CS 6320: Computer Vision Spring 2019

Homework 4

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1

[**10 points**] Show that the following function (all the variables are Boolean) is submodular using the set definition (i.e., $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$:

$$f(x_1, x_2, x_3) = 2x_1 - x_1x_2 - 2x_2x_3 \tag{1}$$

Note that for every pseudo-Boolean function, you can think of an associated set function. In order to show that the a function is submodular, you will need to show that the equation holds for all possible subsets A and B. For the assignment, it is sufficient if you show the results for only 6 possible subsets.

All variables x_1, x_2 and x_3 are Boolean.

A	$f(A)$	B	$f(B)$	$\mathbf{f(A)+f(B)}$	$A \cup B$	$f(A \cup B)$	$A \cap B$	$f(A \cap B)$	$\mathbf{f(A \cup B)+f(A \cap B)}$
(1,0,0)	2	(0,1,0)	0	2	(1,1,0)	1	(0,0,0)	0	2
(1,0,0)	2	(0,0,1)	0	2	(1,0,1)	2	(0,0,0)	0	2
(0,1,0)	0	(0,0,1)	0	0	(0,1,1)	-2	(0,0,0)	0	-2
(1,1,0)	1	(0,0,1)	0	0	(1,1,1)	-1	(0,0,0)	0	-1
(1,0,1)	2	(0,1,0)	0	2	(1,1,1)	-1	(0,0,0)	0	-1
(0,1,1)	-2	(1,0,0)	2	0	(1,1,1)	-1	(0,0,0)	0	-1

Table 1: Submodularity examples.

2

[10 points] Show that the following function (all the variables are Boolean) is NOT submodular using the set definition (i.e., $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$:

$$f(x_1, x_2, x_3) = 2x_1 - x_1x_2 + 2x_2x_3 \tag{2}$$

To show that a given function is not submodular, you will need to show one counterexample. Find some subsets A and B for which the above function does not satisfy submodularity.

All variables x_1, x_2 and x_3 are Boolean.

A	$f(A)$	B	$f(B)$	$\mathbf{f(A)} + \mathbf{f(B)}$	$A \cup B$	$f(A \cup B)$	$A \cap B$	$f(A \cap B)$	$\mathbf{f(A \cup B)} + \mathbf{f(A \cap B)}$
(0,1,0)	0	(0,0,1)	0	0	(0,1,1)	2	(0,0,0)	0	2

Table 2: Counter Example

Clearly, for this example, we have $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ and therefore the function is not sub-modular.

3

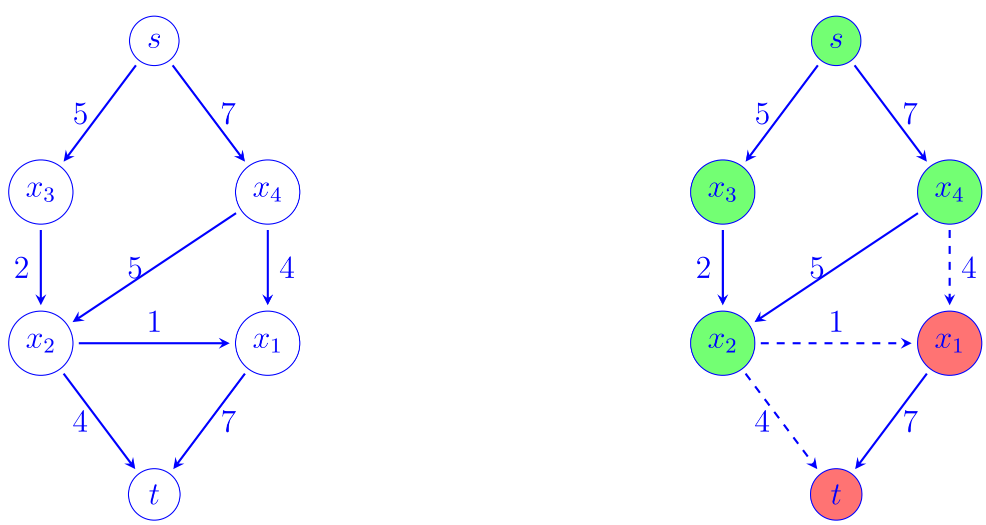
[**10 points**] Consider the following equation:

$$f(x_1, x_2, x_3, x_4) = -2x_1 + 3x_2 + 5x_3 + 7x_4 - x_1x_2 - 2x_2x_3 - 4x_1x_4 - 5x_2x_4 \tag{3}$$

1. Show the maxflow/mincut graph for the above equation.
   We first simplify the expression to bring it in standard form

$$\begin{aligned}
f(x_1, x_2, x_3, x_4) &= -2x_1 + 3x_2 + 5x_3 + 7x_4 - x_1x_2 - 2x_2x_3 - 4x_1x_4 - 5x_2x_4 \\
&= -7x_1 - 4x_2 + 5x_3 + 7x_4 + x_1(1 - x_2) + 2x_2(1 - x_3) + 4x_1(1 - x_4) + 5x_2(1 - x_4) \\
&= -7x_1 - 4x_2 + 5x_3 + 7x_4 + x_1\overline{x_2} + 2x_2\overline{x_3} + 4x_1\overline{x_4} + 5x_2\overline{x_4} \\
&= -7(1 - \overline{x_1}) - 4(1 - \overline{x_2}) + 5x_3 + 7x_4 + x_1\overline{x_2} + 2x_2\overline{x_3} + 4x_1\overline{x_4} + 5x_2\overline{x_4} \\
&= -11 + 7\overline{x_1} + 4\overline{x_2} + 5x_3 + 7x_4 + x_1\overline{x_2} + 2x_2\overline{x_3} + 4x_1\overline{x_4} + 5x_2\overline{x_4} \\
&= -11 + 7\overline{x_1}t + 4\overline{x_2}t + 5\overline{s}x_3 + 7\overline{s}x_4 + \overline{x_2}x_1 + 2\overline{x_3}x_2 + 4\overline{x_4}x_1 + 5\overline{x_4}x_2
\end{aligned} \tag{4}$$

2. Manually identify the best solution (all the variables are Boolean) and show the corresponding cut on the graph. In other words, you will consider all possible states for the Boolean variables and identify the solution that has the minimum value.
   The optimal labels of the nodes are $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$. The graph after optimal cut is shown in the figure 1(b).



(a) Original (b) After Optimal Cut

Figure 1: Graph for the problem - before and after cut.

3. Show that the cost of the cut matches with the cost of the function (without the introduced constant term).
   The value of the function is $f(x_1, x_2, x_3, x_4) = f(1, 0, 0, 0) = -2$. The cost of the cut is 9 and the constant term is -11. Therefore, the optimal solution is $-11 + 9 = -2$ matching the value of the function.

4

[**10 points**] A flow graph is shown in figure 2. There are two terminal nodes (source denoting class 0, and sink denoting class 1). There are three non-terminal nodes.

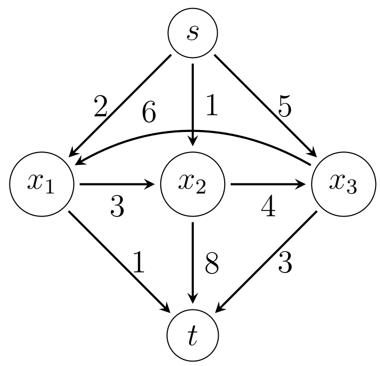


Figure 2: Graph

1. Show the pseudo-Boolean function associated with the above maxflow/mincut graph.

$$
\begin{aligned}
f(x_1,x_2,x_3) &= 2\overline{s}x_1 + \overline{s}x_2 + 5\overline{s}x_3 + 6\overline{x_3}x_1 + 3\overline{x_1}x_2 + 4\overline{x_2}x_3 + \overline{x_1}t + 8\overline{x_2}t + 3\overline{x_3}t \\
&= 2x_1 + x_2 + 5x_3 + 6(1-x_3)x_1 + 3(1-x_1)x_2 + 4(1-x_2)x_3 + (1-x_1) \\
&\qquad + 8(1-x_2) + 3(1-x_3) \\
&= 2x_1 + x_2 + 5x_3 + 6x_1 - 6x_1x_3 + 3x_2 - 3x_1x_2 + 4x_3 - 4x_2x_3 + 1 - x_1 \\
&\qquad + 8 - 8x_2 + 3 - 3x_3 \\
&= 12 + 7x_1 - 4x_2 + 6x_3 - 6x_1x_3 - 3x_1x_2 - 4x_2x_3
\end{aligned} \tag{5}
$$

2. Manually identify the best solution (all the variables are Boolean) and show the corresponding cut on the graph. In other words, you will consider all possible states for the Boolean variables and identify the solution that has the minimum value.

   Manually, the best solution is $(0,1,0)$ and the optimal cut is shown in figure 3.

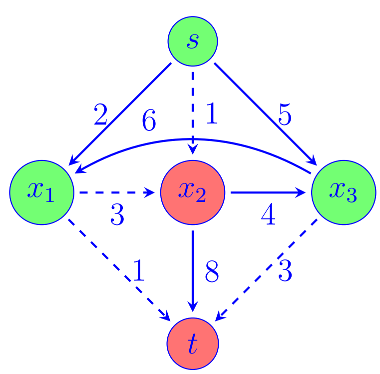
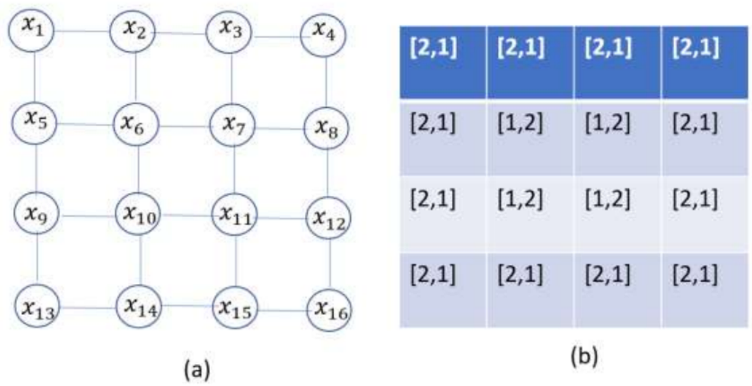


Figure 3: Optimal graph cut.

3. Show that the cost of the cut matches with the cost of the function (without the introduced constant term).
   The cost of the function $f(0,1,0) = 8$. The cost of the corresponding cut is $1+3+1+3=8$. Thus, these two agree with each other.

5

Consider the following image of size 4×4 as shown in figure 4.



[2,1]	[2,1]	[2,1]	[2,1]
[2,1]	[1,2]	[1,2]	[2,1]
[2,1]	[1,2]	[1,2]	[2,1]
[2,1]	[2,1]	[2,1]	[2,1]

Figure 4: (a) We show a 4×4 image with 16 pixels. We use 4-connected neighborhood for adjacent pixels. (b) Unary costs are shown for every pixel in the form $[a, b]$ where a denotes the cost for associated pixel to take the label 0, and b denotes the cost for associated pixel to take the label 1.

The goal is to detect the foreground from the background using maxflow/mincut algorithm. Let the foreground be denoted by the label 0, and let the background be denoted by label 1.

The unary costs for every pixel is given in the table given in figure 4. We use the standard Potts model for the pairwise costs which is given by

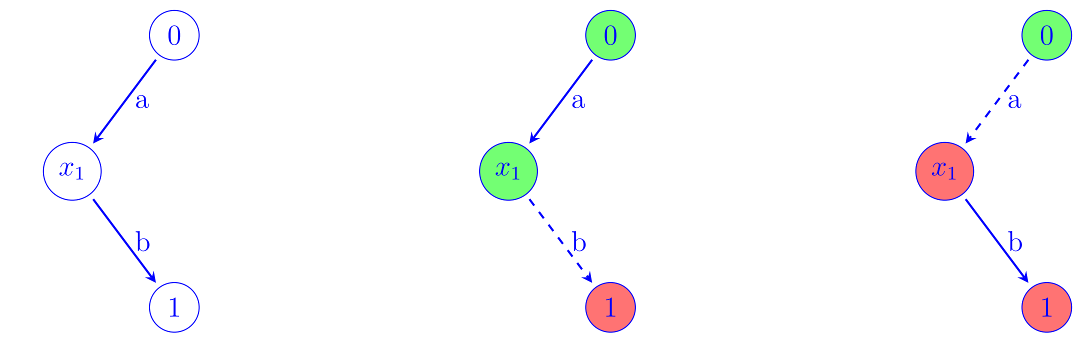
$$\theta_{x_i,x_j}^{l,m} = \begin{cases} 0, & \text{for } l = m \\ 1, & \text{otherwise} \end{cases} \tag{6}$$

where x_i and x_j are adjacent pixels.

1. **[20 points]** Write down the energy function and also show the graph that would be used for the 2-label segmentation. There is no need to solve the problem using maxflow/mincut algorithm, which any of-the-shelf solver can do.

   Cost Assignment :

   We decide the cost assignment to each of the nodes using the following logic. Assume we have only one node x_1 which is to be assigned to either of the nodes 0 or 1. The situation is depicted in figure 5.



(a) Simple Graph (b) Case 1 (c) Case 2

Figure 5: Assigning costs to the nodes. (b) If x_1 assigned as 0, cost of the cut $= b$. (c) If x_1 assigned as 1, cost of the cut $= a$.

From figure 4(b), the cost of assigned of the node to label 0 and 1 should be 2 and 1 respectively which should be same as the optimal cut cost b and a respectively. Hence, we have $b = 2$ and $a = 1$. The Boolean function corresponding to the graph is then

$$f(x_1) = 2\overline{x_1}t + \overline{s}x_1 \tag{7}$$

We could also decide the cost based on the mappings of labels y_1 to the corresponding binary variables x_1 since $y_1 = 1, x_1 = 0$ and $y_1 = 2, x_1 = 1$. The Boolean function now is given by

$$\begin{aligned} f(x_1) &= 2(1 - x_1) + 1x_1 \\ &= 2\overline{x_1}t + \overline{s}x_1 \end{aligned} \tag{8}$$

Clearly, (7) and (8) are in agreement with each other as expected.

Energy Function :
Let $\boldsymbol{x} = [x_1, x_2, \ldots, x_{16}]^T$. Also, the pairwise Potts term $= \overline{x_i}x_j + x_i\overline{x_j} = x_i \oplus x_j$

$$\begin{aligned} f(\boldsymbol{x}) = {} & \overline{s}x_1 + \overline{s}x_2 + \overline{s}x_3 + \overline{s}x_4 + \overline{s}x_5 + 2\overline{s}x_6 + 2\overline{s}x_7 + \overline{s}x_8 + \overline{s}x_9 + 2\overline{s}x_{10} + 2\overline{s}x_{11} + \overline{s}x_{12} + \overline{s}x_{13} \\ & \quad + \overline{s}x_{14} + \overline{s}x_{15} + \overline{s}x_{16} \\ & + 2\overline{x_1}t + 2\overline{x_2}t + 2\overline{x_3}t + 2\overline{x_4}t + 2\overline{x_5}t + \overline{x_6}t + \overline{x_7}t + 2\overline{x_8}t + 2\overline{x_9}t + \overline{x_{10}}t + \overline{x_{11}}t + 2\overline{x_{12}}t + 2\overline{x_{13}}t \\ & \quad + 2\overline{x_{14}}t + 2\overline{x_{15}}t + 2\overline{x_{16}}t \\ & + x_1 \oplus x_2 + x_2 \oplus x_3 + x_3 \oplus x_4 + x_5 \oplus x_6 + x_6 \oplus x_7 + x_7 \oplus x_8 + x_9 \oplus x_{10} + x_{10} \oplus x_{11} \\ & \quad + x_{11} \oplus x_{12} + x_{13} \oplus x_{14} + x_{14} \oplus x_{15} + x_{15} \oplus x_{16} \\ & + x_1 \oplus x_5 + x_2 \oplus x_6 + x_3 \oplus x_7 + x_4 \oplus x_8 + x_5 \oplus x_9 + x_6 \oplus x_{10} + x_7 \oplus x_{11} + x_8 \oplus x_{12} \\ & \quad + x_9 \oplus x_{13} + x_{10} \oplus x_{14} + x_{11} \oplus x_{15} + x_{12} \oplus x_{16} \end{aligned} \tag{9}$$

This can be simplified to

$$
\begin{aligned}
f(\boldsymbol{x}) = & x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + x_8 + x_9 + 2x_{10} + 2x_{11} + x_{12} + x_{13} \\
& \quad + x_{14} + x_{15} + x_{16} \\
& + 2\overline{x_1} + 2\overline{x_2} + 2\overline{x_3} + 2\overline{x_4} + 2\overline{x_5} + \overline{x_6} + \overline{x_7} + 2\overline{x_8} + 2\overline{x_9} + \overline{x_{10}} + \overline{x_{11}} + 2\overline{x_{12}} + 2\overline{x_{13}}t \\
& \quad + 2\overline{x_{14}} + 2\overline{x_{15}} + 2\overline{x_{16}} \\
& + x_1 \oplus x_2 + x_2 \oplus x_3 + x_3 \oplus x_4 + x_5 \oplus x_6 + x_6 \oplus x_7 + x_7 \oplus x_8 + x_9 \oplus x_{10} + x_{10} \oplus x_{11} \\
& \quad + x_{11} \oplus x_{12} + x_{13} \oplus x_{14} + x_{14} \oplus x_{15} + x_{15} \oplus x_{16} \\
& + x_1 \oplus x_5 + x_2 \oplus x_6 + x_3 \oplus x_7 + x_4 \oplus x_8 + x_5 \oplus x_9 + x_6 \oplus x_{10} + x_7 \oplus x_{11} + x_8 \oplus x_{12} \\
& \quad + x_9 \oplus x_{13} + x_{10} \oplus x_{14} + x_{11} \oplus x_{15} + x_{12} \oplus x_{16}
\end{aligned} \tag{10}
$$

Graph :

The graph for segmentation is shown in figure 6.

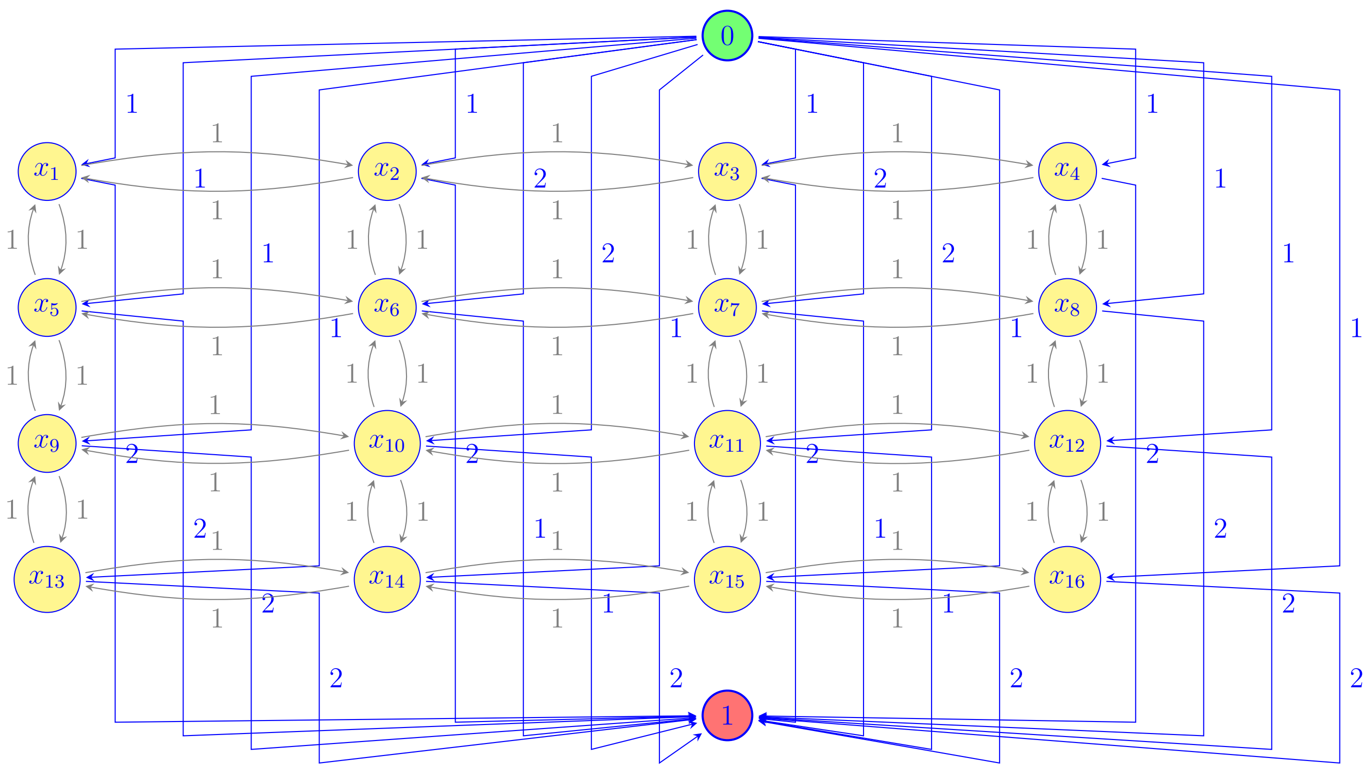


Figure 6: Graph for the problem. The pairwise terms are shown in gray for clarity. The cost of the unary terms are shown on the right side of each edge.

6

Consider a multi-label problem with 2 variables y_1 and y_2 each taking 3 states $\{1, 2, 3\}$.

The unary terms are given in table 3

l	$\theta^l_{y_1}$	$\theta^l_{y_2}$
1	0.5	1.5
2	1.5	1.5
3	1.0	0.0

Table 3: Unary Cost function

We use the standard Potts model for the pairwise costs which is given by

$$\theta^{l,m}_{x_i,x_j} = \begin{cases} 0, & \text{for } l = m \\ 1, & \text{otherwise} \end{cases} \tag{11}$$

where x_i and x_j are the adjacent pixels.

1. **[20 points]** Show the iterations in alpha-expansion till it converges. Note that you will construct a Boolean function in every iteration. You will manually compute the solution for every maxflow/mincut step.

   The pairwise Potts term $= (1 - x_i)x_j + x_i(1 - x_j) = \overline{x_i}x_j + x_i\overline{x_j}$ is denoted by $y_i \oplus y_j$. All variables are initialised by label 1.

   (a) **Iteration 1:** $\alpha = 1$.

   Labels unchanged from $(1, 1)$.

   (b) **Iteration 2:** $\alpha = 2$.
   $y_1 = 1, x_1 = 0$ and $y_1 = 2, x_1 = 1$
   $y_2 = 1, x_2 = 0$ and $y_2 = 2, x_2 = 1$
   Unary cost function $= \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix}$. The cost function is submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.
   Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
   Boolean function is given by

$$\begin{aligned} f &= 0.5(1 - x_1) + 1.5x_1 + 1.5(1 - x_2) + 1.5x_2 + x_1\overline{x_2} + \overline{x_1}x_2 \\ &= 1.5\overline{s}x_1 + 1.5\overline{s}x_2 + 0.5\overline{x_1}t + 1.5\overline{x_2}t + x_1 \oplus x_2 \end{aligned} \tag{12}$$

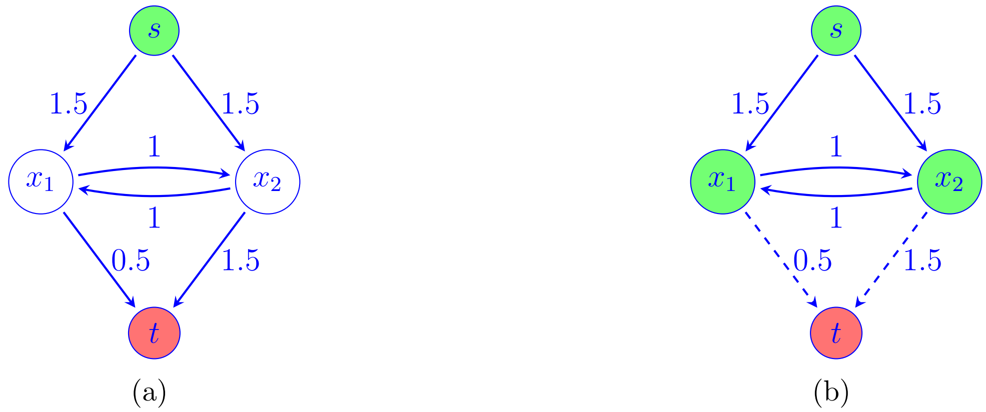


Figure 7: (a) Graph of the boolean function (12). (b) Optimal Cut.

Labels unchanged from $(1, 1)$.

(c) **Iteration 3:** $\alpha = 3$.
$y_1 = 1, x_1 = 0$ and $y_1 = 3, x_1 = 1$
$y_2 = 1, x_2 = 0$ and $y_2 = 3, x_2 = 1$
Unary cost function $= \begin{bmatrix} 0.5 & 1.5 \\ 1.0 & 0.0 \end{bmatrix}$. The cost function is again submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.
Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Boolean function is given by

$$f = \overline{s}x_1 + 0.5\overline{x_1}t + 1.5\overline{x_2}t + x_1 \oplus x_2 \tag{13}$$

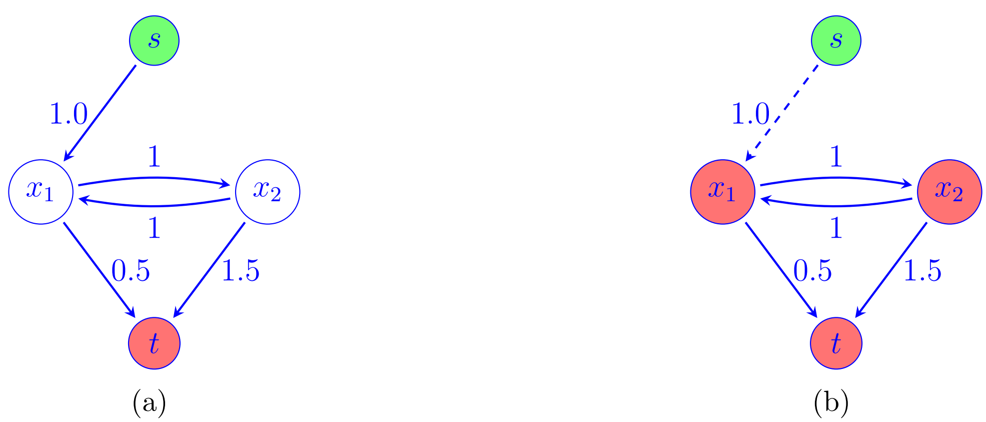


Figure 8: (a) Graph of the boolean function (13). (b) Optimal Cut.

Both labels changed to $(3, 3)$.

(d) **Iteration 4:** $\alpha = 1$.
$y_1 = 3, x_1 = 0$ and $y_1 = 1, x_1 = 1$
$y_2 = 3, x_2 = 0$ and $y_2 = 1, x_2 = 1$
Unary cost function $== \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.5 \end{bmatrix}$
The cost function is again submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.

Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Boolean function is given by

$$f = 0.5\overline{s}x_1 + 1.5\overline{s}x_2 + \overline{x_1}t + x_1 \oplus x_2 \tag{14}$$

Figure 9: (a) Graph of the boolean function (14). (b) Optimal Cut.

Labels unchanged from $(3, 3)$.

(e) **Iteration 5:** $\alpha = 2$.
$y_1 = 3, x_1 = 0$ and $y_1 = 2, x_1 = 1$
$y_2 = 3, x_2 = 0$ and $y_2 = 2, x_2 = 1$
Unary cost function $= \begin{bmatrix} 1.0 & 0.0 \\ 1.5 & 1.5 \end{bmatrix}$
The cost function is again submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.
Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Boolean function is given by

$$f = 1.5\overline{s}x_1 + 1.5\overline{s}x_2 + \overline{x_1}t + x_1 \oplus x_2 \tag{15}$$

Figure 10: (a) Graph of the boolean function (15). (b) Optimal Cut.

Labels unchanged from $(3, 3)$

(f) **Iteration 6:** $\alpha = 3$.
Labels unchanged from $(3, 3)$
Converged : The final labels of (y_1, y_2) are $(3, 3)$.

2. **[20 points]** Show the iterations in alpha-beta swap till it converges. Note that you will construct a Boolean function in every iteration. You will manually compute the solution for every maxflow/mincut step.

The pairwise Potts term $= (1 - x_i)x_j + x_i(1 - x_j) = \overline{x_i}x_j + x_i\overline{x_j}$ is denoted by $x_i \oplus x_j$. All variables are initialised by label 1.

(a) **Iteration 1:** $\alpha = 1, \beta = 2$.
$y_1 = 1, x_1 = 0$ and $y_1 = 2, x_1 = 1$
$y_2 = 1, x_2 = 0$ and $y_2 = 2, x_2 = 1$
Unary cost function $= \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix}$
The cost function is submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.
Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Boolean function is given by

$$f = 1.5\overline{s}x_1 + 1.5\overline{s}x_2 + 0.5\overline{x_1}t + 1.5\overline{x_2}t + x_1 \oplus x_2 \qquad (16)$$

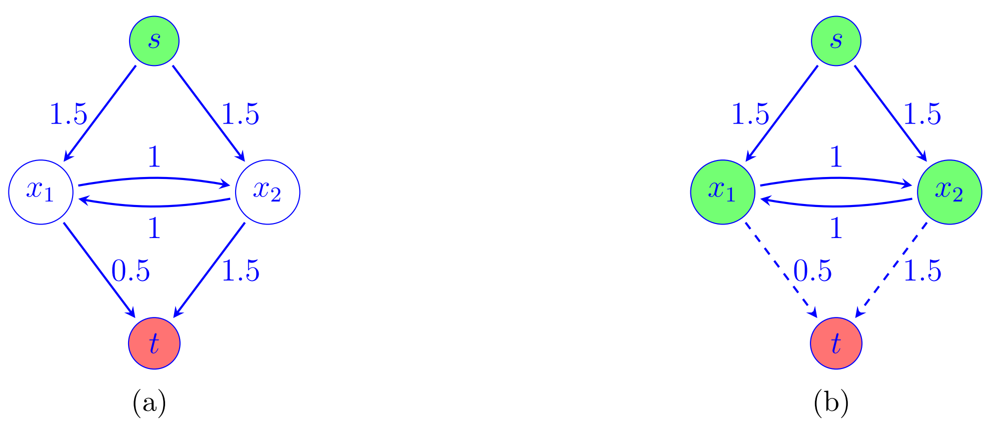


Figure 11: (a) Graph of the boolean function (16). (b) Optimal Cut.

Labels unchanged from $(1, 1)$.

(b) **Iteration 2:** $\alpha = 1, \beta = 3$.
$y_1 = 1, x_1 = 0$ and $y_1 = 3, x_1 = 1$
$y_2 = 1, x_2 = 0$ and $y_2 = 3, x_2 = 1$
Unary cost function $= \begin{bmatrix} 0.5 & 1.5 \\ 1.0 & 0.0 \end{bmatrix}$
The cost function is again submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.

Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Boolean function is given by

$$f = \overline{s}x_1 + 0.5\overline{x_1}t + 1.5\overline{x_2}t + x_1 \oplus x_2 \tag{17}$$

Figure 12: (a) Graph of the boolean function (17). (b) Optimal Cut.

Both labels changed to $(3, 3)$.

(c) **Iteration 3:** $\alpha = 2, \beta = 3$.
$y_1 = 2, x_1 = 0$ and $y_1 = 3, x_1 = 1$
$y_2 = 2, x_2 = 0$ and $y_2 = 3, x_2 = 1$
Unary cost function $= \begin{bmatrix} 1.5 & 1.5 \\ 1.0 & 0.0 \end{bmatrix}$
The cost function is again submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.
Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Boolean function is given by

$$f = 1.0\overline{s}x_1 + 1.5\overline{x_1}t + 1.5\overline{x_2}t + x_1 \oplus x_2 \tag{18}$$

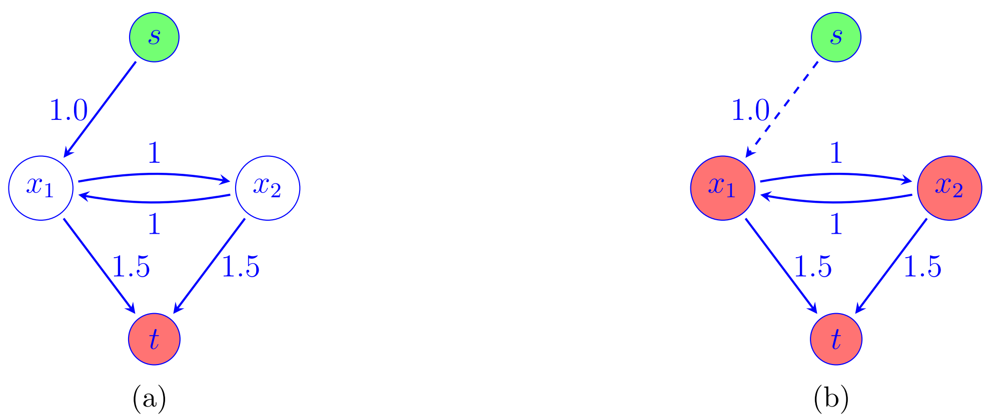


Figure 13: (a) Graph of the boolean function (18). (b) Optimal Cut.

Labels unchanged from $(3, 3)$.

(d) **Iteration 4:** $\alpha = 1, \beta = 2$.
Not possible since both variables have value 3.

(e) **Iteration 5:** $\alpha = 1, \beta = 3$.
$y_1 = 1, x_1 = 0$ and $y_1 = 3, x_1 = 1$
$y_2 = 1, x_2 = 0$ and $y_2 = 3, x_2 = 1$
Unary cost function $= \begin{bmatrix} 0.5 & 1.5 \\ 1.0 & 0.0 \end{bmatrix}$
The cost function is again submodular since sum of main diagonal elements is less than or equal to the sum of elements in the off-diagonal.
Pairwise cost function $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Boolean function is given by

$$f = \overline{s}x_1 + 0.5\overline{x_1}t + 1.5\overline{x_2}t + x_1 \oplus x_2 \tag{19}$$

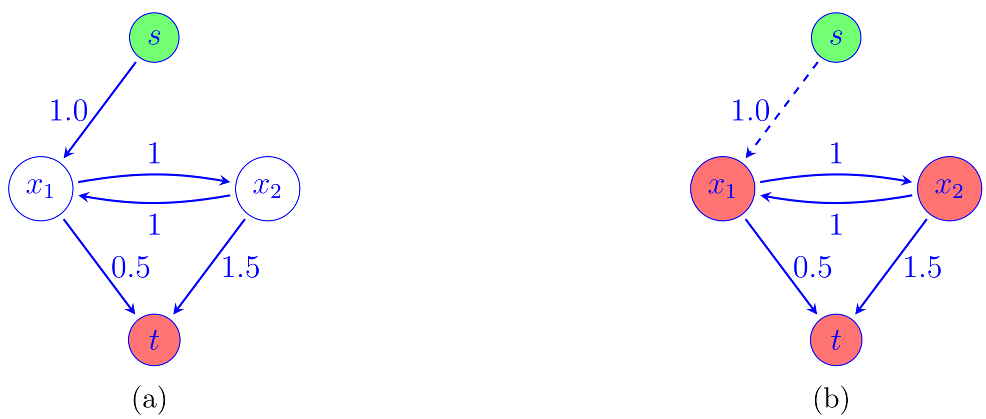


Figure 14: (a) Graph of the boolean function (19). (b) Optimal Cut.

Labels unchanged from $(3, 3)$
Converged : The final labels of (y_1, y_2) are $(3, 3)$.