Master's Theorem

→ $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ $a >= 1;$ $b > 1$

n: Size of the problem

a: Number of subproblems in recursion

n/b: The size of each subproblem. (It is assumed that the size of all the subproblems are the same)

f(n): Cost of work done outside the recursive calls, which basically includes the cost of dividing the problem and merging the solutions of the subproblems

$f(n) = \Theta(n^k (\log n)^p)$
$= \Theta(n^k \log^p n)$

i) $\log_b a$ ii) k

Case 1: If $\log_b a > k$ then $\Theta(n^{\log_b a})$

Case 2: If $\log_b a = k$

- → If $p > -1$ $\Theta(n^k \log^{p+1} n)$
- → If $p = -1$ $\Theta(n^k \log \log n)$
- → If $p < -1$ $\Theta(n^k)$

Case 3: If $\log_b a < k$

- → If $p \geq 0$ $\Theta(n^k \log^p n)$
- → If $p < 0$ $\Theta(n^k)$

i) $T(n) = 2T\left(\frac{n}{2}\right) + 1$

$a = 2$
$b = 2$
$f(n) = \Theta(1)$
$= \Theta(n^0 \log^0 n)$
$k = 0$
$p = 0$

$\log_2 2 = 1 > k = 0$ Case 1

$\Theta(n^{\log_b a}) = \Theta(n)$

ii) $T(n) = 4T\left(\frac{n}{2}\right) + n$ Case 1

$\log_2 4 = 2 > K = 1$ $p = 0$

$\Theta(n^2)$

iii) $T(n) = 8T\left(\frac{n}{2}\right) + n$ Case 1

$\log_2 8 = 3 > k = 1$ $p = 0$

$\Theta(n^3)$

iv) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$\log_2 2 = 1$ $k = 1$ $p = 0$ Case 2

$\Theta(n^k \log^{p+1} n) = \Theta(n \log n)$

v) $T(n) = 4T\left(\frac{n}{2}\right) + n^2$

$\log_2 4 = 2 \quad k = 2 \quad p = 0$ Case 2

$\Theta(n^2 \log n)$

vi) $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \log n$

$\log_2 4 = 2 \quad k = 2 \quad p = 1$

$\Theta(n^2 \log^2 n)$

vii) $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n^1}{\log n}$

$\log_b a = 1 \quad k = 1 \quad p = -1$

$\Theta(n^k \log \log n) = \Theta(n \log \log n)$

viii) $T(n) = 2T\left(\frac{n}{2}\right) + n \log^{-2} n$

$\log_b a = 1 \quad k = 1 \quad p = -2$

$\Theta(n^k) = \Theta(n)$

ix) $T(n) = T\left(\frac{n}{2}\right) + n^2$

$\log_2 1 = 0 \quad k = 2 \quad \Theta(n^2)$

x) $T(n) = 2T\left(\frac{n}{2}\right) + n^2 \log^2 n$

$\log_b a = 1 \quad k = 2 \quad p = 2$

$\Theta(n^k \log^p n) = \Theta(n^2 \log^2 n)$

$f(n) = 2^n$
$g(n) = n!$
$h(n) = n^2 \log n$

$h(n) < f(n) < g(n)$

O - Worst case
Ω - Best case
Θ - Avg case

→ $h(n)$ ka worst case $f(n)$
$f(n)$ ka worst case $g(n)$

$g(n)$ ka best case $f(n)$
$f(n)$ ka best case $h(n)$