Loss Functions

He He

CDS, NYU

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Review

Three spaces for a prediction problem:

- Input space $\mathcal{X}$, e.g. email sender, title etc.
- Action space $\mathcal{A}$, e.g. score of SPAM
- Output space $\mathcal{Y}$, e.g. SPAM or NO SPAM

Loss Function

A **loss function** evaluates an action in the context of the outcome y.

$$\begin{aligned} \ell: \quad \mathcal{A} \times \mathcal{Y} &\to \quad \mathbf{R} \\ (a, y) &\mapsto \ell(a, y) \end{aligned}$$

Contents

Regression Loss Functions

Regression Problems

- Examples:
  - Predicting the stock price given history prices
  - Predicting medical cost of given age, sex, region, BMI etc.
  - Predicting the age of a person based on their photos
- Regression spaces:
  - Input space $\mathcal{X} = \mathrm{R}^d$
  - Action space $\mathcal{A} = \mathrm{R}$
  - Outcome space $\mathcal{Y} = \mathrm{R}$.
- Notation:
  - $\hat{y}$ is the predicted value (the action)
  - y is the actual observed value (the outcome)

Loss Functions for Regression

- In general, loss function may take the form

$$(\hat{y}, y) \mapsto \ell(\hat{y}, y) \in \mathbb{R}$$

- Regression losses usually only depend on the **residual** $r = y - \hat{y}$. $\hat{y} + r = y$
  - what you have to add to your prediction to get the right answer

- Loss $\ell(\hat{y}, y)$ is called **distance-based** if it
  1. only depends on the residual:

$$\ell(\hat{y}, y) = \psi(\overbrace{y - \hat{y}}^{= r}) \quad \text{for some } \psi : \mathbb{R} \to \mathbb{R}$$

  2. loss is zero when residual is 0:

$$\psi(0) = 0$$

Distance-Based Losses are Translation Invariant

- Distance-based losses are translation-invariant. That is,

$$\ell(\hat{y}+b, y+b) = \ell(\hat{y}, y) \qquad \forall b \in \mathrm{R}.$$

- When might you not want to use a translation-invariant loss?

Distance-Based Losses are Translation Invariant

- Distance-based losses are translation-invariant. That is,

$$\ell(\hat{y}+b, y+b) = \ell(\hat{y}, y) \qquad \forall b \in \mathrm{R}.$$

- When might you not want to use a translation-invariant loss?
- Sometimes relative error $\frac{\hat{y}-y}{y}$ is a more natural loss (but not translation-invariant)
- Often you can transform response y so it's translation-invariant (e.g. log transform)

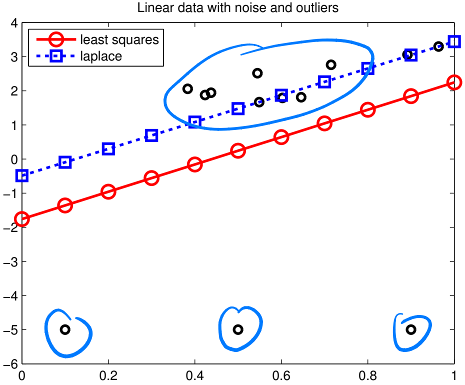
Some Losses for Regression

- **Residual:** $r = y - \hat{y}$ $\quad \ell(y, \hat{y}) = \psi(y - \hat{y}) = \psi(r)$
- **Square** or ℓ_2 Loss: $\ell(r) = r^2$
- **Absolute** or **Laplace** or ℓ_1 Loss: $\ell(r) = |r|$

y	$\hat{y}$	$\|r\| = \|y - \hat{y}\|$	$r^2 = (y - \hat{y})^2$
1	0	1	1
5	0	5	25
10	0	10	100
50	0	50	2500

- Outliers typically have large residuals.
- Square loss much more affected by outliers than absolute loss.

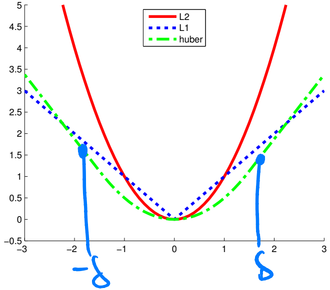
- **Robustness** refers to how affected a learning algorithm is by outliers.



KPM Figure 7.6

Some Losses for Regression

- **Square** or ℓ_2 Loss: $\ell(r) = r^2$ (*not robust*)
- **Absolute** or **Laplace** Loss: $\ell(r) = |r|$ (*not differentiable*)
  - gives **median regression**
- **Huber** Loss: Quadratic for $|r| \leqslant \delta$ and linear for $|r| > \delta$ (*robust and differentiable*)
  - Equal values and slopes at $r = \delta$



KPM Figure 7.6

Classification Loss Functions

The Classification Problem

- Examples:
  - Predict whether the image contains a cat
  - Predict whether the email is SPAM
- Classification spaces:
  - Input space R^d
  - Action space $\mathcal{A} = \mathrm{R}$
  - Outcome space $\mathcal{Y} = \{-1, 1\}$
- Inference:

$$
\begin{aligned}
f(x) > 0 &\implies \text{Predict } 1 \\
f(x) < 0 &\implies \text{Predict } -1
\end{aligned}
$$

The Score Function

- Action space $\mathcal{A} = \mathrm{R}$ Output space $\mathcal{Y} = \{-1, 1\}$
- **Real-valued prediction function** $f : \mathcal{X} \to \mathrm{R}$

Definition

The value $f(x)$ is called the **score** for the input x.

- In this context, f may be called a **score function**.
- Intuitively, magnitude of the score represents the **confidence of our prediction**.

The Margin

Definition

The **margin** (or **functional margin**) for predicted score $\hat{y}$ and true class $y \in \{-1, 1\}$ is $y\hat{y}$.

- The margin is often written as $yf(x)$, where $f(x)$ $= \hat{y}$ is our score function.
- The margin is a measure of how **correct** we are.
  - If y and $\hat{y}$ are the same sign, prediction is **correct** and margin is **positive**.
  - If y and $\hat{y}$ have different sign, prediction is **incorrect** and margin is **negative**.
- We want to **maximize the margin**
- Most classification losses depend only on the margin, which is called **margin-based loss**.

Classification Losses: $0-1$ Loss

- **0-1 loss** for $f : \mathcal{X} \to \{-1, 1\}$:

indicator function

$$\ell(f(x), y) = \mathbb{1}(f(x) \neq y)$$

sign(f(x)) ≠ y

- Empirical risk for $0-1$ loss:

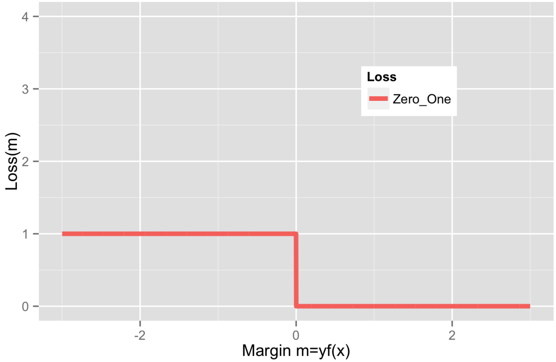
$$\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^{n} 1(y_i f(x_i) \leqslant 0)$$

Minimizing empirical $0-1$ risk not computationally feasible

$\hat{R}_n(f)$ is non-convex, not differentiable (in fact, discontinuous!).
Optimization is **NP-Hard**.

Classification Losses

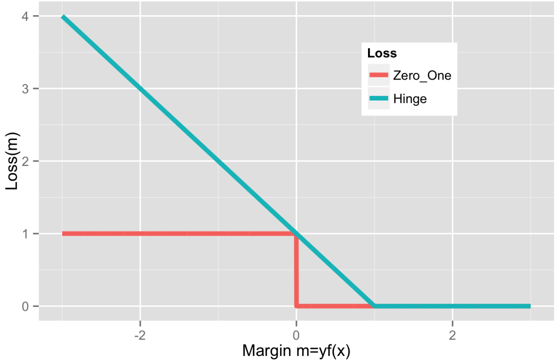
Zero-One loss: $\ell_{0\text{-}1} = 1(m \leqslant 0)$



- x-axis is **margin**: $m > 0 \iff$ correct classification

Classification Losses

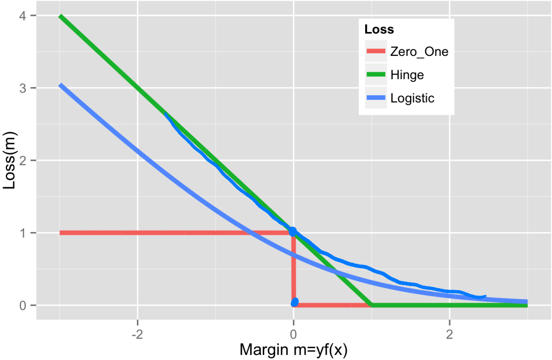
SVM/Hinge loss: $\ell_{\text{Hinge}} = \max(1-m, 0)$



Hinge is a **convex**, **upper bound** on $0-1$ loss. Not differentiable at $m=1$.

Classification Losses

Logistic/Log loss: $\ell_{\text{Logistic}} = \log\left(1 + e^{-m}\right) \times \frac{1}{\log 2}$



Logistic loss is differentiable. Logistic loss always wants more margin (loss never 0).

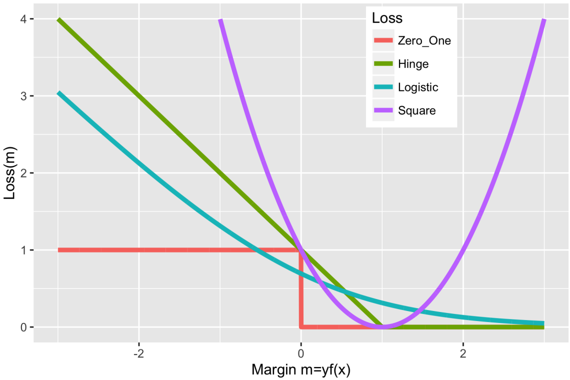
What About Square Loss for Classification?

- Action space $\mathcal{A} = R$ Output space $\mathcal{Y} = \{-1, 1\}$
- Loss $\ell(f(x), y) = (f(x) - y)^2$.
- Turns out, can write this in terms of margin $m = f(x)y$:

$$\ell(f(x), y) = (f(x) - y)^2 = (1 - f(x)y)^2 = (1 - m)^2$$

- Prove using fact that $y^2 = 1$, since $y \in \{-1, 1\}$.

What About Square Loss for Classification?



Heavily penalizes outliers (e.g. mislabeled examples).

May have higher sample complexity (i.e. needs more data) than hinge & logistic[1].

[1] Rosasco et al's "Are Loss Functions All the Same?" http://web.mit.edu/lrosasco/www/publications/loss.pdf