MIT Primes

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G1

We have n fair six-sided dice, labeled 1 through 6. Let p_n be the probability that when rolled, the product of all n numbers shown is at most 6.

Part A

Compute the value of p_2

Proof. S is the set of all sides of the dice. There are 36 unique combinations one can get by rolling 2 die.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Their products are:

1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24
5	10	15	20	25	30
6	12	18	24	30	36

There are 14 cases where $P \leq 6$, so $p_2 = \frac{14}{36} = 38.9\%$ □

Part B

Determine p_n for any integer $n \geq 2$.

Proof. I was unable to figure out the solution by hand, so I wrote a python program.

```python
def find_matches(num: int) -> float:
    from typing import Tuple, Iterable
    sides: int = 6
    limit: int = 6

    import itertools
    combinations: Iterable[Tuple[int]] = (
        itertools.product(
            range(1, sides + 1),
            repeat=num
        )
    )

    import numpy
    matches: int = sum((
        1
        for product in (
            numpy.prod(combination)
            for combination in combinations
        )
        if product <= limit
    ))
    return matches
```

Running the above program with inputs of $1, 2, 3, 4, 5, 6, 7, 8, 9$ mapped to $6, 14, 25, 39, 56, 76, 99, 125, 154$. Using the OEIS, I found that that the sequence is the Second Pentagonal Numbers offset by -1. Knowing this, I conjectured that the matches for each n is:

$$m_n = \frac{3n^2 + n}{2} - 1 \tag{1}$$

$$= \frac{(3n-2)(n+1)}{2} \tag{2}$$

However, m_2 gave 6, the first index in my found sequence, so I needed to shift 1 up by one index, so:

$$m_n = \frac{(3n+1)(n+2)}{2} \tag{3}$$

Our sample space for each n is 6^n, so to get our probability we take 3 and divide it by the sample space, giving:

$$p_n = \frac{m_n}{6^n} = \frac{(3n+1)(n+2)}{2 * 6^n} \tag{4}$$

Setting $n = 2$ into 4 gives $\frac{28}{72}$, which is equivalent to the answer found in Part A of this problem.

$\square$

G3

There are 100 marbles in a bag: 30 red marbles, 60 green marbles and 10 yellow marbles. We select three of them uniformly at random, independently and with replacement (meaning we put the ball back after it is removed). Let E be the event "all three marbles are the same color".

Part A

Find the probability of E.

Proof. E can be defined as the union of three mutually exclusive different events:

$$P(E) = P(R \cup G \cup Y) \tag{5}$$

Their individual probabilities are:

$$P(R) = \left(\frac{30}{100}\right)^3 = \left(\frac{3}{10}\right)^3 = \frac{27}{1000} \tag{6}$$

$$P(G) = \left(\frac{60}{100}\right)^3 = \left(\frac{6}{10}\right)^3 = \frac{216}{1000} \tag{7}$$

$$P(Y) = \left(\frac{10}{100}\right)^3 = \left(\frac{1}{10}\right)^3 = \frac{1}{1000} \tag{8}$$

So, from equation 5 we can say:

$$P(E) = P(R \cup G \cup Y) = P(R) + P(G) + P(Y) - P(R \cap G \cap Y)$$

R, G, P are independent from eachother so their intersection is 0

$$P(E) = P(R) + P(G) + P(Y)$$

$$P(E) = \frac{27}{1000} + \frac{216}{1000} + \frac{1}{1000}$$

$$P(E) = \frac{244}{1000} = 24.4\%$$

□

Part B

Find the probability of E, given that at least one selected marble is red.

Proof. $\square$

G5

Consider the following six points in the coordinate plane:

$$A = (0, 1), B = (0, 3), C = (1, 4), D = (4, 9), E = (6, 7), F = (6, 8)$$

For a point P in the coordinate plane let $S(P) = PA + PB + PC + PD + PE + PF$

Part A

Prove that S(P) is minimized at some point P.

Proof. By the distance formula, we know that:

$$\begin{aligned} S(P) = \sqrt{(P_x - A_x)^2 + (P_y - A_y)^2} + \sqrt{(P_x - B_x)^2 + (P_y - B_y)^2} + \dots \\ + \sqrt{(P_x - F_x)^2 + (P_y - F_y)^2} \end{aligned}$$

By substitution, we can find that the total distance function is equal to:

$$\begin{aligned} S(P) = \sqrt{(x)^2 + (y-1)^2} + \sqrt{(x)^2 + (y-3)^2} + \sqrt{(x-1)^2 + (y-4)^2} \\ + \sqrt{(x-4)^2 + (y-9)^2} + \sqrt{(x-6)^2 + (y-7)^2} + \sqrt{(x-6)^2 + (y-8)^2} \end{aligned} \tag{9}$$

The point will be between the boundaries of the minimum and maximum x-value and the minimum and maximum y-value, so $x \in [0, 6]$ and $y \in [1, 9]$. By the extreme value theorem, we know that $S(P)$ must have a minimum and maximum in those intervals.

□

Part B

Determine the value of that minimum.

Proof. P is a Geometric Median, which cannot be solved analytically. I wrote a python program to solve it:

```python
from typing import List, Tuple

def geometric_median(points: List[Tuple[int, int]]):
    import numpy
    from scipy.optimize import minimize
    from scipy.spatial.distance import cdist

    points_array: numpy.array = (
        numpy.asarray(points)
    )

    def distance_sum(val: numpy.array):
        return cdist(
            [val], points_array
        ).sum()

    return minimize(
        distance_sum,
        points_array.mean(axis=0),
        method="COBYLA").x
```

Running the above program with the given points, with Python 3.6.3 on MacOS, yields:

```python
>>> geometric_median(
>>>     [(0,1),(0,3),(1,4),(4,9),(6,7),(6,8)]
>>> )
array([1.71518827,   4.42938511])
```

Therefore, we can conclude that $P \approx (1.7152, 4.4294)$

Solving $S(P)$ with the found P:

$$S(1.7152, 4.4294) \approx 22.5855$$

$\square$

G6

Let a, b, c be positive real numbers for which $min(ab, bc, ca) \geq 1$.

Part A

Prove that $\log abc \geq \sqrt[3]{\log^3 a + \log^3 b + \log^3 c}$

Proof.

$$\begin{aligned}
\log^3 abc &\geq \log^3 a + \log^3 b + \log^3 c \\
(\log a + \log b + \log c)^3 &\geq \log^3 a + \log^3 b + \log^3 c \\
6 \log a \log b \log c + 3 \log^2 a \log b & \\
+3 \log a \log^2 b + 3 \log^2 a \log c & \\
+3 \log a \log^2 c + \log^3 a & \\
+3 \log b \log^2 c + 3 \log^2 b \log c & \\
+ \log^3 b + \log^3 c &\geq \log^3 a + \log^3 b + \log^3 c \\
\log^2 a \log b + \log^2 a \log c & \\
+ \log a \log^2 b + \log^2 b \log c & \\
+ \log b \log^2 c + \log a \log^2 c & \\
+2 \log a \log b \log c &\geq 0 \\
(\log a + \log b) \times (\log b + \log c) \times & \\
(\log a + \log c) &\geq 0 \\
\log ab \log bc \log ac &\geq 0
\end{aligned}$$

We know from the given that:

$$min(ab, bc, ca) \geq 1$$

$$\log ab, \log bc, \log ca \geq \log 1 = 0$$

Therefore, since all the factors are either greater than or equal to 0, their product must be greater than or equal to 0.

□

Part B

Determine for which triples (a, b, c) the equality holds.

Proof. Set the equation equal to 0, ignoring the inequality.

$$\log ab \log bc \log ac = 0$$

This implies that at least one of the logarithms must be equal to 0. Let's choose $logab$

$$\log ab = 0$$

$$ab = 1$$

If we say $a = n$, then $b = \frac{1}{n}$. Therefore, since $bc \geq 1$, and $ac \geq 1$, $c \geq \frac{1}{b} = n$, or $c \geq \frac{1}{a} = \frac{1}{n}$. The intersection of these two solutions varies depending on the interval of n. So, $c \geq max(\frac{1}{n}, n)$.

The triplet is then:

$$(a, b, c) = \left(n, \frac{1}{n}, \geq max\left(\frac{1}{n}, n\right)\right)$$

□

G7

For each integer $n \geq 1$ let τ_n denote the set of nondegenerate triangles whose side lengths are in $1, \ldots, n$. Moreover, for each triangle $\triangle ABC$, let:

$$D(\triangle ABC) = min(|AB - AC|, |BC - BA|, |CA - CB|) \tag{10}$$

Part A

For each triangle, to maximize the minimum side differences, each triangle should have a constant difference from each side. It doesn't matter if a triangle has one side with a great difference, since D takes a minimum value. Therefore, we can say the sides of the triangle are:

$$a > b > c \tag{11}$$

$$a = b - k = c - 2k \tag{12}$$

$$\begin{aligned} D(a, b, c) &= min(|a - b|, |a - c|, |b - c|) && (13) \\ &= min(|b - k - b|, |c - 2k - c|, |a + k - (a + 2k)|) && (14) \\ &= min(| - 2k|, | - 2k|, | - k|) = k && (15) \end{aligned}$$

Where k is the constant difference between each side. To discover the value of k, one must utilise triangle inequalities:

$$a < b + c \tag{16}$$

$$a < a + k + a + 2k \tag{17}$$

$$-a < 3k \tag{18}$$

$$a > -3k \tag{19}$$

$$a > 3k \tag{20}$$

$$\text{k is a difference, so the negative is absorbed into the variable} \tag{21}$$

$$a = n \tag{22}$$

$$n > 3k \tag{23}$$

$$k < \frac{n}{3} \tag{24}$$

$$k = floor\left(\frac{n}{3}\right) \tag{25}$$

$$\text{The value has to be adjusted by an index of 1} \tag{26}$$

$$\text{because multiples of 3 can be degenerate cases} \tag{27}$$

$$k = floor\left(\frac{n-1}{3}\right) \tag{28}$$

$$\tag{29}$$

Subsituting the k value from 16 into 11 gives:

$$max(D(\triangle\tau_n)) = floor\left(\frac{n-1}{3}\right) \tag{30}$$

Part B

For which n is this maximum value achieved for a unique triangle in τ_n (up to congruence)?

M4

For integers $n \geq 0$, let:

$$a_n = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n+i}{n-i} \binom{2i}{i} \tag{31}$$

Identify the sequence $(a_n)_n$ by name and prove that an is the claimed sequence. (You may use the Online Encyclopedia of Integer Sequences, http://oeis.org/.)

Solution

$(a_n)_n$ is the sequence of the Shroeder Numbers (OEIS A006318), the number of paths between $(0,0)$ and (n,n) by moving one unit up, down, left, or right.

Proof. If 31 is the sequence describing the Shroeder Numbers, it must satisfy the recurrence relation:

$$S_n = S_{n-1} + \sum_{k=1}^{n-1} S_k S_{n-1-k} \tag{32}$$

The closed form of 32 is:

$$S_n =_2 F_1(-n, n+1; 2; -1) \tag{33}$$

33 can be evaluated using the series form of the Gaussian Hypogeometric Function. Since the first term, $-n \leq 0$ for all $n \geq 0$, the series terminates leaving:

$$S_n = \sum_{i=0}^{n} (-1)^n \binom{n}{i} \frac{(n+1)_{(i)}}{2_{(i)}} (-1)^n \tag{34}$$

Simplifying the binomial coeffecients and rising factorials yields:

$$S_n = \sum_{i=0}^{n} \binom{n}{i} \frac{(n+1)_{(i)}}{2_{(i)}} \tag{35}$$

$$= \sum_{i=0}^{n} \frac{n!}{(i!)(n-i)!} \frac{(n+1)_{(i)}}{2_{(i)}} \tag{36}$$

$$= \sum_{i=0}^{n} \frac{n!}{(n-i)!} \binom{n+i}{i} \frac{1}{2_{(i)}} \tag{37}$$

$$= \sum_{i=0}^{n} \frac{n!}{(n-i)!} \frac{(n+i)!}{i!n!} \frac{1}{2_{(i)}} \tag{38}$$

$$= \sum_{i=0}^{n} \frac{(n+i)!}{(n-i)!} \frac{1}{2_{(i)} i!} \tag{39}$$

$$= \sum_{i=0}^{n} \frac{(n+i)!}{(n-i)!(2i)!} \frac{(2i)!}{2_{(i)} i!} \tag{40}$$

$$= \sum_{i=0}^{n} \binom{n+i}{n-i} \frac{(2i)!}{2_{(i)} i!} \tag{41}$$

$$= \sum_{i=0}^{n} \binom{n+i}{n-i} \frac{(2i)!}{(i+1)! i!} \tag{42}$$

$$= \sum_{i=0}^{n} \frac{1}{i+1} \binom{n+i}{n-i} \frac{(2i)!}{i! i!} \tag{43}$$

$$= \sum_{i=0}^{n} \frac{1}{i+1} \binom{n+i}{n-i} \binom{2i}{i} \tag{44}$$

So,

$$S_n = a_n$$

$\square$