R}^n$ onto a lower dimensional subspace $U \subseteq \mathbb{R}^n$ with $\dim(U) = m \geq 1$.

$$1 \le m \le n.$$

Let $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m\}$ be a basis of U.

Let $\pi_U(\vec{x})$ denote the projection of $\vec{x}$ onto U. We know $\pi_U(\vec{x}) \in U$. Let

$$\pi_U(\vec{x}) = \sum_{i=1}^{m} \lambda_i \vec{b}_i$$

Like we did in the 1-dim case, we follow a 3-step procedure to find $\lambda_1, \ldots, \lambda_m$, $\pi_U(x)$ and the projection matrix P_π.

1. Finding $\lambda_1, \lambda_2, \ldots, \lambda_m$

$$\pi_U(\vec{x}) = \sum_{i=1}^{m} \lambda_i \vec{b}_i = \boldsymbol{B\lambda} \quad (1)$$

, where

$n \times 1$

$$\mathbf{B} = \underbrace{\left[\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m \right]}_{n \times m} \in \mathbb{R}^{n \times m}$$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_m \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ \lambda_m \end{bmatrix}_{m \times 1} \qquad \begin{bmatrix} \vdots \\ \lambda_m \end{bmatrix} =$$

Assuming dot product as the inner product, since $\pi_U(\vec{x})$ is closest to $\vec{x}$, we have

$$\langle \vec{b}_1, (x - \pi_U(\vec{x})) \rangle = \vec{b}_1^T (x - \pi_U(\vec{x})) = 0$$

$$\langle \vec{b}_2, (\vec{x} - \pi_U(\vec{x})) \rangle = \vec{b}_2^T (x - \pi_U(\vec{x})) = 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\langle \vec{b}_m, (\vec{x} - \pi_U(\vec{x})) \rangle = \vec{b}_m^T (\vec{x} - \pi_U(\vec{x})) = 0$$

Using ① to replace $\pi_U(x)$ with $\mathbf{B}\lambda$,

$$\vec{b}_1^T (\vec{x} - \mathbf{B}\lambda) = 0$$

$$\vec{b}_2^T (\vec{x} - \mathbf{B}\lambda) = 0$$

$$\vdots \qquad \vdots$$

$$\vec{b}_m^T (\vec{x} - \mathbf{B}\lambda) = 0$$

From here, we obtain the following homogeneous system of linear equations

$$\underset{m\times n}{\begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \\ \vdots \\ \vec{b}_m^T \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} \vec{x} - \mathbf{B}\boldsymbol{\lambda} \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

That is,

$$\underset{m\times n}{\mathbf{B}^T} \underset{n\times 1}{(\vec{x} - \mathbf{B}\boldsymbol{\lambda})} = \underset{m\times 1}{\vec{0}}$$

or

$$\underset{m\times m}{\mathbf{B}^T\mathbf{B}} \underset{m\times 1}{\boldsymbol{\lambda}} = \underset{m\times n}{\mathbf{B}^T} \underset{n\times 1}{\vec{x}}$$

it is symmetric, positive definite and therefore it is invertible

$\underset{1\times n}{\vec{x}^T} \underset{n\times n}{A} \underset{n\times 1}{\vec{x}} \geq 0$ always

and $= 0$ only when $\vec{x} = \vec{0}$

Thus

$$\boxed{\boldsymbol{\lambda} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\vec{x}}$$

2. $\underset{n\times 1}{\pi_U(\vec{x})} = \underset{n\times m}{B}\,\underset{m\times 1}{\lambda}$ (from Eqn ①)

$$= \underbrace{B\,(B^T B)^{-1}\,B^T\vec{x}}$$

3. Projection matrix

$$P_\pi = B\,(B^T B)^{-1}\,B^T$$

<u>Remark</u>.

$m=1$. $\dim(U) = 1$

$\underset{1\times n}{B^T}\,\underset{n\times 1}{B}$ is a scalar.

$B \to \vec{b}$

Therefore, $P_\pi = \dfrac{BB^T}{B^T B} = \dfrac{\vec{b}\,\vec{b}^T}{\|\vec{b}\|^2}$

$\boxed{3:55}$

<u>Remark</u>:

$$\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$$ sys of lin eqns.

Suppose it does not have a solution. This means

Suppose it does not have a solution. This means $\vec{b}$ does not lie in the space spanned by the column vectors of A.

$$\begin{array}{c} x_1 \quad x_2 \quad \cdots \quad x_n \\ \begin{bmatrix} | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{bmatrix} \end{array} \underset{m\times n}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}$$

Let U_A denote the space spanned by the column vectors of A. Project $\vec{b}$ onto U_A to get an approximate solution.

Remark:

If $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m\}$ is an orthonormal basis, then

$$\Pi_U(\vec{x}) = \underbrace{B}_{n\times m}\underbrace{B^T}_{m\times n}\vec{x} \quad (\text{since } B^TB = I)$$

and $\lambda = B^T\vec{x}$

column vectors orthogonal

Gram–Schmidt Orthogonalization

Transform an.. basis $\{\vec{\ }, \vec{\ } \ldots \vec{\ }\}$

Transform any basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of an n-dim v.s. V into an orthogonal/orthonormal basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ of V.

$$\vec{u}_1 = \vec{b}_1$$

$$\vec{u}_2 = \vec{b}_2 - \Pi_{span(\vec{u}_1)}(\vec{b}_2)$$

$$\vec{u}_3 = \vec{b}_3 - \Pi_{span(\vec{u}_1, \vec{u}_2)}(\vec{b}_3)$$

$$\vec{u}_4 = \vec{b}_4 - \Pi_{span(\vec{u}_1, \vec{u}_2, \vec{u}_3)}(\vec{b}_4)$$

$$\vdots$$

$$\vec{u}_k = \vec{b}_k - \Pi_{span(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{k-1})}(\vec{b}_k)$$

What we have got, $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$, is an orthogonal basis of V. To make it orthonormal, divide each vector by its norm.

$$\left\{ \frac{\vec{u}_1}{\|\vec{u}_1\|}, \frac{\vec{u}_2}{\|\vec{u}_2\|}, \ldots, \frac{\vec{u}_n}{\|\vec{u}_n\|} \right\}$$

orthonormal basis of V.

Projection onto affine spaces.

Let V be a n-dim v.s over reals.

$L = \vec{x}_0 + U$, where U is a subspace of V. and $\vec{x}_0$ is any vector in V.

$\to$ affine subspace.

To project a vector $\vec{x}$ onto L:

$$\Pi_L(\vec{x}) = \vec{x}_0 + \Pi_U(\vec{x} - \vec{x}_0)$$

Rotations.

Recall orthogonal matrices — columns are orthonormal, $A^{-1} = A^T$

Transformation using such matrices preserve length and angles between vectors.

Defn. A rotation is a linear mapping

Defn. A rotation is a linear mapping that rotates a plane by an angle θ about the origin.

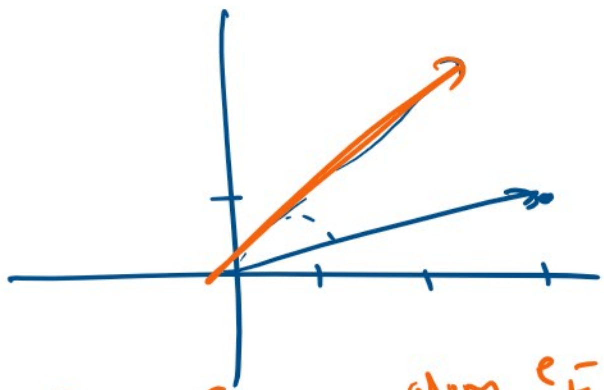
Convention: rotate in counterclockwise direction for any +ve angle θ.

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \text{or} \quad \text{adj} = \text{hyp}\cdot\cos\theta$$

$$=$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

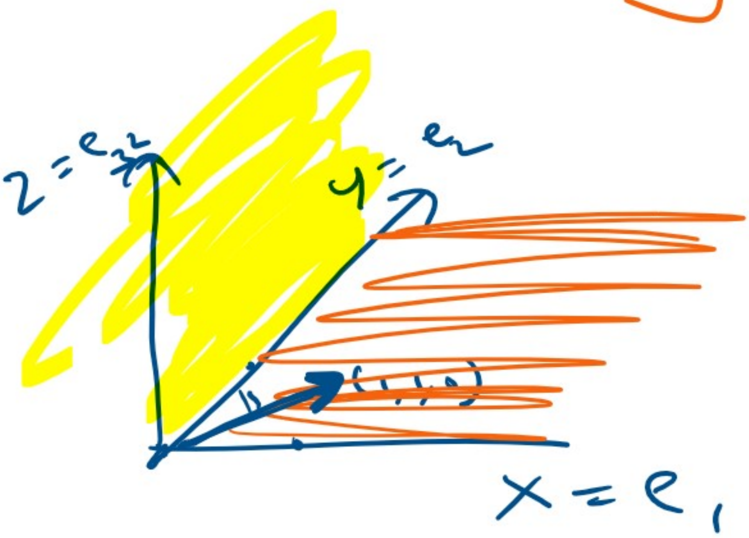


rotation about e_1-axis (rotation along e_2-e_3 plane)

$$\begin{array}{ccc} 1 & 0 & 0 \end{array}$$

$$\begin{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} & \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



To rotate along $(i-j)^{th}$ plane

$$\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{matrix} & i & & j & \end{matrix} \\ \begin{bmatrix} 1 & & & & \\ & \cos\theta & & -\sin\theta & \\ & & 1 & & \\ & \sin\theta & & \cos\theta & \\ & & & & 1 \end{bmatrix}_{n\times n} \end{array}$$