

Loss Functions

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Three spaces for a prediction problem:

- Input space \mathcal{X} , e.g. email sender, title etc.
- Action space \mathcal{A} , e.g. score of SPAM
- Output space \mathcal{Y} , e.g. SPAM or NO SPAM

Loss Function

A **loss function** evaluates an action in the context of the outcome y .

$$\begin{aligned}\ell: \mathcal{A} \times \mathcal{Y} &\rightarrow \mathbb{R} \\ (a, y) &\mapsto \ell(a, y)\end{aligned}$$

Contents

Regression Loss Functions

Regression Problems

- Examples:
 - Predicting the stock price given history prices
 - Predicting medical cost of given age, sex, region, BMI etc.
 - Predicting the age of a person based on their photos
- Regression spaces:
 - Input space $\mathcal{X} = \mathbb{R}^d$
 - Action space $\mathcal{A} = \mathbb{R}$
 - Outcome space $\mathcal{Y} = \mathbb{R}$.
- Notation:
 - \hat{y} is the predicted value (the action)
 - y is the actual observed value (the outcome)

Loss Functions for Regression

- In general, loss function may take the form

$$(\hat{y}, y) \mapsto \ell(\hat{y}, y) \in \mathbb{R}$$

- Regression losses usually only depend on the **residual** $r = y - \hat{y}$.
 - what you have to add to your prediction to get the right answer
- Loss $\ell(\hat{y}, y)$ is called **distance-based** if it
 - ① only depends on the residual:

$$\ell(\hat{y}, y) = \psi(y - \hat{y}) \quad \text{for some } \psi: \mathbb{R} \rightarrow \mathbb{R}$$

- ② loss is zero when residual is 0:

$$\psi(0) = 0$$

Distance-Based Losses are Translation Invariant

- Distance-based losses are translation-invariant. That is,

$$\ell(\hat{y} + b, y + b) = \ell(\hat{y}, y) \quad \forall b \in \mathbb{R}.$$

- When might you not want to use a translation-invariant loss?

Distance-Based Losses are Translation Invariant

- Distance-based losses are translation-invariant. That is,

$$\ell(\hat{y} + b, y + b) = \ell(\hat{y}, y) \quad \forall b \in \mathbb{R}.$$

- When might you not want to use a translation-invariant loss?
- Sometimes relative error $\frac{\hat{y} - y}{y}$ is a more natural loss (but not translation-invariant)
- Often you can transform response y so it's translation-invariant (e.g. log transform)

Some Losses for Regression

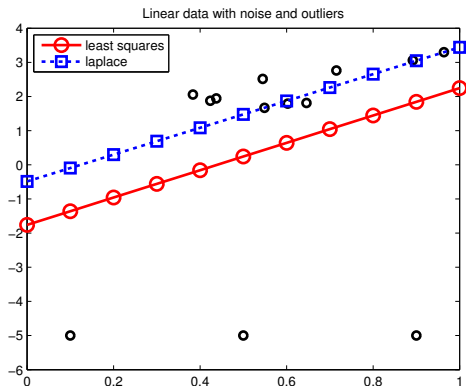
- **Residual:** $r = y - \hat{y}$
- **Square** or ℓ_2 Loss: $\ell(r) = r^2$
- **Absolute** or **Laplace** or ℓ_1 Loss: $\ell(r) = |r|$

y	\hat{y}	$ r = y - \hat{y} $	$r^2 = (y - \hat{y})^2$
1	0	1	1
5	0	5	25
10	0	10	100
50	0	50	2500

- Outliers typically have large residuals.
- Square loss much more affected by outliers than absolute loss.

Loss Function Robustness

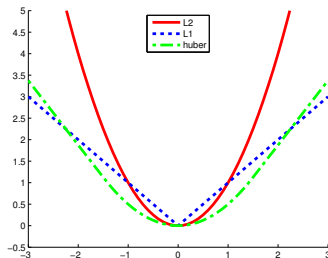
- **Robustness** refers to how affected a learning algorithm is by outliers.



KPM Figure 7.6

Some Losses for Regression

- **Square** or ℓ_2 Loss: $\ell(r) = r^2$ (*not robust*)
- **Absolute** or **Laplace** Loss: $\ell(r) = |r|$ (*not differentiable*)
 - gives **median regression**
- **Huber** Loss: Quadratic for $|r| \leq \delta$ and linear for $|r| > \delta$ (*robust and differentiable*)
 - Equal values and slopes at $r = \delta$



KPM Figure 7.6

Classification Loss Functions

The Classification Problem

- Examples:
 - Predict whether the image contains a cat
 - Predict whether the email is SPAM
- Classification spaces:
 - Input space \mathcal{R}^d
 - Action space $\mathcal{A} = \mathcal{R}$
 - Outcome space $\mathcal{Y} = \{-1, 1\}$
- Inference:

$$f(x) > 0 \implies \text{Predict } 1$$

$$f(x) < 0 \implies \text{Predict } -1$$

The Score Function

- Action space $\mathcal{A} = \mathcal{R}$ Output space $\mathcal{Y} = \{-1, 1\}$
- **Real-valued prediction function** $f : \mathcal{X} \rightarrow \mathcal{R}$

Definition

The value $f(x)$ is called the **score** for the input x .

- In this context, f may be called a **score function**.
- Intuitively, magnitude of the score represents the **confidence of our prediction**.

The Margin

Definition

The **margin** (or **functional margin**) for predicted score \hat{y} and true class $y \in \{-1, 1\}$ is $y\hat{y}$.

- The margin is often written as $yf(x)$, where $f(x)$ is our score function.
- The margin is a measure of how **correct** we are.
 - If y and \hat{y} are the same sign, prediction is **correct** and margin is **positive**.
 - If y and \hat{y} have different sign, prediction is **incorrect** and margin is **negative**.
- We want to **maximize the margin**
- Most classification losses depend only on the margin, which is called **margin-based loss**.

Classification Losses: 0 – 1 Loss

- **0-1 loss** for $f : \mathcal{X} \rightarrow \{-1, 1\}$:

$$\ell(f(x), y) = 1(f(x) \neq y)$$

- Empirical risk for 0 – 1 loss:

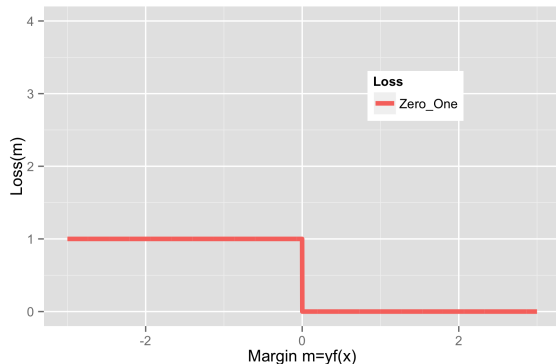
$$\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^n 1(y_i f(x_i) \leq 0)$$

Minimizing empirical 0 – 1 risk not computationally feasible

$\hat{R}_n(f)$ is non-convex, not differentiable (in fact, discontinuous!).
Optimization is **NP-Hard**.

Classification Losses

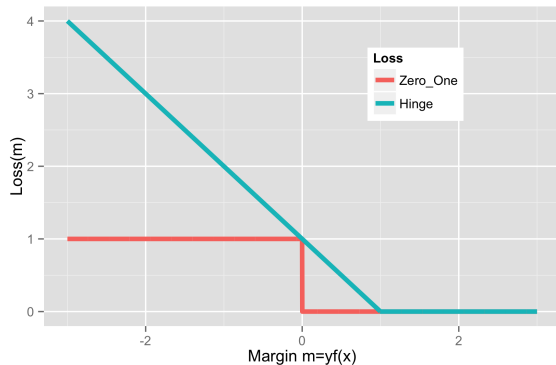
Zero-One loss: $\ell_{0-1} = 1(m \leq 0)$



- x-axis is **margin**: $m > 0 \iff$ correct classification

Classification Losses

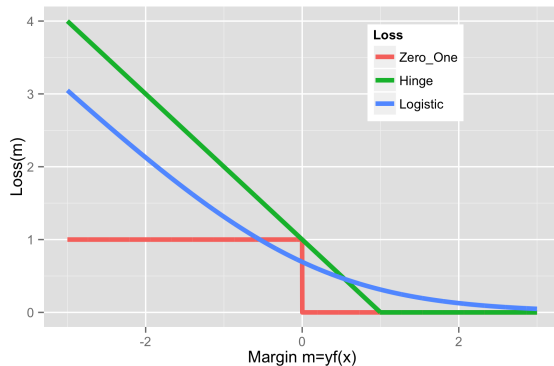
SVM/Hinge loss: $\ell_{\text{Hinge}} = \max(1 - m, 0)$



Hinge is a **convex, upper bound** on 0–1 loss. Not differentiable at $m = 1$.

Classification Losses

Logistic/Log loss: $\ell_{\text{Logistic}} = \log(1 + e^{-m})$



Logistic loss is differentiable. Logistic loss always wants more margin (loss never 0).

What About Square Loss for Classification?

- Action space $\mathcal{A} = \mathbb{R}$ Output space $\mathcal{Y} = \{-1, 1\}$

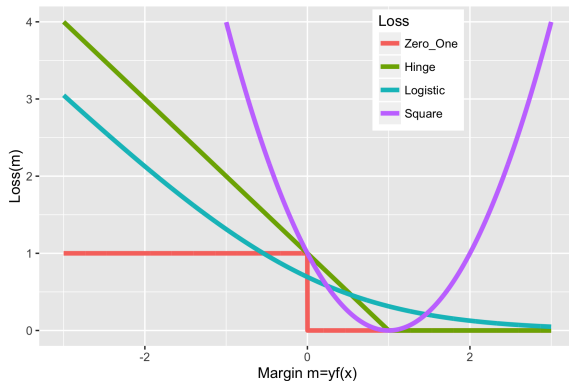
- Loss $\ell(f(x), y) = (f(x) - y)^2$.

- Turns out, can write this in terms of margin $m = f(x)y$:

$$\ell(f(x), y) = (f(x) - y)^2 = (1 - f(x)y)^2 = (1 - m)^2$$

- Prove using fact that $y^2 = 1$, since $y \in \{-1, 1\}$.

What About Square Loss for Classification?



Heavily penalizes outliers (e.g. mislabeled examples).

May have higher sample complexity (i.e. needs more data) than hinge & logistic¹.

¹Rosasco et al's "Are Loss Functions All the Same?" <http://web.mit.edu/lrosasco/www/publications/loss.pdf>