Abstraction Principles in Mathematics
Exam

1. **Ideals of $\mathbb{Z}$:** Prove that every ideal of $\mathbb{Z}$ is principal, as follows. Let I be the ideal, and let $a \in I$ be the smallest positive integer in I. Show that if some other element $b \in I$ is not a multiple of a, then there is a smaller positive integer in I.

2. **Not all ideals are principal:** Prove that the ideal $(2, x)$ of $\mathbb{Z}[x]$ is not principal.

3. **Maximal ideals are prime:** Prove that every maximal ideal is prime. Here are two possible ways to do this, you can pick either, or give your own proof.

   i) Prove that every field is an integral domain. Then prove that if the quotient by an ideal is an integral domain, the ideal is prime. Finally combine these statements to get the required statement.

   ii) Assume I is a maximal ideal, but not prime. Then there are elements a, b such that $ab \in I$ but $a \notin I$ and $b \notin I$. Show that this implies $1 = r + ca$ for some $r \in I$ and $c \in R$ and also $1 = s + db$ for some $s \in I$ and $d \in R$. Use this to show $1 \in I$, which contradicts the fact that I is a proper ideal.