Mechatronics
DEPAZG511

Lecture

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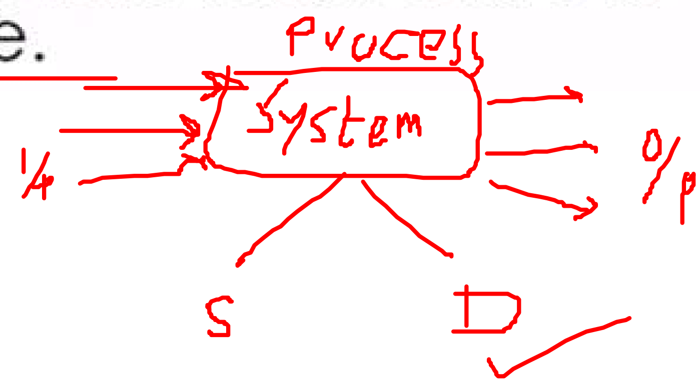
Dynamic response of systems:
1st and 2nd order systems and their performance measures

Response of Systems

- Most important function of a model is to predict what the output will be for a particular input.
- Here we are concerned about
  - ✓ What happens at static situation i.e., when steady state is reached.
  - ✓ How output change with time when there is change in input.
  - ✓ How output change with time when there is change in input with time.



e.g.

- Vehicles on the bridge,
- Fuel consumption in vehicle

Modelling of dynamic system

- These are the systems whose behaviour as a function of time is important for the modeller.
- For example, for an aircraft a static system analysis may be sufficient for the stress in wings during steady flight.
- However for the aircraft to be subjected to the time varying stresses during flight through turbulent air, emergency maneuvers or hard landing a dynamic system analysis has to be done.

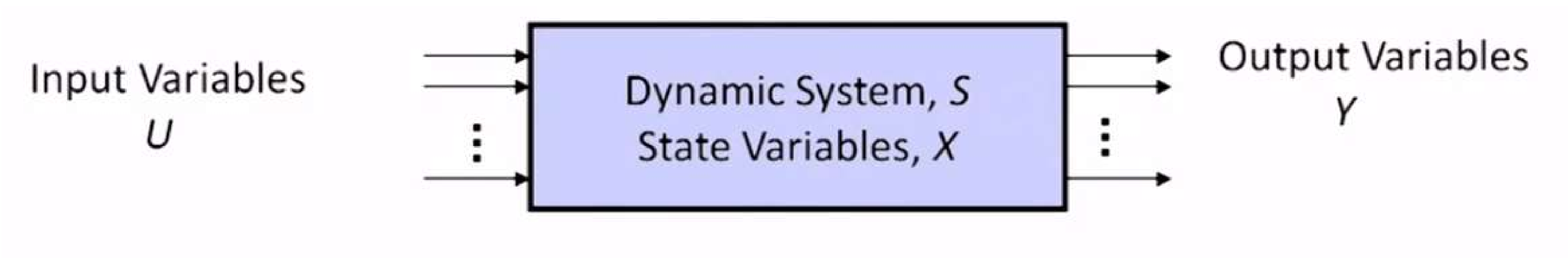
Static
hyd
Ele
Mech
Ele-Mech
thermal

linear
vo

Response of Systems

- $\frac{dx}{dt}$ represents the first order system and it describes the rate at which x varies with time.
- $\frac{d^2x}{dt^2}$ represents the second order system and it describes the rate at which $\frac{dx}{dt}$ varies with time.
- $\frac{d^nx}{dt^n}$ represents the nth order system

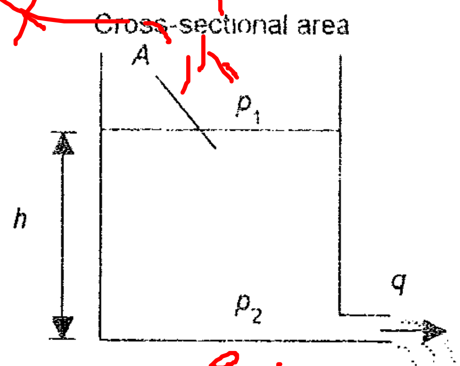
A general dynamic system model



The term **natural response** is used for a system when there is no input to the system forcing the variable to change but it is just changing naturally.

Dynamic response of systems

Natural Response
Initial input condition is Zero

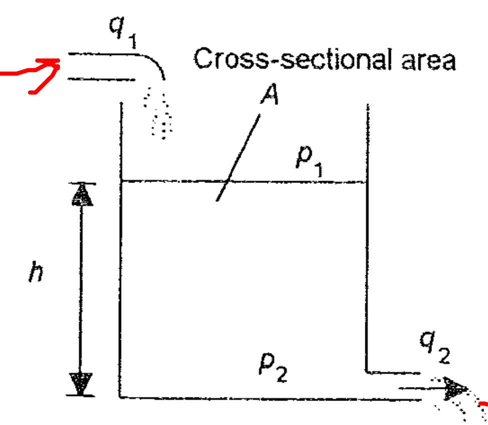


$$RA\frac{dh}{dt} + (\rho g)h = 0$$

$\frac{1}{r}$

$$P_1 - P_2 = \rho g h \propto q = Rq = -RA\frac{dh}{dt}$$

Forced Response – initially some input is present in the system



$$RA\frac{dh}{dt} + (\rho g)h = q_1$$

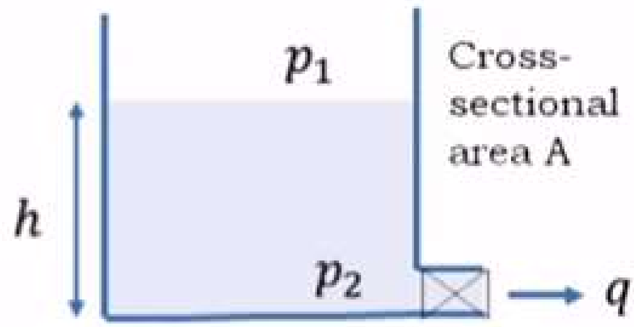
$$q = -\frac{dV}{dt} = -\frac{dAh}{dt}$$

First order system- Natural response

Example of First Order System

Initial input condition is Zero

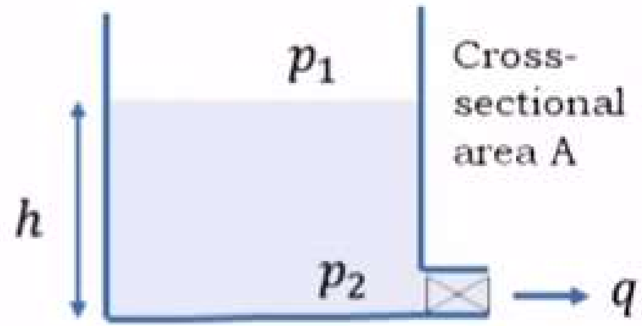
- Natural response
- Consider water flowing out of a tank.
- For this system relation for hydraulic resistance (R)can be written as
- $(p_1 - p_2) = Rq$ → **V= IR**
- $\rho g h = R\left(-\frac{dV}{dt}\right)$
- $\rho g h = RA\left(-\frac{dh}{dt}\right)$



Water flowing out of a tank naturally with no input

First order system- Natural response

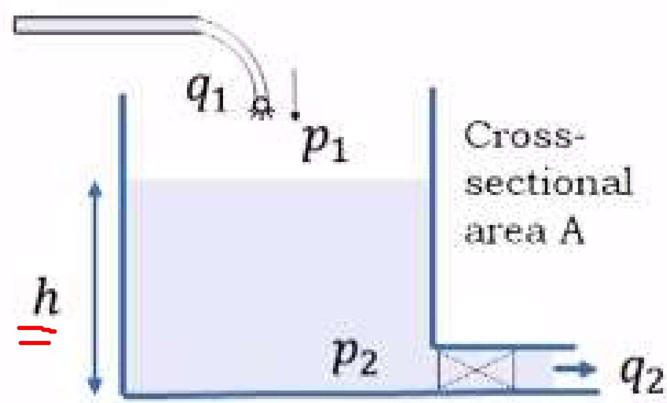
- $RA\frac{dh}{dt} + \rho g h = 0$
- This equation is a first order differential equation.
- Here there is no input to the system forcing the variable to change,
- Thus the response is natural response.



First order system- Force response

- **Initial input condition is non Zero.**
- **Tank acts as a capacitor.**

- In case there is a flow of water in tank
- For capacitor $q_1 - q_2 = C\frac{dp}{dt}$
- For valve
- $(p_1 - p_2) = Rq_2$
- $\rho g h = Rq_2 \Rightarrow q_2 = \frac{\rho g h}{R}$
- $\boxed{q_1 - \frac{\rho g h}{R} = C\frac{dp}{dt}}$



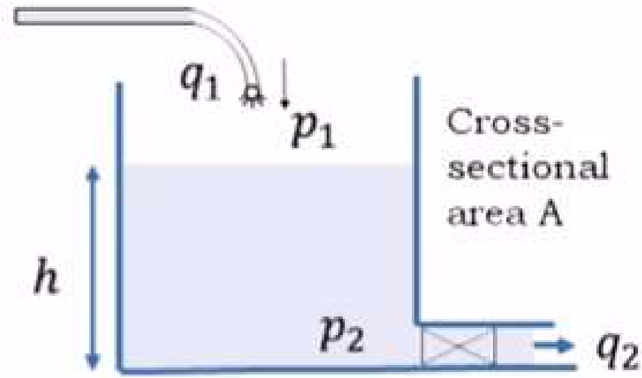
$i/p \longrightarrow o/p$

$q_1 - q_2 \propto p_1 - p_2$

$= C\frac{dp}{dt}$

First order system- Force response

- $q_1 - \frac{\rho g h}{R} = C\frac{dp}{dt}$
- $q_1 - \frac{\rho g h}{R} = \frac{A}{\rho g}\frac{d(\rho g h)}{dt}$
- $q_1 - \frac{\rho g h}{R} = A\frac{dh}{dt}$
- $A\frac{dh}{dt} + \frac{\rho g h}{R} = q_1$ I/P
- This system equation has a forcing function.

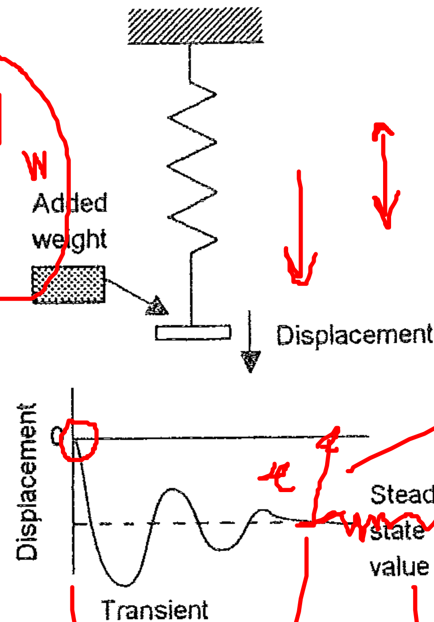


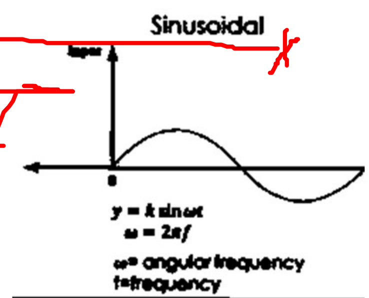
Water flowing out of a tank with forcing input

Dynamic response of systems

Transient and steady state response

Forced





Forms of Inputs

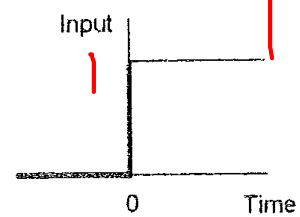


Fig. 10.4 Step input at time 0

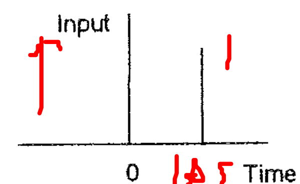


Fig. 10.5 Impulse

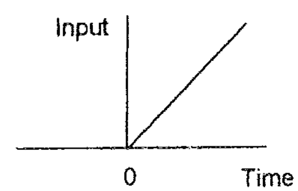


Fig. 10.6 Ramp input at time 0

Dynamic response of systems

Transient and steady-state responses

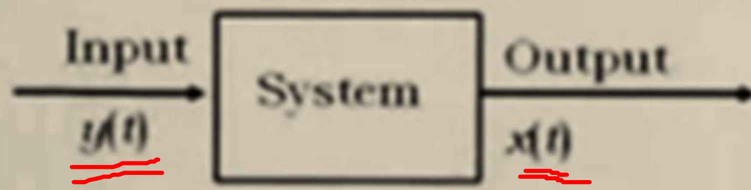
Total response of a control system or element of a system = Transient response (It occurs when there is change in input to system, it dies away quickly) natural + Steady state response (Response which remains after transients have died)

First order system- generalised equation

First Order System

Consider first order system

$$a_1\frac{dx}{dt}+a_0x=b_0y$$



Here a_1, a_0, b_0 are constant

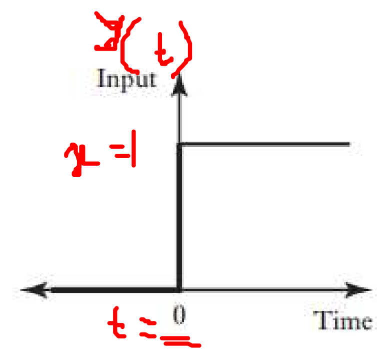
First — natural — with damping
second — forced — without —"—

Natural Response – First order system

Consider a first-order system with $y(t)$ as the input to the system and $x(t)$ the output and which has a forcing input $b_0 y$ and can be described by a differential equation of the form

$$a_1 \frac{dx}{dt} + a_0 x = b_0 y = 0$$

$t = 0$ (i/c)



where a_1, a_0 and b_0 are constants.

For natural response -Initial input condition is Zero
With no input supplied Hence $b_0 y=0$

$$a_1 \frac{dx}{dt} + a_0 x = 0 \quad ✓$$

$\Rightarrow$ sol^n & Response

$$\boxed{\frac{dx}{x} = -\frac{a_0}{a_1} dt}$$

o/p = ?

Integrating this between the initial value of $x = 1$ at $t = 0$, i.e. a unit step input, and x at t gives

$$\ln x = -\frac{a_0}{a_1} t$$

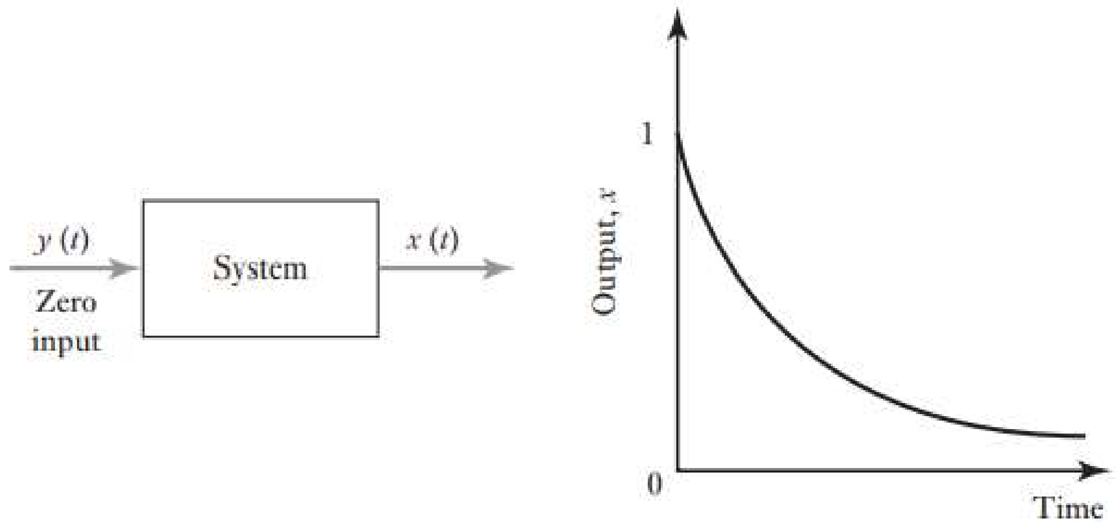
and so we have

$$\boxed{x = e^{-a_0 t/a_1}} \Rightarrow -\frac{a_0}{a_1} = \text{constant}$$

Natural Response – First order system

$$x = e^{-a_0 t / a_1}$$

Figure 19.4 Natural response of a first-order system.



At t=0 output = 1
As t increases, output reduces exponentially.

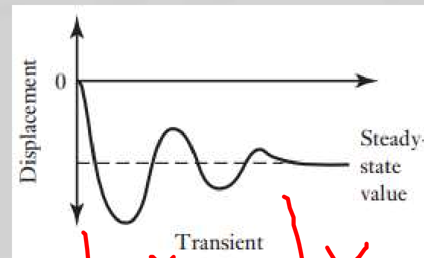
$t = \infty$
$x = 0$

Response with a forcing input- First order system

total response = natural response + forced response

x tr / Natr

forced



u v

- $a_1 \frac{dx}{dt} + a_0 x = b_0 y$ $\neq 0$
- Let solution be $x = u + v$
- u be the transient part of solution
- v be the steady state part of solution, so
- $a_1 \frac{d(u+v)}{dt} + a_0 (u + v) = b_0 y$
- $\left(a_1 \frac{du}{dt} + a_0 u\right) + \boxed{\left(a_1 \frac{dv}{dt} + a_0 v\right) = b_0 y}$
- Let $\left(a_1 \frac{du}{dt} + a_0 u\right) = 0$ and then $\left(a_1 \frac{dv}{dt} + a_0 v\right) = b_0 y$

Response with a forcing input- First order system

total response = natural response + forced response

- So first eq is differential eq with natural response while second is with forcing function.
- For natural response $u = Ae^{-\left(\frac{a_0}{a_1}\right)t}$
- For force response output will depend on input.
- So let input be a step at a time t=0, with step size of step being k

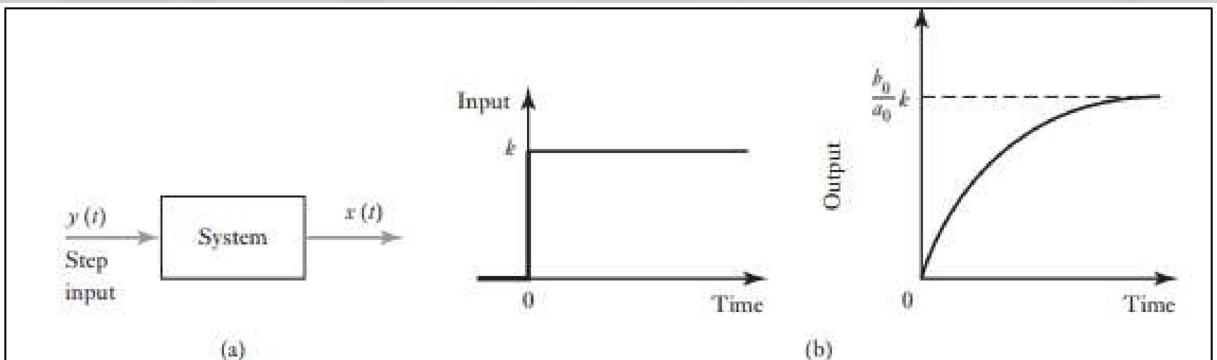
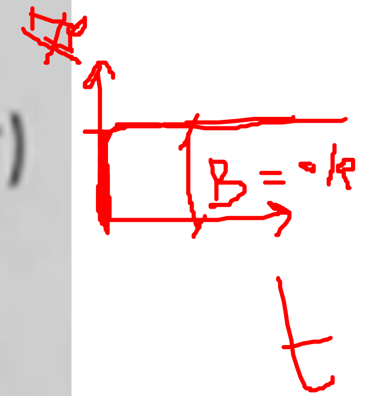


Figure 19.5 (a) Step input, (b) resulting output.

Response with a forcing input- First order system

$a_1 \frac{dB}{dt} + a_0 B = b_0 k$

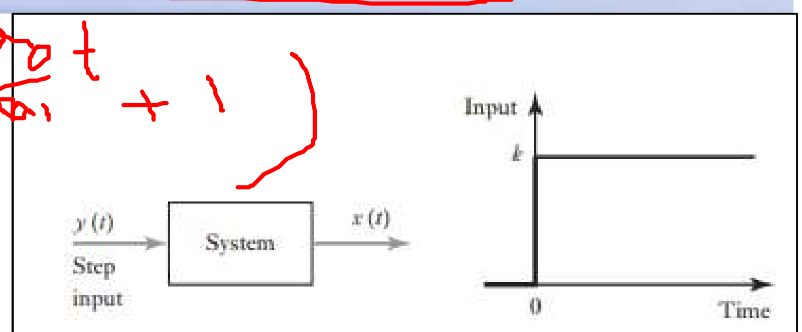
- So for $\left(a_1 \frac{dv}{dt} + a_0 v\right) = b_0 y$
- Let solution be $v = B$ (B=constant)
- $a_0 B = b_0 k$
- $\boxed{B = \frac{b_0}{a_0} k}$
- So solution is $\boxed{v = \frac{b_0}{a_0} k}$
- Thus complete solution is $x = u + v$
- $\boxed{x = Ae^{-\left(\frac{a_0}{a_1}\right)t} + \frac{b_0}{a_0} k}$



Response with a forcing input- First order system

- $x = Ae^{-(\frac{a_0}{a_1})t} + \frac{b_0}{a_0}k$
- A can be found using the initial condition, say at t=0, x=0
- So $\boxed{A = -\frac{b_0}{a_0}k,}$
- So solution $\boxed{x = -\frac{b_0}{a_0}ke^{-(\frac{a_0}{a_1})t} + \frac{b_0}{a_0}k}$

$\frac{b_0}{a_0}k\left(-e^{-\frac{a_0}{a_1}t} + 1\right)$



Response with a forcing input- First order system

- $x = \frac{b_0}{a_0} k \left(1 - e^{-\left(\frac{a_0}{a_1}\right)t}\right)$ $\quad t \longrightarrow \infty$
- So as t →∞, exponential tends to zero, thus steady state response is $x = \frac{b_0}{a_0} k$

o/p = i/p

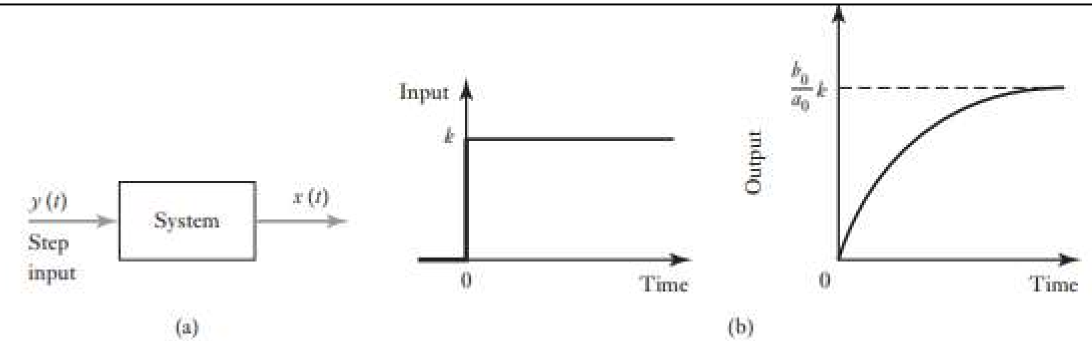


Figure 19.5 (a) Step input, (b) resulting output.

Response with a forcing input- First order system - Example

Example: A DC motor

$V_a - V_b$

- Relation for an armature controlled motor are
- $V_a - K_3\omega = L_a \frac{di_a}{dt} + i_a R_a$

V_b

- $I\frac{d\omega}{dt} = K_4 i_a - R_b\omega$
- We can substitute for i_a in second equation from first in order to get the relation between input V_a and output ω.

Response with a forcing input- First order system - Example

- $\frac{IR_a}{k_3 k_4}\frac{d\omega}{dt} + \omega = \frac{1}{k_3} V_a$
- Compare with
- $a_1 \frac{dx}{dt} + a_0 x = b_0 y$

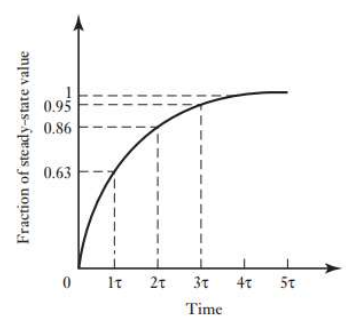
x corresponds to output in case of DC motor ω is output.
a1 = constant
a0 = 1
bo = 1/k3
y corresponds to input in case of DC motor Va is input armature voltage

Time constant

- $x = \frac{b_0}{a_0} k\left(1 - e^{-\left(\frac{a_0}{a_1}\right)t}\right)$ → transient
- x = (Steady state value) $\left(1 - e^{-\left(\frac{a_0}{a_1}\right)t}\right)$ → $k(1 - e^{-1})$
- When time $t = \frac{a_1}{a_0}$, output has risen 0.63 times the steady state value.
- This time is called the time constant $\tau = \frac{a_1}{a_0}$
- So the response of the foirst order system for a step input is x = (Steady state value) $\left(1 - e^{-\frac{t}{\tau}}\right)$

$x = 0.63\,(SS)$

Time t	Fraction of steady-state output
0	0
1τ	0.63
2τ	0.86
3τ	0.95
4τ	0.98
5τ	0.99
∞	1



Second order system

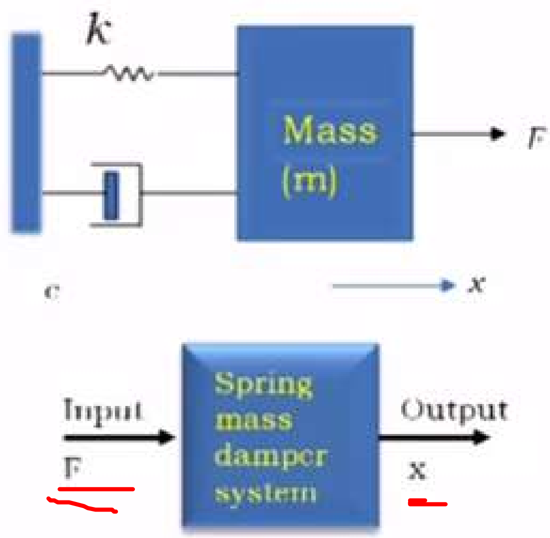
Introduction

- Second order system will have $\frac{d^2x}{dt^2}$ terms
- Many 2nd order system can consist of an inertia, a compliant and a damping terms.
- Examples include spring-mass-damper system, torsional system in mechanical systems and R-L-C circuit in electrical systems.

Second order system

Second Order System

- Relationship between the input force F and output of a displacement x is



Spring–dashpot–mass system

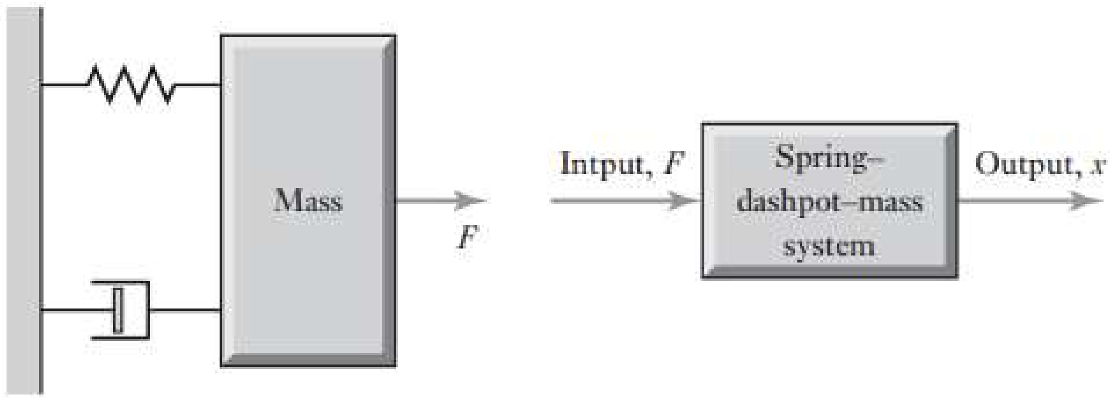
o/p   i/p

- $m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = F$

I + D + S   forcing fn

x corresponds to output
F corresponds to input

Second order system



Such a system was analysed in Section 17.2.2. The equation describing the relationship between the input of force F and the output of a displacement x is

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F}$$

=0

Natural / transient ⇒

forced

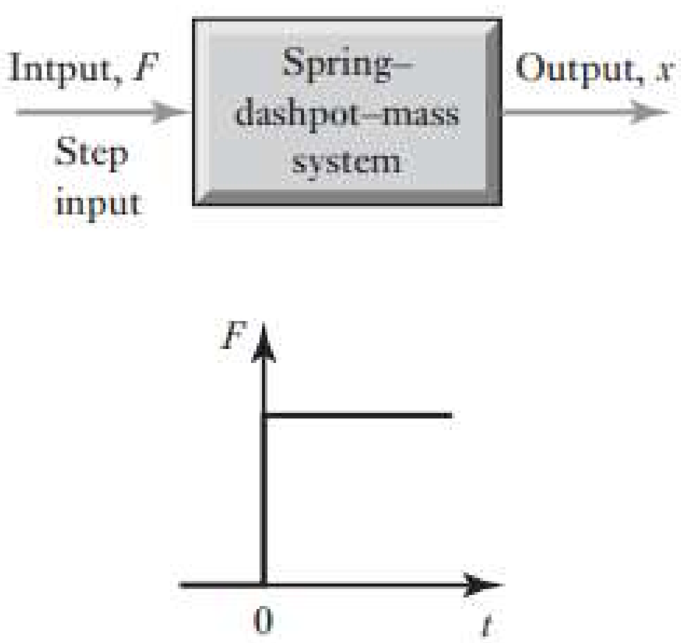
with damping

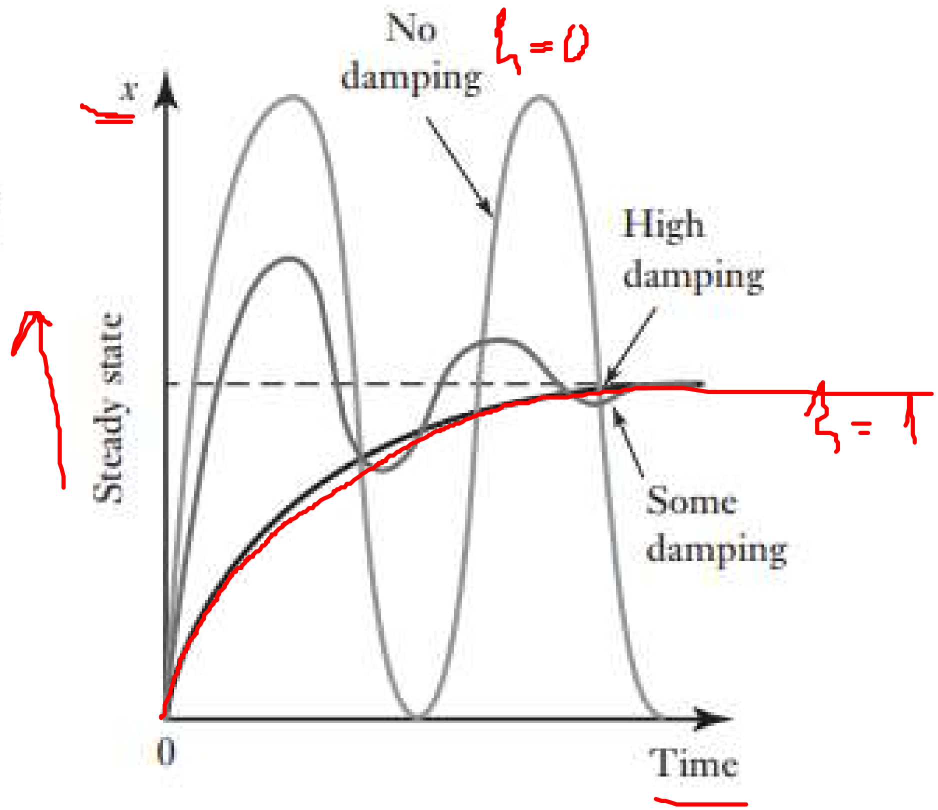
c = 0
F = 0

Second order system

- The variation of x with time depends on amount of damping present in the system.
- If force is applied as step input then
- If no damping is present then mass will freely oscillate.
- Damping causes oscillations to die away until steady displacement of mass is obtained.
- If damping is high there will be no oscillations which means displacement of mass will slowly increase with time and moves towards steady displacement position.

Second order system- effect of damping



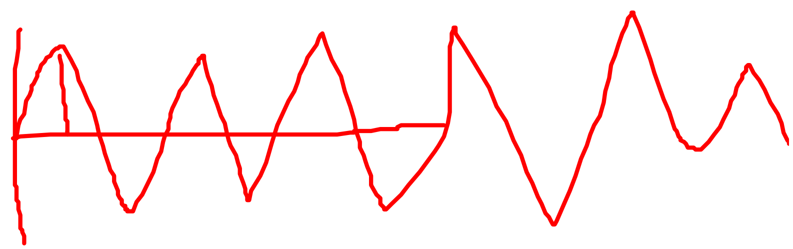


19.4.1 Natural response

Second order system- without damping

- $m\frac{d^2x}{dt^2} + \cancel{c\frac{dx}{dt}} + kx = 0$ I/P
- If c=0 (no damping present)
- We will have continuous oscillations
- So assume the solution to be
- $x = A \sin \omega_n t$ (A =amplitude of oscillations, ωn be angular frequency of free undamped oscillations)

O/P



sine wave time

Undamped vibrations

Second order system- without damping

$$x = A \sin \omega_n t$$

where x is the displacement at a time t, A the amplitude of the oscillation and ω_n the angular frequency of the free undamped oscillations. Differentiating this gives

$$\frac{dx}{dt} = \omega_n A \cos \omega_n t$$

Differentiating a second time gives

$$\frac{d^2x}{dt^2} = -\omega_n^2 A \sin \omega_n t = -\omega_n^2 x$$

This can be reorganised to give the differential equation

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

But for a mass m on a spring of stiffness k we have a restoring force of kx and thus

$$m\frac{d^2x}{dt^2} = -kx$$

This can be written as

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Undamped vibrations

Second order system- without damping

- $x = A \sin \omega_n t$ $\Longrightarrow$ sol^n
- $\frac{dx}{dt} = A\omega_n \cos \omega_n t$
- $\frac{d^2x}{dt^2} = -A\omega_n{}^2 \sin \omega_n t$
- $\frac{d^2x}{dt^2} = -\omega_n{}^2 x$
- $\frac{d^2x}{dt^2} + \omega_n{}^2 x = 0$ $\Longrightarrow$ $m\frac{d^2x}{dt^2} + kx = 0$

Undamped vibrations

Second order system- without damping

- $\frac{d^2x}{dt^2} + \omega_n{}^2 x = 0$ (1)
- In absence of c and F system equation is
- $m\frac{d^2x}{dt^2} + kx = 0$
- $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ (2)
- Comparing (1) and (2)
- $\boxed{\omega_n{}^2 = \frac{k}{m}}$
- $\boxed{x = A \sin\sqrt{\frac{k}{m}}t}$ is the solution of differential equation.

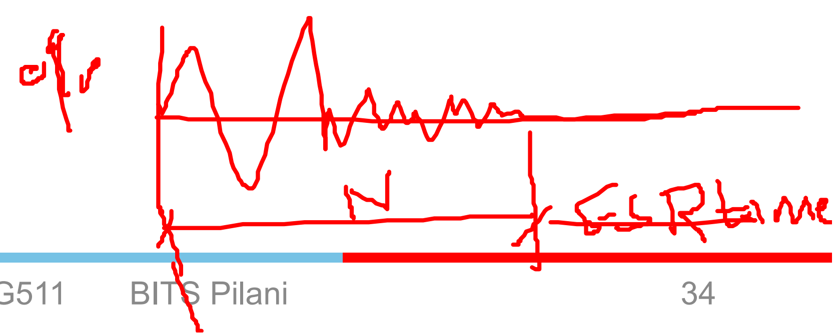
Undamped vibrations

Second order system- with damping forced input

- Motion of the mass is described by

o/p

- $\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F}$ Forcing f^n i/p
- Let the solution consists of transient response and force response
- $x = x_n + x_f$
- $m\frac{d^2(x_n+x_f)}{dt^2} + c\frac{d(x_n+x_f)}{dt} + k(x_n + x_f) = F \neq 0$



Second order system- with damping

- $m\frac{d^2(x_n+x_f)}{dt^2} + c\frac{d(x_n+x_f)}{dt} + k(x_n + x_f) = F$
- $\left(m\frac{d^2x_n}{dt^2} + c\frac{dx_n}{dt} + kx_n\right) + \left(m\frac{d^2x_f}{dt^2} + c\frac{dx_f}{dt} + kx_f\right) = F$
- If $m\frac{d^2x_n}{dt^2} + c\frac{dx_n}{dt} + kx_n = 0;$
- We must have
- $m\frac{d^2x_f}{dt^2} + c\frac{dx_f}{dt} + kx_f = F$

Second order system- with damping

- The equation is
- $m\frac{d^2x_n}{dt^2} + c\frac{dx_n}{dt} + kx_n = 0;$
- Let the solution be
- $x_n = Ae^{st}$ (A and s are constants)
- Substituting in above equation
- $mAs^2e^{st} + cAse^{st} + kAe^{st} = 0$
- $Ae^{st}(ms^2+cs+k) = 0$
- $Ae^{st} \neq 0$, as it will result in $x_n = 0$

$\neq 0$

$\frac{dx_n}{dt} = Ase^{st}$

$\frac{d^2x_n}{dt^2} = As^2e^{st}$

$\neq 0$

Second order system- with damping

- So $ms^2 + cs + k = 0$ (Auxiliary equation) $\Rightarrow$

$$ax^2 + bx + c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- So $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
- $s = -\dfrac{c}{2m} \pm \sqrt{\left(\dfrac{c}{2m}\right)^2 - \dfrac{k}{m}}$
- $s = -\dfrac{c}{2m} \pm \sqrt{\dfrac{k}{m}\left(\dfrac{c^2}{4mk}\right) - \dfrac{k}{m}}$
- But $\boxed{\omega_n^2 = \dfrac{k}{m}}$ and if we define $\zeta^2 = \dfrac{c^2}{4mk}$ then $\zeta = \dfrac{c}{2\sqrt{mk}}$

Second order system- with damping

- So s can be given as
- $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- If $\zeta > 1$ (two different real roots)
- $s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$
- $s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$
- So the general solution is
- $\boxed{x_n = Ae^{s_1 t} + Be^{s_2 t}}$
- System is said to be overdamped.

damping ratio > 1 hence overdamped

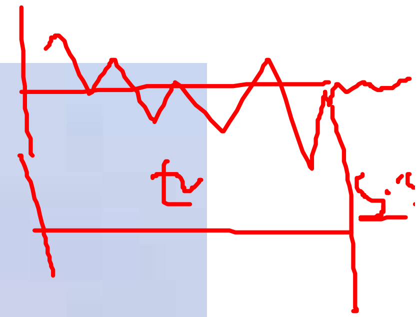
Second order system- with damping

- If $\zeta = 1$ (two equal roots)
- $s_1 = s_2 = -\omega_n$
- $x_n = (At + B)e^{-\omega_n t}$
- System is said to be critical damped
- If $\zeta < 1$ (two roots are complex)
- $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
- So let $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$
- $s = -\zeta\omega_n \pm j\omega_d$

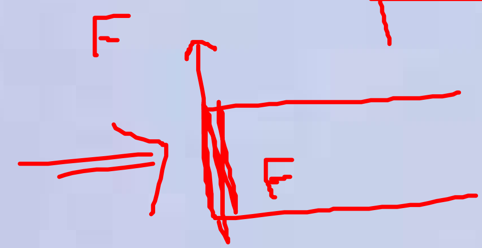
damping ratio = 1 hence Critical damped and < 1 underdamped

Second order system- with forcing function

Solution for forcing equation



- $\cancel{m\frac{d^2x_f}{dt^2}} + \cancel{c\frac{dx_f}{dt}} + kx_f = F \neq 0$



- Let for a step input of size F at time t=0.
- Let the solution be $x_f = A$ (A=constant)
- $kA = F$ so $A = F/k$
- So $x_f = F/k$ → sol^n
- So complete solution can be written as

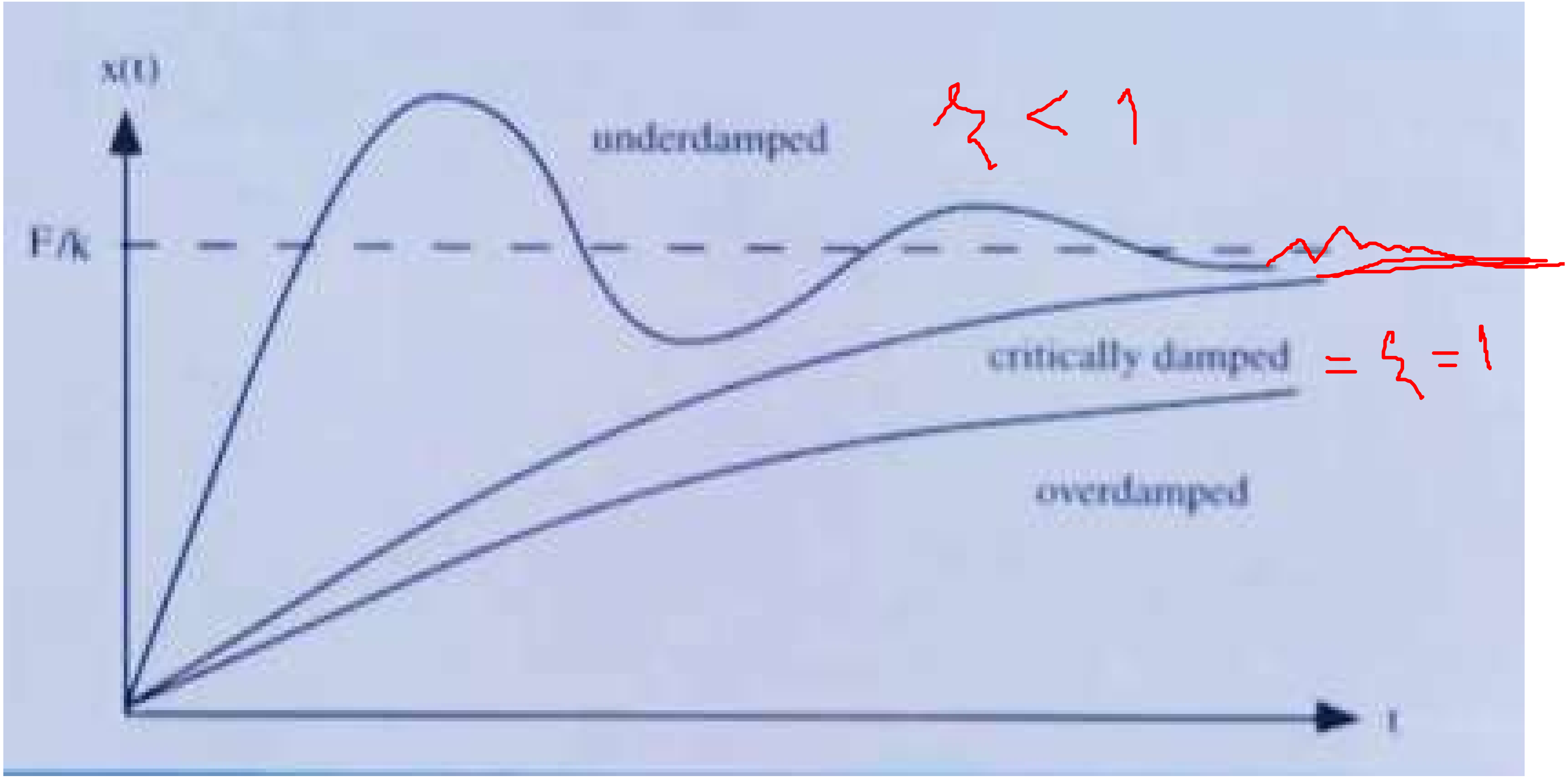
Second order system- with forcing function

- Thus for a 2nd order differential equation of the form
- $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$ $= F$
- For this system ω_n and ζ can be defined as
- $\omega_n^2 = \frac{a_0}{a_2}$ and $\zeta^2 = \frac{a_1^2}{4a_2 a_0}$ $= \frac{c^2}{4km}$

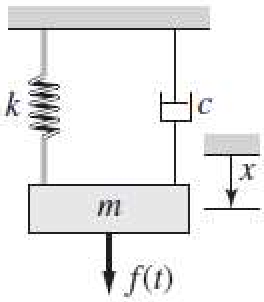
$\frac{k}{m}$

$$\boxed{x = x_h + x_f}$$

Second order system



Behavior of second order system



The characteristic equation is $s^2 + cs + 16 = 0$. The roots are

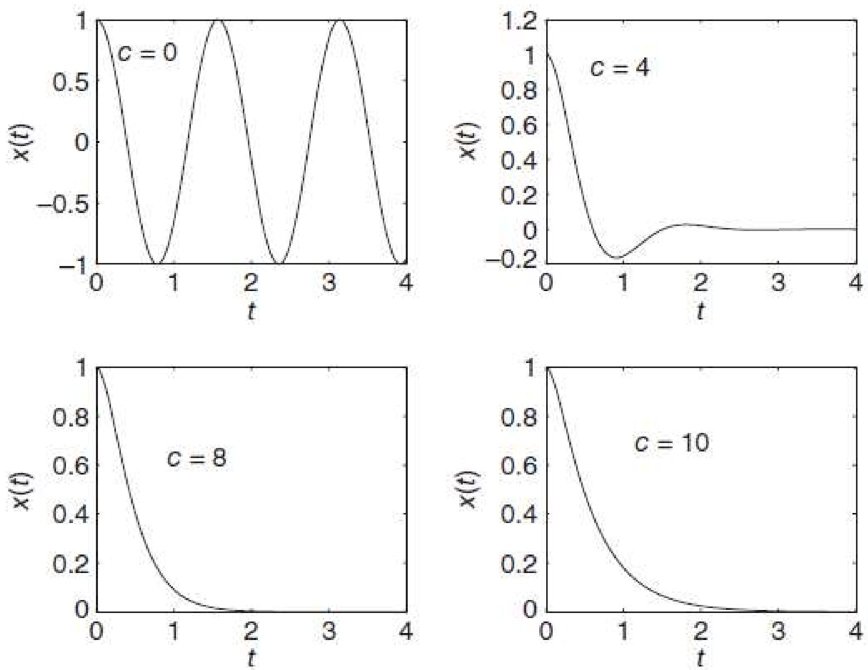
$$s = \frac{-c \pm \sqrt{c^2 - 64}}{2}$$

For $c = 0$ $\quad x(t) = \cos 4t$

For $c = 4$ $\quad x(t) = 1.155e^{-2t} \sin\left(\sqrt{12}t + 1.047\right)$

For $c = 8$ $\quad x(t) = (1 + 4t)e^{-4t}$

For $c = 10$ $\quad x(t) = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}$



System Response

- 1^{st} order
  - Natural response
  - Forced response
  - Total response
    - With damping
    - Without damping
- 2^{nd} order
  - Natural response
  - Forced response
  - Total response
    - With damping
    - Without damping

2^{nd} Order underdamped system with step input

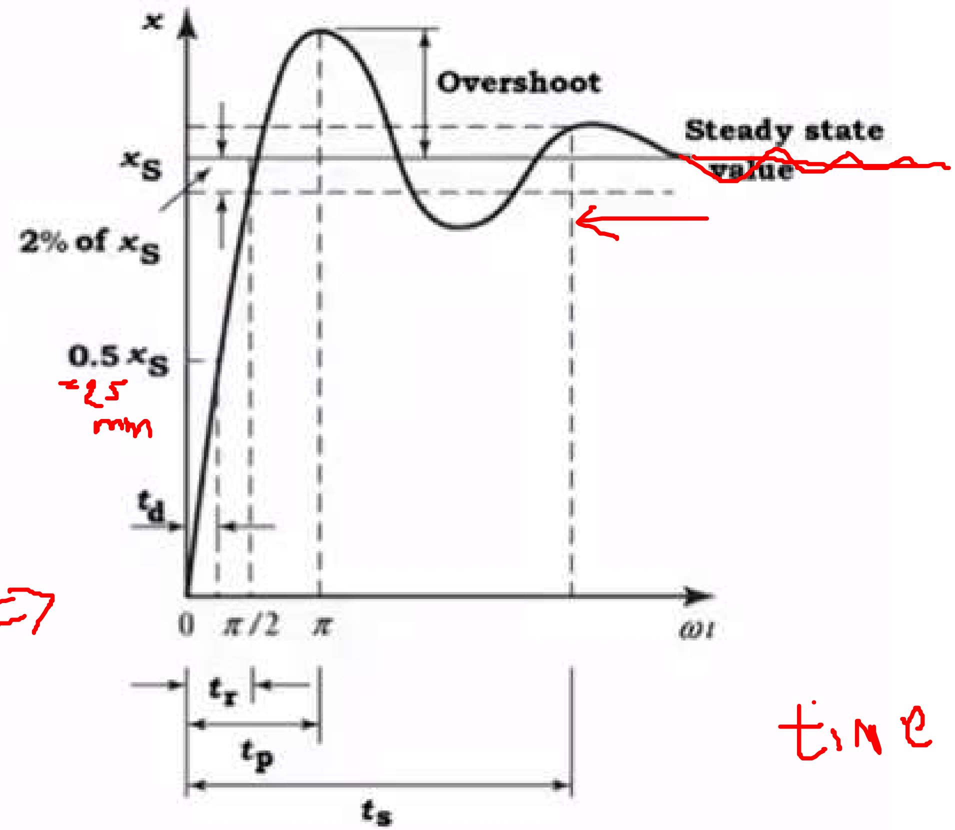
Introduction

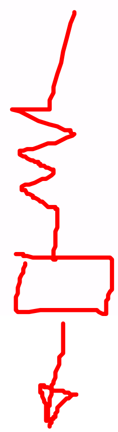
- There are some parameters by which we can specify the performance of an underdamped second order system to a step input.
- These parameters are
- Delay time (t_d)
- Rise time (t_r)
- Peak time (t_p)
- Maximum overshoot (M_p)
- Settling time (t_s)

2^{nd} Order underdamped system with step input

Delay time (t_d)

- It is the time required for the response to reach half the final value at the very first time





50 = δ mm

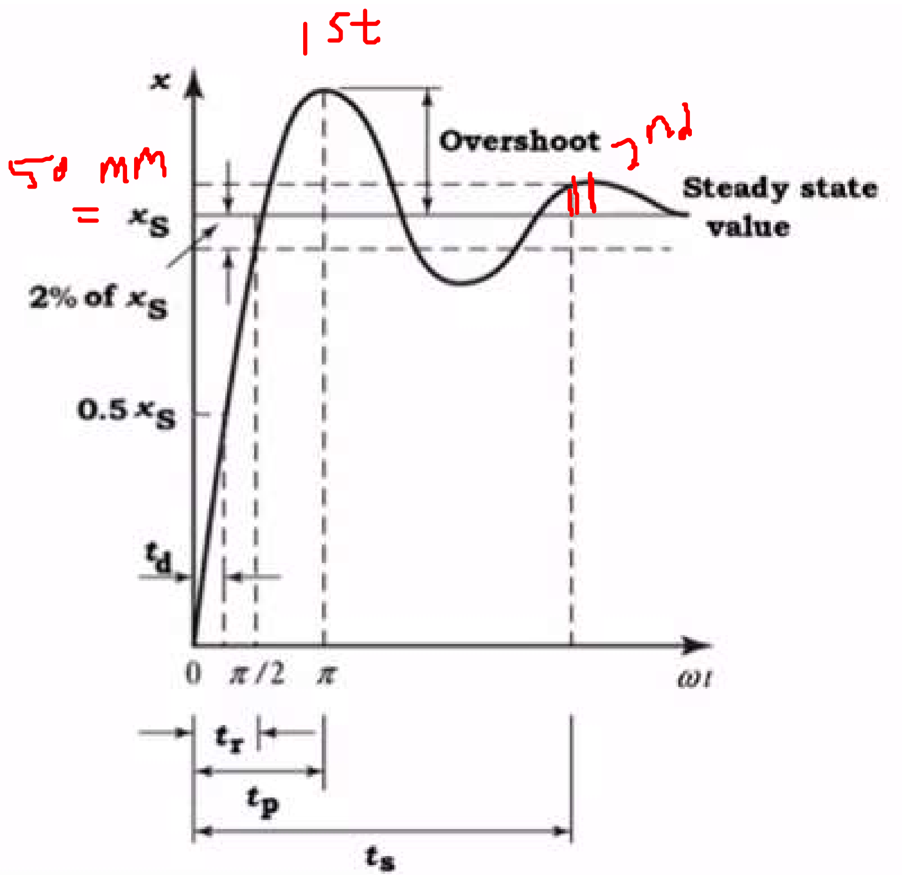
F = 100N

time

2^{nd} Order underdamped system with step input

Rise time (t_r)

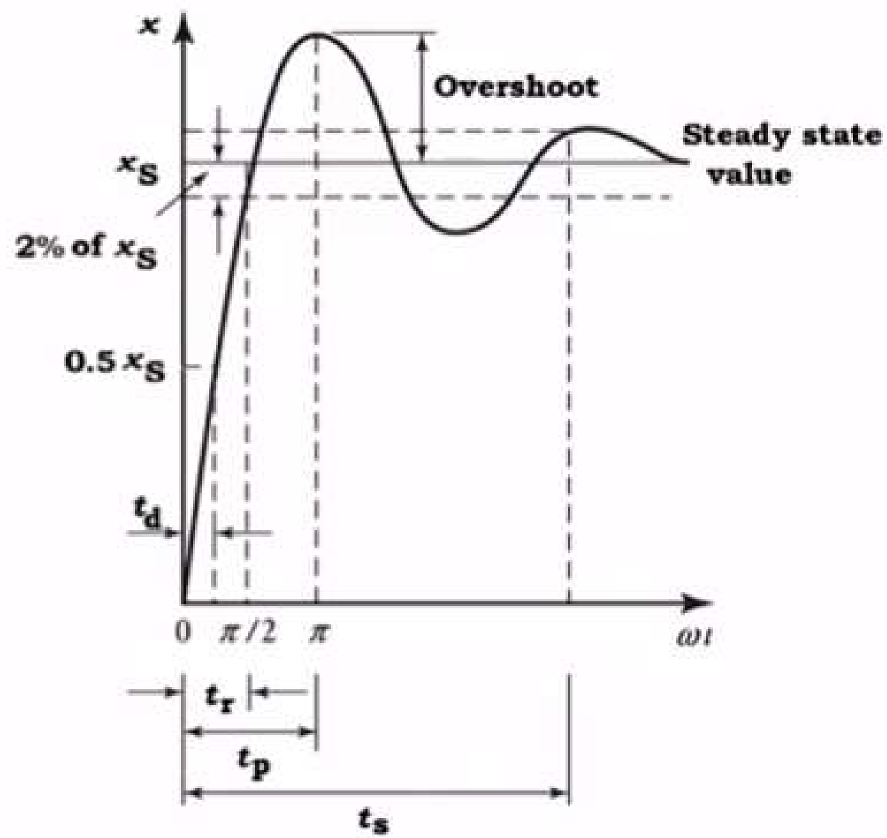
- It is time taken for the response x to rise from 0 to the steady state value x_s.
- It is a measure of how fast a system responds to the input.



2^{nd} Order underdamped system with step input

Peak time

- It is the time for a oscillatory response to complete a quarter of a cycle i.e., $\pi/2$.
- $\omega t_r = \pi/2$
- For overdamped system t_r is considered as the value for rise of response from some % of steady state value (say 10% to another specified percentage say 90%)

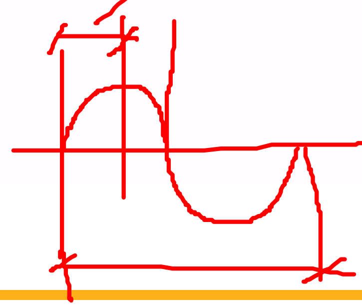


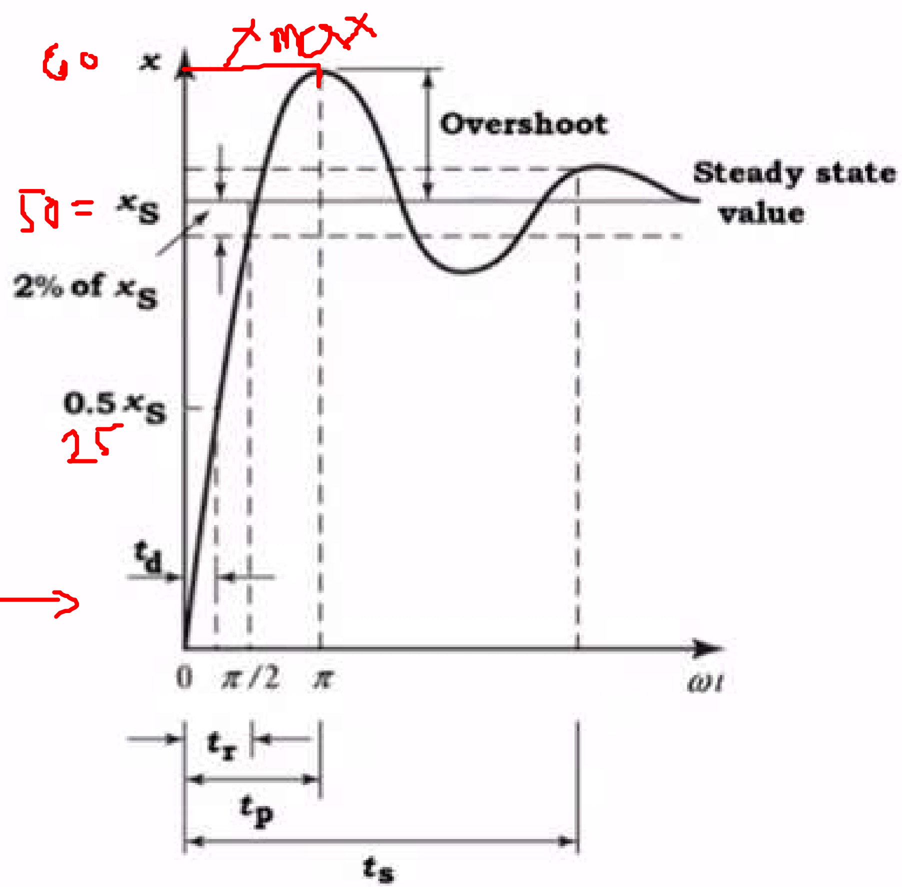
2^{nd} Order underdamped system with step input

Peak time (t_p)

- It is time taken for the response to rise from 0 to the first peak value.
- It is time of the oscillating response to complete one half cycle.
- $\boxed{\omega t_p = \pi}$

$= \frac{\pi}{\omega_n}$

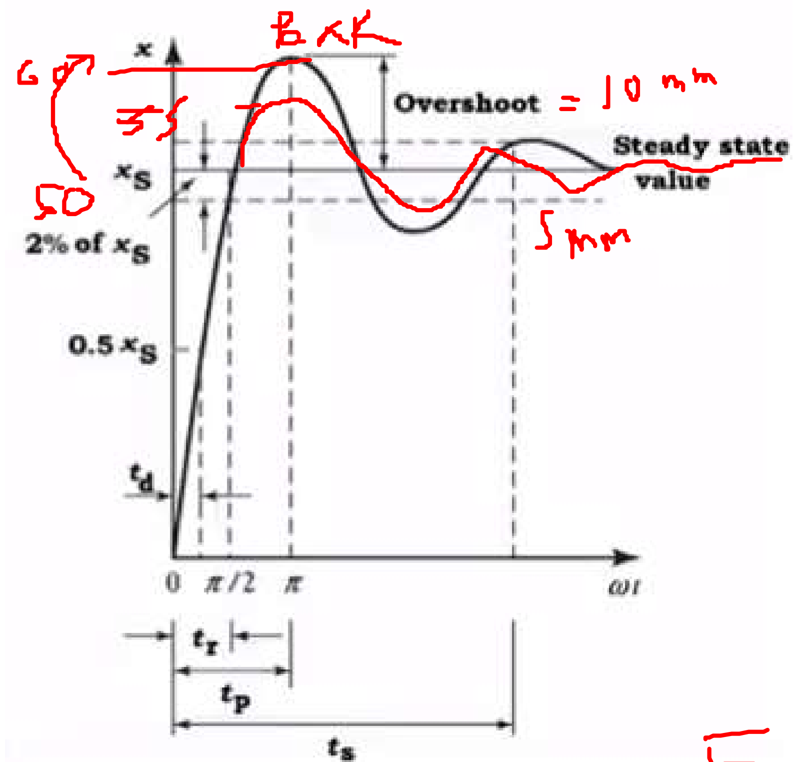




2^{nd} Order underdamped system with step input

Maximum overshoot (M_p)

- It is the maximum amount by which the response exceeds the steady state value.
- It is the amplitude of first peak
- Overshoot is written as % of steady state value.
- The maximum % overshoot directly indicates the relative stability of the system.



$\frac{5}{10}$

2nd Order underdamped system with step input

$\zeta < 1$

- $\% \text{ Maximum Overshoot} = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$
- Percentage peak overshoot

Damping ratio	Percentage overshoot
0.2	52.7
0.4	25.4
0.6	9.5
0.8	1.5

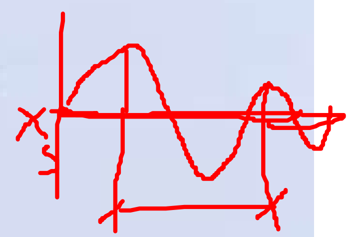
0.625 9 %

$\zeta = 1$ → 0 → SSV

2nd Order underdamped system with step input

Subsidence Ratio or Decrement

- This provides the information about how fast the oscillations decay
- This is defined as ratio of second overshoot to the first overshoot.
- The first overshoot occurs at $\omega t = \pi$
- The second overshoot occurs at $\omega t = 2\pi$



- First overshoot $= x_s e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$

2^{nd} Order underdamped system with step input

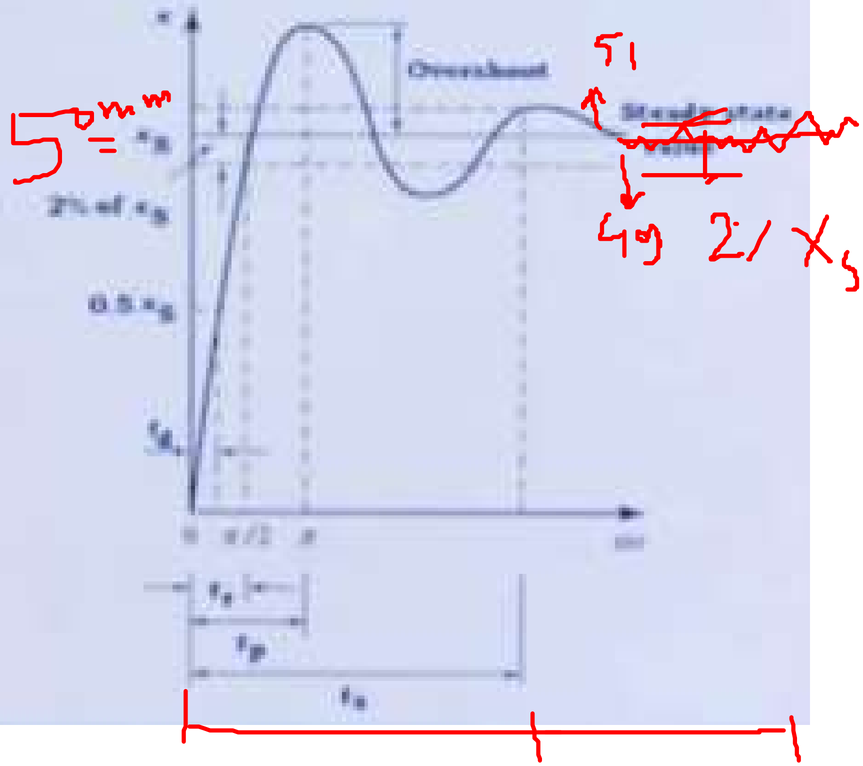
- Second overshoot $= x_s e^{-\left(\frac{\zeta(2\pi)}{\sqrt{1-\zeta^2}}\right)}$
- So decrement $= \dfrac{\text{2nd overshoot}}{\text{1st overshoot}} = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$

2^{nd} Order underdamped system with step input

$\zeta = 0$ to 1

Settling time (t_s)

- It measures the time taken for the oscillations to die away
- It is measured by time taken by response to fall and remain within some specified % of the steady state value.
- Say 2% i.e., the amplitude of oscillations should be less than 2% of x_s.



2nd Order underdamped system with step input

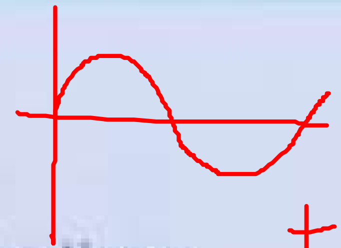
- In this case
- $(x - x_s) = 0.02$ of x_s

$51 - 50 = 2\%\ 5s$

- $0.02\, x_s = x_s\, e^{-\zeta\omega_n t_s}$
- $0.02 = e^{-\zeta\omega_n t_s}$
- $\ln(0.02) = -\zeta\omega_n t_s$
- $-3.9 = -\zeta\omega_n t_s$
- $\boxed{t_s \approx \frac{4}{\zeta\omega_n}}$

2^{nd} Order underdamped system with step input

Number of Oscillations



$t\omega = 2\pi$

$t = \frac{2\pi}{\omega}$

$t_s \rightarrow$ 2% X_s

- It is given by settling time/periodic time.
- If settling time t_s corresponds to 2% of the steady state value.
- No of oscillations $= \dfrac{\frac{4}{\zeta\omega_n}}{\frac{2\pi}{\omega}} = \dfrac{4}{\zeta\omega_n} \times \dfrac{\omega}{2\pi} = \dfrac{4}{\zeta\omega_n} \times \dfrac{\omega_n\sqrt{1-\zeta^2}}{2\pi}$
- $\boxed{\text{No of oscillations} = \dfrac{2\sqrt{1-\zeta^2}}{\pi\zeta}}$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

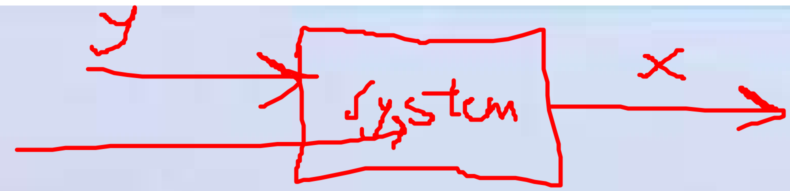
$t_s = \dfrac{4}{\zeta\omega_n}$

$t = 2\pi/\omega_d$

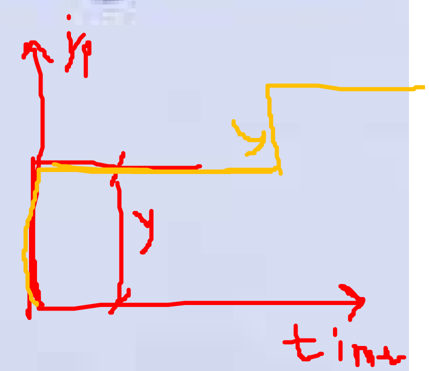
2nd Order underdamped system with step input

Example

behavior



- Let the system equation for step input y be given as
- $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 16x = 10y$ Find i/p
- The underdamped angular frequency
- The damping factor $\zeta < 1$
- The damped angular frequency
- The rise time
- The peak time
- % maximum overshoot
- 2% settling time



Find ω_n, ω_d, ζ | t_r, t_p, %MO | t_s

2^{nd} Order underdamped system with step input

- We know for a 2^{nd} order differential equation of the form
- $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$
- For this system ω_n and ζ can be defined as
- $\omega_n^2 = \frac{a_0}{a_2}$ and $\zeta^2 = \frac{a_1^2}{4a_2a_0}$

- $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 16x = 10y.$
- $a_2=1,\ a_1=5,\ a_0=16$ $\quad b_0 = 10$
- $\omega_n^2 = \frac{16}{1}$, so $\omega_n = 4\ rad/s$
- $\zeta^2 = \frac{a_1^2}{4a_2a_0} = \frac{5^2}{4\times16} = 0.39$
- $\zeta = 0.625$ $\quad < 1$

$\omega_n = \sqrt{\frac{k}{m}}$

ω_d

2^{nd} Order underdamped system with step input

- % Maximum Overshoot $= e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$
- $M_p = e^{-\left(\frac{0.625\pi}{\sqrt{1-(0.625)^2}}\right)} \times 100 = \underline{8\%}$
- 2% settling time
- $t_s \approx \frac{4}{\zeta\omega_n}$
- $t_s \approx \frac{4}{0.625 \times 4} = \underline{1.6 \text{ sec}}$

Thank you