Mechatronics
(Merged - DEZG516/DMZG511/ESZG511)

Lecture

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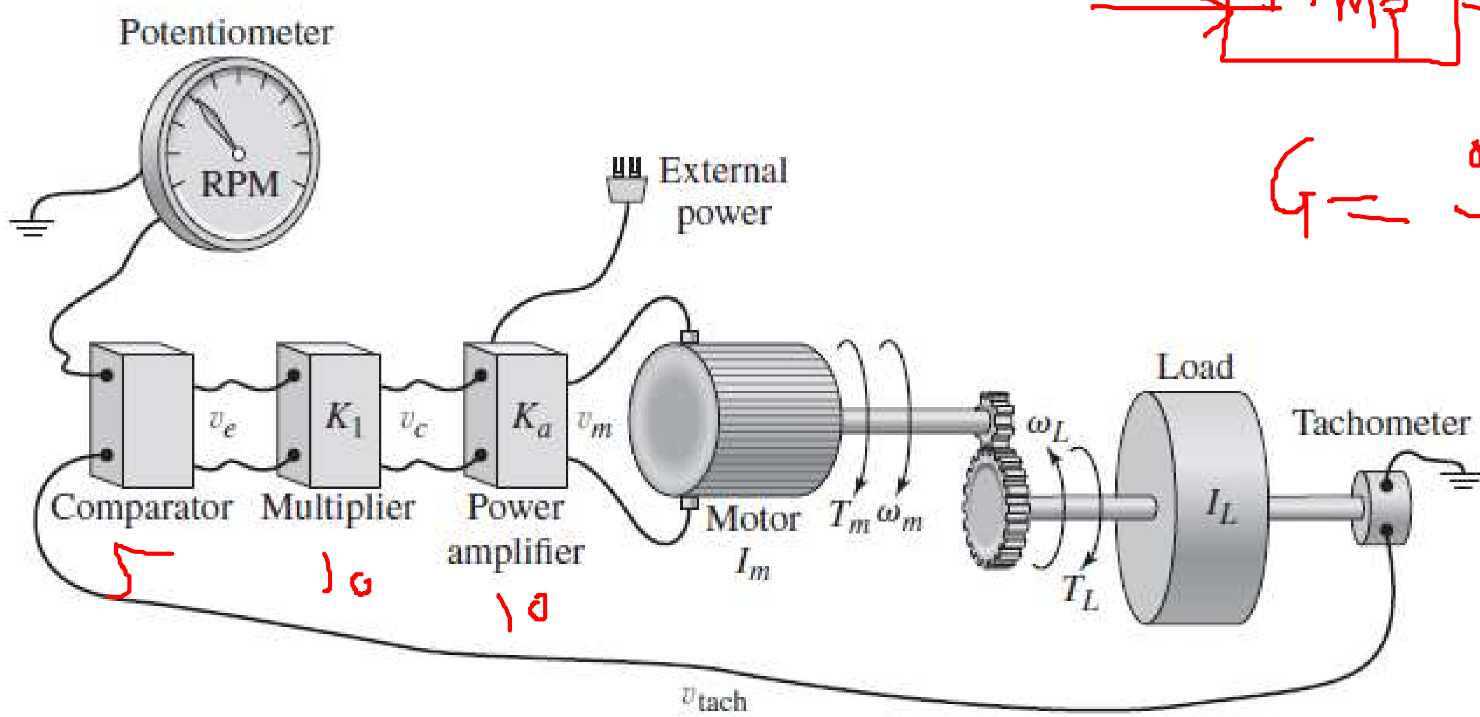
Session 11

Type	Content Ref.	Topic Title	Study/HW Resource Reference
Pre CH			
During CH	T1, R4	Transfer Function, Effect of pole location, Frequency response of systems	T1: Chapter 14,15, R4
Post CH			Chapter end problems

Control systems



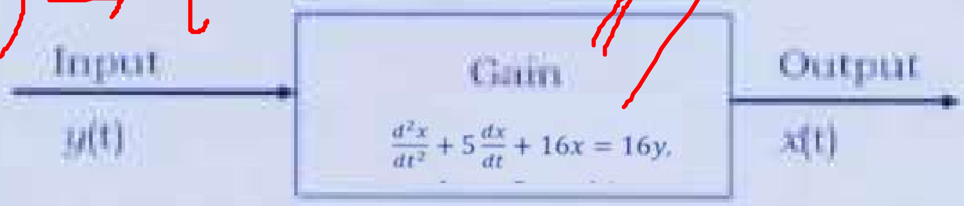
0-5V → Amp → 5V

$G = \frac{o/p}{i/p} = \frac{5}{0.5}$

$= 10$

Transfer functions

- For an amplifier we define gain as
- $\text{Gain} = \dfrac{\text{Output}}{\text{input}}$ $\Rightarrow \eta$



time domain
eq →
algebraic form

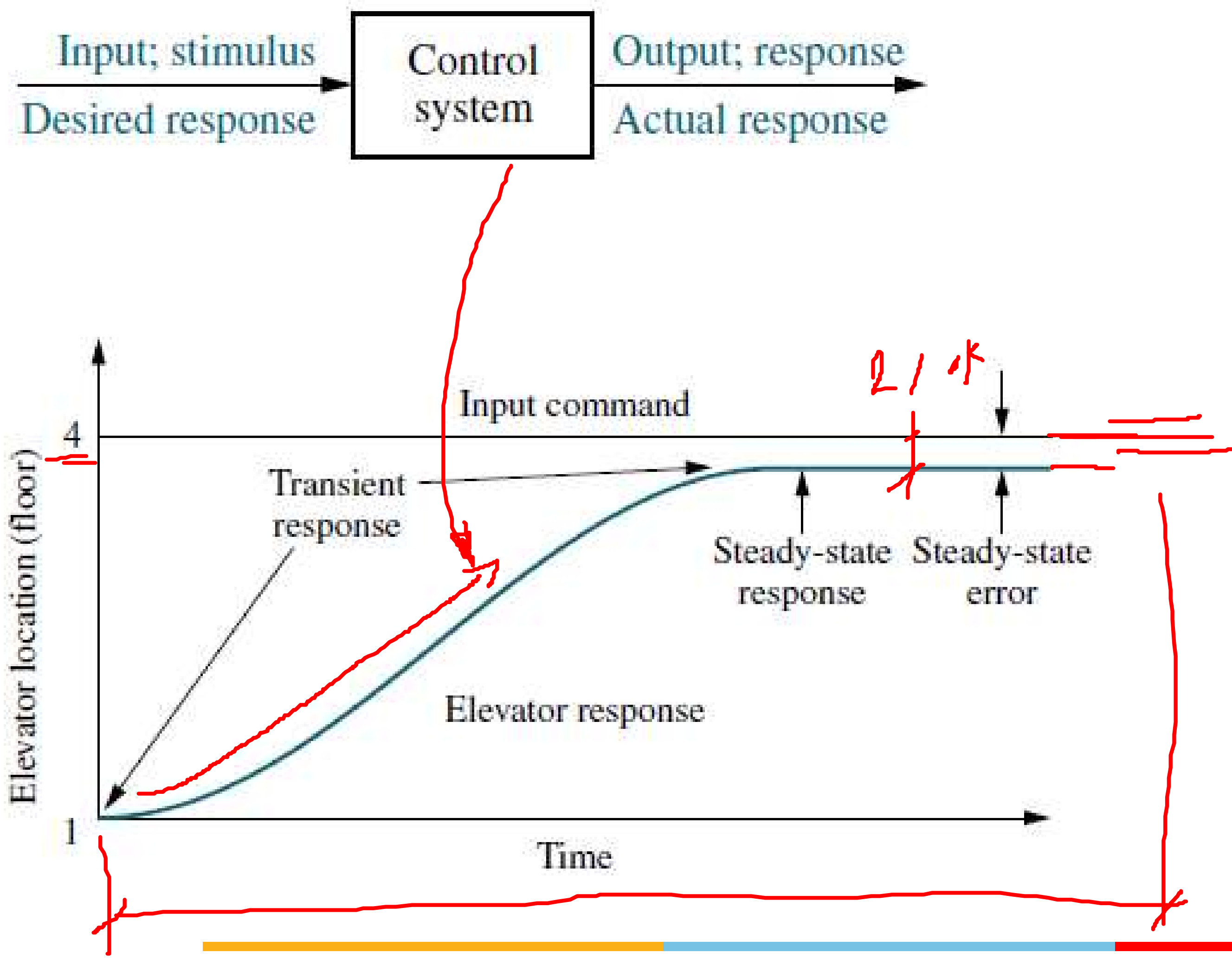
- The physical meaning of this equation is that if the gain of an amplifier is 5, then for an input of 4mV, the output will be 20 mV.
- For many systems the relationship between output and input is not in algebraic form as above.

$$\text{gain} = \frac{\text{output}}{\text{input}} = \frac{x(t)}{y(t)}$$

Transfer functions

- Rather it is in form of differential equation, due to which above formula can not be applied as differential equation describes how system behaves with time.
- This differential equation can be converted into an algebraic equation using Laplace transform.
- We say that conversion has been taken place from time domain to s domain.

Control systems

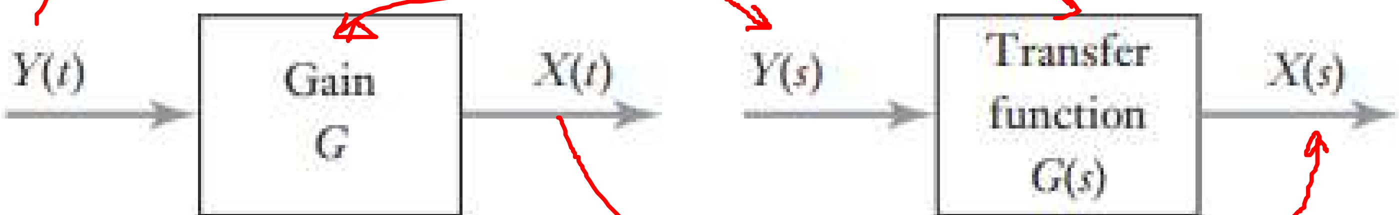


Transfer function

$$\text{transfer function} = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}$$

$$G(s) = \frac{X(s)}{Y(s)}$$

algebrac



time D ⟶ S domain

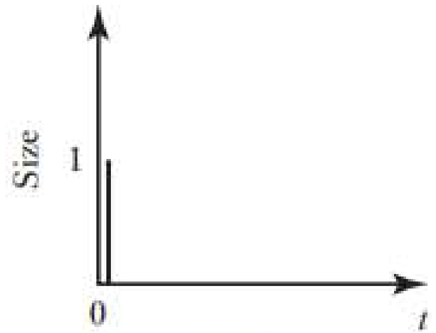
Laplace Transforms

Advantage of Laplace Transform Method

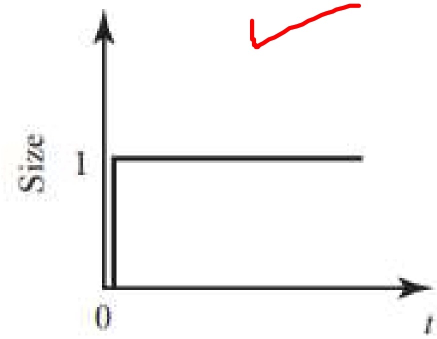
- It allows use of graphical technique for predicting the system performance without solving system differential equation.
- When we solve differential equation using this method transients and steady state components of the solution can be obtained simultaneously.

Laplace Transform

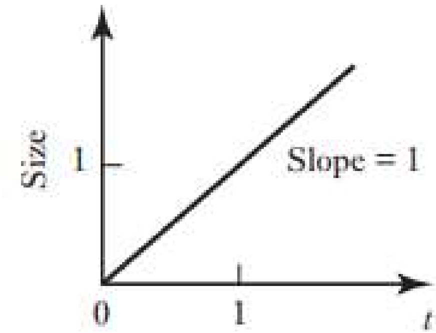
types of i/p



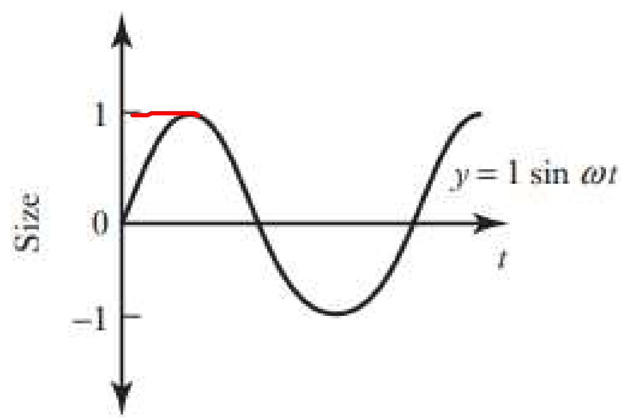
Unit impulse at zero time has the transform of 1



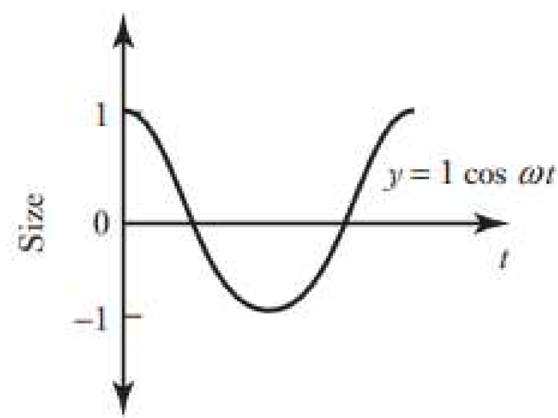
Unit step at zero time has the transform of $1/s$



Unit ramp at zero time has the transform of $1/s^2$



Unit amplitude sine wave has the transform of $\omega/(s^2 + \omega^2)$



Unit amplitude cosine wave has the transform of $s/(s^2 + \omega^2)$

Laplace transform of derivatives

$$\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} = sF(s) - f(0)$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right\} = s^2F(s) - sf(0) - \frac{\mathrm{d}}{\mathrm{d}t}f(0)$$

Laplace transform rules

$af(t)$ has the transform of $aF(s)$

$f(t) + g(t)$ has the transform $F(s) + G(s)$

$$L\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0)$$

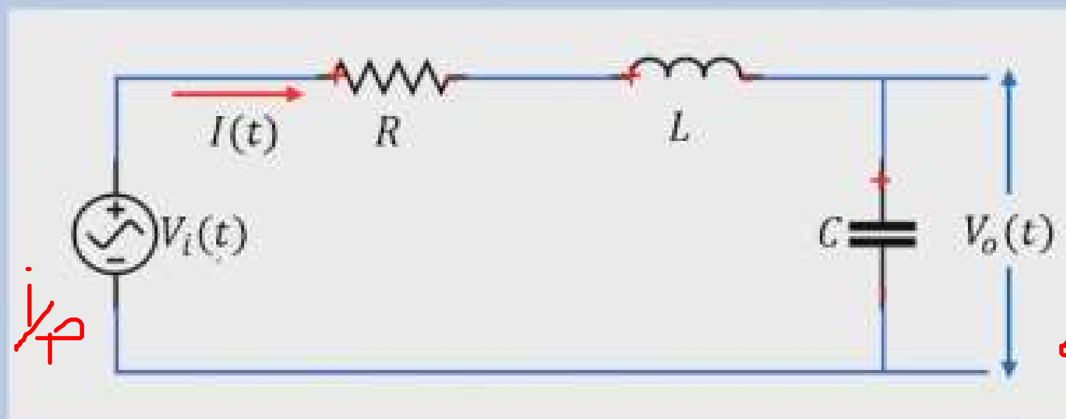
$$L\left\{\frac{d^2}{dt^2}f(t)\right\} = s^2F(s) - sf(0) - \frac{d}{dt}f(0)$$

$= 0$

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

Transfer functions

Transfer Function : Example 1



i/p

o/p

Model Equations:

$$\underline{V_i(t)} = RI(t) + L\frac{dI}{dt} + \frac{1}{C}\int Idt$$ i/p

$$\underline{V_0(t)} = \frac{1}{C}\int Idt$$ o/p

$$TF = \frac{L(o/p)}{L(i/p)}$$

Transfer functions

2. **Input and Output Variables:**
   - Input: $V_i(t)$
   - Output: $V_o(t)$
3. **Laplace Transform:** (assuming initial conditions to be zero)

$$V_i(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \quad \text{— I}$$

$$V_0(s) = \frac{1}{sC}I(s) \quad \text{— II}$$

4. **Transfer Function:**

$$G(s) = \frac{V_0(s)}{V_i(s)} = \frac{\frac{1}{sC}I(s)}{\left(R + sL + \frac{1}{Cs}\right)I(s)} = \frac{\frac{1}{sC}}{\left(R + sL + \frac{1}{Cs}\right)} = \boxed{\frac{1}{s^2LC + sRC + 1}}$$

Transfer functions $= \frac{L(o/p)}{L(i/p)} = \frac{x(s)}{F(s)}$

Example

- Consider mass spring damper system shown.
- Dynamics of system is described by
- $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$
- Taking Laplace transform
- $ms^2X(s) + csX(s) + kX(s) = F(s)$

i/p

x = o/p

$$\boxed{G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}}$$

2 1 0

$$X(s)\left[ms^2 + cs + k\right] = F(s)$$

Laplace Transforms

- Let f(t) be a function of time t such that f(t)=0, for t<0
- s be a complex variable
- $\mathcal{L}$ be an operational symbol indicating the Laplace transform of function
- Laplace transform of function f(t) is given by
- $\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} dt [f(t)] = \int_0^\infty f(t) e^{-st} dt$

Inverse Laplace Transforms

- The process of finding the function f(t) from the Laplace transform F(s) is called inverse Laplace transformation.
- Inverse Laplace transform is represented as $\mathcal{L}^{-1}$
- Inverse Laplace transform can be found from F(s) by inversion integral as
- $\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds, \text{ for } t > 0$
- Here c is the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of F(s)

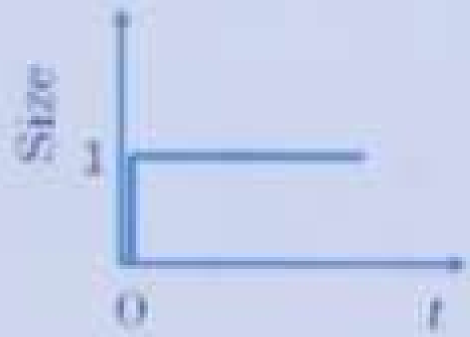
Laplace Transforms

Basic Laplace Transforms for Common Input

- Unit impulse at time t=0 has a transform of 1

- A unit step signal defined by

$$1(t) = 0, \text{for } t < 0$$
$$= 1, \text{for } t > 0$$

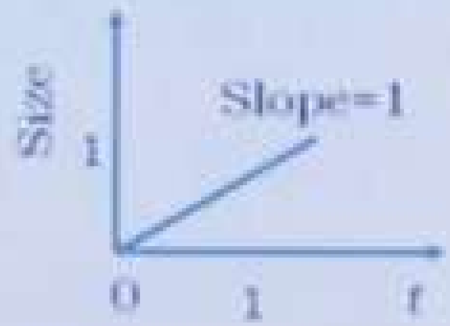


- $\mathcal{L}[1(t)] = \int_0^{\infty} e^{-st} dt = \frac{1}{s}$

Laplace Transforms

- Unit ramp function

$$f(t) = 0, \text{ for } t < 0,$$
$$= t, \text{ for } t \geq 0$$
$$\mathcal{L}[t] = \int_0^\infty t e^{-st} dt = \frac{1}{s^2}$$



- Unit amplitude sine wave signal

$$f(t) = 0, \text{ for } t < 0,$$
$$= \sin \omega t \text{ , for } t \geq 0$$
$$\mathcal{L}[\sin \omega t] = F(s) = \int_0^\infty \sin \omega t \, e^{-st} dt = \frac{\omega}{s^2 + \omega^2}$$

Laplace Transforms

- Unit amplitude cosine wave signal

$$f(t) = 0, \text{for } t < 0,$$
$$= \cos\omega t \text{ , for } t \geq 0$$

$$\mathcal{L}[\cos\omega t] = F(s) = \int_0^\infty \cos\omega t \, e^{-st} dt = \frac{s}{s^2 + \omega^2}$$

Laplace Transforms

Basic rules in working with Laplace transform

- $\mathcal{L}[Af(t)] = AF(s)$
- $\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$
- Laplace transform of 1st derivative of a function is
- $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$,however with transfer function all initial values are taken to be zero.
- $\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0) - \frac{df(0)}{dt}$
- Laplace transform of an integral of a function is
- $\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$

Inverse Laplace Transforms

	Laplace transform	Function of time
1	$\dfrac{1}{s+a}$	e^{-at}
2	$\dfrac{a}{s(s+a)}$	$(1-e^{-at})$
3	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
4	$\dfrac{s}{(s+a)^2}$	$(1-at)e^{-at}$
5	$\dfrac{a}{s^2(s+a)}$	$t-\dfrac{1-e^{-at}}{a}$

Transfer function – 1st order system

$$\boxed{a_1 \frac{dx}{dt} + a_0 x = b_0 y}$$

T domain $\Rightarrow$ S L – domain

where a_1, a_0, b_0 are constants, y is the input and x the output, both being functions of time. The Laplace transform of this, with all initial conditions zero, is

$$a_1 s X(s) + a_0 X(s) = b_0 Y(s) = \frac{X(s)(a_1 s + a_0)}{b_0} = Y(s)$$

y = i/p

x = o/p

and so we can write the transfer function $G(s)$ as

$$\boxed{G(s) = \frac{X(s)}{\cancel{Y(s)}} = \frac{b_0}{a_1 s + a_0} \; 1/s}$$

This can be rearranged to give

$$\boxed{G(s) = \frac{b_0/a_0}{(a_1/a_0)s + 1} = \frac{G}{\tau s + 1}}$$

τ

$$\frac{b_0/a_0}{\frac{a_1}{a_0} s + a_0/a_0}$$

$$TF = \frac{X(s)}{Y(s)}$$

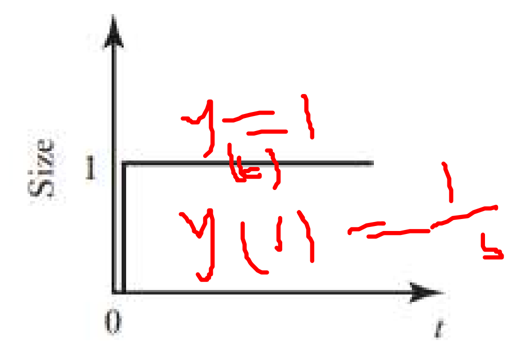
Transfer function – 1st order system

When a first-order system is subject to a unit step input, then $Y(s) = 1/s$ and the output transform $X(s)$ is

$$X(s) = G(s)Y(s) = \frac{G}{s(\tau s + 1)} = G\frac{(1/\tau)}{s(s + 1/\tau)}$$

Hence, since we have the transform in the form $a/s(s + a)$, using the inverse transformation listed as item 2 in the previous section gives

$$\boxed{x = G(1 - e^{-t/\tau})}$$



Unit step at zero time has the transform of $1/s$

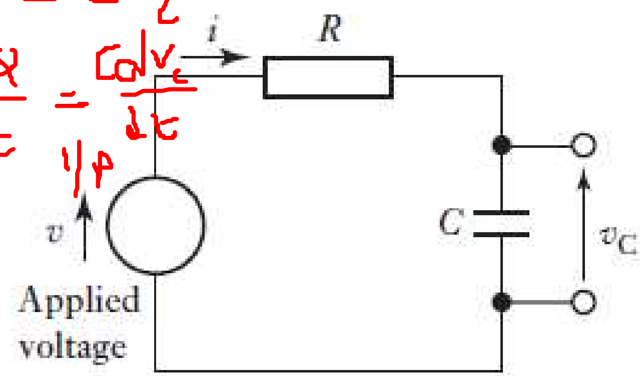
First order system example-1

$$v = RC\frac{dv_C}{dt} + v_C$$

$Q = CV_c$

$I = \frac{dQ}{dt} = C\frac{dV_c}{dt}$

$IR + V_c + \frac{1}{C}\int I$



v_o

L

$\frac{o/p}{i/p}$

The Laplace transform is:

$$V(s) = RCsV_C(s) + V_C(s)$$

$(RCs + 1)V_c(s)$

$\frac{V_c(s)}{V(s)}$

Thus $V(s)$ is the Laplace transform of the input voltage v and $V_C(s)$ is the Laplace transform of the output voltage v_C. Rearranging gives:

$$\boxed{\frac{V_C(s)}{V(s)} = \frac{1}{RCs + 1}}$$

The above equation thus describes the relationship between the input and output of the system when described as s functions.

Example 2 – Transfer Function

Problem

Obtain the transfer functions $X(s)/F(s)$ and $X(s)/G(s)$ for the following equation.

$$5\ddot{x} + 30\dot{x} + 40x = 6f(t) - 20g(t)$$

→ 2nd

Solution

Using the derivative property with zero initial conditions, we can immediately write the equation as

$$5s^2X(s) + 30sX(s) + 40X(s) = 6F(s) - 20G(s)$$

Solve for $X(s)$.

$$X(s) = \boxed{\frac{6}{5s^2 + 30s + 40}F(s)} - \boxed{\frac{20}{5s^2 + 30s + 40}G(s)}$$

When there is more than one input, the transfer function for a specific input can be obtained by temporarily setting the other inputs equal to zero (this is another aspect of the superposition property of linear equations). Thus, we obtain

$$\frac{X(s)}{F(s)} = \frac{6}{5s^2 + 30s + 40} \qquad \frac{X(s)}{G(s)} = -\frac{20}{5s^2 + 30s + 40}$$

First order system problem

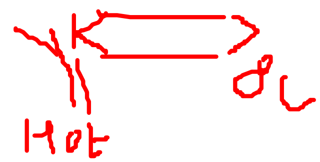
Example

A thermocouple which has a transfer function linking its voltage output V and temperature input of:

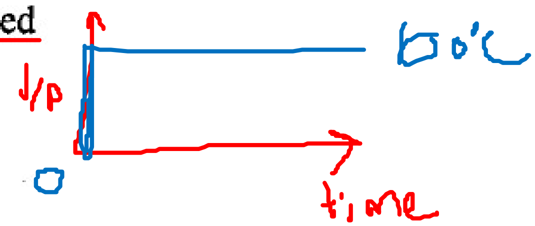
$i/p = T$

$o/p = \Delta V$

$$G(s) = \frac{30 \times 10^{-6}}{10s + 1} \text{ V/°C}$$



Determine the response of the system when it is suddenly immersed in a water bath at 100°C. $\Rightarrow$ Laplace $= \frac{100}{s}$



The output as an s function is:

$$V(s) = G(s) \times \text{input } (s)$$

$$G(s) = \frac{V(s)}{T(s)}$$

The sudden immersion of the thermometer gives a step input of size 100°C and so the input as an s function is $100/s$. Thus:

$$V(s) = \frac{30 \times 10^{-6}}{10s + 1} \times \frac{100}{s} = \frac{30 \times 10^{-4}}{10s(s + 0.1)} = 30 \times 10^{-4} \frac{0.1}{s(s + 0.1}$$

$\rightarrow a$

$$V(s) = G(s)\,T(s)$$

The fraction element is of the form $a/s(s + a)$ and so the output as a function of time is:

$$\boxed{V = 30 \times 10^{-4}\,(1 - e^{-0.1t})\text{ V}}$$

Second Order system

- For a 2nd order system, the relationship between input y and output x is given by differential equation of the form
- $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y,$
- Here a_2, a_1, a_0 and b_0 are constants.
- Taking the Laplace transform of equation with all initial conditions as zero, gives
- $a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s) = b_0 Y(s),$

Second Order system

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

Taking Laplace transform,

$$a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s) = b_0 Y(s)$$

Hence

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$

Second Order system

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

And

$$\omega_n^2 = \frac{a_0}{a_2} \text{ and } \zeta^2 = \frac{a_1^2}{4a_2a_0}$$

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = b_0\omega_n^2 y$$

where ω_n is the natural angular frequency with which the system oscillates and ζ is the damping ratio. The Laplace transform of this equation gives

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The above are the general forms taken by the transfer function for a second-order system.

Second Order system

When a second-order system is subject to a unit step input, i.e. $Y(s) = 1/s$, then the output transform is

$$X(s) = G(s)Y(s) = \frac{b_0\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n)}$$

This can be rearranged as

$$X(s) = \frac{b_0\omega_n^2}{s(s + p_1)(s + p_2)}$$

where p_1 and p_2 are the roots of the equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Hence, using the equation for the roots of a quadratic equation,

$$p = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

and so the two roots p_1 and p_2 are

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \qquad p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Second Order system

$\zeta > 1$ **Over damped**

$$x = \frac{b_0 \omega_n^2}{p_1 p_2}\left[1 - \frac{p_2}{p_2 - p_1} e^{-p_2 t} + \frac{p_1}{p_2 - p_1} e^{-p_1 t}\right]$$

$\zeta = 1$ **Critically damped**

$$x = b_0[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]$$

$\zeta < 1$ **Under damped**

With $\zeta < 1$, then

$$x = b_0\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{(1 - \zeta^2)}t + \phi)\right]$$

where $\cos \phi = \zeta$. This is an under-damped oscillation.

Example- Second Order system

What will be the state of damping of a system having the following transfer function and subject to a unit step input?

$$G(s) = \frac{1}{s^2 + 8s + 16}$$

The output $Y(s)$ from such a system is given by:

$$Y(s) = G(s)X(s)$$

For a unit step input $X(s) = 1/s$ and so the output is given by:

$$Y(s) = \frac{1}{s(s^2 + 8s + 16)} = \frac{1}{s(s+4)(s+4)}$$

The roots of $s^2 + 8s + 16$ are $p_1 = p_2 = -4$. Both the roots are real and the same and so the system is critically damped.

Thank you