CS 6320 Computer Vision

Homework 4 (Due Date – April 9^{th})

This is not a programming assignment, but you are free to use code to automate any of the questions in the assignments instead of manually doing it by hand. There is no need to submit any code for this assignment. The report has to be self-contained providing all the important steps to arrive at your results.

1. Show that the following function (all the variables are Boolean) is submodular using the set definition (i.e., $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$):

   $f(x_1, x_2, x_3) = 2x_1 - x_1x_2 - 2x_2x_3$

   Note that for every pseudo-Boolean function (i.e., $f(x_1, x_2, x_3)$), you can think of an associated set function (i.e., $f(A), f(B)$). In order to show that the above function is submodular, you will need to show that the equation holds for all possible subsets A and B. If you try to enumerate all possible cases, there are 8 possible subsets ({},{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}) for A and the same 8 possible subsets for B. Overall, you will have 64 cases considering all possible cases for A and B. For the assignment, it is sufficient if you prove that the function $f(x_1, x_2, x_3)$ for only six cases (i.e., for six different subset pairs for A and B). For all the six cases, write down the subsets A and B, and also the L.H.S and R.H.S of the submodularity condition (i.e., $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$) to show that the condition holds.

   [10 points]

2. Show that the following function (all the variables are Boolean) is not submodular using the set definition (i.e., $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$):

   $$f(x_1, x_2, x_3) = 2x_1 - x_1x_2 + 2x_2x_3$$

   To show that a given function is not submodular, you will need to show one counterexample. Find some subsets A and B for which the above function does not satisfy submodularity condition.

   [10 points]

3. Show the maxflow/min-cut graph for the following equation:

   $f(x_1, x_2, x_3, x_4) = -2x_1 + 3x_2 + 5x_3 + 7x_4 - x_1x_2 - 2x_2x_3 - 4x_1x_4 - 5x_2x_4$

   Manually identify the **best** solution (all the variables are Boolean) and show the corresponding cut on the graph. In other words, you will consider all possible states for the Boolean variables and identify the solution that has the minimum value. Show that the cost of the cut matches with the cost of the function (without the introduced constant term).

   [10 points]

4. Show the pseudo-Boolean function associated with the following maxflow/min-cut graph shown in Fig. 1.

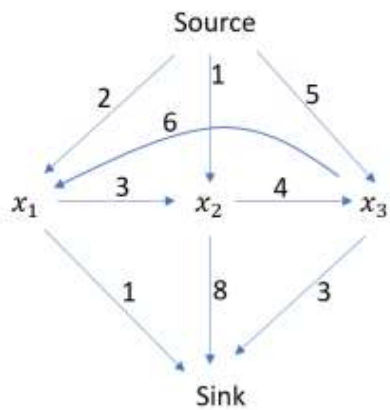
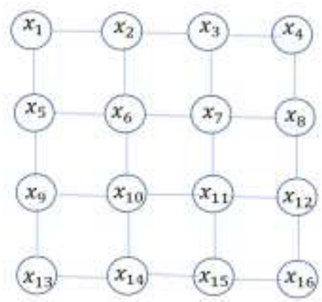


Figure 1: There are two terminal nodes (source denoting class 0, and sink denoting class 1). There are three non-terminal nodes whose values need to be estimated.

Manually identify the **best** solution (all the variables are Boolean) and show the corresponding cut on the graph. In other words, you will consider all possible states for the Boolean variables and identify the solution that has the minimum value. Show that the cost of the cut matches with the cost of the function (without the introduced constant term). [10 points]

5. Consider the following image of size 4x4 as shown in Fig. 2.



(a)

[2,1]	[2,1]	[2,1]	[2,1]
[2,1]	[1,2]	[1,2]	[2,1]
[2,1]	[1,2]	[1,2]	[2,1]
[2,1]	[2,1]	[2,1]	[2,1]

(b)

Figure 2: (a) We show a 4x4 image with 16 pixels. We use a 4-connected neighborhood for adjacent pixels. (b) Unary costs are shown for every pixel in the form [a,b] in the 4x4 table (corresponding to the 4x4 pixel image on the left) where "a" denotes the cost for associated pixel to take the label 0, and "b" denotes the cost for an associated pixel to take the label 1.

The goal is to detect the foreground pixels from the background pixels using maxflow/min-cut algorithm. Let the foreground be denoted by the label 0, and let the background be denoted by label 1. The unary costs for the pixels are given in the table shown in Fig. 2. We use the standard

Potts model for the pairwise costs and it is given by $\{\theta_{x_i x_j}^{lm} = 0\ if\ l = m\ and\ 1\ otherwise\}$, for adjacent pixels x_i and x_j. Write down the energy function and also show the graph that would be used for the 2-label segmentation. There is no need to solve the problem using maxflow/mincut algorithm, which any off-the-shelf solver can do. In other words, you are not required to code this problem or find the optimal solution. Show the maxflow/mincut graph with 2 terminal nodes (source, sink) and 16 non-terminal nodes ($\{x_1, x_2, \ldots, x_{16}\}$). Show all the edges between the nodes and their associated edge costs. In addition, you will also show the equation for the energy function.

[20 points]

6. Consider a multi-label problem with two variables y_1 and y_2 each taking 3 states $\{1,2,3\}$. Let the unary terms be given by $\{\theta_{y_1}^{l} = \{[l=1] \rightarrow 0.5, [l=2] \rightarrow 1.5, [l=3] \rightarrow 1.0\},\ and\ \{\theta_{y_2}^{l} = \{[l=1] \rightarrow 1.5, [l=2] \rightarrow 1.5, [l=3] \rightarrow 0.0\}$. We use Potts pairwise model and it is given by $\{\theta_{y_1 y_2}^{lm} = 0\ if\ l = m,\ and\ 1\ otherwise\}$. In both the subproblems below, initialize both variables to 1 in the beginning. For alpha-expansion, you can iterate through alpha = {1,2,3,1,2,3,…}. For alpha-beta swap, you can iterate through alpha-beta = {(1,2),(1,3),(2,3),(1,2),(1,3),(2,3),…}.

   (a) Show the iterations in alpha-expansion till it converges. Note that you will construct a Boolean function in every iteration. You will manually compute the solution for every maxflow/mincut step.
   (b) Show the iterations in alpha-beta swap till it converges. Note that you will construct a Boolean function in every iteration. You will manually compute the solution for every maxflow/mincut step.

[40 points]

1. for all $A \subseteq B$ or $B \subseteq A$, we can prove that:

$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

$$f(x_1, x_2, x_3) = 2x_1 - x_1x_2 - 2x_2x_3$$

Take $A \subseteq B$ for example. $f(A) = f(A \cap B)$

$f(B) = f(A \cup B)$

thus, we have $f(A) + f(B) = f(A \cup B) + f(A \cap B)$.

for other cases:

A	B	$f(A)+f(B)$	$f(A\cup B)+f(A\cap B)$	
$\{1\}$	$\{2\}$	$2 + 0$	$1 + 0$	True
	$\{3\}$	$2+0$	$2 + 0$	True
	$\{2,3\}$	$2 + (-2)$	$(-1) + 0$	True.
$\{2\}$	$\{1\}$	$0 + 2$	$1+0$	True
	$\{3\}$	$0 + 0$	$(-2)+0$	True
	$\{1,3\}$	$0 + 2$	$(-1)+0$	True
$\{3\}$	$\{1\}$	$0 + 2$	$2 + 0$	True
	$\{2\}$	$0 + 0$	$(-2)+0$	True
	$\{2,1\}$	$0 + 1$	$(-1)+0$	True
$\{1,2\}$	$\{3\}$	$1 + 0$	$(-1)+0$	True
	$\{1,3\}$	$1 + 2$	$(-1)+2$	True
	$\{2,3\}$	$1 + (-2)$	$(-1)+0$	True

A	B	$f(A)+f(B)$	$f(A\cup B)+f(A\cap B)$	
$\{1,3\}$	$\{2\}$	$2+0$	$(-1)+0$	True
	$\{1,2\}$	$2+1$	$(-1)+2$	True
	$\{2,3\}$.	$2+(-2)$	$(-1)+0$	True
$\{2,3\}$	$\{1\}$	$-2+2$	$(-1)+0$	True
	$\{1,2\}$	$-2+1$	$(-1)+0$	True
	$\{1,3\}$	$-2+2$	$(-1)+0$	True

$f(x_1, x_2, x_3) = 2x_1 - x_1x_2 - 2x_2x_3$

Submodular Quadratic Pseudo Boolean function

2. $f(x_1, x_2, x_3) = 2x_1 - x_1x_2 + 2x_2x_3$.

$A = \{1, 2\}$ $\quad$ $f(A)+f(B) = 1+2$

$B = \{1, 3\}$

$f(A\cup B)+f(A\cap B) = 3+2$

$f(A)+f(B)$

for other cases:

A	B	$f(A)+f(B)$	$f(A\cup B)+f(A\cap B)$	
$\{1\}$	$\{2\}$	$2+0$	$1+0$	T
	$\{3\}$	$2+0$	$2+0$	T
$2x_1-x_1x_2+2x_2x_3$	$\{2,3\}$	$2+2$	$3+0$	T
$\{2\}$	$\{1\}$	$0+2$	$1+0$	T
	$\{3\}$ ✓	$0+0$	$2+0$	F
	$\{1,3\}$ ✓	$0+2$	$3+0$	F
$\{3\}$	$\{1\}$	$0+2$	$2+0$	T
	$\{2\}$ ✓	$0+0$	$2+0$	F
	$\{2,1\}$ ✓	$0+1$	$3+0$	F
$\{1,2\}$	$\{3\}$ ✓	$1+0$	$3+0$	F
	$\{1,3\}$ ✓	$1+2$	$3+2$	F
	$\{2,3\}$	$1+2$	$3+0$	T
$\{2,3\}$	$\{1\}$ ✓	$2+0$	$3+0$	F
	$\{1,2\}$	$2+1$	$3+0$	T
	$\{1,3\}$	$2+2$	$3+0$	T
$\{1,3\}$	$\{2\}$ ✓	$2+0$	$3+0$	F
	$\{1,2\}$ ✓	$2+1$	$3+2$	F
	$\{2,3\}$	$2+2$	$3+0$	T

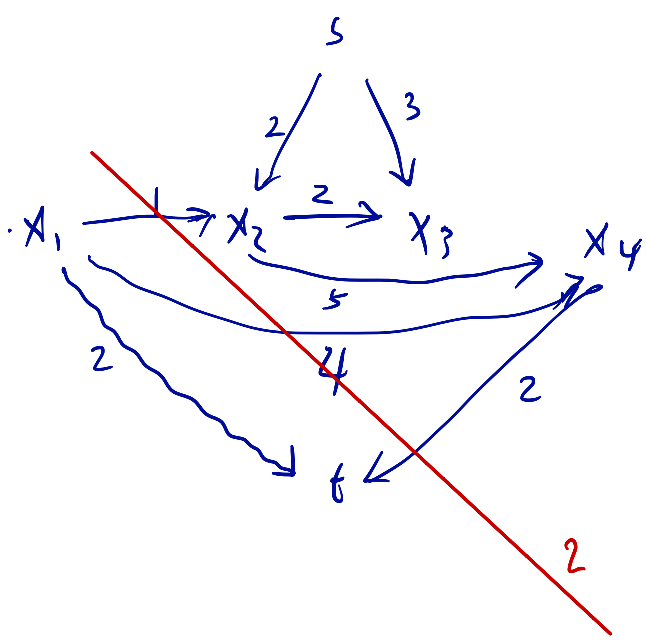
$$3.\ f(x_1, x_2, x_3, x_4) = -2x_1 + 3x_2 + 5x_3 + 7x_4 - x_1x_2 - 2x_2x_3 - 4x_1x_4 - 5x_2x_4$$

$$
\begin{aligned}
&= -2x_1 + 3x_2 + 5x_3 + 7x_4 + (1-x_1)x_2 - x_2 \\
&\quad + 2(1-x_2)x_3 - 2x_3 \\
&\quad + 4(1-x_1)x_4 - 4x_4 \\
&\quad + 5(1-x_2)x_4 - 5x_4
\end{aligned}
$$

$$
\begin{aligned}
&= -2x_1 + 2x_2 + 3x_3 - 2x_4 + \bar{x}_1x_2 + 2\bar{x}_2x_3 + 4\bar{x}_1x_4 \\
&\quad + 5\bar{x}_2x_4
\end{aligned}
$$

$$
\begin{aligned}
&= 2(1-x_1) - 2 + 2x_2 + 3x_3 + 2(1-x_4) - 2 + \bar{x}_1x_2 + 2\bar{x}_2x_3 + 4\bar{x}_1x_4 \\
&\quad + 5\bar{x}_2x_4
\end{aligned}
$$

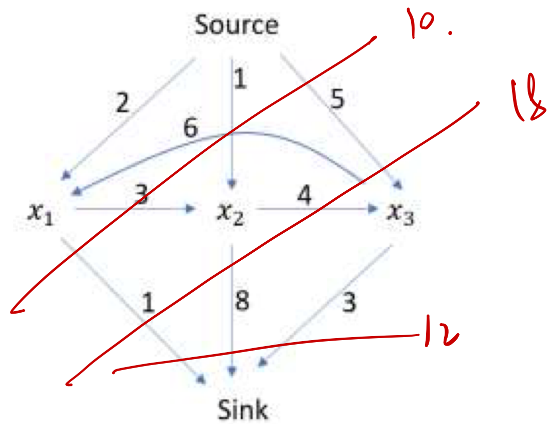
$$
\begin{aligned}
&= -4 + 2\bar{x}_1t + 2sx_2 + 3sx_3 + 2\bar{x}_4t + \bar{x}_1x_2 \\
&\quad + 2\bar{x}_2x_3 + 4\bar{x}_1x_4 + 5\bar{x}_2x_4
\end{aligned}
$$

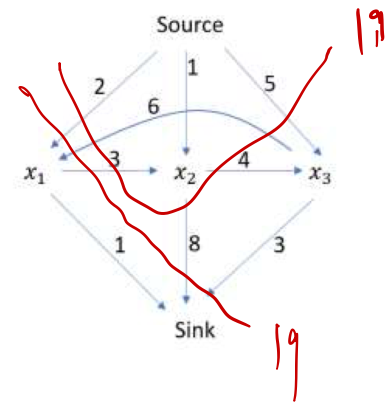


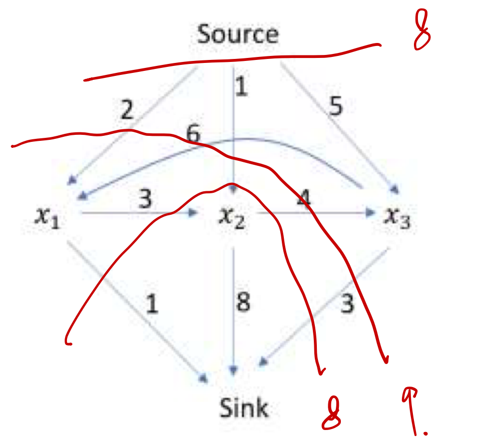
$x_1 = 1$

$x_2 = x_3 = x_4 = 0$

4.







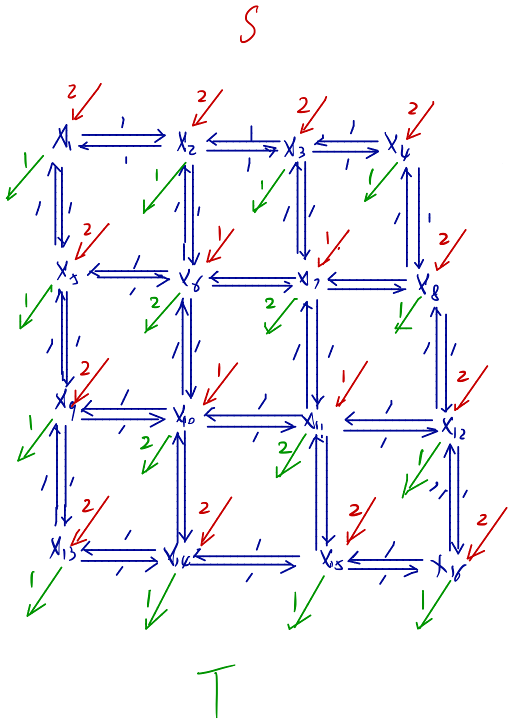
$$f(x_1,x_2,x_3) = 2x_1 + x_2 + 5x_3 + 3\bar{x}_1x_2 + 4\bar{x}_2x_3 + 6\bar{x}_3x_1 + \bar{x}_1 + 8\bar{x}_2 + 3\bar{x}_3$$

$$= 2x_1 + x_2 + 5x_3 + 3(1-x_1)x_2 + 4(1-x_2)x_3 + 6(1-x_3)x_1 + (1-x_1) + 8(1-x_2) + 3(1-x_3)$$

$$= 12 + x_1 - 7x_2 + 2x_3 + 3x_2 + 4x_3 + 6x_1 - 3x_1x_2 - 4x_2x_3 - 6x_1x_3$$

$$= 12 + 7x_1 - 4x_2 + 6x_3 - 3x_1x_2 - 4x_2x_3 - 6x_1x_3$$

5.



$$(1-x_1)x_2 + x_1(1-x_2) = x_1 + x_2 - 2x_1x_2$$

$$\bar{x}_1 x_2 + \bar{x}_2 x_1$$

$$\sum_{(i,j)\in E} x_i + x_j - 2x_i x_j$$

$$f(x_1, x_2, \ldots x_{16}) = \sum_{i=1}^{16} 2x_i - x_6 - x_7 - x_{10} - x_{11}$$

$$+ \sum_{i=1}^{16} \bar{x}_i + \bar{x}_6 + \bar{x}_7 + \bar{x}_{10} + \bar{x}_{11}$$

$$+ \sum_{(i,j)\in E} (x_i + x_j - 2x_i x_j)$$

$$= 20 + \sum_{i=1}^{16} x_i - 2x_6 - 2x_7 - 2x_{10} - 2x_{11}$$

$$+ 2(x_1 + x_4 + x_{13} + x_{16})$$

$$+ 3(x_2 + x_3 + x_5 + x_9 + x_8 + x_{12} + x_{14} + x_{15})$$

$$+ 4(x_6 + x_7 + x_{10} + x_{11}) - 2\sum_{(i,j)\in E} x_i x_j$$

$$= 20 + 3\sum_{i=1}^{16} x_i + (x_2 + x_3 + x_5 + x_9 + x_8 + x_{12} + x_{14} + x_{15}) - 2\sum_{(i,j)\in E} x_i x_j$$

6. unary cost:

	1	2	3
y_1	0.5	1.5	1.0
y_2	1.5	1.5	0

pair cost:

$$\theta_{y_1 y_2}^{lm} = \begin{cases} 0 & l = m \\ 1 & l \neq m \end{cases}$$

cost for expanding labels:

$$E_1 = \sum_{i=0}^{1} \theta_{x_1}^{i} \delta_{x_1}^{i} + \sum_{j=0}^{1} \theta_{x_2}^{j} \delta_{x_2}^{j} + \sum_{i=0}^{1} \sum_{j=0}^{1} \theta_{x_1 x_2}^{ij} \delta_{x_1}^{i} \delta_{x_2}^{j}$$

(1). α-expansion:

We initialize y_1, y_2 with label 1.

$y_1 = 1,\ y_2 = 1$

$\alpha = 2$:

$y_1 = 1 \Leftrightarrow x_1 = 0 \qquad y_2 = 1 \Leftrightarrow x_2 = 0$

$y_1 = 2 \Leftrightarrow x_1 = 1 \qquad y_2 = 2 \Leftrightarrow x_2 = 1.$

We have $\theta_{x_1}^{0} = \theta_{y_1}^{1}$, $\theta_{x_1}^{1} = \theta_{y_1}^{2}$, $\theta_{x_2}^{0} = \theta_{y_2}^{1}$, $\theta_{x_2}^{1} = \theta_{y_2}^{2}$

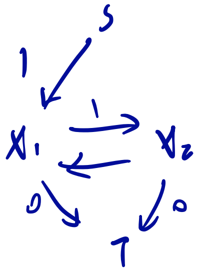
$$\begin{array}{|c|c|} \hline \theta_{x_1 x_2}^{00} & \theta_{x_1 x_2}^{01} \\ \hline \theta_{x_1 x_2}^{01} & \theta_{x_1 x_2}^{11} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \theta_{y_1 y_2}^{11} & \theta_{y_1 y_2}^{21} \\ \hline \theta_{y_1 y_2}^{12} & \theta_{y_1 y_2}^{22} \\ \hline \end{array}$$

$$E = 0.5 \cdot \delta_{x_1}^{0} + 1.5 \cdot \delta_{x_1}^{1} + 1.5 \delta_{x_2}^{0} + 1.5 \delta_{x_2}^{1} + \delta_{x_1}^{0} \delta_{x_2}^{1} + \delta_{x_1}^{1} \delta_{x_2}^{0}$$

$$= 0.5 \cdot \bar{x}_1 + 1.5 x_1 + 1.5 \bar{x}_2 + 1.5 x_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$$= 2 + x_1 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$$= 2 + \bar{5} x_1 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$



$\min(E_1) = 2$ when $x_1 = x_2 = 0$.

$\Rightarrow y_1 = 1, \; y_2 = 1$.

$\lambda = 3$:

$y_1 = 1 \Leftrightarrow x_1 = 0 \qquad y_2 = 1 \Leftrightarrow x_2 = 0$

$y_1 = 3 \Leftrightarrow x_1 = 1 \qquad y_2 = 3 \Leftrightarrow x_2 = 1$.

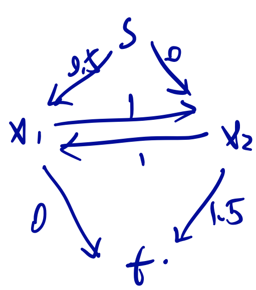
We have $\theta^0_{x_1} = \theta^1_{y_1}$, $\theta^1_{x_1} = \theta^3_{y_1}$, $\theta^0_{x_2} = \theta^1_{y_2}$, $\theta^1_{x_2} = \theta^3_{y_2}$

$$\begin{array}{|c|c|} \hline \theta^{00}_{x_1 x_2} & \theta^{01}_{x_1 x_2} \\ \hline \theta^{01}_{x_1 x_2} & \theta^{11}_{x_1 x_2} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \theta^{11}_{y_1 y_2} & \theta^{31}_{y_1 y_2} \\ \hline \theta^{13}_{y_1 y_2} & \theta^{33}_{y_1 y_2} \\ \hline \end{array}$$

$E_2 = 0.5 \cdot \delta^0_{x_1} + 1 \, \delta^1_{x_1} + 1.5 \, \delta^0_{x_2} + 0 \, \delta^1_{x_2} + \delta^0_{x_1} \delta^1_{x_2} + \delta^1_{x_1} \delta^0_{x_2}$

$= 0.5 \cdot \bar{x}_1 + x_1 + 1.5 \, \bar{x}_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$

$= 0.5 + 0.5 \, x_1 + 1.5 \, \bar{x}_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$



$\min(E_2) = 1$ when $x_1 = x_2 = 1$

$\Rightarrow y_1 = y_2 = 3$.

$\alpha = 1$:

$y_1 = 3 \Leftrightarrow x_1 = 0 \qquad y_2 = 3 \Leftrightarrow x_2 = 0$

$y_1 = 1 \Leftrightarrow x_1 = 1 \qquad y_2 = 1 \Leftrightarrow x_2 = 1.$

cost for expanding labels:

$$E_1 = \sum_{i=0}^{1} \theta_{x_1}^{i} \delta_{x_1}^{i} + \sum_{i=0}^{1} \theta_{x_2}^{i} \delta_{x_2}^{i} + \sum_{i=0}^{1} \sum_{j=0}^{1} \theta_{x_1 x_2}^{ij} \delta_{x_1}^{i} \delta_{x_2}^{j}$$

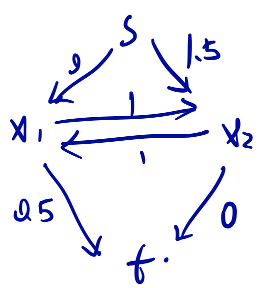
We have $\theta_{x_1}^{0} = \theta_{y_1}^{3}$, $\theta_{x_1}^{1} = \theta_{y_1}^{1}$, $\theta_{x_2}^{0} = \theta_{y_2}^{3}$, $\theta_{x_2}^{1} = \theta_{y_2}^{1}$

$$\begin{array}{|c|c|} \hline \theta_{x_1 x_2}^{00} & \theta_{x_1 x_2}^{01} \\ \hline \theta_{x_1 x_2}^{01} & \theta_{x_1 x_2}^{11} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \theta_{y_1 y_2}^{33} & \theta_{y_1 y_2}^{31} \\ \hline \theta_{y_1 y_2}^{13} & \theta_{y_1 y_2}^{11} \\ \hline \end{array}$$

$$E_2 = 1.0 \cdot \delta_{x_1}^{0} + 0.5 \cdot \delta_{x_1}^{1} + 0 \cdot \delta_{x_2}^{0} + 1.5 \delta_{x_2}^{1} + \delta_{x_1}^{0} \delta_{x_2}^{1} + \delta_{x_1}^{1} \delta_{x_2}^{0}$$

$$= \bar{x}_1 + 0.5 x_1 + 1.5 x_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$$= 0.5 + 0.5 \bar{x}_1 + 1.5 x_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$



$\min(E_2) = 1$ when $x_1 = x_2 = 0$

$\Rightarrow y_1 = y_2 = 3.$

The result is the same as the previous step. So $y_1 = y_2 = 3$.

(2) α-β swap:

	1	2	3
y_1	0.5	1.5	1.0
y_2	1.5	1.5	0

The cost of swap:

$$E = \sum_{i=0}^{1} \theta_{x_1}^{i} \delta_{x_1}^{i} + \sum_{i=0}^{1} \theta_{x_2}^{i} \delta_{x_2}^{i} + \sum_{i=0}^{1}\sum_{j=0}^{1} \theta_{x_1 x_2}^{ij} \delta_{x_1 x_2}^{ij}$$

$y_1 = y_2 = 1$

① $\alpha = 1,\ \beta = 2$:

$y_1 = 1 \Leftrightarrow x_1 = 0 \qquad y_2 = 2 \Leftrightarrow x_2 = 1$

$y_1 = 2 \Leftrightarrow x_1 = 1 \qquad y_2 = 1 \Leftrightarrow x_2 = 0.$

$$\begin{bmatrix} \theta_{x_1 x_2}^{00} & \theta_{x_1 x_2}^{01} \\ \theta_{x_1 x_2}^{10} & \theta_{x_1 x_2}^{11} \end{bmatrix} = \begin{bmatrix} \theta_{y_1 y_2}^{11} & \theta_{y_1 y_2}^{12} \\ \theta_{y_1 y_2}^{21} & \theta_{y_1 y_2}^{22} \end{bmatrix}$$

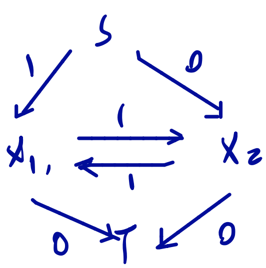
$\theta_{x_1}^{0} = \theta_{y_1}^{1}, \quad \theta_{x_1}^{1} = \theta_{y_1}^{2}, \quad \theta_{x_2}^{1} = \theta_{y_2}^{2}, \quad \theta_{x_2}^{0} = \theta_{y_2}^{1}$

$$E_1 = 0.5 \cdot \delta_{x_1}^{0} + 1.5\, \delta_{x_1}^{1} + 1.5\, \delta_{x_2}^{0} + 1.5\, \delta_{x_2}^{1} + \delta_{x_1}^{1} \delta_{x_2}^{0} + \delta_{x_1}^{0} \delta_{x_2}^{1}$$

$$= 0.5\, \bar{x}_1 + 1.5\, x_1 + 1.5\, \bar{x}_2 + 1.5\, x_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$$= 2 + x_1 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$\min(E_1) = 2$ when $x_1 = x_2 = 0$



$\Rightarrow y_1 = 1,\ y_2 = 1.$

(2) α-β swap:

	1	2	3
y_1	0.5	1.5	1.0
y_2	1.5	1.5	0

The cost of swap:

$$E = \sum_{i=0}^{1} \theta^{i}_{x_1} \delta^{i}_{x_1} + \sum_{i=0}^{1} \theta^{i}_{x_2} \delta^{i}_{x_2} + \sum_{i=0}^{1} \sum_{j=0}^{1} \theta^{ij}_{x_1 x_2} \delta^{ij}_{x_1 x_2}$$

② $y_1=1,\ y_2=1$

$\alpha=1,\ \beta=3$:

$y_1=1 \Leftrightarrow x_1=0 \qquad y_2=3 \Leftrightarrow x_2=1$

$y_1=3 \Leftrightarrow x_1=1 \qquad y_2=1 \Leftrightarrow x_2=0.$

$$\begin{bmatrix} \theta^{00}_{x_1x_2} & \theta^{01}_{x_1x_2} \\ \theta^{10}_{x_1x_2} & \theta^{11}_{x_1x_2} \end{bmatrix} = \begin{bmatrix} \theta^{11}_{y_1y_2} & \theta^{13}_{y_1y_2} \\ \theta^{31}_{y_1y_2} & \theta^{33}_{y_1y_2} \end{bmatrix}$$

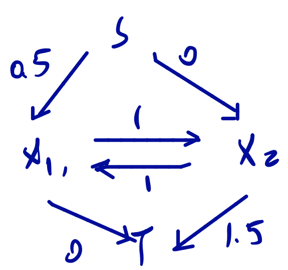
$\theta^{0}_{x_1} = \theta^{1}_{y_1}$, $\theta^{1}_{x_1} = \theta^{3}_{y_1}$, $\theta^{1}_{x_2} = \theta^{3}_{y_2}$, $\theta^{0}_{x_2} = \theta^{1}_{y_2}$

$$E_2 = 0.5 \cdot \delta^{0}_{x_1} + 1.0 \cdot \delta^{1}_{x_1} + 1.5 \cdot \delta^{0}_{x_2} + 0 \cdot \delta^{1}_{x_2} + \delta^{1}_{x_1} \delta^{0}_{x_2} + \delta^{0}_{x_1} \delta^{1}_{x_2}$$

$$= 0.5 \cdot \bar{x}_1 + x_1 + 1.5 \cdot \bar{x}_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$$= 0.5 + 0.5 x_1 + 1.5 \bar{x}_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$$

$\min(E_2) = 1$ when $x_1 = x_2 = 1$



$\Rightarrow y_1 = 3,\ y_2 = 3$

(2) α-β swap:

	1	2	3
y_1	0.5	1.5	1.0
y_2	1.5	1.5	0

The cost of swap:

$$E = \sum_{i=0}^{1} \theta_{x_1}^{i} \delta_{x_1}^{i} + \sum_{i=0}^{1} \theta_{x_2}^{i} \delta_{x_2}^{i} + \sum_{i=0}^{1}\sum_{j=0}^{1} \theta_{x_1 x_2}^{ij} \delta_{x_1 x_2}^{ij}$$

$y_1 = 3, y_2 = 3$

③ $\alpha = 2, \beta = 3$:

$y_1 = 2 \Leftrightarrow x_1 = 0 \qquad y_2 = 3 \Leftrightarrow x_2 = 1$

$y_1 = 3 \Leftrightarrow x_1 = 1 \qquad y_2 = 2 \Leftrightarrow x_2 = 0.$

$$\begin{array}{|c|c|} \hline \theta_{x_1x_2}^{00} & \theta_{x_1x_2}^{01} \\ \hline \theta_{x_1x_2}^{10} & \theta_{x_1x_2}^{11} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \theta_{y_1y_2}^{22} & \theta_{y_1y_2}^{23} \\ \hline \theta_{y_1y_2}^{32} & \theta_{y_1y_2}^{33} \\ \hline \end{array}$$

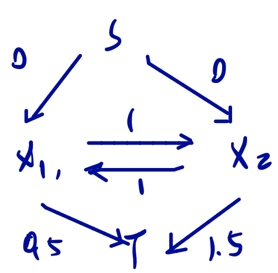
$\theta_{x_1}^{0} = \theta_{y_1}^{2}$, $\theta_{x_1}^{1} = \theta_{y_1}^{3}$, $\theta_{x_2}^{1} = \theta_{y_2}^{3}$. $\theta_{x_2}^{0} = \theta_{y_2}^{2}$

$E_3 = 1.5 \cdot \delta_{x_1}^{0} + 1 \cdot \delta_{x_1}^{1} + 1.5 \cdot \delta_{x_2}^{0} + 0 \cdot \delta_{x_2}^{1} + \delta_{x_1}^{1} \delta_{x_2}^{0} + \delta_{x_1}^{0} \delta_{x_2}^{1}$

$= 1.5\bar{x}_1 + x_1 + 1.5\bar{x}_2 \quad + \bar{x}_1 x_2 + x_1 \bar{x}_2$

$= 1 + 0.5\bar{x}_1 + 1.5\bar{x}_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2$

$\min(E_3) = 1$ when $x_1 = x_2 = 1$



$\Rightarrow y_1 = 3, y_2 = 3.$

As $E_3 = E_2$, should return?

④ $\alpha = 1, \beta = 2.$ $y_1 = 3, y_2 = 3$

(2) α-β swap:

	1	2	3
y_1	0.5	1.5	1.0
y_2	1.5	1.5	0

The cost of swap:

$$E = \sum_{i=0}^{1} \theta^{i}_{x_1} \delta^{i}_{x_1} + \sum_{i=0}^{1} \theta^{i}_{x_2} \delta^{i}_{x_2} + \sum_{i=0}^{1} \sum_{j=0}^{1} \theta^{ij}_{x_1 x_2} \delta^{ij}_{x_1 x_2}$$

$y_1 = 3 \quad y_2 = 3$

⑤ $\alpha = 1, \beta = 3$:

$y_1 = 1 \Leftrightarrow x_1 = 0 \qquad y_2 = 3 \Leftrightarrow x_2 = 1$

$y_1 = 3 \Leftrightarrow x_1 = 1 \qquad y_2 = 1 \Leftrightarrow x_2 = 0.$

$$\begin{bmatrix} \theta^{00}_{x_1 x_2} & \theta^{01}_{x_1 x_2} \\ \theta^{10}_{x_1 x_2} & \theta^{11}_{x_1 x_2} \end{bmatrix} = \begin{bmatrix} \theta^{11}_{y_1 y_2} & \theta^{13}_{y_1 y_2} \\ \theta^{31}_{y_1 y_2} & \theta^{33}_{y_1 y_2} \end{bmatrix}$$

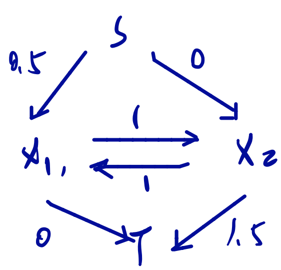
$\theta^{0}_{x_1} = \theta^{1}_{y_1}$, $\theta^{1}_{x_1} = \theta^{3}_{y_1}$, $\theta^{1}_{x_2} = \theta^{3}_{y_2}$. $\theta^{0}_{x_2} = \theta^{1}_{y_2}$

$E_1 = 0.5 \cdot \delta^{0}_{x_1} + 1 \cdot \delta^{1}_{x_1} + 1.5\, \delta^{0}_{x_2} + 0 \cdot \delta^{1}_{x_2} + \delta^{1}_{x_1} \delta^{0}_{x_2} + \delta^{0}_{x_1} \delta^{1}_{x_2}$

$= 0.5\, \bar{x}_1 + x_1 + 1.5\, \bar{x}_2 + x_1 \bar{x}_2 + \bar{x}_1 x_2$

$= 0.5 + 0.5 x_1 + 1.5\, \bar{x}_2 + x_1 \bar{x}_2 + \bar{x}_1 x_2$

$\min(E_1) = 1$ when $x_1 = 0$, $x_2 = 0$



$\Rightarrow y_1 = 3, \; y_2 = 3.$

⑥ $y_1 = 3, \; y_2 = 3,$ repeat ③.

$\alpha = 2, \; \beta = 3.$