Math 440 – Computational Inverse Problems Fall 2017

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Exercise 4

1. A notoriously ill-posed problem is to recover a function from its Laplace transform. If $u : [0, \infty) \to \mathbb{R}$ is a given function, its Laplace transform is defined as

$$\mathscr{L}u(s) = \int_0^\infty e^{-st} u(t) dt, \quad s > 0.$$

Inverse Laplace transforms of elementary functions are tabulated, but if only numerical noisy values are given, the problem of recovering u is a challenge.

(a) Assume that the data arises from noisy observation of the Laplace transform of the function

$$u(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 3/2 - t/2, & \text{if } 1 \leq t < 3, \\ 0, & \text{if } t > 3. \end{cases}$$

To avoid inverse crimes, compute the Laplace transform of u analytically.

(b) Discretization: The larger s is, the less information about the function u the Laplace transform contains, because the exponential in the integral decreases increasingly rapidly. Therefore, it is desirable to have the Laplace transform more densely sampled for small values of s. We assume that the data consist of noisy observations of the Laplace transform,

$$b_j = \mathscr{L}u(s_j) + \varepsilon_j, \quad 1 \leq j \leq m,$$

where

$$\log_{10} s_j = -1 + \frac{2(j-1)}{m-1}, \quad 1 \leq j \leq m,$$

resulting in a denser sampling near $s = 0$. The noise ε_j is assumed to be normally distributed independent noise with variance σ^2,

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epsilon = sigma*randn(m,1);
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To discretize the unknown for solving the inverse problem, assume that we know that u is supported in the interval $[0, T]$ for some T, and therefore

$$\int_0^\infty e^{-st} u(t) dt = \int_0^T e^{-st} u(t) dt.$$

We use the Gauss-Legendre quadrature rule of order n to approximate the integral: If the nodes and the weights of the quadrature rule are (t_j, w_j), $1 \leq j \leq n$, where $0 < t_j < T$, we approximate

$$\int_0^T e^{-s_j t} u(t) dt \approx \sum_{k=1}^n w_k e^{-s_j t_k} u_k, \quad u_k = u(t_k),$$

or, using matrix notation,

$$b = \mathsf{A}u + \varepsilon,$$

where

$$\mathsf{A} \in \mathbb{R}^{m \times n}, \quad a_{jk} = w_k e^{-s_j t_k}, \quad 1 \leq j \leq m, \, 1 \leq k \leq n.$$

(i) Choose $m = 50$, and generate the data points s_j. Then compute the *noiseless exact data* by using the analytically computed Laplace transform of part (a). Call this data vector $b_0 \in \mathbb{R}^m$.

(ii) Download the program `GLquadrature.m` from Canvas. Generate the discretization points t_k, $1 \leq k \leq n$. Use $T = 5$, and $n = 60$.

(iii) Form the forward matrix $\mathsf{A} \in \mathbb{R}^{m \times n}$.

Compute and plot the singular values of your matrix (please use logarithmic scale).

Discretize your true function, computing the values $u(t_j)$ and check that when you multiply this vector with the matrix A, the values coincide with the analytically computed values, up to a presumably very small approximation error (Gauss-Legendre quadrature is $2n - 1$-order accurate!). How big is the relative error? Observe that if the error is large, either your analytic solution is wrong or the discretization has a bug.

Compute also the minimum norm solution with noiseless data to get an idea of the ill-posedness of this problem.

(c) Implement the Kaczmarz iterative solver. Apply the iterations to the inverse Laplace transform. First, use the noiseless data, running a couple of iterations and plotting the results, to get an idea of the convergence rate of the method. Plot the approximate solutions, including the true solution $u(t)$ in the plot to get an idea how close you are.

(d) Add noise to the exact data, and solve the inverse problem with the Kaczmarz iteration, stopping at the discrepancy as the Morozov discrepancy principle suggests. Instead of using $\delta = \|\varepsilon\|$ in the discrepancy criterion, which is kind of cheating since in reality we never know the noise vector, use the estimate

$$\delta = \sigma\sqrt{m}.$$

Try different noise levels, e.g., $\sigma = 10^{-6}, ..10^{-2}$. Show the results, indicating the number of iterations needed to reach the discrepancy level.